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HIGHER LEVEL

Mathematics

2012 edition

DEVELOPED SPECIFICALLY FOR THE

IB DIPLOMA

IBRAHIM WAZIR • TIM GARRY

PETER ASHBOURNE • PAUL BARCLAY • PETER FLYNN • KEVIN FREDERICK • MIKE WAKEFORD

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Dedications

I dedicate this work to the memory of my parents.

My special thanks go to my wife Lody for standing beside me throughout writing this book. She has been my inspiration and motivation for continuing to improve my knowledge and move my career forward. She is my rock, and I dedicate this book to her.

My appreciation and thanks also go to my friend and teacher Ram Mohapatra for his help with the Options section and to Peter Ashbourne for his help with the complex numbers chapter.

My thanks go to all the students and teachers who used the 1st edition and sent us their comments and corrections.

Ibrahim Wazir

My gratitude and deepest love go to my wonderful family – Val, Bethany, Neil and Rhona – for your support, patience and good humour. Some of the considerable time and energy that went into writing and revising two textbooks was borrowed from precious family time. Please forgive me for that. It is time with you, my family, which I most cherish in life.

I also wish to thank my good friend Marty Kehoe for his help and friendly advice; and to all the students that have passed through my classrooms since 1983 – especially students in the past four years who have provided constructive feedback on the first edition.

Tim Garry

Introduction

This textbook comprehensively covers all of the material in the **core syllabus** for the two-year **Mathematics Higher Level** course in the **International Baccalaureate (IB) Diploma Programme.** A new syllabus for each of the IB mathematics courses was issued in early 2012 for which students will first take exams in May 2014. This second edition is specifically designed for the 2014 Higher Level syllabus. Students will first be taught the course with this syllabus in the autumn of 2012. All of the material for the **option syllabus** is contained on a Pearson website and is password protected (see below for more information). Your teacher will specify which one of the four option topics you will study.

Content

As you will see when you look at the table of contents, the six **core syllabus topics** (see margin) are fully covered, though some are split over different chapters in order to group the information as logically as possible. The textbook has been designed so that the chapters proceed in a manner that supports effective learning of the necessary concepts and skills. Thus – although not absolutely necessary – it is recommended that you read and study the chapters in numerical order. It is particularly important that all of the content in the first chapter, ***Fundamentals***, is thoroughly reviewed and understood before studying any of the other chapters. It covers most of the *presumed knowledge* for the course, including essential terminology, notation and techniques that are essential for successful completion of the Mathematics HL course.

IB Mathematics Higher Level Core syllabus topics

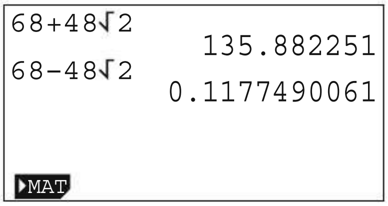
1 Algebra
2 Functions and equations
3 Circular functions and trigonometry
4 Vectors
5 Statistics and probability
6 Calculus

The previous syllabus for Mathematics HL contained a topic on matrices in the core syllabus. This topic is not in the 2014 syllabus, resulting in most of the content on matrices being removed. Matrices is an interesting and practical area of mathematics – so we decided to keep the chapter Matrix Algebra (Chapter 6) from the 1st edition. However, you could skip Chapter 6 and still cover the entire core syllabus.

Other than the final three chapters, each chapter has a set of **exercises** at the end of each section. Also, at the end of each of these chapters (except for Chapter 1) there is a set of **practice questions,** which are designed to give students practice with exam-like questions. Many of the end-of-chapter practice questions are taken from past IB exam papers. Near the end of the book, just before the index, you will find answers to all of the exercises and practice questions that appear in this textbook.

There are numerous **worked examples** throughout the textbook, showing you how to apply the concepts and skills you are studying.

This example appears in Section 5 of Chapter 3 Algebraic Functions, Equations and Inequalities.



Example 31 – Another equation in quadratic form

Find all solutions, expressed exactly, to the equation $w^{\frac{1}{2}} = 4w^{\frac{1}{4}} - 2$.

Solution

$w^{\frac{1}{2}} - 4w^{\frac{1}{4}} + 2 = 0$	Set the equation to zero.
$\left(w^{\frac{1}{4}}\right)^2 - 4\left(w^{\frac{1}{4}}\right) + 2 = 0$	Attempt to write in quadratic form: $at^2 + bt + c = 0$
$t^2 - 4t + 2 = 0$	Make appropriate substitution; in this case, let $w^{\frac{1}{4}} = t$.
$t = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2}$	Trinomial does not factorize; apply quadratic formula.
$t = \dfrac{4 \pm \sqrt{8}}{2} = \dfrac{4 \pm 2\sqrt{2}}{2}$	
$t = 2 \pm \sqrt{2}$	
$w^{\frac{1}{4}} = 2 \pm \sqrt{2}$	Substituting $w^{\frac{1}{4}}$ back in for t; raise both sides to 4th power.

$w = (2 + \sqrt{2})^4$ or $w = (2 - \sqrt{2})^4$

$w = \left((2 + \sqrt{2})^2\right)^2$ or $w = \left((2 - \sqrt{2})^2\right)^2$

$w = (6 + 4\sqrt{2})^2$ or $w = (6 - 4\sqrt{2})^2$

$w = 68 + 48\sqrt{2} \approx 135.882$ or $w = 68 - 48\sqrt{2} \approx 0.117\,749$ (approx. values found with GDC)

Chapter 19 contains three full-length Paper 1 and Paper 2 **sample exams**. Solution keys for these exams are available from the authors' website.

Finally, you will find a **Theory of Knowledge** chapter, which should stimulate you to think more deeply and critically about the nature of knowledge in mathematics and the relationship between mathematics and other subject areas.

Website support

At www.pearsonbacconline.com you will find a selection of free online learning resources supporting the material in this book. More comprehensive support for teachers who adopt the textbook will be available at the authors' website: www.wazir-garry-math.org, which will be regularly updated. You will be required to register before gaining access to materials on the authors' website.

The following will be available from the authors' website:

1 Further practice/mock exams and mark schemes
2 Additional exercises with solutions
3 Internal Assessment ('Mathematical Exploration') notes and guidance
4 Graphing calculators and other technology
5 Instructional activities for students
6 Chapter tests and quizzes.

Worked solutions

Worked solutions for all exercises and practice questions can be accessed from the online e-book for this textbook (more on the e-book below).

HL Option topics

Over 600 textbook pages of material covering the four Higher Level Options can be accessed via the online e-book. All four Options will be presented as e-books when you log in to your e-book account and you can use whichever of them you need. The Options are covered comprehensively, with thorough explanations, worked examples, exercises and practice exam questions. Please note that unauthorized circulation of this material is not permitted.

Online e-book

Included with this textbook is an e-book that contains a digital copy of the textbook. To access this e-book, please follow the instructions on the inside front cover of this book. The textbook on the e-book offers far more than just another copy of the textbook. There are many interactive features on the e-book, which can be accessed by clicking on active links embedded in the pages of the digital version of the textbook. These features include:

1 Additional explanations and examples
2 Practice quizzes for each chapter
3 Dynamic demonstrations of key concepts
4 Audio-video graphing calculator support with activities and tips
5 Worked Solutions for all exercises and practice questions
6 All four Options chapters
7 Software illustrations and simulations.

These interactive resources are designed to support and enhance students' understanding of essential concepts and skills throughout the course. We are profoundly indebted to Peter Ashbourne, Paul Barclay, Peter Flynn, Kevin Frederick and Mike Wakeford – the team of highly experienced and gifted mathematics teachers who created these supplementary student resources on the e-book.

Overview of syllabus changes

As a result of the IB's cyclical curriculum review process, the IB Mathematics HL core syllabus for first exams in May 2014 differs from the previous syllabus in some ways. The following is an overview of the most important changes.

Topic 1 ***Algebra*** remains Topic 1 and has the following addition: *solution of systems of linear equations (maximum of three equations in three unknowns).*

Topic 2 ***Functions and equations*** remains Topic 2 and has the following addition: *sum and product of the roots of polynomial equations.*

Topic 3 ***Circular functions and trigonometry*** remains Topic 3 and is unchanged.

Topic 4 ***Matrices*** has been removed. *Solution of systems of linear equations* is in the *Algebra* topic and *row reduction* for finding the intersection of three planes is still in the *Vectors* topic.

Topic 5 ***Vectors*** is now Topic 4 and the *determinant representation of the vector product* has been removed.

Topic 6 ***Statistics and probability*** is now Topic 5 and *estimation of mean and variance of a population from a sample* has been removed.

Topic 7 ***Calculus*** is now Topic 6 and has the following additions: *informal idea of continuity* and *total distance travelled equals* $\int_a^b \left|v(t)\right| dt$. Also the *solution of first order differential equations by separation of variables* has been removed from this topic and is now in the **Calculus** Option topic.

Certainly, there is a great deal of useful mathematics that cannot 'fit' into the syllabus. We have decided to include a few non-syllabus items in the textbook and have clearly identified any such items as optional.

Internal assessment

This textbook, the online e-book, and the two supporting websites (from Pearson and the authors) provide comprehensive support for the new Internal Assessment component (*Mathematical Exploration*). There is a brief chapter near the end of the textbook on *Mathematical Exploration* in the context of the IA programme for Mathematics HL. Further in-depth information and guidance for teachers adopting the textbook will be provided on the authors' website. We will be updating teacher support and advice for Internal Assessment on our website regularly to address the latest developments, so teachers are encouraged to check from time to time for updates.

Information boxes

As you read this textbook, you will encounter numerous boxes of different colours containing a wide range of helpful information.

Assessment statements

3.6 Solution of triangles.

The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$.

The sine rule: $= \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Area of a triangle as $\frac{1}{2}ab\sin C$.

You will find a box like the one at the bottom of page x at the start of each chapter. They outline the components of the HL core syllabus (indicating syllabus section and sub-section numbers) that will be covered in that chapter.

The green box at right is an example (from Chapter 9) of a 'key' fact drawn out of the main text and highlighted. This makes them useful for quick reference and they also enable you to identify the core learning points within a section.

The process of 'breaking-up' the vector into its components, as we did in the example, is called **resolving** the vector into its components. Notice that the process of resolving a vector is not unique. That is, you can resolve a vector into several pairs of directions.

Beige boxes, like the one below (from Chapter 5), contain interesting information which will add to your wider knowledge but which does not fit within the main body of the text.

Radioactive carbon (carbon-14 or C-14), produced when nitrogen-14 is bombarded by cosmic rays in the atmosphere, drifts down to Earth and is absorbed from the air by plants. Animals eat the plants and take C-14 into their bodies. Humans in turn take C-14 into their bodies by eating both plants and animals. When a living organism dies, it stops absorbing C-14, and the C-14 that is already in the object begins to decay at a slow but steady rate, reverting to nitrogen-14. The half-life of C-14 is 5730 years. Half of the original amount of C-14 in the organic matter will have disintegrated after 5730 years; half of the remaining C-14 will have been lost after another 5730 years, and so forth. By measuring the ratio of C-14 to N-14, archaeologists are able to date organic materials. However, after about 50 000 years, the amount of C-14 remaining will be so small that the organic material cannot be dated reliably.

Margin hints (like the one at right) can be found alongside questions, exercises and worked examples, providing insight into how best to analyze and/or answer a question. They also identify common errors and pitfalls, and suggest approaches that IB examiners like to see.

- **Hint:** Notice here that P(B or C) is *not* the sum of P(B) and P(C) because B and C are not disjoint.

Blue boxes (like the one below) in the main body of the text have important facts, definitions, rules and theorems.

Inequality properties

For three real numbers a, b and c:

1 If $a > b$, and $b > c$, then $a > c$.
2 If $a > b$, and $c > 0$, then $ac > bc$.
3 If $a > b$, and $c < 0$, then $ac < bc$.
4 If $a > b$, then $a + c > b + c$.

Approach

This textbook is designed to be read by you – the student. It is important that you read this textbook *carefully*. Developing your ability to read and understand mathematical explanations will prove to be valuable in your long-term intellectual development, while also helping you to understand the mathematics necessary to be successful in your Mathematics

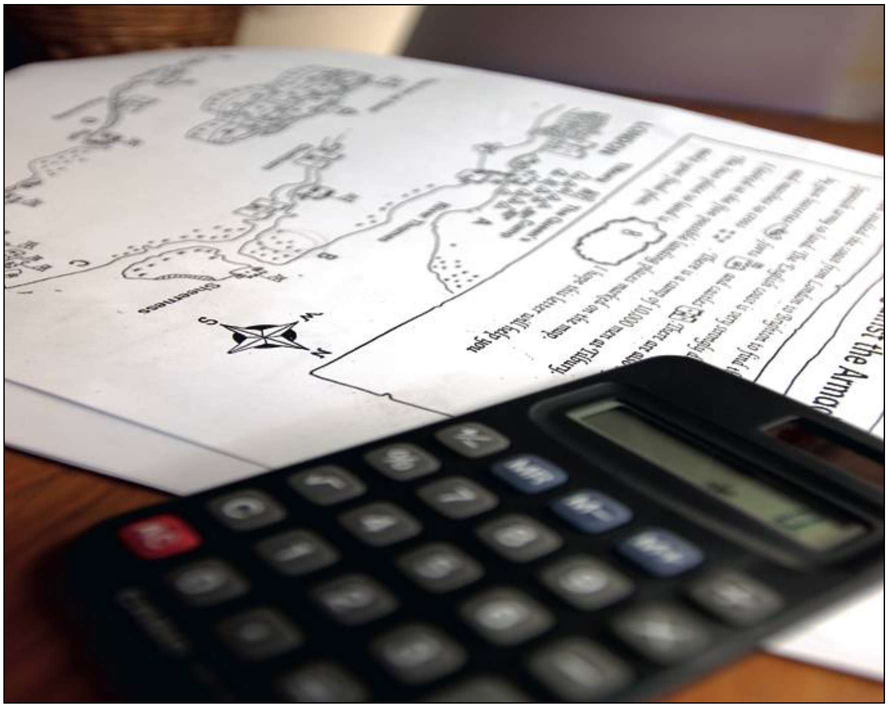
Higher Level course. You should always read a section thoroughly *before* attempting any of the exercises at the end of the section. In preparing this textbook, we have endeavoured to write clear and thorough explanations supported by suitable worked examples. Our primary goal was to present sound mathematics with sufficient rigour and detail at a level appropriate for a student of Higher Level Mathematics.

The positive feedback and constructive comments on the 1st edition, which we received from numerous teachers and students, was very much appreciated. Your comments assisted us greatly in being able to make many improvements and corrections in this 2nd edition. Thank you. We welcome your feedback with regard to any aspects of the textbook and the online e-book. We encourage teachers who adopt the textbook to register at our authors' website and make use of the materials available on it.

Email: info@wazir-garry-math.org
Website: www.wazir-garry-math.org

Ibrahim Wazir and *Tim Garry*

1 Fundamentals



Introduction

This first chapter reviews some of the *presumed knowledge* for the course – that is, mathematical knowledge that you *must* be familiar with before delving fully into the Mathematics Higher Level course (Chapter 2 and beyond). It is not necessary to work through each section in detail; however, it is very important that you read the entire chapter carefully in order to find out what is in it, and to become familiar with terminology, notations, and algebraic techniques used regularly in the course.

1.1 Sets, inequalities, absolute value and properties of real numbers

The language and notation of sets is often convenient for expressing results to a variety of problems in mathematics. We will review basic concepts, some important sets and useful notation. Some set concepts and notation will be applied again to probability problems in Chapter 12.

Sets of numbers and set notation

A set is a collection of objects or **elements**. Typically in mathematics and in this course the elements of a set will be numbers that can be defined by a list or a mathematical rule. Sets are usually denoted by capital letters. The elements, or members, of a set are listed between braces $\{\quad\}$. For example, if the set A consists of the numbers 4, 5 and 6, we write $A = \{4, 5, 6\}$ where 4, 5 and 6 are the elements of set A. Symbolically, we write $4 \in A$, $5 \in A$ and $6 \in A$; read as '4 is an element of set A', or '4 is a member of set A' etc. To express that the number 3 is not an element of set A, we write $3 \notin A$.

The three dots seen in the set $\{1, 2, 3, \ldots\}$ are an **ellipsis** and can have two different interpretations when used as a mathematical notation. When used in set notation, or raised up to show a repeated operation (e.g. $2 + 4 + 6 + \cdots + 48 + 50$), an ellipsis indicates that the numbers continue indefinitely in the same pattern. It should only be used in this way if the pattern is clear. Alternatively, an ellipsis can also be used to indicate that the decimal representation of an irrational number continues indefinitely and does *not* have a repeating pattern. For example, $\pi = 3.141\,592\,65\ldots$.

Sets whose number of elements can be counted are **finite**. If the number of elements in a set cannot be given a specific number then it is **infinite**. When we count objects, we start with the number 1, then 2, 3, etc; that is, the set $\{1, 2, 3, \ldots\}$. This is the set of positive integers (also known as the set of counting numbers) which is given the special symbol $\mathbb{Z}^+$. The number of elements in the set $A = \{4, 5, 6\}$ is three so it is a **finite set**. Even though we can define the set of positive integers in the form of a list, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, it is an **infinite set** because it is not possible to specify how many members are in the set.

Rather than defining a finite set by listing all the elements, we can specify the elements using a rule. For example, the set $B = \{x \mid 4 \leqslant x \leqslant 10, x \in \mathbb{Z}^+\}$ is read as 'B is the set of all x-values such that x is a positive integer between 4 and 10, inclusive'. This is an alternative way of writing $B = \{4, 5, 6, 7, 8, 9, 10\}$. Set notation using a mathematical rule is particularly useful when defining an infinite set, for which it is not possible to list all the elements, or a finite set with a large number of elements with a continuing pattern.

Example 1 – Defining sets

Using set notation and an appropriate mathematical rule, define each of following sets. Also indicate whether the set is finite or infinite.

a) The set of all integers between -8 and 6, not including -8 and 6 (i.e. exclusive).

b) The set of all integer multiples of $\frac{\pi}{4}$ greater than zero and less than or equal to 2π.

c) The set of positive odd integers.

Solution

a) $\{x \mid -8 < x < 6, x \in \mathbb{Z}\}$ finite set

b) $\left\{0 < n \cdot \frac{\pi}{4} \leqslant 2\pi, n \in \mathbb{Z}\right\}$ or $\left\{x \mid x = n \cdot \frac{\pi}{4}, 0 < n \leqslant 8, n \in \mathbb{Z}\right\}$ finite set

c) $\{2k - 1, k \in \mathbb{Z}^+\}$ or $\{2k + 1, k = 0, 1, 2, \ldots\}$ infinite set

Symbol	Set name	Set notation
$\mathbb{C}$	set of complex numbers	$\{a + bi \mid a, b \in \mathbb{R}\}$ where $i^2 = -1$
$\mathbb{R}$	set of real numbers	$\{x \in \mathbb{R}\}$
$\mathbb{R}^+$	set of positive real numbers	$\{x \mid x > 0, x \in \mathbb{R}\}$
$\mathbb{Q}$	set of rational numbers	$\left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$
$\mathbb{Q}^+$	set of positive rational numbers	$\{x \mid x > 0, x \in \mathbb{Q}\}$
$\mathbb{Z}$	set of integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{N}$	set of natural numbers (or whole numbers)	$\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}^+$	set of positive integers (or counting numbers)	$\{1, 2, 3, \ldots\}$

Table 1.1 Some important infinite sets are listed here, indicating their special symbols and how to express them with set notation, if possible.

There is some disagreement in the mathematics community about whether the number zero should be included in the natural numbers. So do not be confused if you see other textbooks indicate that the set of natural numbers, $\mathbb{N}$, does *not* include zero – and is defined as $\mathbb{N} = \{1, 2, 3, \ldots\}$. In IB mathematics the set $\mathbb{N}$ is defined to be the set of positive integers *and* zero, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Now that we have the symbol $\mathbb{N}$ for the set of natural numbers $\{0, 1, 2, 3, \ldots\}$, we can also write the answer to Example 1, part c), the set of positive odd integers, as $\{2k + 1, k \in \mathbb{N}\}$.

A **real number** is any number that can be represented by a point on the real number line (Figure 1.1). Each point on the real number line corresponds to one unique real number, and conversely each real number corresponds to one unique point on the real number line. This kind of relationship is called a **one-to-one correspondence**. The number associated with a point on the real number line is called the **coordinate** of the point.

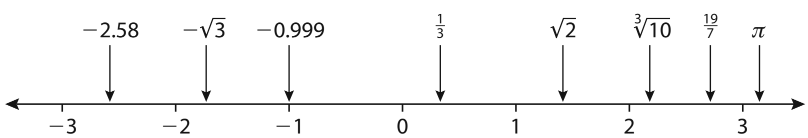


Figure 1.1 The real number line.

The real numbers are a subset of the **complex numbers**. It is likely that you will have limited or no experience with complex numbers or **imaginary numbers**. We will encounter complex and imaginary numbers in Chapter 3 and study them thoroughly in Chapter 10. However, it is worth saying a few introductory words about them at this point. The complex numbers, $\mathbb{C}$, involve a combination of real and imaginary numbers. Any complex number can be written in the form $a + bi$ where a and b are real numbers and $\boldsymbol{i}$ is the imaginary number defined such that $i^2 = -1$. For a complex number $a + bi$, if $b = 0$ then the complex number is a real number (e.g. $5 = 5 + 0i$, and $\sqrt{2} = \sqrt{2} + 0i$), and if $b \neq 0$ then the complex number is an imaginary number (e.g. $5 - 3i$, and $0 + 2i = 2i$). Hence, any complex number is either a real number or an imaginary number (see Figure 1.2).

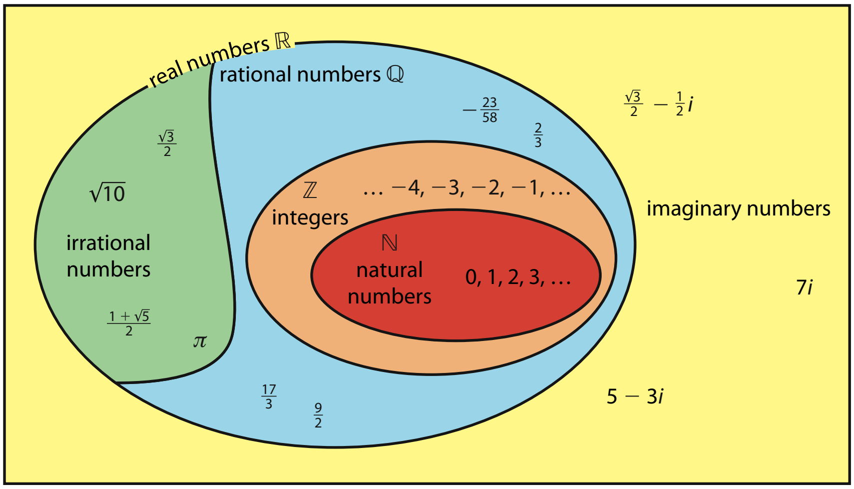
We will see in Chapter 3 that some polynomial equations will have solutions that are imaginary numbers. For the quadratic equation $x^2 + 1 = 0$, we must find x such that $x^2 = -1$. A value for x will not be a real number. The symbol i was invented such that $i^2 = -1$. Hence, $x^2 + 1 = 0$ has two imaginary solutions, $x = i$ and $x = -i$. We define the imaginary number i as $i^2 = -1$ but we are allowed to write $i = \sqrt{-1}$. We will study complex numbers in greater depth in Chapter 10.

● **Hint:** The answer for Example 1, part a), was the set $\{x \mid -8 < x < 6, x \in \mathbb{Z}\}$. The definition for the elements of the set specified that all elements must be integers. If the definition of a set does not specify to which set the elements belong, it is assumed to be the set of real numbers $\mathbb{R}$. For example, the set $\{x \mid -8 < x < 6\}$ would contain all of the real numbers between -8 and 6 exclusive. This is an infinite set, whereas the set $\{x \mid -8 < x < 6, x \in \mathbb{Z}\}$ is finite.

Similarly, any real number is either rational or irrational, with the rational numbers and irrational numbers being subsets of the real numbers (Figure 1.2). We construct the **rational numbers** $\mathbb{Q}$ by taking ratios of integers. Thus, a real number is rational if it can be written as the ratio $\frac{p}{q}$ of any two integers, where $q \neq 0$. The decimal representation of a rational number either repeats or terminates. For example, $\frac{5}{7} = 0.714\,285\,714\,285\ldots = 0.\overline{714\,285}$ (the block of six digits repeats) or $\frac{3}{8} = 0.375$ (the decimal 'terminates' at 5, or alternatively has a repeating zero after the 5).

A real number that cannot be written as the ratio of two integers, such as π and $\sqrt{2}$, is called **irrational**. Irrational numbers have infinite non-repeating decimal representations. For example, $\sqrt{2} \approx 1.414\,213\,5623\ldots$ and $\pi \approx 3.141\,592\,653\,59\ldots$. There is no special symbol for the set of irrational numbers.

Figure 1.2 The diagram depicts the relationships between the different subsets of the complex numbers. The real numbers combined with the imaginary numbers make up the entire set of complex numbers. The rational numbers combined with the irrational numbers make up the entire set of real numbers.



The earliest known use of irrational numbers was in India between 800–500 BCE. The first mathematical proof that a number could not be expressed as the ratio of two integers (i.e. irrational) is usually attributed to the Pythagoreans. The revelation that not all numbers were rational was a great shock to Pythagoras and his followers, given that their mathematics and theories about the physical world were based completely on positive integers and their ratios. Euclid (ca. 325–265 BCE) wrote a proof of the irrationality of $\sqrt{2}$ in his *Elements*, one of the most famous books in mathematics. Euclid's proof is considered to be an elegant proof because it is both simple and powerful. Euclid used a method called **proof by contradiction**, or in Latin, *reductio ad absurdum*. Here is a condensed version of his proof that $\sqrt{2}$ cannot be written as the ratio of two integers. This is equivalent to saying that there is no rational number $\frac{p}{q}$ whose square is 2 where $p, q \in \mathbb{Z}$. The proof begins by assuming that the statement to be proved is false – that is, we assume that there ***is*** a rational number $\frac{p}{q}$ completely simplified (i.e. p and q have no common factor) whose square is 2. Then $\left(\frac{p}{q}\right)^2 = 2$, and it follows that $p^2 = 2q^2$. Hence, p^2 has a factor of 2 which means that p must be an even number. Since p is even, then let's replace p with $2k$, where k is an integer, giving $4k^2 = 2q^2$ leading to $2k^2 = q^2$. Therefore, q^2 has a factor of 2 and so q is also even. This means that p and q both have a factor of 2. But this contradicts the assumption that p and q have no common factors. Therefore, the initial assumption that there is a rational number $\frac{p}{q}$ whose square is 2 leads to a contradiction. It logically follows then that this assumption must be false, i.e. there is no rational number whose square is 2.

Example 2 – Expressing a repeating decimal as a rational number

Express each as a rational number completely simplified.

a) $1.416\,6666\ldots = 1.41\overline{6}$

b) $38.245\,3453\ldots = 38.2\overline{453}$

Solution

a) Let $N = 1.416\,6666\ldots$

Then $1000N = 1416.666\,66\ldots$ and $100N = 141.666\,66\ldots$

Now subtract $100N$ from $1000N$:

$$\begin{array}{r} 1000N = 1416.666\,66\ldots \\ -100N = -141.666\,66\ldots \\ \hline 900N = 1275 \end{array} \qquad \text{This gives } N = \frac{1275}{900} = \frac{25 \times 51}{25 \times 36} = \frac{51}{36} = \frac{3 \times 17}{3 \times 12} = \frac{17}{12}$$

Therefore, $1.41\overline{6} = \frac{17}{12}$.

b) Let $N = 38.245\,3453\ldots$

Then $10\,000N = 382\,453.453\,453\ldots$ and $10N = 382.453\,453\ldots$

Now subtract $10N$ from $10\,000N$:

$$\begin{array}{r} 10\,000N = 382\,453.453\,453\ldots \\ -10N = -382.453\,453\ldots \\ \hline 9990N = 382\,071 \end{array} \qquad \text{This gives } N = \frac{382\,071}{9990} = \frac{3 \times 127\,357}{3 \times 3330} = \frac{127\,357}{3330}$$

Therefore, $38.2\overline{453} = \frac{127\,357}{3300}$.

Note: 382 071 is divisible by 3 because the sum of its digits (21) is divisible by 3. The fraction $\frac{127\,357}{3330}$ cannot be simplified because 127 357 and 3330 share no common factors; $3330 = 2 \times 3 \times 3 \times 5 \times 37$ (prime factorization) and 2, 3, 5 and 37 are not factors of 127 357

Another approach to expressing a repeating decimal as a rational number appears in Chapter 4.

Set relations, operations and diagrams

If every element of a set C is also an element of a set D, then C is a **subset** of set D, and is written symbolically as $C \subseteq D$. If two sets are equal (i.e. they have identical elements), they satisfy the definition of a subset and each would be a subset of the other. For example, if $C = \{2, 4, 6\}$ and $D = \{2, 4, 6\}$, then $C = D$, $C \subseteq D$ and $D \subseteq C$. What is more common is that a subset is a set that is contained in a larger set and does not contain at least one element of the larger set. Such a subset is called a **proper subset** and is denoted with the symbol $\subset$. For example, if $D = \{2, 4, 6\}$ and $E = \{2, 4\}$, then E is a proper subset of D and is written $E \subset D$, but $C \not\subset D$. Other than the set of complex numbers itself, all of the sets listed in Table 1.1 are proper subsets of the complex numbers.

The set of all elements under consideration for a particular situation or problem is called the **universal set**, usually denoted by the symbol U. The

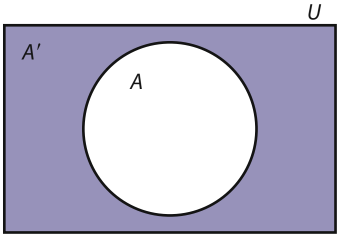


Figure 1.3 Venn diagram for the universal set U, set A, and the complement of A, A' (shaded region).

complement of a given set A is the set of all elements in the universal set that are not elements of set A, and is denoted by the symbol A'. **Venn diagrams** are used to pictorially represent the relationship of sets within a universal set. The universal set, U, is represented by a rectangle and any subset of U is represented by the interior of a circle within the rectangle (see Figure 1.3).

If for a certain problem the universal set is the complex numbers $\mathbb{C}$, then the complement of the real numbers is the imaginary numbers. For problems in secondary school mathematics, and in this course, the universal set will often be a subset of the complex numbers – commonly the real numbers $\mathbb{R}$. If the universal set is the real numbers, then the set of irrational numbers is the complement of the rational numbers $\mathbb{Q}$. See Figure 1.2.

Although the set {2, 3} is equal to the set {3, 2}, the ordered pairs (2, 3) and (3, 2) are not the same. Hence, for the Cartesian product of two sets A and B, in general, $A \times B \neq B \times A$.

The **intersection** of sets A and B, denoted by $A \cap B$ and read 'A intersection B', is the set of all elements that are in both set A *and* set B. The **union** of two sets A and B, denoted by $A \cup B$ and read 'A union B', is the set of all elements that are in set A *or* in set B (or in both). The set that contains no elements is called the **empty set** (or null set) and is denoted by $\varnothing$. Sets whose intersection is the empty set, i.e. they have no elements in common, are **disjoint sets**.

● **Hint:** The symbol for the union of two sets, ∪, can be remembered by connecting it with the first letter in the word 'union'.

The **Cartesian product** of two sets A and B is the set of all **ordered pairs** $\{(a, b)\}$, where $a \in A$ and $b \in B$. It is written as $A \times B = \{(a, b) \mid a \in A, b \in B\}$. For example, if $X = \{1, 2\}$ and $Y = \{3, 4, 5\}$,

then $\quad X \times Y = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

and $\quad Y \times X = \{(3,1), (3,2), (4,1), (4,2), (5,1), (5,2)\}$

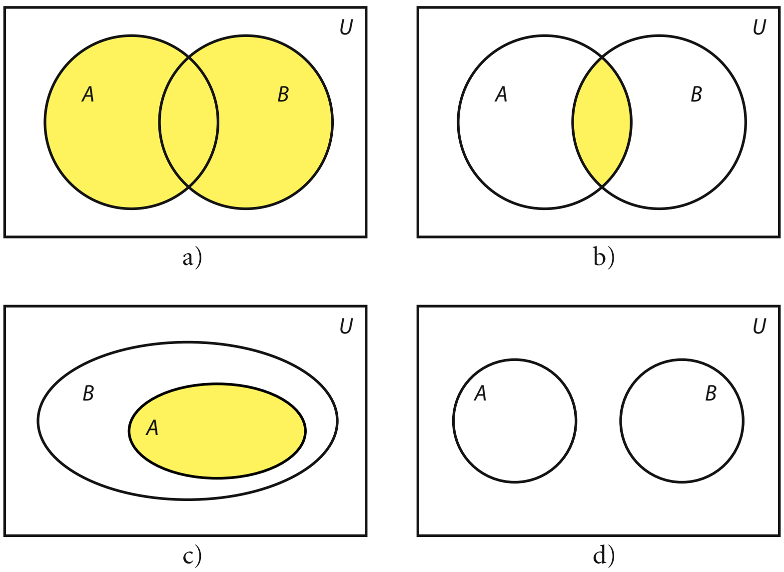


Figure 1.4 a) Union of sets A and B, $A \cup B$
b) Intersection of sets A and B, $A \cap B$
c) Proper subset, $A \subset B$ d) Two disjoint sets, A and B

Venn diagrams are named after the British mathematician, philosopher and writer John Venn (1834–1923). Although he was not the first to use diagrams as an aid to problems in set theory and logic, he was the first to formalize their usage and popularized them in his writings such as in his first book *Symbolic Logic* published in 1881.

Set relations and operations

Subset:	$A \subseteq B$ means that A is a subset of B
Proper subset:	$A \subset B$ means that $A \subseteq B$ but $A \neq B$
Intersection:	$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
Union:	$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}$
Complement:	$A' = \{x \mid x \notin A\}$
Empty set:	$\varnothing$, the set with no elements
Cartesian product:	$A \times B = \{(a, b) \mid a \in A, b \in B\}$

Example 3 – Set operations

Consider that the universal set U is defined to be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, and $A = \{2, 5, 8, 11\}$, $B = \{2, 4, 6, 8, 10, 12\}$, $C = \{2, 3, 5, 7, 11, 13\}$.

a) Find the following:

(i) $A \cap B$ (ii) $A \cup B$ (iii) A'

(iv) $A \cap C$ (v) $A \cap B \cap C$ (vi) $(B \cup C)'$

(vii) $A \cap (B \cup C)'$ (viii) $A \cup B \cup C$

b) Draw a Venn diagram to illustrate the relationship between the sets A, B and C.

Solution

a) (i) $A \cap B = \{2, 8\}$ (ii) $A \cup B = \{2, 4, 5, 6, 8, 10, 11, 12\}$

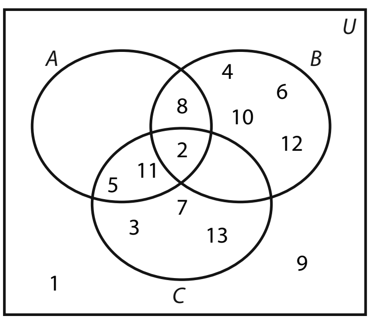
(iii) $A' = \{1, 3, 4, 6, 7, 9, 10, 12, 13\}$ (iv) $A \cap C = \{2, 5, 11\}$

(v) $A \cap B \cap C = \{2\}$ (vi) $(B \cup C)' = \{1, 9\}$

(vii) $A \cap (B \cup C)' = \varnothing$

(viii) $A \cup B \cup C = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13\}$

b)



● **Hint:** When we list the elements of a set we never repeat an element. For example, for $A \cup B$ in Example 3 the numbers 2 and 8 are in both A and B but they are each written once when listing the elements in $A \cup B$.

In Example 3, instead of defining sets U, A, B and C using lists, we could have defined each of the sets using a rule. For example,
$U = \{x \mid 1 \leqslant x \leqslant 13, x \in \mathbb{Z}\}$,
$A = \{x \mid x = 3n - 1, n = 1, 2, 3, 4\}$,
$B = \{x \mid x = 2n, 1 \leqslant n \leqslant 6 \text{ and } n \in \mathbb{Z}\}$,
and
$C = \{x \mid x \leqslant 13, x \text{ is a prime number}\}$.

Inequalities (order relations)

An inequality is a statement involving one of four symbols that indicates an **order relation** between two numbers or algebraic expressions on either side of the symbol. The symbols are

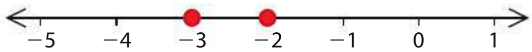
$<$ (less than)

$>$ (greater than)

$\leqslant$ (less than or equal to)

$\geqslant$ (greater than or equal to).

The relation $a > b$ is read 'a is greater than b' and in the geometric representation of numbers on the real number line it means that a lies to the right of b. Since -2 lies to the right of -3 on the number line then $-2 > -3$. The inequality $a > b$ is equivalent to the inequality $b < a$ (b lies to the left of a on the number line), and similarly $-2 > -3$ is equivalent to $-3 < -2$.



Working with inequalities is very important for many of the topics in this course. There are four basic properties for inequalities.

Inequality properties

For three real numbers a, b and c:

1. If $a > b$, and $b > c$, then $a > c$.
2. If $a > b$, and $c > 0$, then $ac > bc$.
3. If $a > b$, and $c < 0$, then $ac < bc$.
4. If $a > b$, then $a + c > b + c$.

The first property is sometimes referred to as the transitive property. The second property for inequalities expresses the fact that an inequality that is multiplied on both sides by a positive number does not change the inequality symbol. For example, given that $x > 6$ then multiplying both sides by $\frac{1}{2}$ gives $\frac{x}{2} > 3$.
The third property tells us that if we multiply both sides of an inequality by a negative number then the inequality symbol is reversed. For example, if $-3x \leqslant 12$ then multiplying both sides by $-\frac{1}{3}$ gives $x \geqslant -4$. The fourth property means that the same quantity being added to both sides will produce an equivalent inequality.

When you solve an inequality the result will be a range of possible values of the variable. The inequalities in the next example are solved by applying the properties for inequalities (stated above) and basic rules for solving linear equations with which you are familiar.

Example 4 – Solving inequalities

Solve each inequality.

a) $6x + 1 > x - 5$ b) $9 - 4x \leqslant 2x - 3$

c) $3(1 - 2x) < 15$ d) $-3 \leqslant 2x - 1 < 9$ e) $-2 \leqslant 4 - 3x < 13$

Solution

a) $6x + 1 > x - 5 \Rightarrow 6x > x - 6 \Rightarrow 5x > -6 \Rightarrow x > -\frac{6}{5}$

b) $9 - 4x \leqslant 2x - 3 \Rightarrow 12 - 4x \leqslant 2x \Rightarrow 12 \leqslant 6x \Rightarrow 2 \leqslant x$ or $x \geqslant 2$
Alternatively, $9 - 4x \leqslant 2x - 3 \Rightarrow -4x \leqslant 2x - 12 \Rightarrow -6x \leqslant -12 \Rightarrow x \geqslant 2$

c) $3(1 - 2x) < 15 \Rightarrow 1 - 2x < 5 \Rightarrow -2x < 4 \Rightarrow x > -2$

d) The inequality $-3 \leqslant 2x - 1 < 9$ is a 'double inequality' containing two separate inequalities $-3 \leqslant 2x - 1$ and $2x - 1 < 9$; we can solve each separately or simultaneously as shown here.

$-3 \leqslant 2x - 1 < 9 \Rightarrow -2 \leqslant 2x < 10 \Rightarrow -1 \leqslant x < 5$

This solution set is read 'x is any real number that is greater than or equal to -1 **and** less than 5'.

e) $-2 < 4 - 3x < 13 \Rightarrow -6 < -3x < 9 \Rightarrow 2 > x > -3 \Rightarrow -3 < x < 2$

• **Hint:** It is improper to write the solution to Example 4 e) as $2 > x > -3$. A double inequality should be written with the lesser quantity on the left and greater on the right, i.e. $-3 < x < 2$ for Example 4 e). A double inequality is the intersection of two sets. For example, the expression $-3 < x < 2$ represents the **intersection** of $x > -3$ **and** $x < 2$; i.e. the numbers greater than -3 **and** less than 2. The **union** of two sets **cannot** be written as a double inequality. Using inequalities to represent the numbers less than 4 **or** greater than 7 must be written as two separate inequalities, $x < 4$ **or** $x > 7$.

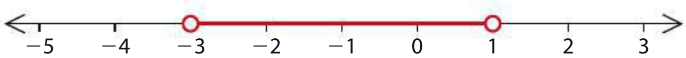
In Chapter 3, we will be solving further inequalities involving linear, quadratic and rational (fractional) expressions.

Intervals on the real number line

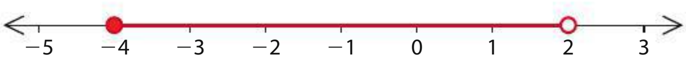
Except when studying complex numbers in Chapter 10 and solving certain polynomial equations in Chapter 3, problems that we encounter in this course will be in the context of the real numbers. For example, the solution set for the inequality in Example 4 c) is the set of all real numbers greater than negative three. Such a set can be represented geometrically by a part, or an **interval**, of the real number line and corresponds to a line segment or a ray. It can be written symbolically by an **inequality** or by **interval notation**. For example, the set of all real numbers x between 2 and 5 inclusive, can be expressed by the inequality $2 \leqslant x \leqslant 5$ or by the interval notation $x \in [2, 5]$. This is an example of a **closed interval** (i.e. both endpoints are included in the set) and corresponds to the line segment with endpoints of $x = 2$ and $x = 5$.

• **Hint:** Unless indicated otherwise, if interval notation is used, we assume that it indicates an infinite set containing any real number within the indicated range. For example, the expression $x \in [-4, 2]$ is read 'x is any real number between -4 and 2 inclusive.'

An example of an **open interval** is $-3 < x < 1$, also written as $x \in \,]-3, 1[$, where both endpoints are *not* included in the set. This set corresponds to a line segment with 'open dots' on the endpoints indicating they are excluded.



If an interval, such as $-4 \leqslant x < 2$, also written as $x \in [-4, 2[$, includes one endpoint but not the other, it is referred to as a **half-open** interval.



• **Hint:** The symbols ∞ (positive infinity) and $-\infty$ (negative infinity) do not represent real numbers. They are simply symbols used to indicate that an interval extends indefinitely in the positive or negative direction.

The three examples of intervals on the real number line given above are all considered **bounded** intervals in that they are line segments with two endpoints (regardless whether included or excluded). The set of all real numbers greater than 2 is an open interval because the one endpoint is excluded and can be expressed by the inequality $x > 2$, also written as $x \in [2, \infty[$. This is also an example of an **unbounded** interval and corresponds to a part of the real number line that is a ray.

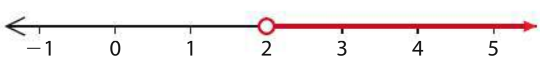


Table 1.2 The nine possible types of intervals – both bounded and unbounded. For all of the examples given, we assume that $a < b$.

Interval notation	Inequality	Interval type	Graph
$x \in [a, b]$	$a \leqslant x \leqslant b$	closed bounded	
$x \in]a, b[$	$a < x < b$	open bounded	
$x \in [a, b[$	$a \leqslant x < b$	half-open bounded	
$x \in]a, b]$	$a < x \leqslant b$	half-open bounded	
$x \in [a, \infty[$	$x \geqslant a$	half-open unbounded	
$x \in]a, \infty[$	$x > a$	open unbounded	
$x \in]-\infty, b]$	$x \leqslant b$	half-open unbounded	
$x \in]-\infty, b[$	$x < b$	open unbounded	
$x \in]-\infty, \infty[$	real number line		

Absolute value (or modulus)

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Since a distance must be positive or zero, then the absolute value of a number is never negative. Note that if a is a negative number then $-a$ will be positive.

Definition of absolute value

If a is a real number, the **absolute value** of a is

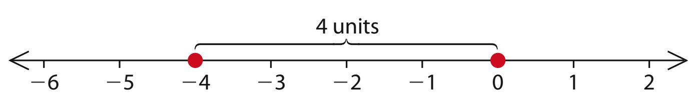
$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Here are four useful properties of absolute value.

Given that a and b are real numbers, then:

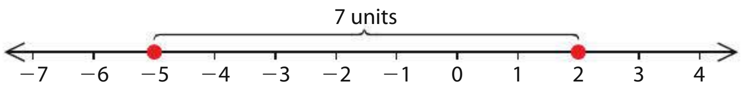
1. $|a| \geqslant 0$ **2.** $|-a| = |a|$ **3.** $|ab| = |a||b|$ **4.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

Absolute value is used to define the distance between two numbers on the real number line. The distance between -4 and 0 on the number line is clearly seen to be 4 units in the figure below.



Also the distance between 4 and 0 is 4 units. Note that $|-4| = 4$ gives the distance between -4 and 0, and $|-4| = 4$ gives the distance between 4 and 0. These observations lead to the geometrical interpretation of absolute value as a distance: $|a|$ is the distance between a and 0 on the real number line.

Now consider the distance between two non-zero numbers on the number line. The distance between -5 and 2 is 7 units as shown below.



Note that $|-5 - 2| = 7$ gives the distance between -5 and 2. It is also true that $|2 - (-5)| = 7$ is the distance between -5 and 2. The examples suggest that the distance between two numbers on the number line is always given by the absolute value of their difference.

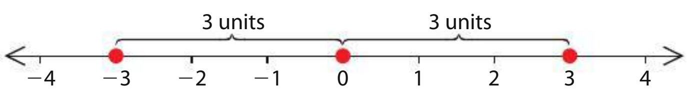
Distance between two points on the real number line

Given that a and b are real numbers, the distance between the points with coordinates a and b on the real number line is $|b-a|$, which is equivalent to $|a-b|$.

The geometric statement that 'the distance between k and 8 is $\frac{5}{2}$' can be expressed algebraically as $|k - 8| = \frac{5}{2}$, or as $|8 - k| = \frac{5}{2}$. The geometric interpretation of absolute value as a distance leads to a helpful method for solving equations involving absolute value expressions.

Example 5 – Absolute value as a distance

How many real numbers have an absolute value equal to 3? Since absolute value can be interpreted as the distance from a number on the real number line to 0 (the origin), then this question is equivalent to asking which real numbers are a distance of 3 units from 0 on the number line? See the figure below.



There are just two numbers whose distance to 0 is 3 units, namely, 3 and -3. In other words the equation $|x| = 3$ has two solutions: $x = 3$ or $x = -3$. This reasoning leads to a general method for solving a linear equation containing the absolute value of a variable expression.

If $|x| = c$, where x is an unknown and c is a positive real number, then $x = c$ or $x = -c$.

Example 6 – Solving absolute value linear equations

Solve for x in each equation.

a) $|x - 5| = 8$ b) $4\left|x + \frac{3}{2}\right| = 9$ c) $\left|\frac{3x - 4}{2}\right| - 7 = 9$

Solution

a) $|x - 5| = 8$

$x - 5 = 8$ or $x - 5 = -8$ — Applying the property that if $|x| = c$, then $x = c$ or $x = -c$.

$x = 13$ or $x = -3$

b) $4\left|x + \frac{3}{2}\right| = 9 \Rightarrow \left|x + \frac{3}{2}\right| = \frac{9}{4}$

$x + \frac{3}{2} = \frac{9}{4}$ or $x + \frac{3}{2} = -\frac{9}{4}$

$x = \frac{3}{4}$ or $x = -\frac{15}{4}$

c) $\left|\frac{3x - 4}{2}\right| - 7 = 9 \Rightarrow \left|\frac{3x - 4}{2}\right| = 16$

$\frac{3x - 4}{2} = 16$ or $\frac{3x - 4}{2} = -16$

$3x - 4 = 32$ or $3x - 4 = -32$

$x = 12$ or $x = -\frac{28}{3}$

We will encounter more sophisticated equations involving absolute value in Chapter 3.

Properties of real numbers

There are four arithmetic operations with real numbers: addition, multiplication, subtraction and division. Since subtraction can be written as addition $(a - b = a + (-b))$, and division can be written as multiplication $\left(\frac{a}{b} = a\left(\frac{1}{b}\right), b \neq 0\right)$, then the properties of the real numbers are defined in terms of addition and multiplication only. In these definitions, $-a$ is the **additive inverse** (or opposite) of a, and $\frac{1}{a}$ is the **multiplicative inverse** (or reciprocal) of a.

Table 1.3 Properties of real numbers.

Property	Rule	Example
commutative property of addition:	$a + b = b + a$	$2x^3 + y = y + 2x^3$
commutative property of multiplication:	$ab = ba$	$(x - 2)3x^2 = 3x^2(x - 2)$
associative property of addition:	$(a + b) + c = a + (b + c)$	$(1 + x) - 5x = 1 + (x - 5x)$
associative property of multiplication:	$(ab)c = a(bc)$	$(3x \cdot 5y)\left(\frac{1}{y}\right) = (3x)\left(5y \cdot \frac{1}{y}\right)$
distributive property:	$a(b + c) = ab + ac$	$x^2(x - 2) = x^2 \cdot x + x^2(-2)$
additive identity property:	$a + 0 = a$	$4y + 0 = 4y$
multiplicative identity property:	$1 \cdot a = a$	$\frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{4}{4} \cdot \frac{2}{3} = \frac{8}{12}$
additive inverse property:	$a + (-a) = 0$	$6y^2 + (-6y^2) = 0$
multiplicative inverse property:	$a \cdot \frac{1}{a} = 1, a \neq 0$	$(y - 3)\left(\frac{1}{y - 3}\right) = 1$

Note: These properties can be applied in either direction as shown in the 'rules' above.

Exercise 1.1

In questions 1–6, use the symbol $\subset$ (proper subset) to write a correct statement involving the two sets.

1 $\mathbb{Z}$ and $\mathbb{Q}$ **2** $\mathbb{N}$ and $\mathbb{Q}$ **3** $\mathbb{C}$ and $\mathbb{R}$

4 $\mathbb{Z}$ and $\mathbb{N}$ **5** $\mathbb{Z}$ and $\mathbb{Z}^+$ **6** $\mathbb{N}$ and $\mathbb{R}$

In questions 7–9, express each repeating decimal as a completely simplified fraction.

7 2.151 515… **8** 11.913 333… **9** $8.\overline{714285}$

In questions 10–15, state the indicated set given that $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{2, 4, 6\}$.

10 $A \cap B$ **11** $A \cup B$ **12** $B \cap C$

13 $A \cup C$ **14** $A \cap C$ **15** $A \cup B \cup C$

16 Consider that the universal set U is defined to be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and $A = \{2n, n \in \mathbb{Z}^+\}$, $B = \{2n - 1, n \in \mathbb{Z}^+\}$, $C = \{3n, n \in \mathbb{Z}^+\}$, $D = \{1, 6, 7, 9\}$.

a) Describe in words the elements of sets A, B and C.

b) Find the following:
(i) $A \cap B$ (ii) $A \cup B$ (iii) A' (iv) B' (v) $A \cap D$
(vi) $B \cap C$ (vii) $B \cap C \cap D$ (viii) $(C \cup D)'$ (ix) $A \cap (C \cap D)'$

c) Draw a Venn diagram to illustrate the relationship between the sets A, C and D.

17 P and Q are subsets of the universal set U with the three sets defined as follows.
$U = \{x \mid -2 \leqslant x \leqslant 4\}$, $P = \{x \mid -2 \leqslant x \leqslant 1\}$, $Q = \{x \mid 0 < x < 3\}$

(Remember: If not specified, the elements in an interval will be an infinite set of real numbers.)

Being careful with the inclusion or exclusion of endpoints, list the following sets.

a) $P \cap Q$ b) $P \cup Q$ c) P' d) Q' e) $(P \cup Q)'$ (f) $(P \cap Q)'$

In questions 18–23, solve the inequality.

18 $\frac{x}{5} > -2$ **19** $3 + 4x \leqslant -9$ **20** $7 - 3x < -3$

21 $6(2 - x) < 2x + 15$ **22** $9 \leqslant 8x - 3 < 11$ **23** $-4 \leqslant 1 - 5x \leqslant 16$

In questions 24–31, determine whether each statement is true for all real numbers x. If the statement is false, then indicate one counterexample, i.e. a value of x for which the statement is false.

24 $2x \geqslant x$ **25** $x^3 + 1 > x^3$ **26** $x^3 + x > x^3$

27 $x^2 \geqslant x$ **28** $x^2 \geqslant 0$ **29** $\sqrt{x} \geqslant 0$

30 $-x \leqslant 0$ **31** $\frac{1}{x} \leqslant x$

In questions 32–37, plot the two real numbers on the real number line, and then find the exact distance between their coordinates.

32 -7 and $\frac{15}{2}$ **33** -2 and -11 **34** 27.4 and 19.2

35 π and 3 **36** -3π and $\frac{2\pi}{3}$ **37** $\frac{61}{7}$ and $-\frac{23}{11}$

In questions 38–43, write an inequality to represent the given interval and state whether the interval is closed, open or half-open. Also state whether the interval is bounded or unbounded.

38 $[-5, 3]$ **39** $]-10, -2]$ **40** $[1, \infty[$

41 $]-\infty, 4]$ **42** $[0, 2\pi[$ **43** $[a, b]$

In questions 44–49, use interval notation to represent the subset of real numbers that is indicated by the inequality.

44 $x > -3$ **45** $-4 < x < 6$ **46** $x \leqslant 10$

47 $0 \leqslant x < 12$ **48** $x < \pi$ **49** $-3 \leqslant x \leqslant 3$

In questions 50–53, use both inequality and interval notation to represent the given subset of real numbers.

50 x is at least 6.

51 x is greater than or equal to 4 and less than 10.

52 x is negative.

53 x is any positive number less than 25.

In questions 54–57, express the inequality, or inequalities, using absolute value.

54 $-6 < x < 6$ **55** $x \leqslant -4$ or $x \geqslant 4$

56 $-\pi \leqslant x \leqslant \pi$ **57** $x < -1$ or $x > 1$

In questions 58–63, evaluate each absolute value expression.

58 $|-13|$ **59** $|7 - 11|$ **60** $-5|-5|$

61 $|-3|-|-8|$ **62** $|\sqrt{3} - 3|$ **63** $\dfrac{-1}{|-1|}$

In questions 64–71, find all values of x that make the equation true.

64 $|x| = 5$ **65** $|x - 3| = 4$

66 $|6 - x| = 10$ **67** $|x + 5| = -2$

68 $|3x + 5| = 1$ **69** $\frac{1}{2}|x - \frac{2}{3}| = 5$

70 $\left|\dfrac{6 - 2x}{3}\right| + \dfrac{2}{5} = 8$ **71** $2\left|\dfrac{x + 2}{2}\right| = 2$

72 For each of the following statements, find at least one counterexample that confirms the statement is false.

a) $|x + y| = |x| + |y|$ b) $|x - y| = |x| - |y|$

73 Using properties of inequalities, prove each of the statements.

a) If $x < y$ and $x > 0$, then $\dfrac{1}{y} < \dfrac{1}{x}$.

b) If $x < 0 < y$, then $\dfrac{1}{y} > \dfrac{1}{x}$.

1.2 Roots and radicals (surds)

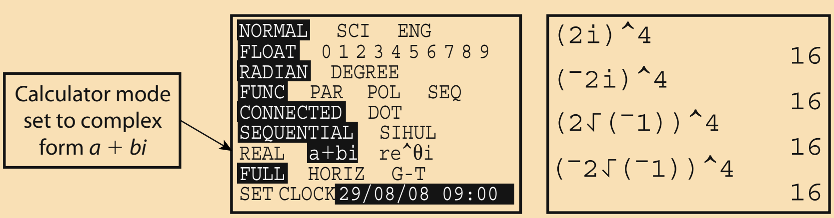
Roots

If a number can be expressed as the product of two equal factors, then that factor is called the **square root** of the number. For example, 7 is the square root of 49 because $7 \times 7 = 49$. Now 49 is also equal to -7×-7, so -7 is also a square root of 49. Every positive real number will have two real number square roots, one positive and one negative. However, there are many instances where we only want the positive square root. The symbol $\sqrt{\ }$ (called the **radical sign**) indicates only the positive square root, referred to as the **principal square root**. Because $4^2 = 16$ and $(-4)^2 = 16$ the square roots of 16 are 4 and -4; but the principal square root of 16 is only positive four, that is $\sqrt{16} = 4$. The negative square root of 16 is written as $-\sqrt{16} = -4$, and when both square roots are wanted we write $\pm\sqrt{16}$. In the real numbers, every positive number has two square roots (one positive and the other negative) but only one principal square root (positive) denoted with the radical sign.

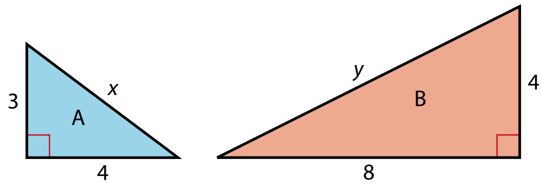
When a number can be expressed as the product of three equal factors, then that factor is called the **cube root** of the number. For example, -4 is the cube root of -64 because $(-4)^3 = -64$. With the radical sign this is written as $\sqrt[3]{-64} = -4$. In the real numbers, every number (positive or negative) has just one cube root. In the notation $\sqrt[n]{a}$, a is called the **radicand** and n is a positive integer called the **index**. The index indicates which root (square root or cube root or 4th root, etc.) is to be extracted. If no index is written it is assumed to be a 2, thereby indicating a square root.

In general, if a real number a can be expressed as the factor b multiplied n times, i.e. $b^n = a$, then that factor b is called the ***n*th root** of a. In the set of real numbers, if n is an even number (e.g. square root, 4th root, 6th root, etc.) then a has two nth roots (positive and negative) with the positive root being the **principal *n*th root**. Because $2^4 = 16$ and $(-2)^4 = 16$, then both 2 and -2 are 4th roots of 16. However, the principal 4th root of 16 is 2, written $\sqrt[4]{16} = 2$. If the index n is an odd number (e.g. cube root, 5th root, etc.) then the sign ($+$ or $-$) of the nth root of a will be the same as the sign of a. For example, the 5th root of 32 is 2, and the 5th root of -32 is -2. With the radical sign these results are written as $\sqrt[5]{32} = 2$ and $\sqrt[5]{-32} = -2$.

Our discussion here on roots and radicals is limited to the real numbers. We will learn in Chapter 10 that if we broaden our consideration to the complex numbers, then any number will have exactly n different nth roots. For example, the number 16 has four 4th roots: 2, -2, $2i$ and $-2i$. Your GDC may have the imaginary number i. Try taking the 4th power of $2i$ and $-2i$ (could also be entered as $2\sqrt{-1}$ and $-2\sqrt{-1}$) on your GDC (see calculator screen images below). You may need to change the mode of your calculator from real to complex.



● **Hint:** There are many words that have more than one meaning in mathematics. The correct interpretation of a word will depend on the situation (context) in which it is being applied. The word *root* is not only used for square root, cube root, *n*th root, etc. but can also mean the solution of an equation. For example, $x = 3$ and $x = -1$ are roots of the equation $x^2 - 2x - 3 = 0$ (see Section 3.5).



$$\begin{aligned} x^2 &= 3^2 + 4^2 \\ x^2 &= 9 + 16 \\ x^2 &= 25 \\ \sqrt{x^2} &= \sqrt{25} \\ x &= \sqrt{25} \\ x &= 5 \end{aligned}$$

$$\begin{aligned} y^2 &= 4^2 + 8^2 \\ y^2 &= 16 + 64 \\ y^2 &= 80 \\ \sqrt{y^2} &= \sqrt{80} \\ y &= \sqrt{80} \end{aligned}$$

● **Hint:** The solution for the hypotenuse of triangle A involves the equation $x^2 = 25$. Because x represents a length that must be positive, we want only the positive square root when taking the square root of both sides of the equation – i.e. $\sqrt{25}$. However, if there were no constraints on the value of x, we must remember that a positive number will have two square roots and we would write $\sqrt{x^2} = |x| = 5 \Rightarrow x = \pm 5$.

Radicals (surds)

Some roots are rational and some are irrational. Consider the two right triangles on the left. By applying Pythagoras' theorem, we find the length of the hypotenuse for triangle A to be exactly 5 (an integer and rational number) and the hypotenuse for triangle B to be exactly $\sqrt{80}$ (an irrational number). An irrational root – e.g. $\sqrt{80}, \sqrt{3}, \sqrt{10}, \sqrt[3]{4}$ – is called a **radical** or **surd**. The only way to express irrational roots exactly is in radical, or surd, form.

It is not immediately obvious that the following expressions are all equivalent.

$$\sqrt{80},\ 2\sqrt{20},\ \frac{16\sqrt{5}}{\sqrt{16}},\ 2\sqrt{2}\sqrt{10},\ \frac{10\sqrt{8}}{\sqrt{10}},\ 4\sqrt{5},\ 5\sqrt{\frac{16}{5}}$$

Square roots occur frequently in several of the topics in this course, so it will be useful for us to be able to simplify radicals and recognise equivalent radicals. Two useful rules for manipulating expressions with radicals are given below.

Simplifying radicals

For $a \geqslant 0$, $b \geqslant 0$ and $n \in \mathbb{Z}^+$, the following rules can be applied:

1 $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ **2** $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Note: Each rule can be applied in either direction.

Example 7 – Simplifying radicals I

Simplify completely:

a) $\sqrt{5} \times \sqrt{5}$ b) $\sqrt{12} \times \sqrt{21}$ c) $\dfrac{\sqrt{48}}{\sqrt{3}}$

d) $\sqrt[3]{12} \times \sqrt[3]{18}$ e) $7\sqrt{2} - 3\sqrt{2}$ f) $\sqrt{5} + 2\sqrt{25} - 3\sqrt{5}$

g) $\sqrt{3}(2 - 2\sqrt{3})$ h) $(1 + \sqrt{2})(1 - \sqrt{2})$

Solution

a) $\sqrt{5} \times \sqrt{5} = \sqrt{5 \cdot 5} = \sqrt{25} = 5$

Note: A special case of the rule $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ when $n = 2$ is $\sqrt{a} \times \sqrt{a} = a$.

b) $\sqrt{12} \times \sqrt{21} = \sqrt{4} \times \sqrt{3} \times \sqrt{3} \times \sqrt{7} = \sqrt{4} \times (\sqrt{3} \times \sqrt{3}) \times \sqrt{7}$

$= 2 \times 3 \times \sqrt{7} = 6\sqrt{7}$

c) $\dfrac{\sqrt{48}}{\sqrt{3}} = \sqrt{\dfrac{48}{3}} = \sqrt{16} = 4$

d) $\sqrt[3]{12} \times \sqrt[3]{18} = \sqrt[3]{12 \cdot 18} = \sqrt[3]{216} = 6$

e) $7\sqrt{2} - 3\sqrt{2} = 4\sqrt{2}$

f) $\sqrt{5} + 2\sqrt{25} - 3\sqrt{5} = 10 - 2\sqrt{5}$

g) $\sqrt{3}(2 - 2\sqrt{3}) = 2\sqrt{3} - 2\sqrt{3}\sqrt{3} = 2\sqrt{3} - 2 \cdot 3 = 2\sqrt{3} - 6$ or $-6 + 2\sqrt{3}$

h) $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - \sqrt{2} + \sqrt{2} - \sqrt{2}\sqrt{2} = 1 - 2 = -1$

The radical $\sqrt{24}$ can be simplified because one of the factors of 24 is 4, and the square root of 4 is rational (i.e. 4 is a perfect square).

$$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$$

Rewriting 24 as the product of 3 and 8 (rather than 4 and 6) would not help simplify $\sqrt{24}$ because neither 3 nor 8 are perfect squares, i.e. there is no integer whose square is 3 or 8.

Example 8 – Simplifying radicals II

Express each in terms of the simplest possible radical.

a) $\sqrt{80}$ b) $\sqrt{\frac{14}{81}}$ c) $\sqrt[3]{24}$ d) $5\sqrt{128}$

e) $\sqrt{x^2}$ f) $\sqrt{20a^4b^2}$ g) $\sqrt[3]{81}$ h) $\sqrt{4+9}$

Solution

a) $\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$

Note: 4 is a factor of 80 and is a perfect square, but 16 is the *largest* factor that is a perfect square

b) $\sqrt{\frac{14}{81}} = \frac{\sqrt{14}}{\sqrt{81}} = \frac{\sqrt{14}}{9}$

c) $\sqrt[3]{24} = \sqrt[3]{8} \times \sqrt[3]{3} = 2\sqrt[3]{3}$

d) $5\sqrt{128} = 5\sqrt{64}\sqrt{2} = 5 \cdot 8\sqrt{2} = 40\sqrt{2}$

e) $\sqrt{x^2} = |x|$

f) $\sqrt{20a^4b^2} = \sqrt{4}\sqrt{5}\sqrt{a^4}\sqrt{b^2} = 2a^2|b|\sqrt{5}$

g) $\sqrt[3]{81} = \sqrt[3]{27}\sqrt[3]{3} = 3\sqrt[3]{3}$

h) $\sqrt{4+9} = \sqrt{13}$

In many cases we prefer not to have radicals in the denominator of a fraction. Recall from Example 7, part a), the special case of the rule $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ when $n = 2$ is $\sqrt{a} \times \sqrt{a} = a$, assuming $a > 0$. The process of eliminating irrational numbers from the denominator is called **rationalizing the denominator**.

Example 9 – Rationalizing the denominator I

Rationalize the denominator of each expression.

a) $\frac{2}{\sqrt{3}}$ b) $\frac{\sqrt{7}}{4\sqrt{10}}$

Solution

a) $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

b) $\frac{\sqrt{7}}{4\sqrt{10}} = \frac{\sqrt{7}}{4\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{70}}{4 \cdot 10} = \frac{\sqrt{70}}{40}$

For any real number a, it would first appear that the rule $\sqrt{a^2} = a$ would be correct, but it is not. What if $a = -3$? Then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3. The correct rule that is true for any real number a is $\sqrt{a^2} = |a|$. Generalizing for any index where n is a positive integer, we need to consider whether n is even or odd. If n is even, then $\sqrt[n]{a^n} = |a|$; and if n is odd, then $\sqrt[n]{a^n} = a$.
For example,
$\sqrt[6]{(-3)^6} = \sqrt[6]{729} = \sqrt[6]{3^6} = 3$;
and
$\sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5$.

Hint: Note that in Example 8 h) the square root of a sum is *not* equal to the sum of the square roots. That is, avoid the error $\sqrt{a+b} \cancel{=} \sqrt{a} + \sqrt{b}$.

Changing a fraction from having a denominator that is irrational to an equivalent fraction where the denominator is rational (rationalizing the denominator) is not always a necessity. For example, expressing the cosine ratio of 45° as $\frac{1}{\sqrt{2}}$ rather than the equivalent value of $\frac{\sqrt{2}}{2}$ is mathematically correct. However, there will be instances where a fraction with a rational denominator will be preferred. It is a useful skill for simplifying some more complex fractions and for recognizing that two expressions are equivalent. For example, $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, or a little less obvious, $\frac{3}{2+\sqrt{5}} = -6 + 3\sqrt{5}$. There are even situations where it might be useful to rationalize the numerator (see Example 11 below).

Recall the algebraic rule $(a + b)(a - b) = a^2 - b^2$. Any pair of expressions fitting the form of $a + b$ and $a - b$ are called a pair of **conjugates**. The result of multiplying a pair of conjugates is always a **difference of two squares**, $a^2 - b^2$, and this can be helpful in some algebraic manipulations – as we will see in the next example.

Example 10 – Rationalizing the denominator II

Express the quotient $\frac{2}{4-\sqrt{3}}$ so that the denominator is a rational number.

Solution

Multiply numerator and denominator by the conjugate of the denominator, $4 + \sqrt{3}$, and simplify:

$$\frac{2}{4-\sqrt{3}} \cdot \frac{4+\sqrt{3}}{4+\sqrt{3}} = \frac{8+2\sqrt{3}}{4^2-(\sqrt{3})^2} = \frac{8+2\sqrt{3}}{16-3} = \frac{8+2\sqrt{3}}{13} \text{ or } \frac{8}{13} + \frac{2\sqrt{3}}{13}$$

Example 11 – Rationalizing the numerator

We will encounter the following situation in our study of calculus.

We are interested to analyze the behaviour of the quotient $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ as the value of h approaches zero. It is not possible to directly substitute zero in for h in the present form of the quotient because that will give an undefined result of $\frac{0}{0}$. Perhaps we can perform the substitution if we rationalize the numerator. We will assume that x and $x + h$ are positive.

Solution

Multiplying numerator and denominator by the conjugate of the numerator and simplifying:

$$\frac{(\sqrt{x+h}-\sqrt{x})}{h} \cdot \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Substituting zero for h into this expression causes no problems. Therefore, as h approaches zero, the expression $\frac{\sqrt{x+h} - \sqrt{x}}{h}$ would appear to approach the expression $\frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Exercise 1.2

In questions 1–15, express each in terms of the simplest possible radical.

1 $\sqrt{h^2} \times \sqrt{h^2}$

2 $\frac{\sqrt{45}}{\sqrt{5}}$

3 $\sqrt{18} \times \sqrt{10}$

4 $\sqrt{\frac{28}{49}}$

5 $\sqrt[3]{4} \times \sqrt[3]{16}$

6 $\sqrt{\frac{15}{20}}$

7 $\sqrt{5}(3 + 4\sqrt{5})$

8 $(2 + \sqrt{6})(2 - \sqrt{6})$

9 $\sqrt{98}$

10 $4\sqrt{1000}$

11 $\sqrt[3]{48}$

12 $\sqrt{12x^3y^3}$

13 $\sqrt[5]{m^5}$

14 $\sqrt{\frac{27}{6}}$

15 $\sqrt{x^{16}(1 + x)^2}$

In questions 16–18, completely simplify the expression.

16 $13\sqrt{7} - 10\sqrt{7}$

17 $\sqrt{72} - 8\sqrt{3} + 3\sqrt{48}$

18 $\sqrt{500} + 5\sqrt{20} - \sqrt{45}$

In questions 19–30, rationalize the denominator, simplifying if possible.

19 $\frac{1}{\sqrt{5}}$

20 $\frac{2}{5\sqrt{2}}$

21 $\frac{6\sqrt{7}}{\sqrt{3}}$

22 $\frac{4}{\sqrt{32}}$

23 $\frac{2}{1 + \sqrt{5}}$

24 $\frac{1}{3 + 2\sqrt{5}}$

25 $\frac{\sqrt{3}}{2 - \sqrt{3}}$

26 $\frac{4}{\sqrt{2} + \sqrt{5}}$

27 $\frac{x - y}{\sqrt{x} + \sqrt{y}}$

28 $\frac{1 + \sqrt{3}}{2 + \sqrt{3}}$

29 $\sqrt{\frac{1}{x^2} - 1}$

30 $\frac{h}{\sqrt{x + h} - \sqrt{x}}$

In questions 31–33, rationalize the numerator, simplifying if possible.

31 $\frac{\sqrt{a} - 3}{a - 9}$

32 $\frac{\sqrt{x} - \sqrt{y}}{x - y}$

33 $\frac{\sqrt{m} - \sqrt{7}}{7 - x}$

1.3 Exponents (indices)

As we've already seen with roots in the previous section, repeated multiplication of identical numbers can be written more efficiently by using exponential notation.

Exponential notation

If b is any real number ($b \in \mathbb{R}$) and n is a positive integer ($n \in \mathbb{Z}^+$), then

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}$$

where n is the **exponent**, b is the **base** and b^n is called the ***n*th power** of b.
Note: n is also called the **power** or **index** (plural: indices).

Integer exponents

We now state seven laws of integer exponents (or indices) that you will have learned in a previous mathematics course. Familiarity with these rules is essential for work throughout this course.

Let a and b be real numbers ($a, b \in \mathbb{R}$) and let m and n be integers ($m, n \in \mathbb{Z}$). Assume that all denominators and bases are not equal to zero. All of the laws can be applied in either direction.

Table 1.4 Laws of exponents (indices) for integer exponents.

	Property	Example	Description
1.	$b^m b^n = b^{m+n}$	$x^2 x^5 = x^7$	multiplying like bases
2.	$\frac{b^m}{b^n} = b^{m-n}$	$\frac{2w^7}{3w^2} = \frac{2w^5}{3}$	dividing like bases
3.	$(b^m)^n = b^{mn}$	$(3^x)^2 = 3^{2x} = (3^2)^x = 9^x$	a power raised to a power
4.	$(ab)^n = a^n b^n$	$(4k)^3 = 4^3k^3 = 64k^3$	the power of a product
5.	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{y}{3}\right)^2 = \frac{y^2}{3^2} = \frac{y^2}{9}$	the power of a quotient
6.	$a^0 = 1$	$(t^2 + 5)^0 = 1$	definition of a zero exponent
7.	$a^{-n} = \frac{1}{a^n}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	definition of a negative exponent

• **Hint:** If the base of an exponential expression is negative, then it is necessary to write it in brackets. The expression such as -3^2 is equivalent to $-(3)^2$. Hence, $(-3)^2 = 9$ but $-3^2 = -9$.

Negative integers and fractions were first used as exponents in the modern conventional notation (as raised numbers, such as 5^{-2}, $x^{\frac{3}{2}}$) by Isaac Newton in a letter in 1676 to a fellow scientist in which he described his derivation of the binomial theorem (Chapter 4 in this book).

The last two laws of exponents listed above – the definition of a zero exponent and the definition of a negative exponent – are often assumed without proper explanation. The definition of a^n as repeated multiplication, i.e. n factors of a, is easily understood when n is a positive integer. So how do we formulate appropriate definitions for a^n when n is negative or zero? These definitions will have to be compatible with the laws for positive integer exponents. If the law stating $b^m b^n = b^{m+n}$ is to hold for a zero exponent, then $b^n b^0 = b^{n+0} = b^n$. Since the number 1 is the identity element for multiplication (multiplicative identity property) then $b^n \cdot 1 = b^n$. Therefore, we must define b^0 as the number 1. If the law $b^m b^n = b^{m+n}$ is to also hold for negative integer exponents, then

$b^n b^{-n} = b^{n-n} = b^0 = 1$. Since the product of b^n and b^{-n} is 1, then they must be reciprocals (multiplicative inverse property). Therefore, we must define b^{-n} as $\frac{1}{b^n}$.

Rational exponents (fractional exponents)

We know that $4^3 = 4 \times 4 \times 4$ and $4^0 = 1$ and $4^{-2} = \frac{1}{4^2} = \frac{1}{4 \times 4}$, but what meaning are we to give to $4^{\frac{1}{2}}$? In order to carry out algebraic operations with expressions having exponents that are rational numbers, it will be very helpful if they follow the laws established for integer exponents. From the law $b^m b^n = b^{m+n}$, it must follow that $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}} = 4^1$. Likewise, from the law $(b^m)^n = b^{mn}$, it follows that $(4^{\frac{1}{2}})^2 = 4^{\frac{1}{2}\cdot 2} = 4^1$. Therefore, we need to define $4^{\frac{1}{2}}$ as the square root of 4 or, more precisely, as the principal (positive) square root of 4, that is, $\sqrt{4}$. We are now ready to use radicals to define a rational exponent of the form $\frac{1}{n}$, where n is a positive integer. If the rule $(b^m)^n = b^{mn}$ is to apply when $m = \frac{1}{n}$, it must follow that $(b^{\frac{1}{n}})^n = b^{\frac{n}{n}} = b^1$. This means that the nth power of $b^{\frac{1}{n}}$ is b and, from the discussion of nth roots in Section 1.2, we define $b^{\frac{1}{n}}$ as the principal nth root of b.

Definition of $b^{\frac{1}{n}}$

If $n \in \mathbb{Z}^+$, then $b^{\frac{1}{n}}$ is the principal nth root of b. Using a radical, this means

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

This definition allows us to evaluate exponential expressions such as the following:

$$36^{\frac{1}{2}} = \sqrt{36} = 6;\ (-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3;\ \left(\frac{1}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{1}{81}} = \frac{1}{3}$$

Now we can apply the definition of $b^{\frac{1}{n}}$ and the rule $(b^m)^n = b^{mn}$ to develop a rule for expressions with exponents of the form not just $\frac{1}{n}$ but of the more general form $\frac{m}{n}$.

$$b^{\frac{m}{n}} = b^{m\frac{1}{n}} = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m};\ \text{or, equivalently, } b^{\frac{m}{n}} = b^{\frac{1}{n}\cdot m} = (b^{\frac{1}{n}})^m = (\sqrt[n]{b})^m$$

This will allow us to evaluate exponential expressions such as $9^{\frac{3}{2}}$, $(-8)^{\frac{5}{3}}$ and $64^{\frac{5}{6}}$.

Definition of a rational exponent

If m and n are positive integers with no common factors, then

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} \text{ or } (\sqrt[n]{b})^m$$

If n is an even number, then we must have $b \geqslant 0$.

The numerator of a rational exponent indicates the power to which the base of the exponential expression is raised, and the denominator indicates the root to be taken. With this definition for rational exponents, we can conclude that all of the laws of exponents stated for integer exponents in Table 1.4 also hold true for rational exponents.

Example 12 – Applying laws of exponents

Evaluate and/or simplify each of the following expressions. Leave only positive exponents.

a) $(3a^2b)^3$ b) $3(a^2b)^3$ c) $(-2)^{-3}$

d) $(x+y)^0$ e) $(3^3)^{\frac{1}{2}} \cdot 9^{\frac{3}{4}}$ f) $\dfrac{m^2n^{-3}}{m^{-5}n^3}$

g) $(-27)^{-\frac{2}{3}}$ h) $8^{\frac{2}{3}}$ i) $(2^x)(2^{3-x})$

j) $(0.04)^{-2}$ k) $\dfrac{\sqrt{a}\sqrt{a^3}}{a^3}\ (a > 0)$ l) $\dfrac{x^{-2}y^3z^{-4}}{(2x^2)^3} \times \dfrac{8}{y^{-2}z^4}$

m) $\sqrt[4]{81a^8b^{12}}$ n) $\dfrac{x^{\frac{3}{2}} + x^{\frac{1}{2}}}{x^{\frac{1}{2}}}\ (x > 0)$ o) $2^{n+3} - 2^{n+1}$

p) $\dfrac{\sqrt{a+b}}{a+b}$ q) $\dfrac{(x+y)^2}{(x+y)^{-2}}$ r) $\dfrac{x^2 + 2^{\frac{3}{2}} - 2(x^2+2)^{\frac{1}{2}}}{x^2}$

• **Hint for (o):** apply $b^m b^n = b^{m+n}$ in other direction.

Solution

a) $(3a^2b)^3 = 3^3(a^2)^3b^3 = 27a^6b^3$

b) $3(a^2b)^3 = 3(a^2)^3b^3 = 3a^6b^3$

c) $(-2)^{-3} = \dfrac{1}{(-2)^3} = -\dfrac{1}{8}$

d) $(x+y)^0 = 1$

e) $(3^3)^{\frac{1}{2}} \cdot 9^{\frac{3}{4}} = 3^{\frac{3}{2}}(3^2)^{\frac{3}{4}} = 3^{\frac{3}{2}} \cdot 3^{\frac{3}{2}} = 3^{\frac{6}{2}} = 3^3 = 27$

f) $\dfrac{m^2n^{-3}}{m^{-5}n^3} = \dfrac{m^2}{m^{-5}} \cdot \dfrac{n^{-3}}{n^3} = \dfrac{m^{2-(-5)}}{1} \cdot \dfrac{1}{n^{3-(-3)}} = \dfrac{m^7}{n^6}$

g) $(-27)^{-\frac{2}{3}} = [(-3)^3]^{-\frac{2}{3}} = (-3)^{3(-\frac{2}{3})} = (-3)^{-2} = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$

h) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = (2)^2 = 4$ or $8^{\frac{2}{3}} = (2^3)^{\frac{3}{2}} = 2^2 = 4$

i) $(2^x)(2^{3-x}) = 2^{x+3-x} = 2^3 = 8$

j) $(0.04)^{-2} = \left(\dfrac{4}{100}\right)^{-2} = \left(\dfrac{1}{25}\right)^{-2} = \left(\dfrac{25}{1}\right)^2 = 625$

k) $\dfrac{\sqrt{a}\sqrt{a^3}}{a^3} = \dfrac{a^{\frac{1}{2}} \cdot a^{\frac{3}{2}}}{a^3} = \dfrac{a^{\frac{1}{2}+\frac{3}{2}}}{a^3} = \dfrac{a^2}{a^3} = \dfrac{1}{a}$

l) $\dfrac{x^{-2}y^3z^{-4}}{(2x^2)^3} \times \dfrac{8}{y^{-2}z^4} = \dfrac{x^{-2}y^3z^{-4}}{8x^6} \times \dfrac{8}{y^{-2}z^4} = \dfrac{y^3}{x^2x^6z^4} \times \dfrac{y^2}{z^4} = \dfrac{y^5}{x^8z^8}$

m) $\sqrt[4]{81a^8b^{12}} = \sqrt[4]{81} \cdot \sqrt[4]{a^8} \cdot \sqrt[4]{b^{12}} = 3a^{\frac{8}{4}}b^{\frac{12}{4}} = 3a^2b^3$

n) $\dfrac{x^{\frac{3}{2}} + x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} + \dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \dfrac{x^{\frac{3}{2}-\frac{1}{2}}}{1} + 1 = x + 1$

o) $2^{n+3} - 2^{n+1} = (2^n)(2^3) - (2^n)(2^1) = 8(2^n) - 2(2^n) = 6(2^n)$

p) $\dfrac{\sqrt{a+b}}{a+b} = \dfrac{(a+b)^{\frac{1}{2}}}{(a+b)^1} = \dfrac{1}{(a+b)^{1-\frac{1}{2}}} = \dfrac{1}{(a+b)^{\frac{1}{2}}} = \dfrac{1}{\sqrt{a+b}}$

q) $\dfrac{(x+y)^2}{(x+y)^{-2}} = (x+y)^{2-(-2)} = (x+y)^4$

Although $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, merely expanding is not 'simplifying'.

r) $\dfrac{(x^2+2)^{\frac{3}{2}} - 2(x^2+2)^{\frac{1}{2}}}{x^2} = \dfrac{(x^2+2)^{\frac{1}{2}}[(x^2+2)^1 - 2]}{x^2} = \dfrac{(x^2+2)^{\frac{1}{2}}[x^2]}{x^2}$

$= (x^2+1)^{\frac{1}{2}}$ or $\sqrt{x^2+1}$

● **Hint:** Note that in Example 12 q) that the square of a sum is **not** equal to the sum of the squares. That is, avoid the error $(x+y)^2 \neq x^2 + y^2$, and in general $(x+y)^n \neq x^n + y^n$.

Exercise 1.3

In questions 1–6, simplify (without your GDC) each expression to a single integer.

1 $16^{\frac{1}{4}}$ **2** $9^{\frac{3}{2}}$ **3** $64^{\frac{2}{3}}$

4 $8^{\frac{4}{3}}$ **5** $32^{\frac{3}{5}}$ **6** $(\sqrt{2})^6$

In questions 7–9, simplify each expression (without your GDC) to a quotient of two integers.

7 $\left(\dfrac{8}{27}\right)^{\frac{2}{3}}$ **8** $\left(\dfrac{9}{16}\right)^{\frac{1}{2}}$ **9** $\left(\dfrac{25}{4}\right)^{\frac{3}{2}}$

In questions 10–13, evaluate (without your GDC) each expression.

10 $(-3)^{-2}$ **11** $(13)^0$ **12** $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ **13** $\left(-\dfrac{3}{4}\right)^{-3}$

In questions 14–34, simplify each exponential expression (leave only positive exponents).

14 $(-xy^3)^2$ **15** $-(xy^3)^2$ **16** $(-2xy^3)^3$

17 $(2x^3y^{-5})(2x^{-1}y^3)^4$ **18** $(4m^2)^{-3}$ **19** $\dfrac{3k^3p^4}{(3k^3)^2p^2}$

20 $(-32)^{\frac{3}{5}}$ **21** $(125)^{\frac{2}{3}}$ **22** $\dfrac{x\sqrt{x}}{\sqrt[3]{x}}$

23 $\dfrac{4a^3b^5}{(2a^2b)^4} \cdot \dfrac{b^{-1}}{a^{-3}}$ **24** $\dfrac{(\sqrt[3]{x})(\sqrt[3]{x^4})}{\sqrt[3]{x^2}}$ **25** $\dfrac{6(a-b)^2}{3a-3b}$

26 $\dfrac{(x+4y)^{\frac{1}{2}}}{2(x+4y)^{-1}}$ **27** $\dfrac{p^2+q^2}{\sqrt{p^2+q^2}}$ **28** $\dfrac{5^{3x+1}}{25}$

29 $\dfrac{x^{\frac{1}{3}} + x^{\frac{1}{4}}}{x^{\frac{1}{2}}}$ **30** $3^{n+1} - 3^{n-2}$ **31** $\dfrac{8^{k+2}}{2^{3k+2}}$

32 $\sqrt[3]{24x^6y^{12}}$ **33** $\dfrac{1}{n}\sqrt{n^2+n^4}$ **34** $\dfrac{x+\sqrt{x}}{1+\sqrt{x}}$

● **Hint:** In question 34 it is incorrect to 'cancel' the term of $\sqrt{x}$ from the numerator and denominator. That is, remember $\dfrac{a+b}{c+b} \neq \dfrac{a}{c}$.

1.4 Scientific notation (standard form)

Exponents provide an efficient way of writing and calculating with very large or very small numbers. The need for this is especially great in science. For example, a light year (the distance that light travels in one year) is 9 460 730 472 581 kilometres and the mass of a single water molecule is 0.000 000 000 000 000 000 000 0056 grams. It is far more convenient and useful to write such numbers in **scientific notation** (also called **standard form**).

Scientific notation

A positive number N is written in scientific notation if it is expressed in the form:

$$N = a \times 10^k, \text{ where } 1 \leqslant a < 10 \text{ and } k \text{ is an integer.}$$

In scientific notation, a light year is about 9.46×10^{12} kilometres. This expression is determined by observing that when a number is multiplied by 10^k and k is **positive**, the decimal point will move k places to the **right**. Therefore, $9.46 \times 10^{12} = 9\underbrace{460\,000\,000\,000}_{\text{12 decimal places}}$. Knowing that when a number is multiplied by 10^k and k is **negative** the decimal point will move k places to the **left** helps us to express the mass of a water molecule as 5.6×10^{-24} grams. This expression is equivalent to $0.\underbrace{000\,000\,000\,000\,000\,000\,000\,0056}_{\text{24 decimal places}}$.

Scientific notation is also a very convenient way of indicating the number of **significant figures** (digits) to which a number has been approximated. A light year expressed to an accuracy of 13 significant figures is 9 460 730 472 581 kilometres. However, many calculations will not require such a high degree of accuracy. For a certain calculation it may be more appropriate to have a light year approximated to 4 significant figures, which could be written as 9 461 000 000 000 kilometres, or more efficiently and clearly in scientific notation as 9.461×10^{12} kilometres.

Not only is scientific notation conveniently compact, it also allows a quick comparison of the magnitude of two numbers without the need to count zeros. Moreover, it enables us to use the laws of exponents to perform otherwise unwieldy calculations.

Example 13 – Scientific notation

Use scientific notation to calculate each of the following.

a) $64\,000 \times 2\,500\,000\,000$

b) $\dfrac{0.000\,000\,78}{0.000\,000\,0012}$

c) $\sqrt[3]{27\,000\,000\,000}$

Solution

a) $64\,000 \times 2\,500\,000\,000 = (6.4 \times 10^4)(2.5 \times 10^9)$

$= 6.4 \times 2.5 \times 10^4 \times 10^9$

$= 16 \times 10^{4+9}$

$= 1.6 \times 10^1 \times 10^{13} = 1.6 \times 10^{14}$

b) $\dfrac{0.000\,000\,78}{0.000\,000\,0012} = \dfrac{7.8 \times 10^{-7}}{1.2 \times 10^{-9}} = \dfrac{7.8}{1.2} \times \dfrac{10^{-7}}{10^{-9}} = 6.5 \times 10^{-7-(-9)}$

$= 6.5 \times 10^2$ or 650

c) $\sqrt[3]{27\,000\,000\,000} = (2.7 \times 10^{10})^{\frac{1}{3}} = (27 \times 10^9)^{\frac{1}{3}} = (27)^{\frac{1}{3}}(10^9)^{\frac{1}{3}}$

$= 3 \times 10^3$ or 3000

Your GDC will automatically express numbers in scientific notation when a large or small number exceeds its display range. For example, if you use your GDC to compute 2 raised to the 64th power, the display (depending on the GDC model) will show the approximation

`1.844674407E19` or `1.844674407 19`

The final digits indicate the power of 10, and we interpret the result as $1.844\,674\,407 \times 10^{19}$. ($2^{64}$ is exactly 18 446 744 073 709 551 616.)

Exercise 1.4

In questions 1–10, write each number in scientific notation, rounding to three significant figures.

1 253.8 **2** 0.007 81 **3** 7 405 239

4 0.000 001 0448 **5** 4.9812 **6** 0.001 991

7 Land area of Earth: 148 940 000 square kilometres

8 Relative density of hydrogen: 0.000 0899 grams per cm^3

9 Mean distance from the Earth to the Sun (a unit of length referred to as the Astronomical Unit, *AU*): 149 597 870.691 kilometres

10 Mass of an electron 0.000 000 000 000 000 000 000 000 000 000 910 938 15 kg

In questions 11–14, write each number in ordinary decimal notation.

11 2.7×10^{-3} **12** 5×10^7

13 9.035×10^{-8} **14** 4.18×10^{12}

In questions 15–22, use scientific notation and the laws of exponents to perform the indicated operations. Give the result in scientific notation rounded to two significant figures.

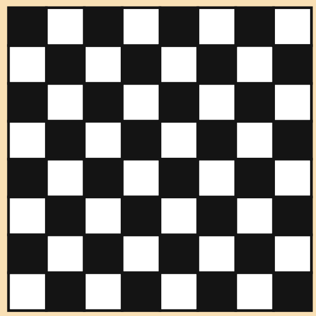
15 $(2.5 \times 10^{-3})(10 \times 10^5)$ **16** $\dfrac{3.2 \times 10^6}{1.6 \times 10^2}$

17 $\dfrac{(1 \times 10^{-3})(3.28 \times 10^6)}{4 \times 10^7}$ **18** $(2 \times 10^3)^4(3.5 \times 10^5)$

19 (0.000 000 03) (6 000 000 000 000) **20** $\dfrac{(1\,000\,000)^2\sqrt{0.000\,000\,04}}{(8\,000\,000\,000)^{\frac{2}{3}}}$

21 $\dfrac{4 \times 10^4}{(6.4 \times 10^2)(2.5 \times 10^{-5})}$ **22** $(5.4 \times 10^2)^5(-1.1 \times 10^{-6})^2$

The wheat and chessboard problem is a mathematical question that is posed as part of a story that has been told in many variations over the centuries. In any version of the story, the question is: If one grain of wheat is placed on the first square of an 8 by 8 chessboard, then two grains of wheat on the second square, four grains on the third square, and so on – each time doubling the grains of rice – then exactly how many grains of wheat in total are on the board after grains are placed on the last square?



1.5 Algebraic expressions

Examples of algebraic expressions are

$$5a^3b^2 \qquad 2x^2 + 7x - 8 \qquad \frac{Gm_1m_2}{r^2} \qquad \frac{t}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

Algebraic expressions are formed by combining variables and constants using addition, subtraction, multiplication, division, exponents and radicals.

The word *algebra* comes from the 9th-century Arabic book *Hisâb al-Jabr w'al-Muqabala*, written by the Islamic mathematician and astronomer Abu Ja'far Muhammad ibn Musa al-Khwarizmi (c. 778–850). The book title refers to transposing and combining terms, two processes used in solving equations. In Latin translations, the title was shortened to *Aljabr*, from which we get the word *algebra*. Al-Khwarizmi worked as a scholar in Baghdad studying and writing about mathematics and science. Some of his works were later translated into Latin, thus helping to establish Hindu-Arabic numerals and algebra concepts into Europe. The word *algorithm* comes from a Latinized version of his name.

Polynomials

An algebraic expression that has only non-negative powers of one or more variables and contains no variable in a denominator is called a **polynomial**.

Definition of a polynomial in the variable x

Given $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$ and $n \geqslant 0$, $n \in \mathbb{Z}^+$, then a **polynomial in x** is a sum of distinct **terms** in the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

where $a_1, a_2, \ldots, a_n$ are the **coefficients**, a_0 is the **constant term** and n (the greatest exponent) is the **degree** of the polynomial.

● **Hint:** Polynomials with one, two and three terms are called **monomials**, **binomials** and **trinomials**, respectively. A polynomial of: degree 1 is '**linear**'; degree 2 is '**quadratic**'; degree 3 is '**cubic**'; degree 4 is '**quartic**' and degree 5 is '**quintic**'. Beyond degree 5 there are no generally accepted names for polynomials. Quadratic polynomials are studied in depth in Chapter 3.

Polynomials are added or subtracted using the properties of real numbers that were discussed in Section 1.1 of this chapter. We do this by combining **like terms** – terms containing the same variable(s) raised to the same power(s) – and applying the distributive property.

For example,

$2x^2y + 6x^2 - 7x^2y = 2x^2y - 7x^2y + 6x^2$ Rearrange terms so the like terms are together.

$= (2 - 7)x^2y + 6x^2$ Apply distributive property: $ab + ac = (b + c)a$.

$= -5x^2y + 6x^2$ No like terms remain, so polynomial is simplified.

Expanding and factorizing polynomials

We apply the distributive property in the other direction, i.e. $a(b + c) = ab + ac$, in order to multiply polynomials. For example,

$$\begin{aligned}(2x - 3)(x + 5) &= 2x(x + 5) - 3(x + 5)\\ &= 2x^2 + 10x - 3x - 15 && \text{Combining like terms } 10x \text{ and } -3x.\\ &= 2x^2 + 7x - 15 && \text{Terms written in descending order of the exponents.}\end{aligned}$$

The process of multiplying polynomials is often referred to as **expanding**. Especially in the case of a polynomial being raised to a power, the number of terms in the resulting polynomial, after applying the distributive property and combining like terms, has increased (expanded) compared to the original number of terms. For example,

$$\begin{aligned}(x + 3)^2 &= (x + 3)(x + 3) && \text{Squaring a 1st degree (linear) binomial.}\\ &= x(x + 3) + 3(x + 3)\\ &= x^2 + 3x + 3x + 9\\ &= x^2 + 6x + 9 && \text{The result is a 2nd degree (quadratic) trinomial.}\end{aligned}$$

and

$$\begin{aligned}(x + 1)^3 &= (x + 1)(x + 1)(x + 1) && \text{Cubing a 1st degree binomial.}\\ &= (x + 1)(x^2 + x + x + 1)\\ &= x(x^2 + 2x + 1) + 1(x^2 + 2x + 1) && \text{Distributive property.}\\ &= x^3 + 2x^2 + x + x^2 + 2x + 1\\ &= x^3 + 3x^2 + 3x + 1 && \text{Result is a 3rd degree (cubic) polynomial with four terms.}\end{aligned}$$

As stated in Section 1.2, pairs of binomials of the form $a + b$ and $a - b$ are called **conjugates**. In most instances, the product of two binomials produces a trinomial. However, the product of a pair of conjugates produces a binomial such that both terms are squares and the second term is negative – referred to as a **difference of two squares**. For example,

$$\begin{aligned}(x + 5)(x - 5) &= x(x - 5) + 5(x - 5) && \text{Multiplying two conjugates;}\\ &= x^2 - 5x + 5x - 25 && \text{distributive property.}\\ &= x^2 - 25 && x^2 - 25 \text{ is a difference of two squares.}\end{aligned}$$

The inverse (or 'undoing') of multiplication (expansion) is **factorization**. If it is helpful for us to rewrite a polynomial as a product, then we need to factorize it – i.e. apply the distributive property in the *reverse* direction $(ab + ac = (b + c)a)$. The previous four examples can be used to illustrate equivalent pairs of factorized and expanded polynomials.

Factorized		**Expanded**
$(2x - 3)(x + 5)$	$=$	$2x^2 + 7x - 15$
$(x + 3)^2$	$=$	$x^2 + 6x + 9$
$(x + 1)^3$	$=$	$x^3 + 3x^2 + 3x + 1$
$(x + 5)(x - 5)$	$=$	$x^2 - 25$

Certain polynomial expansions (products) and factorizations occur so frequently you should be able to quickly recognize and apply them. Here is a list of some of the more common ones. You can verify these identities by performing the multiplication (expanding).

Common polynomial expansion and factorization patterns

Expanding →

Product of two binomials	$(x + a)(x + b)$	$= x^2 + (a + b)x + ab$	Factorizing a trinomial
Product of two binomials	$(ax + b)(cx + d)$	$= acx^2 + (ad + bc)x + bd$	Factorizing a trinomial
Product of two conjugates	$(a + b)(a - b)$	$= a^2 - b^2$	Difference of two squares
Square of sum of 2 terms	$(a + b)^2$	$= a^2 + 2ab + b^2$	Trinomial perfect square
Square of difference of 2 terms	$(a - b)^2$	$= a^2 - 2ab + b^2$	Trinomial perfect square
Cube of a sum of 2 terms	$(a + b)^3$	$= a^3 + 3a^2b + 3ab^2 + b^3$	Perfect cube
Cube of difference of 2 terms	$(a - b)^3$	$= a^3 - 3a^2b + 3ab^2 - b^3$	Perfect cube
	$(a + b)(a^2 - ab + b^2)$	$= a^3 + b^3$	Sum of two cubes
	$(a - b)(a^2 + ab + b^2)$	$= a^3 - b^3$	Difference of two cubes

← Factorizing

These identities are useful patterns into which we can substitute any number or algebraic expression for a, b or x. This allows us to efficiently find products and powers of polynomials and also to factorize many polynomials.

Example 14 – Multiplying polynomials

Find each product.

a) $(x + 2)(x - 7)$ b) $(3x - 4)(4x + 1)$

c) $(6x + y)(6x - y)$ d) $(4h - 5)^2$

e) $(a + 2)^3$ f) $(3x + 2\sqrt{5})(3x - 2\sqrt{5})$

g) $(x^2 - y)^3$ h) $(1 + 3m)^2$

i) $(x + 2i)(x - 2i)$ j) $(x + y + 4)(x + y - 4)$

k) $(-6 - 15w)(w + 2)$ l) $(a - b + c)^2$

Solution

a) This product fits the pattern $(x + a)(x + b) = x^2 + (a + b)x + ab$.

$$(x + 2)(x - 7) = x^2 + (2 - 7)x + (2)(-7) = x^2 - 5x - 14$$

You should be able to perform the middle step 'mentally' without writing it.

b) This product fits the pattern $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$.

$$(3x - 4)(4x + 1) = 12x^2 + (3 - 16)x - 4 = 12x^2 - 13x - 4$$

c) This fits the pattern $(a + b)(a - b) = a^2 - b^2$ where the result is a difference of two squares.

$$(5x^3 + 3y)(5x^3 - 3y) = (5x^3)^2 - (3y)^2 = 25x^6 - 9y^2$$

d) This fits the pattern $(a - b)^2 = a^2 - 2ab + b^2$.

$$(4h - 5)^2 = (4h)^2 - 2(4h)(5) + (5)^2 = 16h^2 - 40h + 25$$

e) This fits the pattern $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$$(a + 2)^3 = (a)^3 + 3(a)^2(2) + 3(a)(2)^2 + (2)^3 = a^3 + 6a^2 + 12a + 8$$

f) This is a pair of conjugates, so they fit the pattern $(a + b)(a - b) = a^2 - b^2$.

$$(3x + 2\sqrt{5})(3x - 2\sqrt{5}) = (3x)^2 - (2\sqrt{5})^2 = 9x^2 - (4 \cdot 5) = 9x - 20$$

Note: As we have observed earlier, the product of two **irrational** conjugates is a single **rational** number. We used this result to simplify fractions with irrational denominators in Section 1.2.

g) This fits the pattern $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

$$(x^2 - 4y)^3 = (x^2)^3 - 3(x^2)^2(4y) + 3(x^2)(4y)^2 - (4y)^3$$
$$= x^6 - 12x^4y + 48x^2y^2 - 64y^3$$

h) This fits the pattern $(a + b)^2 = a^2 + 2ab + b^2$.

$$(1 + 3m^2)^2 = (1)^2 + 2(1)(3m^2) + (3m^2)^2$$
$$= 1 + 6m^2 + 9m^4 \text{ or } 9m^4 + 6m^2 + 1$$

i) This fits the pattern $(a + b)(a - b) = a^2 - b^2$.

$$(x + 2i)(x - 2i) = x^2 - (2i)^2 = x^2 - 4i^2 = x^2 - 4(-1) = x^2 + 4$$

Remember from Section 1.1, that the imaginary number i is defined such that $i^2 = -1$.

j) Initially the product does not seem to fit a pattern and we can find the product simply by applying the distributive property.

$$(x + y + 4)(x + y - 4) = x^2 + xy - 4x + xy + y^2 - 4y + 4x + 4y - 16$$
$$= x^2 + 2xy + y^2 - 16$$

However, upon closer inspection we see that this is a product of two conjugates. This can be made clear with the insertion of brackets.

$$[(x + y) + 4][(x + y) - 4] = (x + y)^2 - 4^2 = x^2 + 2xy + y^2 - 16$$

k) Factor out GCF of -3 from the first factor, and then multiply.

$$(-6 - 9w)(3w + 2) = -3(2 + 3w)(3w + 2) = -3(3w + 2)^2$$
$$= -3(9w^2 + 12w + 4) = -27w^2 - 36w - 12$$

l) By inserting a pair of brackets, this product can be considered as the square of a binomial.

$$(a - b + c)^2 = [(a - b) + c]^2 = (a - b)^2 + 2(a - b)c + c^2$$
$$= a^2 - 2ab + b^2 + 2ac - 2bc + c^2$$

or $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$

Note: It would be incorrect to insert brackets to write $(a - b + c)^2 = [a - (b + c)]^2$ for (l). Why?

● **Hint:** The result in Example 14 i), $(x + 2i)(x - 2i) = x^2 + 4$, shows that imaginary numbers could be used to factorize certain polynomials. However, when we factorize a polynomial in this course we will only look for factors that contain coefficients and/or constants that are rational numbers. For example, we consider both of the polynomials $x^2 - 5$ and $x^2 + 9$ *not* to be factorable, even though $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$, and $x^2 + 9 = (x + 3i)(x - 3i)$.

Example 15 – Factorizing polynomials

Completely factorize the following expressions.

a) $2x^2 - 14x + 24$	b) $2x^2 + x - 15$	c) $8x^7 - 18x$
d) $3y^3 + 24y^2 + 48y$	e) $(x + 3)^2 - y^2$	f) $5x^3y + 20xy^3$
g) $c^3 + 27$	h) $1 - 8h^6$	i) $a^4 - \frac{1}{16}$
j) $15 - x^2 - 2x$	k) $3x^2 + 20x - 7$	l) $y^2 + 5y + \frac{25}{4}$

Solution

a) $2x^2 - 14x + 24 = 2(x^2 - 7x + 12)$ Factor out the greatest common factor (GCF).

$= 2[x^2 + (-3 - 4)x + (-3)(-4)]$ Fits the pattern $(x + a)(x + b) = x^2 + (a + b)x + ab$.

$= 2(x - 3)(x - 4)$ 'Trial and error' to find $-3 - 4 = -7$ and $(-3)(-4) = 12$.

b) The terms have no common factor and the leading coefficient is not equal to one. This factorization requires a logical 'trial and error' approach. There are eight possible factorizations.

$(2x - 1)(x + 15)$ $(2x - 3)(x + 5)$ $(2x - 5)(x + 3)$ $(2x - 15)(x + 1)$

$(2x + 1)(x - 15)$ $(2x + 3)(x - 5)$ $(2x + 5)(x - 3)$ $(2x + 15)(x - 1)$

Testing the middle term in each, you find that the correct factorization is $2x^2 + x - 15 = (2x - 5)(x + 3)$.

c) Factor out GCF then write as difference of two squares in the form $a^2 - b^2 = (a + b)(a - b)$.

$$8x^7 - 18x = 2x(4x^6 - 9) = 2x[(2x^3)^2 - 3^2] = 2x(2x^3 + 3)(2x^3 - 3)$$

d) $3y^3 + 24y^2 + 48y = 3y(y^2 + 8y + 16)$ Factor out the greatest common factor.

$= 3y(y^2 + 2 \cdot 4y + 4^2)$ Fits the pattern $a^2 + 2ab + b^2 = (a + b)^2$.

$= 3y(y + 4)^2$

e) Fits the difference of two squares pattern: $a^2 - b^2 = (a + b)(a - b)$ with $a = x + 3$ and $b = y$.

Therefore, $(x + 3)^2 - y^2 = [(x + 3) + y][(x + 3) - y]$
$= (x + y + 3)(x - y + 3)$

f) $5x^3y + 20xy^3 = 5xy(x^2 + 4y^2)$ We can only factor out the greatest common factor of $5xy$. Although both of the terms x^2 and $4y^2$ are perfect squares, the expression $x^2 + 4y^2$ is not a *difference* of squares – and, hence, it cannot be factorized. The sum of two squares, $a^2 + b^2$, cannot be factorized.

g) This binomial is the sum of two cubes, fitting the pattern $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$$c^3 + 27 = c^3 + 3^3 = (c + 3)(c^2 - 3c + 9)$$

h) This binomial is the difference of two cubes, fitting the pattern $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$$1 - 8h^6 = 1^3 - (2h^2)^3 = (1 - 2h)(1 + 2h^2 + 4h^4)$$

i) This binomial is the difference of two squares – but be sure to factorize completely.

$$a^4 - \tfrac{1}{16} = (a^2 - \tfrac{1}{4})(a^2 + \tfrac{1}{4}) = (a + \tfrac{1}{2})(a - \tfrac{1}{2})(a^2 + \tfrac{1}{4})$$

j) Write the terms in order of descending exponents and then factor out the -1 so that the leading coefficient is positive.

$$15 - x^2 - 2x = -x^2 - 2x + 15 = -(x^2 + 2x - 15) = -(x + 5)(x - 3)$$

k) When searching for factors of a quadratic like $3x^2 + 20x - 7$ we restrict our search to factors with coefficients and constants that are integers. Since 3 is a prime number, then we can start the factorizing by writing $3x^2 + 20x - 7 = (3x + ?)(x + ?)$. We know the two missing numbers have a product of -7, and since 7 is a prime number then the two missing numbers are either -7 and 1, or -1 and 7. With trial and error, it can be determined that $3x^2 + 20x - 7 = (3x - 1)(x + 7)$.

l) This fits the factoring pattern of $a^2 + 2ab + b^2 = (a + b)^2$ (trinomial perfect square). Consider the pattern written as $a^2 + (2b)a + b^2$ and substitute y for a, then $y^2 + (2b)y + b^2$. The last term, $\frac{25}{4}(b^2)$, is the square of $\frac{5}{2}$ which is one-half of 5, the coefficient of the middle term $(2b)$. Thus, $y^2 + 5y + \frac{25}{4} = (y + \frac{5}{2})^2$.

Guidelines for factoring polynomials

1. Factor out the greatest common factor (GCF), if one exists.
2. Determine if the polynomial, or any factors, fit any of the special polynomial patterns – and factor accordingly.
3. Any quadratic trinomial of the form $ax^2 + bx + c$ will require a logical trial and error approach, if it factorizes.

Most polynomials cannot be factored into a product of polynomials with integer or rational coefficients. In fact, factorizing is often difficult even when possible for polynomials with degree 3 or higher. Nevertheless, factorizing is a powerful algebraic technique that can be applied in many situations.

Algebraic fractions

An **algebraic fraction** (or rational expression) is a quotient of two algebraic expressions or two polynomials. Given a certain algebraic fraction, we must assume that the variable can only have values so that the denominator is not zero. For example, for the algebraic fraction $\dfrac{x + 3}{x^2 - 4}$, x cannot be 2 or -2. Most of the algebraic fractions that we will encounter will have numerators and denominators that are polynomials.

- **Hint:** Only common **factors** can be cancelled between the numerator and denominator of a fraction. For example, $\frac{5\times 3}{3}=\frac{5}{1}\times\frac{3}{3}=5\times 1=5$ where the common factors of 3 cancel; that is, $\frac{5\times\cancel{3}}{\cancel{3}}=5$. However, a common error is cancelling common terms that are *not* factors. For example, avoid the following common error: $\frac{5+3}{3}=\frac{5+\cancel{3}}{\cancel{3}}=5$. This is clearly incorrect, because $\frac{5+3}{3}=\frac{8}{3}\neq 5$.

Simplifying algebraic fractions

When trying to simplify algebraic fractions we need to completely factor the numerator and denominator and cancel any common factors.

Example 16 – Cancelling common factors in fractions

Simplify:

a) $\dfrac{2a^2-2ab}{6ab-6b^2}$ b) $\dfrac{1-x^2}{x^2+x-2}$ c) $\dfrac{(x+h)^2-x^2}{h}$

Solution

a) $\dfrac{2a^2-2ab}{6ab-6b^2}=\dfrac{2a\cancel{(a-b)}}{6b\cancel{(a-b)}}=\dfrac{{}^{1}\cancel{2}a}{{}_{3}\cancel{6}b}=\dfrac{a}{3b}$

b) $\dfrac{1-x^2}{x^2+x-2}=\dfrac{(1-x)(1+x)}{(x-1)(x+2)}=\dfrac{-(-1+x)(1+x)}{(x-1)(x+2)}=\dfrac{-\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x+2)}$

$$=-\frac{x+1}{x+2}\ \text{or}\ -\frac{x-1}{x+2}$$

c) $\dfrac{(x+h)^2-x^2}{h}=\dfrac{x^2+2hx+h^2-x^2}{h}=\dfrac{2hx+h^2}{h}=\dfrac{\cancel{h}(2x+h)}{\cancel{h}}=2x+h$

Adding and subtracting algebraic fractions

Before any fractions – numerical or algebraic – can be added or subtracted they must be expressed with the same denominator, preferably the least common denominator. Then the numerators can be added or subtracted according to the rule: $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$.

Example 17 – Working with algebraic fractions

Perform the indicated operation and simplify.

a) $x-\dfrac{1}{x}$ b) $\dfrac{2}{a+b}+\dfrac{3}{a-b}$ c) $\dfrac{2}{x+2}-\dfrac{x-4}{2x^2+x-6}$

Solution

a) $x-\dfrac{1}{x}=\dfrac{x}{1}-\dfrac{1}{x}=\dfrac{x^2}{x}-\dfrac{1}{x}=\dfrac{x^2-1}{x}$ or $\dfrac{(x+1)(x-1)}{x}$

b) $\dfrac{2}{a+b}+\dfrac{3}{a-b}=\dfrac{2}{a+b}\cdot\dfrac{a-b}{a-b}+\dfrac{3}{a-b}\cdot\dfrac{a+b}{a+b}=\dfrac{2(a-b)+3(a+b)}{(a+b)(a-b)}$

$$=\frac{2a-2b+3a+3b}{a^2-b^2}=\frac{5a+b}{a^2-b^2}$$

c) $\dfrac{2}{x+2}-\dfrac{x-4}{2x^2+x-6}=\dfrac{2}{x+2}-\dfrac{x-4}{(2x-3)(x+2)}$

$$=\frac{2}{x+2}\cdot\frac{2x-3}{2x-3}-\frac{x-4}{(2x-3)(x+2)}$$

$$=\frac{2(2x-3)-(x-4)}{(2x-3)(x+2)}$$

$$=\frac{4x-6-x+4}{(2x-3)(x+2)}$$

$$=\frac{3x-2}{(2x-3)(x+2)}\ \text{or}\ \frac{3x-2}{2x^2+x-6}$$

- **Hint:** Although it is true that $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$, be careful to avoid an error here: $\frac{a}{b+c}\neq\frac{a}{b}+\frac{a}{c}$. Also, be sure to only cancel common *factors* between numerator and denominator. It is true that $\frac{ac}{bc}=\frac{a}{b}$ (with the common factor of c cancelling) because $\frac{ac}{bc}=\frac{a}{b}\cdot\frac{c}{c}=\frac{a}{b}\cdot 1=\frac{a}{b}$; but, in general, it is *not* true that $\frac{a+c}{b+c}=\frac{a}{b}$. The term c is not a common factor of the numerator and denominator.

Simplifying a compound fraction

Fractional expressions with fractions in the numerator or denominator, or both, are usually referred to as compound fractions. A compound fraction is best simplified by first simplifying both its numerator and denominator into single fractions, and then multiplying numerator and denominator by the reciprocal of the denominator, i.e. $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{\frac{a}{b}\cdot\frac{d}{c}}{\frac{c}{d}\cdot\frac{d}{c}} = \dfrac{\frac{ad}{bc}}{1} = \dfrac{ad}{bc}$; thereby expressing the compound fraction as a single fraction.

Example 18 – Simplifying compound fractions

Simplify:

a) $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$ b) $\dfrac{\frac{a}{b}+1}{1-\frac{a}{b}}$ c) $\dfrac{x(1-2x)^{-\frac{3}{2}}+(1-2x)^{-\frac{1}{2}}}{1-x}$

Solution

a) $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h} = \dfrac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{\frac{h}{1}} = \dfrac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}} = \dfrac{x-x-h}{x(x+h)}\cdot\dfrac{1}{h}$

$= \dfrac{-\cancel{h}}{x(x+h)}\cdot\dfrac{1}{\cancel{h}} = -\dfrac{1}{x(x+h)}$

b) $\dfrac{\frac{a}{b}+1}{1-\frac{a}{b}} = \dfrac{\frac{a}{b}+\frac{b}{b}}{\frac{b}{b}-\frac{a}{b}} = \dfrac{\frac{a+b}{b}}{\frac{b-a}{b}} = \dfrac{a+b}{\cancel{b}}\cdot\dfrac{\cancel{b}}{b-a} = \dfrac{a+b}{b-a}$

c) $\dfrac{x(1-2x)^{-\frac{3}{2}}+(1-2x)^{-\frac{1}{2}}}{1-x} = \dfrac{(1-2x)^{-\frac{3}{2}}[x+(1-2x)^{1}]}{1-x}$

Factor out the power of $1-2x$ with the *smallest* exponent.

$= \dfrac{(1-2x)^{-\frac{3}{2}}[x+1-2x]}{1-x}$

$= \dfrac{(1-2x)^{-\frac{3}{2}}\cancel{(1-x)}}{\cancel{1-x}}$

$= \dfrac{1}{(1-2x)^{\frac{3}{2}}}$

With rules for rational exponents and radicals we can rewrite the result from c) above, but it's not any *simpler*…

$$\frac{1}{(1-2x)^{\frac{3}{2}}} = \frac{1}{\sqrt{(3x-2)^3}} = \frac{1}{\sqrt{(3x-2)^2}\sqrt{3x-2}} = \frac{1}{|3x-2|\sqrt{3x-2}}$$

Rationalizing the denominator

Recall Example 9 from Section 1.2, where we rationalized the denominator of the numerical fractions $\dfrac{2}{\sqrt{3}}$ and $\dfrac{\sqrt{7}}{4\sqrt{10}}$. Also recall that expressions of the form $a+b$ and $a-b$ are called conjugates and their product is a^2-b^2 (difference of two squares). If a fraction has an irrational denominator of the form $a+b\sqrt{c}$, we can change it to a rational expression by multiplying numerator and denominator by its conjugate $a-b\sqrt{c}$, given that $(a+b\sqrt{c})(a-b\sqrt{c}) = a^2-(b\sqrt{c})^2 = a^2-b^2c$.

Example 19 – Rationalizing the denominator

Rationalize the denominator of each fractional expression.

a) $\dfrac{1}{1-\sqrt{x}}$ $x \geqslant 0, x \neq 1$ b) $\dfrac{x-2}{x+3\sqrt{2}}$

Solution

a) $\dfrac{1}{1-\sqrt{x}} = \dfrac{1}{1-\sqrt{x}} \cdot \dfrac{1+\sqrt{x}}{1+\sqrt{x}} = \dfrac{1+\sqrt{x}}{1-(\sqrt{x})^2} = \dfrac{1+\sqrt{x}}{1-x}$

b) $\dfrac{x-\sqrt{2}}{x+3\sqrt{2}} = \dfrac{x-\sqrt{2}}{x+3\sqrt{2}} \cdot \dfrac{x-3\sqrt{2}}{x-3\sqrt{2}}$

$= \dfrac{x^2-(3\sqrt{2})x-(\sqrt{2})x+3\cdot 2}{x^2-(3\sqrt{2})^2} = \dfrac{x^2-(4\sqrt{2})x+6}{x-18}$

Exercise 1.5

In questions 1–16, expand and simplify.

1 $(x-4)(x+5)$

2 $(3h-1)(2h-3)$

3 $(y+9)(y-9)$

4 $(4x+2)^2$

5 $(2n-5)^2$

6 $(2y-5)^3$

7 $(6a-7b)(6a+7b)$

8 $(2x+3+y)(2x+3-y)$

9 $(ax+b)^3$

10 $(ax+b)^4$

11 $(2+x\sqrt{5})(2-x\sqrt{5})$

12 $(2x-1)(4x^2+2x+1)$

13 $(x+y-z)^2$

14 $(x+yi)(x-yi)$

15 $(m+3)(3-m)$

16 $(1-\sqrt{x^2+1})^2$

In questions 17–36, completely factorize the expression.

17 $12x^2-48$

18 x^3-6x^2

19 x^2+x-12

20 $7-6m-m^2$

21 $x^2-10x+16$

22 y^2+7y+6

23 $3n^2-21n+30$

24 $2x^3+20x^2+18x$

25 a^2-16

26 $3y^2-14y-5$

27 $25n^4-4$

28 $ax^2+6ax+9a$

29 $2n(m+1)^2-(m+1)^2$

30 x^4-1

31 $9-(y-3)^2$

32 $4y^4-10y^3-96y^2$

33 $4x^2-20x+25$

34 $(2x+3)^{-2}+2x(2x+3)^{-3}$

35 $(n-2)^4-(n-2)^3(2n-3)$

36 $m^3-\frac{4}{3}m^2+\frac{4}{9}m$

In questions 37–46, simplify the algebraic fraction.

37 $\dfrac{x+4}{x^2+5x+4}$

38 $\dfrac{3n-3}{6n^2-6n}$

39 $\dfrac{a^2-b^2}{5a-5b}$

40 $\dfrac{x^2+4x+4}{x+2}$

41 $\dfrac{2a-5}{5-2a}$

42 $\dfrac{(2x+h)^2-4x^2}{h}$

43 $\dfrac{(x+1)^3(3x-5)-(x+1)^2(8x+3)}{(x-4)(x+1)^3}$

44 $\dfrac{3y(y+3)-2(2y+1)}{(y+2)^2}$

45 $\dfrac{a-\dfrac{a^2}{b}}{\dfrac{a^2}{b}-a}$

46 $\dfrac{1+\dfrac{1}{1+\dfrac{1}{x-1}}}{1-\dfrac{1}{x-1}}$

In questions 47–60, perform the indicated operation and simplify.

47 $\dfrac{1}{n}-1$

48 $\dfrac{2}{2x-1}-4$

49 $\dfrac{x}{5}-\dfrac{x-1}{3}$

50 $\dfrac{1}{a}-\dfrac{1}{b}$

51 $\dfrac{1}{(x-3)^2}-\dfrac{3}{x-3}$

52 $\dfrac{x}{x+3}+\dfrac{1}{x}$

53 $\dfrac{1}{x+y}+\dfrac{1}{x-y}$

54 $\dfrac{3}{x-2}+\dfrac{5}{2-x}$

55 $\dfrac{2x-6}{x}\cdot\dfrac{3x}{x-3}$

56 $\dfrac{2x+6}{7}\times\dfrac{1}{x^2-9}$

57 $\dfrac{a+b}{b}\cdot\dfrac{1}{a^2-b^2}$

58 $\dfrac{3x^2-3}{6x}\times\dfrac{5x^2}{1-x}$

59 $\dfrac{3}{y+2}+\dfrac{5}{y^2-3y-10}$

60 $\dfrac{8}{9-x^2}\div\dfrac{2x}{x^3-x^2-6x}$

In questions 61–64, rationalize the denominator of each fractional expression.

61 $\dfrac{1}{x-\sqrt{2}}$

62 $\dfrac{5}{2+x\sqrt{3}}$

63 $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

64 $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

1.6 Equations and formulae

Equations, identities and formulae

We will encounter a wide variety of equations in this course. Essentially an equation is a statement equating two algebraic expressions that may be true or false depending upon what value(s) are substituted for the variable(s). The value(s) of the variable(s) that make the equation true are called the **solutions** or **roots** of the equation. All of the solutions to an equation comprise the **solution set** of the equation. An equation that is true for all possible values of the variable is called an **identity**. All of the common polynomial expansion and factorization patterns shown in Section 1.5 are identities. For example, $(a+b)^2 = a^2 + 2ab + b^2$ is true for all values of a and b. The following are also examples of identities.

$$3(x-5) = 2(x+3) + x - 21 \qquad (x+y)^2 - 2xy = x^2 + y^2$$

One of the most famous equations in the history of mathematics, $x^n + y^n = z^n$, is associated with Pierre Fermat (1601–1665), a French lawyer and amateur mathematician. Writing in the margin of a French translation of *Arithmetica*, considered to be the first book of algebra, written by the 3rd-century BC Greek mathematician Diophantus, Fermat conjectured that the equation $x^n + y^n = z^n$ $(x, y, z, n \in \mathbb{Z})$ has no non-zero solutions for the variables x, y and z when the parameter n is greater than two. When $n = 2$ the equation is equivalent to Pythagoras' theorem for which there are an infinite number of integer solutions – Pythagorean triples, such as $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, and their multiples. Fermat claimed to have a proof for his conjecture but that he could not fit it in the margin. All the other margin conjectures in Fermat's copy of *Arithmetica* were proven by the start of the 19th century but this one remained unproven for over 350 years, until the English mathematician Andrew Wiles proved it in 1994.

An equation may be referred to as a **formula** (plural: formulae). These typically contain more than one variable and, often, other symbols that represent specific constants or **parameters** (constants that may change in value but do not alter the properties of the expression). Formulae with which you may be familiar include:

$A = \pi r^2$, $d = rt$, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $V = \frac{4}{3}\pi r^3$

Whereas most equations that we will encounter have numerical solutions, we can solve a formula for a certain variable in terms of other variables – sometimes referred to as changing the subject of a formula.

Example 20 – Changing the subject of a formula

Solve for the indicated variable in each formula.

a) $a^2 + b^2 = c^2$ Solve for b.

b) $T = 2\pi\sqrt{\frac{l}{g}}$ Solve for l.

c) $I = \frac{nR}{R + r}$ Solve for R.

Solution

a) $a^2 + b^2 = c^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b = \pm\sqrt{c^2 - a^2}$
If b is a length then $b = \sqrt{c^2 - a^2}$.

b) $T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow \sqrt{\frac{l}{g}} = \frac{T}{2\pi} \Rightarrow \frac{l}{g} = \frac{T^2}{4\pi^2} \Rightarrow l = \frac{T^2 g}{4\pi^2}$

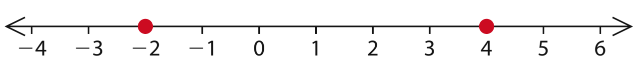
c) $I = \frac{nR}{R + r} \Rightarrow I(R + r) = nR \Rightarrow IR + Ir = nR \Rightarrow IR - nR = -Ir$
$\Rightarrow R(I - n) = -Ir \Rightarrow R = \frac{-Ir}{I - n}$

Note that factorization was required in solving for R in Example 20 c).

Equations and graphs

Two important characteristics of any equation are the number of variables (unknowns) and the type of algebraic expressions it contains (e.g. polynomials, rational expressions, trigonometric, exponential, etc.). Nearly all of the equations in this course will have either one or two variables, and in this introductory chapter we will only discuss equations with algebraic expressions that are polynomials. Solutions for equations with a single variable will consist of individual numbers that can be *graphed* as points on a number line. The **graph** of an equation is a visual representation of the equation's solution set. For example, the solution set of the one-variable equation containing quadratic and linear polynomials $x^2 = 2x + 8$ is $x \in \{-2, 4\}$. The graph of this one-variable equation (Figure 1.5) is depicted below on a one-dimensional coordinate system, i.e. the real number line.

Figure 1.5 Graph of the solution set for the equation $x^2 = 2x + 8$.



The solution set of a two-variable equation will be an **ordered pair** of numbers. An ordered pair corresponds to a location indicated by a point on a two-dimensional coordinate system, i.e. a **coordinate plane**. For example, the solution set of the two-variable **quadratic equation** $y = x^2$ will be an infinite set of ordered pairs (x, y) that satisfy the equation. Four ordered pairs in the solution set are graphed in Figure 1.6 in red. The graph of all the ordered pairs in the solution set form a curve as shown in blue. (Quadratic equations will be covered in detail in Chapter 3.)

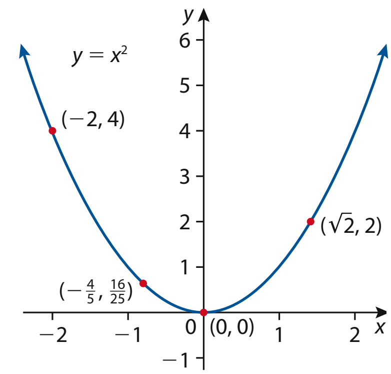


Figure 1.6 Four ordered pairs in the solution set of $y = x^2$ are graphed in red. The graph of all the ordered pairs in the solution set form a curve, as shown in blue.

Equations of lines

A one-variable **linear equation** in x can always be written in the form $ax + b = 0$, $a \neq 0$ and it will have exactly one solution, $x = -\frac{b}{a}$.

An example of a two-variable **linear equation in x and y** is $x - 2y = 2$. The graph of this equation's solution set (an infinite set of ordered pairs) is a **line** (Figure 1.7).

The **slope** m, or **gradient**, of a non-vertical line is defined by the formula $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}}$. Because division by zero is undefined, the slope of a vertical line is undefined. Using the two points $(1, -\frac{1}{2})$ and $(4, 1)$, we compute the slope of the line with equation $x - 2y = 2$ to be

$$m = \frac{1 - \left(-\frac{1}{2}\right)}{4 - 1} = \frac{\frac{3}{2}}{\frac{3}{1}} = \frac{1}{2}.$$

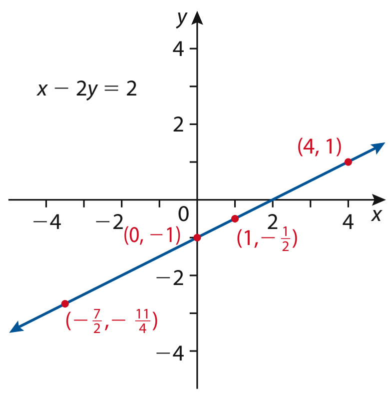


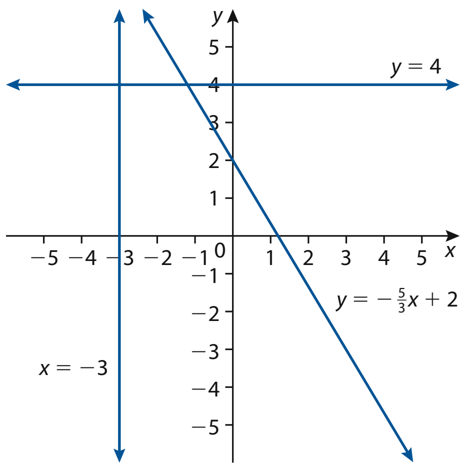
Figure 1.7 The graph of $x - 2y = 2$, ordered pairs shown in red.

If we solve for y, we can rewrite the equation in the form $y = \frac{1}{2}x - 1$. Note that the coefficient of x is the slope of the line and the constant term is the y-coordinate of the point at which the line intersects the y-axis, i.e. the y-intercept. There are several forms in which to write linear equations whose graphs are lines.

Table 1.5 Forms of equations of lines.

Form	Equation	Characteristics
general form	$ax + by + c = 0$	every line has an equation in this form if both a and $b \neq 0$
slope-intercept form	$y = mx + c$	m is the slope; $(0, c)$ is the y-intercept
point-slope form	$y - y_1 = m(x - x_1)$	m is the slope; (x_1, y_1) is a known point on the line
horizontal line	$y = c$	slope is zero; $(0, c)$ is the y-intercept
vertical line	$x = c$	slope is undefined; unless line is y-axis, no y-intercept

Most problems involving equations and graphs fall into two categories: (1) given an equation, determine its graph; and (2) given a graph, or some information about it, find its equation. For lines, the first type of problem is often best solved by using the slope-intercept form, whereas for the second type of problem the point-slope form is usually most useful.



Example 21 – Sketching the graphs of linear equations

Without using a GDC, sketch the line that is the graph of each of the following linear equations written here in general form.

a) $5x + 3y - 6 = 0$

b) $y - 4 = 0$

c) $x + 3 = 0$

Solution

a) Solve for y to write the equation in slope-intercept form. $5x + 3y - 6 = 0 \Rightarrow 3y = -5x + 6 \Rightarrow y = -\frac{5}{3}x + 2$. The line has a y-intercept of $(0, 2)$ and a slope of $-\frac{5}{3}$.

b) The equation $y - 4 = 0$ is equivalent to $y = 4$, whose graph is a horizontal line with a y-intercept of $(0, 4)$.

c) The equation $x + 3 = 0$ is equivalent to $x = -3$, whose graph is a vertical line with no y-intercept; but, it has an x-intercept of $(-3, 0)$.

Example 22 – Finding the equation of a line

a) Find the equation of the line that passes through the point $(3, 31)$ and has a slope of 12. Write the equation in slope-intercept form.

b) Find the linear equation in C and F knowing that when $C = 10$ then $F = 50$, and when $C = 100$ then $F = 212$. Solve for F in terms of C.

Solution

a) Substitute into the point-slope form $y - y_1 = m(x - x_1)$; $x_1 = 3$, $y_1 = 31$ and $m = 12$.

$$y - y_1 = m(x - x_1) \Rightarrow y - 31 = 12(x - 3) \Rightarrow y = 12x - 36 + 31 \Rightarrow y = 12x - 5$$

b) The two points, ordered pairs (C, F), that are known to be on the line are $(10, 50)$ and $(100, 212)$. The variable C corresponds to the variable x and F corresponds to y in the definitions and forms stated above. The slope of the line is $m = \frac{F_2 - F_1}{C_2 - C_1} = \frac{212 - 50}{100 - 10} = \frac{162}{90} = \frac{9}{5}$. Choose one of the points on the line, say $(10, 50)$, and substitute it and the slope into the point-slope form.

$$F - F_1 = m(C - C_1) \Rightarrow F - 50 = \frac{9}{5}(C - 10) \Rightarrow F = \frac{9}{5}C - 18 + 50 \Rightarrow F = \frac{9}{5}C + 32$$

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular. The two lines graphed in Figure 1.8 suggests the following property: Two distinct non-vertical lines are **parallel** if and only if their slopes are equal, $m_1 = m_2$.

The two lines graphed in Figure 1.9 suggests another property: Two non-vertical lines are perpendicular if and only if their slopes are negative reciprocals – that is, $m_1 = -\frac{1}{m_2}$, which is equivalent to $m_1 \cdot m_2 = -1$.

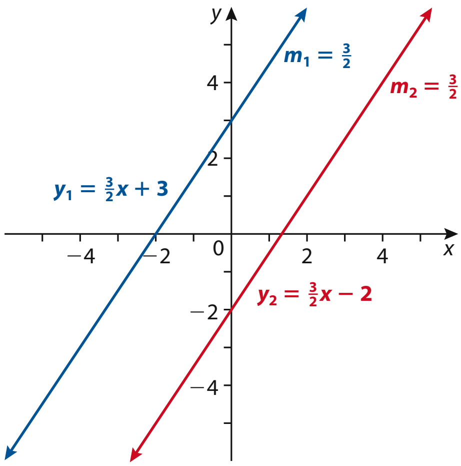


Figure 1.8 Parallel lines.

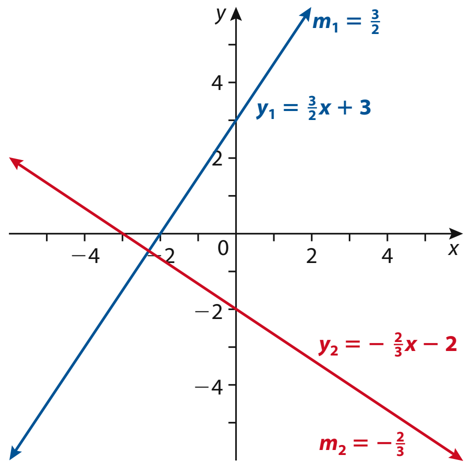


Figure 1.9 Perpendicular lines.

Distances and midpoints

Recall from Section 1.1 that absolute value (modulus) is used to define the **distance** (always positive) between two points on the real number line.

The distance between the points A and B on the real number line is $|B - A|$, which is equivalent to $|A - B|$.

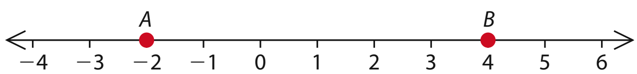


Figure 1.10 The length of the line segment $[AB]$ is AB.

The points A and B are the endpoints of a line segment that is denoted with the notation $[AB]$ and the length of the line segment is denoted AB. In Figure 1.10, the distance between A and B is $AB = |4 -(-2)| = |-2 - 4| = 6$.

The distance between two general points (x_1, y_1) and (x_2, y_2) on a coordinate plane can be found using the definition for distance on a number line and Pythagoras' theorem. For the points (x_1, y_1) and (x_2, y_2), the horizontal distance between them is $|x_1 - x_2|$ and the vertical distance is $|y_1 - y_2|$. As illustrated in Figure 1.11, these distances are the lengths of two legs of a right-angled triangle whose hypotenuse is the distance between the points. If d represents the distance between (x_1, y_1) and (x_2, y_2), then by Pythagoras' theorem $d^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$. Because the square of any number is positive, the absolute value is not necessary, giving us the **distance formula** for two-dimensional coordinates.

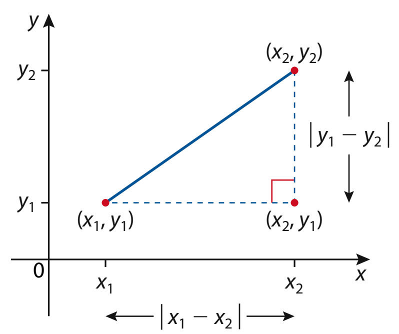


Figure 1.11 Distance between two points on a coordinate plane.

The distance formula

The distance d between the two points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The coordinates of the **midpoint** of a line segment are the average values of the corresponding coordinates of the two endpoints.

The midpoint formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 23 – Using the distance and midpoint formulae

a) Show that the points $P(1, 2)$, $Q(3, 1)$ and $R(4, 8)$ are the vertices of a right triangle.

b) Find the midpoint of the hypotenuse.

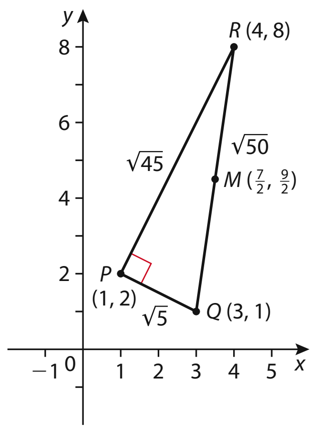


Figure 1.12 Diagram for Example 23.

Solution

a) The three points are plotted and the line segments joining them are drawn in Figure 1.12. Applying the distance formula, we can find the exact lengths of the three sides of the triangle.

$PQ = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$

$QR = \sqrt{(3-4)^2 + (1-8)^2} = \sqrt{1+49} = \sqrt{50}$

$PR = \sqrt{(1-4)^2 + (2-8)^2} = \sqrt{9+36} = \sqrt{45}$

$PQ^2 + PR^2 = QR^2$ because $(\sqrt{5})^2 + (\sqrt{45})^2 = 5 + 45 = 50 = (\sqrt{50})^2$. The lengths of the three sides of the triangle satisfy Pythagoras' theorem, confirming that the triangle is a right-angled triangle.

b) QR is the hypotenuse. Let the midpoint of QR be point M. Using the midpoint formula, $M = \left(\frac{3+4}{2}, \frac{1+8}{2}\right) = \left(\frac{7}{2}, \frac{9}{2}\right)$. This point is plotted in Figure 1.12.

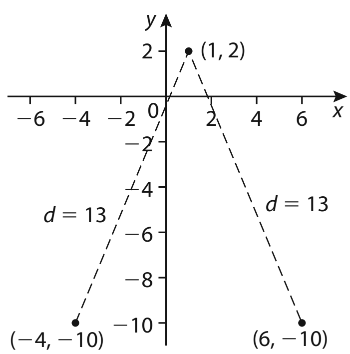


Figure 1.13 The graph shows the two points that are both a distance of 13 from (1, 2).

Example 24 – Using the distance formula

Find x so that the distance between the points $(1, 2)$ and $(x, -10)$ is 13.

Solution

$d = 13 = \sqrt{(x-1)^2 + (-10-2)^2} \Rightarrow 13^2 = (x-1)^2 + (-12)^2$

$\Rightarrow\ 169 = x^2 - 2x + 1 + 144 \Rightarrow x^2 - 2x - 24 = 0$

$\Rightarrow\ (x-6)(x+4) = 0 \Rightarrow x - 6 = 0 \text{ or } x + 4 = 0$

$\Rightarrow\ x = 6 \text{ or } x = -4$

Simultaneous equations

Many problems that we solve with algebraic techniques involve sets of equations with several variables, rather than just a single equation with one or two variables. Such a set of equations is called a set of **simultaneous**

equations because we find the values for the variables that solve all of the equations simultaneously. In this section, we consider only the simplest set of simultaneous equations – a pair of linear equations in two variables. We will take a brief look at three methods for solving simultaneous linear equations. They are:

1. Graphical method
2. Elimination method
3. Substitution method

Although we will only look at pairs of linear equations in this section, it is worthwhile mentioning that the graphical and substitution methods are effective for solving sets of equations where not all of the equations are linear, e.g. one linear and one quadratic equation.

Graphical method

The graph of each equation in a system of two linear equations in two unknowns is a line. The graphical interpretation of the solution of a pair of simultaneous linear equations corresponds to determining what point, or points, lies on both lines. Two lines in a coordinate plane can only relate to one another in one of three ways: (1) intersect at exactly one point, (2) intersect at all points on each line (i.e. the lines are identical), or (3) the two lines do not intersect (i.e. the lines are parallel). These three possibilities are illustrated in Figure 1.14.

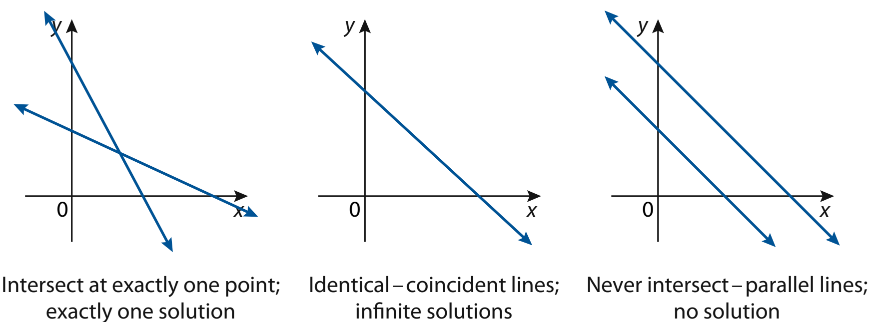


Figure 1.14 Possible relationship between two lines in a coordinate plane.

Although a graphical approach to solving simultaneous linear equations provides a helpful visual picture of the number and location of solutions, it can be tedious and inaccurate if done by hand. The graphical method is far more efficient and accurate when performed on a graphical display calculator (GDC).

Example 25 – Solving simultaneous equations with a GDC

Use the graphical features of a GDC to solve each pair of simultaneous equations.

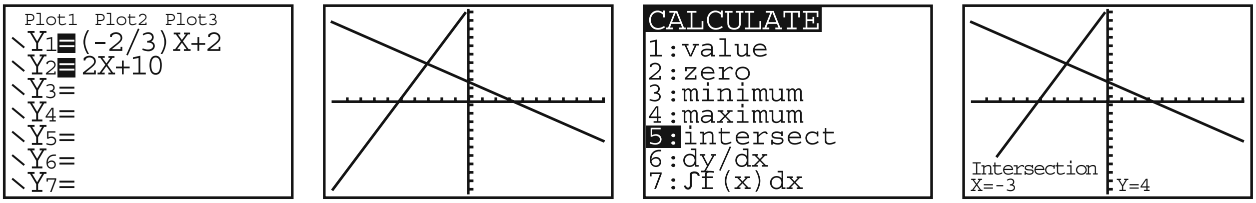
a) $2x + 3y = 6$
   $2x - y = -10$

b) $7x - 5y = 20$
   $3x + y = 2$

Solution

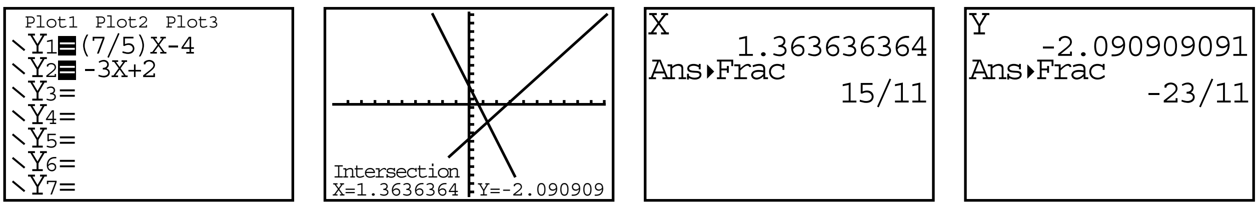
a) First, we will rewrite each equation in slope-intercept form, i.e. $y = mx + c$. This is a necessity if we use our GDC, and is also very useful for graphing by hand (manual).

$$2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2 \text{ and } 2x - y = -10 \Rightarrow y = 2x + 10$$



The intersection point and solution to the simultaneous equations is $x = -3$ and $y = 4$, or $(-3, 4)$. If we manually graphed the two linear equations in a) very carefully using graph paper, we may have been able to determine the exact coordinates of the intersection point. However, using a graphical method without a GDC to solve the simultaneous equations in b) would only allow us to crudely approximate the solution.

b) $7x - 5y = 20 \Rightarrow 5y = 7x - 20 \Rightarrow y = \frac{7}{5}x - 4$ and
$3x + y = 2 \Rightarrow y = -3x + 2$



The solution to the simultaneous equations is $x = \frac{15}{11}$ and $y = -\frac{23}{11}$, or $\left(\frac{15}{11}, -\frac{23}{11}\right)$.

The full power and efficiency of the GDC is used in this example to find the exact solution.

Elimination method

To solve a system using the **elimination method**, we try to combine the two linear equations using sums or differences in order to eliminate one of the variables. Before combining the equations, we need to multiply one or both of the equations by a suitable constant to produce coefficients for one of the variables that are equal (then subtract the equations), or that differ only in sign (then add the equations).

Example 26 – Elimination method

Use the elimination method to solve each pair of simultaneous equations.

a) $5x + 3y = 9$
$2x - 4y = 14$

b) $x - 2y = 7$
$2x - 4y = 5$

Solution

a) We can obtain coefficients for y that differ only in sign by multiplying the first equation by 4 and the second equation by 3. Then we add the equations to eliminate the variable y.

$$\begin{array}{rcl} 5x + 3y = 9 & \rightarrow & 20x + 12y = 36 \\ \underline{2x - 4y = 14} & \rightarrow & \underline{6x - 12y = 42} \\ & & 26x = 78 \\ & & x = \dfrac{78}{26} \\ & & x = 3 \end{array}$$

By substituting the value of 3 for x in either of the original equations we can solve for y.

$5x + 3y = 9 \Rightarrow 5(3) + 3y = 9 \Rightarrow 3y = -6 \Rightarrow y = -2$

The solution is $(3, -2)$.

b) To obtain coefficients for x that are equal, we multiply the first equation by 2 and then subtract the equations to eliminate the variable x.

$$\begin{array}{rcl} x - 2y = 7 & \rightarrow & 2x - 4y = 14 \\ \underline{2x - 4y = 5} & \rightarrow & \underline{2x - 4y = 5} \\ & & 0 = 9 \end{array}$$

Because it is not possible for 0 to equal 9, there is no solution. The lines that are the graphs of the two equations are parallel. To confirm this we can rewrite each of the equations in the form $y = mx + c$.

$$x - 2y = 7 \Rightarrow 2y = x - 7 \Rightarrow y = \tfrac{1}{2}x - \tfrac{7}{2} \text{ and}$$
$$2x - 4y = 5 \Rightarrow 4y = 2x - 5 \Rightarrow y = \tfrac{1}{2}x - \tfrac{5}{2}$$

Both equations have a slope of $\frac{1}{2}$, but different y-intercepts. Therefore, the lines are parallel. This confirms that this pair of simultaneous equations has no solution.

Substitution method

The algebraic method that can be applied effectively to the widest variety of simultaneous equations, including non-linear equations, is the **substitution method**. Using this method, we choose one of the equations and solve for one of the variables in terms of the other variable. We then substitute this expression into the other equation to produce an equation with only one variable, which we can solve directly.

Example 27 – Substitution method

Use the substitution method to solve each pair of simultaneous equations.

a) $3x - y = -9$
$6x + 2y = 2$

b) $-2x + 6y = 4$
$3x - 9y = -6$

Solution

a) Solve for y in the top equation, $3x - y = -9 \Rightarrow y = 3x + 9$, and substitute $3x + 9$ in for y in the bottom equation:
$6x + 2(3x + 9) = 2 \Rightarrow 6x + 6x + 18 = 2 \Rightarrow 12x = -16 \Rightarrow x = -\frac{16}{12} = -\frac{4}{3}$.
Now substitute $-\frac{4}{3}$ for x in either equation to solve for y.
$3\left(-\frac{4}{3}\right) - y = -9 \Rightarrow y = -4 + 9 \Rightarrow y = 5$.
The solution is $x = -\frac{4}{3}, y = 5$, or $\left(-\frac{4}{3}, 5\right)$.

b) Solve for x in the top equation,
$-2x + 6y = 4 \Rightarrow 2x = 6y - 4 \Rightarrow x = 3y - 2$, and substitute $3y - 2$ in for x in the bottom equation:
$3(3y - 2) - 9y = -6 \Rightarrow 9y - 6 - 9y = -6 \Rightarrow 0 = 0$.
The resulting equation $0 = 0$ is true for any values of x and y. The two equations are equivalent, and their graphs will produce identical lines – i.e. coincident lines. Therefore, the solution set consists of all points (x, y) lying on the line $-2x + 6y = 4$ $\left(\text{or } y = \frac{1}{3}x + \frac{2}{3}\right)$.

Exercise 1.6

In questions 1–8, solve for the indicated variable in each formula.

1 $m(h - x) = n$ solve for x

2 $v = \sqrt{ab - t}$ solve for a

3 $A = \frac{h}{2}(b_1 + b_2)$ solve for b_1

4 $A = \frac{1}{2}r^2\theta$ solve for r

5 $\frac{f}{g} = \frac{h}{k}$ solve for k

6 $at = x - bt$ solve for t

7 $V = \frac{1}{3}\pi r^3 h$ solve for r

8 $F = \frac{g}{m_1 k + m_2 k}$ solve for k

In questions 9–12, find the equation of the line that passes through the two given points. Write the line in slope-intercept form ($y = mx + c$), if possible.

9 $(-9, 1)$ and $(3, -7)$

10 $(3, -4)$ and $(10, -4)$

11 $(-12, -9)$ and $(4, 11)$

12 $\left(\frac{7}{3}, -\frac{1}{2}\right)$ and $\left(\frac{7}{3}, \frac{5}{2}\right)$

13 Find the equation of the line that passes through the point $(7, -17)$ and is parallel to the line with equation $4x + y - 3 = 0$. Write the line in slope-intercept form ($y = mx + c$).

14 Find the equation of the line that passes through the point $\left(-5, \frac{11}{2}\right)$ and is perpendicular to the line with equation $2x - 5y - 35 = 0$. Write the line in slope-intercept form ($y = mx + c$).

In questions 15–18, a) find the exact distance between the points, and b) find the midpoint of the line segment joining the two points.

15 $(-4, 10)$ and $(4, -5)$

16 $(-1, 2)$ and $(5, 4)$

17 $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{5}{2}, \frac{4}{3}\right)$

18 $(12, 2)$ and $(-10, 9)$

In questions 19 and 20, find the value(s) of k so that the distance between the points is 5.

19 $(5, -1)$ and $(k, 2)$

20 $(-2, -7)$ and $(1, k)$

In questions 21–23, show that the given points form the vertices of the indicated polygon.

21 Right-angled triangle: $(4, 0)$, $(2, 1)$ and $(-1, -5)$

22 Isosceles triangle: $(1, -3)$, $(3, 2)$ and $(-2, 4)$

23 Parallelogram: $(0, 1)$, $(3, 7)$, $(4, 4)$ and $(1, -2)$

In questions 24–29, use the elimination method to solve each pair of simultaneous equations.

24 $x + 3y = 8$
$x - 2y = 3$

25 $x - 6y = 1$
$3x + 2y = 13$

26 $6x + 3y = 6$
$5x + 4y = -1$

27 $x + 3y = -1$
$x - 2y = 7$

28 $8x - 12y = 4$
$-2x + 3y = 2$

29 $5x + 7y = 9$
$-11x - 5y = 1$

In questions 30–35, use the substitution method to solve each pair of simultaneous equations.

30 $2x + y = 1$
$3x + 2y = 3$

31 $3x - 2y = 7$
$5x - y = -7$

32 $2x + 8y = -6$
$-5x - 20y = 15$

33 $\frac{x}{5} + \frac{y}{2} = 8$
$x + y = 20$

34 $2x - y = -2$
$4x + y = 5$

35 $0.4x + 0.3y = 1$
$0.25x + 0.1y = -0.25$

In questions 36–38, solve the pair of simultaneous equations using any method – elimination, substitution or the graphical features of your GDC.

36 $3x + 2y = 9$
$7x + 11y = 2$

37 $3.62x - 5.88y = -10.11$
$0.08x - 0.02y = 0.92$

38 $2x - 3y = 4$
$5x + 2y = 1$

2 Functions

Assessment statements

2.1 Concept of a function $f: x \rightarrow f(x)$; domain, range, image (value).
One-to-one and many-to-one functions.
Composite functions $f \circ g$; identity function. Inverse function f^{-1} including domain restriction.

2.2 The graph of a function; its equation $y = f(x)$.
Investigation of key features of graphs such as intercepts, horizontal and vertical asymptotes, symmetry and consideration of domain and range.
The graphs of the absolute value functions, $y = |f(x)|$ and $y = f(|x|)$.
The graph of $y = \frac{1}{f(x)}$ from $y = f(x)$.

2.3 Transformations of graphs: translations, stretches, reflections in the axes.
The graph of $y = f^{-1}(x)$ as the reflection in the line $y = x$ of the graph $y = f(x)$.

2.4 The reciprocal function $x \rightarrow \frac{1}{x}$, $x \neq 0$: its graph; its self-inverse nature.

Introduction

The relationship between two quantities – how the value of one quantity depends on the value of another quantity – is the key behind the concept of a function. Functions and how we use them are at the very foundation of many topics in mathematics, and are essential to our understanding of much of what will be covered later in this book. This chapter will look at some general characteristics and properties of functions. We will consider composite and inverse functions, and investigate how the graphs of functions can be transformed by means of translations, stretches and reflections.

2.1 Definition of a function

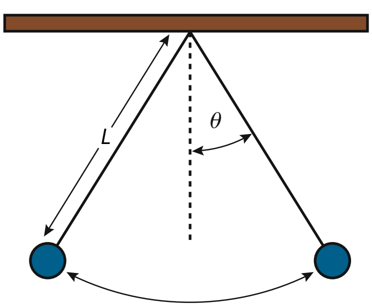
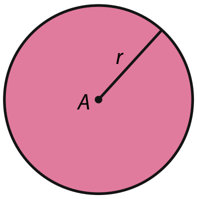


Figure 2.1 A simple pendulum.

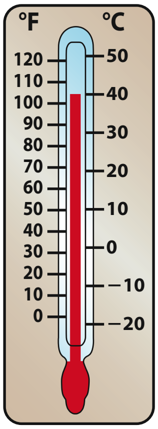
A simple pendulum consists of a heavy object hanging from a string of length L (in metres) and fixed at a pivot point (Figure 2.1). If you displace the suspended object to one side by a certain angle θ from the vertical and release it, the object will swing back and forth under the force of gravity. The period T (in seconds) of the pendulum is the time for the object to return to the point of release and, for a small angle θ, the two variables T and L are related by the formula $T = 2\pi\sqrt{\frac{L}{g}}$ where g is the gravitational field strength (acceleration due to gravity). Therefore, assuming that the force of gravity is constant at a given elevation ($g \approx 9.81\ \mathrm{m\,s^{-2}}$ at sea level), the formula can be used to calculate the value of T for any value of L.

As with the period T and the length L for a pendulum, many mathematical relationships concern how the value of one variable determines the value of a second variable. Other examples include:

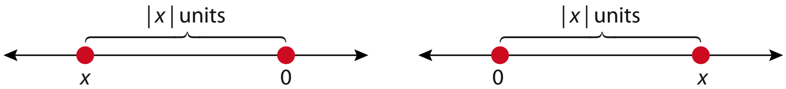
Area of a circle determined by its radius:
$A = \pi r^2$ (π is a constant)



Converting degrees Celsius to degrees Fahrenheit:
$F = \frac{9}{5}C + 32$



Distance that a number is from the origin determined by its absolute value:



In general, suppose that the values of a particular **independent variable**, for example x, determine the values of a **dependent variable** y in such a way that for a specific value of x, a single value of y is determined. Then we say that y is a **function** of x and we write $y = f(x)$ (read 'y equals f of x'), or $y = g(x)$ etc., where the letters f and g represent the name of the function. For the four mathematical relationships that were described above, we have:

Period T is a function of length L: $T = 2\pi\sqrt{\frac{L}{g}}$, or $f(L) = 2\pi\sqrt{\frac{L}{g}}$ where $T = f(L)$.

Area A is a function of radius r: $A = \pi r^2$, or $g(r) = \pi r^2$ where $A = g(r)$.

°F (degrees Fahrenheit) is a function of °C: $F = \frac{9}{5}C + 32$, or $t(C) = \frac{9}{5}C + 32$ where $F = t(C)$.

Distance y from origin is a function of x: $y = |x|$, or $f(x) = |x|$ where $y = f(x)$.

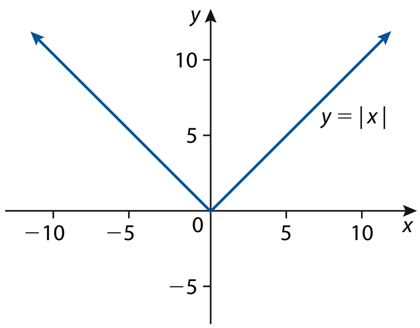
Along with equations, other useful ways of representing a function include a graph of the equation on a **Cartesian coordinate system** (also called

René Descartes

The Cartesian coordinate system is named in honour of the French mathematician and philosopher René Descartes (1596–1650). Descartes stimulated a revolution in the study of mathematics by merging its two major fields – algebra and geometry. With his coordinate system utilizing ordered pairs (*Cartesian coordinates*) of real numbers, geometric concepts could be formulated analytically and algebraic concepts (e.g. relationships between two variables) could be viewed graphically. Descartes initiated something that is very helpful to all students of mathematics – that is, considering mathematical concepts from multiple perspectives: graphical (visual) and analytical (algebraic).

a **rectangular coordinate system**), a **table**, a **set of ordered pairs**, or a **mapping.** These are illustrated below for the absolute value function $y = |x|$.

Graph



● **Hint:** The coordinate system for the graph of an equation has the independent variable on the horizontal axis and the dependent variable on the vertical axis.

Table $y = |x|$

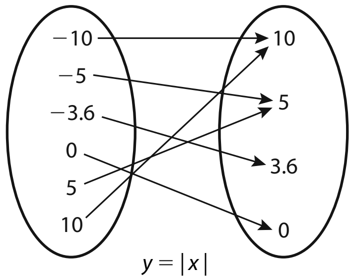
x	y
-10	10
$-\frac{15}{2}$	$\frac{15}{2}$
-5	5
-3.6	3.6
0	0
$\sqrt{2}$	$\sqrt{2}$
5	5
8.3	8.3
10	10

Set of ordered pairs

The graph of the equation $y = |x|$ consists of an infinite set of ordered pairs (x, y) such that each is a solution of the equation. The following set includes some of the ordered pairs on the line:

$\{(-23, 23), (-10, 10), (-\sqrt{7}, \sqrt{7}), (0, 0), (5, 5)\}$.

Mapping



The largest possible set of values for the independent variable (the **input** set) is called the **domain** – and the set of resulting values for the dependent variable (the **output** set) is called the **range**. In the context of a mapping, each value in the domain is mapped to its **image** in the range.

All of the various ways of representing a mathematical function illustrate that its defining characteristic is that it is a rule by which each number in the domain determines a unique number in the range.

Definition of a function

A **function** is a correspondence (**mapping**) between two sets X and Y in which each element of set X corresponds to (maps to) exactly one element of set Y. The **domain** is set X (**independent variable**) and the **range** is set Y (**dependent variable**).

Not all equations represent a function. The solution set for the equation $x^2 + y^2 = 1$ is the set of ordered pairs (x, y) on the circle of radius equal to 1 and centre at the origin (see Figure 2.2). If we solve the equation for y, we get $y = \pm\sqrt{1 - x^2}$. It is clear that any value of x between -1 and 1 will produce two different values of y (opposites). Since at least one value in the domain (x) determines more than one value in the range (y), then the equation does not represent a function. A correspondence between two sets that does not satisfy the definition of a function is called a **relation**.

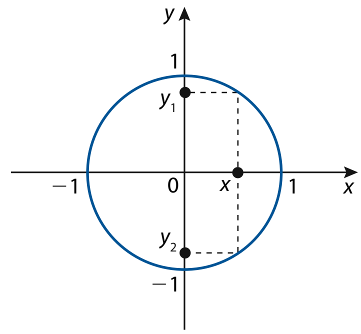
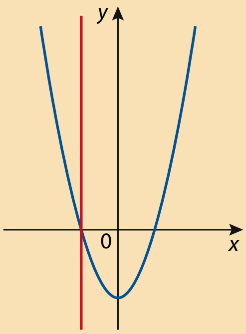


Figure 2.2 Graph of $x^2 + y^2 = 1$.

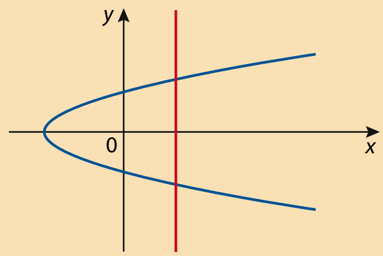
Alternative definition of a function

A **function** is a **relation** in which no two different ordered pairs have the same first coordinate.

A vertical line intersects the graph of a function at no more than one point (vertical line test).



Any vertical line intersects the graph at no more than one point, so y is a function of x.



At least one vertical line intersects the graph at more than one point, so y is *not* a function of x.

Not only are functions important in the study of mathematics and science, we encounter and use them routinely – often in the form of tables. Examples include height and weight charts, income tax tables, loan payment schedules, and time and temperature charts. The importance of functions in mathematics is evident from the many functions that are installed on your GDC.

For example, the keys labelled SIN x^{-1} LN $\sqrt{\ }$ each represent a function, because for each input (entry) there is only one output (answer). The calculator screen image shows that for the function $y = \ln x$, the input of $x = 10$ has only one output of $y \approx 2.302\,585\,093$.

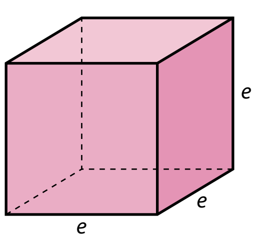
```
ln(10)
            2.302585093
█
```

For many physical phenomena, we observe that one quantity depends on another. The word **function** is used to describe this dependence of one quantity on another – i.e. how the value of an independent variable determines the value of a dependent variable. A common mathematical task is to find how to express one variable as a function of another variable.

Example 1

a) Express the volume V of a cube as a function of the length e of each edge.

b) Express the volume V of a cube as a function of its surface area S.



Solution

a) V as a function of e is $V = e^3$.

b) The surface area of the cube consists of six squares each with an area of e^2. Hence, the surface area is $6e^2$; that is, $S = 6e^2$. We need to write V in terms of S. We can do this by first expressing e in terms of S, and then substituting this expression in for e in the equation $V = e^3$.

$$S = 6e^2 \Rightarrow e^2 = \frac{S}{6} \Rightarrow e = \sqrt{\frac{S}{6}}.$$

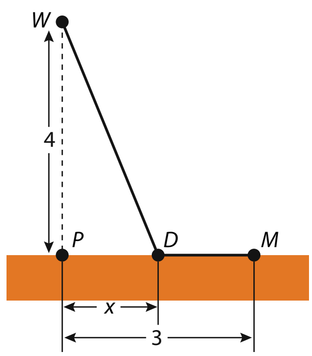
Substituting,

$$V = \left(\sqrt{\frac{S}{6}}\right)^3 = \frac{(S^{\frac{1}{2}})^3}{(6^{\frac{1}{2}})^3} = \frac{S^{\frac{3}{2}}}{6^{\frac{3}{2}}} = \frac{S^1 \cdot S^{\frac{1}{2}}}{6^1 \cdot 6^{\frac{1}{2}}} = \frac{S}{6}\sqrt{\frac{S}{6}}$$

V as a function of S is $V = \frac{S}{6}\sqrt{\frac{S}{6}}$.

Example 2 – Finding a function in terms of a single variable

An offshore wind turbine is located at point W, 4 km offshore from the nearest point P on a straight coastline. A maintenance station is at point M, 3 km down the coast from P. An engineer is returning by boat from the wind turbine. He decides to row to a dock at point D that is located between P and M at an unknown distance x km from point P. The engineer can row 3 km/hr and walk 6 km/hr. Express the total time T (hours) for the trip from the wind turbine to the maintenance station as a function of x (km).



Solution

To get an equation for T in terms of x, we use the fact that time $= \frac{\text{distance}}{\text{rate}}$. We then have

$$T = \frac{\text{distance } WD}{3} + \frac{\text{distance } DM}{6}$$

The distance WD can be expressed in terms of x by using Pythagoras' theorem.

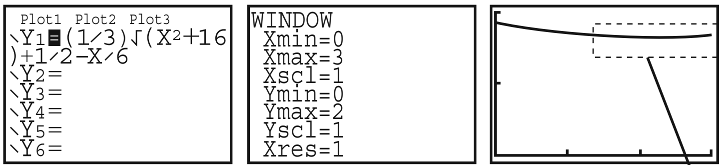
$$WD^2 = x^2 + 4^2 \Rightarrow WD = \sqrt{x^2 + 16}$$

To express T in terms of only the single variable x, we note that $DM = 3 - x$.

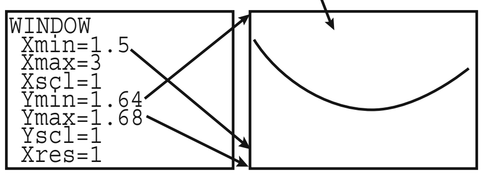
Then the total time T can be written in terms of x by the equation:

$$T = \frac{\sqrt{x^2 + 16}}{3} + \frac{3 - x}{6} \text{ or } T = \frac{1}{3}\sqrt{x^2 + 16} + \frac{1}{2} - \frac{x}{6}$$

Using our graphic display calculator (GDC) to graph the equation gives a helpful picture showing how T changes when x changes. In function graphing mode on a GDC, the independent variable is always x and the dependent variable is always y.



Zooming in on the graph indicates that there is a value for x between 1.5 and 3 that will make the time for the trip a minimum. In Chapter 13, we will use calculus techniques to find the value of x that gives a minimum time for the trip.



Domain and range of a function

The domain of a function may be stated explicitly, or it may be implied by the expression that defines the function. Except in Chapter 10, where we will encounter functions for which the variables can have values that are imaginary numbers, we can assume that any functions that we will work with are **real-valued functions of a real variable**. That is, the domain and range will only contain real numbers or some subset of the real numbers. Therefore, if not explicitly stated otherwise, the domain of a function is the set of all real numbers for which the expression is defined as a real number. For example, if a certain value of x is substituted into the algebraic expression defining a function and it causes division by zero or the square root of a negative number (both undefined in the real numbers) to occur, that value of x cannot be in the domain. The domain of a function may also be implied by the physical context or limitations that exist in a problem. For example, for both functions derived in Example 1 $\left(V = \frac{S}{6}\sqrt{\frac{S}{6}} \text{ and } V = e^3\right)$ the domain is the set of positive real numbers (symbolized by $\mathbb{R}^+$) because neither a length (edge of a cube) nor a surface area (face of a cube) can have a value that is negative or zero. In Example 2 the domain for the function is $0 < x < 3$ because of the constraints given in the problem. Usually the range of a function is not given explicitly and is determined by analyzing the output of the function for all values of the input (domain). The range of a function is often more difficult to find than the domain, and analyzing the graph of a function is very helpful in determining it. A combination of algebraic and graphical analysis is very useful in determining the domain and range of a function.

Example 3 – Domain of a function

Find the domain of each of the following functions.

a) $\{(-6, -3), (-1, 0), (2, 3), (3, 0), (5, 4)\}$

b) Volume of a sphere: $V = \frac{4}{3}\pi r^3$

c) $y = \dfrac{5}{2x - 6}$

d) $y = \sqrt{3 - x}$

Solution

a) The function consists of a set of ordered pairs. The domain of the function consists of all first coordinates of the ordered pairs. Therefore, the domain is the set $x \in \{-6, -1, 2, 3, 5\}$.

b) The physical context tells you that a sphere cannot have a radius that is negative or zero. Therefore, the domain is the set of all real numbers r such that $r > 0$.

c) Since division by zero is not defined for real numbers then $2x - 6 \neq 0$. Therefore, the domain is the set of all real numbers x such that $x \in \mathbb{R}, x \neq 3$.

d) Since the square root of a negative number is not real, then $3 - x \geqslant 0$. Therefore, the domain is all real numbers x such that $x \leqslant 3$.

Example 4 – Domain and range of a function I

What is the domain and range for the function $y = x^2$?

Solution

- *Algebraic analysis*: Squaring any real number produces another real number. Therefore, the domain of $y = x^2$ is the set of all real numbers ($\mathbb{R}$). What about the range? Since the square of any positive or negative number will be positive and the square of zero is zero, the range is the set of all real numbers greater than or equal to zero.
- *Graphical analysis*: For the domain, focus on the x-axis and *horizontally* scan the graph from $-\infty$ to $+\infty$. There are no 'gaps' or blank regions in the graph and the parabola will continue to get 'wider' as x goes to either $-\infty$ or $+\infty$. Therefore, the domain is all real numbers. For the range, focus on the y-axis and *vertically* scan from $-\infty$ or $+\infty$. The parabola will continue 'higher' as y goes to $+\infty$, but the graph does not go below the x-axis. The parabola has no points with negative y-coordinates. Therefore, the range is the set of real numbers greater than or equal to zero. See Figure 2.3.

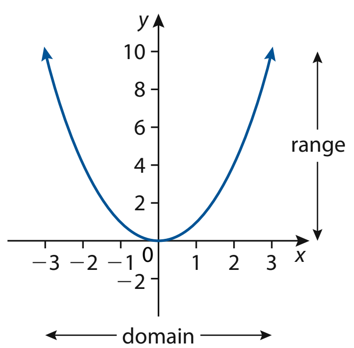


Figure 2.3 The graph of $y = x^2$.

Table 2.1 Different ways of expressing the domain and range of $y = x^2$.

Description in words	Interval notation (both formats)
domain is any real number	domain is $\{x : x \in \mathbb{R}\}$, or domain is $x \in]-\infty, \infty[$
range is any real number greater than or equal to zero	range is $\{y : y \geqslant 0\}$, or range is $y \in [0, \infty[$

Function notation

It is common practice to name a function using a single letter, with f, g and h being the most common. Given that the domain variable is x and the range variable is y, the symbol $f(x)$ denotes the unique value of y that is generated by the value of x. Another notation – sometimes referred to as **mapping notation** – is based on the idea that the function f is the rule that maps x to $f(x)$ and is written $f: x \mapsto f(x)$. For each value of x in the domain, the corresponding unique value of y in the range is called the **function value** at x, or the **image** of x under f. The image of x may be written as $f(x)$ or as y. For example, for the function $f(x) = x^2$: '$f(3) = 9$'; or 'if $x = 3$ then $y = 9$'.

● **Hint:** When asked to determine the domain and range of a function, it is wise for you to conduct both algebraic and graphical analysis – and not rely too much on either approach. For graphical analysis of a function, producing a *comprehensive graph* on your GDC is essential, i.e. a graph that shows all important features of the graph.

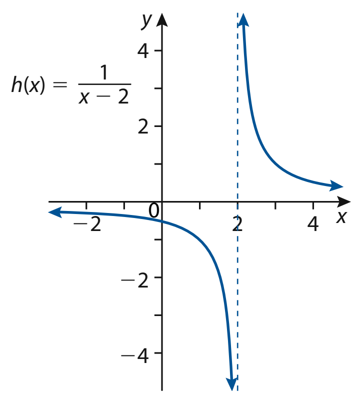
Notation	Description in words
$f(x) = x^2$	'the function f, in terms of x, is x^2'; or, simply, 'f of x equals x^2'
$f: x \mapsto x^2$	'the function f maps x to x^2'
$f(3) = 9$	'the value of the function f when $x = 3$ is 9'; or, simply, 'f of 3 equals 9'
$f: 3 \mapsto 9$	'the image of 3 under the function f is 9'

Table 2.2 Function notation.

● **Hint:** It is common to write $y = f(x)$ and call it a function but this can be considered a misuse of the notation. If we were to be very precise, we would call f the function and $f(x)$ the value of the function at x. But this is often overlooked and we accept writing expressions such as $y = x^2$ or $y = \sin x$ and calling them functions.

Example 5 – Domain and range of a function II

Find the domain and range of the function $h: x \mapsto \frac{1}{x-2}$.



Solution

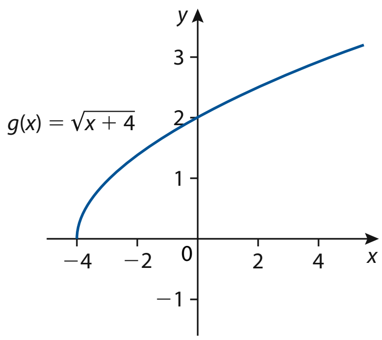
- *Algebraic analysis*: The function produces a real number for all x, except for $x = 2$ when division by zero occurs. Hence, $x = 2$ is the only real number not in the domain. Since the numerator of $\frac{1}{x-2}$ can never be zero, the value of y cannot be zero. Hence, $y = 0$ is the only real number not in the range.
- *Graphical analysis*: A horizontal scan shows a 'gap' at $x = 2$ dividing the graph of the equation into two branches that both continue indefinitely, with no other 'gaps' as $x \to \pm\infty$. Both branches are **asymptotic** (approach but do not intersect) to the vertical line $x = 2$. This line is a **vertical asymptote** and is drawn as a dashed line (it is *not* part of the graph of the equation). A vertical scan reveals a 'gap' at $y = 0$ (x-axis) with both branches of the graph continuing indefinitely, with no other 'gaps' as $y \to \pm\infty$. Both branches are also asymptotic to the x-axis. The x-axis is a **horizontal asymptote**.

  Both approaches confirm the following for $h: x \mapsto \frac{1}{x-2}$:

  The domain is $\{x : x \in \mathbb{R}, x \neq 2\}$ or $x \in\,]-\infty, 2[\cup]2, \infty[$

  The range is $\{y : y \in \mathbb{R}, y \neq 0\}$ or $y \in\,]-\infty, 0[\cup]0, \infty[$

Example 6 – Domain and range of function II



Consider the function $g(x) = \sqrt{x+4}$.

a) Find: (i) $g(7)$ (ii) $g(32)$ (iii) $g(-4)$

b) Find the values of x for which g is undefined.

c) State the domain and range of g.

Solution

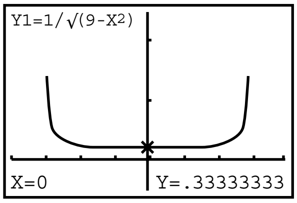
a) (i) $g(7) = \sqrt{7+4} = \sqrt{11} \approx 3.32$ (3 significant figures)
(ii) $g(32) = \sqrt{32+4} = \sqrt{36} = 6$
(iii) $g(-4) = \sqrt{-4+4} = \sqrt{0} = 0$

b) $g(x)$ will be undefined (square root of a negative) when $x + 4 < 0$.
$x + 4 < 0 \Rightarrow x < -4$. Therefore, $g(x)$ is undefined when $x < -4$.

c) It follows from the result in b) that the domain of g is $\{x : x \geqslant -4\}$.
The symbol $\sqrt{}$ stands for the **principal square root** that, by definition, can only give a result that is positive or zero. Therefore, the range of g is $\{y : y \geqslant 0\}$. The domain and range are confirmed by analyzing the graph of the function.

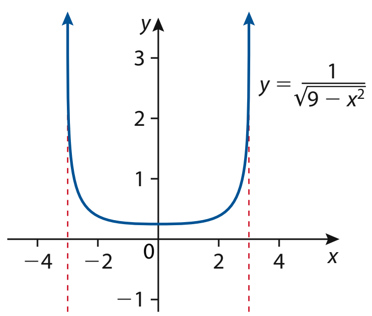
Example 7 – Domain and range of a function III

Find the domain and range of the function
$f(x) = \dfrac{1}{\sqrt{9 - x^2}}$.



Solution

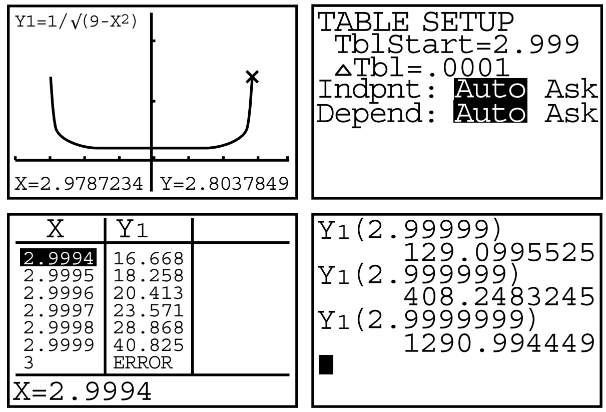
The graph of $y = \dfrac{1}{\sqrt{9-x^2}}$ on a GDC, shown above, agrees with algebraic analysis indicating that the expression $\dfrac{1}{\sqrt{9-x^2}}$ will be positive for all x, and is defined only for $-3 < x < 3$.



Further analysis and tracing the graph reveals that $f(x)$ has a minimum at $\left(0, \frac{1}{3}\right)$. The graph on the GDC (next page) is misleading in that it appears to show that the function has a maximum value (y) of approximately 2.803 7849. Can this be correct? A lack of algebraic thinking and over-reliance on your GDC could easily lead to a mistake. The graph abruptly stops its curve upwards because of low screen resolution.

• **Hint:** As Example 7 illustrates, it is dangerous to completely trust graphs produced on a GDC without also doing some algebraic thinking. It is important to mentally check that the graph shown is comprehensive (shows all important features of the graph), and that the graph agrees with algebraic analysis of the function – e.g. where should the function be zero, positive, negative, undefined, increasing/decreasing without bound, etc.

Function values should get quite large for values of x a little less than 3, because the value of $\sqrt{9-x^2}$ will be small, making the fraction $\dfrac{1}{\sqrt{9-x^2}}$ large. Using your GDC to make a table for $f(x)$, or evaluating the function for values of x very close to -3 or 3, confirms that as x approaches -3 or 3, y increases without bound, i.e. y goes to $+\infty$. Hence, $f(x)$ has vertical asymptotes of $x = -3$ and $x = 3$. This combination of graphical and algebraic analysis leads to the conclusion that the domain of $f(x)$ is $\{x: -3 < x < 3\}$, and the range of $f(x)$ is $\{y: y \geqslant \frac{1}{3}\}$.



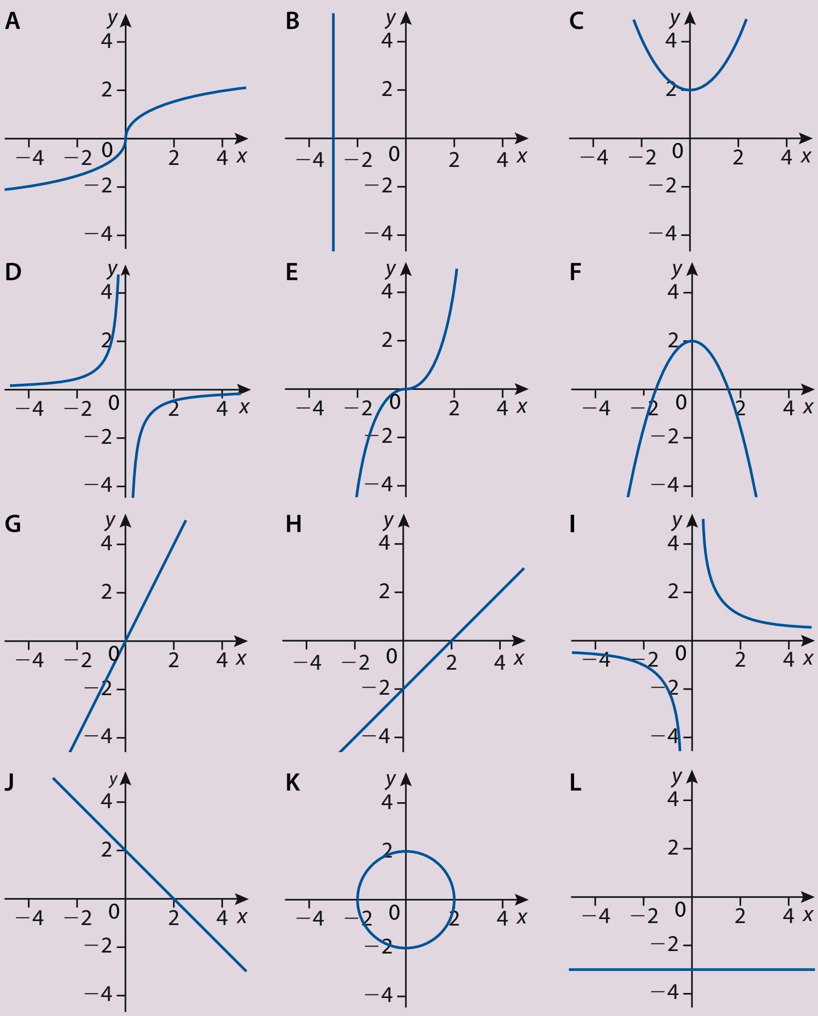
Exercise 2.1

For each equation 1–9, a) match it with its graph (choices are labelled A to L), and b) state whether or not the equation represents a function – with a justification. Assume that x is the independent variable and y is the dependent variable.

1 $y = 2x$ **2** $y = -3$ **3** $x - y = 2$

4 $x^2 + y^2 = 4$ **5** $y = 2 - x$ **6** $y = x^2 + 2$

7 $y^3 = x$ **8** $y = \dfrac{2}{x}$ **9** $x^2 + y = 2$



10 Express the area, A, of a circle as a function of its circumference, C.

11 Express the area, A, of an equilateral triangle as a function of the length, ℓ, of each of its sides.

12 A rectangular swimming pool with dimensions 12 metres by 18 metres is surrounded by a pavement of uniform width x metres. Find the area of the pavement, A, as a function of x.

13 In a right isosceles triangle, the two equal sides have length x units and the hypotenuse has length h units. Write h as a function of x.

14 The pressure P (measured in kilopascals, kPa) for a particular sample of gas is directly proportional to the temperature T (measured in kelvin, K) and inversely proportional to the volume V (measured in litres, ℓ). With k representing the constant of proportionality, this relationship can be written in the form of the equation $P = k\frac{T}{V}$.

a) Find the constant of proportionality, k, if 150 ℓ of gas exerts a pressure of 23.5 kPa at a temperature of 375 K.

b) Using the value of k from part a) and assuming that the temperature is held constant at 375 K, write the volume V as a function of pressure P for this sample of gas.

15 In physics, Hooke's law states that the force F (measured in newtons, N) needed to keep a spring stretched a displacement of x units beyond its natural length is directly proportional to the displacement x. Label the constant of proportionality k (known as the spring constant for a particular spring).

a) Write F as a function of x.

b) If a spring has a natural length of 12 cm and a force of 25 N is needed to keep the spring stretched to a length of 16 cm, find the spring constant k.

c) What force is needed to keep the spring stretched to a length of 18 cm?

In questions 16–23, find the domain of the function.

16 $\{(-6.2, -7), (-1.5, -2), (0.7, 0), (3.2, 3), (3.8, 3)\}$

17 Surface area of a sphere: $S = 4\pi r^2$

18 $f(x) = \frac{2}{5}x - 7$

19 $h: x \mapsto x^2 - 4$

20 $g(t) = \sqrt{3 - t}$

21 $h(t) = \sqrt[3]{t}$

22 $f: x \mapsto \dfrac{6}{x^2 - 9}$

23 $f(x) = \sqrt{\dfrac{1}{x^2} - 1}$

24 Do all linear equations represent a function? Explain.

25 Consider the function $h(x) = \sqrt{x - 4}$.

a) Find: (i) $h(21)$ (ii) $h(53)$ (iii) $h(4)$

b) Find the values of x for which h is undefined.

c) State the domain and range of h.

In questions 26–30, a) find the domain and range of the function, and b) sketch a comprehensive graph of the function clearly indicating any intercepts or asymptotes.

26 $f: x \mapsto \dfrac{1}{x - 5}$

27 $g(x) = \dfrac{1}{\sqrt{x^2 - 9}}$

28 $h(x) = \dfrac{2x - 1}{x + 2}$

29 $p: x \mapsto \sqrt{5 - 2x^2}$

30 $f(x) = \dfrac{1}{x} - 4$

2.2 Composite functions

Composition of functions

Consider the function in Example 6 in the previous section, $f(x) = \sqrt{x + 4}$. When you evaluate $f(x)$ for a certain value of x in the domain (for example, $x = 5$) it is necessary for you to perform computations in two separate steps in a certain order.

$f(5) = \sqrt{5 + 4} \Rightarrow f(5) = \sqrt{9}$ Step 1: compute the sum of $5 + 4$.

$\Rightarrow f(5) = 3$ Step 2: compute the principal square root of 9.

Given that the function has two separate evaluation 'steps', $f(x)$ can be seen as a combination of two 'simpler' functions that are performed in a specified order. According to how $f(x)$ is evaluated (as shown above), the simpler function to be performed first is the rule of 'adding 4' and the second is the rule of 'taking the square root'. If $h(x) = x + 4$ and $g(x) = \sqrt{x}$, we can create (compose) the function $f(x)$ from a combination of $h(x)$ and $g(x)$ as follows:

$f(x) = g(h(x))$

$= g(x + 4)$ Step 1: substitute $x + 4$ for $h(x)$, making $x + 4$ the argument of $g(x)$.

$= \sqrt{x + 4}$ Step 2: apply the function $g(x)$ on the argument $x + 4$.

We obtain the rule $\sqrt{x + 4}$ by first applying the rule $x + 4$ and then applying the rule $\sqrt{x}$. A function that is obtained from 'simpler' functions by applying one after another in this way is called a **composite function**. In the example above, $f(x) = \sqrt{x + 4}$ is the **composition** of $h(x) = x + 4$ followed by $g(x) = \sqrt{x}$. In other words, f is obtained by substituting h into g, and can be denoted in function notation by $g(h(x))$ – read 'g of h of x'.

From the explanation on how f is the composition (or composite) of g and h, you can see why a composite function is sometimes referred to as a 'function of a function'. Also, note that in the notation $g(h(x))$ the function h that is applied first is written 'inside', and the function g that is applied second is written 'outside'.

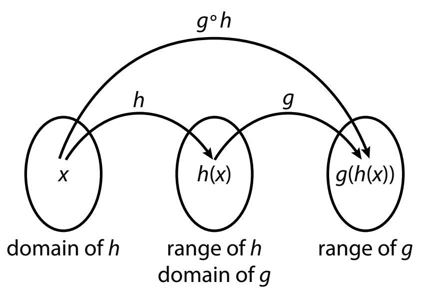


Figure 2.4 Mapping for composite function $g(h(x))$.

We start with a number x in the domain of h and find its image $h(x)$. If this number $h(x)$ is in the domain of g, we then compute the value of $g(h(x))$. The resulting composite function is denoted as $(g \circ h)(x)$. See mapping illustration in Figure 2.4.

Definition of the composition of two functions

The composition of two functions, g and h, such that h is applied first and g second is given by

$$(g \circ h)(x) = g(h(x))$$

The domain of the composite function $g \circ h$ is the set of all x in the domain of h such that $h(x)$ is in the domain of g.

● **Hint:** The notations $(g \circ h)(x)$ and $g(h(x))$ are both commonly used to denote a composite function where h is applied first and then followed by applying g. Since we are reading this from left to right, it is easy to apply the functions in the incorrect order. It may be helpful to read $g \circ h$ as 'g following h', or as 'g composed with h' to emphasize the order in which the functions are applied. Also, in either notation, $(g \circ h)(x)$ or $g(h(x))$, the function applied first is closest to the variable x.

Example 8 – Forming a composition of two functions I

If $f(x) = 3x$ and $g(x) = 2x - 6$, find:

a) $(f \circ g)(5)$ b) Express $(f \circ g)(x)$ as a single function rule (expression).

c) $(g \circ f)(5)$ d) Express $(g \circ f)(x)$ as a single function rule (expression).

e) $(g \circ g)(5)$ f) Express $(g \circ g)(x)$ as a single function rule (expression).

Solution

a) $(f \circ g)(5) = f(g(5)) = f(2 \cdot 5 - 6) = f(4) = 3 \cdot 4 = 12$

b) $(f \circ g)(x) = f(g(x)) = f(2x - 6) = 3(2x - 6) = 6x - 18$
Therefore, $(f \circ g)(x) = 6x - 18$.
Check with result from a): $(f \circ g)(5) = 6 \cdot 5 - 18 = 30 - 18 = 12$

c) $(g \circ f)(5) = g(f(5)) = g(3 \cdot 5) = g(15) = 2 \cdot 15 - 6 = 24$

d) $(g \circ f)(x) = g(f(x)) = g(3x) = 2(3x) - 6 = 6x - 6$
Therefore, $(g \circ f)(x) = 6x - 6$.
Check with result from c): $(g \circ f)(5) = 6 \cdot 5 - 6 = 30 - 6 = 24$

e) $(g \circ g)(5) = g(g(5)) = g(2 \cdot 5 - 6) = g(4) = 2 \cdot 4 - 6 = 2$

f) $(g \circ g)(x) = g(g(x)) = g(2x - 6) = 2(2x - 6) - 6 = 4x - 18$
Therefore, $(g \circ g)(x) = 4x - 18$.
Check with result from e): $(g \circ g)(5) = 4 \cdot 5 - 18 = 20 - 18 = 2$

It is important to notice that in parts b) and d) in Example 8, $f \circ g$ is *not* equal to $g \circ f$. At the start of this section, it was shown how the two functions $h(x) = x + 4$ and $g(x) = \sqrt{x}$ could be combined into the composite function $(g \circ h)(x)$ to create the single function $f(x) = \sqrt{x + 4}$. However, the composite function $(h \circ g)(x)$ – the functions applied in reverse order – creates a different function: $(h \circ g)(x) = h(g(x)) = h(\sqrt{x}) = \sqrt{x} + 4$. Since $\sqrt{x} + 4 \neq \sqrt{x + 4}$, then again $f \circ g$ is *not* equal to $g \circ f$. Is it always true that $f \circ g \neq g \circ f$? The next example will answer that question.

Example 9 – Forming a composition of two functions II

Given $f: x \mapsto 3x - 6$ and $g: x \mapsto \frac{1}{3}x + 2$, find the following:

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

Solution

a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3}x + 2\right) = 3\left(\frac{1}{3}x + 2\right) - 6 = x + 6 - 6 = x$

b) $(g \circ f)(x) = g(f(x)) = g(3x - 6) = \frac{1}{3}(3x - 6) + 2 = x - 2 + 2 = x$

Example 9 shows that it is possible for $f \circ g$ to be equal to $g \circ f$. We will learn in the next section that this occurs in some cases where there is a 'special' relationship between the pair of functions. However, in general, $f \circ g \neq g \circ f$.

Decomposing a composite function

In Examples 8 and 9, we created a single function by forming the composition of two functions. As we did with the function $f(x) = \sqrt{x + 4}$ at the start of this section, it is also important for you to be able to identify two functions that *make up* a composite function, in other words, for you to *decompose* a function into two simpler functions. When you are doing this it is very useful to think of the function which is applied first as the 'inside' function, and the function that is applied second as the 'outside' function. In the function $f(x) = \sqrt{x + 4}$, the 'inside' function is $h(x) = x + 4$ and the 'outside' function is $g(x) = \sqrt{x}$.

● **Hint:** Decomposing composite functions – identifying the component functions that form a composite function – is an important skill when working with certain functions in the topic of calculus. For the composite function $f(x) = (g \circ h)(x)$, g and h are the component functions.

Example 10 – Decomposing a composite function

Each of the following functions is a composite function of the form $(f \circ g)(x)$. For each, find the two component functions f and g.

a) $h: x \mapsto \dfrac{1}{x + 3}$ b) $k: x \mapsto 2^{4x + 1}$ c) $p(x) = \sqrt[3]{x^2 - 4}$

Solution

a) If you were to evaluate the function $h(x)$ for a certain x in the domain, you would first evaluate the expression $x + 3$, and then evaluate the expression $\frac{1}{x}$. Hence, the 'inside' function (applied first) is $y = x + 3$, and the 'outside' function (applied second) is $y = \frac{1}{x}$. Then, with $g(x) = x + 3$ and $f(x) = \frac{1}{x}$, it follows that $h: x \mapsto (f \circ g)(x)$.

b) Evaluating $k(x)$ requires you to first evaluate the expression $4x + 1$, and then evaluate the expression 2^x. Hence, the 'inside' function is $y = 4x + 1$, and the 'outside' function is $y = 2^x$. Then, with $g(x) = 4x + 1$ and $f(x) = 2^x$, it follows that $k: x \mapsto (f \circ g)(x)$.

c) Evaluating $p(x)$ requires you to perform three separate evaluation 'steps': (1) squaring a number, (2) subtracting four, and then (3) taking the cube root. Hence, it is possible to decompose $p(x)$ into three component functions: if $h(x) = x^2$, $g(x) = x - 4$ and $f(x) = \sqrt[3]{x}$, then $p(x) = (f \circ g \circ h)(x) = f(g(h(x)))$. However, for our purposes it is best to decompose the composite function into only two component functions: if $g(x) = x^2 - 4$ and $f(x) = \sqrt[3]{x}$, then $p(x) = (f \circ g)(x)$.

Finding the domain of a composite function

Referring back to Figure 2.4 (shown again here as Figure 2.5), it is important to note that in order for a value of x to be in the domain of the composite function $g \circ h$, two conditions must be met:

(1) x must be in the domain of h, and (2) $h(x)$ must be in the domain of g.

Likewise, it is also worth noting that $g(h(x))$ is in the range of $g \circ h$ only if x is in the domain of $g \circ h$. The next example illustrates these points – and also that, in general, the domains of $g \circ h$ and $h \circ g$ are not necessarily the same.

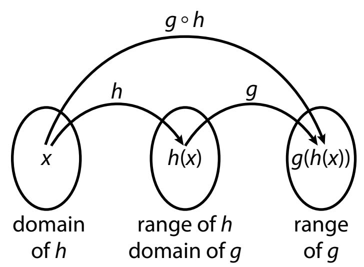


Figure 2.5 Mapping for composite function $g(h(x))$.

Example 11 – Domain and range of a composite function

Let $g(x) = x^2 - 4$ and $h(x) = \sqrt{x}$. Find:

a) $(g \circ h)(x)$ and its domain and range

b) $(h \circ g)(x)$ and its domain and range.

Solution

Firstly, establish the domain and range for both g and h. For $g(x) = x^2 - 4$, the domain is $x \in \mathbb{R}$ and the range is $y \geqslant -4$. For $h(x) = \sqrt{x}$, the domain is $x \geqslant 0$ and the range is $y \geqslant 0$.

a) $(g \circ h)(x) = g(h(x))$

$= g(\sqrt{x})$ — To be in the domain of $g \circ h$, $\sqrt{x}$ must be defined for $x \Rightarrow x \geqslant 0$.

$= (\sqrt{x})^2 - 4$ — Therefore, the domain of $g \circ h$ is $x \geqslant 0$.

$= x - 4$ — Since $x \geqslant 0$, the range for $y = x - 4$ is $y \geqslant -4$.

Therefore, $(g \circ h)(x) = x - 4$, and its domain is $x \geqslant 0$, and its range is $y \geqslant -4$.

b) $(h \circ g)(x) = h(g(x))$ — $g(x) = x^2 - 4$ must be in the domain of h
$x^2 - 4 \geqslant 0 \Rightarrow x^2 \geqslant 4$

$= h(x^2 - 4)$ — Therefore, the domain of $h \circ g$ is $x \leqslant -2$ or $x \geqslant 2$

$= \sqrt{x^2 - 4}$ — and, with $x \leqslant -2$ or $x \geqslant 2$, the range for $y = \sqrt{x^2 - 4}$ is $y \geqslant 0$.

Therefore, $(h \circ g)(x) = \sqrt{x^2 - 4}$, and its domain is $x \leqslant -2$ or $x \geqslant 2$, and its range is $y \geqslant 0$.

Exercise 2.2

1 Let $f(x) = 2x$ and $g(x) = \dfrac{1}{x-3}$, $x \neq 0$.

a) Find the value of (i) $(f \circ g)(5)$ and (ii) $(g \circ f)(5)$.
b) Find the function rule (expression) for (i) $(f \circ g)(x)$ and (ii) $(g \circ f)(x)$.

2 Let $f: x \mapsto 2x - 3$ and $g: x \mapsto 2 - x^2$.
In a)-f), evaluate:

a) $(f \circ g)(0)$ b) $(g \circ f)(0)$ c) $(f \circ f)(4)$
d) $(g \circ g)(-3)$ e) $(f \circ g)(-1)$ f) $(g \circ f)(-3)$

In g)-j), find the expression:

g) $(f \circ g)(x)$ h) $(g \circ f)(x)$
i) $(f \circ f)(x)$ j) $(g \circ g)(x)$

For each pair of functions in questions 3–12, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and state the domain for each.

3 $f(x) = 4x - 1$, $g(x) = 2 + 3x$

4 $f(x) = x^2 + 1$, $g(x) = -2x$

5 $f(x) = \sqrt{x+1}$, $g(x) = 1 + x^2$

6 $f(x) = \dfrac{2}{x+4}$, $g(x) = x - 1$

7 $f(x) = 3x + 5, g(x) = \frac{x - 5}{3}$

8 $f(x) = x^2 - 2x, g(x) = -x^2 - 2x$

9 $f(x) = \frac{2x}{4 - x}, g(x) = \frac{1}{x^2}$

10 $f(x) = 2 - x^3, g(x) = \sqrt[3]{1 - x^2}$

11 $f(x) = \frac{2}{x + 3} - 3, g(x) = \frac{2}{x + 3} - 3$ $[f = g]$

12 $f(x) = \frac{x}{x - 1}, g(x) = x^2 - 1$

13 Let $g(x) = \sqrt{x - 1}$ and $h(x) = 10 - x^2$. Find:

a) $(g \circ h)(x)$ and its domain and range, and

b) $(h \circ g)(x)$ and its domain and range.

14 Let $f(x) = \frac{1}{x}$ and $g(x) = 10 - x^2$. Find:

a) $(f \circ g)(x)$ and its domain and range, and

b) $(g \circ f)(x)$ and its domain and range.

In questions 15–22, determine functions g and h so that $f(x) = g(h(x))$.

15 $f(x) = (x + 3)^2$

16 $f(x) = \sqrt{x - 5}$

17 $f(x) = 7 - \sqrt{x}$

18 $f(x) = \frac{1}{x + 3}$

19 $f(x) = 10^{x + 1}$

20 $f(x) = \sqrt[3]{x - 9}$

21 $f(x) = |x^2 - 9|$

22 $f(x) = \frac{1}{\sqrt{x - 5}}$

In questions 23–26, find the domain for a) the function f, b) the function g, and c) the composite function $f \circ g$.

23 $f(x) = \sqrt{x}, g(x) = x^2 + 1$

24 $f(x) = \frac{1}{x}, g(x) = x + 3$

25 $f(x) = \frac{3}{x^2 - 1}, g(x) = x + 1$

26 $f(x) = 2x + 3, g(x) = \frac{x}{2}$

2.3 Inverse functions

Pairs of inverse functions

If we choose a number and cube it (raise it to the power of 3), and then take the cube root of the result, the answer is the original number. The same result would occur if we applied the two rules in the reverse order. That is, first take the cube root of a number and then cube the result – and again the answer is the original number. Let's write each of these rules as a function with function notation. Write the cubing function as $f(x) = x^3$, and the cube root function as $g(x) = \sqrt[3]{x}$. Now using what we know about composite functions and operations with radicals and exponents, we can write what was described above in symbolic form.

1. Cube a number and then take the cube root of the result:

   $g(f(x)) = \sqrt[3]{x^3} = (x^3)^{\frac{1}{3}} = x^1 = x$

   For example, $g(f(-2)) = \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$

You are already familiar with pairs of **inverse operations**. Addition and subtraction are inverse operations. For example, the rule of 'adding six' ($x + 6$), and the rule of 'subtracting six' ($x - 6$), *undo* each other. Accordingly, the functions $f(x) = x + 6$ and $g(x) = x - 6$ are a pair of inverse functions. Multiplication and division are also inverse operations.

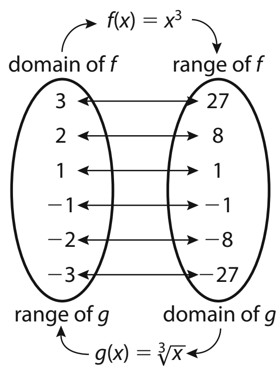
2. Take the cube root of a number and then cube the result:

$$f(g(x)) = (\sqrt[3]{x})^3 = (x^{\frac{1}{3}})^3 = x^1 = x$$

For example, $f(g(27)) = (\sqrt[3]{27})^3 = (3)^3 = 27$

Because function g has this reverse (inverse) effect on function f, we call function g the **inverse** of function f. Function f has the same inverse effect on function g [$g(27) = 3$ and then $f(3) = 27$], making f the inverse function of g. The functions f and g are inverses of each other. The cubing and cube root functions are an example of a pair of **inverse functions**. The mapping diagram for functions f and g in Figure 2.6 illustrates the relationship for a pair of inverse functions where the domain of one is the range for the other.

Figure 2.6 A mapping diagram for the cubing and cube root functions.



The composite of two inverse functions is the function that always produces the same number that was first substituted into the function. This function is called the **identity function** because it assigns each number in its domain to itself, and is denoted by $I(x) = x$.

Definition of the inverse of a function

If f and g are two functions such that $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f, the function g is the *inverse* of the function f. The notation to indicate the function that is the 'inverse of function f' is f^{-1}. Therefore,

$$(f \circ f^{-1})(x) = x \text{ and } (f^{-1} \circ f)(x) = x$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Figure 2.7 shows a mapping diagram for a pair of inverse functions.

It follows from the definition that if g is the inverse of f, it must also be true that f is the inverse of g.

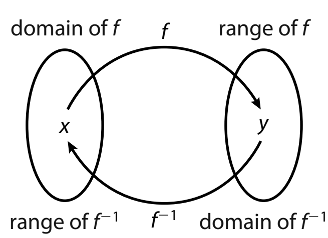


Figure 2.7 $f(x) = y$ and $f^{-1}(y) = x$.

For a pair of inverse functions, f and g, the composite functions $f(g(x))$ and $g(f(x))$ are equal, a 'special' relationship that we learned last section is not generally true for an arbitrary pair of functions.

Note: Remember that the notation $(f \circ g)(x)$ is equivalent to $f(g(x))$.

• **Hint:** Do not mistake the -1 in the notation f^{-1} for an exponent. It is *not* an exponent. If a superscript of -1 is applied to the name of a function, as in f^{-1} or $\sin^{-1}$, then it denotes the function that is the inverse of the named function (e.g. f or sin). If a superscript of -1 is applied to an expression, as in 7^{-1} or $(2x + 5)^{-1}$, then it is an exponent and denotes the reciprocal of the expression.

In general, the functions $f(x)$ and $g(x)$ are a pair of inverse functions if the following two statements are true:

1. $g(f(x)) = x$ for all x in the domain of f.
2. $f(g(x)) = x$ for all x in the domain of g.

Example 12 – Verifying a pair of functions are inverses

Given $h(x) = \frac{x-3}{2}$ and $p(x) = 2x + 3$, show that h and p are a pair of inverse functions.

Solution

Since the domain and range of both $h(x)$ and $p(x)$ is the set of all real numbers, then:

1. For any real number x, $p(h(x)) = p\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = x - 3 + 3 = x$
2. For any real number x, $h(p(x)) = h(2x+3) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x$

Since $p(h(x)) = h(p(x)) = x$ then h and p are a pair of inverse functions.

Returning to our initial example, it is clear that both $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ satisfy the definition of a function because for both f and g every number in its domain determines exactly one number in its range. Since they are a pair of inverse functions then the 'reverse' is also true for both – that is, every number in its range is determined by exactly one number in its range. Such a function is called a **one-to-one function**. The phrase 'one-to-one' is appropriate because each value in the domain corresponds to exactly **one** value in the range, and each value in the range corresponds to exactly **one** value in the domain.

• **Hint:** The mapping diagram for f and g in Figure 2.6 nicely illustrates this 'one-to-one correspondence' between the domain and range for each function.

A one-to-one function
A function is **one-to-one** if each element y in the range is the image of exactly one element x in the domain.

The existence of an inverse function

Determining whether a function is one-to-one is very useful because the inverse of a one-to-one function will also be a function. Analyzing the graph of a function is the most effective way to determine whether a function is one-to-one. Let's look at the graph of the one-to-one function $f(x) = x^3$ shown in Figure 2.8. It is clear that as the values of x increase over the domain (i.e. from $-\infty$ to ∞) that the function values are always increasing. A function that is always increasing, or always decreasing, throughout its domain is one-to-one and has an inverse function.

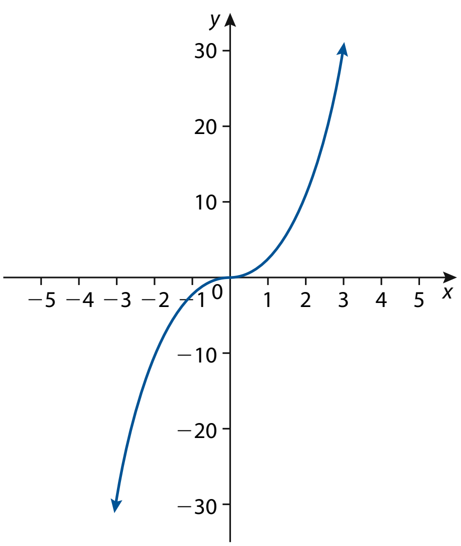


Figure 2.8 Graph of $f(x) = x^3$ which is increasing as x goes from $-\infty$ to ∞.

A function f is an **increasing function** if $x_1 < x_2$ implies $f(x_1) < f(x_2)$, and it is a **decreasing function** if $x_1 < x_2$ implies $f(x_1) > f(x_2)$. If a function is either increasing or decreasing, it is said to be **monotonic**.

Example 13 shows that a function that is not one-to-one (always increasing or always decreasing) can be made so by restricting its domain.

Example 13 – Restricting the domain so that a function is one-to-one

The function $f(x) = x^2$ (Figure 2.9) is not one-to-one for all real numbers. However, the function $g(x) = x^2$ with domain $x \geqslant 0$ (Figure 2.10) is always increasing (one-to-one), and the function $h(x) = x^2$ with domain $x \leqslant 0$ (Figure 2.11) is always decreasing (one-to-one).

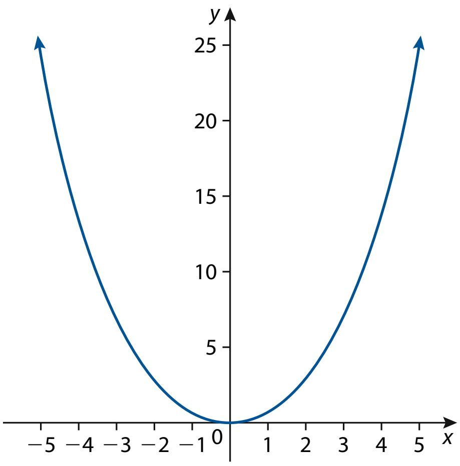


Figure 2.9 $f(x) = x^2$

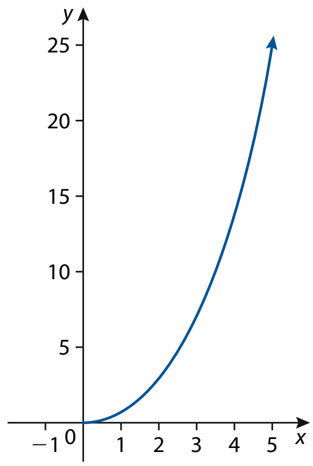


Figure 2.10 $g(x) = x^2, x \geqslant 0$

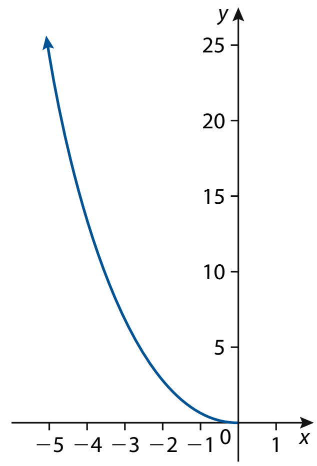
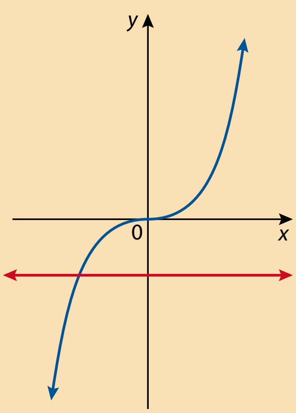


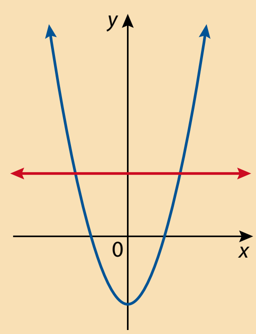
Figure 2.11 $h(x) = x^2, x \leqslant 0$

If a function f is always increasing or always decreasing in its domain (i.e. it is monotonic), then f has an inverse f^{-1}.

No horizontal line can pass through the graph of a one-to-one function at more than one point.



Any horizontal line intersects the graph at no more than one pont, so y is a one-to-one function of x; and its inverse is a function.



At least one horizontal line intersects the graph at more than one point, so y is **not** a one-to-one function of x; and its inverse is not a function.

A function for which at least one element y in the range is the image of more than one element x in the domain is called a **many-to-one function**. Examples of many-to-one functions that we have already encountered are $y = x^2, x \in \mathbb{R}$ and $y = |x|, x \in \mathbb{R}$. As Figure 2.12 illustrates for $y = |x|$, a horizontal line exists that intersects a many-to-one function at more than one point. Thus, the inverse of a many-to-one function will not be a function.

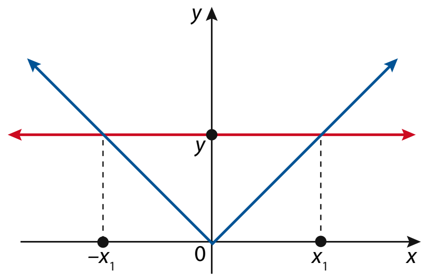


Figure 2.12 Graph of $y = |x|$; an example of a many-to-one function.

Finding the inverse of a function

Example 14 – Finding an inverse function I

The function f is defined for $x \in \mathbb{R}$ by $f(x) = 4x - 8$. Determine if f has an inverse f^{-1}. If not, restrict the domain of f in order to find an inverse function f^{-1}. Verify the result by showing that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. Graph f and its inverse function f^{-1} on the same set of axes.

Solution

Firstly, we recognize that f is an increasing function for $(-\infty, \infty)$ because the graph of $f(x) = 4x - 8$ is a straight line with a constant slope of 4. Therefore, f is a one-to-one function and it has an inverse f^{-1}. To find the equation for f^{-1}, we start by switching the domain (x) and range (y) since the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}, as stated in the definition and depicted in Figure 2.7. Also, recall that $y = f(x)$.

$f(x) = 4x - 8$

$y = 4x - 8$ Write $y = f(x)$.

$x = 4y - 8$ Interchange x and y (i.e. switch the domain and range).

$4y = x + 8$ Solve for y (dependent variable) in terms of x (independent variable).

$y = \frac{1}{4}x + 2$

$f^{-1}(x) = \frac{1}{4}x + 2$ Resulting equation is $y = f^{-1}(x)$.

Verify that f and f^{-1} are inverses by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

$$f\left(\tfrac{1}{4}x + 2\right) = 4\left(\tfrac{1}{4}x + 2\right) - 8 = x + 8 - 8 = x$$

$$f^{-1}(4x - 8) = \tfrac{1}{4}(4x - 8) + 2 = x - 2 + 2 = x$$

This confirms that $y = 4x - 8$ and $y = \frac{1}{4}x + 2$ are inverses of each other.

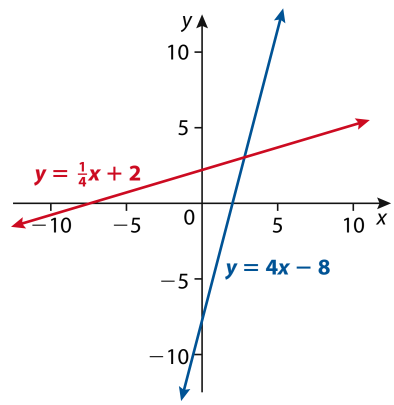


Figure 2.13 Graph of pair of inverse functions for Example 14.

The method of interchanging domain (x) and range (y) to find the inverse function used in Example 14 also gives us a way for obtaining the graph of f^{-1} from the graph of f. Given the reversing effect that a pair of inverse functions have on each other, if $f(a) = b$ then $f^{-1}(b) = a$. Hence, if the ordered pair (a, b) is a point on the graph of $y = f(x)$, then the 'reversed' ordered pair (b, a) must be on the graph of $y = f^{-1}(x)$. Figure 2.14 shows that the point (b, a) can be found by reflecting the point (a, b) about the line $y = x$. Therefore, as Figure 2.15 illustrates, the following statement can be made about the graphs of a pair of inverse functions.

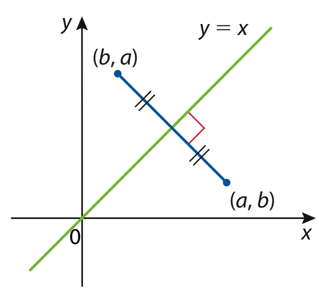


Figure 2.14 The point (b, a) is a reflection over the line $y = x$ of the point (a, b).

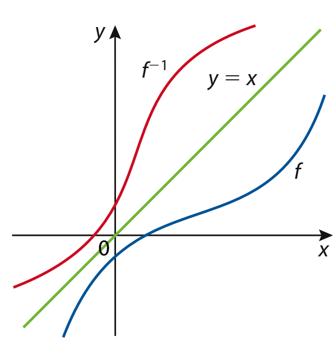


Figure 2.15 Graphs of f and f^{-1} are symmetrical about the line $y = x$.

Graphical symmetry of inverse functions

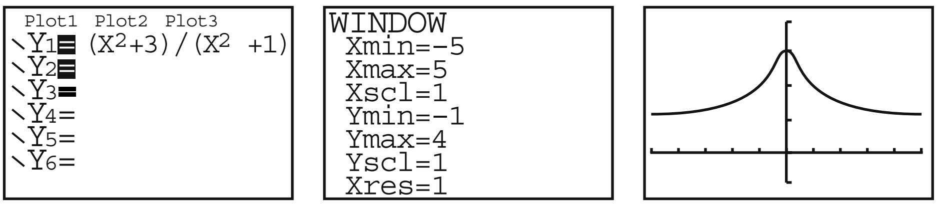
The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

Example 15 – Finding an inverse function II

The function f is defined for $x \in \mathbb{R}$ by $f: x \mapsto \dfrac{x^2+3}{x^2+1}$. Determine if f has an inverse f^{-1}. If not, restrict the domain of f in order to find an inverse function f^{-1}. Graph f and its inverse f^{-1} on the same set of axes.

Solution

A graph of f produced on a GDC reveals that it is not monotonic over its domain $(-\infty, \infty)$. It is increasing for $(-\infty, 0]$, and decreasing for $[0, \infty)$. Therefore, f does not have an inverse f^{-1} for $x \in \mathbb{R}$. It is customary to restrict the domain to the 'largest' set possible. Hence, we can choose to restrict the domain to either $x \in (-\infty, 0]$ (making f an increasing function), or $x \in [0, \infty)$ (making f a decreasing function). Let's change the domain from $x \in \mathbb{R}$ to $x \in [0, \infty)$.



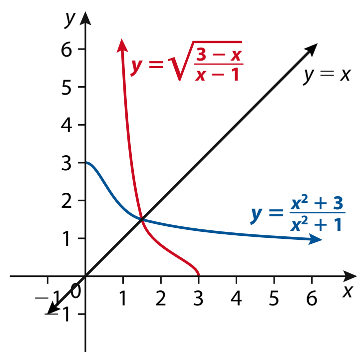
We use a method similar to that in Example 14 to find the equation for f^{-1}. First solve for x in terms of y and then interchange the domain (x) and range (y).

$$f: x \mapsto \frac{x^2+3}{x^2+1} \Rightarrow y = \frac{x^2+3}{x^2+1} \Rightarrow x^2y + y = x^2 + 3 \Rightarrow x^2y - x^2 = 3 - y$$

$$\Rightarrow x^2(y-1) = 3 - y \Rightarrow x^2 = \frac{3-y}{y-1} \Rightarrow x = \pm\sqrt{\frac{3-y}{y-1}} \Rightarrow y = \pm\sqrt{\frac{3-x}{x-1}}$$

Since we chose to restrict the domain of f to $x \in [0, \infty)$, then the range of f^{-1} will be $y \in [0, \infty)$. Therefore, from the working above, the resulting inverse function is $f^{-1}(x) = \sqrt{\dfrac{3-x}{x-1}}$.

Figure 2.16 Graphs of f and f^{-1} for Example 15 show symmetry about the line $y = x$.



Finding the inverse of a function

To find the inverse of a function *f*, use the following steps:

1 Confirm that *f* is one-to-one (although, for this course, you can assume this).
2 Replace $f(x)$ with y.
3 Interchange x and y.
4 Solve for y.
5 Replace y with $f^{-1}(x)$.
6 The domain of f^{-1} is equal to the range of *f*, and the range of f^{-1} is equal to the domain of *f*.

Example 16

Consider the function $f: x \mapsto \sqrt{x+3},\ x \geqslant -3$.

a) Determine the inverse function f^{-1}. b) What is the domain of f^{-1}?

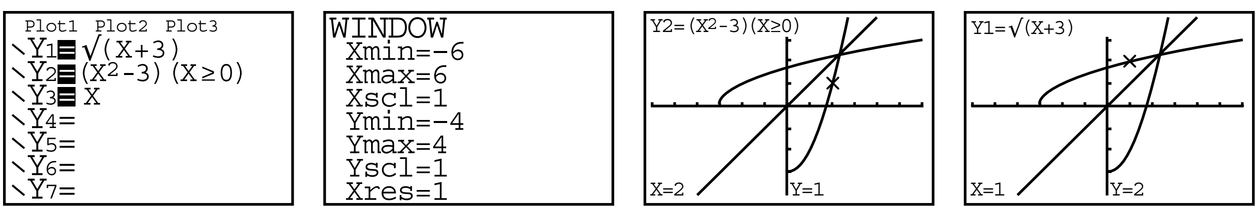
Solution

a) Following the steps for finding the inverse of a function gives:

$y = \sqrt{x+3}$	Replace $f(x)$ with y.
$x = \sqrt{y+3}$	Interchange x and y.
$x^2 = y + 3$	Solve for y (squaring both sides here).
$y = x^2 - 3$	Solved for y.
$f^{-1}: x \mapsto x^2 - 3$	Replace y with $f^{-1}(x)$.

b) The domain explicitly defined for f is $x \geqslant -3$ and since the $\sqrt{\ }$ symbol stands for the principal square root (positive), then the range of f is all positive real numbers, i.e. $y \geqslant 0$. The domain of f^{-1} is equal to the range of f; therefore, the domain of f^{-1} is $x \geqslant 0$.

Graphing $y = \sqrt{x+3}$ and $y = x^2 - 3$ from Example 16 on your GDC visually confirms these results. Note that since the calculator would have automatically assumed that the domain is $x \in \mathbb{R}$, the domain for the equation $y = x^2 - 3$ has been changed to $x \geqslant 0$. In order to show that f and f^{-1} are reflections about the line $y = x$, the line $y = x$ has been graphed and a viewing window has been selected to ensure that the scales are equal on each axis. Using the trace feature of your GDC, you can explore a characteristic of inverse functions – that is, if some point (a, b) is on the graph of f, the point (b, a) must be on the graph of f^{-1}.



Example 17

Consider the function $f(x) = 2(x+4)$ and $g(x) = \dfrac{1-x}{3}$.

a) Find g^{-1} and state its domain and range.

b) Solve the equation $(f \circ g^{-1})(x) = 2$.

Solution

a)

$$y = \frac{1-x}{3}$$ Replace $f(x)$ with y.

$$x = \frac{1-y}{3}$$ Interchange x and y.

$$3x = 1 - y$$ Solve for y.

$$y = -3x + 1$$ Solved for y.

$$g^{-1}(x) = -3x + 1$$ Replace y with $g^{-1}(x)$.

g is a linear function and its domain is $x \in \mathbb{R}$ and its range is $y \in \mathbb{R}$; therefore, for g^{-1} the domain is $x \in \mathbb{R}$ and range is $y \in \mathbb{R}$.

b)

$$(f \circ g^{-1})(x) = f(g^{-1}(x)) = f(-3x+1) = 2$$
$$2[(-3x+1)+4] = 2$$
$$-6x + 2 + 8 = 2$$
$$-6x = -8$$
$$x = \tfrac{4}{3}$$

Exercise 2.3

In questions 1–4, assume that f is a one-to-one function.

1 a) If $f(2) = -5$, what is $f^{-1}(-5)$?

b) If $f^{-1}(6) = 10$, what is $f(10)$?

2 a) If $f(-1) = 13$, what is $f^{-1}(13)$?

b) If $f^{-1}(b) = a$, what is $f(a)$?

3 If $g(x) = 3x - 7$, what is $g^{-1}(5)$?

4 If $h(x) = x^2 - 8x$, with $x \geqslant 4$, what is $h^{-1}(-12)$?

In questions 5–14, show a) algebraically and b) graphically that f and g are inverse functions by verifying that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, and by sketching the graphs of f and g on the same set of axes, with equal scales on the x- and y-axes. Use your GDC to assist in making your sketches on paper.

5 $f: x \mapsto x + 6$; $g: x \mapsto x - 6$

6 $f: x \mapsto 4x$; $g: x \mapsto \dfrac{x}{4}$

7 $f: x \mapsto 3x + 9$; $g: x \mapsto \frac{1}{3}x - 3$

8 $f: x \mapsto \dfrac{1}{x}$; $g: x \mapsto \dfrac{1}{x}$

9 $f: x \mapsto x^2 - 2, x \geqslant 0$; $g: x \mapsto \sqrt{x+2}, x \geqslant -2$

10 $f: x \mapsto 5 - 7x$; $g: x \mapsto \dfrac{5-x}{7}$

11 $f: x \mapsto \dfrac{1}{1+x}$; $g: x \mapsto \dfrac{1-x}{x}$

12 $f: x \mapsto (6-x)^{\frac{1}{2}}$; $g: x \mapsto 6 - x^2, x \geqslant 0$

13 $f: x \mapsto x^2 - 2x + 3, x \geqslant 1$; $g: x \mapsto 1 + \sqrt{x-2}, x \geqslant 2$

14 $f: x \mapsto \sqrt[3]{\dfrac{x+6}{2}}$; $g: x \mapsto 2x^3 - 6$

In questions 15–24, find the inverse function f^{-1} and state its domain.

15 $f(x) = 2x - 3$

16 $f(x) = \dfrac{x + 7}{4}$

17 $f(x) = \sqrt{x}$

18 $f(x) = \dfrac{1}{x + 2}$

19 $f(x) = 4 - x^2, x \geqslant 0$

20 $f(x) = \sqrt{x - 5}$

21 $f(x) = ax + b, a \neq 0$

22 $f(x) = x^2 + 2x, x \geqslant -1$

23 $f(x) = \dfrac{x^2 - 1}{x^2 + 1}, x \leqslant 0$

24 $f(x) = x^3 + 1$

In questions 25–28, determine if f has an inverse f^{-1}. If not, restrict the domain of f in order to find an inverse function. Graph f and its inverse f^{-1} on the same set of axes.

25 $f(x) = \dfrac{2x + 3}{x - 1}$

26 $f(x) = (x - 2)^2$

27 $f(x) = \dfrac{1}{x^2}$

28 $f(x) = 2 - x^4$

29 Use your GDC to graph the function $f(x) = \dfrac{2x}{1 + x^2}, x \in \mathbb{R}$. Find three intervals for which f is a one-to-one function (monotonic) and hence will have an inverse f^{-1} on the interval. The union of all three intervals is all real numbers.

In questions 30–37, use the functions $g(x) = x + 3$ and $h(x) = 2x - 4$ to find the indicated value or the indicated function.

30 $(g^{-1} \circ h^{-1})(5)$

31 $(h^{-1} \circ g^{-1})(9)$

32 $(g^{-1} \circ g^{-1})(2)$

33 $(h^{-1} \circ h^{-1})(2)$

34 $g^{-1} \circ h^{-1}$

35 $h^{-1} \circ g^{-1}$

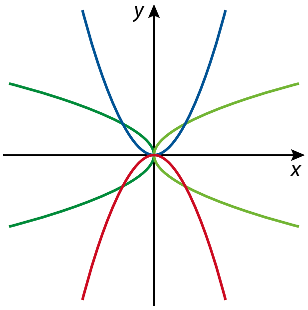
36 $(g \circ h)^{-1}$

37 $(\mathrm{h} \circ g)^{-1}$

38 The reciprocal function in question 8, $f(x) = \dfrac{1}{x}$, is its own inverse (self-inverse). Show that any function in the form $f(x) = \dfrac{a}{x + b} - b, a \neq 0$ is its own inverse.

2.4 Transformations of functions

Even when you use your GDC to sketch the graph of a function, it is helpful to know what to expect in terms of the location and shape of the graph – and even more so if you're not allowed to use your GDC for a particular question. In this section, we look at how certain changes to the equation of a function can affect, or **transform**, the location and shape of its graph. We will investigate three different types of **transformations** of functions that include how the graph of a function can be **translated**, **reflected** and **stretched** (or shrunk). Studying graphical transformations gives us a better understanding of how to efficiently sketch and visualize many different functions. We will also take a closer look at two specific functions: the absolute value function, $y = |x|$, and the reciprocal function, $y = \frac{1}{x}$.



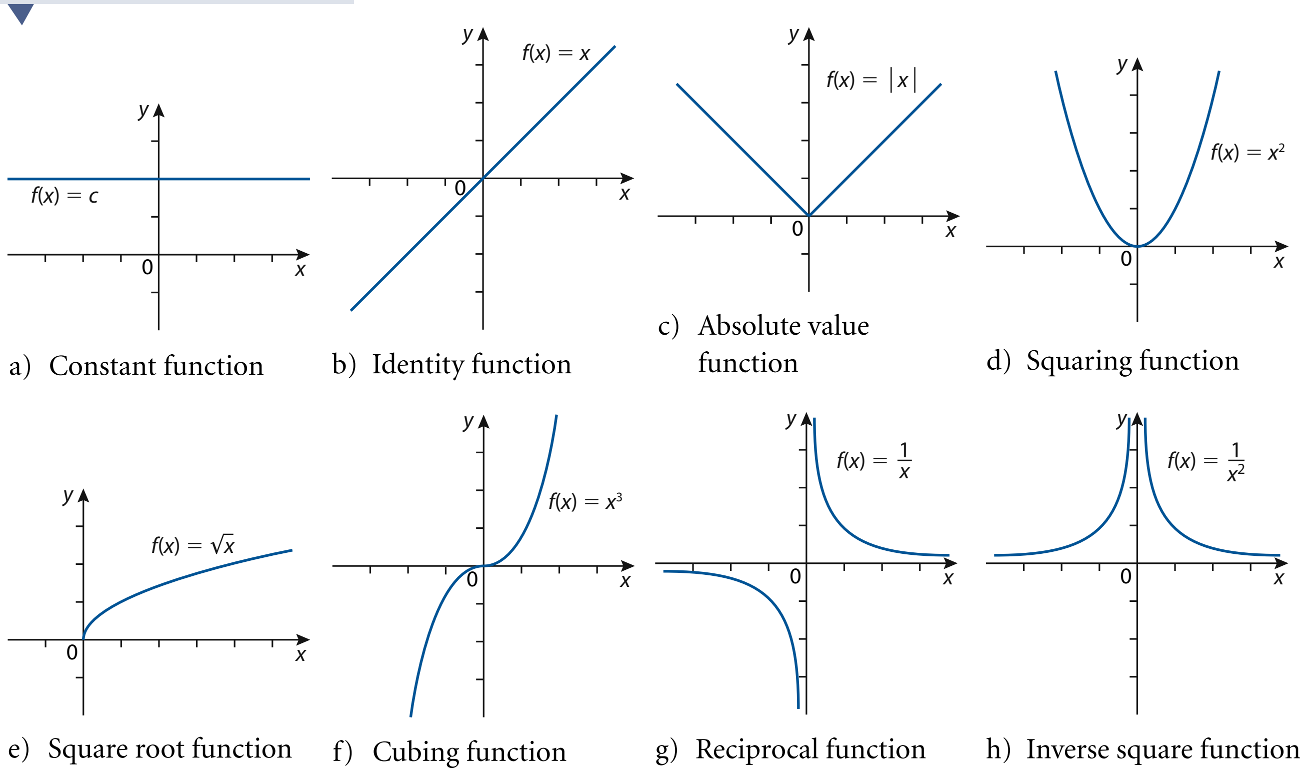
● **Hint:** When analyzing the graph of a function, it is often convenient to express a function in the form $y = f(x)$. As we have done throughout this chapter, we often refer to a function such as $f(x) = x^2$ by the equation $y = x^2$.

Graphs of common functions

It is important for you to be familiar with the location and shape of a certain set of common functions. For example, from your previous knowledge about linear equations, you can determine the location of the linear function $f(x) = ax + b$. You know that the graph of this function is a line whose slope is a and whose y-intercept is $(0, b)$.

The eight graphs in Figure 2.17 represent some of the most commonly used functions in algebra. You should be familiar with the characteristics of the graphs of these common functions. This will help you predict and analyze the graphs of more complicated functions that are derived from applying one or more transformations to these simple functions. There are other important basic functions with which you should be familiar – for

Figure 2.17 Graphs of common functions.



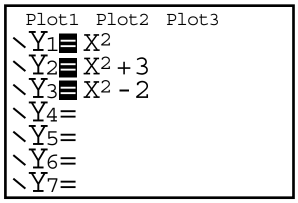
example, exponential, logarithmic and trigonometric functions – but we will encounter these in later chapters.

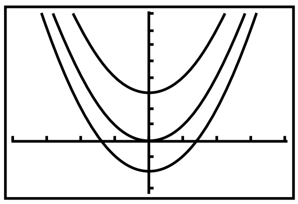
We will see that many functions have graphs that are a transformation (translation, reflection or stretch), or a combination of transformations, of one of these common functions.

• **Hint:** The word *inverse* can have different meanings in mathematics depending on the context. In Section 2.3 of this chapter, *inverse* is used to describe operations or functions that undo each other. However, 'inverse' is sometimes used to denote the **multiplicative inverse** (or **reciprocal**) of a number or function. This is how it is used in the name for the function shown in h) of Figure 2.17. The function in g) is the **inverse function**. (See page 62.)

Vertical and horizontal translations

Use your GDC to graph each of the following three functions: $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 2$. How do the graphs of g and h compare with the graph of f that is one of the common functions displayed in Figure 2.17? The graphs of g and h both appear to have the same shape – it's only the location, or position, that has changed compared to f. Although the curves (parabolas) appear to be getting closer together, their vertical separation at every value of x is constant.





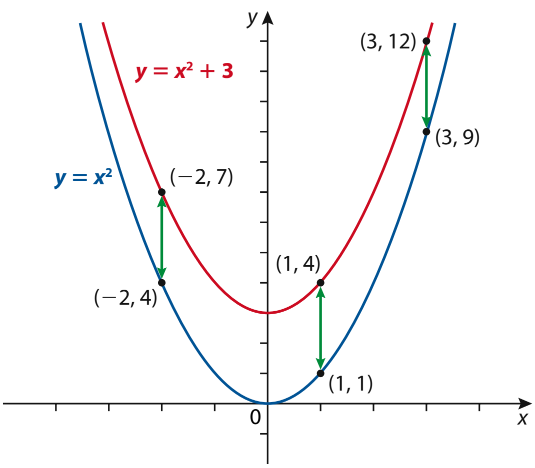


Figure 2.18 Translating $f(x) = x^2$ up.

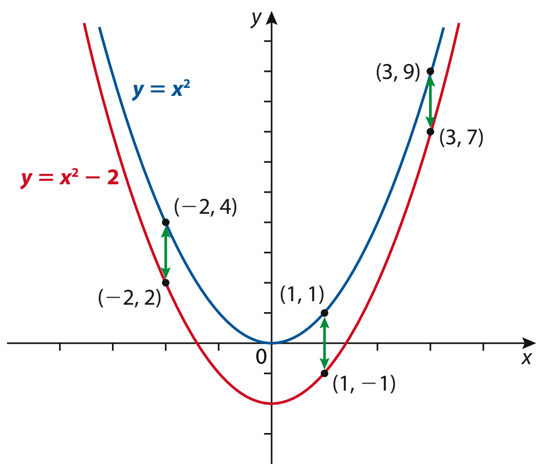


Figure 2.19 Translating $f(x) = x^2$ down.

As Figures 2.18 and 2.19 clearly show, you can obtain the graph of $g(x) = x^2 + 3$ by translating (shifting) the graph of $f(x) = x^2$ *up* three units, and you can obtain the graph of $h(x) = x^2 - 2$ by translating the graph of $f(x) = x^2$ *down* two units.

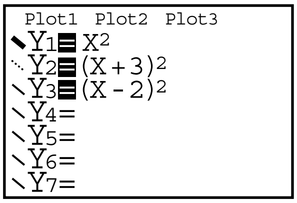
Vertical translations of a function

Given $k > 0$, then:

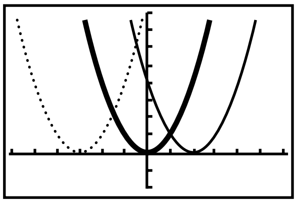
I. The graph of $y = f(x) + k$ is obtained by translating *up* k units the graph of $y = f(x)$.

II. The graph of $y = f(x) - k$ is obtained by translating *down* k units the graph of $y = f(x)$.

Change function g to $g(x) = (x + 3)^2$ and change function h to $h(x) = (x - 2)^2$. Graph these two functions along with the 'parent' function



Note that a different graphing style is assigned to each equation on the GDC.



$f(x) = x^2$ on your GDC. This time we observe that functions g and h can be obtained by a horizontal translation of f.

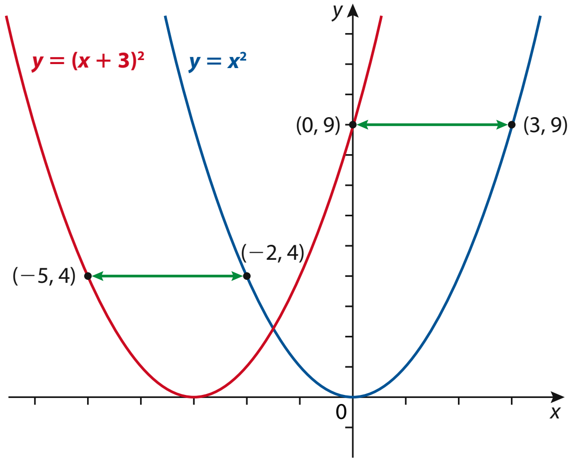


Figure 2.20 Translate $y = x^2$ left 3 units to produce graph of $y = (x + 3)^2$.

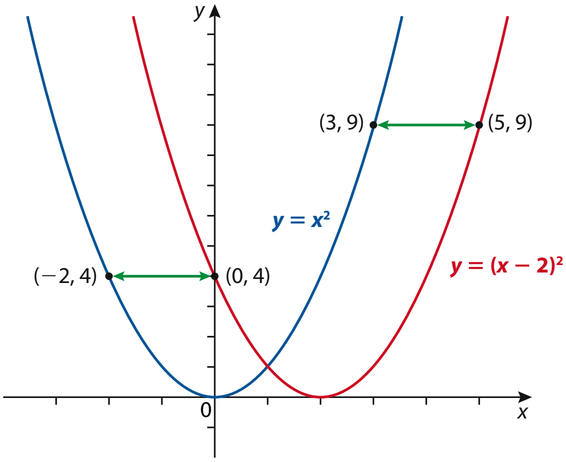


Figure 2.21 Translate $y = x^2$ right 2 units to produce graph of $y = (x - 2)^2$.

As Figures 2.20 and 2.21 clearly show, you can obtain the graph of $g(x) = (x + 3)^2$ by translating the graph of $f(x) = x^2$ three units to the *left*, and you can obtain the graph of $h(x) = (x - 2)^2$ by translating the graph of $f(x) = x^2$ two units to the *right*.

Horizontal translations of a function

Given $h > 0$, then:

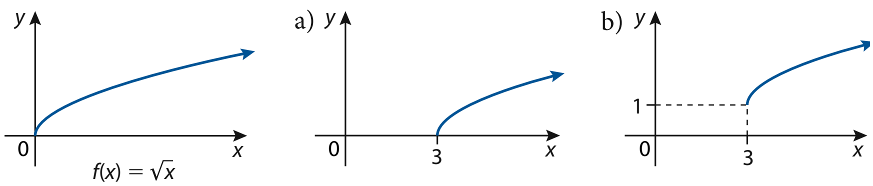
I. The graph of $y = f(x - h)$ is obtained by translating the graph of $y = f(x)$ h units to the *right*.

II. The graph of $y = f(x + h)$ is obtained by translating the graph of $y = f(x)$ h units to the *left*.

Hint: An alternative (and more consistent) approach to vertical and horizontal translations is to think of what number is being added directly to the x- or y-coordinate. For example, the equation for the graph obtained by translating the graph of $y = x^2$ three units up is $y = x^2 + 3$, which can also be written as $y - 3 = x^2$. In this form, negative three is added to the y-coordinate (vertical coordinate), which causes a vertical translation in the *upward* (or positive) direction. Likewise, the equation for the graph obtained by translating the graph of $y = x^2$ two units to the right is $y = (x - 2)^2$. Negative two is added to the x-coordinate (horizontal coordinate), which causes a horizontal translation to the right (or positive direction). There is consistency between vertical and horizontal translations. Assuming that movement up or to the right is considered positive, and that movement down or to the left is negative, then the direction for either type of translation is opposite to the sign ($\pm$) of the number being added to the vertical (y) or horizontal (x) coordinate. In fact, what is actually being translated is the y-axis or the x-axis. For example, the graph of $y - 3 = x^2$ can also be obtained by not changing the graph of $y = x^2$ but instead translating the y-axis three units down – which creates exactly the same effect as translating the graph of $y = x^2$ three units up.

Example 18 – Translations of a graph

The diagrams show how the graph of $y = \sqrt{x}$ is transformed to the graph of $y = f(x)$ in three steps. For each diagram, a) and b), give the equation of the curve.



Solution

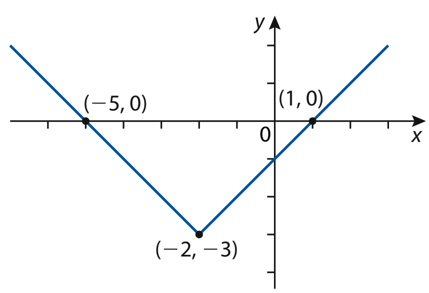
To obtain graph a), the graph of $y = \sqrt{x}$ is translated three units to the right. To produce the equation of the translated graph, -3 is added *inside* the argument of the function $y = \sqrt{x}$. Therefore, the equation of the curve graphed in a) is $y = \sqrt{x - 3}$.

To obtain graph b), the graph of $y = \sqrt{x - 3}$ is translated up one unit. To produce the equation of the translated graph, $+1$ is added *outside* the function. Therefore, the equation of the curve graphed in b) is $y = \sqrt{x - 3} + 1$ (or $y = 1 + \sqrt{x - 3}$).

Note that in Example 18, if the transformations had been performed in reverse order – that is, the vertical translation followed by the horizontal translation – it would produce the same final graph (in part b)) with the same equation. In other words, when applying both a vertical and horizontal translation on a function it does not make any difference which order they are applied (i.e. they are commutative). However, as we will see further on in the chapter, it *can* make a difference to how other sequences of transformations are applied. In general, transformations are *not* commutative.

Example 19

Write the equation of the absolute value function whose graph is shown on the right.

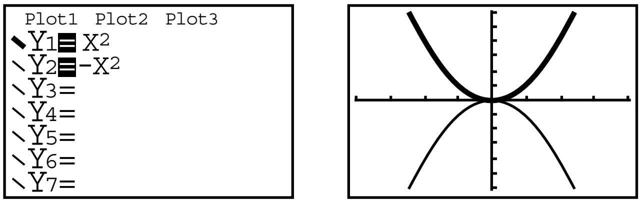


Solution

The graph shown is exactly the same shape as the graph of the equation $y = |x|$ but in a different position. Given that the vertex is $(-2, -3)$, it is clear that this graph can be obtained by translating $y = |x|$ two units left and then three units down. When we move $y = |x|$ two units left we get the graph of $y = |x + 2|$. Moving the graph of $y = |x + 2|$ three units down gives us the graph of $y = |x + 2| - 3$. Therefore, the equation of the graph shown is $y = |x + 2| - 3$. (Note: The two translations applied in reverse order produce the same result.)

Reflections

Use your GDC to graph the two functions $f(x) = x^2$ and $g(x) = -x^2$. The graph of $g(x) = -x^2$ is a reflection in the x-axis of $f(x) = x^2$. This certainly makes sense because g is formed by multiplying f by -1, causing the y-coordinate of each point on the graph of $y = -x^2$ to be the negative of the y-coordinate of the point on the graph of $y = x^2$ with the same x-coordinate.



● **Hint:** The expression $-x^2$ is potentially ambiguous. It is accepted to be equivalent to $-(x)^2$. It is *not* equivalent to $(-x)^2$. For example, if you enter the expression -3^2 into your GDC, it gives a result of -9, *not* $+9$. In other words, the expression -3^2 is consistently interpreted as 3^2 being multiplied by -1. The same as $-x^2$ is interpreted as x^2 being multiplied by -1.

Figures 2.22 and 2.23 illustrate that the graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.

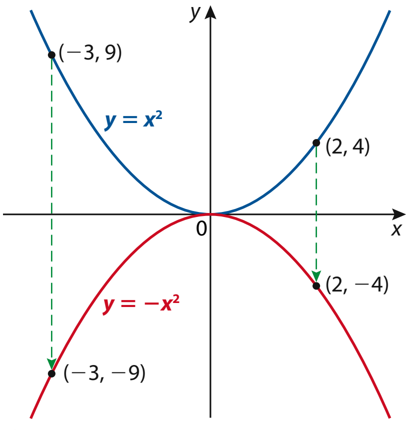


Figure 2.22 Reflecting $y = x^2$ in the x-axis.

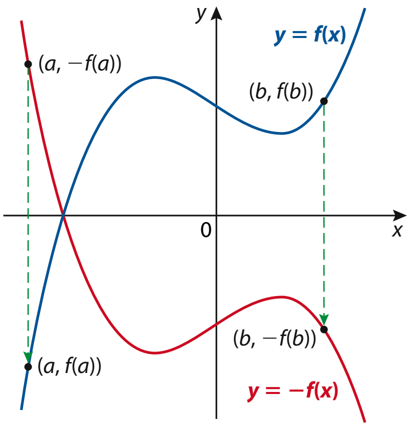


Figure 2.23 Reflecting $f(x)$ in the x-axis.

Graph the functions $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{-x - 2}$. Previously, with $f(x) = x^2$ and $g(x) = -x^2$, g was formed by multiplying the entire function f by -1. However, for $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{-x - 2}$, g is formed by multiplying the variable x by -1. In this case, the graph of $g(x) = \sqrt{-x - 2}$ is a reflection in the y-axis of $f(x) = \sqrt{x - 2}$. This makes sense if you recognize that the y-coordinate on the graph of $y = \sqrt{-x}$ will be the same as the y-coordinate on the graph of $y = \sqrt{x}$, if the value substituted for x in $y = \sqrt{-x}$ is the opposite of the value of x in $y = \sqrt{x}$. For example, if $x = 9$ then $y = \sqrt{9} = 3$; and, if $x = -9$ then $y = \sqrt{-(-9)} = \sqrt{9} = 3$. Opposite values of x in the two functions produce the same y-coordinate for each.

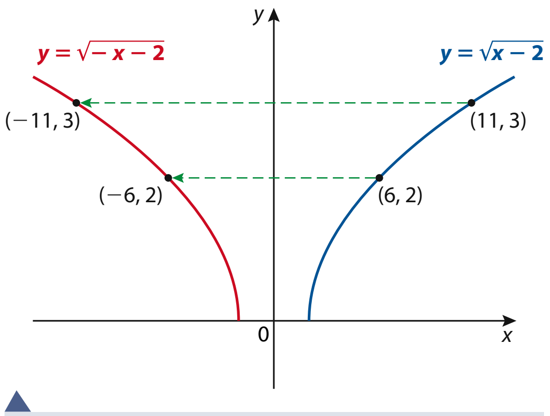


Figure 2.24 Reflecting $y = \sqrt{x-2}$ in the y-axis.

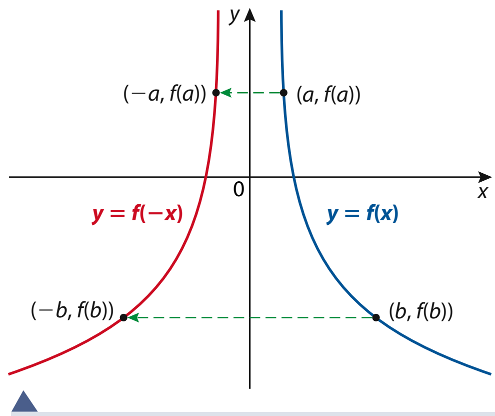


Figure 2.25 Reflecting $f(x)$ in the y-axis.

Figures 2.24 and 2.25 illustrate that the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

Reflections of a function in the coordinate axes

I. The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.
II. The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

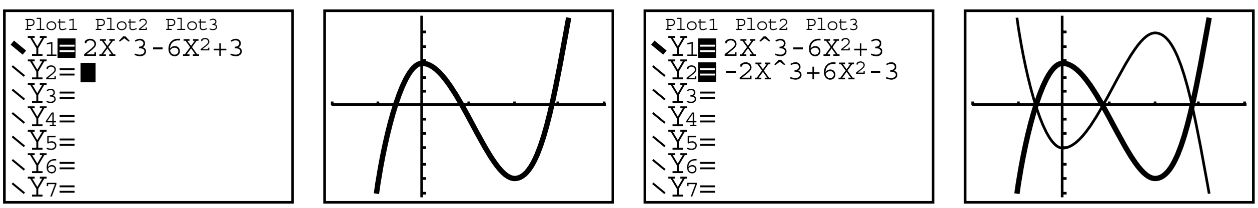
Example 20 – Reflections in the coordinate axes

For $g(x) = 2x^3 - 6x^2 + 3$, find:

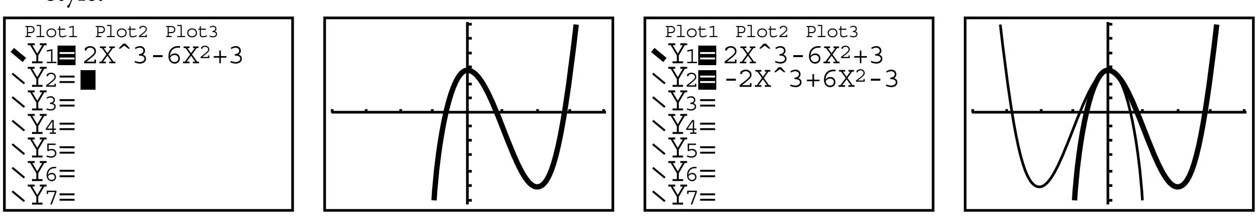
a) the function $h(x)$ that is the reflection of $g(x)$ in the x-axis
b) the function $p(x)$ that is the reflection of $g(x)$ in the y-axis.

Solution

a) Knowing that $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis, then $h(x) = -g(x) = -(2x^3 - 6x^2 + 3) \Rightarrow h(x) = -2x^3 + 6x^2 - 3$ will be the reflection of $g(x)$ in the x-axis. We can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.



b) Knowing that $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis, we need to substitute $-x$ for x in $y = g(x)$. Thus, $p(x) = g(-x) = 2(-x)^3 - 6(-x)^2 + 3 \Rightarrow p(x) = -2x^3 - 6x^2 + 3$ will be the reflection of $g(x)$ in the y-axis. Again, we can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.

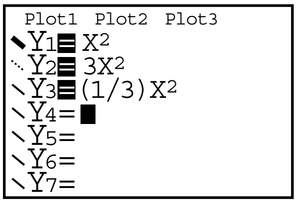


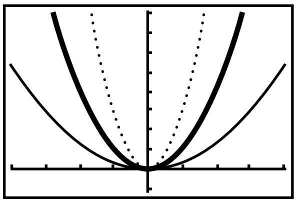
Non-rigid transformations: stretching and shrinking

Horizontal and vertical translations, and reflections in the x- and y-axes are called **rigid transformations** because the shape of the graph does not change – only its position is changed. **Non-rigid transformations** cause the shape of the original graph to change. The non-rigid transformations that we will study cause the shape of a graph to *stretch* or *shrink* in either the vertical or horizontal direction.

Vertical stretch or shrink

Graph the following three functions: $f(x) = x^2$, $g(x) = 3x^2$ and $h(x) = \frac{1}{3}x^2$. How do the graphs of g and h compare to the graph of f? Clearly, the shape of the graphs of g and h is not the same as the graph of f. Multiplying the function f by a positive number greater than one, or less than one, has distorted the shape of the graph. For a certain value of x, the y-coordinate of $y = 3x^2$ is three times the y-coordinate of $y = x^2$. Therefore, the graph of $y = 3x^2$ can be obtained by *vertically stretching* the graph of $y = x^2$ by a factor of 3 (**scale factor 3**). Likewise, the graph of $y = \frac{1}{3}x^2$ can be obtained by *vertically shrinking* the graph of $y = x^2$ by **scale factor** $\frac{1}{3}$.





Figures 2.26 and 2.27 illustrate how multiplying a function by a positive number, a, *greater than one* causes a transformation by which the function *stretches* vertically by scale factor a. A point (x, y) on the graph of $y = f(x)$ is transformed to the point (x, ay) on the graph of $y = af(x)$.

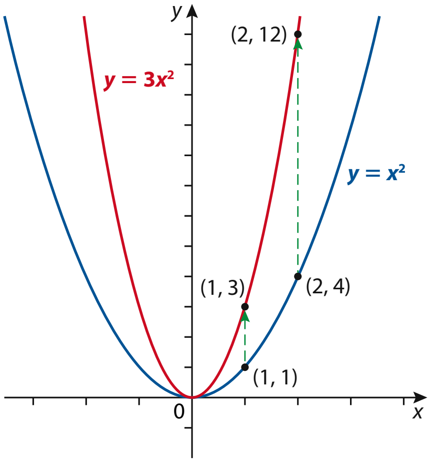


Figure 2.26 Vertical stretch of $y = x^2$ by scale factor 3.

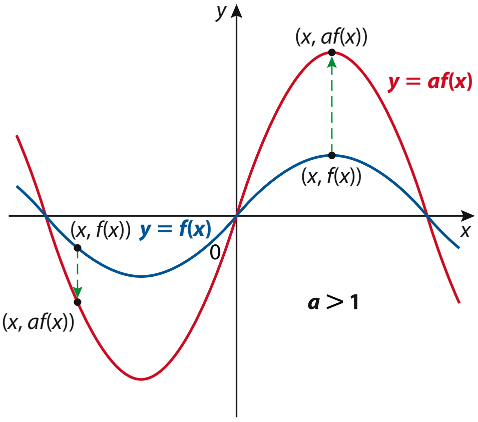


Figure 2.27 Vertical stretch of $f(x)$ by scale factor a.

Figures 2.28 and 2.29 illustrate how multiplying a function by a positive number, a, *greater than zero and less than one* causes the function to *shrink* vertically by scale factor a. A point (x, y) on the graph of $y = f(x)$ is transformed to the point (x, ay) on the graph of $y = af(x)$.

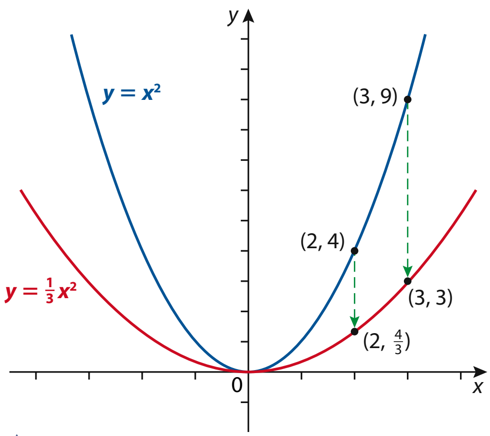


Figure 2.28 Vertical shrink of $y = x^2$ by scale factor $\frac{1}{3}$.

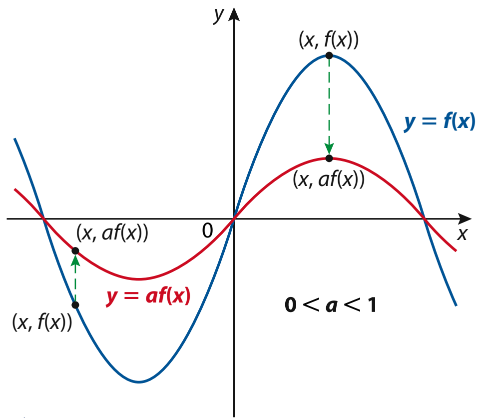


Figure 2.29 Vertical shrink of $f(x)$ by scale factor a.

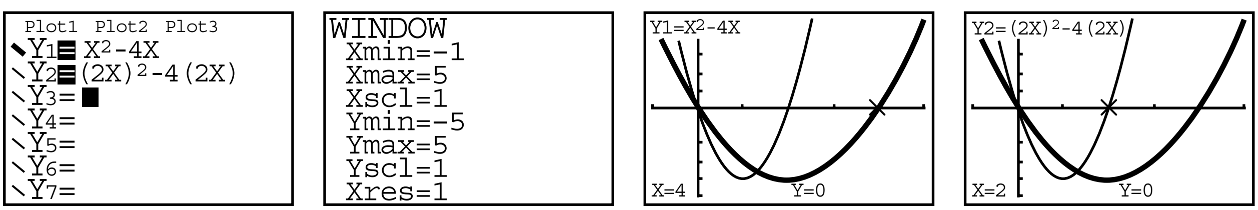
Vertical stretching and shrinking of functions

I. If $a > 1$, the graph of $y = af(x)$ is obtained by *vertically stretching* the graph of $y = f(x)$.

II. If $0 < a < 1$, the graph of $y = af(x)$ is obtained by *vertically shrinking* the graph of $y = f(x)$.

Horizontal stretch or shrink

Let's investigate how the graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$. Given $f(x) = x^2 - 4x$, find another function, $g(x)$, such that $g(x) = f(2x)$. We substitute $2x$ for x in the function f, giving $g(x) = (2x)^2 - 4(2x)$. For the purposes of our investigation, let's leave $g(x)$ in this form. On your GDC, graph these two functions, $f(x) = x^2 - 4x$ and $g(x) = (2x)^2 - 4(2x)$, using the indicated viewing window and graphing f in bold style.



Comparing the graphs of the two equations, we see that $y = g(x)$ is *not* a translation or a reflection of $y = f(x)$. It is similar to the *shrinking* effect that occurs for $y = af(x)$ when $0 < a < 1$, except, instead of a vertical shrinking, the graph of $y = g(x) = f(2x)$ is obtained by *horizontally* shrinking the graph of $y = f(x)$. Given that it is a shrinking – rather than a stretching – the scale factor must be less than one. Consider the point $(4, 0)$ on the graph of $y = f(x)$. The point on the graph of $y = g(x) = f(2x)$ with the same y-coordinate and on the same side of the parabola is $(2, 0)$. The x-coordinate of the point on $y = f(2x)$ is the x-coordinate of the point on $y = f(x)$ multiplied by $\frac{1}{2}$. Use your GDC to confirm this for other pairs of corresponding points on $y = x^2 - 4x$ and $y = (2x)^2 - 4(2x)$ that have the same y-coordinate. The graph of $y = f(2x)$ can be obtained by *horizontally shrinking* the graph of $y = f(x)$ by scale factor $\frac{1}{2}$. This makes sense because if $f(2x_2) = (2x_2)^2 - 4(2x_2)$ and $f(x_1) = x_1^2 - 4x_1$ are to produce the same y-value then $2x_2 = x_1$; and, thus, $x_2 = \frac{1}{2}x_1$. Figures 2.30 and 2.31 illustrate how multiplying the x-variable of a function by a positive number, a, *greater than one* causes the function to *shrink* horizontally by scale factor $\frac{1}{a}$. A point (x, y) on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

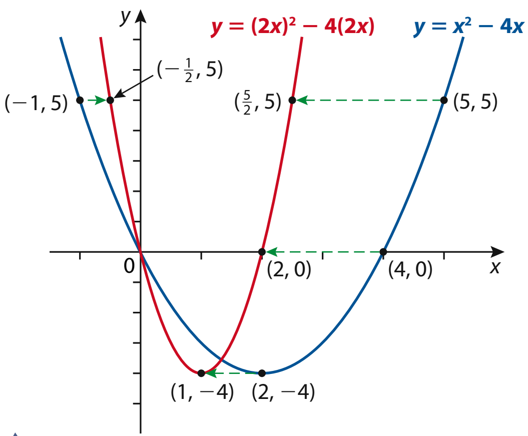


Figure 2.30 Horizontal shrink of $y = x^2 - 4x$ by scale factor $\frac{1}{2}$.

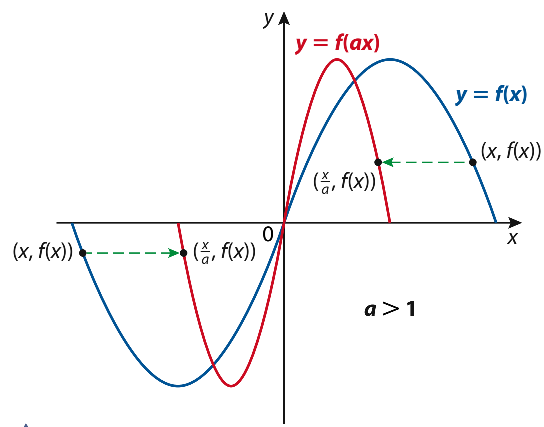
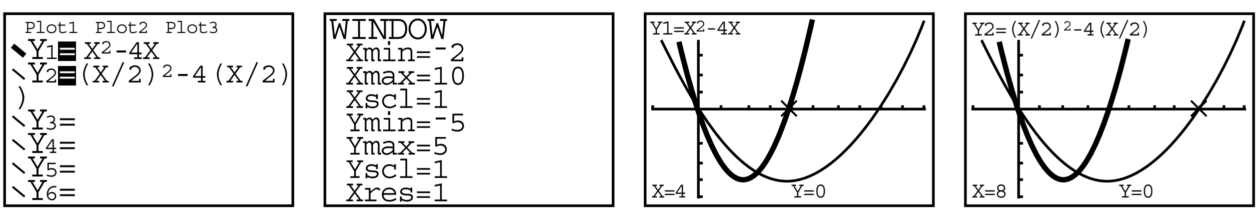


Figure 2.31 Horizontal shrink of $f(x)$ by scale factor $\frac{1}{a}$, $a > 1$.

If $0 < a < 1$, the graph of the function $y = f(ax)$ is obtained by a *horizontal stretching* of the graph of $y = f(x)$ – rather than a shrinking – because the scale factor $\frac{1}{a}$ will be a value greater than 1 if $0 < a < 1$. Now, letting $a = \frac{1}{2}$ and, again using the function $f(x) = x^2 - 4x$, find $g(x)$, such that $g(x) = f\left(\frac{1}{2}x\right)$. We substitute $\frac{x}{2}$ for x in f, giving $g(x) = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$. On your GDC, graph the functions f and g using the indicated viewing window with f in bold.



The graph of $y = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$ is a horizontal stretching of the graph of $y = x^2 - 4x$ by scale factor $\frac{1}{a} = \frac{1}{\frac{1}{2}} = 2$. For example, the point $(4, 0)$ on $y = f(x)$ has been moved horizontally to the point $(8, 0)$ on $y = g(x) = f\left(\frac{x}{2}\right)$.

Figures 2.32 and 2.33 illustrate how multiplying the x-variable of a function by a positive number, a, *greater than zero and less than one* causes the function to *stretch* horizontally by scale factor $\frac{1}{a}$. A point (x, y) on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

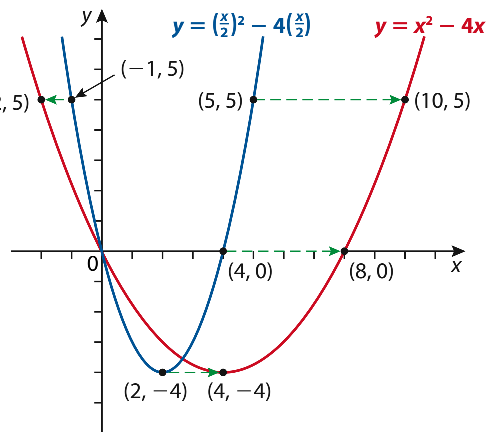


Figure 2.32 Horizontal stretch of $y = x^2 - 4x$ by scale factor 2.

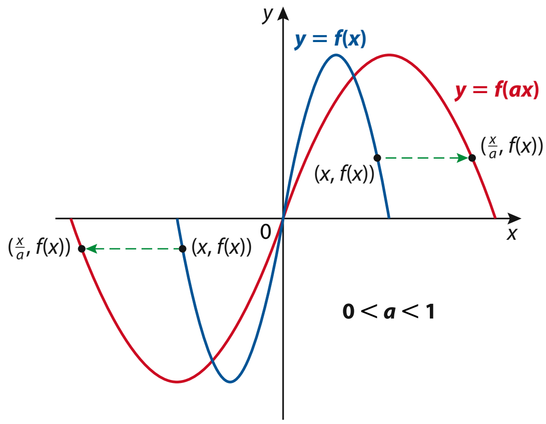


Figure 2.33 Horizontal stretch of $f(x)$ by scale factor $\frac{1}{a}$, $0 < a < 1$.

Horizontal stretching and shrinking of functions

I. If $a > 1$, the graph of $y = f(ax)$ is obtained by *horizontally shrinking* the graph of $y = f(x)$.

II. If $0 < a < 1$, the graph of $y = f(ax)$ is obtained by *horizontally stretching* the graph of $y = f(x)$.

Example 21

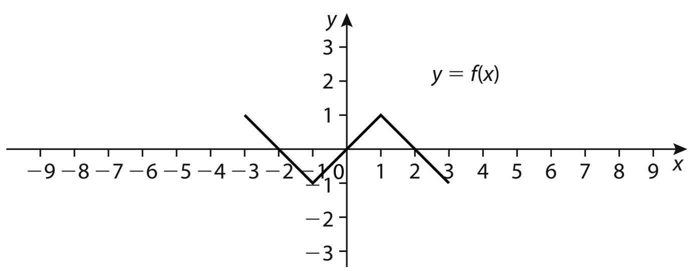
The graph of $y = f(x)$ is shown. Sketch the graph of each of the following two functions.

a) $y = 3f(x)$

b) $y = \frac{1}{3}f(x)$

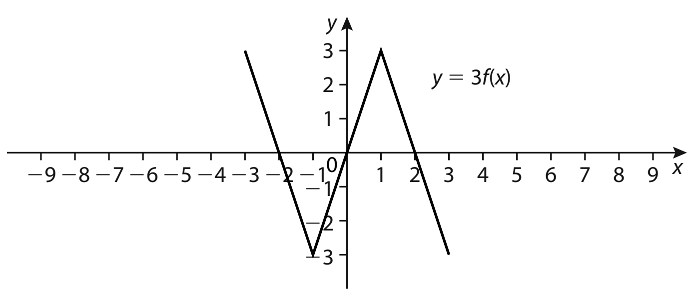
c) $y = f(3x)$

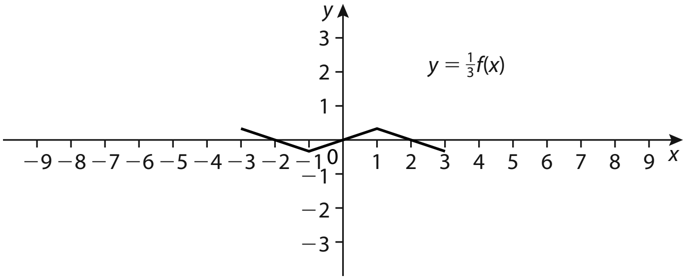
d) $y = f\left(\frac{1}{3}x\right)$



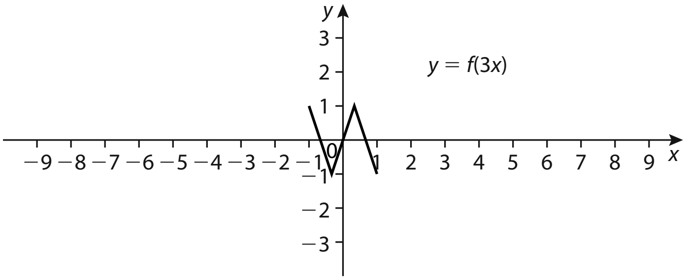
Solution

a) The graph of $y = 3f(x)$ is obtained by vertically stretching the graph of $y = f(x)$ by scale factor 3.

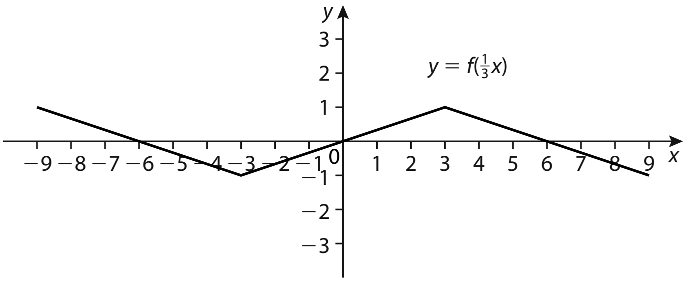




b) The graph of $y = \frac{1}{3}f(x)$ is obtained by vertically shrinking the graph of $y = f(x)$ by scale factor $\frac{1}{3}$.



c) The graph of $y = f(3x)$ is obtained by horizontally shrinking the graph of $y = f(x)$ by scale factor $\frac{1}{3}$.



d) The graph of $y = f\left(\frac{1}{3}x\right)$ is obtained by horizontally stretching the graph of $y = f(x)$ by scale factor 3.

Example 22

Describe the sequence of transformations performed on the graph of $y = x^2$ to obtain the graph of $y = 4x^2 - 3$.

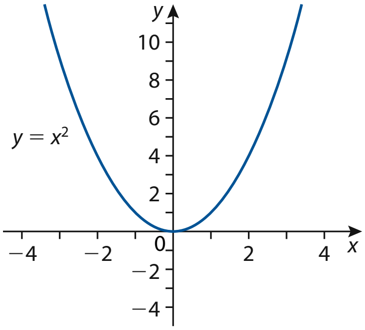
Solution

Step 1: Start with the graph of $y = x^2$.

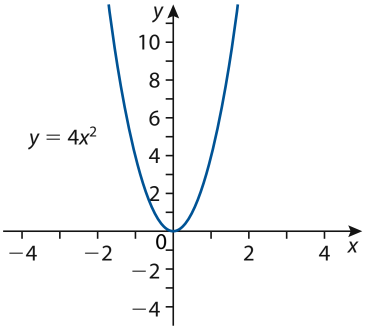
Step 2: Vertically stretch $y = x^2$ by scale factor 4.

Step 3: Vertically translate $y = 4x^2$ three units down.

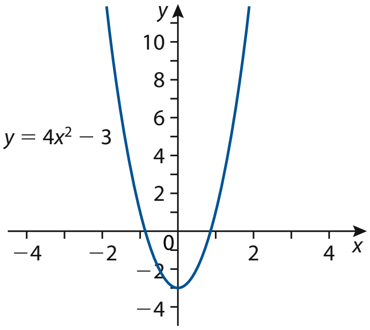
Step 1:



Step 2:



Step 3:



Note that in Example 22, a vertical stretch followed by a vertical translation does not produce the same graph if the two transformations are performed in reverse order. A vertical translation followed by a vertical stretch would generate the following sequence of equations:

Step1: $y = x^2$ Step 2: $y = x^2 - 3$ Step 3: $y = 4(x^2 - 3) = 4x^2 - 12$

This final equation is not the same as $y = 4x^2 - 3$.

When combining two or more transformations, the order in which they are performed can make a difference. In general, when a sequence of transformations includes a vertical/horizontal stretch or shrink, or a reflection through the x-axis, the order may make a difference.

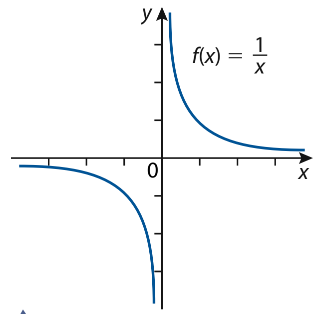


Figure 2.34 The reciprocal function $y = \frac{1}{x}$.

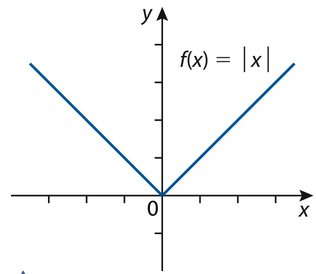


Figure 2.35 The absolute value function $y = |x|$.

Reciprocal and absolute value graphs

Two of the functions that appeared in the set of common functions in Figure 2.17 at the start of this section were the reciprocal function, $f(x) = \frac{1}{x}$, and the absolute value function (Figures 2.34 and 2.35).

Lets investigate how the graph of a given function, say $g(x)$, compares to that of a composite function $f(g(x))$, where the function f is either the reciprocal function or the absolute value function.

Example 23 – Graph of the reciprocal of a function

Given $f(x) = \frac{1}{x}$, $g(x) = -2x + 4$ and $h(x) = x^2 + 2x - 3$, sketch the graphs of the composite functions $f(g(x))$ and $f(h(x))$. Discuss the characteristics of each graph.

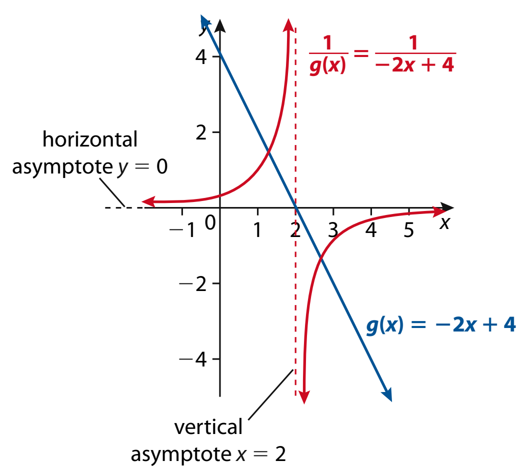
Solution

$$f(g(x)) = \frac{1}{g(x)} \Rightarrow y = \frac{1}{-2x + 4}$$

Clearly the reciprocal of g will be undefined wherever $g(x) = 0$ making the domain of $\frac{1}{g(x)}$ to be $\{x : x \in \mathbb{R}, x \neq 2\}$. Consequently the graph of $\frac{1}{g(x)}$ will have a **vertical asymptote** with equation $x = 2$. The graph of g illustrates that as x approaches the value of 2 ($x \to 2$) from the left side, the value of $g(x)$ is always positive but is converging to zero. Therefore, as $x \to 2$ from the left (or, $x \to 2^-$), the values of $\frac{1}{g(x)}$ become increasingly large in the positive direction. We can express this behaviour symbolically by writing, 'as $x \to 2^-$, $\frac{1}{g(x)} \to +\infty$'. Similarly, as $x \to 2^+$, $\frac{1}{g(x)} \to -\infty$.

Also, the x-axis ($y = 0$) is a **horizontal asymptote** for the

Figure 2.36 Graph of $g(x)$ and its reciprocal.

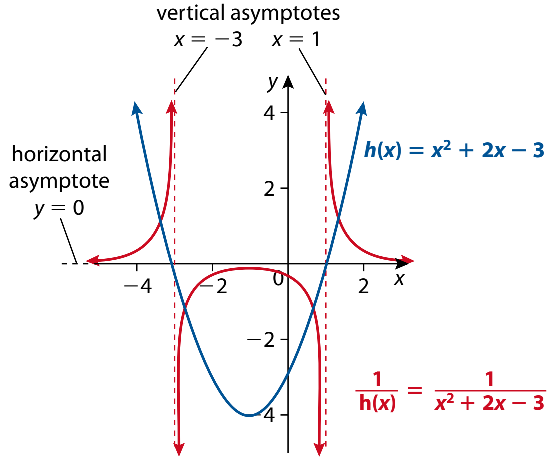


graph of $\dfrac{1}{g(x)}$ because as the value of $g(x)$ becomes very large (either positive or negative), the value of $\dfrac{1}{g(x)}$ converges to zero; or, symbolically, as $x \to \pm\infty$, $\dfrac{1}{g(x)} \to 0$.

$$f(h(x)) = \frac{1}{h(x)} = \frac{1}{x^2 + 2x - 3} = \frac{1}{(x+3)(x-1)}$$

Domain for $\dfrac{1}{h(x)}$ is $\{x : x \in \mathbb{R}, x \neq -3, x \neq 1\}$.

Figure 2.37 Graph of $h(x)$ and its reciprocal.



Since $h(x) = 0$ for $x = -3$ and $x = 1$ we anticipate that the graph of its reciprocal, $\dfrac{1}{h(x)}$, will have vertical asymptotes of $x = -3$ and $x = 1$. This is confirmed by the fact that as $x \to -3^-$, $\dfrac{1}{h(x)} \to +\infty$; as $x \to -3^+$, $\dfrac{1}{h(x)} \to -\infty$; and as $x \to 1^-$, $\dfrac{1}{h(x)} \to -\infty$; as $x \to 1^+$, $\dfrac{1}{h(x)} \to +\infty$.
The graph of $\dfrac{1}{h(x)}$ will also have a horizontal asymptote of $y = 0$ (x-axis) because as $x \to \pm\infty$, $\dfrac{1}{h(x)} \to 0$.

Vertical and horizontal asymptotes

In general, the line $x = c$ is a vertical asymptote of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as x approaches c from either the left or the right. The line $y = c$ is a horizontal asymptote of the graph of f if $f(x)$ approaches c as $x \to \infty$ or $x \to -\infty$.

Example 24 – Graphs of composites with absolute value function

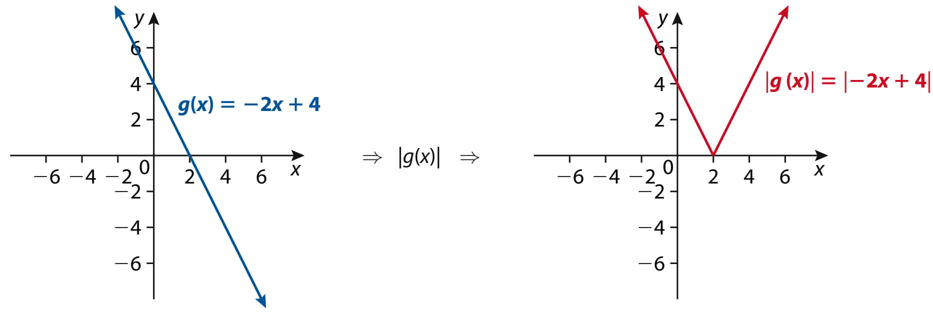
Given $f(x) = |x|$ and using the same functions g and h from Example 23,

a) graph the composite functions $f \circ g$ and $f \circ h$; and

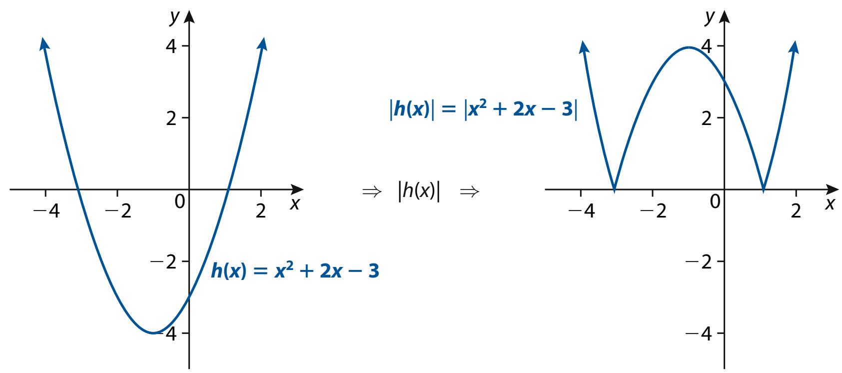
b) graph the composite functions $g \circ f$ and $h \circ f$.

Solution

a) $(f \circ g)(x) = f(-2x + 4) = |-2x + 4|$

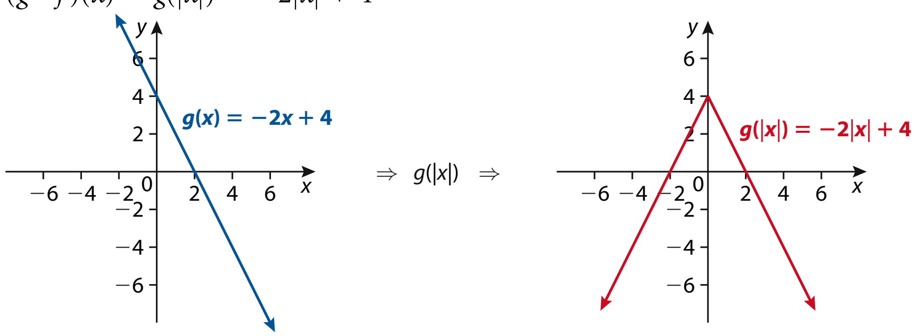


$(f \circ h)(x) = f(x^2 + 2x - 3) = |x^2 + 2x - 3|$

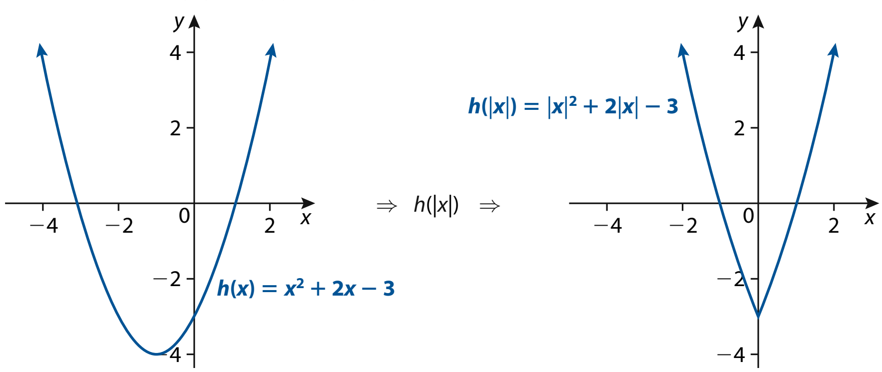


From these two examples with functions $g(x)$ and $h(x)$, we see the change that occurs from the graph of a function to the graph of the **absolute value of the function**. Any portion of the graph of $g(x)$ or $h(x)$ that was below the x-axis gets reflected above the x-axis.

b) $(g \circ f)(x) = g(|x|) = -2|x| + 4$



$(h \circ f)(x) = h(|x|) = |x|^2 + 2|x| - 3$



Similarly to part a) we can see a change from the graph of a function to the graph of the **function of the absolute value**. Any portion of the graph of $g(x)$ or $h(x)$ that was left of the y-axis is eliminated, and any portion that was to the right of the y-axis is reflected to the left of the y-axis. Since the portion that was right of the y-axis remains, the resulting graph is always symmetric about the y-axis.

Summary of transformations on the graphs of functions

Assume that a, h and k are positive real numbers.

Transformed function	Transformation performed on $y = f(x)$
$y = f(x) + k$	vertical translation k units up
$y = f(x) - k$	vertical translation k units down
$y = f(x - h)$	horizontal translation h units right
$y = f(x + h)$	horizontal translation h units left
$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis
$y = af(x)$	vertical stretch ($a > 1$) or shrink ($0 < a < 1$)
$y = f(ax)$	horizontal stretch ($0 < a < 1$) or shrink ($a > 1$)
$y = \|f(x)\|$	portion of graph of $y = f(x)$ below x-axis is reflected above x-axis
$y = f(\|x\|)$	symmetric about y-axis; portion right of y-axis is reflected over y-axis

Exercise 2.4

In questions 1–14, sketch the graph of f, without a GDC or by plotting points, by using your knowledge of some of the basic functions shown in Figure 2.17.

1 $f: x \mapsto x^2 - 6$

2 $f: x \mapsto (x - 6)^2$

3 $f: x \mapsto |x| + 4$

4 $f: x \mapsto |x + 4|$

5 $f: x \mapsto 5 + \sqrt{x - 2}$

6 $f: x \mapsto \dfrac{1}{x - 3}$

7 $f: x \mapsto \dfrac{1}{(x + 5)^2} + 2$

8 $f: x \mapsto -x^3 - 4$

9 $f: x \mapsto -|x - 1| + 6$

10 $f: x \mapsto \sqrt{-x + 3}$

11 $f: x \mapsto 3\sqrt{x}$

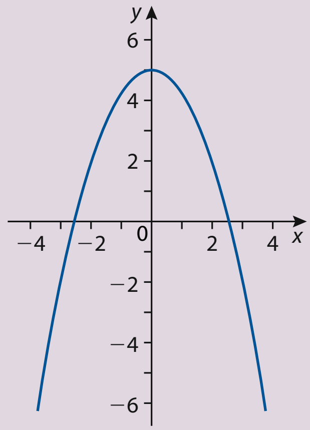
12 $f: x \mapsto \frac{1}{2}x^2$

13 $f: x \mapsto \left(\frac{1}{2}x\right)^2$

14 $f: x \mapsto (-x)^3$

In questions 15–18, write the equation for the graph that is shown.

15



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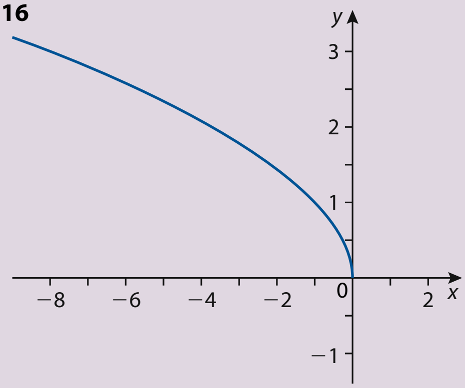

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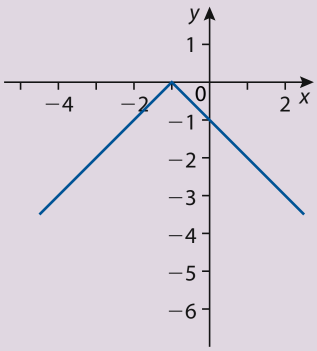
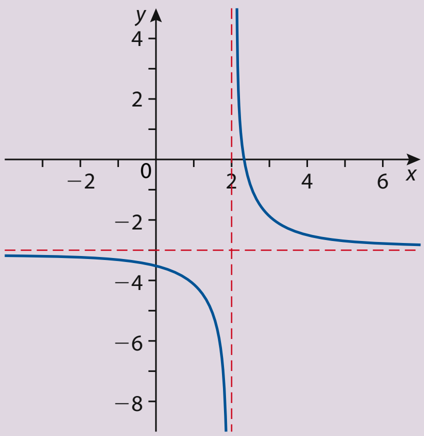
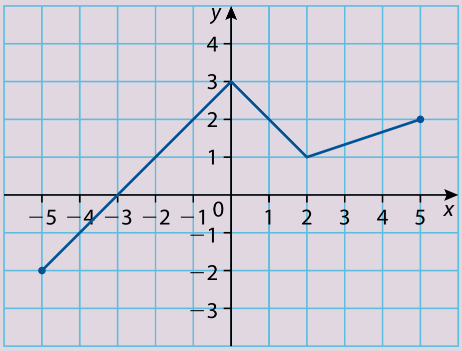


18 Vertical and horizontal asymptotes shown:



19 The graph of f is given. Sketch the graphs of the following functions.

a) $y = f(x) - 3$
b) $y = f(x - 3)$
c) $y = 2f(x)$
d) $y = f(2x)$
e) $y = -f(x)$
f) $y = f(-x)$
g) $y = 2f(x) + 4$



In questions 20–23, specify a sequence of transformations to perform on the graph of $y = x^2$ to obtain the graph of the given function.

20 $g: x \mapsto (x - 3)^2 + 5$

21 $h: x \mapsto -x^2 + 2$

22 $p: x \mapsto \frac{1}{2}(x + 4)^2$

23 $f: x \mapsto [3(x - 1)]^2 - 6$

Without using your GDC, for each function $f(x)$ in questions 24–26 sketch the graph of a) $\frac{1}{f(x)}$, b) $|f(x)|$ and c) $f(|x|)$. Clearly label any intercepts or asymptotes.

24 $f(x) = \frac{1}{2}x - 4$

25 $f(x) = (x - 4)(x + 2)$

26 $f(x) = x^3$

Practice questions

1 Let $f: x \mapsto \sqrt{x - 3}$ and $g: x \mapsto x^2 + 2x$. The function $(f \circ g)(x)$ is defined for all $x \in \mathbb{R}$ **except** for the interval $]a, b[$.

a) Calculate the values of a and b.

b) Find the range of $f \circ g$.

2 Two functions g and h are defined as $g(x) = 2x - 7$ and $h(x) = 3(2 - x)$.

Find: **a)** $g^{-1}(3)$

b) $(h \circ g)(6)$

3 Consider the functions $f(x) = 5x - 2$ and $g(x) = \frac{4-x}{3}$.

a) Find g^{-1}.

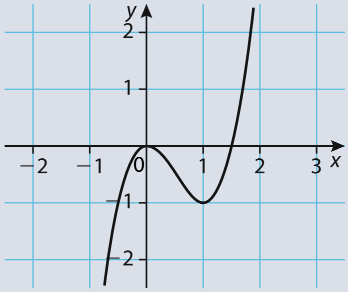
b) Solve the equation $(f \circ g^{-1})(x) = 8$.

4 The functions g and h are defined by $g : x \mapsto x - 3$ and $h : x \mapsto 2x$.

a) Find an expression for $(g \circ h)(x)$.

b) Show that $g^{-1}(14) + h^{-1}(14) = 24$.

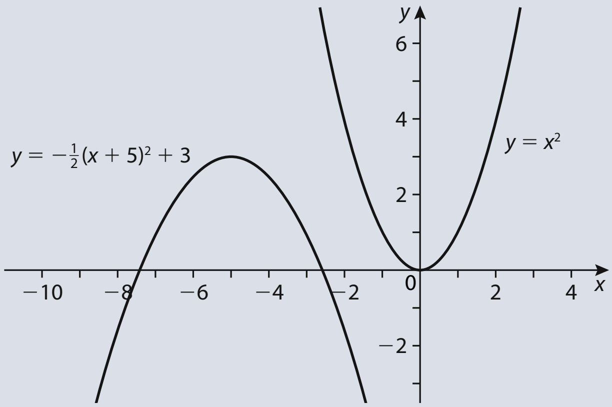
5 The diagram right shows the graph of $y = f(x)$. It has maximum and minimum points at (0, 0) and (1, −1), respectively.



a) Copy the diagram and, on the same diagram, draw the graph of $y = f(x + 1) - \frac{1}{2}$.

b) What are the coordinates of the minimum and maximum points of $y = f(x + 1) - \frac{1}{2}$?

6 The diagram shows parts of the graphs of $y = x^2$ and $y = -\frac{1}{2}(x + 5)^2 + 3$.



The graph of $y = x^2$ may be transformed into the graph of $y = -\frac{1}{2}(x + 5)^2 + 3$ by these transformations.

A reflection in the line $y = 0$, followed by
a vertical stretch by scale factor k, followed by
a horizontal translation of p units, followed by
a vertical translation of q units.

Write down the value of

a) k **b)** p **c)** q.

7 The function f is defined by $f(x) = \frac{4}{\sqrt{16 - x^2}}$, for $-4 < x < 4$.

a) Without using a GDC, sketch the graph of f.

b) Write down the equation of each vertical asymptote.

c) Write down the range of the function f.

8 Let $g : x \mapsto \frac{1}{x}$, $x \neq 0$.

a) Without using a GDC, sketch the graph of g.

The graph of g is transformed to the graph of h by a translation of 4 units to the left and 2 units down.

b) Find an expression for the function h.

c) **(i)** Find the x- and y-intercepts of h.
   (ii) Write down the equations of the asymptotes of h.
   (iii) Sketch the graph of h.

9 Consider $f(x) = \sqrt{x+3}$.

a) Find:

(i) $f(8)$ **(ii)** $f(46)$ **(iii)** $f(-3)$

b) Find the values of x for which f is undefined.

c) Let $g: x \mapsto x^2 - 5$. Find $(g \circ f)(x)$.

10 Let $g(x) = \dfrac{x-8}{2}$ and $h(x) = x^2 - 1$.

a) Find $g^{-1}(-2)$.

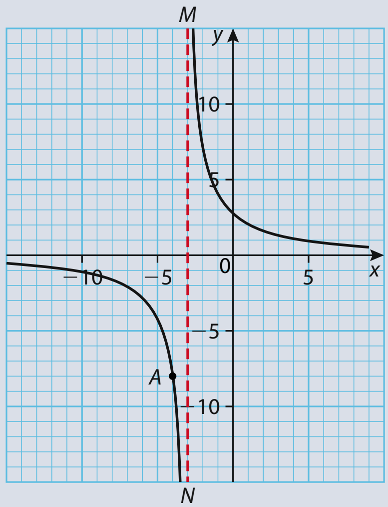
b) Find an expression for $(g^{-1} \circ h)(x)$.

c) Solve $(g^{-1} \circ h)(x) = 22$.

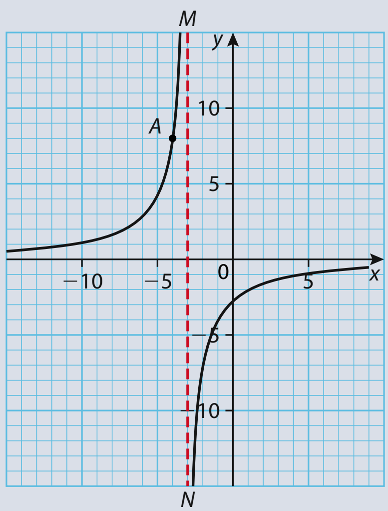
11 Given the functions $f: x \mapsto 3x - 1$ and $g: x \mapsto \dfrac{4}{x}$, find the following:

a) f^{-1} **b)** $f \circ g$ **c)** $(f \circ g)^{-1}$ **d)** $g \circ g$

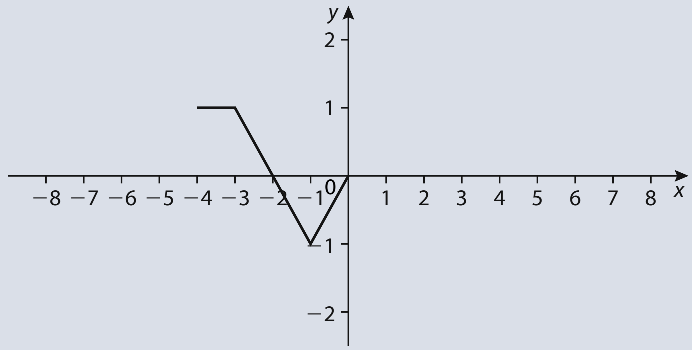
12 a) The diagram shows part of the graph of the function $h(x) = \dfrac{a}{x-b}$. The curve passes through the point $A\,(-4, -8)$. The vertical line (MN) is an asymptote. Find the value of: **(i)** a **(ii)** b.



b) The graph of $h(x)$ is transformed as shown in the diagram right. The point A is transformed to $A'(-4, 8)$. Give a full geometric description of the transformation.



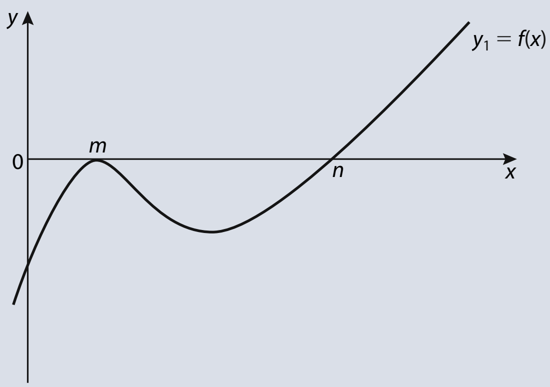
13 The graph of $y = f(x)$ is shown in the diagram.



a) Make two copies of the coordinate system as shown in the diagram but without the graph of $y = f(x)$. On the first diagram sketch a graph of $y = 2f(x)$, and on the second diagram sketch a graph of $y = f(x - 4)$.

b) The point $A(-3, 1)$ is on the graph of $y = f(x)$. The point A' is the corresponding point on the graph of $y = -f(x) - 1$. Find the coordinates of A'.

14 The diagram below shows the graph of $y_1 = f(x)$. The x-axis is a tangent to $f(x)$ at $x = m$ and $f(x)$ crosses the x-axis at $x = n$.



On the same diagram, sketch the graph of $y_2 = f(x - k)$, where $0 < k < n - m$ and indicate the coordinates of the points of intersection of y_2 with the x-axis.

15 Given functions $f: x \mapsto x + 1$ and $g: x \mapsto x^3$, find the function $(f \circ g)^{-1}$.

16 If $f(x) = \dfrac{x}{x + 1}$ for $x \neq -1$ and $g(x) = (f \circ f)(x)$, find

a) $g(x)$

b) $(g \circ g)(2)$.

17 Let $f: x \mapsto \sqrt{\dfrac{1}{x^2} - 2}$. Find

a) the set of real values of x for which f is real and finite;

b) the range of f.

18 The function $f: x \mapsto \dfrac{2x + 1}{x - 1}$, $x \in \mathbb{R}, x \neq 1$. Find the inverse function, f^{-1}, clearly stating its domain.

19 The one-to-one function f is defined on the domain $x > 0$ by $f(x) = \dfrac{2x - 1}{x + 2}$.

a) State the range, A, of f.

b) Obtain an expression for $f^{-1}(x)$, for $x \in A$.

20 The function f is defined by $f : x \mapsto x^3$.

Find an expression for $g(x)$ in terms of x in each of the following cases

a) $(f \circ g)(x) = x + 1$;

b) $(g \circ f)(x) = x + 1$.

21 a) Find the largest set S of values of x such that the function $f(x) = \dfrac{1}{\sqrt{3 - x^2}}$ takes real values.

b) Find the range of the function f defined on the domain S.

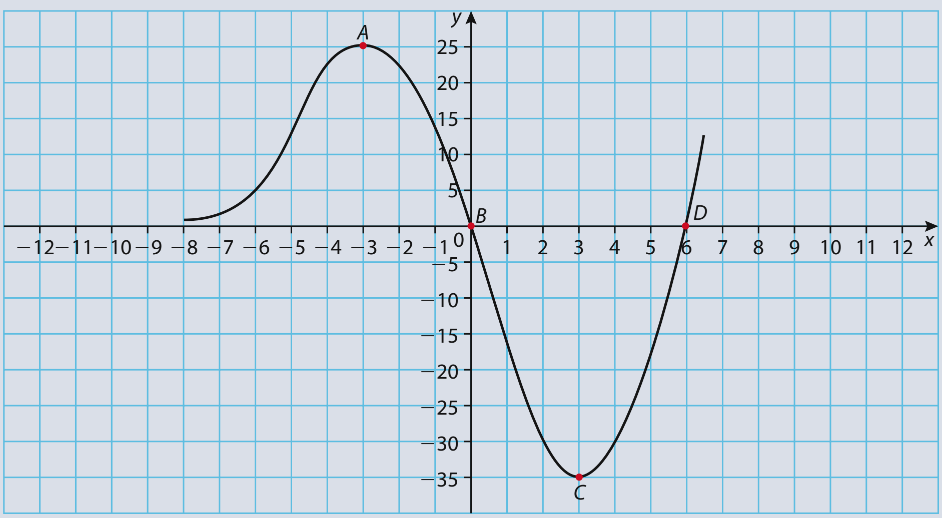
22 Let f and g be two functions. Given that $(f \circ g)(x) = \dfrac{x + 1}{2}$ and $g(x) = 2x - 1$, find $f(x - 3)$.

23 The diagram below shows the graph of $y = f(x)$ which passes through the points A, B, C and D.

Sketch, indicating clearly the images of A, B, C and D, the graphs of

a) $y = f(x - 4)$;

b) $y = f(-3x)$.



3 Algebraic Functions, Equations and Inequalities

Assessment statements

2.1 Odd and even functions (also see Chapter 7).

2.4 The rational function $x \mapsto \dfrac{ax+b}{cx+d}$ and its graph.

2.5 Polynomial functions.
The factor and remainder theorems.
The fundamental theorem of algebra.

2.6 The quadratic function $x \mapsto ax^2 + bx + c$: its graph, axis of symmetry $x = -\dfrac{b}{2a}$.
The solution of $ax^2 + bx + c = 0$, $a \neq 0$.
The quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$.
Solving equations both graphically and algebraically.
Sum and product of the roots of polynomial equations.

2.7 Solution of inequalities $g(x) \geqslant f(x)$; graphical and algebraic methods.

Introduction

A function $x \mapsto f(x)$ is called **algebraic** if, substituting for the number x in the domain, the corresponding number $f(x)$ in the range can be computed using a finite number of **elementary operations** (i.e. addition, subtraction, multiplication, division, and extracting a root). For example, $f(x) = \dfrac{x^2 + \sqrt{9-x}}{2x-6}$ is algebraic. For our purposes in this course, functions can be organized into three categories:

1. Algebraic functions
2. Exponential and logarithmic functions (Chapter 5)
3. Trigonometric and inverse trigonometric functions (Chapter 7)

The focus of this chapter is algebraic functions of a single variable which – given the definition above – are functions that contain polynomials, radicals (surds), rational expressions (quotients), or a combination of these. The chapter will begin by looking at polynomial functions in general and then moves onto a closer look at 2nd degree polynomial functions (quadratic functions). Solving equations containing polynomial functions is an important skill that will be covered. We will also study rational functions, which are quotients of polynomial functions and the associated topic of partial fractions (optional). The chapter will close with methods of solving inequalities and absolute value functions, and strategies for solving various equations.

The concept of a function is a fairly recent development in the history of mathematics. Its meaning started to gain some clarity about the time of René Descartes (1596–1650) when he defined a function to be any positive integral power of x (i.e. x^2, x^3, x^4, etc.). Leibniz (1646–1716) and Johann Bernoulli (1667–1748) developed the concept further. It was Euler (1707–1783) who introduced the now standard function notation $y = f(x)$.

3.1 Polynomial functions

The most common type of algebraic function is a polynomial function where, not surprisingly, the function's rule is given by a polynomial. For example,

$$f(x) = x^3, \qquad h(t) = -2t^2 + 16t - 24, \qquad g(y) = y^5 + y^4 - 11y^3 + 7y^2 + 10y - 8$$

Recalling the definition of a polynomial, we define a polynomial function.

Definition of a polynomial function in the variable x

A **polynomial function** P is a function that can be expressed as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \qquad a_n \neq 0$$

where the non-negative integer n is the **degree** of the polynomial function. The numbers $a_0, a_1, a_2, \dots, a_n$, are real numbers and are the **coefficients** of the polynomial. a_n is the **leading coefficient**, $a_n x^n$ is the **leading term** and a_0 is the **constant term**.

It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used.

Degree	Function form	Function name	Graph
Zero	$P(x) = a$	Constant function	Horizontal line
First	$P(x) = ax + b$	Linear function	Line with slope a
Second	$P(x) = ax^2 + bx + c$	Quadratic function	Parabola (∪-shape, 1 turn)
Third	$P(x) = ax^3 + bx^2 + cx + d$	Cubic function	∿-shape (2 or no turns)

Table 3.1 Features of polynomial functions of low degree.

To identify an individual term in a polynomial function, we use the function name correlated with the power of x contained in the term. For example, the polynomial function $f(x) = x^3 - 9x + 4$ has a *cubic* term of x^3, no *quadratic* term, a *linear* term of $-9x$, and a *constant* term of 4.

For each polynomial function $P(x)$ there is a corresponding **polynomial equation** $P(x) = 0$. When we solve polynomial equations, we often refer to solutions as **roots**.

● **Hint:** When working with a polynomial function, such as $f(x) = x^3 - 9x + 4$, it is common to refer to it in a couple of different ways – either as 'the polynomial $f(x)$', or as 'the function $x^3 - 9x + 4$.'

● **Hint:** The use of the word '**root**' here to denote the solution of a polynomial equation should not be confused with the use of the word in the context of square root, cube root, fifth root, etc.

Zeros and roots

If P is a function and c is a number such that $P(c) = 0$, then c is a **zero** of the function P (or of the polynomial P) and $x = c$ is a **root** of the equation $P(x) = 0$.

Approaches to finding zeros of various polynomial functions will be considered in the first three sections of this chapter.

Graphs of polynomial functions

As we reviewed in Section 1.6, the graph of a first-degree polynomial function (linear function), such as $P(x) = 2x - 5$, is a line (Figure 3.1a). The graph of every second-degree polynomial function (quadratic function) is a parabola (Figure 3.1b). A thorough review and discussion of quadratic functions and their graphs is in the next section.

The simplest type of polynomial function is one whose rule is given by a power of x. In Figure 3.1, the graphs of $P(x) = x^n$ for $n = 1, 2, 3, 4, 5$ and 6 are shown. As the figure suggests, the graph of $P(x) = x^n$ has the same general ∪-shape as $y = x^2$ when n is even, and the same general ∿ shape as $y = x^3$ when n is odd. However, as the degree n increases, the graphs of polynomial functions become flatter near the origin and steeper away from the origin.

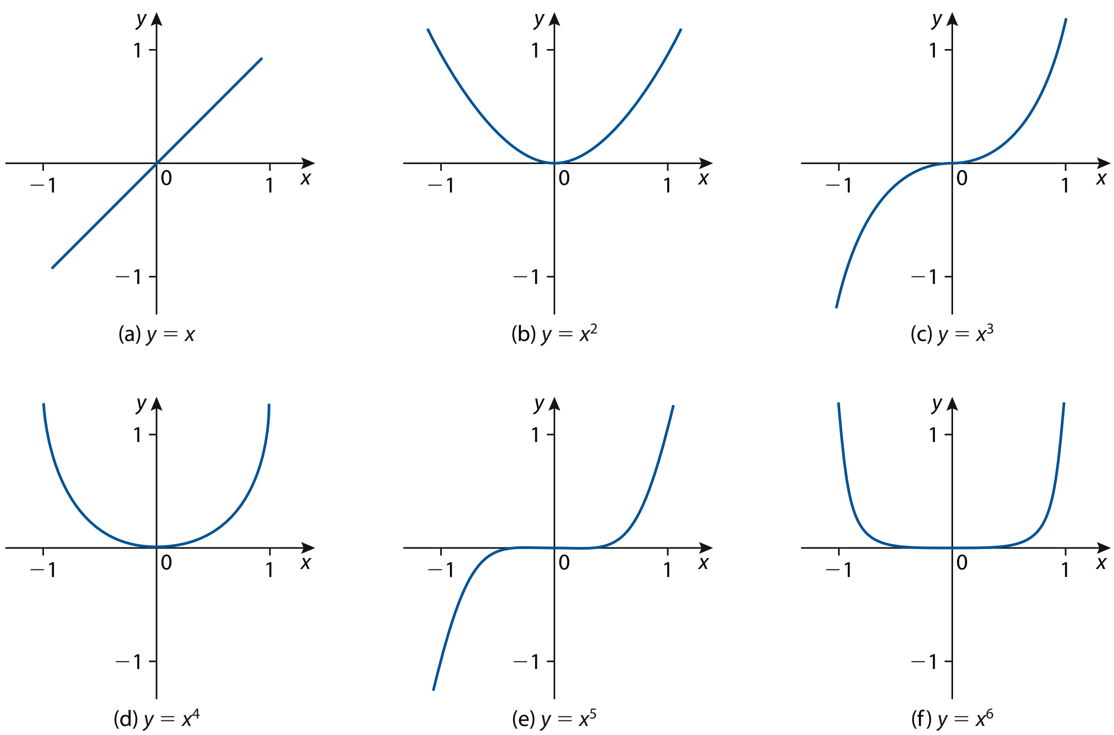
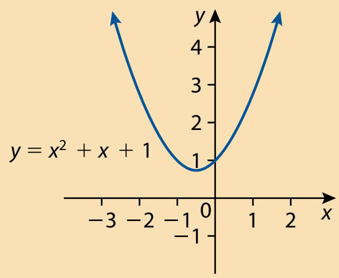


Figure 3.1 Graphs of $P(x) = x^n$ for increasing n.

Another interesting observation is that, depending on the degree of the polynomial function, its graph displays a certain type of symmetry. The graph of $P(x) = x^n$ is symmetric with respect to the origin when n is odd. Such a function is aptly called an **odd function**. The graph of $P(x) = x^n$ is symmetric with respect to the y-axis when n is even. Accordingly any such function is called an **even function**. Formal definitions for odd and even functions will be presented in Chapter 7 when we investigate the graphs of the sine and cosine functions.

Not all polynomial functions are even or odd – that is, not all polynomial functions display rotation symmetry about the origin or reflection symmetry about the y-axis. For example, the graph of the polynomial function $y = x^2 + x + 1$ is neither even nor odd. It has line symmetry, but the line of symmetry is not the y-axis.



Note that the graph of an **even function** may or may not intersect the x-axis (x-intercept). As we will see, where and how often the graph of a function intersects the x-axis is helpful information when trying to determine the value and nature of the roots of a polynomial equation $P(x) = 0$.

The graphs of polynomial functions that are not in the form $P(x) = x^n$ are more difficult to sketch. However, the graphs of all polynomial functions share these properties:

1. It is a smooth curve (i.e. it has no sharp, pointed turns – only smooth, rounded turns).
2. It is continuous (i.e. it has no breaks, gaps or holes).
3. It rises ($P(x) \to \infty$) or falls ($P(x) \to -\infty$) without bound as $x \to +\infty$ or $x \to -\infty$.
4. It extends on forever both to the left ($-\infty$) and to the right ($+\infty$); domain is $\mathbb{R}$.
5. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

The property that is listed third of the five properties of the graphs of polynomial functions is referred to as the **end behaviour** of the function because it describes how the curve *behaves* at the left and right *ends* (i.e. as $x \to +\infty$ and as $x \to -\infty$). The end behaviour of a polynomial function is determined by its degree and by the sign of its leading coefficient. See Exercise 3.1, Q11.

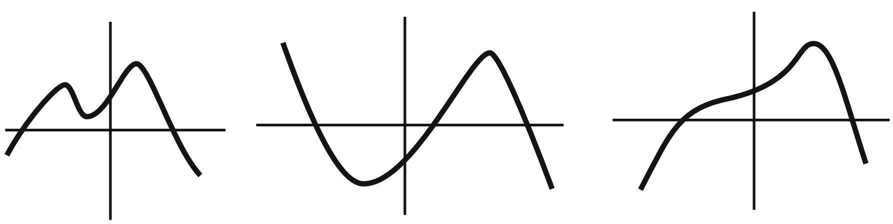


Figure 3.2 The graph of a polynomial function is a smooth, unbroken, continuous curve, such as the ones shown here.

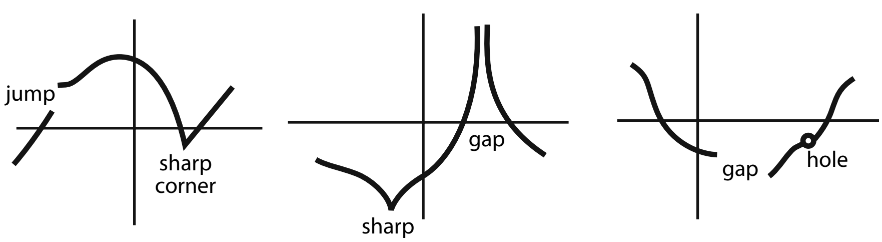


Figure 3.3 There can be no jumps, gaps, holes or sharp corners on the graph of a polynomial function. Thus none of the functions whose graphs are shown here are polynomial functions.

If we wish to sketch the graph of a polynomial function without a GDC, we need to compute some function values in order to locate a few points on the graph. This could prove to be quite tedious if the polynomial function has a high degree. We will now develop a method that provides an efficient procedure for evaluating polynomial functions. It will also be useful in the third section of this chapter for some situations when we divide polynomials. For simplicity, we give the method for a fourth-degree polynomial, but it is applicable to any nth degree polynomial.

Synthetic substitution (Optional)

Suppose we want to find the value of $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ when $x = c$, that is, find $P(c)$. The computation of c^4 may be tricky, so rather than substituting c directly into $P(x)$ we will take a gradual approach that consists of a sequence of multiplications and additions. We define b_4, b_3, b_2, b_1, and R by the following equations.

$$b_4 = a_4 \qquad (1)$$

$$b_3 = b_4c + a_3 \qquad (2)$$

$$b_2 = b_3c + a_2 \qquad (3)$$

$$b_1 = b_2c + a_1 \qquad (4)$$

$$R = b_1c + a_0 \qquad (5)$$

Our goal is to show that the value of $P(c)$ is equivalent to the value of R. Firstly, we substitute the expression for b_3 given by equation (2) into equation (3), and also use equation (1) to replace b_4 with a_4, to produce

$$\begin{aligned} b_2 &= (a_4c + a_3)c + a_2 \\ &= a_4c^2 + a_3c + a_2 \qquad (6) \end{aligned}$$

We now substitute this expression for b_2 in (6) into (4) to give

$$\begin{aligned} b_1 &= (a_4c^2 + a_3c + a_2)c + a_1 \\ &= a_4c^3 + a_3c^2 + a_2c + a_1 \qquad (7) \end{aligned}$$

To complete our goal we substitute this expression for b_1 in (7) into (5) to give

$$\begin{aligned} R &= (a_4c^3 + a_3c^2 + a_2c + a_1)c + a_0 \\ &= a_4c^4 + a_3c^3 + a_2c^2 + a_1c + a_0 \qquad (8) \end{aligned}$$

This is the value of $P(x)$ when $x = c$. If we condense (6), (7) and (8) into one expression, we obtain

$$\begin{aligned} R &= \{[(a_4c + a_3)c + a_2]c + a_1\}c + a_0 \\ &= a_4c^4 + a_3c^3 + a_2c^2 + a_1c + a_0 = P(c) \qquad (9) \end{aligned}$$

Carrying out the computations for equation (9) can be challenging. However, a nice pattern can be found if we closely inspect the expression $\{[(a_4c + a_3)c + a_2]c + a_1\}c + a_0$. Each nested computation involves finding the product of c and one of the coefficients, a_n, (starting with the leading coefficient) and then adding the next coefficient – and repeating this process until the constant term is used. Hence, the actual computation of R is quite straightforward if we arrange the nested computations required for (9) in the following systematic manner.

$$
\begin{array}{c|ccccc}
c & a_4 & a_3 & a_2 & a_1 & a_0 \\
 & & c \times b_4 & c \times b_3 & c \times b_2 & c \times b_1 \\
\hline
 & b_4 & b_3 & b_2 & b_1 & R \quad = P(c)
\end{array}
$$

In this procedure we place c in a small box to the upper left. The coefficients of the polynomial function $P(x)$ are placed in the first line. We start by simply rewriting the leading coefficient below the horizontal line (remember $b_4 = a_4$). The diagonal arrows indicate that we multiply the number in the row below the line by c to obtain the next number in the second row above the line. Each b_n after the leading coefficient is obtained by adding the two numbers in the first and second rows directly above b_n. At the end of the procedure, the last such sum is $R = P(c)$. This method of computing the value of $P(x)$ when $x = c$ is called **synthetic substitution**.

Example 1 – Using synthetic substitution to find function values

Given $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$, find the value of $P(x)$ when $x = -4, -1$ and 2.

Solution

We use the procedure for synthetic substitution just described.

$$
\begin{array}{c|ccccc}
-4 & 2 & 6 & -5 & 7 & -12 \\
 & & -8 & 8 & -12 & 20 \\
\hline
 & 2 & -2 & 3 & -5 & 8 \quad = P(-4)
\end{array}
$$

Therefore, $P(-4) = 8$.

Note: Contrast using synthetic substitution to evaluate $P(-4)$ with using direct substitution.

$$
\begin{aligned}
P(-4) &= 2(-4)^4 + 6(-4)^3 - 5(-4)^2 + 7(-4) - 12 \\
&= 2(256) + 6(-64) - 5(16) - 28 - 12 \\
&= 512 - 384 - 80 - 28 - 12 \\
&= 128 - 108 - 12 \\
&= 8
\end{aligned}
$$

$$
\begin{array}{c|ccccc}
-1 & 2 & 6 & -5 & 7 & -12 \\
 & & -2 & -4 & 9 & -16 \\
\hline
 & 2 & 4 & -9 & 16 & -28 \quad = P(-1)
\end{array}
$$

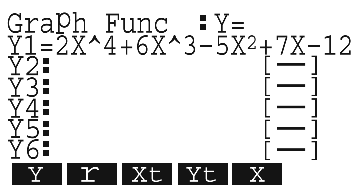
Therefore, $P(-1) = -28$.

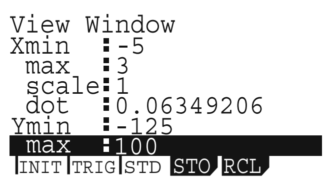
$$\begin{array}{c|ccccc}
2 & 2 & 6 & -5 & 7 & -12 \\
 & & 4 & 20 & 30 & 74 \\
\hline
 & 2 & 10 & 15 & 37 & 62 \quad = P(2)
\end{array}$$

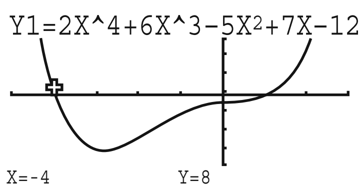
Therefore, $P(2) = 62$.

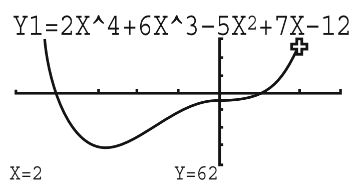
Since the graphs of all polynomial functions are continuous (no gaps or holes), then the function values we computed for the quartic polynomial function in Example 1 can give us information about the location of its zeros (i.e. x-intercepts of the graph). Since $P(-4) = 8$ and $P(-1) = -28$, then the graph of $P(x)$ must cross the x-axis ($P(x) = 0$) at least once between $x = -4$ and $x = -1$. Also, with $P(-1) = -28$ and $P(2) = 62$ there must be at least one x-intercept between $x = -1$ and $x = 2$. Hence, the polynomial equation $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12 = 0$ has at least one real root between -4 and -1, and at least one real root between -1 and 2. In Section 3.3 we will investigate real zeros of polynomial functions and then we will extend the investigation to include imaginary zeros, thereby extending the universal set for solving polynomial equations from the real numbers to complex numbers.

Graphing $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$ on our GDC, we observe that the graph of $P(x)$ does indeed intersect the x-axis between -4 and -1 (just slightly greater than $x = -4$), and again between -1 and 2 (near $x = 1$).









● **Hint:** For some values of x, evaluating $P(x)$ by direct substitution may be quicker than using synthetic substitution. This is certainly true when $x = 0$ or $x = 1$. For example, it is easy to determine that $P(0) = -12$ for the polynomial P in Example 1; and that $P(1) = 2 + 6 - 5 + 7 - 12 = -2$.

Example 2

Use synthetic substitution to find the y-coordinates of the points on the graph of $f(x) = x^3 - 4x^2 + 24$ for $x = -3, -1, 1, 3$ and 5. Sketch the graph of f for $-4 \leqslant x \leqslant 6$.

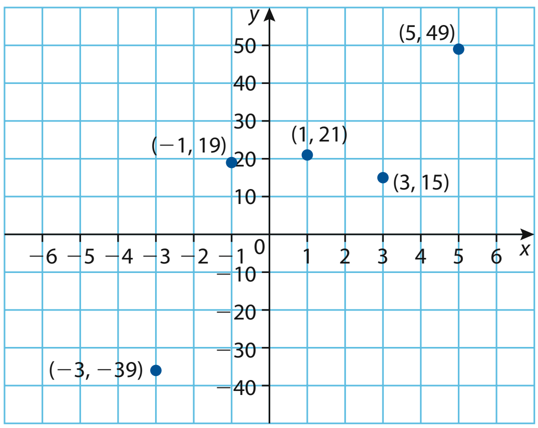
Solution

Important: In order for the method of synthetic substitution to work properly it is necessary to insert 0 for any 'missing' terms in the polynomial. The polynomial $x^3 - 4x^2 + 24$ has no linear term so the top row in the set-up for synthetic substitution must be $1 \ -4 \ 0 \ 24$.

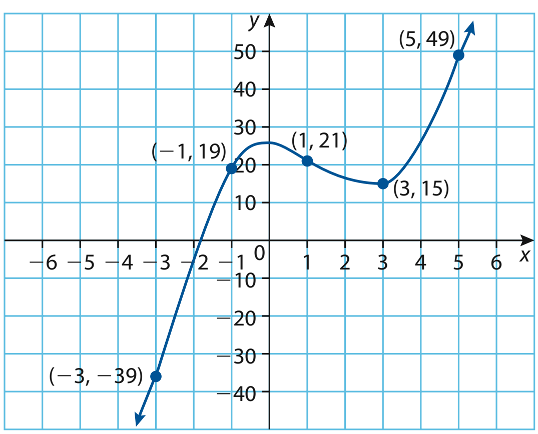
$$\begin{array}{r|rrrr} -3 & 1 & -4 & 0 & 24 \\ & & -3 & 21 & -63 \\ \hline & 1 & -7 & 21 & -39 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & -4 & 0 & 24 \\ & & -1 & 5 & -5 \\ \hline & 1 & -5 & 5 & 19 \end{array} \qquad \begin{array}{r|rrrr} 1 & 1 & -4 & 0 & 24 \\ & & 1 & -3 & -3 \\ \hline & 1 & -3 & -3 & 21 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 1 & -4 & 0 & 24 \\ & & 3 & -3 & -9 \\ \hline & 1 & -1 & -3 & 15 \end{array} \qquad \begin{array}{r|rrrr} 5 & 1 & -4 & 0 & 24 \\ & & 5 & 5 & 25 \\ \hline & 1 & 1 & 5 & 49 \end{array}$$

Therefore, the points $(-3, -39)$, $(-1, 19)$, $(1, 21)$, $(3, 15)$ and $(5, 49)$ are on the graph of f and have been plotted in the coordinate plane below.



Recall that the end behaviour of a polynomial function is determined by its degree and by the sign of its leading coefficient. Since the leading term of f is x^3 then its graph will fall $(y \to -\infty)$ as $x \to -\infty$ and will rise $(y \to \infty)$ as $x \to +\infty$. Also a polynomial function of degree n has at most $n - 1$ turning points; therefore, the graph of f has at most two turning points. Given the coordinates of the five points found with the aid of synthetic substitution, there will clearly be exactly two turning points. The graph of f can now be accurately sketched.



Exercise 3.1

In questions 1–4, use synthetic substitution to evaluate $P(x)$ for the given values of x.

1 $P(x) = x^4 + 2x^3 - 3x^2 - 4x - 20, \quad x = 2,\ x = -3$

2 $P(x) = 2x^5 - x^4 + 3x^3 - 15x - 9, \quad x = -1,\ x = 2$

3 $P(x) = x^5 + 5x^4 + 3x^3 - 6x^2 - 9x + 11, \quad x = -2,\ x = 4$

4 $P(x) = x^3 - (c + 3)x^2 + (3c + 5)x - 5c, \quad x = c,\ x = 2$

5 Given $P(x) = kx^3 + 2x^2 - 10x + 3$, for what value of k is $P(-2) = 15$?

6 Given $P(x) = 3x^4 - 2x^3 - 10x^2 + 3kx + 3$, for what value of k is $x = -\frac{1}{3}$ a zero of $P(x)$?

For questions 7 and 8, do not use your GDC.

7 a) Given $y = 2x^3 + 3x^2 - 5x - 4$, determine the y-value for each value of x such that $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

b) How many times must the graph of $y = 2x^3 + 3x^2 - 5x - 4$ cross the x-axis?

c) Sketch the graph of $y = 2x^3 + 3x^2 - 5x - 4$.

8 a) Given $y = x^4 - 4x^2 - 2x + 1$, determine the y-value for each value of x such that $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

b) How many times must the graph of $y = x^4 - 4x^2 - 2x + 1$ cross the x-axis?

c) Sketch the graph of $y = x^4 - 4x^2 - 2x + 1$.

9 Given $f(x) = x^3 + ax^2 - 5x + 7a$, find a so that $f(2) = 10$.

10 Given $f(x) = bx^3 - 5x^2 + 2bx + 10$, find b so that $f(\sqrt{3}) = -20$.

11 There are four possible end behaviours for a polynomial function $P(x)$. These are:

as $x \to \infty$, $P(x) \to \infty$ and as $x \to -\infty$, $P(x) \to \infty$ or symbolically ($\nwarrow$, $\nearrow$)

as $x \to \infty$, $P(x) \to -\infty$ and as $x \to -\infty$, $P(x) \to \infty$ or symbolically ($\nwarrow$, $\searrow$)

as $x \to \infty$, $P(x) \to -\infty$ and as $x \to -\infty$, $P(x) \to -\infty$ or symbolically ($\swarrow$, $\searrow$)

as $x \to \infty$, $P(x) \to \infty$ and as $x \to -\infty$, $P(x) \to -\infty$ or symbolically ($\swarrow$, $\nearrow$)

a) By sketching a graph on your GDC, state the type of end behaviour for each of the polynomial functions below.

(i) $P(x) = 2x^4 - 6x^3 + x^2 + 4x - 1$

(ii) $P(x) = -2x^4 - 6x^3 + x^2 + 4x - 1$

(iii) $P(x) = -6x^3 + x^2 + 4x - 1$

(iv) $P(x) = 6x^3 + x^2 - 4x - 1$

(v) $P(x) = x^2 - 4x - 1$

(vi) $P(x) = -2x^6 + x^5 + 2x^4 - 3x^3 + 4x^2 - x + 1$

(vii) $P(x) = x^5 + 2x^4 - x^3 + x^2 - x + 1$

(viii) $P(x) = -x^5 + 2x^4 - x^3 + x^2 - x + 1$

b) Use your results from a) to write a general statement about how the leading term of a polynomial function, $a_n x^n$, determines what type of end behaviour the graph of the function will display. Be specific about how the characteristics of the coefficient, a_n, and the power, n, of the leading term affect the function's end behaviour.

3.2 Quadratic functions

A **linear function** is a polynomial function of degree one that can be written in the general form $f(x) = ax + b$ where $a \neq 0$. Linear equations were briefly reviewed in Section 1.6. It is clear that any linear function will have a single solution (root) of $x = -\frac{b}{a}$. In essence, this is a formula that gives the zero of any linear polynomial.

In this section, we will focus on **quadratic functions** – functions consisting of a second-degree polynomial that can be written in the form $f(x) = ax^2 + bx + c$ such that $a \neq 0$. You are probably familiar with the quadratic formula that gives the zeros of any quadratic polynomial. We will also investigate other methods of finding zeros of quadratics and consider important characteristics of the graphs of quadratic functions.

The word *quadratic* comes from the Latin word *quadratus* that means four-sided, to make square, or simply a square. *Numerus quadratus* means a square number. Before modern algebraic notation was developed in the 17th and 18th centuries, the geometric figure of a square was used to indicate a number multiplying itself. Hence, raising a number to the power of two (in modern notation) is commonly referred to as the operation of squaring. *Quadratic* then came to be associated with a polynomial of degree two rather than being associated with the number four, as the prefix quad often indicates (e.g. quadruple).

Definition of a quadratic function

If a, b and c are real numbers, and $a \neq 0$, the function $f(x) = ax^2 + bx + c$ is a **quadratic function**. The graph of f is the graph of the equation $y = ax^2 + bx + c$ and is called a **parabola**.

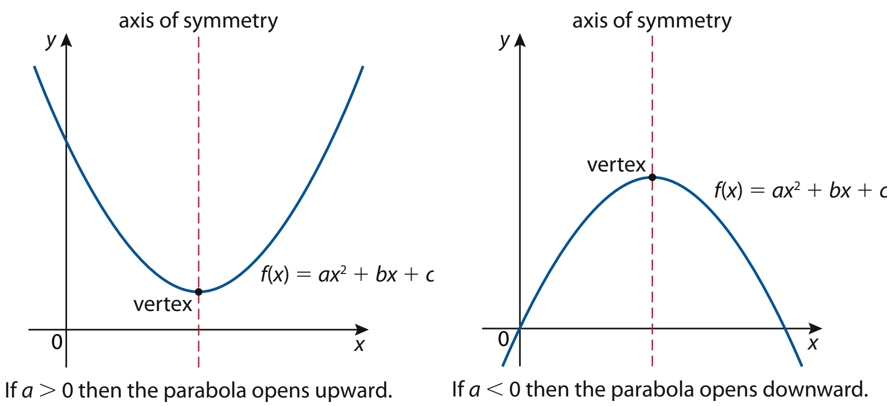


Figure 3.4 'Concave up' and 'concave down' parabolas.

Each parabola is symmetric about a vertical line called its **axis of symmetry.** The axis of symmetry passes through a point on the parabola called the **vertex** of the parabola, as shown in Figure 3.4. If the leading coefficient, a, of the quadratic function $f(x) = ax^2 + bx + c$ is positive, the parabola opens upward (concave up) – and the y-coordinate of the vertex will be a **minimum value** for the function. If the leading coefficient, a, of $f(x) = ax^2 + bx + c$ is negative, the parabola opens downward (concave down) – and the y-coordinate of the vertex will be a **maximum value** for the function.

The graph of $f(x) = a(x - h)^2 + k$

From the previous chapter, we know that the graph of the equation $y = (x + 3)^2 + 2$ can be obtained by translating $y = x^2$ three units to the left and two units up. Being familiar with the shape and position of the graph of $y = x^2$, and knowing the two translations that transform $y = x^2$ to

$y = (x + 3)^2 + 2$, we can easily visualize and/or sketch the graph of $y = (x + 3)^2 + 2$ (see Figure 3.5). We can also determine the axis of symmetry and the vertex of the graph. Figure 3.6 shows that the graph of $y = (x + 3)^2 + 2$ has an axis of symmetry of $x = -3$ and a vertex at $(-3, 2)$. The equation $y = (x + 3)^2 + 2$ can also be written as $y = x^2 + 6x + 11$. Because we can easily identify the vertex of the parabola when the equation is written as $y = (x + 3)^2 + 2$, we often refer to this as the **vertex form** of the quadratic equation, and $y = x^2 + 6x + 11$ as the **general form**.

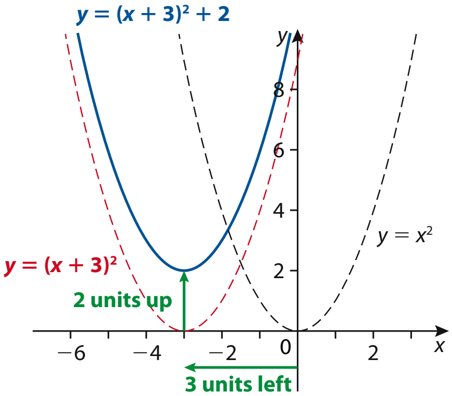


Figure 3.5 Translating $y = x^2$ to give $y = (x + 3)^2 + 2$.

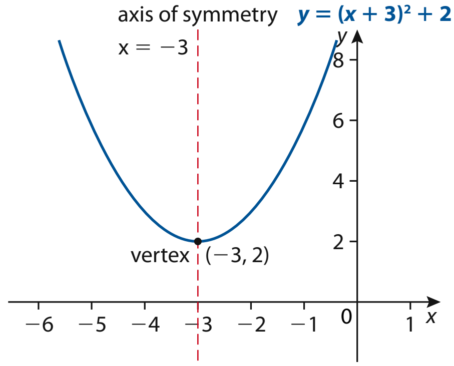


Figure 3.6 The axis of symmetry and the vertex.

• **Hint:** $f(x) = a(x - h)^2 + k$ is sometimes referred to as the **standard form** of a quadratic function.

Vertex form of a quadratic function

If a quadratic function is written in the form $f(x) = a(x - h)^2 + k$, with $a \neq 0$, the graph of f has an axis of symmetry of $x = h$ and a vertex at (h, k).

Completing the square

For visualizing and sketching purposes, it is helpful to have a quadratic function written in vertex form. How do we rewrite a quadratic function written in the form $f(x) = ax^2 + bx + c$ (general form) into the form $f(x) = a(x - h)^2 + k$ (vertex form)? We use the technique of **completing the square.**

For any real number p, the quadratic expression $x^2 + px + \left(\frac{p}{2}\right)^2$ is the square of $\left(x + \frac{p}{2}\right)$. Convince yourself of this by expanding $\left(x + \frac{p}{2}\right)^2$. The technique of *completing the square* is essentially the process of adding a constant to a quadratic expression to make it the square of a binomial. If the coefficient of the quadratic term (x^2) is positive one, the coefficient of the linear term is p, and the constant term is $\left(\frac{p}{2}\right)^2$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2$ and the square is completed.

Remember that the coefficient of the quadratic term (leading coefficient) must be equal to positive one before completing the square.

Example 3

Find the equation of the axis of symmetry and the coordinates of the vertex of the graph of $f(x) = x^2 - 8x + 18$ by rewriting the function in the form $x^2 + px + \left(\frac{p}{2}\right)^2$.

Solution

To complete the square and get the quadratic expression $x^2 - 8x + 18$ in the form $x^2 + px + \left(\frac{p}{2}\right)^2$, the constant term needs to be $\left(\frac{-8}{2}\right)^2 = 16$.

We need to add 16, but also subtract 16, so that we are adding zero overall and, hence, not changing the original expression.

$f(x) = x^2 - 8x + 16 - 16 + 18$ Actually adding zero $(-16 + 16)$ to the right side.

$f(x) = x^2 - 8x + 16 + 2$ $x^2 - 8x + 16$ fits the pattern $x^2 + px + \left(\frac{p}{2}\right)^2$ with $p = -8$.

$f(x) = (x - 4)^2 + 2$ $x^2 - 8x + 16 = (x - 4)^2$

The axis of symmetry of the graph of f is the vertical line $x = 4$ and the vertex is at $(4, 2)$. See Figure 3.7.

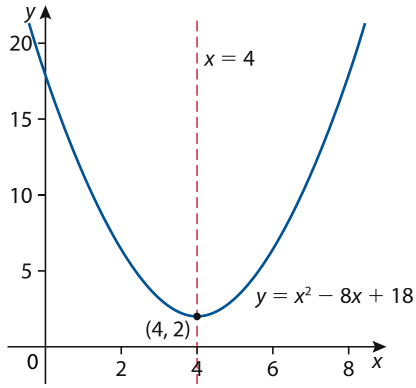


Figure 3.7

Example 4 – Properties of a parabola

For the function $g : x \mapsto -2x^2 - 12x + 7$,

a) find the axis of symmetry and the vertex of the graph

b) indicate the transformations that can be applied to $y = x^2$ to obtain the graph

c) find the minimum or maximum value.

Solution

a) $g : x \mapsto -2\left(x^2 + 6x - \frac{7}{2}\right)$ Factorize so that the coefficient of the quadratic term is +1.

$g : x \mapsto -2\left(x^2 + 6x + 9 - 9 - \frac{7}{2}\right)$ $p = 6 \Rightarrow \left(\frac{p}{2}\right)^2 = 9$; hence, add $+9 - 9$ (zero)

$g : x \mapsto -2\left[(x + 3)^2 - \frac{18}{2} - \frac{7}{2}\right]$ $x^2 + 6x + 9 = (x + 3)^2$

$g : x \mapsto -2\left[(x + 3)^2 - \frac{25}{2}\right]$

$g : x \mapsto -2(x + 3)^2 + 25$ Multiply through by -2 to remove outer brackets.

$g : x \mapsto -2(x - (-3))^2 + 25$ Express in vertex form: $g : x \mapsto a(x - h)^2 + k$

The axis of symmetry of the graph of g is the vertical line $x = -3$ and the vertex is at $(-3, 25)$. See Figure 3.8.

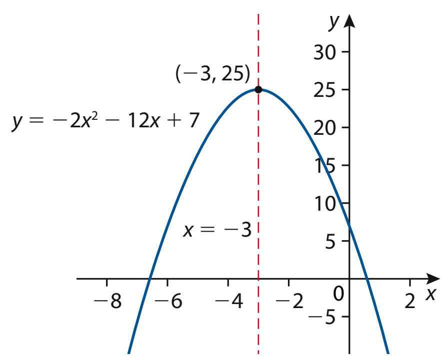


Figure 3.8

b) Since $g : x \mapsto -2x^2 - 12x + 7 = -2(x + 3)^2 + 25$, the graph of g can be obtained by applying the following transformations (in the order given) on the graph of $y = x^2$: horizontal translation of 3 units left;

reflection in the x-axis (parabola opening down); vertical stretch of factor 2; and a vertical translation of 25 units up.

c) The parabola opens down because the leading coefficient is negative. Therefore, g has a maximum and no minimum value. The maximum value is 25 (y-coordinate of vertex) at $x = -3$.

The technique of completing the square can be used to derive the quadratic formula. The following example derives a general expression for the axis of symmetry and vertex of a quadratic function in the general form $f(x) = ax^2 + bx + c$ by completing the square.

Example 5 – Graphical properties of general quadratic functions

Find the axis of symmetry and the vertex for the general quadratic function $f(x) = ax^2 + bx + c$.

Solution

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Factorize so that the coefficient of the x^2 term is $+1$.

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$p = \frac{b}{a} \Rightarrow \left(\frac{p}{2}\right)^2 = \left(\frac{b}{2a}\right)^2$

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = x + \left(\frac{b}{2a}\right)^2$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Multiply through by a.

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}$$

Express in vertex form: $f(x) = a(x - h)^2 + k$

This result leads to the following generalization.

Symmetry and vertex of $f(x) = ax^2 + bx + c$

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the vertical line with the equation $x = -\frac{b}{2a}$ and the vertex has coordinates $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Check the results for Example 4 using the formulae for the axis of symmetry and vertex. For the function $g: x \mapsto -2x^2 - 12x + 7$:

$x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3 \Rightarrow$ axis of symmetry is the vertical line $x = -3$

$c - \frac{b^2}{4a} = 7 - \frac{(-12)^2}{4(-2)} = \frac{56}{8} + \frac{144}{8} = 25 \Rightarrow$ vertex has coordinates $(-3, 25)$

These results agree with the results from Example 4.

Zeros of a quadratic function

A specific value for x is a **zero** of a quadratic function $f(x) = ax^2 + bx + c$ if it is a solution (or **root**) to the equation $ax^2 + bx + c = 0$.

As we will observe, every quadratic function will have two zeros although it is possible for the same zero to occur twice (double zero, or double root). The x-coordinate of any point(s) where f crosses the x-axis (y-coordinate is zero) is a **real zero** of the function. A quadratic function can have one, two or no real zeros as Figure 3.9 illustrates. To find non-real zeros we need to extend our search to the set of complex numbers and we will see that a quadratic function with no real zeros will have two distinct **imaginary zeros**. Finding all zeros of a quadratic function requires you to solve quadratic equations of the form $ax^2 + bx + c = 0$. Although $a \neq 0$, it is possible for b or c to be equal to zero. There are five general methods for solving quadratic equations as outlined in Table 3.2 below.

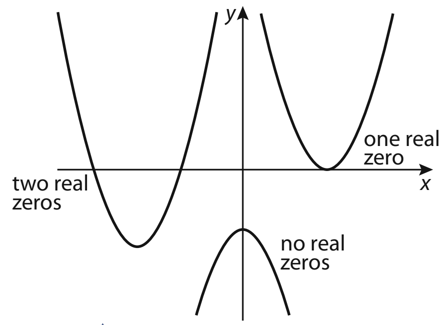
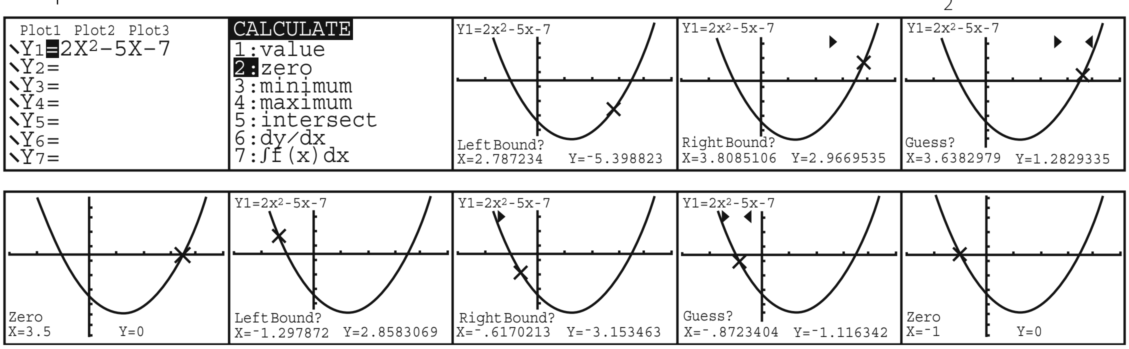


Figure 3.9

Table 3.2 Methods for solving quadratic equations.

Square root	If $a^2 = c$ and $c > 0$, then $a = \pm\sqrt{c}$.	
Examples	$x^2 - 25 = 0$ $x^2 = 25$ $x = \pm 5$	$(x + 2)^2 = 15$ $x + 2 = \pm\sqrt{15}$ $x = -2 \pm\sqrt{15}$
Factorizing	If $ab = 0$, then $a = 0$ or $b = 0$.	
Examples	$x^2 + 3x - 10 = 0$ $(x + 5)(x - 2) = 0$ $x = -5$ or $x = 2$	$x^2 - 7x = 0$ $x(x - 7) = 0$ $x = 0$ or $x = 7$
Completing the square	If $x^2 + px + q = 0$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$ which leads to $\left(x + \frac{p}{2}\right)^2 = -q + \frac{p^2}{4}$ and then the square root of both sides (as above).	
Example	$x^2 - 8x + 5 = 0$ $x^2 - 8x + 16 = -5 + 16$ $(x - 4)^2 = 11$ $x - 4 = \pm\sqrt{11}$ $x = 4 \pm \sqrt{11}$	
Quadratic formula	If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	
Example	$2x^2 - 3x - 4 = 0$ $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$ $x = \frac{3 \pm \sqrt{41}}{4}$	
Graphing	Graph the equation $y = ax^2 + bx + c$ on your GDC. Use the calculating features of your GDC to determine the x-coordinates of the point(s) where the parabola intersects the x-axis. *Note*: This method works for finding real solutions, but **not** imaginary solutions.	
Example	$2x^2 - 5x - 7 = 0$ GDC calculations reveal that the zeros are at $x = \frac{7}{2}$ and $x = -1$	



Sum and product of the roots of a quadratic equation

In the next section, the Factor Theorem formally states the relationship between linear factors of the form $x - \alpha$ and the zeros for *any* polynomial.

Consider the quadratic equation $x^2 + 5x - 24 = 0$. This equation can be solved using factorization as follows.

$$x^2 + 5x - 24 = (x + 8)(x - 3) = 0 \Rightarrow x = -8 \text{ or } x = 3$$

Clearly, if $x - \alpha$ is a factor of the quadratic polynomial $ax^2 + bx + c$, then $x = \alpha$ is a root (solution) of the quadratic equation $ax^2 + bx + c = 0$.

Now let us consider the general quadratic equation $ax^2 + bx + c = 0$, whose roots are $x = \alpha$ and $x = \beta$. Given our observation from the previous paragraph, we can write the quadratic equation with roots α and β as:

$$\begin{aligned} ax^2 + bx + c = (x - \alpha)(x - \beta) &= 0 \\ x^2 - ax - \beta x + \alpha\beta &= 0 \\ x^2 - (\alpha + \beta)x + \alpha\beta &= 0 \end{aligned}$$

Since the equation $ax^2 + bx + c = 0$ can also be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then:

$$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Equating coefficients of both sides, gives the following results.

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Sum and product of the roots of a quadratic equation
For any quadratic equation in the form $ax^2 + bx + c = 0$, the **sum of the roots** of the equation is $-\frac{b}{a}$ and the **product of the roots** is $\frac{c}{a}$. (In the next section, this result is extended to polynomial equations of any degree.)

Example 6

If α and β are the roots of each equation, find the sum, $\alpha + \beta$, and product, $\alpha\beta$, of the roots.

a) $x^2 - 5x + 3 = 0$ b) $3x^2 + 4x - 7 = 0$

Solution

a) For the equation $x^2 - 5x + 3 = 0$, $a = 1$, $b = -5$ and $c = 3$.
Therefore, $\alpha + \beta = -\frac{b}{a} = -\frac{-5}{1} = 5$ and $\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$.

b) For the equation $3x^2 + 4x - 7 = 0$, $a = 3$, $b = 4$ and $c = -7$.
Therefore, $\alpha + \beta = -\frac{b}{a} = -\frac{4}{3}$ and $\alpha\beta = \frac{c}{a} = \frac{-7}{3}$.

If the sum and product of the roots of a quadratic equation are known, then the equation can be written in the following form: $x^2 -$ (sum of roots)x + (product of roots) $= 0$

Example 7

If α and β are the roots of the equation $2x^2 + 6x - 5 = 0$, find a quadratic equation whose roots are:

a) $2\alpha, 2\beta$ b) $\frac{1}{\alpha + 1}, \frac{1}{\beta + 1}$

Solution

For the equation $2x^2 + 6x - 5 = 0$, $a = 2$, $b = 6$ and $c = -5$.

Thus, $\alpha + \beta = -\frac{b}{a} = -\frac{6}{2} = -3$ and $\alpha\beta = \frac{c}{a} = \frac{-5}{2}$.

a) Sum of the new roots $= 2\alpha + 2\beta = 2(\alpha + \beta) = 2(-3) = -6$.

Thus for the new equation, $-\frac{b}{a} = -6$.

Product of the new roots $= 2\alpha \cdot 2\beta = 4\alpha\beta = 4\left(-\frac{5}{2}\right) = -10$.

Thus for the new equation, $\frac{c}{a} = -10$.

The new equation we are looking for can be written as $ax^2 + bx + c = 0$ or $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Therefore, the quadratic equation with roots 2α, 2β is $x^2 - (-6)x - 10 = 0$
$\Rightarrow x^2 + 6x - 10 = 0$

b) Sum of the new roots $\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} = \frac{\beta + 1 + \alpha + 1}{(\alpha + 1)(\beta + 1)}$

$= \frac{\alpha + \beta + 2}{\alpha\beta + \alpha + \beta + 1} = \frac{-3 + 2}{-\frac{5}{2} - 3 + 1} = \frac{-1}{-\frac{9}{2}} = \frac{2}{9}$.

Thus for the new equation, $-\frac{b}{a} = \frac{2}{9}$.

Product of the new roots $\left(\frac{1}{\alpha + 1}\right)\left(\frac{1}{\beta + 1}\right) = \frac{1}{\alpha\beta + \alpha + \beta + 1}$

$= \frac{1}{-\frac{5}{2} - 3 + 1} = \frac{1}{-\frac{9}{2}} = -\frac{2}{9}$.

Thus for the new equation, $\frac{c}{a} = -\frac{2}{9}$.

The new equation we are looking for can be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Therefore, the quadratic equation with roots $\frac{1}{\alpha + 1}, \frac{1}{\beta + 1}$ is $x^2 - \frac{2}{9}x - \frac{2}{9} = 0$ or $9x^2 - 2x - 2 = 0$.

Example 8

Given that the roots of the equation $x^2 - 4x + 2 = 0$ are α and β, find the values of the following expressions.

a) $\alpha^2 + \beta^2$ b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution

With $x^2 - 4x + 2 = 0$, $\alpha + \beta = -\frac{b}{a} = -\frac{-4}{1} = 4$ and $\alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$.

Both of the expressions $\alpha^2 + \beta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ need to be expressed in terms of $\alpha + \beta$ and $\alpha\beta$.

a) $\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta$

Substituting the values for $\alpha + \beta$ and $\alpha\beta$ from above, gives
$\alpha^2 + \beta^2 = 4^2 - 2 \cdot 2 = 16 - 4 = 12$.

b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2}{\alpha^2\beta^2} + \frac{\alpha^2}{\alpha^2\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$

From part a) we know that $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$. Substituting this into the numerator gives:

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$ Then substituting the values for $\alpha + \beta$ and $\alpha\beta$ from above, gives:

$= \frac{4^2 - 2 \cdot 2}{2^2} = \frac{12}{4} = 3$

Therefore, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 3$.

The quadratic formula and the discriminant

The expression that is beneath the radical sign in the quadratic formula, $b^2 - 4ac$, determines whether the zeros of a quadratic function are real or imaginary. Because it acts to 'discriminate' between the types of zeros, $b^2 - 4ac$ is called the **discriminant**. It is often labelled with the Greek letter Δ (delta). The value of the discriminant can also indicate if the zeros are equal and if they are rational.

The discriminant and the nature of the zeros of a quadratic function

For the quadratic function $f(x) = ax^2 + bx + c$, $(a \neq 0)$ where a, b and c are real numbers:

If $\Delta = b^2 - 4ac > 0$, then f has two distinct real zeros, and the graph of f intersects the x-axis twice.

If $\Delta = b^2 - 4ac = 0$, then f has one real zero (double root), and the graph of f intersects the x-axis once (i.e. it is tangent to the x-axis).

If $\Delta = b^2 - 4ac < 0$, then f has two conjugate imaginary zeros, and the graph of f does not intersect the x-axis.

In the special case when a, b and c are integers and the discriminant is the square of an integer (a *perfect square*), the polynomial $ax^2 + bx + c$ has two distinct **rational zeros**.

When the discriminant is zero then the solution of a quadratic function is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$. As mentioned, this solution of $-\frac{b}{2a}$ is called a double zero (or root) which can also be described as a **zero of multiplicity of 2**. If a and b are integers then the zero $-\frac{b}{2a}$ will be rational.

When we solve polynomial functions of higher degree later this chapter, we will encounter zeros of higher multiplicity.

• **Hint:** Remember that the **roots** of a polynomial equation are those values of x for which $P(x) = 0$. These values of x are called the **zeros** of the polynomial P.

Factorable quadratics

If the zeros of a quadratic polynomial are rational – either two distinct zeros or two equal zeros (double zero/root) – then the polynomial is factorable. That is, if $ax^2 + bx + c$ has rational zeros then $ax^2 + bx + c = (mx + n)(px + q)$ where m, n, p and q are rational numbers.

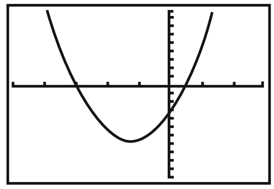
Example 9 – Using discriminant to determine the nature of the roots of a quadratic equation

Use the discriminant to determine how many real roots each equation has. Visually confirm the result by graphing the corresponding quadratic function for each equation on your GDC.

a) $2x^2 + 5x - 3 = 0$ b) $4x^2 - 12x + 9 = 0$ c) $2x^2 - 5x + 6 = 0$

Solution

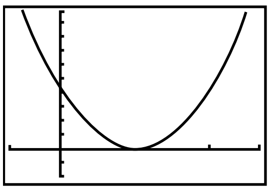
$y = 2x^2 + 5x - 3$

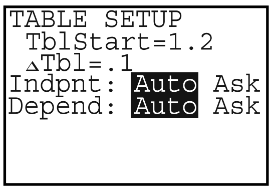


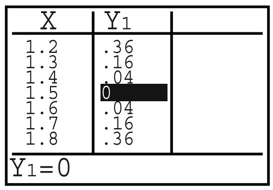
a) The discriminant is $\Delta = 5^2 - 4(2)(-3) = 49 > 0$. Therefore, the equation has two distinct real roots. This result is confirmed by the graph of the quadratic function $y = 2x^2 + 5x - 3$ that clearly shows it intersecting the x-axis twice. Also since $\Delta = 49$ is a perfect square then the two roots are also rational and the quadratic polynomial $2x^2 + 5x - 3 = 0$ is factorable: $2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0$. Thus, the two rational roots are $x = \frac{1}{2}$ and $x = -3$.

b) The discriminant is $\Delta = (-12)^2 - 4(4)(9) = 0$. Therefore, the equation has one rational root (a double root). The graph on the GDC of $y = 4x^2 - 12x + 9$ appears to intersect the x-axis at only one point. We can be more confident with this conclusion by investigating further – for example, tracing or looking at a table of values on the GDC.

$y = 4x^2 - 12x + 9$







Also, since the root is rational ($\Delta = 0$), the polynomial $4x^2 - 12x + 9$ must be factorable.

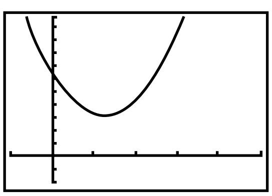
$4x^2 - 12x + 9 = (2x - 3)(2x - 3) = \left[2\left(x - \frac{3}{2}\right)2\left(x - \frac{3}{2}\right)\right] = 4\left(x - \frac{3}{2}\right)^2 = 0$

There are two equal linear factors which means there are two equal rational zeros – both equal to $\frac{3}{2}$ in this case.

c) The discriminant is $\Delta = (-5)^2 - 4(2)(6) = -23 < 0$. Therefore, the equation has no real roots. This result is confirmed by the graph of the quadratic function $y = 2x^2 - 5x + 6$ that clearly shows that the graph does not intersect the x-axis. The equation will have two imaginary roots.

● **Hint:** If a quadratic polynomial has a zero of multiplicity 2($\Delta = 0$), as in Example 6 b), then not only is the polynomial factorable but its factorization will contain two equal linear factors. In such a case then $ax^2 + bx + c = a(x - p)^2$ where $x - p$ is the linear factor and $x = p$ is the rational zero.

$y = 2x^2 - 5x + 6$



Example 10 – The discriminant and number of real zeros

For $4x^2 + 4kx + 9 = 0$, determine the value(s) of k so that the equation has: a) one real zero, b) two distinct real zeros, and c) no real zeros.

Solution

a) For one real zero $\Delta = (4k)^2 - 4(4)(9) = 0 \Rightarrow 16k^2 - 144 = 0$

$\Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

b) For two distinct real zeros $\Delta = (4k)^2 - 4(4)(9) > 0 \Rightarrow 16k^2 > 144$
$\Rightarrow k^2 > 9 \Rightarrow k < -3$ or $k > 3$

c) For no real zeros $\Delta = (4k)^2 - 4(4)(9) < 0 \Rightarrow 16k^2 < 144 \Rightarrow k^2 < 9$
$\Rightarrow k > -3$ and $k < 3 \Rightarrow -3 < k < 3$

Example 11 – Conjugate imaginary solutions

Find the zeros of the function $g: x \rightarrow 2x^2 - 4x + 7$.

Solution

Solve the equation $2x^2 - 4x + 7 = 0$ using the quadratic formula with $a = 2, b = -4, c = 7$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(7)}}{2(2)} = \frac{4 \pm \sqrt{-40}}{4} = \frac{4 \pm \sqrt{4}\sqrt{-1}\sqrt{10}}{4}$$

$$= \frac{4 \pm 2i\sqrt{10}}{4} = 1 \pm \frac{i\sqrt{10}}{2}$$

The two zeros of g are $1 + \frac{\sqrt{10}}{2}i$ and $1 - \frac{\sqrt{10}}{2}i$.

Note that the imaginary zeros are written in the form $a + bi$ (introduced in Section 1.1) and that they clearly are a pair of conjugates, i.e. fitting the pattern $a + bi$ and $a - bi$.

Number of complex zeros of a quadratic polynomial

Every quadratic polynomial has exactly two complex zeros, provided that a zero of multiplicity 2 (two equal zeros) is counted as two zeros.

• **Hint:** Recall from Section 1.1 that the real numbers and the imaginary numbers are distinct subsets of the complex numbers. A complex number can be either real $\left(\text{e.g. } -7, \frac{\pi}{2}, 3 - \sqrt{2}\right)$ or imaginary (e.g. $4i$, $2 + i\sqrt{5}$).

The graph of $f(x) = a(x - p)(x - q)$

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$ then we can easily identify the x-intercepts of the graph of f. Consider that $f(p) = a(p - p)(p - q) = a(0)(p - q) = 0$ and that $f(q) = a(q - p)(q - q) = a(q - p)(0) = 0$. Therefore, the quadratic function $f(x) = a(x - p)(x - q)$ will intersect the x-axis at the points $(p, 0)$ and $(q, 0)$. We need to factorize in order to rewrite a quadratic function in the form $f(x) = ax^2 + bx + c$ to the form $f(x) = a(x - p)(x - q)$. Hence, $f(x) = a(x - p)(x - q)$ can be referred to as the **factorized** form of a quadratic function. Recalling the symmetric nature of a parabola, it is clear that the x-intercepts $(p, 0)$ and $(q, 0)$ will be equidistant from the axis of symmetry (see Figure 3.10). As a result, the equation of the axis of symmetry and the x-coordinate of the vertex of the parabola can be found from finding the average of p and q.

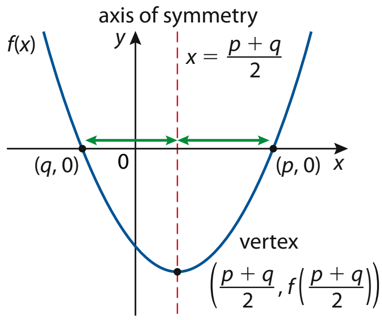


Figure 3.10

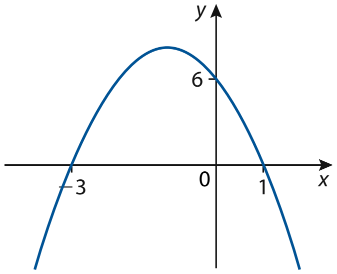
Factorized form of a quadratic function

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$, with $a \neq 0$, the graph of f has x-intercepts at $(p, 0)$ and $(q, 0)$, an axis of symmetry with equation $x = \frac{p+q}{2}$, and a vertex at $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$.

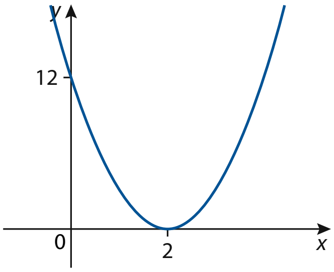
Example 12

Find the equation of each quadratic function from the graph in the form $f(x) = a(x - p)(x - q)$ and also in the form $f(x) = ax^2 + bx + c$.

a)



b)



Solution

a) Since the x-intercepts are -3 and 1 then $y = a(x + 3)(x - 1)$. The y-intercept is 6, so when $x = 0$, $y = 6$. Hence, $6 = a(0 + 3)(0 - 1) = -3a \Rightarrow a = -2$ ($a < 0$ agrees with the fact that the parabola is opening down). The function is $f(x) = -2(x + 3)(x - 1)$, and expanding to remove brackets reveals that the function can also be written as $f(x) = -2x^2 - 4x + 6$.

b) The function has one x-intercept at 2 (double root), so $p = q = 2$ and $y = a(x - 2)(x - 2) = a(x - 2)^2$. The y-intercept is 12, so when $x = 0$, $y = 12$. Hence, $12 = a(0 - 2)^2 = 4a \Rightarrow a = 3$ ($a > 0$ agrees with the parabola opening up). The function is $f(x) = 3(x - 2)^2$. Expanding reveals that the function can also be written as $f(x) = 3x^2 - 12x + 12$.

Example 13

The graph of a quadratic function intersects the x-axis at the points $(-6, 0)$ and $(-2, 0)$ and also passes through the point $(2, 16)$. a) Write the function in the form $f(x) = a(x - p)(x - q)$. b) Find the vertex of the parabola. c) Write the function in the form $f(x) = a(x - h)^2 + k$.

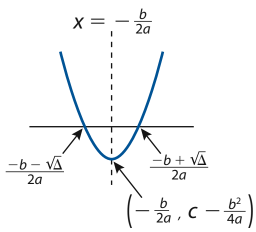
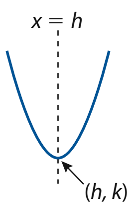
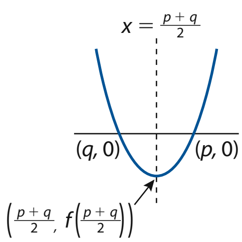
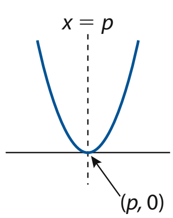
Solution

a) The x-intercepts of -6 and -2 gives $f(x) = a(x + 6)(x + 2)$. Since f passes through $(2, 16)$, then $f(2) = 16 \Rightarrow f(2) = a(2 + 6)(2 + 2) = 16 \Rightarrow 32a = 16 \Rightarrow a = \frac{1}{2}$. Therefore, $f(x) = \frac{1}{2}(x + 6)(x + 2)$.

b) The x-coordinate of the vertex is the average of the x-intercepts. $x = \dfrac{-6 - 2}{2} = -4$, so the y-coordinate of the vertex is $y = f(-4) = \frac{1}{2}(-4 + 6)(-4 + 2) = -2$. Hence, the vertex is $(-4, -2)$.

c) In vertex form, the quadratic function is $f(x) = \frac{1}{2}(x + 4)^2 - 2$.

Table 3.3 Review of properties of quadratics.

Quadratic function, $a \neq 0$	Graph of function	Results
General form $f(x) = ax^2 + bx + c$ $\Delta = b^2 - 4ac$ (discriminant)	Parabola opens up if $a > 0$ Parabola opens down if $a < 0$  If $\Delta \geqslant 0$, f has x-intercept(s) If $\Delta < 0$, f has no x-intercept(s)	Axis of symmetry is $x = -\frac{b}{2a}$ If $\Delta \geqslant 0$, f has x-intercept(s): $\left(\frac{-b \pm \sqrt{\Delta}}{2a}, 0\right)$ Vertex is: $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$
Vertex form $f(x) = a(x - h)^2 + k$		Axis of symmetry is $x = h$ Vertex is (h, k)
Factorized form (two distinct rational zeros) $f(x) = a(x - p)(x - q)$		Axis of symmetry is $x = \frac{p+q}{2}$ x-intercepts are: $(p, 0)$ and $(q, 0)$
Factorized form (one rational zero) $f(x) = a(x - p)^2$		Axis of symmetry is $x = p$ Vertex and x-intercept is $(p, 0)$

Exercise 3.2

For each of the quadratic functions f in questions 1–5, find the following:

a) the axis of symmetry and the vertex, by algebraic methods

b) the transformation(s) that can be applied to $y = x^2$ to obtain the graph of $y = f(x)$

c) the minimum or maximum value of f.

Check your results using your GDC.

1 $f : x \mapsto x^2 - 10x + 32$

2 $f : x \mapsto x^2 + 6x + 8$

3 $f : x \mapsto -2x^2 - 4x + 10$

4 $f : x \mapsto 4x^2 - 4x + 9$

5 $f : x \mapsto \frac{1}{2}x^2 + 7x + 26$

In questions 6–13, solve the quadratic equation using factorization.

6 $x^2 + 2x - 8 = 0$

7 $x^2 = 3x + 10$

8 $6x^2 - 9x = 0$

9 $6 + 5x = x^2$

10 $x^2 + 9 = 6x$

11 $3x^2 + 11x - 4 = 0$

12 $3x^2 + 18 = 15x$

13 $9x - 2 = 4x^2$

In questions 14–19, use the method of completing the square to solve the quadratic equation.

14 $x^2 + 4x - 3 = 0$

15 $x^2 - 4x - 5 = 0$

16 $x^2 - 2x + 3 = 0$

17 $2x^2 + 16x + 6 = 0$

18 $x^2 + 2x - 8 = 0$

19 $-2x^2 + 4x + 9 = 0$

20 Let $f(x) = x^2 - 4x - 1$. a) Use the quadratic formula to find the zeros of the function. b) Use the zeros to find the equation for the axis of symmetry of the parabola. c) Find the minimum or maximum value of f.

In questions 21–24, determine the number of real solutions to each equation.

21 $x^2 + 3x + 2 = 0$

22 $2x^2 - 3x + 2 = 0$

23 $x^2 - 1 = 0$

24 $2x^2 - \frac{9}{4}x + 1 = 0$

25 Find the value(s) of p for which the equation $2x^2 + px + 1 = 0$ has one real solution.

26 Find the value(s) of k for which the equation $x^2 + 4x + k = 0$ has two distinct real solutions.

27 The equation $x^2 - 4kx + 4 = 0$ has two distinct real solutions. Find the set of all possible values of k.

28 Find all possible values of m so that the graph of the function $g: x \mapsto mx^2 + 6x + m$ does not touch the x-axis.

29 Find the range of values of k such that $3x^2 - 12x + k > 0$ for all real values of x. (Hint: Consider what must be true about the zeros of the quadratic equation $y = 3x^2 - 12x + k$.)

30 Prove that the expression $x - 2 - x^2$ is negative for all real values of x.

In questions 31 and 32, find a quadratic function in the form $y = ax^2 + bx + c$ that satisfies the given conditions.

31 The function has zeros of $x = -1$ and $x = 4$ and its graph intersects the y-axis at $(0, 8)$.

32 The function has zeros of $x = \frac{1}{2}$ and $x = 3$ and its graph passes through the point $(-1, 4)$.

33 Find the range of values for k in order for the equation $2x^2 + (3 - k)x + k + 3 = 0$ to have two imaginary solutions.

34 For what values of m does the function $f(x) = 5x^2 - mx + 2$ have two distinct real zeros?

35 The graph of a quadratic function passes through the points $(-3, 10)$, $(\frac{1}{4}, -\frac{9}{16})$ and $(1, 6)$. Express the function in the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$.

36 The maximum value of the function $f(x) = ax^2 + bx + c$ is 10. Given that $f(3) = f(-1) = 2$, find $f(2)$.

37 Find the values of x for which $4x + 1 < x^2 + 4$.

38 Show that there is no real value t for which the equation $2x^2 + (2 - t)x + t^2 + 3 = 0$ has real roots.

39 Show that the two roots of $ax^2 - a^2x - x + a = 0$ are reciprocals of each other.

40 Find the sum and product of the roots for each of the following quadratic equations.

a) $2x^2 + 6x - 5 = 0$ b) $x^2 = 1 - 3x$ c) $4x^2 - 6 = 0$

d) $x^2 + ax - 2a = 0$ e) $m(m - 2) = 4(m + 1)$ f) $3x - \frac{2}{x} = 1$

41 The roots of the equation $2x^2 - 3x + 6 = 0$ are α and β. Find a quadratic equation with integral coefficients whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

42 If α and β are the roots of the equation $3x^2 + 5x + 4 = 0$, find the values of the following expressions.

a) $\alpha^2 + \beta^2$ b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

c) $\alpha^3 + \beta^3$ [Hint: factorise $\alpha^3 + \beta^3$ into a product of a binomial and a trinomial.]

43 Consider the quadratic equation $x^2 + 8x + k = 0$ where k is a constant.

a) Find both roots of the equation given that one root of the equation is three times the other.

b) Find the value of k.

44 The roots of the equation $x^2 + x + 4 = 0$ are α and β.

a) Without solving the equation, find the value of the expression $\frac{1}{\alpha} + \frac{1}{\beta}$.

b) Find a quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

45 If α and β are roots of the quadratic equation $5x^2 - 3x - 1 = 0$, find a quadratic equation with integral coefficients which have the roots:

a) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ b) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

3.3 Zeros, factors and remainders

Finding the zeros of polynomial functions is a feature of many problems in algebra, calculus and other areas of mathematics. In our analysis of quadratic functions in the previous section, we saw the connection between the graphical and algebraic approaches to finding zeros. Information obtained from the graph of a function can be used to help find its zeros and, conversely, information about the zeros of a polynomial function can be used to help sketch its graph. Results and observations from the last section lead us to make some statements about real zeros of all polynomial functions. Later in this section we will extend our consideration to imaginary zeros. The following box summarizes what we have observed thus far about the zeros of polynomial functions.

Real zeros of polynomial functions

If P is a polynomial function and c is a real number, then the following statements are equivalent.

- $x = c$ is a zero of the function P.
- $x = c$ is a solution (or root) of the polynomial equation $P(x) = 0$.
- $x - c$ is a linear factor of the polynomial P.
- $(c, 0)$ is an x-intercept of the graph of the function P.

Polynomial division

As with integers, finding the factors of polynomials is closely related to dividing polynomials. An integer n is **divisible** by another integer m if m is a factor of n. If n is not divisible by m we can use the process of **long division** to find the quotient of the numbers and the remainder. For example, let's use long division to divide 485 by 34.

$$\begin{array}{r} 14 \\ 34\overline{)485} \\ \underline{34} \\ 145 \\ \underline{136} \\ 9 \end{array} \qquad\qquad \text{check:} \quad \begin{array}{rl} 14 & \text{quotient} \\ \underline{\times\ 34} & \text{divisor} \\ 56 & \\ \underline{420} & \\ 476 & \\ \underline{+\ \ 9} & \text{remainder} \\ 485 & \text{dividend} \end{array}$$

The number 485 is the **dividend**, 34 is the **divisor**, 14 is the **quotient** and 9 is the **remainder**. The long division process (or algorithm) stops when a remainder is less than the divisor. The procedure shown above for checking the division result may be expressed as

$$485 = 34 \times 14 + 9$$

or in words as

$$\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}$$

The process of division for polynomials is similar to that for integers. If a polynomial $D(x)$ is a factor of polynomial $P(x)$, then $P(x)$ is divisible by $D(x)$. However, if $D(x)$ is not a factor of $P(x)$ then we can use a **long division algorithm for polynomials** to find a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$. In the same way that the remainder must be less than the divisor when dividing integers, the remainder must be a polynomial of a lower degree than the divisor when dividing polynomials. Consequently, when the divisor is a linear polynomial (degree of 1) the remainder must be of degree 0, i.e. a constant.

• **Hint:** A common error when performing long division with polynomials is to add rather than subtract during each cycle of the process.

Example 14

Find the quotient $Q(x)$ and remainder $R(x)$ when $P(x) = 2x^3 - 5x^2 + 6x - 3$ is divided by $D(x) = x - 2$.

Solution

$$\begin{array}{rlll}
 & 2x^2 - x + 4 & & \\
x - 2 \overline{)} & 2x^3 - 5x^2 + 6x - 3 & & \\
 & \underline{2x^3 - 4x^2} & & \leftarrow 2x^2(x-2) \\
 & \quad - x^2 + 6x & & \leftarrow \text{Subtract} \\
 & \quad \underline{- x^2 + 2x} & & \leftarrow -x(x-2) \\
 & \qquad\quad 4x - 3 & & \leftarrow \text{Subtract} \\
 & \qquad\quad \underline{4x - 8} & & \leftarrow 4(x-2) \\
 & \qquad\qquad\ \ 5 & & \leftarrow \text{Subtract}
\end{array}$$

Thus, the quotient $Q(x)$ is $2x^2 - x + 4$ and the remainder is 5. Therefore, we can write

$$2x^3 - 5x^2 + 6x - 3 = (x - 2)(2x^2 - x + 4) + 5$$

This equation provides a means to check the result by expanding and simplifying the right side and verifying it is equal to the left side.

$$\begin{aligned}
2x^3 - 5x^2 + 6x - 3 &= (x - 2)(2x^2 - x + 4) + 5 \\
&= (2x^3 - x^2 + 4x - 4x^2 + 2x - 8) + 5 \\
&= 2x^3 - 5x^2 + 6x - 3
\end{aligned}$$

Taking the identity $P(x) = D(x) \cdot Q(x) + R(x)$ and dividing both sides by $D(x)$ produces the equivalent identity $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$.

Hence, the result for Example 14 could also be written as

$$\frac{2x^3 - 5x^2 + 6x - 3}{x - 2} = 2x^2 - x + 4 + \frac{5}{x - 2}.$$

Note that writing the result in this manner is the same as rewriting $17 = 5 \times 3 + 2$ as $\frac{17}{5} = 3 + \frac{2}{5}$, which we commonly write as the 'mixed number' $3\frac{2}{5}$.

• **Hint:** When performing long division with polynomials it is necessary to write all polynomials so that the powers (exponents) of the terms are in descending order. Example 12 illustrates that if there are any 'missing' terms then they have a coefficient of zero and a zero must be included in the appropriate location in the division scheme.

Example 15

Divide $f(x) = 4x^3 - 31x - 15$ by $2x + 5$, and use the result to factor $f(x)$ completely.

Solution

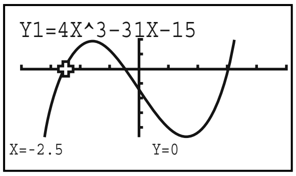
$$\begin{array}{rl}
 & 2x^2 - 5x - 3 \\
2x + 5 \overline{)} & 4x^3 + \ 0x^2 - 31x - 15 \\
 & \underline{4x^3 + 10x^2} \\
 & \quad - 10x^2 - 31x \\
 & \quad \underline{- 10x^2 - 25x} \\
 & \qquad\qquad -6x - 15 \\
 & \qquad\qquad \underline{-6x - 15} \\
 & \qquad\qquad\qquad\quad 0
\end{array}$$

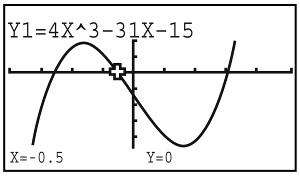
Thus $f(x) = 4x^3 - 31x - 15 = (2x + 5)(2x^2 - 5x - 3)$

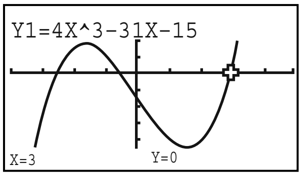
… and factorizing the quadratic quotient (also a factor of $f(x)$), gives

$$\begin{aligned} f(x) = 4x^3 - 31x - 15 &= (2x + 5)(2x^2 - 5x - 3) \\ &= (2x + 5)(2x + 1)(x - 3) \end{aligned}$$

This factorization would lead us to believe that the three zeros of $f(x)$ are $x = -\frac{5}{2}$, $x = -\frac{1}{2}$ and $x = 3$. Graphing $f(x)$ on our GDC and using the 'trace' feature confirms that all three values are zeros of the cubic polynomial.







Division algorithm for polynomials

If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$, and the degree of $D(x)$ is less than or equal to the degree of $P(x)$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$\underset{\text{dividend}}{P(x)} = \underset{\text{divisor}}{D(x)} \cdot \underset{\text{quotient}}{Q(x)} + \underset{\text{remainder}}{R(x)}$$

and where $R(x)$ is either zero or of degree less than the degree of $D(x)$.

Remainder and factor theorems

As illustrated by Examples 14 and 15, we commonly divide polynomials of higher degree by linear polynomials. By doing so we can often uncover zeros of polynomials as occurred in Example 15. Let's look at what happens to the division algorithm when the divisor $D(x)$ is a linear polynomial of the form $x - c$. Since the degree of the remainder $R(x)$ must be less than the degree of the divisor (degree of one in this case) then the remainder will be a constant, simply written as R. Then the division algorithm for a linear divisor is the identity:

$$P(x) = (x - c) \cdot Q(x) + R$$

If we evaluate the polynomial function P at the number $x = c$, we obtain

$$P(c) = (c - c) \cdot Q(c) + R = 0 \cdot Q(c) + R = R$$

Thus the remainder R is equal to $P(c)$, the value of the polynomial P at $x = c$. Because this is true for any polynomial P and any linear divisor $x - c$, we have the following theorem.

The remainder theorem

If a polynomial function $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Example 16

What is the remainder when $g(x) = 2x^3 + 5x^2 - 8x + 3$ is divided by $x + 4$?

Solution

The linear polynomial $x + 4$ is equivalent to $x - (-4)$. Applying the remainder theorem, the required remainder is equal to the value of $g(-4)$.

$$g(-4) = 2(-4)^3 + 5(-4)^2 - 8(-4) + 3 = 2(-64) + 5(16) + 32 + 3$$
$$= -128 + 80 + 35 = -13$$

Therefore, when the polynomial function $g(x)$ is divided by $x + 4$ the remainder is -13.

Figure 3.11 Connection between synthetic substitution and long division.

$$\begin{array}{r|rrrr} -4 & 2 & 5 & -8 & 3 \\ & & -8 & 12 & -16 \\ \hline & 2 & -3 & 4 & -13 \end{array} = g(-4)$$

$$\begin{array}{r} 2x^2 - 3x + 4 \\ x + 4 \overline{)2x^3 + 5x^2 - 8x + 3} \\ \underline{2x^3 + 8x^2 } \\ -3x^2 - 8x \\ \underline{-3x^2 - 12x } \\ 4x + 3 \\ \underline{4x + 16} \\ -13 \end{array}$$

We found the value of $g(-4)$ in Example 16 by directly substituting -4 into $g(x)$. Alternatively, we could have used the efficient method of synthetic substitution that we developed in Section 3.1 to evaluate $g(-4)$.

We could also have found the remainder by performing long division, which is certainly the least efficient method. However, there is a very interesting and helpful connection between the process of long division with a linear divisor and synthetic substitution.

Not only does synthetic substitution find the value of the remainder, but the numbers in the bottom row preceding the remainder (shown in red in Figure 3.11) are the same as the coefficients of the quotient (also in red) found from the long division process. Clearly, synthetic substitution is the most efficient method for finding the remainder *and* quotient when dividing a polynomial by a linear polynomial in the form $x - c$. When this method is used to find a quotient and remainder we refer to it as **synthetic division**.

The numbers in the last row of the synthetic substitution process give both the remainder and the coefficients of the quotient when a polynomial is divided by a linear polynomial in the form $x - c$.

A consequence of the remainder theorem is the factor theorem, which also follows intuitively from our discussion in the previous section about the zeros and factors of quadratic functions. It formalizes the relationship between zeros and linear factors of all polynomial functions with real coefficients.

The factor theorem
A polynomial function $P(x)$ has a factor $x - c$ if and only if $P(c) = 0$.

It is important to understand that the factor theorem is a **biconditional** statement of the form 'A if and only if B'. Such a statement is true in either 'direction'; that is, 'If A then B', and also 'If B then A' – usually abbreviated $A \rightarrow B$ and $B \rightarrow A$, respectively.

To illustrate the efficiency of synthetic division, let's answer the same problem posed in Example 14 (solution reproduced in Figure 3.12) in Example 17.

Example 17

Find the quotient $Q(x)$ and remainder $R(x)$ when $P(x) = 2x^3 - 5x^2 + 6x - 3$ is divided by $D(x) = x - 2$.

Solution

Using synthetic division

$$
\begin{array}{r|rrrr}
2 & 2 & -5 & 6 & -3 \\
 & & 4 & -2 & 8 \\
\hline
 & \underbrace{2 \quad -1 \quad 4}_{\text{coefficients of the quotient}} & & & 5 \leftarrow \text{remainder}
\end{array}
$$

$$
\begin{array}{rll}
 & 2x^2 - x + 4 & \\
x - 2 \,\big) & 2x^3 - 5x^2 + 6x - 3 & \\
 & \underline{2x^3 - 4x^2} & \leftarrow 2x^2(x-2) \\
 & -x^2 + 6x & \leftarrow \text{Subtract} \\
 & \underline{-x^2 + 2x} & \leftarrow -x(x-2) \\
 & 4x - 3 & \leftarrow \text{Subtract} \\
 & \underline{4x - 8} & \leftarrow 4(x-2) \\
 & 5 & \leftarrow \text{Subtract}
\end{array}
$$

The quotient $Q(x)$ is $2x^2 - x + 4$ and the remainder is 5.

Figure 3.12 Solution for Example 14.

Since a divisor of degree 1 is dividing a polynomial of degree 3 then the quotient must be of degree 2 and, with all polynomials written so that their terms are descending in powers (exponents), we know that the numbers in the bottom row of the synthetic division scheme are the coefficients of a quadratic polynomial. Hence, the quotient is $2x^2 - x + 4$ and the remainder is 5.

When one or more zeros of a given polynomial are known, applying the factor theorem and synthetic division is a very effective strategy to aid in finding factors and zeros of the polynomial.

Example 18

Given that $x = -\frac{1}{2}$ and $x = 8$ are zeros of the polynomial function $h(x) = x^4 - \frac{15}{2}x^3 - 30x - 16$, find the other two zeros of $h(x)$.

Solution

From the factor theorem, it follows that $x + \frac{1}{2}$ and $x - 8$ are factors of $h(x)$. Dividing the 4th degree polynomial by the two linear factors in succession will yield a quadratic factor. We can find the zeros of this quadratic factor by using known factorizing techniques or by applying the quadratic formula.

$$
\begin{array}{r|rrrrr}
-\frac{1}{2} & 1 & -\frac{15}{2} & 0 & -30 & -16 \\
 & & -\frac{1}{2} & 4 & -2 & 16 \\
\hline
8 & 1 & -8 & 4 & -32 & 0 \\
 & & 8 & 0 & 32 & \\
\hline
 & 1 & 0 & 4 & 0 &
\end{array}
$$

This row shows that $x^4 - \frac{15}{2}x^3 - 30x - 16 = (x + \frac{1}{2})(x^3 - 8x^2 + 4x - 32)$.

This row shows that $x^3 - 8x^2 + 4x - 32 = (x - 8)(x^2 + 4)$.

• Hint: Example 18 indicates that if we divide the quartic polynomial $x^4 - \frac{15}{2}x^3 - 30x - 16$ by $x^2 + 4$ the remainder will be zero, since $x^2 + 4$ is a factor. Synthetic division *only* works for linear divisors of the form $x - c$ so this division could only be done by using the long division process.

Hence, $x^4 - \frac{15}{2}x^3 - 30x - 16 = (x + \frac{1}{2})(x - 8)(x^2 + 4)$.

The zeros of the quadratic factor $x^2 + 4$ must also be zeros of $h(x)$.

$$x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \pm\sqrt{-4} \Rightarrow x = \pm\sqrt{4}\sqrt{-1} \Rightarrow x = \pm 2i$$

Therefore, the other two remaining zeros of $h(x)$ are $x = 2i$ and $x = -2i$.

Note that the two imaginary zeros, $x = 2i$ and $x = -2i$, of the polynomial in Example 18 are a pair of conjugates. In the previous section we asserted that imaginary zeros of a quadratic polynomial always come in conjugate pairs. Although it is beyond the scope of this book to prove it, we will accept that this is true for imaginary zeros of any polynomial.

Conjugate zeros

If a polynomial P has real coefficients, and if the complex number $z = a + bi$ is a zero of P, then its conjugate $z^* = a - bi$ is also a zero of P.

Example 19

Given that $2 - 3i$ is a zero of the polynomial $5x^3 - 19x^2 + 61x + 13$, find all remaining zeros of the polynomial.

Solution

Firstly, we need to consider what is the maximum number of zeros that the cubic polynomial can have. In the previous section we stated that every quadratic polynomial has exactly two complex zeros. It is reasonable to conjecture that a cubic will have three complex zeros. Since $2 - 3i$ is a zero, then $2 + 3i$ must also be a zero; and the third zero must be a real number. Although not explicitly stated in the remainder and factor theorems, both theorems are true for linear polynomials $x - c$ where the number c is real *or* imaginary, i.e. it can be any complex number. Therefore, the cubic polynomial has factors $x - (2 - 3i)$ and $x - (2 + 3i)$. Rather than attempting to divide the cubic polynomial by one of these factors, let's find the product of these factors and use it as a divisor.

$$\begin{aligned}[x - (2 - 3i)][x - (2 + 3i)] &= [x - 2 + 3i][x - 2 - 3i] \\ &= [(x - 2) + 3i][(x - 2) - 3i] \\ &= (x - 2)^2 - (3i)^2 \\ &= x^2 - 4x + 4 - 9i^2 \\ &= x^2 - 4x + 4 + 9 \\ &= x^2 - 4x + 13\end{aligned}$$

We can only use synthetic division with linear divisors, so we will need to divide $5x^3 - 19x^2 + 61x + 13$ by $x^2 - 4x + 13$ using long division.

$$\begin{array}{r} 5x + 1 \phantom{{}+ 61x + 13} \\ x^2 - 4x + 13\overline{)5x^3 - 19x^2 + 61x + 13} \\ \underline{5x^3 - 20x^2 + 65x} \phantom{{}+ 13} \\ x^2 - 4x + 13 \\ \underline{x^2 - 4x + 13} \\ 0 \end{array}$$

Thus, $5x^3 - 19x^2 + 61x + 13$ also has a linear factor of $5x + 1$ and therefore has a zero of $x = -\frac{1}{5}$.

The zeros of the cubic polynomial are:

$x = 2 - 3i$, $x = 2 + 3i$ and $x = -\frac{1}{5}$.

The cubic polynomial in Example 19 had three complex zeros – one real and two imaginary. The quartic polynomial in Example 18 had four complex zeros – two real and two imaginary. In Example 15, we factored a cubic polynomial into a product of three linear polynomials, so the factor theorem says it will have three real zeros. And in the previous section we concluded that, provided we take into account the multiplicity of a zero (e.g. double root), all quadratic polynomials have two complex zeros – either two real zeros or two imaginary zeros. These examples are illustrations of the following useful fact.

• **Hint:** Although for this course we restrict our study to polynomials with real coefficients, it is worthwhile to note that the statement about the number of complex zeros that exist for a polynomial of degree n also holds true for a polynomial with imaginary coefficients. For example, the 2nd degree polynomial $2ix^2 + 4$ has zeros of $1 + i$ and $-1 - i$ (verify this). Note that these two imaginary zeros are not conjugates. Only if a polynomial's coefficients are real must its imaginary zeros occur in conjugate pairs.

Zeros of polynomials of degree *n*

A polynomial of degree $n > 0$ with complex coefficients has exactly n complex zeros, provided that each zero is counted as many times as its multiplicity.

Since imaginary zeros always exist in conjugate pairs then if a polynomial with real coefficients has any imaginary zeros there can only be an even number of them. It logically follows then that a polynomial with an odd degree has at least one real zero. One consequence of this fact is that the graph of an odd-degree polynomial function must intersect the x-axis at least once. This agrees with our claim in Section 3.1 that the end behaviour of a polynomial function is influenced by its degree. Odd-degree polynomial functions will rise as $x \to \infty$ and fall as $x \to -\infty$ (or the other way around if the leading coefficient is negative) producing the same general ∿ shape as $y = x^3$, and hence will cross the x-axis at least once.

Example 20

Given that $2x + 1$ is a factor of the cubic function $f(x) = 2x^3 - 15x^2 + 24x + 16$

a) completely factorize the polynomial

b) find all of the zeros and their multiplicities

c) sketch its graph for the interval $-1 \leqslant x \leqslant 6$, given that the graph of the function has a turning point at $x = 1$

Solution

a) Remember that synthetic division can only be used for linear divisors of the form $x - c$. Because $2x + 1 = 2\left(x + \frac{1}{2}\right)$, then if $2x + 1$ is a factor $x + \frac{1}{2}$ is also a factor. So we can set up synthetic division with a divisor of $x + \frac{1}{2}$, but we must take the following into account.

$$\begin{aligned} 2x^3 - 15x^2 + 24x + 16 &= (2x + 1)\cdot Q(x) \\ &= 2\left(x + \tfrac{1}{2}\right)\cdot Q(x) \\ &= \left(x + \tfrac{1}{2}\right)\cdot 2Q(x) \end{aligned}$$

$$\frac{2x^3 - 15x^2 + 24x + 16}{x + \frac{1}{2}} = 2Q(x)$$

When the polynomial is divided by $x + \frac{1}{2}$, the quotient will be two times the quotient from dividing by $2x + 1$. Dividing by two will give us the quotient that we want.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -15 & 24 & 16 \\ & & -1 & 8 & -16 \\ \hline & 2 & -16 & 32 & 0 \end{array}$$

Hence, $2x^3 - 15x^2 + 24x + 16 = \left(x + \frac{1}{2}\right)(2x^2 - 16x + 32)$

and $2x^3 - 15x^2 + 24x + 16 = 2\left(x + \frac{1}{2}\right)\frac{1}{2}(2x^2 - 16x + 32)$

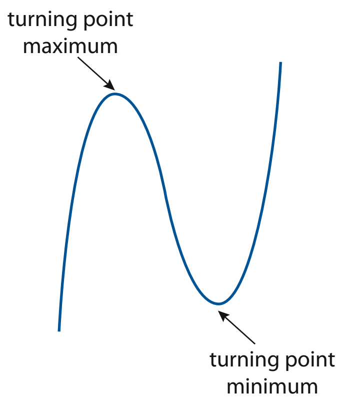
$= (2x + 1)(x^2 - 8x + 16)$ Factorize the quadratic factor.

$= (2x + 1)(x - 4)(x - 4)$ $x^2 - 8x + 16$ fits the pattern $x^2 + 2ax + a^2 = (x + a)^2$

$= (2x + 1)(x - 4)^2$

b) The zeros of $2x^3 - 15x^2 + 24x + 16$ are $x - \frac{1}{2}$ and $x = 4$ (multiplicity of two).

c) Because the polynomial is of degree 3 and its leading coefficient is positive, the end behaviour of the graph will be such that the graph rises as $x \to \infty$ and falls as $x \to -\infty$. That means the general shape of the graph will be a $\mathcal{N}$ shape with one maximum and one minimum as shown right.

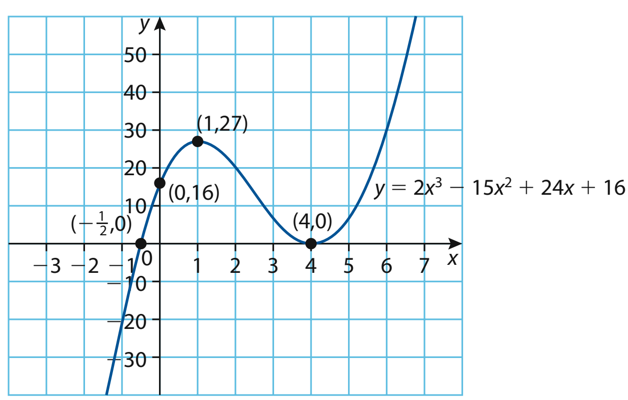


Find the coordinates of the given turning point by evaluating $f(1)$ using synthetic substitution.

$$\begin{array}{r|rrrr} 1 & 2 & -15 & 24 & 16 \\ & & 2 & -13 & 11 \\ \hline & 2 & -13 & 11 & 27 \end{array}$$

$\Rightarrow f(1) = 27$. Hence, the point $(1, 27)$ is on the graph.

Since $f(0) = 16$ then the y-intercept is $(0, 16)$, which means that $(1, 27)$ is a maximum point. Because the zero $x = 4$ has a multiplicity of two, then we know from the previous chapter on quadratic functions that the graph will be tangent to the x-axis at the point $(4, 0)$. The other x-intercept is $\left(-\frac{1}{2}, 0\right)$. We can now make a very accurate sketch of the function.



We know how to find the exact zeros of linear and quadratic functions. The quadratic formula is a general rule that gives the *exact* values of *all* complex zeros of *any* quadratic polynomial using radicals and the coefficients of the polynomial. We also know how to use our GDC to approximate real zeros. In this chapter, we have gained techniques to search for, or verify, the zeros of polynomial functions of degree 3 or higher. This leads us to an important question: Can we find exact values of all complex zeros of any polynomial function of 3rd degree and higher? This question was answered for cubic and quartic polynomials in the 16th century when the Italian mathematician Girolamo Cardano (1501–1576) presented a 'cubic formula' and a 'quartic formula'. These formulae were methods for finding all complex zeros of 3rd degree and 4th degree polynomials using only radicals and coefficients. Cardano's presentation of the formulae depended heavily on the work of other Italian mathematicians. Scipione del Ferro (1465–1526) is given credit as the first to find a general algebraic solution to cubic equations. Cardano's method of solving any cubic was obtained from Niccolo Fontana (1500–1557) known as 'Tartaglia'. Similarly, Cardano solved quartic equations using a method that he learned from his own student Lodovico Ferrari (1522–1565). The methods for solving cubic and quartic equations are quite complicated and are not part of this course. The question of finding formulae for exact zeros of polynomials of degree 5 (quintic) and higher was not resolved until the early 19th century. In 1824, a young Norwegian mathematician, Niels Henrik Abel (1802–1829), proved that it was impossible to find an algebraic formula for a general quintic equation. An even more remarkable discovery was made by the French mathematician Evariste Galois (1811–1832) who died in a pistol duel before turning 21. Galois proved that for any polynomial of degree 5 or greater, it is not possible, except in special cases, to find the exact zeros by using only radicals and the polynomial's coefficients. Mathematicians have developed sophisticated methods of approximating the zeros of polynomial equations of high degree and other types of equations for which there are no algebraic solution methods. These are studied in a branch of advanced mathematics called **numerical analysis**.

There is a theorem called the **fundamental theorem of algebra** that guarantees that *every* polynomial function of non-zero degree with complex coefficients has at least one complex zero. The theorem was first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855). Many of the results in this section on the zeros of polynomials are directly connected with this important theorem.

Example 21

Find a polynomial P with integer coefficients of least degree having zeros of $x = 2$, $x = -\frac{1}{3}$ and $x = 1 - i$.

Solution

Given that $1 - i$ is a zero then its conjugate $1 + i$ must also be a zero. Thus, the required polynomial has four complex zeros, and four corresponding factors. The four factors are:

$x - 2$, $x + \frac{1}{3}$, $x - (1 - i)$ and $x - (1 + i)$

$$P(x) = (x-2)\left(x+\tfrac{1}{3}\right)[x-(1-i)][x-(1+i)]$$

$$= \left(x^2 - \tfrac{5}{3}x - \tfrac{2}{3}\right)[(x-1)+i][(x-1)-i]$$ Multiplying by 3 does not change the zeros …

$$= (3x^2 - 5x - 2)[(x-1)^2 - i^2]$$ … but does guarantee integer coefficients.

$$= (3x^2 - 5x - 2)(x^2 - 2x + 1 + 1)$$

$$= (3x^2 - 5x - 2)(x^2 - 2x + 2)$$

$$= 3x^4 - 6x^3 + 6x^2 - 5x^3 + 10x^2 - 10x - 2x^2 + 4x - 4$$

$$P(x) = 3x^4 - 11x^3 + 14x^2 - 6x - 4$$

Sum and product of the roots of any polynomial equation

In the previous section, we found a way to express the sum and product of the roots of a quadratic equation, $ax^2 + bx + c = 0$, in terms of a, b and c. It is natural to wonder whether a similar method could be found for polynomial equations of degree greater than two.

Using the same approach as in the previous section for quadratic equations, let's consider the general cubic equation $ax^3 + bx^2 + cx + d = 0$ whose roots are $x = \alpha$, $x = \beta$ and $x = \gamma$. It follows that this general cubic equation can be written in the form $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$. Applying the Factor Theorem, it can also be written in the form $(x - \alpha)(x - \beta)(x - \gamma) = 0$. Expanding the brackets gives:

$$\begin{aligned}(x-\alpha)(x-\beta)(x-\gamma) &= x^3 - \alpha x^2 - \beta x^2 - \gamma x^2 + \alpha\beta x + \beta\gamma x + \alpha\gamma x \\ &\quad - \alpha\beta\gamma \\ &= 0\end{aligned}$$

$$x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$$

Equating coefficients for $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ and $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$ gives us the following results for the sum and product of the roots for any cubic equation.

$$\alpha + \beta + \gamma = -\frac{b}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}$$

This result for the sum and product of the roots of any cubic equation looks very similar to that for any quadratic equation. The only difference is that the product of the roots, $\alpha\beta\gamma$, is the opposite of the quotient $\dfrac{\text{constant term}}{\text{leading coefficient}}$.

For the general quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α, β, γ and δ, the factored form of the equation expands as follows:

$$\begin{aligned}&(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = \\ &x^4 - (\alpha+\beta+\gamma+\delta)x^2 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x - \\ &(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) + \alpha\beta\gamma\delta = 0\end{aligned}$$

Since this is equivalent to $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$, then the sum and product of the roots for any quartic equation are:

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \text{ and } \alpha\beta\gamma\delta = \frac{e}{a}.$$

These results for the sum and product of roots for polynomial equations of degree 2 (quadratic), degree 3 (cubic) and degree 4 (quartic) lead to the following result for any polynomial function of degree n that we state without a formal proof.

Sum and product of the roots (zeros) of any polynomial equation

For the **polynomial equation of degree *n*** given by $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$, $a_n \neq 0$ the **sum of the roots** is $-\frac{a_{n-1}}{a_n}$ and the **product of the roots** is $\frac{(-1)^n a_0}{a_n}$.

Example 22

Two of the roots of the equation $x^3 - 3x^2 + kx + 75 = 0$ are opposites. Find the values of all the roots and the constant k.

Solution

Let the three unknown roots be represented by α, $-\alpha$ and β.

Then $\alpha - \alpha + \beta = 3 \Rightarrow \beta = 3$ and $\alpha(-\alpha)\beta = -75 \Rightarrow \alpha(-\alpha)(3) = -75 \Rightarrow -3\alpha^2 = -75 \Rightarrow \alpha^2 = 25 \Rightarrow \alpha = \pm 5$

Therefore, the three roots are 5, -5 and 3.

To find the value of k, write the cubic in factored form and expand.

$$\begin{aligned}(x-3)(x+5)(x-5) = 0 &\Rightarrow (x-3)(x^2-25) = 0\\ &\Rightarrow x^3 - 3x^2 - 25x + 75 = 0\end{aligned}$$

Therefore, $k = -25$.

Example 23

Consider the equation $2x^4 - x^3 - 4x^2 + 10x - 4 = 0$. Given that one of the zeros of the equation is $r_1 = 1 + i$, find the other three zeros r_2, r_3 and r_4.

Solution

There are other strategies (e.g. using factors and polynomial division) but it is more efficient to apply what we know about the sum and product of the roots (zeros) of a polynomial equation.

Firstly, since $r_1 = 1 + i$ is a zero, then its conjugate must also be a zero; hence $r_2 = 1 - i$.

From the fact that the sum of the roots is $-\frac{a_{n-1}}{a_n}$, then $r_1 + r_2 + r_3 + r_4 = -\frac{a_3}{a_4}$.

Substituting in known values gives $1 + i + 1 - i + r_3 + r_4 = -\frac{-1}{2}$

$\Rightarrow 2 + r_3 + r_4 = \frac{1}{2} \Rightarrow r_3 + r_4 = -\frac{3}{2}$

Also, since the product of the roots is $\frac{(-1)^n a_0}{a_n}$, then $r_1r_2r_3r_4 = \frac{(-1)^n a_0}{a_n}$.

Substituting gives:

$$\begin{aligned}(1+i)(1-i)r_3r_4 = \frac{(-1)^4(-4)}{2} &\Rightarrow (1-i^2)r_3r_4 = -2\\ &\Rightarrow 2r_3r_4 = -2\\ &\Rightarrow r_3r_4 = -1\end{aligned}$$

To find r_3 and r_3, we need to use the pair of equations $\begin{cases} r_3 + r_4 = -\frac{3}{2} \\ r_3r_4 = -1 \end{cases}$

Solving for r_3 in the first equation gives $r_3 = -r_4 - \frac{3}{2}$.

Substituting into the other equation gives: $\left(-r_4 - \frac{3}{2}\right)r_4 = -1$

$$\begin{aligned}&\Rightarrow r_4^2 + \frac{3}{2}r_4 - 1 = 0\\ &\Rightarrow 2r_4^2 + 3r_4 - 2 = 0\\ &\Rightarrow (2r_4 - 1)(r_4 + 2) = 0\\ &\Rightarrow r_4 = \frac{1}{2} \text{ or } r_4 = -2\end{aligned}$$

If $r_4 = \frac{1}{2}$, then $r_3 = -\frac{1}{2} - \frac{3}{2} = -2$. $\left[\text{And if } r_4 = -2, \text{ then } r_3 = \frac{1}{2}\right]$
Therefore the other three zeros are $1 - i, \frac{1}{2}$ and -2.

Exercise 3.3

In questions 1–5, two polynomials P and D are given. Use either synthetic division or long division to divide $P(x)$ by $D(x)$, and express $P(x)$ in the form
$P(x) = D(x) \cdot Q(x) + R(x)$.

1 $P(x) = 3x^2 + 5x - 5,\ D(x) = x + 3$

2 $P(x) = 3x^4 - 8x^3 + 9x + 5,\ D(x) = x - 2$

3 $P(x) = x^3 - 5x^2 + 3x - 7,\ D(x) = x - 4$

4 $P(x) = 9x^3 + 12x^2 - 5x + 1,\ D(x) = 3x - 1$

5 $P(x) = x^5 + x^4 - 8x^3 + x + 2,\ D(x) = x^2 + x - 7$

6 Given that $x - 1$ is a factor of the function $f(x) = 2x^3 - 17x^2 + 22x - 7$ factorize f completely.

7 Given that $2x + 1$ is a factor of the function $f(x) = 6x^3 - 5x^2 - 12x - 4$ factorize f completely.

8 Given that $x + \frac{2}{3}$ is a factor of the function $f(x) = 3x^4 + 2x^3 - 36x^2 + 24x + 32$ factorize f completely.

In questions 9–12, find the quotient and the remainder.

9 $\dfrac{x^2 - 5x + 4}{x - 3}$

10 $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$

11 $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

12 $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

In questions 13–16, use synthetic division and the remainder theorem to evaluate $P(c)$.

13 $P(x) = 2x^3 - 3x^2 + 4x - 7, c = 2$

14 $P(x) = x^5 - 2x^4 + 3x^2 + 20x + 3, c = -1$

15 $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, c = -7$

16 $P(x) = x^3 - x + 1, c = \frac{1}{4}$

17 Given that $x = -6$ is a zero of the polynomial $x^3 + 2x^2 - 19x + 30$ find all remaining zeros of the polynomial.

18 Given that $x = 2$ is a double root of the polynomial $x^4 - 5x^3 + 7x^2 - 4$ find all remaining zeros of the polynomial.

19 Find the values of k such that -3 is a zero of $f(x) = x^3 - x^2 - k^2x$.

20 Find the values of a and b such that 1 and 4 are zeros of $f(x) = 2x^4 - 5x^3 - 14x^2 + ax + b$.

In questions 21–23, find a polynomial with real coefficients satisfying the given conditions.

21 Degree of 3; and zeros of -2, 1 and 4

22 Degree of 4; and zeros of -1, 3 (multiplicity of 2) and -2

23 Degree of 3; and 2 is the only zero (multiplicity of 3)

In questions 24–26, find a polynomial of lowest degree with real coefficients and the given zeros.

24 $x = -1$ and $x = 1 - i$

25 $x = 2, x = -4$ and $x = -3i$

26 $x = 3 + i$ and $x = 1 - 2i$

27 Given that $x = 2 - 3i$ is a zero of $f(x) = x^3 - 7x^2 + 25x - 39$ find the other remaining zeros.

28 The polynomial $6x^3 + 7x^2 + ax + b$ has a remainder of 72 when divided by $x - 2$ and is exactly divisible (i.e. remainder is zero) by $x + 1$.

a) Calculate a and b.

b) Show that $2x - 1$ is also a factor of the polynomial and, hence, find the third factor.

29 The polynomial $p(x) = (ax + b)^3$ leaves a remainder of -1 when divided by $x + 1$, and a remainder of 27 when divided by $x - 2$. Find the values of the real numbers a and b.

30 The quadratic polynomial $x^2 - 2x - 3$ is a factor of the quartic polynomial function $f(x) = 4x^4 - 6x^3 - 15x^2 - 8x - 3$. Find all of the zeros of the function f. Express the zeros exactly and completely simplified.

31 $x - 2$ and $x + 2$ are factors of $x^3 + ax^2 + bx + c$, and it leaves a remainder of 10 when divided by $x - 3$. Find the values of a, b and c.

32 Let $P(x) = x^3 + px^2 + qx + r$. Two of the zeros of $P(x) = 0$ are 3 and $1 + 4i$. Find the value of p, q and r.

33 When divided by $(x + 2)$ the expression $5x^3 - 3x^2 + ax + 7$ leaves a remainder of R. When the expression $4x^3 + ax^2 + 7x - 4$ is divided by $(x + 2)$ there is a remainder of $2R$. Find the value of the constant a.

34 The polynomial $x^3 + mx^2 + nx - 8$ is divisible by $(x + 1 + i)$. Find the value of m and n.

35 Given that the roots of the equation $x^3 - 9x^2 + bx - 216 = 0$ are consecutive terms in a geometric sequence, find the value of b and solve the equation.

36 a) Prove that when a polynomial $P(x)$ is divided by $ax - b$ the remainder is $P\left(\frac{b}{a}\right)$.

b) Hence, find the remainder when $9x^3 - x + 5$ is divided by $3x + 2$.

37 Find the sum and product of the roots of the following equations.

a) $x^4 - \frac{2}{3}x^3 + 3x^2 - 2x + 5 = 0$

b) $(x - 2)^3 = x^4 - 1$

c) $\dfrac{3}{x^2 + 2} = \dfrac{2x^2 - x}{2x^5 + 1}$

38 If α, β and γ are the three roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, show that $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$.

39 One of the zeros of the equation $x^3 - 63x + 162 = 0$ is double another zero. Find all three zeros.

40 Find the three zeros of the equation $x^3 - 6x^2 - 24x + 64 = 0$ given that they are consecutive terms in a geometric sequence. [Hint: let the zeros be represented by $\frac{\alpha}{r}$, α, αr where r is the common ratio.]

41 Consider the equation $x^5 - 12x^4 + 62x^3 - 166x^2 + 229x - 130 = 0$. Given that two of the zeros of the equation are $x = 3 - 2i$ and $x = 2$, find the remaining three zeros.

42 Find the value of k such that the zeros of the equation $x^3 - 6x^2 + kx + 10 = 0$ are in arithmetic progression, that is, they can be represented by α, $\alpha + d$ and $\alpha + 2d$ for some constant d. [Hint: use the result from question 38.]

43 Find the value of k if the roots of the equation $x^3 + 3x^2 - 6x + k = 0$ are in geometric progression.

3.4 Rational functions

Another important category of algebraic functions is rational functions, which are functions in the form $R(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials and the domain of the function R is the set of all real numbers except the real zeros of polynomial g in the denominator. Some examples of rational functions are

$$p(x) = \frac{1}{x-5}, \quad q(x) = \frac{x+2}{(x+3)(x-1)}, \quad \text{and} \quad r(x) = \frac{x}{x^2+1}$$

The domain of p excludes $x = 5$, and the domain of q excludes $x = -3$ and $x = 1$. The domain of r is all real numbers because the polynomial $x^2 + 1$ has no real zeros.

Example 24

Find the domain and range of $h(x) = \frac{1}{x-2}$. Sketch the graph of h.

Solution

Because the denominator is zero when $x = 2$, the domain of h is all real numbers except $x = 2$, i.e. $x \in \mathbb{R}, x \neq 2$. Determining the range of the function is a little less straightforward. It is clear that the function could never take on a value of zero because that will only occur if the numerator is zero. And since the denominator can have any value except zero it seems that the function values of h could be any real number except zero. To confirm this and to determine the behaviour of the function (and shape of the graph), some values of the domain and range (pairs of coordinates) are displayed in the tables below.

• **Hint:** A fraction is only zero if its numerator is zero.

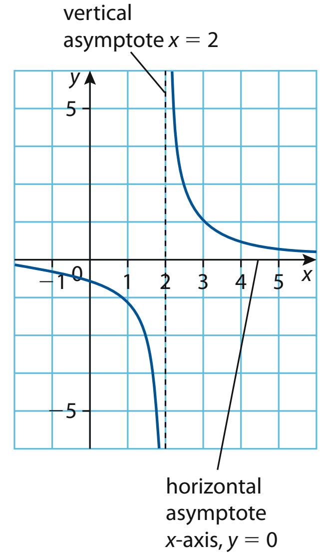
x approaches 2 from the left

x	$h(x)$
−98	−0.01
−8	−0.1
0	−0.5
1	−1
1.5	−2
1.9	−10
1.99	−100
1.999	−1000

x approaches 2 from the right

x	$h(x)$
102	0.01
12	0.1
4	0.5
3	1
2.5	2
2.1	10
2.01	100
2.001	1000

The values in the tables provide clear evidence that the range of h is all real numbers except zero, i.e. $h(x) \in \mathbb{R}, h(x) \neq 0$. The values in the tables also show that as $x \to -\infty$, $h(x) \to 0$ from below (sometimes written $h(x) \to 0^-$) and as $x \to +\infty$, $h(x) \to 0$ from above ($h(x) \to 0^+$). It follows that the line with equation $y = 0$ (the x-axis) is a horizontal asymptote for the graph of h. As $x \to 2$ from the left (sometimes written $x \to 2^-$), $h(x)$ appears to decrease without bound, whereas as $x \to 2$ from the right $(x \to 2^+)$, $h(x)$ appears to increase without bound. This indicates that the graph of h will have a vertical asymptote at $x = 2$. This behaviour is confirmed by the graph at left.



Horizontal and vertical asymptotes

The line $y = c$ is a **horizontal asymptote** of the graph of the function f if at least one of the following statements is true:

- as $x \to +\infty$, then $f(x) \to c^+$
- as $x \to -\infty$, then $f(x) \to c^+$
- as $x \to +\infty$, then $f(x) \to c^-$
- as $x \to -\infty$, then $f(x) \to c^-$

The line $x = d$ is a **vertical asymptote** of the graph of the function f if at least one of the following statements is true:

- as $x \to d^+$, then $f(x) \to +\infty$
- as $x \to d^+$, then $f(x) \to -\infty$
- as $x \to d^-$, then $f(x) \to +\infty$
- as $x \to d^-$, then $f(x) \to -\infty$

Example 25

Consider the function $f(x) = \dfrac{3x^2 - 12}{x^2 + 3x - 4}$. Sketch the graph of f and identify any asymptotes and any x- or y-intercepts. Use the sketch to confirm the domain and range of the function.

Solution

Firstly, let's completely factorize both the numerator and denominator.

$$f(x) = \frac{3x^2 - 12}{x^2 + 3x - 4} = \frac{3(x+2)(x-2)}{(x-1)(x+4)}$$

Axis intercepts:

The x-intercepts will occur where the numerator is zero. Hence, the x-intercepts are $(-2, 0)$ and $(2, 0)$. A y-intercept will occur when $x = 0$.

$f(0) = \dfrac{3(2)(-2)}{(-1)(4)} = 3$, so the y-intercept is $(0, 3)$.

Vertical asymptote(s):

Any vertical asymptote will occur where the denominator is zero, that is, where the function is undefined. From the factored form of f we see that the vertical asymptotes are $x = 1$ and $x = -4$. We need to determine if the graph of f falls $(f(x) \to -\infty)$ or rises $(f(x) \to \infty)$ on either side of each vertical asymptote. It's easiest to do this by simply analyzing what the sign of h will be as x approaches 1 and -4 from both the left and right. For example, as $x \to 1^-$ we can use a test value close to and to the left of 1 (e.g. $x = 0.9$) to check whether $f(x)$ is positive or negative to the left of 1.

$$f(x) = \frac{3(0.9+2)(0.9-2)}{(0.9-1)(0.9+4)} \Rightarrow \frac{(+)(-)}{(-)(+)} \Rightarrow f(x) > 0 \Rightarrow \text{as } x \to 1^-,$$

then $f(x) \to +\infty$ (rises)

As $x \to 1^+$ we use a test value close to and to the right of 1 (e.g. $x = 1.1$) to check whether $f(x)$ is positive or negative to the right of 1.

● **Hint:** The farther the number n is from 0, the closer the number $\frac{1}{n}$ is to 0. Conversely, the closer the number n is to 0, the farther the number $\frac{1}{n}$ is from 0. These facts can be expressed simply as:

$$\frac{1}{\text{BIG}} = \text{little and } \frac{1}{\text{little}} = \text{BIG}$$

They can also be expressed more mathematically using the concept of a limit expressed in limit notation as: $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to 0}\frac{1}{n} = \infty$.
Note: Infinity is not a number, so $\lim_{n\to 0}\frac{1}{n}$ actually does not exist, but writing $\lim_{n\to 0}\frac{1}{n} = \infty$ expresses the idea that $\frac{1}{n}$ increases without bound as n approaches 0.

$$f(x) = \frac{3(1.1+2)(1.1-2)}{(1.1-1)(1.1+4)} \Rightarrow \frac{(+)(-)}{(+)(+)} \Rightarrow f(x) < 0 \Rightarrow \text{as } x \to 1^+,$$

then $f(x) \to -\infty$ (falls)

Conducting similar analysis for the vertical asymptote of $x = -4$, produces:

$$f(x) = \frac{3(-4.1+2)(-4.1-2)}{(-4.1-1)(-4.1+4)} \Rightarrow \frac{(-)(-)}{(-)(-)} \Rightarrow f(x) > 0 \Rightarrow \text{as } x \to 4^-,$$

then $f(x) \to +\infty$ (rises)

$$f(x) = \frac{3(-3.9+2)(-3.9-2)}{(-3.9-1)(-3.9+4)} \Rightarrow \frac{(-)(-)}{(-)(+)} \Rightarrow f(x) < 0 \Rightarrow \text{as } x \to 4^+,$$

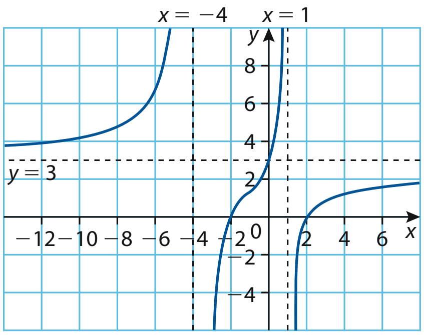
then $f(x) \to -\infty$ (falls)

Horizontal asymptote(s):

A horizontal asymptote (if it exists) is the value that $f(x)$ approaches as $x \to \pm\infty$. To find this value, we divide both the numerator and denominator by the highest power of x that appears in the denominator (x^2 for function f).

$$f(x) = \frac{\frac{3x^2}{x^2} - \frac{12}{x^2}}{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{4}{x^2}} \text{ then, as } x \to \pm\infty, f(x) = \frac{3-0}{1+0-0} = 3$$

Hence, the horizontal asymptote is $y = 3$.



Sketch of graph:

Now we know the behaviour (rising or falling) of the function on either side of each vertical asymptote and that the graph will approach the horizontal asymptote as $x \to \pm\infty$, an accurate sketch of the graph can be made as shown right.

Domain and range:

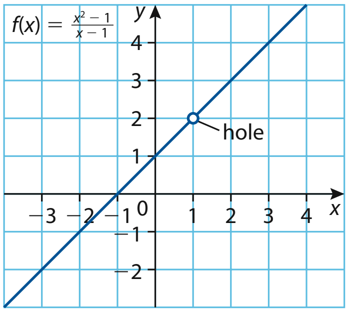
Because the zeros of the polynomial in the denominator are $x = 1$ and $x = -4$, the domain of f is all real numbers except 1 and -4.
From our analysis and from the sketch of the graph, it is clear that between $x = -4$ and $x = 1$ the function takes on all values from $-\infty$ to $+\infty$, therefore the range of f is all real numbers.

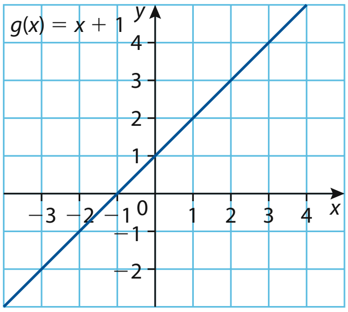
We are in the habit of cancelling factors in algebraic expressions (Section 1.5), such as

$$\frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1$$

However, the function $f(x) = \frac{x^2-1}{x-1}$ and the function $g(x) = x+1$ are **not** the same function. The difference occurs when $x = 1$.

$f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$, which is undefined, and $g(1) = 1 + 1 = 2$. So, 1 is not in the domain of f but it is in the domain of g. As we might expect the graphs of the two functions appear identical, but upon closer inspection it is clear that there is a 'hole' in the graph of f at the point $(1, 2)$. Thus, f is a *discontinuous* function but the polynomial function g is continuous. f and g are different functions.





● **Hint:** Try graphing $\frac{x^2 - 1}{x - 1}$ on your GDC and zooming in closely to the region around the point $(1, 2)$. Can you see the 'hole'?

In working with rational functions, we often assume that every linear factor that appears in both the numerator and in the denominator has been cancelled. Therefore, for a rational function in the form $\frac{f(x)}{g(x)}$, we can usually assume that the polynomial functions f and g have no common factors.

Example 26

Find any asymptotes for the function $p(x) = \frac{x^2 - 9}{x - 4}$.

Solution

The denominator is zero when $x = 4$, thus the line with equation $x = 4$ is a vertical asymptote. Although the numerator $x^2 - 9$ is not divisible by $x - 4$, it does have a larger degree. Some insight into the behaviour of function p may be gained by dividing $x - 4$ into $x^2 - 9$. Since the degree of the numerator is one greater than the degree of the denominator, the quotient will be a linear polynomial. Recalling from the previous section that $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, where Q and R are the quotient and remainder, we can rewrite $p(x)$ as a linear polynomial plus a fraction.

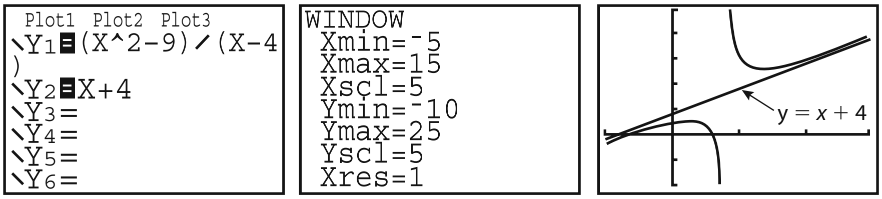
Since the denominator is in the form $x - c$ we can carry out the division efficiently by means of synthetic division.

$$\begin{array}{c|ccc} 4 & 1 & 0 & -9 \\ \hline & & 4 & 16 \\ \hline & 1 & 4 & 7 \end{array}$$

Hence, $p(x) = \frac{x^2 - 9}{x - 4} = x + 4 + \frac{7}{x - 4}$.

As $x \to \pm\infty$, the fraction $\frac{7}{x - 4} \to 0$. This tells us about the end behaviour of function p, namely that the graph of p will get closer and closer to the line $y = x + 4$ as the values of x get further away from the origin. Symbolically, this can be expressed as follows: as $x \to \pm\infty$, $p(x) \to x + 4$.

We can graph both the rational function $p(x)$ and the line $y = x + 4$ on our GDC to visually confirm our analysis.



If a line is an asymptote of a graph but it is neither horizontal nor vertical, it is called an **oblique asymptote** (sometimes called a slant asymptote).

The graph of any rational function of the form $\frac{f(x)}{g(x)}$, where the degree of function f is one more than the degree of function g will have an oblique asymptote.

Using Example 25 as a model, we can set out a general procedure for analyzing a rational function leading to a sketch of its graph and determining its domain and range.

Analyzing a rational function $R(x) = \frac{f(x)}{g(x)}$ given functions f and g have no common factors

1. **Factorize**: Completely factorize both the numerator and denominator.
2. **Intercepts**: A zero of f will be a zero of R and hence an x-intercept of the graph of R. The y-intercept is found by evaluating $R(0)$.
3. **Vertical asymptotes**: A zero of g will give the location of a vertical asymptote (if any). Then perform a sign analysis to see if $R(x) \to +\infty$ or $R(x) \to -\infty$ on either side of each vertical asymptote.
4. **Horizontal asymptote**: Find the horizontal asymptote (if any) by dividing both f and g by the highest power of x that appears in g, and then letting $x \to \pm\infty$.
5. **Oblique asymptotes**: If the degree of f is one more than the degree of g, then the graph of R will have an oblique asymptote. Divide g into f to find the quotient $Q(x)$ and remainder. The oblique asymptote will be the line with equation $y = Q(x)$.
6. **Sketch of graph**: Start by drawing dashed lines where the asymptotes are located. Use the information about the intercepts, whether $Q(x)$ falls or rises on either side of a vertical asymptote, and additional points as needed to make an accurate sketch.
7. **Domain and range**: The domain of R will be all real numbers except the zeros of g. You need to study the graph carefully in order to determine the range. Often, but not always (as in Example 25), the value of the function at the horizontal asymptote will not be included in the range.

End behaviour of a rational function

Let R be the rational function given by

$$R(x) = \frac{f(x)}{g(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

where functions f and g have no common factors. Then the following holds true:

1. If $n < m$, then the x-axis (line $y = 0$) is a horizontal asymptote for the graph of R.
2. If $n = m$, then the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote for the graph of R.
3. If $n > m$, then the graph of R has no horizontal asymptote. However, if the degree of f is one more than the degree of g, then the graph of R will have an oblique asymptote.

Exercise 3.4

In questions 1–10, sketch the graph of the rational function without the aid of your GDC. On your sketch clearly indicate any x- or y-intercepts and any asymptotes (vertical, horizontal or oblique). Use your GDC to verify your sketch.

1 $f(x) = \dfrac{1}{x+2}$

2 $g(x) = \dfrac{3}{x-2}$

3 $h(x) = \dfrac{1-4x}{1-x}$

4 $R(x) = \dfrac{x}{x^2-9}$

5 $p(x) = \dfrac{2}{x^2+2x-3}$

6 $M(x) = \dfrac{x^2+1}{x}$

7 $f(x) = \dfrac{x}{x^2+4x+4}$

8 $h(x) = \dfrac{x^2+2x}{x-1}$

9 $g(x) = \dfrac{2x+8}{x^2-x-12}$

10 $C(x) = \dfrac{x-2}{x^2-4x}$

In questions 11–14, use your GDC to sketch a graph of the function, and state the domain and range of the function.

11 $f(x) = \dfrac{2x^2+5}{x^2-4}$

12 $g(x) = \dfrac{x+4}{x^2+3x-4}$

13 $h(x) = \dfrac{6}{x^2+6}$

14 $r(x) = \dfrac{x^2-2x+1}{x-1}$

In questions 15–18, use your GDC to sketch a graph of the function. Clearly label any x- or y-intercepts and any asymptotes.

15 $f(x) = \dfrac{2x-5}{2x^2+9x-18}$

16 $g(x) = \dfrac{x^2+x+1}{x-1}$

17 $h(x) = \dfrac{3x^2}{x^2+x+2}$

18 $g(x) = \dfrac{1}{x^3-x^2-4x+4}$

19 If a, b and c are all positive, sketch the curve $y = \dfrac{x-a}{(x-b)(x-c)}$ for each of the following conditions:

a) $a < b < c$ b) $b < a < c$ c) $b < c < a$

20 A drug is given to a patient and the concentration of the drug in the bloodstream is carefully monitored. At time $t \geqslant 0$ (in minutes after patient receiving the drug), the concentration, in milligrams per litre (mg/l) is given by the following function.

$$C(t) = \frac{25t}{t^2+4}$$

a) Sketch a graph of the drug concentration (mg/l) versus time (min).

b) When does the highest concentration of the drug occur, and what is it?

c) What eventually happens to the concentration of the drug in the bloodstream?

d) How long does it take for the concentration to drop below 0.5 mg/l?

3.5 Other equations and inequalities

We have studied some approaches to analyzing and solving polynomial equations in this chapter. Some problems lead to equations with expressions that are not polynomials, for example, expressions with radicals, fractions, or absolute value. Problems in mathematics often do not involve equations but inequalities. We need to be familiar with effective methods for solving inequalities involving polynomials – and again, radicals, fractions, or absolute value.

Equations involving a radical

Example 27 – Solving an equation with a single radical expression

Solve for x: $\sqrt{3x+6} = 2x+1$

Solution

Squaring both sides gives $3x + 6 = (2x+1)^2$

$$3x + 6 = 4x^2 + 4x + 1$$
$$4x^2 + x - 5 = 0$$

Factorizing: $(4x+5)(x-1) = 0$

$$x = -\tfrac{5}{4} \text{ or } x = 1$$

Check both solutions in the original equation:

When $x = -\frac{5}{4}$, $\sqrt{3\left(-\frac{5}{4}\right)+6} = 2\left(-\frac{5}{4}\right) + 1 \Rightarrow \sqrt{\frac{9}{4}} = -\frac{3}{2} \Rightarrow \frac{3}{2} \neq -\frac{3}{2}$

Therefore, $x = -\frac{5}{4}$ is *not* a solution.

When $x = 1$, $\sqrt{3(1)+6} = 2(1) + 1 \Rightarrow \sqrt{9} = 3 \Rightarrow 3 = 3$

Therefore, $x = 1$ is the only solution.

If two quantities are equal, for example $a = b$, then it is certainly true that $a^2 = b^2$, and $a^3 = b^3$, etc. However, the converse is not necessarily true. A simple example can illustrate this.

Consider the trivial equation $x = 3$. There is only one value of x that makes the equation true – and that is 3. Now if we take this original equation and square both sides we transform it to the equation $x^2 = 9$. This transformed equation has two solutions, 3 and -3, so it is not equivalent to the original equation. By squaring both sides we gained an extra solution, often called an **extraneous solution**, that satisfies the transformed equation but not the original equation as occurred in Example 27. Whenever you raise both sides of an equation by a power it is imperative that you check all solutions in the original equation.

Every solution of the equation $a = b$ is also a solution of the equation $a^n = b^n$, but it is not necessarily true that every solution of $a^n = b^n$ is a solution of $a = b$.

Example 28 – Solving an equation with two radical expressions

Solve for x in the equation $\sqrt{2x-3} - \sqrt{x+7} = 2$.

Solution

Squaring both sides of the original equation will produce a messy expression on the left side, so it is better to rearrange the terms so that one side of the equation contains only a single radical term.

$$\sqrt{2x-3} = 2 + \sqrt{x+7}$$
$$(\sqrt{2x-3})^2 = (2+\sqrt{x+7})^2$$
$$2x - 3 = 4 + 4\sqrt{x+7} + x + 7$$
$$x - 14 = 4\sqrt{x+7}$$
$$(x-14)^2 = (4\sqrt{x+7})^2$$

Squaring both sides again to eliminate the radical.

$$x^2 - 28x + 196 = 16(x+7)$$
$$x^2 - 44x + 84 = 0$$
$$(x-2)(x-42) = 0$$
$$x = 2 \text{ or } x = 42$$

Check both solutions in the original equation:

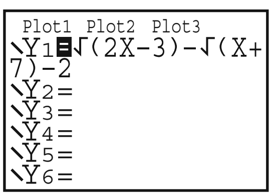
When $x = 2$, $\sqrt{2(2)-3} \stackrel{?}{=} 2 + \sqrt{2+7} \Rightarrow \sqrt{1} \stackrel{?}{=} 2 + \sqrt{9} \Rightarrow 1 \neq 5$

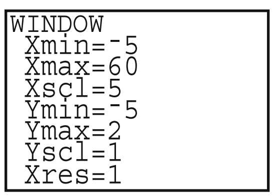
Thus, $x = 2$ is *not* a solution.

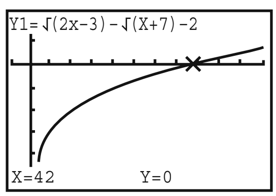
When $x = 42$, $\sqrt{2(42)-3} \stackrel{?}{=} 2 + \sqrt{42+7} \Rightarrow \sqrt{81} \stackrel{?}{=} 2 + \sqrt{49} \Rightarrow 9 = 2 + 7$

Thus, $x = 42$ is a solution.

We can verify the single solution of $x = 42$ using our GDC by graphing the equation $y = \sqrt{2x-3} - \sqrt{x+7} - 2$ and looking for x-intercepts (zeros). Since we are restricted to real number solutions then the smallest possible value for x that can be substituted into the equation is $\frac{3}{2}$. This helps determine a suitable viewing window for the graph on our GDC.







This verifies that $x = 42$ is the only solution to the equivalent equation $\sqrt{2x-3} = 2 + \sqrt{x+7}$.

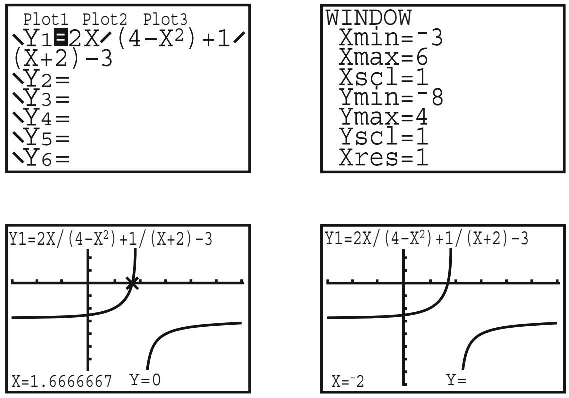
Equations involving fractions

It is also possible for extraneous solutions to appear when solving equations with fractions.

Example 29 – An extraneous root in an equation with fractions

Find all real solutions of the equation $\dfrac{2x}{4-x^2} + \dfrac{1}{x+2} = 3$ and verify solution(s) with a GDC.

Solution



Multiply both sides of the equation by the least common denominator of the fractions, $4 - x^2$.

$$\frac{4 - x^2}{1} \cdot \frac{2x}{4 - x^2} + \frac{(2 - x)(2 + x)}{1} \cdot \frac{1}{x + 2} = 3(4 - x^2)$$

Factorizing $4 - x^2$ gives $(2 - x)(2 + x)$.

$$2x + 2 - x = 12 - 3x^2$$

$$3x^2 + x - 10 = 0$$

$$(3x - 5)(x + 2) = 0$$

$$x = \tfrac{5}{3} \text{ or } x = -2$$

Clearly $x = -2$ cannot be a solution because that would cause division by zero in the original equation.

The GDC images show that the equation $y = \dfrac{2x}{4 - x^2} + \dfrac{1}{x + 2} - 3$ has an x-intercept at $\left(\frac{5}{3}, 0\right)$, confirming the solution $x = \frac{5}{3}$.

● **Hint:** Not only is it possible to *gain* an extraneous solution when solving certain equations, it is also possible to *lose* a correct solution by incorrectly dividing both sides of an equation by a common factor. For example, solve for x in the equaton $4(x + 2)^2 = 3x(x + 2)$. Dividing both sides by $(x + 2)$, gives $4(x + 2) = 3x \Rightarrow 4x + 8 = 3x \Rightarrow x = -8$. However, there are two solutions, $x = -8$ *and* $x = -2$. The solution of $x = -2$ was lost because a factor of $x + 2$ was eliminated from both sides of the original equation. This is a common error to be avoided.

Equations in quadratic form

In Section 3.2 we covered methods of solving quadratic equations. As the three previous examples illustrate, quadratic equations commonly appear in a range of mathematical problems. The methods of solving quadratics can sometimes be applied to other equations. An equation in the form $at^2 + bt + c = 0$, where t is an algebraic expression, is an equation in **quadratic form**. We can solve such equations by substituting for the algebraic expression and then apply an appropriate method for solving a quadratic equation.

Example 30 – A 4th degree polynomial equation in quadratic form

Find all real solutions of the equation $2m^4 - 5m^2 + 2 = 0$.

Solution

The equation can be written as $2(m^2)^2 - 5(m^2) + 2 = 0$ showing it is quadratic in terms of m^2. Let $t = m^2$, and substituting gives $2t^2 - 5t + 2 = 0$. Solve for t, substitute m^2 back in for t, and then solve for m.

$2m^4 - 5m^2 + 2 = 0$

Substitute t for m^2 $\quad 2t^2 - 5t + 2 = 0$

$(2t - 1)(t - 2) = 0$

$t = \frac{1}{2}$ or $t = 2$

Substituting m^2 for t $\quad m^2 = \frac{1}{2}$ or $m^2 = 2$

$m = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$ or $m = \pm\sqrt{2}$

These four solutions – which are two pairs of opposites – can be checked by substituting them directly into the original equation. A value for m will be raised to the 4th and 2nd powers, thus we only need to check one value from each pair of opposites.

When $m = \frac{\sqrt{2}}{2}$, $2\left(\frac{\sqrt{2}}{2}\right)^4 - 5\left(\frac{\sqrt{2}}{2}\right)^2 + 2 = 0 \Rightarrow 2\left(\frac{1}{4}\right) - 5\left(\frac{1}{2}\right) + 2 = 0$

$\Rightarrow \frac{1}{2} - \frac{5}{2} + 2 = 0 \Rightarrow 0 = 0$

When $m = \sqrt{2}$, $2(\sqrt{2})^4 - 5(\sqrt{2})^2 + 2 = 0 \Rightarrow 2(4) - 5(2) + 2 = 0$

$\Rightarrow 8 - 10 + 2 = 0 \Rightarrow 0 = 0$

Therefore, the solutions to the equation are $m = \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \sqrt{2}$ and $-\sqrt{2}$.

Example 31 – Another equation in quadratic form

Find all solutions, expressed exactly, to the equation $w^{\frac{1}{2}} = 4w^{\frac{1}{4}} - 2$.

Solution

$w^{\frac{1}{2}} - 4w^{\frac{1}{4}} + 2 = 0$ Set the equation to zero.

$\left(w^{\frac{1}{4}}\right)^2 - 4\left(w^{\frac{1}{4}}\right) + 2 = 0$ Attempt to write in quadratic form: $at^2 + bt + c = 0$

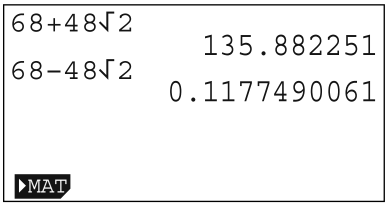
$t^2 - 4t + 2 = 0$ Make appropriate substitution; in this case, let $w^{\frac{1}{4}} = t$.

$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2}$ Trinomial does not factorize; apply quadratic formula.

$t = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2}$

$t = 2 \pm \sqrt{2}$

$w^{\frac{1}{4}} = 2 \pm \sqrt{2}$ Substituting $w^{\frac{1}{4}}$ back in for t; raise both sides to 4th power.



$w = (2 + \sqrt{2})^4$ or $w = (2 - \sqrt{2})^4$

$w = ((2 + \sqrt{2})^2)^2$ or $w = ((2 - \sqrt{2})^2)^2$

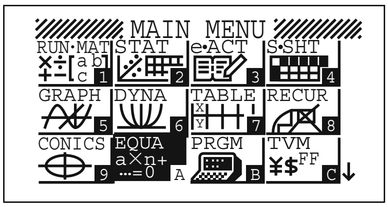
$w = (6 + 4\sqrt{2})^2$ or $w = (6 - 4\sqrt{2})^2$

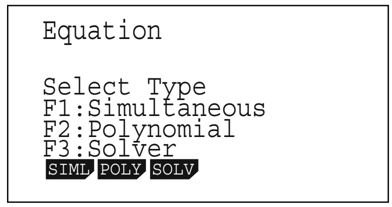
$w = 68 + 48\sqrt{2} \approx 135.882$ or $w = 68 - 48\sqrt{2} \approx 0.117\,749$ (approx. values found with GDC)

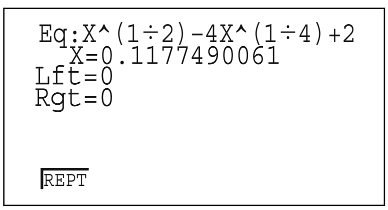
It will be difficult to check these two solutions by substituting them directly into the original equation as we did in the previous example. It will be more efficient to use our GDC.

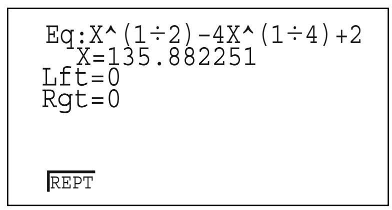
● **Hint:** We will encounter equations in later chapters – for example, equations with logarithms and trigonometric functions – that will be in quadratic form.

Most GDC models have an equation 'solver'. The main limitation of this GDC feature is that it will usually return only approximate solutions. However, even if exact solutions are required, approximate solutions from a GDC are still very helpful as a check of the exact solutions obtained algebraically.









Equations involving absolute value

Equations involving absolute value occur in a range of different topics in mathematics. To solve an equation containing one or more absolute value expressions, we apply the definition from Section 1.1, which states that the absolute value of a real number a, denoted by $|a|$, is given by

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Also recall that in Section 1.1 we stated that $|a|$ is the distance between the coordinate a and the origin on the real number line.

Example 32 – Equation with an absolute value expression

Use an algebraic approach to solve the equation $|2x + 7| = 13$. Check any solution(s) on a GDC.

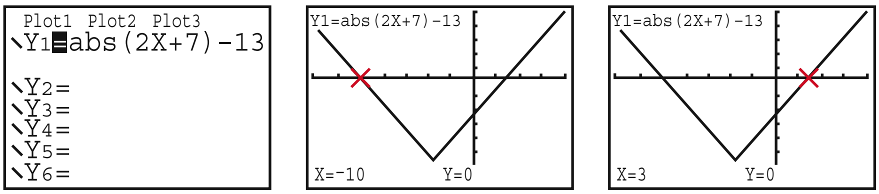
Solution

The expression inside the absolute value symbols must be either 13 or -13, so $2x + 7$ equals 13 or -13. Hence, the given equation is satisfied if either

$$\begin{array}{lll} 2x + 7 = 13 & \text{or} & 2x + 7 = -13 \\ 2x = 6 & & 2x = -20 \\ x = 3 & & x = -10 \end{array}$$

The solutions are $x = 3$ and $x = -10$.

To check the solutions on a GDC, graph the equation $y = |2x + 7| - 13$ and confirm that $x = 3$ and $x = -10$ are the x-intercepts of the graph.



The x-intercepts of the graph of $y = |2x + 7| - 13$ agree with the solutions to the equation.

Example 33 – Equation with two absolute value expressions

Find algebraically the solution(s) to the equation $|2x - 1| = |7 - 3x|$. Check the solution(s) graphically.

Solution

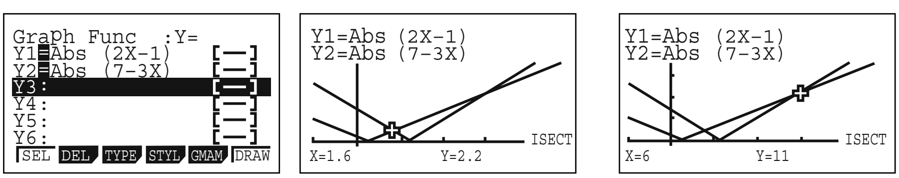
There are four possibilities:

$$2x - 1 = 7 - 3x \text{ or } 2x - 1 = -(7 - 3x) \text{ or } -(2x - 1) = 7 - 3x$$
$$\text{or } -(2x - 1) = -(7 - 3x)$$

The first and last equations are equivalent, and the second and third equations are also equivalent. Thus, it is only necessary to solve the first two equations.

$$\begin{array}{lll} 2x - 1 = 7 - 3x & \text{or} & 2x - 1 = -(7 - 3x) \\ 5x = 8 & & 2x - 1 = -7 + 3x \\ x = \frac{8}{5} & & 6 = x \Rightarrow x = 6 \end{array}$$

To check, we can graph the equations $y_1 = |2x - 1|$ and $y_2 = |7 - 3x|$, and confirm that the x-coordinates of their points of intersection agree with the solutions to the given equation.



Solving inequalities

Working with inequalities is very important for many of the topics in this course. Inequalities were covered in Section 1.1 in the context of order on the real number line. Recall the four important properties for inequalities.

Properties of inequalities

For three real numbers *a, b* and *c:*

1. If $a > b$ and $b > c$, then $a > c$.
2. If $a > b$ and $c > 0$, then $ac > bc$.
3. If $a > b$ and $c < 0$, then $ac < bc$.
4. If $a > b$, then $a + c > b + c$.

Quadratic inequalities

In the topics covered in this course, you will need to be as proficient with solving inequalities as with solving equations. We solved some simple linear inequalities in Section 1.1. Here we will consider strategies for other inequalities – particularly involving quadratic and absolute value expressions.

Example 34 – A quadratic inequality

Find the values of x that solve the inequality $x^2 > x$.

Solution

It is possible to determine the solution set to this inequality by a method of trial and error, or simply using a mental process. That may be successful but generally speaking it is a good idea to attempt to find the solution set by some algebraic method and then check, usually by means of a GDC. For this example, it is tempting to consider dividing both sides by x, but that cannot be done because it is not known whether x is positive or negative. Recall that when multiplying or dividing both sides of an inequality by a negative number it is necessary to reverse the inequality sign (3rd property of inequalities listed above). Instead a better approach is to place all terms on one side of the inequality (with zero on the other side) and then try to factorize.

$$x^2 > x$$
$$x^2 - x > 0$$
$$x(x - 1) > 0$$

Now analyze the signs of the two different factors in a 'sign chart'.

sign chart

		0		1		x
x	$-$	0	$+$		$+$	
$x - 1$	$-$		$-$	0	$+$	
$x(x - 1)$	$+$	0	$-$	0	$+$	

The sign chart indicates that the product of the two factors, $x(x - 1)$, will be positive when x is less than 0 or greater than 1. Therefore, the solution set is $x < 0$ or $x > 1$.

• **Hint:** The solution set, $x < 0$ or $x > 1$, for Example 34 comprises two intervals that do not intersect (disjoint). It is incorrect to write the solution as $0 > x > 1$, or as $1 < x < 0$. Both of these formats imply that the solution set consists of the values of x *between* 0 and 1, but that is not the case. Only write the 'combined' inequality $a < x < b$ if $x > a$ *and* $x < b$ where the two intervals are intersecting *between* a and b.

Inequalities with quadratic polynomials arise in many different contexts. Problems in which we need to analyze the value of the discriminant of a quadratic equation will usually require us to solve a quadratic inequality, as the next example illustrates.

Example 35 – A quadratic from evaluating a discriminant

Given $f(x) = 3kx^2 - (k + 3)x + k - 2$, find the range of values of k for which f has no real zeros.

Solution

The quadratic function f will have no real zeros when its discriminant is negative. Since f is written in the form $ax^2 + bx + c = 0$ then, in terms of the parameter k, $a = 3k$, $b = -(k + 3)$ and $c = k - 2$. Substituting these values into the discriminant, we have the inequality

$(-(k+3))^2 - 4(3k)(k-2) < 0$

$k^2 + 6k + 9 - 12k^2 + 24k < 0$

$-11k^2 + 30k + 9 < 0$ Easier to factorize if leading coefficient is positive.

$11k^2 - 30k - 9 > 0$ Multiply both sides by -1; reverse inequality sign.

$$k = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(11)(-9)}}{2(11)} = \frac{30 \pm \sqrt{1296}}{22} = \frac{30 \pm 36}{22}$$

$$k = \frac{30+36}{22} = \frac{66}{22} = 3 \text{ or } k = \frac{30-36}{22} = -\frac{6}{22} = -\frac{3}{11}$$

The two rational zeros indicate $11k^2 - 30k - 9$ could have been factorized into $(11k + 3)(k - 3)$:

$$(11k + 3)(k - 3) > 0$$

The results of the sign chart indicate that the solution set to the inequality is $k < -\frac{3}{11}$ or $k > 3$. Therefore, any value of k such that $k < -\frac{3}{11}$ or $k > 3$ will cause the function f to have no real zeros.

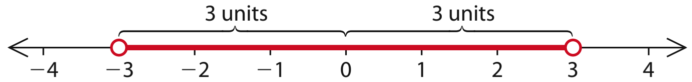
sign chart

k		$-\frac{3}{11}$		3	
$11k + 3$	$-$	0	$+$		$+$
$k - 3$	$-$		$-$	0	$+$
$(11k + 3)(k - 3)$	$+$	0	$-$	0	$+$

Absolute value inequalities

In Section 1.1 we described how absolute value is used to indicate distance on the number line. For example, the equation $|x| = 3$ means that some number x is a distance of 3 units from the origin. The two solutions to this equation are $x = 3$ and $x = -3$. Consequently, the inequality $|x| < 3$ means that x lies *at most* 3 units from the origin, as shown in Figure 3.13.

Figure 3.13



This means that x lies *between* -3 and 3, that is, $-3 < x < 3$. Similarly, the inequality $|x| > 3$ means that x lies 3 *or more* units from the origin. This occurs if x is to the left of -3 (that is, $x < -3$) or if x lies to the right of 3 (that is, $x > 3$).

Properties of absolute value inequalities

For any real numbers x and c such that $c > 0$:

1. $|x| < c$ if and only if $-c < x < c$.
2. $|x| > c$ if and only if $x < -c$ or $x > c$.

Example 36 – Absolute value inequality I

Solve for x: $|3x - 7| \geqslant 8$

Solution

Applying the second property for absolute value inequalities, we have

$$3x - 7 \leqslant -8 \text{ or } 3x - 7 \geqslant 8$$
$$3x \leqslant -1 \text{ or } 3x \geqslant 15$$
$$x \leqslant -\tfrac{1}{3} \text{ or } x \geqslant 5$$

Therefore, the solution set is the union of two half-open intervals $x \leqslant -\frac{1}{3}$ or $x \geqslant 5$, which can also be written in interval notation as $\left]-\infty, -\frac{1}{3}\right] \cup [5, \infty[$.

Example 37 – Absolute value inequality II

Find the values of x which satisfy the inequality $\left|\frac{x}{x+4}\right| < 2$.

Solution

Applying the first property for absolute value inequalities gives

$$-2 < \frac{x}{x+4} < 2$$

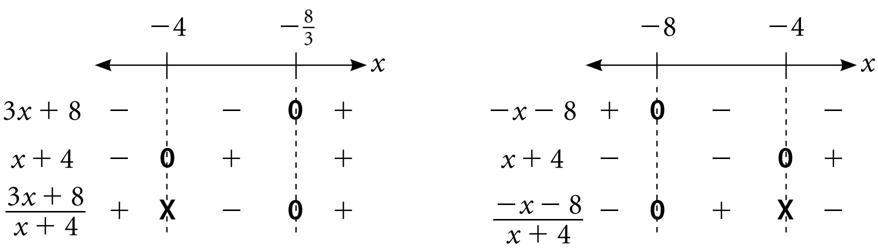
We cannot multiply both sides by $x + 4$ unless we take into account the two different cases: (1) when $x + 4$ is positive (inequality is *not* reversed), and (2) when $x + 4$ is negative (inequality sign *is* reversed). Instead, let's solve the two inequalities in the 'combined' inequality separately by rearranging so that zero is on one side and then analyze where the expression on the other side is zero, positive and negative. This is similar to the approach used in Example 34.

$\frac{x}{x+4} > -2$ and $\frac{x}{x+4} < 2$ the word ‘and’ indicates intersection

$\frac{x}{x+4} + 2 > 0$ and $\frac{x}{x+4} - 2 < 0$

$\frac{x}{x+4} + \frac{2x+8}{x+4} > 0$ and $\frac{x}{x+4} - \frac{2x+8}{x+4} < 0$

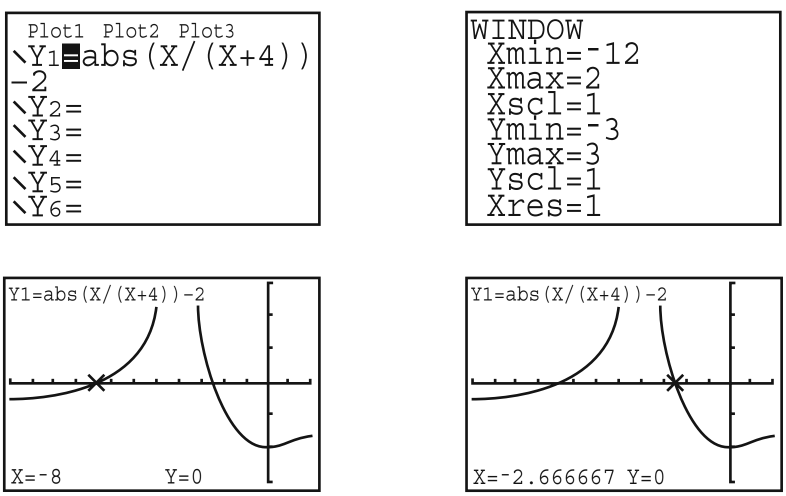
$\frac{3x+8}{x+4} > 0$ and $\frac{-x-8}{x+4} < 0$



$x < -4 \cup x > -\frac{8}{3}$ $\cap$ $x < -8 \cup x > -4$

The solution set for the original ‘combined’ inequality, $-2 < \frac{x}{x+4} < 2$, will be the intersection of the solution sets of the two separate inequalities graphed above on the number line. Thus, the solution set is $x < -8$ or $x > -\frac{8}{3}$.

A graphical check using a GDC can be effectively performed by graphing the equation $y = \left|\frac{x}{x+4}\right| - 2$ and observing where the graph is below the x-axis. The values of x for which this is true will correspond to the solution set for the inequality $\left|\frac{x}{x+4}\right| < 2$.



Example 38 – Algebraic and graphical methods

Solve the inequality $|x - 4| > 2|x - 7|$.

Solution

Method 1 – Algebraic

If $a > 0$, $b > 0$ and $a = b$, then $a^2 = b^2$. Since the expressions on both sides must be positive then we can square both sides and remove the absolute value signs.

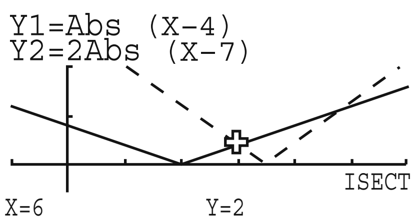
$$(x - 4)^2 > (2(x - 7))^2$$
$$x^2 - 8x + 16 > 4(x^2 - 14x + 49)$$
$$x^2 - 8x + 16 > 4x^2 - 56x + 196$$
$$0 > 3x^2 - 48x + 180$$
$$0 > x^2 - 16x + 60$$
$$(x - 10)(x - 6) < 0$$

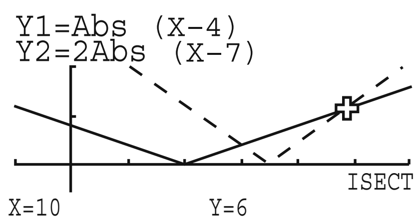
x		6		10	
$x - 10$	$-$		$-$	0	$+$
$x - 6$	$-$	0	$+$		$+$
$(x - 10)(x - 6)$	$+$	0	$-$	0	$+$

Therefore, the solution set is the open interval $6 < x < 10$.

Method 2 - Graphical

We can graph the two equations $y_1 = |x - 4|$ and $y_2 = 2|x - 7|$ and use our GDC to determine for what values of x the graph of y_1 is above the graph of y_2.





The equation $y_2 = 2|x - 7|$ has been graphed in a dashed style. By using the 'intersect' command on the GDC we find that the graph of y_1 is above the graph of y_2 for $6 < x < 10$. Therefore, the solution set is the open interval $6 < x < 10$.

Example 39 – Inequality involving rational expressions

For what values of x is $\frac{x}{x + 8} \leqslant \frac{1}{x - 1}$? Solve algebraically.

Solution

As applied in previous examples, an effective algebraic approach is to rearrange the inequality so that both fractions are on the same side with zero on the other side. Then combine the two fractions into one fraction and analyze where the fraction is zero, positive and negative.

		-8		-2		1		4	
$x+2$	$-$		$-$	0	$+$		$+$		$+$
$x-4$	$-$		$-$		$-$		$-$	0	$+$
$x+8$	$-$	0	$+$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$		$+$
$\frac{(x+2)(x-4)}{(x+8)(x-1)}$	$+$	X	$-$		$+$	X	$-$		$+$

$$\frac{x}{x+8} - \frac{1}{x-1} \leqslant 0$$

$$\frac{x(x-1)-(x+8)}{(x+8)(x-1)} \leqslant 0$$

$$\frac{x^2-2x-8}{(x+8)(x-1)} \leqslant 0$$

$$\frac{(x+2)(x-4)}{(x+8)(x-1)} \leqslant 0$$

Therefore, $\frac{x}{x+8} \leqslant \frac{1}{x-1}$ when $-8 < x \leqslant -2$ or $1 < x \leqslant 4$, which can also be expressed in interval notation as $]-8, -2] \cup]1, 4]$.

Exercise 3.5

In questions 1–22, solve for x in the equation. If possible, find all real solutions and express them exactly. If this is not possible, then solve using your GDC and approximate any solutions to three significant figures. Be sure to check answers and to recognize any extraneous solutions.

1 $\sqrt{x+6} + 2x = 9$

2 $\sqrt{x+7} + 5 = x$

3 $\sqrt{7x+14} - 2 = x$

4 $\sqrt{2x+3} - \sqrt{x-2} = 2$

5 $\frac{5}{x+4} - \frac{4}{x} = \frac{21}{5x+20}$

6 $\frac{x+1}{2x+3} = \frac{5x-1}{7x+3}$

7 $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x+4}$

8 $\frac{2x}{1-x^2} + \frac{1}{x+1} = 2$

9 $x^4 - 2x^2 - 15 = 0$

10 $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 15 = 0$

11 $x^6 - 35x^3 + 216 = 0$

12 $5x^{-2} - x^{-1} - 2 = 0$

13 $|3x+4| = 8$

14 $|x+6| = |3x-24|$

15 $|5x+1| = 2x$

16 $|x-1| + |x| = 3$

17 $\left|\frac{x+1}{x-1}\right| = 3$

18 $\sqrt{x} - \frac{6}{\sqrt{x}} = 1$

19 $\sqrt{4-x} - \sqrt{6+x} = \sqrt{14+2x}$

20 $\frac{6}{x^2+1} = \frac{1}{x^2} + \frac{10}{x^2+4}$

21 $x - \sqrt{x+10} = 0$

22 $6x - 37\sqrt{x} + 56 = 0$

In questions 23–30, find the values of x that solve the inequality.

23 $3x^2 - 4 < 4x$

24 $\frac{2x-1}{x+2} \geqslant 1$

25 $2x^2 + 8x \leqslant 120$

26 $|1-4x| > 7$

27 $|x-3| > |x-14|$

28 $\left|\frac{x^2-4}{x}\right| \leqslant 3$

29 $\frac{x}{x-2} > \frac{1}{x+1}$

30 $\frac{4x-1}{x^2-2x-3} < 3$

31 Find the values of p for which the equation $px^2 - 3x + 1 = 0$ has a) one real solution, b) two real solutions, and c) no real solutions.

32 Given $f(x) = x^2 + x(k - 1) + k^2$, find the range of values of k so that $f(x) > 0$ for all real values of x.

33 Show that both of the following inequalities are true for all real numbers m and n such that $m > n > 0$.

a) $m + \frac{1}{n} > 2$ b) $(m + n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4$

34 Find all of the exact solutions to the equation $(x^2 + x)^2 = 5x^2 + 5x - 6$.

35 If a, b and c are positive and unequal, show that $(a + b + c)^2 < 3(a^2 + b^2 + c^2)$.

36 Find the values of x that solve each inequality.

a) $\left|\frac{2x - 3}{x}\right| < 1$ b) $\frac{3}{x - 1} - \frac{2}{x + 1} < 1$

37 Provide a geometric or algebraic argument to show that $|a + b| \leqslant |a| + |b|$ for all $a, b \in \mathbb{R}$.

3.6 Partial fractions (Optional)

In arithmetic, when we add fractions we find the least common denominator. Then we multiply both the numerator and denominator of each term by what is needed to complete the common denominator. For example:

$$\frac{2}{3} + \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{7} + \frac{5}{7} \cdot \frac{3}{3} = \frac{14 + 15}{21} = \frac{29}{21}$$

$$\frac{2}{3} + \frac{5}{9} + \frac{1}{27} = \frac{2}{3} \cdot \frac{9}{9} + \frac{5}{9} \cdot \frac{3}{3} + \frac{1}{27} = \frac{18 + 15 + 1}{27} = \frac{34}{27}$$

Reversing the process is called expressing each compound fraction as *partial fractions*. That is, given for example the fraction $\frac{29}{21} = \frac{29}{3 \times 7}$, we express it as a sum of two fractions. One fraction has denominator 3 and the other has denominator 7. Hence, we have the name *partial fractions*.

The process of finding the *partial fractions* is a straightforward process. We write:

$\frac{29}{3 \times 7} = \frac{a}{3} + \frac{b}{7}$ and then we solve for two integers a and b.

$$\frac{29}{3 \times 7} = \frac{a}{3} + \frac{b}{7} = \frac{7a + 3b}{21} \Rightarrow 7a + 3b = 29$$

Now by trial and error we can find that $a = 2$ and $b = 5$. Other answers are also possible $(-1, 12)$, $(8, -9)$ …

Notice the situation in the second example. The L.C.M. contains different powers of the same number. Consequently, when finding the partial fractions decomposition you need to consider that all powers less than or equal to the highest one may be present. That is, when we set up the process of decomposing $\frac{24}{27}$ we set it up in the following manner:

$$\frac{24}{27} = \frac{a}{27} + \frac{b}{9} + \frac{c}{3}$$

Then we attempt to find the values of a, b, and c.

In algebra, we carry out that process on the addition of rational expressions. Once again we multiply the numerator and denominator of each term by what was missing from the denominator of that term.

Partial fractions decomposition (PFD)

With partial fractions decomposition, we are going to reverse the process and decompose a rational expression into two or more simpler proper rational expressions. This is a very useful skill in which a single fraction with a factorable denominator is split into the sum of two or more fractions (partial fractions) whose denominators are the factors of the original denominator.

The method of partial fractions decomposition is extremely helpful in evaluating certain integrals as you will see in Section 16.5 (optional).

For example: $\frac{12x - 1}{2x^2 - 5x - 3} = \frac{2}{2x + 1} + \frac{5}{x - 3}$

Example 40

Find the partial fraction decomposition of $\frac{x + 1}{x^2 + 5x + 6}$.

Solution

$\frac{x + 1}{x^2 + 5x + 6} \equiv \frac{x + 1}{(x + 2)(x + 3)}$, and hence we will attempt to find two numbers a and b such that:

$$\frac{x + 1}{x^2 + 5x + 6} \equiv \frac{a}{x + 2} + \frac{b}{x + 3}$$

(Notice that we wrote this as an identity rather than equality because it has to be true for all values of x and not only for a few.)

$$\frac{x + 1}{x^2 + 5x + 6} \equiv \frac{a}{x + 2} + \frac{b}{x + 3} \equiv \frac{a(x + 3) + b(x + 2)}{(x + 2)(x + 3)}$$

Since the denominators of these identical fractions are the same, their numerators must also be the same. That is

$$x + 1 \equiv a(x + 3) + b(x + 2).$$

We have two methods of solution here.

First method

$$x + 1 \equiv a(x + 3) + b(x + 2) \Leftrightarrow x + 1 \equiv (a + b)x + (3a + 2b)$$

For two polynomials to be identical, the coefficients of the same powers must be the same, that is, the coefficient of x on the left must be the same as the coefficient of x on the right and similarly the constant terms. Hence:

$1 = a + b$ and $1 = 3a + 2b$

Now, solving the system with two equations will yield:

$a = -1$ and $b = 2$

Hence, $\dfrac{x+1}{x^2+5x+6} \equiv \dfrac{-1}{x+2} + \dfrac{2}{x+3}$.

This is also called the 'cover-up' method. This method allows the choice of numbers that are not initially in the domain of the original rational expression.

Second method

$x + 1 \equiv a(x+3) + b(x+2)$

Again, since this is an identity, the two sides must be the same for any choice of x. Hence, we can substitute any two numbers for x to get the value of each of a and b, specifically replacing x with -3 yields:

$x + 1 \equiv a(x+3) + b(x+2) \Rightarrow -2 = -b \Rightarrow b = 2.$

Notice how the choice of -3 eliminated the term with a and allowed us to find b directly. Replacing x with -2 yields:

$x + 1 \equiv a(x+3) + b(x+2) \Rightarrow -1 = a.$

This is of course the same result as above. Also notice here how the choice of -2 eliminated the term with b and allowed us to find a directly.

Note: This method is helpful in cases where there are no repeated factors.

The second method is faster whenever applicable. (We will discuss this in more detail later.)

Example 41

Find the PFD for $\dfrac{5x^2+16x+17}{2x^3+9x^2+7x-6}$.

Solution

$$\frac{5x^2+16x+17}{2x^3+9x^2+7x-6} \equiv \frac{5x^2+16x+17}{(2x-1)(x+2)(x+3)} \equiv \frac{a}{2x-1} + \frac{b}{x+2} + \frac{c}{x+3}$$

First method

$$5x^2 + 16x + 17 \equiv a(x+2)(x+3) + b(2x-1)(x+3) + c(2x-1)(x+2)$$
$$\equiv (a+2b+2c)x^2 + (5a+5b+3c)x + 6a - 3b - 2c$$

This leads to this system: $\begin{cases} a + 2b + 2c = 5 \\ 5a + 5b + 3c = 16 \\ 6a - 3b - 2c = 17 \end{cases}$

Using any method of your choice for solving systems of equations, you should have:

$a = 3$, $b = -1$, $c = 2$ and hence:

$$\frac{5x^2+16x+17}{2x^3+9x^2+7x-6} \equiv \frac{3}{2x-1} - \frac{1}{x+2} + \frac{2}{x+3}$$

Second method

$5x^2 + 16x + 17 \equiv a(x + 2)(x + 3) + b(2x - 1)(x + 3) + c(2x - 1)(x + 2)$

$x = -2 \Rightarrow 5 = -5b \Rightarrow b = -1$

$x = -3 \Rightarrow 14 = 7c \Rightarrow c = 2$

$x = \frac{1}{2} \Rightarrow \frac{105}{4} = \frac{35}{4}a \Rightarrow a = 3$

Properties

1 Partial fractions decomposition only works for proper rational expressions, that is, the degree of the numerator must be less than the degree of the denominator. If it is not, then you must perform long division first, and then perform the partial fractions decomposition on the rational part (the remainder over the divisor). After you've done the partial fraction decomposition, just add back in the quotient part from the long division.

2 **Linear factors**: We can only decompose the partial fractions into proper rational expressions. Hence, in each partial fraction, when the denominator is linear, only a constant can be in the numerator. So, for every linear factor in the denominator, you will need a constant in the numerator. See Examples 40 and 41 above.

3 **Repeated linear factors**: If the denominator of the rational expression contains repeated linear factors, then following our discussion in the introduction, the process is as follows.

We need to include a factor in the expansion for each power possible. For example, if we have $(x - 1)^3$, we will need to include $(x - 1)$, an $(x - 1)^2$, and $(x - 1)^3$. Each of those $(x - 1)$ factors would have a constant term in the numerator because $x - 1$ is linear, no matter what power it is raised to.

For example: $\frac{13x^3 - 62x^2 + 101x - 58}{(x - 1)^3(2x - 5)} \equiv \frac{a}{(x - 1)^3} + \frac{b}{(x - 1)^2} + \frac{c}{x - 1} + \frac{d}{2x - 5}$

4 **Irreducible quadratic factors:** If the rational expression we are decomposing contains irreducible quadratic factors in the denominator, then the numerator could have a linear term and/or a constant term. So, for every irreducible quadratic factor in the denominator, you will need a linear term and a constant term in the numerator.

For example: $\frac{-8x^3 + 15x^2 - 26x + 33}{(x - 1)^2(2x^2 + 5)} \equiv \frac{a}{(x - 1)^2} + \frac{b}{x - 1} + \frac{cx + d}{2x^2 + 5}$

Note: It may turn out that any of the numbers a, b, c, or d is zero.

Example 42

Write $\frac{3x - 1}{x^2 + 4x + 4}$ as the sum of partial fractions.

Solution

The first step is to factorise the denominator.

$x^2 + 4x + 4 = (x + 2)^2$

Here the denominator has a repeated linear factor: $\frac{3x - 1}{x^2 + 4x + 4} = \frac{3x - 1}{(x + 2)^2}$

Because there are two (i.e. repeated) linear factors of $x + 2$ in the denominator of the original rational expression then it *must* have a partial fraction with a denominator of $(x + 2)^2$, and it *may* also have a partial fraction with a denominator of $x + 2$.

Thus, we are looking for constants A and B such that:

$$\frac{3x - 1}{(x + 2)^2} \equiv \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

Multiplying both sides of the equation by $(x + 2)^2$ gives:

$$3x - 1 \equiv A(x + 2) + B$$

Essentially, the task is to find the unique values of A and B such that this equation is an identity, i.e. it is true for all values of x for which the original fraction is defined (in this case $x \neq -2$). However, as you recall, the 'cover-up' method allows us to choose 'helpful' values of x including such numbers. For example, in this case, if $x = -2$ then A is eliminated and the value of B can be found directly.

Let $x = -2$: $3x - 1 \equiv A(x + 2) + B \Rightarrow 3(-2) - 1 = A \cdot 0 + B$
$\Rightarrow B = -7$

Let $x = 0$: $3x - 1 \equiv A(x + 2) + B \Rightarrow 3 \cdot 0 - 1 = 2A - 7$
$\Rightarrow 2A = 6$
$\Rightarrow \mathrm{A} = 3$

Therefore, $\dfrac{3x - 1}{x^2 + 4x + 4} = \dfrac{3}{x + 2} - \dfrac{7}{(x + 2)^2}$

Example 43

Write $\dfrac{2}{x^3 + 3x^2 + 2x}$ as the sum of partial fractions.

Solution

We first factorize the denominator and discover that one of the factors is an irreducible quadratic factor:

$$\frac{2}{x^3 + 3x^2 + 2x} = \frac{2}{x(x^2 + 2x + 2)} \equiv \frac{a}{x} + \frac{bx + c}{x^2 + 2x + 2}$$

Simplifying the expression gives:

$$2 \equiv a(x^2 + 2x + 2) + x(bx + c) \Rightarrow 2 \equiv (a + b)x^2 + (2a + c)x + 2a \Rightarrow$$

$$\begin{cases} a + b = 0 \\ 2a + c = 0 \\ 2a = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -1 \\ c = -2 \end{cases}$$

Therefore $\dfrac{2}{x^3 + 2x^2 + 2x} = \dfrac{1}{x} - \dfrac{x + 2}{x^2 + 2x + 2}$.

Exercise 3.6

Decompose each of the following rational expressions into partial fractions.

1 $\dfrac{5x+1}{x^2+x-2}$

2 $\dfrac{x+4}{x^2-2x}$

3 $\dfrac{x+2}{x^2+4x+3}$

4 $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

5 $\dfrac{2x^2+x-12}{x^3+5x^2+6x}$

6 $\dfrac{4x^2+2x-1}{x^3+x^2}$

7 $\dfrac{3}{x^2+x-2}$

8 $\dfrac{5-x}{2x^2+x-1}$

9 $\dfrac{3x+4}{(x+2)^2}$

10 $\dfrac{12}{x^4-x^3-2x^2}$

11 $\dfrac{2}{x^3+x}$

12 $\dfrac{x+2}{x^3+3x}$

13 $\dfrac{3x+2}{x^3+6x}$

14 $\dfrac{2x+3}{x^3+8x}$

15 $\dfrac{x+5}{x^3-4x^2-5x}$

Practice questions

1 Solve for x in the equation $x^2-(a+3b)x+3ab=0$.

2 Find the values of x that solve the following inequality.

$$\frac{3x-2}{5}+3 \geqslant \frac{4x-1}{3}$$

3 For what value of c is the vertex of the parabola $y=3x^2-8x+c$ at $\left(\frac{4}{3}, -\frac{1}{3}\right)$?

4 The quadratic function $f(x)=ax^2+bx+c$ has the following characteristics:

(i) passes through the point (2, 4); **(ii)** has a maximum value of 6 when $x=4$; and **(iii)** has a zero of $x=4+2\sqrt{3}$

Find the values of a, b and c.

5 If the roots of the equation $x^3+5x^2+px+q=0$ are ω, 2ω and $\omega+3$, find the values of ω, p and q.

6 Find all values of m such that the equation $mx^2-2(m+2)x+m+2=0$ has **a)** two real roots; **b)** two real roots (one positive and one negative).

7 $x-1$ and $x+1$ are factors of the polynomial x^3+ax^2+bx+c, and the polynomial has a remainder of 12 when divided by $x-2$. Find the values of a, b and c.

8 Solve the inequality $|x|<5|x-6|$.

9 Find the range of values for k in order for the equation $2x^2+(3-k)x+k+3=0$ to have two imaginary solutions.

10 Consider the rational function $f(x)=\dfrac{2x^2+8x+7}{x^2+4x+5}$. Do not use your GDC for this question.

a) Write $f(x)$ in the form $a-\dfrac{b}{(x+c)^2+d}$.

b) State the values of **(i)** $\lim_{x\to+\infty} f(x)$, and **(ii)** $\lim_{x\to-\infty} f(x)$.

c) State the coordinates of the minimum point on the graph of $f(x)$.

11 Find the values of k so that the equation $(k-2)x^2 + 4x - 2k + 1 = 0$ has two distinct real roots.

12 When the function $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x+1)$ the remainder is -20. Find the value of a.

13 The polynomial $p(x) = (ax+b)^3$ leaves a remainder of -1 when divided by $(x+1)$, and a remainder of 27 when divided by $(x-2)$. Find the values of the real numbers a and b.

14 The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x-2)$ as when divided by $(x+1)$. Find the value of a.

15 When the polynomial $x^4 + ax + 3$ is divided by $(x-1)$, the remainder is 8. Find the value of a.

16 The polynomial $x^3 + ax^2 - 3x + b$ is divisible by $(x-2)$ and has a remainder 6 when divided by $(x+1)$. Find the value of a and of b.

17 The polynomial $x^2 - 4x + 3$ is a factor of $x^3 + (a-4)x^2 + (3-4a)x + 3$. Calculate the value of the constant a.

18 Consider $f(x) = x^3 - 2x^2 - 5x + k$. Find the value of k if $(x+2)$ is a factor of $f(x)$.

19 Find the real number k for which $1 + ki$ $(i = \sqrt{-1})$ is a zero of the polynomial $z^2 + kz + 5$.

20 The equation $kx^2 - 3x + (k+2) = 0$ has two distinct real roots. Find the set of possible values of k.

21 Consider the equation $(1+2k)x^2 - 10x + k - 2 = 0$, $k \in \mathbb{R}$. Find the set of values of k for which the equation has real roots.

22 Find the range of values of m such that for all x

$$m(x+1) \leqslant x^2.$$

23 Find the values of x for which $|5-3x| \leqslant |x+1|$.

24 Solve the inequality $x^2 - 4 + \frac{3}{x} < 0$.

25 Solve the inequality $|x-2| \geqslant |2x+1|$.

26 Let $f(x) = \frac{x+4}{x+1}$, $x \neq -1$ and $g(x) = \frac{x-2}{x-4}$, $x \neq 4$.

Find the set of values of x such that $f(x) \leqslant g(x)$.

27 Solve the inequality $\left|\frac{x+9}{x-9}\right| \leqslant 2$.

28 Given that $2 + i$ is a root of the equation $x^3 - 6x^2 + 13x - 10 = 0$ find the other two roots.

29 Find all values of x that satisfy the inequality $\frac{2x}{|x-1|} < 1$.

4 Sequences and Series

Assessment statements

1.1 Arithmetic sequences and series; sum of finite arithmetic sequences; geometric sequences and series; sum of finite and infinite geometric series.
Sigma notation.

1.3 Counting principles, including permutations and combinations. The binomial theorem: expansion of $(a + b)^n$, $n \in \mathbb{N}$.

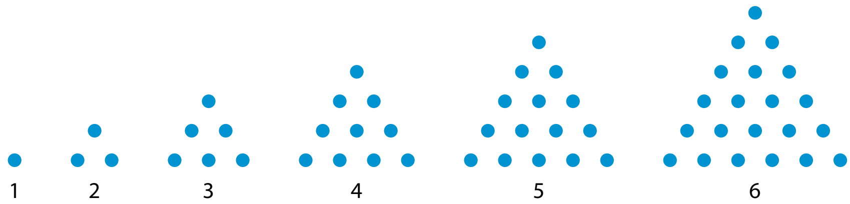
1.4 Proof by mathematical induction.

Introduction

The heights of consecutive bounds of a ball, compound interest, and Fibonacci numbers are only a few of the applications of sequences and series that you have seen in previous courses. In this chapter you will review these concepts, consolidate your understanding and take them one step further.

4.1 Sequences

Take the following pattern as an example:



The first figure represents 1 dot, the second represents 3 dots, etc. This pattern can also be described differently. For example, in function notation:

$$f(1) = 1, f(2) = 3, f(3) = 6, \text{ etc., where the domain is } \mathbb{Z}^+$$

Here are some more examples of sequences:

1 6, 12, 18, 24, 30

2 $3, 9, 27, \ldots, 3^k, \ldots$

3 $\left\{\frac{1}{i^2}; i = 1, 2, 3, \ldots, 10\right\}$

4 $\{b_1, b_2, \ldots, b_n, \ldots\}$, sometimes used with an abbreviation $\{b_n\}$

The first and third sequences are **finite** and the second and fourth are **infinite**. Notice that, in the second and third sequences, we were able to define a rule that yields the nth number in the sequence (called the nth term) as a function of n, the term's number. In this sense, a sequence is a **function** that assigns a **unique** number (a_n) to each positive integer n.

Example 1

Find the first five terms and the 50th term of the sequence $\{b_n\}$ such that $b_n = 2 - \frac{1}{n^2}$.

Solution

Since we know an *explicit* expression for the nth term as a *function* of its number n, we only need to find the value of that function for the required terms:

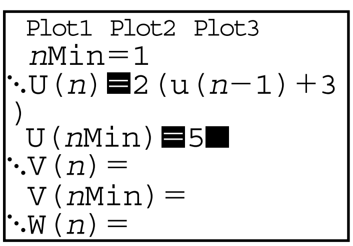
$$b_1 = 2 - \frac{1}{1^2} = 1;\ b_2 = 2 - \frac{1}{2^2} = 1\frac{3}{4};\ b_3 = 2 - \frac{1}{3^2} = 1\frac{8}{9};\ b_4 = 2 - \frac{1}{4^2} = 1\frac{15}{16};$$

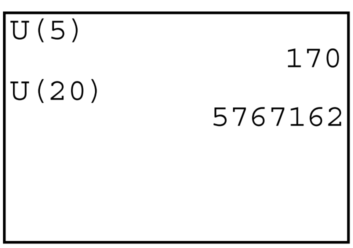
$$b_5 = 2 - \frac{1}{5^2} = 1\frac{24}{25};\ \text{and}\ \ b_{50} = 2 - \frac{1}{50^2} = 1\frac{2499}{2500}.$$

So, informally, **a sequence is an ordered set of real numbers**. That is, there is a first number, a second, and so forth. The notation used for such sets is shown above. The way we defined the function in Example 1 is called the **explicit** definition of a sequence. There are other ways to define sequences, one of which is the **recursive** definition. The following example will show you how this is used.

Example 2

Find the first five terms and the 20th term of the sequence $\{b_n\}$ such that $b_1 = 5$ and $b_n = 2(b_{n-1} + 3)$.





Solution

The defining formula for this sequence is recursive. It allows us to find the nth term b_n if we know the preceding term b_{n-1}. Thus, we can find the second term from the first, the third from the second, and so on. Since we know the first term, $b_1 = 5$, we can calculate the rest:

$$b_2 = 2(\mathbf{b_1} + 3) = 2(\mathbf{5} + 3) = 16$$
$$b_3 = 2(\mathbf{b_2} + 3) = 2(\mathbf{16} + 3) = 38$$
$$b_4 = 2(\mathbf{b_3} + 3) = 2(\mathbf{38} + 3) = 82$$
$$b_5 = 2(\mathbf{b_4} + 3) = 2(\mathbf{82} + 3) = 170$$

Thus, the first five terms of this sequence are 5, 16, 38, 82, 170. However, to find the 20th term, we must first find all 19 preceding terms. This is one of the drawbacks of the recursive definition, unless we can change the definition into explicit form. This can easily be done using a GDC.

Example 3

A Fibonacci sequence is defined recursively as

$$F_n = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ F_{n-1} + F_{n-2} & n > 2 \end{cases}$$

a) Find the first 10 terms of the sequence.

b) Evaluate $S_n = \sum_{i=1}^{n} F_i$ for $n = 1, 2, 3, \ldots, 10$.

c) By observing that $F_1 = F_3 - F_2$, $F_2 = F_4 - F_3$, and so on, derive a formula for the sum of the first n Fibonacci numbers.

Fibonacci numbers are a sequence of numbers named after Leonardo of Pisa, known as Fibonacci (a short form of filius Bonaccio, 'son of Bonaccio').

Solution

a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

b) $S_1 = 1, S_2 = 2, S_3 = 4, S_4 = 7, S_5 = 12, S_6 = 20, S_7 = 33, S_8 = 54,$
$S_9 = 88, S_{10} = 143$

c) Since $F_3 = F_2 + F_1$, then

$$\begin{aligned}
F_1 &= \cancel{F_3} - F_2 \\
F_2 &= \cancel{F_4} - \cancel{F_3} \\
F_3 &= \cancel{F_5} - \cancel{F_4} \\
F_4 &= \cancel{F_6} - \cancel{F_5} \\
\vdots\; & \quad \vdots \qquad \vdots \\
\underline{F_n} &= \underline{F_{n+2} - \cancel{F_{n+1}}} \\
S_n &= F_{n+2} - F_2
\end{aligned}$$

Notice that $S_5 = 12 = F_7 - F_2 = 13 - 1$ and $S_8 = 54 = F_{10} - F_2 = 55 - 1$.

Note: parts a) and b) can be made easy by using a spreadsheet. Here is an example:

	A	B	C	D
1	*F*(*n*)	*S*(*n*)		
2	1	1		
3	1	2		
4	**2**	4		
5	3	7		
6	5	12		
7	8	20		
8	13	33		
9	21	54		
10	34	88		
11	55	**143**		Let this cell be A2 + A3 Then copy it down
12	89	232		
13	144	376		
14	233	609		
15	377	986		Let this cell be B10 + A11 Then copy it down
16	610	1596		
17	987	2583		

Notice that not all sequences have formulae, either recursive or explicit. Some sequences are given only by listing their terms. Among the many kinds of sequences that there are, two types are of interest to us: arithmetic and geometric sequences, which we will discuss in the next two sections.

Exercise 4.1

Find the first five terms of each infinite sequence defined in questions 1–6.

1 $s(n) = 2n - 3$

2 $g(k) = 2^k - 3$

3 $f(n) = 3 \times 2^{-n}$

4 $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 3; \text{ for } n > 1 \end{cases}$

5 $a_n = (-1)^n(2^n) + 3$

6 $\begin{cases} b_1 = 3 \\ b_n = b_{n-1} + 2n; \text{ for } n \geqslant 2 \end{cases}$

Find the first five terms and the 50th term of each infinite sequence defined in questions 7–14.

7 $a_n = 2n - 3$

8 $b_n = 2 \times 3^{n-1}$

9 $u_n = (-1)^{n-1}\dfrac{2n}{n^2 + 2}$

10 $a_n = n^{n-1}$

11 $a_n = 2a_{n-1} + 5$ and $a_1 = 3$

12 $u_{n+1} = \dfrac{3}{2u_n + 1}$ and $u_1 = 0$

13 $b_n = 3 \cdot b_{n-1}$ and $b_1 = 2$

14 $a_n = a_{n-1} + 2$ and $a_1 = -1$

Suggest a recursive definition for each sequence in questions 15–17.

15 $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \frac{1}{192}, \ldots$

16 $\frac{1}{2}a, \frac{2}{3}a^3, \frac{8}{9}a^5, \frac{32}{27}a^7, \ldots$

17 $a - 5k, 2a - 4k, 3a - 3k, 4a - 2k, 5a - k, \ldots$

In questions 18–21, write down a possible formula that gives the nth term of each sequence.

18 $4, 7, 12, 19, \ldots$

19 $2, 5, 8, 11, \ldots$

20 $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

21 $\frac{1}{4}, \frac{3}{5}, \frac{5}{6}, 1, \frac{9}{8}, \ldots$

22 Define $a_n = \dfrac{F_{n+1}}{F_n}$, $n > 1$, where F_n is a member of a Fibonacci sequence.

a) Write the first 10 terms of a_n.

b) Show that $a_n = 1 + \dfrac{1}{a_{n-1}}$

23 Define the sequence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n}\right)$$

a) Find the first 10 terms of this sequence and compare them to Fibonacci numbers.

b) Show that $3 \pm \sqrt{5} = \dfrac{(1 \pm \sqrt{5})^2}{2}$.

c) Use the result in b) to verify that F_n satisfies the recursive definition of Fibonacci sequences.

4.2 Arithmetic sequences

Examine the following sequences and the most likely recursive formula for each of them.

$7, 14, 21, 28, 35, 42, \ldots \qquad a_1 = 7$ and $a_n = a_{n-1} + 7$, for $n > 1$

$2, 11, 20, 29, 38, 47, \ldots \qquad a_1 = 2$ and $a_n = a_{n-1} + 9$, for $n > 1$

$48, 39, 30, 21, 12, 3, -6, \ldots \qquad a_1 = 48$ and $a_n = a_{n-1} - 9$, for $n > 1$

Note that in each case above, every term is formed by adding a constant number to the preceding term. Sequences formed in this manner are called **arithmetic sequences**.

Definition of an arithmetic sequence

A sequence $a_1, a_2, a_3, \ldots$ is an **arithmetic sequence** if there is a constant d for which

$$a_n = a_{n-1} + d$$

for all integers $n > 1$. d is called the **common difference** of the sequence, and $d = a_n - a_{n-1}$ for all integers $n > 1$.

So, for the sequences above, 7 is the common difference for the first, 9 is the common difference for the second and -9 is the common difference for the third.

This description gives us the recursive definition of the arithmetic sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable you to see the expression we are seeking:

$$a_2 = a_1 + d;\ a_3 = a_2 + d = a_1 + d + d = a_1 + 2d;$$
$$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d;\ \ldots$$

So, as you see, you can get to the nth term by adding d to a_1, $(n - 1)$ times, and therefore:

***n*th term of an arithmetic sequence**

The general (nth) term of an arithmetic sequence, a_n, with first term a_1 and common difference d, may be expressed explicitly as

$$a_n = a_1 + (n - 1)d$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Note: The arithmetic sequence can be looked at as a linear function as explained in the introduction to this chapter, i.e. for every increase of one unit in n, the value of the term will increase by d units. As the first term is a_1, the point $(1, a_1)$ belongs to this function. The constant increase d can be considered to be the gradient (slope) of this linear model; hence, the nth term, the dependent variable in this case, can be found by using the *point-slope* form of the equation of a line:

$$y - y_1 = m(x - x_1)$$
$$a_n - a_1 = d(n - 1) \Leftrightarrow a_n = a_1 + (n - 1)d$$

This agrees with our definition of an arithmetic sequence.

Example 4

Find the nth and the 50th terms of the sequence 2, 11, 20, 29, 38, 47, …

Solution

This is an arithmetic sequence whose first term is 2 and common difference is 9. Therefore,

$$a_n = a_1 + (n - 1)d = 2 + (n - 1) \times 9 = 9n - 7$$

$$\Rightarrow a_{50} = 9 \times 50 - 7 = 443$$

Example 5

Find the recursive and the explicit forms of the definition of the following sequence, then calculate the value of the 25th term.

13, 8, 3, -2, …

Solution

This is clearly an arithmetic sequence, since we observe that -5 is the common difference.

Recursive definition: $a_1 = 13$

$$a_n = a_{n-1} - 5$$

Explicit definition: $a_n = 13 - 5(n - 1) = 18 - 5n$, and

$$a_{25} = 18 - 5 \times 25 = -107$$

Example 6

Find a definition for the arithmetic sequence whose first term is 5 and fifth term is 11.

Solution

Since the fifth term is given, using the explicit form, we have

$$a_5 = a_1 + (5 - 1)d \Rightarrow 11 = 5 + 4d \Rightarrow d = \tfrac{3}{2}$$

This leads to the general term,

$a_n = 5 + \frac{3}{2}(n - 1)$, or, equivalently, the recursive form

$$\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + \frac{3}{2}, n > 1 \end{cases}$$

● **Hint:** Definition: In a finite arithmetic sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3 \ldots, a_{k-1}$ are called **arithmetic means** between a_1 and a_k.

Example 7

Insert four arithmetic means between 3 and 7.

Solution

Since there are four means between 3 and 7, the problem can be reduced to a situation similar to Example 6 by considering the first term to be 3 and the sixth term to be 7. The rest is left as an exercise for you!

Exercise 4.2

1 Insert four arithmetic means between 3 and 7.

2 Say whether each given sequence is an arithmetic sequence. If yes, find the common difference and the 50th term; if not, say why not.

a) $a_n = 2n - 3$

b) $b_n = n + 2$

c) $c_n = c_{n-1} + 2$, and $c_1 = -1$

d) $u_n = 3u_{n-1} + 2$

e) 2, 5, 7, 12, 19, …

f) 2, −5, −12, −19, …

For each arithmetic sequence in questions 3–8, find:

a) the 8th term

b) an explicit formula for the *n*th term

c) a recursive formula for the *n*th term.

3 −2, 2, 6, 10, …

4 29, 25, 21, 17, …

5 −6, 3, 12, 21, …

6 10.07, 9.95, 9.83, 9.71, …

7 100, 97, 94, 91, …

8 $2, \frac{3}{4}, -\frac{1}{2}, -\frac{7}{4}, \ldots$

9 Find five arithmetic means between 13 and −23.

10 Find three arithmetic means between 299 and 300.

11 In an arithmetic sequence, $a_5 = 6$ and $a_{14} = 42$. Find an explicit formula for the *n*th term of this sequence.

12 In an arithmetic sequence, $a_3 = -40$ and $a_9 = -18$. Find an explicit formula for the *n*th term of this sequence.

In each of questions 13–17, the first 3 terms and the last term of an arithmetic sequence are given. Find the number of terms.

13 3, 9, 15, …, 525

14 9, 3, −3, …, −201

15 $3\frac{1}{8}, 4\frac{1}{4}, 5\frac{3}{8}, \ldots, 14\frac{3}{8}$

16 $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots, 2\frac{5}{6}$

17 $1 - k, 1 + k, 1 + 3k, \ldots, 1 + 19k$

18 Find five arithmetic means between 15 and −21.

19 Find three arithmetic means between 99 and 100.

20 In an arithmetic sequence, $a_3 = 11$ and $a_{12} = 47$. Find an explicit formula for the *n*th term of this sequence.

21 In an arithmetic sequence, $a_7 = -48$ and $a_{13} = -10$. Find an explicit formula for the *n*th term of this sequence.

22 The 30th term of an arithmetic sequence is 147 and the common difference is 4. Find a formula for the *n*th term.

23 The first term of an arithmetic sequence is −7 and the common difference is 3. Is 9803 a term of this sequence? If so, which one?

24 The first term of an arithmetic sequence is 9689 and the 100th term is 8996. Show that the 110th term is 8926. Is 1 a term of this sequence? If so, which one?

25 The first term of an arithmetic sequence is 2 and the 30th term is 147. Is 995 a term of this sequence? If so, which one?

4.3 Geometric sequences

Examine the following sequences and the most likely recursive formula for each of them.

7, 14, 28, 56, 112, 224, …	$a_1 = 7$ and $a_n = a_{n-1} \times 2$, for $n > 1$
2, 18, 162, 1458, 13 122, …	$a_1 = 2$ and $a_n = a_{n-1} \times 9$, for $n > 1$
48, −24, 12, −6, 3, −1.5, …	$a_1 = 48$ and $a_n = a_{n-1} \times -0.5$, for $n > 1$

Note that in each case above, every term is formed by multiplying a constant number with the preceding term. Sequences formed in this manner are called **geometric sequences**.

Definition of a geometric sequence

A sequence $a_1, a_2, a_3, \ldots$ is a **geometric sequence** if there is a constant r for which

$$a_n = a_{n-1} \times r$$

for all integers $n > 1$. r is called the **common ratio** of the sequence, and $r = a_n \div a_{n-1}$ for all integers $n > 1$.

So, for the sequences above, 2 is the common ratio for the first, 9 is the common ratio for the second and −0.5 is the common ratio for the third.

This description gives us the recursive definition of the geometric sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable you to see the expression we are seeking:

$$a_2 = a_1 \times r;\ a_3 = a_2 \times r = a_1 \times r \times r = a_1 \times r^2;$$
$$a_4 = a_3 \times r = a_1 \times r^2 \times r = a_1 \times r^3;\ \ldots$$

So, as you see, you can get to the nth term by multiplying a_1 with r, $(n - 1)$ times, and therefore:

***n*th term of geometric sequence**

The general (nth) term of a geometric sequence, a_n, with common ratio r and first term a_1, may be expressed explicitly as

$$a_n = a_1 \times r^{(n-1)}$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Example 8

a) Find the geometric sequence with $a_1 = 2$ and $r = 3$.

b) Describe the sequence 3, −12, 48, −192, 768, …

c) Describe the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

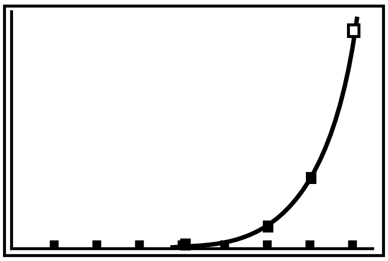
d) Graph the sequence $a_n = \frac{1}{4} \cdot 3^{n-1}$

Solution

a) The geometric sequence is 2, 6, 18, 54, …, $2 \times 3^{n-1}$. Notice that the ratio of a term to the preceding term is 3.

b) This is a geometric sequence with $a_1 = 3$ and $r = -4$. The nth term is $a_n = 3 \times (-4)^{n-1}$. Notice that, when the common ratio is negative, the terms of the sequence alternate in sign.

c) The nth term of this sequence is $a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1}$. Notice that the ratio of any two consecutive terms is $\frac{1}{2}$. Also, notice that the terms decrease in value.



d) The graph of the geometric sequence is shown on the left. Notice that the points lie on the graph of the function $y = \frac{1}{4} \cdot 3^{x-1}$.

Example 9

At 8:00 a.m., 1000 mg of medicine is administered to a patient. At the end of each hour, the concentration of medicine is 60% of the amount present at the beginning of the hour.

a) What portion of the medicine remains in the patient's body at noon if no additional medication has been given?

b) If a second dosage of 1000 mg is administered at 10:00 a.m., what is the total concentration of the medication in the patient's body at noon?

Solution

a) We use the geometric model, as there is a constant multiple by the end of each hour. Hence, the concentration at the end of any hour after administering the medicine is given by:

$a_n = a_1 \times r^{(n-1)}$, where n is the number of hours

Thus, at noon $n = 5$, and $a_5 = 1000 \times 0.6^{(5-1)} = 129.6$.

b) For the second dosage, the amount of medicine at noon corresponds to $n = 3$, and $a_3 = 1000 \times 0.6^{(3-1)} = 360$.

So, the concentration of medicine is $129.6 + 360 = 489.6$ mg.

Compound interest

Interest compounded annually

When we borrow money we pay interest, and when we invest money we receive interest. Suppose an amount of €1000 is put into a savings account that bears an annual interest of 6%. How much money will we have in the bank at the end of four years?

It is important to note that the 6% interest is given annually and is added to the savings account, so that in the following year it will also earn interest, and so on.

Time in years	Amount in the account
0	1000
1	$1000 + 1000 \times 0.06 = 1000(1 + 0.06)$
2	$1000(1 + 0.06) + (1000(1 + 0.06)) \times 0.06 = 1000(1 + 0.06)(1 + 0.06) = 1000(1 + 0.06)^2$
3	$1000(1 + 0.06)^2 + (1000(1 + 0.06)^2) \times 0.06 = 1000(1 + 0.06)^2(1 + 0.06) = 1000(1 + 0.06)^3$
4	$1000(1 + 0.06)^3 + (1000(1 + 0.06)^3) \times 0.06 = 1000(1 + 0.06)^3(1 + 0.06) = \mathbf{1000(1 + 0.06)^4}$

Table 4.1 Compound interest.

This appears to be a geometric sequence with five terms. You will notice that the number of terms is five, as both the beginning and the end of the first year are counted. (Initial value, when time = 0, is the first term.)

In general, if a **principal** of P euros is invested in an account that yields an interest rate r (expressed as a decimal) annually, and this interest is added at the end of the year, every year, to the principal, then we can use the geometric sequence formula to calculate the **future value** A, which is accumulated after t years.

If we repeat the steps above, with

$$A_0 = P = \text{initial amount}$$
$$r = \text{annual interest rate}$$
$$t = \text{number of years}$$

it becomes easier to develop the formula:

Table 4.2 Compound interest formula.

Time in years	Amount in the account
0	$A_0 = P$
1	$A_1 = P + Pr = P(1 + r)$
2	$A_2 = A_1(1 + r) = P(1 + r)^2$
$\vdots$	
t	$A_t = P(1 + r)^t$

Notice that since we are counting from 0 to t, we have $t + 1$ terms, and hence using the geometric sequence formula,

$$a_n = a_1 \times r^{(n-1)} \Rightarrow A_t = A_0 \times (1 + r)^t$$

Interest compounded *n* times per year

Suppose that the principal P is invested as before but the interest is paid n times per year. Then $\frac{r}{n}$ is the interest paid every compounding period. Since every year we have n periods, for t years, we have nt periods. The amount A in the account after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 10

€1000 is invested in an account paying compound interest at a rate of 6%. Calculate the amount of money in the account after 10 years if

a) the compounding is annual

b) the compounding is quarterly

c) the compounding is monthly.

Solution

a) The amount after 10 years is

$A = 1000(1 + 0.06)^{10} = €1790.85.$

b) The amount after 10 years quarterly compounding is

$A = 1000\left(1 + \frac{0.06}{4}\right)^{40} = €1814.02.$

c) The amount after 10 years monthly compounding is

$A = 1000\left(1 + \frac{0.06}{12}\right)^{120} = €1819.40.$

Example 11

You invested €1000 at 6% compounded quarterly. How long will it take this investment to increase to €2000?

Solution

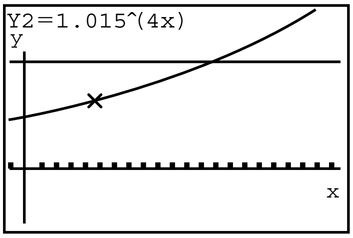
Let $P = 1000$, $r = 0.06$, $n = 4$ and $A = 2000$ in the compound interest formula:

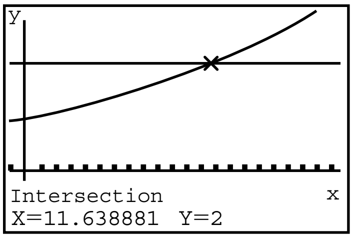
$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Then solve for t:

$$2000 = 1000\left(1 + \frac{0.06}{4}\right)^{4t} \Rightarrow 2 = 1.015^{4t}$$

Using a GDC, we can graph the functions $y = 2$ and $y = 1.015^{4t}$ and then find the intersection between their graphs.





As you can see, it will take the €1000 investment 11.64 years to double to €2000. This translates into approximately 47 quarters.

You can check your work to see that this is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{47} = €2013.28$$

Later in the book, you will learn how to solve the problem algebraically.

Example 12

You want to invest €1000. What interest rate is required to make this investment grow to €2000 in 10 years if interest is compounded quarterly?

Solution

Let $P = 1000$, $n = 4$, $t = 10$ and $A = 2000$ in the compound interest formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Now solve for r:

$$2000 = 1000\left(1 + \frac{r}{4}\right)^{40} \Rightarrow 2 = \left(1 + \frac{r}{4}\right)^{40} \Rightarrow 1 + \frac{r}{4} = \sqrt[40]{2} \Rightarrow r = 4(\sqrt[40]{2} - 1)$$
$$= 0.0699$$

So, at a rate of 7% compounded quarterly, the €1000 investment will grow to at least €2000 in 10 years.

You can check to see whether your work is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.07}{4}\right)^{40} = €2001.60$$

Population growth

The same formulae can be applied when dealing with population growth.

Example 13

The city of Baden in Lower Austria grows at an annual rate of 0.35%. The population of Baden in 1981 was 23 140. What is the estimate of the population of this city for 2013?

Solution

This situation can be modelled by a geometric sequence whose first term is 23 140 and whose common ratio is 1.0035. Since we count the population of 1981 among the terms, the number of terms is 33.

2013 is equivalent to the 33rd term in this sequence. The estimated population for Baden is, therefore,

$$\text{Population (2013)} = a_{31} = 23\,140(1.0035)^{32} = 25\,877$$

Note: Later in the book, more realistic population growth models will be explored and more efficient methods will be developed, as well as the ability to calculate interest that is continuously compounded.

Exercise 4.3

In each of questions 1–15 determine whether the sequence in each question is arithmetic, geometric, or neither. Find the common difference for the arithmetic ones and the common ratio for the geometric ones. Find the common difference or ratio and the 10th term for each arithmetic or geometric one as appropriate.

1 $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots$

2 $a_n = 3n - 3$

3 $b_n = 2^{n+2}$

4 $c_n = 2c_{n-1} - 2$, and $c_1 = -1$

5 $u_n = 3u_{n-1}, u_1 = 4$

6 2, 5, 12.5, 31.25, 78.125, …

7 2, −5, 12.5, −31.25, 78.125, …

8 2, 2.75, 3.5, 4.25, 5, …

9 $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$

10 52, 55, 58, 61, …

11 −1, 3, −9, 27, −81, …

12 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, …

13 3, 6, 12, 18, 21, 27, …

14 6, 14, 20, 28, 34, …

15 2.4, 3.7, 5, 6.3, 7.6, …

For each arithmetic or geometric sequence in questions 16–32 find

a) the 8th term

b) an explicit formula for the nth term

c) a recursive formula for the nth term.

16 $-3, 2, 7, 12, \ldots$

17 $19, 15, 11, 7, \ldots$

18 $-8, 3, 14, 25, \ldots$

19 $10.05, 9.95, 9.85, 9.75, \ldots$

20 $100, 99, 98, 97, \ldots$

21 $2, \frac{1}{2}, -1, -\frac{5}{2}, \ldots$

22 $3, 6, 12, 24, \ldots$

23 $4, 12, 36, 108, \ldots$

24 $5, -5, 5, -5, \ldots$

25 $3, -6, 12, -24, \ldots$

26 $972, -324, 108, -36, \ldots$

27 $-2, 3, -\frac{9}{2}, \frac{27}{4} \ldots$

28 $35, 25, \frac{125}{7}, \frac{625}{49}, \ldots$

29 $-6, -3, -\frac{3}{2}, -\frac{3}{4}, \ldots$

30 $9.5, 19, 38, 76, \ldots$

31 $100, 95, 90.25, \ldots$

32 $2, \frac{3}{4}, \frac{9}{32}, \frac{27}{256}, \ldots$

33 Insert 4 geometric means between 3 and 96.

● **Hint:** Definition: In a finite geometric sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3, \ldots, a_{k-1}$ are called *geometric means* between a_1 and a_k.

34 Find 3 geometric means between 7 and 4375.

35 Find a geometric mean between 16 and 81.

● **Hint:** This is also called the *mean proportional*.

36 Find 4 geometric means between 7 and 1701.

37 Find a geometric mean between 9 and 64.

38 The first term of a geometric sequence is 24 and the fourth term is 3, find the fifth term and an expression for the nth term.

39 The first term of a geometric sequence is 24 and the third term is 6, find the fourth term and an expression for the nth term.

40 The common ratio in a geometric sequence is $\frac{2}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.

41 Which term of the geometric sequence 6, 18, 54, … is 118 098?

42 The 4th term and the 7th term of a geometric sequence are 18 and $\frac{729}{8}$. Is $\frac{59049}{128}$ a term of this sequence? If so, which term is it?

43 The 3rd term and the 6th term of a geometric sequence are 18 and $\frac{243}{4}$. Is $\frac{19683}{64}$ a term of this sequence? If so, which term is it?

44 Jim put €1500 into a savings account that pays 4% interest compounded semiannually. How much will his account hold 10 years later if he does not make any additional investments in this account?

45 At her daughter Jane's birth, Charlotte set aside £500 into a savings account. The interest she earned was 4% compounded quarterly. How much money will Jane have on her 16th birthday?

46 How much money should you invest now if you wish to have an amount of €4000 in your account after 6 years if interest is compounded quarterly at an annual rate of 5%?

47 In 2007, the population of Switzerland was estimated to be 7554 (in thousands). How large would the Swiss population be in 2012 if it grows at a rate of 0.5% annually?

48 The common ratio in a geometric sequence is $\frac{3}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.

49 Which term of the geometric sequence 7, 21, 63, … is 137 781?

50 Tim put €2500 into a savings account that pays 4% interest compounded semiannually. How much will his account hold 10 years later if he does not make any additional investments in this account?

51 At her son William's birth, Jane set aside £1000 into a savings account. The interest she earned was 6% compounded quarterly. How much money will William have on his 18th birthday?

4.4 Series

The word 'series' in common language implies much the same thing as 'sequence'. But in mathematics when we talk of a series, we are referring in particular to sums of terms in a sequence, e.g. for a sequence of values a_n, the corresponding series is the sequence of S_n with

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n$$

If the terms are in an arithmetic sequence, we call the sum an **arithmetic series**.

Sigma notation

Most of the series we consider in mathematics are **infinite** series. This name is used to emphasize the fact that the series contain infinitely many terms. Any sum in the series S_k will be called a partial sum and is given by

$$S_k = a_1 + a_2 + \dots + a_{k-1} + a_k$$

For convenience, this partial sum is written using the sigma notation:

$$S_k = \sum_{i=1}^{i=k} a_i = a_1 + a_2 + \dots + a_{k-1} + a_k$$

Sigma notation is a concise and convenient way to represent long sums. Here, the symbol Σ is the Greek capital letter *sigma* that refers to the initial letter of the word 'sum'. So, the expression $\sum_{i=1}^{i=k} a_i$ means the sum of all the terms a_i, where i takes the values from 1 to k. We can also write $\sum_{i=m}^{n} a_i$ to mean the sum of the terms a_i, where i takes the values from m to n. In such a sum, m is called the lower limit and n the upper limit.

Example 14

Write out what is meant by:

a) $\sum_{i=1}^{5} i^4$ b) $\sum_{r=3}^{7} 3^r$ c) $\sum_{j=1}^{n} x_j p(x_j)$

Solution

a) $\sum_{i=1}^{5} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4$

b) $\sum_{r=3}^{7} 3^r = 3^3 + 3^4 + 3^5 + 3^6 + 3^7$

c) $\sum_{j=1}^{n} x_j p(x_j) = x_1 p(x_1) + x_2 p(x_2) + \ldots + x_n p(x_n)$

Example 15

Evaluate $\sum_{n=0}^{5} 2^n$

Solution

$$\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$$

Example 16

Write the sum $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots + \frac{99}{100}$ in sigma notation.

Solution

We notice that each term's numerator and denominator are consecutive integers, so they take on the absolute value of $\frac{k}{k+1}$ or any equivalent form. We also notice that the signs of the terms alternate and that we have 99 terms. To take care of the sign, we use some power of (-1) that will start with a positive value. If we use $(-1)^k$, the first term will be negative, so we can use $(-1)^{k+1}$ instead. We can, therefore, write the sum as

$$(-1)^{1+1}\frac{1}{2} + (-1)^{2+1}\frac{2}{3} + (-1)^{3+1}\frac{3}{4} + \ldots + (-1)^{99+1}\frac{99}{100} = \sum_{k=1}^{99}(-1)^{k+1}\frac{k}{k+1}$$

Properties of the sigma notation

There are a number of useful results that we can obtain when we use sigma notation.

1 For example, suppose we had a sum of constant terms

$$\sum_{i=1}^{5} 2$$

What does this mean? If we write this out in full, we get

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 5 \times 2 = 10.$$

In general, if we sum a constant n times then we can write

$$\sum_{i=1}^{n} k = k + k + \dots + k = n \times k = nk.$$

2 Suppose we have the sum of a constant times i. What does this give us? For example,

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5 \times (1 + 2 + 3 + 4 + 5) = 75.$$

However, this can also be interpreted as follows

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5 \times (1 + 2 + 3 + 4 + 5) = 5\sum_{i=1}^{5} i$$

which implies that

$$\sum_{i=1}^{5} 5i = 5\sum_{i=1}^{5} i$$

In general, we can say

$$\begin{aligned}\sum_{i=1}^{n} ki &= k \times 1 + k \times 2 + \dots + k \times n \\ &= k \times (1 + 2 + \dots + n) \\ &= k\sum_{i=1}^{n} i\end{aligned}$$

3 Suppose that we need to consider the summation of two different functions, such as

$$\begin{aligned}\sum_{k=1}^{n} (k^2 + k^3) &= (1^2 + 1^3) + (2^2 + 2^3) + \dots + n^2 + n^3 \\ &= (1^2 + 2^2 + \dots + n^2) + (1^3 + 2^3 + \dots + n^3) \\ &= \sum_{k=1}^{n} (k^2) + \sum_{k=1}^{n} (k^3)\end{aligned}$$

In general,

$$\sum_{k=1}^{n} (f(k) + g(\mathrm{k})) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$$

Arithmetic series

In arithmetic series, we are concerned with adding the terms of arithmetic sequences. It is very helpful to be able to find an easy expression for the partial sums of this series.

Let us start with an example:

Find the partial sum for the first 50 terms of the series

$$3 + 8 + 13 + 18 + \dots$$

We express S_{50} in two different ways:

$$\begin{aligned} S_{50} &= 3 + 8 + 13 + \dots + 248, \quad \text{and} \\ S_{50} &= 248 + 243 + 238 + \dots + 3 \\ \hline 2S_{50} &= 251 + 251 + 251 + \dots + 251 \end{aligned}$$

There are 50 terms in this sum, and hence

$$2S_{50} = 50 \times 251 \Rightarrow S_{50} = \frac{50}{2}(251) = 6275.$$

This reasoning can be extended to any arithmetic series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and a common difference d. We can construct the series in two ways: Forward, by adding d to a_1 repeatedly, and backwards by subtracting d from a_n repeatedly. We get the following two expressions for the sum:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$$

and

$$S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_1 = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d)$$

By adding, term by term vertically, we get

$$\begin{array}{rcl} S_n &=& a_1 + (a_1 + \mathbf{d}) + (a_1 + \mathbf{2d}) + \ldots + (a_1 + \mathbf{(n-1)d}) \\ S_n &=& a_n + (a_n - \mathbf{d}) + (a_n - \mathbf{2d}) + \ldots + (a_n - \mathbf{(n-1)d}) \\ \hline & & \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \\ 2S_n &=& (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) \end{array}$$

Since we have n terms, we can reduce the expression above to

$2S_n = n(a_1 + a_n)$, which can be reduced to

$S_n = \frac{n}{2}(a_1 + a_n)$, which in turn can be changed to give an interesting perspective of the sum,

i.e. $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ is n times the average of the first and last terms!

If we substitute $a_1 + (n-1)d$ for a_n then we arrive at an alternative formula for the sum:

$$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d) = \frac{n}{2}(2a_1 + (n-1)d)$$

Sum of an arithmetic series

The sum, S_n, of n terms of an arithmetic series with common difference d, first term a_1, and nth term a_n is:

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{or} \quad S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Example 17

Find the partial sum for the first 50 terms of the series

$$3 + 8 + 13 + 18 + \ldots$$

Solution

Using the second formula for the sum, we get

$$S_{50} = \frac{50}{2}(2 \times 3 + (50 - 1)5) = 25 \times 251 = 6275.$$

Using the first formula requires that we know the nth term. So, $a_{50} = 3 + 49 \times 5 = 248$, which now can be used:

$$S_{50} = 25(3 + 248) = 6275.$$

Geometric series

As is the case with arithmetic series, it is often desirable to find a general expression for the nth partial sum of a geometric series.

Let us start with an example:

Find the partial sum for the first 20 terms of the series

$$3 + 6 + 12 + 24 + \ldots$$

We express S_{20} in two different ways and subtract them:

$$\begin{array}{lllllllll} S_{20} & = 3 + 6 & + & 12 & + & \ldots & + & 1\,572\,864 & \\ 2S_{20} & = \quad\;\; 6 & + & 12 & + & \ldots & + & 1\,572\,864 & + 3\,145\,728 \\ \hline -S_{20} & = 3 & & & & & & & - 3\,145\,728 \\ \Rightarrow S_{20} & = 3\,145\,725 & & & & & & & \end{array}$$

This reasoning can be extended to any geometric series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be a geometric sequence with first term a_1 and a common ratio $r \neq 1$. We can construct the series in two ways as before and using the definition of the geometric sequence, i.e. $a_n = a_{n-1} \times r$, then

$$\begin{aligned} S_n &= a_1 + \mathbf{a_2} + \mathbf{a_3} + \ldots + \mathbf{a_{n-1}} + \mathbf{a_n}, \text{ and} \\ rS_n &= ra_1 + ra_2 + ra_3 + \ldots + ra_{n-1} + ra_n \\ &= \mathbf{a_2} + \mathbf{a_3} + \ldots + \mathbf{a_{n-1}} + \mathbf{a_n} + ra_n \end{aligned}$$

Now, we subtract the first and last expressions to get

$$S_n - rS_n = a_1 - ra_n \Rightarrow S_n(1-r) = a_1 - ra_n \Rightarrow \mathbf{S_n = \frac{a_1 - ra_n}{1-r}}; r \neq 1.$$

This expression, however, requires that r, a_1, as well as a_n be known in order to find the sum. However, using the nth term expression developed earlier, we can simplify this sum formula to

$$S_n = \frac{a_1 - r\mathbf{a_n}}{1-r} = \frac{a_1 - r\mathbf{a_1 r^{n-1}}}{1-r} = \frac{a_1(1-r^n)}{1-r}; r \neq 1.$$

Sum of a geometric series

The sum, S_n, of n terms of a geometric series with common ratio r ($r \neq 1$) and first term a_1, is:

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \left[\text{equivalent to } S_n = \frac{a_1(r^n - 1)}{r-1}\right]$$

Example 18

Find the partial sum for the first 20 terms of the series $3 + 6 + 12 + 24 + \ldots$ in the opening example for this section.

Solution

$$S_{20} = \frac{3(1-2^{20})}{1-2} = \frac{3(1 - 1\,048\,576)}{-1} = 3\,145\,725$$

Infinite geometric series

Consider the series

$$\sum_{k=1}^{n} 2\left(\tfrac{1}{2}\right)^{k-1} = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

Consider also finding the partial sums for 10, 20 and 100 terms. The sums we are looking for are the partial sums of a geometric series. So,

$$\sum_{k=1}^{10} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}} \approx 3.996$$

$$\sum_{k=1}^{20} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} \approx 3.999\,996$$

$$\sum_{k=1}^{100} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{100}}{1 - \frac{1}{2}} \approx 4$$

As the number of terms increases, the partial sum appears to be approaching the number 4. This is no coincidence. In the language of limits,

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = \lim_{n\to\infty} 2 \times \frac{1 - \left(\frac{1}{2}\right)^{k}}{1 - \frac{1}{2}} = 2 \times \frac{1-0}{\frac{1}{2}} = 4, \text{since } \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n} = 0.$$

This type of problem allows us to extend the usual concept of a 'sum' of a **finite** number of terms to make sense of sums in which an **infinite** number of terms is involved. Such series are called **infinite series.**

One thing to be made clear about infinite series is that they are not true sums! The associative property of addition of real numbers allows us to extend the definition of the sum of two numbers, such as $a + b$, to three or four or n numbers, but not to an infinite number of numbers. For example, you can add any specific number of 5s together and get a real number, but if you add an *infinite* number of 5s together, you cannot get a real number! The remarkable thing about infinite series is that, in some cases, such as the example above, the sequence of partial sums (which are true sums) approach a finite limit L. The limit in our example is 4. This we write as

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n) = L.$$

We say that the series **converges** to L, and it is convenient to define L as the **sum of the infinite series.** We use the notation

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L.$$

We can, therefore, write the limit above as

$$\sum_{k=1}^{\infty} 2\left(\frac{1}{2}\right)^{k-1} = \lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = 4.$$

If the series does not have a limit, it **diverges** and does not have a sum.

We are now ready to develop a general rule for **infinite geometric series.** As you know, the sum of the geometric series is given by

$$S_n = \frac{a_1 - r\boldsymbol{a_n}}{1 - r} = \frac{a_1 - r\boldsymbol{a_1 r^{n-1}}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}; r \neq 1.$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and

$$S_n = S = \lim_{n\to\infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}.$$

We will call this **the sum of the infinite geometric series**. In all other cases the series diverges. The proof is left as an exercise.

$$\sum_{k=1}^{\infty} 2\left(\frac{1}{2}\right)^{k-1} = \frac{2}{1-\frac{1}{2}} = 4, \text{ as already shown.}$$

Sum of an infinite geometric series

The sum, S_∞, of an infinite geometric series with first term a_1, such that the common ratio r satisfies the condition $|r| < 1$ is given by:

$$S_\infty = \frac{a_1}{1-r}$$

Example 19

A rational number is a number that can be expressed as a quotient of two integers. Show that $0.\overline{6} = 0.666\ldots$ is a rational number.

Solution

$$0.\overline{6} = 0.666\ldots = 0.6 + 0.06 + 0.006 + 0.0006 + \ldots$$
$$= \frac{6}{10} + \frac{6}{10}\cdot\frac{1}{10} + \frac{6}{10}\cdot\left(\frac{1}{10}\right)^2 + \frac{6}{10}\cdot\left(\frac{1}{10}\right)^3 + \ldots$$

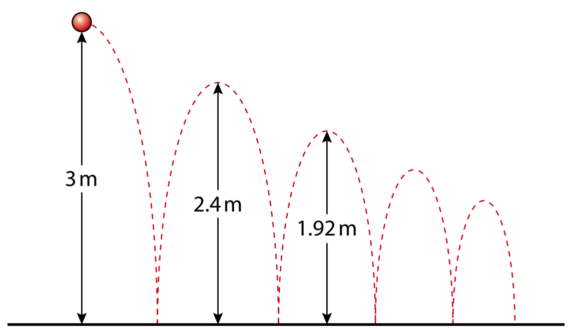
This is an infinite geometric series with $a_1 = \frac{6}{10}$ and $r = \frac{1}{10}$; therefore,

$$0.\overline{6} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{6}{10}\cdot\frac{10}{9} = \frac{2}{3}$$

Example 20

If a ball has elasticity such that it bounces up 80% of its previous height, find the total vertical distances travelled down and up by this ball when it is dropped from an altitude of 3 metres. Ignore friction and air resistance.

Solution



After the ball is dropped the initial 3 m, it bounces up and down a distance of 2.4 m. Each bounce after the first bounce, the ball travels 0.8 times the previous height twice – once upwards and once downwards. So, the total vertical distance is given by

$$h = 3 + 2(2.4 + (2.4 \times 0.8) + (2.4 \times 0.8^2) + \ldots) = 3 + 2 \times l$$

The amount in parenthesis is an infinite geometric series with $a_1 = 2.4$ and $r = 0.8$. The value of that quantity is

$$l = \frac{2.4}{1-0.8} = 12.$$

Hence, the total distance required is

$$h = 3 + 2(12) = 27 \text{ m}.$$

Applications of series to compound interest calculations

Annuities

An **annuity** is a sequence of equal periodic payments. If you are saving money by depositing the same amount at the end of each compounding period, the annuity is called **ordinary annuity**. Using geometric series you can calculate the **future value (FV)** of this annuity, which is the amount of money you have after making the last payment.

You invest €1000 at the end of each year for 10 years at a fixed annual interest rate of 6%. See table below.

Year	Amount invested	Future value
10	1000	1000
9	1000	$1000(1 + 0.06)$
8	1000	$1000(1 + 0.06)^2$
⋮		
1	1000	$1000(1 + 0.06)^9$

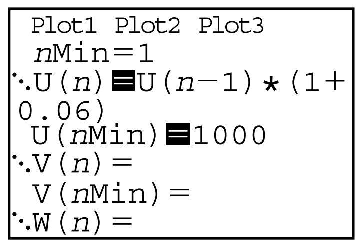
Table 4.3 Calculating the future value.

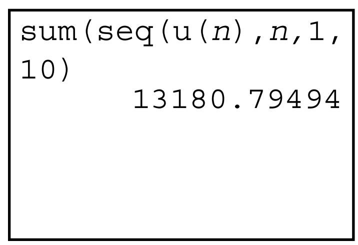
The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = 1000 + 1000(1 + 0.06) + 1000(1 + 0.06)^2 + \ldots + 1000(1 + 0.06)^9$$

This sum is a partial sum of a geometric series with $n = 10$ and $r = 1 + 0.06$. Hence,

$$FV = \frac{1000(1 - (1 + 0.06)^{10})}{1 - (1 + 0.06)} = \frac{1000(1-(1 + 0.06)^{10})}{-0.06} = 13\,180.79.$$





This result can also be produced with a GDC, as shown.

We can generalize the previous formula in the same manner. Let the periodic payment be R and the periodic interest rate be i, i.e. $i = \frac{r}{n}$. Let the number of periodic payments be m.

Period	Amount invested	Future value
m	R	R
$m - 1$	R	$R(1 + i)$
$m - 2$	R	$R(1 + i)^2$
⋮		
1	R	$R(1 + i)^{m-1}$

Table 4.4 Calculating the future value – formula.

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{m-1}$$

This sum is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence,

$$FV = \frac{R(1 - (1+i)^m)}{1 - (1+i)} = \frac{R(1 - (1+i)^m)}{-i} = R\left(\frac{(1+i)^m - 1}{i}\right)$$

Note: If the payment is made at the beginning of the period rather than at the end, the annuity is called **annuity due** and the future value after m periods will be slightly different. The table for this situation is given below.

Table 4.5 Calculating the future value (annuity due).

Period	Amount invested	Future value
m	R	$R(1+i)$
$m-1$	R	$R(1+i)^2$
$m-2$	R	$R(1+i)^3$
⋮		
1	R	$R(1+i)^m$

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = R(1+i) + R(1+i)^2 + \ldots + R(1+i)^{m-1} + R(1+i)^m$$

This sum is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence,

$$FV = \frac{R(1 + i(1 - (1+i)^m))}{1 - (1+i)} = \frac{R(1 + i - (1+i)^{m+1})}{-i} = R\left(\frac{(1+i)^{m+1} - 1}{i} - 1\right)$$

If the previous investment is made at the beginning of the year rather than at the end, then in 10 years we have

$$FV = R\left(\frac{(1+i)^{m+1} - 1}{i} - 1\right) = 1000\left(\frac{(1+0.06)^{10+1} - 1}{0.006} - 1\right) = 13\,971.64.$$

Exercise 4.4

1 Find the sum of the arithmetic series $11 + 17 + \ldots + 365$.

2 Find the sum:

$2 - 3 + \frac{9}{2} - \frac{27}{4} + \ldots - \frac{177\,147}{1024}$

3 Evaluate $\sum_{k=0}^{13} (2 - 0.3k)$.

4 Evaluate $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \ldots$

5 Evaluate $\frac{1}{3} + \frac{\sqrt{3}}{12} + \frac{1}{16} + \frac{\sqrt{3}}{64} + \frac{3}{256} + \ldots$

6 Express each repeating decimal as a fraction:

a) $0.\overline{52}$ b) $0.4\overline{53}$ c) $3.01\overline{37}$

7 At the beginning of every month, Maggie invests £150 in an account that pays 6% annual rate. How much money will there be in the account after six years?

In questions 8–10, find the sum.

8 $9 + 13 + 17 + \ldots + 85$

9 $8 + 14 + 20 + \ldots + 278$

10 $155 + 158 + 161 + \ldots + 527$

11 The kth term of an arithmetic sequence is $2 + 3k$. Find, in terms of n, the sum of the first n terms of this sequence.

12 How many terms should we add to exceed 678 when we add $17 + 20 + 23 \ldots$?

13 How many terms should we add to exceed 2335 when we add $-18 - 11 - 4 \ldots$?

14 An arithmetic sequence has a as first term and $2d$ as common difference, i.e., a, $a + 2d$, $a + 4d$, …. The sum of the first 50 terms is T. Another sequence, with first term $a + d$, and common difference $2d$, is combined with the first one to produce a new arithmetic sequence. Let the sum of the first 100 terms of the new combined sequence be S. If $2T + 200 = S$, find d.

15 Consider the arithmetic sequence 3, 7, 11, …, 999.

a) Find the number of terms and the sum of this sequence.

b) Create a new sequence by removing every third term, i.e., 11, 23,…. Find the sum of the terms of the remaining sequence.

16 The sum of the first 10 terms of an arithmetic sequence is 235 and the sum of the second 10 terms is 735. Find the first term and the common difference.

In questions 17–19, use your GDC or a spreadsheet to evaluate each sum.

17 $\displaystyle\sum_{k=1}^{20} (k^2 + 1)$

18 $\displaystyle\sum_{i=3}^{17} \frac{1}{i^2 + 3}$

19 $\displaystyle\sum_{n=1}^{100} (-1)^n \frac{3}{n}$

20 Find the sum of the arithmetic series

$13 + 19 + \ldots + 367$

21 Find the sum

$2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots - \frac{4096}{177\,147}$

22 Evaluate $\displaystyle\sum_{k=0}^{11} (3 + 0.2k)$.

23 Evaluate $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots$

24 Evaluate $\frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{3}} + \frac{1}{3} + \frac{\sqrt{2}}{3\sqrt{3}} + \frac{2}{9} + \ldots$

In questions 25–27, find the first four partial sums and then the nth partial sum of each sequence.

25 $u_n = \frac{3}{5^n}$

26 $v_n = \frac{1}{n^2 + 3n + 2}$ Hint: Show that $v_n = \frac{1}{n+1} - \frac{1}{n+2}$

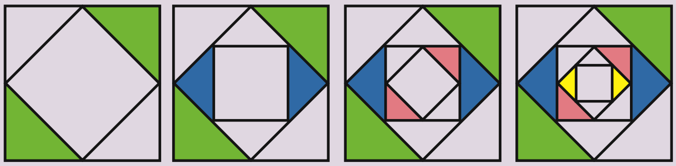
27 $u_n = \sqrt{n+1} - \sqrt{n}$

28 A ball is dropped from a height of 16 m. Every time it hits the ground it bounces 81% of its previous height.

a) Find the maximum height it reaches *after* the 10th bounce.

b) Find the total distance travelled by the ball till it rests. (Assume no friction and no loss of elasticity).

29

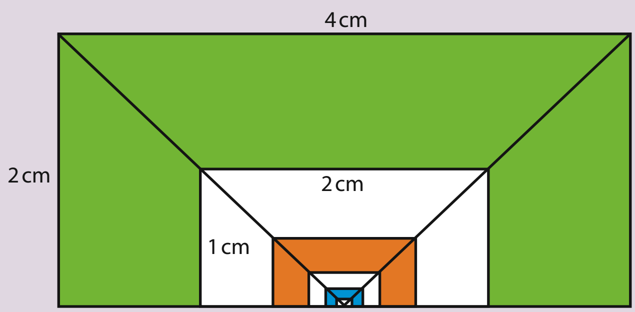


The sides of a square are 16 cm in length. A new square is formed by joining the midpoints of the adjacent sides and two of the resulting triangles are coloured as shown.

a) If the process is repeated 6 more times, determine the total area of the shaded region.

b) If the process is repeated indefinitely, find the total area of the shaded region.

30



The largest rectangle has dimensions 4 by 2, as shown; another rectangle is constructed inside it with dimensions 2 by 1. The process is repeated. The region surrounding every other inner rectangle is shaded, as shown.

a) Find the total area for the three regions shaded already.

b) If the process is repeated indefinitely, find the total area of the shaded regions.

In questions 31–34, find each sum.

31 $7 + 12 + 17 + 22 + \ldots + 337 + 342$

32 $9486 + 9479 + 9472 + 7465 + \ldots + 8919 + 8912$

33 $2 + 6 + 18 + 54 + \ldots + 3\,188\,646 + 9\,565\,938$

34 $120 + 24 + \frac{24}{5} + \frac{24}{25} + \ldots + \frac{24}{78\,125}$

4.5 Counting principles

Simple counting problems

This section will introduce you to some of the basic principles of counting. In Section 4.6 you will apply some of this in justifying the binomial theorem and in Chapter 12 you will use these principles to tackle many probability problems. We will start with two examples.

Example 21

Nine paper chips each carrying the numerals 1–9 are placed in a box. Two chips are chosen such that the first chip is chosen, the number is recorded and the chip is put back in the box, then the second chip is drawn. The numbers on the chips are added. In how many ways can you get a sum of 8?

Solution

To solve this problem, count the different number of ways that a total of 8 can be obtained:

1st chip	1	2	3	4	5	6	7
2nd chip	7	6	5	4	3	2	1

From this list, it is clear that you can have 7 different ways of receiving a sum of 8.

Example 22

Suppose now that the first chip is chosen, the number is recorded and the chip is *not* put back in the box, then the second chip is drawn. In how many ways can you get a sum of 8?

Solution

To solve this problem too, count the different number of ways that a total of 8 can be obtained:

1st chip	1	2	3	5	6	7
2nd chip	7	6	5	3	2	1

From this list, it is clear that you can have 6 different ways of receiving a sum of 8.

The difference between the two situations is described by saying that the first random selection is done **with replacement**, while the second is **without replacement**, which ruled out the use of two 4s.

Fundamental principle of counting

The above examples show you simple counting principles in which you can list each possible way that an event can happen. In many other cases, listing the ways an event can happen may not be feasible. In such cases we need to rely on counting principles. The most important of which is the **fundamental principle of counting**, also known as the multiplication principle. Consider the following situations:

Example 23

You can make a sandwich from one of three types of bread and one of four kinds of cheese, with or without pickles. How many different kinds of sandwiches can be made?

Solution

With each type of bread you can have 4 sandwiches. There are 12 possible sandwiches altogether. These are without pickles; if you want sandwiches with pickles, then you have 24 possible ones. That is, there are $3 \times 4 \times 2 = 24$ possible sandwiches.

Example 24

How many 3-digit even numbers are there?

Solution

The first digit cannot be zero, since the number has to be a 3-digit number, so there are 9 ways the hundred's digit can be. There is no condition on what the ten's digit should be, so we have 10 possibilities, and to be even, the number must end with 0, 2, 4, 6, or 8. Therefore, we have $9 \times 10 \times 5 = 450$ 3-digit even numbers.

Examples 23 and 24 are examples of the following principle:

Fundamental principle of counting

If there are m ways an event can occur followed by n ways a second event can occur, then there are a total of $(m)(n)$ ways that the two can occur.

This principle can be extended to more than two events or processes:

If there are k events than can happen in $n_1, n_2, \ldots, n_k$ ways, then the whole sequence can happen in

$$n_1 \times n_2 \times \ldots \times n_k \text{ ways.}$$

Example 25

A large school issues special coded identification cards that consist of two letters of the alphabet followed by three numerals. For example, AB 737 is such a code. How many different ID cards can be issued if the letters or numbers can be used more than once?

Solution

As the letters can be used more than once, then each letter position can be filled in 26 different ways, i.e. the letters can be filled in $26 \times 26 = 676$ ways. Each number position can be filled in 10 different ways; hence, the numerals can be filled in $10 \times 10 \times 10 = 1000$ different ways. So, the code can be formed in $676 \times 1000 = 676\,000$ different ways.

Permutations

One major application of the fundamental principle is in determining the number of ways the n objects can be arranged. Consider the following situation for example. You have 5 books you want to put on a shelf: maths (M), physics (P), English (E), biology (B), and history (H). In how many ways can you do this?

To find this out, number the positions you want to place the books in as shown

1 2 3 4 5

If we decide to put the maths book in position 1, then there are four different ways of putting a book in position 2.

M *P*	*M* *E*	*M* *B*	*M* *H*
1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5

Since we can put any of the 5 books in the first position, then there will be $5 \times 4 = 20$ ways of shelving the first two books. Once you place the books in positions 1 and 2, the third book can be any one of three books left.

M *P* *E*	*M* *P* *B*	*M* *P* *H*
1 2 3 4 5	1 2 3 4 5	1 2 3 4 5

Once you use three books, there are two books for the fourth position and only one way of placing the fifth book. So, the number of ways of arranging all 5 books is

$$5 \times 4 \times 3 \times 2 \times 1 = 120 = 5!$$

Factorial notation

The product of the first n positive integers is denoted by $n!$ and is called ***n* factorial:**

$$n! = 1 \times 2 \times 3 \times 4 \ldots (n-2) \times (n-1) \times n$$

We also define $0! = 1$.

Permutations

An arrangement is called a **permutation**. It is the reorganization of objects or symbols into distinguishable sequences. When we place things in order, we say we have made an arrangement. When we change the order, we say we have changed the arrangement. So each of the arrangements that can be made by taking *some* or *all* of a number of things is known as a **permutation**.

● **Hint:** A permutation of n different objects can be understood as an ordering (arrangement) of the objects such that one object is first, one is second, one is third, and so on.

Number of permutations of *n* objects

The previous set up can be applied to n objects rather than only 5. The number of ways of filling in the first position can be done in n ways.

n	$n-1$	$n-2$	$n-3$	…	1
1	2	3	4		n

Once the first position is filled, the second position can be filled by any of the $n - 1$ objects left, and hence using the fundamental principle there will be $n \cdot (n-1)$ different ways for filling the first two positions. Repeating the same procedure till the nth position is filled is therefore

$$n \cdot (n-1) \cdot (n-2) \ldots 2 \cdot 1 = n!$$

Frequently, we are engaged in arranging a **subset** of the whole collection rather than the entire collection. For example, suppose we want to shelve 3 of the books rather than all 5 of them. The discussion will be analogous to the previous situation. However, we have to limit our search to the first three positions only, i.e. the number of ways we can shelve three out of the 5 books is

$$5 \times 4 \times 3 = 60$$

To change this product into factorial notation, we do the following:

$$5 \times 4 \times 3 = 5 \times 4 \times 3 \times \frac{2!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2!} = \frac{5!}{2!}$$
$$= \frac{5!}{(5-3)!}$$

This leads us to the following general result.

Number of permutations of *n* objects taken *r* at a time

The number of permutations of n objects taken r at a time is

$${}^nP_r = {}_nP_r = P^n_r = P(n, r) = \frac{n!}{(n-r)!};\ n \geqslant r$$

To verify the formula above, you can proceed in the same manner as with the permutation of n objects.

n	$n-1$	$n-2$	$n-3$		$n-(r-1)$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	…	$\downarrow$
1	2	3	4		r

When you arrive to the rth position, you would have used $r - 1$ objects already, and hence you are left with $n - (r - 1) = n - r + 1$ objects to fill this position. So, the number of ways of arranging n objects taken r at a time is

$${}^nP_r = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)$$

Here again, to make the expression more manageable, we can write it in factorial notation:

$$\begin{aligned} {}^nP_r &= n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) \\ &= n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) \frac{(n-r)!}{(n-r)!} \\ &= \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) \cdot (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \end{aligned}$$

Example 26

15 drivers are taking part in a Formula 1 car race. In how many different ways can the top 6 positions be filled?

Solution

Since the drivers are all different, this is a permutation of 15 'objects' taken 6 at a time.

$$^{15}P_6 = \frac{15!}{(15-6)!} = 3\,603\,600$$

This can also be easily calculated using a GDC.

```
15 nPr 6
                3603600
15!/9!
                3603600
```

Combinations

A **combination** is a selection of some or all of a number of different objects. It is an unordered collection of unique sizes. In a permutation, the order of occurrence of the objects or the arrangement is important, but in combination the order of occurrence of the objects is not important. In that sense, a combination of r objects out of n objects is a subset of the set of n objects.

For example, there are 24 permutations of three letters out of ABCD, while there are only 4 combinations! Here is why:

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

For one combination, ABC for example, there are $3! = 6$ permutations. This is true for all combinations. So, the number of permutations is 6 times the number of combinations, i.e.

$$^4P_3 = 3!\,^4C_3$$

where $^4C_3$ is the number of combinations of the 4 letters taken 3 at a time.

According to the previous result, we can write

$$^4C_3 = \frac{^4P_3}{3!} = \frac{\frac{4!}{(4-3)!}}{3!} = \frac{4!}{3!(4-3)!}$$

The last result can also be generalized to n elements combined r at a time. (The ISO notation for this quantity, which is also used by the IB is $\binom{n}{r}$. In this book, we will follow the ISO notation.)

$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$. This symmetry is obvious as when we pick r objects, we leave $n - r$ objects behind, and hence the number of ways of choosing r objects is the same as the number of ways of $n - r$ objects not chosen.

Every subset of r objects (combination), gives rise to $r!$ permutations. So, if you have $\binom{n}{r}$ combinations, these will result in $r!\binom{n}{r}$ permutations. Therefore,

$$^nP_r = r!\binom{n}{r} \Leftrightarrow \boxed{\binom{n}{r} = \frac{^nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\,r!}}$$

```
45 nCr 6
             8145060
█
```

Example 27

A lottery has 45 numbers. If you buy a ticket, then you choose 6 of these numbers. How many different choices does this lottery have?

Solution

Since 6 numbers will have to be chosen and order is not an issue here, this is a combination case. The number of possible choices is

$$\binom{45}{6} = 8\ 145\ 060.$$

This can also be calculated using a GDC.

Example 28

In poker, a deck of 52 cards is used, and a 'hand' is made up of 5 cards.

a) How many hands are there?

b) How many hands are there with 3 diamonds and 2 hearts?

Solution

a) Since the order is not important, as a player can reorder the cards after receiving them, this is a combination of 52 cards taken 5 at a time:

$\binom{52}{5} = 2\ 598\ 960.$

```
52 nCr 5
             2598960
13 nCr 3
                 286
13 nCr 2
                  78
█
```

b) Since there are 13 diamonds and we want 3 of them, there are $\binom{13}{3} = 286$ ways to get the 3 diamonds. Since there are 13 hearts and we want 2 of them, there are $\binom{13}{2} = 78$ ways to get the 2 hearts. Since we want them both to occur at the same time, we use the fundamental counting principle and multiply 286 and 78 together to get 22 308 possible hands.

Example 29

A code is made up of 6 different digits. How many possible codes are there?

Solution

Since there are 10 digits and we are choosing 6 of them, and since the order we use these digits makes a difference in the code, then this is a permutation case. The number of possible codes is

$${}^{10}P_6 = 151\ 200.$$

Exercise 4.5

1 Evaluate each of the following expressions.

a) ${}^{5}P_5$ b) $5!$ c) ${}^{20}P_1$ d) ${}^{8}P_3$

2 Evaluate each of the following expressions.

a) $\binom{5}{5}$ b) $\binom{5}{0}$ c) $\binom{10}{3}$ d) $\binom{10}{7}$

3 Evaluate each of the following expressions.

a) $\binom{7}{3} + \binom{7}{4}$ b) $\binom{8}{4}$ c) $\binom{10}{6} + \binom{10}{7}$ d) $\binom{11}{7}$

4 Evaluate each of the following expressions.

a) $\binom{8}{5} - \binom{8}{3}$ b) $11 \cdot 10!$ c) $\binom{10}{3} - \binom{10}{7}$ d) $\binom{10}{1}$

5 Tell whether each of the following expressions is true.

a) $\frac{10!}{5!} = 2!$ b) $(5!)^2 = 25!$ c) $\binom{101}{8} = \binom{101}{93}$

6 You are buying a computer and have the following choices: three types of HD, two types of DVD players, four types of graphic cards. How many different systems can you choose from?

7 You are going to a restaurant with a set menu. They have three starters, four main meals, two drinks, and three deserts. How many different choices are available for you to choose your meal from?

8 A school is in need of three teachers: PE, maths, and English. They have 8 applicants for the PE position, 3 applicants for the maths position and 13 applicants for English. How many different combinations of choices do they have?

9 You are given a multiple choice test where each question has four possible answers. The test is made up of 12 questions and you are guessing at random. In how many ways can you answer all the questions on the test?

10 The test in question 9 is divided into two parts, the first six are true/false questions and the last six are multiple choice as described. In how many different ways can you answer all questions on that test?

11 Passwords on a network are made up of two parts. One part consists of three letters of the alphabet, not necessarily different, and five digits, also not necessarily different. How many passwords are possible on this network?

12 How many 5-digit numbers can be made if the units digit cannot be 0?

13 Four couples are to be seated in a theatre row. In how many different ways can they be seated if

a) no restrictions are made

b) every two members of each couple like to sit together?

14 Five girls and three boys should go through a doorway in single file. In how many orders can they do that if

a) there are no constraints

b) the girls must go first?

15 Write all the permutations of the letters in JANE.

16 Write all the permutations of the letters in MAGIC taken three at a time.

17 A computer code is made up of three letters followed by four digits.

a) In how many ways is the code possible?

b) If 97 of the three-letter combinations cannot be used because they are offensive, how many codes are still possible?

18 A local bridge club has 17 members, 10 females and 7 males. They have to elect three officers: president, deputy, and treasurer. In how many ways is this possible if

a) there are no restrictions

b) the president is a male

c) the deputy must be a male, the president can be any gender, but the treasurer must be a female

d) the president and deputy are of the same gender

e) all three officers are not the same gender.

19 The research and development department for a computer manufacturer has 26 employees: 8 mathematicians, 12 computer scientists, and 6 electrical engineers. They need to select three employees to be leaders of the group. In how many ways can they do this if

a) the three officers are of the same specialization

b) at least one of them must be an engineer

c) two of them must be mathematicians?

20 A 'combination' lock has three numbers, each in the range 1 to 50.

a) How many different combinations are possible?

b) How many combinations do not have duplicates?

c) How many have the first and second numbers matching?

d) How many have exactly two of the numbers matching?

21 In how many ways can five married couples be seated around a circle so that spouses sit together?

22 a) How many subsets of $\{1, 2, 3, \ldots, 9\}$ have two elements?

b) How many subsets of $\{1, 2, 3, \ldots, 9\}$ have an odd number of elements?

23 Nine seniors and 12 juniors make up the maths club at a school. They need four members for an upcoming competition.

a) How many 4-member teams can they form?

b) How many of these 4-member teams have the same number of juniors and seniors?

c) How many of these 4-member teams have more juniors than seniors?

24 This problem uses the same data as question 23 above. Tim, a junior, is the strongest 'mathlete' among his group while senior Gwen is the strongest among her group. Either Tim or Gwen must be on the team, but they cannot both be on the team. Answer the same questions as above.

25 A shipment of 100 hard disks contains 4 defective disks. We choose a sample of 6 disks for inspection.

a) How many different possible samples are there?

b) How many samples could contain all 4 defective disks? What percentage of the total is that?

c) How many samples could contain at least 1 defective disk? What percentage of the total is that?

26 There are three political parties represented in a parliament: 10 conservatives, 8 liberals, and 4 independents. A committee of 6 members is needed to be set up.

a) How many different committees are possible?

b) How many committees with equal representation are possible?

27 How many ways are there for 9 boys and 6 girls to stand in a line so that no two girls stand next to each other?

4.6 The binomial theorem

A binomial is a polynomial with two terms. For example, $x + y$ is a binomial. In principle, it is easy to raise $x + y$ to any power; but raising it to high powers would be tedious. We will find a formula that gives the expansion of $(x + y)^n$ for any positive integer n. The proof of the binomial theorem is given in Section 4.7.

Let us look at some special cases of the expansion of $(x + y)^n$:

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

There are several things that you will have noticed after looking at the expansion:

- There are $n + 1$ terms in the expansion of $(x + y)^n$.
- The degree of each term is n.
- The powers on x begin with n and decrease to 0.
- The powers on y begin with 0 and increase to n.
- The coefficients are symmetric.

For instance, notice how the exponents of x and y behave in the expansion of $(x + y)^5$.

The exponents of x decrease:

$(x + y)^5 = x^{\boxed{5}} + 5x^{\boxed{4}}y + 10x^{\boxed{3}}y^2 + 10x^{\boxed{2}}y^3 + 5x^{\boxed{1}}y^4 + x^{\boxed{0}}y^5$

The exponents of y increase:

$(x + y)^5 = x^5y^{\boxed{0}} + 5x^4y^{\boxed{1}} + 10x^3y^{\boxed{2}} + 10x^2y^{\boxed{3}} + 5xy^{\boxed{4}} + y^{\boxed{5}}$

Using this pattern, we can now proceed to expand any binomial raised to power n: $(x + y)^n$. For example, leaving a blank for the missing coefficients, the expansion for $(x + y)^7$ can be written as

$(x+y)^7$

$= \square x^7 + \square x^6y + \square x^5y^2 + \square x^4y^3 + \square x^3y^4 + \square x^2y^5 + \square xy^6 + \square y^7$

To finish the expansion we need to determine these coefficients. In order to see the pattern, let us look at the coefficients of the expansion we started the section with.

$(x+y)^0$	1							row 0
$(x+y)^1$	1	1						row 1
$(x+y)^2$	1	2	1					row 2
$(x+y)^3$	1	3	3	1				row 3
$(x+y)^4$	1	4	6	4	1			row 4
$(x+y)^5$	1	5	10	10	5	1		row 5
$(x+y)^6$	1	6	15	20	15	6	1	row 6
	column 0	column 1	column 2	column 3	column 4	column 5	column 6	

A triangle like the one above is known as Pascal's triangle. Notice how the first and **second** terms in row **3** give you the **second** term in row **4**; the third and **fourth** terms in row **3** give you the **fourth** term of row **4**; the second and **third** terms in row **5** give you the **third** term in row **6**; and the fifth and **sixth** terms in row **5** give you the **sixth** term in row **6**, and so on. So now we can state the key property of Pascal's triangle.

Pascal's triangle was known to Persian and Chinese mathematicans in the 13th century.

Pascal's triangle

Every entry in a row is the sum of the term directly above it and the entry diagonally above and to the left of it. When there is no entry, the value is considered zero.

Take the last entry in row 5, for example; there is no entry directly above it, so its value is $0 + 1 = 1$.

From this property it is easy to find all the terms in any row of Pascal's triangle from the row above it. So, for the expansion of $(x + y)^7$, the terms are found from row 6 as follows:

$$\begin{array}{ccccccccccccccccc} 0 & \rightarrow & 1 & \rightarrow & 6 & \rightarrow & 15 & \rightarrow & 20 & \rightarrow & 15 & \rightarrow & 6 & \rightarrow & 1 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

So, $(x + y)^7 = x^7 + \boxed{7}\,x^6y + \boxed{21}\,x^5y^2 + \boxed{35}\,x^4y^3 + \boxed{35}\,x^3y^4 + \boxed{21}\,x^2y^5$
$+ \boxed{7}\,xy^6 + y^7$.

Note: Several sources use a slightly different arrangement for Pascal's triangle. The common usage considers the triangle as isosceles and uses the principle that every two entries add up to give the entry diagonally below them, as shown in the following diagram.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \end{array}$$

Example 30

Use Pascal's triangle to expand $(2k - 3)^5$.

Solution

We can find the expansion above by replacing x by $2k$ and y by -3 in the binomial expansion of $(x + y)^5$.

Using the fifth row of Pascal's triangle for the coefficients will give us the following:

$$\mathbf{1}(2k)^5 + \mathbf{5}(2k)^4(-3) + \mathbf{10}(2k)^3(-3)^2 + \mathbf{10}(2k)^2(-3)^3 + \mathbf{5}(2k)(-3)^4 + \mathbf{1}(-3)^5 = 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243.$$

Pascal's triangle is an easy and useful tool in finding the coefficients of the binomial expansion for relatively small values of n. It is not very efficient doing that for large values of n. Imagine you want to evaluate $(x + y)^{20}$. Using Pascal's triangle, you will need the terms in the 19th row and the 18th row and so on. This makes the process tedious and not practical.

Luckily, we have a formula that can find the coefficients of any Pascal's triangle row. This formula is the binomial formula, whose proof is beyond the scope of this book. Every entry in Pascal's triangle is denoted by $\binom{n}{r}$, which is also known as the binomial coefficient.

In $\binom{n}{r}$, n is the row number and r is the column number.

The factorial notation makes many formulae involving the multiplication of consecutive positive integers shorter and easier to write. That includes the binomial coefficient.

The binomial coefficient

With n and r as non-negative integers such that $n \geqslant r$, the binomial coefficient $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 31

Find the value of a) $\binom{7}{3}$ b) $\binom{7}{4}$ c) $\binom{7}{0}$ d) $\binom{7}{7}$

Solution

a) $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \cdot 5 \cdot 6 \cdot 7}{(1 \cdot 2 \cdot 3)(\cancel{1 \cdot 2 \cdot 3 \cdot 4})} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 35$

b) $\binom{7}{4} = \frac{7!}{3!(7-4)!} = \frac{7!}{4!3!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4} \cdot 5 \cdot 6 \cdot 7}{(\cancel{1 \cdot 2 \cdot 3 \cdot 4})(1 \cdot 2 \cdot 3)} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 35$

c) $\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{\cancel{7}!}{0!\cancel{7}!} = \frac{1}{1} = 1$

d) $\binom{7}{7} = \frac{7!}{7!(7-7)!} = \frac{\cancel{7}!}{\cancel{7}!0!} = \frac{1}{1} = 1$

● **Hint:** Your calculator can do the tedious work of evaluating the binomial coefficient. If you have a TI, the binomial coefficient appears as ${}_nC_r$, which is another notation frequently used in mathematical literature.

```
7 nCr 3
                35
7 nCr 4
                35
7 nCr 0
                 1
█
```

Although the binomial coefficient $\binom{n}{r}$ appears as a fraction, all its results where n and r are non-negative integers are positive integers. Also, notice the **symmetry** of the coefficient in the previous examples. This is a property that you are asked to prove in the exercises:

$$\binom{n}{r} = \binom{n}{n-r}$$

Example 32

Calculate the following:

$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5}, \binom{6}{6}$

Solution

$\binom{6}{0} = \mathbf{1}, \binom{6}{1} = \mathbf{6}, \binom{6}{2} = \mathbf{15}, \binom{6}{3} = \mathbf{20}, \binom{6}{4} = \mathbf{15}, \binom{6}{5} = \mathbf{6}, \binom{6}{6} = \mathbf{1}$

The values we calculated above are precisely the entries in the sixth row of Pascal's triangle.

We can write Pascal's triangle in the following manner:

$$\begin{array}{lllllll}
\binom{0}{0} \\
\binom{1}{0} & \binom{1}{1} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\cdots & \cdots & \cdots & \cdots \\
\binom{n}{0} & \binom{n}{1} & \cdots & \cdots & \cdots & \cdots & \binom{n}{n}
\end{array}$$

Example 33

Calculate $\binom{n}{r-1} + \binom{n}{r}$.

• **Hint:** You will be able to provide reasons for the steps after you do the exercises!

This is called Pascal's rule.

Solution

$$\begin{aligned}
\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\
&= \frac{n! \cdot r}{r \cdot (r-1)!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r)! \cdot (n-r+1)} \\
&= \frac{n! \cdot r}{r!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r+1)!} \\
&= \frac{n! \cdot r + n! \cdot (n-r+1)}{r!(n-r+1)!} = \frac{n!(r+n-r+1)}{r!(n-r+1)!} \\
&= \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = \binom{n+1}{r}
\end{aligned}$$

If we read the result above carefully, it says that the sum of the terms in the nth row $(r-1)$th and rth columns is equal to the entry in the $(n+1)$th row and rth column. That is, the two entries on the left are adjacent entries in the nth row of Pascal's triangle and the entry on the right is the entry in the $(n+1)$th row directly below the rightmost entry. This is precisely the principle behind Pascal's triangle!

Using the binomial theorem

We are now prepared to state the binomial theorem. The proof of the theorem is optional and will require mathematical induction. We will develop the proof in Section 4.7.

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \binom{n}{3} x^{n-3}y^3 + \ldots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n$$

In a compact form, we can use sigma notation to express the theorem as follows:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Example 34

Use the binomial theorem to expand $(x+y)^7$.

Solution

$$\begin{aligned}(x+y)^7 &= \binom{7}{0}x^7 + \binom{7}{1}x^{7-1}y + \binom{7}{2}x^{7-2}y^2 + \binom{7}{3}x^{7-3}y^3 + \binom{7}{4}x^{7-4}y^4 \\ &\quad + \binom{7}{5}x^{7-5}y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7 \\ &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7\end{aligned}$$

Example 35

Find the expansion for $(2k-3)^5$.

Solution

$$\begin{aligned}(2k-3)^5 &= \binom{5}{0}(2k)^5 + \binom{5}{1}(2k)^4(-3) + \binom{5}{2}(2k)^3(-3)^2 + \binom{5}{3}(2k)^2(-3)^3 \\ &\quad + \binom{5}{4}(2k)(-3)^4 + \binom{5}{5}(-3)^5 \\ &= 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243\end{aligned}$$

Example 36

Find the term containing a^3 in the expansion $(2a-3b)^9$.

Note: Why is the binomial theorem related to the number of combinations of n elements taken r at a time?

Consider evaluating $(x+y)^n$. In doing so, you have to multiply $(x+y)$ n times by itself. As you know, one term has to be x^n. How to get this term? x^n is the result of multiplying x in each of the n factors $(x+y)$ and that can only happen in one way. However, consider the term containing x^r. To have a power of r over the x, means that the x in each of r factors has to be multiplied, and the rest will be the $n-r$ y-terms. This can happen in $\binom{n}{r}$ ways. Hence, the coefficient of the term x^ry^{n-r} is $\binom{n}{r}$.

Solution

To find the term, we do not need to expand the whole expression.

Since $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$, the term containing a^3 is the term where

$n - i = 3$, i.e. when $i = 6$. So, the required term is

$$\binom{9}{6}(2a)^{9-6}(-3b)^6 = 84 \cdot 8a^3 \cdot 729b^6 = 489\,888a^3b^6.$$

Example 37

Find the term independent of x in $\left(4x^3 - \frac{2}{x^2}\right)^5$.

Solution

The phrase 'independent of x' means the term with no x variable, i.e. the constant term. A constant is equivalent to the product of a number and x^0, since $x^0 = 1$. We are looking for the term in the expansion such that the resulting power is zero. In terms of i, each term in the expansion is given by

$$\binom{5}{i}(4x^3)^{5-i}(-2x^{-2})^i$$

Thus, for the constant term:

$$3(5-i) - 2i = 0 \Rightarrow 15 - 5i = 0 \Rightarrow i = 3$$

Therefore, the term independent of x is:

$$\binom{5}{3}(4x^3)^2(-2x^{-2})^3 = 10 \cdot 16x^6(-8x^{-6}) = -1280$$

Example 38

Find the coefficient of b^6 in the expansion of $\left(2b^2 - \frac{1}{b}\right)^{12}$.

Solution

The general term is

$$\binom{12}{i}(2b^2)^{12-i}\left(-\frac{1}{b}\right)^i = \binom{12}{i}(2)^{12-i}(b^2)^{12-i}\left(-\frac{1}{b}\right)^i$$

$$= \binom{12}{i}(2)^{12-i}b^{24-2i}b^{-i}(-1)^i = \binom{12}{i}(2)^{12-i}b^{24-3i}(-1)^i$$

$24 - 3i = 6 \Rightarrow i = 6$. So, the coefficient in question is $\binom{12}{6}(2)^6(-1)^6 = 59\,136$.

Exercise 4.6

1 Use Pascal's triangle to expand each binomial.

a) $(x + 2y)^5$ b) $(a - b)^4$ c) $(x - 3)^6$

d) $(2 - x^3)^4$ e) $(x - 3b)^7$ f) $\left(2n + \frac{1}{n^2}\right)^6$

g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$

2 Evaluate each expression.

a) $\binom{8}{3}$ b) $\binom{18}{5} - \binom{18}{13}$ c) $\binom{7}{4}\binom{7}{3}$

d) $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

e) $\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6}$

3 Use the binomial theorem to expand each of the following.

a) $(x + 2y)^7$ b) $(a - b)^6$ c) $(x - 3)^5$

d) $(2 - x^3)^6$ e) $(x - 3b)^7$ f) $\left(2n + \frac{1}{n^2}\right)^6$

g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$ h) $(1 + \sqrt{5})^4 + (1 - \sqrt{5})^4$

i) $(\sqrt{3} + 1)^8 - (\sqrt{3} - 1)^8$ j) $(1 + i)^8$, where $i^2 = -1$

k) $(\sqrt{2} - i)^6$, where $i^2 = -1$

4 Consider the expression $\left(x - \frac{2}{x}\right)^{45}$.

a) Find the first three terms of this expansion.

b) Find the constant term if it exists or justify why it does not exist.

c) Find the last three terms of the expansion.

d) Find the term containing x^3 if it exists or justify why it does not exist.

5 Prove that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ and $n \geqslant k$.

6 Prove that for any positive integer n,

$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n - 1$ • **Hint:** $2^n = (1 + 1)^n$

7 Consider all $n, k \in \mathbb{N}$ and $n \geqslant k$.

a) Verify that $k! = k(k - 1)!$

b) Verify that $(n - k + 1)! = (n - k + 1)\,(n - k)!$

c) Justify the steps given in the proof of $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ in the examples.

8 Find the value of the expression:

$$\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \ldots + \binom{6}{6}\left(\frac{2}{3}\right)^6$$

9 Find the value of the expression:

$$\binom{8}{0}\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right) + \binom{8}{2}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^2 + \ldots + \binom{8}{8}\left(\frac{3}{5}\right)^8$$

10 Find the value of the expression:

$$\binom{n}{0}\left(\frac{1}{7}\right)^n + \binom{n}{1}\left(\frac{1}{7}\right)^{n-1}\left(\frac{6}{7}\right) + \binom{n}{2}\left(\frac{1}{7}\right)^{n-2}\left(\frac{6}{7}\right)^2 + \ldots + \binom{n}{n}\left(\frac{6}{7}\right)^n$$

11 Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{x}\right)^6$.

12 Find the term independent of x in the expansion of $\left(3x - \frac{2}{x}\right)^8$.

13 Find the term independent of x in the expansion of $\left(2x - \frac{3}{x^3}\right)^8$.

14 Find the first three terms of the expansion of $(1 + x)^{10}$ and use them to find an approximation to

a) 1.01^{10} b) 0.99^{10}

15 Show that $\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \binom{n+2}{r+1}$ and interpret your result on the entries in Pascal's triangle.

16 Express each repeating decimal as a fraction:

a) $0.\overline{7}$ b) $0.3\overline{45}$ c) $3.21\overline{29}$

17 Find the coefficient of x^6 in the expansion of $(2x - 3)^9$.

18 Find the coefficient of x^3b^4 in $(ax + b)^7$.

19 Find the constant term of $\left(\frac{2}{z^2} - z\right)^{15}$.

20 Expand $(3n - 2m)^5$.

21 Find the coefficient of r^{10} in $(4 + 3r^2)^9$.

4.7 Mathematical induction

Domino effect

In addition to playing games of strategy, another familiar activity using dominoes is to place them on edge in lines, then topple the first tile, which falls on and topples the second, which topples the third, etc., resulting in all of the tiles falling. Arrangements of millions of tiles have been made that have taken many minutes to fall.

The Netherlands has hosted an annual domino toppling competition called *Domino Day* since 1986. The record, achieved in 2006, is 4 079 381 dominoes.

Similar phenomena of chains of small events each causing similar events leading to an eventual grand result, by analogy, are called *domino effects*. The phenomenon also has some theoretical bearing to familiar applications like the amplifier, digital signals, or information processing.

Induction

In mathematics, we have a parallel in **mathematical induction**, which is a method for proving a statement that is maintained about every natural number. For example,

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$$

This claims that the sum of consecutive numbers from 1 to n is half the product of the last term, n, and the integer after it.

We want to prove that this will be true for $n = 1$, $n = 2$, $n = 3$, and so on. Now we can test the formula for any given number, say $n = 3$:

$$1 + 2 + 3 = \tfrac{1}{2} \cdot 3 \cdot 4 = 6, \text{ which is true.}$$

It is also true for $n = 4$:

$$1 + 2 + 3 + 4 = \tfrac{1}{2} \cdot 4 \cdot 5 = 10$$

But how are we to prove this rule for every value of n?

The method of proof is shown to the right. It is called the principle of mathematical induction.

Mathematical induction

1) When a statement is true for the natural number $n = k$, then it is also true for its successor, $n = k + 1$; and
2) the statement is true for $n = 1$; then the statement is true for every natural number n.

Note: The order of the steps varies from one source to the other. We present you with both arrangements.

When the statement is true for $n = 1$, then according to 1), it will also be true for $n = 2$. But that implies it will be true for $n = 3$; which implies it will be true for $n = 4$. And so on. It will be true for every natural number.

To prove a statement by induction, then, we must prove parts 1) and 2) above.

The hypothesis of Step 1) – 'The statement is true for $n = k$' – is called the **induction assumption**, or the **induction hypothesis**. It is what we assume when we prove a theorem by induction.

Example 39

Prove that the sum of the first n natural numbers is given by this formula:

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$$

We will call this statement $S(n)$, because it depends on n.

Proof

We will do Steps 1) and 2) above. First, we will assume that the statement is true for $n = k$; that is, we will assume that $S(k)$ is true:

$$S(k)\text{: } 1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2} \qquad (1)$$

This is the induction assumption. Assuming this, we must prove that $S(k+1)$ is also true. That is, we must show:

$$S(k+1): 1+2+3+\ldots+(k+1)=\frac{(k+1)((k+1)+1)}{2} \qquad (2)$$

To do that, we will simply add the next term $(k+1)$ to both sides of the induction assumption, equation (1), and then simplify:

$$\begin{aligned} S(k+1): 1+2+3+\ldots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

This is equation (2), which is the first thing we wanted to show.

It is extremely important to note that mathematical induction can be used to prove results obtained in some other way. It is *not* a tool for discovering formulae or theorems.

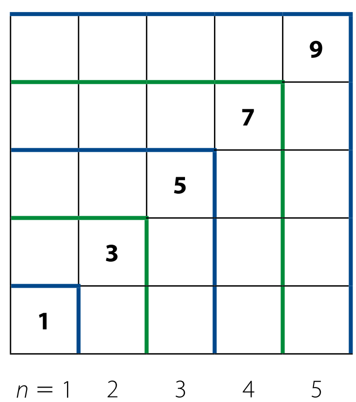
Next, we must show that the statement is true for $n = 1$. We have

$$S(1): 1=\frac{1(1+1)}{2}$$

The formula therefore is true for $n = 1$. We have now fulfilled both conditions of the principle of mathematical induction. $S(n)$ is therefore true for every natural number.

Example 40

In an investigation to find the sum of the first n positive *odd* integers, we can do the following: Investigate the sums of the first few odd integers and then try to come up with a conjecture. Then mathematical induction will provide us with a tool to prove the conjecture.



For $n = 1$, the sum is $1 = 1$.

For $n = 2$, the sum is $1 + 3 = 4$.

For $n = 3$, the sum is $1 + 3 + 5 = 9$.

For $n = 4$, the sum is $1 + 3 + 5 + 7 = 16$.

For $n = 5$, the sum is $1 + 3 + 5 + 7 + 9 = 25$.

It is clear that the number of integers you add, and the sum, are related, i.e. the sum of n such integers is n^2.

n	1	2	3	4	5	6	...	n
SUM	1	4	9	16	25	36	...	n^2

Solution

Let $S(n)$ denote the statement that the sum of the first n odd positive integers is n^2.

First, we must complete the basis step, i.e. we must show that $S(1)$ is true. Then we must carry out the inductive step, i.e. we have to *show* that $S(k + 1)$ is true whenever $S(k)$ is assumed true.

Basis step: $S(1)$, which means that the sum of the first odd integer is 1^2. This is obvious as the sum of 1 is 1!

Inductive step: We must show that the implication $S(k) \Rightarrow S(k + 1)$ is true, regardless of the choice of k. To that end, we start with an assumption that $S(k)$ is true for any choice of k; i.e.

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2.$$

Now, we must show that $S(k + 1)$ is true.

$$S(k + 1): 1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2,$$
(the $(k + 1)$th odd integer is $2(k + 1) - 1 = 2k + 1$)

The left-hand side can be written as

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2,$$

Therefore, $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$, which is nothing but $S(k + 1)$.

This shows that $S(k + 1)$ follows from $S(k)$. Since $S(1)$ is true, and the implication $S(k) \Rightarrow S(k + 1)$ is true for all positive integers k, the mathematical induction principle shows that $S(n)$ is true for all positive integers n.

Example 41

Prove that $3^n < n!$ for all integers $n > 6$.

Solution

Let $S(n)$ be the statement that $3^n < n!$

Basis step: To prove this inequality the basis step must be $S(7)$. Note that $S(6)$: $3^6 = 729 < 6! = 720$ is not true! $S(7)$: $3^7 = 2187 < 7! = 5040$ is true.

In general, a proof by mathematical induction that a statement $S(n)$ is true for every positive integer $n \geqslant 1$ consists of two steps:

BASIS STEP: The statement $S(1)$ is *shown* to be true.

INDUCTIVE STEP: The implication $S(k) \Rightarrow S(k + 1)$ is *shown* to be true for any positive integer k.

Not all statements are true for all positive integers $n \geqslant 1$. In such cases, a variation of the mathematical induction principle is used: A statement $S(n)$ is true for every positive integer $n \geqslant n_0$ consists of two steps:

BASIS STEP: The statement $S(n_0)$ is shown to be true.

INDUCTIVE STEP: The implication $S(k) \Rightarrow (k+1)$ is shown to be true for any positive integer $k \geqslant n_0$. For example, $2^n < n!$ can only be true for $n \geqslant 4$.

Note: The nth odd positive integer is $2n - 1$. This is so because we are adding '2' a total of $n - 1$ times to 1; i.e. $1 + 2(n - 1) = 2n - 1$.

Note: In a proof by mathematical induction, we do not assume that $S(k)$ is true for all positive integers! We only show that if it is assumed that $S(k)$ is true, then $S(k + 1)$ is also true.

Inductive step: Assume $S(k)$ is true, i.e. assume that $3^k < k!$ is true. We must show that $S(k+1)$ is also true, i.e. we must show that $3^{k+1} < (k+1)!$

On the assumption that $3^k < k!$, multiply both sides of this inequality by 3.

$3 \cdot 3^k < 3 \cdot k!$, and since $k > 6$, then $3 < k + 1$; hence,

$$\begin{aligned} 3 \cdot 3^k &< 3 \cdot k! \\ &< (k+1) \cdot k! \\ &= (k+1)! \\ \Rightarrow 3^{k+1} &< (k+1)! \end{aligned}$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true. This completes the inductive step of the proof.

Therefore, $3^n < n!$ for all integers $n > 6$.

> **Note:** When we use mathematical induction to prove a statement $S(n)$, we show that $S(1)$ is true. Then we know that $S(2)$ is true, since $S(1) \Rightarrow S(1+1)$.
>
> Further, we know that $S(3)$ is true, since $S(2) \Rightarrow S(2+1)$. Continuing along these lines, we see that $S(n)$ is true for every positive integer n.

Example 42

Show that in an arithmetic sequence where $a_n = a_{n-1} + d$, the nth term can be given by the formula

$$a_n = a_1 + (n-1)d.$$

Solution

Let $S(n)$ be the statement that $a_n = a_1 + (n-1)d$.

Basis step: To prove this formula the basis step must be $S(1)$.
$S(1)$: $a_1 = a_1 + (1-1)d = a_1$ is true.

Inductive step: Assume $S(k)$ is true, i.e. assume that $a_k = a_1 + (k-1)d$ is true. We must show that $S(k+1)$ is also true, i.e. we must show that $a_{k+1} = a_1 + (k+1-1)d = a_1 + kd$.

On the assumption that $a_k = a_1 + (k-1)d$:

$a_{k+1} = a_k + d$ by definition of an arithmetic sequence; hence,

$$a_{k+1} = \underbrace{a_k}_{} + d = \underbrace{a_1 + (k-1)d}_{} + d = a_1 + kd$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true. This completes the inductive step of the proof.

Therefore, $a_k = a_1 + (k-1)d$ for all integers n.

Example 43

Show that in an arithmetic series: $S_n = \frac{n}{2}(2a_1 + (n-1)d)$.

Solution

Let $P(n)$ be the statement that $S_n = \frac{n}{2}(2a_1 + (n-1)d)$.

Basis step: To prove this formula the basis step must be $P(1)$.
$P(1)$: $S_1 = \frac{1}{2}(2a_1 + (1-1)d) = a_1$ is true. $(S_1 = a_1)$

Inductive step: Assume $P(k)$ is true, i.e. assume that $S_k = \frac{k}{2}(2a_1 + (k-1)d)$ is true. We must show that $P(k+1)$ is also true, i.e. we must show that

$$S_{k+1} = \frac{k+1}{2}(2a_1 + (k+1-1)d) = \frac{k+1}{2}(2a_1 + kd).$$

On the assumption that $S_k = \frac{k}{2}(2a_1 + (k-1)d)$:

$S_{k+1} = S_k + a_{k+1}$ by definition of an arithmetic series; hence,

$$S_{k+1} = \underbrace{S_k}_{} + \overbrace{a_{k+1}}^{} = \underbrace{\frac{k}{2}(2a_1 + (k-1)d)}_{} + \overbrace{a_1 + kd}^{}$$

By combining like terms and simplifying, the expression (page 194) can be reduced to

$$\begin{aligned} S_{k+1} &= \frac{k}{2}\cdot 2a_1 + \frac{k}{2}(k-1)d + a_1 + kd = (k+1)a_1 + \frac{k}{2}(k-1)d + kd \\ &= \frac{(k+1)}{2}\cdot 2a_1 + \frac{k(k+1)}{2}d = \frac{k+1}{2}(2a_1 + kd) \end{aligned}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true. This completes the inductive step of the proof.

Therefore, $S_n = \frac{n}{2}(2a_1 + (n-1)d)$ for all integers n.

Notice here that we are using $P(n)$ rather than $S(n)$. The use of the name does not influence the method!

Example 44

Show that 3 divides $n^3 + 2n$ for all non-negative integers n.

Solution

Let $P(n)$ be the statement that '3 divides $n^3 + 2n$'.

Basis step: To prove this formula the basis step must be $P(0)$.

$P(0)$: is true since $0^3 + 2(0) = 0$ is a multiple of 3. (If you are not convinced, you can try $P(1)$: $1^3 + 2(1) = 3$ is a multiple of 3.)

Inductive step: Assume $P(k)$ is true, i.e. assume that 3 divides $k^3 + 2k$. We must prove that

$P(k+1)$ is true, i.e. 3 divides $(k+1)^3 + 2(k+1)$.

Note that

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3k^2 + 3k + 1 + 2$

$= (k^3 + 2k) + 3(k^2 + k + 1)$

Since both terms in this sum are multiples of 3 – the first by the induction hypothesis and the second because it is 3 times an integer – it follows that the sum is a multiple of 3.

Hence, $(k+1)^3 + 2(k+1)$ is a multiple of 3.

This shows that $P(k+1)$ is true whenever $P(k)$ is true. This completes the inductive step of the proof.

Therefore, 3 divides $n^3 + 2n$ for all non-negative integers n.

Example 45

Show, using mathematical induction, that for all non-negative integers n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Solution

Let $P(n)$ be the statement that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

Basis step: To prove this formula, the basis step must be $P(0)$.

$P(0)$: is true since $\binom{0}{0} = 2^0 = 1$ is true. Moreover, $P(1)$ is also true since $\binom{1}{0} + \binom{1}{1} = 2^1$ is true!

Inductive step: Assume $P(k)$ is true, i.e. assume that

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k-1} + \binom{k}{k} = 2^k.$$

Recall from Section 4.5 that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ which we claim to be the basis of Pascal's triangle. Using this fact, we can perform the following addition:

$$\begin{array}{ccccccccccccccc}
 & & \binom{k}{0} & + & \binom{k}{1} & + & \binom{k}{2} & + \ldots + & \binom{k}{k-2} & + & \binom{k}{k-1} & + & \binom{k}{k} & = 2^k \\
\binom{k}{0} & + & \binom{k}{1} & + & \binom{k}{2} & + & \ldots & + & \binom{k}{k-1} & + & \binom{k}{k} & & & = 2^k \\
\hline
\binom{k}{0} & + & \binom{k+1}{1} & + & \binom{k+1}{2} & + & \ldots & + & \binom{k+1}{k-1} & + & \binom{k+1}{k} & + & \binom{k}{k} & = 2 \cdot 2^k
\end{array}$$

However, $\binom{k}{0} = \binom{k+1}{0} = \binom{k}{k} = \binom{k+1}{k+1} = 1$, so the last result can be written as

$$\binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \ldots + \binom{k+1}{k} + \binom{k+1}{k+1} = 2 \cdot 2^k = 2^{k+1}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true. This completes the inductive step of the proof.

Therefore, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$ for all non-negative integers n.

Proof of the binomial theorem (optional)

Before we get into the proof, we need to state a few properties of the summation notation.

1. *Change of limits* property: If $f(i)$ is an expression used in the summation process, then the following is true:

   $$\sum_{i=k}^{m} f(i) = \sum_{i=k+r}^{m+r} f(i-r)$$

   For example, suppose we need to find $10^2 + 11^2 + \ldots + 49^2$ using summation notation. We can either write it as

   $$\sum_{i=10}^{49} i^2 \quad \text{or} \quad \sum_{i=1}^{40} (i+9)^2. \text{ Here } r = -9.$$

2. Another useful property is the following:

   $$\begin{aligned}\sum_{i=k}^{m} f(i) &= f(k) + f(k+1) + \ldots + f(m)\\ &= f(k) + \sum_{i=k+1}^{m} f(i)\end{aligned}$$

   Or

   $$\begin{aligned}\sum_{i=k}^{m} f(i) &= f(k) + f(k+1) + \ldots + f(m-1) + f(m)\\ &= \sum_{i=k}^{m-1} f(i) + f(m)\end{aligned}$$

The binomial theorem

Let $P(n)$ be the statement that $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i;\ \forall n \geqslant 0$.

Basis step: To prove this formula the basis step must be $P(0)$.

$P(0)$ is true since $(a+b)^0 = 1 = \sum_{i=0}^{0} \binom{0}{i} a^{n-i} b^i = \binom{0}{0} a^{0-0} b^0 = 1 \cdot 1 \cdot 1 = 1$.

Also, $P(1)$ is true since

$(a+b)^1 = a + b$

• **Hint:** The symbol $\forall$ stands for the universal quantifier: 'For all n'.

$$= \sum_{i=0}^{1} \binom{1}{i} a^{n-i} b^i$$

$$= \binom{1}{0} a^{1-0} b^0 + \binom{1}{1} a^{1-1} b^1$$

$$= 1 \cdot a \cdot 1 + 1 \cdot 1 \cdot b$$

$$= a + b.$$

Inductive step: Assume $P(k)$ is true, i.e. assume that $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$. We must prove that $P(k+1)$ is true, i.e. $(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$.

$$(a+b)^{k+1} = (a+b)(a+b)^k = (a+b) \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

and using the distributive property, we get

$$RHS = a \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i + b \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

$$= \sum_{i=0}^{k} \binom{k}{i} a \cdot a^{k-i} b^i + \sum_{i=0}^{k} \binom{k}{i} b \cdot a^{k-i} b^i$$

$$= \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1}$$

Now, using property 2 on page 197,

$$RHS = \binom{k}{0} a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^{i+1} + \binom{k}{k} b^{k+1}$$

Moreover, using property 1, we have

$$RHS = \binom{k}{0} a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=1}^{k} \binom{k}{i-1} a^{k-(i-1)} b^{(i-1)+1} + \binom{k}{k} b^{k+1}$$

$$= \binom{k}{0} a^{k+1} + \left\{ \sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^i \right\} + \binom{k}{k} b^{k+1}$$

Now, you observe that the terms inside the brackets have a common factor, so

$$RHS = \binom{k}{0} a^{k+1} + \sum_{i=1}^{k} \left\{ \binom{k}{i} + \binom{k}{i-1} \right\} a^{k+1-i} b^i + \binom{k}{k} b^{k+1}$$

Finally, using Pascal's property along with the fact that

$$\binom{k}{0} = \binom{k+1}{0} = \binom{k}{k} = \binom{k+1}{k+1} = 1, \text{ we have}$$

$$RHS = \binom{k+1}{0}a^{k+1} + \sum_{i=1}^{k}\binom{k+1}{i}a^{k+1-i}b^i + \binom{k+1}{k+1}b^{k+1}$$

$$= \sum_{i=0}^{k+1}\binom{k+1}{i}a^{k+1-i}b^i$$

This shows that $P(k + 1)$ is true whenever $P(k)$ is true. This completes the inductive step of the proof.

Therefore, $(a + b)^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i; \forall n \geqslant 0.$

Exercise 4.7

1 Find a formula for the sum of the first n even positive integers and prove it using mathematical induction.

2 Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

$a_1 = 1, a_n = 3a_{n-1}; n \geqslant 1$

Show that $a_n = 3^{n-1}$ for all positive integers n.

3 Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

$a_1 = 1, a_n = a_{n-1} + 4; n \geqslant 2$

Show that $a_n = 4n - 3$ for all positive integers $n > 1$.

4 Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

$a_1 = 1, a_n = 2a_{n-1} + 1; n \geqslant 2$

Show that $a_n = 2^n - 1$ for all positive integers $n > 1$.

5 Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

$a_1 = \frac{1}{2}, a_n = a_{n-1} + \frac{1}{n(n+1)}; n \geqslant 2$

Show that $a_n = \frac{n}{n+1}$ for all positive integers $n > 1$.

6 Find a formula for $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}$ and then use mathematical induction to prove your formula.

7 Show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all non-negative integers n.

8 Show, using mathematical induction, that in a geometric sequence $a_n = a_1r^{n-1}$.

9 Show, using mathematical induction, that in a geometric series $S_n = \frac{a - ar^n}{1 - r}$.

10 Prove that $2^n < n!$ for all positive integers larger than 3.

11 Prove that $2^n > n^2$ for all positive integers larger than 4.

12 Show that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots n \cdot n! = (n + 1)! - 1$.

13 Show that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$ for all positive integers n.

14 Show that $n^3 - n$ is divisible by 3 for all positive integers n.

15 Show that $n^5 - n$ is divisible by 5 for all positive integers n.

16 Show that $n^3 - n$ is divisible by 6 for all positive integers n.

17 Show that $n^2 + n$ is an even number for all integers n.

18 Show that $5^n - 1$ is divisible by 4 for all integers n.

19 Show that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ for every positive integer n and where a and b are real numbers.

20 Prove each of the following statements.

a) $\sum_{i=1}^{n}(2i + 4) = n^2 + 5n$ for each positive integer n.

b) $\sum_{i=1}^{n}(2\cdot 3^{i-1}) = 3^{n-1}$ for each positive integer n.

c) $\sum_{i=1}^{n}\dfrac{1}{(2i-1)(2i+1)} = \dfrac{n}{2n+1}$ for each positive integer n.

Practice questions

1 In an arithmetic sequence, the first term is 4, the 4th term is 19 and the nth term is 99. Find the common difference and the number of terms n.

2 How much money should you invest now if you wish to have an amount of €3000 in your account after 6 years if interest is compounded quarterly at an annual rate of 6%?

3 Two students, Nick and Charlotte, decide to start preparing for their IB exams 15 weeks ahead of the exams. Nick starts by studying for 12 hours in the first week and plans to increase the amount by 2 hours per week. Charlotte starts with 12 hours in the first week and decides to increase her time by 10% every week.

a) How many hours did each student study in week 5?

b) How many hours in total does each student study for the 15 weeks?

c) In which week will Charlotte exceed 40 hours per week?

d) In which week does Charlotte catch up with Nick in the number of hours spent on studying per week?

4 Two diet schemes are available for relatively overweight people to lose weight. Plan A promises the patient an initial weight loss of 1000 g the first month, with a steady loss of an additional 80 g every month after the first. So, the second month the patient will lose 1080 g and so on for a maximum duration of 12 months.

Plan B starts with a weight loss of 1000 g the first month and an increase in weight loss by 6% more every following month.

a) Write down the amount of grams lost under Plan B in the second and third months.

b) Find the weight lost in the 12th month for each plan.

c) Find the total weight loss during a 12-month period under

(i) Plan A **(ii)** Plan B.

5 Planning on buying your first car in 10 years, you start a savings plan where you invest €500 at the beginning of the year for 10 years. Your investment scheme offers a fixed rate of 6% per year compounded annually.

Calculate, giving your answers to the nearest euro (€),

(a) how much the first €500 is worth at the end of 10 years

(b) the total value your investment will give you at the end of the 10 years.

6 The first three terms of an arithmetic sequence are 6, 9.5, 13.

a) What is the 40th term of the sequence?

b) What is the sum of the first 103 terms of the sequence?

7 $\{a^n\}$ is defined as follows

$a_n = \sqrt[3]{(8 - a^3_{n-1})}$

a) Given that $a_1 = 1$, evaluate a_2, a_3, a_4. Describe $\{a_n\}$.

b) Given that $a_1 = 2$, evaluate a_2, a_3, a_4. Describe $\{a_n\}$.

8 A marathon runner plans her training programme for a 20 km race. On the first day she plans to run 2 km, and then she wants to increase her distance by 500 m on each subsequent training day.

a) On which day of her training does she first run a distance of 20 km?

b) By the time she manages to run the 20 km distance, what is the total distance she would have run for the whole training programme?

9 In the nation of Telefonica, cellular phones were first introduced in the year 2000. During the first year, the number of people who bought a cellular phone was 1600. In 2001, the number of new participants was 2400, and in 2002 the new participants numbered 3600.

a) You notice that the trend is a geometric sequence; find the common ratio.

Assuming that the trend continues,

b) how many participants will join in 2012?

c) in what year would the number of new participants first exceed 50 000?

Between 2000 and 2002, the total number of participants reaches 7600.

d) What is the total number of participants between 2000 and 2012?

During this period, the total adult population of Telefonica remains at approximately 800 000.

e) Use this information to suggest a reason why this trend in growth would not continue.

10 In an arithmetic sequence, the first term is 25, the fourth term is 13 and the nth term is $-11\,995$. Find the common difference d and the number of terms n.

11 The midpoints M, N, P, Q of the sides of a square of side 1 cm are joined to form a new square.

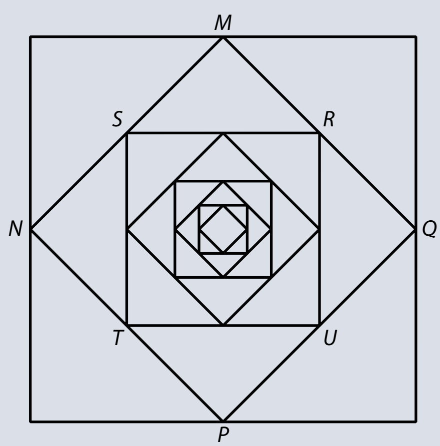
a) Show that the side of the second square $MNPQ$ is $\frac{\sqrt{2}}{2}$.

b) Find the area of square $MNPQ$.

A new third square $RSTU$ is constructed in the same manner.

c) (i) Find the area of the third square just constructed.

(ii) Show that the areas of the squares are in a geometric sequence and find its common ratio.



The procedure continues indefinitely.

d) (i) Find the area of the tenth square.

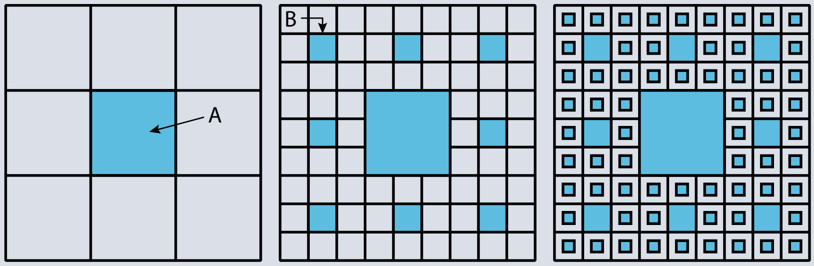
(ii) Find the sum of the areas of all the squares.

12 Tim is a dedicated swimmer. He goes swimming once every week. He starts the first week of the year by swimming 200 metres. Each week after that he swims 20 m more than the previous week. He does that all year long (52 weeks).

a) How far does he swim in the final week?

b) How far does he swim altogether?

13 The diagram below shows three iterations of constructing squares in the following manner: A square of side 3 units is given, then it is divided into nine smaller squares as shown and the middle square is shaded. Each of the unshaded squares is in turn divided into nine squares and the process is repeated. The area of the first shaded square is 1 unit.



a) Find the area of each of the squares A and B.

b) Find the area of any small square in the third diagram.

c) Find the area of the shaded regions in the second and third iterations.

d) If the process is continued indefinitely, find the area left unshaded.

14 The table below shows four series of numbers. One series is an arithmetic one, one is a converging geometric series, one is a diverging geometric series and the fourth is neither geometric nor arithmetic.

Series		Type of series
(i)	$2 + 22 + 222 + 2222 + \dots$	
(ii)	$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \dots$	
(iii)	$0.8 + 0.78 + 0.76 + 0.74 + \dots$	
(iv)	$2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \dots$	

a) Complete the table by stating the type of each series.

b) Find the sum of the infinite geometric series above.

15 Two IT companies offer 'apparently' similar salary schemes for their new appointees. Kell offers a starting salary of €18 000 per year and then an annual increase of €400 every year after the first. YBO offers a starting salary of €17 000 per year and an annual increase of 7% for the rest of the years after the first.

a) **(i)** Write down the salary paid during the second and third years for each company.

(ii) Calculate the total amount that an employee working for 10 years will accumulate in each company.

(iii) Calculate the salary paid during the tenth year for each company.

b) Tim works at Kell and Merijayne works at YBO.

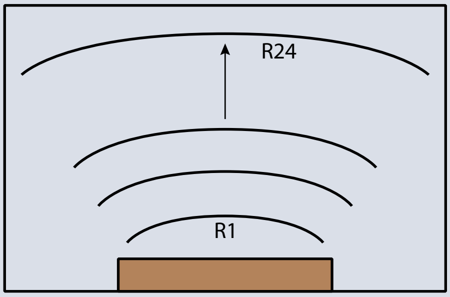
(i) When would Merijayne start earning more than Tim?

(ii) What is the minimum number of years that Merijayne requires so that her total earnings exceed Tim's total earnings?

16 A theatre has 24 rows of seats. There are 16 seats in the first row and each successive row increases by 2 seats, 1 on each side.

a) Calculate the number of seats in the 24th row.

b) Calculate the number of seats in the whole theatre.



17 The amount of €7000 is invested at 5.25% annual compound interest.

a) Write down an expression for the value of this investment after t full years.

b) Calculate the minimum number of years required for this amount to become €10 000.

c) For the same number of years as in part b), would an investment of the same amount be better if it were at a 5% rate compounded quarterly?

18 With S_n denoting the sum of the first n terms of an arithmetic sequence, we are given that $S_1 = 9$ and $S_2 = 20$.

a) Find the second term.

b) Calculate the common difference of the sequence.

c) Find the fourth term.

19 The second term of an arithmetic sequence is 7. The sum of the first four terms of the arithmetic sequence is 12. Find the first term, a, and the common difference, d, of the sequence.

20 Given that

$(1 + x)^5 (1 + ax)^6 \equiv 1 + bx + 10x^2 + \dots\dots\dots + a^6 x^{11}$,

find the values of $a, b \in \mathbb{Z}$, where $a \neq 0$.

21 The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$. If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of this sequence.

22 Using mathematical induction, prove that the number $2^{2n} - 3n - 1$ is divisible by 9, for $n = 1, 2, \dots$.

23 An arithmetic sequence has 5 and 13 as its first two terms respectively.

a) Write down, in terms of n, an expression for the nth term, an.

b) Find the number of terms of the sequence which are less than 400.

24 Find the coefficient of x^7 in the expansion of $(2 + 3x)^{10}$, giving your answer as a whole number.

25 The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term u_n.

26 Mr Blue, Mr Black, Mr Green, Mrs White, Mrs Yellow and Mrs Red sit around a circular table for a meeting. Mr Black and Mrs White must not sit together.

Calculate the number of different ways these six people can sit at the table without Mr Black and Mrs White sitting together.

27 Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, ….

28 The coefficient of x in the expansion of $\left(x + \frac{1}{a(x)^2}\right)^7$ is $\frac{7}{3}$. Find the possible values of a.

29 The sum of an infinite geometric sequence is $\frac{27}{2}$, and the sum of the first three terms is 13. Find the first term.

30 In how many ways can six different coins be divided between two students so that each student receives at least one coin?

31 Find the sum to infinity of the geometric series $-12 + 8 - \frac{16}{3}$.

32 The nth term, u_n, of a geometric sequence is given by $u_n = 3(4)^{n+1}$, $n \in \mathbb{Z}^+$.

a) Find the common ratio r.

b) Hence, or otherwise, find S_n, the sum of the first n terms of this sequence.

33 Consider the infinite geometric series

$$1 + \left(\frac{2x}{3}\right) + \left(\frac{2x}{3}\right)^2 + \left(\frac{2x}{3}\right)^3 + \ldots$$

a) For what values of x does the series converge?

b) Find the sum of the series if $x = 1.2$.

34 How many four-digit numbers are there which contain at least one digit 3?

35 Consider the arithmetic series $2 + 5 + 8 + \ldots$.

a) Find an expression for S_n, the sum of the first n terms.

b) Find the value of n for which $S_n = 1365$.

36 Find the coefficient of x^3 in the binomial expansion of $\left(1 - \frac{1}{2}x\right)^8$.

37 Find $\sum_{r=1}^{50} \ln(2^r)$, giving the answer in the form $a \ln 2$, where $a \in \mathbb{Q}$.

38 A sequence $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = 2$, $u_{n+1} = 3u_n - 2u_{n-1}$ where $n \in \mathbb{Z}^+$.

a) Find u_2, u_3, and u_4.

b) **(i)** Express u_n in terms of n.

(ii) Verify that your answer to part b)(i) satisfies the equation

$u_{n+1} = 3u_n - 2u_{n-1}$.

39 A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

a) the common ratio;

b) the first term.

40 The first four terms of an arithmetic sequence are 2, $a - b$, $2a + b + 7$, and $a - 3b$, where a and b are constants. Find a and b.

41 A committee of four children is chosen from eight children. The two oldest children cannot both be chosen. Find the number of ways the committee may be chosen.

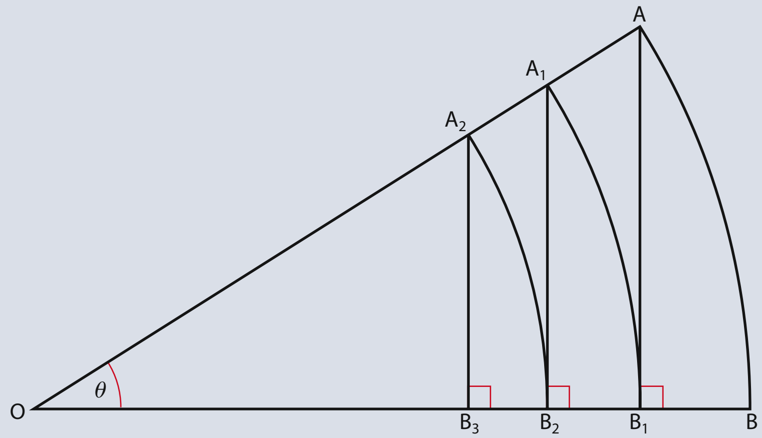
42 The three terms a, 1, b are in arithmetic progression. The three terms 1, a, b are in geometric progression. Find the value of a and of b given that $a \neq b$.

43 The diagram on the following page shows a sector AOB of a circle of radius 1 and centre O, where $\mathrm{A\hat{O}B} = \theta$.

The lines $(\mathrm{AB_1})$, $(\mathrm{A_1B_2})$, $(\mathrm{A_2B_3})$ are perpendicular to OB. $\mathrm{A_1B_1}$, $\mathrm{A_2B_2}$ are all arcs of circles with centre O.

Calculate the sum to infinity of the arc lengths

$\mathrm{AB} + \mathrm{A_1B_1} + \mathrm{A_2B_2} + \mathrm{A_3B_3} + \ldots$



44 The sum of the first n terms of a series is given by

$S_n = 2n^2 - n$, where $n \in \mathbb{Z}^+$.

a) Find the first three terms of the series.

b) Find an expression for the nth term of the series, giving your answer in terms of n.

45 a) Find the expansion of $(2 + x)^5$, giving your answer in ascending powers of x.

b) By letting $x = 0.01$ or otherwise, find the exact value of 2.01^5.

46 A sum of $5000 is invested at a compound interest rate of 6.3% per annum.

a) Write down an expression for the value of the investment after n full years.

b) What will be the value of the investment at the end of five years?

c) The value of the investment will exceed $10 000 after n full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of n.

47 Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

48 The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m.

5 Exponential and Logarithmic Functions

Assessment statements

1.2 Exponents and logarithms.
Laws of exponents; laws of logarithms. Change of base.

2.4 The function $x \mapsto a^x$, $a > 0$.
The inverse function $x \mapsto \log_a x$, $x > 0$.
Graphs of $y = a^x$ and $y = \log_a x$.
The exponential function $x \mapsto e^x$.
The logarithmic function $x \mapsto \ln x$, $x > 0$.

2.6 Solutions of $a^x = b$ using logarithms.

Introduction

A variety of functions have already been considered in this text (see Figure 2.17 in Section 2.4): polynomial functions (e.g. linear, quadratic and cubic functions), functions with radicals (e.g. square root function), rational functions (e.g. inverse and inverse square functions) and the absolute value function. This chapter examines exponential and logarithmic functions.

Exponential functions help us model a wide variety of physical phenomena. The natural exponential function (or simply, *the* exponential function), $f(x) = e^x$, is one of the most important functions in calculus. Exponential functions and their applications – especially to situations involving growth and decay – will be covered at length.

Logarithms, which were originally invented as a computational tool, lead to logarithmic functions. These functions are closely related to exponential functions and play an equally important part in calculus and a range of applications. We will learn that certain exponential and logarithmic functions are inverses of each other.

5.1 Exponential functions

Characteristics of exponential functions

We begin our study of exponential functions by comparing two algebraic expressions that represent two seemingly similar but very different functions. The two expressions $y = x^2$ and $y = 2^x$ are similar in that they both contain a **base** and an **exponent** (or power). In $y = x^2$, the base is

• **Hint:** Another word for exponent is **index** (plural: **indices**).

the variable x and the exponent is the constant 2. In $y = 2^x$, the base is the constant 2 and the exponent is the variable x.

The quadratic function $y = x^2$ is in the form 'variable base$^{\text{constant power}}$', where the base is a variable and the exponent is an integer greater than or equal to zero (non-negative integer). Any function in this form is called a **power function**.

The function $y = 2^x$ is in the form 'constant base$^{\text{variable power}}$', where the base is a positive real number (not equal to one) and the exponent is a variable. Any function in this form is called an **exponential function**.

To illustrate a fundamental difference between exponential functions and power functions, consider the function values for $y = x^2$ and $y = 2^x$ when x is an integer from 0 to 10. Table 5.1 showing these results displays clearly how the values for the exponential function eventually increase at a significantly faster rate than the power function.

x	$y = x^2$	$y = 2^x$
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128
8	64	256
9	81	512
10	100	1024

Table 5.1 Contrast between power function and exponential function.

Another important point to make is that power functions can easily be defined (and computed) for any real number. For any power function $y = x^n$, where n is any positive integer, y is found by simply taking x and repeatedly multiplying it n times. Hence, x can be any real number. For example, for the power function $y = x^3$, if $x = \pi$, then $y = \pi^3 \approx 31.006\,276\,68\ldots$. Since a power function like $y = x^3$ is defined for all real numbers, we can graph it as a continuous curve so that every real number is the x-coordinate of some point on the curve. What about the exponential function $y = 2^x$? Can we compute a value for y for any real number x? Before we try, let's first consider x being any rational number and recall the following laws of exponents (indices) that were covered in Section 1.3.

Laws of exponents

For $b > 0$ and $m, n \in \mathbb{Q}$ (rational numbers):

$b^m \cdot b^n = b^{m+n}$ $\quad \frac{b^m}{b^n} = b^{m-n}$ $\quad (b^m)^n = b^{mn}$ $\quad b^0 = 1$ $\quad b^{-m} = \frac{1}{b^m}$

Also, in Section 1.3, we covered the definition of a rational exponent.

Rational exponent

For $b > 0$ and $m, n \in \mathbb{Z}$ (integers):

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$$

From these established facts, we are able to compute b^x $(b > 0)$ when x is any rational number. For example, $b^{4.7} = b^{\frac{47}{10}}$ represents the 10th root of b raised to the 47th power, i.e. $\sqrt[10]{b^{47}}$. Now, we would like to define b^x when x is any real number such as π or $\sqrt{2}$. We know that π has a non-terminating, non-repeating decimal representation that begins $\pi = 3.141\,592\,653\,589\,793\ldots$. Consider the sequence of numbers

$$b^3, b^{3.1}, b^{3.14}, b^{3.141}, b^{3.1415}, b^{3.14159}, \ldots$$

To demonstrate just how quickly $y = 2^x$ increases, consider what would happen if you were able to repeatedly fold a piece of paper in half 50 times. A typical piece of paper is about five thousandths of a centimetre thick. Each time you fold the piece of paper the thickness of the paper doubles, so after 50 folds the thickness of the folded paper is the height of a stack of 2^{50} pieces of paper. The thickness of the paper after being folded 50 times would be $2^{50} \times 0.005$ cm – which is more than 56 million kilometres (nearly 35 million miles)! Compare that with the height of a stack of 50^2 pieces of paper that would be a meagre $12\frac{1}{2}$ cm – only 0.000 125 km.

Every term in this sequence is defined because each has a rational exponent. Although it is beyond the scope of this text, it can be proved that each number in the sequence gets closer and closer to a certain real number – defined as b^{π}. Similarly, we can define other irrational exponents in such a way that the laws of exponents hold for all real exponents. Table 5.2 shows a sequence of exponential expressions approaching the value of 2^{π}.

Table 5.2 Approaching the value of 2^{π}.

x	2^x (12 s.f.)
3	8.000 000 000 00
3.1	8.574 187 700 29
3.14	8.815 240 927 01
3.141	8.821 353 304 55
3.1415	8.824 411 082 48
3.141 59	8.824 961 595 06
3.141 592	8.824 973 829 06
3.141 5926	8.824 977 499 27
3.141 592 65	8.824 977 805 12

Your GDC will give an approximate value for 2^{π} to at least 10 significant figures, as shown below.

```
2^π
          8.824977827
```

Graphs of exponential functions

Using this definition of irrational powers, we can now construct a complete graph of any exponential function $f(x) = b^x$ such that b is a number greater than zero ($b \neq 1$) and x is any real number.

Example 1

Graph each exponential function by plotting points.

a) $f(x) = 3^x$ b) $g(x) = \left(\frac{1}{3}\right)^x$

Solution

We can easily compute values for each function for integral values of x from -3 to 3. Knowing that exponential functions are defined for all real numbers – not just integers – we can sketch a smooth curve in Figure 5.1, filling in between the ordered pairs shown in the table.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

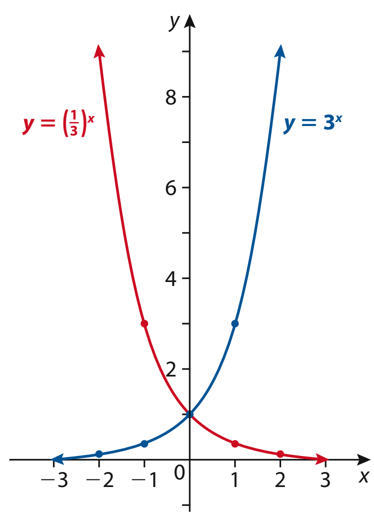


Figure 5.1

Remember that in Section 2.4 we established that the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis. It is clear from the table and the graph in Figure 5.1 that the graph of function g is a reflection of function f about the y-axis. Let's use some laws of exponents to show that $g(x) = f(-x)$.

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x} = f(-x)$$

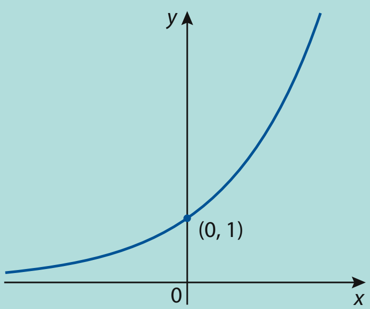
It is useful to point out that both of the graphs, $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$, pass through the point $(0, 1)$ and have a horizontal asymptote of $y = 0$ (x-axis). The same is true for the graph of all exponential functions in the form $y = b^x$ given that $b \neq 1$. If $b = 1$, then $y = 1^x = 1$ and the graph is a horizontal line rather than a constantly increasing or decreasing curve.

Exponential functions

If $b > 0$ and $b \neq 1$, the **exponential function** with base b is the function defined by

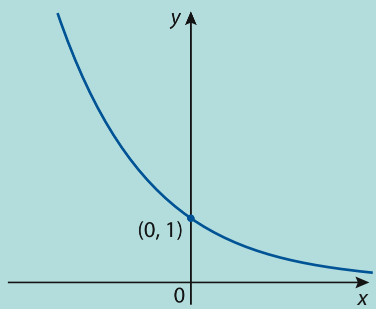
$$f(x) = b^x$$

The **domain** of f is the set of real numbers ($x \in \mathbb{R}$) and the **range** of f is the set of positive real numbers ($y > 0$). The graph of f passes through $(0, 1)$, has the x-axis as a **horizontal asymptote**, and, depending on the value of the base of the exponential function b, will either be a continually increasing **exponential growth curve** or a continually decreasing **exponential decay curve**.



$f(x) = b^x$ for $b > 1$
as $x \to \infty$, $f(x) \to \infty$

f is an increasing function
exponential growth curve



$f(x) = b^x$ for $0 < b < 1$
as $x \to \infty$, $f(x) \to 0$

f is a decreasing function
exponential decay curve

The graphs of all exponential functions will display a characteristic growth or decay curve. As we shall see, many natural phenomena exhibit exponential growth or decay. Also, the graphs of exponential functions behave **asymptotically** for either very large positive values of x (decay curve) or very large negative values of x (growth curve). This means that there will exist a horizontal line that the graph will approach, but not intersect, as either $x \to \infty$ or as $x \to -\infty$.

Transformations of exponential functions

Recalling from Section 2.4 how the graphs of functions are translated and reflected, we can efficiently sketch the graph of many exponential functions.

Example 2

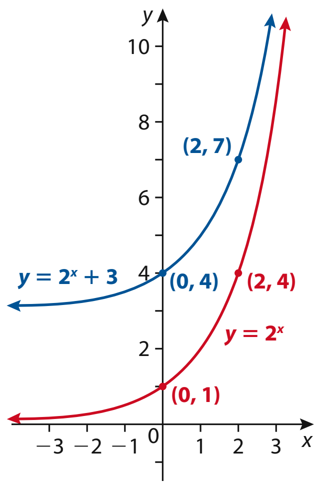
Using the graph of $f(x) = 2^x$, sketch the graph of each function. State the domain and range for each function and the equation of its horizontal asymptote.

a) $g(x) = 2^x + 3$ b) $h(x) = 2^{-x}$ c) $p(x) = -2^x$

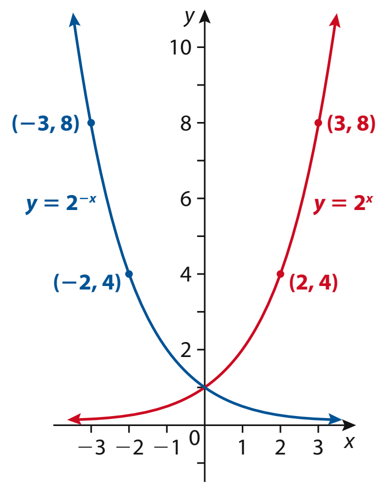
d) $r(x) = 2^{x-4}$ e) $v(x) = 3(2^x)$

Solution

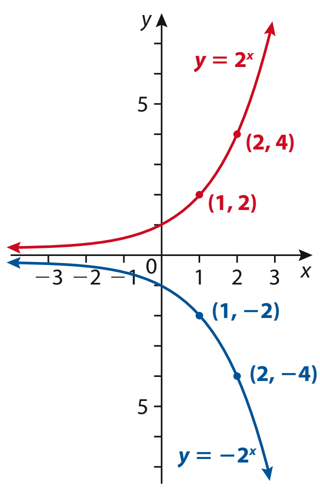
a) The graph of $g(x) = 2^x + 3$ can be obtained by translating the graph of $f(x) = 2^x$ vertically three units up. For function g, the domain is x is any real number ($x \in \mathbb{R}$) and the range is $y > 3$. The horizontal asymptote for g is $y = 3$.



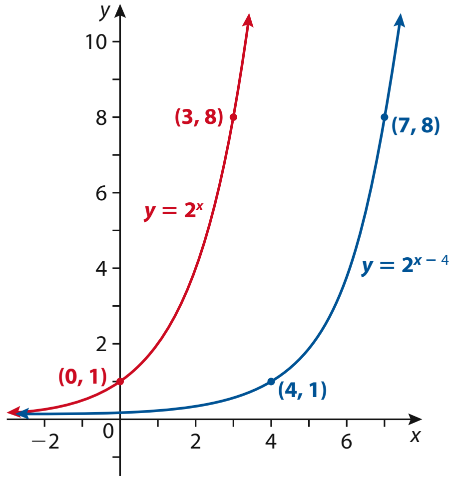
b) The graph of $h(x) = 2^{-x}$ can be obtained by reflecting the graph of $f(x) = 2^x$ across the y-axis. For function h, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



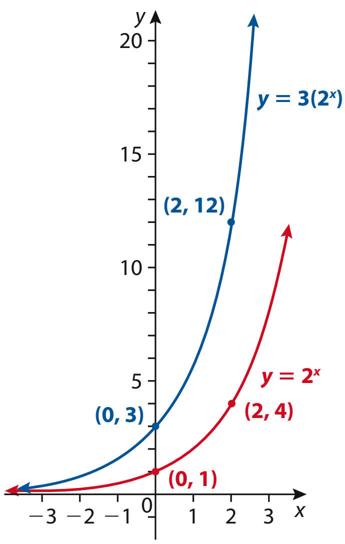
c) The graph of $p(x) = -2^x$ can be obtained by reflecting the graph of $f(x) = 2^x$ across the x-axis. For function p, the domain is $x \in \mathbb{R}$ and the range is $y < 0$. The horizontal asymptote is $y = 0$ (x-axis).



d) The graph of $r(x) = 2^{x-4}$ can be obtained by translating the graph of $f(x) = 2^x$ four units to the right. For function r, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



e) The graph of $v(x) = 3(2^x)$ can be obtained by a vertical stretch of the graph of $f(x) = 2^x$ by scale factor 3. For function v, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



Note that for function p in part c) of Example 2 the horizontal asymptote is an **upper bound** (i.e. no function value is equal to or greater than $y = 0$). Whereas, in parts a), b), d) and e) the horizontal asymptote for each function is a **lower bound** (i.e. no function value is equal to or less than the y-value of the asymptote).

5.2 Exponential growth and decay

Mathematical models of growth and decay

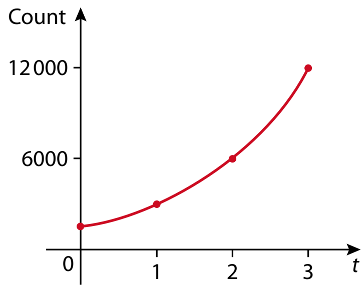
Exponential functions are well suited as a mathematical model for a wide variety of steadily increasing or decreasing phenomena of many kinds, including population growth (or decline), investment of money with compound interest and radioactive decay. Recall from the previous chapter that the formula for finding terms in a geometric sequence (repeated multiplication by common ratio r) is an exponential function. Many instances of growth or decay occur geometrically (repeated multiplication by a growth or decay factor).

Exponential models

Exponential models are equations of the form $A(t) = A_0b^t$, where $A_0 \neq 0$, $b > 0$ and $b \neq 1$. $A(t)$ is the **amount after time *t***. $A(0) = A_0b^0 = A_0(1) = A_0$, so A_0 is called the **initial amount** or value (often the value at time $(t) = 0$). If $b > 1$, then $A(t)$ is an **exponential growth model**. If $0 < b < 1$, then $A(t)$ is an **exponential decay model**. The value of b, the base of the exponential function, is often called the **growth or decay factor**.

Example 3

A sample count of bacteria in a culture indicates that the number of bacteria is doubling every hour. Given that the estimated count at 15:00 was 12 000 bacteria, find the estimated count three hours earlier at 12:00 and write an exponential growth function for the number of bacteria at any hour t.



Solution

Consider the time at 12:00 to be the starting, or initial, time and label it $t = 0$ hours. Then the time at 15:00 is $t = 3$. The amount at any time t (in hours) will double after an hour so the growth factor, b, is 2. Therefore, $A(t) = A_0(2)^t$. Knowing that $A(3) = 12\,000$, compute A_0: $12\,000 = A_0(2)^3 \Rightarrow 12\,000 = 8A_0 \Rightarrow A_0 = 1500$. Therefore, the estimated count at 12:00 was 1500, and the growth function for number of bacteria at time t is $A(t) = 1500(2)^t$.

Radioactive carbon (carbon-14 or C-14), produced when nitrogen-14 is bombarded by cosmic rays in the atmosphere, drifts down to Earth and is absorbed from the air by plants. Animals eat the plants and take C-14 into their bodies. Humans in turn take C-14 into their bodies by eating both plants and animals. When a living organism dies, it stops absorbing C-14, and the C-14 that is already in the object begins to decay at a slow but steady rate, reverting to nitrogen-14. The half-life of C-14 is 5730 years. Half of the original amount of C-14 in the organic matter will have disintegrated after 5730 years; half of the remaining C-14 will have been lost after another 5730 years, and so forth. By measuring the ratio of C-14 to N-14, archaeologists are able to date organic materials. However, after about 50 000 years, the amount of C-14 remaining will be so small that the organic material cannot be dated reliably.

Radioactive material decays at exponential rates. The **half-life** is the amount of time it takes for a given amount of material to decay to half of its original amount. An exponential function that models decay with a known value for the half-life, h, will be of the form $A(t) = A_0\left(\frac{1}{2}\right)^h$, where the decay factor is $\frac{1}{2}$ and h represents the number of half-lives that have occurred (i.e. the number of times that A_0 is multiplied by $\frac{1}{2}$). If t represents the amount of time, the number of half-lives will be $\frac{t}{h}$. For example, if the half-life of a certain material is 25 days and the amount of time that has passed since measuring the amount A_0 is 75 days, then the number of half-lives is

$k = \frac{t}{h} = \frac{75}{25} = 3$, and the amount of material remaining is equal to

$$A_0\left(\frac{1}{2}\right)^3 = \frac{A_0}{8}.$$

Half-life formula

If a certain initial amount, A_0, of material decays with a half-life of h, the amount of material that remains at time t is given by the exponential decay model $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$. The time units (e.g. seconds, hours, years) for h and t must be the same.

Example 4

The half-life of radioactive carbon-14 is approximately 5730 years. How much of a 10 g sample of carbon-14 remains after 15 000 years?

Solution

The exponential decay model for the carbon-14 is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$.

What remains of 10 g after 15 000 years is given by

$$A(15\,000) = 10\left(\frac{1}{2}\right)^{\frac{15\,000}{5730}} \approx 1.63 \text{ g}.$$

Compound interest

Recall from Chapter 4 that exponential functions occur in calculating compound interest. If an initial amount of money P, called the **principal**, is invested at an interest rate r per time period, then after one time period the amount of interest is $P \times r$ and the total amount of money is $A = P + Pr = P(1 + r)$. If the interest is added to the principal, the new principal is $P(1 + r)$, and the total amount after another time period is $A = P(1 + r)(1 + r) = P(1 + r)^2$. In the same way, after a third time period the amount is $A = P(1 + r)^3$. In general, after k periods the total amount is $A = P(1 + r)^k$, an exponential function with growth factor $1 + r$. For example, if the amount of money in a bank account is earning interest at a rate of 6.5% per time period, the growth factor is $1 + 0.065 = 1.065$. Is it possible for r to be negative? Yes, if an amount (not just money) is decreasing. For example, if the population of a town is decreasing by 12% per time period, the decay factor is $1 - 0.12 = 0.88$.

For compound interest, if the annual interest rate is r and interest is compounded (number of times added in) n times per year, then each time period the interest rate is $\frac{r}{n}$, and there are $n \times t$ time periods in t years.

Compound interest formula

The exponential function for calculating the amount of money after t years, $A(t)$, where P is the initial amount or principal, the annual interest rate is r and the number of times interest is compounded per year is n, is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 5

An initial amount of 1000 euros is deposited into an account earning $5\frac{1}{4}\%$ interest per year. Find the amounts in the account after eight years if interest is compounded annually, semi-annually, quarterly, monthly and daily.

Solution

We use the exponential function associated with compound interest with values of $P = 1000$, $r = 0.0525$ and $t = 8$ to complete the results in Table 5.3.

Compounding	n	Amount after 8 years
Annual	1	$1000\left(1 + \frac{0.0525}{1}\right)^{8} = 1505.83$
Semi-annual	2	$1000\left(1 + \frac{0.0525}{2}\right)^{2(8)} = 1513.74$
Quarterly	4	$1000\left(1 + \frac{0.0525}{4}\right)^{4(8)} = 1517.81$
Monthly	12	$1000\left(1 + \frac{0.0525}{12}\right)^{12(8)} = 1520.57$
Daily	365	$1000\left(1 + \frac{0.0525}{365}\right)^{365(8)} = 1521.92$

Table 5.3 Compound interest calculations.

Example 6

A new car is purchased for \$22 000. If the value of the car decreases (depreciates) at a rate of approximately 15% per year, what will be the approximate value of the car to the nearest whole dollar in $4\frac{1}{2}$ years?

Solution

The decay factor for the exponential function is $1 - r = 1 - 0.15 = 0.85$. In other words, after each year the car's value is 85% of what it was one year before. We use the exponential decay model $A(t) = A_0 b^t$ with values $A_0 = 22\,000$, $b = 0.85$ and $t = 4.5$.

$$A(4.5) = 22\,000(0.85)^{4.5} \approx 10\,588$$

The value of the car will be approximately \$10 588.

Exercise 5.1 and 5.2

1 a) Write the equation for an exponential equation with base $b > 0$.

b) Given $b \neq 1$, state the domain and range of this function.

c) Sketch the general shape of the graph of this exponential function for each of two cases:

(i) $b > 1$ (ii) $0 < b < 1$

For questions 2–7, sketch a graph of the function and state its domain, range, y-intercept and the equation of its horizontal asymptote.

2 $f(x) = 3^{x+4}$

3 $g(x) = -2^x + 8$

4 $h(x) = 4^{-x} - 1$

5 $p(x) = \dfrac{1}{2^x - 1}$

6 $q(x) = 3(3^{-x}) - 3$

7 $k(x) = 2^{-|x-2|} + 1$

8 If a general exponential function is written in the form $f(x) = a(b)^{x-c} + d$, state the domain, range, y-intercept and the equation of the horizontal asymptote in terms of the parameters a, b, c and d.

9 Using your GDC and a graph-viewing window with Xmin $= -2$, Xmax $= 2$, Ymin $= 0$ and Ymax $= 4$, sketch a graph for each exponential equation on the same set of axes.

a) $y = 2^x$ b) $y = 4^x$ c) $y = 8^x$

d) $y = 2^{-x}$ e) $y = 4^{-x}$ f) $y = 8^{-x}$

10 Write equations that are equivalent to the equations in 9 d), e) and f) but have an exponent of positive x rather than negative x.

11 If $1 < a < b$, which is steeper: the graph of $y = a^x$ or $y = b^x$?

12 The population of a city triples every 25 years. At time $t = 0$, the population is 100 000. Write a function for the population $P(t)$ as a function of t. What is the population after:

a) 50 years b) 70 years c) 100 years?

13 An experiment involves a colony of bacteria in a solution. It is determined that the number of bacteria doubles approximately every 3 minutes and the initial number of bacteria at the start of the experiment is 10^4. Write a function for the number of bacteria $N(t)$ as a function of t (in minutes). Approximately how many bacteria are there after:

a) 3 minutes b) 9 minutes c) 27 minutes d) one hour?

14 A bank offers an investment account that will double your money in 10 years.

a) Express $A(t)$, the amount of money in the account after t years, in the form $A(t) = A_0(r)^t$.

b) If interest was added into the account just once at the end of each year (simple interest), then find the annual interest rate for the account (to 3 significant figures).

15 If \$10 000 is invested at an annual interest rate of 11%, compounded quarterly, find the value of the investment after the given number of years.

a) 5 years b) 10 years c) 15 years

16 A sum of \$5000 is deposited into an investment account that earns interest at a rate of 9% per year compounded monthly.

a) Write the function $A(t)$ that computes the value of the investment after t years.

b) Use your GDC to sketch a graph of $A(t)$ with values of t on the horizontal axis ranging from $t = 0$ years to $t = 25$ years.

c) Use the graph on your GDC to determine the minimum number of years (to the nearest whole year) for this investment to have a value greater than \$20 000.

17 If \$10 000 is invested at an annual interest rate of 11% for a period of five years, find the value of the investment for the following compounding periods.

a) annually b) monthly c) daily d) hourly

18 Imagine a bank account that has the fantastic annual interest rate of 100%. If you deposit \$1 into this account, how much will be in the account exactly one year later, for the following compounding periods?

a) annually b) monthly c) daily d) hourly e) every minute

19 Each year for the past eight years, the population of deer in a national park increases at a steady rate of 3.2% per year. The present population is approximately 248 000.

a) What was the approximate number of deer one year ago?

b) What was the approximate number of deer eight years ago?

20 Radioactive carbon-14 has a half-life of 5730 years. The remains of an animal are found 20 000 years after it died. About what percentage (to 3 significant figures) of the original amount of carbon-14 (when the animal was alive) would you expect to find?

21 Once a certain drug enters the bloodstream of a human patient, it has a half-life of 36 hours. An amount of the drug, A_0, is injected in the bloodstream at 12:00 on Monday. How much of the drug will be in the bloodstream of the patient five days later at 12:00 on Friday?

22 An open can is filled with 1000 ml of fluid that evaporates at a rate of 30% per week.

a) Write a function, $A(w)$, that gives the amount of fluid after w weeks.

b) Use your GDC to find how many weeks (whole number) it will take for the volume of fluid to be less than 1 ml.

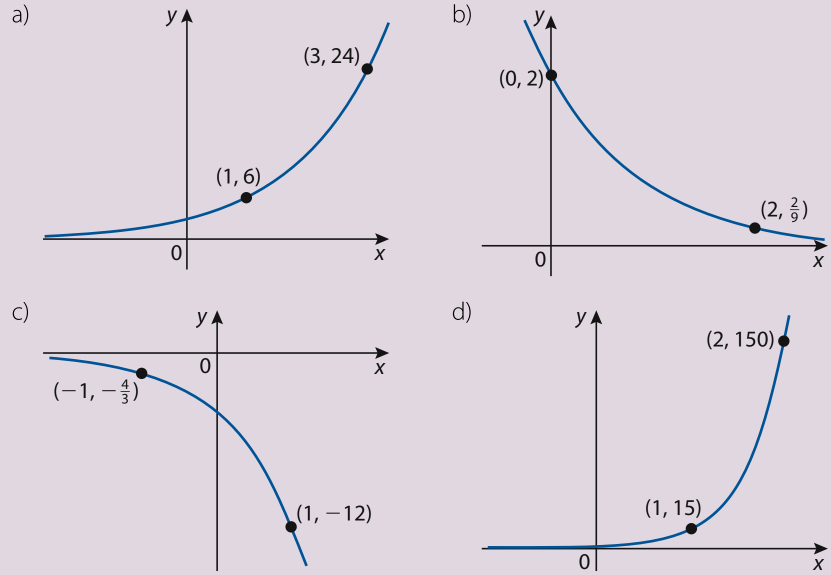
23 Why are exponential functions of the form $f(x) = b^x$ defined so that $b > 0$?

24 You are offered a highly paid job that lasts for just one month – exactly 30 days. Which of the following payment plans, I or II, would give you the largest salary? How much would you get paid?

I One dollar on the first day of the month, two dollars on the second day, three dollars on the third day, and so on (getting paid one dollar more each day) until the end of the 30 days. (You would have a total of \$55 after 10 days.)

II One cent ($0.01) on the first day of the month, two cents ($0.02) on the second day, four cents on the third day, eight cents on the fourth day, and so on (each day getting paid double from the previous day) until the end of the 30 days. (You would have a total of $10.23 after 10 days.)

25 Each exponential function graphed below can be written in the form $f(x) = k(a)^x$. Find the value of a and k for each.



5.3 The number e

Recalling the definition of an exponential function, $f(x) = b^x$, we recognize that b is any positive constant and x is any real number. Graphs of $y = b^x$ for a few values where $b \geqslant 1$ are shown in Figure 5.2. As noted in the first section of this chapter, all the graphs pass through the point $(0, 1)$.

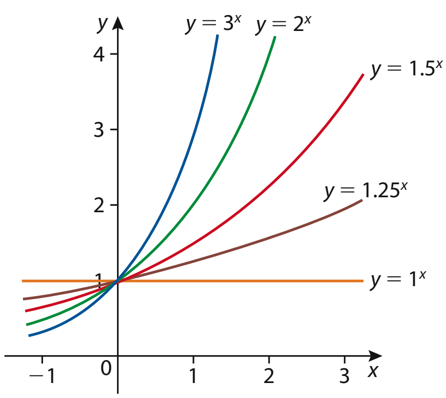


Figure 5.2 Graphs of $y = b^x$ for some values when $b \geqslant 1$.

The question arises: what is the *best* number to choose for the base b? There is a good argument for $b = 10$ since we most commonly use a base 10 number system. Your GDC will have the expression 10^x as a built-in command. The base $b = 2$ is also plausible because a binary number system (base 2) is used in many processes, especially in computer systems. However, the most important base is an irrational number that is denoted with the letter e. As we will see, the value of e approximated to six significant figures is 2.718 28. The importance of e will be clearer when we get to calculus topics. The number π – another very useful irrational number – has a natural geometric significance as the ratio of circumference to diameter for any circle. The number e also occurs in a 'natural' manner. We will illustrate this two different ways: first, by considering the **rate of change** of an exponential function, and secondly, by revisiting compound interest and considering **continuous change** rather than **incremental change**.

The 'discovery' of the constant e is attributed to Jakob Bernoulli (1654–1705). He was a member of the famous Bernoulli family of distinguished mathematicians, scientists and philosophers. This included his brother Johann (1667–1748), who made important developments in calculus, and his nephew Daniel (1700–1782), who is most well known for Bernoulli's principle in physics. The constant e is of enormous mathematical significance – and it appears 'naturally' in many mathematical processes. Jakob Bernoulli first observed e when studying sequences of numbers in connection to compound interest problems.

Rate of change (slope) of an exponential function

Since exponential functions (and associated logarithmic functions) are very important in calculus, the criteria we will use to determine the best value for b will be based on considering the slope of the curve $y = b^x$. In calculus we are interested in the rate of change (i.e. slope of the graph) of functions. Our goal to is to find a value for b such that the slope of the graph of $y = b^x$ at any value of x is equal to the function value y. We could investigate this by trial and error – and with a GDC this might prove fruitful – but it would not guarantee us an exact value and it could prove inefficient. Let's narrow our investigation to studying the slope of the curves at the point (0, 1) which is convenient because it is shared by all the curves.

To obtain a good estimate for the value of e we will use the diagram in Figure 5.3 where the scale on the x- and y-axes are equal and $P(0, 1)$ is the y-intercept of the graph of $y = e^x$. Q is a point on $y = e^x$ close to point P with coordinates (h, e^h). PR and RQ are parallel to the x- and y-axes, respectively, and they intersect at point $R(h, 1)$. The slope of the curve is always changing. It is not constant as with a straight line. As we will justify more thoroughly in our study of differential calculus in Chapter 13, the slope of a curve at a point will be equal to the slope of the line tangent to the curve at that point. PS is the tangent line to the curve at P, intersecting RQ at S. Thus, we are looking for the value e such that the slope of the tangent line PS is equal to 1. It follows that $\frac{RS}{PR} = 1$ and because $PR = h$ then $RS = h$. Since we have set Q close to P then we can assume that h is very small. Therefore, $RS \approx RQ$ and $\frac{RQ}{RS} \approx 1$. The value of $\frac{RQ}{RS}$ will get closer and closer to the value of 1 as h gets smaller (i.e. as Q gets closer to P). Since the y-coordinate of R is 1, then $RQ = e^h - 1$. Substituting h for RS and $e^h - 1$ for RQ into $\frac{RQ}{RS} \approx 1$, gives $\frac{e^h - 1}{h} \approx 1$. We wish to obtain an estimate for e so we multiply through by h to get $e^h - 1 \approx h$ leading to $e^h \approx h + 1$. To isolate e we raise both sides to the $\frac{1}{h}$ power, finally producing, $e \approx (1 + h)^{\frac{1}{h}}$.

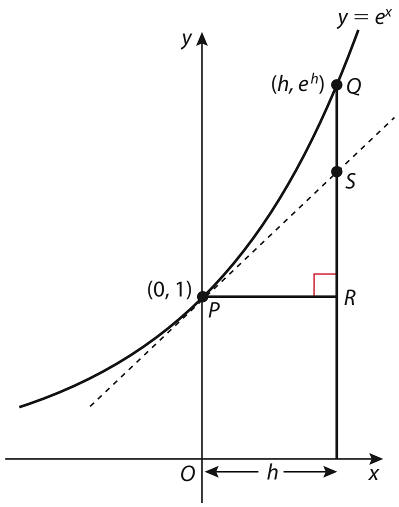


Figure 5.3 Graph of $y = e^x$; slope of the tangent line PS is equal to 1.

h	$e \approx (1 + h)^{\frac{1}{h}}$
0.1	2.593742…
0.01	2.704814…
0.001	2.716924…
0.0001	2.718146…
0.00001	2.718268…
0.000001	2.718280…
0.0000001	2.718282…

Table 5.4 Values for $e \approx (1 + h)^{\frac{1}{h}}$ as h approaches zero (accuracy to 7 significant figures).

Given that h is made small enough, the expression above should give a good estimation of the value of e. Using the approximation $e \approx (1 + h)^{\frac{1}{h}}$, Table 5.4 shows values for e as h approaches zero.

To an accuracy of six significant figures, it appears that the value of e is approximately 2.718 28.

Definition of *e* (I) $\quad e = \lim_{h \to 0}(1 + h)^{\frac{1}{h}}$

The definition is read 'e equals the limit of $(1 + h)^{\frac{1}{h}}$ as n goes to zero.'

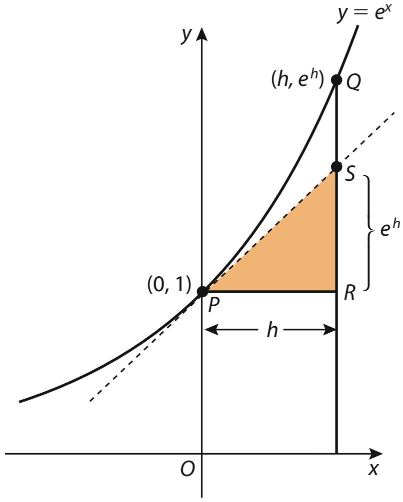


Figure 5.4 At $x = 0$, the rate of change of $y = e^x$ is equal to 1.

Geometrically speaking, as point Q gets closer to point P ($h \to 0$), and also closer to point S, we wanted the slope of the tangent line at $(0, 1)$, $\frac{RS}{PR}$, to be equal to 1. This is the same as saying that we wanted $\frac{e^h - 1}{h} \to 1$ as $h \to 0$ (see coloured triangle in Figure 5.4). The value of e approximated to increasing accuracy in Table 5.4 is the number that makes this happen. A non-geometrical way of describing this feature of the graph is to say that the **rate of change** (slope) of the function $y = e^x$ at $x = 0$ is equal to 1.

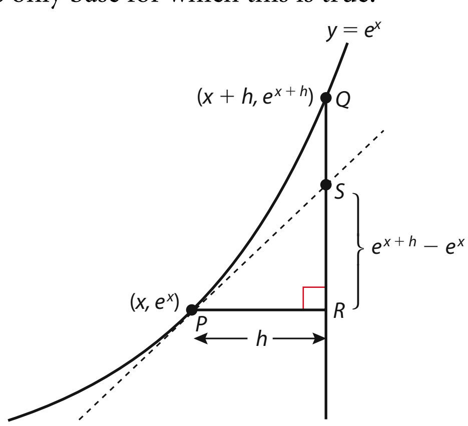
The rate of change of $y = e^x$ at a *general* value of x can be similarly obtained by fixing point P on the curve with coordinates (x, e^x) and a nearby point Q with coordinates $(x + h, e^{x + h})$. See Figure 5.5 below.

Then the rate of change of the function at point P is $\frac{e^{x + h} - e^x}{h}$ as $h \to 0$. We cannot evaluate the limit of $\frac{e^{x + h} - e^x}{h}$ as $h \to 0$ directly by substituting 0 for h. By applying some algebra and knowing that $\frac{e^h - 1}{h} \to 1$ as $h \to 0$, we can evaluate the required limit.

As $h \to 0$, $\frac{e^{x + h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x(e^h - 1)}{h} = e^x\left[\frac{e^h - 1}{h}\right] \to e^x \cdot 1 = e^x$

Therefore, for any value of x, the rate of change of the function $y = e^x$ is e^x. In other words, the rate of change of the function at any value in the domain (x) is equal to the corresponding value of the range (y). This is the amazing feature of $y = e^x$ that makes e the most useful and 'natural' choice for the base of an exponential function, and the irrational number $e \approx 2.718\,28\ldots$ is the only base for which this is true.

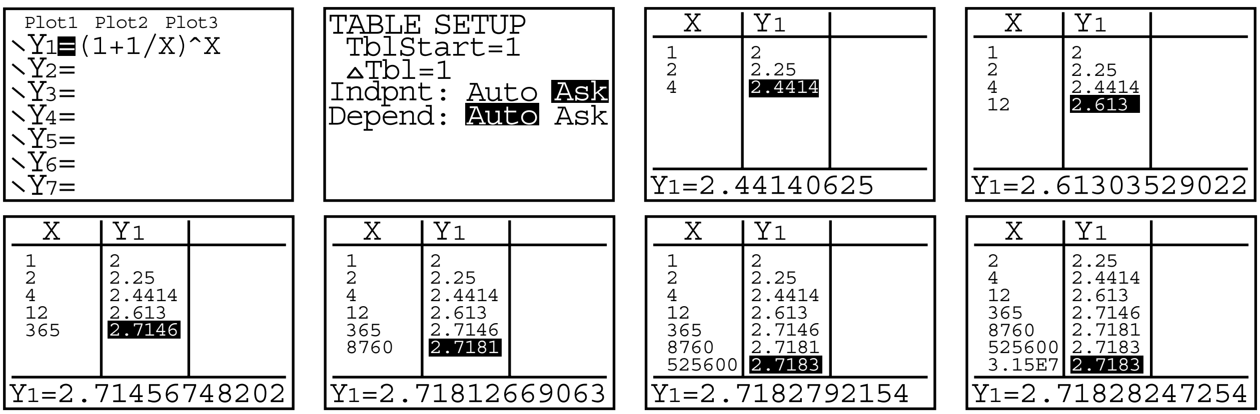
Figure 5.5 The rate of change of $y = e^x$ at a general value of x.



Continuously compounded interest

In the previous section and in Chapter 4, we computed amounts of money resulting from an initial amount (principal) with interest being compounded (added in) at discrete intervals (e.g. yearly, monthly, daily). In the formula that we used, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, n is the number of times that interest is compounded per year. Instead of adding interest only at discrete intervals, let's investigate what happens if we try to add interest continuously – that is, let the value of n increase without bound ($n \to \infty$).

Consider investing just \$1 at a very generous annual interest rate of 100%. How much will be in the account at the end of just one year? It depends on how often the interest is compounded. If it is only added at the end of the year ($n = 1$), the account will have \$2 at the end of the year. Is it possible to compound the interest more often to get a one-year balance of \$2.50 or of \$3.00? We use the compound interest formula with $P = \$1$, $r = 1.00$ (100%) and $t = 1$, and compute the amounts for increasing values of n. $A(1) = 1\left(1 + \frac{1}{n}\right)^{n \cdot 1} = \left(1 + \frac{1}{n}\right)^{n}$. This can be done very efficiently on your GDC by entering the equation $y = \left(1 + \frac{1}{x}\right)^{x}$ to display a table showing function values of increasing values of x.



As the number of compounding periods during the year increases, the amount at the end of the year appears to approach a limiting value.

As $n \to \infty$, the quantity of $\left(1 + \frac{1}{n}\right)^{n}$ approaches the number e. To 13 decimal places, e is approximately 2.718 281 828 4590.

Table 5.5

Compounding	n	$A(1) = \left(1 + \frac{1}{n}\right)^{n}$
Annual	1	2
Semi-annual	2	2.25
Quarterly	4	2.441 406 25…
Monthly	12	2.613 035 290 22…
Daily	365	2.714 567 482 02…
Hourly	8 760	2.718 126 690 63…
Every minute	525 600	2.718 279 2154…
Every second	31 536 000	2.718 282 472 54…

Leonhard Euler (1701–1783) was the dominant mathematical figure of the 18th century and is one of the most influential and prolific mathematicians of all time. Euler's collected works fill over 70 large volumes. Nearly every branch of mathematics has significant theorems that are attributed to Euler.

Euler proved mathematically that the limit of $\left(1 + \frac{1}{n}\right)^n$ as n goes to infinity is precisely equal to an irrational constant which he labelled e. His mathematical writings were influential not just because of the content and quantity but also because of Euler's insistence on clarity and efficient mathematical notation. Euler introduced many of the common algebraic notations that we use today. Along with the symbol e for the base of natural logarithms (1727), Euler introduced $f(x)$ for a function (1734), i for the square root of negative one (1777), π for pi, Σ for summation (1755), and many others. His introductory algebra text, written originally in German (Euler was Swiss), is still available in English translation. Euler spent most of his working life in Russia and Germany. Switzerland honoured Euler by placing his image on the 10 Swiss franc banknote.

Definition of *e* (II)

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$$

The definition is read as 'e equals the limit of $\left(1 + \frac{1}{n}\right)^n$ as n goes to infinity'.

Note that the two definitions that we have provided for the number e are equivalent. Take our first limit definition for e: $e = \lim_{h \to 0}(1 + h)^{\frac{1}{h}}$. Let $\frac{1}{h} = n$, it follows that $h = \frac{1}{n}$ and as $h \to 0$ then $n \to \infty$. Substituting $\frac{1}{n}$ for h, n for $\frac{1}{h}$, and evaluating the limit as $n \to \infty$ transforms $\lim_{h \to 0}(1 + h)^{\frac{1}{h}}$ to $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$, which is our second limit definition for e.

As the number of compoundings, n, increase without bound, we approach continuous compounding – where interest is being added continuously. In the formula for calculating amounts resulting from compound interest, letting $m = \frac{n}{r}$ produces

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^m\right]^{rt}$$

Now if $n \to \infty$ and the interest rate r is constant, then $\frac{n}{r} = m \to \infty$. From the limit definition of e, we know that if $m \to \infty$, then $\left(1 + \frac{1}{m}\right)^m \to e$.

Therefore, for continuous compounding, it follows that

$A(t) = P\left[\left(1 + \frac{1}{m}\right)^n\right]^{rt} = P[e]^{rt}$.

This result is part of the reason that e is the best choice for the base of an exponential function modelling change that occurs continuously (e.g. radioactive decay) rather than in discrete intervals.

Continuous compound interest formula

An exponential function for calculating the amount of money after t years, $A(t)$, for interest compounded continuously, where P is the initial amount or principal and r is the annual interest rate, is given by $A(t) = Pe^{rt}$.

Example 7

An initial investment of 1000 euros earns interest at an annual rate of $5\frac{1}{4}\%$. Find the total amount after 10 years if the interest is compounded:
a) annually (simple interest), b) quarterly, and c) continuously.

Solution

a) $A(t) = P(1 + r)^t = 1000(1 + 0.0525)^{10} = 1669.10$ euros

b) $A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.0525}{4}\right)^{4(10)} = 1684.70$ euros

c) $A(t) = Pe^{rt} = 1000e^{0.0525(10)} = 1690.46$ euros

The natural exponential function and continuous change

For many applications involving continuous change, the most suitable choice for a mathematical model is an exponential function with a base having the value of e.

The natural exponential function

The natural exponential function is the function defined as

$$f(x) = e^x$$

As with other exponential functions, the domain of the natural exponential function is the set of all real numbers ($x \in \mathbb{R}$), and its range is the set of positive real numbers ($y > 0$). The natural exponential function is often referred to as *the* exponential function.

The formula developed for continuously compounded interest does not apply only to applications involving adding interest to financial accounts. It can be used to model growth or decay of a quantity that is changing *geometrically* (i.e. repeated multiplication by a constant ratio, or growth/decay factor) and the change is continuous, or approaching continuous. Another version of a formula for continuous change, which we will learn more about in calculus, is stated below:

Continuous exponential growth/decay

If an initial quantity C (when $t = 0$) increases or decreases continuously at a rate r over a certain time period, the amount $A(t)$ after t time periods is given by the function $A(t) = Ce^{rt}$. If $r > 0$, the quantity is increasing (growing). If $r < 0$, the quantity is decreasing (decaying).

Example 8

The cost of the new Boeing 787 Dreamliner airplane will be 150 million US dollars when purchased new. The airplane will lose value at a continuous rate. This is modeled by the continuous decay function $C(t) = 150e^{-0.053t}$ where $A(t)$ is the value of the airplane (in millions) after t years.

a) How much (to the nearest million dollars) would a Dreamliner jet be worth precisely five years after being purchased?

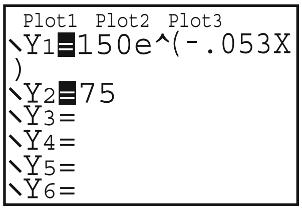
b) If a Dreamliner jet is purchased in 2010, what would be the first year that the jet is worth less than half of its original cost?

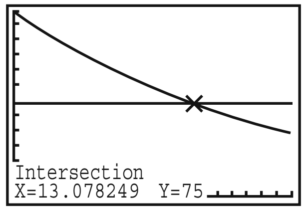
c) Find the value of b (to 4 s.f.) for a discrete decay model, $D(t) = 150b^t$, so that $D(t)$ is a suitable model to describe the same decay as $C(t)$.

Solution

a) $C(5) = 150e^{-0.053(5)} \approx 115$. The value is approximately \$115 million after five years.

b) Using a GDC, we graph the decay equation $y = 150e^{-0.053x}$ and the horizontal line $y = 75$ and determine the intersection point.





The x-coordinate of the intersection point is approximately 13.08. At the start of 2013, the jet's value is not yet half of its original value. Therefore, the first year that the jet is worth less than half of its original cost is 2014.

c) One way to find the value of b so that $D(t) = 150b^t$ serves as a reasonable substitute for $C(t) = 150e^{-0.053t}$ is to compute some function values for $C(t)$ and use them to compute the relative change from one year to the next.

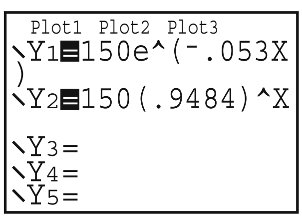
$$C(1) = 150e^{-0.053(1)} \approx 142.2570$$
$$C(2) = 150e^{-0.053(2)} \approx 134.9137$$
$$C(3) = 150e^{-0.053(3)} \approx 127.9495$$

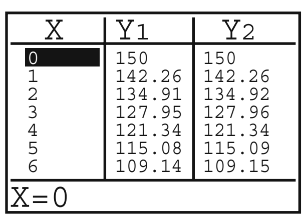
Relative change from year 1 to year 2: $\dfrac{134.9137 - 142.2570}{142.2570} \approx -0.051\,62$

Compute relative change from year 2 to year 3 to make sure it agrees with result above.

Relative change from year 2 to year 3: $\dfrac{127.9495 - 134.9137}{134.9137} \approx -0.051\,62$

The annual rate of decay, b, is the fraction of what remains after each year. Thus, $b = 1 - 0.051\,62 = 0.948\,38$; and to 4 s.f. the annual rate of decay is $b \approx 0.9484$. Therefore, the discrete decay model is $D(t) = 150(0.9484)^t$.





X	Y1	Y2
0	150	150
1	142.26	142.26
2	134.91	134.92
3	127.95	127.96
4	121.34	121.34
5	115.08	115.09
6	109.14	109.15

X=0

To check that the two decay models give similar results for each year, we can use a GDC to display a table of values for both models side by side for easy comparison.

Exercise 5.3

For questions 1–6, sketch a graph of the function and state its: a) domain and range; b) coordinates of any x-intercept(s) and y-intercept; c) and the equation of any asymptote(s).

1 $f(x) = e^{2x-1}$ **2** $g(x) = e^{-x} + 1$

3 $h(x) = -2e^x$

4 $p(x) = e^{x^2} - e$

5 $h(x) = \dfrac{1}{1 - e^x}$

6 $h(x) = e^{|x+2|} - 1$

7 a) State a definition of the number e as a limit.

b) Evaluate $\left(1 - \frac{1}{n}\right)^n$ for $n = 100$, $n = 10\,000$ and $n = 1\,000\,000$.

c) To 5 significant figures, what appears to be the value of $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n$? How does this number relate to the number e?

8 Use your GDC to graph the curve $y = \left(1 + \frac{1}{x}\right)^x$ and the horizontal line $y = 2.72$. Use a graph window so that x ranges from 0 to 20 and y ranges from 0 to 3. Describe the behaviour of the graph of $y = \left(1 + \frac{1}{x}\right)^x$. Will it ever intersect the graph of $y = 2.72$? Explain.

9 Two different banks, Bank A and Bank B, offer accounts with exactly the same annual interest rate of 6.85%. However, the account from Bank A has the interest compounded monthly whereas the account from Bank B compounds the interest continuously. To decide which bank to open an account with, you calculate the **amount of interest** you would earn after three years from an initial deposit of 500 euros in each bank's account. It is assumed that you make no further deposits and no withdrawals during the three years. How much interest would you earn from each of the accounts? Which bank's account earns more – and how much more?

10 Dina wishes to deposit \$1000 into an investment account and then withdraw the total in the account in five years. She has the choice of two different accounts. *Blue Star account*: interest is earned at an annual interest rate of 6.13% compounded weekly (52 weeks in a year). *Red Star account*: interest is earned at an annual interest rate of 5.95% compounded continuously. Which investment account – *Blue Star* **or** *Red Star* – will result in the greatest total at the end of five years? What is the total after five years for this account? How much more is it than the total for the other account?

11 Strontium-90 is a radioactive isotope of strontium. Strontium-90 decays according to the function $A(t) = Ce^{-0.0239t}$, where t is time in years and C is the initial amount of strontium-90 when $t = 0$. If you have 1 kilogram of strontium-90 to start with, how much (approximated to 3 significant figures) will you have after:

a) 1 year?

b) 10 years?

c) 100 years?

d) 250 years?

12 A radioactive substance decays in such a way that the mass (in kilograms) remaining after t days is given by the function $A(t) = 5e^{-0.0347t}$.

a) Find the mass (i.e. initial mass) at time $t = 0$.

b) What **percentage** of the initial mass remains after 10 days?

c) On your GDC and then on paper, draw a graph of the function $A(t)$ for $0 \leqslant t \leqslant 50$.

d) Use one of your graphs to approximate, to the nearest whole day, the half-life of the radioactive substance.

13 Which of the given interest rates and compounding periods would provide the better investment?

a) $8\frac{1}{2}\%$ per year, compounded semi-annually

b) $8\frac{1}{4}\%$ per year, compounded quarterly

c) 8% per year, compounded continuously

14 In certain conditions the bacterium that causes cholera, *Vibrios cholerae*, can grow rapidly in number. In a laboratory experiment a culture of *Vibrios cholerae* is started with 20 bacterium. The bacterium's growth is modeled with the following continuous growth model $A(t) = 20e^{0.068t}$ where $A(t)$ is the number of bacteria after t minutes.

a) Determine the value of r for the discrete growth model $B(t) = 20(r)^t$, so that $B(t)$ is equivalent to $A(t)$.

b) For both of these models, by what percentage does the number of bacteria grow each minute?

15 By comparing the graph of each of the following equations to the graph of $y = e^x$, determine if the slope of the tangent line at the point (0, 1) for the graph of each equation is less than or greater than 1.

a) $y = 2^x$

b) $y = \left(\frac{5}{2}\right)^x$

c) $y = \left(\frac{11}{4}\right)^x$

d) $y = 3^x$

16 Consider that £1000 is invested at 4.5% interest compounded continuously.

a) How much money is in the account after 10 years? After 20 years?

b) Use your GDC to determine how many years (to nearest tenth of a year) it takes for the initial investment to double to £2000.

c) If £5000 is invested at the same rate of interest also compounded continuously, how many years (to nearest tenth) would it take to double?

d) Are the answers to b) and c) the same or different? Why?

5.4 Logarithmic functions

In Example 7 of the previous section, we used the equation $A(t) = 1000e^{0.0525t}$ to compute the amount of money in an account after t years. Now suppose we wish to determine how much time, t, it takes for the initial investment of 1000 euros to double. To find this we need to solve the following equation for t: $2000 = 1000e^{0.0525t} \Rightarrow 2 = e^{0.0525t}$. The unknown t is in the exponent. At this point in the book, we do not have an algebraic method to solve such an equation, but developing the concept of a **logarithm** will provide us with the means to do so.

John Napier (1550–1617) was a Scottish landowner, scholar and mathematician who 'invented' logarithms – a word he coined which derives from two Greek words: *logos* – meaning ratio, and *arithmos* – meaning number. Logarithms made numerical calculations much easier in areas such as astronomy, navigation, engineering and warfare. English mathematician Henry Briggs (1561–1630) came to Scotland to work with Napier and together they perfected logarithms, which included the idea of using the base ten. After Napier died in 1617, Briggs took over the work on logarithms and published a book of tables in 1624. By the second half of the 17th century, the use of logarithms had spread around the world. They became as popular as electronic calculators in our time. The great French mathematician Pierre-Simon Laplace (1749–1827) even suggested that the logarithms of Napier and Briggs doubled the life of astronomers, because it so greatly reduced the labours of calculation. In fact, without the invention of logarithms it is difficult to imagine how Kepler and Newton could have made their great scientific advances. In 1621, an English mathematician and clergyman, William Oughtred (1574–1660) used logarithms as the basis for the invention of the slide rule. The slide rule was a very effective calculation tool that remained in common use for over three hundred years.

The inverse of an exponential function

For $b > 1$, an exponential function with base b is increasing for all x, and for $0 < b < 1$ an exponential function is decreasing for all x. It follows from this that all exponential functions must be one-to-one. Recall from Section 2.3 that a one-to-one function passes both a vertical line test and a horizontal line test. We demonstrated that an inverse function would exist for any one-to-one function. Therefore, an exponential function with base b such that $b > 0$ and $b \neq 1$ will have an inverse function, which is given in the following definition. Also recall from Section 2.3 that the domain of a function f is the range of its inverse function f^{-1}, and, similarly, the range of f is the domain of f^{-1}. The domain and range are switched around for a function and its inverse.

Definition of a logarithmic function

For $b > 0$ and $b \neq 1$, the **logarithmic function** $y = \log_b x$ (read as 'logarithm with base b of x') is the inverse of the exponential function with base b.

$$y = \log_b x \text{ if and only if } x = b^y$$

The domain of the logarithmic function $y = \log_b x$ is the set of positive real numbers ($x > 0$) and its range is all real numbers ($y \in \mathbb{R}$).

Logarithmic expressions and equations

When evaluating logarithms, note that *a logarithm is an exponent*. This means that the value of $\log_b x$ is the exponent to which b must be raised to obtain x. For example, $\log_2 8 = 3$ because 2 must be raised to the power of 3 to obtain 8 – that is, $\log_2 8 = 3$ if and only if $2^3 = 8$.

We can use the definition of a logarithmic function to translate a logarithmic equation into an exponential equation and vice versa. When doing this, it is helpful to remember, as the definition stated, that in either form – logarithmic or exponential – the base is the same.

logarithmic equation	**exponential equation**
exponent $\downarrow$	exponent $\downarrow$
$y = \log_b(x)$	$x = b^y$
$\uparrow$ base	$\uparrow$ base

Example 9

Find the value of each of the following logarithms.

a) $\log_7 49$ b) $\log_5(\frac{1}{5})$ c) $\log_6 \sqrt{6}$ d) $\log_4 64$ e) $\log_{10} 0.001$

Solution

For each logarithmic expression in a) to e), we set it equal to y and use the definition of a logarithmic function to obtain an equivalent equation in exponential form. We then solve for y by applying the logical fact that if $b > 0$, $b \neq 1$ and $b^y = b^k$ then $y = k$.

a) Let $y = \log_7 49$ which is equivalent to the exponential equation $7^y = 49$. Since $49 = 7^2$, then $7^y = 7^2$. Therefore, $y = 2 \Rightarrow \log_7 49 = 2$.

b) Let $y = \log_5(\frac{1}{5})$ which is equivalent to the exponential equation $5^y = \frac{1}{5}$. Since $\frac{1}{5} = 5^{-1}$, then $5^y = 5^{-1}$. Therefore, $y = -1 \Rightarrow \log_5(\frac{1}{5}) = -1$.

c) Let $y = \log_6 \sqrt{6}$ which is equivalent to the exponential equation $6^y = \sqrt{6}$. Since $\sqrt{6} = 6^{\frac{1}{2}}$, then $6^y = 6^{\frac{1}{2}}$. Therefore, $y = \frac{1}{2} \Rightarrow \log_6 \sqrt{6} = \frac{1}{2}$.

d) Let $y = \log_4 64$ which is equivalent to the exponential equation $4^y = 64$. Since $64 = 4^3$, then $4^y = 4^3$. Therefore, $y = 3 \Rightarrow \log_4 64 = 3$.

e) Let $y = \log_{10} 0.001$ which is equivalent to the exponential equation $10^y = 0.001$. Since $0.001 = \dfrac{1}{1000} = \dfrac{1}{10^3} = 10^{-3}$, then $10^y = 10^{-3}$. Therefore, $y = -3 \Rightarrow \log_{10} 0.001 = -3$.

Example 10

Find the domain of the function $f(x) = \log_2(4 - x^2)$.

Solution

From the definition of a logarithmic function the domain of $y = \log_b x$ is $x > 0$, thus for $f(x)$ it follows that $4 - x^2 > 0 \Rightarrow (2 + x)(2 - x) > 0 \Rightarrow -2 < x < 2$.

Hence, the domain is $-2 < x < 2$.

Properties of logarithms

As with all functions and their inverses, their graphs are reflections of each other over the line $y = x$. Figure 5.6 illustrates this relationship for exponential and logarithmic functions, and also confirms the domain and range for the logarithmic function stated in the previous definition.

Notice that the points (0, 1) and (1, 0) are mirror images of each other over the line $y = x$. This corresponds to the fact that since $b^0 = 1$ then $\log_b 1 = 0$. Another pair of mirror image points, $(1, b)$ and $(b, 1)$, highlight the fact that $\log_b b = 1$.

Notice also that since the x-axis is a horizontal asymptote of $y = b^x$, the y-axis is a vertical asymptote of $y = \log_b x$.

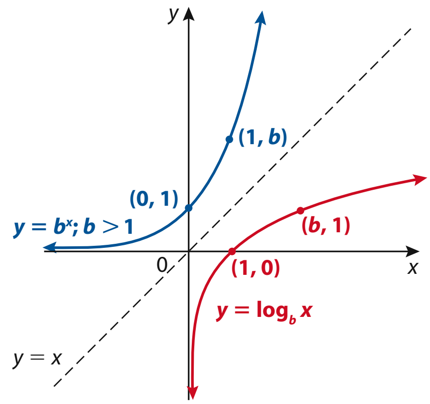


Figure 5.6 Reflection of $y = \log_b x$ over the line $y = x$.

In Section 2.3, we established that a function f and its inverse function f^{-1} satisfy the equations

$$f^{-1}(f(x)) = x \qquad \text{for } x \text{ in the domain of } f$$
$$f(f^{-1}(x)) = x \qquad \text{for } x \text{ in the domain of } f^{-1}$$

When applied to $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, these equations become

$$\log_b(b^x) = x \qquad x \in \mathbb{R}$$
$$b^{\log_b x} = x \qquad x > 0$$

Properties of logarithms I

For $b > 0$ and $b \neq 1$, the following statements are true:

1. $\log_b 1 = 0$ (*because* $b^0 = 1$)
2. $\log_b b = 1$ (*because* $b^1 = b$)
3. $\log_b(b^x) = x$ (*because* $b^x = b^x$)
4. $b^{\log_b x} = x$ (*because* $\log_b x$ *is the power to which b must be raised to get* x)

The logarithmic function with base 10 is called the **common logarithmic function**. On calculators and on your GDC, this function is denoted by **log**. The value of the expression $\log_{10} 1000$ is 3 because 10^3 is 1000. Generally, for common logarithms (i.e. base 10) we omit writing the base of 10. Hence, if **log** is written with no base indicated, it is assumed to have a base of 10. For example, $\log 0.01 = -2$.

Common logarithm: $\log_{10} x = \log x$

As with exponential functions, the most widely used logarithmic function – and the other logarithmic function supplied on all calculators – is the logarithmic function with the base of e. This function is known as the **natural logarithmic function** and it is the inverse of the natural exponential function $y = e^x$. The natural logarithmic function is denoted by the symbol **ln**, and the expression $\ln x$ is read as 'the natural logarithm of x'.

Natural logarithm: $\log_e x = \ln x$

Example 11

Evaluate the following expressions:

a) $\log\left(\frac{1}{10}\right)$ b) $\log(\sqrt{10})$ c) $\log 1$ d) $10^{\log 47}$ e) $\log 50$

f) $\ln e$ g) $\ln\left(\frac{1}{e^3}\right)$ h) $\ln 1$ i) $e^{\ln 5}$ j) $\ln 50$

Solution

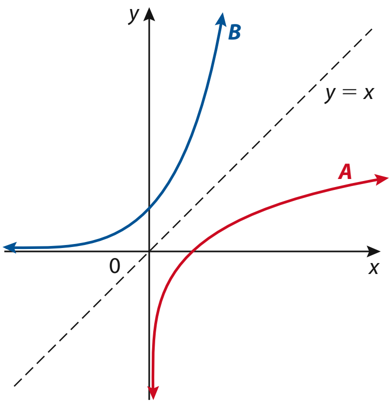
a) $\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$ b) $\log(\sqrt{10}) = \log(10^{\frac{1}{2}}) = \frac{1}{2}$

c) $\log 1 = \log(10^0) = 0$ d) $10^{\log 47} = 47$

e) $\log 50 \approx 1.699$ (using GDC) f) $\ln e = 1$

g) $\ln\left(\frac{1}{e^3}\right) = \ln(e^{-3}) = -3$ h) $\ln 1 = \ln(e^0) = 0$

i) $e^{\ln 5} = 5$ j) $\ln 50 \approx 3.912$ (using GDC)



Example 12

The diagram shows the graph of the line $y = x$ and two curves. Curve A is the graph of the equation $y = \log x$. Curve B is the reflection of curve A in the line $y = x$.

a) Write the equation for curve B.

b) Write the coordinates of the y-intercept of curve B.

Solution

a) Curve A is the graph of $y = \log x$, the common logarithm with base 10, which could also be written as $y = \log_{10} x$. Curve B is the inverse of $y = \log_{10} x$, since it is the reflection of it in the line $y = x$. Hence, the equation for curve B is the exponential equation $y = 10^x$.

b) The y-intercept occurs when $x = 0$. For curve B, $y = 10^0 = 1$. Therefore, the y-intercept for curve B is $(0, 1)$.

The logarithmic function with base b is the inverse of the exponential function with base b. Therefore, it makes sense that the laws of exponents (Section 1.3) should have corresponding properties involving logarithms. For example, the exponential property $b^0 = 1$ corresponds to the logarithmic property $\log_b 1 = 0$. We will state and prove three further important logarithmic properties that correspond to the following three exponential properties.

1. $b^m \cdot b^n = b^{m+n}$

2. $\dfrac{b^m}{b^n} = b^{m-n}$

3. $(b^m)^n = b^{mn}$

Properties of logarithms II

Given $M > 0$, $N > 0$ and k is any real number, the following properties are true for logarithms with $b > 0$ and $b \neq 1$.

Property		Description
1. $\log_b(MN)$	$= \log_b M + \log_b N$	the log of a product is the sum of the logs of its factors
2. $\log_b\left(\frac{M}{N}\right)$	$= \log_b M - \log_b N$	the log of a quotient is the log of the numerator minus the log of the denominator
3. $\log_b(M^k)$	$= k \log_b M$	the log of a number raised to an exponent is the exponent times the log of the number

Any of these properties can be applied in either direction.

Proofs

Property 1: Let $x = \log_b M$ and $y = \log_b N$.

The corresponding exponential forms of these two equations are

$$b^x = M \text{ and } b^y = N$$

Then, $\log_b(MN) = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y$.

It's given that $x = \log_b M$ and $y = \log_b N$; hence, $x + y = \log_b M + \log_b N$.

Therefore, $\log_b(MN) = \log_b M + \log_b N$.

Property 2: Again, let $x = \log_b M$ and $y = \log_b N \Rightarrow b^x = M$ and $b^y = N$.

Then, $\log_b\left(\frac{M}{N}\right) = \log_b\left(\frac{b^x}{b^y}\right) = \log_b(b^{x-y}) = x - y$.

With $x = \log_b M$ and $y = \log_b N$, then $x - y = \log_b M - \log_b N$.

Therefore, $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$.

Property 3: Let $x = \log_b M \Rightarrow b^x = M$.

Now, let's take the logarithm of M^k and substitute b^x for M:

$\log_b(M^k) = \log_b[(b^x)^k] = \log_b(b^{kx}) = kx$

It's given that $x = \log_b M$; hence, $kx = k\log_b M$.

Therefore, $\log_b(M^k) = k\log_b M$.

● **Hint:** The notation $f(x)$ uses brackets *not* to indicate multiplication but to indicate the argument of the function f. The symbol f is the name of a function, not a variable – it is not multiplying the variable x. Therefore, $f(x + y)$ is NOT equal to $f(x) + f(y)$. Likewise, the symbol **log** is also the name of a function. Therefore, $\log_b(x + y)$ is not equal to $\log_b(x) + \log_b(y)$. Other mistakes to avoid include incorrectly simplifying quotients or powers of logarithms. Specifically, $\frac{\log_b x}{\log_b y} \neq \log\left(\frac{x}{y}\right)$ and $(\log_b x)^k \neq k(\log_b x)$.

Example 13

Use the properties of logarithms to write each logarithmic expression as a sum, difference, and/or constant multiple of simple logarithms (i.e. logarithms without sums, products, quotients or exponents).

a) $\log_2(8x)$　　b) $\ln\left(\frac{3}{y}\right)$　　c) $\log(\sqrt{7})$

d) $\log_b\left(\frac{x^3}{y^2}\right)$　　e) $\ln(5e^2)$　　f) $\log\left(\frac{m+n}{n}\right)$

Solution

a) $\log_2(8x) = \log_2 8 + \log_2 x = 3 + \log_2 x$

b) $\ln\left(\frac{3}{y}\right) = \ln 3 - \ln y$

c) $\log(\sqrt{7}) = \log(7^{\frac{1}{2}}) = \frac{1}{2}\log 7$

d) $\log_b\left(\frac{x^3}{y^2}\right) = \log_b(x^3) - \log_b(y^2) = 3\log_b x - 2\log_b y$

e) $\ln(5e^2) = \ln 5 + \ln(e^2) = \ln 5 + 2\ln e = \ln 5 + 2(1) = 2 + \ln 5$
　$(2 + \ln 5 \approx 3.609$ using GDC)

f) $\log\left(\frac{m+n}{m}\right) = \log(m + n) - \log m$

(Remember: $\log(m + n) \neq \log m + \log n$)

Example 14

Write each expression as the logarithm of a single quantity.

a) $\log 6 + \log x$
b) $\log_2 5 + 2\log_2 3$
c) $\ln y - \ln 4$
d) $\log_b 12 - \frac{1}{2}\log_b 9$
e) $\log_3 M + \log_3 N - 2\log_3 P$
f) $\log_2 80 - \log_2 5$

Solution

a) $\log 6 + \log x = \log(6x)$

b) $\log_2 5 + 2\log_2 3 = \log_2 5 + \log_2(3^2) = \log_2 5 + \log_2 9 = \log_2(5 \cdot 9)$
$= \log_2 45$

c) $\ln y - \ln 4 = \ln\left(\frac{y}{4}\right)$

d) $\log_b 12 - \frac{1}{2}\log_b 9 = \log_b 12 - \log_b(9^{\frac{1}{2}}) = \log_b 12 - \log_b(\sqrt{9})$
$= \log_b 12 - \log_b 3 = \log_b\left(\frac{12}{3}\right) = \log_b 4$

e) $\log_3 M + \log_3 N - 2\log_3 P = \log_3(MN) - \log_3(P^2) = \log_3\left(\frac{MN}{P^2}\right)$

f) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$ (because $2^4 = 16$)

Change of base

The answer to part f) of Example 14 was $\log_2 16$ which we can compute to be exactly 4 because we know that $2^4 = 16$. The answer to part e) of Example 13 was $2 + \ln 5$ which we approximated to 3.609 using the natural logarithm function key (**ln**) on our GDC. But, what if we wanted to compute an approximate value for $\log_2 45$, the answer to part b) of Example 14? Our GDC can only evaluate common logarithms (base 10) and natural logarithms (base e). To evaluate logarithmic expressions and graph logarithmic functions to other bases we need to apply a **change of base formula.**

Change of base formula

Let a, b and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_b x$ can be expressed in terms of logarithms to any other base a as follows:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Proof

$y = \log_b x \Rightarrow b^y = x$ Convert from logarithmic form to exponential form.

$\log_a x = \log_a(b^y)$ If $b^y = x$, then log of each with same bases must be equal.

$\log_a x = y\log_a b$ Applying the property $\log_b(M^k) = k\log_b M$.

$y = \frac{\log_a x}{\log_a b}$ Divide both sides by $\log_a b$.

$\log_b x = \frac{\log_a x}{\log_a b}$ Substitute $\log_b x$ for y.

To apply the change of base formula, let $a = 10$ or $a = e$. Then the logarithm of any base b can be expressed in terms of either common logarithms or natural logarithms. For example:

$$\log_2 x = \frac{\log x}{\log 2} \quad \text{or} \quad \frac{\ln x}{\ln 2}$$

$$\log_5 x = \frac{\log x}{\log 5} \quad \text{or} \quad \frac{\ln x}{\ln 5}$$

$$\log_2 45 = \frac{\log 45}{\log 2} = \frac{\ln 45}{\ln 2} \approx 5.492 \quad \text{(using GDC)}$$

Example 15

Use the change of base formula and common or natural logarithms to evaluate each logarithmic expression. Start by making a rough mental estimate. Approximate your answer to 4 significant figures.

a) $\log_3 30$

b) $\log_9 6$

Solution

a) The value of $\log_3 30$ is the power to which 3 is raised to obtain 30. Because $3^3 = 27$ and $3^4 = 81$, the value of $\log_3 30$ is between 3 and 4, and will be much closer to 3 than 4 – perhaps around 3.1. Using the change of base formula and common logarithms, we obtain $\log_3 30 = \dfrac{\log 30}{\log 3} \approx 3.096$. This agrees well with the mental estimate.

After computing the answer on your GDC, use your GDC to also check it by raising 3 to the answer and confirming that it gives a result of 30.

```
log(30)/log(3)
             3.095903274
3^Ans
                      30
■
```

b) The value of $\log_9 6$ is the power to which 9 is raised to obtain 6. Because $9^{\frac{1}{2}} = \sqrt{9} = 3$ and $9^1 = 9$, the value of $\log_9 6$ is between $\frac{1}{2}$ and 1 – perhaps around 0.75. Using the change of base formula and natural logarithms, we obtain $\log_9 6 = \dfrac{\ln 6}{\ln 9} \approx 0.815$. This agrees well with the mental estimate.

```
ln(6)/ln(9)
             .8154648768
9^Ans
                       6
■
```

Exercise 5.4

In questions 1–9, express each logarithmic equation as an exponential equation.

1 $\log_2 16 = 4$ **2** $\ln 1 = 0$ **3** $\log 100 = 2$

4 $\log 0.01 = -2$ **5** $\log_7 343 = 3$ **6** $\ln\left(\frac{1}{e}\right) = -1$

7 $\log 50 = y$ **8** $\ln x = 12$ **9** $\ln(x + 2) = 3$

In questions10–18, express each exponential equation as a logarithmic equation.

10 $2^{10} = 1024$ **11** $10^{-4} = 0.0001$ **12** $4^{-\frac{1}{2}} = \frac{1}{2}$

13 $3^4 = 81$ **14** $10^0 = 1$ **15** $e^x = 5$

16 $2^{-3} = 0.125$ **17** $e^4 = y$ **18** $10^{x+1} = y$

In questions 19–38, find the exact value of the expression without using your GDC.

19 $\log_2 64$ **20** $\log_4 64$ **21** $\log_2\left(\frac{1}{8}\right)$ **22** $\log_3(3^5)$

23 $\log_{16} 8$ **24** $\log_{27} 3$ **25** $\log_{10} 0.001$ **26** $\ln e^{13}$

27 $\log_8 1$ **28** $10^{\log 6}$ **29** $\log_3\left(\frac{1}{27}\right)$ **30** $e^{\ln\sqrt{2}}$

31 $\log 1000$ **32** $\ln(\sqrt{e})$ **33** $\ln\left(\frac{1}{e^2}\right)$ **34** $\log_3 (81^{22})$

35 $\log_4 2$ **36** $3^{\log_3 18}$ **37** $\log_5(\sqrt[3]{5})$ **38** $10^{\log \pi}$

In questions 39–46, use a GDC to evaluate the expression, correct to 4 significant figures.

39 $\log 50$ **40** $\log\sqrt{3}$ **41** $\ln 50$ **42** $\ln\sqrt{3}$

43 $\log 25$ **44** $\log\left(\frac{1+\sqrt{5}}{2}\right)$ **45** $\ln 100$ **46** $\ln(100^3)$

In questions 47–52, find the domain of each function.

47 $f(x) = \log(x - 2)$ **48** $g(x) = \ln(x^2)$ **49** $h(x) = \log(x) - 2$

50 $y = \log_7(8 - 5x)$ **51** $y = \sqrt{x + 2} - \log_3(9 - 3x)$

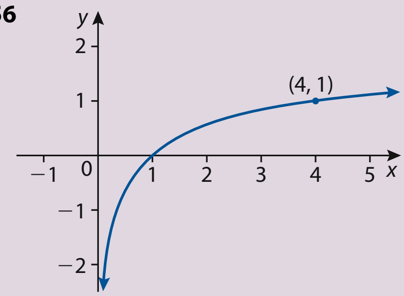
52 $y = \sqrt{\ln(1 - x)}$

In 53–55, find the domain *and* range of each function.

53 $y = \frac{1}{\ln x}$ **54** $y = |\ln(x - 1)|$ **55** $y = \frac{x}{\log x}$

For questions 56–59, find the equation of the function that is graphed in the form $f(x) = \log_b x$.

56



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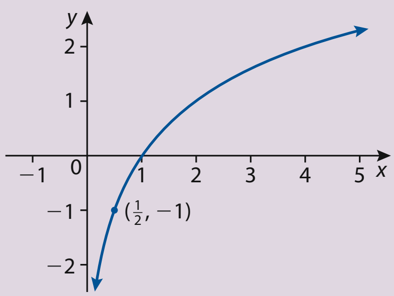

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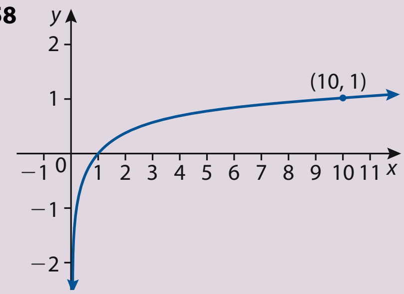

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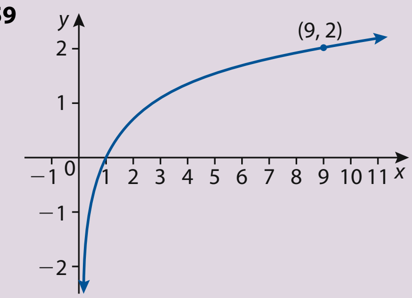


In questions 60–65, use properties of logarithms to write each logarithmic expression as a sum, difference and/or constant multiple of simple logarithms (i.e. logarithms without sums, products, quotients or exponents).

60 $\log_2(2m)$

61 $\log\left(\frac{9}{x}\right)$

62 $\ln\sqrt[5]{x}$

63 $\log_3(ab^3)$

64 $\log[10x(1+r)^t]$

65 $\ln\left(\frac{m^3}{n}\right)$

In 66–71, write each expression in terms of $\log_b p$, $\log_b q$ and $\log_b r$.

66 $\log_b pqr$

67 $\log_b\left(\frac{p^2q^3}{r}\right)$

68 $\log_b\sqrt[4]{pq}$

69 $\log_b\sqrt{\frac{qr}{p}}$

70 $\log_b\frac{p\sqrt{q}}{r}$

71 $\log_b\frac{(pq)^3}{\sqrt{r}}$

In 72–77, write each expression as the logarithm of a single quantity.

72 $\log(x^2)+\log\left(\frac{1}{x}\right)$

73 $\log_3 9+3\log_3 2$

74 $4\ln y-\ln 4$

75 $\log_b 12-\frac{1}{2}\log_b 9$

76 $\log x-\log y-\log z$

77 $2\ln 6-1$ • **Hint:** $\ln(?)=1$

In questions 78–81, use the change of base formula and common or natural logarithms to evaluate each logarithmic expression. Approximate your answer to 3 significant figures.

78 $\log_2 1000$

79 $\log_{\frac{1}{2}} 40$

80 $\log_6 40$

81 $\log_5(0.75)$

In questions 82 and 83, use the change of base formula to evaluate $f(20)$.

82 $f(x)=\log_2 x$

83 $f(x)=\log_5 x$

84 Use the change of base formula to prove the following statement.

$$\log_b a=\frac{1}{\log_a b}$$

85 Show that $\log e=\frac{1}{\ln 10}$.

86 The relationship between the number of decibels dB (one variable) and the intensity I of a sound (in watts per square metre) is given by the formula $dB=10\log\left(\frac{I}{10^{-16}}\right)$. Use properties of logarithms to write the formula in simpler form. Then find the number of decibels of a sound with an intensity of 10^{-4} watts per square metre.

87 a) Given the exponential function $f(x) = 5(2)^x$, show that $f(x)$ varies linearly with x; that is, find the linear equation in terms of x that is equal to $f(x)$.

b) Prove that for any exponential function in the form $f(x) = ab^x$, the function $\log(f(x))$ is linear and can be written in the form $\log(f(x)) = mx + c$. Find the constants m (slope) and c (y-intercept) in terms of $\log a$ and $\log b$.

5.5 Exponential and logarithmic equations

Solving exponential equations

At the start of the previous section, we wanted to find a way to determine how much time t (in years) it would take for an investment of 1000 euros to double, if the investment earns interest at an annual rate of $5\frac{1}{4}\%$. Since the interest is compounded continuously, we need to solve this equation: $2000 = 1000e^{0.0525t} \Rightarrow 2 = e^{0.0525t}$. The equation has the variable t in the exponent. With the properties of logarithms established in the previous section, we now have a way to algebraically solve such equations. Along with these properties, we need to apply the logic that if two expressions are equal then their logarithms must also be equal. That is, if $m = n$, then $\log_b m = \log_b n$.

Example 16

Solve the equation for the variable t. Give your answer accurate to 3 significant figures.

$$2 = e^{0.0525t}$$

Solution

$$2 = e^{0.0525t}$$

$\ln 2 = \ln(e^{0.0525t})$ Take natural logarithm of both sides.

$\ln 2 = 0.0525t$ Apply the property $\log_b(b^x) = x$ and $\ln e = 1$.

$$t = \frac{\ln 2}{0.0525} \approx 13.2$$

With interest compounding continuously at an annual interest rate of $5\frac{1}{4}\%$, it takes about 13.2 years for the investment to double.

This example serves to illustrate a general strategy for solving exponential equations. To solve an exponential equation, first isolate the exponential expression and take the logarithm of both sides. Then apply a property of logarithms so that the variable is no longer in the exponent and it can be isolated on one side of the equation. By taking the logarithm of both sides of an exponential equation, we are making use of the inverse relationship between exponential and logarithmic functions. Symbolically, this method can be represented as follows – solving for x:

(i) If $b = 10$ or e: $y = b^x \Rightarrow \log_b y = \log_b b^x \Rightarrow \log_b y = x$

(ii) If $b \neq 10$ or e:

$$y = b^x \Rightarrow \log_a y = \log_a b^x \Rightarrow \log_a y = x \log_a b \Rightarrow x = \frac{\log_a y}{\log_a b}$$

Example 17

Solve for x in the equation $3^{x-4} = 24$. Approximate the answer to 3 significant figures.

Solution

$3^{x-4} = 24$

$\log(3^{x-4}) = \log 24$ Take common logarithm of both sides.

$(x-4)\log 3 = \log 24$ Apply the property $\log_b(M^k) = k \log_b M$.

$x - 4 = \dfrac{\log 24}{\log 3}$ Divide both sides by log 3. $\left[\text{Note: } \dfrac{\log 24}{\log 3} \neq \log 8\right]$

$x = \dfrac{\log 24}{\log 3} + 4$

$x \approx 6.89$ Using GDC .

● **Hint:** We could have used natural logarithms instead of common logarithms to solve the equation in Example 17. Using the same method but with natural logarithms, we get

$$x = \frac{\ln 24}{\ln 3} + 4 \approx 6.89.$$

Recall Example 11 in Section 4.3 in which we solved an exponential equation graphically, because we did not yet have the tools to solve it algebraically. Let's solve it now using logarithms.

Example 18

You invested €1000 at 6% compounded quarterly. How long will it take this investment to increase to €2000?

Solution

Using the compound interest formula from Section 4.3, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, with $P =$ €1000, $r = 0.06$ and $n = 4$, we need to solve for t when $A(t) = 2P$.

$2P = P\left(1 + \dfrac{0.06}{4}\right)^{4t}$ Substitute $2P$ for $A(t)$.

$2 = 1.015^{4t}$ Divide both sides by P.

$\ln 2 = \ln(1.015^{4t})$ Take natural logarithm of both sides.

$\ln 2 = 4t \ln 1.015$ Apply the property $\log_b(M^k) = k \log_b M$.

$t = \dfrac{\ln 2}{4 \ln 1.015}$

$t \approx 11.6389$ Evaluated on GDC.

The investment will double in 11.64 years – about 11 years and 8 months.

```
ln(2)/(4ln(1.015
))
       11.63888141
```

● **Hint:** Be sure to use brackets appropriately when entering the expression $\dfrac{\ln 2}{4 \ln 1.015}$ on your GDC. Following the rules for order of operations, your GDC will give an incorrect result if entered as shown here.

```
ln(2)/ 4ln(1.015)
        .0025799999
```

missing brackets

Example 19

The bacteria that cause 'strep throat' will grow in number at a rate of about 2.3% per minute. To the nearest whole minute, how long will it take for these bacteria to double in number?

Solution

Let t represent time in minutes and let A_0 represent the number of bacteria at $t = 0$.

Using the exponential growth model from Section 5.2, $A(t) = A_0 b^t$, the growth factor, b, is $1 + 0.023 = 1.023$ giving $A(t) = A_0(1.023)^t$. The same equation would apply to money earning 2.3% annual interest with the money being added (compounded) once per year rather than once per minute. So, our mathematical model assumes that the number of bacteria increase incrementally, with the number increasing by 2.3% at the end of each minute. To find the doubling time, find the value of t so that $A(t) = 2A_0$.

$2A_0 = A_0(1.023)^t$ Substitute $2A_0$ for $A(t)$.

$2 = 1.023^t$ Divide both sides by A_0.

$\ln 2 = \ln(1.023^t)$ Take natural logarithm of both sides.

$\ln 2 = t \ln 1.023$ Apply the property $\log_b(M^k) = k \log_b M$.

$$t = \frac{\ln 2}{\ln 1.023} \approx 30.482$$

The number of bacteria will double in about 30 minutes.

Alternative solution

What if we assumed continuous growth instead of incremental growth? We apply the continuous exponential growth model from Section 5.3: $A(t) = Ce^{rt}$ with initial amount C and $r = 0.023$.

$2C = Ce^{0.023t}$ Substitute $2C$ for $A(t)$.

$2 = e^{0.023t}$ Divide both sides by C.

$\ln 2 = \ln(e^{0.023t})$ Take natural logarithm of both sides.

$\ln 2 = 0.023t$ Apply the property $\log_b(b^x) = x$.

$$t = \frac{\ln 2}{0.023} \approx 30.137$$

Continuous growth has a slightly shorter doubling time, but rounded to the nearest minute it also gives an answer of 30 minutes.

Example 20

\$1000 is invested in an investment account that earns interest at an annual rate of 10% compounded monthly. Calculate the minimum number of years needed for the amount in the account to exceed \$4000.

Solution

We use the exponential function associated with compound interest,

$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ with $P = 1000$, $r = 0.1$ and $n = 12$.

$$4000 = 1000\left(1 + \frac{0.1}{12}\right)^{12t} \Rightarrow 4 = (1.008\overline{3})^{12t} \Rightarrow \log 4 = \log[(1.008\overline{3})^{12t}] \Rightarrow$$

$$\log 4 = 12t\log(1.008\overline{3}) \Rightarrow t = \frac{\log 4}{12\log(1.008\overline{3})} \approx 13.92 \text{ years}$$

The minimum number of years needed for the account to exceed \$4000 is 14 years.

Example 21

A 20 g sample of radioactive iodine decays so that the mass remaining after t days is given by the equation $A(t) = 20e^{-0.087t}$, where $A(t)$ is measured in grams. After how many days (to the nearest whole day) is there only 5 g remaining?

Solution

$$5 = 20e^{-0.087t} \Rightarrow \frac{5}{20} = e^{-0.087t} \Rightarrow \ln 0.25 = \ln(e^{-0.087t}) \Rightarrow$$

$$\ln 0.25 = -0.087t \Rightarrow t = \frac{\ln 0.25}{-0.087} \approx 15.93$$

After about 16 days there is only 5 g remaining.

Example 22 – An equation in quadratic form

Solve for x in the equation $3^{2x} - 18 = 3^{x+1}$. Express any answers *exactly*.

Solution

The key to solving this equation is recognizing that it can be written in *quadratic form*. In Section 3.5, we solved equations of the form $at^2 + bt + c = 0$, where t is an algebraic expression. This is not immediately clear for this equation. We need to apply some laws of exponents to show that the equation is quadratic for the expression 3^x.

$3^{2x} - 18 = 3^{x+1}$

$(3^x)^2 - 3^1 \cdot 3^x - 18 = 0$ Applying rules $b^{mn} = (b^m)^n$ and $b^{m+n} = b^m b^n$.

Substituting a single variable, for example y, for the expression 3^x clearly makes the equation quadratic in terms of 3^x. We solve first for y and then solve for x after substituting 3^x back for y.

$$y^2 - 3y - 18 = 0$$
$$(y + 3)(y - 6) = 0$$
$$y = -3 \text{ or } y = 6$$
$$3^x = -3 \text{ or } 3^x = 6$$

$3^x = -3$ has no solution. Raising a positive number to a power cannot produce a negative number.

$$3^x = 6$$
$$\ln(3^x) = \ln 6 \qquad \text{Take logarithm of both sides.}$$
$$x \ln 3 = \ln 6$$

Therefore, the one solution to the equation is exactly $x = \frac{\ln 6}{\ln 3}$.

- **Hint:** There are a couple of common algebra errors to avoid in the working for Example 22.
- If $3^x = -3$, then it does not follow that $x = -1$. An exponent of -1 indicates reciprocal.
- If $x = \frac{\ln 6}{\ln 3}$, it does ***not*** follow that $x = \ln 2$. The rule $\log m - \log n = \log\left(\frac{m}{n}\right)$ does not apply to the expression $\frac{\ln 6}{\ln 3}$.

Solving logarithmic equations

A logarithmic equation is an equation where the variable appears within the argument of a logarithm. For example, $\log x = \frac{1}{2}$ or $\ln x = 4$. We can solve both of these logarithmic equations directly by applying the definition of a logarithmic function (Section 5.4):

$$y = \log_b x \text{ if and only if } x = b^y$$

The logarithmic equation $\log x = \frac{1}{2}$ is equivalent to the exponential equation $x = 10^{\frac{1}{2}} = \sqrt{10}$, which leads directly to the solution. Likewise, the equation $\ln x = 4$ is equivalent to $x = e^4 \approx 54.598$. Both of these equations could have been solved by means of another method that makes use of the following two facts:

$$\text{(i) if } a = b \text{ then } n^a = n^b; \quad \text{and (ii) } b^{\log_b x} = x$$

To understand (ii) above, remember that a **logarithm is an exponent**. The value of $\log_b x$ is the exponent to which b is raised to give x. And b is being raised to this value; hence, the expression $b^{\log_b x}$ is equivalent to x. Therefore, another method for solving the logarithmic equation $\ln x = 4$ is to **exponentiate** both sides, i.e. use the expressions on either side of the equal sign as exponents for exponential expressions with equal bases. The base needs to be the base of the logarithm. For example,

$$\ln x = 4 \Rightarrow e^{\ln x} = e^4 \Rightarrow x = e^4$$

Example 23

Solve for x: $\log_3(2x - 5) = 2$

Solution

$$\log_3(2x-5) = 2 \Rightarrow 3^{\log_3(2x-5)} = 3^2 \qquad \text{Exponentiate both side with base} = 3.$$
$$2x - 5 = 9 \qquad \text{Applying property } b^{\log_b x} = x.$$
$$2x = 14$$
$$x = 7$$

Example 24

Solve for x in terms of k: $\log_2(5x) = 3 + k$

Solution

$\log_2(5x) = 3 + k \Rightarrow 2^{\log_2(5x)} = 2^{3+k}$ Exponentiate both sides with base = 2.

$5x = 2^3 \cdot 2^k$ Law of exponents $b^m \cdot b^n = b^{m+n}$ used 'in reverse'.

$x = \frac{8}{5}(2^k)$

For some logarithmic equations, it is necessary to first apply a property, or properties, of logarithms to simplify combinations of logarithmic expressions before solving.

Example 25

Solve for x: $\log_2 x + \log_2(10 - x) = 4$

Solution

$\log_2 x + \log_2(10 - x) = 4$

$\log_2[x(10 - x)] = 4$ Property of logarithms: $\log_b M + \log_b N = \log_b(MN)$.

$10x - x^2 = 2^4$ Changing from logarithmic form to exponential form.

$x^2 - 10x + 16 = 0$

$(x - 2)(x - 8) = 0$

$x = 2$ or $x = 8$

When solving logarithmic equations, you should be careful to always check if the *original* equation is a true statement when any solutions are substituted in for the variable. For Example 25, both of the solutions $x = 2$ and $x = 8$ produce true statements when substituted into the original equations. Sometimes 'extra' (extraneous) invalid solutions (met in Chapter 3) are produced, as illustrated in the next example.

Example 26

Solve for x: $\ln(x - 2) + \ln(2x - 3) = 2\ln x$

Solution

$\ln(x - 2) + \ln(2x - 3) = 2\ln x$

$\ln[(x - 2)(2x - 3)] = \ln x^2$ Properties of logarithms.

$\ln(2x^2 - 7x + 6) = \ln x^2$

$e^{\ln(2x^2 - 7x + 6)} = e^{\ln x^2}$ Exponentiate both sides.

$2x^2 - 7x + 6 = x^2$

$x^2 - 7x + 6 = 0$

$(x - 6)(x - 1) = 0$ Factorize.

$x = 6$ or $x = 1$

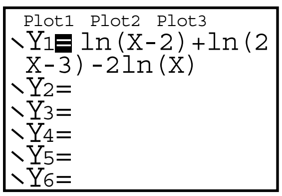
Substituting these two *possible* solutions indicates that $x = 1$ is not a valid solution. The reason is that if you try to substitute 1 for x into the original equation, we are not able to evaluate the expression $\ln(2x - 3)$ because we can only take the logarithm of a positive number. Therefore, $x = 6$ is the only solution. $x = 1$ is an extraneous solution that is not valid.

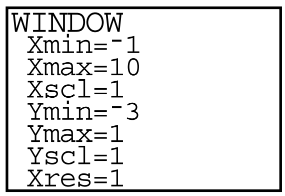
Solving, or checking the solutions to, a logarithmic equation on your GDC will help you avoid, or determine, extraneous solutions. To solve Example 26 on your GDC, a useful approach is to first set the equation equal to zero. Then graph the expression (after setting it equal to y) and observe where the graph intersects the x-axis (i.e. $y = 0$).

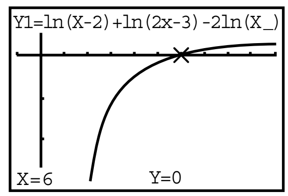
Graphical solution for Example 26:

$\ln(x - 2) + \ln(2x - 3) = 2\ln x \Rightarrow \ln(x - 2) + \ln(2x - 3) - 2\ln x = 0$

Graph the equation $y = \ln(x - 2) + \ln(2x - 3) - 2\ln x$ on your GDC and find x-intercepts.







The graph only intersects the x-axis at $x = 6$ and not at $x = 1$. Hence, $x = 6$ is the only valid solution and $x = 1$ is an extraneous solution.

Exponential and logarithmic inequalities

In Section 3.5, we covered methods of solving a variety of inequalities. These methods can also be applied to solving inequalities involving exponential and logarithmic functions. It is important to consider the domain of any functions in the inequality, and to check any solutions in the original inequality in case any extraneous solutions occur.

Example 27

Find the solution set to the inequality: $2\log_3 x - 1 < 0$.

Solution

Due to the domain of the logarithmic function, all solutions must be positive.

Method 1 (algebraic solution)

Solve the equation $2\log_3 x - 1 = 0$ and find the exact solution.

$$2\log_3 x = 1 \Rightarrow \log_3 x = \tfrac{1}{2} \Rightarrow x = 3^{\frac{1}{2}} = \sqrt{3}$$

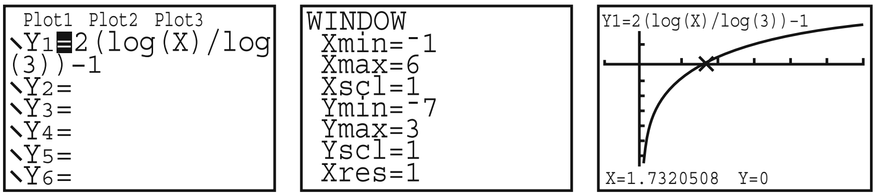
Substitute 'test' values, x_1 and x_2, into the original inequality such that $0 < x_1 < \sqrt{3}$ and $x_2 > \sqrt{3}$.

Let $x_1 = 1$: $\log_3 1 - 1 = 0 - 1 = -1 < 0$ (true)
Let $x_2 = 9$: $\log_3 9 - 1 = 2 - 1 = 1 \times 0$ (false)

Therefore, the solution set is $0 < x < \sqrt{3}$.

Method 2 (graphical solution)

Graph the equation $y = 2\log_3 x - 1$ on your GDC and use it to determine the portion of the graph that is less than zero (i.e. below the x-axis). But, how do we input the expression $\log_3 x$ on the GDC? We can use the change of base formula to write $\log_3 x = \dfrac{\log x}{\log 3}$.



The y-axis is a vertical asymptote. The graph indicates that the solution set is $0 < x < 1.732\,0508$. Although the graphical method is efficient and effective it does not give an exact result.

Example 28

Solve: $(e^x - 2)(e^x + 6) \leqslant 3e^x$

Solution

The fact that the left side is factorized is not helpful because the other side of the inequality is not zero. So we need to expand the left side and rearrange terms to get zero on the right side.

$$(e^x - 2)(e^x + 6) \leqslant 3e^x$$

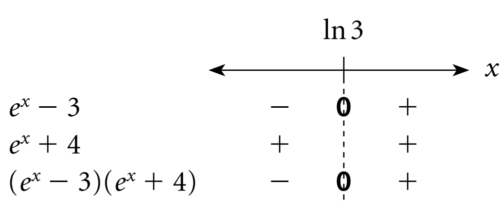
$$(e^x)^2 + 4e^x - 12 \leqslant 3e^x$$

$$e^{2x} + e^x - 12 \leqslant 0$$ Now factorize this expression.

$$(e^x - 3)(e^x + 4) \leqslant 0$$ Find where each factor is zero and construct a sign chart.

$e^x - 3 = 0 \Rightarrow e^x = 3 \Rightarrow x = \ln 3$
and $\quad e^x + 4 = 0 \Rightarrow e^x = -4 \Rightarrow$ no solution

Since $x = \ln 3$ is the only zero of the expression $(e^x - 3)(e^x + 4)$ we only need to test x-values on either side of $x = \ln 3$. The factor $e^x + 4$ will always be positive.



Therefore, the solution set is $x \leqslant \ln 3$.

Exercise 5.5

In questions 1–12, solve for x. Give x accurate to 3 significant figures.

1 $10^x = 5$ **2** $4^x = 32$ **3** $8^{x-6} = 60$

4 $2^{x+3} = 100$ **5** $\left(\frac{1}{5}\right)^x = 22$ **6** $e^x = 15$

7 $10^x = e$ **8** $3^{2x-1} = 35$ **9** $2^{x+1} = 3^{x-1}$

10 $2e^{10x} = 19$ **11** $6^{\frac{x}{2}} = 5^{1-x}$ **12** $\left(1 + \frac{0.05}{12}\right)^{12x} = 3$

In questions 13–16, solve for x. Give answers **exactly**.

13 $4^x - 2^{x+1} = 48$ • **Hint:** write 4 as 2^2 **14** $2^{2x+1} - 2^{x+1} + 1 = 2^x$

15 $6^{2x+1} - 17(6^x) + 12 = 0$ **16** $3^{2x+1} + 3 = 10(3^x)$

17 \$5000 is invested in an account that pays 7.5% interest per year, compounded quarterly.

a) Find the amount in the account after three years.

b) How long will it take for the money in the account to double? Give the answer to the nearest quarter of a year.

18 How long will it take for an investment of €500 to triple in value if the interest is 8.5% per year, compounded continuously. Give the answer in number of years accurate to 3 significant figures.

19 A single bacterium begins a colony in a laboratory dish. If the colony doubles every hour, after how many hours does the colony first have more than one million bacteria?

20 Find the least number of years for an investment to double if interest is compounded annually with the following interest rates.

a) 3% b) 6% c) 9%

21 A new car purchased in 2005 decreases in value by 11% per year. When is the first year that the car is worth less than one-half of its original value?

22 Uranium-235 is a radioactive substance that has a half-life of 2.7×10^5 years.

a) Find the amount remaining from a 1 g sample after a thousand years.

b) How long will it take a 1 g sample to decompose until its mass is 700 milligrams (i.e. 0.7 g)? Give the answer in years accurate to 3 significant figures.

23 The stray dog population in a town is growing exponentially with about 18% more stray dogs each year. In 2008, there are 16 stray dogs.

a) Find the projected population of stray dogs after five years.

b) When is the first year that the number of stray dogs is greater than 70?

24 Initially a water tank contains one thousand litres of water. At the time $t = 0$ minutes, a tap is opened and water flows out of the tank. The volume, V litres, which remains in the tank after t minutes is given by the following exponential function: $V(t) = 1000(0.925)^t$.

a) Find the value of V after 10 minutes.

b) Find how long, to the nearest second, it takes for half of the initial amount of water to flow out of the tank.

c) The tank is considered 'empty' when only 5% of the water remains. From when the tap is first opened, how many whole minutes have passed before the tank can first be considered empty?

25 The mass m kilograms of a radioactive substance at time t days is given by $m = 5e^{-0.13t}$.

a) What is the initial mass?

b) How long does it take for the substance to decay to 0.5 kg? Give the answer in days accurate to 3 significant figures.

In questions 26–36, solve for x in the logarithmic equation. Give exact answers and be sure to check for extraneous solutions.

26 $\log_2(3x - 4) = 4$

27 $\log(x - 4) = 2$

28 $\ln x = -3$

29 $\log_{16} x = \frac{1}{2}$

30 $\log \sqrt{x + 2} = 1$

31 $\ln(x^2) = 16$

32 $\log_2(x^2 + 8) = \log_2 x + \log_2 6$

33 $\log_3(x - 8) + \log_3 x = 2$

34 $\log 7 - \log(4x + 5) + \log(2x - 3) = 0$

35 $\log_3 x + \log_3(x - 2) = 1$

36 $\log x^8 = (\log x)^4$

In questions 37–40, solve each inequality.

37 $5 \log x + 2 > 0$

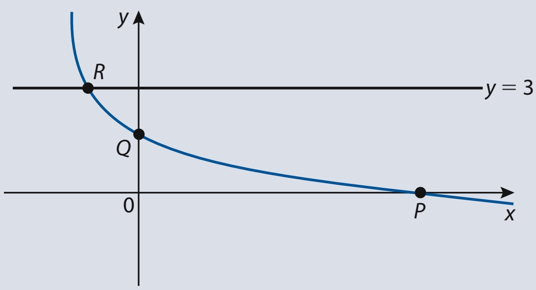
38 $2\log x^2 - 3\log x < \log 8x - \log 4x$

39 $(e^x - 2)(e^x - 3) < 2e^x$

40 $3 + \ln x > e^x$

Practice questions

1 A portion of the graph $y = 2 - \log_3(x + 1)$ is shown. It intersects the x-axis at point P, the y-axis at point Q and the line $y = 3$ at point R. Find the following:



a) The x-coordinate of point P.
b) The y-coordinate of point Q.
c) The coordinates of point R.

2 The amount $A(t)$, in grams, of a certain radioactive substance remaining after t years decays by the formula $A(t) = A_0e^{-0.0045t}$, where A_0 is the initial amount.

a) If 5 grams are left after 800 years, how many grams were present initially?
b) What is the half-life of the substance?

3 **a)** Find expressions for the nth term and the sum to n terms of the following arithmetic series, $\ln y + \ln y^2 + \ln y^3 + \ldots$ where $y > 0$.

b) Hence, find expressions for the nth term and the sum to n terms of the following arithmetic series, $\ln(xy) + \ln(xy^2) + \ln(xy^3) + \ldots$ where $x > 0$ and $y > 0$.

4 Solve, for x, the equation $\log_2(5x^2 - x - 2) = 2 + 2\log_2 x$.

5 If $\log_2 4\sqrt{2} = x$, $\log_z y = 4$, and $y = 4x^2 - 2x - 6 + z$, find y.

6 Find the **exact** values of t for which $2e^{3t} - 7e^{2t} + 7e^t = 2$.

7 Find the **exact** solution(s) to the equation $8e^2 - 2e\ln x = (\ln x)^2$.

8 Find the exact value of x for each equation.

a) $\log_3 x - 4\log_x 3 + 3 = 0$

b) $\log_2(x - 5) + \log_2(x + 2) = 3$

9 Express each as a single logarithm.

a) $2\log a + 3\log b - \log c$

b) $3\ln x - \frac{1}{2}\ln y + 1$

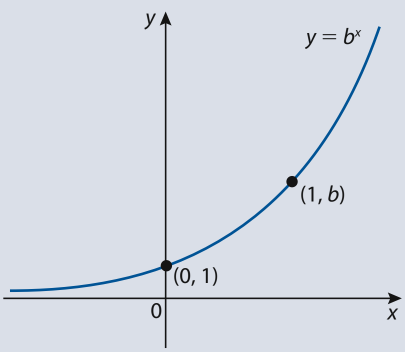
10 A piece of wood is recovered from an ancient building during an archaeological excavation. The formula $A(t) = A_0 e^{-0.000\,124t}$ is used to determine the age of the wood, where A_0 is the amount of carbon in any living tree, $A(t)$ is the amount of carbon in the wood being dated and t is the age of the wood in years. For the ancient piece of wood it is found that $A(t)$ is 79% of the amount of the carbon in a living tree. How old is the piece of wood, to the nearest 100 years?

11 The graph of the equation $y = \log_3(2x - 3) - 4$ intersects the x-axis at the point $(c, 0)$. Without using your GDC, find the exact value of c.

12 The graph of $y = b^x$, $b > 1$ is shown. On separate coordinate planes, sketch the graphs of

a) $y = b^{-x}$

b) $y = b^{1-x}$



13 Radium decays exponentially and its half-life is 1600 years.
If A_0 represents the initial amount of radium in a sample and $A(t)$ represents the amount remaining after t years, then $A(t) = A_0 e^{-kt}$.

a) Find the value of k approximated to four significant figures.

b) Find what percentage of the original amount of radium will be remaining after 4000 years.

14 Solve the equation $e^{-x} - x + 1 = 0$.

15 Find the set of values of x for which $|0.1x^2 - 2x + 3| < \log_{10} x$.

16 Determine the values of x that satisfy the inequality $\dfrac{xe^x}{x^2 - 1} \geqslant 1$.

17 **a)** Solve the equation $2(4^x) + 4^{-x} = 3$.

b) **(i)** Solve the equation $a^x = e^{2x+1}$ where $a > 0$, giving your answer for x in terms of a.

(ii) For what value of a does the equation have no solution?

18 The solution of $2^{2x+3} = 2^{x+1} + 3$ can be expressed in the form $a + \log_2 b$ where $a, b \in \mathbb{Z}$. Find the value of a and of b.

19 Solve $2(\ln x)^2 = 3\ln x - 1$ for x. Give your answers in **exact** form.

20 A sum of \$100 is invested.

a) If the interest is compounded annually at a rate of 5% per year, find the total value V of the investment after 20 years.

b) If the interest is compounded monthly at a rate of $\frac{5}{12}$% per month, find the minimum number of months for the value of the investment to exceed V.

21 Solve the equation $9\log_5 x = 25\log_x 5$ expressing your answer in the form $5^{\frac{p}{q}}$, where $p, q \in \mathbb{Z}$.

22 Solve $|\ln(x + 3)| = 1$. Give your answers in **exact** form.

23 Solve the equation $\left|e^{2x} - \dfrac{1}{x+2}\right| = 2$.

24 An experiment is carried out in which the number n of bacteria in a liquid, is given by the formula $n = 650e^{kt}$, where t is the time in minutes after the beginning of the experiment and k is a constant. The number of bacteria doubles every 20 minutes. Find the exact value of k.

25 The function f is defined for $x > 2$ by $f(x) = \ln x + \ln(x - 2) - \ln(x^2 - 4)$.

a) Express $f(x)$ in the form $\ln\left(\dfrac{x}{x+a}\right)$.

b) Find an expression for $f^{-1}(x)$.

26 **a)** The function f is defined by $f: x \mapsto e^x - 1 - x$.

(i) Use your GDC to find the minimum value of f.

(ii) Prove that $e^x \geqslant 1 + x$ for all real values of x.

b) Use mathematical induction to prove that

$$(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1 \text{ for all integers } n \geqslant 1$$

c) Use the results of parts a) and b) to prove that

$$e^{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)} > n$$

d) Find a value of n for which

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > 100$$

6 Matrix Algebra

Assessment statement

1.9 Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases when there is a unique solution, an infinity of solutions or no solution.

Note: Sections 6.1 to 6.3 are not required for examinations. However, it is highly recommended that you review these sections because of their important applications. Sections 6.1 and 6.2 can be omitted. Special attention must be paid to the determinant concept in Section 6.3 because it will be used later in the book.

In Section 6.4 the Gauss-Jordan elimination method is required in its 'raw' form, i.e. using equations. However, for reasons of efficiency, and if you were to use a GDC to solve a system of equations, the matrix form is more appropriate. Even though it is not required for examination purposes, in exams, any 'mathematically sound' method is accepted.

Introduction

Ever since their first emergence, matrices have been and remain significant mathematical tools. Uses of matrices span several areas from simply solving systems of simultaneous linear equations, to describing atomic structure, designing computer game graphics, analyzing relationships, coding, and operations research, to mention a few. If you have ever used a spreadsheet such as Excel or Lotus, or have ever created a table, then you have used a matrix. Matrices make the presentation of data understandable and help make calculations easy to perform. For example, your teacher's grade book may look something like this:

Student	Quiz 1	Quiz 2	Test 1	Test 2	Homework	Grade
Tim	70	80	86	82	95	A
Maher	89	56	80	60	55	C
…	…	…	…	…	…	…

If we want to know Tim's grade on Test 2, we simply follow along the row 'Tim' to the column 'Test 2' and find that he received a score of 82. Take a look at the matrix below about the sale of cameras in a store according to location and type.

	City	Donau	Neubau	Moedling
Nikon	153	98	74	56
Canon	211	120	57	29
Olympus	82	31	12	5
Other	308	242	183	107

If we want to know how many Canon cameras were sold in the Neubau shop, we follow along the row 'Canon' to the column 'Neubau' and find that 57 Canons were sold.

6.1 Basic definitions

What is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers.

Matrix $[A]$ is denoted by

$$A = \underbrace{\begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \\ \uparrow \quad\;\; \uparrow \quad\;\; \dots \quad\;\; \uparrow \end{array}}_{n \text{ columns}} \left.\begin{array}{c} \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array}\right\} m \text{ rows}$$

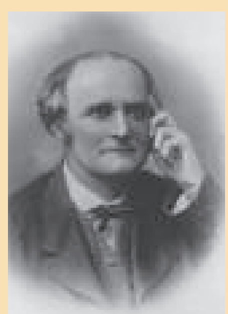
Row i of A has n elements and is $(a_{i1} \quad a_{i2} \quad \dots \quad a_{in})$.

Column j of A has m elements and is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

The number of rows and columns of the matrix define its size (order). So, a matrix that has m rows and n columns is said to have an $m \times n$ (m by n) size (order). A matrix A with $m \times n$ order (size) is sometimes denoted as $[A]_{m \times n}$ or $[A]_{mn}$ to show that A is a matrix with m rows and n columns. (Some authors use $[a_{ij}]$ to represent a matrix.) The sales matrix has a 4×4 order. When $m = n$, the matrix is said to be a **square matrix** with order n, so the sales matrix is a square matrix of order 4.

Every entry in the matrix is called an **entry** or **element** of the matrix, and is denoted by a_{ij}, where i is the row number and j is the column number of that element. The ordered pair (i, j) is also called the **address** of the element. So, in the grades matrix example, the entry (2, 4) is 60, the student Maher's grade on Test 2, while (2, 4) in the sales matrix example is 29, Canon's sales in the Moedling shop.

Arthur Cayley (1821–1895)



Arthur Cayley entered Trinity College, Cambridge in 1838. While still an undergraduate, he published three papers in the *Cambridge Mathematical Journal*. Cayley graduated as Senior Wrangler in 1842 and won the first Smith's prize. Winning a fellowship enabled him to teach for four years at Cambridge. He published 28 papers in the *Cambridge Mathematical Journal* during these years. Since a fellowship had limited tenure, Cayley needed to find a profession. He spent 14 years as a lawyer but, although very skilled in his legal specialty, he always considered it as a means to make money so that he could pursue mathematics. During these 14 years as a lawyer he published around 250 mathematical papers.

His published work comprises over 900 papers and notes covering several fields of modern mathematics. The most important aspect of his work was in developing the algebra of matrices.

Vectors

A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

Row vector

If a matrix has one row, it is called a row vector.

$B = (b_1\ b_2\ \dots\ b_m)$ is a row vector with dimension m.

$B = (1\ 2)$ could represent the position of a point in a plane and is an example of a row vector of dimension 2.

Column vector

If a matrix has one column, it is called a column vector.

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ is a column vector with dimension n.

$C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ again could represent the position of a point in a plane and is an example of a column vector of dimension 2.

As you see, vectors can be represented by row or column matrices.

Submatrix

If some row(s) and/or column(s) of a matrix A are deleted, the remaining matrix is called a submatrix of A.

For example, if we are interested in the sales of the three main types of cameras in the central part of the city, we can represent them with the following *submatrix* of the original matrix:

$$\begin{pmatrix} 153 & 98 \\ 211 & 120 \\ 82 & 31 \end{pmatrix}$$

Zero matrix

A matrix for which all entries are equal to zero ($a_{ij} = 0$ for all i and j).

$$(0\quad 0), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ are zero matrices.

Diagonal

A square matrix where all entries except the diagonal entries are zero is called a **diagonal matrix**.

In a square matrix, the entries $a_{11}, a_{22}, \dots, a_{nn}$ are called the **diagonal elements** of the matrix. Sometimes the diagonal of the matrix is also called the **principal** or **main** of the matrix.

$$\begin{pmatrix} 153 & 0 & 0 & 0 \\ 0 & 120 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 107 \end{pmatrix}$$

What is the diagonal in our sales matrix? Here, $a_{11} = 153$, $a_{22} = 120$, $a_{33} = 12$ and $a_{44} = 107$.

Triangular matrix

You can use a matrix to present data showing distances between different cities.

	Graz	Salzburg	Innsbruck	Linz
Vienna	191	298	478	185
Graz		282	461	220
Salzburg			188	135
Innsbruck				320

Table 6.1

The data in Table 6.1 can be represented by a triangular matrix (upper triangular in this case).

$$\begin{pmatrix} 191 & 298 & 478 & 185 \\ 0 & 282 & 461 & 220 \\ 0 & 0 & 188 & 135 \\ 0 & 0 & 0 & 320 \end{pmatrix}$$

In a triangular matrix, the entries on one side of its diagonal are all zero.

Definition of a triangular matrix

A triangular matrix is a square matrix with order n for which $a_{ij} = 0$ when $i > j$ (upper triangular) or, alternatively, when $i < j$ (lower triangular).

Another way of representing the distance data is given by the following matrix:

	Vienna	*Graz*	*Salzburg*	*Innsbruck*	*Linz*
Vienna	0	191	298	478	185
Graz	191	0	282	461	220
Salzburg	298	282	0	188	135
Innsbruck	478	461	188	0	320
Linz	185	220	1325	320	0

Again the data in the table can be represented by a matrix called a **symmetric** matrix. In such matrices, $a_{ij} = a_{ji}$ for all i and j. All symmetric matrices are square!

$$\begin{pmatrix} 0 & 191 & 298 & 478 & 185 \\ 191 & 0 & 282 & 461 & 220 \\ 298 & 282 & 0 & 188 & 135 \\ 478 & 461 & 188 & 0 & 320 \\ 185 & 220 & 135 & 320 & 0 \end{pmatrix}$$

6.2 Matrix operations

When are two matrices considered to be equal?

Two matrices A and B are equal if the size of A and B is the same (number of rows and columns are the same for A and B) and $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $\begin{pmatrix} 2 & x \\ x^2 - 4 & 7 \end{pmatrix}$ can only be equal if $x = 3$ and $x^2 - 4 = 5$, which can only be true if $x = 3$.

How do you add/subtract two matrices?

Two matrices A and B can be added only if they have the *same size*. If C is the sum of the two matrices, then we write

$$C = A + B$$

where $c_{ij} = a_{ij} + b_{ij}$, i.e. we add 'corresponding' terms, one by one.

For example,

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} x & y \\ a & b \end{pmatrix} = \begin{pmatrix} 2+x & 3+y \\ 5+a & 7+b \end{pmatrix}$$

Subtraction is done similarly:

$$\begin{pmatrix} 2 & 3 & 1 \\ 5 & 7 & 0 \end{pmatrix} - \begin{pmatrix} x & y & 8 \\ a & b & 2 \end{pmatrix} = \begin{pmatrix} 2-x & 3-y & -7 \\ 5-a & 7-b & -2 \end{pmatrix}$$

The operations of addition and subtraction of matrices obey all rules of addition and subtraction of real numbers. That is,

$A + B = B + A$; $A + (B + C) = (A + B) + C$; $A - (B + C) = A - B - C$.

How do we multiply a scalar by a matrix?

A scalar is any object that is not a matrix. The multiplication by a scalar is straightforward. You multiply each term of the matrix by the scalar.

If A is an $m \times n$ matrix, and c is a scalar, the scalar product of c and A is another matrix $B = cA$ such that every entry b_{ij} of B is a multiple of its corresponding A entry, i.e. $b_{ij} = c \times a_{ij}$.

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the following example, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix}$$

Matrix multiplication

At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Matrix multiplication

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix, $AB = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

This definition means that each entry with an address ij AB is obtained by multiplying the entries in the ith row of A by the *corresponding* entries in the jth column of B and then adding the results. The following shows the process in detail:

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 1

Find $C = AB$ if $A = \begin{pmatrix} 3 & -5 & 2 \\ 2 & 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 1 & 5 \\ 5 & 8 & -4 & 0 \\ -9 & 10 & 5 & 3 \end{pmatrix}$.

Solution

A is a **2** $\times$ **3** matrix and B is a **3** $\times$ **4** matrix, so the product must be a **2** $\times$ **4** matrix. Every entry in the product is the result of multiplying the entries in the rows of A and columns of B. For example:

$$c_{12} = \sum_{k=1}^{3} a_{1k}b_{k2} = (a_{11} \quad a_{12} \quad a_{13}) \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = (3 \quad -5 \quad 2) \begin{pmatrix} -2 \\ 8 \\ 10 \end{pmatrix}$$

$$= 3 \times (-2) - 5 \times 8 + 2 \times 10 = -26$$

or

$$c_{23} = \sum_{k=1}^{3} a_{2k}b_{k3} = (a_{21} \quad a_{22} \quad a_{23}) \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = (2 \quad 1 \quad 7) \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

$$= 2 \times 1 + 1 \times (-4) + 7 \times 5 = 33$$

The operation is repeated eight times to get

$$C = AB = \begin{pmatrix} -34 & -26 & 33 & 21 \\ -52 & 74 & 33 & 31 \end{pmatrix}$$

This product can also be found using a GDC.

```
[A][B]
[[-34 -26 33 21...
 [-52 -74 33 31...
█
```

For the product of two matrices to be defined, the number of columns in the first matrix should be the same as the number of rows in the second matrix.

$$\begin{array}{ccccc} A & & B & = & AB \\ m \times n & & n \times p & & m \times p \end{array}$$

equal

order of AB

Examples – matrix multiplication

a) $\begin{pmatrix} 5 & 0 & 3 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 14 \\ 11 & -13 \end{pmatrix}$

$\quad 2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$

b) $\begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix}$

$\quad 2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

c) $\begin{pmatrix} 5 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & -\frac{3}{7} & \frac{3}{7} \\ -\frac{10}{7} & -\frac{9}{7} & \frac{16}{7} \\ \frac{4}{7} & \frac{5}{7} & -\frac{5}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$3 \times 3 \qquad 3 \times 3 \qquad 3 \times 3$

As you see from part b) above, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not create a new value when it is multiplied by another matrix. This is why it is called the **identity** matrix of order 2.

The identity matrix

A $n \times n$ diagonal matrix where $a_{ij} = 1$ and $i = j$ is called the identity matrix of order n.

Examples – identity matrices

a) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

c) $\begin{pmatrix} a & b & c & m \\ d & e & f & n \\ g & h & i & p \\ j & k & l & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c & m \\ d & e & f & n \\ g & h & i & p \\ j & k & l & q \end{pmatrix}$

Sometimes, the identity matrix is denoted by I_n, where n is the order. So, in parts a) and b) above, the identity is I_3, and in c) it is I_4.

Examples – comparing AB with BA

a) $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} = (11)$

$1 \times 3 \qquad 3 \times 1 \qquad 1 \times 1$

b) $\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 6 \\ 10 & -5 & 15 \\ 8 & -4 & 12 \end{pmatrix}$

$3 \times 1 \qquad 1 \times 3 \qquad 3 \times 3$

Notice the difference between the products in parts a) and b). Matrix multiplication, in general, is **not commutative**. It is usually not true that $AB = BA$.

Let $A = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix}$, then $AB = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 39 \\ -8 & 25 \end{pmatrix}$

but

$$BA = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -6 \\ 28 & 16 \end{pmatrix} \Rightarrow AB \neq BA$$

However, if we let

$$A = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}, \text{ then } AB = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 36 & 24 \\ 20 & 32 \end{pmatrix} \text{ and}$$

$$BA = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 36 & 24 \\ 20 & 32 \end{pmatrix} \Rightarrow AB = BA$$

Thus, in general, $AB \neq BA$. However, for some matrices A and B, it may happen that $AB = BA$.

Example 2

Find the average sales in each of the regions (City, Donau, Neubau and Moedling), given the following information.

	City	Donau	Neubau	Moedling
Nikon	153	98	74	56
Canon	211	120	57	29
Olympus	82	31	12	5
Other	308	242	183	107

The average selling price for each make of camera is as follows:
Nikon €1200, Canon €1100, Olympus €900, Other €600

Solution

We set up a matrix multiplication in which the individual camera sales are multiplied by the corresponding price. Since the rows represent the sales of the different makes of camera, create a row matrix of the different prices and perform the multiplication.

$$\begin{pmatrix} 1200 & 1100 & 900 & 600 \end{pmatrix}\begin{pmatrix} 153 & 98 & 74 & 56 \\ 211 & 120 & 57 & 29 \\ 82 & 31 & 12 & 5 \\ 308 & 242 & 183 & 107 \end{pmatrix} = \begin{pmatrix} 674\,300 & 422\,700 & 272\,100 & 167\,800 \end{pmatrix}$$

So, the regions' sales are:

	City	Donau	Neubau	Moedling
Sales	674 300	422 700	272 100	167 800

Remember that we are multiplying a 1×4 matrix with a 4×4 matrix and hence we get a 1×4 matrix.

Exercise 6.1 and 6.2

1 Consider the following matrices

$$A = \begin{pmatrix} -2 & x \\ y-1 & 3 \end{pmatrix}, B = \begin{pmatrix} x+1 & -3 \\ 4 & y-2 \end{pmatrix}$$

a) Evaluate each of the following
 (i) $A + B$ (ii) $3A - B$.

b) Find x and y such that $A = B$.

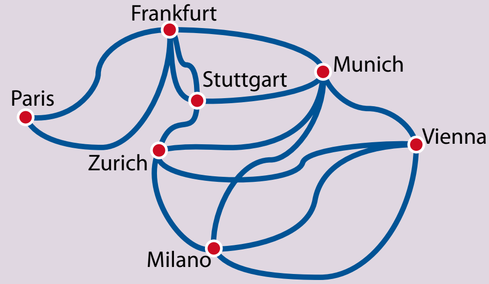
c) Find x and y such that $A + B$ is a diagonal matrix.

d) Find AB and BA.

2 Solve for the variables.

a) $\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$ b) $\begin{pmatrix} 2 & p \\ 3 & q \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

3 The diagram below shows the major highways connecting some European cities: Vienna (V), Munich (M), Frankfurt (F), Stuttgart (S), Zurich (Z), Milano (L) and Paris (P).



a) Write the number of *direct* routes between each pair of cities into a matrix as started below:

$$\begin{array}{c} \\ V \\ M \\ F \\ S \\ Z \\ L \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc} V & M & F & S & Z & L & P \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{array}$$

b) Multiply the matrix from part a) by itself and interpret what it signifies.

4 Consider the following matrices:

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 0 & -3 & 2 \\ 7 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} m & -2 \\ 3m & -1 \\ 2 & 3 \end{pmatrix}, C = \begin{pmatrix} x-1 & 5 & y \\ 0 & -x & y+1 \\ 2x+y & x-3y & 2y-x \end{pmatrix}$$

a) Find $A + C$. b) Find AB. c) Find BA.

d) Solve for x and y if $A = C$. e) Find $B + C$.

f) Solve for m if $3B + 2\begin{pmatrix} -1 & m^2 \\ -5 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 17 & 1 \\ 2m+2 & 7 \end{pmatrix}$.

5 Find a, b and c so that the following equation is true:

$$2 \cdot \begin{pmatrix} a-1 & b \\ c+2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 8 & c+9 \end{pmatrix}$$

6 Find x and y such that:

$$\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} x-11 & 1-x \\ -5 & x+2y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7 Find m and n if

$$\begin{pmatrix} m^2-1 & m+2 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 3 & n+1 \\ 5 & n-5 \end{pmatrix}.$$

8 There are two supermarkets in your area. Your shopping list consists of 2 kg of tomatoes, 500 g of meat and 3 litres of milk. Prices differ between the different shops, and it is difficult to switch between stores to make certain you are paying the least amount of money. A better strategy is to check and see where you pay less on *average*! The prices of the different items are given below. Which shop should you go to?

Product	Price in shop A	Price in shop B
Tomato	€1.66/kg	€1.58/kg
Meat	€2.55/100 g	€2.6/100 g
Milk	€0.90/litre	€0.95/litre

9 Consider the matrices

$A = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix}$.

a) Find $A + (B + C)$ and $(A + B) + C$.

b) Make a conjecture about the addition of 2×2 matrices observed in a) above and prove it.

c) Find $A(BC)$ and $(AB)C$.

d) Make a conjecture about the multiplication of 2×2 matrices observed in c) above and prove it.

10 A company stores and sells air conditioning units, electric heaters and humidifiers. Row matrix A represents the number of each unit sold last year, and matrix B represents the profit margin for each unit. Find AB and describe what the product represents.

$A = (235 \quad 562 \quad 117)$, $B = \begin{pmatrix} €120 \\ €95 \\ €56 \end{pmatrix}$

11 Find r and s such that the following equation is true: $rA + B = A$, where

$A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & -6 \\ s - 8 & -14 \end{pmatrix}$.

12 Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

a) Find:
(i) A^2 (ii) A^3 (iii) A^4 (iv) A^n

Let $B = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$.

b) Find:
(i) B^2 (ii) B^3 (iii) B^4 (iv) B^n

13 Solve for x and y such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}$.

14 Solve for x and y such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix}$.

15 Solve for x such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$.

16 Solve for x and y such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$.

6.3 Applications to systems

There is a wide range of applications of matrices in solving systems of equations. Recall from your algebra that the equation of a straight line can take the form

$$ax + by = c$$

where a, b and c are constants and x and y are variables. We call this equation a **linear equation in two variables**. Similarly, the equation of a plane in three-dimensional space has the form

$$ax + by + cz = d$$

where a, b, c and d are constants. We call this equation a **linear equation in three variables**.

A **solution** of a linear equation in n variables (in this case two or three) is an ordered set of real numbers (x_0, y_0, z_0) so that the equation in question is satisfied when these values are substituted for the corresponding variables. For example, the equation

$$x + 2y = 4$$

is satisfied when $x = 2$ and $y = 1$. Some other solutions are $x = -4$ and $y = 4$, $x = 0$ and $y = 2$, and $x = -2$ and $y = 3$.

The set of all solutions of a linear equation is its **solution set**, and when this set is found, the equation is said to have been **solved**. To describe the entire solution set we often use a **parametric representation** as illustrated in the following examples.

Example 3

Solve the linear equation $x + 2y = 4$.

Solution

To find the solution set of an equation in two variables, we solve for one variable in terms of the other. For instance, if we solve for x, we obtain

$$x = 4 - 2y.$$

In this form, y is **free**, in the sense that it can take on any real value, while x is not free, since its value depends on that of y. To represent this solution set in general terms, we introduce a third variable, for example, t, called a **parameter**, and by letting $y = t$ we represent the solution set as

$$x = 4 - 2t,\ y = t,\ t \text{ is any real number.}$$

Particular solutions can then be obtained by assigning values to the parameter t. For instance, $t = 1$ yields the solution $x = 2$ and $y = 1$, and $t = 3$ yields the solution $x = -2$ and $y = 3$.

Note that the solution set of a linear equation can be represented parametrically in several ways. For instance, in this example, if we solve for y in terms of x, the parametric representation would take the following form:

$$x = m,\ y = 2 - \tfrac{1}{2}m,\ m \text{ is a real number.}$$

Also, by choosing $m = 2$, one particular solution would be $(x, y) = (2, 1)$, and by choosing $m = -2$, another particular solution would be $(-2, 3)$.

Example 4

Solve the linear equation $3x + 2y - z = 3$.

Solution

Choosing x and y as the *free* variables, we solve for z.

$$z = 3x + 2y - 3$$

Letting $x = p$ and $y = q$, we obtain the parametric representation:

$$x = p,\ y = q,\ z = 3x + 2y - 3,\ p \text{ and } q \text{ any real numbers.}$$

A particular solution $(x, y, z) = (1, 1, 2)$.

Parametric representation is very important when we study vectors and lines later on in the book.

Systems of linear equations – refresher

A system of k equations in n variables is a set of k linear equations in the same n variables. For example,

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

is a system of two linear equations in two variables, while

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \end{aligned}$$

is a system with two equations and three variables, and

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

is a system with three equations and three variables.

A **solution** of a system of equations is an ordered set of numbers x_0, y_0, … which satisfy every equation in the system. For example, $(3, -1)$ is a solution of

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

Both equations in the system are satisfied when $x = 3$ and $y = -1$ are substituted into the equations. On the contrary, $(0, 1)$ is not a solution of the system, even though it satisfies the first equation, as it does not satisfy the second.

As you already know, there are several ways of finding solutions to systems. In this chapter, we will consider using matrix methods to solve systems of equations.

Taking our example above, notice how we can write the system of equations in matrix form:

$$\begin{cases} 2x + 3y = 3 \\ x - y = 4 \end{cases} \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The representation of the system of equations in this way enables us to use matrix operations in solving systems. This matrix equation can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow AX = C$$

where A is the coefficient matrix, X is the variables matrix and C is the constants matrix. However, to solve this equation, the inverse of a matrix has to be defined as the solution of the system in the form

$$X = A^{-1}C$$

where A^{-1} is the inverse of the matrix A.

Matrix inverse (Optional)

To solve the equation $2x = 6$ for x, we need to multiply both sides of the equation by $\frac{1}{2}$:

$\frac{1}{2} \times 2x = \frac{1}{2} \times 6 \Rightarrow x = 3$. This is so, because $\frac{1}{2} \times 2 = 2 \times \frac{1}{2} = 1$.

$\frac{1}{2}$ is called the multiplicative inverse of 2. The inverse of a matrix is defined in a similar manner and plays a similar role in solving a matrix equation, such as $AX = C$.

Inverse of a matrix

A square matrix B is the inverse of a square matrix A if $AB = BA = I$, where I is the identity matrix.

The notation A^{-1} is used to denote the inverse of a matrix A. Thus, $B = A^{-1}$. Note that only square matrices can have multiplicative inverses.

Example – matrix inverse

$A = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}$ are multiplicative inverses since

$$AB = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 21 - 20 & -35 + 35 \\ 12 - 12 & -20 + 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 21 - 20 & 15 - 15 \\ -28 + 28 & -20 + 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Finding the inverse can also be achieved using a GDC.

```
[A]-1
                [[3   -5]
                  [-4  7 ]]
[A]-1[A]
                   [[1  0]
                     [0  1]]
█
```

There are a few methods available for finding the inverse of a 2×2 matrix. We will be using the following method only, since the other methods are beyond the scope of this textbook.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume $A^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and then solve the following matrix equation for e, f, g and h in terms of a, b, c and d.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we can set up two systems to solve for the required variables, i.e.:

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} ae + bg &= 1 \\ ce + dg &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} dae + \mathbf{dbg} &= d \\ bce + \mathbf{bdg} &= 0 \end{aligned}\right\} \Rightarrow e = \frac{d}{ad - bc},\ g = \frac{-c}{ad - bc}$$

$$\left.\begin{aligned} af + bh &= 0 \\ cf + dh &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} daf + \mathbf{dbh} &= 0 \\ bcf + \mathbf{bdh} &= b \end{aligned}\right\} \Rightarrow f = \frac{-b}{ad - bc},\ h = \frac{a}{ad - bc}$$

Therefore, $A^{-1} = \begin{pmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{pmatrix}$ or $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Example 5

Find the inverse of $\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$.

Solution

Here $a = 4$, $b = 7$, $c = 3$ and $d = 5$, so $ad - bc = -1$. Thus,

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 5 & -7 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 7 \\ 3 & -4 \end{pmatrix}.$$

```
[A]
                [[4  7]
                 [3  5]]
[A]-1
               [[-5  7]
                [3  -4]]
```

The determinant

The number $ad - bc$ is called the **determinant** of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The notation we will use for this number is **det *A***, so $\det A = ad - bc$.

The determinant plays an important role in determining whether a matrix has an inverse or not.

If the determinant is zero, i.e. $ad - bc = 0$, the matrix does not have an inverse. If a matrix has no inverse, it is called a **singular matrix**; if it is invertible, it is called **non-singular**.

Example 6

Solve the system of equations.

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

Solution

In matrix form, the system can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} -15 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

```
[A]-1 [C]
                [[3  ]
                 [-1]]
```

Solving systems of equations in three variables follows similar procedures. However, finding the inverse of a 3×3 matrix will be delegated to the GDC at this level. As in the case of a 2×2 matrix, the existence of an inverse for a 3×3 matrix depends on the value of its determinant.

The determinant of a 3×3 matrix A can be achieved in one of two ways:

1. $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \det A = a(ei - fh) - b(di - fg) + c(dh - eg)$

   For example, if

   $$A = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix} \Rightarrow \det A = 5(18 + 10) - 1(-12 + 35) - 4(4 + 21) = 17$$

```
[A]
      [[5 1  -4]
       [2 -3 -5]
       [7 2  -6]]
det([A])
                17
```

2. A practical method is to use a 'special' set up as follows:

   $$\det A = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right| = \color{red}{aei + bfg + cdh} - \color{green}{gec - hfa - idb}$$

   This is done by 'copying' the first two columns and adding them to the end of the matrix, multiplying down the main diagonals and adding the products, and then multiplying up the second diagonals and subtracting them from the previous product, as shown. In the example above:

   $$\left|\begin{array}{ccc|cc} 5 & 1 & -4 & 5 & 1 \\ 2 & -3 & -5 & 2 & -3 \\ 7 & 2 & -6 & 7 & 2 \end{array}\right|$$

$= \color{red}{5(-3)(-6) + 1(-5)(7) + (-4) \cdot 2 \cdot 2} \color{green}{- 7(-3)(-4) - 2(-5) \cdot 5 - (-6) \cdot 2 \cdot 1}$

$= \color{red}{90 - 35 - 16} \color{green}{- 84 + 50 + 12}$

$= 152 - 135$

$= 17$

   In fact, this arrangement is simply a reordering of the calculations involved in the previous method.

Example 7

Solve the system of equations.

$$\begin{aligned} 5x + y - 4z &= 5 \\ 2x - 3y - 5z &= 2 \\ 7x + 2y - 6z &= 5 \end{aligned}$$

Solution

We write this system in matrix form:

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

Since det $A \neq 0$, we can find the solution in the same way we did for the 2×2 matrix, i.e.

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

Using a GDC:

```
[A]-1 [C]
              [[3 ]
               [-2]
               [2 ]]
```

To check your work, you can store the answer matrix as D and then substitute the values into the system:

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 - 2 - 8 \\ 6 + 6 - 10 \\ 21 - 4 - 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}, \text{ or}$$

```
[A] [D]
              [[5]
               [2]
               [5]]
```

Area of a triangle

An interesting application of determinants that you may find helpful is finding the area of a triangle whose vertices are given as points in a coordinate plane. The following result will become obvious as you study Chapter 14.

Area of a triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is equal to $\left|\frac{1}{2}|A|\right|$ where

$$A = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

Example 8

Find the area of triangle ABC whose vertices are $A(1, 3)$, $B(5, -1)$ and $C(-2, 5)$.

Solution

We let $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (5, -1)$, and $(x_3, y_3) = (-2, 5)$. To find the area, we evaluate the determinant:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix} = -4.$$

Using this value, we can conclude that the area of the triangle is given by:

$$\text{Area} = \left| \tfrac{1}{2} \begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix} \right| = \left| \tfrac{1}{2} \cdot -4 \right| = 2$$

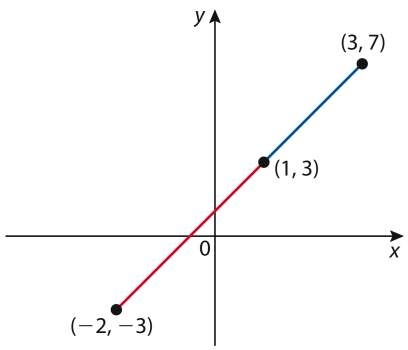
● **Hint:** Try using determinants to find the area of triangle ABC with $A(2, 3)$, $B(12, 3)$, and $C(12, 9)$. Confirm your answer by using the usual area formula of a triangle, $\frac{1}{2}(\textit{base} \times \textit{height})$.

Lines in planes

In our previous discussion, what if the three points are collinear? The answer is very simple. The triangle would collapse into a line segment and the area becomes zero. This fact helps us develop two techniques that are very helpful in dealing with questions of collinearity and equations of lines.

For example, take the points $A(-2, -3)$, $B(1, 3)$ and $C(3, 7)$. Find the area of 'triangle' ABC.

$$\text{Area} = \left| \tfrac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 1 & 3 & 1 \\ 3 & 7 & 1 \end{vmatrix} \right| = \left| \tfrac{1}{2} \cdot -0 \right| = 0$$



This result can be stated in general as given below:

Test for collinearity

The three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example 9

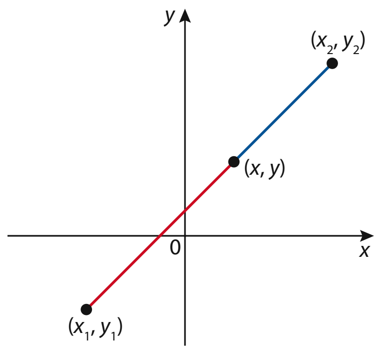
Determine whether the points $(-2, 3)$, $(2, 5)$ and $(5, 7)$ lie on the same line.

Solution

By setting up the matrix as suggested by the rule above, we have

$$\begin{vmatrix} -2 & 3 & 1 \\ 2 & 5 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 2 \neq 0.$$

Because the value of the determinant is not equal to zero, the points cannot lie on a line.



Two-point equation of a line

The test for collinearity leads us to the following result, which enables us to find the equation of a line containing two points. Consider two points (x_1, y_1), (x_2, y_2) which lie on a given line. To find the equation of the line through these two points, we introduce a general point (x, y) on the line. These three points (x_1, y_1), (x_2, y_2) and (x, y) are collinear, and hence they satisfy the determinant equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which gives us the equation of the line in the form:

$$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - y_1x_2) = 0$$

which in turn is of the form: $Ax + By + C = 0$.

Example 10

Find the equation of the line through $(-2, 3)$ and $(3, 7)$.

Solution

Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ -2 & 3 & 1 \\ 3 & 7 & 1 \end{vmatrix} = (3 - 7)x + (3 + 2)y + (-14 - 9) = 0$$

$$-4x + 5y - 23 = 0$$

Exercise 6.3

1 Consider the matrix M which satisfies the matrix equation

$$\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix} M = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.$$

a) Write out the inverse of matrix $\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}$.

b) Hence, write M as a product of two matrices.

c) Evaluate M.

d) Now consider the equation containing the matrix N:

$$N\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

(i) Write N as a product of two matrices.

(ii) Evaluate N.

e) Write a short paragraph describing your work on this problem.

2 Find the matrix E in the following equation:

$$\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} E \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$$

3 a) Prove that the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{pmatrix}$ should have an inverse.

b) Write out A^{-1}.

c) Hence, solve the system of equations:

$$\begin{cases} 2x - 3y + z = 4.2 \\ x + y - 3z = -1.1 \\ 3x - 2y - 3z = 2.9 \end{cases}$$

4 Find the inverse for each matrix.

a) $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ b) $B = \begin{pmatrix} a & 1 \\ a + 2 & \frac{3}{a} + 1 \end{pmatrix}$

5 For what values of x is the following matrix singular?

$A = \begin{pmatrix} x + 1 & 3 \\ 3x - 1 & x + 3 \end{pmatrix}$

6 Find n such that $\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix}$ is the inverse of $\begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix}$.

7 Consider the two matrices $A = \begin{pmatrix} 4 & 2 \\ 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

a) Find X such that $XA = B$.

b) Find Y such that $AY = B$.

c) Is $X = Y$? Explain.

8 Consider the two matrices

$P = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 5 & 4 \\ 1 & 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -1 & 1 \\ 4 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix}$.

a) Find PQ and QP.

b) Find P^{-1}, Q^{-1}, $P^{-1}Q^{-1}$, $Q^{-1}P^{-1}$ $(PQ)^{-1}$, and $(QP)^{-1}$.

c) Write a few sentences about your observations in parts a) and b).

9 Consider the matrices A and B.

$A = \begin{pmatrix} 3 & -2 & 1 \\ -4 & 1 & -3 \\ 1 & -5 & 1 \end{pmatrix}$; $B = \begin{pmatrix} -29 \\ 37 \\ -24 \end{pmatrix}$

a) Find the matrix C if $AC = B$.

b) Solve the system of equations:

$$\begin{cases} 3x - 2y + z = -29 \\ 4x - y + 3z = -37 \\ -x + 5y - z = 24 \end{cases}$$

10 Solve the matrix equation

$$\begin{pmatrix} 2 & 2 + x \\ 5 & 4 + x \end{pmatrix}\begin{pmatrix} 3 & x \\ x - 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & x \\ x - 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 + x \\ 5 & 4 + x \end{pmatrix}$$

11 Consider the matrices A and B below. Find x and y such that $AB = BA$.

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix};\ B = \begin{pmatrix} 2-x & 1 \\ 5x & y \end{pmatrix}$$

12 Consider the matrices A and B below. Find x and y such that $AB = BA$.

$$A = \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix};\ B = \begin{pmatrix} 1-x & x \\ 5x & y \end{pmatrix}$$

13 Consider the matrices A and B below. Find x and y such that $AB = BA$.

$$A = \begin{pmatrix} 3+x & 1 \\ -5 & 2 \end{pmatrix};\ B = \begin{pmatrix} y-x & x \\ 5x-y+1 & y+x \end{pmatrix}$$

14 In each case, you are given two points in the plane. Use matrix methods to find an equation of a line that contains the given points.

a) $A(-5, -6)$, $B(3, 11)$

b) $A(5, -2)$, $B(3, -2)$

c) $A(-5, 3)$, $B(-5, 8)$

15 Find the area of the parallelogram with the given points as three of its vertices:

a) $A(-5, -6)$, $B(3, 11)$, $C(8, 1)$

b) $A(3, -5)$, $B(3, 11)$, $C(8, 11)$

c) $A(4, -6)$, $B(-3, 9)$, $C(7, 7)$

16 Find x such that the area of triangle ABC is 10 square units.

a) $A(x, -6)$, $B(3, 11)$, $C(8, 3)$

b) $A(-5, x)$, $B(3, x+2)$, $C(x^2+2x-3, 1)$

17 Find the value of k such that the points P, Q, and R are collinear.

a) $P(2, -5)$, $Q(4, k)$, $R(5, -2)$

b) $P(-6, 2)$, $Q(-5, k)$, $R(-3, 5)$

18 Exploration:

Consider the matrix $A = \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix}$. Define $f(x) = \det(xI - A)$ where x is any real number and I is the identity matrix.

a) Find $\det(A)$.

b) Expand $f(x)$ and compare the constant term to your answer in a).

c) How is the coefficient of x in the expansion of $f(x)$ related to A?

d) Find $f(A)$ and simplify it.

e) Now repeat parts a)–d) with matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

● **Hint:** $f(x)$ is called the characteristic polynomial of A.

19 Exploration:

Consider the matrix $A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 3 & 2 \\ 5 & 5 & -4 \end{pmatrix}$. Define $f(x) = \det(xI - A)$ where x is any real number and I is the identity matrix.

a) Find $\det(A)$.

b) Expand $f(x)$ and compare the constant term to your answer in a).

c) How is the coefficient of x^2 in the expansion of $f(x)$ related to A?

d) Find $f(A)$ and simplify it.

e) Now repeat parts a)–d) with matrix $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

6.4 Further properties and applications

Pages 267–269 are optional material. You can choose not to work on them. However, starting with Gauss-Jordan elimination (on page 269) the material is required in examinations.

In question 8 of Exercise 6.3, you were asked to make some observations concerning the answers to parts a) and b). The purpose is for you to discover some properties of inverse matrices.

Let us take the following matrices, for example:

Consider the two matrices A and B, where $A = \begin{pmatrix} -1 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -2 & -1 \end{pmatrix}$,

$B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{pmatrix}$.

Find A^{-1}, B^{-1}, AB, BA, $(AB)^{-1}$, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$, and $(BA)^{-1}$.

As shown below,

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{12} & -\frac{7}{12} \\ \frac{2}{3} & \frac{1}{12} & \frac{5}{12} \end{pmatrix}, B^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix}$$

```
 [[0.0  .3  .3 ]
  [-.3  .1  -.6]
  [.7   .1  .4 ]]
Ans▸Frac
[[0.0   1/4   1/4...
 [-1/3  1/12  -7/...
 [2/3   1/12  5/1...
```

```
[[1.0   -2.0 1.0...
 [-1.0  1.0  0.0...
 [.7    0.0  -.3...
Ans▸Frac
[[1.0   -2.0 1.0...
 [-1.0  1.0  0.0...
 [2/3   0.0  -1/...
```

Also,

$$AB = \begin{pmatrix} 4 & 9 & 6 \\ 7 & 16 & 18 \\ -3 & -8 & -6 \end{pmatrix}, BA = \begin{pmatrix} 8 & -1 & 1 \\ 11 & 1 & 2 \\ 13 & 4 & 5 \end{pmatrix}$$

```
[[4.0   9.0  6.0...
 [7.0   16.0 18....
 [-3.0  -8.0 -6....
[B][A]
[[8.0   -1.0 1.0...
 [11.0  1.0  2.0...
 [13.0  4.0  5.0...
```

```
([A][B])-1
 [[1.3  .2   1.8]
  [-.3  -.2  -.8]
  [-.2  .1   .0 ]]
```

```
Ans▸Frac
[[4/3   1/6   11/...
 [-1/3  -1/6  -5/...
 [-2/9  5/36  1/3...
```

$$(AB)^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{1}{6} & \frac{11}{6} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{5}{6} \\ -\frac{2}{9} & \frac{5}{36} & \frac{1}{36} \end{pmatrix}, \text{ also}$$

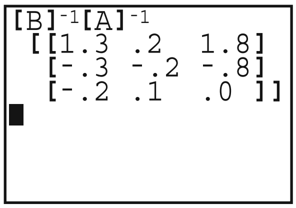
```
[A]-1[B]-1
[[-.1  .3    -.1]
 [-.8  .7    -.1]
 [.9   -1.3  .5 ]]
```

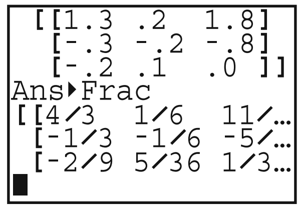
```
[[-.1  .3    -.1]
 [-.8  .7    -.1]
 [.9   -1.3  .5 ]]
Ans▸Frac
[[-1/12   1/4   -...
 [-29/36  3/4   -...
 [31/36   -5/4  1...
```

$$A^{-1}B^{-1} = \begin{pmatrix} -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{29}{36} & \frac{3}{4} & -\frac{5}{36} \\ \frac{31}{36} & -\frac{5}{4} & \frac{19}{36} \end{pmatrix}.$$

This last result shows that $(AB)^{-1} \neq A^{-1}B^{-1}$. However, as you notice below $(AB)^{-1} = B^{-1}A^{-1}$:

$$B^{-1}A^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{1}{6} & \frac{11}{6} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{5}{6} \\ -\frac{2}{9} & \frac{5}{36} & \frac{1}{36} \end{pmatrix}.$$





Finally, we also have

$$(BA)^{-1} = \begin{pmatrix} -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{29}{36} & \frac{3}{4} & -\frac{5}{36} \\ \frac{31}{36} & -\frac{5}{4} & \frac{19}{36} \end{pmatrix}.$$

This in turn is nothing but $A^{-1}B^{-1}$.

So, in general we have the following result:

If A and B are non-singular matrices of order n, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

The proof of this theorem is straightforward:

To show that $B^{-1}A^{-1}$ is the inverse of AB, we need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})\,(AB) = I.$$

Now, $(AB)(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$

Similarly, $(B^{-1}A^{-1})\,(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I.$

Hence, AB is non-singular (invertible) and its inverse is $B^{-1}A^{-1}$.

The following properties will be listed without proof:

$(A^{-1})^{-1} = A$

$(cA)^{-1} = \frac{1}{c}A^{-1}; c \neq 0$

$\det(AB) = \det A \cdot \det B$

This last result is helpful in proving the following property.

If A is non-singular, then $\det A^{-1} = \frac{1}{\det A}$.

Proof: Since $AA^{-1} = I$, then

$$\det(AA^{-1}) = \det I \Rightarrow \det A \cdot \det A^{-1} = 1 \Rightarrow \det A^{-1} = \frac{1}{\det A}.$$

In the previous section, we solved a system of equations using inverse matrices. However, that method works as long as the system is consistent with a unique solution. In many cases, the solution either has an infinite number of solutions or is inconsistent. There is another method of solution which we want to introduce you to.

Some terminology

As we have seen before, it is usual to represent a system of equations using matrix notation. In the previous section you learned how to solve a system of equations by writing the system in matrix form. For example, to solve the system

$$\begin{cases} 2x + 3y - 4z = 8 \\ \quad\;\; 2y + 4z = -3 \\ x \qquad\; - 2z = 4 \end{cases}$$

we wrote

$$\begin{pmatrix} 2 & 3 & -4 \\ 0 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 4 \end{pmatrix}$$

The first matrix is called the **coefficient matrix** (or **matrix of coefficients**) and the matrix on the right is called the **constants matrix** or the **answers matrix**. If the system has a unique solution then it can be solved. As you see, the method is limited and it has a strict constraint. Thanks to a slightly different arrangement, we can use matrices to arrive at our solution regardless of whether it is unique, has an infinite number of solutions, or simply no solution. To that end we need to write the system as follows:

$$\left(\begin{array}{ccc|c} 2 & 3 & -4 & 8 \\ 0 & 2 & 4 & -3 \\ 1 & 0 & -2 & 4 \end{array}\right)$$

This is called the **augmented** matrix of the system. It is customary to put a bar between the coefficients and the answers. However, this bar is not necessary and we will not be using it in this book. Just remember that the last column is the answers column!

Gauss-Jordan elimination

The idea behind this method is very simple. We successively apply certain simple operations to the system of equations reducing them into a special form that is easy to solve. The operations are called **elementary row**

operations and they can be applied to the system without changing the solution to the system. That is, the solution to the reduced system (**reduced row echelon form**) is the same as that for the original system. We can apply the operations either to the system itself or to its augmented matrix. Since the latter is easier to work with, we recommend that you first write the augmented matrix, reduce it, and then write the equivalent system to read the solution from.

There are three types of **elementary row operations**.

1. **Multiply any row by non-zero real number**.
2. **Interchange any two rows**.
3. **Add a multiple of one row to another row**.

Note: The order with which we apply the operations is not unique!

We will demonstrate the method with an example.

Consider the following system and its associated matrix:

$$\begin{cases} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ 2x + 4y + 6z = 6 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 4 & 6 & 6 \end{array}\right)$$

Switch row 1 and row 2 – type 2 operation:

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + y - z = 2 \\ 2x + 4y + 6z = 6 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 2 & 4 & 6 & 6 \end{array}\right)$$

Multiply row 3 by $\frac{1}{2}$ – type 1 operation:

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + y - z = 2 \\ x + 2y + 3z = 3 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & 3 & 3 \end{array}\right)$$

Multiply row 1 by -2 and add it to row 2, and multiply row 1 by -1 and add it to row 3 – type 3 operations:

$$\begin{cases} x + 3y + 2z = 1 \\ \quad - 5y - 5z = 0 \\ \quad - y + z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -5 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Notice here that row 1 did not change and rows 2 and three were replaced with the result of the elementary operation.

Multiply row 2 by $-\frac{1}{5}$:

$$\begin{cases} x + 3y + 2z = 1 \\ \quad y + z = 0 \\ \quad - y + z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Now, add row 2 to row 3, and multiply row 2 by -3 and add it to row 1:

$$\begin{cases} x \quad - z = 1 \\ \quad y + z = 0 \\ \quad 2z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

Now multiply row 3 by $\frac{1}{2}$:

$$\begin{cases} x \quad\quad - z = 1 \\ \quad\ y + z = 0 \\ \quad\quad\quad z = 1 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Lastly, add row 3 to row 1, and multiply row 3 by -1 and add it to row 2:

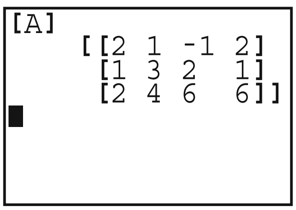
$$\begin{cases} x \quad\quad\quad = 2 \\ \quad y \quad\quad = -1 \\ \quad\quad\quad z = 1 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

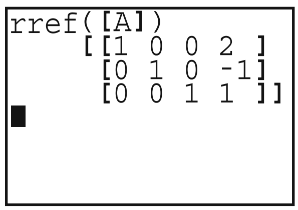
As you notice, from this last system it is easy to read the solution of $(2, -1, 1)$. You can verify that this solution is also the solution to the original system.

The simplified matrix is in its reduced row echelon form (to be defined later).

Of course, when we do the work, we do not have to show the processes in parallel. We just perform the operation on the matrix and then translate it into the equation form.

Note: This whole operation can easily be performed using a GDC.





Example 11

Solve the following system:

$$\begin{cases} x + y + 2z = 1 \\ x + \quad\ \ z = 2 \\ \quad\ y + \ z = 0 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{cases} x + y + 2z = 1 \\ x + y \quad\ z = 2 \\ \quad\ y + \ z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Multiply row 1 with -1 and add to row 2:

$$\begin{cases} x + y + 2z = 1 \\ \quad -y - z = 1 \\ \quad\ \ y + z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Add row 2 to row 1 and row 2 to row 3:

$$\begin{cases} x \quad\quad + z = 2 \\ \quad -y - z = 1 \\ \quad\quad\quad 0 = 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

At this stage, work can stop because if you write the last row as an equation, it reads

$$0x + 0y + 0z = 1.$$

This statement cannot be true for any value, and hence the system is inconsistent.

```
[B]
      [[1  1  2  1]
       [1  0  1  2]
       [0  1  1  0]]
```

```
rref([A])
      [[1  0  1  0]
       [0  1  0  0]
       [0  0  0  1]]
```

Example 12

Solve the following system:

$$\begin{cases} 2x + y - z = 4 \\ x + 3y + 7z = 7 \\ 2x + 4y + 8z = 10 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{cases} 2x + y - z = 4 \\ x + 3y + 7z = 7 \\ 2x + 4y + 8z = 10 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & 3 & 7 & 7 \\ 2 & 4 & 8 & 10 \end{pmatrix}$$

$$\begin{cases} x + 3y + 7z = 7 \\ 2x + y - z = 4 \\ 2x + 4y + 8z = 10 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 2 & 1 & -1 & 4 \\ 2 & 4 & 8 & 10 \end{pmatrix} \quad R_1 \Leftrightarrow R_2$$

$$\begin{cases} x + 3y + 7z = 7 \\ -5y - 15z = -10 \\ 3y + 9z = 6 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 0 & -5 & -15 & -10 \\ 0 & 3 & 9 & 6 \end{pmatrix} \quad \begin{cases} -R_2 + R_3 \\ -2R_1 + R_2 \end{cases}$$

$$\begin{cases} x + 3y + 7z = 7 \\ y + 3z = 2 \\ y + 3z = 2 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix} \quad \begin{cases} -\frac{1}{5}R_2 \\ \frac{1}{3}R_3 \end{cases}$$

$$\begin{cases} x - 2z = 1 \\ y + 3z = 2 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} -R_2 + R_3 \\ -3R_2 + R_1 \end{cases}$$

Since the last row is all zeros, there is not much that we can do. The conclusion is that this last row is true for any choice of values for the variables. Now we are left with a system of two equations and three variables.

$$\begin{cases} x - 2z = 4 \\ y + 3z = 2 \end{cases}$$

We need to solve for two of the variables in terms of the third. A wise choice here would be to solve for x and y in terms of z. That is,

$$x = 1 + 2z, y = 2 - 3z.$$

This means that for every choice of a value for z, we have a corresponding solution for the system. For example, if $z = 0$, then the solution would be $(1, 2, 0)$, for $z = 2$, the solution is $(5, -4, 2)$, and so on. This means that we have an infinite number of solutions. It is customary to present the solution in terms of a parameter, t for example. We let $z = t$, and our general solution would then be

$$(1 + 2t, 2 - 3t, t).$$

So, what is a **reduced row echelon form** (rref)?

We are confident that by now, you have a feel for what it is:

A matrix is in rref if it satisfies the following properties:

1. If there are any rows consisting entirely of zeros, they appear at the bottom of the matrix.
2. In any non-zero row, the first non-zero entry is 1. This entry is called the **pivot** of the row.
3. For any consecutive rows, the pivot of the lower row must be to the right of the pivot of the preceding row.
4. Any column that contains a pivot, has zeros everywhere else.

See the demonstration below; A is in rref while B is not.

$$A = \begin{pmatrix} \boxed{1} & 0 & 3 & 0 & 5 & 8 \\ 0 \rightarrow & \boxed{1} & 4 & 0 & 4 & 2 \\ 0 & 0 \rightarrow & 0 \rightarrow & \boxed{1} & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 6 & 7 \\ 0 & 0 & 1 \leftarrow & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Curve fitting

Another application of matrices (systems) is to help fit specific models to sets of points.

Example 13

Fit a quadratic model to pass through the points $(-1, 10)$, $(2, 4)$, and $(3, 14)$.

Solution

The problem is to find parameters a, b, and c that will force the curve representing the function $f(x) = ax^2 + bx + c$ to contain the given points. This means

$$f(-1) = 10, f(2) = 4, \text{ and } f(3) = 14.$$

Since we need to find the three unknown parameters, we need three equations which are offered by the conditions above:

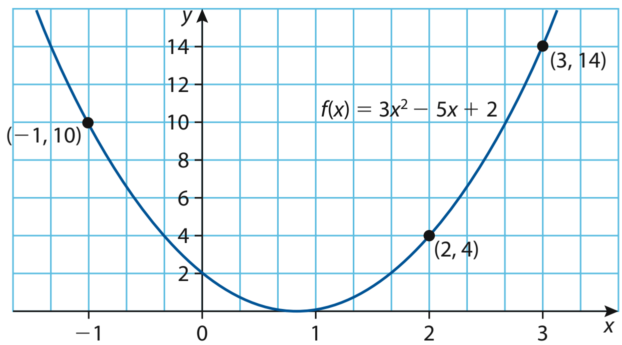
$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
f(-1) &= a - b + c = 10 \\
f(2) &= 4a + 2b + c = 4 \\
f(3) &= 9a + 3b + c = 14
\end{aligned}
$$

This is clearly a system of three equations which can be solved using matrix methods, among other methods of course.

Using *rref*, we get the following result:

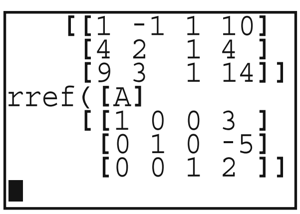
$$
\left(\begin{array}{rrr|r} 1 & -1 & 1 & 10 \\ 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 14 \end{array}\right) \Leftrightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right)
$$

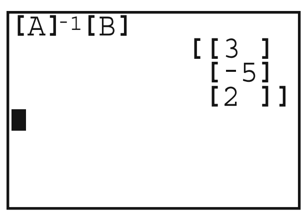
Which means that $a = 3$, $b = -5$, and $c = 2$; so the function is $f(x) = 3x^2 - 5x + 2$.



Equivalently, we can use the inverse matrix directly:

$$
\begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}
$$





Exercise 6.4

1 Given the matrix $A = \begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$ find the value of the real number m such that $\det(A - mI) = 0$, where I is the 2×2 multiplication identity matrix.

2 a) Find the values of a and b, given that the matrix $A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$.

b) For the values of a and b found in part a), solve the system of linear equations:

$x + 2y - 2z = 5$

$3x + by + z = 0$

$-x + y - 3z = a - 1$

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3 Find the value(s) of m so that the matrix $\begin{pmatrix} 1 & m & 1 \\ 3 & 1-m & 2 \\ m & -3 & m-1 \end{pmatrix}$ is singular.

4 Solve each system of equations. If a solution does not exist, justify why not.

a) $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

b) $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

c) $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$

d) $\begin{cases} 3x - 2y + z = -29 \\ -4x + y - 3z = 37 \\ x - 5y + z = -24 \end{cases}$

e) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

f) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 1 \end{cases}$

g) $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

h) $\begin{cases} x - 3y - 2z = 8 \\ -2x + 7y + 3z = -19 \\ x - y - 3z = 3 \end{cases}$

5 a) Find the values of k such that the following matrix is not singular

$$A = \begin{pmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix}.$$

b) Find the value(s) of k such that A is the inverse of B, where

$$B = \begin{pmatrix} k-3 & -3 & k \\ 3 & k+2 & -1 \\ -2 & -4 & 1 \end{pmatrix}.$$

c) For the value of k found in b), apply elementary row operations to reduce the

matrix $\begin{pmatrix} 1 & 1 & k-1 & 1 & 0 & 0 \\ k & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & -3 & 0 & 0 & 1 \end{pmatrix}$ into $\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$ where

$a, \dots, i$ are to be determined.

6 a) Find the values of k such that the following matrix is not singular.

$$A = \begin{pmatrix} \frac{2}{5} & -\frac{17}{5} & \frac{k+9}{5} \\ -\frac{1}{5} & \frac{21}{5} & -\frac{13}{5} \\ k-2 & 3 & -2 \end{pmatrix}$$

b) Find the value(s) of k such that A is the inverse of B, where

$$B = \begin{pmatrix} k+1 & 1 & k \\ 2 & k+2 & -3 \\ 3 & 6 & -5 \end{pmatrix}.$$

c) For the value of k found in b), apply elementary row operations to reduce the matrix $\begin{pmatrix} 2 & -17 & k+9 & 1 & 0 & 0 \\ -1 & 21 & -13 & 0 & 1 & 0 \\ 5(k-2) & 15 & -10 & 0 & 0 & 1 \end{pmatrix}$ into $\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$ where $a, \ldots, i$ are to be determined.

7 Use elementary row operations to transform the matrix $[A \vdots I]$ to a matrix in the form $[I \vdots B]$. Comment on the relationship between A and B and support your conclusion.

a) $\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$ b) $\left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 8 & 6 & 0 & 0 & 1 \end{array}\right)$

8 Determine the function f so that the curve representing it contains the indicated points.

a) $f(x) = ax^2 + bx + c$ to contain $(-1, 5)$, $(2, -1)$, and $(4, 35)$.

b) $f(x) = ax^2 + bx + c$ to contain $(-1, 12)$ and $(2, -3)$.

● **Hint:** there is more than one curve!

c) $f(x) = ax^3 + bx^2 + cx + d$ to contain the points $(-1, 5)$, $(1, -3)$, $(2, 5)$, and $(3, 45)$. [optional material]

d) $f(x) = ax^3 + bx^2 + cx + d$ to contain the points $(-3, 4)$, $(-1, 4)$, and $(2, 4)$.

9 Consider the following system of equations:

$$\begin{cases} 2x + y + 3z = -5 \\ 3x - y + 4z = 2 \\ 5x + 7z = m - 5 \end{cases}$$

Find the value(s) of m for which this system is consistent. For the value of m found, find the most general solution of the system.

10 Consider the following system of equations:

$$\begin{cases} -3x + 2y + 3z = 1 \\ 4x - y - 5z = -5 \\ x + y - 2z = m - 3 \end{cases}$$

Find the value(s) of m for which this system is consistent. For the value of m found, find the most general solution of the system.

11 Consider the matrix $A = \begin{pmatrix} 3 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$.

a) Find $\det(A)$.

b) Use the third elementary row operation to transform the matrix A into matrix B in triangular form (i.e. **add a multiple of one row to another row**).

c) Find $\det(B)$.

d) Use a GDC to find $\det(C)$ for $C = \begin{pmatrix} 2 & 1 & -3 & 5 \\ 4 & 3 & -4 & -6 \\ 6 & -8 & 5 & 7 \\ -6 & -5 & 3 & 4 \end{pmatrix}$.

e) Repeat b) and c) for C.

Practice questions

1 If $\begin{pmatrix} 2x & 3 \\ -4x & x \end{pmatrix}$ and det $A = 14$, find x.

2 Let $M = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}$, where $a \in \mathbb{Z}$.

a) Find M^2 in terms of a.

b) If M^2 is equal to $\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$, find the value of a.

Using this value of a, find M^{-1} and hence solve the system of equations:

$$\begin{aligned} -x + 2y &= -3 \\ 2x - y &= 3 \end{aligned}$$

3 Two matrices are given, where $A = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$. Find B.

4 The matrices A, B, and X are given, where

$A = \begin{pmatrix} 3 & 1 \\ -5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$, $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$.

Find the values of a, b, c and d such that $AX + X = B$.

5 $A = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$ is a 2×2 matrix.

a) Write out A^{-1}.

b) **(i)** If $XA + B = C$, where B, C, and X are 2×2 matrices, express X in terms of A^{-1}, B, and C.

(ii) Find X if $B = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$.

6 Given $A = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix}$,

a) write out $A + B$;

b) find AB.

7 a) Write out the inverse of the matrix $\begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$.

b) Hence, solve the system of simultaneous equations:

$$\begin{aligned} x - 3y + z &= 1 \\ 2x + 2y - z &= 2 \\ x - 5y + 3z &= 3 \end{aligned}$$

8 Given the two matrices C and D, where

$C = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$,

the matrix Q is given such that $3Q = 2C - D$.

b) Find Q.

b) Find CD.

c) Find D^{-1}.

9 a) Find the values of a and b given that the matrix $A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$.

b) For the values of a and b found in part a), solve the system of linear equations:

$$x + 2y - 2z = 5$$
$$3x + by + z = 0$$
$$-x + y - 3z = a - 1$$

10 a) Given matrices A, B, C for which $AB = C$ and $\det A \neq 0$, express B in terms of A and C.

b) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$.

(i) Find the matrix DA.

(ii) Find B if $AB = C$.

c) Find the coordinates of the point of intersection of the planes $x + 2y + 3z = 5$, $2x - y + 2z = 7$ and $3x - 3y + 2z = 10$. (This can be answered after Chapter 14.)

11 a) Find the determinant of the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}$.

b) Find the value of λ for which the following system of equations can be solved.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ \lambda \end{pmatrix}$$

c) For this value of λ, find the general solution to the system of equations.

12 The square matrix X is such that $X^3 = 0$. Show that the inverse of the matrix $(I - X)$ is $I + X + X^2$.

7 Trigonometric Functions and Equations

Assessment statements

2.1 Odd and even functions (also see Chapter 3).

3.1 The circle: radian measure of angles; length of an arc; area of a sector.

3.2 The circular functions $\sin x$, $\cos x$ and $\tan x$: their domains and ranges; their periodic nature; and their graphs.

Definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.
Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.
Exact values of sin, cos and tan of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ and their multiples.
Definition of the reciprocal trigonometric ratios $\sec\theta$, $\csc\theta$ and $\cot\theta$.
Pythagorean identities: $\cos^2\theta + \sin^2\theta = 1$; $1 + \tan^2\theta = \sec^2\theta$; $1 + \cot^2\theta = \csc^2\theta$.

3.3 Compound angle identities.
Double angle identities.

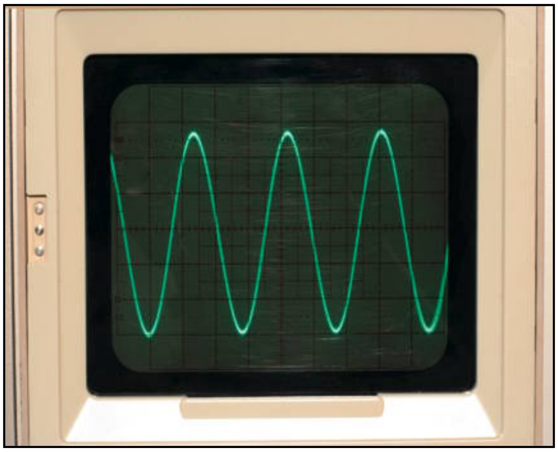
3.4 Composite functions of the form $f(x) = a\sin(b(x + c)) + d$.

3.5 The inverse functions $x \mapsto \arcsin x$, $x \mapsto \arccos x$, $x \mapsto \arctan x$; their domains and ranges; their graphs.

3.6 Algebraic and graphical methods of solving trigonometric equations in a finite interval including the use of trigonometric identities and factorization.

Introduction

The word *trigonometry* comes from two Greek words, *trigonon* and *metron*, meaning 'triangle measurement'. Trigonometry developed out of the use and study of triangles, in surveying, navigation, architecture and astronomy, to find relationships between lengths of sides of triangles and measurement of angles. As a result, trigonometric functions were initially defined as functions of angles – that is, functions with angle measurements as their domains. With the development of calculus in the seventeenth century and the growth of knowledge in the sciences, the application of trigonometric functions grew to include a wide variety of periodic (repetitive) phenomena such as wave motion, vibrating strings, oscillating pendulums, alternating electrical current and biological cycles. These applications of trigonometric functions require their domains to be sets of real numbers without reference to angles or triangles. Hence, trigonometry can be approached from two different perspectives – **functions**



The oscilloscope shows the graph of pressure of sound wave versus time for a high-pitched sound. The graph is a repetitive pattern that can be expressed as the sum of different 'sine' waves. A sine wave is any transformation of the graph of the trigonometric function $y = \sin x$ and takes the form $y = a\sin[b(x + c)] + d$.

of angles, or **functions of real numbers**. The first perspective is the focus of the next chapter where trigonometric functions will be defined in terms of the **ratios of sides of a right triangle**. The second perspective is the focus of this chapter, where trigonometric functions will be defined in terms of a real number that is the **length of an arc along the unit circle**. While it is possible to define trigonometric functions in these two different ways, they assign the same value (interpreted as an angle, an arc length, or simply a real number) to a particular real number. Although this chapter will not refer much to triangles, it seems fitting to begin by looking at angles and arc lengths – geometric objects indispensable to the two different ways of viewing trigonometry.

7.1 Angles, circles, arcs and sectors

Angles

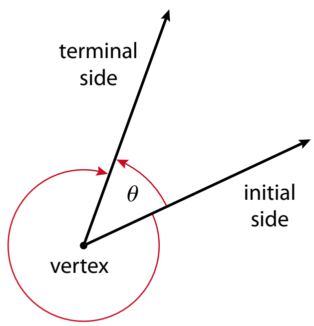


Figure 7.1 Components of an angle.

An **angle** in a plane is made by rotating a ray about its endpoint, called the **vertex** of the angle. The starting position of the ray is called the **initial side** and the position of the ray after rotation is called the **terminal side** of the angle (Figure 7.1). An angle having its vertex at the origin and its initial side lying on the positive x-axis is said to be in **standard position** (Figure 7.2a). A **positive angle** is produced when a ray is rotated in an anticlockwise direction, and a **negative angle** when a ray is rotated in a clockwise direction. Two angles in standard position whose terminal sides are in the same location – regardless of the direction or number of rotations – are called **coterminal angles**. Greek letters are often used to represent angles, and the direction of rotation is indicated by an arc with an arrow at its endpoint. The x- and y-axes divide the coordinate plane into four quadrants (numbered with Roman numerals). Figure 7.2b shows a positive angle α (alpha) and a negative angle β (beta) that are coterminal in quadrant III.

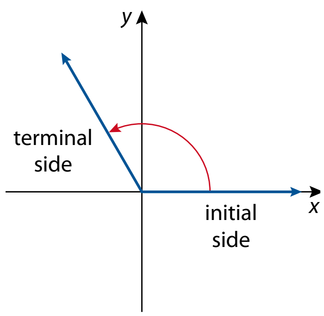


Figure 7.2a Standard position of an angle.

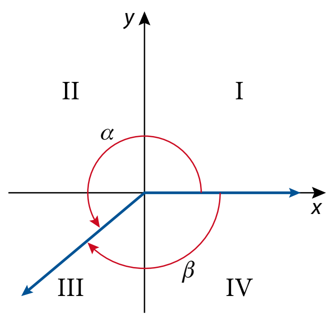


Figure 7.2b Coterminal angles.

Measuring angles: degree measure and radian measure

Perhaps the most natural unit for measuring large angles is the **revolution**. For example, most cars have an instrument (a tachometer) that indicates the number of revolutions per minute (rpm) at which the engine is operating. However, to measure smaller angles, we need a smaller unit. A common unit for measuring angles is the **degree**, of which there are 360 in one revolution. Hence, the unit of one degree (1°) is defined to be 1/360 of one anticlockwise revolution about the vertex.

The convention of having 360 degrees in one revolution can be traced back around 4000 years to ancient Babylonian civilizations. The number system most widely used today is a base 10, or **decimal**, system. Babylonian mathematics used a base 60, or **sexagesimal**, number system. Although 60 may seem to be an awkward number to have as a base, it does have certain advantages. It is the smallest number that has 2, 3, 4, 5 and 6 as factors – and it also has factors of 10, 12, 15, 20 and 30. But why 360 degrees? We're not certain but it may have to do with the Babylonians assigning 60 divisions to each angle in an equilateral triangle and exactly six equilateral triangles can be arranged around a single point. That makes $6 \times 60 = 360$ equal divisions in one full revolution. There are few numbers as small as 360 that have so many different factors. This makes the degree a useful unit for dividing one revolution into an equal number of parts. 120 degrees is $\frac{1}{3}$ of a revolution, 90 degrees is $\frac{1}{4}$ of a revolution, 60 degrees is $\frac{1}{6}$, 45 degrees is $\frac{1}{8}$, and so on.

There is another method of measuring angles that is more natural. Instead of dividing a full revolution into an arbitrary number of equal divisions (e.g. 360), consider an angle that has its vertex at the centre of a circle (a **central angle**) and subtends (or intercepts) a part of the circle, called an **arc of the circle**. Figure 7.3 shows three circles with radii of different lengths ($r_1 < r_2 < r_3$) and the same central angle θ (theta) subtending (intercepting) the arc lengths s_1, s_2 and s_3. Regardless of the size of the circle (i.e. length of the radius), the ratio of arc length (s) to radius (r) for a given circle will be constant. For the angle θ in Figure 7.3, $\frac{s_1}{r_1} = \frac{s_2}{r_2} = \frac{s_3}{r_3}$. Because this ratio is an arc length divided by another length (radius), it is just an ordinary real number and has no units.

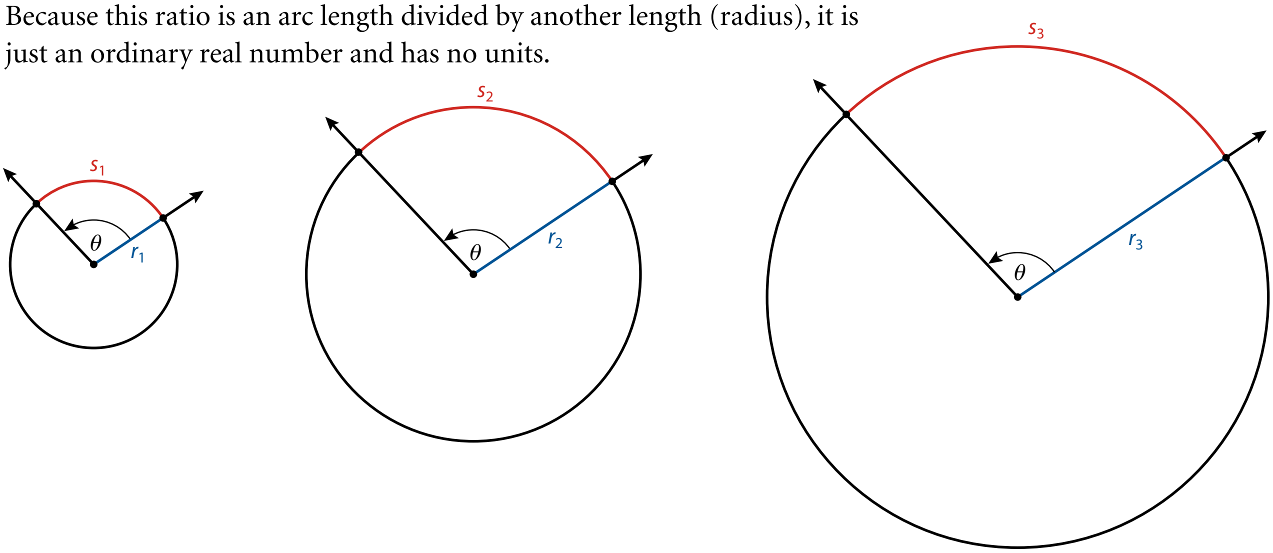


Figure 7.3 Different circles with the same central angle θ subtending different arcs, but the ratio of arc length to radius remains constant.

Minor and major arcs

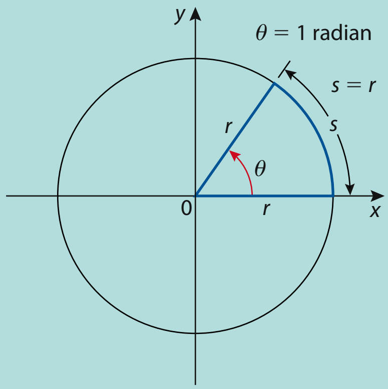
If a central angle is **less** than 180°, the subtended arc is referred to as a **minor arc**. If a central angle is **greater** than 180°, the subtended arc is referred to as a **major arc**.

The ratio $\frac{s}{r}$ indicates how many radius lengths, r, fit into the length of the arc s. For example, if $\frac{s}{r} = 2$, the length of s is equal to two radius lengths. This accounts for the name **radian** and leads to the following definition.

When the measure of an angle is, for example, 5 radians, the word 'radians' does not indicate units (as when writing centimetres, seconds or degrees) but indicates the *method* of angle measurement. If the measure of an angle is in units of degrees, we must indicate this by word or symbol. For example, $\theta = 5$ degrees or $\theta = 5°$. However, when radian measure is used it is customary to write no units or symbol. For example, a central angle θ that subtends an arc equal to five radius lengths (radians) is simply given as $\theta = 5$.

Radian measure

One **radian** is the measure of a central angle θ of a circle that subtends an arc s of the circle that is exactly the same length as the radius r of the circle. That is, when $\theta = 1$ radian, arc length = radius.



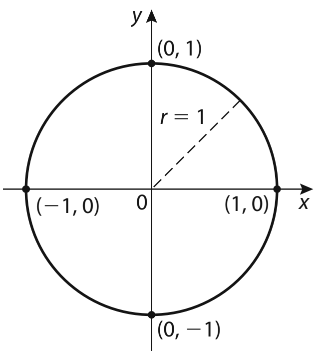


Figure 7.4 The unit circle.

The unit circle

When an angle is measured in radians it makes sense to draw it, or visualize it, so that it is in standard position. It follows that the angle will be a central angle of a circle whose centre is at the origin, as shown above. As Figure 7.3 illustrated, it makes no difference what size circle is used. The most practical circle to use is the circle with a radius of one unit so the radian measure of an angle will simply be equal to the length of the subtended arc.

Radian measure: $\theta = \frac{s}{r}$ If $r = 1$, then $\theta = \frac{s}{1} = s$.

The circle with a radius of one unit and centre at the origin $(0, 0)$ is called the **unit circle** (Figure 7.4). The equation for the unit circle is $x^2 + y^2 = 1$. Because the circumference of a circle with radius r is $2\pi r$, a central angle of one full anticlockwise revolution (360°) subtends an arc on the unit circle equal to 2π units. Hence, if an angle has a degree measure of 360°, its radian measure is exactly 2π. It follows that an angle of 180° has a radian measure of exactly π. This fact can be used to convert between degree measure and radian measure, and vice versa.

Conversion between degrees and radians

Because $180° = \pi$ radians, $1° = \frac{\pi}{180}$ radians, and 1 radian $= \frac{180°}{\pi}$. An angle with a radian measure of 1 has a degree measure of approximately 57.3° (to 3 significant figures).

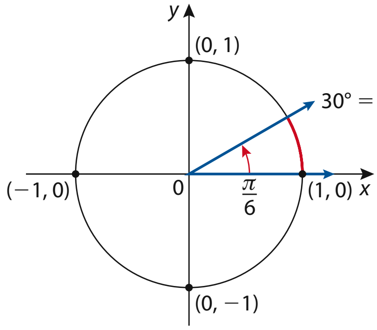
Example 1

The angles of 30° and 45°, and their multiples, are often encountered in trigonometry. Convert 30° and 45° to radian measure and sketch the corresponding arc on the unit circle. Use these results to convert 60° and 90° to radian measure.

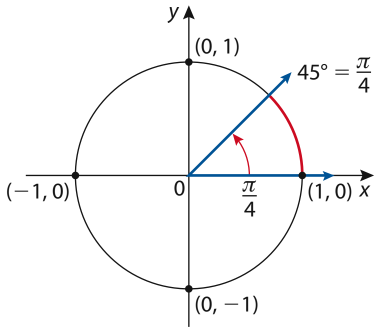
Solution

(Note that the 'degree' units cancel.)

$$30° = 30°\left(\frac{\pi}{180°}\right) = \frac{30°}{180°}\pi = \frac{\pi}{6}$$



$$45° = 45°\left(\frac{\pi}{180°}\right) = \frac{45°}{180°}\pi = \frac{\pi}{4}$$



Since $60° = 2(30°)$ and $30° = \frac{\pi}{6}$, then $60° = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$. Similarly, $90° = 2(45°)$ and $45° = \frac{\pi}{4}$, so $90° = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$.

● **Hint:** It is very helpful to be able to quickly recall the results from Example 1: $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$ and $90° = \frac{\pi}{2}$. Of course, not all angles are multiples of 30° or 45° when expressed in degrees, and not all angles are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ when expressed in radians. However, these 'special' angles often appear in problems and applications. Knowing these four facts can help you to quickly convert mentally between degrees and radians for many common angles. For example, to convert 225° to radians, apply the fact that $225° = 5(45°)$. Since $45° = \frac{\pi}{4}$, then $225° = 5(45°) = 5\left(\frac{\pi}{4}\right) = \frac{5\pi}{4}$.

As another example, convert $\frac{11\pi}{6}$ to degrees: $\frac{11\pi}{6} = 11\left(\frac{\pi}{6}\right) = 11(30°) = 330°$.

Example 2

a) Convert the following radian measures to degrees. Express exactly, if possible. Otherwise, express accurate to 3 significant figures.

(i) $\frac{4\pi}{3}$ (ii) $-\frac{3\pi}{2}$ (iii) 5 (iv) 1.38

b) Convert the following degree measures to radians. Express exactly, if possible. Otherwise, express accurate to 3 significant figures.

(i) 135° (ii) −150° (iii) 175° (iv) 10°

Solution

a) (i) $\frac{4\pi}{3} = 4\left(\frac{\pi}{3}\right) = 4(60°) = 240°$

(ii) $-\frac{3\pi}{2} = -\frac{3}{2}(\pi) = -\frac{3}{2}(180°) = -270°$

(iii) $5\left(\frac{180°}{\pi}\right) \approx 286.479° \approx 286°$

(iv) $1.38\left(\frac{180°}{\pi}\right) \approx 79.068° \approx 79.1°$

● **Hint:** All GDCs will have a degree mode and a radian mode. Before doing any calculations with angles on your GDC, be certain that the mode setting for angle measurement is set correctly. Although you may be more familiar with degree measure, as you progress further in mathematics – and especially in calculus – radian measure is far more useful.

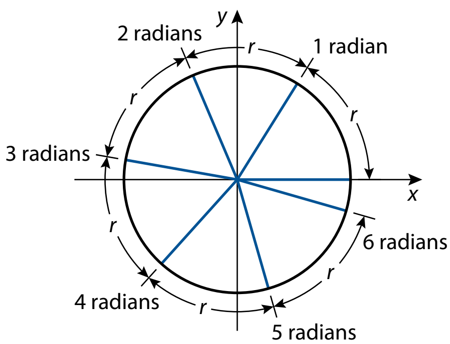
b) (i) $135° = 3(45°) = 3\left(\frac{\pi}{4}\right) = \frac{3\pi}{4}$

(ii) $-150° = -5(30°) = -5\left(\frac{\pi}{6}\right) = -\frac{5\pi}{6}$

(iii) $175°\left(\frac{\pi}{180°}\right) \approx 3.0543 \approx 3.05$

(iv) $10°\left(\frac{\pi}{180°}\right) \approx 0.174\,53 \approx 0.175$

Figure 7.5 Arcs with lengths equal to the radius placed along circumference of a circle.



Because 2π is approximately 6.28 (3 significant figures), there are a little more than six radius lengths in one revolution, as shown in Figure 7.5.

Figure 7.6 Degree measure and radian measure for common angles.

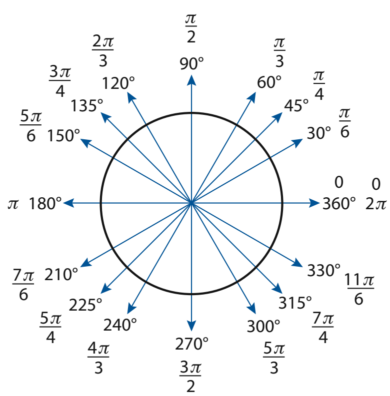


Figure 7.6 shows all of the angles between 0° and 360° inclusive, that are multiples of 30° or 45°, and their equivalent radian measure. You will benefit by being able to convert quickly between degree measure and radian measure for these common angles.

Arc length

For any angle θ, its radian measure is given by $\theta = \frac{s}{r}$. Simple rearrangement of this formula leads to another formula for computing arc length.

Arc length

For a circle of radius r, a central angle θ subtends an arc of the circle of length s given by

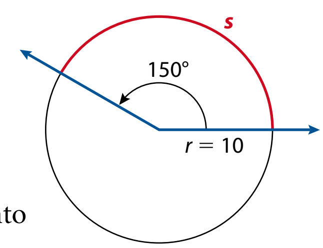
$$s = r\theta$$

where θ is in radian measure.

Example 3

A circle has a radius of 10 cm. Find the length of the arc of the circle subtended by a central angle of 150°.

Solution



To use the formula $s = r\theta$, we must first convert 150° to radian measure.

$$150° = 150°\left(\frac{\pi}{180°}\right) = \frac{150\pi}{180} = \frac{5\pi}{6}$$

Given that the radius, r, is 10 cm, substituting into the formula gives

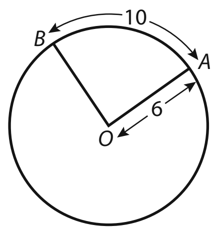
$$s = r\theta \Rightarrow s = 10\left(\frac{5\pi}{6}\right) = \frac{25\pi}{3} \approx 26.179\,94 \text{ cm}$$

The length of the arc is approximately 26.18 cm (4 significant figures).

Note that the units of the product $r\theta$ are the same as the units of r because in radian measure θ has no units.

Example 4

The diagram shows a circle of centre O with radius $r = 6$ cm. Angle AOB subtends the minor arc AB such that the length of the arc is 10 cm. Find the measure of angle AOB in degrees to 3 significant figures.



Solution

From the arc length formula, $s = r\theta$, we can state that $\theta = \frac{s}{r}$. Remember that the result for θ will be in radian measure. Therefore, angle $AOB = \frac{10}{6} = \frac{5}{3}$ or $1.\overline{6}$ radians. Now, we convert to degrees: $\frac{5}{3}\left(\frac{180°}{\pi}\right) \approx 95.49297°$. The degree measure of angle AOB is approximately 95.5°.

Geometry of a circle

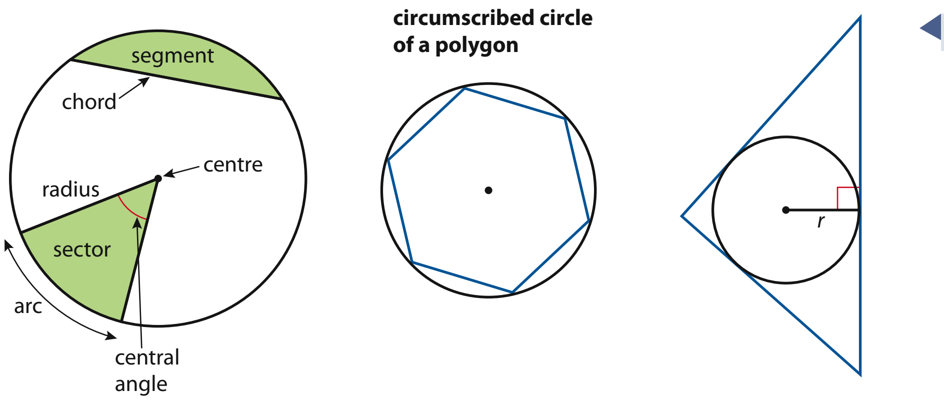


Figure 7.7 Circle terminology.

Sector of a circle

A **sector of a circle** is the region bounded by an arc of the circle and the two sides of a central angle (Figure 7.8). The ratio of the area of a sector to the area of the circle (πr^2) is equal to the ratio of the length of the subtended arc to the circumference of the circle ($2\pi r$). If s is the arc length and A is the area of the sector, we can write the following proportion: $\frac{A}{\pi r^2} = \frac{s}{2\pi r}$. Solving for A gives $A = \frac{\pi r^2 s}{2\pi r} = \frac{1}{2}rs$. From the formula for arc length we have $s = r\theta$, with θ the radian measure of the central angle. Substituting $r\theta$ for s gives the area of a sector to be $A = \frac{1}{2}rs = \frac{1}{2}r(r\theta) = \frac{1}{2}r^2\theta$. This result makes sense because, if the sector is the entire circle, $\theta = 2\pi$ and area $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2(2\pi) = \pi r^2$, which is the formula for the area of a circle.

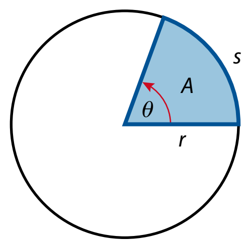


Figure 7.8 Sector of a circle.

Area of a sector

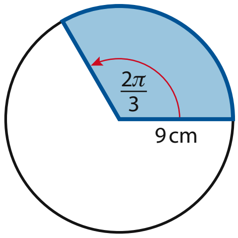
In a circle of radius r, the area of a sector with a central angle θ measured in radians is

$$A = \frac{1}{2}r^2\theta$$

Example 5

A circle of radius 9 cm has a sector whose central angle has radian measure $\frac{2\pi}{3}$. Find the exact values of the following: a) the length of the arc subtended by the central angle, and b) the area of the sector.

Solution



a) $s = r\theta \Rightarrow s = 9\left(\frac{2\pi}{3}\right) = 6\pi$

The length of the arc is exactly 6π cm.

b) $A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(9)^2\left(\frac{2\pi}{3}\right) = 27\pi$

The area of the sector is exactly $27\pi\,\text{cm}^2$.

• **Hint:** The formula for arc length, $s = r\theta$, and the formula for area of a sector, $A = \frac{1}{2}r^2\theta$, are true only when θ is in radians.

Exercise 7.1

In questions 1–9, find the exact radian measure of the angle given in degree measure.

1 60° **2** 150° **3** −270°

4 36° **5** 135° **6** 50°

7 −45° **8** 400° **9** −480°

In questions 10–18, find the degree measure of the angle given in radian measure. If possible, express exactly. Otherwise, express accurate to 3 significant figures.

10 $\frac{3\pi}{4}$ **11** $-\frac{7\pi}{2}$ **12** 2

13 $\frac{7\pi}{6}$ **14** −2.5 **15** $\frac{5\pi}{3}$

16 $\frac{\pi}{12}$ **17** 1.57 **18** $\frac{8\pi}{3}$

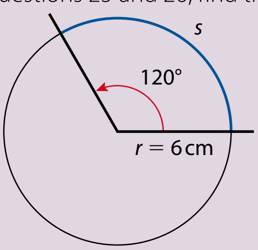
In questions 19–24, the measure of an angle in standard position is given. Find two angles – one positive and one negative – that are coterminal with the given angle. If no units are given, assume the angle is in radian measure.

19 30° **20** $\frac{3\pi}{2}$ **21** 175°

22 $-\frac{\pi}{6}$ **23** $\frac{5\pi}{3}$ **24** 3.25

In questions 25 and 26, find the length of the arc s in the figure.

25



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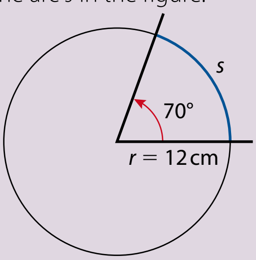
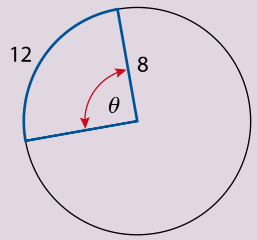
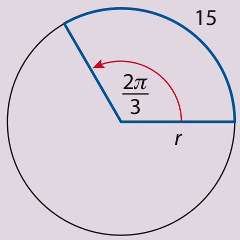


27 Find the angle θ in the figure in both radian measure and degree measure.

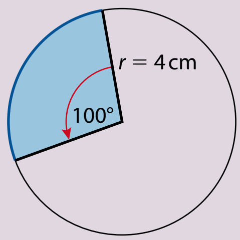


28 Find the radius r of the circle in the figure.



In questions 29 and 30, find the area of the sector in each figure.

29



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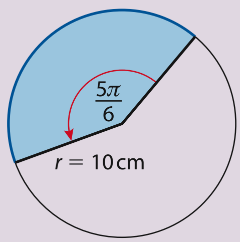


31 An arc of length 60 cm subtends a central angle α in a circle of radius 20 cm. Find the measure of α in both degrees and radians, approximate to 3 significant figures.

32 Find the length of an arc that subtends a central angle with radian measure of 2 in a circle of radius 16 cm.

33 The area of a sector of a circle with a central angle of 60° is 24 cm^2. Find the radius of the circle.

34 A bicycle with tyres 70 cm in diameter is travelling such that its tyres complete one and a half revolutions every second. That is, the **angular velocity** of a wheel is 1.5 revolutions per second.

a) What is the angular velocity of a wheel in radians per second?

b) At what speed (in km/hr) is the bicycle travelling along the ground? (This is the **linear velocity** of the bicycle.)

35 A bicycle with tyres 70 cm in diameter is travelling along a road at 25 km/hr. What is the angular velocity of a wheel of the bicycle in radians per second?

36 Given that ω is the angular velocity in radians/second of a point on a circle with radius r cm, express the linear velocity, v, in cm/second, of the point as a function in terms of ω and r.

37 A chord of 26 cm is in a circle of radius 20 cm. Find the length of the arc the chord subtends.

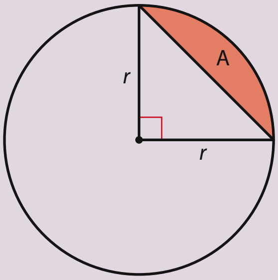
38 A circular irrigation system consists of a 400 metre pipe that is rotated around a central pivot point. If the irrigation pipe makes one full revolution around the pivot point in a day, then how much area, in square metres, does it irrigate each hour?

39 a) Find the radius of a circle circumscribed about a regular polygon of 64 sides if one side is 3 cm.

b) What is the difference between the circumference of the circle and the perimeter of the polygon?

40 What is the area of an equilateral triangle that has an inscribed circle with an area of 50π cm^2, and a circumscribed circle with an area of 200π cm^2?

41 In the diagram, the sector of a circle is subtended by two perpendicular radii. If the area of the sector is **A** square units, then find an expression for the area of the circle in terms of **A**.



7.2 The unit circle and trigonometric functions

Several important functions can be described by mapping the coordinates of points on the real number line onto the points of the unit circle. Recall from the previous section that the unit circle has its centre at $(0, 0)$, it has a radius of one unit and its equation is $x^2 + y^2 = 1$.

A wrapping function: the real number line and the unit circle

Suppose that the real number line is tangent to the unit circle at the point $(1, 0)$ – and that zero on the number line matches with $(1, 0)$ on the circle, as shown in Figure 7.9. Because of the properties of circles, the real number line in this position will be perpendicular to the x-axis. The scales on the number line and the x- and y-axes need to be the same. Imagine that the real number line is flexible like a string and can wrap around the circle, with zero on the number line remaining fixed to the point $(1, 0)$ on the unit circle. When the top portion of the string moves along the circle, the wrapping is anticlockwise ($t > 0$), and when the bottom portion of the string moves along the circle, the wrapping is clockwise ($t < 0$). As the string wraps around the unit circle, each real number t on the string is mapped onto a point (x, y) on the circle. Hence, the real number line from 0 to t makes an arc of length t starting on the circle at $(1, 0)$ and ending at the point (x, y) on the circle. For example, since the circumference of the unit circle is 2π, the number $t = 2\pi$ will be wrapped anticlockwise around the circle to the point $(1, 0)$. Similarly, the number $t = \pi$ will be wrapped anticlockwise halfway around the circle to the point $(-1, 0)$ on the circle. And the number $t = -\frac{\pi}{2}$ will be wrapped clockwise one-quarter of the way around the circle to the point $(0, -1)$ on the circle. Note that each number t on the real number line is mapped (corresponds) to *exactly one* point on the unit circle, thereby satisfying the definition of a function (Section 2.1) – consequently this mapping is called a **wrapping function**.

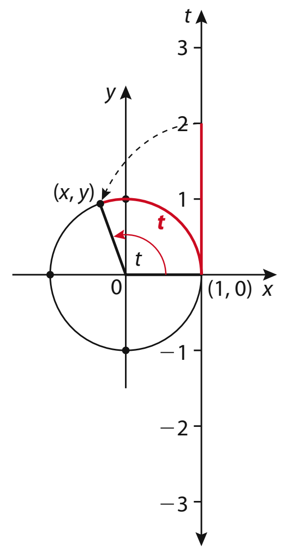


Figure 7.9 The wrapping function.

Before we leave our mental picture of the string (representing the real number line) wrapping around the unit circle, consider any pair of points on the string that are exactly 2π units from each other. Let these two points represent the real numbers t_1 and $t_1 + 2\pi$. Because the circumference of the unit circle is 2π, these two numbers will be mapped to the same point on the unit circle. Furthermore, consider the infinite number of points whose distance from t_1 is any integer multiple of 2π, i.e. $t_1 + k \cdot 2\pi$, $k \in \mathbb{Z}$, and again all of these numbers will be mapped to the same point on the unit circle. Consequently, the wrapping function is not a one-to-one function as defined in Section 2.3. Output for the function (points on the unit circle) are unchanged by the addition of any integer multiple of 2π to any input value (a real number). Functions that behave in such a repetitive (or cyclic) manner are called **periodic**.

We are surrounded by periodic functions. A few examples include: the average daily temperature variation during the year; sunrise and the day of the year; animal populations over many years; the height of tides and the position of the Moon; and an electrocardiogram, which is a graphic tracing of the heart's electrical activity.

Definition of a periodic function

A function f such that $f(x) = f(x + p)$ is a **periodic function**. If p is the least positive constant for which $f(x) = f(x + p)$ is true, p is called the **period** of the function.

Trigonometric functions

From our discussions about functions in Chapter 2, any function will have a domain (input) and range (output) that are sets having individual numbers as elements. We use the individual coordinates x and y of the points on the unit circle to define six **trigonometric functions**: the **sine**, **cosine**, **tangent**, **cosecant**, **secant** and **cotangent** functions. The names of these functions are often abbreviated in writing (but not speaking) as **sin**, **cos**, **tan**, **csc**, **sec**, **cot**, respectively.

When the real number t is wrapped to a point (x, y) on the unit circle, the value of the y-coordinate is assigned to the sine function; the x-coordinate is assigned to the cosine function; and the ratio of the two coordinates $\frac{y}{x}$ is assigned to the tangent function. Sine, cosine and tangent are often referred to as the **basic trigonometric functions**. The other three, cosecant, secant and cotangent, are each a reciprocal of one of the basic trigonometric functions and thus, are often referred to as the **reciprocal trigonometric functions.** All six are defined by means of the length of an arc on the unit circle as follows.

● **Hint:** To help you remember these definitions, note that the functions in the bottom row are the reciprocals of the function directly above in the top row.

Definition of the trigonometric functions

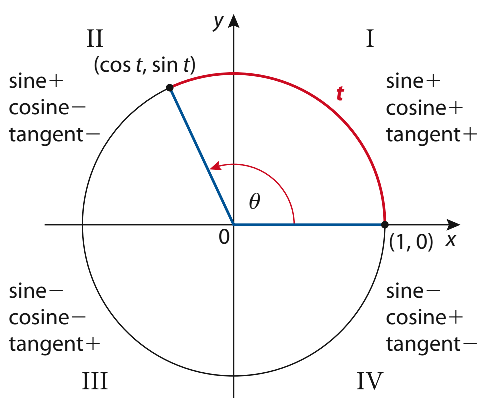
Let t be any real number and (x, y) a point on the unit circle to which t is mapped. Then the function definitions are:

$\sin t = y$ $\qquad \cos t = x$ $\qquad \tan t = \frac{y}{x}, x \neq 0$

$\csc t = \frac{1}{y}, y \neq 0$ $\qquad \sec t = \frac{1}{x}, x \neq 0$ $\qquad \cot t = \frac{x}{y}, y \neq 0$

● **Hint:** Most calculators do not have keys for cosecant, secant and cotangent. You have to use the sine, cosine or tangent keys and the appropriate quotient. Because cosecant is the reciprocal of sine, to evaluate $\csc \frac{\pi}{3}$, for example, you need to evaluate $\frac{1}{\sin \frac{\pi}{3}}$. There is a key on your GDC labelled $\sin^{-1}$. It is **not** the reciprocal of sine but represents the inverse of the sine function, also denoted as the arcsine function (abbreviated arcsin). This is the same for $\cos^{-1}$ and $\tan^{-1}$. We will learn about these three inverse trigonometric functions in the last section of this chapter.

Figure 7.10 Signs of the trigonometric functions depend on the quadrant where the arc t terminates.



On the unit circle: $x = \cos t$, $y = \sin t$.

● **Hint:** When sine, cosine and tangent are defined as circular functions based on the unit circle, radian measure is used. The values for the domain of the sine and cosine functions are real numbers that are arc lengths on the unit circle. As we know from the previous section, the arc length on the unit circle subtends an angle in standard position, whose radian measure is equivalent to the arc length (see Figure 7.10).

Because the definitions for the sine, cosine and tangent functions given here do not refer to triangles or angles, but rather to a real number representing an arc length on the unit circle, the name **circular functions** is also given to them. In fact, from this chapter's perspective that these functions are *functions of real numbers* rather than *functions of angles*, 'circular' is a more appropriate adjective than 'trigonometric'. Nevertheless, trigonometric is the more common label and will be used throughout the book.

Let's use the definitions for these three trigonometric, or circular, functions to evaluate them for some 'easy' values of t.

Example 6

Evaluate the sine, cosine and tangent functions for the following values of t.

a) $t = 0$ b) $t = \frac{\pi}{2}$ c) $t = \pi$

d) $t = \frac{3\pi}{2}$ e) $t = 2\pi$

Solution

Evaluating the sin, cos and tan functions for any value of t involves finding the coordinates of the point on the unit circle where the arc of length t will 'wrap to' (or terminate), starting at the point $(1, 0)$. It is useful to remember that an arc of length π is equal to one-half of the circumference of the unit circle. All of the values for t in this example are positive, so the arc length will wrap along the unit circle in an anticlockwise direction.

a) An arc of length $t = 0$ has no length so it 'terminates' at the point $(1, 0)$. By definition:

$\sin 0 = y = 0$ $\cos 0 = x = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$ $\csc 0 = \frac{1}{y} = \frac{1}{0}$ is undefined

$\sec 0 = \frac{1}{x} = \frac{1}{1} = 1$ $\cot 0 = \frac{x}{y} = \frac{1}{0}$ is undefined

b) An arc of length $t = \frac{\pi}{2}$ is equivalent to one-quarter of the circumference of the unit circle (Figure 7.11) so it terminates at the point $(0, 1)$.
By definition:

$\sin \frac{\pi}{2} = y = 1$ $\cos \frac{\pi}{2} = x = 0$

$\tan \frac{\pi}{2} = \frac{y}{x} = \frac{1}{0}$ is undefined $\csc \frac{\pi}{2} = \frac{1}{y} = 1$

$\sec \frac{\pi}{2} = \frac{1}{x}$ is undefined $\cot \frac{\pi}{2} = \frac{x}{y} = 0$

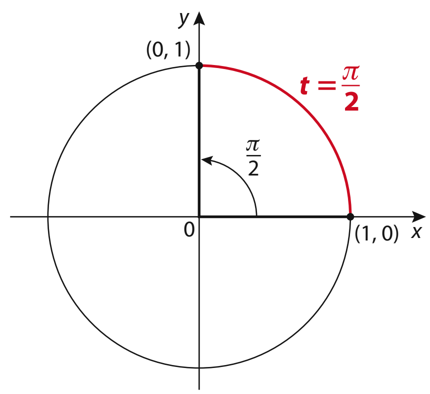


Figure 7.11 Arc length of $\frac{\pi}{2}$ or one-quarter of an anticlockwise revolution.

c) An arc of length $t = \pi$ is equivalent to one-half of the circumference of the unit circle (Figure 7.12) so it terminates at the point $(-1, 0)$. By definition:

$\sin \pi = y = 0$ $\cos \pi = x = -1$

$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$ $\csc \pi = \frac{1}{y}$ is undefined

$\sec \pi = \frac{1}{-1} = -1$ $\cot \pi = \frac{x}{y}$ is undefined

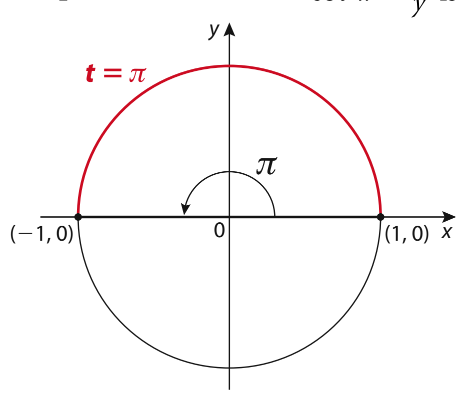


Figure 7.12 Arc length of π, one-half of an anticlockwise revolution.

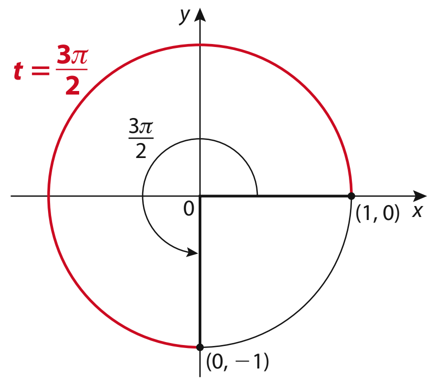
d) An arc of length $t = \frac{3\pi}{2}$ is equivalent to three-quarters of the circumference of the unit circle (Figure 7.13), so it terminates at the point $(0, -1)$. By definition:

$\sin\frac{3\pi}{2} = y = -1$ $\qquad$ $\cos\frac{3\pi}{2} = x = 0$

$\tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0}$ is undefined $\qquad$ $\csc\frac{3\pi}{2} = \frac{1}{y} = -1$

$\sec\frac{3\pi}{2} = \frac{1}{x}$ is undefined $\qquad$ $\cot\frac{3\pi}{2} = \frac{x}{y} = 0$

Figure 7.13 Arc length of $\frac{3\pi}{2}$, three-quarters of an anticlockwise revolution.



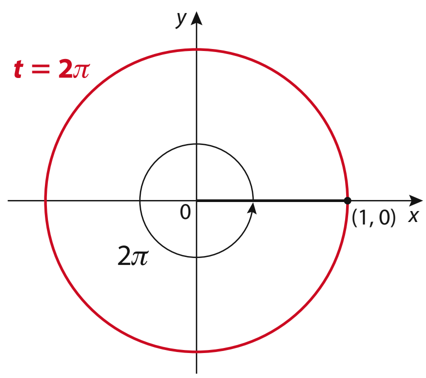
e) An arc of length $t = 2\pi$ terminates at the same point as arc of length $t = 0$ (Figure 7.14), so the values of the trigonometric functions are the same as found in part a):

$\sin 0 = y = 0$ $\qquad$ $\cos 0 = x = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$ $\qquad$ $\csc 0 = \frac{1}{y}$ is undefined

$\sec 0 = \frac{1}{x} = 1$ $\qquad$ $\cot 0 = \frac{x}{y}$ is undefined

Figure 7.14 Arc length of 2π, one full anticlockwise revolution.



Domain and range of trigonometric functions

If s and t are coterminal arcs (i.e. terminate at the same point), then the trigonometric functions of s are equal to those of t. That is, $\sin s = \sin t$, $\cos s = \cos t$, etc.

Because every real number t corresponds to exactly one point on the unit circle, the domain for both the sine function and the cosine function is the set of all real numbers. In Example 6, the tangent function and the three reciprocal trigonometric functions were sometimes undefined. Hence, the domain for these functions cannot be all real numbers. From the definitions of the functions, it is clear that the tangent and secant functions will be undefined when the x-coordinate of the arc's terminal point is zero. Therefore, the domain of the tangent and secant functions is all real numbers but **not** including the infinite set of numbers generated by adding any integer multiple of π to $\frac{\pi}{2}$. For example, $\frac{\pi}{2} + \pi = \frac{3\pi}{2}$ and $\frac{\pi}{2} - \pi = -\frac{\pi}{2}$ (see Figure 7.15), thus the tangent and secant of $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$ are undefined. Similarly, the cotangent and cosecant functions will be undefined when the y-coordinate of the arc's terminal point is zero. Therefore, the domain of the cotangent and cosecant functions is all real numbers but **not** including all of the integer multiples of π.

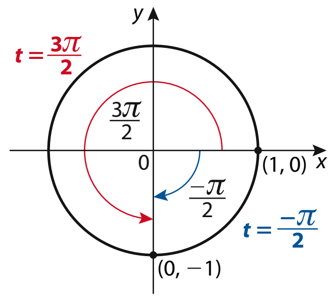


Figure 7.15

Domains of the six trigonometric functions

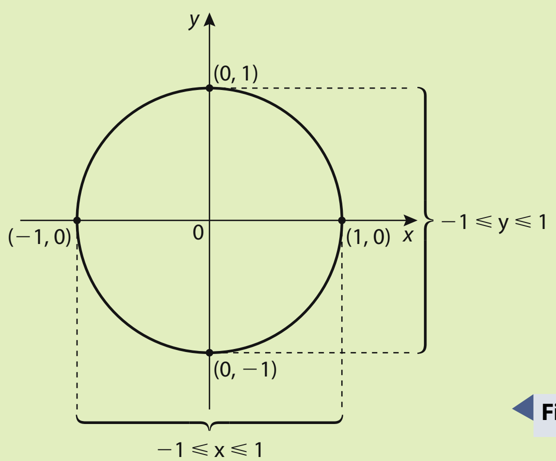


Figure 7.16

$f(t) = \sin t$ and $f(t) = \cos t$

domain: $\{t : t \in \mathbb{R}\}$

$f(t) = \tan t$ and $f(t) = \sec t$

domain: $\left\{t : t \in \mathbb{R}, t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

$f(t) = \cot t$ and $f(t) = \csc t$

domain: $\{t : t \in \mathbb{R}, t \neq k\pi, k \in \mathbb{Z}\}$

To determine the range of the sine and cosine functions, consider the unit circle shown in Figure 7.16. Because $\sin t = y$ and $\cos t = x$ and (x, y) is on the unit circle, we can see that $-1 \leqslant y \leqslant 1$ and $-1 \leqslant x \leqslant 1$. Therefore, $-1 \leqslant \sin t \leqslant 1$ and $-1 \leqslant \cos t \leqslant 1$. The range for the tangent function will not be bounded as for sine and cosine. As t approaches values where $x = \cos t = 0$, the value of $\frac{y}{x} = \tan t$ will become very large – either negative or positive, depending on which quadrant t is in. Therefore, $-\infty < \tan t < \infty$; or, in other words, $\tan t$ can be any real number.

Domain and range of sine, cosine and tangent functions

$f(t) = \sin t$	domain: $\{t : t \in \mathbb{R}\}$	range: $-1 \leqslant f(t) \leqslant 1$
$f(t) = \cos t$	domain: $\{t : t \in \mathbb{R}\}$	range: $-1 \leqslant f(t) \leqslant 1$
$f(t) = \tan t$	domain: $\left\{t : t \in \mathbb{R}, t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$	range: $f(t) \in \mathbb{R}$

From our previous discussion of periodic functions, we can conclude that all three of these trigonometric functions are periodic. Given that the sine and cosine functions are generated directly from the wrapping function, the period of each of these functions is 2π. That is,

$$\sin t = \sin(t + k \cdot 2\pi),\ k \in \mathbb{Z} \text{ and } \cos t = \cos(t + k \cdot 2\pi),\ k \in \mathbb{Z}$$

Since the cosecant and secant functions are reciprocals, respectively, of sine and cosine, the period of cosecant and secant will also be 2π.

Initial evidence from Example 6 indicates that the period of the tangent function is π. That is,

$$\tan t = \tan(t + k \cdot \pi),\ k \in \mathbb{Z}$$

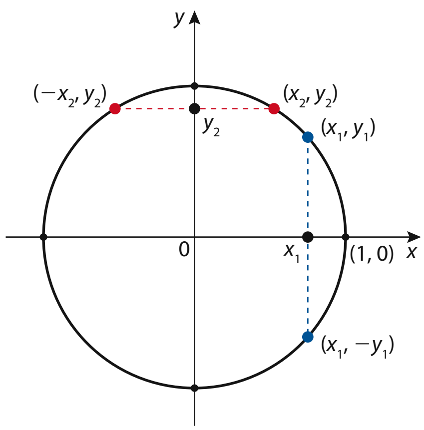
We will establish these results graphically in the next section. Also note that since these functions are periodic then they are not one-to-one functions.

This is an important fact with regard to establishing inverse trigonometric functions (Section 7.6).

Evaluating trigonometric functions

In Example 6, the unit circle was divided into four equal arcs corresponding to t values of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π. Let's evaluate the sine, cosine and tangent functions for further values of t that would correspond to dividing the unit circle into eight equal arcs. The symmetry of the unit circle dictates that any points on the unit circle which are reflections about the x-axis will have the same x-coordinate (same value of sine), and any points on the unit circle which are reflections about the y-axis will have the same y-coordinate, as shown in Figure 7.17.

Figure 7.17



Example 7

Evaluate the sine, cosine and tangent functions for $t = \frac{\pi}{4}$, and then use that result to evaluate the same functions for $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$ and $t = \frac{7\pi}{4}$.

Solution

When an arc of length $t = \frac{\pi}{4}$ is wrapped along the unit circle starting at $(1, 0)$, it will terminate at a point (x_1, y_1) in quadrant I that is equidistant from $(1, 0)$ and $(0, 1)$. Since the line $y = x$ is a line of symmetry for the unit circle, (x_1, y_1) is on this line. Hence, the point (x_1, y_1) is the point of intersection of the unit circle $x^2 + y^2 = 1$ with the line $y = x$. Let's find the coordinates of the intersection point by solving this pair of simultaneous equations by substituting x for y into the equation $x^2 + y^2 = 1$.

$x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

Rationalizing the denominator gives $x = \pm\frac{\sqrt{2}}{2}$ and, since the point is in the first quadrant, $x = \frac{\sqrt{2}}{2}$. Given that the point is on the line $y = x$ then $y = \frac{\sqrt{2}}{2}$. Therefore, the arc of length $t = \frac{\pi}{4}$ will terminate at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle. Using the symmetry of the unit circle, we can also determine the points on the unit circle where arcs of length $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$ and $t = \frac{7\pi}{4}$ terminate. These arcs and the coordinates of their terminal points are given in Figure 7.18.

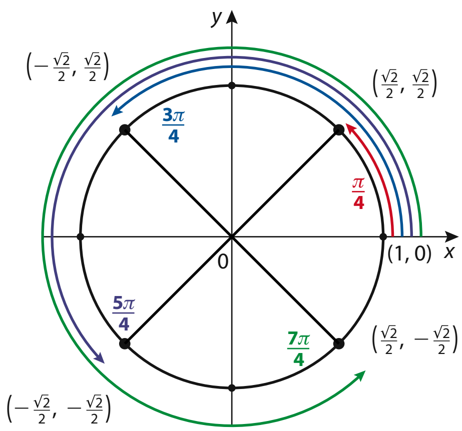


Figure 7.18

Using the coordinates of these points, we can now evaluate the trigonometric functions for $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$. By definition:

$$t = \frac{\pi}{4}:\ \sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$t = \frac{3\pi}{4}:\ \sin\frac{3\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{3\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \tan\frac{3\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

$$t = \frac{5\pi}{4}:\ \sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

$$t = \frac{7\pi}{4}:\ \sin\frac{7\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \cos\frac{7\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{7\pi}{4} = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

We can use a method similar to that of Example 7 to find the point on the unit circle where an arc of length $t = \frac{\pi}{6}$ terminates in the first quadrant. Then we can again apply symmetry about the line $y = x$ and the y- and x-axes to find points on the circle corresponding to arcs whose lengths are multiples of $\frac{\pi}{6}$, e.g. $\frac{2\pi}{6} = \frac{\pi}{3}$, $\frac{4\pi}{6} = \frac{2\pi}{3}$, etc. Arcs whose lengths are multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ correspond to eight equally spaced points and twelve equally spaced points, respectively, around the unit circle, as shown in Figures 7.19 and 7.20. The coordinates of these points give us the sine, cosine and tangent values for common values of t.

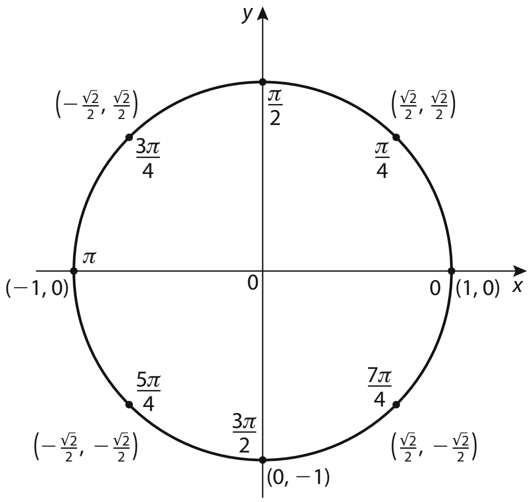


Figure 7.19 Arc lengths that are multiples of $\frac{\pi}{4}$ divide the unit circle into eight equally spaced points.

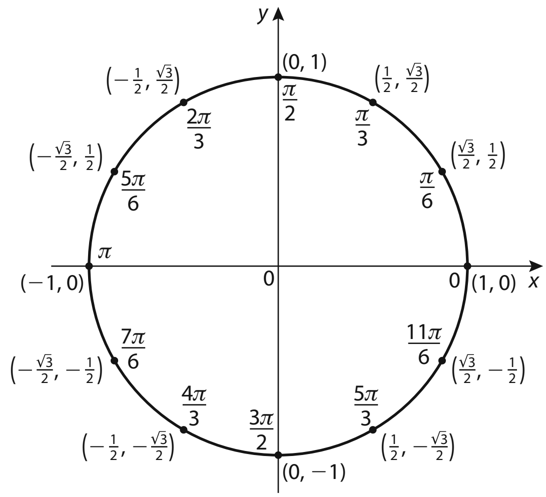


Figure 7.20 Arc lengths that are multiples of $\frac{\pi}{6}$ divide the unit circle into twelve equally spaced points.

The tangent, cosecant, secant and cotangent functions can all be expressed in terms of the sine and/or cosine functions. The following four identities follow directly from the definitions for the trigonometric functions.

$\tan t = \frac{\sin t}{\cos t}$ $\csc t = \frac{1}{\sin t}$

$\sec t = \frac{1}{\cos t}$ $\cot t = \frac{\cos t}{\sin t}$

You will find it very helpful to know from memory the exact values of sine and cosine for numbers that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Use the unit circle diagrams shown in Figures 7.19 and 7.20 as a guide to help you do this and to visualize the location of the terminal points of different arc lengths. With the symmetry of the unit circle and a point's location in the coordinate plane telling us the sign of x and y (see Figure 7.10), we only need to remember the sine and cosine of common values of t in the first quadrant and on the positive x- and y-axes. These are organized in Table 7.1.

Table 7.1 The trigonometric functions evaluated for special values of t.

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	undefined	1	undefined
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

If t is not a multiple of one of these common values, the values of the trigonometric functions for that number can be found using your GDC.

• **Hint:** Memorize the values of sin t and cos t for the values of t that are highlighted in the red box in Table 7.1. These values can be used to derive the values of all six trigonometric functions for any multiple of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$.

For any arc s on the unit circle ($r = 1$) the arc length formula from the previous section, $s = r\theta$, shows us that each real number t not only measures an arc along the unit circle but also measures a central angle in radians. That is, $t = r\theta = 1 \cdot \theta = \theta$ in radian measure. Therefore, when you are evaluating a trigonometric function it does not make a difference whether the argument of the function is considered to be a real number (i.e. length of an arc) or an angle in radians.

Example 8

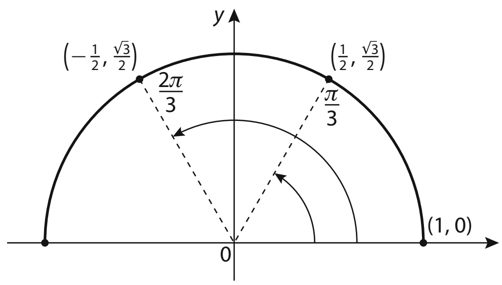
Find the following function values. Find the exact value, if possible. Otherwise, find the approximate value accurate to 3 significant figures.

a) $\sin\frac{2\pi}{3}$ b) $\cos\frac{5\pi}{4}$ c) $\tan\frac{11\pi}{6}$

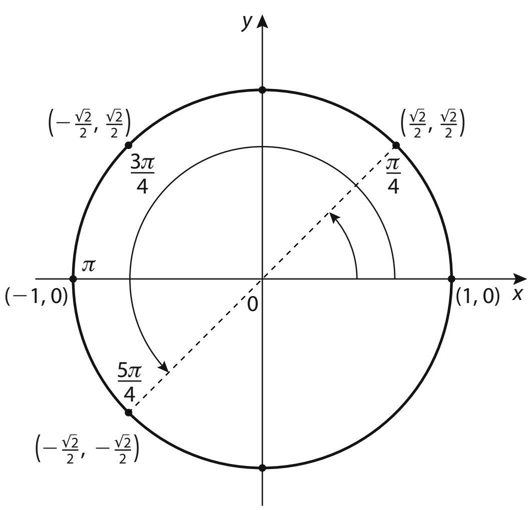
d) $\csc\frac{13\pi}{6}$ e) $\sec 3.75$

Solution

a) The terminal point for $\frac{2\pi}{3}$ is in the second quadrant and is the reflection in the y-axis of the terminal point for $\frac{\pi}{3}$, whose y-coordinate is $\frac{\sqrt{3}}{2}$. Therefore, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

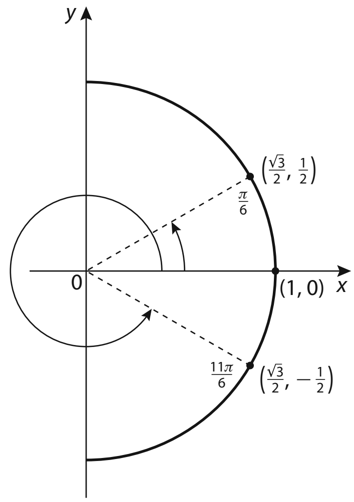


b) $\frac{5\pi}{4}$ is in the third quadrant. Hence, its x-coordinate and cosine must be negative. All of the odd multiples of $\frac{\pi}{4}$ have terminal points with x- and y-coordinates of $\pm\frac{\sqrt{2}}{2}$. Therefore, $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.



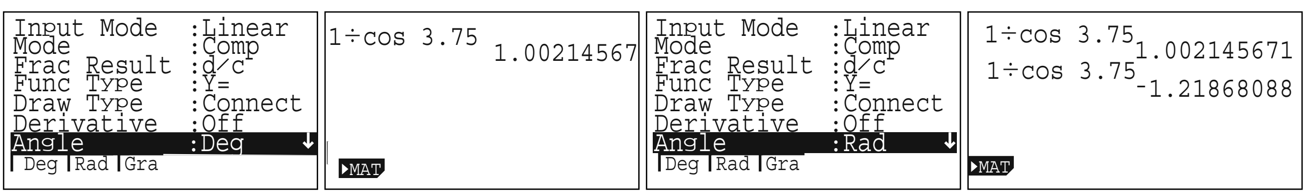
c) $\frac{11\pi}{6}$ is in the fourth quadrant, so its tangent will be negative. Its terminal point is the reflection in the x-axis of the terminal point for $\frac{\pi}{6}$, whose coordinates are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore,

$$\tan\frac{11\pi}{6} = \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$



d) $\frac{13\pi}{6}$ is more than one revolution. Because $\frac{13\pi}{6} = \frac{\pi}{6} + 2\pi$ and the period of the cosecant function is 2π [i.e. $\csc t = \csc(t + k \cdot 2\pi)$, $k \in \mathbb{Z}$], then $\csc\frac{13\pi}{6} = \csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$.

e) To evaluate sec 3.75 you must use your GDC. An arc of length 3.75 will have its terminal point in the third quadrant since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, meaning $\pi < 3.75 < \frac{3\pi}{2}$. Hence, cos 3.75 must be negative, and because the secant function is the reciprocal of cosine, then sec 3.75 is also negative. This fact indicates that the result in the second GDC image below must be incorrect with the GDC wrongly set to 'degree' mode. Changing to 'radian' mode allows for the correct result to be computed. To an accuracy of three significant figures, sec $3.75 \approx -1.22$.



Have you ever wondered how your calculator computes a value for a trigonometric function – such as cos 0.75? Evaluating an algebraic function (Chapter 3) is relatively straightforward because, by definition, it consists of a finite number of elementary operations (i.e. addition, subtraction, multiplication, division, and extracting a root). It is not so straightforward to evaluate non-algebraic functions like exponential, logarithmic and trigonometric functions and efforts by mathematicians to do so have led to some sophisticated approximation techniques using **power series** that are studied in further calculus. A power series is an infinite series that can be thought of as a polynomial with an infinite number of terms. You will learn about the theory and application of power series if your Mathematics HL class covers the *Option: Infinite series and differential equations*. If you look in the Mathematics HL Information (Formulae) Booklet in the Topic 10 section (for series and differential equations) you will see the power series (infinite polynomial) approximation for some functions including the cosine function.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ where $n! = 1 \cdot 2 \cdot 3 \ldots n$ [$n!$ is read 'n factorial']

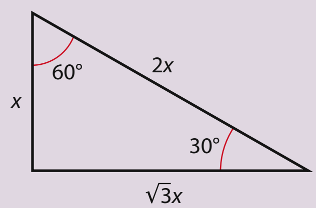
Exploiting the fact that polynomial functions are easy to evaluate, we can easily program a calculator to compute enough terms of the power series to obtain a result to the required accuracy. For example, if we use the first three terms of the power series for cosine to find cos 0.75, we get

$\cos 0.75 = 1 - \frac{0.75^2}{2!} + \frac{0.75^4}{4!} = 0.731\,933\,593\,75$. Compare this to the value obtained using your GDC.

Several important mathematicians in the 17th and 18th centuries, including Isaac Newton, James Gregory, Gottfried Leibniz, Leonhard Euler and Joseph Fourier, contributed to the development of using power series to represent non-algebraic functions. However, the two names most commonly associated with power series are the English mathematician Brook Taylor (1685–1731) and the Scottish mathematician Colin Maclaurin (1698–1746).

Exercise 7.2

1 a) By knowing the ratios of sides in any triangle with angles measuring 30°, 60° and 90° (see figure), find the coordinates of the points on the unit circle where an arc of length $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$ terminate in the first quadrant.

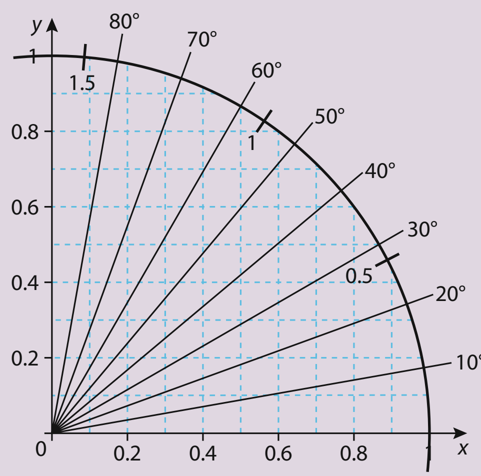


b) Using the result from a) and applying symmetry about the unit circle, find the coordinates of the points on the unit circle corresponding to arcs whose lengths are $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$.

Draw a large unit circle and label all of these points with their coordinates and the measure of the arc that terminates at each point.

Questions 2–9

The figure of quadrant I of the unit circle shown right indicates angles in intervals of 10 degrees and also indicates angles in radian measure of 0.5, 1 and 1.5. Use the figure and the definitions of the sine and cosine functions to approximate the function values to one decimal place in questions 2–9. Check your answers with your GDC (be sure to be in the correct angle measure mode).



2 cos 50°
3 sin 80°
4 cos 1
5 sin 0.5
6 tan 70°
7 cos 1.5
8 sin 20°
9 tan 1

In questions 10–18, t is the length of an arc on the unit circle starting from (1, 0). a) State the quadrant in which the terminal point of the arc lies. b) Find the coordinates of the terminal point (x, y) on the unit circle. Give exact values for x and y, if possible. Otherwise, approximate values to 3 significant figures.

10 $t = \frac{\pi}{6}$
11 $t = \frac{5\pi}{3}$
12 $t = \frac{7\pi}{4}$
13 $t = \frac{3\pi}{2}$
14 $t = 2$
15 $t = -\frac{\pi}{4}$
16 $t = -1$
17 $t = -\frac{5\pi}{4}$
18 $t = 3.52$

In questions 19–27, state the exact value of the sine, cosine and tangent of the given real number.

19 $\frac{\pi}{3}$
20 $\frac{5\pi}{6}$
21 $-\frac{3\pi}{4}$
22 $\frac{\pi}{2}$
23 $-\frac{4\pi}{3}$
24 3π
25 $\frac{3\pi}{2}$
26 $-\frac{7\pi}{6}$
27 $t = 1.25\pi$

In questions 28–31, use the periodic properties of the sine and cosine functions to find the exact value of $\sin x$ and $\cos x$.

28 $x = \frac{13\pi}{6}$
29 $x = \frac{10\pi}{3}$
30 $x = \frac{15\pi}{4}$
31 $x = \frac{17\pi}{6}$

32 Find the exact function values, if possible. Do not use your GDC.

a) $\cos\frac{5\pi}{6}$
b) sin 315°
c) $\tan\frac{3\pi}{2}$
d) $\sec\frac{5\pi}{3}$
e) csc 240°

33 Find the exact function values, if possible. Otherwise, use your GDC to find the approximate value accurate to three significant figures.

a) sin 2.5
b) cot 120°
c) $\cos\frac{5\pi}{4}$
d) sec 6
e) $\tan\pi$

In questions 34–41, specify in which quadrant(s) an angle θ in standard position could be given the stated conditions.

34 $\sin\theta > 0$

35 $\sin\theta > 0$ and $\cos\theta < 0$

36 $\sin\theta < 0$ and $\tan\theta > 0$

37 $\cos\theta < 0$ and $\tan\theta < 0$

38 $\cos\theta > 0$

39 $\sec\theta > 0$ and $\tan\theta > 0$

40 $\cos\theta > 0$ and $\csc\theta < 0$

41 $\cot\theta < 0$

7.3 Graphs of trigonometric functions

The graph of a function provides a useful visual image of its behaviour. For example, from the previous section we know that trigonometric functions are periodic, i.e. their values repeat in a regular manner. The graphs of the trigonometric functions should provide a picture of this periodic behaviour. In this section, we will graph the sine, cosine and tangent functions and transformations of the sine and cosine functions.

```
sin(2.53)
          .5741721484
sin(2.53+2π)
          .5741721484
sin(2.53+4π)
          .5741721484
```

The period of $y = \sin x$ is 2π.

Graphs of the sine and cosine functions

Since the period of the sine function is 2π, we know that two values of t (domain) that differ by 2π (e.g. $\frac{\pi}{6}$ and $\frac{13\pi}{6}$ in Example 8d) will produce the same value for y (range). This means that any portion of the graph of $y = \sin t$ with a t-interval of length 2π (called one **period** or **cycle** of the graph) will repeat. Remember that the domain of the sine function is all real numbers, so one period of the graph of $y = \sin t$ will repeat indefinitely in the positive and negative direction. Therefore, in order to construct a complete graph of $y = \sin t$, we need to graph just one period of the function, that is, from $t = 0$ to $t = 2\pi$, and then repeat the pattern in both directions.

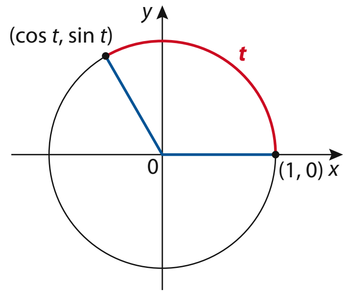
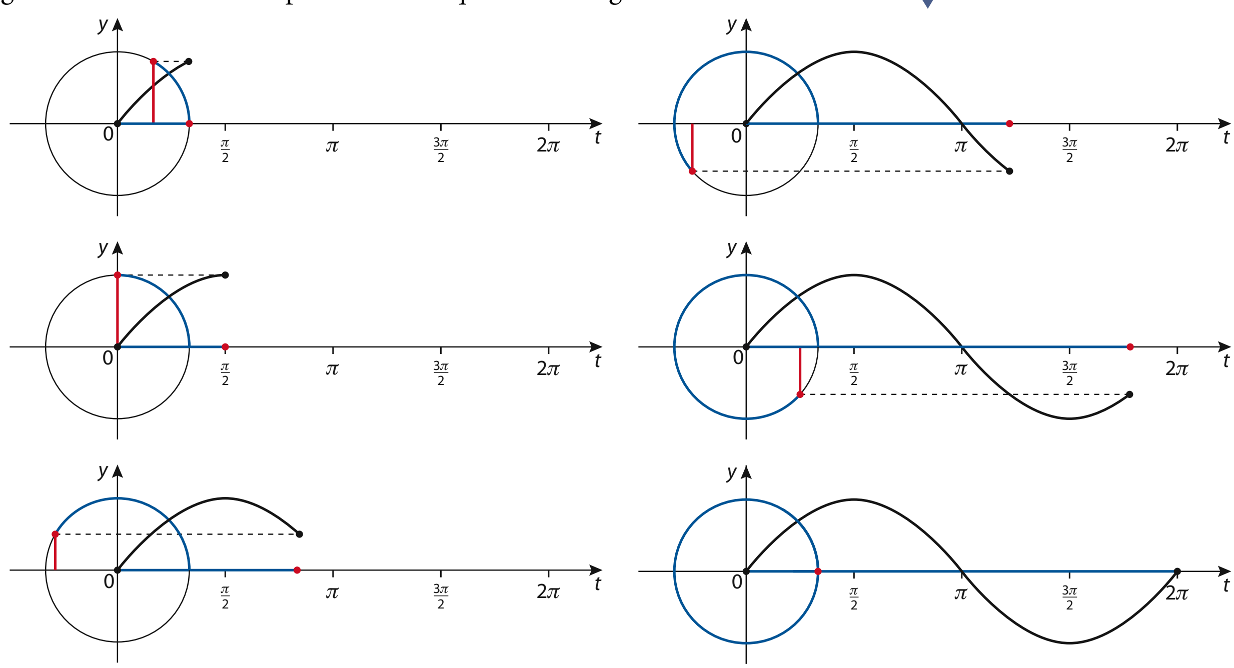


Figure 7.21 Coordinates of terminal point of arc t gives the values of $\cos t$ and $\sin t$.

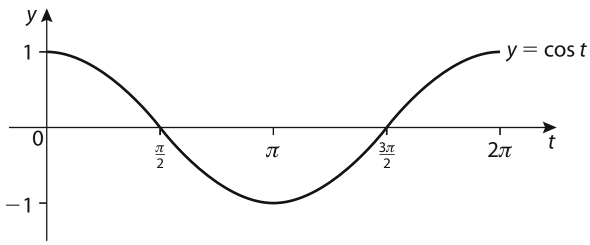
We know from the previous section that $\sin t$ is the y-coordinate of the terminal point on the unit circle corresponding to the real number t (Figure 7.21). In order to generate one period of the graph of $y = \sin t$, we need to record the y-coordinates of a point on the unit circle and the corresponding value of t as the point travels anticlockwise one revolution, starting from the point $(1, 0)$. These values are then plotted on a graph with t on the horizontal axis and y (i.e. $\sin t$) on the vertical axis. Figure 7.22 illustrates this process in a sequence of diagrams.

Figure 7.22 Graph of the sine function for $0 \leqslant t \leqslant 2\pi$ generated from a point travelling along the unit circle.



As the point $(\cos t, \sin t)$ travels along the unit circle, the x-coordinate (i.e. $\cos t$) goes through the same cycle of values as the y-coordinate ($\sin t$). The only difference is that the x-coordinate begins at a different value in the cycle – when $t = 0$, $y = 0$, but $x = 1$. The result is that the graph of $y = \cos t$ is the exact same shape as $y = \sin t$ but it has been shifted to the left $\frac{\pi}{2}$ units. The graph of $y = \cos t$ for $0 \leqslant t \leqslant 2\pi$ is shown in Figure 7.23.

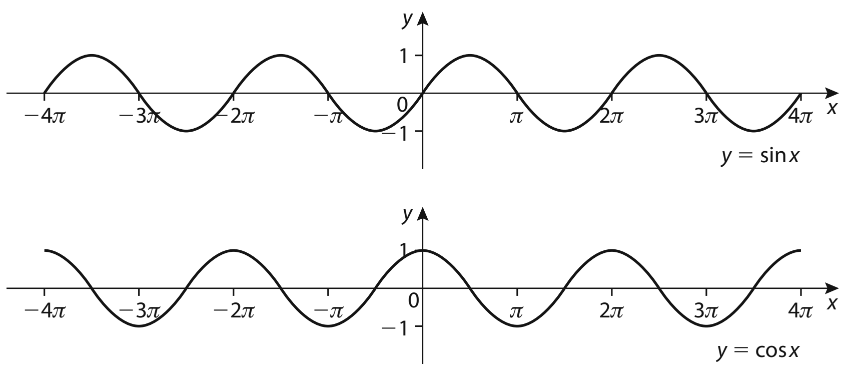
Figure 7.23 Graph of $y = \cos t$ for $0 \leqslant t \leqslant 2\pi$.



The convention is to use the letter x to denote the variable in the domain of the function. Hence, we will use the letter x rather than t and write the trigonometric functions as $y = \sin x$, $y = \cos x$ and $y = \tan x$.

Because the period for both the sine function and cosine function is 2π, to graph $y = \sin x$ and $y = \cos x$ for wider intervals of x we simply need to repeat the shape of the graph that we generated from the unit circle for $0 \leqslant x \leqslant 2\pi$ (Figures 7.22 and 7.23). Figure 7.24 shows the graphs of $y = \sin x$ and $y = \cos x$ for $-4\pi \leqslant x \leqslant 4\pi$.

Figure 7.24 $y = \sin x$ and $y = \cos x$, $0 \leqslant x \leqslant 4\pi$.



Aside from their periodic behaviour, these graphs reveal further properties of the graphs of $y = \sin x$ and $y = \cos x$. Note that the sine function has a maximum value of $y = 1$ for all $x = \frac{\pi}{2} + k \cdot 2\pi$, $k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = -\frac{\pi}{2} + k \cdot 2\pi$, $k \in \mathbb{Z}$. The cosine function has a maximum value of $y = 1$ for all $x = k \cdot 2\pi$, $k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = \pi + k \cdot 2\pi$, $k \in \mathbb{Z}$. This also confirms – as established in the previous section – that both functions have a domain of all real numbers and a range of $-1 \leqslant y \leqslant 1$.

Closer inspection of the graphs, in Figure 7.24, shows that the graph of $y = \sin x$ has rotational symmetry about the origin – that is, it can be rotated one-half of a revolution about (0, 0) and it remains the same. This graph symmetry can be expressed with the identity: $\sin(-x) = -\sin x$. For example, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\left[\sin\left(\frac{\pi}{6}\right)\right] = -\left[\frac{1}{2}\right] = -\frac{1}{2}$. A function that is symmetric about the origin is called an **odd function**. The graph of $y = \cos x$ has line symmetry in the y-axis – that is, it can be reflected in the line $x = 0$ and it remains the same. This graph symmetry can be expressed with the identity: $\cos(-x) = \cos x$. For example, $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. A function that is symmetric about the y-axis is called an **even function**.

> **Odd and even functions**
>
> A function is **odd** if, for each x in the domain of f, $f(-x) = -f(x)$.
>
> The graph of an odd function is symmetric with respect to the origin (rotational symmetry).
>
> A function is **even** if, for each x in the domain of f, $f(-x) = f(x)$.
>
> The graph of an even function is symmetric with respect to the y-axis (line symmetry).

Recall that odd and even functions were first discussed in Section 3.1.

Graphs of transformations of the sine and cosine functions

In Section 2.4, we learned how to transform the graph of a function by horizontal and vertical translations, by reflections in the coordinate axes, and by stretching and shrinking – both horizontal and vertical. The following is a review of these transformations.

Review of transformations of graphs of functions

Assume that a, b, c and d are real numbers.

To obtain the graph of:	From the graph of $y = f(x)$:
$y = f(x) + d$	Translate d units up for $d > 0$, d units down for $d < 0$.
$y = f(x + c)$	Translate c units left for $c > 0$, c units right for $c < 0$.
$y = -f(x)$	Reflect in the x-axis.
$y = af(x)$	Vertical stretch ($a > 1$) or shrink ($0 < a < 1$) of factor a.
$y = f(-x)$	Reflect in the y-axis.
$y = f(bx)$	Horizontal stretch ($0 < b < 1$) or shrink ($b > 1$) of factor $\frac{1}{b}$.

In this section, we will look at the composition of sine and cosine functions of the form

$$f(x) = a\sin[b(x + c)] + d \quad \text{and} \quad f(x) = a\cos[b(x + c)] + d$$

Example 9

Sketch the graph of each function on the interval $-\pi \leqslant x \leqslant 3\pi$.

a) $f(x) = 2\cos x$

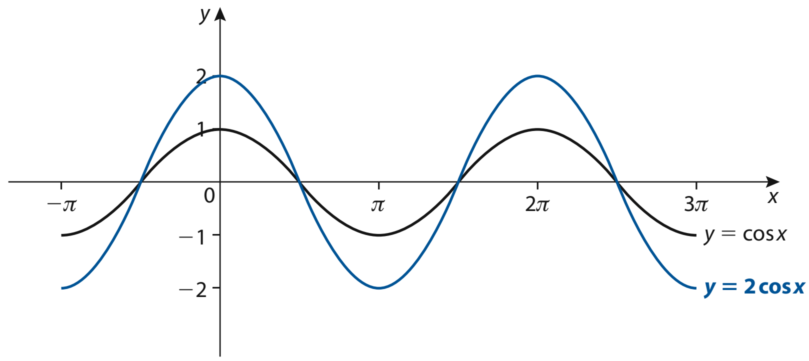
b) $g(x) = \cos x + 3$

c) $h(x) = 2\cos x + 3$

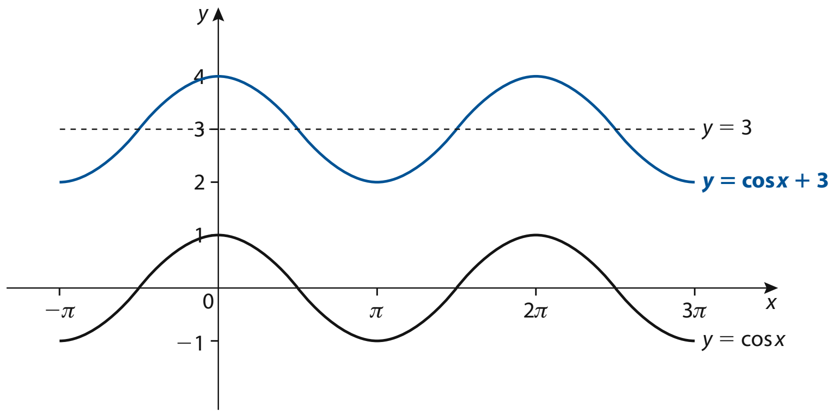
d) $p(x) = \frac{1}{2}\sin x - 2$

Solution

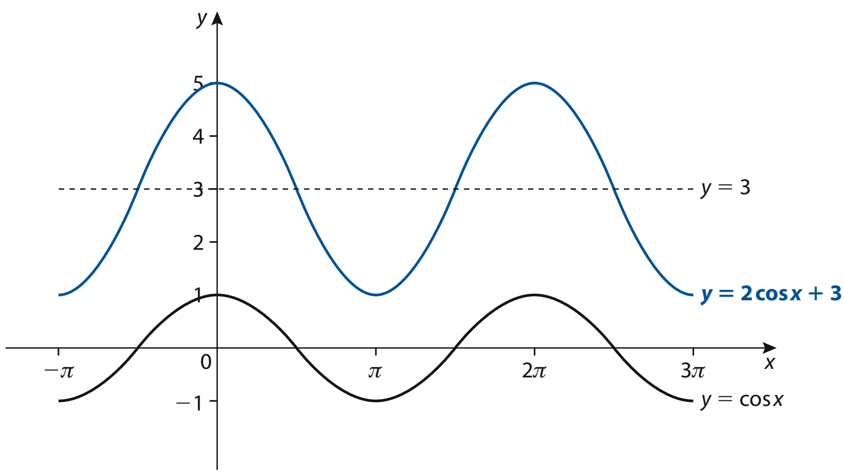
a) Since $a = 2$, the graph of $y = 2\cos x$ is obtained by vertically stretching the graph of $y = \cos x$ by a factor of 2.



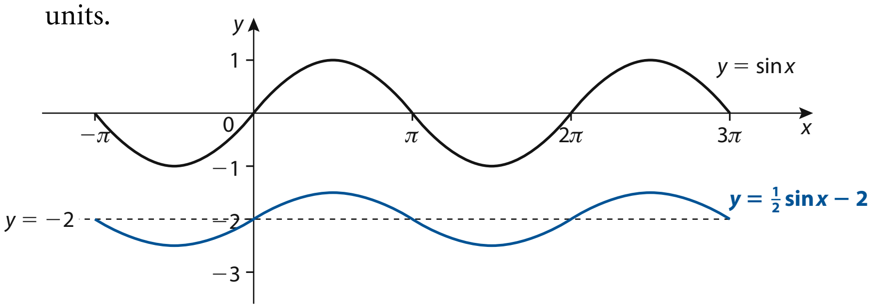
b) Since $d = 3$, the graph of $y = \cos x + 3$ is obtained by translating the graph of $y = \cos x$ three units up.



c) We can obtain the graph of $y = 2\cos x + 3$ by combining both of the transformations to the graph of $y = \cos x$ performed in parts a) and b) – namely, a vertical stretch of factor 2 and a translation three units up.



d) The graph of $y = \frac{1}{2}\sin x - 2$ can be obtained by vertically shrinking the graph of $y = \sin x$ by a factor of $\frac{1}{2}$ and then translating it down two units.



In part a), the graph of $y = 2\cos x$ has many of the same properties as the graph of $y = \cos x$: same period, and the maximum and minimum values occur at the same values of x. However, the graph ranges between -2 and 2 instead of -1 and 1. This difference is best described by referring to the **amplitude** of each graph. The amplitude of $y = \cos x$ is 1 and the amplitude of $y = 2\cos x$ is 2. The amplitude of a sine or cosine graph is not always equal to its maximum value. In part b), the amplitude of $y = \cos x + 3$ is 1; in part c), the amplitude of $y = 2\cos x + 3$ is 2; and the amplitude of $y = \frac{1}{2}\sin x - 2$ is $\frac{1}{2}$. For all three of these, the graphs oscillate about the horizontal line $y = d$. How *high* and *low* the graph oscillates with respect to the mid-line, $y = d$, is the graph's amplitude. With respect to the general form $y = af(x)$, changing the amplitude is equivalent to a vertical stretching or shrinking. Thus, we can give a more precise definition of amplitude in terms of the parameter a.

Amplitude of the graph of sine and cosine functions

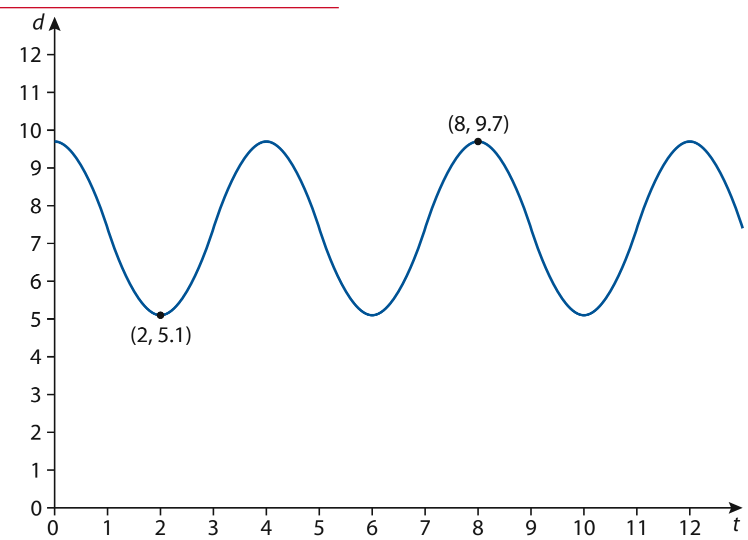
The graphs of $f(x) = a\sin[b(x + c)] + d$ and $f(x) = a\cos[b(x + c)] + d$ have an **amplitude** equal to $|a|$.

Example 10

Waves are produced in a long tank of water. The depth of the water, d metres, at t seconds, at a fixed location in the tank, is modelled by the function $d(t) = M\cos\left(\frac{\pi}{2}t\right) + K$, where M and K are positive constants. On the right is the graph of $d(t)$ for $0 \leqslant t \leqslant 12$ indicating that the point $(2, 5.1)$ is a minimum and the point $(8, 9.7)$ is a maximum.

a) Find the value of K and the value of M.

b) After $t = 0$, find the first time when the depth of the water is 9.7 metres.



Solution

a) The constant K is equivalent to the constant d in the general form of a cosine function: $f(x) = a\cos[b(x + c)] + d$. To find the value of K and the equation of the horizontal mid-line, $y = K$, find the average of the function's maximum and minimum value: $K = \frac{9.7 + 5.1}{2} = 7.4$.

The constant M is equivalent to the constant a whose absolute value is the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|M| = 9.7 - 7.4 = 2.3$. Thus, $M = 2.3$ or $M = -2.3$. Try $M = 2.3$ by evaluating the function at one of the known values:

$d(2) = 2.3\cos\left(\frac{\pi}{2}(2)\right) + 7.4 = 2.3\cos\pi + 7.4 = 2.3(-1) + 7.4 = 5.1.$

This agrees with the point $(2, 5.1)$ on the graph. Therefore, $M = 2.3$.

b) Maximum values of the function $(d(8) = 9.7)$ occur at values of t that differ by a value equal to the period. From the graph, we can see that the difference in t-values from the minimum $(2, 5.1)$ to the maximum $(8, 9.7)$ is equivalent to one-and-a-half periods. Therefore, the period is 4 and the first time after $t = 0$ at which $d = 9.7$ is $t = 4$.

All four of the functions in Example 9 had the same period of 2π, but the function in Example 10 had a period of 4. Because $y = \sin x$ completes one period from $x = 0$ to $x = 2\pi$, it follows that $y = \sin bx$ completes one period from $bx = 0$ to $bx = 2\pi$. This implies that $y = \sin bx$ completes one period from $x = 0$ to $x = \frac{2\pi}{b}$. This agrees with the period for the function $d(t) = 2.3\cos\left(\frac{\pi}{2}t\right) + 7.4$ in Example 10: period $= \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1}\cdot\frac{2}{\pi} = 4$.

Note that the change in amplitude and vertical translation had no effect on the period. We should also expect that a horizontal translation of a sine or cosine curve should not affect the period. The next example looks at a function that is horizontally translated (shifted) and has a period different from 2π.

Example 11

Sketch the function $f(x) = \sin\left(2x + \frac{2\pi}{3}\right)$.

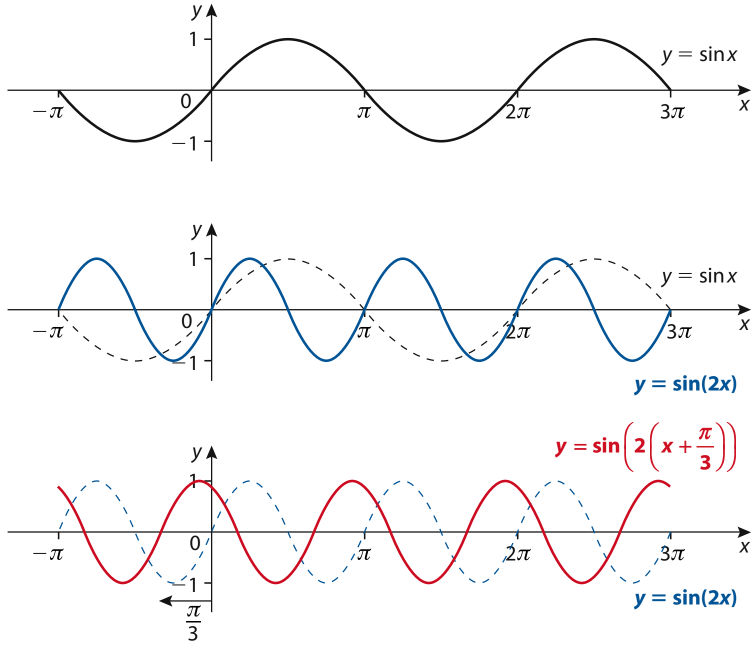
Solution

To determine how to transform the graph of $y = \sin x$ to obtain the graph of $y = \sin\left(2x + \frac{2\pi}{3}\right)$, we need to make sure the function is written in the form $f(x) = a\sin[b(x + c)] + d$. Clearly, $a = 1$ and $d = 0$, but we will need to factorize a 2 from the expression $2x + \frac{2\pi}{3}$ to get $f(x) = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$.

According to our general transformations from Chapter 2, we expect that the graph of f is obtained by first performing a horizontal shrinking of factor $\frac{1}{2}$ to the graph of $y = \sin x$ and then a translation to the left $\frac{\pi}{3}$ units (see Section 2.4).

> Transformations of the graphs of trigonometric functions follow the same rules as for other functions. The rules were established in Section 2.4 and summarized on page 84.

The graphs on the next page illustrate the two-stage sequence of transforming $y = \sin x$ to $y = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$.



Note: A horizontal translation of a sine or cosine curve is often referred to as a **phase shift**. The equations $y = \sin\left(x + \frac{\pi}{3}\right)$ and $y = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$ both underwent a phase shift of $-\frac{\pi}{3}$.

Period and horizontal translation (phase shift) of sine and cosine functions

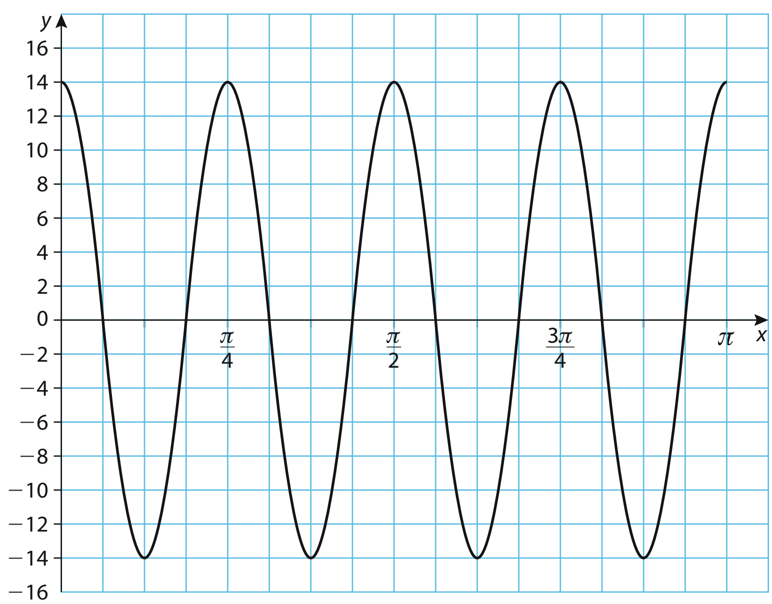
Given that b is a positive real number, $y = a\sin[b(x + c)] + d$ and $y = a\cos[b(x + c)] + d$ have a **period** of $\frac{2\pi}{b}$ and a horizontal translation (**phase shift**) of $-c$.

Example 12

The graph of a function in the form $y = a\cos bx$ is given in the diagram right.

a) Write down the value of a.

b) Calculate the value of b.



Solution

a) The amplitude of the graph is 14.
Therefore, $a = 14$.

b) From inspecting the graph we can see that the period is $\frac{\pi}{4}$.

$$\text{Period} = \frac{2\pi}{b} = \frac{\pi}{4}$$

$$b\pi = 8\pi \Rightarrow b = 8.$$

Example 13

For the function $f(x) = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2}$:

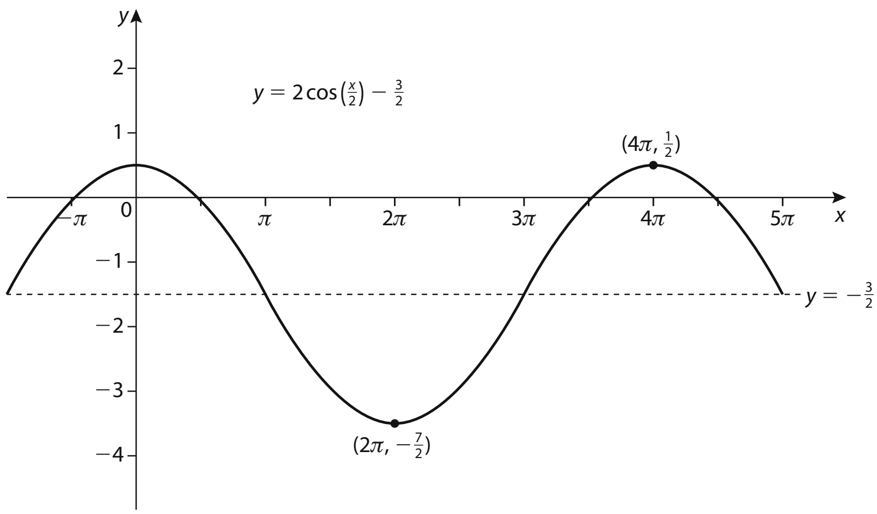
a) Sketch the function for the interval $-\pi \leqslant x \leqslant 5\pi$. Write down its amplitude and period.

b) Determine the domain and range for $f(x)$.

c) Write $f(x)$ as a trigonometric function in terms of sine rather than cosine.

Solution

a) $a = 2 \Rightarrow$ amplitude $= 2$; $b = \frac{1}{2} \Rightarrow$ period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$. To obtain the graph of $y = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2}$, we perform the following transformations on $y = \cos x$: (i) a horizontal stretch by factor $\frac{1}{\frac{1}{2}} = 2$, (ii) a vertical stretch by factor 2, and (iii) a vertical translation down $\frac{3}{2}$ units.



b) The domain is all real numbers. The function will reach a maximum value of $d + a = -\frac{3}{2} + 2 = \frac{1}{2}$, and a minimum value of $d - a = -\frac{3}{2} - 2 = -\frac{7}{2}$.

Hence, the range is $-\frac{7}{2} \leqslant y \leqslant \frac{1}{2}$.

c) The graph of $y = \cos x$ can be obtained by translating the graph of $y = \sin x$ to the left $\frac{\pi}{2}$ units. Thus, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, or, in other words, any cosine function can be written as a sine function with a phase shift $= -\frac{\pi}{2}$. Therefore, $f(x) = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2} = 2\sin\left(\frac{x}{2} + \frac{\pi}{2}\right) - \frac{3}{2}$.

Horizontal translation (phase shift) identities

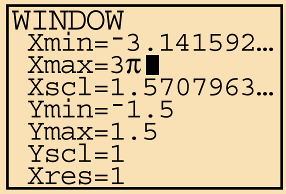
The following are true for all values of x:

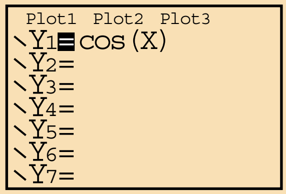
$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \qquad \sin x = \cos\left(x - \frac{\pi}{2}\right)$$

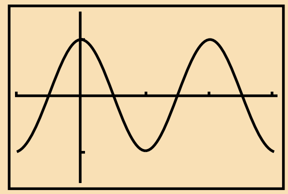
$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

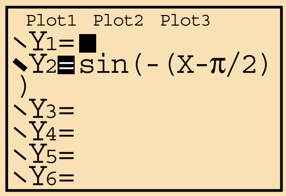
The identity $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ is equivalent to the identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ because $\sin\left(\frac{\pi}{2} - x\right) = \sin\left[-\left(x - \frac{\pi}{2}\right)\right]$ and the graph of $y = \sin\left[-\left(x - \frac{\pi}{2}\right)\right]$ can be obtained by first translating $y = \sin x$ to the right $\frac{\pi}{2}$ units, and then reflecting the graph in the y-axis. This produces the same graph as $y = \cos x$. This can be confirmed nicely on your GDC as shown. Therefore, $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. In fact, it is also true that $\sin x = \cos\left(\frac{\pi}{2} - x\right)$. Clearly, $x + \left(\frac{\pi}{2} - x\right) = \frac{\pi}{2}$. If the domain ($x$) values were being treated as angles, then x and $\frac{\pi}{2} - x$ would be complementary angles.

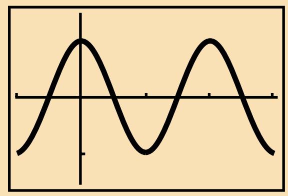
This is why cosine is considered the co-function of sine. Two trigonometric functions f and g are co-functions if the following are true for all x: $f(x) = g\left(\frac{\pi}{2} - x\right)$ and $f\left(\frac{\pi}{2} - x\right) = g(x)$.





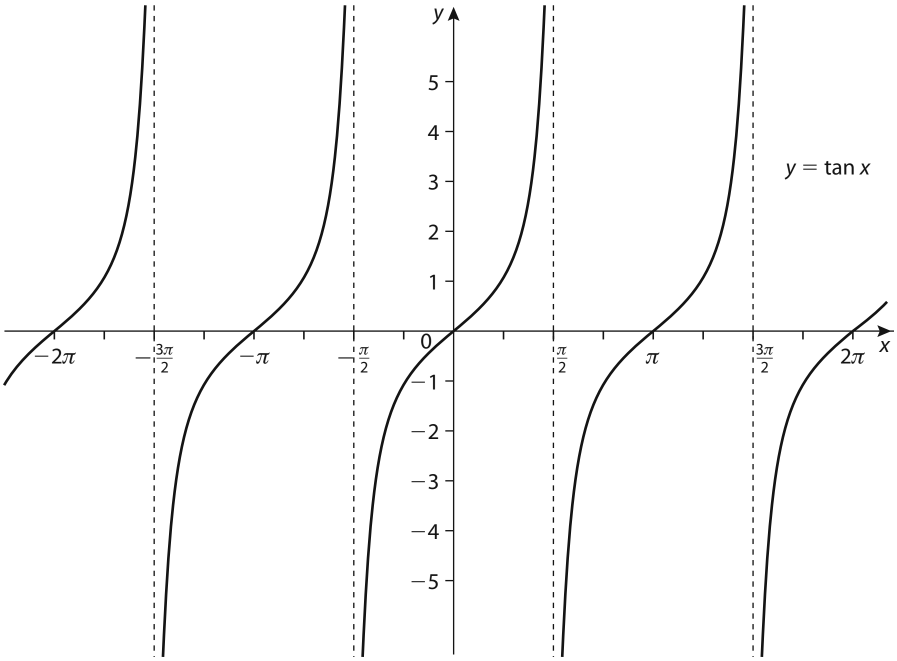






Graph of the tangent function

From work done earlier in this chapter, we expect that the behaviour of the tangent function will be significantly different from that of the sine and cosine functions. In Section 7.2, we concluded that the function $f(x) = \tan x$ has a domain of all real numbers such that $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, and that its range is all real numbers. Also, the results for Example 6 in Section 7.2 led us to speculate that the period of the tangent function is π. This makes sense since the identity $\tan x = \frac{\sin x}{\cos x}$ informs us that $\tan x$ will be zero whenever $\sin x = 0$, which occurs at values of x that differ by π (visualize arcs on the unit circle whose terminal points are either $(1, 0)$ or $(-1, 0)$). The values of x for which $\cos x = 0$ cause $\tan x$ to be undefined ('gaps' in the domain) also differ by π (the points $(0, 1)$ or $(0, -1)$ on the unit circle). As x approaches these values where $\cos x = 0$, the value of $\tan x$ will become very large – either very large negative or very large positive. Thus, the graph of $y = \tan x$ has vertical asymptotes at $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Consequently, the graphical behaviour of the tangent function will not be a wave pattern such as that produced by the sine and cosine functions, but rather a series of separate curves that repeat every π units. Figure 7.25 shows the graph of $y = \tan x$ for $-2\pi \leqslant x \leqslant 2\pi$.



The graph gives clear confirmation that the period of the tangent function is π, that is, $\tan x = \tan(x + k \cdot \pi)$, $k \in \mathbb{Z}$.

The graph of $y = \tan x$ has rotational symmetry about the origin – that is, it can be rotated one-half of a revolution about $(0, 0)$ and it remains the same. Hence, like the sine function, tangent is an odd function and $\tan(-x) = -\tan x$.

Figure 7.25 $y = \tan x$ for $-2\pi \leqslant x \leqslant 2\pi$.

Although the graph of $y = \tan x$ can undergo a vertical stretch or shrink, it is meaningless to consider its amplitude since the tangent function has no maximum or minimum value. However, other transformations can affect the period of the tangent function.

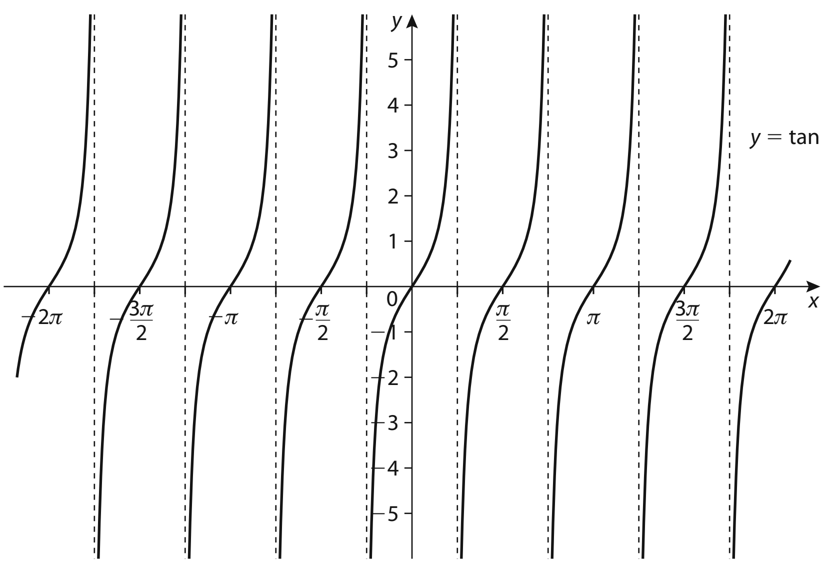
Example 14

Sketch each function.

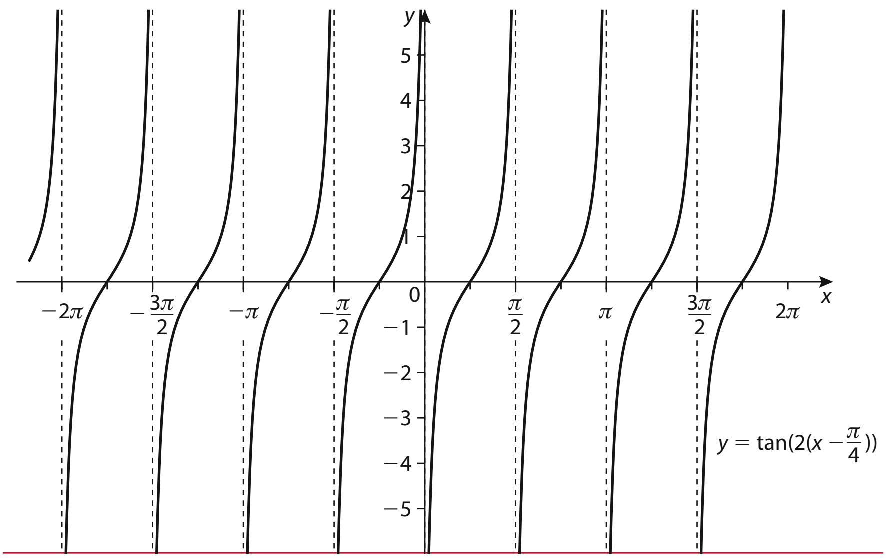
a) $f(x) = \tan 2x$ b) $g(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$

Solution

a) An equation in the form $y = f(bx)$ indicates a horizontal shrinking of $f(x)$ by a factor of $\frac{1}{b}$. Hence, the period of $y = \tan 2x$ is $\frac{1}{2} \cdot \pi = \frac{\pi}{2}$.



b) The graph of $y = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$ is obtained by first performing a horizontal shrinking of the graph of $y = \tan x$ by a factor of $\frac{1}{2}$ and then translating the graph to the right $\frac{\pi}{4}$ units. As for $f(x) = \tan 2x$ in part a), the period of $g(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$ is $\frac{\pi}{2}$.



Exercise 7.3

In questions 1–9, without using your GDC, sketch a graph of each equation on the interval $-\pi \leqslant x \leqslant 3\pi$.

1 $y = 2\sin x$

2 $y = \cos x - 2$

3 $y = \frac{1}{2}\cos x$

4 $y = \sin\left(x - \frac{\pi}{2}\right)$

5 $y = \cos(2x)$

6 $y = 1 + \tan x$

7 $y = \sin\left(\frac{x}{2}\right)$

8 $y = \tan\left(x + \frac{\pi}{2}\right)$

9 $y = \cos\left(2x - \frac{\pi}{4}\right)$

For each function in questions 10–12:

a) Sketch the function for the interval $-\pi \leqslant x \leqslant 5\pi$. Write down its amplitude and period.

b) Determine the domain and range for $f(x)$.

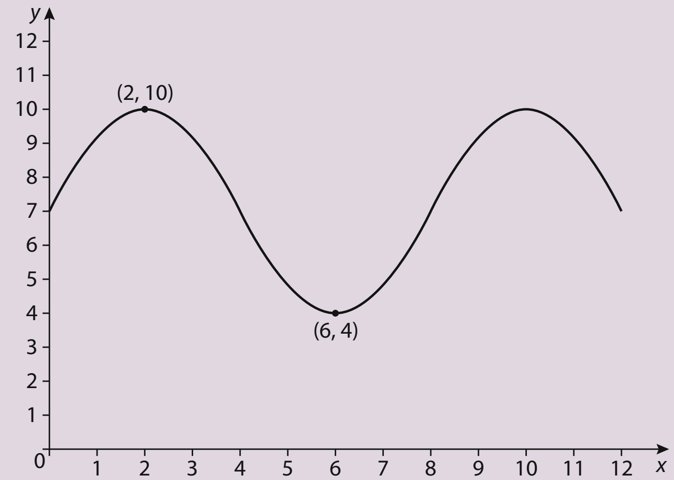
10 $f(x) = \frac{1}{2}\cos x - 3$

11 $g(x) = 3\sin(3x) - \frac{1}{2}$

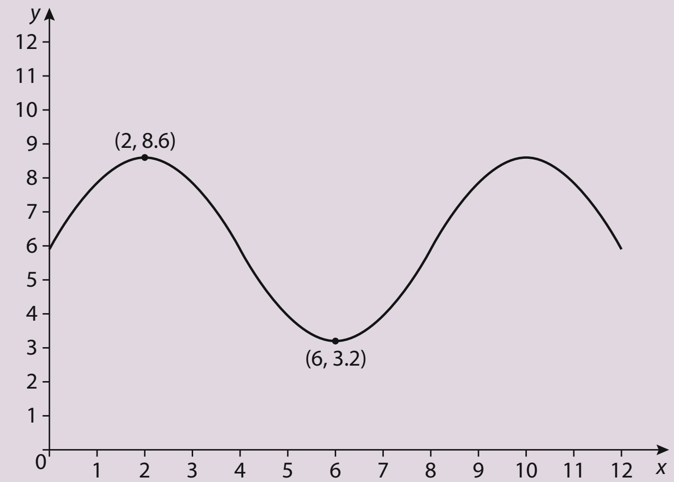
12 $g(x) = 1.2\sin\left(\frac{x}{2}\right) + 4.3$

In questions 13 and 14, a graph of a trigonometric equation is shown, on the interval $0 \leqslant x \leqslant 12$, that can be written in the form $y = A\sin\left(\frac{\pi}{4}x\right) + B$. Two points – one a minimum and the other a maximum – are indicated on the graph. Find the value of A and B for each.

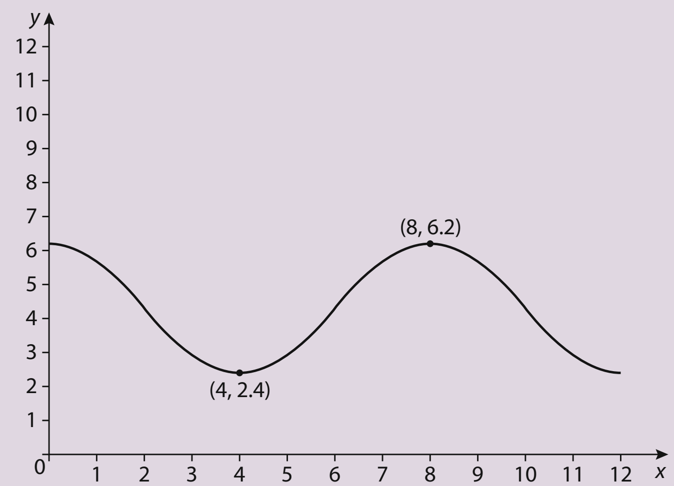
13



14



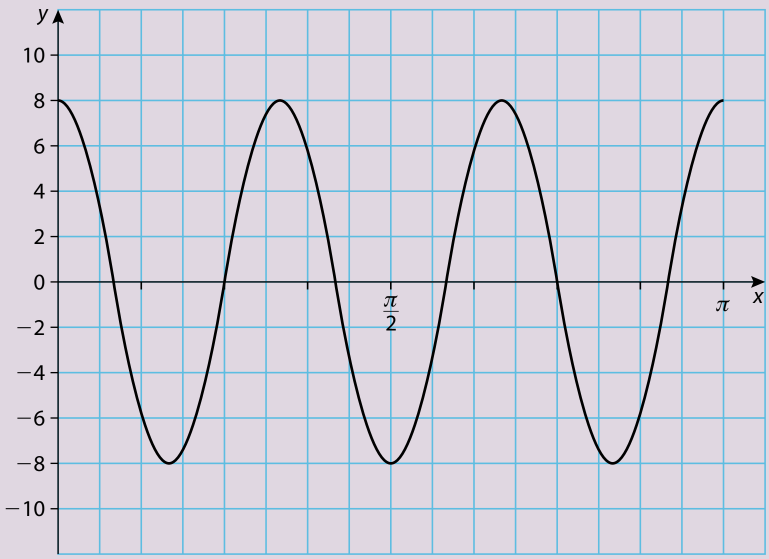
15 A graph of a trigonometric equation is shown below, on the interval $0 \leqslant x \leqslant 12$, that can be written in the form $y = A\cos\left(\frac{\pi}{4}x\right) + B$. Two points – one a minimum and the other a maximum – are indicated on the graph. Find the value of A and of B for each.



16 The graph of a function in the form $y = p\cos qx$ is given in the diagram below.

a) Write down the value of p.

b) Calculate the value of q.



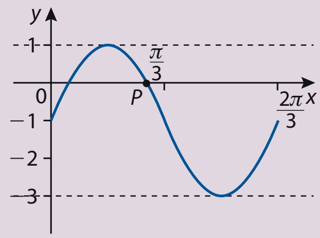
17 a) With help from your GDC, sketch the graphs of the three reciprocal trigonometric functions $y = \csc x$, $y = \sec x$ and $y = \cot x$ for the interval $0 \leqslant x \leqslant 2\pi$. Include any vertical asymptotes as dashed lines.

b) The domain of all of the trigonometric functions is stated in Section 7.2. State the range for each of the three reciprocal trigonometric functions.

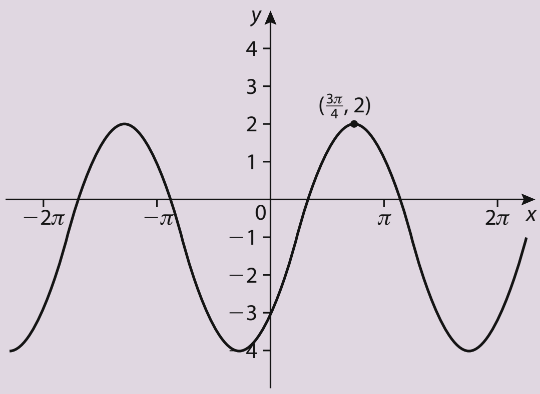
18 The diagram shows part of the graph of a function whose equation is in the form $y = a\sin(bx) + c$.

a) Write down the values of a, b and c.

b) Find the exact value of the x-coordinate of the point P, the point where the graph crosses the x-axis as shown in the diagram.



19 The graph below represents $y = a\sin(x + b) + c$, where a, b, and c are constants. Find values for a, b, and c.



7.4 Trigonometric equations

The primary focus of this section is to give an overview of concepts and strategies for solving **trigonometric equations**. In general, we will look at finding solutions by means of applying algebraic techniques (analytic solution) and/or by analyzing a graph (graphical solution). The following are all examples of trigonometric equations:

$$\csc x = 2, \sin^2\theta + \cos^2\theta = 1,\ 2\cos(3x - \pi) = 1,$$
$$\sec^2\alpha - 2\tan\alpha - 4 = 0, \tan 2\theta = \frac{2\tan}{1 - \tan^2\theta}$$

The mathematical symbol $\equiv$ is used to indicate that an equation has the special property of being an **identity**. It is not consistently used. You will notice that it is not used in the identities listed in the IB Information (Formulae) Booklet for Mathematics HL. The trigonometric identities required for this course are covered in the next section of this chapter.

The equations $\sin^2\theta + \cos^2\theta = 1$ and $\tan 2\theta = \frac{2\tan}{1 - \tan^2\theta}$ are examples of special equations called **identities** (Section 7.5). As we learned in Section 1.6, an identity is an equation that is true for all possible values of the variable. The other equations are true for only certain values or for none. Trigonometric identities will be covered thoroughly in the next section. They will prove to be an indispensable tool for obtaining analytic solutions to certain trigonometric equations. In this chapter, however, we will be applying methods similar to that used to solve equations encountered earlier in this book

The unit circle and exact solutions to trigonometric equations

When you are asked to solve a trigonometric equation, there are two important questions you need to consider:

1. Is it possible, or required, to express any solution(s) exactly?
2. For what interval of the variable are all solutions to be found?

With regard to the first question, exact solutions are only attainable, in most cases, if they are an integer multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Although we are primarily interested in finding numerical solutions (rather than angles in degrees), the language of angles is convenient. Recall from the first section of this chapter that if angles are given using radian measure, then angles between 0 and $\frac{\pi}{2}$ have their terminal sides in quadrant I, angles between $\frac{\pi}{2}$ and π have their terminal sides in quadrant II, and so on. Consequently, we will sometimes refer to a solution of an equation being, for example, a 'number in quadrant I', meaning a number that can be interpreted as either the length of an arc on the unit circle or a central angle in radian measure between 0 and $\frac{\pi}{2}$. As explained in Section 7.2, trigonometric domain values that are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ commonly occur and it is important to be familiar with the exact trigonometric function values for these numbers (Table 7.1).

Concerning the second question, for most trigonometric equations there are infinitely many solutions. For example, the solutions to the equation

$\sin x = \frac{1}{2}$ are any number (arc or central angle) in quadrants I or II positioned so that the terminal point on the unit circle has a y-coordinate of $\frac{1}{2}$ (Figure 7.26). There are an infinite set of numbers that do this, being $\frac{\pi}{6}$ plus any multiple of 2π (quadrant I) or $\frac{5\pi}{6}$ plus any multiple of 2π (quadrant II). This infinite set is concisely written as $x = \frac{\pi}{6} + k \cdot 2\pi$ or $x = \frac{5\pi}{6} + k \cdot 2\pi, k \in \mathbb{Z}$. However, for this course the number of solutions to any trigonometric equation will be limited to a finite set by the fact that the solution set will always be restricted to a specified interval. For the equation $\sin x = \frac{1}{2}$, if the solution set is restricted to the interval $0 \leqslant x < 2\pi$, then the solutions are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. If the solution set is restricted to the interval $-2\pi < x < 2\pi$, then the solutions are $-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}$ and $\frac{5\pi}{6}$. If the solution set is restricted to the interval $0 \leqslant x < 4\pi$, then the solutions are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ and $\frac{17\pi}{6}$. Figure 7.27 illustrates how the graph of $y = \sin x$ can be used to locate the solutions for the equation $\sin x = \frac{1}{2}$ for different intervals of x. When asked to solve a trigonometric equation, a solution interval will always be given, as in the example below.

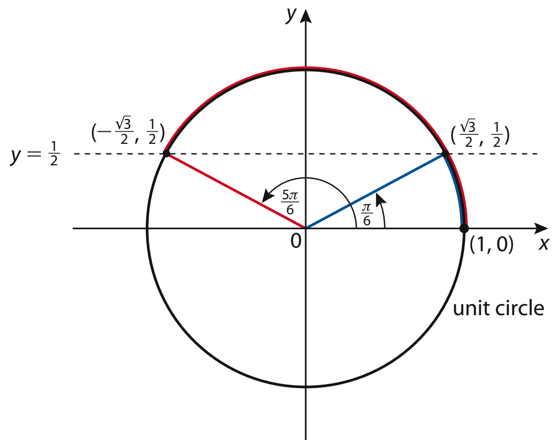


Figure 7.26 Solution to $\sin x = \frac{1}{2}$, $0 \leqslant x < 2\pi$.

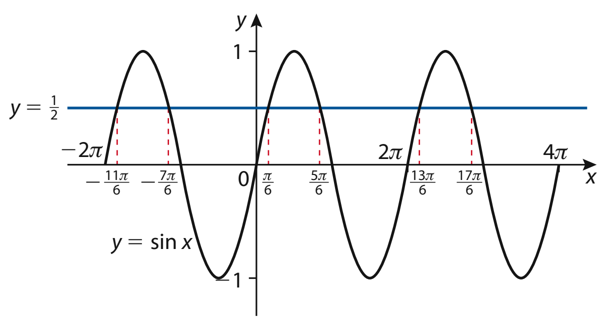


Figure 7.27 Points of intersection between $y = \sin x$ and $y = \frac{1}{2}$.

● **Hint:** As explained here, if the solution set for the equation $\sin x = \frac{1}{2}$ is not restricted, then the **general solution** is $x = \frac{\pi}{6} + k \cdot 2\pi$ or $x = \frac{5\pi}{6} + k \cdot 2\pi, k \in \mathbb{Z}$. This infinite solution corresponds to all of the points of intersection between the graphs of $y = \sin x$ and $y = \frac{1}{2}$ as they will repeatedly intersect as the graphs extend indefinitely in both directions (Figure 7.27). It is recommended that you are familiar with how to use a parameter (k in this case) to write the general solution for an equation with an infinite solution set, though it is not required for this course.

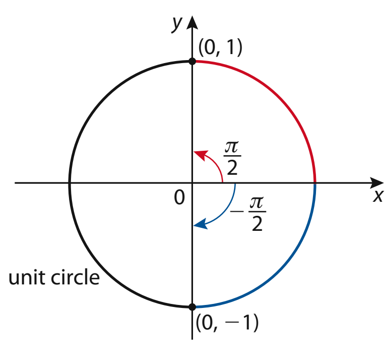
Example 15

Find the exact solution(s) to the equation $\sin x \cos x = 2\cos x$ for $-\pi < x < \pi$.

Solution

There is a temptation to divide both sides by $\cos x$, but as pointed out in Section 3.5, this can result in losing a solution to the equation. In fact, for this equation, both solutions would be lost. Instead, set the equation equal to zero and factorize out the common factor of $\cos x$.

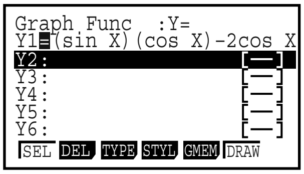
$$\sin x \cos x - 2\cos x = 0$$
$$\cos x(\sin x - 2) = 0$$
$$\cos x = 0 \quad \text{or} \quad \sin x = 2$$

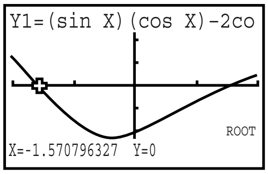


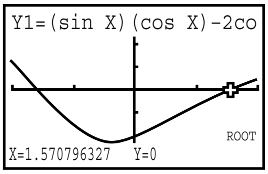
2 is outside the range of the sine function so there is no solution to $\sin x = 2$. Solutions to $\cos x = 0$ occur for arcs (angles) that terminate where the x-coordinate is 0. For the solution interval $-\pi < x < \pi$, this

occurs where the unit circle intersects the y-axis as shown in the diagram. Therefore this analytic solution gives the exact solutions of $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$.

Your GDC can be a very effective tool for searching for solutions graphically. However, it can be limited when exact solutions are requested. The sequence of GDC images below show a graphical solution for the equation in Example 15.







The GDC gives the two solutions in the interval $-\pi < x < \pi$ as $x = -1.570\,796\,327$ and $x = 1.570\,796\,327$. These values are approximations (to 10 significant figures) of the irrational numbers, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, and confirms that they are the correct solutions. If exact solutions are required then you need to first attempt an analytic solution, and then a graphical confirmation can be performed.

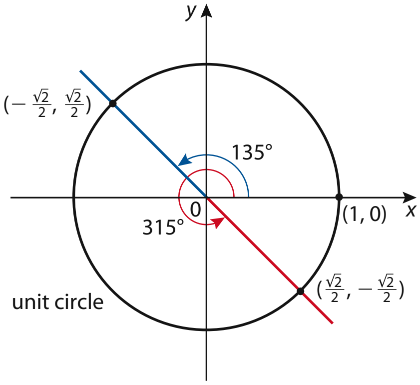
Example 16

Find the exact solution(s) to the equation $\tan(\theta) + 1 = 0$ for $0 \leqslant x < 360°$.

● **Hint:** The expression $\tan x + 1$ is not equivalent to $\tan(x + 1)$. In the first expression, x alone is the argument of the function, and in the second expression, $x + 1$ is the argument of the function. It is a good habit to use brackets to make it absolutely clear what is, or is not, the argument of a function. For example, there is no ambiguity if $\tan x + 1$ is written as $\tan(x) + 1$, or as $1 + \tan x$.

Solution

Since the solution interval is expressed in terms of degrees, it is necessary to give any solution as an angle in degree measure. Solutions to this equation are values of θ such that $\tan\theta = -1$. Applying the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have $\frac{\sin\theta}{\cos\theta} = -1$. We need to find any angles θ such that $\sin\theta$ and $\cos\theta$ have opposite signs. This occurs in quadrant II at $\theta = 135°$ and in quadrant IV at $\theta = 315°$ as shown in the diagram.



It is possible to arrive at exact answers that are not multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$, as the next example illustrates.

Example 17

Find the exact solution(s) to the equation $\cos^2\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ for $0 \leqslant x < 2\pi$.

Solution

The expression $\cos^2\left(x - \frac{\pi}{3}\right)$ can also be written as $\left[\cos\left(x - \frac{\pi}{3}\right)\right]^2$. The first step is to take the square root of both sides – remembering that every positive number has two square roots – which gives $\cos\left(x - \frac{\pi}{3}\right) = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. All of the odd integer multiples of $\frac{\pi}{4}\left(\ldots -\frac{3\pi}{4}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots\right)$ have a cosine equal to either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$.

That is, $x - \frac{\pi}{3} = \frac{\pi}{4} + k \cdot \frac{\pi}{2}$. Now, solve for x.

$x = \frac{\pi}{3} + \frac{\pi}{4} + k \cdot \frac{\pi}{2} = \frac{7\pi}{12} + k \cdot \frac{6\pi}{12}$. The last step is to substitute in different integer values for k to generate all the possible values for x so that $0 \leqslant x < 2\pi$.

When $k = 0$: $x = \frac{7\pi}{12}$; when $k = 1$: $x = \frac{7\pi}{12} + \frac{6\pi}{12} = \frac{13\pi}{12}$;

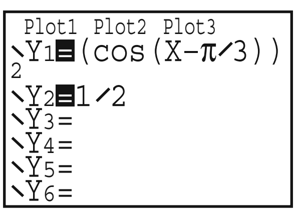
when $k = 2$: $x = \frac{7\pi}{12} + \frac{12\pi}{12} = \frac{19\pi}{12}$;

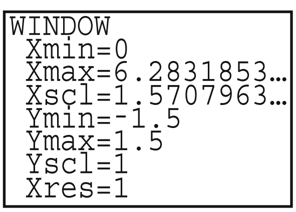
when $k = 3$: $x = \frac{7\pi}{12} + \frac{18\pi}{12} = \frac{25\pi}{12}$; however, $\frac{25\pi}{12} > 2\pi \ldots$ but,

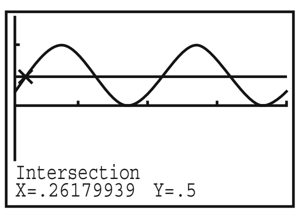
when $k = -1$: $x = \frac{7\pi}{12} - \frac{6\pi}{12} = \frac{\pi}{12}$.

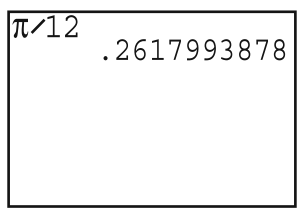
Therefore, there are four exact solutions in the interval $0 \leqslant x < 2\pi$, and they are: $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$ or $\frac{19\pi}{12}$.

• **Hint:** As we did at the end of Example 15, check the solutions to trigonometric equations with your GDC. The sequence of GDC images here verifies that $x = \frac{\pi}{12}$ is the first solution to the equation in Example 17.









When entering the equation $x = \cos^2\left(x - \frac{\pi}{3}\right)$ into your GDC (as shown in the first GDC image), you will have to enter it in the form $y = \left[\cos\left(x - \frac{\pi}{3}\right)\right]^2$. Be aware that $\cos^2\left(x - \frac{\pi}{3}\right)$ is not equivalent to $\cos\left(x - \frac{\pi}{3}\right)^2$. The expression $\cos\left(x - \frac{\pi}{3}\right)^2$ indicates that the quantity $x - \frac{\pi}{3}$ is squared first and then the cosine of the resulting value is found. However, the expression $y = \cos\left(x - \frac{\pi}{3}\right)$. indicates that the cosine of $x - \frac{\pi}{3}$ is found first and then that value is squared.

Graphical solutions to trigonometric equations

If exact solutions are not required then a graphical solution using your GDC is a very effective way to find approximate solutions to trigonometric equations. Unless instructed to do otherwise, you should give approximate solutions to an accuracy of three significant figures.

Example 18

Find all solutions to the equation $3\tan x = 2\cos x$ in the interval $0 \leqslant x < 2\pi$.

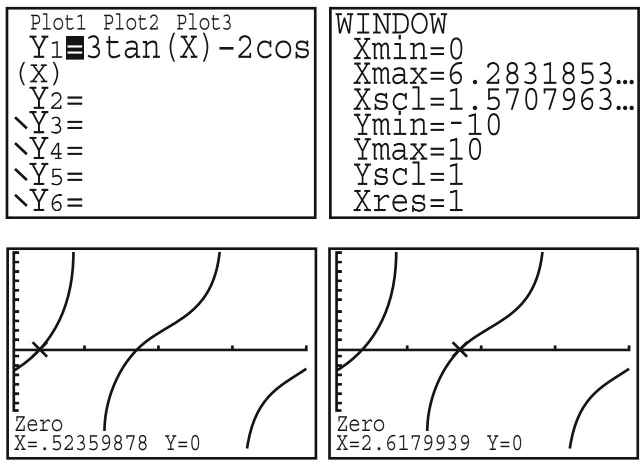
Solution

Graph the equation $y = 3\tan x - 2\cos x$ and find all of its zeros (x-intercepts) in the interval $0 \leqslant x < 2\pi$. Because the domain of the tangent function is $\left\{x : x \in \mathbb{R}, x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$, then we expect there to be 'gaps' (and vertical asymptotes) in the graph at $x = \frac{\pi}{2}$ and at $x = \frac{3\pi}{2}$.

It is possible to solve the equation in Example 18 analytically. See Exercise 7.4, question 30.

The exact solutions are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. The GDC image shows their approximate values agree with the solutions found in the example.

π/6	.5235987756
5π/6	2.617993878



This sequence of GDC images indicates approximate solutions of $x \approx 0.524$ and $x \approx 2.62$ to an accuracy of three significant figures.

A graphical approach is effective and appropriate when it is very difficult, or not possible, to find exact solutions.

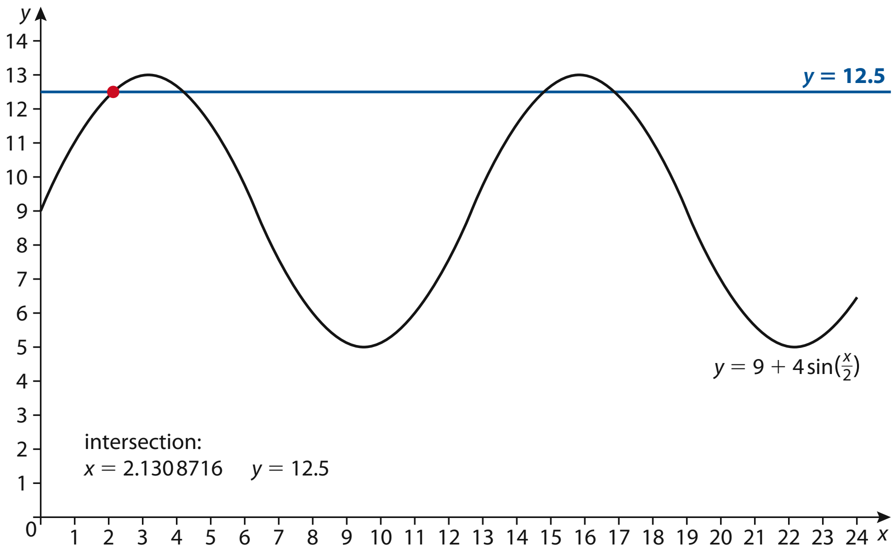
Example 19

The peak height, h metres, of ocean waves during a storm is given by the equation $h = 9 + 4\sin\left(\frac{t}{2}\right)$, where t is the number of hours after midnight.

A tsunami alarm is triggered when the peak height goes above 12.5 metres. Find the value of t when the alarm first sounds.

Solution

Graph the equations $y = 9 + 4\sin\left(\frac{x}{2}\right)$ and $y = 12.5$ and find the first point of intersection for $x > 0$.



Using the Intersect command on the GDC indicates that the first point of intersection has an x-coordinate of approximately 2.13. Therefore, the alarm will first sound when $t \approx 2.13$ hours.

Analytic solutions to trigonometric equations

An analytical approach requires you to devise a solution strategy utilizing algebraic methods that you have applied to other types of equations – such as quadratic equations. Trigonometric equations that demand an analytic approach will often, but not always, result in exact solutions. Although our approach for equations in this section focuses on algebraic techniques, it is important to use graphical methods to support or confirm our analytical solutions.

Example 20

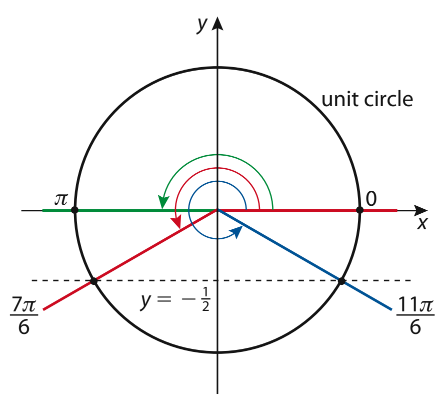
Solve $2\sin^2 x + \sin x = 0$ for $0 \leqslant x < 2\pi$.

Solution

Factorizing gives $\quad \sin x(2\sin x + 1) = 0$

$$\sin x = 0 \text{ or } \sin x = -\tfrac{1}{2}$$

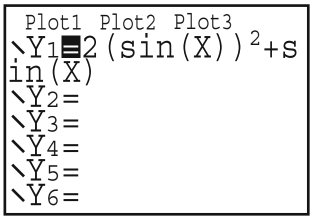
Solutions to $\sin x = 0$ are where the angle is on the x-axis; and solutions to $\sin x = -\frac{1}{2}$ are angles in quadrant III and IV such that their intersection point with the unit circle has y-coordinate of $-\frac{1}{2}$.

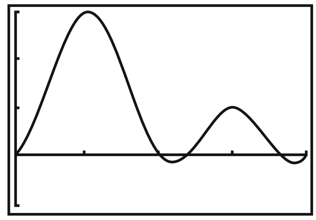


$$\text{for } \sin x = 0\text{: } x = 0, \pi \qquad \text{for } \sin x = -\tfrac{1}{2}\text{: } x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Therefore, the solutions are $x = 0, \pi, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

● **Hint:** Although exact answers were not demanded in Example 20, given our knowledge of the unit circle and familiarity with the sine of common values $\left(\text{i.e. multiples of } \frac{\pi}{6} \text{ and } \frac{\pi}{4}\right)$, we are able to give exact answers without any difficulty. It would have been acceptable to give approximate solutions using your GDC, but it is worth recognizing that this would have required considerable more effort than providing exact solutions. Entering and graphing the equation $y = 2\sin^2 x + \sin x$ on your GDC (see GDC images) would not be the most efficient or appropriate solution method, but if sufficient time is available it is an effective way to confirm your exact solutions. [Note that $\sin^2 x$ must be entered in a GDC as $(\sin x)^2$.]





● **Hint:** As we will see in the next section, it is often the case that an analytic solution is not possible unless a substitution is made using a suitable trigonometric identity.

The next example illustrates how the application of a trigonometric identity can be helpful to rewrite the equation in a way that allows us to solve it algebraically. The next section will introduce many further trigonometric identities and examples of using them to assist in solving trigonometric equations.

Example 21

Solve $3\cos x + \cot x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Solution

Since the structure of this equation is such that an expression is set equal to zero, it would be nice to be able to use the same algebraic technique as the previous example – that is, factorize and solve for when each factor is zero. However, it is not possible to factorize the expression $3\cos x + \cot x$, and rewriting the equation as $3\cos x = -\cot x$ does not help. Are there any expressions in the equation for which we can substitute an equivalent expression that will make the equation accessible to an algebraic solution? We do not have any equivalent expressions for $\cos x$, but we do have an identity for $\cot x$. Since $\cot x$ is the reciprocal of $\tan x$ we know that $\cot x = \dfrac{\cos x}{\sin x}$. Let's see what happens when we substitute $\dfrac{\cos x}{\sin x}$ for $\cot x$.

$$3\cos x + \frac{\cos x}{\sin x} = 0$$

Now, get a common denominator.

$$\frac{3\sin x\cos x}{\sin x} + \frac{\cos x}{\sin x} = 0$$

$$\frac{3\sin x\cos x + \cos x}{\sin x} = 0$$

Noting that $\sin x \neq 0$, multiply both sides by $\sin x$. A fraction equals zero when the denominator equals zero.

$$3\sin x\cos x + \cos x = 0$$

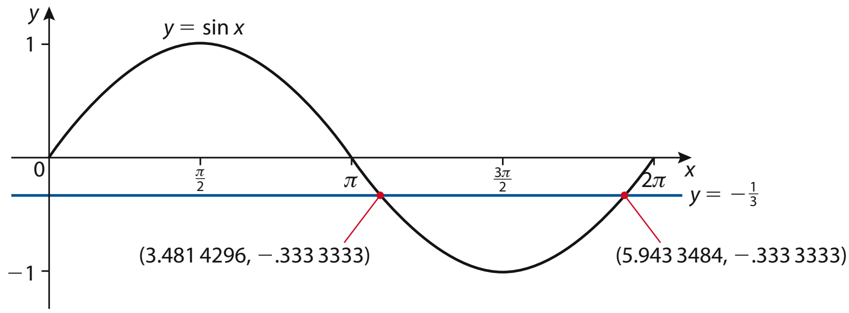
$$\cos x(3\sin x + 1) = 0$$

Factorize.

$$\cos x = 0 \quad \text{or} \quad \sin x = -\tfrac{1}{3}$$

$$\text{For } \cos x = 0\text{: } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

We know that solutions to $\cos x = 0$ are angles on the y-axis giving the two exact solutions of $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Although we know solutions to $\sin x = -\frac{1}{3}$ are angles in quadrants III and IV, we do not know their exact values. So, we will need to use our GDC to find approximate solutions to $\sin x = -\frac{1}{3}$ for $0 \leqslant x \leqslant 2\pi$.



Thus, for $\sin x = -\frac{1}{3}$: $x \approx 3.48$ or $x \approx 5.94$ (3 significant figures)

Therefore, the full solution set for the equation is $x = \frac{\pi}{2}, \frac{3\pi}{2}$; $x \approx 3.48, 5.94$.

● **Hint:** A strategy that often proves fruitful is to try and rewrite a trigonometric equation in terms of just one trigonometric function. If that is not possible, then try and rewrite it in terms of only the sine and cosine functions. This strategy was used in Example 21.

Exercise 7.4

In questions 1–12, find the exact solution(s) for $0 \leqslant x < 2\pi$. Verify your solution(s) with your GDC.

1 $\cos x = \frac{1}{2}$

2 $2\sin x + 1 = 0$

3 $1 - \tan x = 0$

4 $\sqrt{3} = 2\sin x$

5 $2\sin^2 x = 1$

6 $4\cos^2 x = 3$

7 $\tan^2 x - 1 = 0$

8 $4\cos^2 x = 1$

9 $\tan x(\tan x + 1) = 0$

10 $\sin x \cos x = 0$

11 $5 - \sec x = 3$

12 $\csc^2 x = 2$

In questions 13–20, use your GDC to find approximate solution(s) for $0 \leqslant x < 2\pi$. Express solutions accurate to 3 significant figures.

13 $\sin x = 0.4$

14 $3\cos x + 1 = 0$

15 $\tan x = 2$

16 $\sec 2x = 3.46$

17 $\cos(x - 1) = -0.38$

18 $3\tan^2 x = 1$

19 $\csc(2x - 3) = \frac{3}{2}$

20 $3\cot x = 10$

In questions 21–24, given that k is any integer, list all of the possible values for x that are in the specified interval.

21 $\frac{\pi}{2} + k \cdot \pi$, $-3\pi \leqslant x \leqslant 3\pi$

22 $\frac{\pi}{6} + k \cdot 2\pi$, $-2\pi \leqslant x \leqslant 2\pi$

23 $\frac{7\pi}{12} + k \cdot \pi$, $0 \leqslant x < 2\pi$

24 $\frac{\pi}{4} + k \cdot \frac{\pi}{4}$, $0 \leqslant x < 4\pi$

In questions 25–32, find the **exact** solutions for the indicated interval. The interval will also indicate whether the solutions are given in degree or radian measure. Write a complete analytic solution.

25 $\cos\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}, 0 \leqslant x < 2\pi$

26 $\tan(\theta + \pi) = 1, -\pi \leqslant \theta \leqslant \pi$

27 $\sin 2x = \frac{\sqrt{3}}{2}, 0 \leqslant x < 360°$

28 $\sin^2\left(\alpha + \frac{\pi}{2}\right) = \frac{3}{4}, -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$

29 $2\cos^2\theta - 5\cos\theta - 3 = 0, 0 \leqslant \theta < 2\pi$

30 $3\tan x = 2\cos x, 0 \leqslant x < 2\pi$

31 $2\cos(x + 90°) = \sqrt{2}, 0 \leqslant x < 360°$

32 $9\sec^2\theta = 12, 0 \leqslant \theta < \pi$

33 The number, N, of empty birds' nests in a park is approximated by the function $N = 74 + 42\sin\left(\frac{\pi}{12}t\right)$, where t is the number of hours after midnight. Find the value of t when the number of empty nests first equals 90. Approximate the answer to 1 decimal place.

34 In Edinburgh, the number of hours of daylight on day D is modelled by the function $H = 12 + 7.26\sin\left[\frac{2\pi}{365}(D - 80)\right]$, where D is the number of days after December 31 (e.g. January 1 is $D = 1$, January 2 is $D = 2$, and so on). Do not use your GDC on part a).

a) Which days of the year have 12 hours of daylight?

b) Which days of the year have about 15 hours of daylight?

c) How many days of the year have more than 17 hours of daylight?

In questions 35–42, solve the equation for the stated solution interval. Find exact solutions when possible, otherwise give solutions to three significant figures. Verify solutions with your GDC.

35 $2\cos^2 x + \cos x = 0, 0 \leqslant x < 2\pi$

36 $2\sin^2\theta - \sin\theta - 1 = 0, 0 \leqslant \theta < 2\pi$

37 $\tan^2 x - \tan x = 2, -90° \leqslant x \leqslant 90°$

38 $3\cos^2 x - 6\cos x = 2, -\pi < x \leqslant \pi$

39 $2\sin\beta = 3\cos\beta, 0 \leqslant \beta \leqslant 180°$

40 $\sin^2 x = \cos^2 x, 0 \leqslant x \leqslant \pi$

41 $\sec^2 x + 2\sec x + 4 = 0, 0 \leqslant x < 2\pi$

42 $\sin x \tan x = 3\sin x, 0 \leqslant x < 360°$

7.5 Trigonometric identities

The **co-function identities** for sine and cosine were established in Section 7.3 by means of investigating horizontal shifts of graphs of the sine and cosine functions. Similarly we can prove co-function identities for secant and cosecant, and for tangent and cotangent. These appear in Table 7.2 on the next page.

You will recall that an identity is an equation that is true for all values of the variable for which the expressions in the equation are defined. Several trigonometric identities have been introduced earlier in this chapter. They are reviewed here (Table 7.2) and a number of important new identities are presented and proved in this section.

Trigonometric identities are used in a variety of ways. For example, one of the reciprocal identities is applied whenever the cosecant, secant or cotangent function is evaluated on a calculator. The following uses of trigonometric identities will be illustrated in this section.

1. Evaluate trigonometric functions.

2. Simplify trigonometric expressions.
3. Prove other trigonometric identities.
4. Solve trigonometric equations.

The first portion of this section is devoted to developing some further trigonometric identities that are organized into three groups: Pythagorean identities, compound angle identities, and double angle identities.

Reciprocal identities:		
$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$	$\cot x = \frac{1}{\tan x}$
Tangent and cotangent identities:		
$\tan x = \frac{\sin x}{\cos x}$	$\cot x = \frac{\cos x}{\sin x}$	
Odd/even function identities:		
$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$	$\sec(-x) = \cos x$	$\cot(-x) = -\tan x$
Co-function identities:		
$\sin\left(\frac{\pi}{2} - x\right) = \cos x$	$\sec\left(\frac{\pi}{2} - x\right) = \csc x$	$\tan\left(\frac{\pi}{2} - x\right) = \cot x$
$\cos\left(\frac{\pi}{2} - x\right) = \sin x$	$\csc\left(\frac{\pi}{2} - x\right) = \sec x$	$\cot\left(\frac{\pi}{2} - x\right) = \tan x$

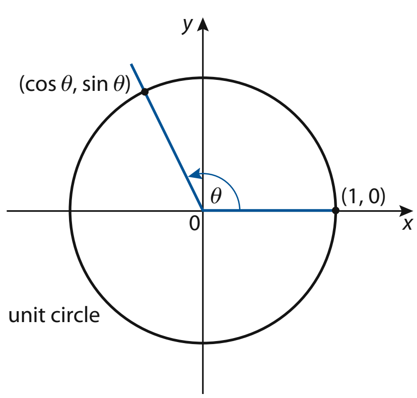
Table 7.2 Summary of fundamental trigonometric identities.

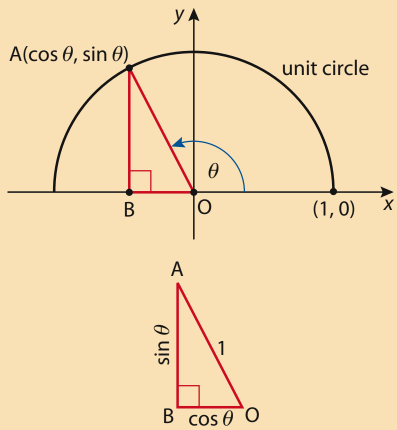
It was confirmed in Section 7.3 that sine and tangent are **odd functions** and that cosine is an **even function**. We will accept without proof that if a function is odd, then its reciprocal is also odd; and the same is true for even functions. Therefore, cosecant and cotangent are odd functions, and secant is an even function.

Pythagorean identities

At the start of the previous section, it was stated that the equation $\sin^2\theta + \cos^2\theta = 1$ is an identity; that is, it's true for all possible values of θ. Let's prove that this is the case.

Recall from Section 7.1 that the equation for the unit circle is $x^2 + y^2 = 1$. That is, the coordinates (x, y) of any point on the circle satisfy the equation $x^2 + y^2 = 1$. As we learned in Section 7.2, if θ is any real number that represents a central angle (in radian measure) of the unit circle that terminates at (x, y), then $x = \cos\theta$ and $y = \sin\theta$. Substituting directly into the equation for the circle gives $\sin^2\theta + \cos^2\theta = 1$. Therefore, the equation $\sin^2\theta + \cos^2\theta = 1$ is true for any real number x.





The identity $\sin^2\theta + \cos^2\theta = 1$ is referred to as a *Pythagorean* identity because it can be derived directly from Pythagoras' theorem. As Figure 7.28 illustrates, for any point angle θ with its terminal side intersecting the unit circle at point **A** (except for a point on the x- or y-axis), a perpendicular segment can be drawn to a point **B** on the y-axis thereby constructing right triangle **ABO**. Side **AB** is equal to $\sin\theta$ and side **OB** is equal to $\cos\theta$. The hypotenuse **AO** is a radius of the unit circle so its length is one. Hence, by Pythagoras' theorem: $\sin^2\theta + \cos^2\theta = 1$.

Figure 7.28

Hint: Graph the equation $y = \sin^2 x + \cos^2 x$ on your GDC with the y-axis ranging from -2 to 2 and the x-axis ranging from -2π to 2π (radian mode) or $-360°$ to $360°$ (degree mode). What do you observe?

Phrases such as 'prove the identity' and 'verify the identity' are often used. Both mean, 'prove that the given equation is an identity'. We do this by performing a series of algebraic manipulations to show that the expression on one side of the equation can be transformed into the expression on the other side, or that both expressions can be transformed into some third expression. When verifying that an equation is an identity, you should not perform an operation to both sides of the equation; for example, multiplying both sides of the equation by a quantity. This can only be done if it is known that the two sides of the equation are equal, but this is exactly what we are trying to verify in the process of 'proving an identity.'

Example 22

Prove that $1 + \tan^2\theta = \sec^2\theta$ is an identity.

Solution

There is more of an opportunity to perform algebraic manipulations on the left side than the right side. Thus, our task is to transform the expression $1 + \tan^2\theta$ into the expression $\sec^2\theta$.

$$1 + \tan^2\theta = \sec^2\theta$$

Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ substitute $\frac{\sin^2\theta}{\cos^2\theta}$ for $\tan^2\theta$.

$$1 + \frac{\sin^2\theta}{\cos^2\theta} =$$

Find a common denominator.

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} =$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} =$$

Apply the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

$$\frac{1}{\cos^2\theta} =$$

Because $\frac{1}{\cos\theta} = \sec\theta$, then $\frac{1}{\cos^2\theta} = \sec^2\theta$.

$$\sec^2\theta = \sec^2\theta$$

Q.E.D.

Q.E.D. is an abbreviation for the Latin phrase '*quod erat demonstrandum*' which means 'that which was to be proved (or demonstrated)'. It is often written at the end of a proof to indicate that its conclusion has been reached.

Another identity than can be proved in a manner similar to the identity in Example 22 is $1 + \cot^2\theta = \csc^2\theta$.

Pythagorean identities

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

The Pythagorean identities are sometimes used in radical forms such as $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ or $\tan\theta = \pm\sqrt{\sec^2\theta - 1}$ where the sign (+ or −) depends on θ (which quadrant it is in).

Example 23

a) Express $2\cos^2 x + \sin x$ in terms of $\sin x$ only.

b) Solve the equation $2\cos^2 x + \sin x = -1$ for x in the interval $0 \leqslant x \leqslant 2\pi$, expressing your answer(s) exactly.

Solution

a) $2\cos^2 x + \sin x = 2(1 - \sin^2 x) + \sin x$ $\quad$ Using Pythagorean identity:
$= 2 - 2\sin^2 x + \sin x$ $\quad$ $\cos^2 x = 1 - \sin^2 x$.

b) $2\cos^2 x + \sin x = -1$

$2 - 2\sin^2 x + \sin x = -1$ $\quad$ Substitute result from a).

$2\sin^2 x - \sin x - 3 = 0$ $\quad$ (Alternatively: let $\sin x = y$, then $2y^2 - y - 3 = 0$)

$(2\sin x - 3)(\sin x + 1) = 0$ $\quad$ Factorize. (alt: $(2y - 3)(y + 1) = 0$)

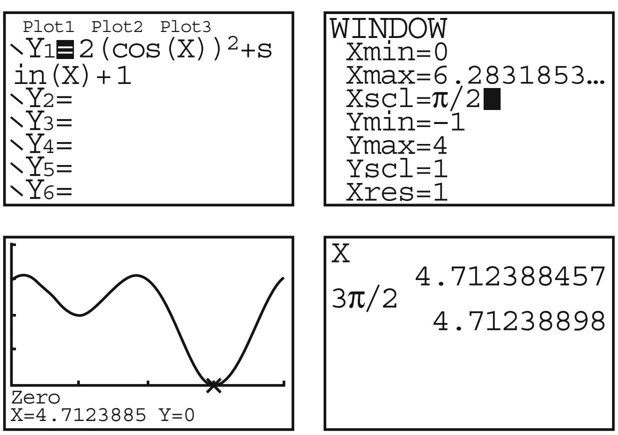
$\sin x = \frac{3}{2}$ or $\sin x = -1$ $\quad$ (Alt: $y = \frac{3}{2}$ or $y = -1 \Rightarrow \sin x = \frac{3}{2}$ or $\sin x = -1$)

For $x = \frac{3}{2}$: no solution because $\frac{3}{2}$ is not in the range of the sine function.

For $\sin x = -1$: $x = \dfrac{3\pi}{2}$.

Therefore, there is only one solution in $0 \leqslant x \leqslant 2\pi$: $x = \dfrac{3\pi}{2}$.

Use your GDC to check this result by rewriting $2\cos^2 x + \sin x = -1$ as $2\cos^2 x + \sin x + 1 = 0$ and then graph $y = 2\cos^2 x + \sin x + 1$; confirming a single zero at $x = \dfrac{3\pi}{2}$ in the interval $x \in [0, 2\pi]$.



Compound angle identities (sum and difference identities)

In this section we develop trigonometric identities known as the compound angle identities for sine, cosine and tangent. These contain the expressions $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$, $\cos(\alpha-\beta)$, $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$. We first find a formula for $\cos(\alpha+\beta)$.

On first reaction you might wonder whether $\cos(\alpha+\beta)=\cos\alpha+\cos\beta$. Often it is easier to prove a mathematical statement false than to prove it true. One counter-example is sufficient to prove a statement false. Let $\alpha=\frac{\pi}{3}$ and $\beta=\frac{\pi}{6}$. Does $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\frac{\pi}{3}+\cos\frac{\pi}{6}$?

• **Hint:** As will occur in Chapter 8, Greek letters such as α (alpha), β (beta), or θ (theta) are frequently used to name angles. In the development of the formula for $\cos(\alpha+\beta)$, α and β are arcs along the unit circle, but they could just as well be representing the central angle (in radian measure) that cuts off (subtends) the arc.

$\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\left(\frac{2\pi}{6}+\frac{\pi}{6}\right)=\left(\frac{3\pi}{6}\right)=\cos\left(\frac{\pi}{2}\right)=0$

and $\cos\frac{\pi}{3}+\cos\frac{\pi}{6}=\frac{1}{2}+\frac{\sqrt{3}}{2}=\frac{1+\sqrt{3}}{2}$.

Thus, the answer is 'no'; $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\neq\cos\frac{\pi}{3}+\cos\frac{\pi}{6}$.

Although $\cos(\alpha+\beta)=\cos\alpha+\cos\beta$ may be true for some values (e.g. it's true for $\alpha=\frac{\pi}{2}$ and $\beta=\frac{3\pi}{4}$), it's not true for **all** possible values of α and β, and therefore, it is **not** an identity.

Derivation of identity for the cosine of the sum of two numbers

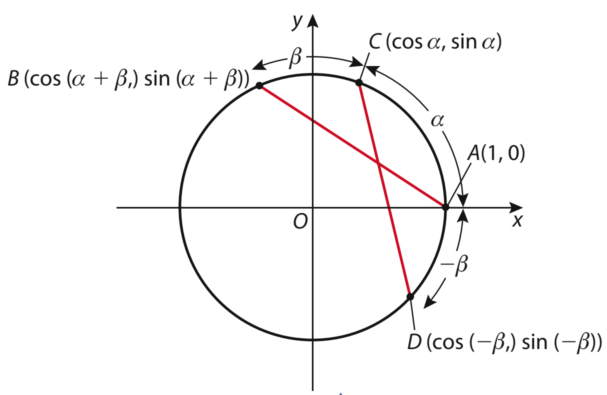


Figure 7.29

To find a formula for $\cos(\alpha+\beta)$, we use Figure 7.29 showing the four points A, B, C and D on the unit circle and the two chords AB and CD. The arc lengths α, β and $-\beta$ have been marked. The coordinates of A, B, C and D in terms of sines and cosines of the arcs are also indicated. The coordinates of point D are $(\cos(-\beta), \sin(-\beta))$, but we can apply the odd/even identities to write the coordinates of D more simply as $(\cos\beta, -\sin\beta)$. Observe that the arc length from A to B is equal to the arc length from D to C because they both have a length equal to $\alpha+\beta$. Since equal arcs on a circle determine equal chords, it must follow that $AB=CD$. Using the respective coordinates for A, B, C and D, we can express $AB=CD$ using the distance formula as

$$\sqrt{(\cos(\alpha+\beta)-1)^2+\sin^2(\alpha+\beta)}=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha+\sin\beta)^2}$$

Squaring both sides and expanding, gives

$$\cos^2(\alpha+\beta)-2\cos(\alpha+\beta)+1+\sin^2(\alpha+\beta)$$
$$=\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha+2\sin\alpha\sin\beta+\sin^2\beta$$
$$[\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)]-2\cos(\alpha+\beta)+1$$
$$=(\cos^2\alpha+\sin^2\alpha)+(\sin^2\beta+\cos^2\beta)-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$

Applying the Pythagorean identity $\sin^2\theta+\cos^2\theta=1$, we can replace three expressions with 1:

$$1-2\cos(\alpha+\beta)+1=1+1-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$

Subtracting 2 from each side and dividing both sides by -2, gives

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This is the **identity for the cosine of the sum of two numbers**.

Previously we were only able to find exact values of a trigonometric function for certain 'special' numbers, i.e. multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

Example 24 – Using the sum identity for cosine

Find the exact values for a) $\cos\frac{5\pi}{12}$, and b) $\cos 75°$.

Solution

a) $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$

Applying the identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{6}$, gives $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Therefore, $\cos\frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$.

Derivation of identity for the cosine of the difference of two numbers

We can use the identity for the cosine of the sum of two numbers and the fact that cosine is an even function and sine is an odd function to derive the formula for $\cos(\alpha + \beta)$.

Let's replace β with $-\beta$ in $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\cos[\alpha + (-\beta)] = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

Substituting $-\sin\beta$ for $\sin(-\beta)$, and $\cos\beta$ for $\cos(-\beta)$, gives

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

This is the **identity for the cosine of the difference of two numbers**.

Example 25 – Using the sum and difference identities for cosine

Given that A and B are numbers representing arcs or angles that are in the first quadrant, and $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, find the exact values of a) $\cos(A + B)$ and b) $\cos(A - B)$.

Solution

We are given the exact values for $\sin A$ and $\cos B$, but we also need exact values for $\sin B$ and $\cos A$ in order to use the sum and difference identities for cosine.

Since B is in the first quadrant then $B > 0$ and re-arranging one of the Pythagorean identities, we have

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169}} = \frac{5}{13}.$$

Similarly, $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$

a) Substituting into the identity for the cosine of the sum of two numbers, gives

$$\cos(A + B) = \cos A \cos B - \sin A \sin B = \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) = \frac{16}{65}.$$

Therefore, $\cos(A + B) = \frac{16}{65}.$

b) Substituting into the identity for the cosine of the difference of two numbers, gives

$$\cos(A - B) = \cos A \cos B + \sin A \sin B = \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) = \frac{56}{65}.$$

Therefore, $\cos(A - B) = \frac{56}{65}.$

• **Hint:** Notice that in Example 25, we obtained $\cos(A + B)$ and $\cos(A - B)$ without finding the actual values of A and B.

Derivation of identities for the sine of the sum/difference of two numbers

The identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ can be used to derive an identity for $\sin(\alpha + \beta)$. Substituting $\frac{\pi}{2}$ for α and $(\alpha + \beta)$ for β, gives

$$\cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$
$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$

Now using the co-function identities $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, we have,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

This is the **identity for the sine of the sum of two numbers.**

By replacing β with $-\beta$, in the identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, we get

$$\sin(\alpha - \beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

Applying the odd/even identities for $\cos(-\beta)$ and $\sin(-\beta)$, produces

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

This is the **identity for the sine of the difference of two numbers.**

Derivation of identities for the tangent of the sum/difference of two numbers

To produce an identity for $\sin(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$, we start with the fundamental identity that the tangent is the quotient of sine and cosine. We have

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \quad \text{given } \cos(\alpha + \beta) \neq 0$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

So that the identity involves $\tan\alpha$ and $\tan\beta$, we divide the numerator and denominator by $\cos\alpha\cos\beta$, with the assumption that $\cos\alpha\cos\beta \neq 0$.

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

This is the **identity for the tangent of the sum of two numbers.**

If in this identity β is replaced with $-\beta$, we get

$$\tan[\alpha + (-\beta)] = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$

Tangent is an odd function, so $\tan(-\beta) = -\tan\beta$. Making this substitution, gives

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

This is the **identity for the tangent of the difference of two numbers.**

Compound angle identities

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

● **Hint:** The compound angle identities are also referred to as the 'sum and difference identities', or the 'addition and subtraction identities'.

Example 26 – Using the sum identity for tangent

If $\tan(A + B) = \frac{1}{7}$ and $\tan A = 3$, find the value of $\tan B$.

Solution

Using the identity for the tangent of the sum of two numbers, we write

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Substituting $\frac{1}{7}$ for $\tan(A + B)$, and 3 for $\tan A$.

$$\frac{1}{7} = \frac{3 + \tan B}{1 - 3\tan B}$$

Cross-multiply and solve for $\tan B$.

$$21 + 7\tan B = 1 - 3\tan B$$

$$10\tan B = -20$$

$$\tan B = -2$$

Note that, similar to Example 25, we found the exact value of $\tan B$ without finding the actual value of B. In fact, we're not even certain which quadrant B is in, only that it must be in either quadrant II or IV since $\tan B < 0$.

Double angle identities

Is $\sin 2\theta = 2\sin\theta$ an identity? Clearly, it is not – as the counter-example $\theta = \frac{\pi}{6}$ shows.

$$\sin\left(2\cdot\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \text{ and } 2\sin\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$$

A direct consequence of the compound angle identities developed in the past few pages are formulas for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, that is, **double angle identities**. For example, the formula for $\sin 2\theta$ can be derived by taking the identity for the sine of two numbers and by letting $\alpha = \beta = \theta$.

$$\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

Similarly, for $\cos 2\theta$ we have,

$$\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

By applying the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can write the double angle identity for $\cos 2\theta$ in two other useful ways.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

● **Hint:** The double angle identity for the tangent function does not hold if $\theta = \frac{\pi}{4} + k\cdot\frac{\pi}{2}$, where k is any integer, because for these values of θ the denominator is zero. The identity also does not hold if $\theta = \frac{\pi}{2} + k\cdot\pi$, where k is any integer, because for these values $\tan\theta$ does not exist. Nevertheless, the equation is still an identity because it is true for all values of θ for which both sides are defined.

To derive the formula for expressing $\tan 2\theta$ in terms of $\tan\theta$, we take the same approach and start with the identity for the tangent of the sum of two numbers and let $\alpha = \beta = \theta$.

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}$$

We now have a useful set of identities for the sine, cosine and tangent of twice an angle (or number).

Double angle identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Now let's look at some further applications of the trigonometric identities we have established, especially for solving more sophisticated equations.

Example 27

Solve the equation $\cos 2x + \cos x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Solution

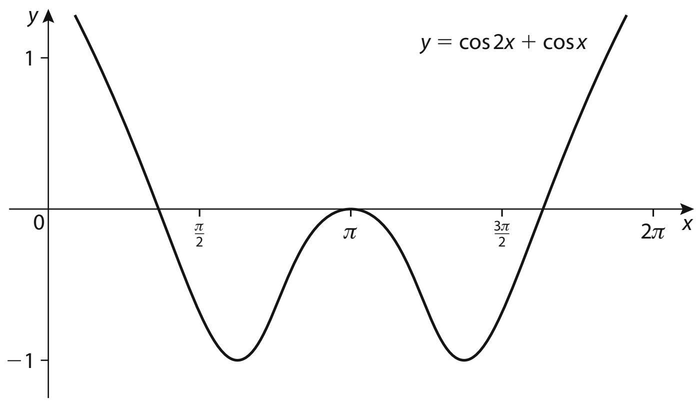
Taking an initial look at the graph of $y = \cos 2x + \cos x$ suggests that there are possibly three solutions in the interval $x \in [0, 2\pi]$. Although the expression $\cos 2x + \cos x$ contains terms with only the cosine function, it is not possible to perform any algebraic operations on them because they have different arguments. In order to solve algebraically, we need both cosine functions to have arguments of x (rather than $2x$). There are three different double angle identities for $\cos 2x$. It is best to have the equation in terms of one trigonometric function, so we choose to substitute $2\cos^2 x - 1$ for $\cos 2x$.

$\cos 2x + \cos x = 0 \Rightarrow 2\cos^2 x - 1 + \cos x = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0 \Rightarrow \cos x = \frac{1}{2}$ or $\cos x = -1$

For $\cos x = \frac{1}{2}$: $x = \frac{\pi}{3}, \frac{5\pi}{3}$; for $\cos x = -1$: $x = \pi$.

Therefore, all of the solutions in the interval $0 \leqslant x \leqslant 2\pi$ are: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.



Example 28

Solve the equation $2\sin 2x = 3\cos x$ for $0 \leqslant x \leqslant \pi$.

Solution

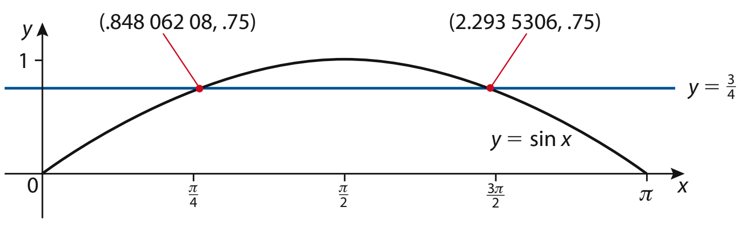
$2\sin 2x = 3\cos x$	
$2(2\sin x\cos x) = 3\cos x$	Using double angle identity for sine.
$4\sin x\cos x = 3\cos x$	Do not divide by $\cos x$; solution(s) may be eliminated.
$4\sin x\cos x - 3\cos x = 0$	Set equal to zero to prepare for solving by factorization.
$\cos x(4\sin x - 3) = 0$	Factorize.

$\cos x = 0$ or $\sin x = \frac{3}{4}$

For $\cos x = 0$: $x = \frac{\pi}{2}$.

For $\sin x = \frac{3}{4}$: $x \approx 0.848$ or 2.29.

Approximate solutions are found using the Intersect command on the GDC. All solutions in interval $0 \leqslant x \leqslant \pi$ are: $x = \frac{\pi}{2}$; $x \approx 0.848, 2.29$.



The next example illustrates how trigonometric identities can be applied to find exact values to trigonometric expressions.

Example 29

Given that $\cos x = \frac{1}{4}$ and that $0 < x < \frac{\pi}{2}$, find the *exact* values of

a) $\sin x$ b) $\sin 2x$

Solution

a) Given $0 < x < \frac{\pi}{2}$ it follows that $\sin x > 0$, because the arc with length x will terminate in the first quadrant. The Pythagorean identity is useful when relating $\sin x$ and $\cos x$.

$$\sin^2 x = 1 - \cos^2 x \Rightarrow \sin x = \sqrt{1 - \cos^2 x}$$

$$\Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

b) $\sin 2x = 2 \sin x \cos x = 2\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = \frac{\sqrt{15}}{8}$

Example 30

Prove the following identity.

$$\frac{\cos A}{\cos A - \sin A} + \frac{\sin A}{\cos A + \sin A} = 1 + \tan 2A$$

● **Hint:** An effective approach to proving identities is to try and work exclusively on one side of the equation. Choosing the side that has an expression that is more 'complicated' is often an efficient path to transform the expression to the one on the other side by means of algebraic manipulations and substitutions. If you do choose to simplify both sides, be careful to work on each side independent of the other. In other words, as mentioned previously, do not perform an operation to both sides (e.g. multiplying both sides by the same quantity). This is only valid if it is known that both sides are equal but this is precisely what you are trying to prove.

Solution

Although we could apply a double angle identity to $\tan 2A$ on the right side it would not help to simplify the expression. The left side appears riper for simplification given that the common denominator of the two fractions is $\cos^2 A - \sin^2 A$ which is equivalent to $\cos 2A$.

$$\frac{\cos A}{\cos A - \sin A} \cdot \frac{\cos A + \sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A + \sin A} \cdot \frac{\cos A - \sin A}{\cos A - \sin A} = \text{RHS}$$

Find a common denominator.

$$\frac{\cos^2 A + \sin A \cos A}{\cos^2 A - \sin^2 A} + \frac{\sin A \cos A - \sin^2 A}{\cos^2 A - \sin^2 A} = \text{RHS}$$

Multiply conjugates $(a + b)(a - b) = a^2 - b^2$.

$$\frac{\cos^2 A - \sin^2 A + 2 \sin A \cos A}{\cos^2 A - \sin^2 A} = \text{RHS}$$

$$\frac{\cos 2A + 2 \sin A \cos A}{\cos 2A} = \text{RHS}$$

Substitute $\cos 2A$ for $\cos^2 A - \sin^2 A$.

Observing that the right-hand side (RHS) has a term equal to 1 directs us to split the left side into two fractions since one of the terms in the numerator is equal to the denominator.

$$\frac{\cos 2A}{\cos 2A} + \frac{2 \sin A \cos A}{\cos 2A} = \text{RHS}$$

$$1 + \frac{\sin 2A}{\cos 2A} = \text{RHS}$$

Substitute $\sin 2A$ for $2 \sin A \cos A$.

$1 + \tan 2A = 1 + \tan 2A$ Q.E.D. Apply tangent identity $\tan x = \frac{\sin x}{\cos x}$.

Table 7.3 Summary of trigonometric identities.

Reciprocal identities		
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$
Tangent and cotangent identities		
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Odd/even function identities		
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \cos\theta$	$\cot(-\theta) = -\tan\theta$
Co-function identities		
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
Pythagorean identities		
$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$
Compound angle identities		
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$		
Double angle identities		
$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$ $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$		

Exercise 7.5

In questions 1–6, use a compound angle identity to find the **exact** value of the expression.

1 $\cos\frac{7\pi}{12}$ **2** $\sin 165°$

3 $\tan\frac{\pi}{12}$ **4** $\sin\left(-\frac{5\pi}{12}\right)$

5 $\cos 255°$ **6** $\cot 75°$

7 a) Find the **exact** value of $\cos\frac{\pi}{12}$.

b) By writing $\cos\frac{\pi}{12}$ as $\cos\left(2 \cdot \frac{\pi}{24}\right)$ and using a double angle identity for cosine, find the **exact** value of $\cos\frac{\pi}{24}$.

In questions 8–10, prove the co-function identity using the compound angle identities.

8 $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ **9** $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ **10** $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

11 Given that $\sin x = \frac{3}{5}$ and that $0 < x < \frac{\pi}{2}$, find the exact values of

a) $\cos x$ b) $\cos 2x$ c) $\sin 2x$

12 Given that $\cos x = -\frac{2}{3}$ and that $\frac{\pi}{2} < x < \pi$, find the exact values of

a) $\sin x$ b) $\sin 2x$ c) $\cos 2x$

In questions 13–16, find the exact values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$ subject to the given conditions.

13 $\sin\theta = \frac{2}{3}, \frac{\pi}{2} < \theta < \pi$

14 $\cos\theta = -\frac{4}{5}, \pi < \theta < \frac{3\pi}{2}$

15 $\tan\theta = 2, 0 < \theta < \frac{\pi}{2}$

16 $\sec\theta = -4, \csc\theta > 0$

In questions 17–20, use a compound angle identity to write the given expression as a function of x alone.

17 $\cos(x - \pi)$

18 $\sin\left(x - \frac{\pi}{2}\right)$

19 $\tan(x + \pi)$

20 $\cos\left(x + \frac{\pi}{2}\right)$

In questions 21–24, use identities to find an equivalent expression involving only sines and cosines, and then simplify it.

21 $\sec\theta + \sin\theta$

22 $\frac{\sec\theta\csc\theta}{\tan\theta\sin\theta}$

23 $\frac{\sec\theta + \csc\theta}{2}$

24 $\frac{1}{\cos^2\theta} + \frac{1}{\cot^2\theta}$

In questions 25–32, simplify each expression.

25 $\cos\theta - \cos\theta\sin^2\theta$

26 $\frac{1 - \cos^2\theta}{\sin^2\theta}$

27 $\cos 2\theta + \sin^2\theta$

28 $\frac{\sin^2\theta}{\cos^2\theta} + \frac{1}{\cot^2\theta}$

29 $\sin(\alpha + \beta) + \sin(\alpha - \beta)$

30 $\frac{1 + \cos 2A}{2}$

31 $\cos(\alpha + \beta) + \cos(\alpha - \beta)$

32 $2\cos^2\theta - \cos 2\theta$

In questions 33–46, prove each identity.

33 $\frac{\cos 2\theta}{\cos\theta + \sin\theta} = \cos\theta - \sin\theta$

34 $(1 - \cos\alpha)(1 + \sec\alpha) = \sin\alpha\tan\alpha$

35 $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$

36 $\cos^4\theta - \sin^4\theta = \cos 2\theta$

37 $\cot\theta - \tan\theta = 2\cot 2\theta$

38 $\frac{\cos\beta - \sin\beta}{\cos\beta + \sin\beta} = \frac{\cos 2\beta}{1 + \sin 2\beta}$

39 $\dfrac{1}{\sec\theta(1-\sin\theta)} = \sec\theta + \tan\theta$

40 $(\tan A - \sec A)^2 = \dfrac{1-\sin A}{1+\sin A}$

41 $\dfrac{\tan 2x \tan x}{\tan 2x - \tan x} = \sin 2x$

42 $\dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 1} = \tan\theta$

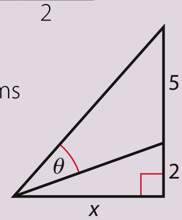
43 $\dfrac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha - \dfrac{\sin\alpha}{1+\cos\alpha}$

44 $\dfrac{1+\cos\beta}{\sin\beta} + \dfrac{\sin\beta}{1+\cos\beta} = 2\csc\beta$

45 $\dfrac{\cot x - 1}{1-\tan x} = \dfrac{\csc x}{\sec x}$

46 $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

47 Given the figure shown right, find an expression in terms of x for the value of $\tan\theta$.



In questions 48–57, solve each equation for x in the given interval. Give answers exactly, if possible. Otherwise, give answers accurate to three significant figures.

48 $2\sin^2 x - \cos x = 1, 0 \leqslant x < 2\pi$

49 $\sec^2 x = 8\cos x, -\pi < x \leqslant \pi$

50 $2\cos x + \sin 2x = 0, -180° < x \leqslant 180°$

51 $2\sin x = \cos 2x, 0 \leqslant x < 2\pi$

52 $\cos 2x = \sin^2 x, 0 \leqslant x < 2\pi$

53 $2\sin x\cos x + 1 = 0, 0 \leqslant x < 2\pi$

54 $\cos^2 x - \sin^2 x = -\frac{1}{2}, 0 \leqslant x \leqslant \pi$

55 $\sec^2 x - \tan x - 1 = 0, 0 \leqslant x < 2\pi$

56 $\tan 2x + \tan x = 0, 0 \leqslant x < 2\pi$

57 $2\sin 2x\cos 3x + \cos 3x = 0, 0 \leqslant x \leqslant 180°$

58 Find an identity for $\sin 3x$ in terms of $\sin x$.

59 a) By squaring $\sin^2 x + \cos^2 x$, prove that $\sin^4 x + \cos^4 x = \frac{1}{4}(\cos 4x + 3)$.

b) Hence, or otherwise, solve the equation $\sin^4 x + \cos^4 x = \frac{1}{2}$ for $0 \leqslant x < 2\pi$.

● **Hint:** For question 46, first prove that $\sin^2 x = 1 - \dfrac{\cos 2x}{2}$, then make a suitable substitution for x. This identity is called the **half-angle identity** for sine. Can you find the corresponding half-angle identity for cosine?

7.6 Inverse trigonometric functions

In Section 2.3, we learned that if a function f is one-to-one then f has an inverse f^{-1}. A defining characteristic of a one-to-one function is that it is always increasing or always decreasing in its domain. Also, recall that no horizontal line can pass through the graph of a one-to-one function at more than one point. It is evident that none of the trigonometric functions are one-to-one functions given their periodic nature. Therefore, the inverse of any of the trigonometric functions over their domain is not a function.

Defining the inverse sine function

Recall that the domain of $y = \sin x$ is all real numbers ($\mathbb{R}$) and its range is the set of all real numbers in the closed interval $-1 \leqslant y \leqslant 1$. The sine function is not one-to-one and hence its inverse is not a function, since more than one value of x corresponds to the same value of y. For example, $\sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = \frac{1}{2}$. That is, for $y = \sin x$ there are an infinite number of ordered pairs with a y-coordinate of $\frac{1}{2}$ (see Figure 7.30).

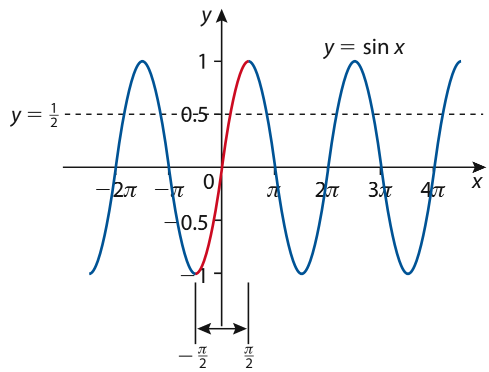


Figure 7.30 A horizontal line, $y = \frac{1}{2}$ shown here, can intersect the graph of $y = \sin x$ more than once, thus indicating that the inverse of $y = \sin x$ is not a function. The portion of the graph (in red) from $-\frac{\pi}{2}$ to $-\frac{\pi}{2}$ is used to define the inverse and only intersects a horizontal line once.

Examples 13 and 15 in Section 2.3, showed us that a function that is not one-to-one can often be made so by restricting its domain. Consequently, even though there is no inverse function for the sine function for all $\mathbb{R}$, we can define the inverse sine function if we restrict its domain so that it is one-to-one (and passes the horizontal line test). We have an unlimited number of ways of restricting the domain but it seems sensible to select an interval of x including zero, and it's standard to restrict the domain to the 'largest' set possible. Consider restricting the domain of $y = \sin x$ to the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. In this interval, $y = \sin x$ is always increasing and takes on every value from -1 to 1 exactly once. Thus, the function $y = \sin x$ with domain $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ is one-to-one and its inverse is a function. We have the following definition:

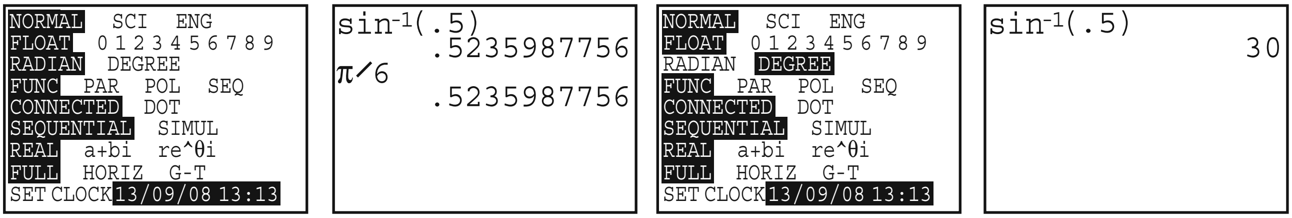
The equation $y = \arcsin x$ is interpreted, 'y is the arc whose sine is x', or 'y is the angle whose sine is x,' or 'y is the real number whose sine is x.' Any GDC labels the inverse sine function as $\sin^{-1} x$. The symbols $y = \arcsin x$ and $y = \sin^{-1} x$ are both commonly used to indicate the inverse sine function, but a disadvantage of writing $y = \sin^{-1} x$ is that it can be confused with $y = (\sin x)^{-1} = \frac{1}{\sin x} = \csc x$.

Inverse sine function

The inverse sine function, denoted by $x = \arcsin x$ or $y = \sin^{-1} x$, is the function with a domain of $-1 \leqslant x \leqslant 1$ and a range of $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ defined by

$$y = \arcsin x \quad \text{if and only if} \quad x = \sin y$$

Thus, $\arcsin x$ (or $\sin^{-1} x$) is the number in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x. For example, $\arcsin\frac{1}{2} = \frac{\pi}{6}$ because the one number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$ is $\frac{\pi}{6}$. Your GDC is programmed such that it will give the same result. If your GDC is in radian mode it will give the approximate value of $\frac{\pi}{6}$ to several significant figures, and if it is in degree mode, it will give the exact result of 30°. See the GDC images on the next page.



From the graphical symmetry of inverse functions, the graph of $y = \arcsin x$ is a reflection of $y = \sin x$ about the line $y = x$, as shown in Figures 7.31 and 7.32.

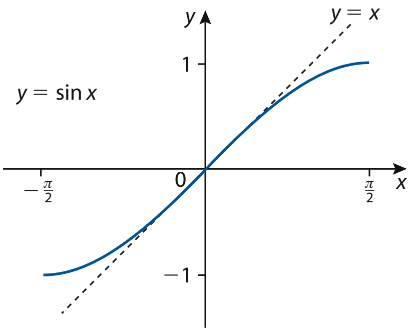


Figure 7.31 The graph of $y = \sin x$ with domain restricted to $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$.

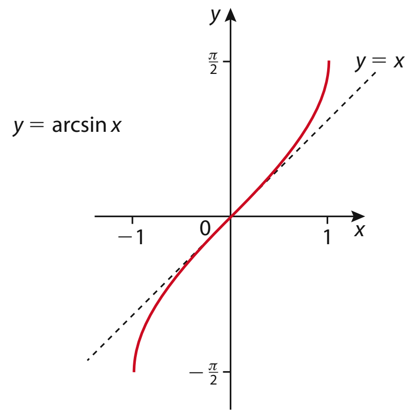


Figure 7.32 The graph of $y = \arcsin x$.

Defining the inverse cosine and inverse tangent functions

The inverse cosine function and inverse tangent function can be defined by following a parallel procedure to that used for defining the inverse sine function. The graphs of $y = \cos x$ and $y = \tan x$ (Figures 7.33 and 7.34) clearly show that neither function is one-to-one and consequently their inverses are not functions. Consider restricting the domain of the cosine function to the closed interval $0 \leqslant x \leqslant \pi$ (Figure 7.33) and restricting the domain of the tangent function to the open interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (Figure 7.34). The interval for tangent cannot include the endpoints, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because tangent is undefined for these values. For these domain restrictions cosine and tangent will attain each of its function values exactly once. Hence, with these restrictions, both cosine and tangent will be one-to-one and their inverses will be functions.

Figure 7.34 The graph of $y = \tan x$ with the portion of the graph (in red) from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (exclusive) used to define its inverse.

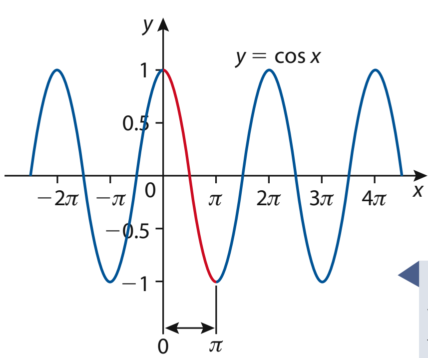
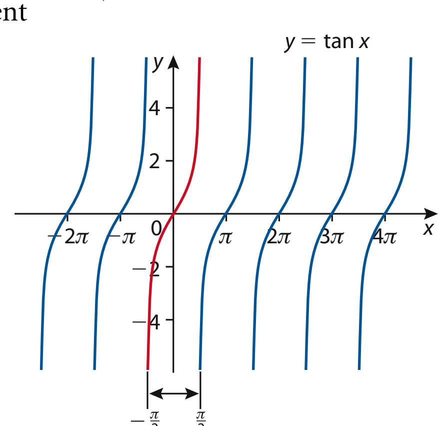


Figure 7.33 The graph of $y = \cos x$ with portion of the graph (in red) from 0 to π (inclusive) used to define its inverse.



Inverse cosine function

The inverse cosine function, denoted by $y = \arccos x$, or $y = \cos^{-1} x$, is the function with a domain of $-1 \leqslant x \leqslant 1$ and a range of $0 \leqslant y \leqslant \pi$ defined by

$$y = \arccos x \quad \text{if and only if} \quad x = \cos y$$

Inverse tangent function

The inverse tangent function, denoted by $y = \arctan x$, or $y = \tan^{-1} x$, is the function with a domain of $\mathbb{R}$ and a range of $-\frac{\pi}{2} < y < \frac{\pi}{2}$ defined by

$$y = \arctan x \quad \text{if and only if} \quad x = \tan y$$

The graphs of $y = \cos x$ (for the appropriate interval) and $y = \arccos x$ are shown in Figures 7.35 and 7.36.

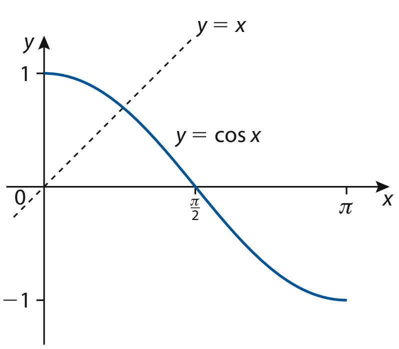


Figure 7.35 The graph of $y = \cos x$ with domain restricted to $0 \leqslant x \leqslant \pi$.

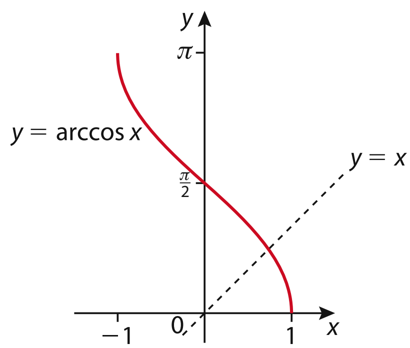


Figure 7.36 The graph of $y = \arccos x$.

The inverse cotangent, secant and cosecant functions are rarely used (and are not in the Maths Higher Level syllabus) so definitions will not be given for them here.

The graphs of $y = \tan x$ (for the appropriate interval) and $y = \arctan x$ are shown in Figures 7.37 and 7.38.

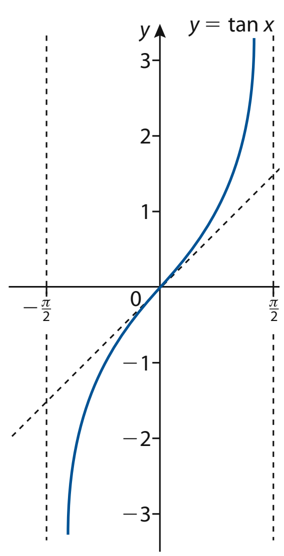


Figure 7.37 The graph of $y = \tan x$ with domain restricted to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

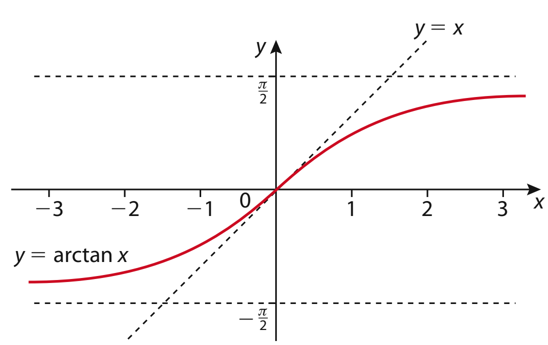


Figure 7.38 The graph of $y = \arctan x$.

● **Hint:** Unless specifically instructed otherwise, we will assume that the result of evaluating an inverse trigonometric function will be a real number that can be interpreted as either an arc length on the unit circle or an angle in radian measure. If the result is to be an angle in degree measure then the instructions will explicitly request this.

Example 30

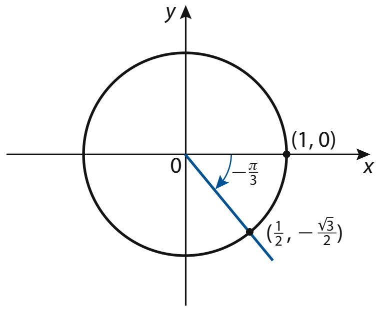
Without using your GDC, find the exact value of each expression.

a) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ b) $\arccos 1$ c) $\arctan\sqrt{3}$ d) $\arcsin\frac{3}{2}$

Solution

a) The expression $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ can be interpreted as 'the number y such that $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ whose sine is $-\frac{\sqrt{3}}{2}$' or 'the number in quadrant I or IV whose sine is $-\frac{\sqrt{3}}{2}$.' We know sine function values are negative in quadrants III and IV, so the number we are looking for is in quadrant IV. The diagram shows that the required number is $-\frac{\pi}{3}$. An angle of $-\frac{\pi}{3}$ in standard position will intersect the unit circle at a point whose y-coordinate is $-\frac{\sqrt{3}}{2}$.

Therefore, $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.



b) The range of the function $y = \arccos x$ is $0 \leqslant y \leqslant \pi$. Thus we are looking for a number in quadrant I or II whose cosine is 1. The number we are looking for is 0, because an angle of measure 0 in standard position will intersect the unit circle at a point whose x-coordinate is 1. Therefore, $\arccos 1 = 0$.

c) The range of the function $y = \arctan x$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Thus we are looking for a number in quadrant I or IV for which the ratio $\frac{\text{sine}}{\text{cosine}}$ is equal to $\sqrt{3}$. It must be in quadrant I because in quadrant IV tangent values are negative. Familiarity with the sine and cosine values for common angles covered earlier in this chapter helps us to recognize that the required ratio will be $\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$. The required number is $\frac{\pi}{3}$ because it is in the first quadrant with $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$.

Therefore, $\arctan\sqrt{3} = \frac{\pi}{3}$.

d) The domain of the function $y = \arccos x$ is $-1 \leqslant x \leqslant 1$, but $\frac{3}{2}$ is not in this interval. There is no number whose sine is $\frac{3}{2}$. Therefore, $\arcsin\frac{3}{2}$ is not defined.

Compositions of trigonometric and inverse trigonometric functions

Recall from Chapter 2 that for a pair of inverse functions the following two properties hold true.

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}; and $f^{-1}(f(x)) = x$ for all x in the domain of f.

It follows that the following properties hold true for the inverse sine, cosine and tangent functions.

● **Hint:** Note that the inverse property $\arcsin(\sin\beta) = \beta$ does **not** hold true when $\beta = \frac{3\pi}{4}$.

$\arcsin\left(\sin\frac{3\pi}{4}\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

and

$\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

The property $\arcsin(\sin\beta) = \beta$ is not valid for values of β outside the interval $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$. Similarly, the property $\arccos(\cos\beta) = \beta$ is not valid for values of β outside the interval $0 \leqslant \beta \leqslant \pi$; and $\arctan(\tan\beta) = \beta$ is not valid for values of β outside the interval $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

Inverse properties

If $-1 \leqslant \alpha \leqslant 1$, then $\sin(\arcsin\alpha) = \alpha$; and if $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$, then $\arcsin(\sin\beta) = \beta$.

If $-1 \leqslant \alpha \leqslant 1$, then $\cos(\arccos\alpha) = \alpha$; and if $0 \leqslant \beta \leqslant \pi$ then $\arccos(\cos\beta) = \beta$.

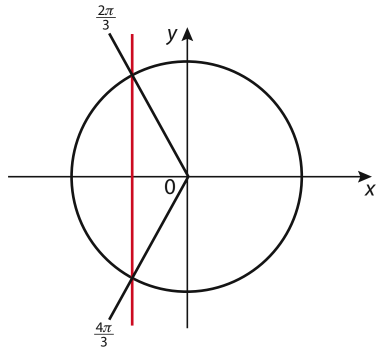
If $\alpha \in \mathbb{R}$, then $\tan(\arctan\alpha) = \alpha$; and if $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, then $\arctan(\tan\beta) = \beta$.

Example 31

Find the exact values, if possible, for the following expressions.

a) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ b) $\tan(\arctan(-7))$ c) $\sin(\arcsin\sqrt{3})$

Solution

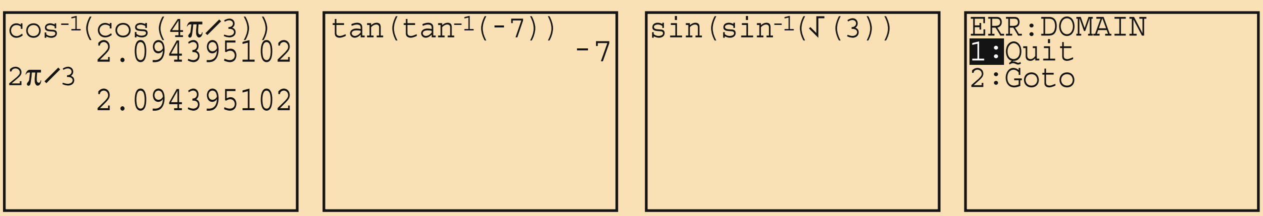


a) $\frac{4\pi}{3}$ is not in the range of the $\cos^{-1}$, or arccos, function $0 \leqslant \beta \leqslant \pi$. However, using the symmetry of the unit circle we know that $\frac{4\pi}{3}$ has the same cosine as $\frac{2\pi}{3}$ (see figure) which is in the interval $0 \leqslant \beta \leqslant \pi$. Thus, $\cos^{-1}\left(\cos\frac{4\pi}{3}\right) = \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$.

b) -7 is in the range of the tangent function (and in the domain of the arctangent function), so the inverse property applies. Therefore, $\tan(\arctan(-7)) = -7$.

c) $\sqrt{3}$ is not in the range of the sine function $-1 \leqslant \alpha \leqslant 1$, so $\arcsin\sqrt{3}$ is not defined. It follows that $\sin(\arcsin\sqrt{3})$ is not defined.

All of the results in Example 31 can be quickly verified on your GDC as shown below. Be sure to be in radian mode.



Example 32

Without using your GDC, find the exact value of each expression.

a) $\cos\left[\sin^{-1}\left(-\frac{8}{17}\right)\right]$

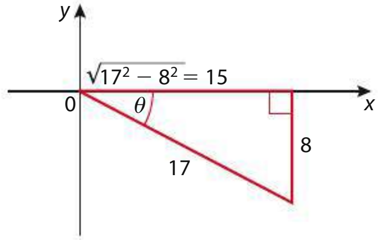
b) $\arcsin\left(\tan\frac{3\pi}{4}\right)$

c) $\sec\left[\arctan\left(\frac{3}{5}\right)\right]$

Solution

a) If we let $\theta = \sin^{-1}\left(-\frac{8}{17}\right)$, then $\sin\theta = -\frac{8}{17}$. Because $\sin\theta$ is negative, then θ must be an angle (arc) in quadrant IV. From a simple sketch of an appropriately labeled triangle in quadrant IV, we can determine $\cos\theta = \cos\left(\sin^{-1}\left(-\frac{8}{17}\right)\right)$.

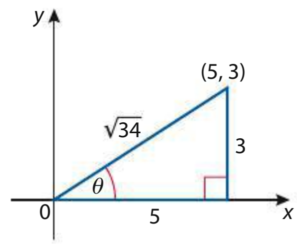
Therefore, $\cos\left(\sin^{-1}\left(-\frac{8}{17}\right)\right) = \frac{15}{17}$.



b) $\arcsin\left(\tan\frac{3\pi}{4}\right) = \arcsin(-1) = -\frac{\pi}{2}$

c) If we let $\theta = \arctan\left(\frac{3}{5}\right)$ then $\tan\theta = \frac{3}{5}$. Because $\tan\theta > 0$ then θ must be in quadrant I. Consequently, we can construct a right triangle containing θ in quadrant I by drawing a line from the origin to the point (5, 3), as shown in the diagram. The hypotenuse is $\sqrt{25 + 9} = \sqrt{34}$.

Therefore, $\sec\left[\arctan\left(\frac{3}{5}\right)\right] = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{5}{\sqrt{34}}} = \frac{\sqrt{34}}{5}$.

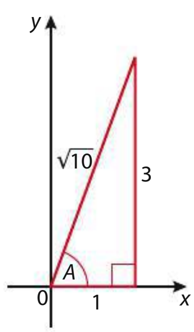


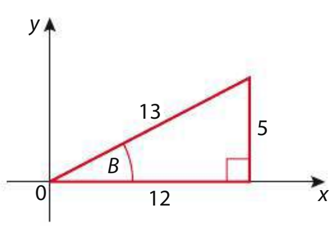
Example 33

If $C = \arctan 3 + \arcsin\left(\frac{5}{13}\right)$, find the exact value of $\cos C$.

Solution

Let $A = \arctan 3$ and $B = \arcsin\left(\frac{5}{13}\right)$. Thus, $C = A + B$ and a strategy for finding $\cos C$ is to use the following compound angle identity: $\cos C = \cos(A + B) = \cos A\cos B - \sin A\sin B$. We know that $\sin B = \frac{5}{13}$. We need to find exact values for $\cos A$, $\cos B$ and $\sin A$. The range for $\arctan x$ is $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the range for $\arcsin x$ is $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, and since $\tan A = 3 > 0$ and $\sin B = \frac{5}{13} > 0$, both A and B are in quadrant I.





$\sin A = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$ $\qquad$ $\sin B = \frac{5}{13}$

$\cos A = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$ $\qquad$ $\cos B = \frac{12}{13}$

Hence, $\cos C = \cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{\sqrt{10}}{10}\right)\left(\frac{12}{13}\right) - \left(\frac{3\sqrt{10}}{10}\right)\left(\frac{5}{13}\right)$$

$$= \frac{(12 - 15)\sqrt{10}}{130}$$

$$= \frac{-3\sqrt{10}}{130}$$

Therefore, $\cos C = \dfrac{-3\sqrt{10}}{130}$.

Example 34

Find all solutions, accurate to three significant figures, to the equation $3 \sin 2\theta = 1$ in the interval $0 \leqslant \theta < 2\pi$.

Solution

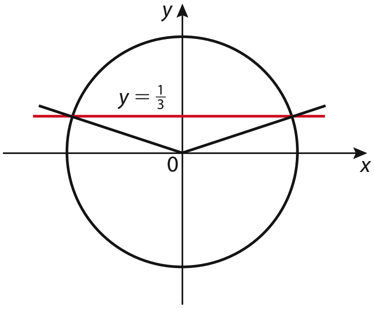
A reasonable idea is to apply a double angle identity and substitute $2 \sin\theta \cos\theta$ for $\sin 2\theta$. Although a substitution like this proved to be an effective technique in the previous section, it is not always the best strategy. In this case, the transformed equation becomes $6 \sin\theta \cos\theta = 1$ which would prove difficult to solve. A better approach is

$$3 \sin 2\theta = 1$$

$$\sin 2\theta = \tfrac{1}{3}$$

$$2\theta = \arcsin\left(\tfrac{1}{3}\right)$$

$$\theta = \frac{1}{2}\arcsin\left(\tfrac{1}{3}\right)$$



There is one angle in quadrant I with a sine equal to $\frac{1}{3}$ and one angle in quadrant II with a sine equal to $\frac{1}{3}$ (see figure). None of the common angles has a sine equal to $\frac{1}{3}$, so we will need to use the inverse sine ($\sin^{-1}$) on our GDC to obtain an approximate answer. Since the range of the inverse sine function, $\sin^{-1}$, is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ your GDC's computation of $\sin^{-1}\left(\frac{1}{3}\right)$ will only give the angle (arc) in quadrant I. From the symmetry of the unit circle, we can obtain the angle in quadrant II by subtracting the angle in quadrant I from π. The GDC images below show the computation to find both answers – and a check of the two answers.

```
sin⁻¹(1/3)
          .3398369095
.5*Ans
          .1699184547
Ans→A
          .1699184547
```

```
sin⁻¹(1/3)
          .3398369095
.5(π-Ans)
          1.400877872
Ans→B
          1.400877872
```

```
3sin(2A)
                    1
3sin(2B)
                    1
```

Therefore, $\theta \approx 0.170$ or $\theta \approx 1.40$ accurate to 3 significant figures.

To an observer, the apparent size of an object depends on the distance from the observer to the object. The farther an object is from an observer, the smaller its apparent size. For example, although the Sun's diameter is 400 times wider than our Moon's diameter, the two objects appear to have the same diameter as viewed from the Earth (see Figure 7.39). Thus, during a total solar eclipse, the Moon blocks out the Sun. Also, if an object is sufficiently above or below the horizontal position of the observer, the apparent size of the object will also decrease if you move close to the object. Thus for this situation, there will be a distance for which the angle subtended at the eye of the observer is a maximum (Example 35).

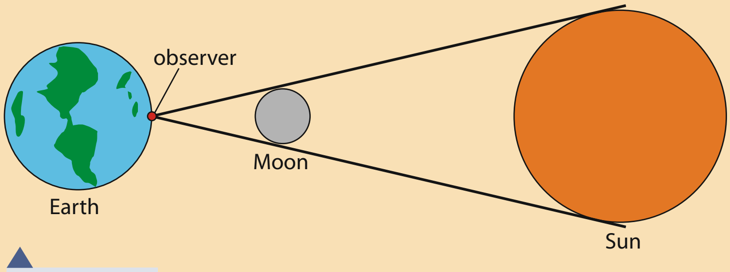


Figure 7.39

On the surface of the Earth the angle subtended by the moon and the Sun is nearly the same. It is approximately 0.54 degrees for the Moon and 0.52 degrees for the Sun. The Sun is 400 times wider than the Moon and coincidentally 400 times further from the Earth than the Moon.

Example 35

A painting that is 125 cm from top to bottom is hanging on the wall of a gallery such that it's base is 250 cm from the floor. Pablo is standing x cm from the wall from which the painting is hung. Pablo's eyes are 170 cm from the floor and from where he stands the painting subtends an angle α degrees. a) Write a function for α in terms of x. b) Find α, accurate to four significant figures, for the following values of x: (i) $x = 75$ cm; (ii) $x = 125$ cm; and (iii) $x = 175$ cm. c) Using a GDC, approximate to the nearest cm, how far Pablo should stand from the wall so that the subtended angle α is a maximum.

Solution

a) The figure shows α, the angle subtended by the painting, and β, the angle subtended by the part of the wall above eye level and below the painting. Let θ be the sum of these two angles. Hence, $\theta = \alpha + \beta$ and $\alpha = \theta - \beta$. From the compound angle identity for tangent, we have

$$\tan\alpha = \frac{\tan\theta - \tan\beta}{1 + \tan\theta\tan\beta}$$

From the right triangles in the figure, we can determine that

$$\tan\beta = \frac{80}{x} \quad \text{and} \quad \tan\theta = \frac{205}{x}$$

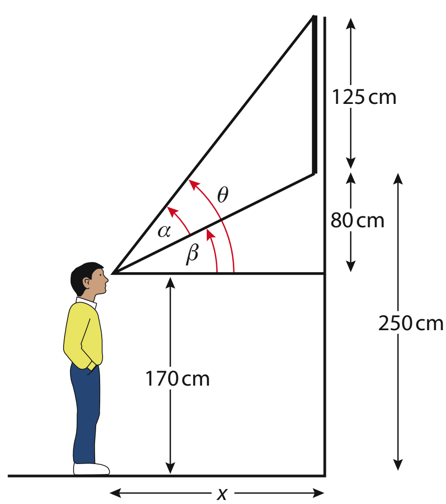
Substituting these into the expression for $\tan\alpha$, gives

$$\tan\alpha = \frac{\frac{205}{x} - \frac{80}{x}}{1 + \left(\frac{205}{x}\right)\left(\frac{80}{x}\right)}$$

$$\tan\alpha = \frac{\frac{125}{x}}{1 + \left(\frac{205}{x}\right)\left(\frac{80}{x}\right)} \cdot \frac{x^2}{x^2}$$

$$\tan\alpha = \frac{125x}{x^2 + 16\,400}$$

Therefore, $\alpha = \tan^{-1}\left(\frac{125x}{x^2 + 16\,400}\right)$.

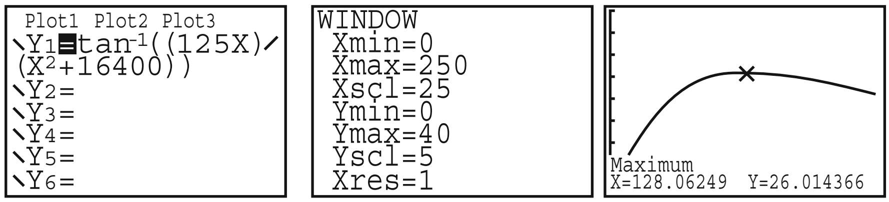


b) (i) For $x = 75$ cm: $\alpha = \tan^{-1}\left(\frac{125 \cdot 75}{75^2 + 16\,400}\right) \approx \tan^{-1}(0.425\,6527)$ $\approx 23.06°$.

(ii) For $x = 125$ cm: $\alpha = \tan^{-1}\left(\frac{125 \cdot 125}{125^2 + 16\,400}\right) \approx \tan^{-1}(0.487\,9001)$ $\approx 26.01°$.

(iii) For $x = 175$ cm: $\alpha = \tan^{-1}\left(\frac{125 \cdot 175}{175^2 + 16\,400}\right) \approx \tan^{-1}(0.465\,1781)$ $\approx 24.95°$.

c) Graph the function found in a). On the GDC, it will be entered as $y = \tan^{-1}\left(\frac{125x}{x^2 + 16\,400}\right)$. Find the value of x that gives the maximum value for y (subtended angle α) by either tracing or using a 'maximum' command on the calculator. See the GDC images below.



Therefore, if Pablo stands 128 cm away from the wall the painting will subtend the widest possible angle at his eye – or, in other words, give him the 'best' view of the painting.

Exercise 7.6

In questions 1–6, find the exact value (in radian measure) of each expression without using your GDC.

1 arcsin 1   **2** $\arccos\left(\frac{1}{\sqrt{2}}\right)$ **3** $\arctan(-\sqrt{3})$

4 $\arccos\left(-\frac{1}{2}\right)$ **5** arctan 0   **6** $\arcsin\left(\frac{-\sqrt{3}}{2}\right)$

In questions 7–20, without using your GDC, find the exact value, if possible, for each expression. Verify your result with your GDC.

7 $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **8** $\cos^{-1}\left(\cos\frac{3}{2}\right)$

9 tan(arctan 12)   **10** $\cos\left(\arccos\frac{2\pi}{3}\right)$

11 $\arctan\left(\tan\left(-\frac{3\pi}{4}\right)\right)$ **12** $\sin(\arcsin \pi)$

13 $\sin\left(\arctan\frac{3}{4}\right)$ **14** $\cos\left(\arcsin\left(\frac{7}{25}\right)\right)$

15 $\arcsin\left(\tan\frac{\pi}{3}\right)$ **16** $\tan^{-1}\left(2\sin\frac{\pi}{3}\right)$

17 $\cos\left(\arctan\left(\frac{1}{2}\right)\right)$ **18** $\cos(\sin^{-1}(0.6))$

19 $\sin\left(\arccos\left(\frac{3}{5}\right) + \arctan\left(\frac{5}{12}\right)\right)$ **20** $\cos\left(\tan^{-1}3 + \sin^{-1}\left(\frac{1}{3}\right)\right)$

In questions 21–26, rewrite the expression as an algebraic expression in terms of x.

21 $\cos(\arcsin x)$ **22** $\tan(\arccos x)$

23 $\cos(\tan^{-1} x)$

24 $\sin(2\cos^{-1} x)$

25 $\tan\left(\frac{1}{2}\arccos x\right)$

26 $\sin(\arcsin x + 2\arctan x)$

27 Show that $\arcsin\frac{4}{5} + \arcsin\frac{5}{13} = \arccos\frac{16}{65}$.

28 Show that $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.

29 Find x if $\tan^{-1} x + \tan^{-1}(1 - x) = \tan^{-1}\frac{4}{3}$.

In questions 30–37, solve for x in the indicated interval.

30 $5\cos(2x) = 2, 0 \leqslant x \leqslant \pi$

31 $\tan\left(\frac{x}{2}\right) = 2, 0 < x \leqslant 2\pi$

32 $2\cos x - \sin x = 0, 0 < x \leqslant 2\pi$

33 $3\sec^2 x = 2\tan x + 4, 0 < x \leqslant 2\pi$

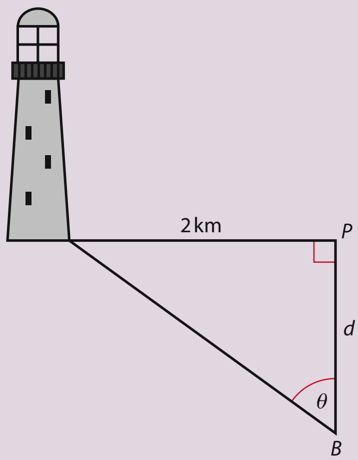
34 $2\tan^2 x - 3\tan x + 1 = 0, 0 \leqslant x \leqslant \pi$

35 $\tan x \csc x = 5, 0 < x \leqslant 2\pi$

36 $\tan 2x + 3\tan x = 0, 0 < x \leqslant 2\pi$

37 $2\cos^2 x - 3\sin 2x = 2, 0 \leqslant x \leqslant \pi$

38 An offshore lighthouse is located 2 km from a straight coastline. The lighthouse has a revolving light. Let θ be the angle that the beam of light from the lighthouse makes with the coastline; and P is the point on the coast the shortest distance from the lighthouse (see figure). If d is the distance in km from P to the point B where the beam of light is hitting the coast, express θ as a function of d. Sketch a complete graph of this function and indicate the portion of the graph that sufficiently represents the given situation.



39 The screen in a movie cinema is 7 metres from top to bottom and is positioned 3 metres above the horizontal floor of the cinema. The first row of seats is 2.5 metres from the wall that the screen is on and the rows are each 1 metre apart. You decide to sit in the row where you get the 'best' view, that is, where the angle subtended at your eyes by the screen is a maximum. When you are sitting in one of the cinema's seats your eyes are 1.2 metres above the horizontal floor.

a) Let x be the distance that you are from the wall that the screen is on, and θ is the angle subtended at your eyes by the screen.
   (i) Draw a clear diagram to represent all the information given.
   (ii) Find a function for θ in terms of x.
   (iii) Sketch a graph of the function.
   (iv) Use your GDC to find the value of x that gives a maximum for θ. In which row should you sit?

b) Suppose that, starting with the first row of seats, the floor of the cinema is sloping upwards at an angle of 20° above the horizontal. Again, the first row of seats is 2.5 metres from the wall that the screen is on and the rows are each 1 metre apart measured along the sloping floor. Let x be the distance from where the first row starts and your seat in the cinema.
   (i) Draw a clear diagram to represent all the information given.
   (ii) Find a function for θ in terms of x.
   (iii) Sketch a graph of the function.
   (iv) Use your GDC to find the value of x that gives a maximum for θ. In which row should you sit?

Practice questions

1 A toy on an elastic string is attached to the top of a doorway. It is pulled down and released, allowing it to bounce up and down. The length of the elastic string, L centimetres, is modelled by the function $L = 110 + 25\cos(2\pi t)$, where t is time in seconds after release.

a) Find the length of the elastic string after 2 seconds.

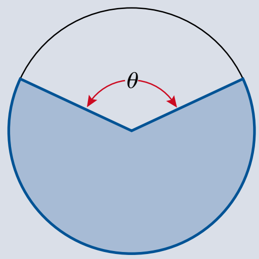
b) Find the minimum length of the string.

c) Find the first time after release that the string is 85 cm.

d) What is the period of the motion?

2 Find the exact solution(s) to the equation $2\sin^2 x - \cos x + 1 = 0$ for $0 \leqslant x \leqslant 2\pi$.

3 The diagram shows a circle of radius 6 cm.
The perimeter of the shaded sector is 25 cm.
Find the radian measure of the angle θ.



4 Consider the two functions $f(x) = \cos 4x$ and $g(x) = \cos\left(\frac{x}{2}\right)$.

a) Write down: **(i)** the minimum value of the function f

(ii) the period of g.

b) For the equation $f(x) = g(x)$, find the number of solutions in the interval $0 \leqslant x \leqslant \pi$.

5 A reflector is attached to the spoke of a bicycle wheel. As the wheel rolls along the ground, the distance, d centimetres, that the reflector is above the ground after t seconds is modelled by the function

$$d = p + q\cos\left(\frac{2\pi}{m}t\right), \text{ where } p, q \text{ and } m \text{ are constants.}$$

The distance d is at a maximum of 64 cm at $t = 0$ seconds and at $t = 0.5$ seconds, and is at a minimum of 6 cm at $t = 0.25$ seconds and at $t = 0.75$ seconds. Write down the value of:

a) p **b)** q **c)** m.

6 Find all solutions to $1 + \sin 3x = \cos(0.25x)$ such that $x \in [0, \pi]$.

7 Find all solutions to both trigonometric equations in the interval $x \in [0, 2\pi]$. Express the solutions exactly.

a) $2\cos^2 x + 5\cos x + 2 = 0$ **b)** $\sin 2x - \cos x = 0$

8 The value of x is in the interval $\frac{\pi}{2} < x < \pi$ and $\cos^2 x = \frac{8}{9}$. Without using your GDC, find the exact values for the following:

a) $\sin x$ **b)** $\cos 2x$ **c)** $\sin 2x$

9 The depth, d metres, of water in a harbour varies with the tides during each day. The first high (maximum) tide after midnight occurs at 5:00 a.m. with a depth of 5.8 m. The first low (minimum) tide occurs at 10:30 a.m. with a depth of 2.6 m.

a) Find a trigonometric function that models the depth, d, of the water t hours after midnight.

b) Find the depth of the water at 12 noon.

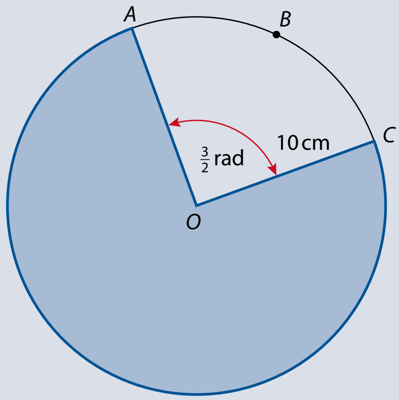
c) A large boat needs at least 3.5 m of water to dock in the harbour. During what time interval after 12 noon can the boat dock safely?

10 Solve the equation $\tan^2 x + 2\tan x - 3 = 0$ for $0 \leqslant x \leqslant \pi$. Give solutions exactly, if possible. Otherwise, give solutions to 3 significant figures.

11 The following diagram shows a circle of centre O and radius 10 cm. The arc ABC subtends an angle of $\frac{3}{2}$ radians at the centre O.

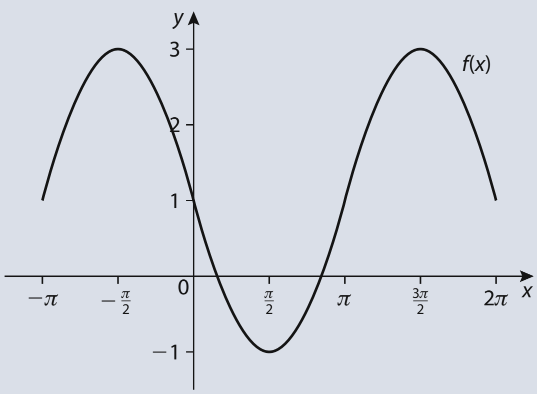
a) Find the length of the arc ACB.

b) Find the area of the shaded region.



12 Consider the function $f(x) = \frac{5}{2}\cos\left(2x - \frac{\pi}{2}\right)$. For what values of k will the equation $f(x) = k$ have no solutions?

13 A portion of the graph of $y = k + a\sin x$ is shown below. The graph passes through the points $(0, 1)$ and $\left(\frac{3\pi}{2}, 3\right)$. Find the value of k and a.



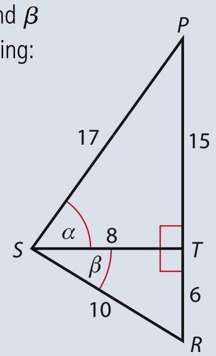
14 The angle α satisfies the equation $2\tan^2\alpha - 5\sec\alpha - 10 = 0$ where α is in the second quadrant. Find the **exact** value of $\sec\alpha$.

15 Triangles PTS and RTS are right-angled at T with angles α and β as shown in the diagram. Find the exact values of the following:

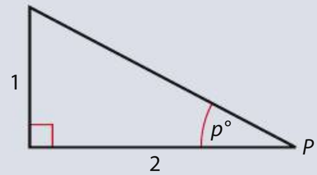
a) $\sin(\alpha + \beta)$

b) $\cos(\alpha + \beta)$

c) $\tan(\alpha + \beta)$



16 The diagram shows a right triangle with legs of length 1 unit and 2 units as shown. The angle at vertex P has a degree measure of $p°$. Find the exact values of $\sin 2p°$ and $\sin 3p°$.



17 The obtuse angle B is such that $\tan B = -\frac{5}{12}$. Find the values of

a) $\sin B$ **b)** $\cos B$ **c)** $\sin 2B$ **d)** $\cos 2B$

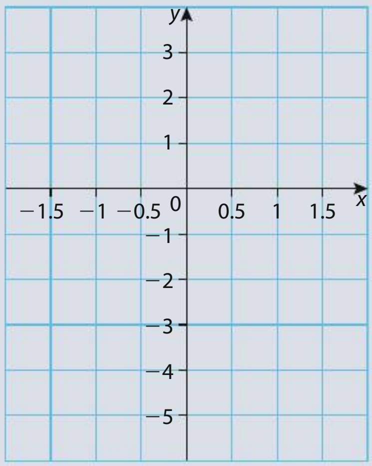
18 Given that $\tan 2\theta = \frac{3}{4}$, find the possible values of $\tan \theta$.

19 If $\sin(x - \alpha) = k \sin(x + \alpha)$ express $\tan x$ in terms of k and α.

20 Solve $\tan^2 2\theta = 1$, in the interval $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

21 Let f be the function $f(x) = x \arccos x + \frac{1}{2}x$ for $-1 \leqslant x \leqslant 1$ and g the function $g(x) = \cos 2x$ for $-1 \leqslant x \leqslant 1$.

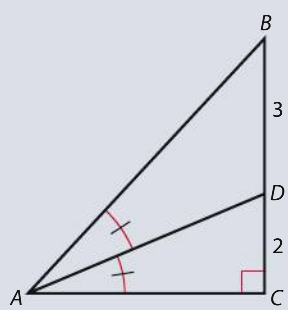
a) On the grid below, sketch the graph of f and of g.



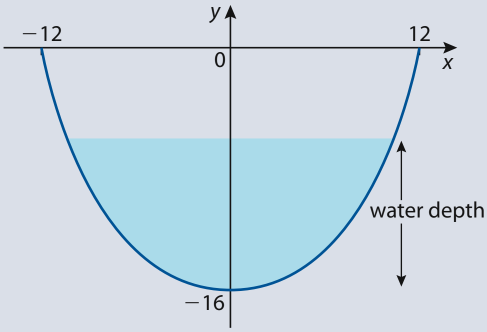
b) Write down the solution of the equation $f(x) = g(x)$.

c) Write down the range of g.

22 Let ABC be a right-angled triangle, where $\hat{C} = 90°$. The line (AD) bisects $B\hat{A}C$, $BD = 3$, and $DC = 2$, as shown in the diagram. Find $D\hat{A}C$.



23 The diagram below shows the boundary of the cross section of a water channel.



The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm. The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm. Find the width of the water surface in the channel when the water depth is 10 cm. Give your answer in the form $a \arccos b$, where $a, b \in \mathbb{R}$.

Questions 17–23 © International Baccalaureate Organization

8 Triangle Trigonometry

Assessment statements

3.6 Solution of triangles.

The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$.

The sine rule: $= \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, including the ambiguous case.

Area of a triangle as $\frac{1}{2}ab\sin C$.

Applications in two and three dimensions.

Introduction

In this chapter, we approach trigonometry from a **right triangle** perspective where trigonometric functions will be defined in terms of the **ratios of sides of a right triangle**. Over two thousand years ago, the Greeks developed trigonometry to make helpful calculations for surveying, navigating, building and other practical pursuits. Their calculations were based on the angles and lengths of sides of a right triangle. The modern development of trigonometry, based on the length of an arc on the unit circle, was covered in the previous chapter. We begin a more classical approach by introducing some terminology regarding right triangles.

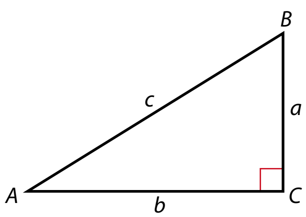


Figure 8.1 Conventional triangle notation.

• **Hint:** In IB notation, [AC] denotes the line segment connecting points A and C. The notation AC represents the *length* of this line segment. Also, the notation $A\hat{B}C$ denotes the angle with its vertex at point B, with one side of the angle containing the point A and the other side containing point C.

8.1 Right triangles and trigonometric functions of acute angles

Right triangles

The conventional notation for triangles is to label the three vertices with capital letters, for example A, B and C. The same capital letters can be used to represent the measure of the angles at these vertices. However, we will often use a Greek letter, such as α (alpha), β (beta) or θ (theta) to do so. The corresponding lower-case letters, a, b and c, represent the lengths of the sides opposite the vertices. For example, b represents the length of the side opposite angle B, that is, the line segment AC, or [AC] (Figure 8.1).

In a right triangle, the longest side is opposite the right angle (i.e. measure of 90°) and is called the **hypotenuse**, and the two shorter sides adjacent to the right angle are often called the **legs** (Figure 8.2). Because the sum of the three angles in any triangle in plane geometry is 180°, then the two non-right angles are both **acute angles** (i.e. measure between 0 and 90 degrees). It also follows that the two acute angles in a right triangle are a pair of **complementary angles** (i.e. have a sum of 90°).

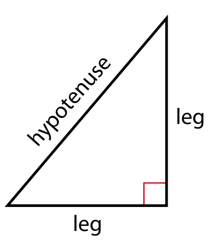


Figure 8.2 Right triangle terminology.

Trigonometric functions of an acute angle

We can use properties of similar triangles and the definitions of the sine, cosine and tangent functions from Chapter 7 to define these functions in terms of the sides of a right triangle.

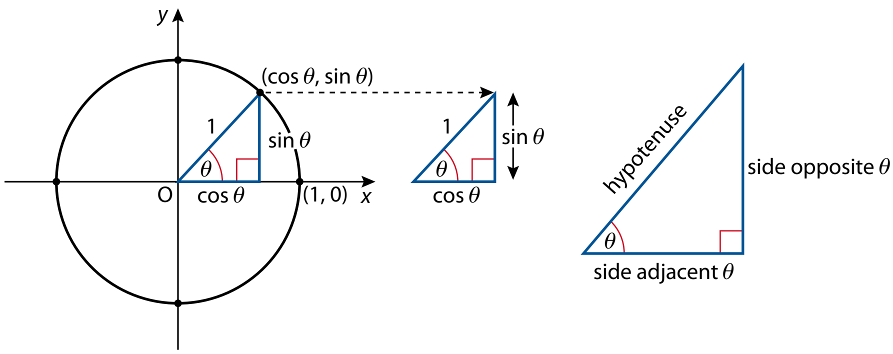


Figure 8.3 Trigonometric functions defined in terms of sides of similar triangles.

The right triangles shown in Figure 8.3 are **similar triangles** because corresponding angles have equal measure – each has a right angle and an acute angle of measure θ. It follows that the ratios of corresponding sides are equal, allowing us to write the following three proportions involving the sine, cosine and tangent of the acute angle θ.

$$\frac{\sin\theta}{1} = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \frac{\cos\theta}{1} = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}}$$

The definitions of the trigonometric functions in terms of the sides of a right triangle follow directly from these three equations.

Right triangle definition of the trigonometric functions

Let θ be an **acute angle** of a right triangle, then the sine, cosine and tangent functions of the angle θ are defined as the following ratios in the right triangle:

$$\sin\theta = \frac{\text{side opposite angle } \theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{side adjacent angle } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{side opposite angle } \theta}{\text{side adjacent angle } \theta}$$

It follows that the sine, cosine and tangent of an acute angle are positive.

It is important to understand that properties of similar triangles are the foundation of right triangle trigonometry. Regardless of the size (i.e. lengths of sides) of a right triangle, so long as the angles do not change, the ratio of any two sides in the right triangle will remain *constant*. All the right triangles in Figure 8.4 have an acute angle with a measure of 30° (thus, the other acute angle is 60°). For each triangle, the ratio of the side opposite the 30° angle to the hypotenuse is exactly $\frac{1}{2}$. In other words, the sine of 30° is always $\frac{1}{2}$. This agrees with results from the previous chapter, knowing that an angle of 30° is equivalent to $\frac{\pi}{6}$ in radian measure.

Thales of Miletus (circa 624–547) was the first of the Seven Sages, or wise men of ancient Greece, and is considered by many to be the first Greek scientist, mathematician and philosopher. Thales visited Egypt and brought back knowledge of astronomy and geometry. According to several accounts, Thales, with no special instruments, determined the height of Egyptian pyramids. He applied formal geometric reasoning. Diogenes Laertius, a 3rd-century biographer of ancient Greek philosophers, wrote: 'Hieronymus says that [Thales] even succeeded in measuring the pyramids by observation of the length of their shadow at the moment when our shadows are equal to our own height.' Thales used the geometric principle that the ratios of corresponding sides of similar triangles are equal.

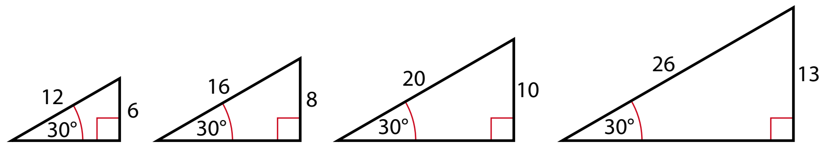


Figure 8.4 Corresponding ratios of a pair of sides for similar triangles are equal.

For any right triangle, the sine ratio for 30° is always $\frac{1}{2}$: $\sin 30° = \frac{1}{2}$.

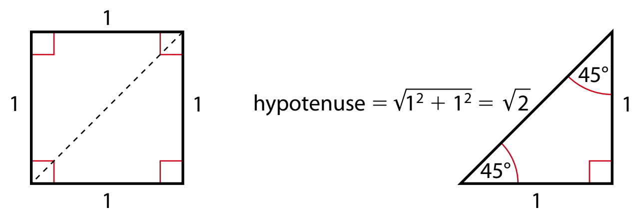
The trigonometric functions of acute angles are not always rational numbers such as $\frac{1}{2}$. We will see in upcoming examples that the sine of 60° is exactly $\frac{\sqrt{3}}{2}$.

Geometric derivation of trigonometric functions for 30°, 45° and 60°

We can use Pythagoras' theorem and properties of triangles to find the exact values for the most common acute angles: 30°, 45° and 60°.

Sine, cosine and tangent values for 45°

Derivation



Consider a square with each side equal to one unit. Draw a diagonal of the square, forming two isosceles right triangles. From geometry, we know that the diagonal will bisect each of the two right angles forming two isosceles right triangles, each with two acute angles of 45°. The isosceles right triangles have legs of length one unit and, from Pythagoras' theorem, a hypotenuse of exactly $\sqrt{2}$ units. The trigonometric functions are then calculated as follows:

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

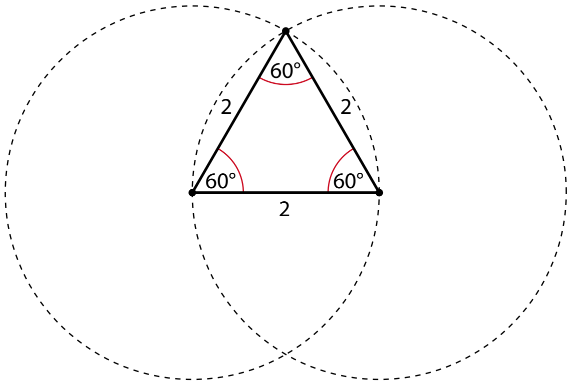
(Multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$ to rationalize the denominator.)

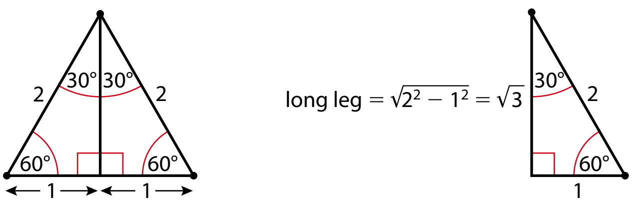
$$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$

Sine, cosine and tangent values for 30° and 60°

Derivation





Start with a line segment of length two units. Using each endpoint as a centre and the segment as a radius, construct two circles. The endpoints of the original line segment and the point of intersection of the two circles are the vertices of an equilateral triangle. Each side has a length of two units and the measure of each angle is 60°. From geometry, the altitude drawn from one of the vertices bisects the angle at that vertex and also bisects the opposite side to which it is perpendicular. Two right triangles are formed that have acute angles of 30° and 60°, a hypotenuse of two units, and a short leg of one unit. Using Pythagoras' theorem, the long leg is $\sqrt{3}$ units. The trigonometric functions of 30° and 60° are then calculated as follows:

$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$ (Rationalizing the denominator.)

The geometric derivation of the values of the sine, cosine and tangent functions for the 'special' acute angles 30°, 45° and 60° agree with the results from the previous chapter. The results for these angles – in both degree and radian measure – are summarised in the box below.

Values of sine, cosine and tangent for common acute angles

$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan 45° = \tan\frac{\pi}{4} = 1$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

● Hint: It is important that you are able to recall – without a calculator – the exact trigonometric values for these common angles.

Observe that $\sin 30° = \cos 60° = \frac{1}{2}$, $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$. Complementary angles (sum of 90°) have equal function values for sine and cosine. That is, for all angles x measured in degrees, $\sin x = \cos(90° - x)$ or $\sin(90° - x) = \cos x$. As noted in Chapter 7, it is for this reason that sine and cosine are called co-functions.

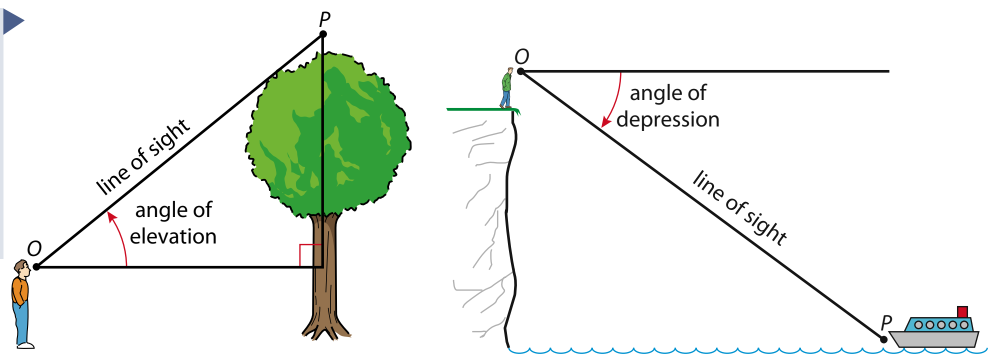
Solution of right triangles

Every triangle has three sides and three angles – six different parts. The ancient Greeks knew how to solve for all of the unknown angles and sides in a right triangle given that either the length of two sides, or the length of one side and the measure of one angle, were known. To **solve a right triangle** means to find the measure of any unknown sides or angles. We can accomplish this by applying Pythagoras' theorem and trigonometric functions. We will utilize trigonometric functions in two different ways when solving for missing parts in right triangles – to find the length of a side, and to find the measure of an angle. Solving right triangles using the sine, cosine and tangent functions is essential to finding solutions to problems in fields such as astronomy, navigation, engineering and architecture. In Sections 8.3 and 8.4, we will see how trigonometry can also be used to solve for missing parts in triangles that are not right triangles.

Angles of depression and elevation

An imaginary line segment from an observation point O to a point P (representing the location of an object) is called the **line of sight** of P. If P is above O, the acute angle between the line of sight of P and a horizontal line passing through O is called the **angle of elevation** of P. If P is below O, the angle between the line of sight and the horizontal is called the **angle of depression** of P. This is illustrated in Figure 8.5.

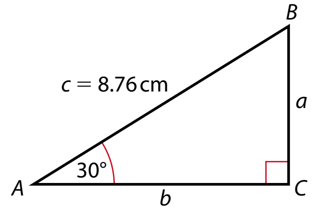
Figure 8.5 An angle of elevation or depression is always measured from the horizontal. Also, note that for each diagram, the angle of elevation from O to P is equal to the angle of depression from P to O.



Example 1

Solve triangle ABC given $c = 8.76$ cm and angle $A = 30°$, where the right angle is at C. Give exact answers when possible, otherwise give to an accuracy of 3 significant figures.

Solution



Knowing that the conventional notation is to use a lower-case letter to represent the length of a side opposite the vertex denoted with the corresponding upper-case letter, we sketch triangle ABC indicating the known measurements.

From the definition of sine and cosine functions, we have

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{8.76} \qquad \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{8.76}$$

$$a = 8.76 \sin 30° \qquad b = 8.76 \cos 30°$$

$$a = 8.76\left(\frac{1}{2}\right) = 4.38 \qquad b = 8.76\left(\frac{\sqrt{3}}{2}\right) \approx 7.586\,382\,537 \approx 7.59$$

Therefore, $a = 4.38$ cm, $b \approx 7.59$ cm, and it's clear that angle $B = 60°$.

We can use Pythagoras' theorem to check our results for a and b.

$$a^2 + b^2 = c^2 \Rightarrow \sqrt{a^2 + b^2} = 8.76$$

Be aware that the result for a is exactly 4.38 cm (assuming measurements given for angle A and side c are exact), but the result for b can only be approximated. To reduce error when performing the check, we should use the most accurate value (i.e. most significant figures) possible for b. The most effective way to do this on our GDC is to use results that are stored to several significant figures, as shown in the GDC screen image.

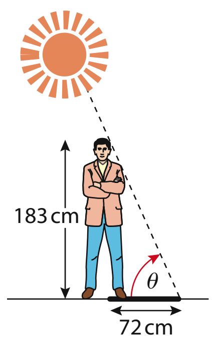
```
8.76(√(3)/2)
              7.586382537
Ans→B
              7.586382537
√(4.38²+B²)
                     8.76
```

Example 2

A man who is 183 cm tall casts a 72 cm long shadow on the horizontal ground. What is the angle of elevation of the sun to the nearest tenth of a degree?

Solution

In the diagram, the angle of elevation of the sun is labelled θ.



● **Hint:** As noted earlier, the notation for indicating the inverse of a function is a superscript of negative one. For example, the inverse of the cosine function is written as $\cos^{-1}$. The negative one is *not* an exponent, so it does not denote reciprocal. Do not make this error: $\cos^{-1} x \neq \frac{1}{\cos x}$.

$$\tan\theta = \frac{183}{72}$$

$$\theta = \tan^{-1}\left(\frac{183}{72}\right)$$

$$\theta \approx 68.5°$$

```
tan⁻¹(183/72)
       68.52320902
```

GDC computation in degree mode

The angle of elevation of the sun is approximately 68.5°.

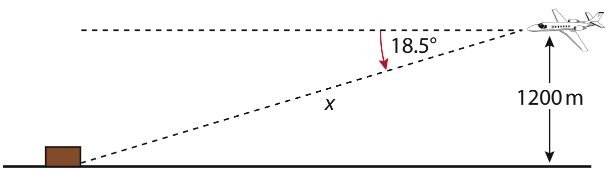
Example 3

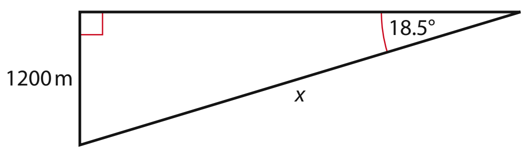
During a training exercise, an air force pilot is flying his jet at a constant altitude of 1200 metres. His task is to fire a missile at a target. At the moment he fires his missile he is able to see the target at an angle of depression of 18.5°. Assuming the missile travels in a straight line, what distance will the missile cover (to the nearest metre) from the jet to the target?

Solution

Draw a diagram to represent the information and let x be the distance that the missile travels from the plane to the target. A right triangle can be 'extracted' from the diagram with one leg 1200 metres, the angle opposite that leg is 18.5°, and the hypotenuse is x. Applying the sine ratio, we can write the equation $\sin 18.5° = \frac{1200}{x}$.

Then $x = \frac{1200}{\sin 18.5°} \approx 3781.85$. Hence, the missile travels approximately 3782 metres.

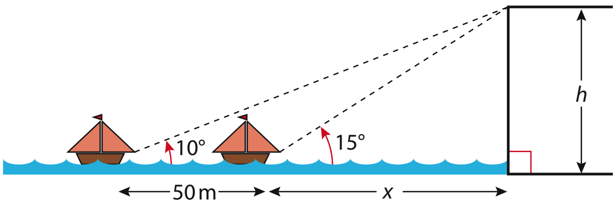




Example 4

A boat is sailing directly towards a cliff. The angle of elevation of a point on the top of the cliff and straight ahead of the boat increases from 10° to 15° as the ship sails a distance of 50 metres. Find the height of the cliff.

Solution



Draw a diagram that accurately represents the information with the height of the cliff labelled h metres and the distance from the base of the cliff to the later position of the boat labelled x metres. There are two right triangles that can be 'extracted' from the diagram. From the smaller right triangle, we have

$$\tan 15° = \frac{h}{x} \Rightarrow h = x \tan 15°$$

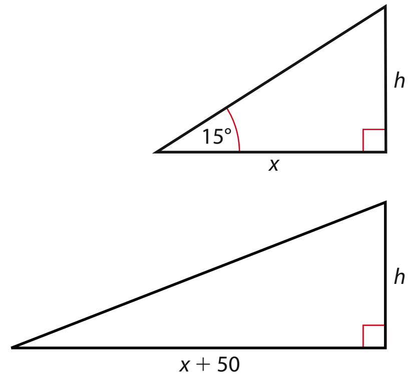
From the larger right triangle, we have

$$\tan 10° = \frac{h}{x + 50} \Rightarrow h = (x + 50)\tan 10°$$

We can solve for x by setting the two expressions for h equal to each other.

Then we can solve for h by substitution.

$$x\tan 15° = (x + 50)\tan 10°$$
$$x\tan 15° = x\tan 10° + 50\tan 10°$$
$$x(\tan 15° - \tan 10°) = 50\tan 10°$$
$$x = \frac{50\tan 10°}{\tan 15° - \tan 10°} \approx 96.225$$



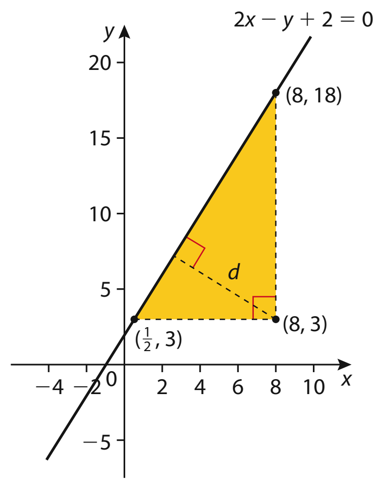
Substituting this value for x into $h = x\tan 15°$, gives

$$h \approx 96.225\tan 15° \approx 25.783$$

Therefore, the height of the cliff is approximately 25.8 metres.

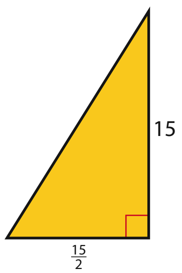
Example 5

Using a suitable right triangle, find the exact minimum distance from the point (8, 3) to the line with the equation $2x - y + 2 = 0$.



Solution

Graph the line with equation $2x - y + 2 = 0$. The minimum distance from the point (8, 3) to the line is the length of the line segment drawn from the point *perpendicular* to the line. This minimum distance is labelled d in the diagram. d is also the height of the large yellow triangle formed by drawing vertical and horizontal line segments from (8, 3) to the line.

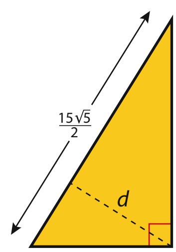


The area of the right triangle is

$$A = \frac{1}{2}\left(\frac{15}{2}\right)(15) = \frac{225}{4}.$$

The area of the triangle can also be found by using the hypotenuse as the base and the distance d as the height. By Pythagoras' theorem, we have

$$\text{hypotenuse} = \sqrt{\left(\frac{15}{2}\right)^2 + 15^2} = \sqrt{\frac{1125}{4}} = \frac{\sqrt{225}\sqrt{5}}{\sqrt{4}} = \frac{15\sqrt{5}}{2}$$



Thus the area can also be expressed as $A = \frac{1}{2}\left(\frac{15\sqrt{5}}{2}\right)d$. We can solve for d by equating the two results for the area of the triangle.

$$\frac{1}{2}\left(\frac{15\sqrt{5}}{2}\right)d = \frac{225}{4}$$
$$\frac{15\sqrt{5}}{4}d = \frac{225}{4}$$
$$d = \frac{225}{4} \cdot \frac{4}{15\sqrt{5}}$$
$$d = \frac{15}{\sqrt{5}} = \frac{15}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{15\sqrt{5}}{5} = 3\sqrt{5}$$

Therefore, the minimum distance from the point (8, 3) to the line with equation $2x - y + 2 = 0$ is $3\sqrt{5}$ units.

Exercise 8.1

For each question 1–9, a) sketch a right triangle corresponding to the given trigonometric function of the acute angle θ, b) find the exact value of the other five trigonometric functions, and c) use your GDC to find the degree measure of θ and the other acute angle (approximate to 3 significant figures).

1 $\sin\theta = \frac{3}{5}$ **2** $\cos\theta = \frac{5}{8}$ **3** $\tan\theta = 2$

4 $\cos\theta = \frac{7}{10}$ **5** $\cot\theta = \frac{1}{3}$ **6** $\sin\theta = \frac{\sqrt{7}}{4}$

7 $\sec\theta = \frac{11}{\sqrt{61}}$ **8** $\tan\theta = \frac{9}{10}$ **9** $\csc\theta = \frac{4\sqrt{65}}{65}$

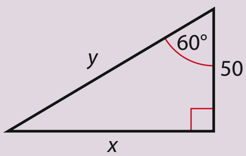
In questions 10–15, find the exact value of θ in degree measure ($0 < \theta < 90°$) and in radian measure $\left(0 < \theta < \frac{\pi}{2}\right)$ without using your GDC.

10 $\cos\theta = \frac{1}{2}$ **11** $\sin\theta = \frac{\sqrt{2}}{2}$ **12** $\tan\theta = \sqrt{3}$

13 $\csc\theta = \frac{2\sqrt{3}}{3}$ **14** $\cot\theta = 1$ **15** $\cos\theta = \frac{\sqrt{3}}{2}$

In questions 16–21, solve for x and y. Give your answer exact or to 3 s.f.

16



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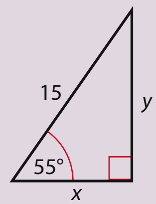

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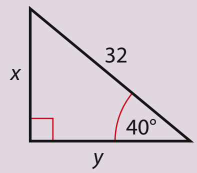

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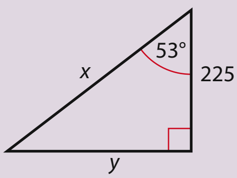

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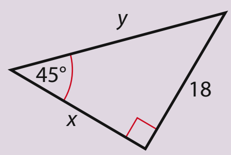

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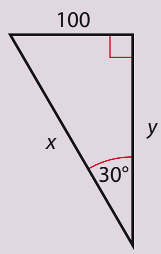
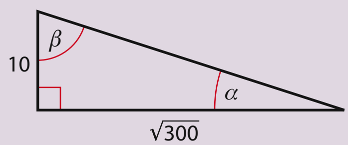


In questions 22–25, find the degree measure of the angles α and β. If possible, give an exact answer – otherwise, approximate to three significant figures.

22



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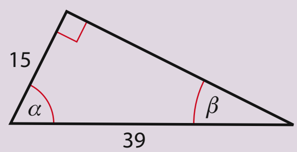

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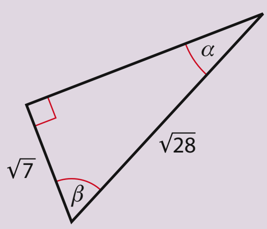

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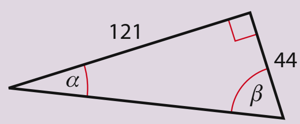
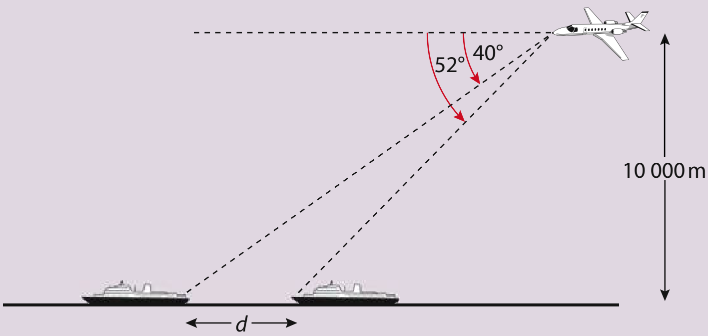


26 The tallest tree in the world is reputed to be a giant redwood named *Hyperion* located in Redwood National Park in California, USA. At a point 41.5 metres from the centre of its base and on the same elevation, the angle of elevation of the top of the tree is 70°. How tall is the tree? Give your answer to three significant figures.

27 The Eiffel Tower in Paris is 300 metres high (not including the antenna on top). What will be the angle of elevation of the top of the tower from a point on the ground (assumed level) that is 125 metres from the centre of the tower's base?

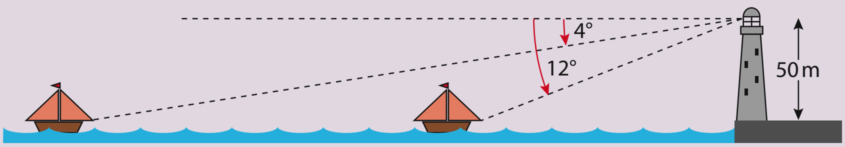
28 A 1.62-metre tall woman standing 3 metres from a streetlight casts a 2-metre long shadow. What is the height of the streetlight?

29 A pilot measures the angles of depression to two ships to be 40° and 52° (see the figure). If the pilot is flying at an elevation of 10 000 metres, find the distance between the two ships.

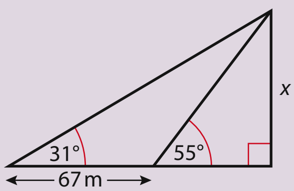


30 Find the measure of all the angles in a triangle with sides of length 8 cm, 8 cm and 6 cm.

31 From a 50-metre observation tower on the shoreline, a boat is sighted at an angle of depression of 4° moving directly toward the shore at a constant speed. Five minutes later the angle of depression of the boat is 12°. What is the speed of the boat in kilometres per hour?



32 Find the length of x indicated in the diagram. Approximate your answer to 3 significant figures.

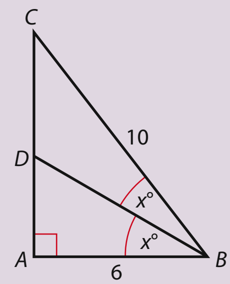


33 A support wire for a tower is connected from an anchor point on level ground to the top of the tower. The straight wire makes a 65° angle with the ground at the anchor point. At a point 25 metres farther from the tower than the wire's anchor point and on the same side of the tower, the angle of elevation to the top of the tower is 35°. Find the wire length to the nearest tenth of a metre.

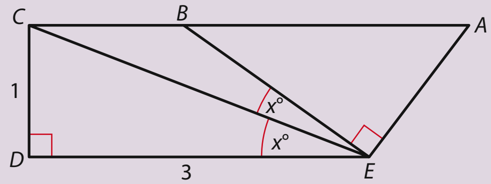
34 A 30-metre high building sits on top of a hill. The angles of elevation of the top and bottom of the building from the same spot at the base of the hill are measured to be 55° and 50° respectively. Relative to its base, how high is the hill to the nearest metre?

35 The angle of elevation of the top of a vertical pole as seen from a point 10 metres away from the pole is double its angle of elevation as seen from a point 70 metres from the pole. Find the height (to the nearest tenth of a metre) of the pole above the level of the observer's eyes.

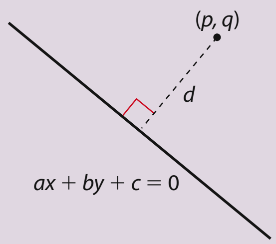
36 Angle ABC of a right triangle is bisected by segment BD. The lengths of sides AB and BC are given in the diagram. Find the exact length of BD, expressing your answer in simplest form.



37 In the diagram, $D\hat{E}C = C\hat{E}B = x°$ and $C\hat{D}E = B\hat{E}A = 90°$, $CD = 1$ unit and $DE = 3$ units. By writing $D\hat{E}A$ in terms of $x°$, find the exact value of $\cos(D\hat{E}A)$.

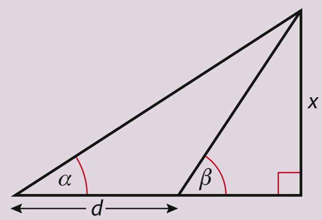


38 For any point with coordinates (p, q) and any line with equation $ax + by + c = 0$, find a formula in terms of a, b, c, p and q that gives the minimum (perpendicular) distance, d, from the point to the line.

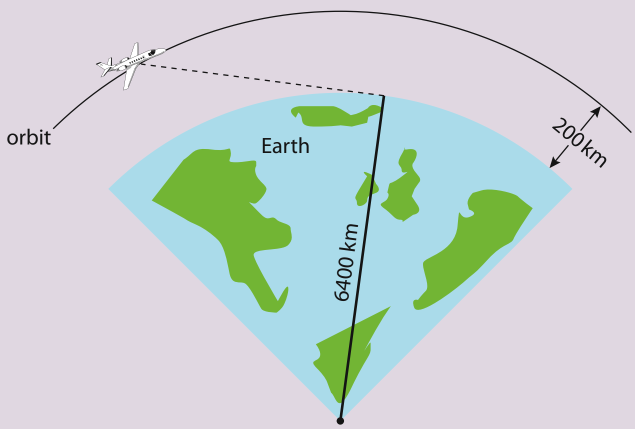


39 Show that the length x in the diagram is given by the formula $x = \dfrac{d}{\cot\alpha - \cot\beta}$.

● **Hint:** First try expressing the formula using the tangent ratio.



40 A spacecraft is travelling in a circular orbit 200 km above the surface of the Earth. Find the angle of depression (to the nearest degree) from the spacecraft to the horizon. Assume that the radius of the Earth is 6400 km. The 'horizontal' line through the spacecraft from which the angle of depression is measured will be parallel to a line tangent to the surface of the Earth directly below the spacecraft.



8.2 Trigonometric functions of any angle

In this section, we will extend the trigonometric ratios to all angles allowing us to solve problems involving any size angle.

Defining trigonometric functions for any angle in standard position

Consider the point $P(x, y)$ on the terminal side of an angle θ in standard position (Figure 8.6) such that r is the distance from the origin O to P. If θ is an acute angle then we can construct a right triangle POQ (Figure 8.7) by dropping a perpendicular from P to a point Q on the x-axis, and it follows that:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}\ (x \neq 0)$$

$$\csc\theta = \frac{r}{y}\ (y \neq 0) \qquad \sec\theta = \frac{r}{x}\ (x \neq 0) \qquad \cot\theta = \frac{x}{y}\ (y \neq 0)$$

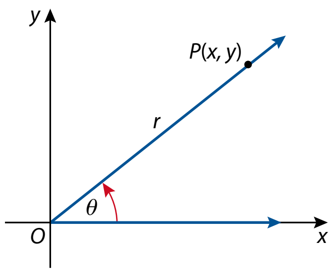


Figure 8.6

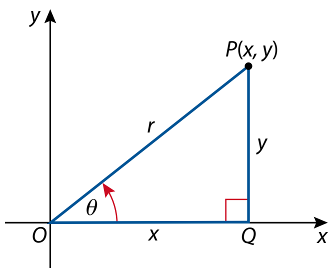


Figure 8.7

Extending this to angles other than acute angles allows us to define the trigonometric functions for any angle – positive or negative. It is important to note that the values of the trigonometric ratios do not depend on the choice of the point $P(x, y)$. If $P'(x', y')$ is any other point on the terminal side of angle θ, as in Figure 8.8, then triangles POQ and $P'OQ'$ are similar and the trigonometric ratios for corresponding angles are equal.

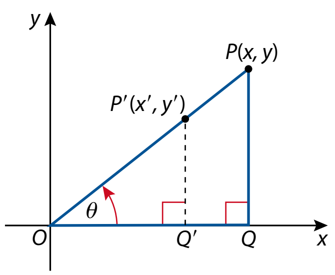
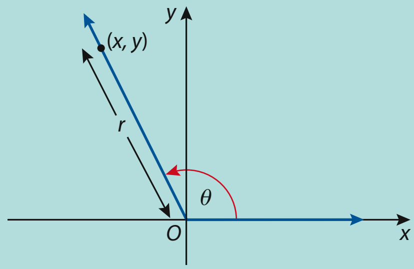


Figure 8.8

Definition of trigonometric functions

Let θ be any angle (in degree or radian measure) in standard position, with (x, y) any point on the terminal side of θ, and $r = \sqrt{x^2 + y^2}$, the distance from the origin to the point (x, y), as shown below.



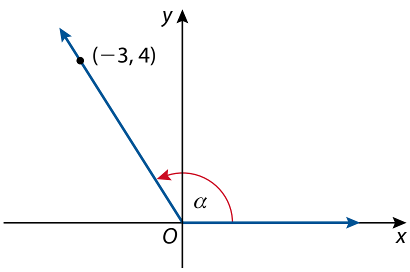
Then the trigonometric functions are defined as follows:

$\sin\theta = \frac{y}{r}$ $\quad\cos\theta = \frac{x}{r}$ $\quad\tan\theta = \frac{y}{x}\ (x \neq 0)$

$\csc\theta = \frac{r}{y}\ (y \neq 0)$ $\quad\sec\theta = \frac{r}{x}\ (x \neq 0)$ $\quad\cot\theta = \frac{x}{y}\ (y \neq 0)$

Example 6

Find the sine, cosine and tangent of an angle α that contains the point $(-3, 4)$ on its terminal side when in standard position.



Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

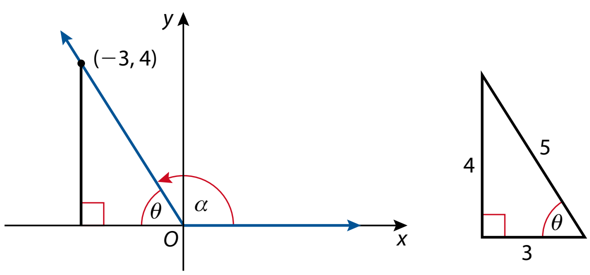
$$\text{Then,}\quad \sin\alpha = \frac{y}{r} = \frac{4}{5}$$

$$\cos\alpha = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$$

$$\tan\alpha = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

Note that for the angle α in Example 6, we can form a right triangle by constructing a line segment from the point $(-3, 4)$ perpendicular to the x-axis, as shown in Figure 8.9. Clearly, $\theta = 180° - \alpha$. Furthermore, the values of the sine, cosine and tangent of the angle θ are the same as that for the angle α, except that the *sign* may be different.

Figure 8.9



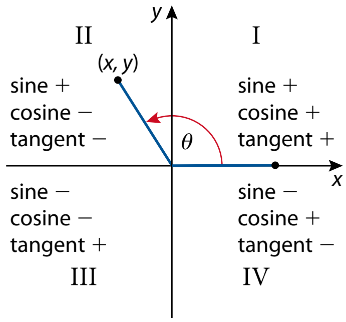


Figure 8.10 Sign of trigonometric function values depends on the quadrant in which the terminal side of the angle lies.

Whether the trigonometric functions are defined in terms of the length of an arc or in terms of an angle, the signs of trigonometric function values are determined by the quadrant in which the arc or angle lies, when in standard position (Figure 8.10).

Example 7

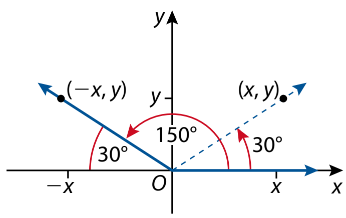
Find the sine, cosine and tangent of the obtuse angle that measures 150°.

Solution

The terminal side of the angle forms a 30° angle with the x-axis. The sine values for 150° and 30° will be exactly the same, and the cosine and tangent values will be the same but of opposite sign. We know that

$\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$ and $\tan 30° = \frac{\sqrt{3}}{3}$.

Therefore, $\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$ and $\tan 150° = -\frac{\sqrt{3}}{3}$.



Example 7 illustrates three trigonometric identities for angles whose sum is 180° (i.e. a pair of supplementary angles). The following are true for any acute angle θ:

$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$

$\csc(180° - \theta) = \csc\theta$

$\sec(180° - \theta) = -\sec\theta$

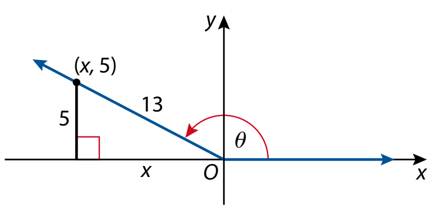
$\cot(180° - \theta) = -\cot\theta$

Example 8

Given that $\sin\theta = \frac{5}{13}$ and $90° < \theta < 180°$, find the exact values of $\cos\theta$ and $\tan\theta$.

Solution

θ is an angle in the second quadrant. It follows from the definition $\sin\theta = \frac{y}{r}$ that with θ in standard position there must be a point on the terminal side of the angle that is 13 units from the origin (i.e. $r = 13$) and which has a y-coordinate of 5, as shown in the diagram.



Using Pythagoras' theorem, $|x| = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$. Because θ is in the second quadrant, the x-coordinate of the point must be negative, thus $x = -12$.

Therefore, $\cos\theta = \frac{-12}{13} = -\frac{12}{13}$, and $\tan\theta = \frac{5}{-12} = -\frac{5}{12}$.

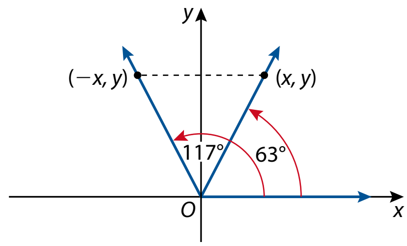
Example 9

a) Find the acute angle with the same sine ratio as (i) 135°, and (ii) 117°.
b) Find the acute angle with the same cosine ratio as (i) 300°, and (ii) 342°.

Solution

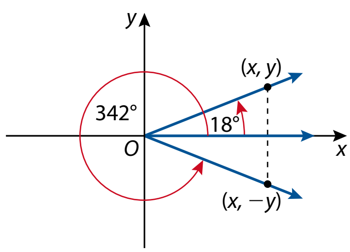
a) (i) Angles in the first and second quadrants have the same sine ratio. Hence, the identity $\sin(180° - \theta) = \sin\theta$. Since $180° - 135° = 45°$, then $\sin 135° = \sin 45°$.

(ii) Since $180° - 117° = 63°$, then $\sin 117° = \sin 63°$.



b) (i) Angles in the first and fourth quadrants have the same cosine ratio. Hence, the identity $\cos(360° - \theta) = \cos\theta$. Since $360° - 300° = 60°$, then $\cos 300° = \cos 60°$.

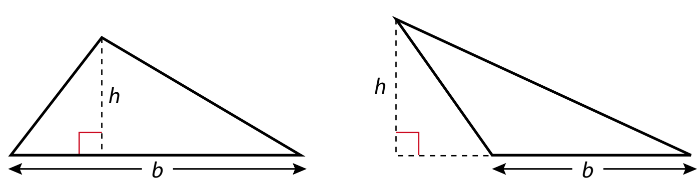
(ii) Since $360° - 342° = 18°$, then $\cos 342° = \cos 18°$.



Areas of triangles

You are familiar with the standard formula for the area of a triangle, area $= \frac{1}{2} \times$ base $\times$ height (or area $= \frac{1}{2}bh$), where the base, b, is a side of the triangle and the height, h, (or altitude) is a line segment perpendicular to the base (or the line containing it) and drawn to the vertex opposite to the base, as shown in Figure 8.11.

Figure 8.11



If the lengths of two sides of a triangle and the measure of the angle between these sides (often called the included angle) are known, then the triangle is unique and has a fixed area. Hence, we should be able to calculate the area from just these measurements, i.e. from knowing two sides and the included angle. This calculation is quite straightforward if the triangle is a right triangle (Figure 8.12) and we know the lengths of the two legs on either side of the right angle.

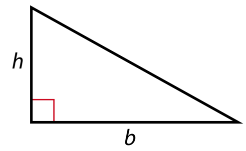


Figure 8.12 A right triangle.

Let's develop a general area formula that will apply to any triangle – right, acute or obtuse. For triangle *ABC* shown in Figure 8.13, suppose we know the lengths of the two sides *a* and *b* and the included angle *C*. If the length of the height from *B* is *h*, the area of the triangle is $\frac{1}{2}bh$. From right triangle trigonometry, we know that $\sin C = \frac{h}{a}$, or $h = a\sin C$. Substituting $a\sin C$ for h, area $= \frac{1}{2}bh = \frac{1}{2}b(a\sin C) = \frac{1}{2}ab\sin C$.

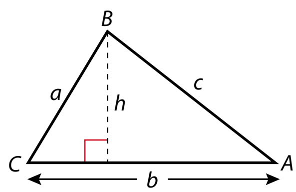


Figure 8.13 An acute triangle.

If the angle *C* is obtuse, then from Figure 8.14 we see that $\sin(180° - C) = \frac{h}{a}$. So, the height is $h = a\sin(180° - C)$. However, $\sin(180° - C) = \sin C$. Thus, $h = a\sin C$ and, again, area $= \frac{1}{2}ab\sin C$.

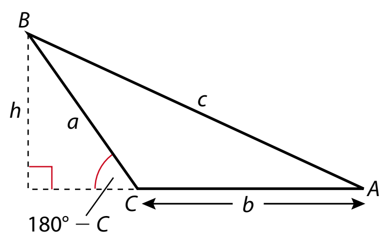


Figure 8.14 An obtuse triangle.

• **Hint:** Note that the procedure for finding the area of a triangle from a pair of sides and the included angle can be performed three different ways. For any triangle labelled in the manner of the triangles in Figures 8.13 and 8.14, its area is expressed by any of the following three expressions.

$$\text{Area of } \triangle = \tfrac{1}{2}ab\sin C$$
$$= \tfrac{1}{2}ac\sin B$$
$$= \tfrac{1}{2}bc\sin A$$

These three equivalent expressions will prove to be helpful for developing an important formula for solving non-right triangles in the next section.

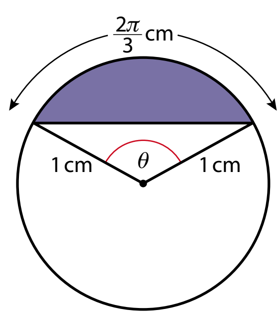
Area of a triangle

For a triangle with sides of lengths *a* and *b* and included angle *C*,

$$\text{Area of } \triangle = \tfrac{1}{2}ab\sin C$$

Example 10

The circle shown has a radius of 1 cm and the central angle θ subtends an arc of length of $\frac{2\pi}{3}$ cm. Find the area of the shaded region.

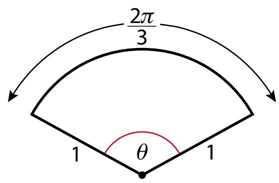


The region bounded by an arc of a circle and the chord connecting the endpoints of the arc is called a **segment** of the circle (see figure for Example 10).

Solution

The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$ (Section 7.1), where θ is the central angle in radian measure. Since the radius of the circle is one, the length of the arc subtended by θ is the same as the radian measure of θ.

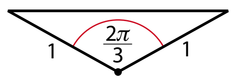
Thus, area of sector $= \frac{1}{2}(1)^2\left(\frac{2\pi}{3}\right) = \frac{\pi}{3}$ cm^2.



The area of the triangle formed by the two radii and the chord is equal to

$$\frac{1}{2}(1)(1)\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}\text{ cm}^2.$$

$$\left[\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$$



The area of the shaded region is found by subtracting the area of the triangle from the area of the sector.

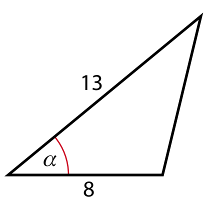
Area $= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$ or $\frac{4\pi - 3\sqrt{3}}{12}$ or approximately 0.614 cm^2 (3 s.f.).

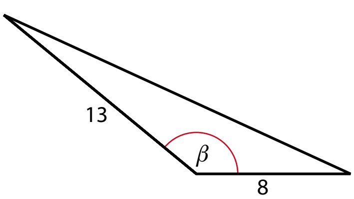
Example 11

Show that it is possible to construct two different triangles with an area of 35 cm^2 that have sides measuring 8 cm and 13 cm. For each triangle, find the measure of the (included) angle between the sides of 8 cm and 13 cm to the nearest tenth of a degree.

Solution

We can visualize the two different triangles with equal areas – one with an acute included angle (α) and the other with an obtuse included angle (β).





$$\text{Area} = \tfrac{1}{2}(\text{side})(\text{side})(\text{sine of included angle}) = 35\text{ cm}^2$$

$$= \tfrac{1}{2}(8)(13)(\sin\alpha) = 35$$

$$52\sin\alpha = 35$$

$$\sin\alpha = \frac{35}{52}$$

$$\alpha = \sin^{-1}\left(\frac{35}{52}\right)$$ Recall that the GDC will only give the acute angle with sine ratio of $\frac{35}{52}$.

$$\alpha \approx 42.3°$$ Round to the nearest tenth.

Knowing that $\sin(180° - \alpha) = \sin\alpha$, the obtuse angle β is equal to $180° - 42.3° = 137.7°$.

Check this answer by computing on your GDC:

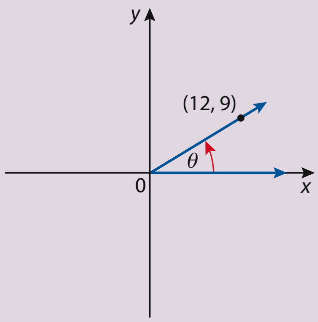
$\frac{1}{2}(8)(13)(\sin 137.7°) \approx 34.997 \approx 35$ cm^2.

Therefore, there are two different triangles with sides 8 cm and 13 cm and area of 35 cm^2 – one with an included angle of 42.3° and the other with an included angle of 137.7°.

Exercise 8.2

In questions 1–4, find the exact value of the sine, cosine and tangent functions of the angle θ.

1



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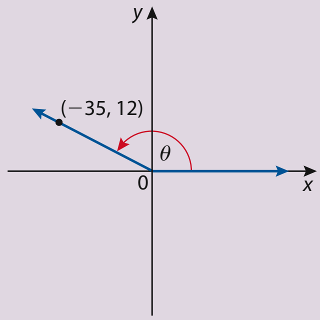

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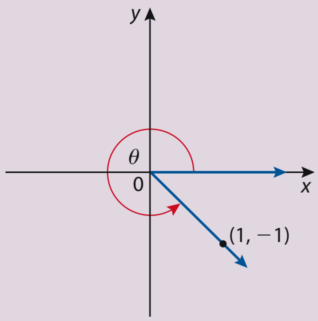

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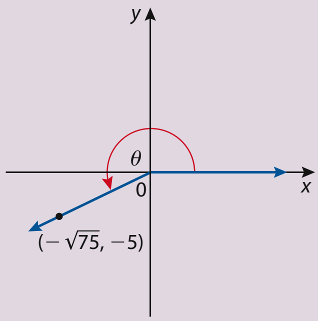


5 Without using your GDC, determine the exact values of all six trigonometric functions for the following angles.

a) $120°$ b) $135°$ c) $330°$ d) $270°$ e) $240°$

f) $\frac{5\pi}{4}$ g) $-\frac{\pi}{6}$ h) $\frac{7\pi}{6}$ i) $-60°$ j) $-\frac{3\pi}{2}$

k) $\frac{5\pi}{3}$ l) $-210°$ m) $-\frac{\pi}{4}$ n) π o) 4.25π

6 Given that $\cos\theta = \frac{8}{17}$ and $0° < \theta < 90°$, find the exact values of the other five trigonometric functions.

7 Given that $\tan\theta = -\frac{6}{5}$ and $\sin\theta < 0$, find the exact values of $\sin\theta$ and $\cos\theta$.

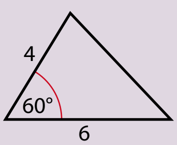
8 Given that $\sin\theta = 0$ and $\cos\theta < 0$, find the exact values of the other five trigonometric functions.

9 If $\sec\theta = 2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the exact values of the other five trigonometric functions.

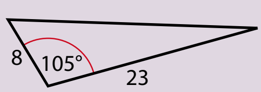
10 a) Find the acute angle with the same sine ratio as (i) 150°, and (ii) 95°.
b) Find the acute angle with the same cosine ratio as (i) 315°, and (ii) 353°.
c) Find the acute angle with the same tangent ratio as (i) 240°, and (ii) 200°.

11 Find the area of each triangle. Express the area exactly, or, if not possible, express it accurate to 3 s.f.

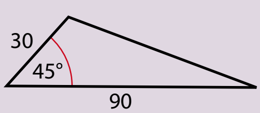
a)



b)



c)

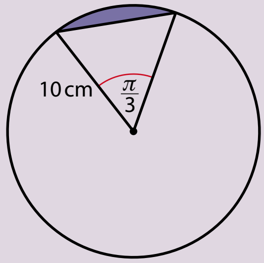


12 Triangle ABC has an area of 43 cm^2. The length of side AB is 12 cm and the length of side AC is 15 cm. Find the degree measure of angle A.

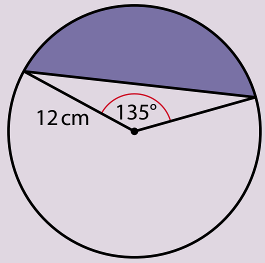
13 A chord AB subtends an angle of 120° at O, the centre of a circle with radius 15 cm. Find the area of a) the sector AOB, and b) the triangle AOB.

14 Find the area of the shaded region (called a *segment*) in each circle.

a)

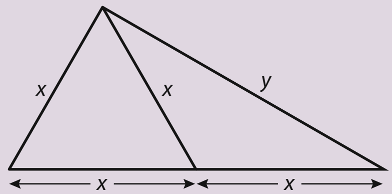


b)

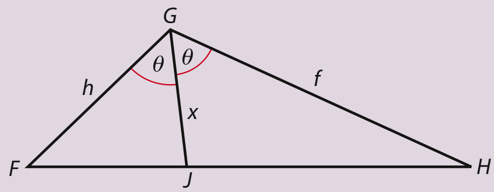


15 Two adjacent sides of a parallelogram have lengths a and b and the angle between these two sides is θ. Express the area of the parallelogram in terms of a, b and θ.

16 For the diagram shown, express y in terms of x.

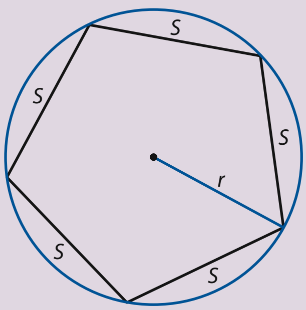


17 In the diagram, GJ bisects $F\hat{G}H$ such that $F\hat{G}J = H\hat{G}J = \theta$. Express x in terms of h, f and $\cos\theta$.



18 If s is the length of each side of a regular polygon with n sides and r is the radius of the circumscribed circle, show that $s = 2r\sin\left(\frac{180°}{n}\right)$. (Note: A *regular* polygon has all sides equal.)

The figure shows a regular pentagon ($n = 5$) with each side of length s circumscribed by a circle with radius r.



19 Suppose a triangle has two sides of lengths 6 cm and 8 cm and an included angle x.

a) Express the area of the triangle as a function of x.

b) State the domain and range of the function and sketch its graph for a suitable interval of x.

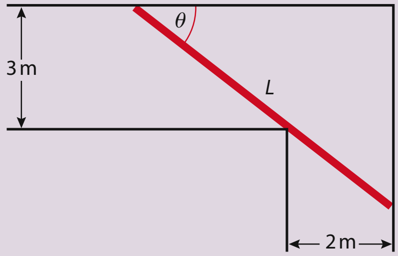
c) Find the exact coordinates of the maximum point of the function. What type of triangle corresponds to this maximum? Explain why this triangle gives a maximum area.

20 A long metal rod is being carried down a hallway 3 metres wide. At the end of the hall there is a right-angled turn into a narrower hallway 2 metres wide. The angle that the rod makes with the outer wall is θ (see figure on the next page).

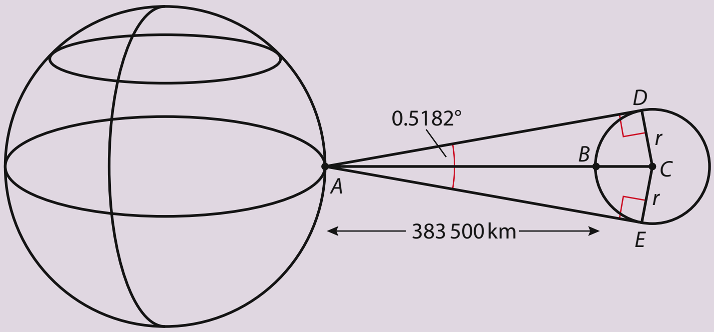
a) Show that the length, L, of the rod is given by the function $L(\theta) = 3\csc\theta + 2\sec\theta$.

b) On your GDC, graph the function L for the interval $0 < \theta < \frac{\pi}{2}$.

c) Using the built-in features of your GDC, find the minimum value of the function L. Explain why this is the length of the longest rod that can be carried around the corner.



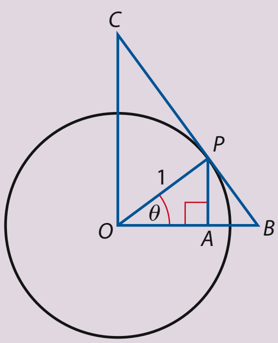
21 As viewed from the surface of the Earth (A), the angle subtended by the full Moon ($D\hat{A}E$) is 0.5182°. Given that the distance from the Earth's surface to the Moon's surface (AB) is approximately 383 500 kilometres, find the radius, r, of the Moon to three significant figures.



22 a) Given that $\sin\theta = x$, find $\sec\theta$ in terms of x.

b) Given that $\tan\beta = y$, find $\sin\beta$ in terms of y.

23 The figure shows the unit circle with angle θ in standard position. Segment BC is tangent to the circle at P and $B\hat{O}C$ is a right angle. Each of the six trigonometric functions of θ is equal to the length of a line segment in the figure. For example, we know from the previous section (and previous chapter) that $\sin\theta = AP$. For each of the five other trigonometric functions, find a line segment in the figure whose length equals the function value of θ.



8.3 The law of sines

In Section 8.1 we used techniques from right triangle trigonometry to solve right triangles when an acute angle and one side are known, or when two sides are known. In this section and the next, we will study methods for finding unknown lengths and angles in triangles that are not right triangles. These general methods are effective for solving problems involving any kind of triangle – right, acute or obtuse.

Possible triangles constructed from three given parts

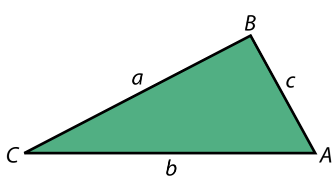
As mentioned in the previous paragraph, we've solved right triangles by either knowing an acute angle and one side, or knowing two sides. Since the triangles also have a right angle, each of those two cases actually involved knowing three different parts of the triangle – either two angles and a side, or two sides and an angle. We need to know at least three parts of a triangle in order to solve for other unknown parts. Different arrangements of the three known parts can be given. Before solving for unknown parts, it is helpful to know whether the three known parts determine a unique triangle, more than one triangle, or none. The table below summarizes the five different arrangements of three parts and the number of possible triangles for each. You are encouraged to confirm these results on your own with manual or computer generated sketches.

Possible triangles formed with three known parts

Known parts	Number of possible triangles
Three angles (AAA)	Infinite triangles (not possible to solve)
Three sides (SSS) (sum of any two must be greater than the third)	One unique triangle
Two sides and their included angle (SAS)	One unique triangle
Two angles and any side (ASA or AAS)	One unique triangle
Two sides and a non-included angle (SSA)	No triangle, one triangle or two triangles

ASA, AAS and SSA can be solved using the **law of sines**, whereas SSS and SAS can be solved using the **law of cosines** (next section).

The law of sines (or sine rule)



In the previous section, we showed that we can write three equivalent expressions for the area of any triangle for which we know two sides and the included angle.

$$\text{Area of } \triangle = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}bc\sin A$$

If each of these expressions is divided by $\frac{1}{2}abc$,

$$\frac{\frac{1}{2}ab\sin C}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}abc}$$

we obtain three equivalent ratios – each containing the sine of an angle divided by the length of the side opposite the angle.

The law of sines

If A, B and C are the angle measures of any triangle and a, b and c are, respectively, the lengths of the sides opposite these angles, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

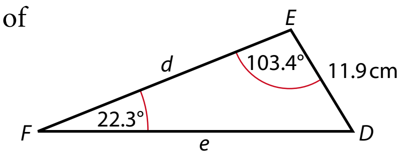
Alternatively, the law of sines can also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Solving triangles given two angles and any side (ASA or AAS)

If we know two angles and any side of a triangle, we can use the law of sines to find any of the other angles or sides of the triangle.

Example 12

Find all of the unknown angles and sides of triangle DEF shown in the diagram. Approximate all measurements to 1 decimal place.



Solution

The third angle of the triangle is

$$D = 180° - E - F = 180° - 103.4° - 22.3° = 54.3°.$$

Using the law of sines, we can write the following proportion to solve for the length e:

$$\frac{\sin 22.3°}{11.9} = \frac{\sin 103.4°}{e}$$

$$e = \frac{11.9 \sin 103.4°}{\sin 22.3°} \approx 30.507 \text{ cm}$$

We can write another proportion from the law of sines to solve for d:

$$\frac{\sin 22.3°}{11.9} = \frac{\sin 54.3°}{d}$$

$$d = \frac{11.9 \sin 54.3°}{\sin 22.3°} \approx 25.467 \text{ cm}$$

Therefore, the other parts of the triangle are $D = 54.3°$, $e \approx 30.5$ cm and $d \approx 25.5$ cm.

- **Hint:** When using your GDC to find angles and lengths with the law of sines (or the law of cosines), remember to store intermediate answers on the GDC for greater accuracy. By not rounding until the final answer, you reduce the amount of round-off error.

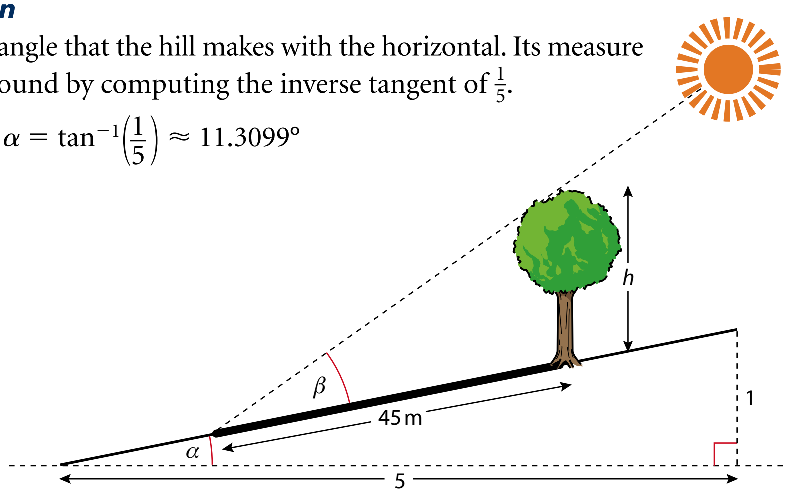
Example 13

A tree on a sloping hill casts a shadow 45 m along the side of the hill. The gradient of the hill is $\frac{1}{5}$ (or 20%) and the angle of elevation of the sun is 35°. How tall is the tree to the nearest tenth of a metre?

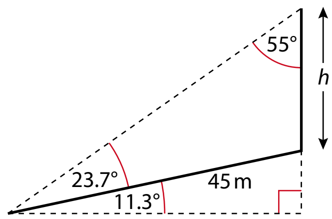
Solution

α is the angle that the hill makes with the horizontal. Its measure can be found by computing the inverse tangent of $\frac{1}{5}$.

$$\alpha = \tan^{-1}\left(\frac{1}{5}\right) \approx 11.3099°$$



The height of the tree is labelled h. The angle of elevation of the sun is the angle between the sun's rays and the horizontal. In the diagram, this angle of elevation is the sum of α and β. Thus, $\beta \approx 35° - 11.3099° \approx 23.6901°$. For the larger right triangle with $\alpha + \beta = 35°$ as one of its acute angles, the other acute angle – and the angle in the obtuse triangle opposite the side of 45 m – must be 55°. Now we can apply the law of sines for the obtuse triangle to solve for h.



$$\frac{\sin 23.7°}{h} = \frac{\sin 55°}{45} \Rightarrow h = \frac{45 \sin 23.7°}{\sin 55°} \approx 22.0809$$

Therefore, the tree is approximately 22.1 m tall.

Two sides and a non-included angle (SSA) – the ambiguous case

The arrangement where we are given the lengths of two sides of a triangle and the measure of an angle not between those two sides can produce three different results: no triangle, one unique triangle or two different triangles. Let's explore these possibilities with the following example.

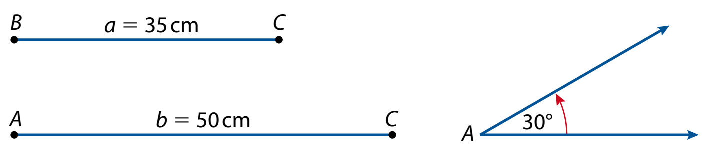
Example 14

Find all of the unknown angles and sides of triangle ABC where $a = 35$ cm, $b = 50$ cm and $A = 30°$. Approximate all measurements to 1 decimal place.

Solution

Figure 8.15 shows the three parts we have from which to try and construct a triangle.

Figure 8.15



We attempt to construct the triangle, as shown in Figure 8.16. We first draw angle A with its initial side (or base line of the triangle) extended. We then measure off the known side $b = AC = 50$. To construct side a (opposite angle A), we take point C as the centre and with radius $a = 35$ we draw an arc of a circle. The points on this arc are all possible positions for vertex B – one of the endpoints of side a, or BC. Point B must be on the base line, so B can be located at any point of intersection of the circular arc and the base line. In this instance, with these particular measurements for the two sides and non-included angle, there are two points of intersection, which we label B_1 and B_2.

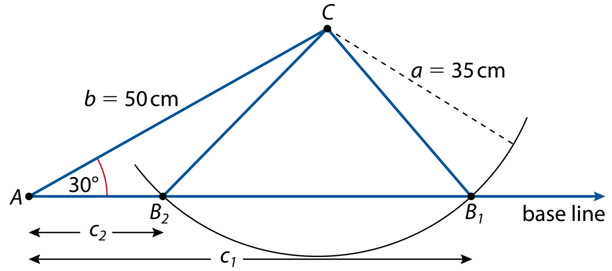


Figure 8.16

Therefore, we can construct two different triangles, triangle AB_1C (Figure 8.17) and triangle AB_2C (Figure 8.18). The angle B_1 will be acute and angle B_2 will be obtuse. To complete the solution of this problem, we need to solve each of these triangles.

- Solve triangle AB_1C:

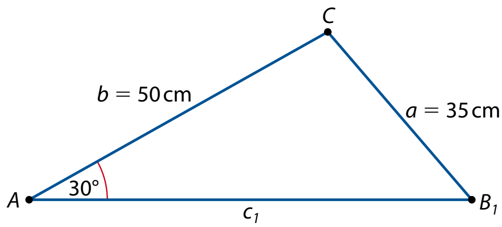


Figure 8.17

We can solve for acute angle B_1 using the law of sines:

$$\frac{\sin 30°}{35} = \frac{\sin B_1}{50}$$

$$\sin B_1 = \frac{50 \sin 30°}{35} = \frac{50(0.5)}{35}$$

$$\mathrm{B}_1 = \sin^{-1}\left(\frac{5}{7}\right) \approx 45.5847°$$

Then, $C \approx 180° - 30° - 45.5847° \approx 104.4153°$.

With another application of the law of sines, we can solve for side c_1:

$$\frac{\sin 30°}{35} = \frac{\sin 104.4153°}{c_1}$$

$$c_1 = \frac{35 \sin 104.4153°}{\sin 30°} \approx \frac{35(0.96852)}{0.5} \approx 67.7964 \text{ cm}$$

Therefore, for triangle AB_1C, $B_1 \approx 45.6°$, $C \approx 104.4°$ and $c_1 \approx 67.8$ cm.

- Solve triangle AB_2C:

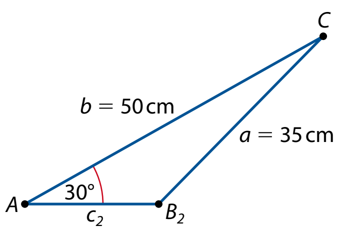


Figure 8.18

Solving for obtuse angle B_2, using the law of sines, gives the same result as above, except we know that $90° < B_2 < 180°$.

We also know that $\sin(180° - \theta) = \sin\theta$.

Thus, $B_2 = 180° - B_1 \approx 180° - 45.5847° \approx 134.4153°$.

Then, $C \approx 180° - 30° - 134.4153° \approx 15.5847°$.

With another application of the law of sines, we can solve for side c_2:

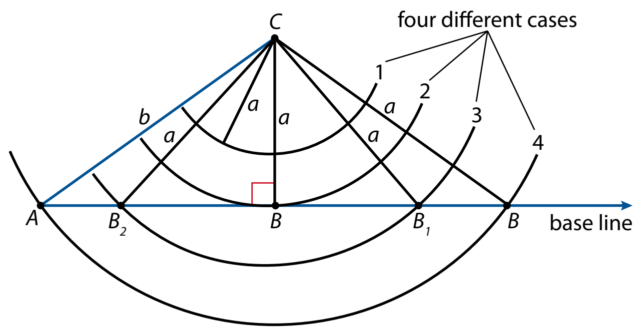
$$\frac{\sin 30°}{35} = \frac{\sin 15.5847°}{c_2}$$

$$c_2 \approx \frac{35 \sin 15.5847°}{\sin 30°} \approx \frac{35(0.26866)}{0.5} \approx 18.8062 \text{ cm}$$

Therefore, for triangle AB_2C, $B_2 \approx 134.4°$, $C \approx 15.6°$ and $c_2 \approx 18.8$ cm.

Now that we have solved this specific example, let's take a more general look and examine all the possible conditions and outcomes for the SSA arrangement. In general, we are given the lengths of two sides – call them a and b – and a non-included angle – for example, angle A that is opposite side a. From these measurements, we can determine the number of different triangles. Figure 8.19 shows the four different possibilities (or cases) when angle A is acute. The number of triangles depends on the length of side a.

Figure 8.19 Four distinct cases for SSA when angle *A* is acute.



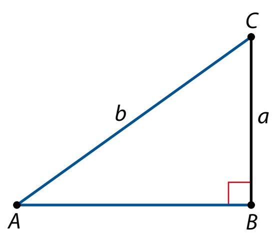


Figure 8.20 Case 2 for SSA: $a = b \sin A$, one right angle.

In case 2, side a is perpendicular to the base line resulting in a single right triangle, shown in Figure 8.20. In this case, clearly $\sin A = \frac{a}{b}$ and $a = b \sin A$. In case 1, the length of a is shorter than it is in case 2, i.e $b \sin A$. In case 3, which occurred in Example 14, the length of a is longer than $b \sin A$, but less than b. And, in case 4, the length of a is greater than b. These results are summarized in the table below. Because the number of triangles may be none, one or two, depending on the length of a (the side opposite the given angle), the SSA arrangement is called the ambiguous case.

The ambiguous case (SSA)

Given the lengths of sides a and b and the fact that the non-included angle A is acute, the following four cases and resulting triangles can occur.

Length of a	Number of triangles	Case in Figure 8.19
$a < b \sin A$	No triangle	1
$a = b \sin A$	One right triangle	2
$b \sin A < a < b$	Two triangles	3
$a \geqslant b$	One triangle	4

The situation is considerably simpler if angle A is obtuse rather than acute. Figure 8.21 shows that if $a > b$ then there is only one possible triangle, and if $a \leqslant b$ then no triangle that contains angle A is possible.

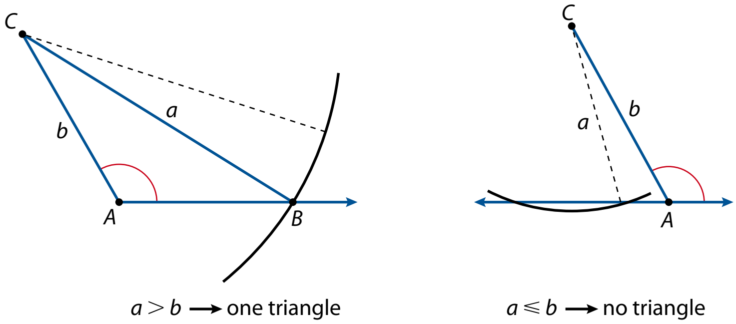


Figure 8.21 Angle A is obtuse.

Example 15

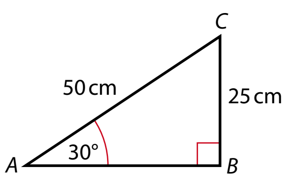
For triangle ABC, if side $b = 50$ cm and angle $A = 30°$, find the values for the length of side a that will produce: a) no triangle, b) one triangle, c) two triangles. This is the same SSA information given in Example 14 with the exception that side a is not fixed at 35 cm, but is allowed to vary.

Solution

Because this is a SSA arrangement and given A is an acute angle, then the number of different triangles that can be constructed is dependent on the length of a. First calculate the value of $b \sin A$:

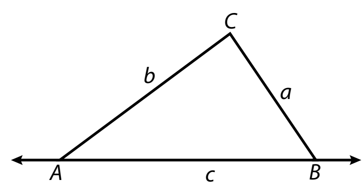
$$b \sin A = 50 \sin 30° = 50(0.5) = 25 \text{ cm}$$

Thus, if a is exactly 25 cm then triangle ABC is a right triangle, as shown in the figure.



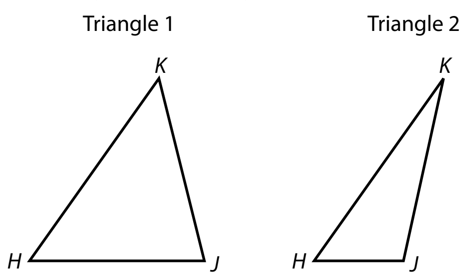
a) If $a < 25$ cm, there is no triangle.
b) If $a = 25$ cm, or $a \geqslant 50$ cm, there is one unique triangle.
c) If $25 \text{ cm} < a < 50$ cm, there are two different possible triangles.

• **Hint:** It is important to be familiar with the notation for line segments and angles commonly used in IB exam questions. For example, the line segment labelled b in the diagram (below) is denoted as [AC] in IB notation. Angle A, the angle between [BA] and [AC], is denoted as $B\hat{A}C$. Also, the line containing points A and B is denoted as (AB).



Example 16

The diagrams below show two different triangles both satisfying the conditions: $HK = 18$ cm, $JK = 15$ cm, $J\hat{H}K = 53°$.



a) Calculate the size of $H\hat{J}K$ in Triangle 2.
b) Calculate the area of Triangle 1.

Solution

a) From the law of sines, $\frac{\sin(H\hat{J}K)}{18} = \frac{\sin 53°}{15} \Rightarrow \sin(H\hat{J}K) = \frac{18\sin 53°}{15}$

$\approx 0.958\,36 \Rightarrow \sin^{-1}(0.958\,36) \approx 73.408°$

However, $H\hat{J}K > 90° \Rightarrow H\hat{J}K \approx 180° - 73.408° \approx 106.592°$.

Therefore, in Triangle 2 $H\hat{J}K \approx 107°$ (3 s.f.).

b) In Triangle 1, $H\hat{J}K < 90° \Rightarrow H\hat{J}K \approx 73.408°$

$\Rightarrow H\hat{K}J \approx 180° - (73.408° + 53°) \approx 53.592°$

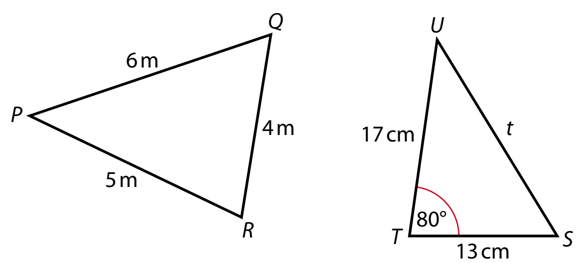
Area $= \frac{1}{2}(18)(15)\sin(53.592°) \approx 108.649\text{ cm}^2$.

Therefore, the area of Triangle 1 is approximately 109 cm^2 (3 s.f.).

8.4 The law of cosines

Two cases remain in our list of different ways to arrange three known parts of a triangle. If three sides of a triangle are known (SSS arrangement), or two sides of a triangle and the angle between them are known (SAS arrangement), then a unique triangle is determined. However, in both of these cases, the law of sines cannot solve the triangle.

Figure 8.22



For example, it is not possible to set up an equation using the law of sines to solve triangle *PQR* or triangle *STU* in Figure 8.22.

- Trying to solve $\triangle PQR$: $\frac{\sin P}{4} = \frac{\sin R}{6} \Rightarrow$ two unknowns; cannot solve for angle P or angle R.
- Trying to solve $\triangle STU$: $\frac{\sin 80°}{t} = \frac{\sin U}{13} \Rightarrow$ two unknowns; cannot solve for angle U or side t.

The law of cosines (or cosine rule)

We will need the **law of cosines** to solve triangles with these kinds of arrangements of sides and angles. To derive this law, we need to place a general triangle *ABC* in the coordinate plane so that one of the vertices is at the origin and one of the sides is on the positive x-axis. Figure 8.23 shows both an acute triangle *ABC* and an obtuse triangle *ABC*. In either case, the coordinates of vertex C are $x = b\cos C$ and $y = b\sin C$. Because c is the distance from A to B, then we can use the distance formula to write

$c = \sqrt{(b\cos C - a)^2 + (b\sin C - 0)^2}$	Distance between $(b\cos C, b\sin C)$ and $(a, 0)$.
$c^2 = (b\cos C - a)^2 + (b\sin C - 0)^2$	Squaring both sides.
$c^2 = b^2\cos^2 C - 2ab\cos C + a^2 + b^2\sin^2 C$	Expand.
$c^2 = b^2(\cos^2 C + \sin^2 C) - 2ab\cos C + a^2$	Factor out b^2 from two terms.
$c^2 = b^2 - 2ab\cos C + a^2$	Apply trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.
$c^2 = a^2 + b^2 - 2ab\cos C$	Rearrange terms.

This equation gives one form of the law of cosines. Two other forms are obtained in a similar manner by having either vertex A or vertex B, rather than C, located at the origin.

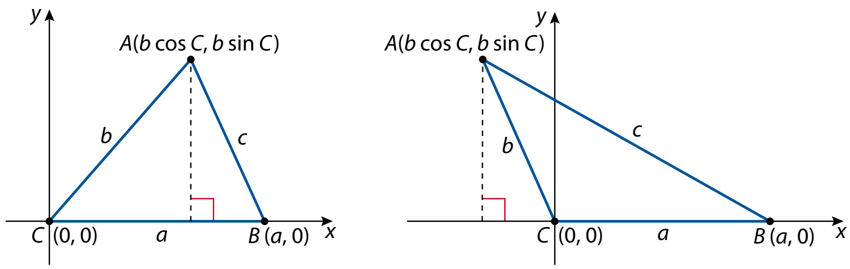


Figure 8.23 Deriving the cosine rule.

The law of cosines

In any triangle ABC with corresponding sides a, b and c:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

It is helpful to understand the underlying pattern of the law of cosines when applying it to solve for parts of triangles. The pattern relies on choosing one particular angle of the triangle and then identifying the two sides that are adjacent to the angle and the one side that is opposite to it. The law of cosines can be used to solve for the chosen angle or the side opposite the chosen angle.

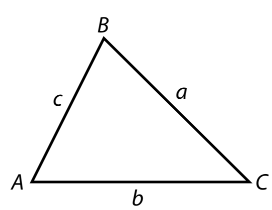
side opposite the chosen angle

chosen angle

$$c^2 = a^2 + b^2 - 2ab\cos C$$

one side adjacent to the chosen angle

other side adjacent to the chosen angle

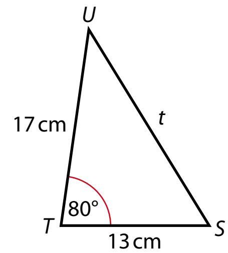


Solving triangles given two sides and the included angle (SAS)

If we know two sides and the included angle, we can use the law of cosines to solve for the side opposite the given angle. Then it is best to solve for one of the two remaining angles using the law of sines.

Example 17

Find all of the unknown angles and sides of triangle *STU*, one of the triangles shown earlier in Figure 8.22. Approximate all measurements to 1 decimal place.



Solution

We first solve for side t, opposite the known angle $S\hat{T}U$, using the law of cosines:

$$t^2 = 13^2 + 17^2 - 2(13)(17)\cos 80°$$

$$t = \sqrt{13^2 + 17^2 - 2(13)(17)\cos 80°}$$

$$t \approx 19.5256$$

Now use the law of sines to solve for one of the other angles, say $S\hat{T}U$:

$$\frac{\sin T\hat{S}U}{17} = \frac{\sin 80°}{19.5256}$$

$$\sin T\hat{S}U = \frac{17\sin 80°}{19.5256}$$

$$T\hat{S}U = \sin^{-1}\left(\frac{17\sin 80°}{19.5256}\right)$$

$$T\hat{S}U \approx 59.0288°$$

Then, $S\hat{U}T \approx 180° - (80° + 59.0288°) \approx 40.9712°$.

Therefore, the other parts of the triangle are $t \approx 19.5$ cm, $T\hat{S}U \approx 59.0°$ and $S\hat{U}T \approx 41.0°$.

You may have noticed that the formula for the law of cosines looks similar to the formula for Pythagoras' theorem. In fact, Pythagoras' theorem can be considered a special case of the law of cosines. When the chosen angle in the law of cosines is 90°, and since $\cos 90° = 0$, the law of cosines becomes Pythagoras' theorem.

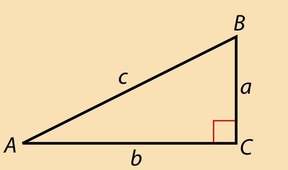
If angle $C = 90°$, then

$c^2 = a^2 + b^2 - 2ab\cos C$

$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos 90°$

$\Rightarrow c^2 = a^2 + b^2 - 2ab(0)$

$\Rightarrow c^2 = a^2 + b^2$ or $a^2 + b^2 = c^2$



• **Hint:** As previously mentioned, remember to store intermediate answers on the GDC for greater accuracy. By not rounding until the final answer, you reduce the amount of round-off error. The GDC screen images below show the calculations in the solution for Example 17 above.

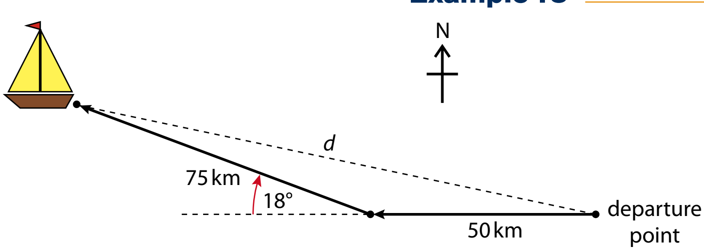
```
√(13²+17-2(13)(
17)cos(80))
        19.52556031
Ans→T
        19.52556031
```

```
Ans→T
        19.52556031
sin⁻¹(17sin(80)/T
)
        59.02884098
Ans→S
        59.02884098
```

```
sin⁻¹(17sin(80)/T
)
        59.02884098
Ans→S
        59.02884098
180-(80+S)
        40.97115902
```

Example 18

A ship travels 50 km due west, then changes its course 18° northward, as shown in the diagram. After travelling 75 km in that direction, how far is the ship from its point of departure? Give your answer to the nearest tenth of a kilometre.



Solution

Let d be the distance from the departure point to the position of the ship. A large obtuse triangle is formed by the three distances of 50 km, 75 km and d km. The angle opposite side d is $180° - 18° = 162°$. Using the law of cosines, we can write the following equation to solve for d:

$$d^2 = 50^2 + 75^2 - 2(50)(75)\cos 162°$$
$$d = \sqrt{50^2 + 75^2 - 2(50)(75)\cos 162°} \approx 123.523$$

Therefore, the ship is approximately 123.5 km from its departure point.

Solving triangles given three sides (SSS)

Given three line segments such that the sum of the lengths of any two is greater than the length of the third, then they will form a unique triangle. Therefore, if we know three sides of a triangle we can solve for the three angle measures. To use the law of cosines to solve for an unknown angle, it is best to first rearrange the formula so that the chosen angle is the subject of the formula.

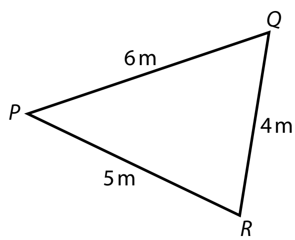
Solve for angle C in:

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow 2ab\cos C = a^2 + b^2 - c^2 \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Then, $C = \cos^{-1}\left(\dfrac{a^2 + b^2 - c^2}{2ab}\right)$.

Example 19

Find all of the unknown angles of triangle PQR, the second triangle shown earlier in Figure 8.22. Approximate all measurements to 1 decimal place.



Solution

Note that the smallest angle will be opposite the shortest side. Let's first solve for the smallest angle – thus, writing the law of cosines with chosen angle P:

$$P = \cos^{-1}\left(\frac{5^2 + 6^2 - 4^2}{2(5)(6)}\right) \approx 41.4096°$$

Now that we know the measure of angle P, we have two sides and a non-included angle (SSA), and the law of sines can be used to find the other non-included angle. Consider the sides $QR = 4$, $RP = 5$ and the angle $P \approx 41.4096°$. Substituting into the law of sines, we can solve for angle Q that is opposite RP.

$$\frac{\sin Q}{5} = \frac{\sin 41.4096°}{4}$$

$$\sin Q = \frac{5\sin 41.4096°}{4}$$

$$Q = \sin^{-1}\left(\frac{5\sin 41.4096°}{4}\right) \approx 55.7711°$$

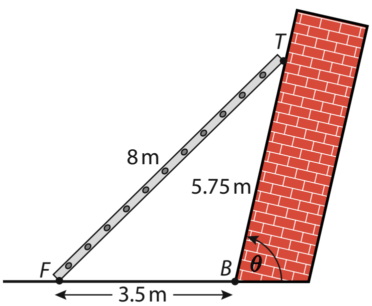
Then, $R \approx 180° - (41.4096° + 55.7711°) \approx 82.8192°$.

Therefore, the three angles of triangle PQR are $P \approx 41.4°$, $Q \approx 55.8°$ and $R \approx 82.8°$.

Example 20

A ladder that is 8 m long is leaning against a non-vertical wall that slopes away from the ladder. The foot of the ladder is 3.5 m from the base of the wall, and the distance from the top of the ladder down the wall to the ground is 5.75 m. To the nearest tenth of a degree, what is the acute angle at which the wall is inclined to the horizontal?

Solution



Let's start by drawing a diagram that accurately represents the given information. θ marks the acute angle of inclination of the wall. Its supplement is $F\hat{B}T$. From the law of cosines:

$$\cos F\hat{B}T = \frac{3.5^2 + 5.75^2 - 8^2}{2(3.5)(5.75)}$$

$$F\hat{B}T = \cos^{-1}\left(\frac{3.5^2 + 5.75^2 - 8^2}{2(3.5)(5.75)}\right) \approx 117.664°$$

$$\theta \approx 180° - 117.664° \approx 62.336°$$

Therefore, the angle of inclination of the wall is approximately 62.3°.

Exercise 8.3 and 8.4

In questions1–6, state the number of distinct triangles (none, one, two or infinite) that can be constructed with the given measurements. If the answer is one or two triangles, provide a sketch of each triangle.

1 $A\hat{C}B = 30°$, $A\hat{B}C = 50°$ and $B\hat{A}C = 100°$

2 $A\hat{C}B = 30°$, $AC = 12$ cm and $BC = 17$ cm

3 $A\hat{C}B = 30°$, $AB = 7$ cm and $AC = 14$ cm

4 $A\hat{C}B = 47°$, $BC = 20$ cm and $A\hat{B}C = 55°$

5 $B\hat{A}C = 25°$, $AB = 12$ cm and $BC = 7$ cm

6 $AB = 23$ cm, $AC = 19$ cm and $BC = 11$ cm

In questions 7–15, solve the triangle. In other words, find the measurements of all unknown sides and angles. If two triangles are possible, solve for both.

7 $B\hat{A}C = 37°$, $A\hat{B}C = 28°$ and $AC = 14$

8 $A\hat{B}C = 68°$, $A\hat{C}B = 47°$ and $AC = 23$

9 $B\hat{A}C = 18°$, $A\hat{C}B = 51°$ and $AC = 4.7$

10 $A\hat{C}B = 112°$, $A\hat{B}C = 25°$ and $BC = 240$

11 $BC = 68$, $A\hat{C}B = 71°$ and $AC = 59$

12 $BC = 16$, $AC = 14$ and $AB = 12$

13 $BC = 42$, $AC = 37$ and $AB = 26$

14 $BC = 34$, $A\hat{B}C = 43°$ and $AC = 28$

15 $AC = 0.55$, $B\hat{A}C = 62°$ and $BC = 0.51$

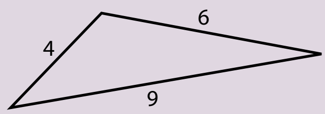
16 Find the lengths of the diagonals of a parallelogram whose sides measure 14 cm and 18 cm and which has one angle of 37°.

17 Find the measures of the angles of an isosceles triangle whose sides are 10 cm, 8 cm and 8 cm.

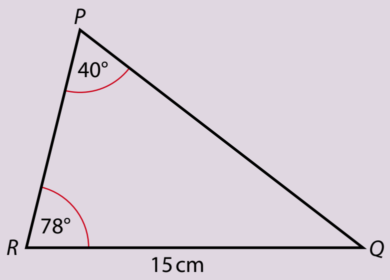
18 Given that for triangle *DEF*, $E\hat{D}F = 43°$, $DF = 24$ and $FE = 18$, find the two possible measures of $D\hat{F}E$.

19 A tractor drove from a point *A* directly north for 500 m, and then drove north-east (i.e. bearing of 45°) for 300 m, stopping at point *B*. What is the distance between points *A* and *B*?

20 Find the measure of the smallest angle in the triangle shown.



21 Find the area of triangle *PQR*.

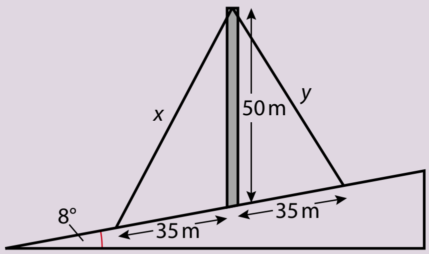


In questions 22 and 23, find a value for the length of *BC* so that the number of possible triangles is: a) one, b) two and c) none.

22 $B\hat{A}C = 36°$, $AB = 5$

23 $B\hat{A}C = 60°$, $AB = 10$

24 A 50 m vertical pole is to be erected on the side of a sloping hill that makes a 8° angle with the horizontal (see diagram). Find the length of each of the two supporting wires (x and y) that will be anchored 35 m uphill and downhill from the base of the pole.



25 The lengths of the sides of a triangle *ABC* are $x - 2$, x and $x + 2$. The largest angle is 120°.

a) Find the value of x.

b) Show that the area of the triangle is $\frac{15\sqrt{3}}{4}$.

c) Find $\sin A + \sin B + \sin C$ giving your answer in the form $\frac{p\sqrt{q}}{r}$ where $p, q, r \in \mathbb{R}$.

26 Find the area of a triangle that has sides of lengths 6, 7 and 8 cm.

27 Let a, b and c be the sides of a triangle where c is the longest side.

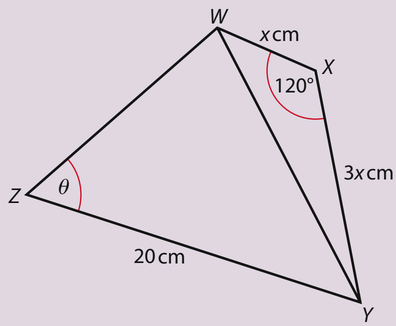
a) If $c^2 > a^2 + b^2$, then what is true about triangle ABC?

b) If $c^2 < a^2 + b^2$, then what is true about triangle ABC?

c) Use the cosine rule to prove each of your conclusions for a) and b).

28 Consider triangle DEF with $E\hat{D}F = 43.6°$, $DE = 19.3$ and $EF = 15.1$. Find DF.

29 In the diagram, $WX = x$ cm, $XY = 3x$ cm, $YZ = 20$ cm, $\sin\theta = \frac{4}{5}$ and $W\hat{X}Y = 120°$.



a) If the area of triangle WZY is 112 cm^2, find the length of [WZ].

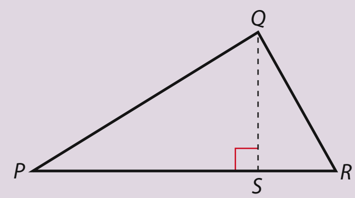
b) Given that θ is an acute angle, state the value of $\cos\theta$ and hence find the length of [WY].

c) Find the **exact** value of x.

d) Find the degree measure of $X\hat{Y}Z$ to three significant figures.

30 In triangle FGH, $FG = 12$ cm, $FH = 15$ cm, and $\angle G$ is twice the size of $\angle H$. Find the approximate degree measure of $\angle H$ to three significant figures.

31 In triangle PQR, $QR = p$, $PR = q$, $PQ = r$ and [QS] is perpendicular to [PR].

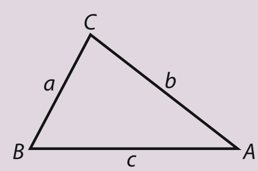


a) Show that $RS = q - r\cos P$.

b) Hence, by using Pythagoras' theorem in the triangle QRS, prove the cosine rule for the triangle PQR.

c) If $P\hat{Q}R = 60°$, use the cosine rule to show that $p = \frac{1}{2}\left(r \pm \sqrt{4q^2 - 3r^2}\right)$.

32 For triangle ABC we can express its area, $\mathbf{A}$, as $\mathbf{A} = \frac{1}{2}ab\sin C$. The cosine rule can be used to write the expression $c^2 = a^2 + b^2 - 2ab\cos C$.



a) Using these two expressions show that $16\mathbf{A}^2 = 4a^2b^2 - (a^2 + b^2 - c^2)$.

• **Hint:** use the Pythagorean identity $\sin^2 C + \cos^2 C = 1$.

b) The perimeter of the triangle is equal to $a + b + c$. Let s be the *semi-perimeter*, that is $s = \dfrac{a+b+c}{2}$. Using the result from a) and that $2s = a + b + c$, show that $16\mathbf{A}^2 = 2s(2s - 2c)(2s - 2a)(2s - 2b)$.

c) Finally, show that the result in b) gives $\mathbf{A} = \sqrt{s(s-a)(s-b)(s-c)}$. This notable result expresses the area of a triangle in terms of *only* the length of its three sides. Although quite possibly known before his time, the formula is attributed to the ancient Greek mathematician and engineer, Heron of Alexandria (ca. 10–70 AD) and is thus called **Heron's formula**. The first written reference to the formula is Heron's proof of it in his book *Metrica*, written in approximately 60 AD.

8.5 Applications

There are some additional applications of triangle trigonometry – both right triangles and non-right triangles – that we should take some time to examine.

Equations of lines and angles between two lines

Recall from Section 1.6, the slope m, or gradient, of a non-vertical line is defined as $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}}$.

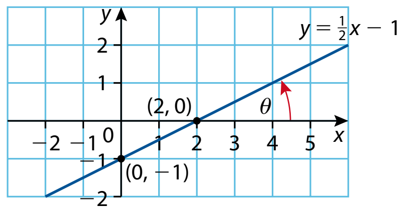


Figure 8.24

The equation of the line shown in Figure 8.24 has a slope $m = \frac{1}{2}$ and a y-intercept of $(0, -1)$. So, the equation of the line is $y = \frac{1}{2}x - 1$. We can find the measure of the acute angle θ between the line and the x-axis by using the tangent function (Figure 8.25).

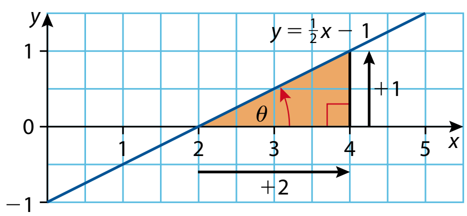


Figure 8.25

$\theta = \tan^{-1}(m) = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.6°$.

Clearly, the slope, m, of this line is equal to $\tan\theta$. If we know the angle between the line and the x-axis, and the y-intercept $(0, c)$, we can write the equation of the line in slope-intercept form ($y = mx + c$) as $y = (\tan\theta)x + c$.

Before we can generalize for any non-horizontal line, let's look at a line with a negative slope.

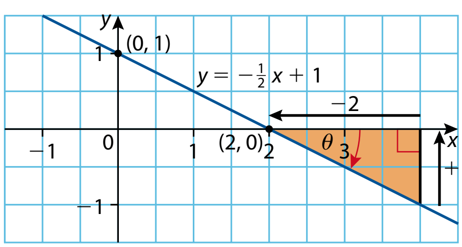


Figure 8.26

The slope of the line is $-\frac{1}{2}$. In order for $\tan\theta$ to be equal to the slope of the line, the angle θ must be the angle that the line makes with the x-axis in the positive direction, as shown in Figure 8.26. In this example, $\theta = \tan^{-1}(m) = \tan^{-1}\left(-\frac{1}{2}\right) \approx -26.6°$.

Remember, an angle with a negative measure indicates a clockwise rotation from the initial side to the terminal side of the angle.

Equations of lines intersecting the x-axis

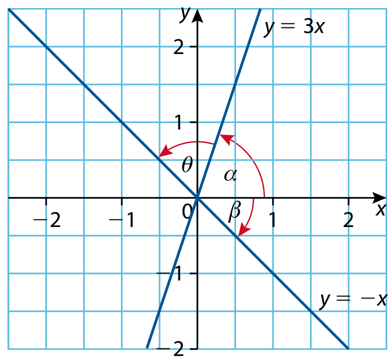
If a line has a y-intercept of $(0, c)$ and makes an angle of θ with the positive direction of the x-axis, such that $-90° < \theta < 90°$, then the slope (gradient) of the line is $m = \tan\theta$ and the equation of the line is $y = (\tan\theta)x + c$. Note: The angle this line makes with any horizontal line will be θ.

Let's use triangle trigonometry to find the angle between any two intersecting lines – not just for a line intersecting the x-axis. Realize that any pair of intersecting lines that are not perpendicular will have both an acute angle and an obtuse angle between them. When asked for an angle between two lines, the convention is to give the acute angle.

Example 21

Find the acute angle between the lines $y = 3x$ and $y = -x$.

Solution



The angle between the line $y = 3x$ and the positive x-axis is α, and the angle between the line $y = -x$ and the positive x-axis is β.

$$\alpha = \tan^{-1}(3) \approx 71.565°$$
$$\beta = \tan^{-1}(-1) = -45°$$

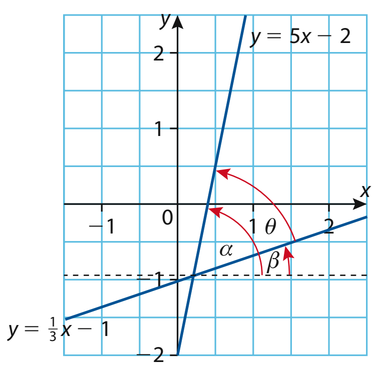
The obtuse angle between the two lines is
$\alpha - \beta \approx 71.565° - (-45°) \approx 116.565°$.

Therefore, the acute angle θ between the two lines is
$\theta = 180° - 116.565° \approx 63.4°$.

Example 22

Find the acute angle between the lines $y = 5x - 2$ and $y = \frac{1}{3}x - 1$.

Solution



A horizontal line is drawn through the point of intersection.

The angle between $y = 5x - 2$ and this horizontal line is α, and the angle between $y = \frac{1}{3}x - 1$ and this horizontal line is β.

$$\alpha = \tan^{-1}(5) \approx 78.690° \quad \text{and} \quad \beta = \tan^{-1}\left(\tfrac{1}{3}\right) = 18.435°$$

The acute angle θ between the two lines is
$\theta = \alpha - \beta \approx 78.690° - 18.435° \approx 60.3°$.

We can generalize the procedure for finding the angle between two lines as follows.

Angle between two lines

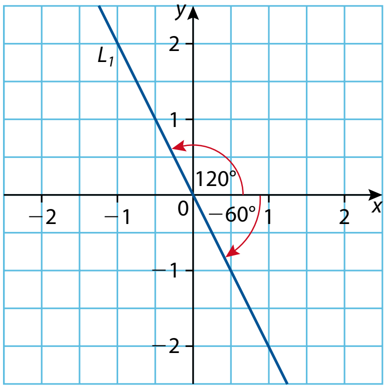
Given two non-vertical lines with equations of $y_1 = m_1x + c_1$ and $y_2 = m_2x + c_2$, the angle between the two lines is $|\tan^{-1}(m_1) - \tan^{-1}(m_2)|$. Note: This angle may be acute or obtuse.

Example 23

a) Find the exact equation of line L_1 that passes through the origin and makes an angle of $-60°$ with the positive direction of the x-axis (or 120°).

b) The equation of line L_2 is $7x + y + 1 = 0$. Find the acute angle between the lines L_1 and L_2.

Solution

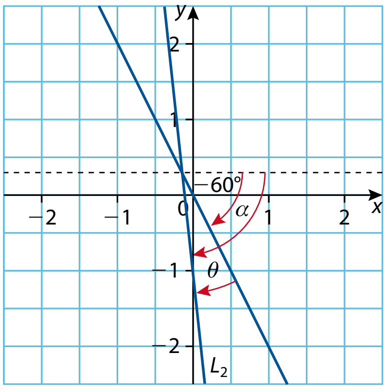


a) The equation of the line is given by $y = (\tan\theta)x$

$$\Rightarrow y = [\tan(-60°)]x = \left[\frac{\sin(-60°)}{\cos(-60°)}\right]x = \left[\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right]x = (-\sqrt{3})x$$

Therefore, the equation of L_1 is $y = -\sqrt{3}x$ or $y = -x\sqrt{3}$.

Note: $\tan(-60°) = \tan 120° = (-\sqrt{3})$.



b) L_2: $7x + y + 1 = 0 \Rightarrow y = -7x - 1$

θ is the acute angle between the lines L_1 and L_2.

$\theta = |\tan^{-1}(m_1) - \tan^{-1}(m_2)| = |\tan^{-1}(-\sqrt{3}) - \tan^{-1}(-7)|$

$\Rightarrow \theta \approx |-60° - (-81.870°)| \approx |-21.87°|$

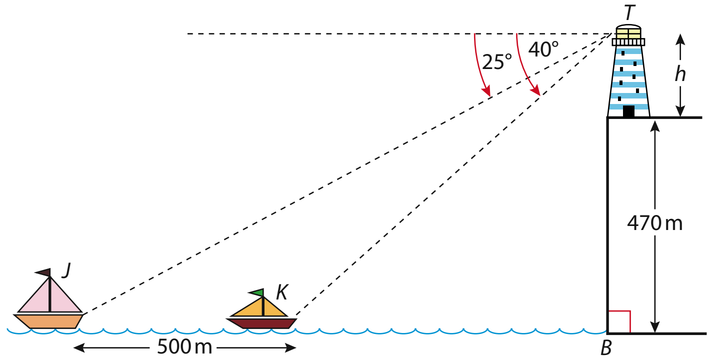
Therefore, the acute angle between the lines is approximately 21.9° (3 s.f.).

Further applications involving the solution of triangles

Many problems that involve distances and angles are represented by diagrams with multiple triangles – right and otherwise. These diagrams can be confusing and difficult to interpret correctly. In these situations, it is important to carry out a careful analysis of the given information and diagram – this will usually lead to drawing additional diagrams. Often we can extract a triangle, or triangles, for which we have enough information to allow us to solve the triangle(s).

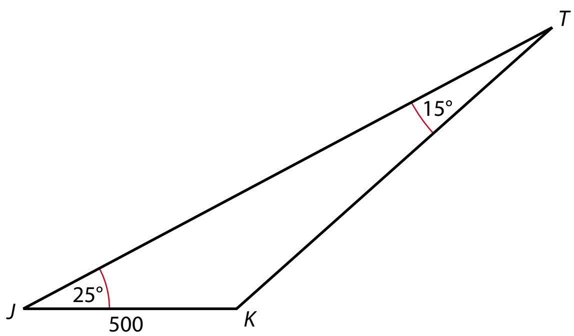
Example 24

Two boats, J and K, are 500 m apart. A lighthouse is on top of a 470 m cliff. The base, B, of the cliff is in line horizontally with $[JK]$. From the top, T, of the lighthouse, the angles of depression of J and K are, respectively, 25° and 40°. Find, correct to the nearest metre, the height, h, of the lighthouse from its base on the clifftop ground to the top T.



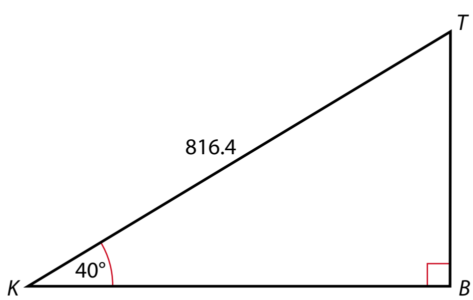
Solution

First, extract obtuse triangle JKT and apply the law of sines to solve for the side KT, which is also the hypotenuse of the right triangle KBT.



$$\frac{\sin 25°}{KT} = \frac{\sin 15°}{500} \Rightarrow KT = \frac{500 \sin 25°}{\sin 15°} \approx 816.436 \text{ m}$$

We can now use the right triangle KBT to find the side BT – which is equal to the height of the cliff plus the height of the lighthouse.



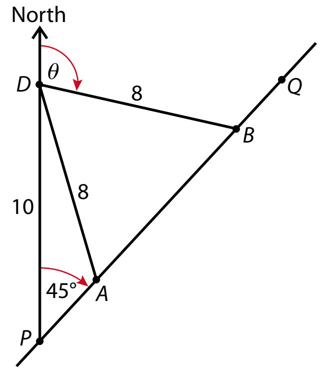
$$\sin 40° = \frac{BT}{816.436} \Rightarrow BT = 816.436 \sin 40° \approx 524.795 \text{ m}$$

Then, $h \approx 524.795 - 470 \approx 54.795$ m.

Therefore, the height of the lighthouse is 54.8 m.

Example 25

The diagram shows a point P that is 10 km due south of a point D. A straight road PQ is such that the (compass) bearing of Q from P is 045°. A and B are two points on this road which are both 8 km from D. Find the bearing of B from D, approximated to 3 s.f.



Solution

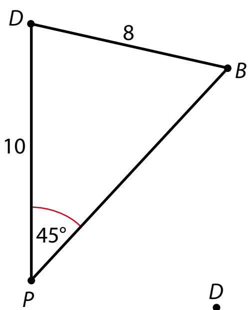
The angle θ in the diagram is the bearing of B from D. A strategy that will lead to finding θ is:

(1) Extract triangle PDB and use the law of sines to solve for $D\hat{B}P$.

(2) Triangle ADB is isosceles (two sides equal), so $D\hat{A}B = D\hat{B}P$; and since the sum of angles in triangle ADB is 180°, we can solve for $A\hat{D}B$.

(3) We can solve for $D\hat{A}P$ because it is supplementary to $D\hat{A}B$, and then we can find the third angle in triangle APD.

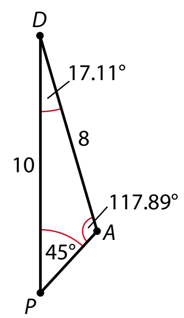
(4) Since $\theta + A\hat{D}B + A\hat{D}P = 180°$, we can solve for θ.

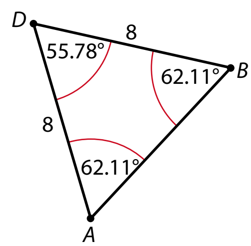


$$\frac{\sin D\hat{B}P}{10} = \frac{\sin 45°}{8}$$

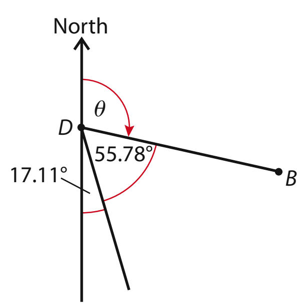
$$\sin D\hat{B}P = \frac{10\sin 45°}{8}$$

$$D\hat{B}P = \sin^{-1}\left(\frac{10\sin 45°}{8}\right) \approx 62.11°$$





$$D\hat{A}B = D\hat{B}P \approx 62.11°$$
$$A\hat{D}B \approx 180° - 2(62.11°) \approx 55.78°$$
$$P\hat{A}D \approx 180° - 62.11° \approx 117.89°$$
$$A\hat{D}P \approx 180° - (45° + 117.89°) \approx 17.11°$$



$$\theta \approx 180° - (17.11° + 55.78°) \approx 107.11°$$

Therefore, the bearing of B from D is approximately 107° to an accuracy of 3 s.f.

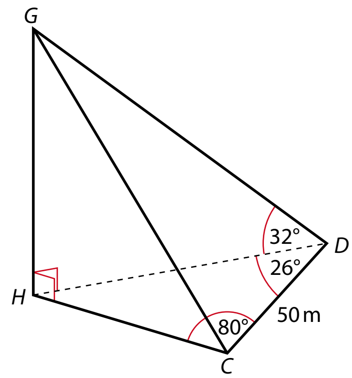
Compass bearings are measured **clockwise** from north.

Three-dimensional trigonometry problems

Of course, not all applications of triangle trigonometry are restricted to just two dimensions. In many problems, it is necessary to calculate lengths and angles in three-dimensional structures. As in the preceding section, it is very important to carefully analyze the three-dimensional diagram and to extract any relevant triangles in order to solve for the necessary angle or length.

Example 26

The diagram shows a vertical pole GH that is supported by two wires fixed to the horizontal ground at C and D. The following measurements are indicated in the diagram: $CD = 50$ m, $G\hat{D}H = 32°$, $H\hat{D}C = 26°$ and $H\hat{C}D = 80°$.



Find a) the distance between H and D, and b) the height of the pole GH.

Solution

a) In triangle HDC: $D\hat{H}C = 180° - (80° + 26°) = 74°$.

Now apply the law of sines:

$$\frac{\sin 80°}{HD} = \frac{\sin 74°}{50} \Rightarrow HD = \frac{50 \sin 80°}{\sin 74°} \approx 51.225 \text{ m}$$

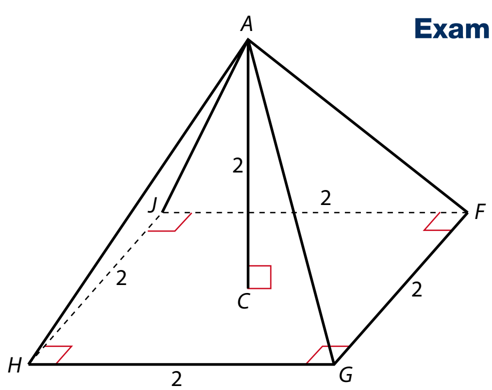
Therefore, the distance from H to D is 51.2 m accurate to 3 s.f.

b) Using the right triangle GHD:

$$\tan 32° = \frac{GH}{51.225} \Rightarrow GH = 51.225 \tan 32° \approx 32.009 \text{ m}$$

Therefore, the height of the pole is 32.0 m accurate to 3 s.f.

Example 27



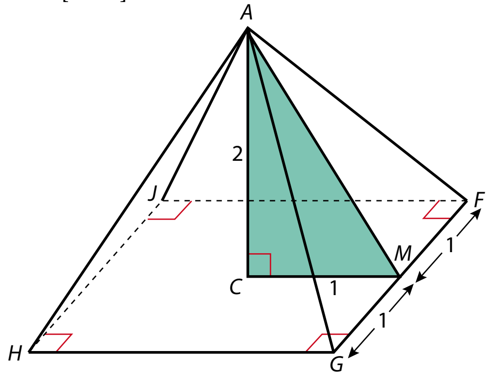
The figure shown is a pyramid with a square base. It is a *right* pyramid, so the line segment (i.e. the height) drawn from the top vertex A perpendicular to the base will intersect the square base at its centre C. If each side of the square base has a length of 2 cm and the height of the pyramid is also 2 cm, find:

a) the measure of $A\hat{G}F$

b) the total surface area of the pyramid.

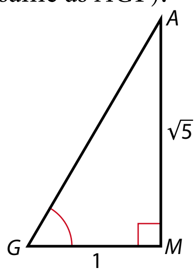
Solution

a) Label the midpoint of $[GF]$ as point M and draw two line segments, $[CM]$ and $[AM]$. Since C is the centre of the square base then $CM = 1$ cm. Extract right triangle ACM and use Pythagoras' theorem to find the length of $[AM]$.



$AM = \sqrt{1^2 + 2^2} = \sqrt{5}$ $[AM]$ is perpendicular to $[GF]$.

Extract right triangle AMG and use the tangent ratio to find $A\hat{G}M$ (same as $A\hat{G}F$):



$$\tan(A\hat{G}M) = \frac{\sqrt{5}}{1}$$
$$A\hat{G}M = \tan^{-1}(\sqrt{5}) \approx 65.905°$$

Therefore, $A\hat{G}M = A\hat{G}F \approx 65.9°$.

b) The total surface area comprises the square base plus four identical lateral faces that are all isosceles triangles. Triangle AGM is one-half the area of one of these triangular faces.

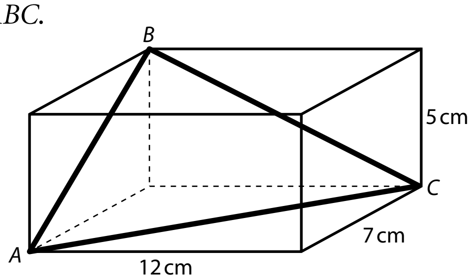
Area of triangle $AGM = \frac{1}{2}(1)(\sqrt{5}) = \frac{\sqrt{5}}{2}$

$\Rightarrow$ area of triangle $AGF = 2\left(\frac{\sqrt{5}}{2}\right) = \sqrt{5}$

Surface area $=$ area of square base $+$ area of four lateral faces
$= 2^2 + 4\sqrt{5} = 4 + 4\sqrt{5} \approx 12.94\text{ cm}^2$

Example 28

For the rectangular box shown, find a) the measure of $A\hat{B}C$, and b) the area of triangle ABC.



Solution

a) Each of the three sides of triangle ABC is the hypotenuse of a right triangle. Using Pythagoras' theorem:

$$AC = \sqrt{7^2 + 12^2} = \sqrt{49 + 144} = \sqrt{193} = 13.892$$

$$AB = \sqrt{5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74} \approx 8.602$$

$$BC = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Apply the law of cosines to find $A\hat{B}C$, using exact lengths of the sides of the triangle.

$$\cos A\hat{B}C = \frac{(\sqrt{74})^2 + 13^2 - (\sqrt{193})^2}{2(\sqrt{74})(13)} \Rightarrow A\hat{B}C = \cos^{-1}\left[\frac{74 + 169 - 193}{2(\sqrt{74})(13)}\right] \approx 77.082°$$

Therefore, the measure of $A\hat{B}C$ is approximately 77.1° to 3 s.f.

b) Area of triangle $= \frac{1}{2}(AB)(BC)\sin A\hat{B}C = \frac{1}{2}(\sqrt{74})(13)\sin(77.082°)$
$\approx 54.499\,96\ \text{cm}^2$

Therefore, the area of triangle ABC is approximately 54.5 cm².

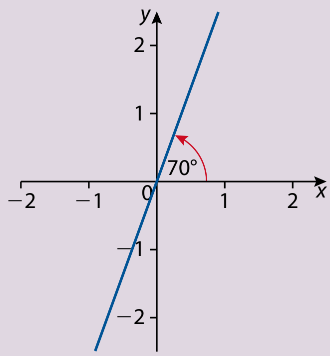
Exercise 8.5

In questions 1–4, determine:

a) the slope (gradient) of the line (approximate to 3 s.f. if not exact)

b) the equation of the line.

1



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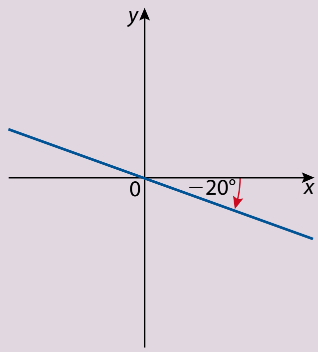

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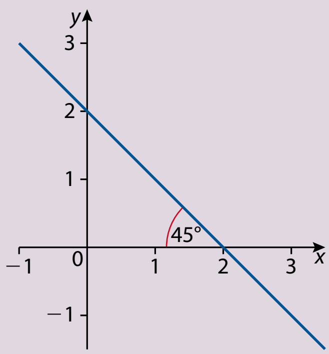

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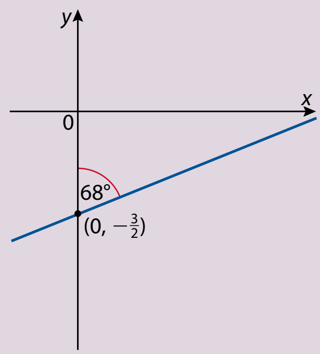


In questions 5–7, find the acute angle that the line through the given pair of points makes with the x-axis.

5 (1, 4) and (−1, 2)

6 (−3, 1) and (6, −5)

7 $\left(2, \frac{1}{2}\right)$ and (−4, −10)

In questions 8 and 9, find the acute angle between the two given lines.

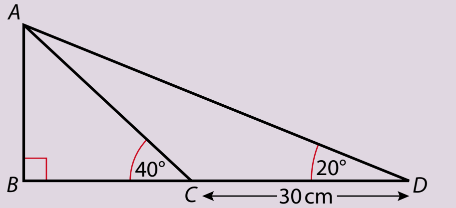
8 $y = -2x$ and $y = x$

9 $y = -3x + 5$ and $y = 2x$

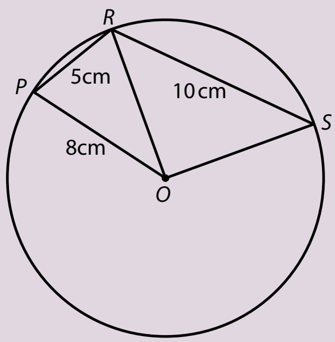
10 a) Find the exact equation of line L_1 that passes through the origin and makes an angle of 30° with the positive direction of the x-axis.

b) The equation of line L_2 is $x + 2y = 6$. Find the acute angle between L_1 and L_2.

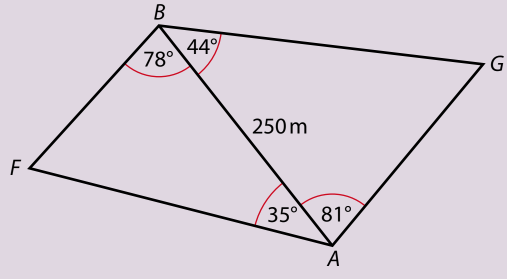
11 Calculate AB given $CD = 30$ cm, and the angle measures given in the diagram.



12 The circle with centre O and radius of 8 cm has two chords PR and RS, such that $PR = 5$ cm and $RS = 10$ cm. Find each of the angles $P\hat{R}O$ and $S\hat{R}O$, and then calculate the area of the triangle PRS.



13 A forester was conducting a survey of a tropical jungle that was mostly inaccessible on foot. The points F and G indicate the location of two rare trees. To find the distance between points F and G, a line AB of length 250 m is measured out so that F and G are on opposite sides of AB. The angles between the line segment AB and the line of sight from each endpoint of AB to each tree are measured, and are shown in the diagram. Calculate the distance between F and G.

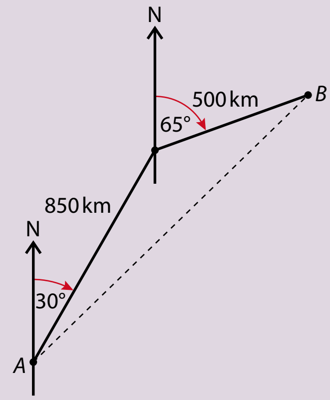


14 Calculate the distance between the tips of the hands of a large clock on a building at 10 o'clock if the minute hand is 3 m long and the hour hand is 2.25 m long.

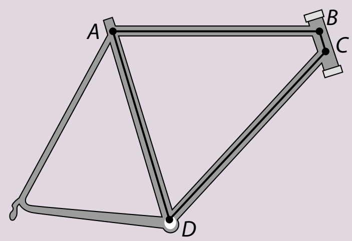
15 An airplane takes off from point *A*. It flies 850 km on a bearing of 030°. It then changes direction to a bearing of 065° and flies a further 500 km and lands at point *B*.

a) What is the straight line distance from *A to B*?

b) What is the bearing from *A* to *B*?

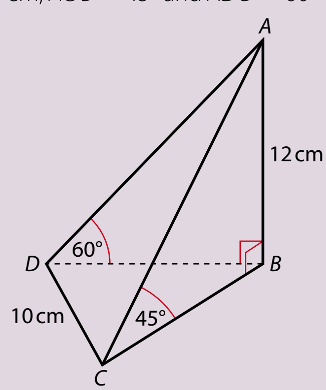


16 The traditional bicycle frame consists of tubes connected together in the shape of a triangle and a quadrilateral (four-sided polygon). In the diagram, *AB*, *BC*, *CD* and *AD* represent the four tubes of the quadrilateral section of the frame. A frame maker has prepared three tubes such that $AD = 53$ cm, $AB = 55$ cm and $BC = 11$ cm. If $D\hat{A}B = 76°$ and $A\hat{B}C = 97°$, what must be the length of tube *CD*? Give your answer to the nearest tenth of a centimetre.



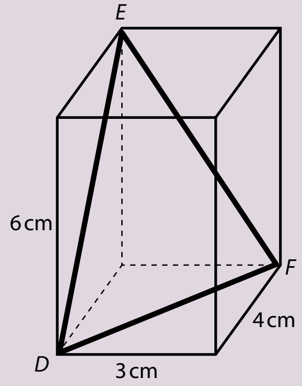
17 The tetrahedron shown in the diagram has the following measurements.

$AB = 12$ cm, $DC = 10$ cm, $A\hat{C}B = 45°$ and $A\hat{D}B = 60°$

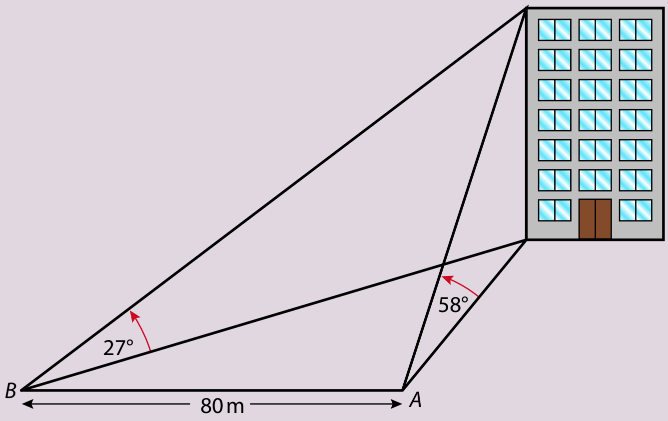


AB is perpendicular to the triangle *BCD*. Find the area of each of the four triangular faces: *ABC*, *ABD*, *BCD* and *ACD*.

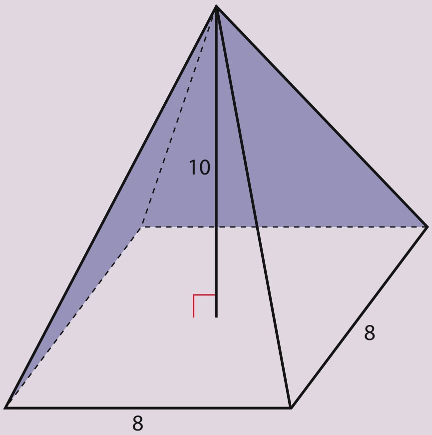
18 Find the measure of angle *DEF* in the rectangular box.



19 At a point *A*, due south of a building, the angle of elevation from the ground to the top of a building is 58°. At a point *B* (on level ground with *A*), 80 m due west of *A*, the angle of elevation to the top of the building is 27°. Find the height of the building.



20 A right pyramid has a square base with sides of length 8 cm. The height of the pyramid is 10 cm. Calculate the angle between two adjacent lateral faces. In other words, find the dihedral angle between two planes each containing one of two adjacent lateral faces. There are four lateral faces that are isosceles triangles and one square base. Two adjacent lateral faces are shaded in the diagram.



● **Hint:** *AB* lies in the plane P_1 and *AC* lies in a second plane P_2 (see Figure 8.27). If *AB* and *AC* are both perpendicular to the line of intersection of the planes, then $B\hat{A}C$ is the angle between the planes. This angle is often called the dihedral angle of the planes.

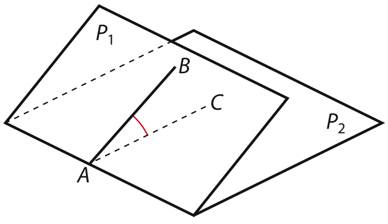
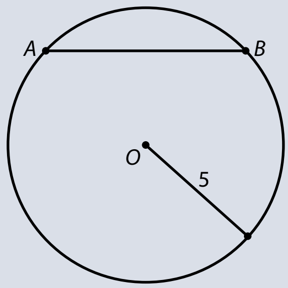


Figure 8.27 Dihedral angle *BAC* of planes P_1 and P_2.

Practice questions

1 The shortest distance from a chord [AB] to the centre O of a circle is 3 units. The radius of the circle is 5 units. Find the exact value of $\sin A\hat{O}B$.



2 In a right triangle, $\tan\theta = \frac{3}{7}$. Find the exact value of $\sin 2\theta$ and $\cos 2\theta$.

3 A triangle has sides of length 4, 5 and 7 units. Find, to the nearest tenth of a degree, the size of the largest angle.

4 If A is an obtuse angle in a triangle and $\sin A = \frac{5}{13}$, calculate the exact value of $\sin 2A$.

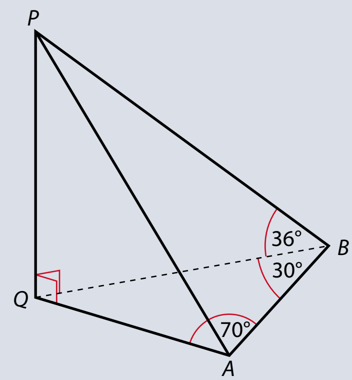
5 The diagram shows a vertical pole PQ, which is supported by two wires fixed to the horizontal ground at A and B.

$BQ = 40\text{ m}$

$P\hat{B}Q = 36°$

$B\hat{A}Q = 70°$

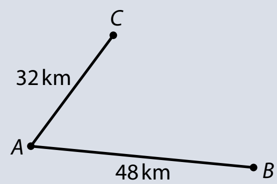
$A\hat{B}Q = 30°$



Find: **a)** the height of the pole PQ

b) the distance between A and B.

6 Town A is 48 km from town B and 32 km from town C, as shown in the diagram.

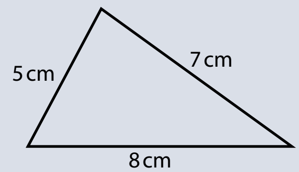


Given that town B is 56 km from town C, find the size of the angle $C\hat{A}B$ to the nearest tenth of a degree.

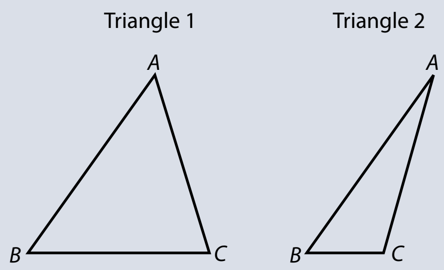
7 The following diagram shows a triangle with sides 5 cm, 7 cm and 8 cm.

Find: **a)** the size of the smallest angle, in degrees

b) the area of the triangle.



8 The diagrams below show two different triangles, both satisfying the conditions: $AB = 20$ cm, $AC = 17$ cm, $A\hat{B}C = 50°$.

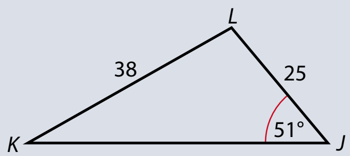


a) Calculate the size of $A\hat{C}B$ in Triangle 2.

b) Calculate the area of Triangle 1.

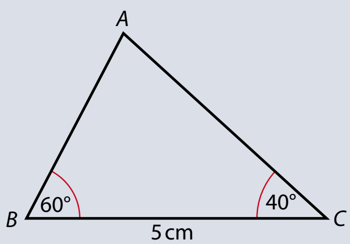
9 Two boats A and B start moving from the same point P. Boat A moves in a straight line at 20 km/h and boat B moves in a straight line at 32 km/h. The angle between their paths is 70°. Find the distance between the two boats after 2.5 hours.

10 In triangle JKL, $JL = 25$, $KL = 38$ and $K\hat{J}L = 51°$, as shown in the diagram.



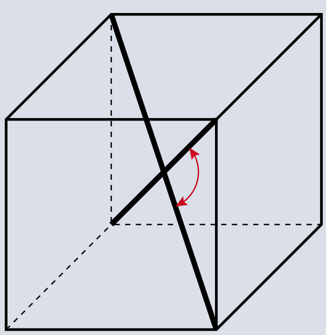
Find $J\hat{K}L$, giving your answer correct to the nearest degree.

11 The following diagram shows a triangle ABC, where $BC = 5$ cm, $A\hat{B}C = 60°$ and $A\hat{C}B = 40°$.



a) Calculate AB. **b)** Find the area of the triangle.

12 Find the measure of the acute angle between a pair of diagonals of a cube.

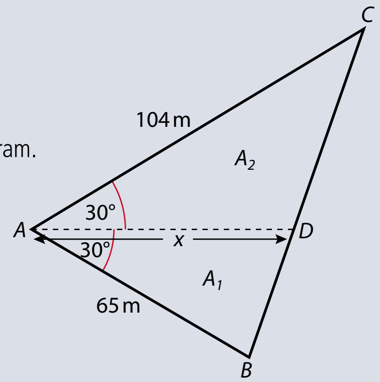


13 A farmer owns a triangular field ABC. One side of the triangle, $[AC]$, is 104 m, a second side, $[AB]$, is 65 m and the angle between these two sides is 60°.

a) Use the cosine rule to calculate the length of the third side, $[BC]$, of the field.

b) Given that $\sin 60° = \frac{\sqrt{3}}{2}$, find the area of the field in the form $p\sqrt{3}$, where p is an integer.

Let D be a point on $[BC]$ such that $[AD]$ bisects the 60° angle. The farmer divides the field into two parts, A_1 and A_2, by constructing a straight fence $[AD]$ of length x m, as shown in the diagram.



c) **(i)** Show that the area of A_1 is given by $\frac{65x}{4}$.

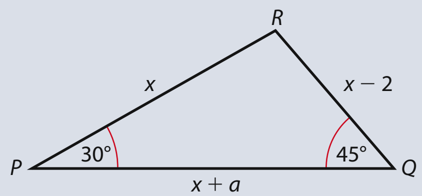
(ii) Find a similar expression for the area of A_2.

(iii) Hence, find the value of x in the form $q\sqrt{3}$, where q is an integer.

d) **(i)** Explain why $\sin A\hat{D}C = \sin A\hat{D}B$.

(ii) Use the result of part **(i)** and the sine rule to show that $\frac{BD}{DC} = \frac{5}{8}$.

14 The lengths of the sides of a triangle PQR are $x - 2$, x and $x + a$ where $a > 0$. Angle P is 30° and angle Q is 45°, as shown in the diagram.

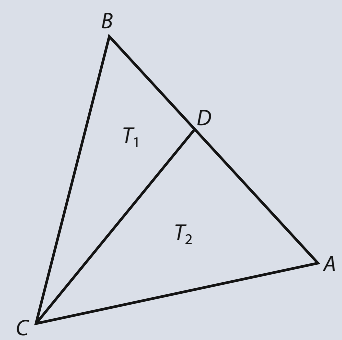


a) Find the exact value of x.

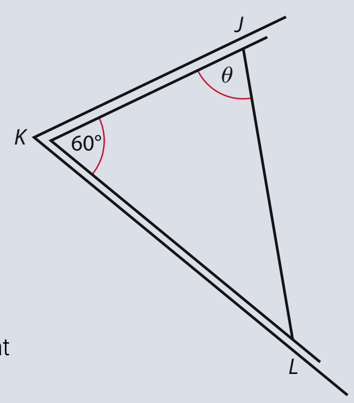
b) Find the exact area of triangle PQR.

15 Given a triangle ABC, a line segment $[CD]$ is drawn from vertex C to a point D on side $[AB]$. Triangle ABC is divided into two triangular regions by $[CD]$. The areas of the regions are denoted as T_1 and T_2 (see diagram). Prove that for any triangle

ABC the ratio of the areas $\frac{T_1}{T_2}$ is equal to the ratio of the lengths $\frac{BD}{AD}$.



16 One corner, K, of a field consists of two stone walls, $[KJ]$ and $[KL]$, at an angle of 60° to each other. A 30-metre wooden fence $[JL]$ is to be built to create a triangular enclosure JKL, as shown in the diagram.



a) If $K\hat{J}L$ is denoted by θ, state the range of possible values for θ.

b) Show that the area of triangle JKL is given by $300\sqrt{3}\sin\theta\sin(\theta + 60°)$.

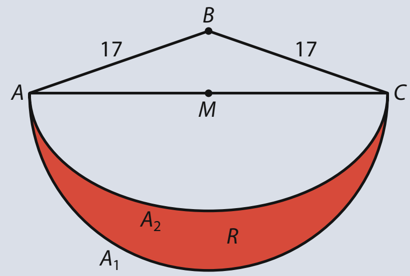
c) Use your GDC to determine the value of θ that gives the maximum area for the enclosure.

17 The diagram shows the triangle ABC with $AB = BC = 17$ cm and $AC = 30$ cm. The midpoint of AC is M. The circular arc A_1 is half the circle (semicircle) with centre M. Another circular arc A_2 is drawn with centre B. The shaded region R is bounded by the arcs A_1 and A_2. Find the following:

a) the area of triangle ABC

b) the measure of $A\hat{B}C$ in radians

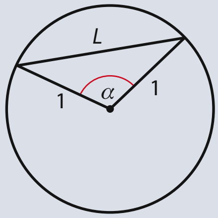
c) the area of the shaded region R.



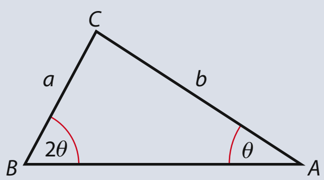
18 **a)** In the diagram, radii drawn to endpoints of a chord of the unit circle determine a central angle α. Show that the length of the chord is equal to $L = \sqrt{2 - 2\cos\alpha}$.

b) By using the substitution $\theta = \frac{\alpha}{2}$ in the double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$, derive a formula for $\sin\frac{\alpha}{2}$, that is a half-angle formula for the sine function.

c) Use the result in **a)** and your result in **b)** to show that the length of the chord is equal to $L = 2\sin\left(\frac{\alpha}{2}\right)$.



19 In triangle ABC, $A\hat{B}C = 2\theta$ and $B\hat{A}C = \theta$. Determine an expression for $\cos\theta$ in terms of a and b.



9 Vectors

Assessment statements

4.1 Vectors as displacements in the plane.
Components of a vector; column representation.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Algebraic and geometric approaches to the following topics:
the sum and difference of two vectors; the zero vector; the vector $-\mathbf{v}$;
multiplication by a scalar, $k\mathbf{v}$;
magnitude of a vector, $|\mathbf{v}|$;
unit vectors; base vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;
position vectors $\overrightarrow{OA} = \mathbf{a}$;
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

4.2 The scalar product of two vectors.
Properties of the scalar product.
Perpendicular vectors; parallel vectors.
The angle between two vectors.

4.3 Representation of a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
The angle between two lines.
(See also Chapter 14.)

Introduction

Vectors are an essential tool in physics and a very significant part of mathematics. Historically, their primary application was to represent forces, and the operation called '**vector addition**' corresponds to the combining of various forces. Many other applications in physics and other fields have been found since. In this chapter, we will discuss what vectors are and how to add, subtract and multiply them by scalars; we will also examine why vectors are useful in everyday life and how they are used in real-life applications. Then we will discuss scalar products.

Control panel of a passenger jet cockpit.

9.1 Vectors as displacements in the plane

We can represent physical quantities like temperature, distance, area, speed, density, pressure and volume by a single number indicating magnitude or size. These are called **scalar quantities**. Other physical quantities possess the properties of magnitude and direction. We define the force needed to pull a truck up a 10° slope by its **magnitude** and **direction**. Force, displacement, velocity, acceleration, lift, drag, thrust and weight are quantities that cannot be described by a single number. These are called **vector quantities**. Distance and displacement, for example, have distinctly different meanings; so do speed and velocity. Speed is a scalar quantity that refers to 'how fast an object is moving'.

Velocity is a vector quantity that refers to 'the rate at which an object *changes its position*'. When evaluating the velocity of an object, we must keep track of direction. It would not be enough to say that an object has a velocity of 55 km/h; we must include direction information in order to fully describe the velocity of the object. For instance, you must describe the object's velocity as being 55 km/h east. This is one of the essential differences between speed and velocity. Speed is a **scalar** quantity and does not keep track of direction; velocity is a **vector** quantity and is direction-conscious.

Thus, an aeroplane moving westward with a speed of 600 km/h has a velocity of 600 km/h west. Note that speed has no direction (it is scalar) and velocity, at any instant, is simply the speed with a direction.

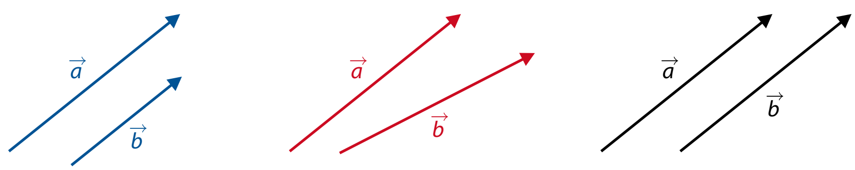
We represent vector quantities with **directed line segments** (Figure 9.1).

The directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B. We use the notation $\overrightarrow{AB}$ to indicate that the line segment represents a vector quantity. We use $|\overrightarrow{AB}|$ to represent the **magnitude** of the directed line segment. The terms **size**, **length** or **norm** are also used. The direction of $\overrightarrow{AB}$ is from A to B. $\overrightarrow{BA}$ has the same length but the opposite direction to $\overrightarrow{AB}$ and hence cannot be equal to it.

Two directed line segments that have the same magnitude and direction are equivalent. For example, the directed line segments in Figure 9.2 are all equivalent.

We call the set of all directed line segments equivalent to a given directed line segment $\overrightarrow{AB}$ a **vector v**, and write $\mathbf{v} = \overrightarrow{AB}$. We denote vectors by lower-case, boldface letters such as **a**, **u**, and **v**.

We say that two vectors **a** and **b** are equal if their corresponding directed line segments are equivalent.



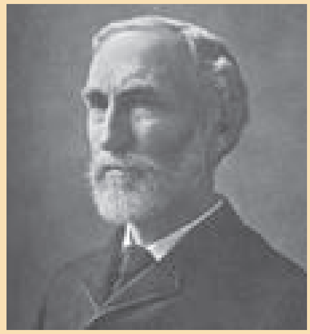
Vectors $\vec{a}$ and $\vec{b}$ have the same direction but different magnitudes $\Rightarrow \vec{a} \neq \vec{b}$.

Vectors $\vec{a}$ and $\vec{b}$ have equal magnitudes but different directions $\Rightarrow \vec{a} \neq \vec{b}$.

Vectors $\vec{a}$ and $\vec{b}$ have equal magnitudes and the same direction $\Rightarrow \vec{a} = \vec{b}$.

Figure 9.3

The notion of vector, as presented here, is due to the mathematician-physicist J. Williard Gibbs (1839–1903) of Yale University. His book *Vector Analysis* (1881) made these ideas accessible to a wide audience.



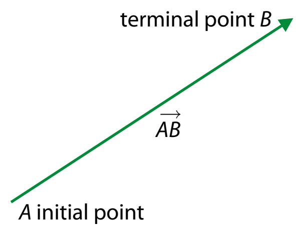


Figure 9.1

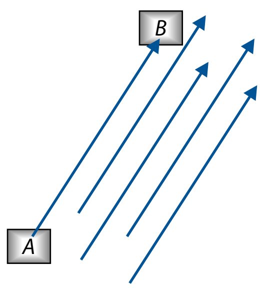


Figure 9.2

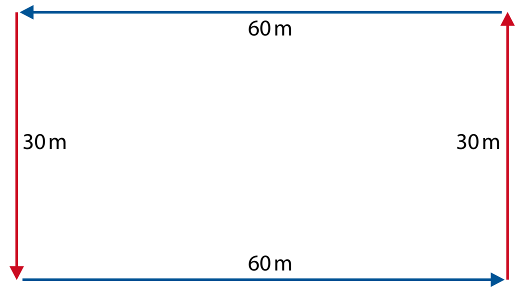
• **Hint:** Note: When we handwrite vectors, we cannot use boldface, so the convention is to use the arrow notation.

Definition 1: Two vectors **u** and **v** are equal if they have the same magnitude and the same direction.

Definition 2: The negative of a vector **u**, denoted by $-\mathbf{u}$, is a vector with the same magnitude but opposite direction.

Example 1

Marco walked around the park as shown in the diagram. What is Marco's displacement at the end of his walk?



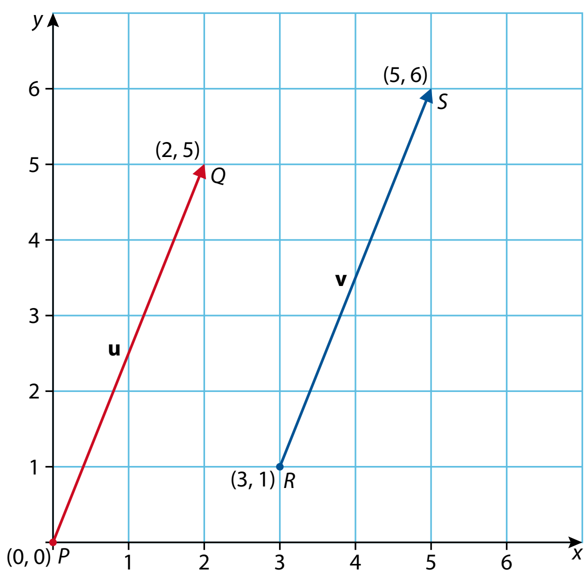
Solution

Even though he walked a total distance of 180 m, his displacement is zero since he returned to his original position. So, his displacement is **0**.

This is a displacement and hence direction is also important, not only magnitude. The 30 m south 'cancelled' the 30 m north, and the 60 m east is cancelled by the 60 m west.

Vectors can also be looked at as displacement/translation in the plane. Take, for example, the directed segments PQ and RS as representing the vectors **u** and **v**, respectively. The points $P(0, 0)$, $Q(2, 5)$, $R(3, 1)$ and $S(5, 6)$ are shown in Figure 9.4.

Figure 9.4



We can prove that these two vectors are equal.

The directed line segments representing the vectors have the same direction, since they both have a slope of $\frac{5}{2}$.

They also have the same magnitude, as:

$$|\overrightarrow{PQ}| = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ and}$$

$$|\overrightarrow{RS}| = \sqrt{(5-3)^2 + (6-1)^2} = \sqrt{29}$$

Component form

The directed line segment with the origin as its initial point is the most convenient way of representing a vector. This representation of the vector is said to be in **standard position**. In Figure 9.4, **u** is in standard position. A vector in standard position can be uniquely represented by the coordinates of its terminal point (u_1, u_2). This is called the **component form of a vector u**, written as $\mathbf{u} = (u_1, u_2)$.

The coordinates u_1 and u_2 are the **components** of the vector **u**. In Figure 9.4, the components of the vector **u** are 2 and 5.

If the initial and terminal points of the vector are the same, the vector is a **zero vector** and is denoted by $\mathbf{0} = (0, 0)$.

If **u** is a vector in the plane with initial point (0, 0) and terminal point (u_1, u_2), the **component form** of **u** is $\mathbf{u} = (u_1, u_2)$.
Note: The component form is also written as $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

So, a vector in the plane is also an ordered pair (u_1, u_2) of real numbers. The numbers u_1 and u_2 are the components of **u**. The vector $\mathbf{u} = (u_1, u_2)$ is also called the **position vector** of the point (u_1, u_2).

If the vector **u** is not in standard position and is represented by a directed segment AB, then it can be written in its component form, observing the following fact:

$\mathbf{u} = (u_1, u_2) = (x_2 - x_1, y_2 - y_1)$, where $A(x_1, y_1)$ and $B(x_2, y_2)$ (Figure 9.5).

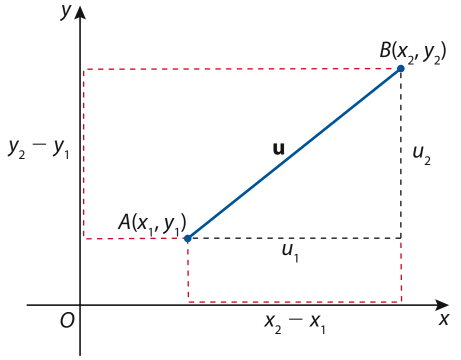


Figure 9.5

The length of vector **u** can be given using Pythagoras' theorem and/or the distance formula:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2

a) Find the components and the length of the vector between the points $P(-2, 3)$ and $Q(4, 7)$.

b) $\overrightarrow{RS}$ is another representation of the vector **u** where $R(7, -3)$. Find the coordinates of S.

Solution

a) $\overrightarrow{PQ} = (4 - (-2), 7 - 3) = (6, 4)$

$|\overrightarrow{PQ}| = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

b) Let S have coordinates (x, y). Therefore,

$\overrightarrow{RS} = (x - 7, y + 3)$.

But,

$\overrightarrow{RS} = \overrightarrow{PQ} \Rightarrow x - 7 = 6$ and $y + 3 = 4 \Rightarrow x = 13, y = 1$.

So, S has coordinates $(13, 1)$.

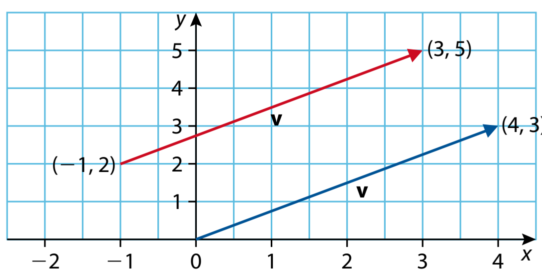
Example 3

The directed segment from $(-1, 2)$ to $(3, 5)$ represents a vector **v.** Find the length of vector **v**, draw the vector in standard position and find the opposite of the vector in component form.

Solution

The length of vector **v** can be found using the distance formula:

$$|\mathbf{v}| = \sqrt{(3 + 1)^2 + (5 - 2)^2} = 5$$



The opposite of this vector can be represented by $-\mathbf{v} = (-4, -3)$.

9.2 Vector operations

Two of the most basic and important operations are scalar multiplication and vector addition.

Scalar multiplication

In working with vectors, numbers are considered scalars. In this discussion, scalars will be limited to real numbers only. Geometrically, the product of a vector **u** and a scalar k, $\mathbf{v} = k\mathbf{u}$, is a vector that is $|k|$ times as long as **u**. If

k is positive, $\mathbf{v}$ has the same direction as $\mathbf{u}$, and when k is negative, $\mathbf{v}$ has the opposite direction to $\mathbf{u}$ (Figure 9.6).

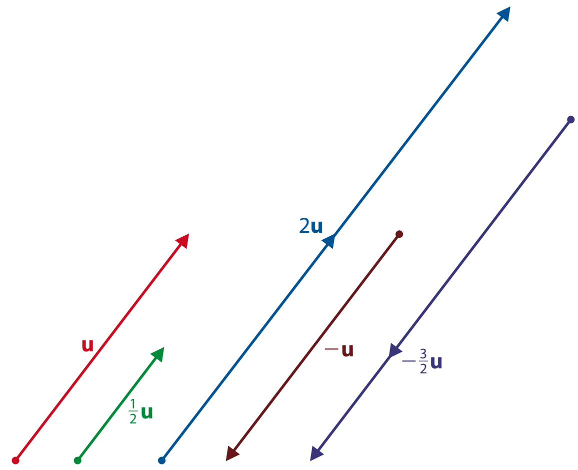


Figure 9.6

Consequence: It becomes clear from this discussion that for two vectors to be parallel, it is necessary and sufficient that one of them is a scalar multiple of the other. That is, if $\mathbf{v}$ and $\mathbf{u}$ are parallel, then $\mathbf{v} = k\mathbf{u}$; and vice versa, if $\mathbf{v} = k\mathbf{u}$, then $\mathbf{v}$ and $\mathbf{u}$ are parallel.

In terms of their components, the operation of scalar multiplication is straightforward.

If $\mathbf{u} = (u_1, u_2)$ then $\mathbf{v} = k\mathbf{u} = k(u_1, u_2) = (ku_1, ku_2)$.

Example 4

Find the magnitude of each vector.

a) $\mathbf{u} = (3, -4)$ b) $\mathbf{v} = (6, -8)$ c) $\mathbf{w} = (7, 0)$ d) $\mathbf{z} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Solution

a) $|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$

b) $|\mathbf{v}| = \sqrt{6^2 + (-8)^2} = 10$ Notice that $\mathbf{v} = 2\mathbf{u}$ and so $|\mathbf{v}| = 2|\mathbf{u}|$.

c) $|\mathbf{w}| = \sqrt{7^2 + 0^2} = 7$

d) $|\mathbf{z}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ This is also called a unit vector as you will see later.

Vector addition

There are two equivalent ways of looking at the addition of vectors geometrically. One is the triangular method and the other is the parallelogram method.

Let $\mathbf{u}$ and $\mathbf{v}$ denote two vectors. Draw the vectors such that the terminal point of $\mathbf{u}$ and initial point of $\mathbf{v}$ coincide. The vector joining the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$ is the sum (resultant) of vectors $\mathbf{u}$ and $\mathbf{v}$ and is denoted by $\mathbf{u} + \mathbf{v}$ (Figure 9.7).

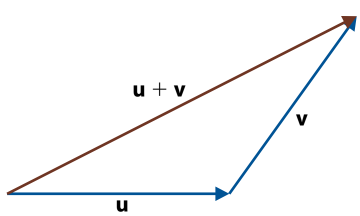


Figure 9.7

Another equivalent way of looking at the sum also gives us the grounds to say that vector addition is commutative.

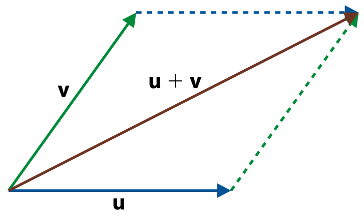


Figure 9.8

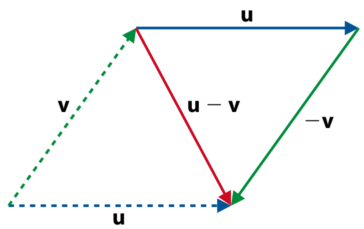


Figure 9.9

Let **u** and **v** denote two vectors. Draw the vectors such that the initial point of **u** and initial point of **v** coincide. The vector joining the common initial point of **u** and **v** to the opposite corner of the parallelogram, formed by the vectors as its adjacent sides, is the sum (resultant) of vectors **u** and **v** and is denoted by $\mathbf{u} + \mathbf{v}$ (Figure 9.8).

The difference of two vectors is an extremely important rule that will be used later in the chapter.

As Figure 9.9 shows, it is an extension of the addition rule. An easy way of looking at it is through a combination of the parallelogram rule and the triangle rule. We draw the vectors **u** and **v** in the usual way, then we draw $-\mathbf{v}$ starting at the terminal point of **u** and we add $\mathbf{u} + (-\mathbf{v})$ to get the difference $\mathbf{u} - \mathbf{v}$. As it turns out, the difference of the two vectors **u** and **v** is the diagonal of the parallelogram with its initial point the terminal of **v** and its terminal point the terminal point of **u**.

Example 5

Consider the vectors $\mathbf{u} = (2, -3)$ and $\mathbf{w} = (1, 3)$.

a) Write down the components of $\mathbf{v} = 2\mathbf{u}$.

b) Find $|\mathbf{u}|$ and $|\mathbf{v}|$ and compare them.

c) Draw the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, 2\mathbf{w}, \mathbf{u} + \mathbf{w}, \mathbf{v} + 2\mathbf{w}, \mathbf{u} - \mathbf{w}, \mathbf{v} - 2\mathbf{w}$.

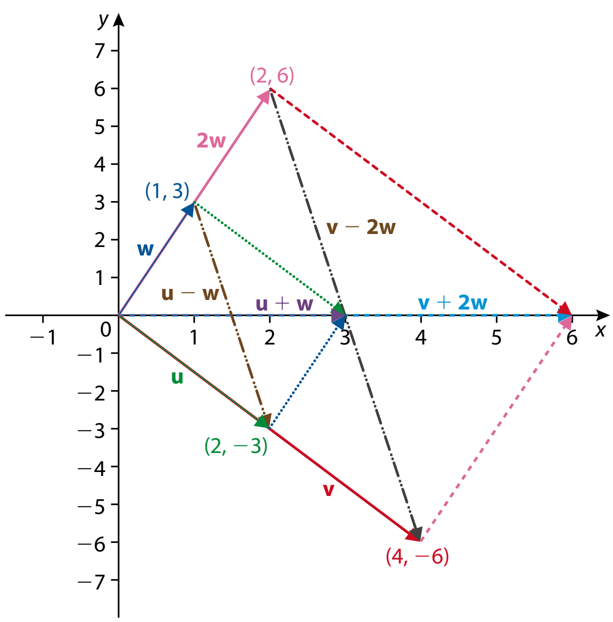
d) Comment on the results of c) above.

Solution

a) $\mathbf{v} = 2(2, -3) = (4, -6)$

b) $|\mathbf{u}| = \sqrt{4 + 9} = \sqrt{13}$, $|\mathbf{v}| = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. Clearly, $|\mathbf{v}| = 2|\mathbf{u}|$.

c)



d) We observe that $\mathbf{u} + \mathbf{w} = (3, 0)$ which turns out to be $(1 + 2, 3 - 3)$, the sum of the corresponding components. We observe the same for $\mathbf{v} + 2\mathbf{w} = (6, 0)$, which in turn is $(2 + 4, 6 - 6)$.

We also observe that $\mathbf{v} + 2\mathbf{w} = 2\mathbf{u} + 2\mathbf{w} = 2(\mathbf{u} + \mathbf{w})$, and $\mathbf{v} - 2\mathbf{w}$ is parallel to $\mathbf{u} - \mathbf{w}$ and is twice its length!

Can you draw more observations?

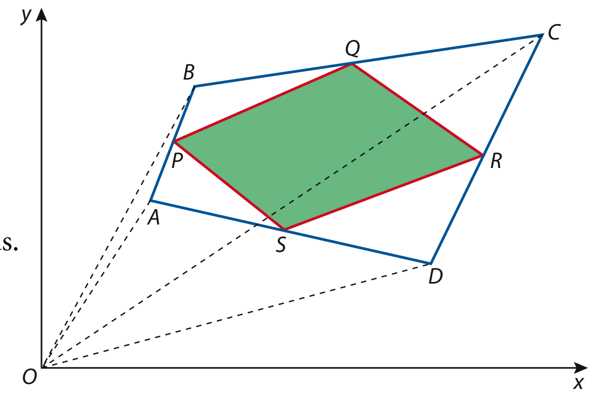
Example 6

ABCD is a quadrilateral with vertices that have position vectors **a**, **b**, **c**, and **d** respectively. *P*, *Q*, *R*, and *S* are the midpoints of the sides.

a) Express each of the following in terms of **a**, **b**, **c**, and **d**:

$\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{AP}$, and $\overrightarrow{OP}$

b) Prove that *PQRS* is a parallelogram using vector methods.



Solution

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \mathbf{d} - \mathbf{c}$

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{b} + \mathbf{a})$

b) One way of proving *PQRS* is a parallelogram is to show a pair of opposite sides parallel and congruent.

You can show that $\overrightarrow{OQ} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\overrightarrow{OR} = \frac{1}{2}(\mathbf{d} + \mathbf{c})$, and $\overrightarrow{OS} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$ as we did for $\overrightarrow{OP}$.

Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{b} + \mathbf{a}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$, and

$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \frac{1}{2}(\mathbf{d} + \mathbf{c}) - \frac{1}{2}(\mathbf{d} + \mathbf{a}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$.

Therefore, $\overrightarrow{PQ} = \overrightarrow{SR}$, and since they are opposite sides of the quadrilateral, so it is a parallelogram.

Base vectors in the coordinate plane

As you have seen before, vectors can also be represented in a coordinate system using their component form. This is a very useful tool that helps make many applications of vectors simple and easy. At the heart of the component approach to vectors we find the 'base' vectors **i** and **j**.

i is a vector of magnitude 1 with the direction of the positive *x*-axis and **j** is a vector of magnitude 1 with the direction of the positive *y*-axis. These vectors and any vector that has a magnitude of 1 are called **unit vectors**. Since vectors of same direction and length are equal, each vector **i** and **j** may be drawn at any point in the plane, but it is usually more convenient to draw them at the origin, as shown in Figure 9.10.

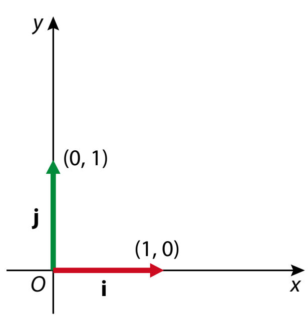


Figure 9.10

Now, the vector $k\mathbf{i}$ has magnitude k and is parallel to the vector **i**. Similarly, the vector $m\mathbf{j}$ has magnitude m and is parallel to **j**.

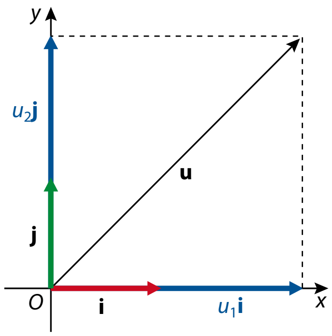


Figure 9.11

Consider the vector $\mathbf{u} = (u_1, u_2)$. This vector, in standard position, has an ***x*-component** u_1 and ***y*-component** u_2 (Figure 9.11).

Since the vector $\mathbf{u}$ is the diagonal of the parallelogram with adjacent sides $u_1\mathbf{i}$ and $u_2\mathbf{j}$, then it is the sum of the two vectors, i.e. $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. It is customary to say that $u_1\mathbf{i}$ is the **horizontal component** and $u_2\mathbf{j}$ is the **vertical component** of $\mathbf{u}$.

The previous discussion shows that it is always possible to express any vector in the plane as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

This form of representation of vectors opens the door to a rich world of vector applications.

Vector addition and subtraction in component form

Consider the two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$.

If vector $\mathbf{u}$ has components (u_1, u_2), then its component form is: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$

(i) Vector sum $\mathbf{u} + \mathbf{v}$

$$\mathbf{u} + \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) + (v_1\mathbf{i} + v_2\mathbf{j}) = (u_1\mathbf{i} + v_1\mathbf{i}) + (u_2\mathbf{j} + v_2\mathbf{j})$$
$$= (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j}$$

For example, to add the two vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$, it is enough to add the corresponding components:

$$\mathbf{u} + \mathbf{v} = (2 + 5)\mathbf{i} + (4 - 3)\mathbf{j} = 7\mathbf{i} + \mathbf{j}$$

(ii) Vector difference $\mathbf{u} - \mathbf{v}$

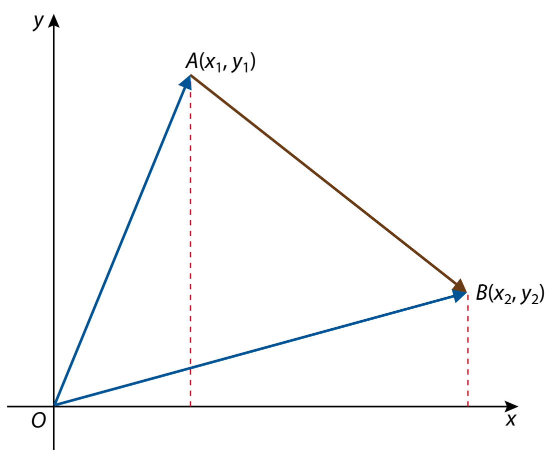
$$\mathbf{u} - \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) - (v_1\mathbf{i} + v_2\mathbf{j}) = (u_1\mathbf{i} - v_1\mathbf{i}) + (u_2\mathbf{j} - v_2\mathbf{j})$$
$$= (u_1 - v_1)\mathbf{i} + (u_2 - v_2)\mathbf{j}$$

For example, to subtract the two vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$, it is enough to subtract the corresponding components:

$$\mathbf{u} - \mathbf{v} = (2 - 5)\mathbf{i} + (4 + 3)\mathbf{j} = -3\mathbf{i} + 7\mathbf{j}$$

This interpretation of the difference gives us another way of finding the components of any vector in the plane, even if it is not in standard position (Figure 9.12).

Figure 9.12



Consider the vector $\overrightarrow{AB}$ where the position vectors of its endpoints are given by the vectors $\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j}$.

As we have seen in section 9.1, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$.
This result was given in Section 9.1 as a definition.

- Many of the laws of ordinary algebra are also valid for vector algebra. These laws are:
  - Commutative law for addition: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
  - Associative law for addition: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

    The verification of the associative law is shown in Figure 9.13.

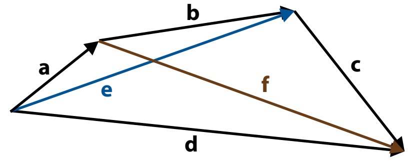


Figure 9.13

If we add **a** and **b** we get a vector **e**. And similarly, if **b** is added to **c**, we get **f**.

Now $\mathbf{d} = \mathbf{e} + \mathbf{c} = \mathbf{a} + \mathbf{f}$. Replacing **e** with $(\mathbf{a} + \mathbf{b})$ and **f** with $(\mathbf{b} + \mathbf{c})$, we get $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ and we see that the law is verified.

  - Commutative law for multiplication: $m\mathbf{a} = \mathbf{a}m$
  - Distributive law (1): $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$, where m and n are two different scalars.
  - Distributive law (2): $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

These laws allow the manipulation of vector quantities in much the same way as ordinary algebraic equations.

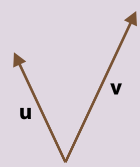
Two vectors **u** and **v** are parallel iff $\mathbf{v} = k\mathbf{u}$. This also means that in component form:

$$\frac{v_1}{u_1} = \frac{v_2}{u_2} = k$$

Exercise 9.1 and 9.2

1 Consider the vectors **u** and **v** given.
Sketch each indicated vector.

a) $2\mathbf{u}$
b) $-\mathbf{v}$
c) $\mathbf{u} + \mathbf{v}$
d) $\mathbf{2u} - \mathbf{v}$
e) $\mathbf{v} - 2\mathbf{u}$



For questions 2–5, consider the points A and B given and answer the following questions:

a) Find $|\overrightarrow{AB}|$.
b) Find the components of the vector $\mathbf{u} = \overrightarrow{AB}$ and sketch it in standard position.
c) Write the vector $\mathbf{v} = \dfrac{1}{|\overrightarrow{AB}|} \cdot \mathbf{u}$ in component form.
d) Find $|\mathbf{v}|$.
e) Sketch the vector **v** and compare it to **u**.

2 $A(3, 4)$ and $B(7, -1)$

3 $A(-2, 3)$ and $B(5, 1)$

4 $A(3, 5)$ and $B(0, 5)$

5 $A(2, -4)$ and $B(2, 1)$

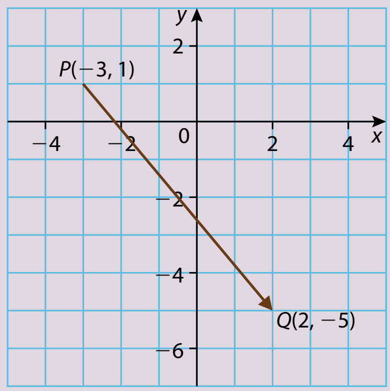
6 Consider the vector shown.

a) Write down the component representation of the vector.

b) Find the length of the vector.

c) Sketch the vector in standard position.

d) Find a vector equal to this one with initial point $(-1, 1)$.



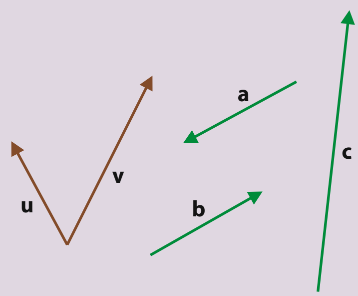
For questions 7–9, the initial point P and terminal point Q are given. Answer the same questions as in question 6.

7 $P(3, 2)$, $Q(7, 8)$

8 $P(2, 2)$, $Q(7, 7)$

9 $P(-6, -8)$, $Q(-2, -2)$

10 Which of the vectors **a**, **b**, or **c** in the figure shown right is equivalent to $\mathbf{u} - \mathbf{v}$?
Which is equivalent to $\mathbf{v} + \mathbf{u}$?



11 Find the terminal point of $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ if the initial point is $(-2, 1)$.

12 Find the initial point of $\mathbf{v} = (-3, 1)$ if the terminal point is $(5, 0)$.

13 Find the terminal point of $\mathbf{v} = (6, 7)$ if the initial point is $(-2, 1)$.

14 Find the initial point of $\mathbf{v} = 2\mathbf{i} + 7\mathbf{j}$ if the terminal point is $(-3, 2)$.

15 Consider the vectors $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$.

a) Find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u} + 3\mathbf{v}$ and $2\mathbf{u} - 3\mathbf{v}$.

b) Find $|\mathbf{u} + \mathbf{v}|$, $|\mathbf{u} - \mathbf{v}|$, $|\mathbf{u}| + |\mathbf{v}|$ and $|\mathbf{u}| - |\mathbf{v}|$.

c) Find $|2\mathbf{u} + 3\mathbf{v}|$, $|2\mathbf{u} - 3\mathbf{v}|$, $2|\mathbf{u}| + 3|\mathbf{v}|$ and $2|\mathbf{u}| - 3|\mathbf{v}|$.

16 Let $\mathbf{u} = (1, 5)$ and $\mathbf{v} = (3, -4)$. Find the vector $\mathbf{x}$ such that $2\mathbf{u} - 3\mathbf{x} + \mathbf{v} = 5\mathbf{x} - 2\mathbf{v}$.

17 Find **u** and **v** if $\mathbf{u} - 2\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{u} + 3\mathbf{v} = \mathbf{i} + \mathbf{j}$.

18 Find the lengths of the diagonals of the parallelogram whose sides are the vectors $2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.

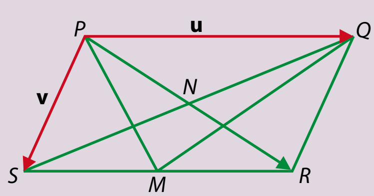
19 Vectors **u** and **v** form two sides of parallelogram $PQRS$, as shown. Express each of the following vectors in terms of **u** and **v**.

a) $\overrightarrow{PR}$

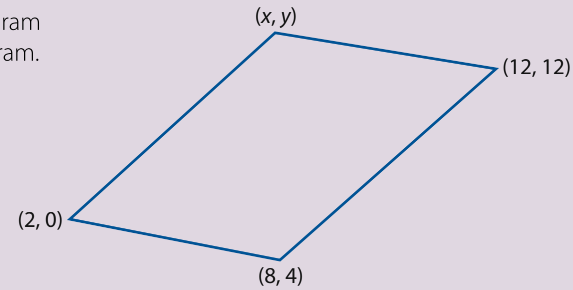
b) $\overrightarrow{PM}$, where M is the midpoint of $[RS]$

c) $\overrightarrow{QS}$

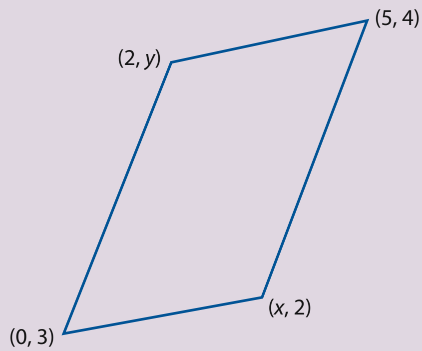
d) $\overrightarrow{QN}$



20 Find (x, y) so that the diagram at the right is a parallelogram.



21 Find x and y in the parallelogram shown right.



22 Find the scalars r and s such that

$\begin{pmatrix} 8 \\ 46 \end{pmatrix} = r\begin{pmatrix} 1 \\ 9 \end{pmatrix} + s\begin{pmatrix} 1 \\ -4 \end{pmatrix}.$

Note: $\begin{pmatrix} 8 \\ 46 \end{pmatrix}$ is said to be written as a linear combination of $\begin{pmatrix} 1 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

23 Write $(4, 7)$ as a linear combination of $(2, 3)$ and $(2, 1)$.

24 Write $(5, -5)$ as a linear combination of $(1, -1)$ and $(-1, 1)$.

25 Write $(-11, 0)$ as a linear combination of $(2, 5)$ and $(3, 2)$.

26 Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j}$. Show that, if $\mathbf{w}$ is any vector in the plane, then it can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. (You can generalize the result to any two non-zero, non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$.)

9.3 Unit vectors and direction angles

Consider the vector $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$. To find the magnitude of this vector, $|\mathbf{u}|$, we use the distance formula:

$$|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$$

If we divide the vector $\mathbf{u}$ by $|\mathbf{u}| = 5$, i.e. we multiply the vector $\mathbf{u}$ by the reciprocal of its magnitude, we get another vector that is parallel to $\mathbf{u}$, since they are scalar multiples of each other. The new vector is

$$\frac{\mathbf{u}}{5} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

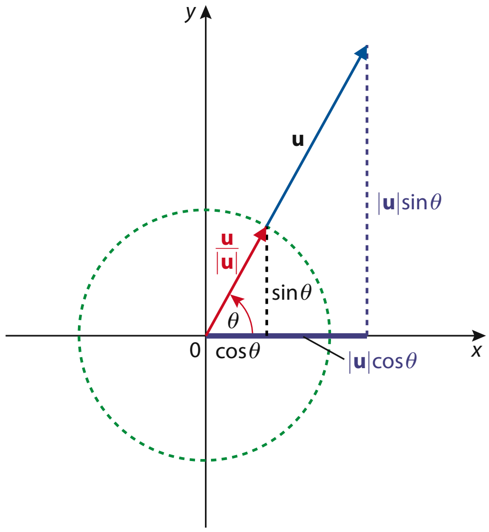
This vector is a unit vector in the same direction as $\mathbf{u}$, because

$$\left|\frac{\mathbf{u}}{5}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$$

Therefore, to find a unit vector in the same direction as a given vector, we divide that vector by its own magnitude.

This is tightly connected to the concept of the **direction angle** of a given vector. The **direction angle** of a vector (in standard position) is the angle it makes with the positive x-axis (Figure 9.14).

Figure 9.14



So, the vector **u** can be expressed in terms of the unit vector parallel to it in the following manner:

$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} = (|\mathbf{u}|\cos\theta)\mathbf{i} + (|\mathbf{u}|\sin\theta)\mathbf{j} = |\mathbf{u}|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$ where $u_1 = |\mathbf{u}|\cos\theta$ and $u_2 = |\mathbf{u}|\sin\theta$. This fact implies two important tools that help us:

1. find the direction of a given vector
2. find vectors of any magnitude parallel to a given vector.

To find a unit vector parallel to a vector **u**, we simply find the vector $\frac{\mathbf{u}}{|\mathbf{u}|}$:

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{u}}{\sqrt{u_1^2 + u_2^2}} = \left(\frac{u_1}{\sqrt{u_2^1 + u_2^2}}, \frac{u_2}{\sqrt{u_2^1 + u_2^2}}\right)$$

Applications of unit vectors and direction angles

Given a vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, find the direction angle of this vector and another vector, whose magnitude is m, that is parallel to the vector **u.**

1. To help determine the direction angle, we observe the following:

   $u_1 = |\mathbf{u}|\cos\theta$ and $u_2 = |\mathbf{u}|\sin\theta$

   This implies that $\frac{u_2}{u_1} = \frac{|\mathbf{u}|\sin\theta}{|\mathbf{u}|\cos\theta} = \tan\theta$.

   So, $\tan^{-1}\theta$ is the reference angle for the direction angle in question. To know what the direction angle is, it is best to look at the numbers u_1 and u_2 in order to determine which quadrant the vector is in. The following example (Example 6) will clarify this point.

2. To find a vector of magnitude m parallel to **u**, we must first find the unit vector in the direction of **u** and then we multiply it by the scalar m.

   The unit vector in the direction of **u** is $\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{|\mathbf{u}|}(u_1\mathbf{i} + u_2\mathbf{j})$, and the vector of magnitude m in this direction will be

   $$m\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{m}{\sqrt{u_1^2 + u_2^2}}(u_1\mathbf{i} + u_2\mathbf{j}).$$

Example 7

Find the direction angle (to the nearest degree) of each vector, and find a vector of magnitude 7 that is parallel to each.

a) $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j}$

b) $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j}$

c) $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j}$

Solution

a) The direction angle for **u** is θ, as shown in Figure 9.15.

$\tan\theta = \frac{2}{2} = 1 \Rightarrow \theta = 45°$

A vector of magnitude 7 that is parallel to **u** is

$$7\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{7}{\sqrt{2^2 + 2^2}}(2\mathbf{i} + 2\mathbf{j}) = \frac{7}{2\sqrt{2}}(2\mathbf{i} + 2\mathbf{j}) = \frac{7}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

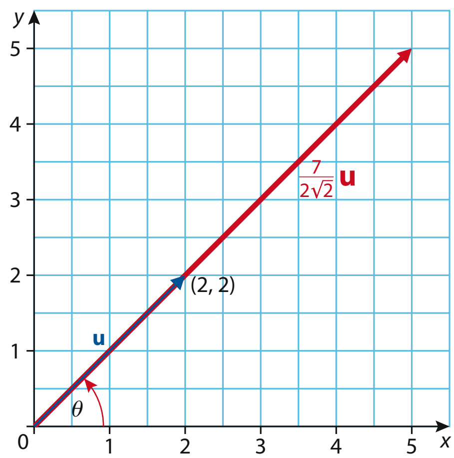


Figure 9.15

b) The direction angle for **v** is $180° - \theta$, as shown in Figure 9.16.

$\tan\theta = \frac{-3}{3} = -1 \Rightarrow \theta = 180° - 45° = 135°$

A vector of magnitude 7 that is parallel to **v** is

$$7\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7}{\sqrt{3^2 + 3^2}}(-3\mathbf{i} + 3\mathbf{j}) = \frac{7}{3\sqrt{2}}(-3\mathbf{i} + 3\mathbf{j}) = \frac{7}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}).$$

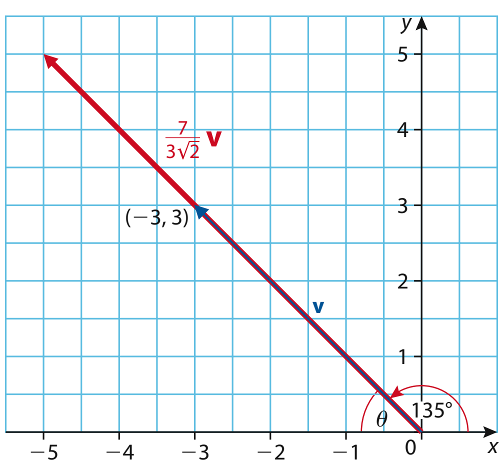


Figure 9.16

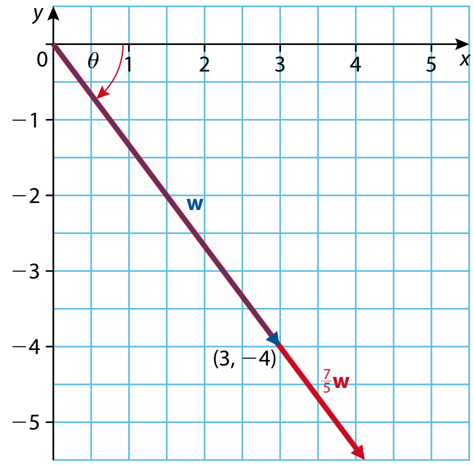
c) The direction angle for $\mathbf{w}$ is θ, as shown in Figure 9.17.

$\tan\theta = \frac{-4}{3} \Rightarrow \theta \approx -53°$

A vector of magnitude 7 that is parallel to $\mathbf{w}$ is

$$7\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{7}{\sqrt{3^2 + (-4)^2}}(3\mathbf{i} - 4\mathbf{j}) = \frac{7}{5}(3\mathbf{i} - 4\mathbf{j}).$$

Figure 9.17



Using vectors to model force, displacement and velocity

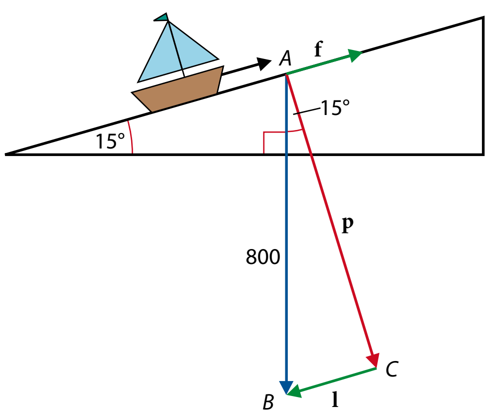
The force on an object can be represented by a vector. We can think of the force as a push or pull on an object such as a person pulling a box along a plane or the weight of a truck which is a downward pull of the Earth's gravity on the truck. If several forces act on an object, the **resultant** force experienced by the object is the vector sum of the forces.

Force

Example 8

What force is required to pull a boat of 800 N up a ramp inclined at 15° from the horizontal? Friction is ignored in this case.

Solution



The process of 'breaking-up' the vector into its components, as we did in the example, is called **resolving** the vector into its components. Notice that the process of resolving a vector is not unique. That is, you can resolve a vector into several pairs of directions.

The situation can be shown on a diagram. The weight is represented by the vector $\overrightarrow{AB}$. The weight of the boat has two components – one perpendicular to the ramp, which is the force responsible for keeping the boat on the ramp and preventing it from tumbling down (**p**). The other force is parallel to the ramp, and is the force responsible for pulling the boat down the ramp (**l**). Therefore, the force we need, **f**, must counter **l**.

In triangle ABC:

$$\sin\angle A = |\mathbf{l}|/800 \Rightarrow |\mathbf{l}| = 800\sin\angle A = 800\sin 15° = 207.06.$$

We need an upward force of 207.06 N along the ramp to move the boat.

Example 9

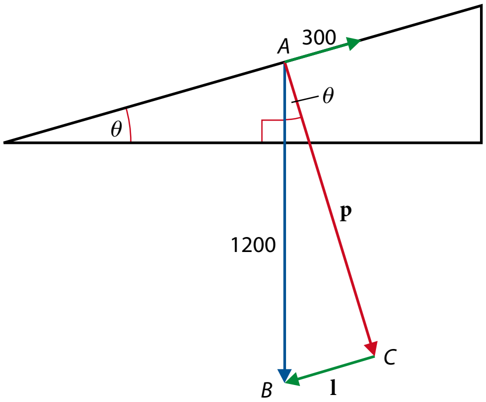
In many countries, it is a requirement that disabled people have access to all places without needing the help of others. Consider an office building whose entrance is 40 cm above ground level. Assuming, on average, that the weight of a person including the equipment used is 1200 N, answer the following questions:

a) At what angle should the ramp designed for disabled persons be set if, on average, the force that a person can apply using their hands is 300 N?

b) How long should the ramp be?

Solution

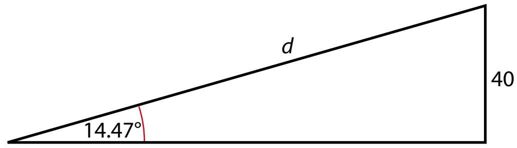
a)



As the diagram above shows, $|\mathbf{l}| = 300$, and

$$\sin\angle A = \frac{|\mathbf{l}|}{1200} = \frac{300}{1200} \Rightarrow \angle A = \sin^{-1} 0.25 \approx 14.47°.$$

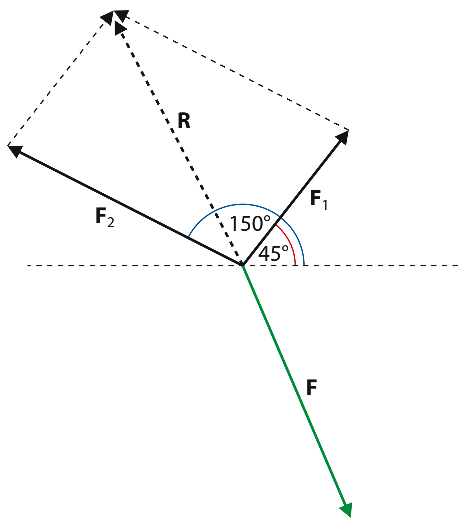
b) The length d of the ramp can be found using right triangle trigonometry:



$$\sin 14.47 = \frac{40}{d} \Rightarrow d = \frac{40}{\sin 14.47} \approx \frac{40}{0.25} = 160 \text{ cm}$$

Resultant force

Two forces $\mathbf{F}_1$ with magnitude 20 N and $\mathbf{F}_2$ with magnitude 40 N are acting on an object at equilibrium as shown in the diagram. Find the force $\mathbf{F}$ required to keep the object at equilibrium.



We will write the vectors for $\mathbf{F}_1$ and $\mathbf{F}_2$ in component form:

$$\mathbf{F}_1 = (20\cos 45°)\mathbf{i} + (20\sin 45°)\mathbf{j} = 10\sqrt{2}\mathbf{i} + 10\sqrt{2}\mathbf{j}$$
$$\mathbf{F}_1 = (40\cos 150°)\mathbf{i} + (40\sin 150°)\mathbf{j} = -20\sqrt{3}\mathbf{i} + 20\mathbf{j}$$

Now, the resultant force $\mathbf{R}$ is

$$\mathbf{R} = (10\sqrt{2}\mathbf{i} + 10\sqrt{2}\mathbf{j}) + (-20\sqrt{3}\mathbf{i} + 20\mathbf{j})$$
$$= (10\sqrt{2} - 20\sqrt{3})\mathbf{i} + (10\sqrt{2} + 20)\mathbf{j}$$

Finally, the force $\mathbf{F}$ required to keep the object at equilibrium is

$$\mathbf{F} = -\mathbf{R} = (-10\sqrt{2} + 20\sqrt{3})\mathbf{i} - (10\sqrt{2} + 20)\mathbf{j}$$

Vectors can be used to help tackle displacement situations. For example, an object at a position defined by the position vector (**a**, **b**) and a velocity vector (**c**, **d**) has a position vector (**a**, **b**) + t(**c**, **d**) after time t.

Displacement and velocity

Note: In navigation, the convention is that the **course** or **bearing** of a moving object is the angle that its direction makes with the north direction measured clockwise. So, for example, a ship going east has a bearing of 90°.

The velocity of an object can be represented by a vector whose direction is the direction of motion and whose magnitude is the speed of the object.

When external forces interfere with the motion, such as wind, stream, and friction, then objects will move under the influence of the **resultant forces**.

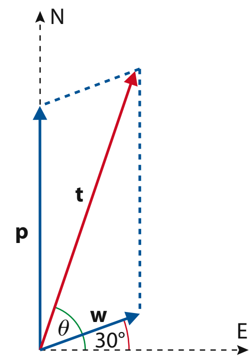
Example 10

An aeroplane heads in a northerly direction with a speed of 450 km/h. The wind is blowing in the direction of N 60° E with a speed of 60 km/h.

a) Write down the component forms of the plane's air velocity and the wind velocity.

b) Find the true velocity of the plane.

c) Find the true speed and direction of the plane.



Solution

Let **p** be the vector for the plane's air velocity, **w** the wind's velocity, and **t** the true velocity.

a) $\mathbf{p} = 0\mathbf{i} + 450\mathbf{j}$

$$\mathbf{w} = (60\cos 30°)\mathbf{i} + (60\sin 30°)\mathbf{j} = 30\sqrt{3}\mathbf{i} + 30\mathbf{j}$$

b) The true velocity of the plane is the resultant of the two forces above, therefore

$$\mathbf{t} = \mathbf{p} + \mathbf{w} = (0\mathbf{i} + 450\mathbf{j}) + (30\sqrt{3}\mathbf{i} + 30\mathbf{j}) = 30\sqrt{3}\mathbf{i} + 480\mathbf{j}.$$

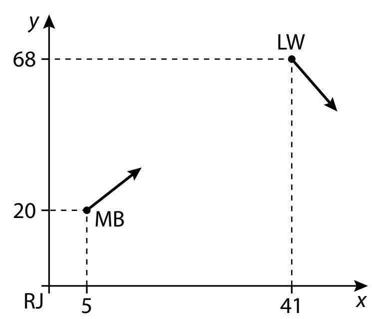
c) The true speed is given by the magnitude of **t**,

$$|\mathbf{t}| = \sqrt{(30\sqrt{3})^2 + 480^2} \approx 482.8 \text{ km/h}.$$

The direction is determined by the angle θ that the true velocity makes with the horizontal. From our discussion earlier, this can be found by using the property that $\tan\theta = \dfrac{480}{30\sqrt{3}} \approx 9.24$, and so $\theta \approx 83.8°$. So, we can now give the true direction of the plane as N 6.2° E.

Example 11

The position vector of a ship (MB) from its starting position at a port RJ is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix} + t\begin{pmatrix} 12 \\ 16 \end{pmatrix}$. Distances are in kilometres and speeds are in km/h. t is time after 00 hour.



a) Find the position of the MB after 2 hours.

b) What is the speed of the MB?

c) Another ship (LW) is at sea in a location $\begin{pmatrix} 41 \\ 68 \end{pmatrix}$ relative to the same port. LW has stopped for some reason. Show that if LW does not start to move, the two ships will collide. Find the time of the potential collision.

d) To avoid collision, LW is ordered to leave its position and start moving at a velocity of $\begin{pmatrix} 15 \\ -36 \end{pmatrix}$ one hour after MB started. Find the position vector of LW.

e) How far apart are the two ships after two hours since the start of MB?

Solution

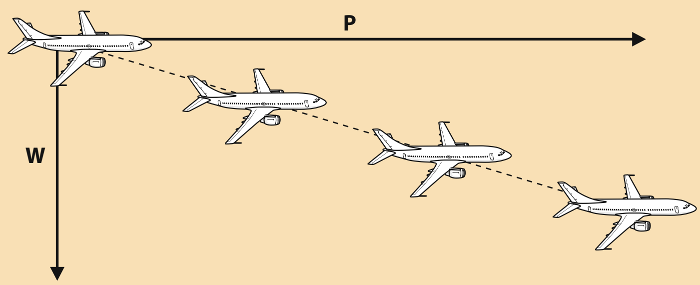
a) MB is at a position with vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix} + 2\begin{pmatrix} 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 29 \\ 52 \end{pmatrix}$.

b) Since the velocity of the ship is $\begin{pmatrix} 12 \\ 16 \end{pmatrix}$, the speed is $\left|\begin{pmatrix} 12 \\ 16 \end{pmatrix}\right| = \sqrt{12^2 + 16^2}$ $= 20$ km/h.

c) The collision can happen if the position vectors of the two ships are equal: $\begin{pmatrix} 5 \\ 20 \end{pmatrix} + t\begin{pmatrix} 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 41 \\ 68 \end{pmatrix} \Rightarrow 5 + 12t = 41$ and $20 + 16t = 68 \Rightarrow 12t = 36$ and $16t = 48 \Rightarrow t = 3$. After 3 hours, at 03:00, a collision could happen.

d) Since LW started one hour later, its position vector is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 41 \\ 68 \end{pmatrix} + (t - 1)\begin{pmatrix} 15 \\ -36 \end{pmatrix}, t \geqslant 1$.

e) MB is at $\begin{pmatrix} 29 \\ 52 \end{pmatrix}$ and LW is at $\begin{pmatrix} 41 \\ 68 \end{pmatrix} + (2 - 1)\begin{pmatrix} 15 \\ -36 \end{pmatrix} = \begin{pmatrix} 56 \\ 32 \end{pmatrix}$. The distance between them is $\sqrt{(56 - 29)^2 + (32 - 52)^2} = \sqrt{1129} = 33.6$ km.



When the wind is strong and is acting in a direction different from that of the airplane and if you watch the plane from the ground you will notice that the 'nose' of the plane is in a direction (air velocity) different from the motion of the plane's 'true' velocity.

Exercise 9.3

1 Find the direction angle for each vector.

a) $\mathbf{u} = (2, 0)$

b) $\mathbf{v} = (0, 3)$

c) $\mathbf{w} = (-3, 0)$

d) $\mathbf{u} + \mathbf{v}$

e) $\mathbf{v} + \mathbf{w}$

2 Find the magnitude and direction angle for each vector.

a) $\mathbf{u} = (3, 2)$ b) $\mathbf{v} = (-3, -2)$ c) $2\mathbf{u}$

d) $3\mathbf{v}$ e) $2\mathbf{u} + 3\mathbf{v}$ f) $2\mathbf{u} - 3\mathbf{v}$

3 Find the magnitude and direction angle for each vector.

a) $\mathbf{u} = (-4, 7)$ b) $\mathbf{v} = (2, 5)$ c) $3\mathbf{u}$

d) $-2\mathbf{v}$ e) $3\mathbf{u} + 2\mathbf{v}$ f) $\mathbf{u} - \mathbf{v}$

4 Write each of the following vectors in component form. θ is the angle that the vector makes with the positive horizontal axis.

a) $|\mathbf{u}| = 310, \theta = 62°$ b) $|\mathbf{u}| = 43.2, \theta = 19.6°$

c) $|\mathbf{u}| = 12, \theta = 135°$ d) $|\mathbf{u}| = 240, \theta = 300°$

5 Find the coordinates of a point D such that $\overrightarrow{AB} = 2\overrightarrow{CD}$ where $A(2, 1)$, $B(4, 7)$, and $C(-1, 1)$.

6 Find the unit vector in the same direction as **u** in each of the following cases.

a) $\mathbf{u} = (3, 4)$

b) $\mathbf{u} = 2\mathbf{i} - 5\mathbf{j}$

7 Find a unit vector in the plane making an angle θ with the positive x-axis where

a) $\theta = 150°$

b) $\theta = 315°$

8 Find a vector of magnitude 7 that is parallel to $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.

9 Find a vector of magnitude 3 that is parallel to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$.

10 Find a vector of magnitude 7 that is perpendicular to $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.

11 Find a vector of magnitude 3 that is perpendicular to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$.

12 A plane is flying on a bearing of 170° at a speed of 840 km/h. The wind is blowing in the direction N 120° E with a strength of 60 km/h.

a) Find the vector components of the plane's still-air velocity and the wind's velocity.

b) Determine the true velocity (ground) of the plane in component form.

c) Write down the true speed and direction of the plane.

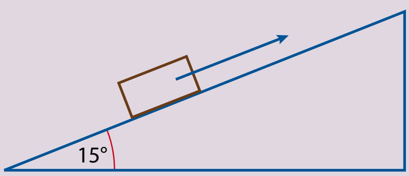
Note: In navigation, the convention is that the **course** or **bearing** of a moving object is the angle that its direction makes with the north direction measured clockwise. So, for example, a ship going east has a bearing of 090°.

13 A plane is flying on a compass heading of 340° at 520 km/h. The wind is blowing with the bearing 320° at 64 km/h.

a) Find the component form of the velocities of the plane and the wind.

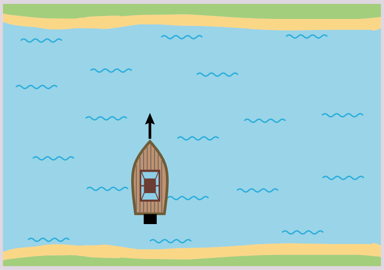
b) Find the actual ground speed and direction of the plane.

14

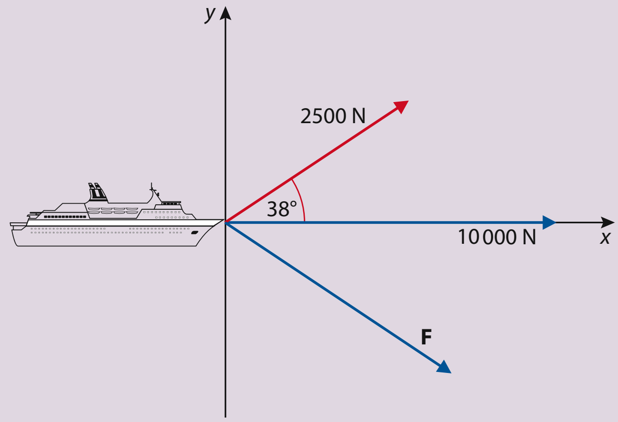


A box is being pulled up a 15° inclined plane. The force needed is 25 N. Find the horizontal and vertical components of the force vector and interpret each of them.

15 A motor boat with the power to steer across a river at 30 km/h is moving such that the bow is pointed in a northerly direction. The stream is moving eastward at 6 km/h. The river is 1 km wide. Where on the opposite side will the boat meet the land?



16 A force of 2500 N is applied at an angle of 38° to pull a 10 000 N ship in the direction given. What force **F** is needed to achieve this?

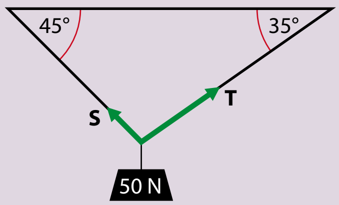


17 A boat is observed to have a bearing of 072°. The speed of the boat relative to still water is 40 km/h. Water is flowing directly south. The boat appears to be heading directly east.

a) Express the velocity of the boat with respect to the water in component form.

b) Find the speed of the water stream and the true speed of the boat.

18 A 50 N weight is suspended by two strings as shown. Find the tensions **T** and **S** in the strings.



19 A runner runs in a westerly direction on the deck of a cruise ship at 8 km/h. The cruise ship is moving north at a speed of 35 km/h. Find the velocity of the runner relative to the water.

20 The boat in question 15 wants to reach a point exactly north of the starting point. In which direction should the boat be steered in order to achieve this objective?

21 Forces **F** = (−10, 3), **G** = (−4, 1) and **H** = (4, −10) act on a point **P**. Find the additional force required to keep the system in equilibrium.

22 A wind is blowing due west at 60 km/h. A small plane with air speed of 300 km/h is trying to maintain a course due north. In what direction should the pilot steer the plane to keep the targeted course? How fast is the plane moving?

23 The points $P(2, 2)$, $Q(10, 2)$ and $R(12, 6)$ are three vertices of a parallelogram. Find the fourth vertex S if

a) P and R are vertices of the same diagonal

b) P and R are vertices of a common side.

24 Show, using vector operations, that the diagonals of a parallelogram intersect each other.

25 Show, using vector operations, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has half its length.

26 Prove that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

27 An athlete is rowing a boat at a speed of 30 m per minute across a small river 150 m wide. The athlete keeps the boat heading perpendicular to the banks of the river.

a) How far down the river does the boat reach the opposite side if the river is flowing at a rate of 10 m/minute?

b) How long does the trip last?

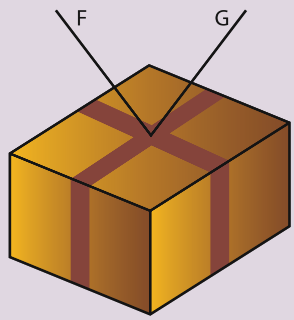
c) At what angle must the athlete steer the boat in order to reach a point directly opposite the starting point on the other side of the river? How long does the trip take?

28 A jet heads in the direction N 30° E at a speed of 400 km/h. The jet experiences a 20 km/h crosswind flowing due east. Find

a) the true velocity **p** of the jet,

b) the true speed and direction of the jet.

29 A box is carried by two strings F and G as shown right. The string F makes an angle of 45° with the horizontal while G makes an angle of 30°. The forces in F and G have a magnitude of 200 N each. The weight of the box is 300 N. What is the magnitude of the resultant force on the box and in which direction does it move?



9.4 Scalar product of two vectors

The multiplication of two vectors is not uniquely defined: in other words, it is unclear whether the product will be a vector or not. For this reason there are two types of vector multiplication:

The **scalar** or **dot product** of two vectors, which results in a scalar; and the **vector** or **cross product** of two vectors, which results in a vector.

In this chapter, we shall discuss only the scalar or dot product. We will discuss the vector product in Chapter 14.

The **scalar product of two vectors**, **a** and **b** denoted by $\mathbf{a} \cdot \mathbf{b}$, is defined as the product of the magnitudes of the vectors times the cosine of the angle between them:
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

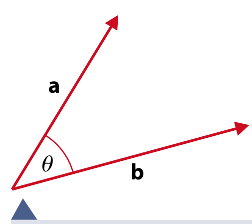


Figure 9.18

This is illustrated in Figure 9.18.

Note that the result of a dot product is a scalar, not a vector. The rules for scalar products are given in the following list:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} \\ 0 \cdot \mathbf{a} &= \mathbf{a} \cdot 0 = 0 \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{a} &= |\mathbf{a}|^2 \\ k(\mathbf{a} \cdot \mathbf{b}) &= k\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot k\mathbf{b}, \text{ with } k \text{ any scalar.} \end{aligned}$$

The first properties follow directly from the definition:

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, and $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\cos\theta$, and, since multiplication of real numbers is commutative, it follows that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ The third property will be proved later in this section. Proofs of the rest of the properties are left as exercises.

Using the definition, it is immediately clear that for two non-zero vectors **u** and **v**, if **u** and **v** are perpendicular, the dot product is zero. This is so, because
$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 90° = |\mathbf{u}||\mathbf{v}| \times 0 = 0$.

The converse is also true: if $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are perpendicular,
$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow |\mathbf{u}||\mathbf{v}|\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$.

Using the definition, it is also clear that for two non-zero vectors **u** and **v**, if **u** and **v** are parallel then the dot product is equal to $\pm\,|\mathbf{u}||\mathbf{v}|$. This is so, because
$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 0° = |\mathbf{u}||\mathbf{v}| \times 1 = |\mathbf{u}||\mathbf{v}|$, or
$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 180° = |\mathbf{u}||\mathbf{v}| \times (-1) = -|\mathbf{u}||\mathbf{v}|$.

The converse is also true: if $\mathbf{u} \cdot \mathbf{v} = \pm\,|\mathbf{u}||\mathbf{v}|$, the vectors are parallel, since
$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \Rightarrow |\mathbf{u}||\mathbf{v}|\cos\theta = \pm\,|\mathbf{u}||\mathbf{v}| \Rightarrow \cos\theta = \pm 1 \Rightarrow \theta = 0°$ or $\theta = 180°$.

Another interpretation of the dot product

Projection

(*This subsection is optional – it is beyond the scope of the IB syllabus, but very helpful in clarifying the concept of dot products.*)

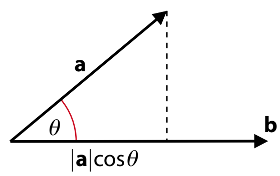


Figure 9.19

The quantity $|\mathbf{a}|\cos\theta$ is called the projection of the vector **a** on vector **b** (Figure 9.19). So, the dot product $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}||\mathbf{a}|\cos\theta = |\mathbf{b}|(|\mathbf{a}|\cos\theta)$
$= |\mathbf{b}| \times$ (the projection of **a** on **b**).

This fact is used in proving the third property on the list on page 419.

If we let B and C stand for the projections of **b** and **c** on **a**, we have
$\mathbf{a}(\mathbf{b} + \mathbf{c}) = |\mathbf{a}|(\mathrm{B} + \mathrm{C}) = |\mathbf{a}|\mathrm{B} + |\mathbf{a}|\mathrm{C} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
This is called the **distributive property** of scalar products over vector addition. See Figure 9.20.

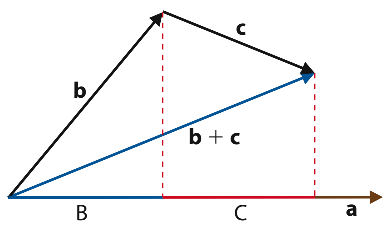


Figure 9.20

With this result, we can develop another definition for the dot product that is more useful in the calculation of this product.

Theorem

If vectors are expressed in component form, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, then $\mathbf{u} \cdot \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) \cdot (v_1\mathbf{i} + v_2\mathbf{j}) = u_1v_1 + u_2v_2$.

Proof

$\mathbf{u} \cdot \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) \cdot (v_1\mathbf{i} + v_2\mathbf{j}) = u_1v_1\mathbf{i}^2 + u_1v_2\mathbf{ij} + u_2v_1\mathbf{ji} + u_2v_2\mathbf{j}^2$

However, $\mathbf{i}^2 = \mathbf{j}^2 = 1$ and $\mathbf{ij} = \mathbf{ji} = 0$. (Proof is left as an exercise for you.)

Therefore, $\mathbf{u} \cdot \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) \cdot (v_1\mathbf{i} + v_2\mathbf{j}) = u_1v_1 + u_2v_2$.

For example, to find the scalar product of the two vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$, it is enough to add the products' corresponding components:

$$\mathbf{u} \cdot \mathbf{v} = 2 \times 5 + 4 \times (-3) = -2$$

If we start the definition of the scalar product as $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$, we can deduce the other definition.

Start with the law of cosines which you learned in Chapter 8. Consider the diagram opposite and apply the law to finding BC in triangle ABC.

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

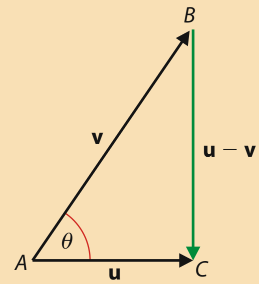
Using the fact that $\mathbf{u} \cdot \mathbf{u} = u_1u_1 + u_2u_2 = \mathbf{u}^2$,

$$\begin{aligned}|\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v})^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})\\ &= u^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + v^2 = u^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + v^2\\ &= |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2\end{aligned}$$

Now, comparing the two results

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

$$\Rightarrow -2(\mathbf{u} \cdot \mathbf{v}) = -2|\mathbf{u}|\,|\mathbf{v}|\cos\theta \Rightarrow \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$



Example 12

Find the dot product of $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$.

Solution

$$\mathbf{u} \cdot \mathbf{v} = 2 \times 3 - 3 \times 2 = 0$$

What does this tell us about the two vectors?

The angle between two vectors

The basic definition of the scalar product offers us a method for finding the angle between two vectors.

Since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

Note: When the vectors **u** and **v** are given in component form, then the angle cosine can be directly calculated with

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{u_1v_1 + u_2v_2}{\sqrt{u_1^2 + u_2^2}\,\sqrt{v_1^2 + v_2^2}}$$

Example 13

Find the angle between the following two vectors:

$$\mathbf{v} = -3\mathbf{i} + 3\mathbf{j} \text{ and } \mathbf{w} = 2\mathbf{i} - 4\mathbf{j}$$

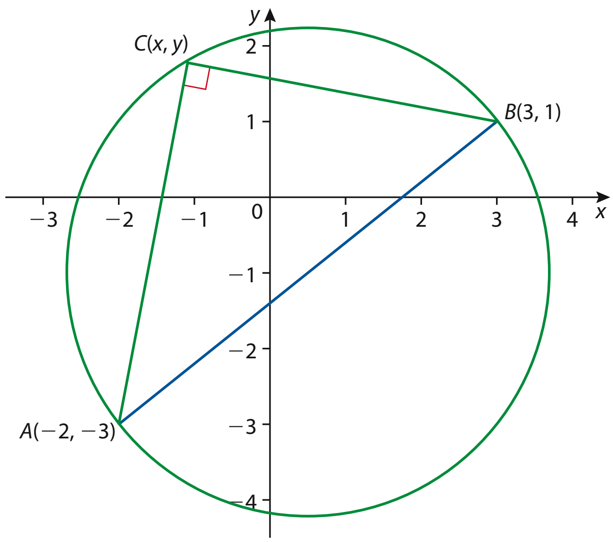
Solution

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{-3 \times 2 + 3 \times -4}{\sqrt{(-3)^2 + 3^2} \times \sqrt{2^2 + 4^2}} = \frac{-18}{\sqrt{18}\sqrt{20}} \Rightarrow \theta = 161.57°$$

Example 14

Consider the segment $[AB]$ with $A(-2, -3)$ and $B(3, 1)$. Use dot products to find the equation of the circle whose diameter is AB.

Solution



Consider any point $C(x, y)$ on the graph. Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$. For the point C to be on the circle, the angle at C must be a right angle. Hence, the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are perpendicular.

For perpendicular vectors, the dot product must be zero.

$$\overrightarrow{AC} = (x + 2, y + 3), \overrightarrow{BC} = (x - 3, y - 1)$$

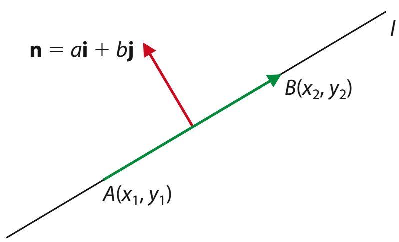
$$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0 \Rightarrow (x + 2)(x - 3) + (y + 3)(y - 1) = 0$$

$$\Rightarrow x^2 - x + y^2 + 2y = 9$$

Example 15

Show that the vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is orthogonal (perpendicular) to the line l with equation $ax + by + c = 0$.

Solution



Consider two points A and B on the line with the coordinates as shown.

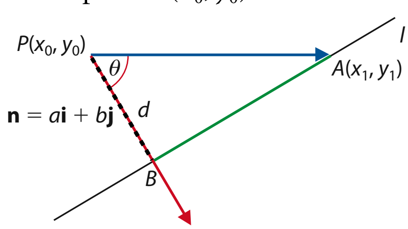
$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ and

$\mathbf{n} \cdot \overrightarrow{AB} = (a, b) \cdot (x_2 - x_1, y_2 - y_1) = (ax_2 + by_2) - (ax_1 + by_1)$, but A and B are on the line, so

$ax_2 + by_2 = -c$ and $ax_1 + by_1 = -c \Rightarrow \mathbf{n} \cdot \overrightarrow{AB} = -c + c = 0.$

Example 16

Find the distance from the point $P(x_0, y_0)$ to the line l with equation $ax + by + c = 0$.



Solution

The required distance, d, can be found using triangle PAB.

$$d = \left|\left|\overrightarrow{PA}\right|\cos\theta\right| = \left|\left|\overrightarrow{PA}\right|\frac{\overrightarrow{PA}\cdot\mathbf{n}}{\left|\overrightarrow{PA}\right||\mathbf{n}|}\right| = \left|\frac{\overrightarrow{PA}\cdot\mathbf{n}}{|\mathbf{n}|}\right|, \left(\frac{\overrightarrow{PA}\cdot\mathbf{n}}{|\mathbf{n}|} \text{ is called the component of } \overrightarrow{PA} \text{ along } \mathbf{n}.\right)$$

Now,

$$\overrightarrow{PA} = (x_1 - x_0, y_1 - y_0) \Rightarrow \overrightarrow{PA}\cdot\mathbf{n} = a(x_1 - x_0) + b(y_1 - y_0)$$

$$\Rightarrow \overrightarrow{PA}\cdot\mathbf{n} = ax_1 + by_1 - ax_0 - by_0 = -c - ax_0 - by_0$$

Therefore, $d = \left|\frac{\overrightarrow{PA}\cdot\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-c - ax_0 - by_0}{\sqrt{a^2 + b^2}}\right| = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$.

So, for example, the distance from $A(2, -3)$ to the line with equation $5x + 3y = 2$ is

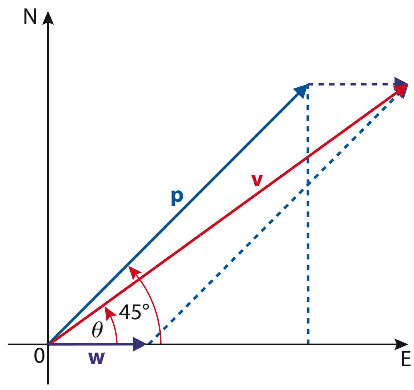
$$d = \frac{|5(2) + 3(-3) - 2|}{\sqrt{5^2 + 3^2}} = \frac{1}{\sqrt{34}} = \frac{\sqrt{34}}{34}.$$

Example 17

The instrument panel in a plane indicates that its airspeed (the speed of the plane relative to the surrounding air) is 200 km/h and that its compass heading (the direction in which the plane's nose is pointing) is N 45° E. There is a steady wind blowing from the west at 50 km/h. Because of the wind, the plane's *true* velocity is different from the panel reading. Find the true velocity of the plane. Also, find its true speed and direction.

Solution

A diagram can help clarify the situation.



The plane velocity $\mathbf{p}$ can be expressed in its component form:

$$x = |\mathbf{p}|\cos 45° = 200\cos 45° = 100\sqrt{2},$$

$$y = |\mathbf{p}|\sin 45° = 200\sin 45° = 100\sqrt{2},$$

so $\mathbf{p}$ can be written as $\mathbf{p} = (100\sqrt{2}, 100\sqrt{2})$.

The wind velocity $\mathbf{w}$ can also be expressed in component form:

$$\mathbf{w} = (50, 0)$$

So, the true velocity, $\mathbf{v} = (100\sqrt{2} + 50, 100\sqrt{2})$.

To find the true speed, we find the magnitude of the resultant found above:

$$|\mathbf{v}| = \sqrt{(100\sqrt{2} + 50)^2 + (100\sqrt{2})^2} \approx 238 \text{ km/h}$$

To find the true direction, we find θ and calculate the *heading* of the plane:

$$\tan\theta = \frac{100\sqrt{2}}{100\sqrt{2} + 50} \approx 0.739 \Rightarrow \theta \approx 36.5°,$$

so the true direction is N 53.5° E.

Exercise 9.4

1 Find (i) $\mathbf{u} \cdot \mathbf{v}$ and (ii) the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree.

a) $\mathbf{u} = \mathbf{i} + \sqrt{3}\,\mathbf{j}$, $\mathbf{v} = \sqrt{3}\,\mathbf{i} - \mathbf{j}$

b) $\mathbf{u} = (2, 5)$, $\mathbf{v} = (4, 1)$

c) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

d) $\mathbf{u} = 2\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \sqrt{3}\,\mathbf{j}$

e) $\mathbf{u} = (-3, 0)$, $\mathbf{v} = (0, 7)$

f) $\mathbf{u} = (3, 0)$, $\mathbf{v} = (\sqrt{3}, 1)$

g) $\mathbf{u} = -6\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 2\sqrt{3}\mathbf{j}$

h) $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = -4\mathbf{i} - 4\mathbf{j}$

2 Using the vectors $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} + 5\mathbf{j}$, find each of the indicated results.

a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

b) $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

c) $\mathbf{u}(\mathbf{v} \cdot \mathbf{w})$

d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

e) $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{w})$

f) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

g) Looking at a)–d) write one paragraph to summarize what you learned!

3 Determine whether $\mathbf{u}$ is orthogonal, parallel or neither to $\mathbf{v}$:

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} 2\sqrt{3} \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

● **Hint:** The work done by any force is defined as the product of the force multiplied by the distance it moves a certain object. In other words, it is the product of the force multiplied by the displacement of the object. As such, work is the dot product between the force and displacement $\mathbf{W} = \mathbf{F} \cdot \mathbf{D}$.

4 Find the work done by the force $\mathbf{F}$ in moving an object between points M and N.

a) $\mathbf{F} = 400\mathbf{i} - 50\mathbf{j}$, $M(2, 3)$, $N(12, 43)$

b) $\mathbf{F} = 30\mathbf{i} + 150\mathbf{j}$, $M(0, 30)$, $N(15, 70)$

c) $\mathbf{F} = \begin{pmatrix} 5 \\ 25 \end{pmatrix}$, $M(0, 0)$, $N(1, 6)$

5 Find the interior angles of the triangle ABC.

a) $A(1, 2)$, $B(3, 4)$, $C(2, 5)$

b) $A(3, 4)$, $B(-1, -7)$, $C(-8, -2)$

c) $A(3, -5)$, $B(1, -9)$, $C(-7, -9)$

6 Find a vector perpendicular to $\mathbf{u}$ in each case below. (Answers are not unique!)

a) $\mathbf{u} = (3, 5)$

b) $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j}$

7 Use the dot product to find the equation of a circle whose diameter is $[AB]$.

a) $A(1, 2)$, $B(3, 4)$

b) $A(3, 4)$, $B(-1, -7)$

8 Decide whether the triangle ABC is right-angled using vector algebra:
$A(1, -3)$, $B(2, 0)$, $C(6, -2)$

9 Find t such that $\mathbf{a} = t\mathbf{i} - 3\mathbf{j}$ is perpendicular to $\mathbf{b} = 5\mathbf{i} + 7\mathbf{j}$.

10 For what value(s) of b are the vectors $(-6, b)$ and (b, b^2) perpendicular?

11 Find a unit vector that makes an angle of 60° with $\mathbf{u} = (3, 4)$.

12 Find t such that $\mathbf{a} = t\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$ make an angle of $\frac{3}{4}\pi$ radians.

13 Use the dot product to prove that the diagonals of a rhombus are perpendicular to each other.

14 Find the component of $\mathbf{u}$ along $\mathbf{v}$ if

a) $\mathbf{u} = (0, 7)$, $\mathbf{v} = (6, 8)$

b) $\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$

15 A young man pulls a sled horizontally by exerting a force of 16 N on the rope that is tied to its front end. The rope makes an angle of 45° with the horizontal. Find the work done in pulling the sled 55 m.

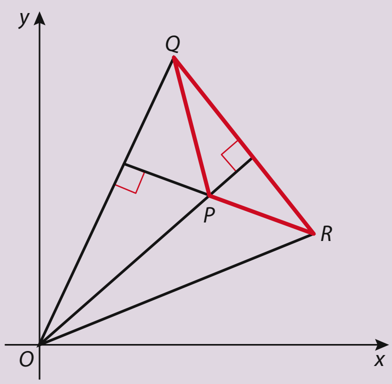
16 Find the distance from the point P to the line l in each case:

a) $P(0, 0)$, l: $3x - 4y + 5 = 0$

b) $P(2, 2)$, l: $3x - 2y = 2$

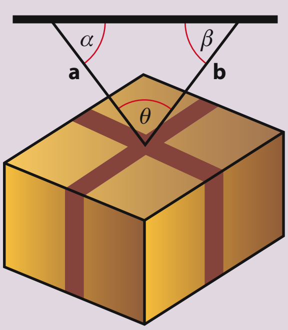
c) $P(1, 5)$, l: $5x - 3y = 11$

17 Given three points in the plane P, Q, and R such that $\overrightarrow{OP} \perp \overrightarrow{QR}$ and $\overrightarrow{OQ} \perp \overrightarrow{PR}$, use scalar product to show that $\overrightarrow{OR} \perp \overrightarrow{PQ}$.



18 Two vectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} x \\ 1 \end{pmatrix}$ have an angle of 30° between them. Find the possible values of x.

19 A weight of 1000 N is supported by two forces $\mathbf{a} = (-200, 400)$ and $\mathbf{b} = (200, 600)$. The weight is in equilibrium. Find the angles α, β, and θ.



20 Show that the vector $|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ bisects the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

Practice questions

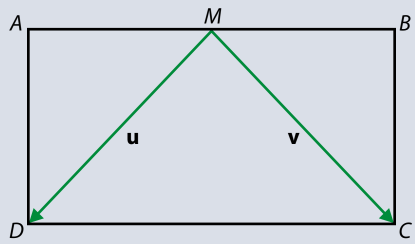
1 $ABCD$ is a rectangle with M the midpoint of $[AB]$. $\mathbf{u}$ and $\mathbf{v}$ represent the vectors joining M to D and C respectively. Express each of the following vectors in terms of $\mathbf{u}$ and $\mathbf{v}$.

a) $\overrightarrow{DC}$

b) $\overrightarrow{AM}$

c) $\overrightarrow{BC}$

d) $\overrightarrow{AC}$



2 Consider the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$.

a) Find the component form of the vector $\mathbf{w} = 2\mathbf{u} + \mathbf{v}$.

b) Find the vector $\mathbf{z}$ which has a magnitude of 6 units and same direction as $\mathbf{w}$.

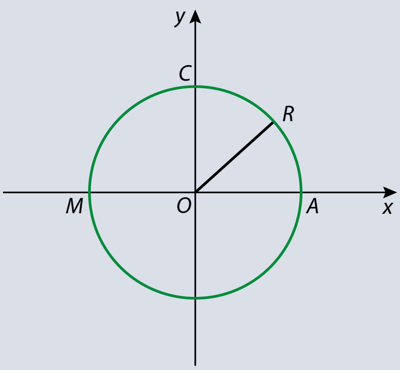
3 M and A are the ends of the diameter of a circle with centre at the origin. The radius of the circle is 15 cm and $\overrightarrow{OR} = \begin{pmatrix} 10 \\ 5\sqrt{5} \end{pmatrix}$.

a) Verify that R lies on the circle.

b) Find the vector $\overrightarrow{AR}$.

c) Find the cosine of $\angle OAR$.

d) Find the area of $\triangle MAR$.



4 Quadrilateral $MARC$ has vertices with coordinates $M(0, 0)$, $A(6, 2)$, $R(11, 4)$ and $C(3, 8)$.

a) Find the vectors $\overrightarrow{MR}$ and $\overrightarrow{AC}$.

b) Find the angle between the diagonals of quadrilateral $MARC$.

c) Let the vector $\mathbf{u}$ be the vector joining the midpoints of $[MA]$ and $[AR]$, and $\mathbf{v}$ be the vector joining the midpoints of $[RC]$ and $[CM]$. Compare $\mathbf{u}$ and $\mathbf{v}$ to $\overrightarrow{MR}$, and hence show that the quadrilateral connecting the midpoints of the sides of $MARC$ form a parallelogram.

5 Vectors $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$ are given. Find the scalars m and n such that $m(\mathbf{u} + \mathbf{v}) - 5\mathbf{i} + 7\mathbf{j} = n(\mathbf{u} - \mathbf{v})$.

6 Vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement in the eastern direction while vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement north. Distances are in kilometres.

Two crews of workers are laying gas pipes in a north-south direction across the North Sea. Consider the base port where the crews leave to start work as the origin (0, 0).

At 07:00 the crews left the base port with their motor boats to two different locations. The crew called 'Marco' travel at a velocity of $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ and the crew called 'Tony' travel at a velocity of $\begin{pmatrix} 18 \\ -8 \end{pmatrix}$. Speeds are in km/h.

a) Find the speed of each boat.

b) Find the position vectors of each crew at 07:30.

c) Hence, or otherwise, find the distance between the vehicles at 07:30.

d) At 07:30 'Tony' stops and the crew begins laying pipes towards the north. 'Marco' continues travelling in the same direction at the same speed until it is exactly north of 'Tony'. At this point, 'Marco' stops and the crew then begins laying pipes towards the south. At what time does 'Marco' start work?

e) Each crew lays an average of 400 m of pipe in an hour. If they work non-stop until their lunch break at 12:30, what is the distance between them at this time?

f) How long would 'Marco' take to return to base port from its lunchtime position, assuming it travelled in a straight line and with the same average speed as on the morning journey? (Give your answer to the nearest minute.)

7 Triangle *TRI* is defined as follows:

$\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\overrightarrow{TR} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $\overrightarrow{TR} \cdot \overrightarrow{IR} = 0$, and $\overrightarrow{TI} = k\mathbf{j}$ where k is a scalar and $\mathbf{j}$ is the unit vector in the y-direction.

a) Draw an accurate diagram of $\triangle TRI$.

b) Write the vector $\overrightarrow{IR}$.

8 Vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement in the eastern direction while vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement north. Distances are in kilometres.

The position vector of a plane for AUA airlines from its starting position in Vienna is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} + t\begin{pmatrix} 360 \\ 480 \end{pmatrix}$. Speeds are in km/h and t is time after 00 hour.

a) Find the position of the AUA plane after 2 hours.

b) What is the speed of the plane?

c) A plane for LH airline started at the same time from a location $\begin{pmatrix} -155 \\ 1300 \end{pmatrix}$ relative to Vienna and moving with a velocity vector $\begin{pmatrix} 480 \\ -360 \end{pmatrix}$, flying at the same height as the AUA plane. Show that if the LH plane does not change route, the two planes will collide. Find the time of the potential collision.

d) To avoid collision, the LH plane is ordered to leave its position and start moving at a velocity of $\begin{pmatrix} 450 \\ -390 \end{pmatrix}$ one hour after it started. Find the position vector of the LH plane at that time.

e) How far apart are the two planes after two hours?

9 For what value(s) of n are the vectors $\begin{pmatrix} 3n \\ 2n+3 \end{pmatrix}$ and $\begin{pmatrix} 2n-1 \\ 4-2n \end{pmatrix}$ perpendicular. Otherwise, show that it is not possible.

10 Let α be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j},\ \mathbf{b} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} \text{ and } 0 < \theta < \frac{\pi}{4}.$$

Express α in terms of θ.

11 Given two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$, find the value of $\mathbf{a}\cdot\mathbf{b}$.

10 Complex Numbers

Assessment statements

1.5 Complex numbers: the number $i = \sqrt{-1}$; the term's real part, imaginary part, conjugate, modulus and argument.
Cartesian form $z = a + ib$.
Sums, products and quotients of complex numbers.

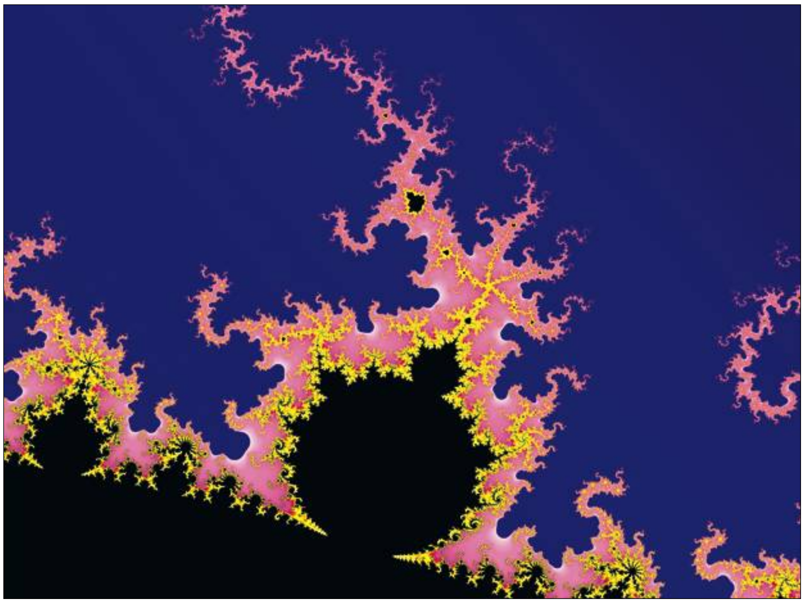
1.6 Modulus–argument (polar) form $z = r(\cos\theta + i\sin\theta) = r\text{cis}(\theta) = re^{i\theta}$.
The complex plane.

1.7 De Moivre's theorem.
Powers and roots of a complex number.

1.8 Conjugate roots of polynomial equations with real coefficients.

Introduction

You have already met complex numbers in Chapters 1 and 3. This chapter will broaden your understanding to include trigonometric representation of complex numbers and some applications.



Fractals can be generated using complex numbers.

Solving a linear equation of the form

$$ax + b = 0, \text{ with } a \neq 0$$

is a straightforward procedure if we are using the set of real numbers. The situation, as you already know, is different with quadratic equations. For example, as you have seen in Chapter 3, solving the quadratic equation

$x^2 + 1 = 0$ *over the set of real numbers* is not possible. The square of any real number has to be non-negative, i.e.

$(x^2 \geqslant 0 \Leftrightarrow x^2 + 1 \geqslant 1) \Rightarrow x^2 + 1 > 0$ for any choice of a real number x.

This means that $x^2 + 1 = 0$ is impossible for every real number x. This forces us to introduce a new set where such a solution is possible.

The situation with finding a solution to $x^2 + 1 = 0$ is analogous to the following scenario: For a child in the first or second grade, a question such as $5 + ? = 9$ is manageable. However, a question such as $5 + ? = 2$ is impossible because the student's knowledge is *restricted* to the set of positive integers.

However, at a later stage when the same student is faced with the same question, he/she can solve it because their scope has been *extended* to include negative numbers too.

Also, at early stages an equation such as

$$x^2 = 5$$

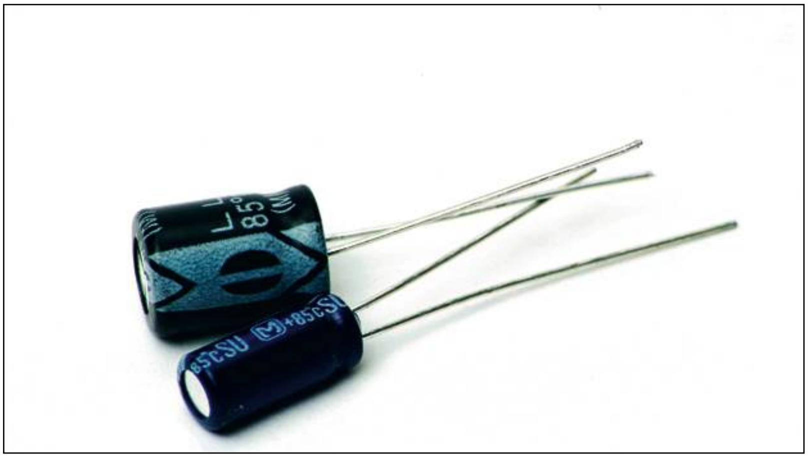
cannot be solved till the student's knowledge of sets is extended to include irrational numbers where he/she can recognize numbers such as $x = \pm\sqrt{5}$.

The situation is much the same for $x^2 + 1 = 0$. We *extend* our number system to include numbers such as $\sqrt{-1}$; i.e. a number whose square is -1.

Numbers such as $\sqrt{-1}$ are not intuitive and many mathematicians in the past resisted their introduction, so they are called **imaginary numbers**.

Thanks to Euler's (1707–1783) seminal work on imaginary numbers, they now feature prominently in the number system. Euler skilfully employed them to obtain many interesting results. Later, Gauss (1777–1855) represented them as points in the plane and renamed them as **complex numbers**, using them to obtain various significant results in number theory.

10.1 Complex numbers, sums, products and quotients



Electronic components like capacitors are used in AC circuits. Their effects are represented using complex numbers.

As you have seen in the introduction, the development of complex numbers had its origin in the search for methods of solving polynomial equations. The quadratic formula

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

had been used earlier than the 16th century to solve quadratic equations – in more primitive notations, of course. However, mathematicians stopped short of using it for cases where $b^2 - 4ac$ was negative. The use of the formula in cases where $b^2 - 4ac$ is negative depends on two principles (in addition to the other principles inherent in the set of real numbers, such as associativity and commutativity of multiplication).

1. $\sqrt{-1} \cdot \sqrt{-1} = -1$
2. $\sqrt{-k} = \sqrt{k} \cdot \sqrt{-1}$ **for any real number** $k > 0$

Example 1

Multiply $\sqrt{-36} \cdot \sqrt{-49}$.

Solution

First we simplify each square root using rule 2.

$$\sqrt{-36} = \sqrt{36} \cdot \sqrt{-1} = 6 \cdot \sqrt{-1}$$

$$\sqrt{-49} = \sqrt{49} \cdot \sqrt{-1} = 7 \cdot \sqrt{-1}$$

And hence using rule 1 with the other obvious rules:

$$\sqrt{-36} \cdot \sqrt{-49} = 6 \cdot \sqrt{-1} \cdot 7 \cdot \sqrt{-1} = 42 \cdot \sqrt{-1} \cdot \sqrt{-1} = -42$$

To deal with the quadratic formula expressions that consist of combinations of real numbers and square roots of negative numbers, we can apply the rules of binomials to numbers of the form

$$a + b\sqrt{-1}$$

where a and b are real numbers. For example, to add $5 + 7\sqrt{-1}$ to $2 - 3\sqrt{-1}$ we combine 'like' terms as we do in polynomials:

$$\begin{aligned}(5 + 7\sqrt{-1}) + (2 - 3\sqrt{-1}) &= 5 + 2 + 7\sqrt{-1} - 3\sqrt{-1}\\ &= (5 + 2) + (7 - 3)\sqrt{-1} = 7 + 4\sqrt{-1}\end{aligned}$$

Similarly, to multiply these numbers we use the binomial multiplication procedures:

$$\begin{aligned}(5 + 7\sqrt{-1}) \cdot (2 - 3\sqrt{-1}) &= 5 \cdot 2 + (7\sqrt{-1}) \cdot (-3\sqrt{-1}) + 5 \cdot (-3\sqrt{-1})\\ &\quad + (7\sqrt{-1}) \cdot 2\\ &= 10 - 21 \cdot (\sqrt{-1})^2 - 15 \cdot \sqrt{-1} + 14 \cdot \sqrt{-1}\\ &= 10 - 21 \cdot (-1) + (-15 + 14)\sqrt{-1}\\ &= 31 - \sqrt{-1}\end{aligned}$$

Euler introduced the symbol i for $\sqrt{-1}$.

A **pure imaginary number** is a number of the form ki, where k is a real number and i, the **imaginary unit**, is defined by $i^2 = -1$.

Note: In some cases, especially in engineering sciences, the number i is sometimes denoted as j.

Note: With this definition of i, a few interesting results are immediately apparent. For example,

$i^3 = i^2 \cdot i = -1 \cdot i = -i$, and

$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$, and so

$i^5 = i^4 \cdot i = 1 \cdot i = i$, and also

$i^6 = i^4 \cdot i^2 = i^2 = -1$; $i^7 = -i$, and finally $i^8 = 1$.

This leads you to be able to evaluate any positive integer power of i using the following property:

$$i^{4n+k} = i^k, k = 0, 1, 2, 3.$$

So, for example $i^{2122} = i^{2120+2} = i^2 = -1$.

Example 2

Simplify

a) $\sqrt{-36} + \sqrt{-49}$ b) $\sqrt{-36} \cdot \sqrt{-49}$

Solution

a) $\sqrt{-36} + \sqrt{-49} = \sqrt{36}\sqrt{-1} + \sqrt{49}\sqrt{-1}$
$= 6i + 7i = 13i$

b) $\sqrt{-36} \cdot \sqrt{-49} = 6i \cdot 7i = 42i^2$
$= 42(-1) = -42$

Gauss introduced the idea of complex numbers by giving them the following definition.

A **complex number** is a number that can be written in the form $a + bi$ where a and b are real numbers and $i^2 = -1$. a is called the **real part** of the number and b is the **imaginary part**.

Notation

It is customary to denote complex numbers with the variable z.

$z = 5 + 7i$ is the complex number with real part 5 and imaginary part 7 and $z = 2 - 3i$ has 2 as real part and -3 as imaginary.

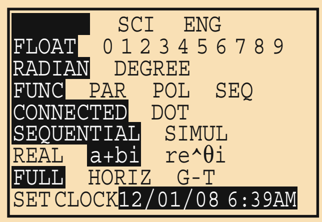
It is usual to write **Re**(z) for the real part of z and **Im**(z) for the imaginary part. So, Re$(2 + 3i) = 2$ and Im$(2 + 3i) = 3$.

Note that both the real and imaginary parts are real numbers!

We do not define $i = \sqrt{-1}$ for a reason. It is the convention in mathematics that when we write $\sqrt{9}$ then we mean the non-negative square root of 9, namely 3. We do not mean -3! i does not belong to this category since we cannot say that $\boldsymbol{i}$ is the positive square root of -1, i.e. $i > 0$. If we do, then $-1 = i \cdot i > 0$, which is false, and if we say $i < 0$, then $-i > 0$, and $-1 = -i \cdot -i > 0$, which is also false. Actually $-i$ is also a square root of -1 because $-i \cdot -i = i^2 = -1$.

With this in mind, we can use a 'convention' which calls i the **principal** square root of -1 and write $i = \sqrt{-1}$.

A GDC can be set up to do basic complex number operations. For example, if you have a TI-84 Plus, the set up is as follows.



Algebraic structure of complex numbers

Gauss' definition of the complex numbers triggers the following understanding of the set of complex numbers as an extension to our number sets in algebra.

The set of *complex numbers* $\mathbb{C}$ is the set of ordered pairs of real numbers $\mathbb{C} = \{z = (x, y): x, y \in \mathbb{R}\}$, with the following additional structure:

Equality

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal if their corresponding components are equal: $(x_1, y_1) = (x_2, y_2)$ if $x_1 = x_2$ and $y_1 = y_2$. That is, *two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.*

This is equivalent to saying: $a + bi = c + di \Leftrightarrow a = c$ and $b = d$.

For example, if $2 - (y - 2)i = x + 3 + 5i$, then x must be -1 and y must be -3. **Explain why.**

An interesting application of the way equality works is in finding the square roots of complex numbers without a need for the trigonometric forms developed later in the chapter.

Find the square root(s) of $z = 5 + 12i$. Let the square root of z be $x + yi$, then $(x + yi)^2 = 5 + 12i \Rightarrow x^2 - y^2 + 2xyi = 5 + 12i \Rightarrow x^2 - y^2 = 5$ and $2xy = 12 \Rightarrow xy = 6 \Rightarrow y = \frac{6}{x}$, and when we substitute this value in $x^2 - y^2 = 5$, we have $x^2 - \left(\frac{6}{x}\right)^2 = 5$. This simplifies to $x^4 - 5x^2 - 36 = 0$ which yields $x^2 = -4$ or $x^2 = 9, \Rightarrow x = \pm 3$. This leads to $x = \pm 2i$, that is, the two square roots of $5 + 12i$ are $3 + 2i$ or $-3 - 2i$.

```
(3+2i)2
              5+12i
(-3-2i)2
              5+12i
```

Addition and subtraction for complex numbers are defined as follows:

Addition

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_2 + y_2)$

This is equivalent to saying: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Multiplication

$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

This is equivalent to using the binomial multiplication on $(a + bi)(c + di)$:

$$(a + bi) \cdot (c + di) = ac + bdi^2 + adi + bci = ac - bd + (ad + bc)i$$

Addition and multiplication of complex numbers inherit most of the properties of addition and multiplication of real numbers:

$z + w = w + z$ and $zw = wz$ (Commutativity)

$z + (u + v) = (z + u) + v$ and $z(uv) = (zu)v$ (Associativity)

$z(u + v) = zu + zv$ (Distributive property)

A number of complex numbers take up unique positions. For example, the number $(0, 0)$ has the properties of 0:

$$(x, y) + (0, 0) = (x, y) \text{ and } (x, y)(0, 0) = (0, 0).$$

It is therefore normal to identify it with 0. The symbol is exactly the same symbol used to identify the 'real' 0. So, the real and complex zeros are the same number.

Another complex number of significance is $(1, 0)$. This number plays an important role in multiplication that stems from the following property:

$$(x, y)\,(1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y)$$

For complex numbers, $(1, 0)$ behaves like the identity for multiplication for real numbers. Again, it is normal to write $(1, 0) = 1$.

The third number of significance is $(0, 1)$. It has the notable characteristic of having a negative square, i.e.

$$(0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)$$

Using the definition above, $(0, 1) = 0 + 1i = i$. So, the last result should be no surprise to us since we know that

$$i \cdot i = -1 = (-1, 0).$$

Since (x, y) represents the complex number $x + yi$, then every real number x can be written as $x + 0i = (x, 0)$. The set of real numbers is therefore a subset of the set of complex numbers. They are the complex numbers whose imaginary part is 0. Similarly, pure imaginary numbers are of the form $0 + yi = (0, y)$. They are the complex numbers whose real part is 0.

Notation

So far, we have learned how to represent a complex number in two forms:

$$(x, y) \text{ and } x + yi.$$

Now, from the properties above

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0)(0, 1)$$

(Check the truth of this equation.)

This last equation justifies why we can write $(x, y) = x + yi$.

Example 3

Simplify each expression.

a) $(4 - 5i) + (7 + 8i)$

b) $(4 - 5i) - (7 + 8i)$

c) $(4 - 5i)(7 + 8i)$

Solution

a) $(4 - 5i) + (7 + 8i) = (4 + 7) + (-5 + 8)i = 11 + 3i$

b) $(4 - 5i) - (7 + 8i) = (4 - 7) + (-5 - 8)i = -3 - 13i$

c) $(4 - 5i)(7 + 8i) = (4 \cdot 7 - (-5) \cdot 8) + (4 \cdot 8 + (-5) \cdot 7)i = 68 - 3i$

```
(4-5i)/(8i)
          -.625-.5i
Ans▸Frac
          -5/8-1/2i
(4-5i)*(7+8i)
               68-3i
```

Division

Multiplication can be used to perform division of complex numbers.

The **division** of two complex numbers, $\frac{a + bi}{c + di}$, involves finding a complex number $(x + yi)$ satisfying $\frac{a + bi}{c + di} = x + yi$; hence, it is sufficient to find the unknowns x and y.

Example 4

Find the quotient $\dfrac{2+3i}{1+2i}$.

Solution

Let $\dfrac{2+3i}{1+2i} = x + iy$. Hence, using multiplication and the equality of complex numbers,

$2 + 3i = (1 + 2i)(x + iy) \Leftrightarrow 2 + 3i = x - 2y + i(2x + y)$

$$\Leftrightarrow \begin{cases} 2 = x - 2y \\ 3 = 2x + y \end{cases} \Rightarrow x = \frac{8}{5}, y = \frac{1}{5}$$

Thus, $\dfrac{2+3i}{1+2i} = \dfrac{8}{5} - \dfrac{1}{5}i$.

```
(2+3i)/(1+2i)
          1.6-.2i
Ans▸Frac
        8/5-1/5i
```

Now, in general, $\dfrac{a+bi}{c+di} = x + yi \Leftrightarrow a + bi = (x + yi)(c + di)$.

With the multiplication as described above:

$$a + bi = (cx - dy) + (dx + cy)i$$

Again by applying the equality of complex numbers property above we get a system of two equations that can be solved.

$$\begin{cases} cx - dy = a \\ dx + cy = b \end{cases} \Rightarrow x = \frac{ac + bd}{c^2 + d^2}; y = \frac{bc - ad}{c^2 + d^2}$$

The denominator $c^2 + d^2$ resulted from multiplying $c + di$ by $c - di$, which is its conjugate.

Conjugate

With every complex number $(a + bi)$ we associate another complex number $(a - bi)$ which is called its conjugate. The conjugate of number z is most often denoted with a bar over it, sometimes with an asterisk to the right of it, occasionally with an apostrophe and even less often with the plain symbol Conj as in

$$\bar{z} = z^* = z' = \text{Conj}(z).$$

In this book, we will use z^* for the conjugate.

The importance of the conjugate stems from the following property

$$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$$

which is a non-negative real number. So the product of a complex number and its conjugate is always a real number.

> Although the conjugate notation z^* will be used in the book, in your own work you can use any notation you feel comfortable with. You just need to understand that the IB questions use this one.

Example 5

Find the conjugate of z and verify the property mentioned above.

a) $z = 2 + 3i$

b) $z = 5i$

c) $z = 11$

Solution

a) $z^* = 2 - 3i$, and $(2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13$.

b) $z^* = -5i$, and $(5i)(-5i) = -5i^2 = (-5)(-1) = 5$.

c) $z^* = 11$, and $11 \cdot 11 = 121$.

So, the method used in dividing two complex numbers can be achieved by multiplying the quotient by a fraction whose numerator and denominator are the conjugate $c - di$.

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 6

Find each quotient and write your answer in standard form.

a) $\dfrac{4-5i}{7+8i}$

b) $\dfrac{4-5i}{8i}$

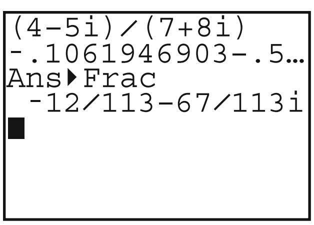
c) $\dfrac{4-5i}{7}$

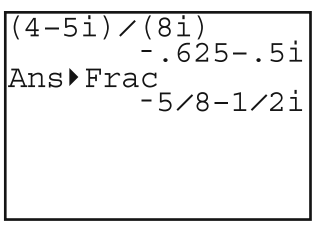
Solution

a) $\dfrac{4-5i}{7+8i} = \dfrac{4-5i}{7+8i} \cdot \dfrac{7-8i}{7-8i} = \dfrac{28-40+(-32-35)i}{49+64} = -\dfrac{12}{113} - \dfrac{67}{113}i$

b) $\dfrac{4-5i}{8i} = \dfrac{4-5i}{8i} \cdot \dfrac{-8i}{-8i} = \dfrac{-32i-40}{64} = -\dfrac{5}{8} - \dfrac{1}{2}i$

c) $\dfrac{4-5i}{7} = \dfrac{4}{7} - \dfrac{5}{7}i$





Example 7

Solve the system of equations and express your answer in Cartesian form.

$$(1 + i)z_1 - iz_2 = -3$$
$$2z_1 + (1 - i)z_2 = 3 - 3i$$

Solution

Multiply the first equation by 2, and the second equation by $(1 + i)$.

$$2(1 + i)z_1 - 2iz_2 = -6 \qquad (\mathbf{1})$$

$$2(1 + i)z_1 + (1 + i)(1 - i)z_2 = (1 + i)(3 - 3i)$$

$$2(1 + i)z_1 + 2z_2 = 6 \qquad (\mathbf{2})$$

By subtracting (**2**) from (**1**), we get

$$(-2 - 2i)z_2 = -12$$

And hence

$$z_2 = \frac{-12}{-2 - 2i} = 3 - 3i$$

$$z_1 = \frac{-3 + i(3 - 3i)}{1 + i} = \frac{3}{2} + \frac{3}{2i}$$

Properties of conjugates

Here is a theorem that lists some of the important properties of conjugates. In the next section, we will add a few more to the list.

Theorem

Let z, z_1 and z_2 be complex numbers, then

(1) $(z^*)^* = z$

(2) $z^* = z$ if and only if z is real.

(3) $(z_1 + z_2)^* = z_1^* + z_2^*$ The conjugate of the sum is the sum of conjugates.

(4) $(-z)^* = -z^*$

(5) $(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$ The conjugate of the product is the product of conjugates.

(6) $(z^{-1})^* = (z^*)^{-1}$, if $z \neq 0$.

The product can be extended to powers of complex numbers, i.e. $(z^2)^* = (z \cdot z)^* = z^* \cdot z^* = (z^*)^2$. This result can be generalized for any non-negative integer power n, i.e. $(z^n)^* = (z^*)^n$ and can be proved by mathematical induction.

The basis case, when $n = 0$, is obviously true: $(z^0)^* = 1 = (z^*)^0$.

Now assume $(z^k)^* = (z^*)^k$. $(z^{k+1})^* = (z^k z)^* = (z^k)^* z^* = (z^*)^k z^*$ (using the product rule).

Therefore, $(z^{k+1})^* = (z^*)^k z^* = (z^*)^{k+1}$.

So, since if the statement is true for $n = k$, it is also true for $n = k + 1$, then by the principle of mathematical induction it is true for all $n \geq 0$.

Proof

(1) and (2) are obvious. For (1), $((a + bi)^*)^* = (a - bi)^* = a + bi$, and for (2), $a - bi = a + bi \Rightarrow 2bi = 0 \Rightarrow b = 0$.

(3) is proved by straightforward calculation:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}(z_1 + z_2)^* &= ((x_1 + iy_1) + (x_2 + iy_2))^* = ((x_1 + x_2) + i(y_1 + y_2))^* \\ &= (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = z_1{}^* + z_2{}^*.\end{aligned}$$

(4) can now be proved using the above results:

$$\begin{aligned} & (z + (-z))^* = 0^* = 0 \\ \text{but,}\quad & (z + (-z))^* = 0^* = z^* + (-z)^*, \\ \text{so}\quad & z^* + (-z)^* = 0, \text{ and } (-z)^* = -z^*.\end{aligned}$$

Also (5) is proved by straightforward calculation:

$$\begin{aligned}(z_1 \cdot z_2)^* &= ((x_1 + iy_1) \cdot (x_2 + iy_2))^* = ((x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2))^* \\ &= (x_1x_2 - y_1y_2) - i(y_1x_2 + x_1y_2) \\ &= (x_1 - iy_1) \cdot (x_2 - iy_2) = z_1{}^* \cdot z_2{}^*\end{aligned}$$

And finally, (6):

$(z(z^{-1}))^* = 1^* = 1$

but, $(z(z^{-1}))^* = z^*(z^{-1})^*$, so $z^*(z^{-1})^* = 1$,

and $(z^{-1})^* = \dfrac{1}{z^*} = (z^*)^{-1}$.

Conjugate zeros of polynomials

In Chapter 3, you used the following result without proof.

If c is a root of a polynomial equation with real coefficients, then c^* is also a root.

Theorem: If c is a root of a polynomial equation with real coefficients, then c^* is also a root of the equation.

We give the proof for $n = 3$, but the method is general.

$$P(x) = ax^3 + bx^2 + dx + e$$

Since c is a root of $P(x) = 0$, we have

$$ac^3 + bc^2 + dc + e = 0$$

$$\Rightarrow (ac^3 + bc^2 + dc + e)^* = 0 \qquad \text{Since } 0^* = 0.$$

$$\Rightarrow (ac^3)^* + (bc^2)^* + (dc)^* + e^* = 0 \qquad \text{Sum of conjugates theorem.}$$

$$\Rightarrow a(c^*)^3 + b(c^*)^2 + d(c^*) + e = 0 \qquad \text{Result of product conjugate.}$$

$$\Rightarrow (c^*) \text{ is a root of } P(x) = 0.$$

Example 8

$1 + 2i$ is a zero of the polynomial $P(x) = x^3 - 5x^2 + 11x - 15$. Find all other zeros.

Solution

Since the polynomial has real coefficients, then $1 - 2i$ is also a zero. Hence, using the factor theorem, $P(x) = (x - (1 + 2i))(x - (1 - 2i))(x - c)$, where c is a real number to be found.

Now, $P(x) = (x^2 - 2x + 5)(x - c)$. c can either be found by division or by factoring by trial and error. In either case, $c = 3$.

Example 9[1]

$1 + 2i$ is a zero of the polynomial
$P(x) = x^3 + (i - 2)x^2 + (2i + 5)x + 8 + i$.
Find all other zeros.

[1] Not included in present IB syllabus.

Solution

Since the polynomial does not have real coefficients, then $1 - 2i$ is not necessarily also a zero. To find the other zeros, we can perform synthetic substitution

$$\begin{array}{c|cccc} 1+2i & 1 & i-2 & 2i+5 & 8+i \\ & & 1+2i & -7+i & -8-i \\ \hline & 1 & -1+3i & -2+3i & 0 \end{array}$$

This shows that $P(x) = (x - 1 - 2i)(x^2 + (-1 + 3i)x - 2 + 3i)$. The second factor can be factored into $(x + 1)(x - 2 + 3i)$ giving us the other two zeros as -1 and $2 - 3i$.

Note: $x^2 + (-1 + 3i)x - 2 + 3i = 0$ can be solved using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 - 3i \pm \sqrt{(-1 + 3i)^2 - 4(-2 + 3i)}}{2}$$

$$= \frac{1 - 3i \pm \sqrt{-8 - 6i + 8 - 12i}}{2} = \frac{1 - 3i \pm \sqrt{-18i}}{2}$$

To find $\sqrt{-18i}$ we let $(a + bi)^2 = -18i \Rightarrow a^2 - b^2 + 2abi = -18i$, then equating the real parts and imaginary parts to each other: $a^2 - b^2 = 0$ and $2ab = -18$ will yield $\sqrt{-18i} = \pm 3 \mp 3i$, and hence

$$x = \frac{1 - 3i \pm \sqrt{-18i}}{2} - \frac{1 - 3i \pm (\pm 3 \mp 3i)}{2}$$

which will yield $x = -1$ or $x = 2 - 3i$.

Exercise 10.1

Express each of the following numbers in the form $a + bi$.

1 $5 + \sqrt{-4}$ **2** $7 - \sqrt{-7}$ **3** -6

4 $-\sqrt{49}$ **5** $\sqrt{-81}$ **6** $-\sqrt{\frac{-25}{16}}$

Perform the following operations and express your answer in the form $a + bi$.

7 $(-3 + 4i) + (2 - 5i)$ **8** $(-3 + 4i) - (2 - 5i)$

9 $(-3 + 4i)(2 - 5i)$ **10** $3i - (2 - 4i)$

11 $(2 - 7i)(3 + 4i)$ **12** $(1 + i)(2 - 3i)$

13 $\frac{3 + 2i}{2 + 5i}$ **14** $\frac{2 - i}{3 + 2i}$

15 $\left(\frac{2}{3} - \frac{1}{2}i\right) + \left(\frac{1}{3} + \frac{1}{2}i\right)$ **16** $\left(\frac{2}{3} - \frac{1}{2}i\right)\left(\frac{2}{3} + \frac{1}{2}i\right)$

17 $\left(\frac{2}{3} - \frac{1}{2}i\right) \div \left(\frac{1}{3} + \frac{1}{2}i\right)$ **18** $(2 + i)(3 - 2i)$

19 $\frac{1}{i}(3 - 7i)$ **20** $(2 + 5i) - (-2 - 5i)$

21 $\frac{13}{5 - 12i}$ **22** $\frac{12i}{3 + 4i}$

23 $3i\left(3 - \frac{2}{3}i\right)$

24 $(3 + 5i)(6 - 10i)$

25 $\dfrac{39 - 52i}{24 + 10i}$

26 $(7 - 4i)^{-1}$

27 $(5 - 12i)^{-1}$

28 $\dfrac{3}{3 - 4i} + \dfrac{2}{6 + 8i}$

29 $\dfrac{(7 + 8i)(2 - 5i)}{5 - 12i}$

30 $\dfrac{5 - \sqrt{-144}}{3 + \sqrt{-16}}$

31 Let $z = a + bi$. Find a and b if $(2 + 3i)z = 7 + i$.

32 $(2 + yi)(x + i) = 1 + 3i$, where x and y are real numbers. Solve for x and y.

33 a) Evaluate $(1 + i\sqrt{3})^3$.

b) Prove that $(1 + i\sqrt{3})^{6n} = 8^{2n}$, *where* $n \in Z^+$.

c) Hence, find $(1 + i\sqrt{3})^{48}$.

34 a) Evaluate $(-\sqrt{2} + i\sqrt{2})^2$.

b) Prove that $(-\sqrt{2} + i\sqrt{2})^{4k} = (-16)^k$, *where* $k \in Z^+$.

c) Hence, find $(-\sqrt{2} + i\sqrt{2})^{46}$.

35 If z is a complex number such that $|z + 4i| = 2|z + i|$, find the value of $|z|$.
($|z| = \sqrt{x^2 + y^2}$ where $z = x + iy$.)

36 Find the complex number z and write it in the form $a + bi$ if $z = 3 + \dfrac{2i}{2 - i\sqrt{2}}$.

37 Find the values of the two real numbers x and y such that $(x + iy)(4 - 7i) = 3 + 2i$.

38 Find the complex number z and write it in the form $a + bi$ if $i(z + 1) = 3z - 2$.

39 Find the complex number z and write it in the form $a + bi$ if $\dfrac{2 - i}{1 + 2i}\sqrt{z} = 2 - 3i$.

40 Find the values of the two real numbers x and y such that $(x + iy)^2 = 3 - 4i$.

41 a) Find the values of the two real numbers x and y such that $(x + iy)^2 = -8 + 6i$.

b) Hence, solve the following equation

$$z^2 + (1 - i)z + 2 - 2i = 0.$$

42 If $z \in \mathbb{C}$, find all solutions to the equation $z^3 - 27i = 0$.

43 Given that $z = \frac{1}{2} + 2i$ is a zero of the polynomial $f(x) = 4x^3 - 16x^2 + 29x - 51$, find the other zeros.

44 Find a polynomial function with integer coefficients and lowest possible degree that has $\frac{1}{2}$, -1 and $3 + i\sqrt{2}$ as zeros.

45 Find a polynomial function with integer coefficients and lowest possible degree that has -2, -2 and $1 + i\sqrt{3}$ as zeros.

46 Given that $z = 5 + 2i$ is a zero of the polynomial $f(x) = x^3 - 7x^2 - x + 87$, find the other zeros.

47 Given that $z = 1 - i\sqrt{3}$ is a zero of the polynomial $f(x) = 3x^3 - 4x^2 + 8x + 8$, find the other zeros.

48 Let $z \in \mathbb{C}$. If $\dfrac{z}{z^*} = a + bi$, show that $|a + bi| = 1$.

49 Given that $z = (k + i)^4$ where k is a real number, find all values of k such that

a) z is a real number

b) z is purely imaginary.

50 Solve the system of equations.

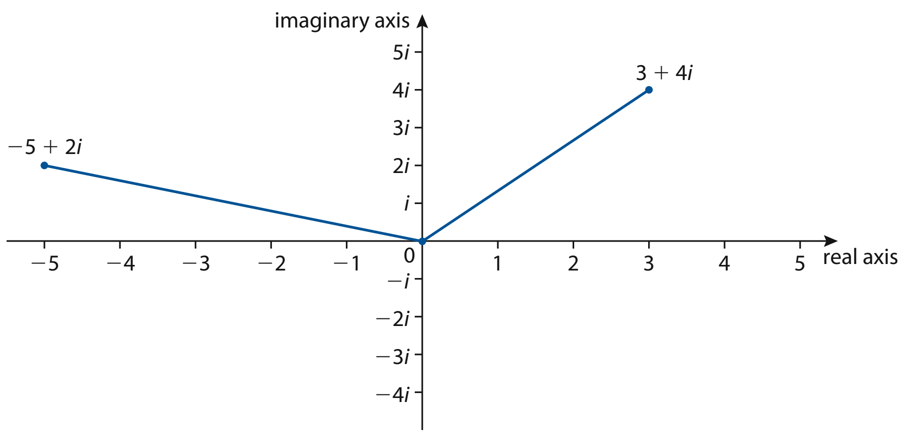
$$iz_1 + 2z_2 = 3 - i$$
$$2z_1 + (2 + i)z_2 = 7 + 2i$$

51 Solve the system of equations.

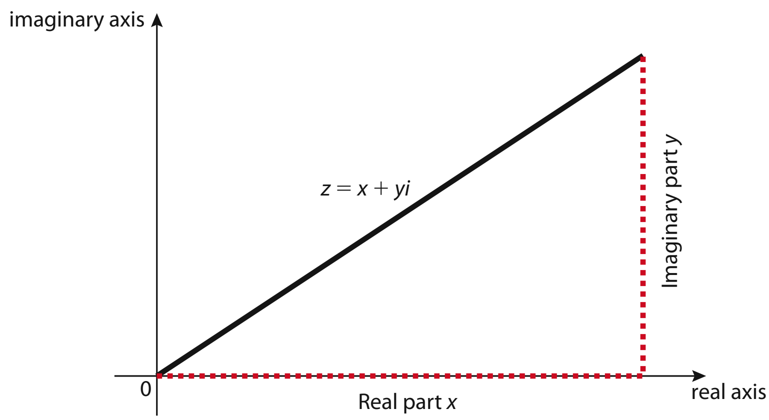
$$iz_1 - (1 + i)z_2 = 3$$
$$(2 + i)z_1 + iz_2 = 4$$

10.2 The complex plane

Our definition of complex numbers as ordered pairs of real numbers enables us to look at them from a different perspective. Every ordered pair (x, y) determines a unique complex number $x + yi$, and vice versa. This correspondence is embodied in the geometric representation of complex numbers. Looking at complex numbers as points in the plane equipped with additional structure changes the plane into what we call **complex plane**, or **Gauss plane**, or **Argand plane** (**diagram**). The complex plane has two axes, the horizontal axis is called the **real axis**, and the vertical axis is the **imaginary axis**. Every complex number $z = x + yi$ is represented by a point (x, y) in the plane. The real part is measured along the real axis and the imaginary part along the imaginary axis.



The diagram above illustrates how the two complex numbers $3 + 4i$ and $-5 + 2i$ are plotted in the complex plane.



Let us consider the sum of two complex numbers:

$$z_1 = x_1 + y_1 i, \text{ and } z_2 = x_2 + y_2 i$$

As we have defined addition before:

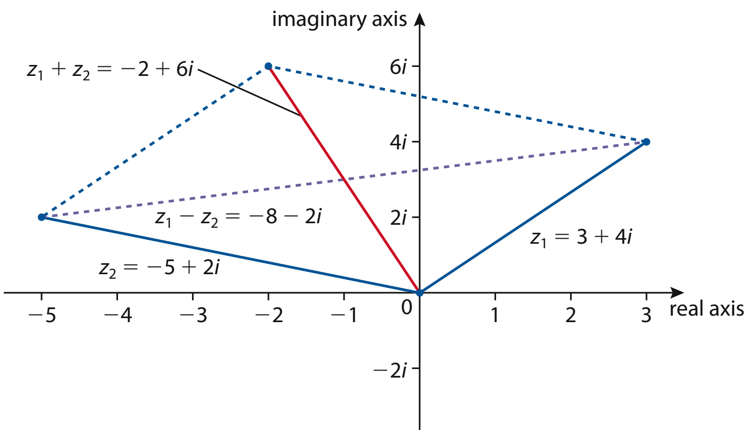
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

This suggests that we consider complex numbers as vectors; i.e. we regard the complex number $z = x + iy$ as a vector in standard form whose terminal point is the complex number (x, y).

Since we are representing the complex numbers by vectors, this results in some analogies between the two sets. So, adding two complex numbers or subtracting them, or multiplying by a scalar, are similar in both sets.

Example 10

Consider the complex numbers $z_1 = 3 + 4i$ and $z_2 = -5 + 2i$. Find $z_1 + z_2$ and $z_1 - z_2$.



Note here that the vector representing the sum, $-2 + 6i$, is the diagonal of the parallelogram with sides representing $3 + 4i$ and $-5 + 2i$, while the vector representing the difference is the second diagonal of the parallelogram.

The length, norm, of a vector also has a parallel in complex numbers. You recall that for a vector $\mathbf{v} = (x, y)$ the length of the vector is

$$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$

For complex numbers, the **modulus** or **absolute value** (or magnitude) of the complex number $z = x + yi$ is

$$|z| = \sqrt{x^2 + y^2}.$$

It follows immediately that since

$$z^* = x - yi \Rightarrow |z^*| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}, \text{ then}$$
$$|z^*| = |z|.$$

Also of interest is the following result.

$z \cdot z^* = (x + iy)(x - iy) = x^2 + y^2$,
$|z|^2 = x^2 + y^2$, and $|z^*|^2 = x^2 + y^2$
$\Rightarrow z \cdot z^* = |z|^2 = |z^*|^2$

For example:

$(3 + 4i)(3 - 4i) = 9 + 16 = 25 = \left(\sqrt{3^2 + 4^2}\right)^2$

Example 11

Calculate the moduli of the following complex numbers

a) $z_1 = 5 - 6i$ b) $z_2 = 12 + 5i$

Solution

a) $|z_1| = |5 - 6i| = \sqrt{5^2 + 6^2} = \sqrt{61}$

b) $|z_2| = |12 + 5i| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$

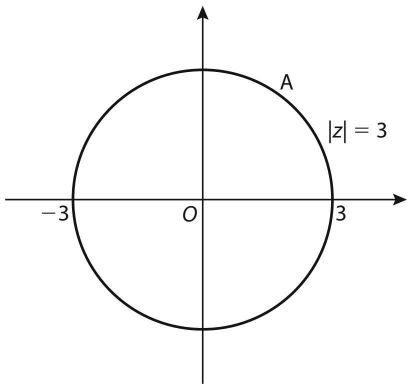
Example 12

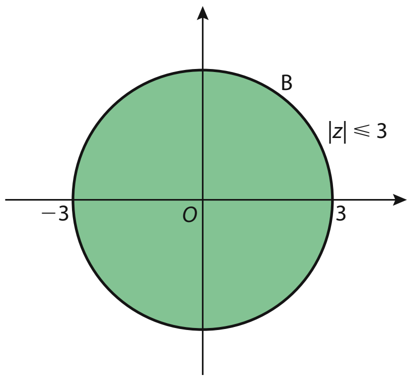
Graph each set of complex numbers.

a) $A = \{z \mid |z| = 3\}$ b) $B = \{z \mid |z| \leqslant 3\}$

Solution

a) A is the set of complex numbers whose distance from the origin is 3 units. So, the set is a circle with radius 3 and centre $(0, 0)$ as shown.





b) B is the set of complex numbers whose distance from the origin is less than or equal to 3. So, the set is a disk of radius 3 and centre at the origin.

Another important property is the following result:

$$|z_1z_2| = |z_1||z_2|$$

Proof:

$$|z_1z_2| = |(x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}$$
$$= \sqrt{(x_1x_2)^2 - 2x_1x_2y_1y_2 + (y_1y_2)^2 + (x_1y_2)^2 + 2x_1y_2x_2y_1 + (x_2y_1)^2}$$
$$= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2}$$

But,

$$|z_1||z_2| = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$
$$= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2}$$

And so the result follows.

Example 13

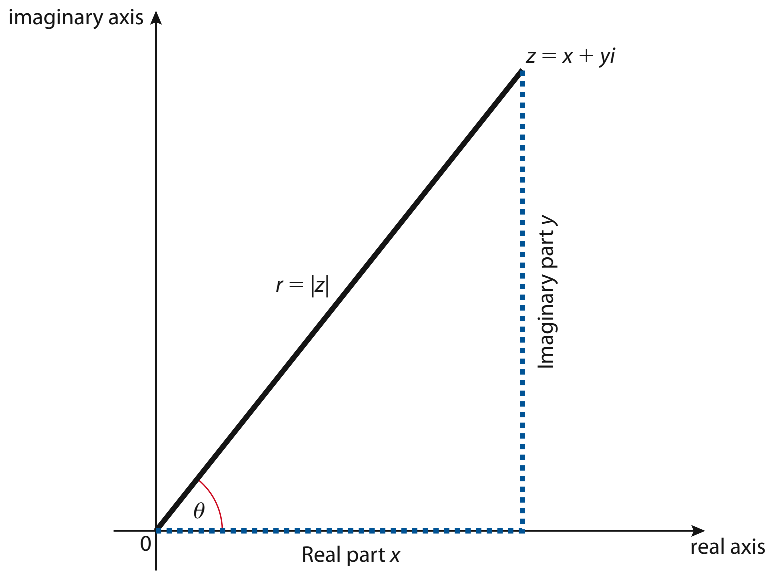
Evaluate $|(3 + 4i)(5 + 12i)|$.

Solution

$|(3 + 4i)(5 + 12i)| = |3 + 4i|\,|5 + 12i| = \sqrt{9 + 16}\,\sqrt{25 + 144} = 5 \times 13 = 65,$

or $|(3 + 4i)(5 + 12i)| = |-33 + 56i| = \sqrt{(-33)^2 + 56^2} = \sqrt{4255} = 65$

Trigonometric/polar form of a complex number



We know by now that every complex number $z = x + yi$ can be considered as an ordered pair (x, y). Hence, using our knowledge of vectors, we can introduce a new form for representing complex numbers – the trigonometric form (also known as polar form).

The trigonometric form uses the modulus of the complex number as its distance from the origin, $r \geqslant 0$, and θ the angle the 'vector' makes with the real axis.

Clearly $x = r\cos\theta$ and $y = r\sin\theta$; $r = \sqrt{x^2 + y^2}$; and $\tan\theta = \frac{y}{x}$.

Therefore, $z = x + yi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$.

The angle θ is called the **argument** of the complex number, $\arg(z)$.

$\text{Arg}(z)$ is not unique. However, all values differ by a multiple of 2π.

Note:

The trigonometric form is called 'modulus-argument' by the IB. Please keep that in mind. Also this trigonometric form is abbreviated, for ease of writing, as follows:

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$. (cis θ stands for $\cos\theta + i\sin\theta$.)

Example 14

Write the following numbers in trigonometric form.

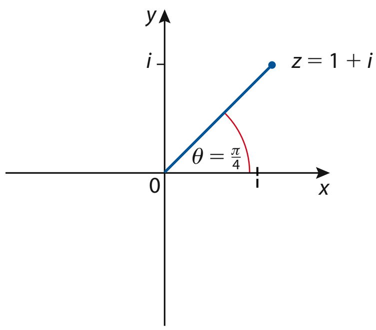
a) $z = 1 + i$ b) $z = \sqrt{3} - i$

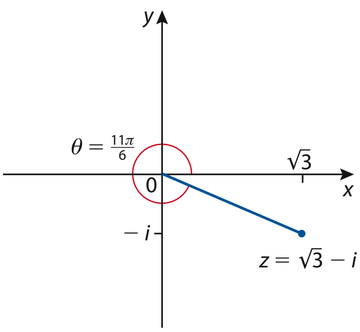
c) $z = -5i$ d) $z = 17$

Solution

a) $r = \sqrt{1^2 + 1^2} = \sqrt{2}$; $\tan\theta = \frac{1}{1} = 1$.
Hence, by observing the real and imaginary parts being positive, we can conclude that the argument must be $\theta = \frac{\pi}{4}$.

$\therefore z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$





b) $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$; $\tan\theta = \frac{-1}{\sqrt{3}}$. The real part is positive, the imaginary part is negative, and the point is therefore in the fourth quadrant, so $\theta = \frac{11\pi}{6}$.

$\therefore z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = 2\operatorname{cis}\frac{11\pi}{8}$

We can also use $\theta = -\frac{\pi}{6}$.

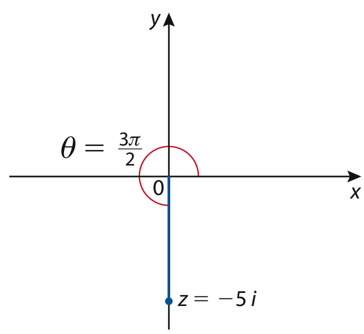
c) $r = 5$ and $\theta = \frac{3\pi}{2}$ since it is on the negative side of the imaginary axis.

$\therefore z = 5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

We can also use $\theta = -\frac{\pi}{2}$.

d) $r = 17$ and $\theta = 0$

$\therefore z = 17\,(\cos 0 + i\sin 0)$



Example 15

Convert each complex number into its rectangular form.

a) $z = 3\cos 150° + 3i\sin 150°$ b) $z = 12\operatorname{cis}\frac{4\pi}{3}$

c) $z = 6(\cos 50° + i\sin 50°)$ d) $z = 15\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Solution

a) $z = 3\left(\frac{-\sqrt{3}}{2}\right) + 3i\left(\frac{1}{2}\right) = \frac{-3\sqrt{3}}{2} + \frac{3i}{2}$

b) $z = 12\cos\frac{4\pi}{3} + 12i\sin\frac{4\pi}{3} = 12 \cdot \frac{-1}{2} + 12i \cdot -\frac{\sqrt{3}}{2} = -6 - \frac{6i}{\sqrt{3}}$

c) $z = 6\cos 50° + 6i\sin 50° = 6 \cdot 0.643 + 6i \cdot 0.766 = 3.857 + 4.596i$

d) $z = 15(0 + i) = 15i$

Multiplication

The trigonometric form of the complex number offers a very interesting and efficient method for multiplying complex numbers.

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in trigonometric form. Then

$$\begin{aligned} z_1z_2 &= (r_1(\cos\theta_1 + i\sin\theta_1))(r_2(\cos\theta_2 + i\sin\theta_2)) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)]. \end{aligned}$$

Now, using the addition formulae for sine and cosine, we have

$$\boxed{z_1z_2 = r_1r_2[(\cos(\theta_1 + \theta_2)) + i(\sin(\theta_1 + \theta_2))]}$$

This formula says: *To multiply two complex numbers written in trigonometric form, we multiply the moduli and add the arguments.*

The analogy between complex numbers and vectors stops at multiplication. As you recall, multiplication of vectors is not 'well defined' in the sense that there are two products – the scalar product which is a scalar, not a vector, and the vector product (discussed later) which is a vector but is not in the plane! Complex number products are complex numbers!

Example 16

Let $z_1 = 2 + 2i\sqrt{3}$ and $z_2 = -1 - i\sqrt{3}$.

a) Evaluate z_1z_2 by using their standard forms (rectangular or Cartesian).

b) Evaluate z_1z_2 by using their trigonometric forms and verify that the two results are the same.

Solution

a) $z_1z_2 = (2 + 2i\sqrt{3})(-1 - i\sqrt{3}) = (-2 + 6) + (-2\sqrt{3} - 2\sqrt{3})i = 4 - 4i\sqrt{3}$

b) Converting both to trigonometric form, we get

$z_1 = 4\operatorname{cis}\frac{\pi}{3}$ and $z_2 = 2\operatorname{cis}\frac{4\pi}{3}$, then

$$\begin{aligned} z_1z_2 &= 4 \cdot 2\left(\operatorname{cis}\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)\right) = 8\operatorname{cis}\left(\frac{5\pi}{3}\right) = 8\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) \\ &= 8\left(\frac{1}{2} + i\left(\frac{-\sqrt{3}}{2}\right)\right) = 4 - 4i\sqrt{3}. \end{aligned}$$

Note: You may observe here that multiplying z_1 by z_2 resulted in a new number whose magnitude is twice that of z_1 and is rotated by an angle of $\frac{4\pi}{3}$. Alternatively, you can see it as multiplying z_2 by z_1 which results in a complex number whose magnitude is 4 times that of z_2 and is rotated by an angle of $\frac{\pi}{3}$.

Example 17

Let $z_1 = -2 + 2i$ and $z_2 = 3\sqrt{3} - 3i$.

Convert to trigonometric form and multiply.

Solution

$$z_1 = 2\sqrt{2}\,\text{cis}\,\frac{3\pi}{4} \text{ and } z_2 = 6\,\text{cis}\,\frac{11\pi}{6}\text{, then}$$

$$z_1 z_2 = 12\sqrt{2}\left(\text{cis}\left(\frac{3\pi}{4} + \frac{11\pi}{6}\right)\right) = 12\sqrt{2}\,\text{cis}\left(\frac{31\pi}{12}\right) = 12\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12} + 2\pi\right)$$

$$= 12\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right) = 12\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

Note: You can simplify this answer further to get an exact rectangular form.

$$z_1 z_2 = 12\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = 12\sqrt{2}\left(\cos\left(\frac{3\pi + 4\pi}{12}\right) + i\sin\frac{3\pi + 4\pi}{12}\right)$$

$$= 12\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right)$$

$$= 12\sqrt{2}\left(\left(\frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right)\right)$$

$$= 12\sqrt{2}\left(\frac{\sqrt{2} - \sqrt{6}}{4} + i\frac{\sqrt{2} + \sqrt{6}}{4}\right) = (6 - 6\sqrt{3}) + i(6 + 6\sqrt{3})$$

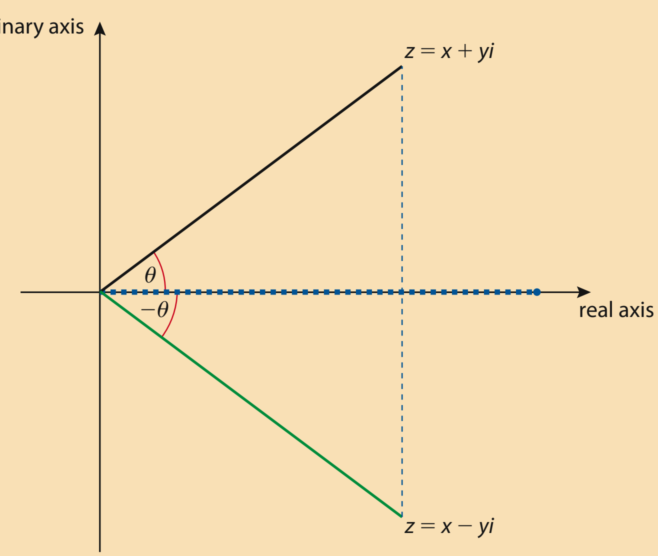
Note: By comparing the Cartesian form of the product to the polar form, i.e. $12\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ and $12\sqrt{2}\left(\frac{\sqrt{2} - \sqrt{6}}{4} + i\frac{\sqrt{2} - \sqrt{6}}{4}\right)$, we can conclude that $\cos\frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$ and $\sin\frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$.

This observation gives us a way of using complex number multiplication in order to find exact values of some trigonometric functions.

You may have noticed that the conjugate of a complex number $z = r(\cos\theta + i\sin\theta)$ is $z^* = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta))$.

Also, $z \cdot z^* = r(\cos\theta + i\sin\theta)\cdot r(\cos\theta - i\sin\theta)$
$= r^2(\cos^2\theta + \sin^2\theta)$
$= r^2$.

Graphically, a complex number and its conjugate are reflections of each other in the real axis. See the figure opposite.



Division of complex numbers

A similar approach gives us the rules for division of complex numbers.

Let

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be two complex numbers written in trigonometric form. Then

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos^2\theta_2 + \sin^2\theta_2)}\right)$$

$$= \frac{r_1}{r_2}\left(\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{1}\right).$$

Now, using the subtraction formulas for sine and cosine, we have

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos(\theta_1 - \theta_2)) + i(\sin(\theta_1 - \theta_2))]}$$

This formula says: *To divide two complex numbers written in trigonometric form, we divide the moduli and subtract the arguments.*

In particular, if we take $z_1 = 1$ and $z_2 = z$ (i.e. $\theta_1 = 0$ and $\theta_2 = \theta$), we will have the following result.

If $z = r(\cos\theta + i\sin\theta)$ then $\boxed{\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos(\theta) - i\sin(\theta))}$

Example 18

Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$.

a) Convert into trigonometric form.

b) Evaluate $\frac{1}{z_2}$.

c) Evaluate $\frac{z_1}{z_2}$.

d) Use the results above to find the exact values of $\sin\frac{5\pi}{12}$ and $\cos\frac{5\pi}{12}$.

Solution

a) $z_1 + \sqrt{2}\operatorname{cis}\frac{\pi}{4}$; $z_2 = 2\operatorname{cis}\frac{11\pi}{6} = 2\operatorname{cis}\frac{-\pi}{6}$

b) $\frac{1}{z_2} = \frac{1}{2}\operatorname{cis}\left(-\frac{-\pi}{6}\right) = \frac{1}{2}\operatorname{cis}\frac{\pi}{6}$

c) $\frac{z_1}{z_2}$ can be found by either multiplying z_1 by $\frac{1}{z_2}$, or by using division as shown above.

$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = \left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right) \cdot \left(\frac{1}{2}\operatorname{cis}\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$, or

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}{2\operatorname{cis}\frac{-\pi}{6}} = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{\pi}{4} - \frac{-\pi}{6}\right) = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$$

d) $\dfrac{z_1}{z_2} = \dfrac{1+i}{\sqrt{3}-i} \cdot \dfrac{\sqrt{3}+i}{\sqrt{3}+i} = \dfrac{\sqrt{3}-1+(\sqrt{3}+1)i}{4}$

Comparing this to part c).

$\dfrac{\sqrt{3}-1}{4} = \dfrac{\sqrt{2}}{2}\cos\dfrac{5\pi}{12} \Rightarrow \cos\dfrac{5\pi}{12} = \dfrac{\sqrt{3}-1}{4} \cdot \dfrac{2}{\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}.$

Also, $\dfrac{\sqrt{3}+1}{4} = \dfrac{\sqrt{2}}{2}\sin\dfrac{5\pi}{12} \Rightarrow \sin\dfrac{5\pi}{12} = \dfrac{\sqrt{3}+1}{4} \cdot \dfrac{2}{\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}.$

Exercise 10.2

In questions 1–14, write the complex number in polar form with argument θ, such that $0 \leqslant \theta < 2\pi$.

1 $2 + 2i$

2 $\sqrt{3} + i$

3 $2 - 2i$

4 $\sqrt{6} - i\sqrt{2}$

5 $2 - 2i\sqrt{3}$

6 $-3 + 3i$

7 $4i$

8 $-3\sqrt{3} - 3i$

9 $i + 1$

10 -15

11 $(4 + 3i)^{-1}$

12 $i(3 + 3i)$

13 π

14 ei

In questions 15–24, find z_1z_2 and $\dfrac{z_1}{z_2}$.

15 $z_1 = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}, z_2 = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

16 $z_1 = \cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}, z_2 = \cos\dfrac{7\pi}{6} + i\sin\dfrac{7\pi}{6}$

17 $z_1 = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}, z_2 = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$

18 $z_1 = \cos\dfrac{13\pi}{12} + i\sin\dfrac{13\pi}{12}, z_2 = \cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}$

19 $z_1 = 3\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right), z_2 = \dfrac{2}{3}\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

20 $z_1 = 3\sqrt{2}\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right), z_2 = 2\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$

21 $z_1 = \cos 135° + i\sin 135°, z_2 = \cos 90° + i\sin 90°$

22 $z_1 = 3(\cos 120° + i\sin 120°), z_2 = 2(\cos 240° + i\sin 240°)$

23 $z_1 = \dfrac{5}{8}(\cos 225° + i\sin 225°), z_2 = \dfrac{\sqrt{3}}{2}(\cos 330° + i\sin 330°)$

24 $z_1 = 3\sqrt{2}(\cos 315° + i\sin 315°), z_2 = 2(\cos 300° + i\sin 300°)$

In questions 25–30, write z_1 and z_2 in polar form, and then find the reciprocals $\dfrac{1}{z_1}, \dfrac{1}{z_2}$, the product z_1z_2, and the quotient $\dfrac{z_1}{z_2}$ $(-\pi < \theta < \pi)$.

25 $z_1 = \sqrt{3} + i$ and $z_2 = 2 - 2i\sqrt{3}$

26 $z_1 = \sqrt{6} + i\sqrt{2}$ and $z_2 = 2\sqrt{3} - 6i$

27 $z_1 = 4\sqrt{3} + 4i$ and $z_2 = -3 - 3i$

28 $z_1 = i\sqrt{3}$ and $z_2 = -\sqrt{2} - i\sqrt{6}$

29 $z_1 = \sqrt{5} + i\sqrt{5}$ and $z_2 = 2i\sqrt{2}$

30 $z_1 = 1 + i\sqrt{3}$ and $z_2 = 2\sqrt{3}$

31 Consider the complex number z where $|z - i| = |z + 2i|$.

a) Show that $\text{Im}(z) = -\frac{1}{2}$.

b) Let z_1 and z_2 be the two possible values of z, such that $|z| = 1$.

(i) Sketch a diagram to show the points which represent z_1 and z_2 in the complex plane.

(ii) Find $\arg(z_1)$ and $\arg(z_2)$.

32 Use the Argand diagram to show that $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

33 If $z = \sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin 2\frac{\pi}{3}\right)$, express each of the following complex numbers in Cartesian form.

a) $\dfrac{3}{\sqrt{3} + z}$ b) $\dfrac{2z}{3 + z^2}$ c) $\dfrac{3 - z^2}{3 + z^2}$

34 Find the modulus and argument (amplitude) of each of the complex numbers

$$z_1 = 2\sqrt{3} - 2i,\ z_2 = 2 + 2i \text{ and } z_3 = (2\sqrt{3} - 2i)(2 + 2i).$$

35 If the numbers in question 34 represent the vertices of a triangle in the Argand diagram, find the area of that triangle.

36 Identify, in the complex plane, the set of points that correspond to the following equations.

a) $|z| = 3$

b) $z^* = -z$

c) $z + z^* = 8$

d) $|z - 3| = 2$

e) $|z - 1| + |z - 3| = 2$

37 Identify, in the complex plane, the set of points that correspond to the following inequations.

a) $|z| \leqslant 3$

b) $|z - 3i| \geqslant 2$

10.3 Powers and roots of complex numbers

The formula established for the product of two complex numbers can be applied to derive a special formula for the nth power of a complex number.

Let $z = r(\cos\theta + i\sin\theta)$, now

$$\begin{aligned} z^2 &= (r(\cos\theta + i\sin\theta))(r(\cos\theta + i\sin\theta)) \\ &= r^2((\cos\theta\cos\theta - \sin\theta\sin\theta) + i(\sin\theta\cos\theta + \cos\theta\sin\theta)) \\ &= r^2((\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)) = r^2(\cos 2\theta + i\sin 2\theta). \end{aligned}$$

Similarly,

$$\begin{aligned} z^3 &= z \cdot z^2 = (r(\cos\theta + i\sin\theta))(r^2(\cos 2\theta + i\sin 2\theta)) \\ &= r^3(\cos(\theta + 2\theta) + i\sin(\theta + 2\theta)) = r^3(\cos 3\theta + i\sin 3\theta). \end{aligned}$$

In general, we obtain the following theorem, named after the French mathematician A. De Moivre (1667–1754).

Note: As a matter of fact, de Moivre stated 'his' formula only implicitly. Its standard form is due to Euler and was generalized by him to any real n.

De Moivre's theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta).$$

The theorem: To find the nth power of any complex number written in trigonometric form, we take the nth power of the modulus and multiply the argument with n.

Proof

The proof of this theorem follows as an application of mathematical induction.

Let $P(n)$ be the statement $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Basis step:

To prove this formula the basis step must be $P(1)$.

$P(1)$: is true since

$z^1 = r^1(\cos\theta + i\sin\theta)$, which is given!

[If you are not convinced, you can try

$P(2)$: $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$, which we showed above.]

Inductive step:

Assume that $P(k)$ is true, i.e.

$z^k = r^k(\cos k\theta + i\sin k\theta)$. We need to show that $P(k+1)$ is also true.

So we have to show that $z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$.

Now,

$$\begin{aligned} z^{k+1} &= z^k \cdot z = (\cos k\theta + i\sin k\theta)(r(\cos\theta + i\sin\theta)) \text{ by assumption} \\ &= r^k r[(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)] \\ &= r^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)] \text{ by addition formulae for sine and cosine} \\ &= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta) \end{aligned}$$

Therefore, by the principle of mathematical induction, since the theorem is true for $n = 1$, and whenever it is true for $n = k$, it was proved true for $n = k + 1$, then the theorem is true for positive integers n.

Note: In fact the theorem is valid for all real numbers n. However, the proof is beyond the scope of this course and this book and therefore we will consider the theorem true for all real numbers without proof at the moment.

Example 19

Find $(1 + i)^6$.

Solution

We convert the number into polar form first.

$(1 + i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Now we can apply De Moivre's theorem.

$$(1 + i)^6 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^6 = (\sqrt{2})^6\left(\cos\left(6\cdot\frac{\pi}{4}\right) + i\sin\left(6\cdot\frac{\pi}{4}\right)\right)$$
$$= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 8(-i) = -8i$$

Imagine you wanted to use the binomial theorem to evaluate the power.

$$(1 + i)^6 = 1 + 6i + 15i^2 + 20i^3 + 15i^4 + 6i^5 + i^6$$
$$= 1 + 6i - 15 - 20i + 15 + 6i - 1 = 8i$$

When the powers get larger, we are sure you will appreciate De Moivre!

Applications of De Moivre's theorem

Several applications of this theorem prove very helpful in dealing with trigonometric identities and expressions.

For example, when $n = -1$, the theorem gives the following result.

$z^{-1} = r^{-1}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta)$

Also,

$z^{-n} = (z^{-1})^n = (r^{-1}(\cos(-\theta) + i\sin(-\theta)))^n = r^{-n}(\cos(-n\theta) + i\sin(-n\theta))$.

If we take the case when $r = 1$, then

$z^n = \cos n\theta + i\sin n\theta$ and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\Rightarrow z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$.

These relationships are quite helpful in allowing us to write powers of $\cos\theta$ and $\sin\theta$ in terms of cosines and sines of multiples of θ.

Example 20

Find $\cos^3\theta$ in terms of first powers of the cosine function.

Solution

Starting with

$$\left(z + \frac{1}{z}\right)^3 = (2\cos\theta)^3$$

and expanding the left-hand side, we get

$$z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} = 8\cos^3\theta \Rightarrow z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right) = 8\cos^3\theta$$
$$\Updownarrow \qquad \Updownarrow$$
$$\Rightarrow 2\cos 3\theta + 3(2\cos\theta) = 8\cos^3\theta$$
$$\Rightarrow \cos^3\theta = \frac{1}{8}(2\cos 3\theta + 3(2\cos\theta))$$
$$= \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

Example 21

Simplify the following expression:

$$\frac{(\cos 6\theta + i\sin 6\theta)(\cos 3\theta + i\sin 3\theta)}{\cos 4\theta + i\sin 4\theta}$$

Solution

$$\frac{(\cos 6\theta + i\sin 6\theta)(\cos 3\theta + i\sin 3\theta)}{\cos 4\theta + i\sin 4\theta}$$
$$= \frac{(\cos \theta + i\sin \theta)^6(\cos \theta + i\sin \theta)^3}{(\cos \theta + i\sin \theta)^4}$$

Using the laws of exponents, we have

$$\frac{(\cos \theta + i\sin \theta)^6(\cos \theta + i\sin \theta)^3}{(\cos \theta + i\sin \theta)^4} = (\cos \theta + i\sin \theta)^5$$
$$= \cos 5\theta + i\sin 5\theta.$$

*n*th roots of a complex number

De Moivre's theorem is an essential tool for finding *n*th roots of complex numbers.

An *n*th root of a given number z is a number w that satisfies the following relation

$$w^n = z.$$

For example, $w = 1 + i$ is a 6th root of $z = -8i$ because, as you have seen above,

$$(1 + i)^6 = -8i, \text{ or}$$

$w = -\sqrt{3} + i$ is a 10th root of $512 + 512i\sqrt{3}$.

This is also because $w^{10} = (-\sqrt{3} + i)^{10} = 512 + 512i\sqrt{3}$.

How to find the *n*th roots:

To find them, we apply the definition of an *n*th root as mentioned above.

Let $w = s(\cos \alpha + i\sin \alpha)$ be an *n*th root of $z = r(\cos \theta + i\sin \theta)$. This means that $w^n = z$, i.e.

$$(s(\cos \alpha + i\sin \alpha))^n = r(\cos \theta + i\sin \theta) \Rightarrow$$
$$s^n(\cos n\alpha + i\sin n\alpha) = r(\cos \theta + i\sin \theta)$$

However, two complex numbers are equal if their moduli are equal, that is,

$$s^n = r \Leftrightarrow s = \sqrt[n]{r} = r^{\frac{1}{n}}.$$

Also,

$$\cos n\alpha = \cos \theta \text{ and } \sin n\alpha = \sin \theta.$$

From your trigonometry chapters, you recall that both sine and cosine functions are periodic of period 2π each; hence,

$$\begin{cases} \cos n\alpha = \cos \theta \\ \sin n\alpha = \sin \theta \end{cases} \Rightarrow n\alpha = \theta + 2k\pi,\ k = 0, 1, 2, \ldots$$

This leads to

$$\alpha = \frac{\theta + 2k\pi}{n} = \frac{\theta}{n} + \frac{2k\pi}{n};\ k = 0, 1, 2, 3, ..., n - 1.$$

Notice that we stop the values of k at $n - 1$. This is so because for values larger than or equal to n, principal arguments for these roots will be identical to those for $k = 0$ till $n - 1$.

nth roots of a complex number

Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer, then z has n distinct nth roots

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$

where $k = 1, 2, 3, \ldots, n - 1$.

Note: Each of the n nth roots of z has the same modulus $\sqrt[n]{r} = r^{\frac{1}{n}}$. Thus all these roots lie on a circle in the complex plane whose radius is $\sqrt[n]{r} = r^{\frac{1}{n}}$. Also, since the arguments of consecutive roots differ by $\frac{2\pi}{n}$, then the roots are also equally spaced on this circle.

Example 22

Find the cube roots of $z = -8 + 8i$.

Solution

$r = 8\sqrt{2}$ and $\theta = \frac{3\pi}{4}$, so the roots are

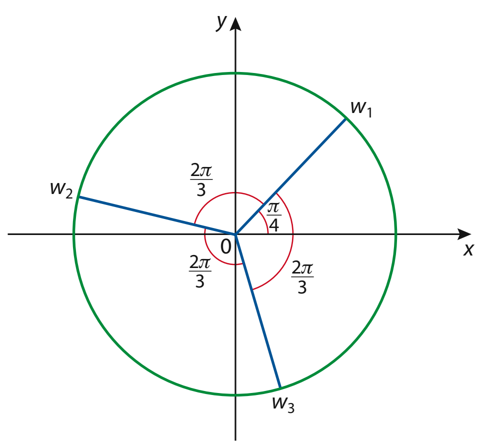
$$w = \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) = \sqrt[3]{(8\sqrt{2})}\left(\cos\left(\frac{\frac{3\pi}{4}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4}}{3} + \frac{2k\pi}{3}\right)\right)$$

$$= 2\left(\sqrt[6]{2}\right)\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right);\ k = 0, 1, 2$$

$$w_1 = 2\left(\sqrt[6]{2}\right)\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

$$w_2 = 2\left(\sqrt[6]{2}\right)\left(\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right) = 2\sqrt[6]{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right)$$

$$w_3 = 2\left(\sqrt[6]{2}\right)\left(\cos\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)\right) = 2\sqrt[6]{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right)$$



Notice how the arguments are distributed equally around a circle with radius $2\left(\sqrt[6]{2}\right)$. The difference between any two arguments is $\frac{2\pi}{3}$.

Notice that if you try to go beyond $k = 2$, then you get back to w_1.

$$w_4 = 2\sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + \frac{6\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{6\pi}{3}\right)\right) = 2\sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + 2\pi\right) + i\sin\left(\frac{\pi}{4} + 2\pi\right)\right)$$

$$= 2\sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = w_1$$

Also, if you raise any of the roots to the third power, you will eventually get z; for example,

$$(w_2)^3 = \left[2\sqrt[6]{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right)\right]^3 = 8\sqrt{2}\left(\cos\left(\frac{33\pi}{12}\right) + i\sin\left(\frac{33\pi}{12}\right)\right)$$

$$= 8\sqrt{2}\left(\cos\left(\frac{11\pi}{4}\right) + i\sin\left(\frac{11\pi}{4}\right)\right) = 8\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = z$$

Example 23

Find the six sixth roots of $z = -64$ and graph these roots in the complex plane.

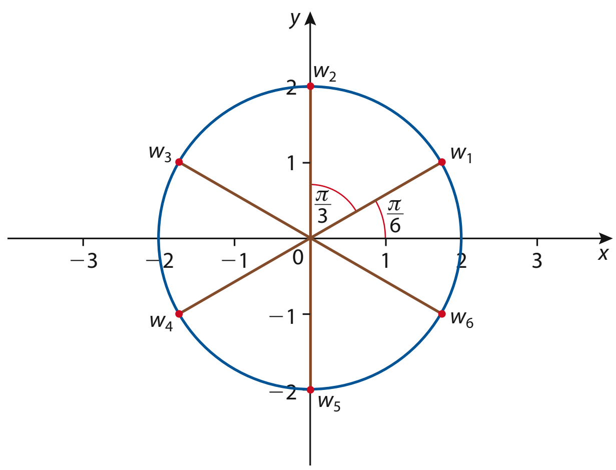
Solution

Here $r = 64$ and $\theta = \pi$. So the roots are

$$w = s\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$

$$= \sqrt[6]{64}\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6} + \frac{k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)\right);\ k = 0, 1, 2, 3, 4, 5$$



$$w_1 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$w_2 = 2\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

$$w_3 = 2\left(\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

$$w_4 = 2\left(\cos\left(\frac{\pi}{6} + \frac{3\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{3\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$$

$$w_5 = 2\left(\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$$

$$w_6 = 2\left(\cos\left(\frac{\pi}{6} + \frac{5\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{5\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$$

*n*th roots of unity

The rules we established can be applied to finding the nth roots of 1 (unity). Since 1 is a real number, then in polar/trigonometric form it has a modulus of 1 and an argument of 0. We can write it as

$$1 = 1(\cos 0 + i\sin 0).$$

Now applying the rules above, 1 has n distinct nth roots given by

$$\begin{aligned} z_k &= \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \\ &= \sqrt[n]{1}\left(\cos\left(\frac{0}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{0}{n} + \frac{2k\pi}{n}\right)\right) \\ &= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right); k = 0, 1, 2, ..., n - 1 \end{aligned}$$

Or in degrees,

$$z_k = \cos\left(\frac{360k}{n}\right) + i\sin\left(\frac{360k}{n}\right); k = 0, 1, 2, ..., n - 1$$

Example 24

Find

a) the square roots of unity

b) the cube roots of unity.

Solution

a) Here $k = 2$, and therefore the two roots are

$$z_k = \cos\left(\frac{360k}{2}\right) + i\sin\left(\frac{360k}{2}\right); k = 0, 1$$

$$z_0 = \cos\left(\frac{0}{2}\right) + i\sin\left(\frac{0}{2}\right) = 1$$

$$z_1 = \cos\left(\frac{360}{2}\right) + i\sin\left(\frac{360}{2}\right) = \cos 180 + i\sin 180 = -1$$

b) Here $k = 3$, and the three roots are

$$z_k = \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right); k = 0, 1, 2, 3$$

$$z_0 = \cos\left(\frac{0}{3}\right) + i\sin\left(\frac{0}{3}\right) = 1$$

$$z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Euler's formula

The material in this part depends on work that you will do in the Analysis option. Otherwise, you will have to accept the result without proof.

In the options section on infinite series, we have the following results.

Taylor's (Maclaurin's) series expansion for $\sin x$, $\cos x$ and e^x are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_0^\infty \frac{x^n}{n!}$$

Now if you add

$$\sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \ldots$$

and compare the result to e^x expansion, we notice a stark similarity in the terms, except for the 'discrepancy' in the signs! The signs in the sum alternate in a way where pairs of terms alternate! This property is typical of powers of i.

Look at $i, i^2, i^3, i^4, i^5, i^6, i^7, i^8, \ldots = i, \boxed{-1, -i}, \boxed{1, i}, \boxed{-1, -i}, 1, \ldots$

This suggests expanding e^{ix}

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \ldots$$

$$= 1 + \frac{i^2x^2}{2!} + \frac{i^4x^4}{4}! + \frac{i^6x^6}{6!} + ix + \frac{i^3x^3}{3!} + \frac{i^5x^5}{5!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + i\left(x + \frac{i^2x^3}{3!} + \frac{i^4x^5}{5!} + \ldots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= \cos x + i \sin x$$

Since, for any complex number

$z = x + iy = r(\cos\theta + i\sin\theta)$ and since $e^{i\theta} = \cos\theta + i\sin\theta$, then
$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

This is known as Euler's formula.

Example 25

Evaluate each of the following

a) $e^{i\pi}$ b) $e^{i\frac{\pi}{2}}$

Solution

a) $e^{i\pi} = \cos\pi + i\sin\pi = -1$

b) $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

Example 26

Use Euler's formula to prove DeMoivre's theorem.

Solution

$$(r(\cos\theta + i\sin\theta))^n = (re^{i\theta})^n = r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

Example 27

Find the real and imaginary parts of the complex numbers:

a) $z = 3e^{i\frac{\pi}{6}}$ b) $z = 7e^{2i}$

Solution

a) Since $|z| = 3$ and $\arg(z) = \frac{\pi}{6}$, $\mathrm{Re}(z) = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$ and $\mathrm{Im}(z) = 3\sin\frac{\pi}{6} = \frac{3}{2}$.

b) Since $|z| = 7$ and $\arg(z) = 2$, $\mathrm{Re}(z) = 7\cos 2$ and $\mathrm{Im}(z) = 7\sin 2$.

Example 28

Express $z = 5 + 5i$ in exponential form.

Solution

$|z| = 5\sqrt{2}$ and $\tan\theta = \frac{5}{5} = 1 \Rightarrow \theta = \frac{\pi}{4}$, therefore $z = 5\sqrt{2}\,e^{i\frac{\pi}{4}}$.

Example 29

Evaluate $(5 + 5i)^6$ and express your answer in rectangular form.

Solution

Let $z = 5 + 5i$. From the example above, $z = 5\sqrt{2}\,e^{i\frac{\pi}{4}}$; hence,

$$z^6 = \left(5\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^6 = (5\sqrt{2})^6 e^{i\frac{\pi}{4}\times 6} = 125\,000\,e^{i\frac{3\pi}{2}} = -125\,000i.$$

Alternatively,

$$(5 + 5i)^6 = \left(5\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^6 = (5\sqrt{2})^6\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = -125\,000i.$$

Example 30

Simplify the following expression:

$$\frac{(\cos 6\theta + i\sin 6\theta)(\cos 3\theta + i\sin 3\theta)}{\cos 4\theta + i\sin 4\theta}$$

Solution

$$\frac{(\cos 6\theta + i\sin 6\theta)(\cos 3\theta + i\sin 3\theta)}{\cos 4\theta + i\sin 4\theta} = \frac{e^{6i\theta} \cdot e^{3i\theta}}{e^{4i\theta}} = e^{5i\theta} = \cos 5\theta + i\sin 5\theta$$

Example 31

Use Euler's formula to find the cube roots of i.

Solution

$$i = e^{i\left(\frac{\pi}{2} + 2k\pi\right)} \Rightarrow i^{\frac{1}{3}} = \left(e^{i\left(\frac{\pi}{2} + 2k\pi\right)}\right)^{\frac{1}{3}} = e^{i\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)};\ k = 0, 1, 2$$

Therefore,

$$z_0 = e^{i\left(\frac{\pi}{6}\right)} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$z_1 = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = e^{i\left(\frac{5\pi}{6}\right)} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$z_2 = e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = e^{i\left(\frac{3\pi}{2}\right)} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

As you notice here, Euler's formula provides us with a very powerful tool to perform otherwise extremely laborious calculations.

Exercise 10.3

In questions 1–6, write the complex number in Cartesian form.

1 $z = 4e^{-i\frac{2\pi}{3}}$

2 $z = 3e^{2\pi i}$

3 $z = 3e^{0.5\pi i}$

4 $z = 4\,\text{cis}\left(\frac{7\pi}{12}\right)$ (exact value)

5 $z = 13e^{\frac{\pi i}{3}}$

6 $z = 3e^{1+\frac{\pi}{3}i}$

In questions 7–16, write each complex number in exponential form.

7 $2 + 2i$

8 $\sqrt{3} + i$

9 $\sqrt{6} - i\sqrt{2}$

10 $2 - 2i\sqrt{3}$

11 $-3 + 3i$

12 $4i$

13 $-3\sqrt{3} - 3i$

14 $i(3 + 3i)$

15 π

16 ei

In questions 17–25, find each complex number. Express in exact rectangular form when possible.

17 $(1 + i)^{10}$

18 $(\sqrt{3} - i)^6$

19 $(3 + 3i\sqrt{3})^9$

20 $(2 - 2i)^{12}$

21 $(\sqrt{3} - i\sqrt{3})^8$

22 $(-3 + 3i)^7$

23 $(\sqrt{3} - i\sqrt{3})^{-8}$

24 $(-3\sqrt{3} - 3i)^{-7}$

25 $2(\sqrt{3} + i)^7$

In questions 26–30, find each root and graph them in the complex plane.

26 The square roots of $4 + 4i\sqrt{3}$.

27 The cube roots of $4 + 4i\sqrt{3}$.

28 The fourth roots of -1.

29 The sixth roots of i.

30 The fifth roots of $-9 - 9i\sqrt{2}$.

In questions 31–36, solve each equation.

31 $z^5 - 32 = 0$

32 $z^8 + i = 0$

33 $z^3 + 4\sqrt{3} - 4i = 0$

34 $z^4 - 16 = 0$

35 $z^5 + 128 = 128i$

36 $z^6 - 64i = 0$

In questions 37–40, use De Moivre's theorem to simplify each of the following expressions.

37 $(\cos(9\beta) + i\sin(9\beta))(\cos(5\beta) - i\sin(5\beta))$

38 $\dfrac{(\cos(6\beta) + i\sin(6\beta))(\cos(4\beta) + i\sin(4\beta))}{(\cos(3\beta)) + i\sin(3\beta))}$

39 $(\cos(9\beta) + i\sin(9\beta))^{\frac{1}{3}}$

40 $\sqrt[n]{(\cos(2n\beta) + i\sin(2n\beta))}$

41 Use $e^{i\theta}$ to prove that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

42 Use De Moivre's theorem to show that $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$.

43 Use De Moivre's theorem to show that $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$.

44 Use De Moivre's theorem to show that $\cos^4\alpha = \frac{1}{8}(\cos 4\alpha + 4\cos 2\alpha + 3)$.

45 Let $z = \cos 2\alpha + i\sin 2\alpha$.

a) Show that $z + \dfrac{1}{z} = 2\cos 2\alpha$ and that $2i\sin 2\alpha = z - \dfrac{1}{z}$.

b) Find an expression for $\cos 2n\alpha$ and $\sin 2n\alpha$ in terms of z.

46 Let the cubic roots of 1 be 1, ω and ω^2. Simplify $(1 + 3\omega)(1 + 3\omega^2)$.

47 a) Show that the fourth roots of unity can be written as 1, β, β^2, and β^3.

b) Simplify $(1 + \beta)(1 + \beta^2 + \beta^3)$.

c) Show that $\beta + \beta^2 + \beta^3 = -1$.

48 a) Show that the fifth roots of unity can be written as 1, α, α^2, α^3 and α^4.

b) Simplify $(1 + \alpha)(1 + \alpha^4)$.

c) Show that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$.

49 Show that $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$ is real and find its value for $n = 18$.

50 Given that $z = (2a + 3i)^3$, and $a \in \mathbb{R}^+$, find the values of a such that $\arg z = 135°$.

Practice questions

1 Let $z = x + yi$. Find the values of x and y if $(1 - i)z = 1 - 3i$.

2 Let x and y be real numbers, and ω be one of the complex solutions of the equation $z^3 = 1$. Evaluate:

a) $1 + \omega + \omega^2$

b) $(\omega x + \omega^2 y)(\omega y + \omega^2 x)$

3 **a)** Evaluate $(1 + i)^2$ where $i = \sqrt{-1}$.

b) Prove, by mathematical induction, that $(1 + i)^{4n} = (-4)^n$, where $n \in \mathbb{N}^+$.

c) Hence or otherwise, find $(1 + i)^{32}$.

4 Let $z_1 = \dfrac{\sqrt{6} - i\sqrt{2}}{2}$ and $z_2 = 1 - i$.

a) Write z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

b) Show that $\dfrac{z_1}{z_2} = \cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}$.

c) Find the value of $\dfrac{z_1}{z_2}$ in the form $a + bi$, where a and b are to be determined exactly in radical (surd) form. Hence or otherwise, find the exact values of $\cos\dfrac{\pi}{12}$ and $\sin\dfrac{\pi}{12}$.

5 Let $z_1 = a\left(\cos\dfrac{\pi}{4} + \mathrm{i}\sin\dfrac{\pi}{4}\right)$ and $z_2 = b\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

Express $\left(\dfrac{z_1}{z_2}\right)^3$ in the form $z = x + yi$.

6 If z is a complex number and $|z + 16| = 4|z + 1|$, find the value of $|z|$.

7 Find the values of a and b, where a and b are real, given that $(a + bi)(2 - i) = 5 - i$.

8 Given that $z = (b + i)^2$, where b is real and positive, find the exact value of b when $\arg z = 60°$.

9 The complex number z satisfies $i(z + 2) = 1 - 2z$, where $i = \sqrt{-1}$. Write z in the form $z = a + bi$, where a and b are real numbers.

10 a) Express $z^5 - 1$ as a product of two factors, one of which is linear.

b) Find the zeros of $z^5 - 1$, giving your answers in the form

$$r(\cos\theta + i\sin\theta) \text{ where } r > 0 \text{ and } -\pi < \theta \leqslant \pi.$$

c) Express $z^4 + z^3 + z^2 + z + 1$ as a product of two real quadratic factors.

11 a) Express the complex number $8i$ in polar form.

b) The cube root of $8i$ which lies in the first quadrant is denoted by z. Express z

(i) in polar form

(ii) in Cartesian form.

12 Consider the complex number $z = \dfrac{\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - i\sin\frac{\pi}{24}\right)^4}$.

a) (i) Find the modulus of z.

(ii) Find the argument of z, giving your answer in radians.

b) Using De Moivre's theorem, show that z is a cube root of one, i.e. $z = \sqrt[3]{1}$.

c) Simplify $(1 + 2z)(2 + z^2)$, expressing your answer in the form $a + bi$, where a and b are exact real numbers.

13 The complex number z satisfies the equation $\sqrt{z} = \dfrac{2}{1 - i} + 1 - 4i$.

Express z in the form $x + iy$ where $x, y \in \mathbb{Z}$.

14 a) Prove, using mathematical induction, that for a positive integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \text{ where } i^2 = -1.$$

b) The complex number z is defined by $z = \cos\theta + i\sin\theta$.

(i) Show that $\dfrac{1}{z} = \cos(-\theta) + i\sin(-\theta)$.

(ii) Deduce that $z^n + z^{-n} = 2\cos n\theta$.

c) (i) Find the binomial expansion of $(z + z^{-1})^5$.

(ii) Hence, show that $\cos^5\theta = \frac{1}{16}(a\cos 5\theta + b\cos 3\theta + c\cos\theta)$, where a, b and c are positive integers to be found.

15 Consider the equation $2(p + iq) = q - ip - 2(1 - i)$, where p and q are both real numbers. Find p and q.

16 Consider $z^5 - 32 = 0$.

(i) Show that $z_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$ is one of the complex roots of this equation.

(ii) Find z_1^2, z_1^3, z_1^4 and z_1^5 giving your answer in the modulus argument form.

(iii) Plot the points that represent z_1, z_1^2, z_1^3, z_1^4 and z_1^5 in the complex plane.

(iv) The point $z^n{}_1$ is mapped to $z^{n+1}{}_1$ by a composition of two linear transformations, where $n = 1, 2, 3, 4$. Give a full geometric description of the two transformations.

17 A complex number z is such that $|z| = |z - 3i|$.

a) Show that the imaginary part of z is $\frac{3}{2}$.

b) Let z_1 and z_2 be the two possible values of z, such that $|z| = 3$.

(i) Sketch a diagram to show the points which represent z_1 and z_2 in the complex plane, where z_1 is in the first quadrant.

(ii) Show that $\arg(z_1) = \frac{\pi}{6}$.

(iii) Find $\arg(z_2)$.

c) Given that $\arg\left(\frac{z_1^k z_2}{2i}\right) = \pi$, find a value for k.

18 Given that $(a + i)(2 - bi) = 7 - i$, find the value of a and of b, where $a, b \in \mathbb{Z}$.

19 Consider the complex number $z = \cos\theta + i\sin\theta$.

a) Using De Moivre's theorem show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

20 Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

a) Find an expression for z, the common ratio of this series.

b) Show $|z| < 1$.

c) Write down an expression for the sum to infinity of this series.

d) **(i)** Express your answer to part **c)** in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence, show that

$$\cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

21 Let $P(z) = z^3 + az^2 + bz + c$, where a, b and $c \in \mathbb{R}$. Two of the roots of $P(z) = 0$ are -2 and $(-3 + 2i)$. Find the value of a, of b and of c.

22 Given that $|z| = 2\sqrt{5}$, find the complex number z that satisfies the equation

$$\frac{25}{z} - \frac{15}{z^*} = 1 - 8i.$$

23 Solve the simultaneous system of equations giving your answers in $x + iy$ form:

$$iz_1 + 2z_2 = 3$$

$$z_1 + (1 - i)z_2 = 4$$

24 **a)** Solve the equation $x^2 - 4x + 8 = 0$. Denote its two roots by z_1 and z_2 and express them in exponential form with z_1 in the first quadrant.

b) Find the value of $\frac{z_1^4}{z_2^2}$ and write it in the form $x + yi$.

c) Show that $z_1^4 = z_2^4$.

d) Find the value of $\frac{z_1}{z_2} + \frac{z_2}{z_1}$.

e) For what values of n is z_1^n real?

25 **a)** Show that $z = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a root of the equation $x^7 - 1 = 0$.

b) Show that $z^7 - 1 = (z - 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$ and deduce that $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$.

c) Show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

11 Statistics

Assessment statements

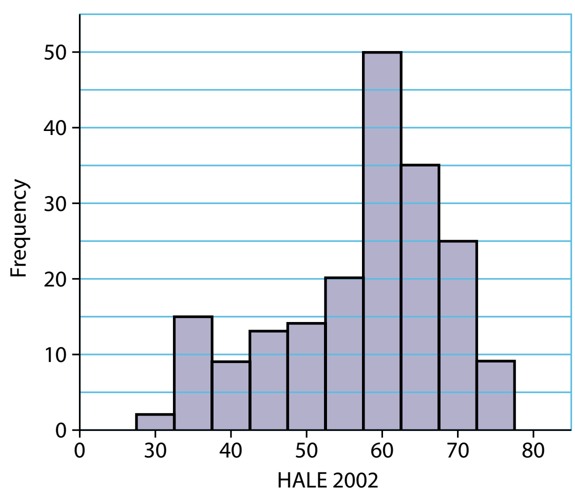
5.1 Concepts of population, sample, random sample and frequency distribution of discrete and continuous data.
Grouped data: use of mid-interval values, interval width, upper and lower interval boundaries.
Mean, variance, standard deviation.

Introduction

You will almost inevitably encounter statistics in one form or another on a daily basis. Here is an example:

The World Health Organization (WHO) collects and reports data pertaining to worldwide population health on all 192 UN member countries. Among the indicators reported is the **health-adjusted life expectancy** (HALE), which is based on life expectancy at birth, but includes an adjustment for time spent in poor health. It is most easily understood as the equivalent number of years in full health that a newborn can expect to live, based on current rates of ill-health and mortality. According to WHO rankings, lost years due to disability are substantially higher in poorer countries. Several factors contribute to this trend including injury, blindness, paralysis, and the debilitating effects of tropical disease.

W More information on HALE can be found by visiting www.pearsonhotlinks.com, enter the ISBN or title of this book and select weblink 1.



Of the 192 countries ranked by WHO, Japan has the highest life expectancy (75 years) and the lowest ranking country is Sierra Leone (29 years).

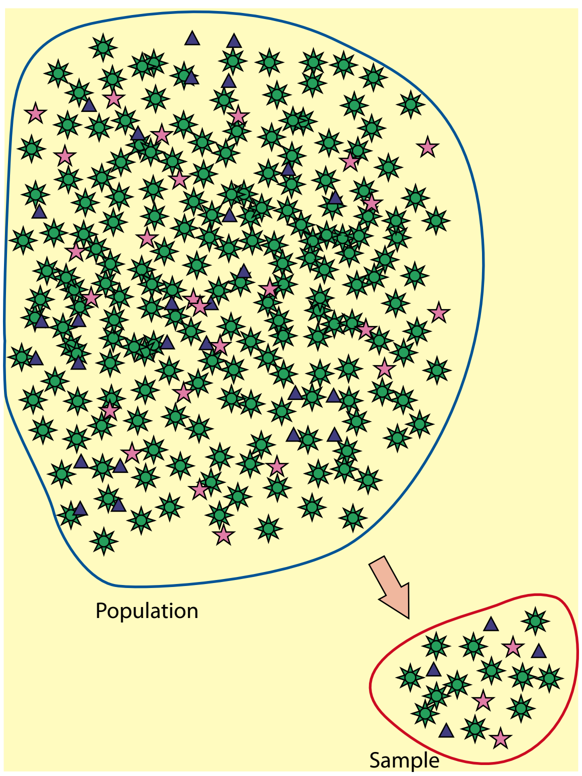
Reports similar to this one are commonplace in publications of several organizations, newspapers and magazines, and on the internet.

Questions that come to mind as we read such a report include: How did the researchers collect the data? How can we be sure that these results are reliable? What conclusions should be drawn from this report? The increased frequency with which statistical techniques are used in all fields, from business to agriculture to social and natural sciences, leads to the need for statistical literacy – familiarity with the goals and methods of these techniques – to be a part of any well-rounded educational programme.

Since statistical methods for summary and analysis provide us with powerful tools for making sense out of the data we collect, in this chapter we will first start by introducing two basic components of most statistical problems – population and sample – and then delve into the methods of presenting and making sense of data.

In the language of statistics, one of the most basic concepts is sampling. In most statistical problems, we draw a specified number of measurements or data – a sample – from a much larger body of measurements, called the population. On the basis of our observation of the data in the well-chosen sample, we try to describe or predict the behaviour of the population.

A **population** is any entire collection of people, animals, plants or things from which we may collect data. It is the entire group we are interested in, which we wish to describe or draw conclusions about. In order to make any generalizations about a population, a **sample**, that is meant to be representative of the population, is often studied. For each population there are many possible samples.



For example, a report on the effect the economic status (ES) has on healthy children's postures stated that:

'…ES, independent of overt malnutrition, affects height, weight, … with some gender differences in healthy children. Influence of income on height and weight show sexual dimorphism, a slight but significant effect is observed only in boys. MPH (mid-parental height) is the most prominent variable effecting height in healthy children. Higher height … observed in higher income groups suggest that secular trend in growth still exists, at least in boys, in a country of favorable economic development.'

Source: *European Journal of Clinical Nutrition* (2007) **61**, 752–758

The population is the 3-tuple measurement (economic status, height, weight) of all children of age 3–18 in Turkey. The sample is the set of measurements of the 428 boys and 386 girls that took part in the study. Notice that the population and sample are the measurements and not the people! The boys and girls are 'experimental units' or subjects in this study.

In this chapter we will present some basic techniques in **descriptive statistics** – the branch of statistics concerned with describing sets of measurements, both samples and populations.

11.1 Graphical tools

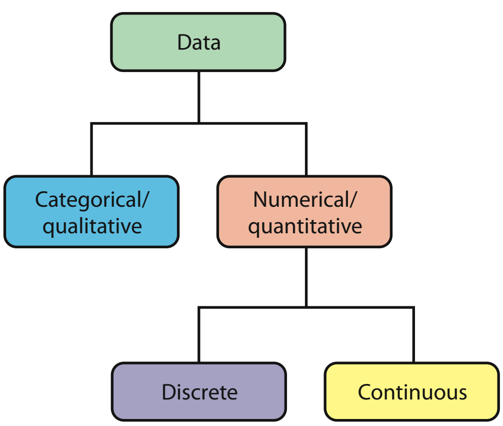
Once you have collected a set of measurements, how can you display this set in a clear, understandable and readable form? First, you must be able to define what is meant by measurement or 'data' and to categorize the types of data you are likely to encounter. We begin by introducing some definitions of the new terms in the statistical language that you need to know.

A **variable** is a characteristic that changes or varies over time and/or for different objects under consideration.

For example, if you are measuring the height of adults in a certain area, the height is a variable that changes with time for an individual and from person to person. When a variable is actually measured, a set of measurements or **data** will result. So, if you gather the heights of the students at your school, the set of measurements you get is a **data set**.

As the process of data collection begins, it becomes clear that often the number of data collected is so large that it is difficult for the statistician to see the findings of the data. The statistician's objective is to summarize succinctly, bringing out the important characteristics of the numbers and values in such a way that a clear and accurate picture emerges.

There are several ways of summarizing and describing data. Among them are tables and graphs and numerical measures.

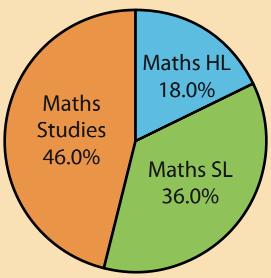


Frequently we use pie charts as a way of summarizing a set of categorical data or displaying the different values of a given variable (e.g., percentage distribution). This type of chart is a circle divided into a series of segments. Each segment represents a particular category. The area of each segment is the same proportion of a circle as the category is of the total data set.

Pie charts usually show the component parts of a whole. Often you will see a segment of the drawing separated from the rest of the pie in order to emphasize an important piece of information.

For example, in a large school, there are 230 students in the Maths Studies class, 180 students in the Standard Level maths class and 90 students in the HL mathematics class.

The pie chart for this data is given below.



Classification of variables

Numerical or categorical

When classifying data, there are two major classifications: numerical or categorical data.

NUMERICAL (QUANTITATIVE) DATA – Quantitative variables measure a numerical quantity or amount on each experimental unit. Quantitative data yields a numerical response.

Examples: Yearly income of company presidents, the heights of students at school, the length of time it takes students to finish their lunch at school, and the total score you receive on exams, are all numerical.

Moreover, there are two types of numerical data:

DISCRETE – responses which arise from counting.

Example: Number of courses students take in a day.

CONTINUOUS – responses which arise from measuring.

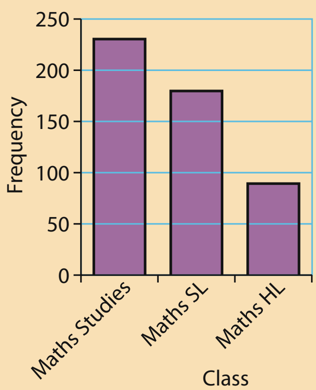
Example: Time it takes a student to travel from home to school.

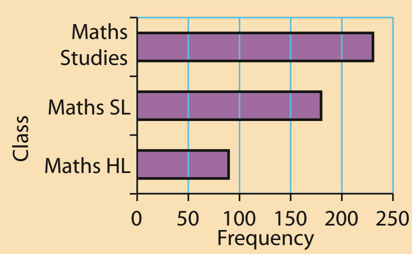
CATEGORICAL (QUALITATIVE) DATA – Qualitative variables measure a quality or characteristic of the experimental unit. Categorical data yields a qualitative response, i.e. data is kind or type rather than quantity.

Examples: Categorizing students into first year IB or second year IB; into Maths Studies SL, Maths SL, Further Maths SL, or Maths HL; or political affiliation, will result in qualitative variables and data.

Bar graphs are one of the many techniques also used to present data in a visual form so that the reader may readily recognize patterns or trends. A bar graph may be either horizontal or vertical. The important point to note about bar graphs is their bar length or height – the greater their length or height, the greater their value.

Bar graphs usually present categorical and numeric variables grouped in class intervals. They consist of an axis and a series of labelled horizontal or vertical bars. The bars depict frequencies of different values of a variable or simply the different values themselves. The student data in the previous box can be represented by a bar graph as shown below.





Notice here that the parts do not need to show the component parts of a whole. The key is to show their relative heights.

When data is first collected, there are some simple ways of beginning to organize the data. These include an ordered array and the stem-and-leaf display – not required.

- Data in raw form (as collected):
  24, 26, 24, 21, 27, 27, 30, 41, 32, 38
- Data in ordered array from smallest to largest (an ordered array is an arrangement of data in either ascending or descending order):
  21, 24, 24, 26, 27, 27, 30, 32, 38, 41

Suppose a consumer organization was interested in studying weekly food and living expenses of college students. A survey of 80 students yielded the following expenses to the nearest euro:

Table 11.1

38	50	55	60	46	51	58	64	50	49	48	65	58	61	65	53
39	51	56	61	48	53	59	65	54	54	54	59	65	66	47	49
40	51	56	62	47	55	60	63	60	59	59	50	46	45	54	47
41	52	57	64	50	53	58	67	67	66	65	58	54	52	55	52
44	52	57	64	51	55	61	68	67	54	55	48	57	57	66	66

The first step in the analysis is a summary of the data, which should show the following information:

- What values of the variable have been measured?
- How often has each value occurred?

A stem-and-leaf plot, or stem plot, is a technique used to classify and organize data as they are collected.

225	250	213	216	183
211	200	246	243	231
209	209	225	200	217
224	230	237	185	235
258	225	232	216	227
216	256	226	271	217
196	243	232	230	246
228	200	216	219	
200	224	209	191	

A stem-and-leaf plot looks something like a bar graph. Each number in the data is broken down into a stem and a leaf, thus the name. Here is a set of data representing the lives of 43 light bulbs of a certain type.

The stem of the number, in this case, consists of the multiples of 10. For example, 183, 18 is the stem, and 3 is the leaf. The leaf of the number will always be a single digit.

The stem-and-leaf plot shows how the data are spread–that is, highest number, lowest number, most common number and outliers and it preserves the individual values.

Once you have decided that a stem-and-leaf plot is the best way to show your data, draw it as follows:

On the left-hand side, write down the thousands, hundreds or tens (all digits except the last one). These will be your stems.

Draw a line to the right of these stems.

On the other side of the line, write down the ones (the last digit of a number). These will be your leaves.

Stem-and-leaf display	
18	3 5
19	1 6
20	0 0 0 0 9 9 9
21	1 3 6 6 6 6 7 7 9
22	4 4 5 5 5 6 7 8
23	0 0 1 2 2 5 7
24	3 3 6 6
25	0 6 8
26	
27	1

For example, if the observed value is 25, then the stem is 2 and the leaf is the 5. If the observed value is 369, then the stem is 36 and the leaf is 9. Where observations are accurate to one or more decimal places, such as 23.7, the stem is 23 and the leaf is 7. If the range of values is too great, the number 23.7 can be rounded up to 24 to limit the number of stems.

Such summaries can be done in many ways. The most useful are the frequency distribution and the histogram. There are other methods of presenting data, some of which we will discuss later. The rest are not within the scope of this book.

Frequency distribution (table)

A **frequency distribution** is a table used to organize data. The left column (called classes or groups) includes numerical intervals on a variable being studied. The right column is a list of the frequencies, or number of observations, for each class. Intervals normally are of equal size, must cover the range of the sample observations, and are non-overlapping (Table 11.2).

There are some general rules for preparing frequency distributions that make it easier to summarize data and to communicate results.

Construction of a frequency distribution (table)

If classes are described with discrete limits such as '30–34', '35–39', '40–44'…, then the boundaries are midway between the neighbouring class limits / end points. That is, the classes above will be considered as '29.5, but less than 34.5', '34.5, but less than 39.5', '39.5, but less than 44.5' etc. Here, the boundaries are 29.5, 34.5, 39.5, 44.5. Each class width is 5. See Example 3.

Rule 1: Intervals (classes) must be inclusive and non-overlapping; each observation must belong to one and only one class interval. Consider a frequency distribution for the living expenses of the 80 college students. If the frequency distribution contains the intervals '35–40' and '40–45', to which of these two classes would a person spending €40 belong?

The boundaries, or endpoints, of each class must be clearly defined. For our example, appropriate intervals would be '35 but less than 40' and '40 but less than 45'.

Rule 2: Determine k, the number of classes. Practice and experience are the best guidelines for deciding on the number of classes. In general, the number of classes could be between 5 and 10. But this is not an absolute rule. Practitioners use their judgement in these issues. If the number of classes is too few, some characteristics of the distribution will be hidden, and if too many, some characteristics will be lost with the detail.

In some cases, we do not necessarily create intervals with the same width. Look at the end of this section for an example.

Rule 3: Intervals should be the same width, w. The width is determined by the following:

$$\text{interval width} = \frac{\text{largest number} - \text{smallest number}}{\text{number of intervals}}$$

Both the number of intervals and the interval width should be rounded upward, possibly to the next largest integer. The above formula can be used when there are no natural ways of grouping the data. If this formula is used, the interval width is generally rounded to a convenient whole number to provide for easy interpretation.

In the example of the weekly living expenses of students, a reasonable grouping with nice round numbers was that of '35 but less than 40' and '40 but less than 45', etc.

Living expenses (€)	Number of students	Percentage of students
35 but < 40	2	2.50
40 but < 45	3	3.75
45 but < 50	11	13.75
50 but < 55	21	26.25
55 but < 60	19	23.75
60 but < 65	11	13.75
65 but < 70	13	16.25
Total	**80**	**100.00**

Table 11.2 Frequency and percentage frequency distributions of the weekly expenses of 80 students.

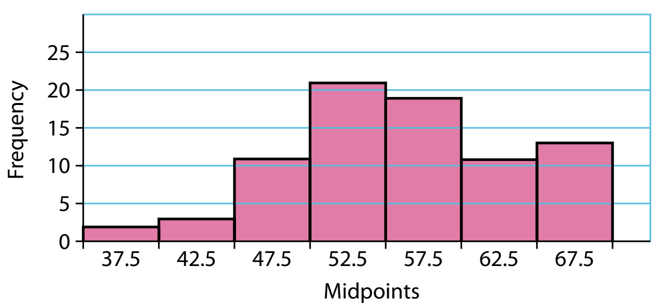
Grouping the data in a table like this one enables us to see some of its characteristics. For example, we can observe that there are few students who spend as little as 35 to 45 euros, while the majority of the students spend more than €45. Grouping the data will also cause some loss of detail, as we do not see from the table what the real values in each class are.

In the table above, the impression we get is that the class midpoint, also known as the mid-interval value, will represent the data in that interval. For example, 37.5 will represent the data in the first class, while 62.5 will represent the data in the 60 to 65 class. 35 and 40 are known as the **interval boundaries**.

Graphically, we have a tool that helps visualize the distribution. This tool is the **histogram**.

Histogram

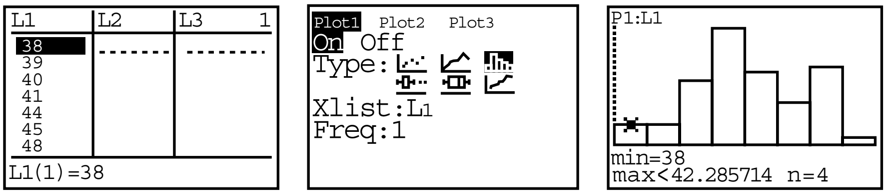
A histogram is a graph that consists of vertical bars constructed on a horizontal line that is marked off with intervals for the variable being displayed. The intervals correspond to those in a frequency distribution table. The height of each bar is proportional to the number of observations in that interval. The number of observations can also be displayed above the bars.



By looking at the histogram, it becomes visually clear that our observations above are true. From the histogram we can also see that the distribution is not symmetric.

To get a histogram on your GDC:

- Enter your data into a list
- Go to **StatPlot** and change it as shown below
- Graph



Cumulative and relative cumulative frequency distributions

A **cumulative frequency distribution** contains the total number of observations whose values are less than the upper limit for each interval. It is constructed by adding the frequencies of all frequency distribution intervals up to and including the present interval. A **relative cumulative frequency distribution** converts all cumulative frequencies to cumulative percentages.

In our example above, the following is a cumulative distribution and a relative (percentage) cumulative distribution.

Table 11.3 Cumulative frequency and cumulative relative frequency distributions of the weekly expenses of 80 students.

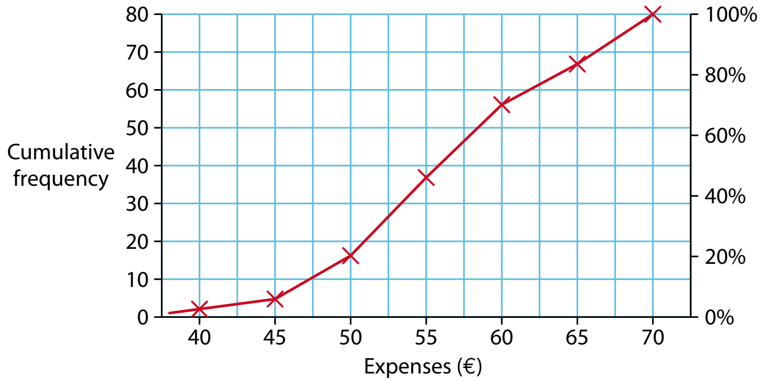
Living expenses (€)	Number of students	Cumulative number of students	Percentage of students	Cumulative percentage of students
35 but < 40	2	2	2.50	2.50
40 but < 45	3	5	3.75	6.25
45 but < 50	11	16	13.75	20.00
50 but < 55	21	37	26.25	46.25
55 but < 60	19	56	23.75	70.00
60 but < 65	11	67	13.75	83.75
65 but < 70	13	80	16.25	100.00
	80		**100.00**	

Notice how every cumulative frequency is added to the frequency in the next interval to give you the next cumulative frequency. The same is true for the relative frequencies.

As we will see later, cumulative frequencies and their graphs help in analyzing data that are given in group form.

Cumulative line graph/cumulative frequency graph

Sometimes called an **ogive**, this is a line that connects points that are the cumulative percentage of observations below the upper limit of each class in a cumulative frequency distribution.



Notice how the height of each line at the upper boundary represents the cumulative frequency for that interval. For example, at 50 the height is 16 and at 60 it is 56.

Example 1

Here is the WHO data in raw form.

29	36	40	44	48	52	54	56	59	60	61	61	62	63	64	66	68	71	72	73	63	64	66	68
31	36	41	44	49	52	54	57	59	60	61	62	62	64	64	66	68	71	72	75	63	64	66	68
33	36	41	44	49	52	55	57	59	60	61	62	62	64	65	66	69	71	72	35	38	43	47	71
34	37	41	45	49	53	55	58	59	60	61	62	63	64	65	66	69	71	73	36	40	44	48	71
34	37	42	45	50	53	55	58	59	60	61	62	63	64	65	67	70	71	73	50	54	56	59	72
35	37	42	45	50	53	55	58	59	60	61	62	63	64	65	67	70	71	73	51	54	56	59	72
35	37	43	46	50	54	55	58	59	60	61	62	63	64	65	67	70	71	73	60	60	61	62	73
35	38	43	46	50	54	55	58	59	60	61	62	63	64	65	67	70	72	73	60	61	61	62	73

Prepare a frequency table starting with 25 and with a class interval of 5. Then draw a histogram of the data and a cumulative frequency graph.

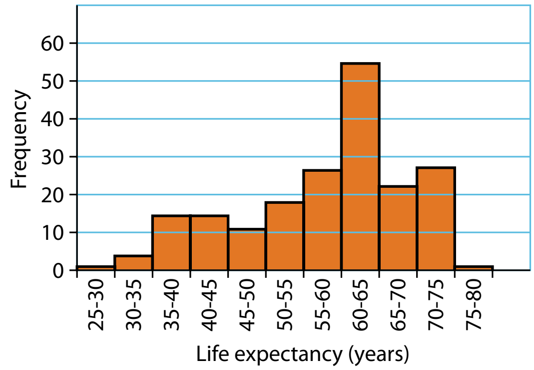
Solution

We first sort the data and then make sure we count every number in one class only.

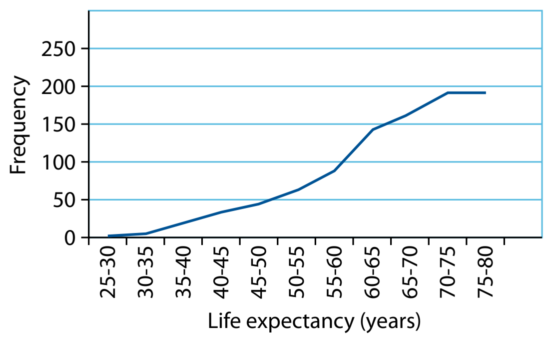
Life expectancy (years)[1]	Number of countries	Life expectancy (years)	Number of countries
25–30	1	55–60	26
30–35	4	60–65	54
35–40	14	65–70	22
40–45	14	70–75	27
45–50	11	75–80	1
50–55	18		

[1] 25–30 contains all observations larger than or equal to 25 but less than 30.

The histogram created by Excel is shown on the next page. Since we have classes of equal width, the height and the area give the same impression about the frequency of the class interval. For example, the class of 60–65 contains almost twice as much as the class of 55–60, and the height of the histogram is also twice as high. So is the area. Similarly, the height of the 65–70 class is double that of the 45–50 class.



Life expectancy (years)	Number of countries	Cumulative number of countries	Life expectancy (years)	Number of countries	Cumulative number of countries
25–30	1	1	55–60	26	88
30–35	4	5	60–65	54	142
35–40	14	19	65–70	22	164
40–45	14	33	70–75	27	191
45–50	11	44	75–80	1	192
50–55	18	62			



Histograms when class widths are unequal

In some cases, the class widths are not equal. The basic idea behind the histogram is that the area of each 'bar' reflects the frequency of the class. Hence, using the frequency along the vertical axis is a practical thing. However, when the classes have different widths, this practice will be misleading. An alternative for the usual representation is to use the 'frequency density'. The idea behind it is simple: the area of each bar must represent the frequency of the class. So, the height of each bar is measured by its density, which is the frequency of the class per unit of the class size.

This can be found by taking the height of each bar as:

$$\textit{Class height} = \textit{frequency density} = \frac{\textit{frequency}}{\textit{class width}}.$$

This means that the area of each bar $= \textit{width} \times \textit{height}$

$$= \cancel{\textit{width}} \times \frac{\textit{frequency}}{\cancel{\textit{width}}}$$

$$= \textit{frequency}$$

Note: The modal class in a grouped frequency distribution is the class with the largest frequency density.

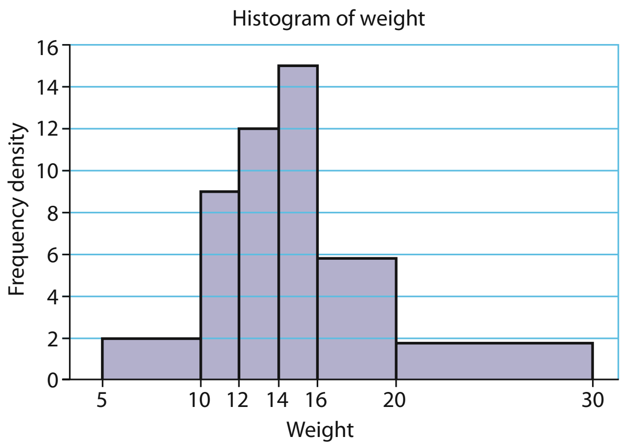
Example 2

The following table gives the weights (in Newtons) of young children visiting a pediatrician's practice in a certain week.

Weight	5–10	10–12	12–14	14–16	16–20	20–30
Frequency	10	18	24	30	22	16

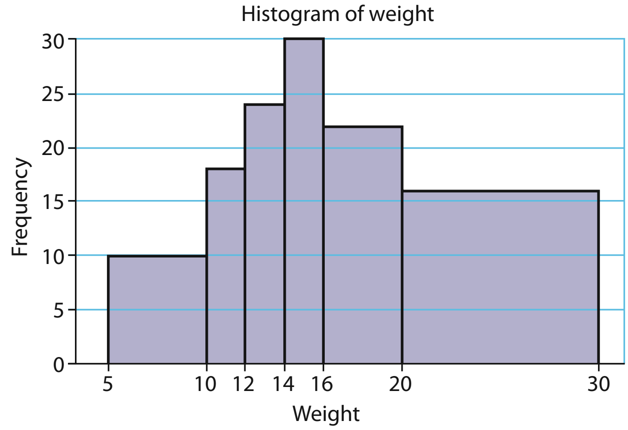
To draw a meaningful histogram, we find the frequency density for each class.

Weight	5–10	10–12	12–14	14–16	16–20	20–30
Class width	5	2	2	2	4	10
Frequency	10	18	24	30	22	16
Frequency density	2	9	12	15	5.5	1.6



The modal class here is the class 14–16 as it has the largest frequency density of 15.

Look at the histogram below. Notice that if we were to draw the histogram using the frequency itself, the histogram would have given us the wrong representation of the relative size of the classes 5–10, 16–20, and 20–30.



Example 3

The ages (to the nearest year) of visitors to the Prater (Amusement park) in Vienna on a Sunday in July are given in the table below.

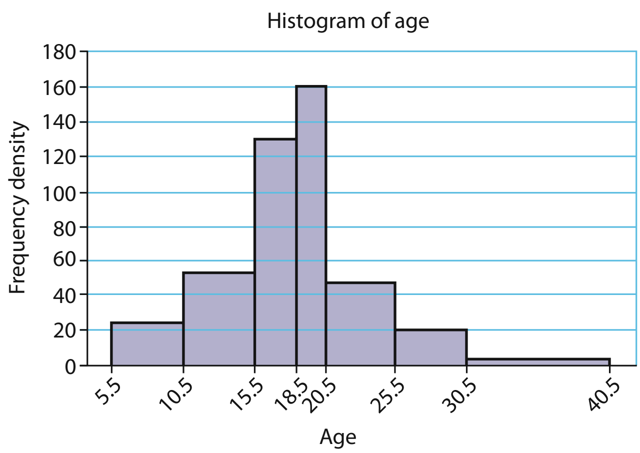
Age	6–10	11–15	16–18	19–20	21–25	26–30	31–40
Frequency	120	265	390	320	240	100	45

Draw a histogram of the data in the table.

We first represent classes by their boundaries and change the frequencies into densities.

Age	5.5–10.5	10.5–15.5	15.5–18.5	18.5–20.5	20.5–25.5	25.5–30.5	30.5–40.5
Class width	5	5	3	2	5	5	10
Frequency	120	265	390	320	240	100	45
Density	24	53	130	160	48	20	4.5

Here is the histogram. The modal class here is the class 18.5–20.5 with a frequency density of 160.

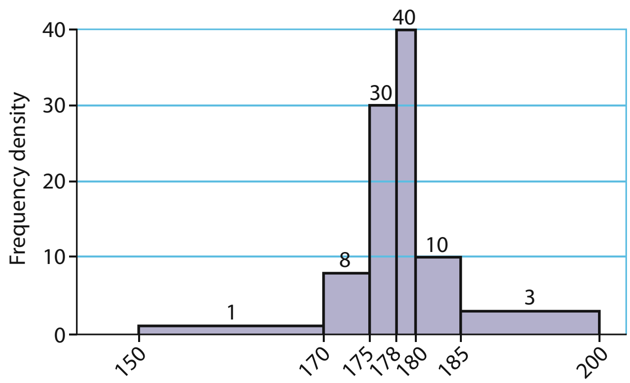


Note: The frequency of each class is given by:

$$\textit{width} \times \textit{height} = \textit{width} \times \textit{density} = \cancel{\textit{width}} \times \frac{\textit{frequency}}{\cancel{\textit{width}}} = \textit{frequency}$$

Example 4

The histogram below represents the heights of students (in centimetres) in a high school. Write out the frequency table for this distribution.



Remember that the frequency is equal to the product of the class width and the frequency density.

Hence, for the 150–170 class the frequency is $20 \times 1 = 20$, and for the 170–175 class the frequency is $5 \times 8 = 40$. Therefore, the frequency distribution is given by:

Height	150–170	170–175	175–178	178–180	180–185	185–200
Frequency	20	40	90	80	50	45

Exercise 11.1

1 Identify the experimental units, sensible population and sample on which each of the following variables is measured. Then indicate whether the variable is quantitative or qualitative.

a) Gender of a student
b) Number of errors on a final exam for 10th-grade students
c) Height of a newborn child
d) Eye colour for children aged less than 14
e) Amount of time it takes to travel to work
f) Rating of a country's leader: excellent, good, fair, poor
g) Country of origin of students at international schools

2 State what you expect the shapes of the distributions of the following variables to be: uniform, unimodal, bimodal, symmetric, etc. Explain why.

a) Number of goals shot by football players during last season.
b) Weights of newborn babies in a major hospital during the course of 10 years.
c) Number of countries visited by a student at an international school.
d) Number of emails received by a high school student at your school per week.

3 Identify each variable as quantitative or qualitative:

a) Amount of time to finish your extended essay.
b) Number of students in each section of IB Maths HL.

c) Rating of your textbook as excellent, good, satisfactory, terrible.
d) Country of origin of each student on Maths HL courses.

4 Identify each variable as discrete or continuous:
a) Population of each country represented by HL students in your session of the exam.
b) Weight of IB Maths HL exams printed every May since 1976.
c) Time it takes to mark an exam paper by an examiner.
d) Number of customers served at a bank counter.
e) Time it takes to finish a transaction at a bank counter.
f) Amount of sugar used in preparing your favourite cake.

5 Grade point averages (GPA) in several colleges are on a scale of 0–4. Here are the GPAs of 45 students at a certain college.

1.8	1.9	1.9	2.0	2.1	2.1	2.1	2.2	2.2	2.3	2.3	2.4	2.4	2.4	2.5
2.5	2.5	2.5	2.5	2.5	2.6	2.6	2.6	2.6	2.6	2.7	2.7	2.7	2.7	2.7
2.8	2.8	2.8	2.9	2.9	2.9	3.0	3.0	3.0	3.1	3.1	3.1	3.2	3.2	3.4

Prepare a frequency histogram, a relative frequency histogram and a cumulative frequency graph. Describe the data in two to three sentences.

6 The following are the grades of an IB course with 40 students (two sections) on a 100-point test. Use the graphical methods you have learned so far to describe the grades.

61	62	93	94	91	92	86	87	55	56
63	64	86	87	82	83	76	77	57	58
94	95	89	90	67	68	62	63	72	73
87	88	68	69	65	66	75	76	84	85

7 The length of time (months) between repeated speeding violations of 50 young drivers are given in the table below:

2.1	1.3	9.9	0.3	32.3	8.3	2.7	0.2	4.4	7.4
9	18	1.6	2.4	3.9	2.4	6.6	1	2	14.1
14.7	5.8	8.2	8.2	7.4	1.4	16.7	24	9.6	8.7
19.2	26.7	1.2	18	3.3	11.4	4.3	3.5	6.9	1.6
4.1	0.4	13.5	5.6	6.1	23.1	0.2	12.6	18.4	3.7

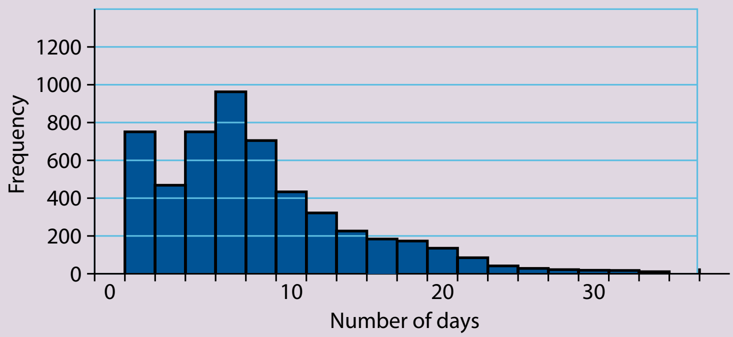
a) Construct a histogram for the data.
b) Would you describe the shape as symmetric?
c) The law in this country requires that the driving licence be taken away if the driver repeats the violation within a period of 10 months. Use a cumulative frequency graph to estimate the fraction of drivers who may lose their licence.

8 To decide on the number of counters needed to be open during rush hours in a supermarket, the management collected data from 60 customers for the time they spent waiting to be served. The times, in minutes, are given in the following table.

3.6	0.7	5.2	0.6	1.3	0.3	1.8	2.2	1.1	0.4
1	1.2	0.7	1.3	0.7	1.6	2.5	0.3	1.7	0.8
0.3	1.2	0.2	0.9	1.9	1.2	0.8	2.1	2.3	1.1
0.8	1.7	1.8	0.4	0.6	0.2	0.9	1.8	2.8	1.8
0.4	0.5	1.1	1.1	0.8	4.5	1.6	0.5	1.3	1.9
0.6	0.6	3.1	3.1	1.1	1.1	1.1	1.4	1	1.4

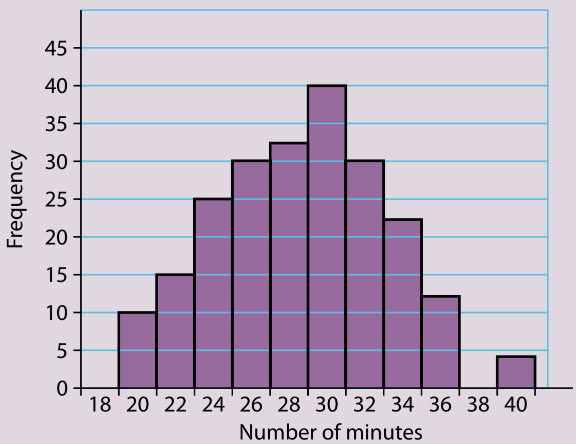
a) Construct a relative frequency histogram for the times.
b) Construct a cumulative frequency graph and estimate the number of customers who have to wait 2 minutes or more.

9 The histogram below shows the number of days spent by heart patients in Austrian hospitals in the 2003–2005 period.



a) Describe the data in a few sentences.
b) Draw a cumulative frequency graph for the data.
c) What percentage of the patients stayed less than 6 days?

10 One of the authors exercises on almost a daily basis. He records the length of time he exercises on most of the days. Here is what he recorded for 2006.



a) What is the longest time he has spent doing his exercises?
b) What percentage of the time did he exercise more than 30 minutes?
c) Draw a cumulative frequency graph for his exercise time.

11 Radar devices are installed at several locations on a main highway. Speeds, in km/h, of 400 cars travelling on that highway are measured and summarized in the following table.

Speed	60–75	75–90	90–105	105–120	120–135	Over 135
Frequency	20	70	110	150	40	10

a) Construct a frequency table for the data.
b) Draw a histogram to illustrate the data.
c) Draw a cumulative frequency graph for the data.
d) The speed limit in this country is 130 km/h. Use your graph in c) to estimate the percentage of the drivers driving faster than this limit.

12 Electronic components used in the production of CD players are manufactured in a factory and their measures must be very accurate. Here are the measures of a sample of 400 such components.

Length (mm)	Less than 5.00	5.00–5.05	5.05–5.10	5.10–5.15	5.15–5.20	More than 5.20
Frequency	16	100	123	104	48	9

a) Construct a cumulative relative frequency graph for the data.
b) The components must have a length between 5.01 and 5.18, and any component beyond these measures must be scrapped. Use your graph to estimate the percentage of components that must be scrapped from this production facility.

13 The waiting time, in seconds, of 300 customers at a supermarket cash register are recorded in the table below.

Time	<60	60–120	120–180	180–240	240–300	300–360	>360
Frequency	12	15	42	105	66	45	15

a) Draw a histogram of the data.
b) Construct a cumulative frequency graph of the data.
c) Use the cumulative frequency graph to estimate the waiting time that is exceeded by 25% of the customers.

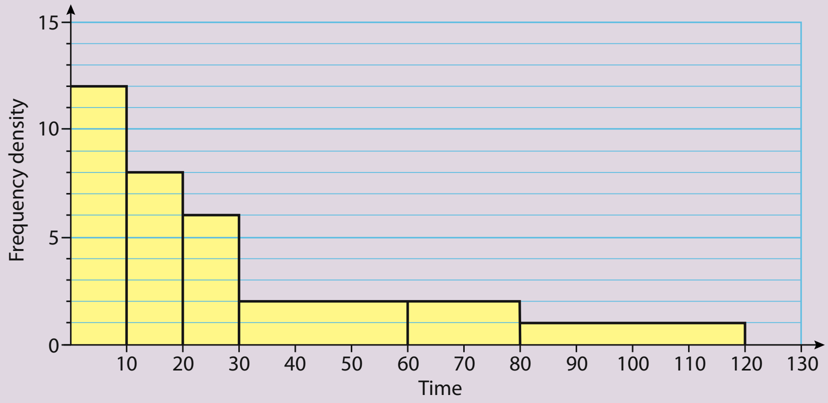
14 The time to solve a puzzle given to a large number of students is given below. Draw a histogram to illustrate the situation.

Time (seconds)	5–10	10–20	20–30	30–45	45–60	More than 60
Frequency	20	120	70	150	20	0

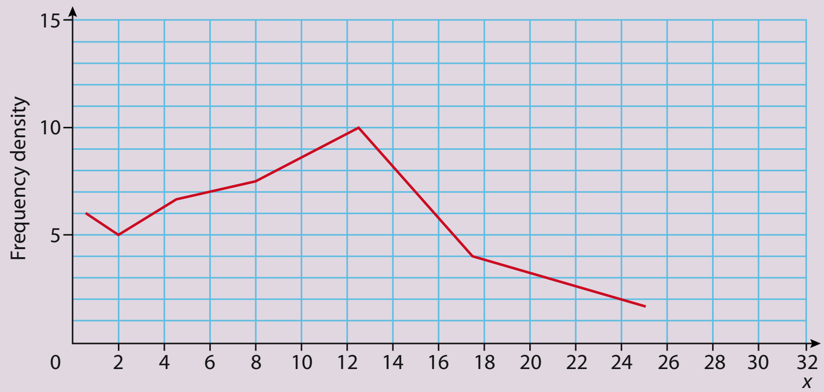
15 Post offices weigh the letters customers send before they decide on the amount of postage required. The table below lists the masses (in grams) of letters processed by a post office in a large city on a certain day. (Any letter heavier than 2000 g is considered a parcel.) Draw a histogram to illustrate the situation.

Mass	1–200	201–400	401–600	601–800	801–1000	1001–2000
Frequency	3220	450	130	96	54	40

16 In a study to determine the relative frequency of delays at a major airport, the following histogram has been produced. Develop the frequency distribution of the study. Flights more than 2 hours late were considered as atypical.

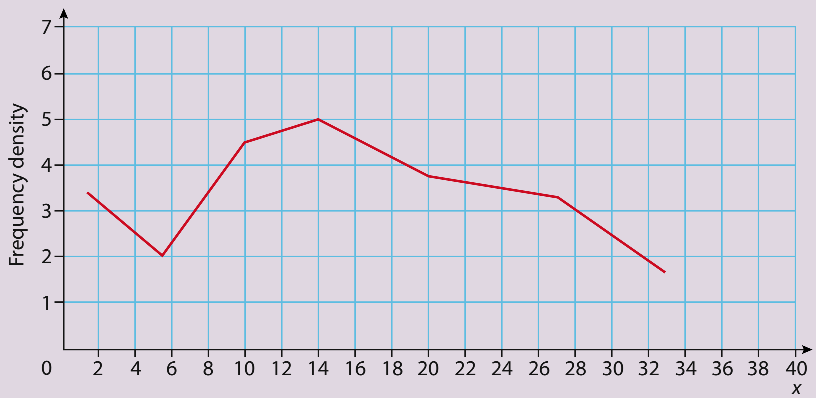


17 Copy and complete the frequency distribution for the data represented by the frequency polygon below.



x	$0 \leqslant x < 1$	$1 \leqslant x < 3$	$3 \leqslant x < 6$	$6 \leqslant x < 10$	$10 \leqslant x < 15$	$15 \leqslant x < 20$	$20 \leqslant x < 30$
Frequency	6						

18 Write out the frequency distribution for the data represented by the frequency polygon below. The lowest boundary is 0.



19 Find the value of m, n, p, and q in the frequency density calculation table below.

x	20–24	25–34	35–42	43–49	50–60
Frequency	m	n	20	21	p
Frequency density	2	4.5	q	3	3

11.2 Measures of central tendency

Summarizing data can help us understand them, especially when the number of data is large. This section presents several ways to summarize quantitative data by a typical value (a measure of location, such as the **mean**, **median** or **mode**) and a measure of how well the typical value represents the list (a measure of spread, such as the **range**, **interquartile range** or **standard deviation**). When looking at raw data, rather than looking at tables and graphs, it may be of interest to use summary measures to describe the data. The farthest we can reduce a set of data, and still retain any information at all, is to summarize the data with a single value. Measures of location do just that – they try to capture with a single number what is typical of the data. What single number is most representative of an entire list of numbers? We cannot say without defining 'representative' more precisely. We will study three common measures of location: the mean, the median and the mode. The mean, median and mode are all 'most representative', but for different, related notions of representativeness.

- The **median** is the number that divides the (ordered) data in half. At least half the data is equal to or smaller than the median, and at least half the data is equal to or greater than the median. (In a histogram, the median is that middle value that divides the histogram into two equal areas.)
- The **mode** of a set of data is the most common value among the data.
- The **mean** (more precisely, the arithmetic mean) is commonly called the average. It is the sum of the data, divided by the number of data:

$$\text{mean} = \frac{\text{sum of data}}{\text{number of data}} = \frac{\text{total}}{\text{number of data}}$$

When these measures are computed for a population, they are called **parameters**. When they are computed for a sample, they are called **statistics**.

Statistic and parameter

A statistic is a descriptive measure computed from a sample of data. A parameter is a descriptive measure computed from an entire population of data.

Measures of central tendency provide information about a 'typical' observation in the data, or locate the data set.

The mean and the median

The most common measure of central tendency is the arithmetic mean, usually referred to simply as the 'mean' or the 'average.'

Example 5

The following are the five closing prices of the NASDAQ Index for the first business week in November 2007. This is a sample of size $n = 5$ for the closing prices from the entire 2007 population: 2794.83, 2810.38, 2795.18, 2825.18, 2748.76.

What is the average closing price?

Solution

$$\text{Average} = \frac{2794.83 + 2810.38 + 2795.18 + 2825.18 + 2748.76}{5} = 2794.87.$$

This is called the sample mean. A second measure of central tendency is the median, which is the value in the middle position when the measurements are ordered from smallest to largest. The median of this data can only be calculated if we first sort them in ascending order:

2748.76 2794.83 **2795.18** 2810.38 2825.18

↑

The **arithmetic mean** or **average** of a set of n measurements (data set) is equal to the sum of the measurements divided by n.

Notation

The sample mean: $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$, where n is the sample size.

This is a **statistic**.

The population mean: $\mu = \dfrac{\sum_{i=1}^{N} x_i}{N} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_N}{N}$, where N is the population size. This is a **parameter**.

It is important to observe that you normally do not know the mean of the population μ and that you usually estimate it with the sample mean $\bar{x}$.

The **median** of a set of n measurements is the value of x that falls in the middle position when the data is sorted in ascending order.

In the previous example, we calculated the sample median by finding the third measurement to be in the middle position. However, in a different situation, where the number of measurements is even, the process is slightly different.

Let us assume that you took six tests last term and your marks were, in ascending order, 52, 63, 74, 78, 80, 89.

52 63 $\boxed{74 \quad 78}$ 80 89

↑

There are two 'middle' observations here. To find the median, choose a value halfway between the two middle observations:

$$m = \frac{74 + 78}{2} = 76$$

Note: The position of the median can be given by $\frac{n+1}{2}$. If this number ends with a decimal, you need to average the adjacent values.

In the NASDAQ Index case, we have five observations, the position of the median is then at $\frac{5+1}{2} = 3$, which we found. In the grades example, the position of the median score is at $\frac{6+1}{2} = 3.5$, and hence we average the numbers at positions **three** and **four**.

Although both the mean and median are good measures for the centre of a distribution, the median is less sensitive to extreme values or **outliers**. For example, the value 52 in the previous example is lower than all your test scores and is the only failing score you have. The median, 76, is not affected by this outlier even if it were much lower than 52. Assume, for example, that your lowest score is 12 rather than 52. The median calculation

$$12 \qquad 63 \qquad \boxed{74 \qquad 78} \qquad 80 \qquad 89$$

still gives the same median of 76. If we were to calculate the mean of the original set, we would get

$$\bar{x} = \frac{\sum x}{6} = \frac{436}{6} = 72.\overline{6}.$$

While the new mean, with 12 as the lowest score, is

$$\bar{x} = \frac{\sum x}{6} = \frac{396}{6} = 66.$$

Clearly, the low outlier 'pulled' the mean towards it while leaving the median untouched. However, because the mean depends on every observation and uses all the information in the data, it is generally, wherever possible, the preferred measure of central tendency.

A third way to locate the centre of a distribution is to look for the value of x that occurs with the highest frequency. This measure of the centre is called the **mode**.

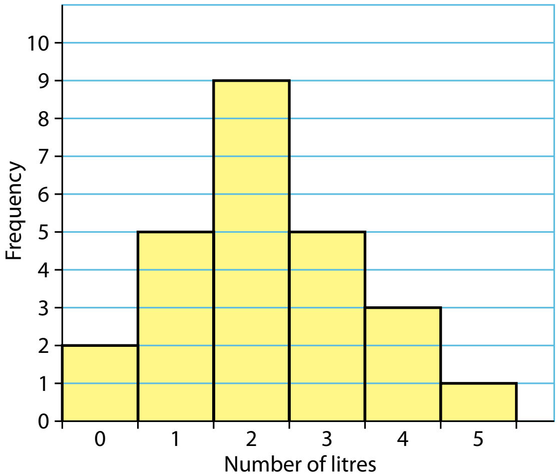
Example 6

Here is a table listing the frequency distribution of 25 families in Lower Austria that were polled in a marketing survey to list the number of litres of milk consumed during a particular week.

Number of litres	Frequency	Relative frequency
0	2	0.08
1	5	0.20
2	9	0.36
3	5	0.20
4	3	0.12
5	1	0.04

Find the frequency histogram.

Solution

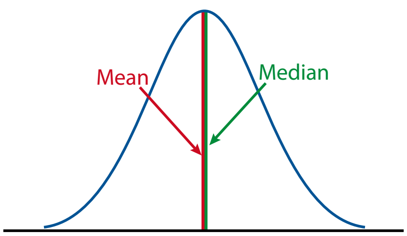


The histogram (Example 3) shows a relatively symmetric shape with a modal class at $x = 2$. Apparently, the mean and median are not far from each other. The median is the 13th observation, which is 2, and the mean is calculated to be 2.2.

For lists, the **mode** is the most common (frequent) value. A list can have more than one mode. For histograms, the mode is a relative maximum.

Shape of the distribution

An examination of the shape of a distribution will illustrate how the distribution is centred around the mean. Distributions are either symmetric or they are not symmetric, in which case the shape of the distribution is described as asymmetric or skewed.



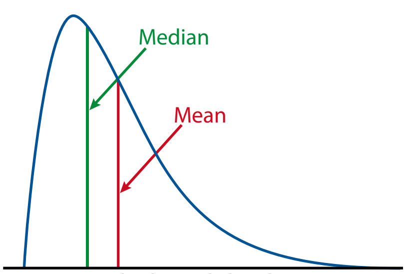
Symmetric distribution

Symmetry

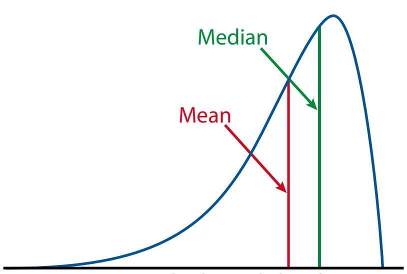
The shape of a distribution is said to be **symmetric** if the observations are balanced, or evenly distributed, about the mean. In a symmetric distribution, *the mean and the median are equal.*

Skewness

A distribution is **skewed** if the observations are not symmetrically distributed above and below the mean.



'Positively skewed' distribution



'Negatively skewed' distribution

A **positively skewed** (or skewed to the right) distribution has a tail that extends to the right in the direction of positive values. A **negatively skewed** (or skewed to the left) distribution has a tail that extends to the left in the direction of negative values.

Looking back at the WHO data, we can clearly see that the data is skewed to the left. Few countries have low life expectancies. The bulk of the countries have life expectancies between approximately 50 and 65 years.

The average HALE is $\mu = \frac{\sum x}{n} = \frac{11028}{192} = 57.44$. Looking at the raw data, it does not appear sensible to search for the mode, as there are very few of them (61, 59, 60 or 62). However, after grouping the data into classes, we can see that the modal class is 60–65.

As there are 192 observations, which means that the median is at $\frac{n+1}{2} = \frac{192+1}{2} = 96.5$, we take the average of the 96th and 97th observations, which are Palau and Moldova with 60 each. So, the median is 60!

Knowing the median, we could say that a typical life expectancy is 60 years. How much does this really tell us? How well does this median describe the real situation? After all, not all countries have the same 60 years HALE. Whenever we find the centre of data, the next step is always to ask how well it actually summarizes the data.

When we describe a distribution numerically, we always report a measure of its **spread** along with its centre.

Exercise 11.2

1 You are given eight measurements: 5, 4, 7, 8, 6, 6, 5, 7.

a) Find $\bar{x}$.
b) Find the median.
c) Based on the previous results, is the data symmetric or skewed? Explain and support your conclusion with an appropriate graph.

2 You are given ten measurements: 5, 7, 8, 6, 12, 7, 8, 11, 4, 10.

a) Find $\bar{x}$.
b) Find the median.
c) Find the mode.

3 The following table gives the number of DVD players owned by a sample of 50 typical families in a large city in Germany.

Number of DVD players	0	1	2	3
Number of households	12	24	8	6

Find the average and the median number of DVD players. Which measure is more appropriate here? Explain.

4 Ten of the Fortune 500 large businesses that lost money in 2006 are listed below:

Company	Loss ($ million)	Company	Loss ($ million)
Vodafone	39 093	General Motors	10 567
Kodak	1362	Japan Airlines	417
UAL	21 167	Japan Post	3
Mitsubishi Motors	814	AMR	861
Visteon	270	Karstadt Quelle	393

Calculate the mean and median of the losses. Which measure is more appropriate in this case? Explain.

5 Even on a crucial examination, students tend to lose focus while writing their tests. In a psychology experiment, 20 students were given a 10-minute quiz and the number of seconds they spent 'on task' were recorded. Here are the results:

350	380	500	460	480	400	370	380	450	530
520	460	390	360	410	470	470	490	390	340

Find the mean and median of the time spent on task. If you were writing a report to describe these times, which measure of central tendency would you use and why?

6 At 5:30 p.m. during the holiday season, a toy shop counted the number of items sold and the revenue collected for that day. The result was $n = 90$ toys with a total revenue of $\sum x = €4460$.

a) Find the average amount spent on each toy that day.

Shortly before the shop closed at 6 p.m., two new purchases of €74 and €60 were made.

b) Calculate the new mean of the sales per toy that day.

7 A farmer has 144 bags of new potatoes weighing 2.15 kg each. He also has 56 bags of potatoes from last year with an average weight of 1.80 kg. Find the mean weight of a bag of potatoes available from this farmer.

8 The following are the grades earned by 25 students on a 50-mark test in statistics.

26, 27, 36, 38, 23, 26, 20, 35, 19, 24, 25, 27, 34, 27, 26, 42, 46, 18, 22, 23, 24, 42, 46, 33, 40.

a) Calculate the mean of the grades.

b) Draw a stem plot of the grades. Use the plot to estimate where the median is.

c) Draw a histogram of the grades.

d) Develop a cumulative frequency graph of the grades. Use your graph to estimate the median.

9 The following are data concerning the injuries in road accidents in the UK classified by severity.

Year	Fatal	Serious	Slight
1970	758	7860	13 515
1975	699	6912	13 041
1980	644	7218	13 926
1985	550	6507	13 587
1990	491	5237	14 443
1995	361	4071	12 102
2000	297	3007	11 825
2005	264	2250	10 922

a) Draw bar graphs for the total number of injuries and describe any patterns you observe.

b) Draw pie charts for the different types of injuries for the years 1970, 1990 and 2005.

10 The following data report the car driver casualties in Durham county in the UK in 2006.

a) Draw a histogram of the data.

b) Estimate the mean of the data.

c) Develop a cumulative frequency graph and use it to estimate the median of the data.

Age	Number
15–19	103
20–24	125
25–29	103
30–34	80
35–39	88
40–44	96
45–49	78
50–54	60
55–59	45
60–64	33
65–69	17
70–74	13
75–79	26

11 Use the data in question 9 of Exercise 11.1 to estimate the median and the mean of the number of days in hospital by heart patients.

12 Use the data in question 10 of Exercise 11.1 to estimate the median and the mean of the exercise time of the author for 2006.

13 Use the data in question 11 of Exercise 11.1 to estimate the median and the mean speed of cars on the highway.

14 Use the data in question 12 of Exercise 11.1 to estimate the median and the mean length of components at this facility.

15 Use the data in question 13 of Exercise 11.1 to estimate the median and the mean of the waiting time for customers at this supermarket.

16 a) Given that $\sum_{i=1}^{40} x_i = 1664$, find $\bar{x}$.

b) Given that $\sum_{i=1}^{20} (x_i - 20) = 1664$, find $\bar{x}$.

17 For a large class of 60 students, 12 points are added to each grade to boost the student's score on a relatively difficult test.

a) Knowing that $\sum(x + 12) = 4404$, find the mean score (without the 12 points) of this group of 60 students.

b) Another section of the class has 40 students and their average score is 67.4. Find the average of the whole group of 100 students.

11.3 Measures of variability

Measures of location summarize what is typical of elements of a list, but not every element is typical. Are all the elements close to each other? Are most of the elements close to each other? What is the biggest difference between elements? On average, how far are the elements from each other? The answer lies in the measures of spread or variability.

It is possible that two data sets have the same mean, but the individual observations in one set could vary more from the mean than do the observations in the second set. It takes more than the mean alone to describe data. Measures of variability (also called measures of dispersion or spread), which include the range, the variance, the standard deviation, interquartile range and the coefficient of variation, will help to summarize the data.

Range

The range in a set of data is the difference between the largest and smallest observations.

Consider the expense data given at the beginning of this chapter. Also consider the same data when the largest value of 68 is replaced by 120. What is the range for these two sets of data?

Table 11.4

	Expense data	Expense data with outlier
Minimum	38	38
Maximum	68	120
Range	30	82

Notice that the range is a *single number*, not an interval of values as you might think from its use in common speech. The maximum of the HALE data is 79 and the minimum is 29, so the range is 50.

Range doesn't take into account how the data is distributed and is, of course, affected by extreme values (outliers) as you see above.

Variance and standard deviation

Note: For an SL treatment of this topic, see our SL book. In this chapter, we will gear the discussion to HL notation.

The most comprehensive measures of dispersion are those in terms of the average deviation from some location parameter.

Variance

The sample variance, s^2, is the sum of the squared differences between each observation and the sample mean divided by the sample size minus 1.

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

- **Hint:** Discussing the reason we define the sample variance in this manner is beyond the scope of this book. The use of $n - 1$ in the denominator has to deal with the use of the sample variance as an estimate of the population variance. Such an estimate has to be unbiased, and this sample variance is the most unbiased estimate of the population variance. However, the IB syllabus uses a different definition of this variance.

The IB variance is listed as s_n^2 and is evaluated as follows:

$$s_n^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}$$

s^2 is called the *unbiased estimate* of the population variance σ^2 and is denoted as s_{n-1}^2. However, it is not required for the current IB syllabus.

It is obvious that $s_{n-1}^2 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1} = \dfrac{n}{n-1} \cdot \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} = \dfrac{n}{n-1} \cdot$

Or $s_n^2 = \dfrac{n-1}{n} \cdot s_{n-1}^2$ in case you want to use s_x^2 from your GDC.

With your calculators you should also be careful as the listed s_x in TI and Casio GDCs corresponds to s_{n-1}^2. So, when you use your GDC, make sure you use what is called σ_x.

The population variance, σ^2, is the sum of the squared differences between each observation and the population mean divided by the population size, N.

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{N}$$

The variance is a measure of the variation about the mean squared. In order to bring the measure down to the data measurements, the square root is taken and the measure looked at is the standard deviation.

The standard deviation measures the **standard amount of deviation** or **spread** around the mean.

Standard deviation

The sample standard deviation, s_n, is the (positive) square root of the variance, and is defined as:

$$s_n = \sqrt{s_n^2} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}} \qquad \text{(IB)}$$

$$s_{n-1} = \sqrt{s_{n-1}^2} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}} = \sqrt{\frac{n}{n-1}} \cdot s_n$$

$$\text{or, } s_n = \sqrt{\frac{n-1}{n}} \cdot s_{n-1}$$

The population standard deviation is:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{N}}$$

These are measures of variation about the mean.

When does $s = 0$? Answer: When all the data take on the same value and there is no variability about the mean.

When is s large? Answer: When there is a large amount of variability about the mean.

Consider the following example:

In business, investors invest their money in stocks whose prices fluctuate with market conditions. Stocks are considered risky if they have high fluctuations. Here are the closing prices of two stocks traded on Vienna's stock market for the first seven business days in September 2007:

Stock A	Stock B
4	1
4.25	3
5	2.5
4.75	5
5.75	7
5.25	6.5
6	10
$\bar{x}_A = 5$ Median (A) $= 5$	$\bar{x}_B = 5$ Median (B) $= 5$

Even though the two stocks have similar central values, they do behave differently. It is obvious that stock B is more variable and it becomes more obvious when we calculate the standard deviations.

We will calculate the standard deviation manually in this example to demonstrate the process. You do not have to do this manually all the time!

$$s_A^2 = \frac{\sum_{i=1}^{7}(x_i - 5)^2}{7} = \frac{(4-5)^2 + (4.25-5)^2 + (5-5)^2 + (4.75-5)^2 + (5.75-5)^2 + (5.25-5)^2 + (6-5)^2}{7} = 0.464$$

$$s_B^2 = \frac{\sum_{i=1}^{7}(x_i - 5)^2}{7} = \frac{(1-5)^2 + (3-5)^2 + (2.5-5)^2 + (5-5)^2 + (7-5)^2 + (6.5-5)^2 + (10-5)^2}{7} = 8.21$$

This means that the standard deviations are $s_A = 0.681$ and $s_B = 2.866$. Stock B is 4.2 times as variable as stock A.

Note: When computing s_n^2 or s_{n-1}^2 manually, you may find the following shortcut of some use:

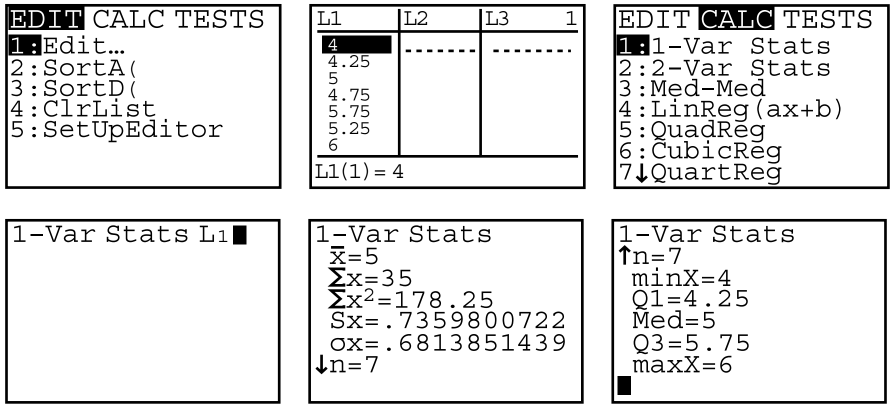
$$s_n^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n} = \frac{\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{n} = \frac{\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i\bar{x} + \sum_{i=1}^{n}\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \frac{2\bar{x}\sum_{i=1}^{n} x_i}{n} + \frac{\sum_{i=1}^{n}\bar{x}^2}{n} = \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\bar{x}\sum_{i=1}^{n}\frac{x_i}{n} + \frac{n\bar{x}^2}{n} = \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$$

$$s_{n-1}^2 = \left(\frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2\right) \cdot \frac{n}{n-1} = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1}$$

However, remember that once you have a good understanding of the standard deviation, you will rely on a GDC or software to do most of the calculation for you.

Here is how you can use your TI GDC:



The s_x used by your GDC gives $s_{n-1} = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$ and $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$ which is the s_n used in IB exams.

The screenshots also show you that the GDC gives you $\sum x^2$, which can be used if you want to find the variance by hand.

$$s_n^2 = \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2 = \frac{178.25}{7} - 5^2 = 0.464 \Rightarrow s_n = 0.681$$

$$s_{n-1} = \sqrt{\frac{7}{6}} \times 0.681 = 0.736, \text{ or}$$

$$s_{n-1}^2 = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1} = \frac{178.25 - 7.(5)^2}{6} = \frac{3.25}{6} = 0.542$$

$$\Rightarrow s_{n-1} = 0.736$$

The interquartile range and measures of non-central tendency

To understand another measure of spread known as the **interquartile range**, it is first necessary to define percentiles and quartiles.

Percentiles and quartiles

Data must first be in ascending order.

Percentiles separate large ordered data sets into hundredths. The pth percentile is a number such that p per cent of the observations are at or below that number.

Quartiles are descriptive measures that separate large ordered data sets into four quarters.

To score in the 90th percentile indicates 90% of the test scores were less than or equal to your score. An excellent performance! You scored in the upper 10% of all persons taking the test.

- **First quartile, Q_1**
  The first quartile, Q_1, is another name for the 25th percentile. The first quartile divides the ordered data such that 25% of the observations are at or below this value. Q_1 is located in the $0.25(n + 1)$st position when the data is in ascending order. That is,

$$Q_1 = \frac{n+1}{4} \text{ ordered observation}$$

- **Third quartile, Q_3**
  The third quartile, Q_3, is another name for the 75th percentile. The third quartile divides the ordered data such that 75% of the observations are at or below this value. Q_3 is located in the $0.75(n + 1)$st position when the data is in ascending order. That is,

$$Q_3 = \frac{3(n+1)}{4} \text{ ordered observation}$$

- **The median**
  The median is the 50th percentile, or the second quartile, Q_2.

A measure which helps to measure variability and is not affected by extreme values is the interquartile range. It avoids the problem of extreme values by just looking at the range of the middle 50% of the data.

Interquartile range

The interquartile range (IQR) measures the spread in the middle 50% of the data; it is the difference between the observations at the 25th and the 75th percentiles:

$$\mathbf{IQR = Q_3 - Q_1}$$

If we consider the student expense data in Table 11.1 (on page 467) and once again look at that same data with the outlier 120 replacing the largest value 68, we have the following results:

	Expense data	Expense data with outlier
Minimum	38	38
Q_1	50	50
Median	55	55
Q_3	61	61
Maximum	68	120
Range	30	82
IQR	11	11

● **Hint:** The first quartile is also called the lower quartile. The third quartile is also called the upper quartile.

A practical method to calculate the quartiles is to split the data into two halves at the median. (When n is odd, include the median in both halves!) The first quartile is the median of the first half and the third quartile is the median of the second half. For example, with the stocks data, {4, 4.25, 4.75, 5, 5.25, 5.75, 6}, $n = 7$, the median is the fourth observation, 5. The first quartile is then the median of {4, 4.25, 4.75, 5}, which is 4.5, and the third quartile is the median of {5, 5.25, 5.75, 6}, which is 5.5.

Range doesn't take into account how the data is distributed and is, of course, affected by extreme values. We clearly saw that in Table 11.4 (page 487). However, the IQR evidently does not have that problem.

Five-number summary

Five-number summary refers to the five descriptive measures: minimum, first quartile, median, third quartile, maximum.
Clearly, $X_{minimum} < Q_1 < \text{Median} < Q_3 < X_{maximum}$.

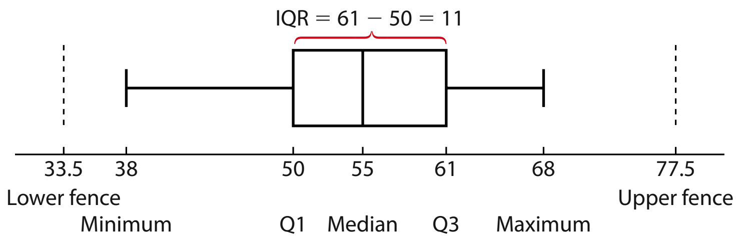
Box-and-whisker plot

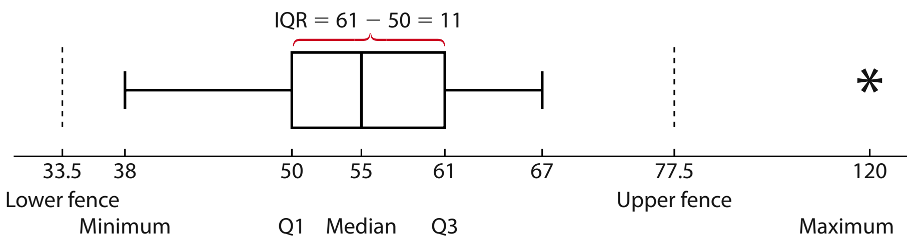
Whenever we have a five-number summary, we can put the information together in one graphical display called a **box plot**, also known as a **box-and-whisker** plot. In the student expenditure data, the IQR is €11. This is evident in the box plot below, where the IQR is the difference between 50 and 61.

Let us make a box plot with the student expense data.

- Draw an axis spanning the range of the data. Mark the numbers corresponding to the median, minimum, maximum, and the lower and upper quartiles.
- Draw a rectangle with lower end at Q1 and upper end at Q3, as shown below.
- To help us consider outliers, mark the points corresponding to lower and upper fences. Mark them with a dotted line since they are not part of the box. The fences are constructed at the following positions:
  - Lower fence: $Q_1 - 1.5 \times \text{IQR}$ (Here it is $50 - 1.5(11) = 33.5$.)
  - Upper fence: $Q_3 + 1.5 \times \text{IQR}$ (Here it is $61 + 1.5(11) = 77.5$.)

  Any point beyond the lower or upper fence is considered an **outlier**.
- Mark any outlier with an asterisk (*) on the graph (shown below).
- Extend horizontal lines called 'whiskers' from the ends of the box to the smallest and largest observations that are not outliers. In the first case these are 38 and 68, while in the second they are 38 and 67.

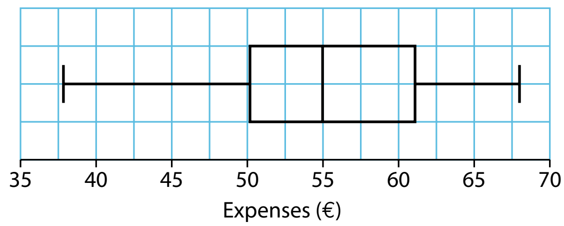




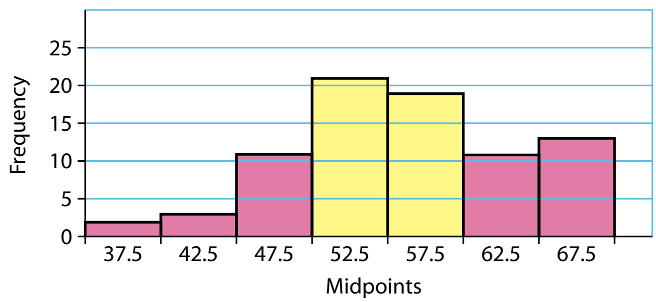
An outlier is an *unusual* observation. It lies at an abnormal distance from the rest of the data. There is no unique way of describing what an outlier is. A common practice is to consider any observation that is further than 1.5 IQR from the first or third quartile as an outlier. Outliers are important in statistical analysis. Outliers may contain important information not shared with the rest of the data. Statisticians look very carefully at outliers because of their influence on the shapes of distributions and their effect on the values of the other statistics, such as the mean and standard deviation.

Here is a box plot of the data done by a software package.

Box plot of student expense data

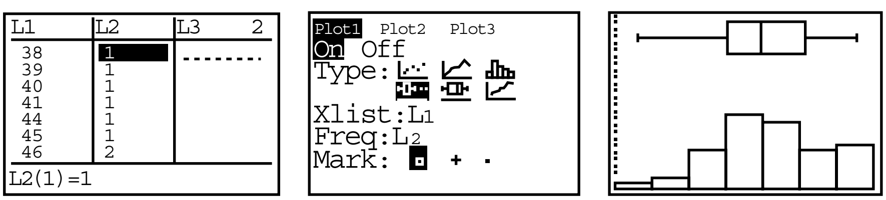


As you can see, the box contains the middle 50% of the data. The width of the box is nothing but the IQR! Now we know that the middle 50% of the students' expenditure is €11. This seems, at times, as a reasonable summary of the spread of the distribution, as you can see in the histogram below.

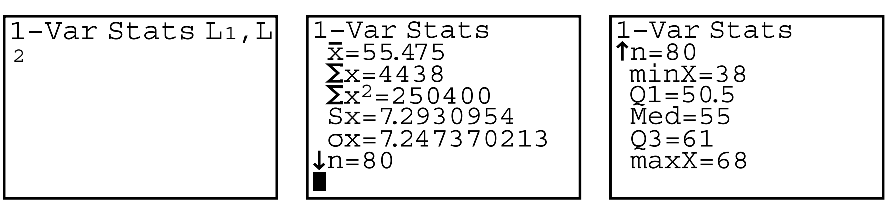


If you locate the IQR on the histogram, you can also get another visual indication of the spread of the data.

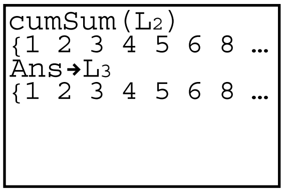
How to use your GDC for histograms and box plots:

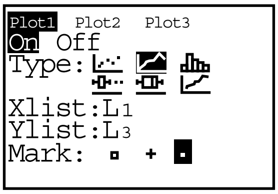


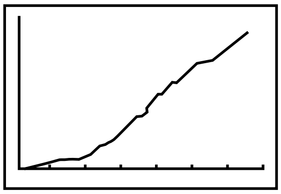
For grouped data:

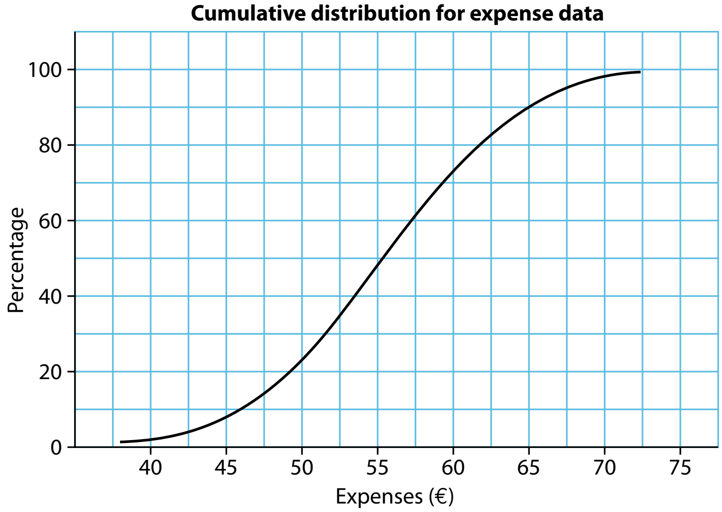


An ogive can also be produced:

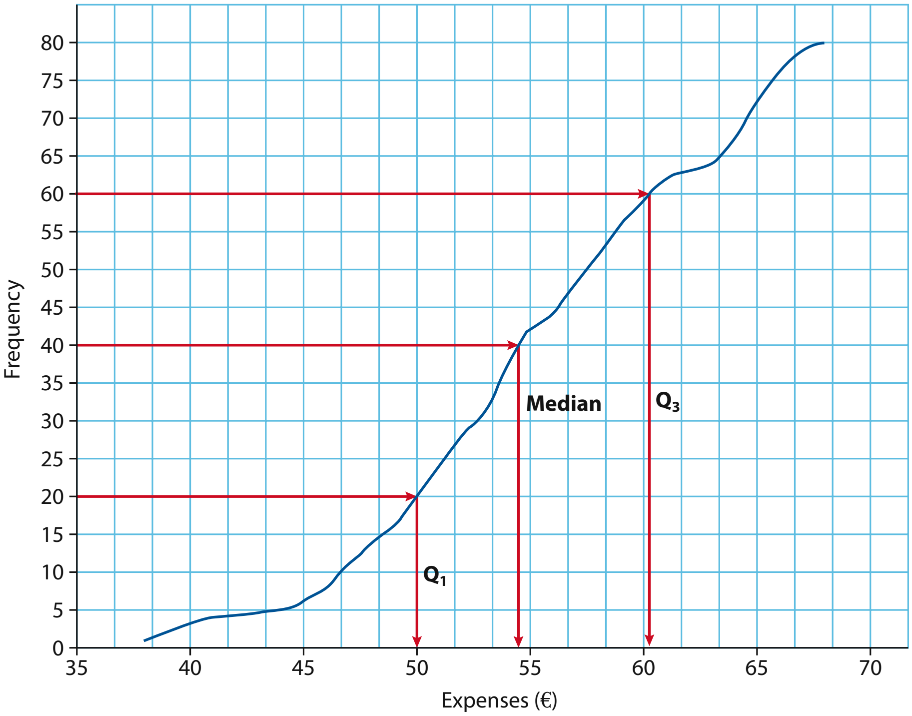








This is a realistic ogive.



Notice how we locate the first quartile. Since there are 80 observations, the first quartile is approximately at the $\frac{n+1}{4} = \frac{81}{4} \approx 20$th position, which appears to be around 50.

The median is at the $\frac{n+1}{2} = \frac{81}{2} \approx$ 40th–41st position, i.e. approximately at 55.

Similarly, the third quartile is at $\frac{3(n+1)}{4} = \frac{243}{4} \approx$ 61st, which happens here at approximately 61!

The calculation of the mean and variance for grouped data is essentially the same as for raw data. The difference lies in the use of frequencies to save typing (writing) all numbers. Here is a comparison:

Statistic	Raw data	Grouped data	Grouped data with intervals
$\bar{x}$	$\bar{x} = \frac{\sum_{all\,x} x}{n}$	$\bar{x} = \frac{\sum_{all\,x} x_i \cdot f(x_i)}{n} = \frac{\sum_{all\,x} x_i \cdot f(x_i)}{\sum f(x_i)}$	$\bar{x} = \frac{\sum_{all\,x} m_i \cdot f(m_i)}{n} = \frac{\sum_{all\,x} m_i \cdot f(m_i)}{\sum f(m_i)}$
s_n^2	$s_n^2 = \frac{\sum_{all\,x} (x_i - \bar{x})^2}{n}$	$s_n^2 = \frac{\sum_{all\,x} (x_i - \bar{x})^2 \cdot f(x_i)}{n} = \frac{\sum_{all\,x} (x_i - \bar{x})^2 \cdot f(x_i)}{\sum f(x_i)}$	$s_n^2 = \frac{\sum_{all\,x} (m_i - x)^2 \cdot f(m_i)}{n} = \frac{\sum_{all\,x} (m_i - \bar{x})^2 \cdot f(m_i)}{\sum f(m_i)}$

where x_i = data point

$f(x_i)$ = frequency of x_i

m_i = interval midpoint (mid mark or mid value)

$f(m_i)$ = frequency of interval i

$\sum f(x_i), \sum f(m_i)$ = total number of data points

For the grouped data reproduced here, this is how we estimate the mean and variance:

Living expenses	Midpoint m	Number of students $f(m)$	$m_i \times f(m_i)$	$(m_i - \bar{x})^2$	$(m_i - \bar{x})^2 \times f(m_i)$
35 but <40	37.5	2	75	344.5	688.9
40 but <45	42.5	3	127.5	183.9	551.6
45 but <50	47.5	11	522.5	73.3	806.0
50 but <55	52.5	21	1102.5	12.7	266.1
55 but <60	57.5	19	1092.5	2.1	39.4
60 but <65	62.5	11	687.5	41.5	456.2
65 but <70	67.5	13	877.5	130.9	1701.4
Totals		$\sum f(m_i) = 80$	$\sum_{all\,x} m_i \cdot f(m_i) = 4485$	$\sum_{all\,x} (m_i - \bar{x})^2 \cdot f(m_i) = 4509.6$	
		Mean	$\frac{4485}{80} = 56.06$	**Variance**	$\frac{4509.6}{80} = 56.37$
				Standard deviation	7.51

The numbers here are estimates of the mean and the variance and eventually the standard deviation. As you will notice, they are not equal to the values we calculated earlier, but are close. The reason for this is that, with grouping, we lost the detail in each interval. For example, the interval between 45 and 50 is represented by the midpoint 47.5. In essence, we are assuming that every number in the interval is equal to 47.5.

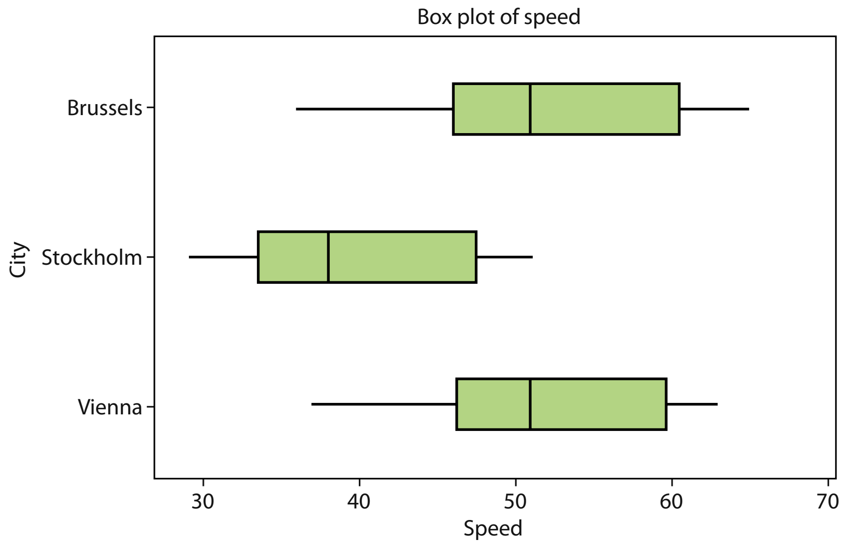
Example 7

Speed limits in some European cities are set to 50 km/h. Drivers in various cities react to such limits differently. The results of the survey to compare drivers' behaviour in Brussels, Vienna and Stockholm are given in the accompanying table. Use box plots to compare the results.

Vienna	62	60	59	50	61	63	53	46	58	49	51	37	47	51	63	52	44	50	45	44
Brussels	64	61	63	57	49	49	46	58	45	60	51	36	65	45	47	46				
Stockholm	43	44	34	35	31	34	29	33	36	38	45	47	29	48	51	49	48			

Solution

Parallel box plots may be an appropriate tool to enable a comparison between the three data sets.



It appears that, on average, drivers in Brussels and Vienna tend to be on the 'speedy' side. The median in both cities is higher than 50, which means than more than 50% of the drivers in the two cities tend not to respect the speed limit. The variation in both cities is comparable with Brussels having a slightly wider range than Vienna. Almost all drivers in Stockholm appear to adhere to the 50 km/h limit. The median is around 40 km/h and the third quartile about 47, which means that more than 75% of the drivers in this city drive at a speed less than the 50 km/h limit.

Shape, centre and spread

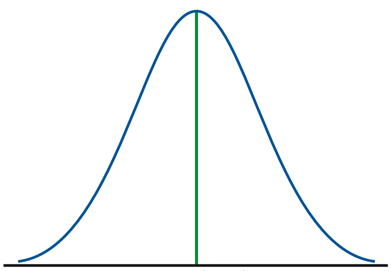
Statistics is about variation, so spread is an important fundamental concept. Measures of spread help us to precisely analyze what we do not know! If the values we are looking at are scattered very far from the centre, the IQR and the standard deviation will be large. If these are large, our central values will not represent the data well. That is why we always report spread with any central value.

A practical way of seeing the significance of the standard deviation can be demonstrated with the following (optional) observations:

Empirical rule:

If the data is close to being symmetrical, as in the figure left, the following is true:

- The interval $\mu \pm \sigma$ contains approximately 68% of the measurements.
- The interval $\mu \pm 2\sigma$ contains approximately 95% of the measurements.
- The interval $\mu \pm 3\sigma$ contains approximately 99.7% of the measurements.



Symmetric distribution

The empirical rule usually indicates if an observation is very far from the expected or not. Take the following example:

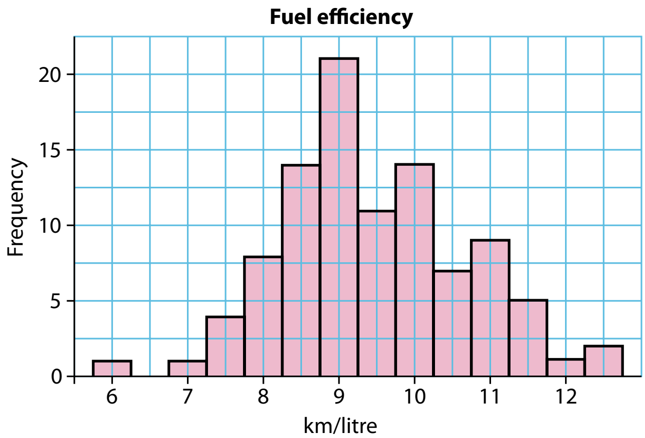
I have recorded my car's fuel efficiency over the last 98 times that I have filled the tank with gasoline. Opposite is the data expressing how many kilometeres per litre the car travelled:

km/litre	Frequency	km/litre	Frequency
6.0	1	10.0	14
7.0	1	10.5	7
7.5	4	11.0	9
8.0	8	11.5	5
8.5	14	12.0	1
9.0	21	12.5	2
9.5	11		

The summary measures are:

Mean	9.454
σ	1.223
Median	9.25
Q_1	8.5
Q_3	10.125
IQR	1.625

The histogram shows that the distribution is almost symmetric. The possible outlier has little effect on the mean and standard deviation. That is why the mean and median are almost the same. Looking at the box plot, you can see that there is one outlier.



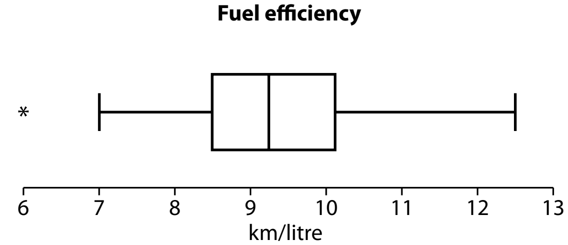
Chebyshev's rule:
This rule is similar to the empirical rule in the sense that it tries to interpret the standard deviation. Contrary to the empirical rule, it applies to any data set, regardless of the shape of the distribution.

Rule: For any number $k > 1$, at least $\left(1 - \frac{1}{k^2}\right)$ of the measurements will fall within k standard deviations of the mean. [i.e. within the interval $(\overline{x} - k\sigma, \overline{x} + k\sigma)$.]

a) No useful information is provided on the fraction of measurements that fall within 1 standard deviation from the mean – $(\overline{x} - \sigma, \overline{x} + \sigma)$.

b) At least $\frac{8}{9}$ will fall within 3 standard deviations from the mean – $(\overline{x} - 3\sigma, \overline{x} + 3\sigma)$.

c) At least $\frac{3}{4}$ will fall within 2 standard deviations from the mean – $(\overline{x} - 2\sigma, \overline{x} + 2\sigma)$.



The confirmation is below:

$9.25 - 1.5 \times 1.625 = 6.8$, which is why 6 is considered as an outlier.

$10.125 + 1.5 \times 1.625 = 12.6$, and hence no outliers on this side.

If we use the empirical rule, we can expect about 99.7% of the data to lie within three standard deviations of the mean, i.e. $9.454 - 3 \times 1.223 = 5.8$ and $9.454 + 3 \times 1.223 = 13.1$. In fact, you see all the data is within the specified interval, including the potential outlier!

Question: What should you be able to tell about a quantitative variable?

Answer: Report the shape of its distribution, and include a centre and a spread.

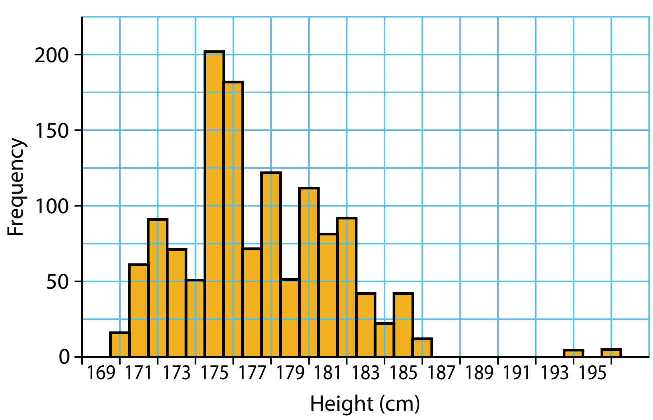
Question: Which central measure and which measure of spread?

Answer: The rules are:

- If the shape is skewed, report the median and IQR. You *may* want to include the mean and standard deviation, but you should point out that the mean and median differ as this difference is a sign that the data is skewed. A histogram can help.
- If the shape is symmetrical, report the mean and standard deviation. You may report the median and IQR as well.
- If there are clear outliers, report the data with and without the outliers. The differences may be revealing.

Example 8

The records of a large high school show the heights of their students for the year 2006.



a) Which statistics would best represent the data here? Why?

b) Calculate the mean and standard deviation.

c) Develop a cumulative frequency graph of the data.

d) Use your result of c) above to estimate the median, Q_1, Q_3 and IQR.

e) Are there any outliers in the data? Why?

f) Write a few sentences describing the distribution.

Solution

a) The data appears to have outliers and is slightly skewed to the right. The most appropriate measure is the median, since the mean is influenced by the extreme values.

b) To calculate the mean and standard deviation, we will set up a table that will facilitate the calculation.

Height (cm) x_i	Number of students $f(x)$	$x_i \times f(x_i)$	$(x_i - \overline{x})^2$	$(x_i - \overline{x})^2 \times f(x_i)$
170	15	2550	51.84	777.6
171	60	10 260	38.44	2306.4
172	90	15 480	27.04	2433.6
173	70	12 110	17.64	1234.8
174	50	8700	10.24	512
175	200	35 000	4.84	968
176	180	31 680	1.44	259.2
177	70	12 390	0.04	2.8
178	120	21 360	0.64	76.8
179	50	8950	3.24	162
180	110	19 800	7.84	862.4
181	80	14 480	14.44	1155.2
182	90	16 380	23.04	2073.6
183	40	7320	33.64	1345.6
184	20	3680	46.24	924.8
185	40	7400	60.84	2433.6
186	10	1860	77.44	774.4
194	2	388	282.24	564.5
196	3	588	353.44	1060.3
Totals	$\sum f(x_i) = 1300$	$\sum_{\text{all } x} x_i \cdot f(x_i) = 230\,376$	$\sum_{\text{all } x} (x_i - \overline{x})^2 \cdot f(x_i) = 19\,927.6$	
	Mean	$\frac{230\,376}{1300} = 177.2$	**Variance**	$\frac{19\,927.4}{1300} = 15.33$
			Standard deviation	3.92

Note: Using the alternative formula for the variance will also give the same result. (Due to rounding, answers will differ slightly.)

$$s_n^2 = \frac{\sum_{i=1}^{n} x_i^2 \times f(x_i)}{n} - \bar{x}^2 = \frac{40\,845\,390}{1300} - 177.2123^2 = 15.3315 \Rightarrow$$

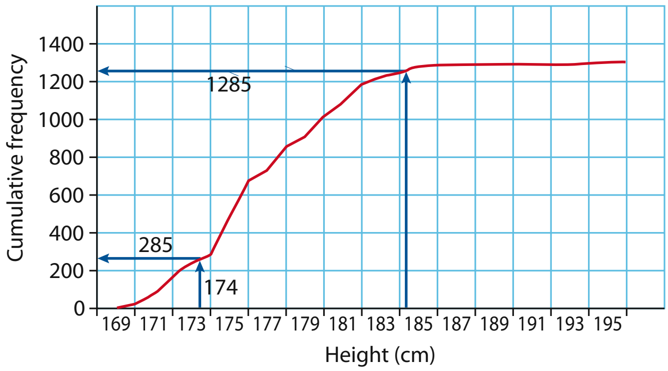
$$s_n = \sqrt{15.3315} = 3.92$$

c) To develop the cumulative frequency graph, we first need to develop the cumulative frequency table. This is done by accumulating the frequencies as shown below.

x	$f(x)$	Cum $f(x)$	x	$f(x)$	Cum $f(x)$
170	15	15	184	20	1245
171	60	75	185	40	1285
172	90	165	186	10	1295
173	70	235	187	0	1295
174	50	285	188	0	1295
175	200	485	189	0	1295
176	180	665	190	0	1295
177	70	735	191	0	1295
178	120	855	192	0	1295
179	50	905	193	0	1295
180	110	1015	194	2	1297
181	80	1095	195	0	1297
182	90	1185	196	3	1300
183	40	1225			

The cumulative frequency table is constructed such that the cumulative frequency corresponding to any measurement is the number of observations that are less than or equal to its value. So, for example, the cumulative frequency corresponding to a height of 174 cm is 285, which consists of the 50 observations with height 174 cm and the 235 observations for heights less than 174 cm.

The cumulative frequency graph plots the observations on the horizontal axis against their cumulative frequencies on the vertical axis, as shown below.



d) The median is the observation between $\frac{1300}{2} = 650$th and 651st observations, since the number is even. From the cumulative table, we can see that the median is in the 176 interval. So the median is 176.

Q_1 is at $\frac{1301}{4} \approx 325$th observation. From the table, as 174 has a cumulative frequency of 285, and 175 has a cumulative frequency of 485, then Q1 has to be **175**.

Also, Q_3 is at $\frac{3 \times 1301}{4} \approx 976$th observation. So, similarly, it is **180**.

IQR $= 180 - 175 =$ **5**.

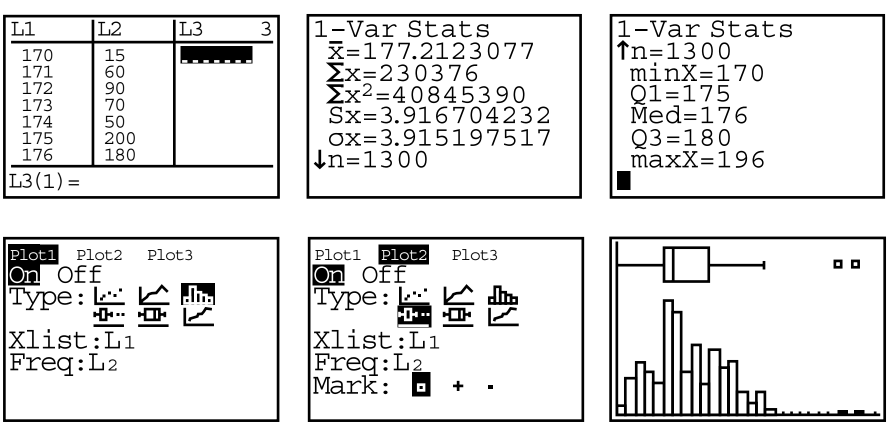
e) To check for outliers, we can calculate the lengths of the whiskers.

Lower fence: $175 - 1.5 \times 5 = 167.5$, which is lower than the minimum value, so there are no outliers on the left.

Upper fence: $180 + 1.5 \times 5 = 187.5$. So we have five outliers, two at 194 cm and three at 196 cm.

f) The distribution appears to be bimodal with two modes at 175 and 176. It is slightly skewed to the right with a few extreme values at 194 and 196. This is further confirmed by the fact that the mean of 177.2 is higher than the median of 176.

Note: Here are the calculations using a GDC:



Exercise 11.3

1 The pulse rates of 15 patients chosen at random from visitors of a local clinic are given below:

72, 80, 67, 68, 80, 68, 80, 56, 76, 68, 71, 76, 60, 79, 71

a) Calculate the mean and standard deviation of the pulse rate of the patients at the clinic.

b) Draw a box plot of the data and indicate the values of the different parts of the box.

c) Check if there are any outliers.

2 The number of passengers on 50 flights from Washington to London on a commercial airline were:

165	173	158	171	177	156	178	210	160	164
141	127	119	146	147	155	187	162	185	125
163	179	187	174	166	174	139	138	153	142
153	163	185	149	154	154	180	117	168	182
130	182	209	126	159	150	143	198	189	218

a) Calculate the mean and standard deviation of the number of passengers on this airline between the two cities.

b) Set up a stem plot for the data and use it to find the median of the number of passengers.

c) Develop a cumulative frequency graph. Estimate the median, and first and third quartiles. Draw a box plot.

d) Find the IQR and use it to check whether there are any outliers.

e) Use the empirical rule to check for outliers.

3 At a school, 100 students took a 'mock' IB exam using paper 3. The paper was marked out of 60 marks. Here are the results

Marks	0–9	10–19	20–29	30–39	40–49	50–60
No. of students	5	9	16	24	27	19

a) Draw a cumulative frequency curve.

b) Estimate the median and quartiles.

4 130 first-year IB students were given a placement test to decide whether they go for SL or HL. The times for these students to finish the test are given in the table below:

Time	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
No. of students	8	12	24	29	19	16	12	8	2

a) Develop a cumulative frequency curve.

b) Estimate the median and the IQR.

c) 20 students did not manage to finish the test after 120 minutes and had to hand it in uncompleted. Estimate the median finishing time for all 130 students.

5 The mean score of 26 students on a 40-point paper is 22. The mean for another group of 84 other students is 32. Find the mean of the combined group of 110 students.

6 The scores on a 100-mark test of a sample of 80 students in a large school are given below.

Score	59–63	63–67	67–71	71–75	75–79	79–83	83–87
No. of students	6	10	18	24	10	8	4

a) Find the mean and standard deviation of the scores of all students.

b) A bonus of 13 points is to be added to these scores. What is the new value of the mean and standard deviation?

7 *Cats* is a famous musical. In a large theatre in Vienna (1744 capacity), during a period of 10 years, it played 1000 performances. The manager of the group kept a record of the empty seats on the days it played. Here is the table.

Number of empty seats	1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Days	15	50	100	170	260	220	90	45	30	20

a) Copy and complete the following cumulative frequency table for the above information.

Number of empty seats	$x \leqslant 10$	$x \leqslant 20$	$x \leqslant 30$	$x \leqslant 40$	$x \leqslant 50$	$x \leqslant 60$	$x \leqslant 70$	$x \leqslant 80$	$x \leqslant 90$	$x \leqslant 100$
Days	15		165			815				1000

b) Draw a cumulative frequency graph of this distribution. Use 1 unit on the vertical axis to represent the number of 100 days and 1 unit on the horizontal axis to represent every 10 seats.

c) Use the graph from b) to answer the following questions:

(i) Find an estimate of the median number of empty seats.

(ii) Find an estimate for the first quartile, third quartile and the IQR.

(iii) The days the number of empty seats was less than 35 seats were considered bumper days (lots of profit). How many days were considered bumper days?

(iv) The highest 15% of the days with empty seats were categorized as loss days. What is the number of empty seats above which a day is claimed as a loss?

8 Aptitude tests sometimes use jigsaw puzzles to test the ability of new applicants to perform precision assembly work in electronic instruments. One such company that produces the computerized parts of video and CD players gave the following results:

Time to finish the puzzle (nearest second)	Number of employees
30–35	16
35–40	24
40–45	22
45–50	26
50–55	38
55–60	36
60–65	32
65–70	18

a) Draw a histogram of the data.

b) Draw a cumulative frequency curve and estimate the median and IQR.

c) Calculate the estimates of the mean and standard deviation of all such participants.

9 The heights of football players at a given school are given in the table below:

Height	Frequency	Height	Frequency	Height	Frequency	Height	Frequency	Height	Frequency	Height	Frequency
152	2	160	7	168	18	175	5	183	9	191	4
155	6	163	5	170	7	178	11	185	4	193	1
157	9	165	20	173	12	180	8	188	2		

a) Find the five-number summary for this data.

b) Display the data with a box plot and a histogram.

c) Find the mean and standard deviation of the data.

d) Describe the data with a few sentences.

e) Draw a cumulative frequency graph and estimate the height of the player that is in the 90th percentile.

f) 10 players' data were missing when we collected the data. The average height of the 10 players is 182. Find the average height of all the players, including the last 10.

10 Consider 10 data measures.

a) If the mean of the first 9 measures is 12 and the 10th measure is 12, what is the mean of the 10 measures?

b) If the mean of the first 9 measures is 11, and the 10th measure is 21, what is the mean of the 10 measures?

c) If the mean of the first 9 measures is 11, and the mean of the 10 measures is 21, what is the value of the 10th measure?

11 Suppose that the mean of a set of 10 data points is 30.

a) It is discovered that a data point having a value of 25 was incorrectly entered as 15. What should be the revised value of the mean?

b) Suppose an additional point of value 32 was added. Will this increase or decrease the value of the mean?

12 Half the values of a sample are equal to 20, one-sixth are equal to 40, and one-third are equal to 60. What is the sample mean?

13 The seven numbers 7, 10, 12, 17, 21, x and y have a mean $\mu = 12$ and a variance $\sigma^2 = \frac{172}{7}$. Find x and y given that $x < y$.

14 A sample of 25 observations was taken out of a large population of measurements. If it is given that $\sum_{i=1}^{25} x_i = 278$ and $\sum_{i=1}^{n} x_i^2 = 3682$, estimate the mean and the variance of the population of measurements.

15 Use the data in question 9 of Exercise 11.1 to estimate the IQR and the standard deviation of the number of days in hospital by heart patients.

16 Use the data in question 10 of Exercise 11.1 to estimate the IQR and the standard deviation of the exercise time of the author for 2006.

17 Use the data in question 11 of Exercise 11.1 to estimate the IQR and the standard deviation of the speed of cars on the highway.

18 Use the data in question 12 of Exercise 11.1 to estimate the IQR and the standard deviation of the length of components at this facility.

19 Use the data in question 13 of Exercise 11.1 to estimate the IQR and the standard deviation of the waiting time for customers at this supermarket.

Practice questions

1 Given that μ is the mean of a data set $y_1, y_2, \ldots, y_{30}$, and you know that

$$\sum_{i=1}^{30} y_i = 360 \text{ and } \sum_{i=1}^{30} (y_i - \mu)^2 = 925, \text{ find}$$

a) the value of μ

b) the standard deviation of the set.

2 Laura made a survey of some students at school asking them about the time it takes each of them to come to school every morning. She scribbled the numbers on a piece of paper and, unfortunately, could not read the number of students who spend 40 minutes on their trip to school. The average number of minutes she had originally found was 34 minutes. Find out how many students spend 40 minutes on their trip.

Time in minutes	10	20	30	40	50
Number of students with this time	1	2	5	**?**	3

3 The following table gives 50 measurements of the time it took a certain reaction to be done in a laboratory experiment.

3.1	5.1	4.9	1.8	2.8	5.6	3.6	2.2	2.5	3.4
4.5	2.5	3.5	3.6	3.7	5.1	4.1	4.8	4.9	1.6
2.9	3.6	2.1	6.1	3.5	4.7	4	3.9	3.7	3.9
2.7	4.3	4	5.7	4.4	3.7	3.7	4.6	4.2	4
3.8	5.6	6.2	4.9	2.5	4.2	2.9	3.1	2.8	3.9

a) Construct a frequency table and histogram starting at 1.6 and with interval length of 0.5.

b) What fraction of the measurements is less than 5.1?

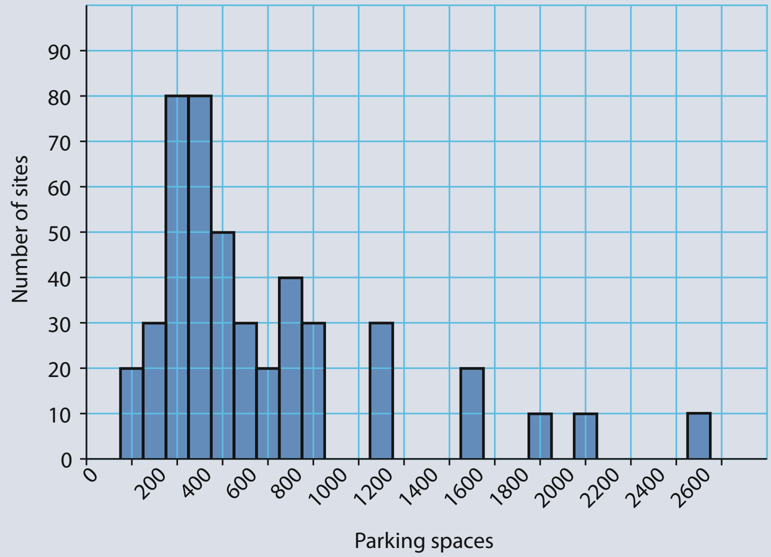
c) Estimate, from your histogram, the median of this data set.

d) Estimate the mean and standard deviation using your frequency table.

e) Construct a cumulative frequency graph.

f) From your cumulative frequency graph, estimate each of the five numbers in the five-number summary.

4 In large cities around the world, governments offer parking facilities for public use. The histogram below gives a picture of the number of parking sites available with the capacity of each, in a number of cities chosen at random.



a) Which statistics would best represent the data here? Why?

b) Calculate the mean and standard deviation.

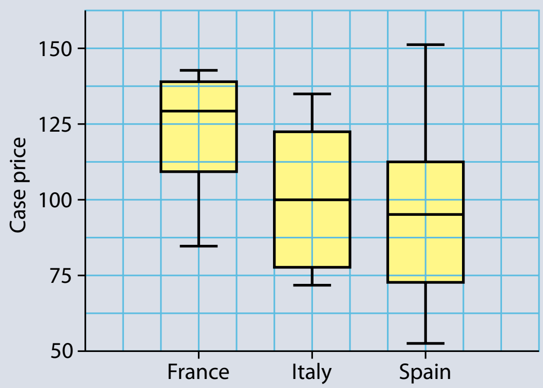
c) Develop a cumulative frequency graph of the data.

d) Use your result from **c)** above to estimate the median, Q_1, Q_3 and IQR.

e) Are there any outliers in the data? Why?

f) Write a few sentences describing the distribution.

5 The box plots display the case prices (in €) of red wines produced in France, Italy and Spain.



a) Which country appears to produce the most expensive red wine? The cheapest?

b) In which country are the red wines generally more expensive?

c) Write a few sentences comparing the pricing of red wines in the three countries.

6

112.72	53.55	54.12	54.33	58.79	59.26	60.39	62.45	52.22	52.52	52.58	52.85
54.06	51.34	51.93	52.09	52.14	52.24	52.24	52.53	53.5	51.82	51.93	52
52.78	52.82	50.28	50.49	51.28	51.28	51.52	51.62	52.4	52.43	49.83	50.46
50.95	51.07	51.11	49.45	49.45	49.73	49.76	49.93	50.19	50.32	50.63	48.64
49.79	50.19	50.62	50.96	49.09	49.16	49.29	49.74	49.74	49.75	49.84	49.76
52.9	52.91	53.4	52.18	52.57	52.72	50.56	50.87	50.9	49.32	49.7	

The table shows the record for the times (seconds) of the 71 male swimmers in the 100 m swim on the first day during the Summer Olympics 2000 in Sydney.

a) Calculate the mean time and the standard deviation.

b) Calculate the median and IQR.

c) Explain the differences between these two sets of measures.

7 In a survey of universities in major cities in the world, the percentage of first-year students who graduate on time (some require 4 years and some 5 years) was reported. The summary statistics are given below.

Number of universities surveyed	120	Mean percentage	69
Median percentage	70	Standard deviation	9.8
Minimum	42	Maximum	86
Range	44	Q_1	60.25
Q_3	75.75		

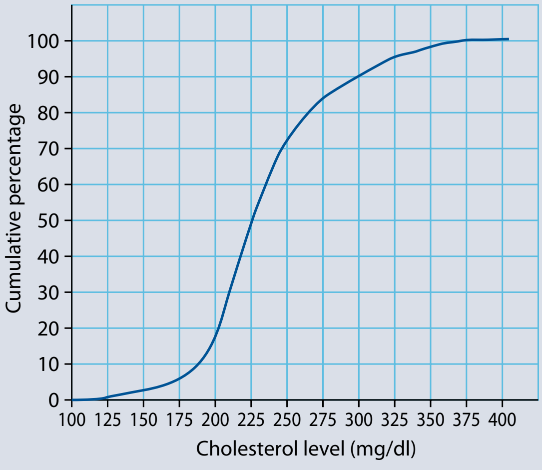
a) Is this distribution symmetric? Explain.

b) Check for outliers.

c) Create a box plot of the data.

d) Describe the data in a short paragraph.

8 The International Heart Association studies, among other factors, the influence of cholesterol level (in mg/dl) on the conditions of heart patients. In a study of 2000 subjects, the following cumulative relative frequency graph was recorded.



a) Estimate the median cholesterol level of heart patients in the study.

b) Estimate the first and third quartiles, and the 90th and 10th percentiles.

c) Estimate the IQR. Also estimate the number of patients in the middle 50% of this distribution.

d) Create a box plot of the data.

e) Give a short description of the distribution.

9 Many of the streets in Vienna, Austria have a speed limit of 30 km/h. On one Sunday evening the police registered the speed of cars passing an important intersection, in order to give speeding tickets when drivers exceeded the limit. Here is a random sample of 100 cars recorded that evening.

26	46	39	41	44	37	38	35	34	31
27	47	39	41	44	37	38	35	34	32
27	47	39	41	44	37	38	35	34	32
27	48	39	41	44	27	38	35	34	32
29	48	40	41	45	37	38	36	34	33
30	48	40	41	45	37	38	36	35	33
30	48	40	42	45	38	39	36	35	33
30	49	40	42	46	38	39	36	35	33
30	50	41	42	46	38	39	36	35	33
31	54	41	43	46	38	39	36	35	33

a) Prepare a frequency table for the data.
b) Draw a histogram of the data and describe the shape.
c) Calculate, showing all work, the mean and standard deviation of the data.
d) Prepare a cumulative frequency table of the data.
e) Find the median, Q_1, Q_3 and IQR.
f) Are there any outliers in the data? Explain using an appropriate diagram.

10 The following is the data collected from 50 industrial countries chosen at random in 2001. The data represents the per capita gasoline consumption in these countries. The Netherlands' consumption was at 1123 litres per capita while Italy stood at 2220 litres per capita.

2062	2076	1795	1732	2101	2211	1748	1239	1936	1658
1639	1924	2086	1970	2220	1919	1632	1894	1934	1903
1714	1689	1123	1671	1950	1705	1822	1539	1976	1999
2017	2055	1943	1553	1888	1749	2053	1963	2053	2117
1600	1795	2176	1445	1727	1751	1714	2024	1714	2133

a) Calculate the mean, median, standard deviation, Q1, Q3 and IQR.
b) Are there any outliers?
c) Draw a box plot.
d) What consumption levels are within 1 standard deviation from the mean?
e) Germany, with a consumption level of 2758 litres per capita, was not included in the sample. What effect on the different statistics calculated would adding Germany have? Do not recalculate the statistics.

11 90 students on a statistics course were given an experiment where each reported, to the nearest minute, the time, x, it took them to commute to school on a specific day. The teachers then reported back that the total travelling time for the course participants was $\sum x = 4460$ minutes.

a) Find the mean number of minutes the students spent travelling to school that day.

Four students who were absent when the data was first collected reported that they spent 35, 39, 28 and 32 minutes, respectively.

b) Calculate the new mean including these four students.

12 Two thousand students at a large university take the final statistics examination, which is marked on a 100-scale, and the distribution of marks received is given in the table below.

Marks	1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	81–90	91–100
Number of candidates	30	100	200	340	520	440	180	90	60	40

a) Complete the table below so that it represents the cumulative frequency for each interval.

Marks	$\leqslant 10$	$\leqslant 20$	$\leqslant 30$	$\leqslant 40$	$\leqslant 50$	$\leqslant 60$	$\leqslant 70$	$\leqslant 80$	$\leqslant 90$	$\leqslant 100$
Number of candidates	30	130				1630				

b) Draw a cumulative frequency graph of the distribution, using a scale of 1 cm for 100 students on the vertical axis and 1 cm for 10 marks on the horizontal axis.

c) Use your graph from **b)** to answer parts **(i)**–**(iii)** below.

(i) Find an estimate for the median score.

(ii) Candidates who scored less than 35 were required to retake the examination. How many candidates had to retake the exam?

(iii) The highest-scoring 15% of candidates were awarded a distinction. Find the mark above for which a distinction was awarded.

13 At a conference of 100 mathematicians there are 72 men and 28 women. The men have a mean height of 1.79 m and the women have a mean height of 1.62 m. Find the mean height of the 100 mathematicians.

14 The mean of the population $x_1, x_2, \ldots, x_{25}$ is m. Given that $\sum_{i=1}^{25} x_i = 300$ and $\sum_{i=1}^{25} (x_i - m)^2 = 625$, find

a) the value of m

b) the standard deviation of the population.

15 The table shows the scores of competitors in a competition.

Score	10	20	30	40	50
Number of competitors with this score	1	2	5	k	3

The mean score is 34. Find the value of k.

16 A survey is carried out to find the waiting times for 100 customers at a supermarket.

Waiting time (seconds)	Number of customers
0–30	5
30–60	15
60–90	33
90–120	21
120–150	11
150–180	7
180–210	5
210–240	3

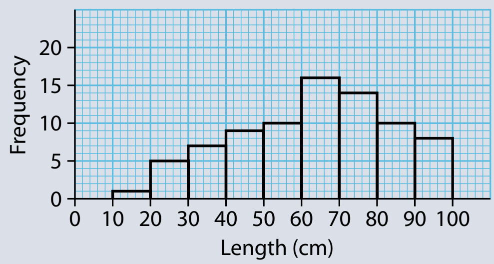
a) Calculate an estimate for the mean of the waiting times, by using an appropriate approximation to represent each interval.

b) Construct a cumulative frequency table for this data.

c) Use the cumulative frequency table to draw, on graph paper, a cumulative frequency graph, using a scale of 1 cm per 20 seconds waiting time for the horizontal axis and 1 cm per 10 customers for the vertical axis.

d) Use the cumulative frequency graph to find estimates for the median and the lower and upper quartiles.

17 The following diagram represents the lengths, in cm, of 80 plants grown in a laboratory.



a) How many plants have lengths in cm between

(i) 50 and 60?

(ii) 70 and 90?

b) Calculate estimates for the mean and the standard deviation of the lengths of the plants.

c) Explain what feature of the diagram suggests that the median is different from the mean.

d) The following is an extract from the cumulative frequency table.

Length in cm less than	Cumulative frequency
.	.
50	22
60	32
70	48
80	62
.	.

Use the information in the table to estimate the median. Give your answer to 2 significant figures.

18 The table below represents the weights, W, in grams, of 80 packets of roasted peanuts.

Weight (W)	$80 < W \leqslant 85$	$85 < W \leqslant 90$	$90 < W \leqslant 95$	$95 < W \leqslant 100$	$100 < W \leqslant 105$	$105 < W \leqslant 110$	$110 < W \leqslant 115$
Number of packets	5	10	15	26	13	7	4

a) Use the midpoint of each interval to find an estimate for the standard deviation of the weights.

b) Copy and complete the following cumulative frequency table for the above data.

Weight (W)	$W \leqslant 85$	$W \leqslant 90$	$W \leqslant 95$	$W \leqslant 100$	$W \leqslant 105$	$W \leqslant 110$	$W \leqslant 115$
Number of packets	5	15					80

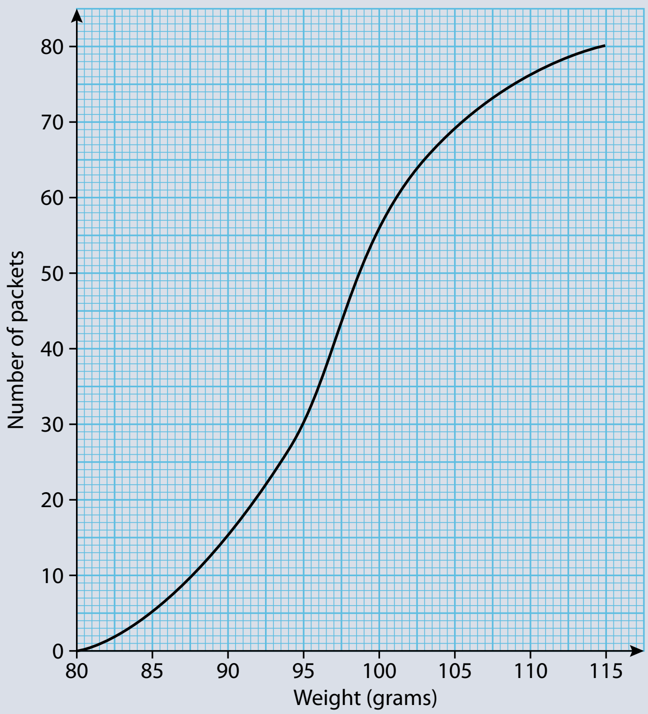
c) A cumulative frequency graph of the distribution is shown below, with a scale of 2 cm for 10 packets on the vertical axis and 2 cm for 5 grams on the horizontal axis.

Use the graph to estimate

(i) the median

(ii) the upper quartile (that is, the third quartile).

Give your answers to the nearest gram.



d) Let $W_1, W_2, \ldots, W_{80}$ be the individual weights of the packets, and let $\overline{W}$ be their mean. What is the value of the sum

$$(W_1 - \overline{W}) + (W_2 - \overline{W}) + (W_3 - \overline{W}) + \ldots + (W_{79} - \overline{W}) + (W_{80} - \overline{W})?$$

e) One of the 80 packets is selected at random. Given that its weight satisfies $85 < W \leqslant 110$, find the probability that its weight is greater than 100 grams.

19 The speeds, in km h^{-1}, of cars passing a point on a highway are recorded in the following table.

Speed v	Number of cars
$v \leqslant 60$	0
$60 < v \leqslant 70$	7
$70 < v \leqslant 80$	25
$80 < v \leqslant 90$	63
$90 < v \leqslant 100$	70
$100 < v \leqslant 110$	71
$110 < v \leqslant 120$	39
$120 < v \leqslant 130$	20
$130 < v \leqslant 140$	5
$v > 140$	0

a) Calculate an estimate of the mean speed of the cars.

b) The following table gives some of the cumulative frequencies for the information above.

Speed v	Cumulative frequency
$v \leqslant 60$	0
$v \leqslant 70$	7
$v \leqslant 80$	32
$v \leqslant 90$	95
$v \leqslant 100$	a
$v \leqslant 110$	236
$v \leqslant 120$	b
$v \leqslant 130$	295
$v \leqslant 140$	300

(i) Write down the values of a and b.

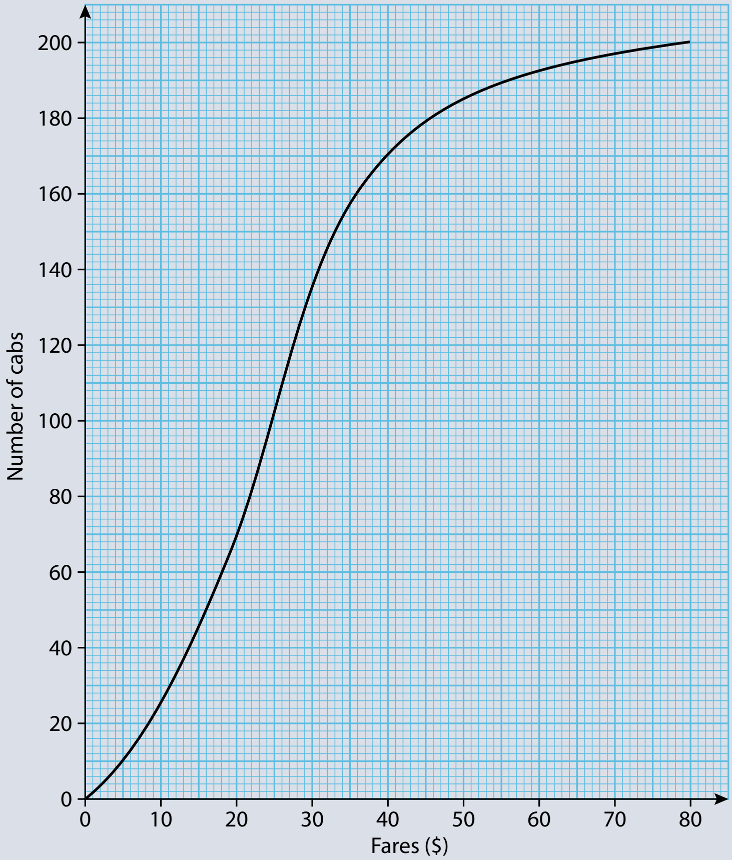
(ii) On graph paper, construct a cumulative frequency **curve** to represent this information. Use a scale of 1 cm for 10 km h^{-1} on the horizontal axis and a scale of 1 cm for 20 cars on the vertical axis.

c) Use your graph to determine

(i) the percentage of cars travelling at a speed in excess of 105 km h^{-1}

(ii) the speed which is exceeded by 15% of the cars.

20 A taxi company has 200 taxi cabs. The cumulative frequency curve below shows the fares in dollars ($) taken by the cabs on a particular morning.



a) Use the curve to estimate

(i) the median fare

(ii) the number of cabs in which the fare taken is $35 or less.

The company charges 55 cents per kilometre for distance travelled. There are no other charges. Use the curve to answer the following.

b) On that morning, 40% of the cabs travel less than a km. Find the value of a.

c) What percentage of the cabs travel more than 90 km on that morning?

21 Three positive integers a, b and c, where $a < b < c$, are such that their median is 11, their mean is 9 and their range is 10. Find the value of a.

22 In a suburb of a large city, 100 houses were sold in a three-month period. The following **cumulative frequency table** shows the distribution of selling prices (in thousands of dollars).

Selling price P ($ thousand)	$P \leqslant 100$	$P \leqslant 200$	$P \leqslant 300$	$P \leqslant 400$	$P \leqslant 500$
Total number of houses	12	58	87	94	100

a) Represent this information on a cumulative frequency **curve**, using a scale of 1 cm to represent $50 000 on the horizontal axis and 1 cm to represent 5 houses on the vertical axis.

b) Use your curve to find the interquartile range.

The price information is represented in the following frequency distribution.

Selling price P (\$ thousand)	Total number of houses
$0 < P \leqslant 100$	12
$100 < P \leqslant 200$	46
$200 < P \leqslant 300$	29
$300 < P \leqslant 400$	a
$400 < P \leqslant 500$	b

c) Find the values of a and b.

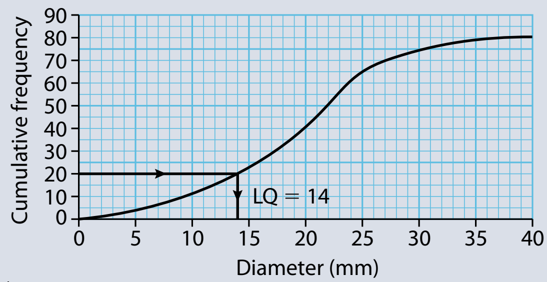
d) Use mid-interval values to calculate an estimate for the mean selling price.

e) Houses which sell for more than \$350 000 are described as *De Luxe*.

(i) Use your graph to estimate the number of *De Luxe* houses sold. Give your answer to the nearest integer.

(ii) Two *De Luxe* houses are selected at random. Find the probability that **both** have a selling price of more than \$400 000.

23 A student measured the diameters of 80 snail shells. His results are shown in the following cumulative frequency graph. The lower quartile (LQ) is 14 mm and is marked clearly on the graph.



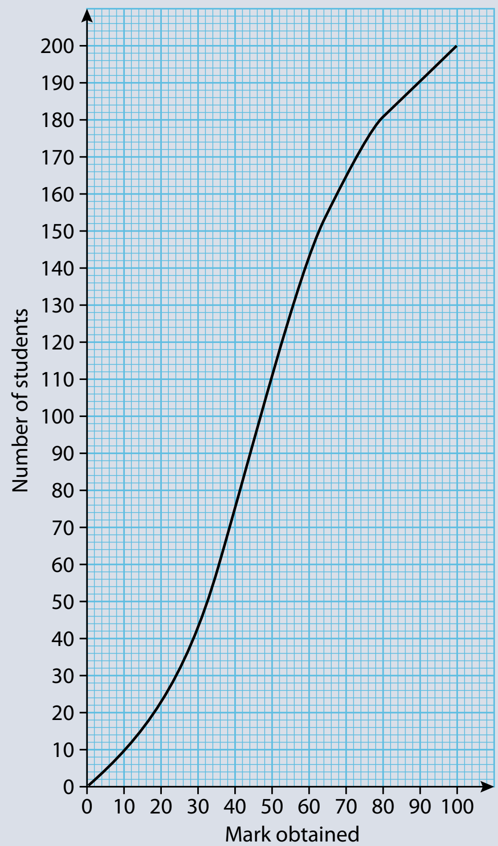
a) On the graph, mark clearly and write down the value of

(i) the median **(ii)** the upper quartile.

b) Write down the interquartile range.

24 The cumulative frequency curve right shows the marks obtained in an examination by a group of 200 students.

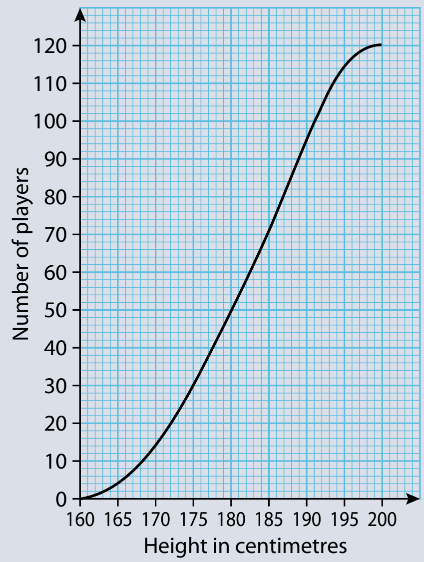
a) Use the cumulative frequency curve to complete the frequency table on the next page.



Mark (x)	$0 \leqslant x < 20$	$20 \leqslant x < 40$	$40 \leqslant x < 60$	$60 \leqslant x < 80$	$80 \leqslant x < 100$
Number of students	22				20

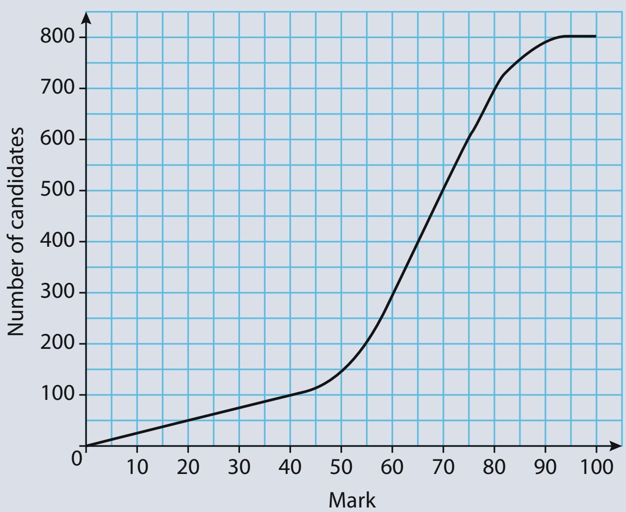
b) Forty per cent of the students fail. Find the pass mark.

25 The cumulative frequency curve right shows the heights (in centimetres) of 120 basketball players.
Use the curve to estimate
a) the median height
b) the interquartile range.



26 Let a, b, c and d be integers such that $a < b$, $b < c$ and $c = d$.
The mode of these four numbers is 11.
The range of these four numbers is 8.
The mean of these four numbers is 8.
Calculate the value of each of the integers a, b, c and d.

27 A test, to be marked out of 100, is completed by 800 students. The cumulative frequency graph for the marks is given below.



a) Write down the number of students who scored 40 marks or less on the test.
b) The middle 50% of test results lie between marks a and b, where $a < b$. Find a and b.

28 x and y are integers with $x < y$. The set of numbers $\{x, y, 10, 12, 16, 16, 18, 18\}$ have a mean of 13 and a variance σ^2 of 21. Find x and y.

12 Probability

Assessment statements

5.2 Concepts of trial, outcome, equally likely outcomes, sample space (U) and event.
The probability of an event A as $P(A) = n(A)/n(U)$.
The complementary events as A and A' (not A);
$P(A) + P(A') = 1$.
Use of Venn diagrams, tree diagrams and tables of outcomes to solve problems.

5.3 Combined events, the formula: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
$P(A \cap B) = 0$ for mutually exclusive events.

5.4 Conditional probability; the definition: $P(A|B) = P(A \cap B)/P(B)$.
Independent events; the definition: $P(A|B) = P(A) = P(A|B')$.
Use of Bayes' theorem for a maximum of three events.

Introduction

Now that you have learned to describe a data set in Chapter 11, how can you use sample data to draw conclusions about the populations from which you drew your samples?

The techniques we use in drawing conclusions are part of what we call **inferential statistics**, which is a part of one of the HL options. Inferential statistics uses **probability** as one of its tools. To use this tool properly, you must first understand how it works. This chapter will introduce you to the language and basic tools of probability.

The variables we discussed in Chapter 11 can now be redefined as **random variables**, whose values depend on the chance selection of the elements in the sample. Using probability as a tool, you will be able to create **probability distributions** that serve as models for random variables. You can then describe these using a mean and a standard deviation as you did in Chapter 11.

12.1 Randomness

Probability is the study of randomness and uncertainty.

The reasoning in statistics rests on asking, 'How often would this method give a correct answer if I used it very many times?' When we produce data by random sampling or by experiments, the laws of probability enable us to answer the question, 'What would happen if we did this many times?'

What does 'random' mean? In ordinary speech, we use 'random' to denote things that are unpredictable. Events that are **random** are not perfectly predictable, but *they have long-term regularities* that we can describe and quantify using probability. In contrast, **haphazard** events *do not necessarily have long-term regularities*. Take, for example, the tossing of an unbiased coin and observing the number of heads that appear. This is random behaviour.

Please distinguish between random and haphazard (chaos). At first glance they might seem to be the same because neither of their outcomes can be anticipated with certainty.

When you throw the coin, there are only two outcomes, heads or tails. Figure 12.1 shows the results of the first 50 tosses of an experiment that tossed the coin 5000 times. Two sets of trials are shown. The red graph shows the result of the first trial: the first toss was a head followed by a tail, making the proportion of heads to be 0.5. The third toss was also a tail, so the proportion of heads is 0.33, then 0.25. On the other hand, the other set of trials, shown in green, starts with a series of tails, then a head, which raises the proportion to 0.2, etc.

The proportion of heads is quite variable at first. However, in the long run, and as the number of tosses increases, the proportion of heads stabilizes around 0.5. We say that 0.5 is the **probability** of a head.

Figure 12.1

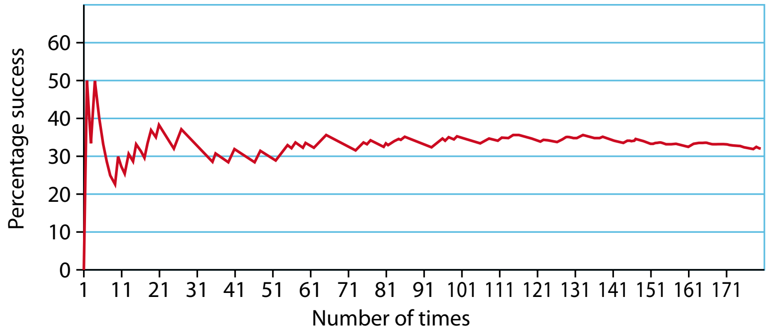
It is important that you know that the proportion of heads in a small number of tosses can be far from the probability. Probability describes only what happens in the long run. How a fair coin lands when it is tossed is an example of a random event. One cannot predict perfectly whether the coin will land heads or tails. However, in repeated tosses, the fraction of times the coin lands heads will tend to settle down to a limit of 50%. The outcome of an individual toss is not perfectly predictable, but the long-term average behaviour is predictable. Thus, it is reasonable to consider the outcome of tossing a fair coin to be random.

Imagine the following scenario:

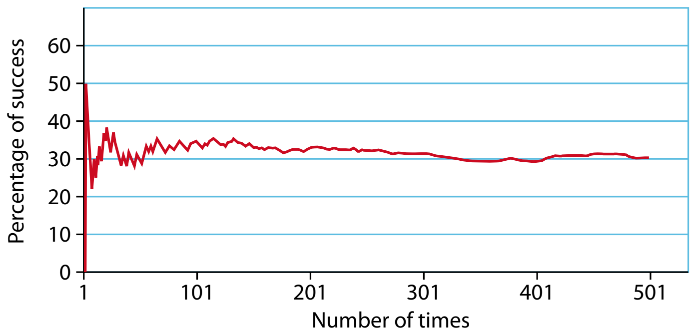
I drive every day to school. Shortly before school, there is a traffic light. It appears that it is always red when I get there. I collected data over the course of one year (180 school days) and considered the green light to be a 'success'. Here is a partial table of the collected data.

Day	1	2	3	4	5	6	7	...
Light	red	green	red	green	red	red	red	...
Percentage green	0	50	33.3	50	40	33.3	28.6	...

The first day it was red, so the proportion of success is 0% (0 out of 1); the second day it was green, so the frequency is now 50% (1 out of 2); the third day it was red again, so 33.3% (1 out of 3), and so on. As we collect more data, the new measurement becomes a smaller and smaller fraction of the accumulated frequency, so, in the long run, the graph settles to the real chance of finding it green, which in this case is about 30%. The graph is shown below.



Actually, if you run a simulation for a longer period, you can see that it really stabilizes around 30%. See graph below.



You have to observe here that the randomness in the experiment is not in the traffic light itself, as it is controlled by a timer. In fact, if the system works well, it may turn green at the same time every day. The randomness of the event is the time I arrive at the traffic light.

The French Count Buffon (1707–1788) tossed a coin 4040 times and received 2048 heads, i.e. a proportion of 50.69%. Also, the English statistician Karl Pearson (1857–1936) tossed a coin 24 000 times and received 12 012 heads, a 50.05% proportion for heads.

Count Buffon

If we ask for the probability of finding the traffic light green in the above example, our answer will be about 30%. We base our answer on knowing that, in the long run, the fraction of time that the traffic light was green is 30%. We could also say that the **long-run relative frequency** of the green light settles down to about 30%.

12.2 Basic definitions

Data is obtained by observing either uncontrolled events in nature or controlled situations in a laboratory. We use the term **experiment** to describe either method of data collection.

An **experiment** is the process by which an observation (or measurement) is obtained. A **random** (chance) **experiment** is an experiment where there is uncertainty concerning which of two or more possible outcomes will result.

Tossing a coin, rolling a die and observing the number on the top surface, counting cars at a traffic light when it turns green, measuring daily rainfall in a certain area, etc. are a few experiments in this sense of the word.

A description of a random phenomenon in the language of mathematics is called a **probability model**. For example, when we toss a coin, we cannot know the outcome in advance. What *do* we know? We are willing to say that the outcome will be either heads or tails. Because the coin appears to be balanced, we believe that each of these outcomes has probability 0.50. This description of coin tossing has two parts:

- A list of possible outcomes.
- A probability for each outcome.

This two-part description is the starting point for a probability model. We will begin by describing the outcomes of a random phenomenon and learn how to assign probabilities to the outcomes by using one of the definitions of probability.

The **sample space S** of a random experiment (or phenomenon) is the set of all possible outcomes.

The notation for sample space could also be **U** (IB notation) or any other letter.

For example, for one toss of a coin, the sample space is

$$S = \{\text{heads, tails}\}, \text{ or simply } \{h, t\}$$

Example 1

Toss a coin twice (or two coins once) and record the results. What is the sample space?

Solution

$$S = \{hh, ht, th, tt\}$$

Example 2

Toss a coin twice (or two coins once) and count the number of heads showing. What is the sample space?

Solution

$S = \{0, 1, 2\}$

A **simple event** is the outcome we observe in a single repetition (trial) of the experiment.

For example, an experiment is throwing a die and observing the number that appears on the top face. The simple events in this experiment are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$ and $\{6\}$. Of course, the set of all these simple events is the sample space of the experiment.

Set theory provides a foundation for all of mathematics. The language of probability is much the same as the language of set theory. Logical statements can be interpreted as statements about sets. This will enable us later to introduce a method of understanding how to set up probability problems that we need to tackle.

We are now ready to define an **event**. There are several ways of looking at it, which in essence are all the same.

An **event** is an **outcome** or a **set of outcomes** of a random experiment.

With this understanding, we can also look at the event as a subset of the sample space or as a collection of simple events.

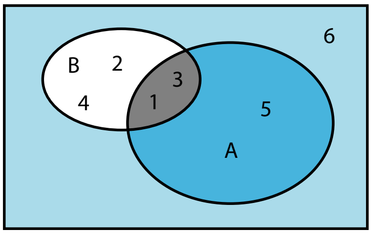
Example 3

When rolling a standard six-sided die, what are the sets of event A 'observe an odd number', and event B 'observe a number less than 5'.

Solution

Event A is the set $\{1, 3, 5\}$. Event B is the set $\{1, 2, 3, 4\}$.

Sometimes it helps to visualize an experiment using some tools of set theory. Basically, there are several similarities between the ideas of set theory and probability, and it is very helpful when we see the connection. A simple but powerful diagram is the **Venn diagram**. The diagram shows the outcomes of the die rolling experiment.



In general, in this book, we will use a rectangle to represent the sample space and closed curves to represent events, as shown in Example 3.

To understand the definitions more clearly, let's look at the following additional example.

Example 4

Suppose we choose one card at random from a deck of 52 playing cards, what is the sample space S?

Solution

S = {A♣, 2♣, … K♣, A♦, 2♦, … K♦, A♥, 2♥, … K♥, A♠, 2♠, … K♠}

Some events of interest:

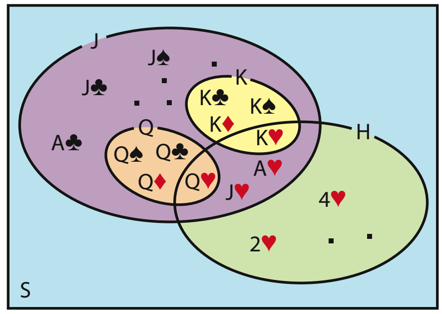
K = event of king = {K♣, K♦, K♥, K♠}

H = event of heart = {A♥, 2♥, … K♥}

J = event of jack or better

= {J♣, J♦, J♥, J♠, Q♣, Q♦, Q♥, Q♠, K♣, K♦, K♥, K♠, A♣, A♦, A♥, A♠}

Q = event of queen = {Q♣, Q♦, Q♥, Q♠}



Some useful set theory results

Set operations have a number of properties, which are basic consequences of the definitions.

Some examples are:

$$\boxed{\begin{array}{l} A \cup B = B \cup A \\ (A')' = A \\ A \cap S = A \\ A \cup S = S \\ A \cap A' = \varnothing \\ A \cup A' = S \end{array}}$$

S is the sample space and $\varnothing$ is the empty set.

Two mainly valuable properties are known as De Morgan's laws, which state that:

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

And finally $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

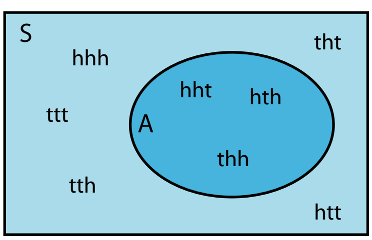
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example 5

Toss a coin three times and record the results. Show the event 'observing two heads' as a Venn diagram.

Solution

The sample space is made up of 8 possible outcomes such as hhh, hht, tht, etc.

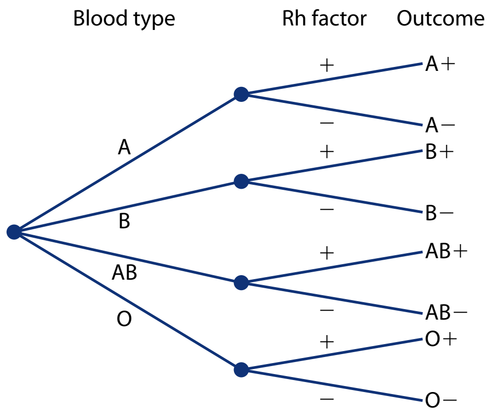


Observing exactly two heads is an event with three elements: {hht, hth, thh}.

Tree diagrams, tables and grids

In an experiment to check the blood types of patients, the experiment can be thought of as a two-stage experiment: first we identify the type of the blood and then we classify the Rh factor + or −.

The simple events in this experiment can be counted using another tool, the **tree diagram**, which is extremely powerful and helpful in solving probability problems.

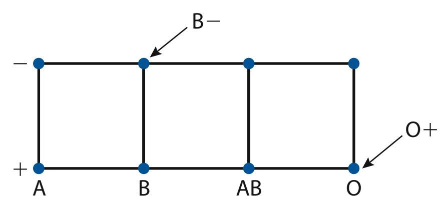


Our sample space in this experiment is the set {A+, A−, B+, B−, AB+, AB−, O+, O−} as we can read from the last column.

This data can also be arranged in a **probability table**:

	Blood type			
Rh factor	A	B	AB	O
Positive	A+	B+	AB+	O+
Negative	A−	B−	AB−	O−

Or using a 2-dimensional grid as shown right:



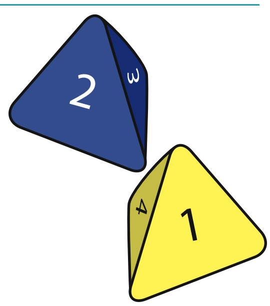
Example 6

Two tetrahedral dice, one blue and one yellow, are rolled. List the elements of the following events:

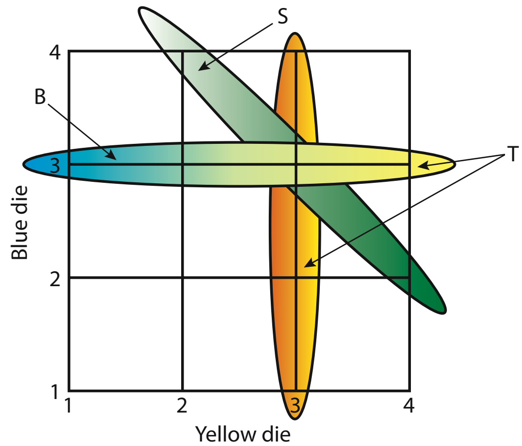
T = {3 appears on at least one die}

B = {the blue die is a 3}

S = {sum of the dice is a six}



Solution



$T = \{(1, 3), (2, 3), (3, 3), (4, 3), (3, 4), (3, 2), (3, 1)\}$
$B = \{(1, 3), (2, 3), (3, 3), (4, 3)\}$
$S = \{(2, 4), (3, 3), (4, 2)\}$

Exercise 12.1 and 12.2

1 In a large school, a student is selected at random. Give a reasonable sample space for answers to each of the following questions:
- a) Are you left-handed or right-handed?
- b) What is your height in centimetres?
- c) How many minutes did you study last night?

2 We throw a coin and a standard six-sided die and we record the number and the face that appear in that order. For example, (5, h) represents a 5 on the die and a head on the coin. Find the sample space.

3 We draw cards from a deck of 52 playing cards.
- a) List the sample space if we draw one card at a time.
- b) List the sample space if we draw two cards at a time.
- c) How many outcomes do you have in each of the experiments above?

4 Tim carried out an experiment where he tossed 20 coins together and observed the number of heads showing. He repeated this experiment 10 times and got the following results:

11, 9, 10, 8, 13, 9, 6, 7, 10, 11

- a) Use Tim's data to get the probability of obtaining a head.
- b) He tossed the 20 coins for the 11th time. How many heads should he expect to get?
- c) He tossed the coins 1000 times. How many heads should he expect to see?

5 In the game 'Dungeons and Dragons', a four-sided die with sides marked with 1, 2, 3 and 4 spots is used. The intelligence of the player is determined by rolling the die twice and adding 1 to the sum of the spots.
- a) What is the sample space for rolling the die twice? (Record the spots on the 1st and 2nd throws.)
- b) What is the sample space for the intelligence of the player?

6 A box contains three balls, blue, green and yellow. You run an experiment where you draw a ball, look at its colour and then replace it and draw a second ball.

a) What is the sample space of this experiment?

b) What is the event of drawing yellow first?

c) What is the event of drawing the same colour twice?

7 Repeat the same exercise as in question 6 above, without replacing the first ball.

8 Nick flips a coin three times and each time he notes whether it is heads or tails.

a) What is the sample space of this experiment?

b) What is the event that heads occur more often than tails?

9 Franz lives in Vienna. He and his family decided that their next vacation will be to either Italy or Hungary. If they go to Italy, they can fly, drive or take the train. If they go to Hungary, they will drive or take a boat. Letting the outcome of the experiment be the location of their vacation and their mode of travel, list all the points in the sample space. Also list the sample space of the event 'fly to destination.'

10 A hospital codes patients according to whether they have health insurance or no insurance, and according to their condition. The condition of the patient is rated as good (g), fair (f), serious (s), or critical (c). The clerk at the front desk marks 0, for non-insured patients, and 1 for insured, and uses one of the letters for the condition. So, (1, c) means an insured patient with critical condition.

a) List the sample space of this experiment.

b) What is the event 'not insured, in serious or critical condition'?

c) What is the event 'patient in good or fair condition'?

d) What is the event 'patient has insurance'?

11 A social study investigates people for different characteristics. One part of the study classifies people according to gender (G_1 = female, G_2 = male), drinking habits (K_1 = abstain, K_2 = drinks occasionally, K_3 = drinks frequently), and marital status (M_1 = married, M_2 = single, M_3 = divorced, M_4 = widowed).

a) List the elements of an appropriate sample space for observing a person in this study.

b) Define the following events:
A = the person is a male, B = the person drinks, and C = the person is single
List the elements of each A, B and C.

c) Interpret the following events in the context of this situation:

$$A \cup B;\ A \cap C;\ C';\ A \cap B \cap C;\ A' \cap B.$$

12 Cars leaving the highway can take a right turn (R), left turn (L), or go straight (S). You are collecting data on traffic patterns at this intersection and you group your observations by taking four cars at a time every 5 minutes.

a) List a few outcomes in your sample space U. How many are there?

b) List the outcomes in the event that all cars go in the same direction.

c) List the outcomes that only two cars turn right.

d) List the outcomes that only two cars go in the same direction.

13 You are collecting data on traffic at an intersection for cars leaving a highway. Your task is to collect information about the size of the vehicle: truck (*T*), bus (*B*), car (*C*). You also have to record whether the driver has the safety belt on (*SY*) or no safety belt (*SN*), as well as whether the headlights are on (*O*) or off (*F*).

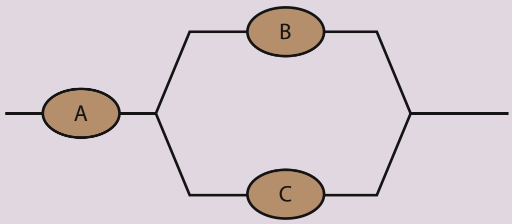
a) List the outcomes of your sample space, U.

b) List the outcomes of the event *SY* that the driver has the safety belt on.

c) List the outcomes of the event C that the vehicle you are recording is a car.

d) List the outcomes of the event in $C \cap SY$, C', and $C \cup SY$.

14 Many electric systems use a built in 'back-up' system so that the equipment using the system will work even if some parts fail. Such a system is given in the diagram below.



Two parts of this system are installed 'in parallel', so that the system will work if at least one of them works. If we code a working system by 1 and a failing system by 0, then one of the outcomes would be (1, 0, 1), which means parts A and C work while B failed.

a) List the outcomes of your sample space, U.

b) List the outcomes of the event X that exactly 2 of the parts work.

c) List the outcomes of the event Y that at least 2 of the parts work?

d) List the outcomes of the event Z that the system functions.

e) List the outcomes of the events: Z', $X \cup Z$, $X \cap Z$, $Y \cup Z$, and $Y \cap Z$.

15 Your school library has 5 copies of George Polya's *How To Solve It* book. Copies 1 and 2 are first-edition, and copies 3, 4 and 5 are second edition. You are searching for a first-edition book, and you will stop when you find a copy. For example, if you find copy 2 immediately, then the outcome is 2. Outcome 542 represents the outcome that a first edition was found on the third attempt.

a) List the outcomes of your sample space, U.

b) List the outcomes of the event A that two books must be searched.

c) List the outcomes of the event B that at least two books must be searched.

d) List the outcomes of the event C that copy 1 is found.

12.3 Probability assignments

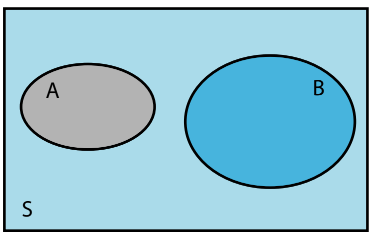
There are a few theories of probability that assign meaning to statements like 'the probability that A occurs is p%'. In this book, we will primarily examine only the **relative frequency theory**. In essence, we will follow the idea that probability is 'the long-run proportion of repetitions on which an event occurs'. This allows us to 'merge' two concepts into one.

- Equally likely outcomes
  In the theory of equally likely outcomes, probability has to do with symmetries and the indistinguishability of outcomes. If a given experiment or trial has n possible outcomes among which there is no preference, they are equally likely. The probability of each outcome is then $\frac{100\%}{n}$ or $\frac{1}{n}$. For example, if a coin is balanced well, there is no reason for it to land heads in preference to tails when it is tossed, so,

accordingly, the probability that the coin lands heads is equal to the probability that it lands tails, and both are $\frac{100\%}{2} = 50\%$. Similarly, if a die is fair, the chance that when it is rolled it lands with the side with 1 on top is the same as the chance that it shows 2, 3, 4, 5 or 6: $\frac{100\%}{6}$ or $\frac{1}{6}$.
In the theory of equally likely outcomes, probabilities are between 0% and 100%. If an event consists of more than one possible outcome, the chance of the event is the number of ways it can occur divided by the total number of things that could occur. For example, the chance that a die lands showing an even number on top is the number of ways it could land showing an even number (2, 4 or 6) divided by the total number of things that could occur (6, namely showing 1, 2, 3, 4, 5 or 6).

- Frequency theory
  In the frequency theory, probability is the limit of the relative frequency with which an event occurs in repeated trials. Relative frequencies are always between 0% and 100%. According to the frequency theory of probability, 'the probability that A occurs is p%' means that if you repeat the experiment over and over again, independently and under essentially identical conditions, the percentage of the time that A occurs will converge to p. For example, to say that the chance a coin lands heads is 50% means that if you toss the coin over and over again, independently, the ratio of the number of times the coin lands heads to the total number of tosses approaches a limiting value of 50%, as the number of tosses grows. Because the ratio of heads to tosses is always between 0% and 100%, when the probability exists it must be between 0% and 100%.

In all theories, probability is on a scale of 0% to 100%. 'Probability' and 'chance' are synonymous.



Using Venn diagrams and the 'equally likely' concept, we can say that the probability of any event is the number of elements in an event A divided by the total number of elements in the sample space **S**. This is equivalent to saying: $P(A) = \frac{n(A)}{n(S)}$, where $n(A)$ represents the number of outcomes in A and $n(S)$ represents the total number of outcomes. So, in Example 5, the probability of observing exactly two heads is: $P(2 \text{ heads}) = \frac{3}{8}$.

Probability rules

Regardless of which theory we subscribe to, the probability rules apply.

Rule 1

Any probability is a number between 0 and 1, i.e. the probability $P(A)$ of any event A satisfies $0 \leqslant P(A) \leqslant 1$. If the probability of any event is 0, the event *never* occurs. Likewise, if the probability is 1, it *always* occurs. In rolling a standard die, it is impossible to get the number 9, so $P(9) = 0$. Also, the probability of observing any integer between 1 and 6, inclusive, is 1.

No matter how little a chance you think an event has, there is no such thing as negative probability.

Rule 2

All possible outcomes together must have a probability of 1, i.e. the probability of the sample space **S** is 1: $P(S) = 1$. Informally, this is sometimes called the 'something has to happen rule'.

No matter how large a chance you think an event has, there is no such thing as a probability larger than 1!

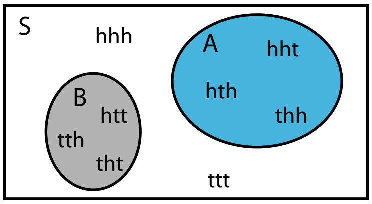
Rule 3

If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities. Two events that have no outcomes in common, and hence can never occur together, are called **disjoint** events or **mutually exclusive** events.

$$P(A \text{ or } B) = P(A) + P(B)$$

This is the **addition rule for mutually exclusive events.**

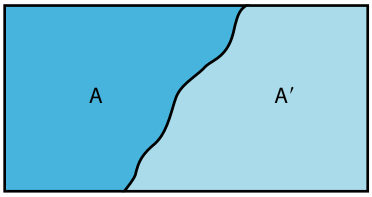
For example, in tossing three coins, the events of getting exactly two heads or exactly two tails are disjoint, and hence the probability of getting exactly two heads or two tails is $\frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$.



Additionally, we can always add the probabilities of **outcomes** because they are always disjoint. A trial cannot come out in two different ways at the same time. This will give you a way to check whether the probabilities you assigned are *legitimate*.

Rule 4

Suppose that the probability that you receive a 7 on your IB exam is 0.2, then the probability of *not* receiving a 7 on the exam is 0.8. The event that contains the outcomes **not in *A*** is called the **complement** of *A*, and is denoted by ***A*′**.



$P(A') = 1 - P(A)$, or $P(A) = 1 - P(A')$.

You have to be careful with these rules. By the 'something has to happen' rule, the total of the probabilities of all possible outcomes **must be 1**. This is so because they are disjoint, and their sum covers all the elements of the sample space. Suppose someone reports the following probabilities for students in your high school (4 years). If the probability that a grade 1, 2, 3 or 4 student is chosen at random from the high school is 0.24, 0.24, 0.25 and 0.19 respectively, with no other possibilities, you should know immediately that there is something wrong. These probabilities add up to 0.92. Similarly, if someone claims that these probabilities are 0.24, 0.28, 0.25, 0.26 respectively, there is also something wrong. These probabilities add up to 1.03, which is more than 1.

Example 7

Data for traffic violations was collected in a certain country and a summary is given below:

Age group	18–20 years	21–29 years	30–39 years	Over 40 years
Probability	0.06	0.47	0.29	0.18

What is the probability that the offender is a) in the youngest age group, b) between 21 and 40, and c) younger than 40?

Solution

Each probability is between 0 and 1, and the probabilities add up to 1. Therefore, this is a legitimate assignment of probabilities.

a) The probability that the offender is in the youngest group is 6%.

b) The probability that the driver is in the group 21 to 39 years is $0.47 + 0.29 = 0.76$.

c) The probability that a driver is younger than 40 years is $1 - 0.18 = 0.82$.

Example 8

It is a striking fact that when people create codes for their cellphones, the first digits follow distributions very similar to the following one:

First digit	0	1	2	3	4	5	6	7	8	9
Probability	0.009	0.300	0.174	0.122	0.096	0.078	0.067	0.058	0.051	0.045

a) Find the probabilities of the following three events:

$A = \{\text{first digit is } 1\}$

$B = \{\text{first digit is more than } 5\}$

$C = \{\text{first digit is an odd number}\}$

b) Find the probability that the first digit is (i)1 or greater than 5, (ii) not 1, and (iii) an odd number or a number larger than 5.

Solution

a) From the table:

$$P(A) = 0.300$$

$$\begin{aligned} P(B) &= P(6) + P(7) + P(8) + P(9) \\ &= 0.067 + 0.058 + 0.051 + 0.045 \\ &= 0.221 \end{aligned}$$

$$\begin{aligned} P(C) &= P(1) + P(3) + P(5) + P(7) + P(9) \\ &= 0.300 + 0.122 + 0.078 + 0.058 + 0.045 \\ &= 0.603 \end{aligned}$$

b) (i) Since A and B are mutually exclusive, by the addition rule, the probability that the first digit is 1 or greater than 5 is

$$P(A \text{ or } B) = 0.300 + 0.\,221 = 0.521.$$

(ii) Using the complement rule, the probability that the first digit is not 1 is

$$P(A') = 1 - P(A) = 1 - 0.300 = 0.700.$$

(iii) The probability that the first digit is an odd number or a number larger than 5:

$$\begin{aligned} P(B \text{ or } C) &= P(1) + P(3) + P(5) + P(6) + P(7) + P(8) + P(9) \\ &= 0.300 + 0.122 + 0.078 + 0.067 + 0.058 + 0.051 \\ &\quad + 0.045 \\ &= 0.721 \end{aligned}$$

● **Hint:** Notice here that P(B or C) is *not* the sum of P(B) and P(C) because B and C are not disjoint.

Equally likely outcomes

In some cases we are able to assume that individual outcomes are equally likely because of some balance in the experiment. Tossing a balanced coin renders heads or tails equally likely, with each having a probability of 50%, and rolling a standard balanced die gives the numbers from 1 to 6 as equally likely, with each having a probability of $\frac{1}{6}$.

Suppose in Example 8 we consider all the digits to be equally likely to happen, then our table would be:

First digit	0	1	2	3	4	5	6	7	8	9
Probability	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

$\mathrm{P}(A) = 0.1$

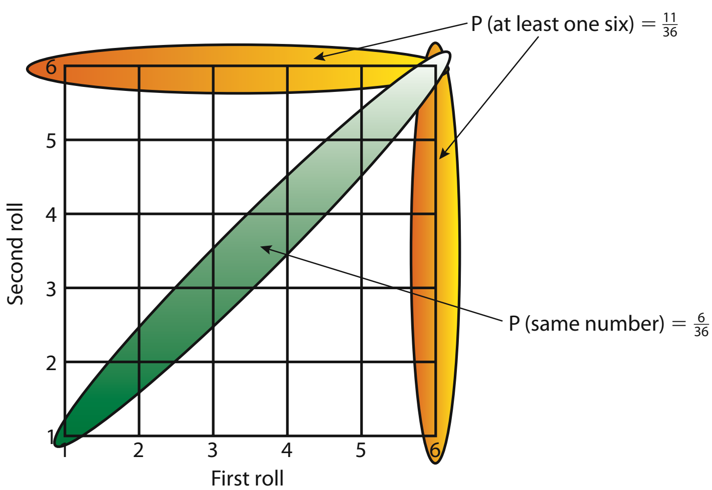
$\mathrm{P}(B) = \mathrm{P}(6) + \mathrm{P}(7) + \mathrm{P}(8) + \mathrm{P}(9) = 4 \times 0.1 = 0.4$

$\mathrm{P}(C) = \mathrm{P}(1) + \mathrm{P}(3) + \mathrm{P}(5) + \mathrm{P}(7) + \mathrm{P}(9) = 5 \times 0.1 = 0.5$

Also, by the complement rule, the probability that the first digit is not 1 is

$$\mathrm{P}(A') = 1 - \mathrm{P}(A) = 1 - 0.1 = 0.9.$$

2-dimensional grids are also very helpful tools that are used to visualize 2-stage or sequential probability models. For example, consider rolling a normal unbiased cubical die twice. Here are some events and how to use the grid in calculating their probabilities:



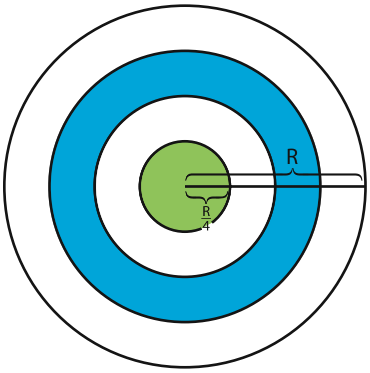
If we are interested in the probability that at least one roll shows a 6, we count the points on the column corresponding to 6 on the first roll and the points on the row corresponding to 6 on the second roll observing naturally that the point in the corner should not be counted twice.

If we are interested in the number showing on both rolls to be the same, then we count the points on the diagonal as shown.

Finally, if we are interested in the probability that the first roll shows a number larger than the second roll, then we pick the points below the diagonal.

Hence, $\mathrm{P}(\text{first number} > \text{second number}) = \frac{15}{36}$.

Geometric probability



Some cases give rise to interpreting events as areas in the plane. Take for example shooting at a circular target at random. What is the probability of hitting the central part?

The probability of hitting the central part is given by

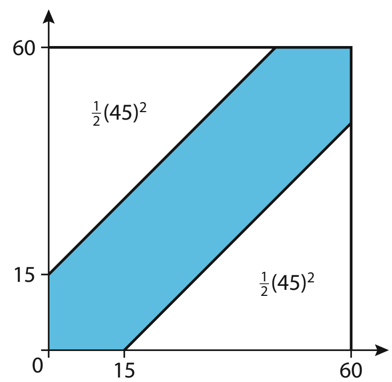
$$\mathrm{P} = \frac{\pi\left(\frac{R}{4}\right)^2}{\pi R^2} = \frac{1}{16}.$$

Example 9

Lydia and Rania agreed to meet at the 'museum quarter' between 12:00 and 13:00. The first person to arrive will wait 15 minutes. If the second person does not show up, the first person will leave and they meet afterwards. Assuming that their arrivals are at random, what is the probability that they meet?

Solution

If Lydia arrives x minutes after 12:00 and Rania arrives y minutes after 12:00, then the condition for them to meet is $|x - y| \leqslant 15$, and $x \leqslant 60$, $y \leqslant 60$.



Geometrically, the outcomes of their 'encounter' region is given in the shaded region in the diagram right.

The area for each triangle is $\frac{1}{2}bh = \frac{1}{2}(45)^2$, so, the shaded area is $60^2 - 45^2$.

The probability they meet is therefore $\dfrac{60^2 - 45^2}{60^2} = \dfrac{7}{16}$.

Probability calculation for equally likely outcomes using counting principles

In an experiment where all outcomes are equally likely, the theoretical probability of an event A is given by

$$\mathrm{P}(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ is the number of outcomes that make up the event A, and $n(S)$ is the total number of outcomes in the sample space.

The new ideas we want to discuss here involve the calculation of $n(A)$ and $n(S)$. Such calculations will involve what you learned in Chapter 4 about counting principles.

Example 10

In a group of 18 students, eight are females. What is the probability of choosing five students

a) with all girls?

b) with three girls and two boys?

c) with at least one boy?

Solution

The total number of outcomes is the number of ways we can choose 5 out of the 18 students. So

$$n(S) = \binom{18}{5} = 8586.$$

a) This event will require that we pick our group from among the 8 girls. So,

$$n(A) = \binom{8}{5} = 56 \Rightarrow \mathrm{P}(A) = \frac{56}{8568} = 0.0065.$$

b) This event will require that we pick three out of the 8 girls, and at the same time, we pick 2 out of the 10 boys. So, using the multiplication principle,

$$n(B) = \binom{8}{3} \cdot \binom{10}{2} = 56.45 = 2520 \Rightarrow \mathrm{P}(B) = \frac{2520}{8568} = 0.294.$$

Note: Did you observe that $\binom{8}{5} = \binom{8}{3}$? Why?

c) This event can be approached in two ways:

- To have at least 1 boy means that we can have 1, 2, 3, 4 or 5 boys. These are mutually exclusive, so the probability in question is the sum

$$\mathrm{P}(C) = \frac{\binom{10}{1}\binom{8}{4} + \binom{10}{2}\binom{8}{3} + \binom{10}{3}\binom{8}{2} + \binom{10}{4}\binom{8}{1} + \binom{10}{5}\binom{8}{0}}{\binom{18}{5}} = \frac{8512}{8568} = 0.9935$$

- To recognize that at least 1 boy is the complement of no boys at all, i.e. 0 boys or all 5 girls.

$$\mathrm{P}(C) = 1 - \mathrm{P}(A) = 1 - 0.0065 = 0.9935.$$

Example 11

A deck of playing cards has 52 cards. In a game, the player is given five cards. Find the probability of the player having

a) three cards of one denomination and two cards of another (three 7s and two Js for example).

This game can be played at two stages, First, the player is given five cards, and then he/she can decide to exchange some of the cards. (The cards exchanged are discarded and not returned to the deck!)

A player was given the following hand: Q♠, Q♦, Q♥, 4♣, 9♠. She decided to change the last two cards. Find the probability of the player having

b) three cards of one denomination and two cards of another

c) four queens.

Solution

a) The sample space consists of all possible 5-card hands that can be given out:

$$n(S) = \binom{52}{5} = 2\,598\,960$$

Call the event of interest A.

As there are 13 denominations in the deck of cards then there are 13 choices for the first required denomination. Once a denomination is chosen, say 9,
then there are $\binom{4}{3}$ ways of choosing 3 cards out of the four. Using the multiplication rule, there are $13 \cdot \binom{4}{3}$ ways of choosing 3 cards of the first denomination. We are now left with 12 possible denominations for the second one, each can give us $\binom{4}{2}$ ways of getting two of the cards, and hence using the multiplication rule, there are $12 \cdot \binom{4}{2}$ ways of choosing the cards for the second denomination. Again using the multiplication rule we will have $\left[13 \cdot \binom{4}{3}\right]\left[12 \cdot \binom{4}{2}\right] = 3744$ ways of choosing the first and second denominations.

The requested probability is then

$$\mathrm{P}(A) = \frac{3744}{2\,598\,960} \approx 0.001\,44.$$

b) Since we have 3 queens, then we need only look for 2 cards of a different denomination. Now, there are only 47 cards left in the deck because we had 5 already. So the sample space has $n(S) = \binom{47}{2} = 1081$ ways of getting the rest of the 5 cards. The other cards could be two 4's, two 9's or two of the rest of the 10 denominations.

We have $\binom{3}{2} = 3$ ways of getting two 4's since 4♣ is already discarded. We also have $\binom{3}{2} = 3$ ways to get two 9's. Or, for each of the other 10 denominations (no Q, no 4 and no 9), we have $\binom{4}{2} = 6$ different ways of getting two of them, i.e. we have $10 \cdot \binom{4}{2} = 60$ different ways of getting two cards of the same denomination other than Q, 4 or 9.

So, the total number of ways of getting two cards of the same denomination is $3 + 3 + 60 = 66$ ways.

So, the required probability is $\mathrm{P}(A) = \dfrac{n(A)}{n(S)} = \dfrac{66}{1081} \approx 0.0611.$

c) To have 4 Q's we only have to look for one, and there is only one way of getting the missing Q♣. That leaves us with one card to be chosen from the 46 cards left. 46 ways!

Therefore, $P(A) = \frac{46}{1081} \approx 0.0426$.

Exercise 12.3

1 In a simple experiment, chips with integers 1–20 inclusive were placed in a box and one chip was picked at random.

a) What is the probability that the number drawn is a multiple of 3?

b) What is the probability that the number drawn is not a multiple of 4?

2 The probability an event *A* happens is 0.37.

a) What is the probability that it does not happen?

b) What is the probability that it may or may not happen?

3 You are playing with an ordinary deck of 52 cards by drawing cards at random and looking at them.

a) Find the probability that the card you draw is
   (i) the ace of hearts
   (ii) the ace of hearts or any spade
   (iii) an ace or any heart
   (iv) not a face card.

b) Now you draw the ten of diamonds, put it on the table and draw a second card. What is the probability that the second card is
   (i) the ace of hearts?
   (ii) not a face card?

c) Now you draw the ten of diamonds, return it to the deck and draw a second card. What is the probability that the second card is
   (i) the ace of hearts?
   (ii) not a face card?

4 On Monday morning, my class wanted to know how many hours students spent studying on Sunday night. They stopped schoolmates at random as they arrived and asked each, 'How many hours did you study last night?' Here are the answers of the sample they chose on Monday, 14 January, 2008.

Number of hours	0	1	2	3	4	5
Number of students	4	12	8	3	2	1

a) Find the probability that a student spent less than three hours studying Sunday night.

b) Find the probability that a student studied for two or three hours.

c) Find the probability that a student studied less than six hours.

5 We throw a coin and a standard six-sided die and we record the number and the face that appear. Find

a) the probability of having a number larger than 3

b) the probability that we receive a head and a 6.

6 A die is constructed in a way that a 1 has the chance to occur twice as often as any other number.

a) Find the probability that a 5 appears.

b) Find the probability an odd number will occur.

7 You are given two fair dice to roll in an experiment.

a) Your first task is to report the numbers you observe.

(i) What is the sample space of your experiment?

(ii) What is the probability that the two numbers are the same?

(iii) What is the probability that the two numbers differ by 2?

(iv) What is the probability that the two numbers are not the same?

b) In a second stage, your task is to report the sum of the numbers that appear.

(i) What is the probability that the sum is 1?

(ii) What is the probability that the sum is 9?

(iii) What is the probability that the sum is 8?

(iv) What is the probability that the sum is 13?

8 The blood types of people can be one of four types: O, A, B or AB. The distribution of people with these types differs from one group of people to another. Here are the distributions of blood types for randomly chosen people in the US, China and Russia.

Country \ Blood type	O	A	B	AB
US	0.43	0.41	0.12	?
China	0.36	0.27	0.26	0.11
Russia	0.39	0.34	?	0.09

a) What is the probability of type AB in the US?

b) Dirk lives in the US and has type B blood. What is the probability that a randomly chosen US citizen can donate blood to Dirk? (Type B can only receive from O and B.)

c) What is the probability of randomly choosing an American and a Chinese (independently) with type O blood?

d) What is the probability of randomly choosing an American, a Chinese and a Russian (independently) with type O blood?

e) What is the probability of randomly choosing an American, a Chinese and a Russian (independently) with the same blood type?

9 In each of the following situations, state whether or not the given assignment of probabilities to individual outcomes is legitimate. Give reasons for your answer.

a) A die is loaded such that the probability of each face is according to the following assignment (x is the number of spots on the upper face and P(x) is its probability.)

x	1	2	3	4	5	6
P(x)	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	0

b) A student at your school categorized in terms of gender and whether they are diploma candidates or not.
P(female, diploma candidate) $=$ 0.57, P(female, not a diploma candidate) $=$ 0.23,
P(male, diploma candidate) $=$ 0.43, P(male, not a diploma candidate) $=$ 0.18.

c) Draw a card from a deck of 52 cards (x is the suit of the card and P(x) is its probability).

x	Hearts	Spades	Diamonds	Clubs
P(x)	$\frac{12}{52}$	$\frac{15}{52}$	$\frac{12}{52}$	$\frac{13}{52}$

10 In Switzerland, there are three 'official' mother tongues, German, French and Italian. You choose a Swiss at random and ask, 'What is your mother tongue?' Here is the distribution of responses:

Language	German	French	Italian	Other
Probability	0.58	0.24	0.12	?

a) What is the probability that a Swiss person's mother tongue is not one of the official ones?

b) What is the probability that a Swiss person's mother tongue is not German?

c) What is the probability that you choose two Swiss independent of each other and they both have German mother tongue?

d) What is the probability that you choose two Swiss independent of each other and they both have the same mother tongue?

11 The majority of email messages are now 'spam.' Choose a spam email message at random. Here is the distribution of topics:

Topic	Adult	Financial	Health	Leisure	Products	Scams
Probability	0.165	0.142	0.075	0.081	0.209	0.145

a) What is the probability of choosing a spam message that does not concern these topics?

Parents are usually concerned with spam messages with 'adult' content and scams.

b) What is the probability that a randomly chosen spam email falls into one of the other categories?

12 Consider n to be a positive integer. Let $f(x) = \dfrac{\binom{n}{x+1}}{\binom{n}{x}}$, where x is also a positive integer. Determine the values of x (in terms of n) for which $f(x) < 1$.

13 Determine n in each of the following cases:

a) $\binom{n}{2} = 190$ b) $\binom{n}{4} = \binom{n}{8}$

14 An experiment involves rolling a pair of dice, 1 white and 1 red, and recording the numbers that come up. Find the probability

a) that the sum is greater than 8

b) that a number greater than 4 appears on the white die

c) that at most a total of 5 appears.

15 Three books are picked from a shelf containing 5 novels, 3 science books and a thesaurus. What is the probability that

a) the thesaurus is selected?

b) two novels and a science book are selected?

16 Five cards are chosen at random from a deck of 52 cards. Find the probability that the set contains

a) 3 kings

b) 4 hearts and 1 diamond.

17 A class consists of 10 girls and 12 boys. A team of 6 members is to be chosen at random. What is the probability that the team contains

a) one boy?

b) more boys than girls?

18 A committee of six people is to be chosen from a group of 15 people that contains two married couples.

a) What is the probability that the committee will include both married couples?

b) What is the probability that the committee will include the three youngest members in the group?

19 The computer department at your school has received a shipment of 25 printers, of which 10 are colour laser printers and the rest are black-and-white laser models. Six printers are selected at random to be checked for defects. What is the probability that

a) exactly 3 of them are colour lasers?

b) at least 3 are colour lasers?

20 The city of Graz just bought 30 buses to put in service in their public transport network. Shortly after the first week in service, 10 buses developed cracks on their instruments panel.

a) How many ways are there to select a sample of 6 buses for a thorough inspection?

b) What is the probability that half of the chosen buses have cracks?

c) What is the probability that at least half of the chosen buses have cracks?

d) What is the probability that at most half of the chosen buses have cracks?

21 A small factory has a 24-hour production facility. They employ 30 workers on the day shift (08:00–16:00), 22 workers on the evening shift (16:00–24:00) and 15 workers on the morning shift (00:00–08:00). A quality control consultant is to select 9 workers for in-depth interviews.

a) What is the probability that all 9 come from the day shift?

b) What is the probability that all 9 come from the same shift?

c) What is the probability that at least two of the shifts are represented?

d) What is the probability that at least one of the shifts is unrepresented?

22 a) A box contains 8 chips numbered 1 to 8. Two are chosen at random and their numbers are added together. What is the probability that their sum is 7?

b) A box contains 20 chips numbered 1 to 20. Two are chosen at random. What is the probability that the numbers on the two chips differ by 3?

c) A box contains 20 chips numbered 1 to 20. Two are chosen at random. What is the probability that the numbers on the two chips differ by more than 3?

23 Tim and Val want to meet for dinner at their favourite restaurant. They reserved a table for between 20:00 and 22:00. The first person to arrive will order a salad and spend 30 minutes before ordering the main meal. If the second person does not arrive within the 30 minutes, the first person will pay the bill and leave. What is the probability that they manage to eat dinner together at the restaurant?

24 Bus 48A and Tram 49 serve different routes in the city. They share one stop next to my house. They stop at this station every 20 minutes. Every stay is 3 minutes long. Assuming their arrivals at the hour are random, find the probability that both are at the stop at 12:00 on Monday.

25 During a dinner party, Magda plans on opening six bottles of wine. Her supply includes 8 French, 10 Australian and 12 Italian wines. She sends her sister Mara to choose the bottles. Mara has no knowledge of the wine types and picks the bottles at random.

a) What is the probability that two of each type get selected?

b) What is the probability that all served bottles are of the same type?

c) What is the probability of serving only Italian and French wines?

26 A wooden cube has its faces painted green. We cut the cube into 1000 small cubes of equal size. We mix the small cubes thoroughly. If we draw one cube at random, what is the probability that the cube

a) has two faces coloured green?

b) has three coloured faces?

c) does not have a coloured face at all?

12.4 Operations with events

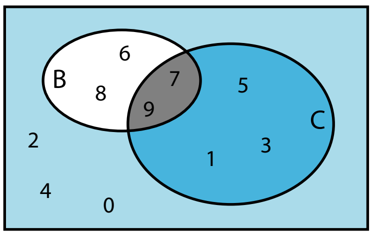
In Example 8, we talked about the following events:

$B = \{\text{first digit is more than } 5\}$

$C = \{\text{first digit is an odd number}\}$

We also claimed that these two events are not disjoint. This brings us to another concept for looking at combined events.

The **intersection** of two events *B* and *C*, denoted by the symbol $B \cap C$ or simply *BC*, is the event containing all outcomes common to *B* and *C*.



Here $B \cap C = \{7, 9\}$ because these outcomes are in both *B* and *C*. Since the intersection has outcomes common to the two events *B* and *C*, they are not mutually exclusive.

The probability of $B \cap C$ is $0.058 + 0.045 = 0.103$. Recall from Example 8 that we said that the probability of *B* or *C* is not simply the sum of the probabilities. That brings us to the next concept. How can we find the probability of *B* or *C* when they are not mutually exclusive? To answer this question, we need to define another operation.

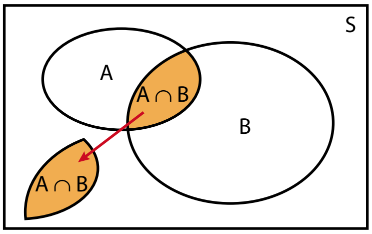
The **union** of two events *B* and *C*, denoted by the symbol $B \cup C$, is the event containing all the outcomes that belong to *B* or to *C* or to both.

Here $B \cup C = \{1, 3, 5, 6, 7, 8, 9\}$. In calculating the probability of $B \cup C$, we observe that the outcomes 7 and 9 are counted twice. To remedy the situation, if we decide to add the probabilities of *B* and *C*, we subtract one of the incidents of double counting. So, $P(B \cup C) = 0.221 + 0.603 - 0.103 = 0.721$, which is the result we received with direct calculation. In general, we can state the following probability rule:

Rule 5

For any two events *A* and *B*, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

As you see from the diagram below, $P(A \cap B)$ has been added twice, so the 'extra' one is subtracted to give the probability of $(A \cup B)$.



This general probability addition rule applies to the case of mutually exclusive events too. Consider any two events A and B. The probability of A or B is given by

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B), \text{ since } P(A \cap B) = 0. \end{aligned}$$

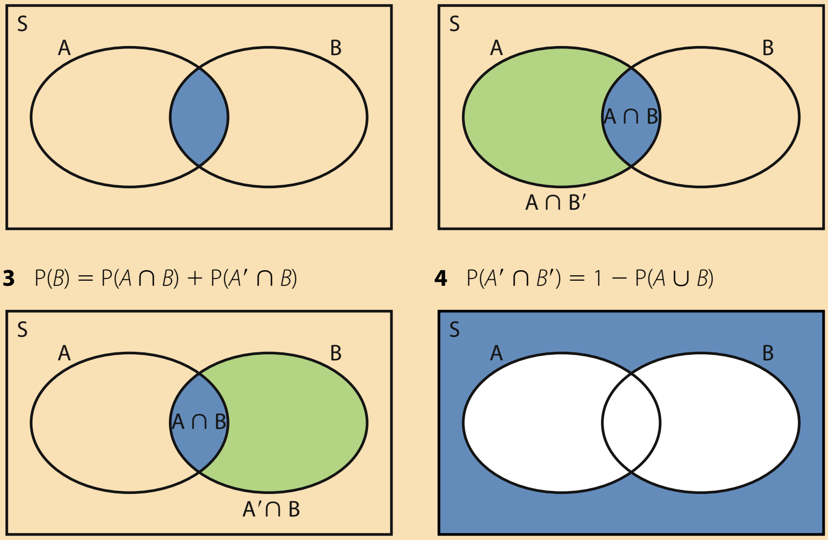
Some useful results

1 $P(A \cap B) = P(B \cap A)$

2 $P(A) = P(A \cap B) + P(A \cap B')$

3 $P(B) = P(A \cap B) + P(A' \cap B)$

4 $P(A' \cap B') = 1 - P(A \cup B)$



Rule 6

The simple multiplication rule.

Consider the following situation: In a large school, 55% of the students are male. It is also known that the percentage of smokers among males and females in this school is the same, 22%. What is the probability of selecting a student at random from this population and the student is a male smoker?

Applying common sense only, we can think of the problem in the following manner. Since the proportion of smokers is the same in both groups, smoking and gender are independent of each other in the sense that knowing that the student is a male does not influence the probability that he smokes!

The chance we pick a male student is 55%. From those 55% of the population, we know that 22% are smokers, so by simple arithmetic the chance that we select a male smoker is $0.22 \times 0.55 = 12.1\%$.

This is an example of the multiplication rule for independent events.

Two events A and B are **independent** if knowing that one of them occurs does not change the probability that the other occurs.

The **multiplication rule for independent events:** If two events A and B are independent, then $P(A \cap B) = P(A) \times P(B)$.

Example 12

Reconsider the situation with the traffic light at the beginning of this chapter. The probability that I find the light green is 30%. What is the probability that I find it green on two consecutive days?

Solution

We will assume that my arrival and finding the light green is a random event, and that if it turns green on one day it does not influence how it turns the next day. In that case our calculation is very simple:

$$\begin{aligned} P(\text{green the first and second day}) &= P(\text{green first day}) \times P(\text{green second day}) \\ &= 0.30 \times 0.30 = 0.09. \end{aligned}$$

This rule can also be extended to more than two independent events. For example, on the assumption of independence, what is the chance that I find the light green five days of the week?

$$P(\text{green on five days}) = 0.3 \times 0.3 \times 0.3 \times 0.3 \times 0.3 = 0.002\,43$$

Do not confuse independent with disjoint. 'Disjoint' means that if one of the events occurs then the other does not occur; while 'independent' means that knowing one of the events occurs does not influence the probability of whether the other occurs or not!

Example 13

Computers bought from a well-known producer require repairs quite frequently. It is estimated that 17% of computers bought from the company require one repair job during the first month of purchase, 7% will need repairs twice during the first month, and 4% require three or more repairs.

a) What is the probability that a computer chosen at random from this producer will need
   (i) no repairs?
   (ii) no more than one repair?
   (iii) some repair?

b) If you buy two such computers, what is the probability that
   (i) neither will require repair?
   (ii) both will need repair?

Solution

a) Since all of the events listed are disjoint, the addition rule can be used.

(i) P(no repairs) $= 1 - $ P(some repairs) $= 1 - (0.17 + 0.07 + 0.04)$
$= 1 - (0.28) = 0.72$

(ii) P(no more than one repair) $=$ P(no repairs or one repair)
$= 0.72 + 0.17 = 0.89$

(iii) P(some repairs) $=$ P(one or two or three or more repairs)
$= 0.17 + 0.07 + 0.04 = 0.28$

b) Since repairs on the two computers are independent from one another, the multiplication rule can be used. Use the probabilities of events from part a) in the calculations.

(i) P(neither will need repair) $= (0.72)(0.72) = 0.5184$

(ii) P(both will need repair) $= (0.28)(0.28) = 0.0784$

Conditional probability

In probability, conditioning means incorporating new restrictions on the outcome of an experiment: updating probabilities to take into account new information. This section describes conditioning, and how conditional probability can be used to solve complicated problems. Let us start with an example.

Example 14

A public health department wanted to study the smoking behaviour of high school students. They interviewed 768 students from grades 10–12 and asked them about their smoking habits. They categorized the students into three categories: smokers (more than 1 pack of 20 cigarettes per week), occasional smokers (less than 1 pack per week), and non-smokers. The results are summarized below:

	Smoker	Occasional	Non-smoker	Total
Male	127	73	214	414
Female	99	66	189	354
Total	226	139	403	768

If we select a student at random from this study, what is the probability that we select a) a girl, b) a male smoker, and c) a non-smoker?

Solution

a) P(female) $= \frac{354}{768} = 0.461$

So, 46.1% of our sample are females.

b) Since we have 127 boys categorized as smokers, the chance of a male smoker will be

P(male smoker) $= \frac{127}{768} = 0.165.$

c) P(non-smoker) $= \frac{403}{768} = 0.525$

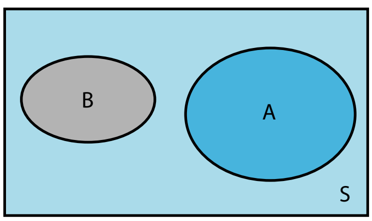
In the above example, what if we know that the selected student is a girl? Does that influence the probability that the selected student is a non-smoker? Yes, it does!

Knowing that the selected student is a female changes our choices. The 'revised' sample space is not made up of all students anymore. It is only the female students. The chance of finding a non-smoker among the females is $\frac{189}{354} = 0.534$, i.e. 53.4% of the females are non-smokers as compared to the 52.5% of non-smokers in the whole population.

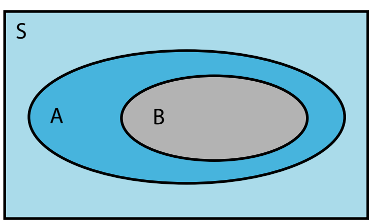
This probability is called a conditional probability, and we write this as $\text{P(non-smoker}\,|\,\text{female)} = \frac{189}{354}$.

We read this as, '*Probability of selecting a non-smoker* **given that** *we have selected a female*'.

The conditional probability of A given B, $\mathrm{P}(A|B)$, is the probability of the event A, updated on the basis of the knowledge that the event B occurred. Suppose that A is an event with probability $\mathrm{P}(A) = p \neq 0$, and that $A \cap B = \varnothing$ (A and B are disjoint). Then if we learn that B occurred we know A did not occur, so we should revise the probability of A to be zero, $\mathrm{P}(A|B) = 0$ (the conditional probability of A given B is zero).



On the other hand, suppose that $A \cap B = B$ (B is a subset of A, so B implies A). Then if we learn that B occurred we know A must have occurred as well, so we should revise the probability of A to be 100%, $\mathrm{P}(A|B) = 1$(the conditional probability of A given B is 100%).



Remember that the probability we assign to an event can change if we know that some other event has occurred. This idea is the key to understanding conditional probability.

Imagine the following scenario:
You are playing cards and your opponent is about to give you a card. What is the probability that the card you receive is a queen?

As you know, there are 52 cards in the deck, 4 of these cards are queens. So, assuming that the deck was thoroughly shuffled, the probability of receiving a queen is

$$\mathrm{P(queen)} = \frac{4}{52} = \frac{1}{13}.$$

This calculation assumes that you know nothing about any cards already dealt from the deck.

Suppose now that you are looking at the five cards you have in your hand, and one of them is a queen. You know nothing about the other 47 cards except that exactly three queens are among them. The probability of being given a queen as the next card, given what you know, is

$$\mathrm{P(queen}\,|\,1\text{ queen in hand}) = \frac{3}{47} \neq \frac{1}{13}.$$

So, knowing that there is one queen among your five cards changes the probability of the next card being a queen.

Consider Example 14 again. We want to express the table frequencies as relative frequencies or probabilities. Our table will look like this:

	Smoker	Occasional	Non-smoker
Male	0.165	0.095	0.279
Female	0.129	0.086	0.246

To find the probability of selecting a student at random and finding that student is a female non-smoker, we look at the intersection of the female row with the non-smoking column and find that this probability is 0.246.

Looking at this calculation from a different perspective, we can think about it in the following manner:

We know that the percentage of females in our sample is 46.1, and among those females, in Example 14, we found that 53.4% of those are non-smokers. So, the percentage of female non-smokers in the population is the 53.4% of those 46.1% females, i.e. $0.534 \times 0.461 = 0.246$.

In terms of events, this can be read as:

$$\begin{aligned}\mathrm{P(non\text{-}smoker}\,|\,\mathrm{female}) \times \mathrm{P(female)} &= \mathrm{P(female\ and\ non\text{-}smoker)}\\ &= \mathrm{P(female} \cap \mathrm{non\text{-}smoker)}.\end{aligned}$$

The previous discussion is an example of the **multiplication rule** of any two events A and B.

Multiplication rule

Given any events A and B, the probability that both events happen is given by

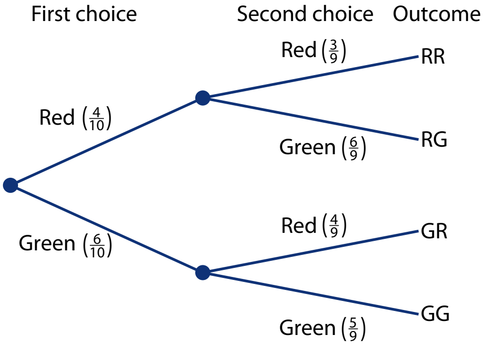
$$\mathrm{P}(A \cap B) = \mathrm{P}(A|B) \times \mathrm{P}(B).$$

Example 15

In a psychology lab, researchers are studying the colour preferences of young children. Six green toys and four red toys (identical apart from colour) are placed in a container. The child is asked to select two toys at random. What is the probability that the child chooses two red toys?

Solution

To solve this problem, we will use a tree diagram.



As you notice, every entry on the 'branches' has a conditional probability. So, Red on the second choice is actually either Red|Red or Red|Green. We are interested in RR, so the probability is

$$P(RR) = P(R) \times P(R|R) = \frac{4}{10} \times \frac{3}{9} = 13.3\%.$$

If $P(A \cap B) = P(A|B) \times P(B)$, as discussed above, and if $P(B) \neq 0$, we can rearrange the multiplication rule to produce a definition of the conditional probability $P(A|B)$ in terms of the 'unconditional' probabilities $P(A \cap B)$ and $P(B)$.

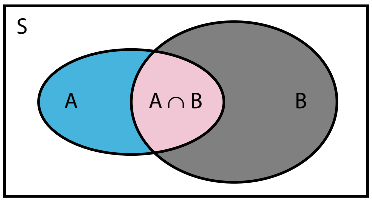
When $P(B) \neq 0$, the **conditional probability** of A given B is $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Why does this formula make sense?

First of all, note that it does agree with the intuitive answers we found above. If $A \cap B = \emptyset$, $P(A \cap B) = 0$, so $P(A|B) = 0/P(B) = 0$;

and if $A \cap B = B$, $P(A|B) = P(B)/P(B) = 100\%$.

Now, if we learn that B occurred, we can restrict attention to just those outcomes that are in B, and disregard the rest of S, so we have a new sample space that is just B (see diagram below).



For A to have occurred in addition to B, requires that $A \cap B$ occurred, so the conditional probability of A given B is $P(A \cap B)/P(B)$, just as we defined it above.

Example 16

In an experiment to study the phenomenon of colour blindness, researchers collected information concerning 1000 people in a small town and categorized them according to colour blindness and gender. Here is a summary of the findings:

	Male	Female	Total
Colour-blind	40	2	42
Not colour-blind	470	488	958
Total	510	490	1000

What is the probability that a person is colour-blind given that the person is a woman?

Solution

To answer this question, we notice that we do not have to search the whole population for this event. We limit our search to the women. We have 490 women. As we only need to consider women, then when we search for colour blindness, we only look for the women who are colour-blind, i.e. the intersection. Here we only have two women. Therefore, the chance we get a colour-blind person given the person is a woman is

$$\mathrm{P}(C|W) = \frac{\mathrm{P}(C \cap W)}{\mathrm{P}(W)} = \frac{n(C \cap W)}{n(W)} = 0.004,$$ where C is for colour-blind and W for woman.

Notice here that we used the frequency rather than the probability. However, these are equivalent since dividing by $n(S)$ will transform the frequency into a probability.

$$\frac{n(C \cap W)}{n(W)} = \frac{\frac{n(C \cap W)}{n(S)}}{\frac{n(W)}{n(S)}} = \frac{\mathrm{P}(C \cap W)}{\mathrm{P}(W)} = \mathrm{P}(C|W)$$

Example 17

AUA, a national airline, is known for its punctuality. The probability that a regularly scheduled flight departs on time is $\mathrm{P}(D) = 0.83$, the probability that it arrives on time is $\mathrm{P}(A) = 0.92$, and the probability that it arrives and departs on time, $\mathrm{P}(A \cap D) = 0.78$. Find the probability that a flight

a) arrives on time given that it departed on time

b) departs on time given that it arrived on time.

Solution

a) The probability that a flight arrives on time given that it departed on time is

$$\mathrm{P}(A|D) = \frac{\mathrm{P}(A \cap D)}{\mathrm{P}(D)} = \frac{0.78}{0.83} = 0.94.$$

b) The probability that a flight departs on time given that it arrived on time

$$\mathrm{P}(D|A) = \frac{\mathrm{P}(D \cap A)}{\mathrm{P}(A)} = \frac{0.78}{0.92} = 0.85.$$

Independence

Two events are **independent** if learning that one occurred does not affect the chance that the other occurred. That is, if $P(A|B) = P(A)$, and vice versa.

This means that if we apply our definition to the general multiplication rule, then

$$P(A \cap B) = P(A|B) \times P(B) = P(A) \times P(B)$$

which is the multiplication rule for independent events we studied earlier.

These results give us some helpful tools in checking the independence of events.

> Two events are **independent** if and only if either $P(A \cap B) = P(A) \times P(B)$, or $P(A|B) = P(A)$. Otherwise, the events are **dependent**.

Example 18

Take another look at the AUA situation in Example 17. Are the events of arriving on time (A) and departing on time (D) independent?

Solution

We can answer this question in two different ways:

a) $P(A) = 0.92$ and we found that $P(A|D) = 0.94$. Since the two values are not the same, we can say that the two events are not independent.

b) Alternately, $P(A \cap D) = 0.78$ and
$P(A) \times P(D) = 0.92 \times 0.83 = 0.76 \neq P(A \cap D)$.

Example 19

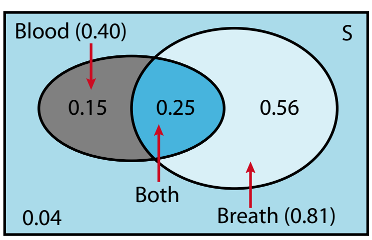
In many countries, the police stop drivers on suspicion of drunk driving. The stopped drivers are given a breath test, a blood test or both. In a country where this problem is vigorously dealt with, the police records show the following:

81% of the drivers stopped are given a breath test, 40% a blood test, and 25% both tests.

a) What is the probability that a suspected driver is given
   (i) a test?
   (ii) exactly one test?
   (iii) no test?

b) Are giving the two tests independent?

Solution



A Venn diagram can help explain the solution.

a) (i) The probability that a driver receives a test means that he/she receives either a blood test, a breath test or both tests. The probability as such can be calculated directly from the diagram, or by applying the addition rule. The diagram shows that if 81% receive the breath test and 25% are also given the blood test, then 56% do not receive a blood test. Similarly, 15% of the blood test receivers do not get a breath test. So, the probability of receiving a test is $0.56 + 0.25 + 0.15 = 0.96$.

Also, if we apply the addition rule,

$$\begin{aligned} P(\text{breath or blood}) &= P(\text{breath}) + P(\text{blood}) - P(\text{both}) \\ &= 0.81 + 0.40 - 0.25 = 0.96. \end{aligned}$$

(ii) To receive exactly one test is to receive a blood test or a breath test, but not both! So, from the Venn diagram it is clear that this probability is $0.15 + 0.56 = 0.71$. To approach it differently, since we know that the union of the two events still contains the intersection, we can subtract the probability of the intersection from that of the union: $0.96 - 0.25 = 0.71$.

(iii) To receive no test is equivalent to the complement of the union of the events. Hence, $P(\text{no test}) = 1 - P(1\text{ test}) = 1 - 0.96 = 0.04$.

b) To check for independence, we can use any of the two methods we tried before. Since all the necessary probabilities are given, we can use the product rule. If they were independent, then

$P(\text{both tests}) = P(\text{breath}) \times P(\text{blood}) = 0.81 \times 0.40 = 0.324$, but $P(\text{both tests}) = 0.25$. Therefore, the events of receiving a breath test and a blood test are not independent.

Example 20

Jane and Kate frequently play tennis with each other. When Jane serves first, she wins 60% of the time, and the same pattern occurs with Kate. They alternate the serve of course. They usually play for a prize, which is a chocolate bar. The first one who loses on her serve will have to buy the chocolate. Jane serves first.

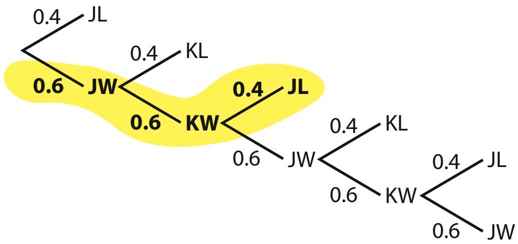
a) Find the probability that Jane pays on her second serve.

b) Find the probability that Jane eventually pays for the chocolate.

c) Find the probability that Kate pays for the chocolate.

Solution

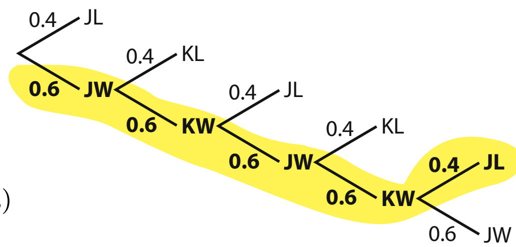
A tree diagram can help in tackling this problem. Let JW stand for Jane winning her serve, and JL for Jane losing her serve and hence paying. KW and KL are defined similarly.



a) For Jane to pay on her second serve, she should win her first serve, Kate must also win her first serve, and then Jane loses her second serve. See diagram above. The probability this happens is

$$P(JW) \cdot P(KW) \cdot P(JL) = 0.6 \times 0.6 \times 0.4 = 0.4 \cdot (0.6)^2 = 0.144.$$

b) For Jane to pay, she needs to be the first one to lose on her serve. This means she loses on the first serve or the second or the third, and so on. So, the probability that she pays is



$$\begin{aligned} P(J \text{ pays}) &= P(JL) + P(JW) \cdot P(KW) \cdot P(JL) \\ &\quad + P(JW) \cdot P(KW) \cdot P(JW) \cdot P(KW) \cdot P(JL) \\ &= 0.4 + 0.4 \cdot (0.6)^2 + 0.4 \cdot (0.6)^4 + \ldots \end{aligned}$$

This appears to be the sum of an infinite geometric series with $(0.6)^2$ as a common ratio; hence,

$$P(J \text{ pays}) = 0.4 + 0.4 \cdot (0.6)^2 + 0.4 \cdot (0.6)^4 + \ldots = \frac{0.4}{1 - (0.6)^2} = 0.625.$$

c) Obviously,

$$P(K \text{ pays}) = 1 - P(J \text{ pays}) = 1 - 0.625 = 0.375.$$

This also gives us the opportunity to look at it differently. For Kate to pay, she needs to lose on her first serve, i.e. 0.6×0.4, or on her second, third, etc.

$$\begin{aligned} P(K \text{ pays}) &= 0.6 \times 0.4 + 0.6 \times (0.6)^2 \times 0.4 + 0.6 \times (0.6)^4 \times 0.4 + \ldots \\ &= \frac{0.6 \times 0.4}{1 - (0.6)^2} = 0.375 \end{aligned}$$

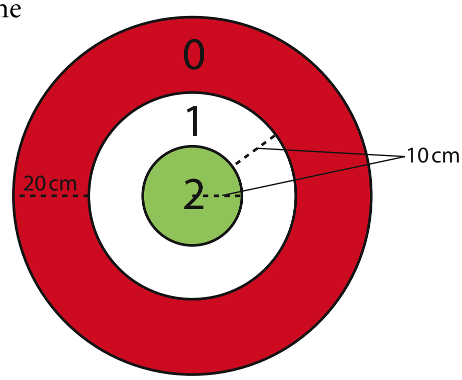
Example 21

A target for a dart game is shown here. The radius of the board is 40 cm and it is divided into three regions as shown. You score 2 points if you hit the centre, 1 point for the middle region and 0 points for the outer region.

a) What is the probability of scoring a 1 in one attempt?

b) What is the probability of scoring a 2 in one attempt?

c) How many attempts are necessary so that the probability of scoring at least one 2 is at least 50%?



Solution

a) $P(1) = \dfrac{\pi(20^2 - 10^2)}{\pi(40^2)} = \dfrac{3}{16}$

b) $\mathrm{P}(2) = \dfrac{\pi(10^2)}{\pi(40^2)} = \dfrac{1}{16}$

c) Let the number of attempts be n.

P(at least one 2) $= 1 - $ P(no 2 in n attempts) $= 1 - \left(\dfrac{15}{16}\right)^n$

For this probability to be at least 50%, then

$$1 - \left(\frac{15}{16}\right)^n \geqslant 0.5 \Leftrightarrow \left(\frac{15}{16}\right)^n \leqslant 0.5$$

$$\Rightarrow n \geqslant \ln\left(\frac{15}{16}\right) \leqslant \ln(0.5)$$

$$\Rightarrow n \geqslant \frac{\ln(0.5)}{\ln\left(\frac{15}{16}\right)} \quad \left\{\text{since } \ln\left(\frac{15}{16}\right) < 0\right\}$$

$$\Rightarrow n \leqslant 10.74$$

So, 11 attempts are required.

Exercise 12.4

1 Events A and B are given such that $\mathrm{P}(A) = \frac{3}{4}$, $\mathrm{P}(A \cup B) = \frac{4}{5}$ and $\mathrm{P}(A \cap B) = \frac{3}{10}$. Find P($B$).

2 Events A and B are given such that $\mathrm{P}(A) = \frac{7}{10}$, $\mathrm{P}(A \cup B) = \frac{9}{10}$ and $\mathrm{P}(A \cap B) = \frac{3}{10}$. Find

a) $\mathrm{P}(B)$ b) $\mathrm{P}(B' \cap A)$ c) $\mathrm{P}(B \cap A')$

d) $\mathrm{P}(B' \cap A')$ e) $\mathrm{P}(B|A')$

3 If events A and B are given such that $\mathrm{P}(A) = \frac{1}{3}$, $\mathrm{P}(A \cup B) = \frac{4}{9}$ and $\mathrm{P}(B) = \frac{2}{9}$, show that A and B are neither independent nor mutually exclusive.

4 Events A and B are given such that $\mathrm{P}(A) = \frac{3}{7}$ and $\mathrm{P}(A \cap B) = \frac{3}{10}$. If A and B are independent, find $\mathrm{P}(A \cup B)$.

5 Driving tests in a certain city are not easy to pass the first time you take them. After going through training, the percentage of new drivers passing the test the first time is 60%. If a driver fails the first test, there is a chance of passing it on a second try, two weeks later. 75% of the second-chance drivers pass the test. Otherwise, the driver has to retrain and take the test after 6 months. Find the probability that a randomly chosen new driver will pass the test without having to wait 6 months.

6 People with O-negative blood type are universal donors, i.e. they can donate blood to individuals with any blood type. Only 8% of people have O-negative.

a) One person randomly appears to give blood. What is the probability that he/she does not have O-negative?

b) Two people appear independently to give blood. What is the probability that
 (i) both have O-negative?
 (ii) at least one of them has O-negative?
 (iii) only one of them has O-negative?

c) Eight people appear randomly to give blood. What is the probability that at least one of them has O-negative?

7 PIN numbers for cellular phones usually consist of four digits that are not necessarily different.

a) How many possible PINs are there?

b) You don't want to consider the pins that start with 0. What is the probability that a PIN chosen at random does not start with a zero?

c) What is the probability that a PIN contains at least one zero?

d) Given a PIN with at least one zero, what is the probability that it starts with a zero?

8 An urn contains six red balls and two blue ones. We make two draws and each time we put the ball back after marking its colour.

a) What is the probability that at least one of the balls is red?

b) Given that at least one is red, what is the probability that the second one is red?

c) Given that at least one is red, what is the probability that the second one is blue?

9 Two dice are rolled and the numbers on the top face are observed.

a) List the elements of the sample space.

b) Let x represent the sum of the numbers observed. Copy and complete the following table.

x	2	3	4	5	6	7	8	9	10	11	12
P(x)		$\frac{1}{18}$									

c) What is the probability that at least one die shows a 6?

d) What is the probability that the sum is at most 10?

e) What is the probability that a die shows 4 or the sum is 10?

f) Given that the sum is 10, what is the probability that one of the dice is a 4?

10 A large school has the following numbers categorized by class and gender:

Grade / Gender	Grade 9	Grade 10	Grade 11	Grade 12	Total
Male	180	170	230	220	800
Female	200	130	190	180	700

a) What is the probability that a student chosen at random will be a female?

b) What is the probability that a student chosen at random is a male grade 12 student?

c) What is the probability that a female student chosen at random is a grade 12 student?

d) What is the probability that a student chosen at random is a grade 12 or female student?

e) What is the probability that a grade 12 student chosen at random is a male?

f) Are gender and grade independent of each other? Explain.

11 Some young people do not like to wear glasses. A survey considered a large number of teenage students as to whether they needed glasses to correct their vision and whether they used the glasses when they needed to. Here are the results.

		Used glasses when needed	
		Yes	No
Need glasses for correct vision	Yes	0.41	0.15
	No	0.04	0.40

a) Find the probability that a randomly chosen young person from this group
  (i) is judged to need glasses
  (ii) needs to use glasses but does not use them.

b) From those who are judged to need glasses, what is the probability that he/she does not use them?

c) Are the events of using and needing glasses independent?

12 Fill in the missing entries in the following table.

P(A)	P(B)	Conditions for events A and B	P($A \cap B$)	P($A \cup B$)	P($A \mid B$)
0.3	0.4	Mutually exclusive			
0.3	0.4	Independent			
0.1	0.5			0.6	
0.2	0.5		0.1		

13 In a large graduating class, there are 100 students taking the IB examination. 40 students are doing Maths/SL, 30 students are doing Physics/SL and 12 are doing both.

a) A student is chosen at random. Find the probability that this student is doing Physics/SL given that he/she is doing Maths/SL.

b) Are doing Physics/SL and Maths/SL independent?

14 A market chain in Germany accepts only Mastercard and Visa. It estimates that 21% of its customers use Mastercard, 57% use Visa and 13% use both cards.

a) What is the probability that a customer will have an acceptable credit card?

b) What proportion of their customers has neither card?

c) What proportion of their customers has exactly one acceptable card?

15 132 of 300 patients at a hospital are signed up for a special exercise program which consists of a swimming class and an aerobics class. Each of these 132 patients takes at least one of the two classes. There are 78 patients in the swimming class and 84 in the aerobics class. Find the probability that a randomly chosen patient at this hospital is

a) not in the exercise program

b) enrolled in both classes.

16 An ordinary unbiased 6-sided die is rolled three times. Find the probability of rolling

a) three twos

b) at least one two

c) exactly one two.

17 An athlete is shooting arrows at a target. She has a record of hitting the centre 30% of the time. Find the probability that she hits the centre

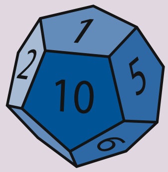
a) with her second shot

b) exactly once with her first three shots

c) at least once with her first three shots.

18 Two unbiased dodecahedral (12 faces) dice are thrown. The scores are the numbers on the top side.

Find the probability that



a) at least one twelve shows

b) a sum of 12 shows on both dice

c) a total score of at least 20 shows on both dice

d) given that a twelve shows on one die, a total score of at least 20 is achieved.

19 For question 18, define the following two events:

$A = \{$at least one of the numbers is a 10$\}$

$B = \{$the sum of the numbers is at most 15$\}$

Describe the following events, list their elements and find their probabilities:

a) $A \cap B$ b) $A \cup B$ c) $(A \cap B)'$

d) $(A \cup B)'$ e) $A' \cup B'$ f) $A' \cap B'$

g) $(A \cap B') \cup (A' \cap B)$

20 Consider any events A, B, and C. Prove each of the following:

a) $\mathrm{P}(A \cap B) \geqslant \mathrm{P}(A) + \mathrm{P}(B) - 1$

b) $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$

21 Three fair 6-sided dice are rolled.

a) Find the probability that triples are rolled.

b) Given that the roll is a sum of 8 or less, find the probability that triples are rolled.

c) Find the probability that at least one six appears.

d) Given that the dice all have different numbers, find the probability that at least one six appears.

22 You are given four coins: one has two heads, one has two tails, and the other two are normal. You choose a coin at random and toss it. The result is tails. What is the probability that the opposite face is heads?

23 George and Kassanthra play a game in which they roll two unbiased cubical dice. The first one who rolls a sum of 6 wins. Kassanthra rolls the dice first.

a) What is the probability that Kassanthra wins on her second roll?

b) What is the probability that George wins on his second roll?

c) What is the probability that Kassanthra wins?

24 A small repair shop for washing machines has the following demand for their services:

On 10% of the days they have no requests, they have one request on 30% of the days, and two requests 50% of the time.

a) On Monday, what is the chance of more than two requests?

b) What is the chance of no requests for a whole (5 day) week?

25 A small class has five boys and six girls. A group of four students are to be selected at random to interview a new school director.

a) Find the probability that the group contains at least one boy.

b) Find the probability that the majority of the group is girls.

c) Given that the group contains at least one boy, what is the chance that the boys are in the majority?

26 A construction company is bidding on three projects: B_1, B_2 and B_3. From previous experience they have the following probabilities of winning the bids: $P(B_1) = 0.22$, $P(B_2) = 0.25$ and $P(B_3) = 0.28$. Winning the bids is not independent one from another. The joint probabilities are given below.

	B_1	B_2	B_3
B_1		0.11	0.05
B_2	0.11		0.07
B_3	0.05	0.07	

Also, $P(B_1 \cap B_2 \cap B_3) = 0.01$. Find the following probabilities:

a) $P(B_1 \cup B_2)$

b) $P(B'_1 \cap B'_2)$

c) $P(B'_1 \cap B'_2) \cup B_3$

d) $P(B'_1 \cap B'_2 \cap B_3)$

e) $P(B_2 \cap B_3 | B_1)$

f) $P(B_2 \cup B_3 | B_1)$

27 Circuit boards used in electronic equipment go through more than one layer of inspection. The process of finding faults in the solder joints on these boards is highly subjective and prone to disagreements among inspectors. In a batch of 20 000 joints, Nick found 1448 faulty joints while David found 1502 faulty ones. All in all, among both inspectors, 2390 joints were judged to be faulty. Find the probability that a randomly chosen joint is

a) judged to be faulty by neither of the two inspectors

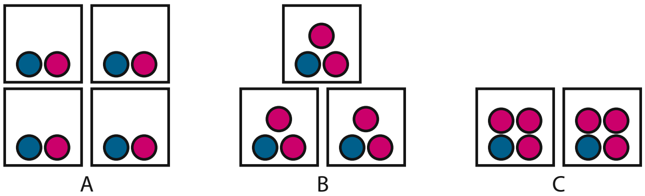
b) judged to be defective by David but not Nick.

12.5 Bayes' theorem

Lie detectors (polygraphs), drug and alcohol tests, and disease screening tests are among the many applications where the results are frequently cautiously scrutinized. Tests with high precision rates are open to error. Bayes' theorem helps us understand and analyze the results of such tests.

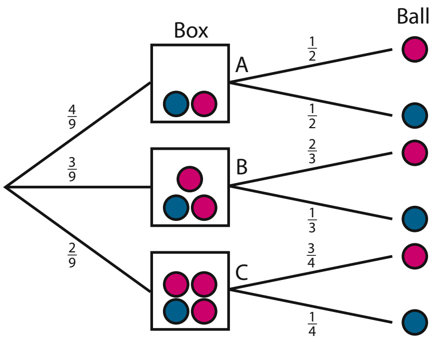
To understand the theorem, we start this section with an example.

Suppose I have 9 indistinguishable boxes with blue and red balls as shown in the diagram (next page). They are of three types: Type A contain 2 balls each – one red and one blue, B contain 3 balls each – one blue and two red, and C contain 4 balls each – one blue and three red. I mix up the boxes and choose one at random, and then I pick a ball from that box, also at random. I show you the ball. What is the probability that you can guess the box type it was drawn from if the ball is red?



Letting R represent red ball and A, B or C representing box types, the question is to find

$$P(A|R), P(B|R), \text{ or } P(C|R).$$



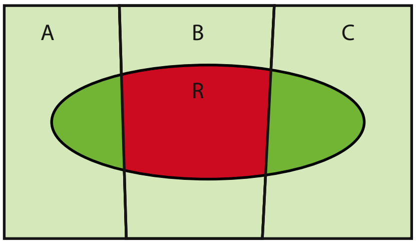
Recalling information from conditional probability, your task is to calculate

$$P(A|R) = \frac{P(A \cap R)}{P(R)}.$$

Since you already know $P(A)$ and $P(R|A)$, then

$$P(A \cap R) = P(R|A) \cdot P(A) = \tfrac{4}{9} \cdot \tfrac{1}{2} = \tfrac{2}{9}.$$

However, to find $P(R)$, we need to see what constitutes the event R.



As you see in the diagram right, A, B and C are mutually exclusive, and hence

$$R = (R \cap A) \cup (R \cap B) \cup (R \cap C) \Rightarrow$$

$$P(R) = P(R \cap A) + P(R \cap B) + P(R \cap C), \text{ and so}$$

$$\begin{aligned} P(R) &= P(R \cap A) + P(R \cap B) + P(R \cap C) \\ &= P(R|A) \cdot P(A) + P(R|B) \cdot P(B) + P(R|C) \cdot P(C) \\ &= \tfrac{1}{2} \cdot \tfrac{4}{9} + \tfrac{2}{3} \cdot \tfrac{3}{9} + \tfrac{3}{4} \cdot \tfrac{2}{9} = \tfrac{11}{18} \end{aligned}$$

And now

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{\frac{2}{9}}{\frac{11}{18}} = \tfrac{4}{11}, \text{ and similarly}$$

$$P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{\frac{2}{3} \cdot \frac{3}{9}}{\frac{11}{18}} = \tfrac{4}{11}, \text{ and finally}$$

$$P(C|R) = \frac{P(C \cap R)}{P(R)} = \frac{\frac{3}{4} \cdot \frac{2}{9}}{\frac{11}{18}} = \tfrac{3}{11}.$$

So, A or B could more likely be the source of a red ball than C.

Conditional probability typically deals with the probability of an event when you have information about something that happened earlier. In conditional probability, we assume that the first event is known (selecting a box) and ask for the probability of the second event (colour of ball). Bayes' rule deals with a reverse situation. It assumes that the second event is known (red ball) and asks for the probability of the first event.

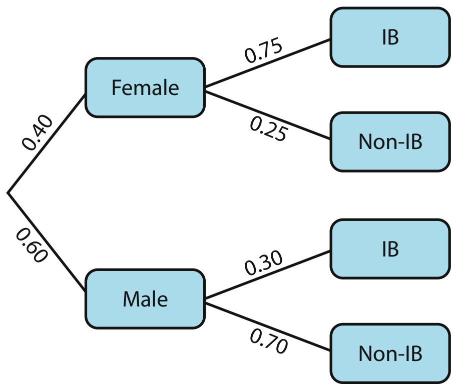
Let us look at another example to see how we apply Bayes' theorem. We will then make a formal statement of the rule.

Example 22

60% of the students at a university are male and 40% are female. Records show that 30% of the males have IB diplomas while 75% of the females have IB diplomas. A student is selected and found to have a diploma. What is the probability that the student is a female?

Solution

Let the event female be called F, event male M and event IB diploma I. The question asks for $\mathrm{P}(F|I)$.



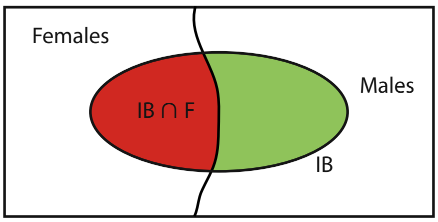
$$\mathrm{P}(F|I) = \frac{\mathrm{P}(F \cap I)}{\mathrm{P}(I)} = \frac{\mathrm{P}(F) \cdot \mathrm{P}(I|F)}{\mathrm{P}(I)}$$

Since $I = (F \cap I) \cup (M \cap I)$, then

$$\mathrm{P}(I) = 0.40 \times 0.75 + 0.60 \times 0.30 = 0.48, \text{ and so}$$

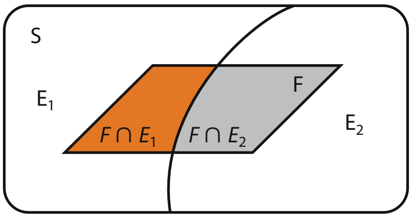
$$\mathrm{P}(F|I) = \frac{\mathrm{P}(F \cap I)}{\mathrm{P}(I)} = \frac{0.40 \times 0.75}{0.48} = 0.625.$$

Note: Another interpretation of this number is the proportion of females among the IB holders.



Bayes' theorem – simple case

Let S be a sample space with E_1 and E_2 mutually exclusive events that *partition* this sample space. Let F be a non-empty event in this sample space. Then



$$P(E_1|F) = \frac{P(E_1 \cap F)}{P(F)} = \frac{P(E_1 \cap F)}{P(E_1 \cap F) + P(E_2 \cap F)}$$

$$= \frac{P(E_1) \cdot P(F|E_1)}{P(E_1) \cdot P(F|E_1) + P(E_2) \cdot P(F|E_2)}$$

Note:
The probability of an event in terms of its mutually exclusive and exhaustive subsets is usually called the **total probability**:
$P(F) = P(E_1) \cdot P(F|E_1) + P(E_2) \cdot P(F|E_2)$

Sometimes, this rule is stated differently:

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E \cap F)}{P(E \cap F) + P(E' \cap F)}$$

$$= \frac{P(E_1) \cdot P(F|E)}{P(E) \cdot P(F|E) + P(E') \cdot P(F|E')}$$

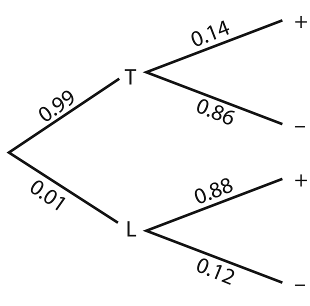
where E' is the complement of E.

Example 23

Lie detectors (polygraphs) are not considered reliable in court. They are, however, administered to employees in sensitive positions. One such test gives a positive reading (person is lying) when a person is lying 88% of the time and a negative reading (person telling the truth) when a person is telling the truth 86% of the time. In a security-related question, the vast majority of subjects have no reason to lie so that 99% of the subjects will tell the truth. An employee produces a positive response on the polygraph. What is the probability that this employee is actually telling the truth?

Solution

Let $+$ denote the event of the test being positive (subject is lying), and $-$ denote the event of the test being negative (subject telling the truth). Let L represent the person is lying, and T the person is telling the truth. Hence, we have



$$P(+|L) = 0.88, P(-|L) = 0.12, P(-|T) = 0.86, P(+|T) = 0.14$$

$$P(+) = P(+ \cap L) + P(+ \cap T) = P(+|L) \cdot P(L) + P(+|T) \cdot P(T)$$

$$= 0.88 \times 0.01 + 0.14 \times 0.99$$

And finally,

$$P(T|+) = \frac{P(+|T) \cdot P(T)}{P(+|L) \cdot P(L) + P(+|T) \cdot P(T)} = \frac{0.14 \times 0.99}{0.88 \times 0.01 + 0.14 \times 0.99} = 0.94.$$

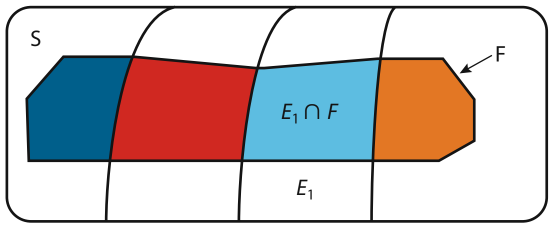
Thus, in screening this population of mostly innocent people, 94% of the positive polygraph readings will be misleading!

Bayes' theorem – general rule

Let S be a sample space with $E_1, E_2, \ldots, E_n$ mutually exclusive events that *partition* this sample space. Let F be a non-empty event in this sample space. Then

$$P(E_i|F) = \frac{P(E_i \cap F)}{P(F)} = \frac{P(E_i \cap F)}{P(E_1 \cap F) + P(E_2 \cap F) + \ldots + P(E_n \cap F)}$$

$$= \frac{P(E_i) \cdot P(F|E_i)}{P(E_1) \cdot P(F|E_1) + P(E_2) \cdot P(F|E_2) + \ldots + P(E_n) \cdot P(F|E_n)}$$

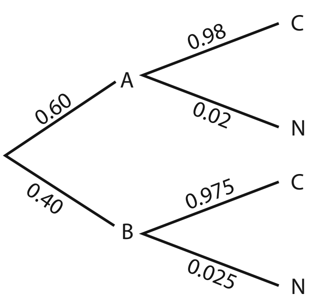


Example 24

A paper factory produces high-quality paper using two machines. Like any process, the final product is checked for quality. 98% of the time machine A produces paper that conforms to the accepted norms. 97.5% of machine B's paper conforms to norms. Machine A produces 60% of the paper in this company. You pick a paper at random and it does not conform to the norms, what is the chance it was produced by A?

Solution

Let the non-conforming paper be called N. Since the paper is produced by these two machines, then the non-conforming paper has to be from the output of one of these machines, i.e.



$$N = (N \cap A) \cup (N \cap B),$$
$$P(N) = P(N \cap A) + P(N \cap B) = P(N|A) \cdot P(A) + P(N|B) \cdot P(B)$$
$$= 0.02 \times 0.60 + 0.025 \times 0.40 = 0.022$$

So, using Bayes' theorem,

$$P(A|N) = \frac{P(A \cap N)}{P(N)} = \frac{P(N|A) \cdot P(A)}{P(N)} = \frac{0.02 \times 0.60}{0.022} = 54.5\%.$$

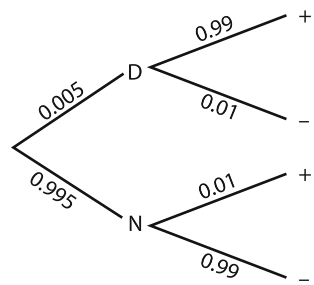
Example 25

In many countries the law requires that a driver's licence is withdrawn if he/she is found to have more that 0.05% blood alcohol concentration (BAC). Suppose that the police use a test that will correctly identify a 'drunk' driver (testing positive) 99% of the time, and will correctly identify a sober driver (testing negative) 99% of the time. Let's assume that 0.5% of the drivers in your city drive under the influence of alcohol. We want to know the probability that given a positive test a driver is actually 'drunk'.

Solution

Let D be the event of being drunk and N indicate being not drunk. Let $+$ be the event of a positive test. We need to know $P(D|+)$.

The question is to find the proportion of the drunk drivers who test positive out of all of those who test positive.



$P(D)$, or the probability that the driver is drunk, regardless of any other information, is 0.005, since 0.5% of the drivers drink and drive.

$P(N)$, or the probability that the driver is sober, is $1 - P(D)$, or 0.995.

$P(+|D)$, or the probability that the test is positive given that the driver is drunk, is 0.99, since the test is 99% accurate.

$P(+|N)$, or the probability that the test is positive, given that the driver is not drunk, is 0.01, since the test will produce a *false positive* for 1% of sober drivers.

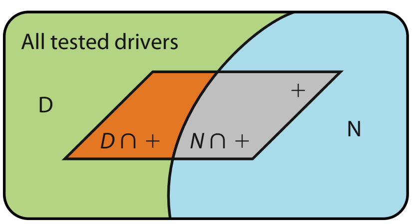
$P(+)$, or the probability of a positive test event, regardless of other information, is 0.0149 or 1.49%, which is found by adding the probability that a *true positive* result will appear ($99\% \times 0.5\% = 0.495\%$) plus the probability that a *false positive* will appear ($1\% \times 99.5\% = 0.995\%$).

Given this information, we can compute the $P(D|+)$ of a driver who tested positive actually being a drunk driver:

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|N)\,P(N)}$$

$$= \frac{0.99 \times 0.005}{0.99 \times 0.005 + 0.01 \times 0.995} = 0.3322$$

Even with the high precision of the test, the probability that a driver who tested positive in fact did have a high BAC is only about 33%, so it is essentially more likely that the driver is not a drunk driver.



Example 26

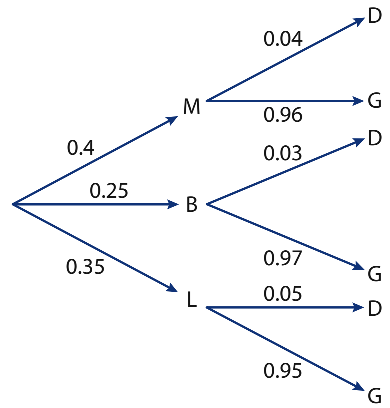
A computer manufacturer receives hard disks from three different suppliers. Marco supplies 40% of the disks, Berto supplies 25%, while Lukas supplies the rest. Disks from Marco have a defective rate of 4%, those from Berto 3%, while Lukas' have a 5% rate.

a) A disk is checked at random. What is the probability that it is defective?

b) A disk is checked and found defective. What is the probability that it was supplied by Lukas?

Solution

Let M represent Marco, B represent Berto and L represent Lukas. Also let D represent defective items and G the non-defective ones. A tree diagram may be helpful here.



a) A defective disk could be supplied by any of the three suppliers, hence:

$$\begin{aligned} P(D) &= P(M \cap D) + P(B \cap D) + P(L \cap D) \\ &= P(M) \cdot P(D \mid M) + P(B) \cdot P(D \mid B) + P(L) \cdot P(D \mid L) \\ &= 0.4 \times 0.04 + 0.25 \times 0.03 + 0.35 \times 0.05 \\ &= 0.041 \end{aligned}$$

b) This is a Bayes theorem application. We are reversing the order of events:

$$\begin{aligned} P(L \mid D) &= \frac{P(L \cap D)}{P(D)} \\ &= \frac{P(L) \cdot P(D \mid L)}{P(M) \cdot P(D \mid M) + P(B) \cdot P(D \mid B) + P(L) \cdot P(D \mid L)} \\ &= \frac{0.35 \times 0.05}{0.041} \\ &= 0.427 \end{aligned}$$

Exercise 12.5

1 In a sample space S, we have the following events and some associated probabilities:

$$P(E) = \tfrac{2}{3},\ P(A \mid E) = \tfrac{3}{50} \text{ and } P(A \mid E') = \tfrac{1}{25}$$

a) Represent the information using a tree diagram.

b) Find $P(A)$.

c) Find $P(E \mid A)$.

2 The rate of prostate cancer among men in 2002 was approximately 26 cases per 100 000 people. Diagnosing this type of cancer saves the lives of about 70% of those treated. At a hospital, the probability of diagnosing a person with this cancer correctly is 78% and the probability of diagnosing a person without this cancer as having the disease is 6%.

a) What is the probability of diagnosing a person as having this cancer?

b) What percentage of those diagnosed with this cancer actually have it?

3 Police in a small town plan to enforce speed limits by installing 'radar traps' at the two main town entrances: east end and west end. Traffic statistics show that 40% of the cars entering the town use the east entrance and the rest use the west one. The east entrance traps are operated 40% of the time and the west entrance are operated 60% of the time. Assuming that the proportion of 'speeders' is the same at both entrances, what is the probability that:

a) a speeding driver is spotted passing one of the traps?

b) a speeding driver who got spotted has actually come from the west end?

4 Coloured balls are placed in three boxes as follows:

	Box		
	1	2	3
Green	4	8	6
Red	6	2	8
Blue	10	6	6

A box is selected at random from which a ball is randomly drawn.

a) What is the probability that the ball is green?

b) Given that the ball is green, what is the probability that it was drawn from box 2?

5 Two coins are in your pocket. When tossed, one of the coins is biased with 0.6 probability of landing heads, while the other coin is unbiased. You select one of the coins at random and toss it.

a) What is the probability it lands heads?

b) Given that it lands on tails, what is the probability that it was the unbiased coin?

6 When answering a question on a multiple choice test, a student is given 5 choices, one of which is correct. The test is so designed that the choices are very close and the probability of getting the correct answer, when you know the material, is 0.6. In a class where 70% of the students are well prepared, a randomly chosen student answers the question correctly. What is the probability that the student really knew the material?

7 Nigel is a student at Wigley College and lives in the dorms. To avoid coming late to his morning classes he usually sets his alarm clock. 85% of the time he manages to remember and set his alarm. When the alarm goes off he manages to go to his morning classes 90% of the time. If the alarm is not set, he still manages to get up and go to class on 60% of the days.

a) What percentage of the days does he manage to get to his morning classes?

b) He made it to class one day. What is the chance that he did that without having set the alarm?

8 Marco plays tennis. In this game, a player has two 'serves'. If the first serve is successful, the game continues. If the first serve is not successful, the player is given another chance. If the second serve fails, then the player loses the point.

Marco is successful with his first serve 60% of the time and 95% successful with his second serve. When his first serve is successful he goes on to win the point 75% of the time, and when it takes him two serves, he wins the point 50% of the time.

a) What is the probability that Marco wins the point?

b) If Marco wins a point, what is the probability that he succeeded with the first serve?

9 In Vienna, conventional wisdom has it that in February days are snowy or fine. 80% of the time a fine day follows a fine day. 40% of the time a snowy day is followed by a fine day. The forecast for the first of February to be a fine day is 0.75.

a) Find the probability that 2nd February is fine.

b) Given that 2nd February turns out to be snowy, what is the probability that the 1st of February was a fine day?

10 It is known that 33% of people over the age of 50 around the world have some kind of arthritis. A test has been developed to detect arthritis in individuals. This test was given to a large group of individuals with confirmed cases and a positive test result was achieved in 87% of the cases. That same test gave a positive test to 4% of individuals that do not have arthritis.

If this test is given to an individual at random and it tests positive, what is the probability that the individual has this disease?

11 (Relatively challenging!) A high school has a large graduating class. The table below shows how the students are categorized according to their college plans and gender.

	Local university	University abroad
Male	51%	4%
Female	16%	29%

5% of these students are in the IB mathematics/HL class. 72% of the HL class will attend local university and the rest are going abroad.

a) What percentage of the graduating class are mathematics/HL students planning on studying locally?

b) What percentage of the non-mathematics/HL students are going to study at a local university?

c) Among the students studying locally, what percentage are mathematics/HL students?

d) Among the students studying abroad, what percentage are mathematics/HL students?

12 When Olympic athletes are tested for illegal drug use (doping), the results of a single test are used to ban the athlete from competition. In an experiment on 1000 athletes, 100 were using the testosterone drug. During the medical examination, the available test would positively identify 50% of the users. It would also falsely identify 9% of the non-users as users.

If an athlete tests positive, what is the probability that he/she is really doping?

13 An engineering company employs three architects that are responsible for cost estimates of new projects. Antonio makes 30% of the estimates, Richard makes 20% and Marco 50%. Like all estimates, there are usually some errors in these estimates. The record of percentages of 'serious' errors that cost the company thousands of euros shows Antonio at 3%, Richard at 2%, and Marco at 1%. Which of the three engineers is probably responsible for most of the serious errors?

14 At a small airport, if an aircraft is present at 10 km distance from the runway, radar detects it and generates an alarm signal 99% of the time. If an aircraft is not present, the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05.

a) What is the probability that the radar gives an alarm signal?

b) Given that there is no alarm signal, what is the probability that an aircraft is there?

15 You enter a chess tournament where your probability of winning a game is 0.3 against half the players (novices), 0.4 against a quarter of the players (experienced) and 0.5 against the remaining quarter of the players (masters). You play a game against a randomly chosen opponent.

a) What is the probability of winning?

b) Given that you won, what is the probability that the game was against a master?

16 Driving tests in a certain city are relatively easy to pass the first time you take them. After going through training, the percentage of new drivers passing the test the first time is 80%. If a driver fails the first test, there is chance of passing it on a second harder test, two weeks later. 50% of the second-chance drivers pass the test. If the second test is unsuccessful, a third attempt, a week later, is given and 30% of the participants pass it. Otherwise, the driver has to retrain and take the test after 1 year.

a) Find the probability that a randomly chosen new driver will pass the test without having to wait one year.

b) Find the probability that a randomly chosen new driver that passed the test did so on the second attempt.

17 A school has 250 employees categorized by task and gender in the following table.

	Teaching	Administrative	Support
Male	84	14	52
Female	56	26	18

An employee is randomly selected. Let A be the event that he/she is an administrative staff member, T teaching staff, S support, M male, and F female.

a) Write down the following probabilities: $P(F)$, $P(F \cap T)$, $P(F \cup A')$, $P(F'|A)$.

b) Which events are independent of F, which are mutually exclusive to F. Justify your choices.

c) (i) Given that 90% of teachers, as well as 80% of the administrative staff and 30% of the support staff, own cars, find the probability that a staff member chosen at random owns a car.

(ii) Knowing that the randomly chosen staff member owns a car, find the probability that he/she is a teacher.

18 Car insurance companies categorize drivers as high risk, medium risk and low risk. (For your information only: Teens and seniors are considered high risk!)

20% of the drivers insured by 'First Insurance' are high risk, and 50% are medium risk driver. The company's actuaries estimate the chance that each class of driver will have at least one accident in the coming 12 months as follows: High risk 6%, medium 3% and low 1%.

a) Find the probability that a randomly chosen driver is a high-risk driver that will have an accident in a 12-month period.

b) Find the probability that a randomly chosen driver insured by this company will have an accident in the next 12 months.

c) A customer has a claim for an accident. What is the probability that he/she is a high-risk driver?

Practice questions

1 Two independent events A and B are given such that $P(A) = k$, $P(B) = k + 0.3$ and $P(A \cap B) = 0.18$

a) Find k.
b) Find $P(A \cup B)$.
c) Find $P(A' | B')$.

2 Many airport authorities test prospective employees for drug use, with the intent of improving efficiency and reducing accidents. This procedure has plenty of opponents who claim that it creates difficulties for some classes of people and that it prevents others from getting these jobs even if they were not drug users. The claim depends on the fact that these tests are not 100% accurate. To test this claim, let us assume that a test is 98% accurate in the sense that it identifies a person as a user or non-user 98% of the time. Each job applicant takes this test twice. The tests are done at separate times and are designed to be independent of each other. What is the probability that

a) a non-user fails both tests?
b) a drug user is detected (i.e. he/she fails at least one test)?
c) a drug user passes both tests?

3 Communications satellites are difficult to repair when something goes wrong. One satellite works on solar energy and has two systems that provide electricity: the main system with a probability of failure of 0.002, and a back-up system that works independently of the main one. It has a failure rate of 0.01. What is the probability that the systems do not fail at the same time?

4 In a group of 200 students taking the IB examination, 120 take Spanish, 60 take French and 10 take both.

a) If a student is selected at random, what is the probability that he/she
  (i) takes either French or Spanish?
  (ii) takes either French or Spanish but not both?
  (iii) does not take any French or Spanish?
b) Given that a student takes the Spanish exam, what is the chance that he/she takes French?

5 In a factory producing disk drives for computers, there are three machines that work independently to produce one of the components. In any production process, machines are not 100% fault free. The production after one 'run' from these machines is listed below.

	Defective	Non-defective
Machine I	6	120
Machine II	4	80
Machine III	10	150

a) A component is chosen at random from the produced lot. Find the probability that the chosen component is
  (i) from machine I
  (ii) a defective component from machine II
  (iii) non-defective or from machine I
  (iv) from machine I given that it is defective.
b) Is the quality of the component dependent on the machine used?

6 At a school, the students are organizing a lottery to raise money for the needy in their community. The lottery tickets they have consist of small coloured envelopes inside which there is a small note. The note says: 'You won a prize!' or 'Sorry, try another ticket.' The envelopes have several colours. They have 70 red envelopes that contain two prizes, and the rest (130 tickets) contain four other prizes.

a) You want to help this class and you buy a ticket hoping that it does not have a prize. Additionally, you don't like the red colour. You pick your ticket at random by closing your eyes. What is the probability that your wish comes true?

b) You are surprised – you picked a red envelope. What is the probability that you did not win a prize?

7 You are given two events A and B with the following conditions:

$$P(A \mid B) = 0.30, \; P(B \mid A) = 0.60, \; P(A \cap B) = 0.18$$

a) Find $P(B)$.

b) Are A and B independent? Why?

c) Find $P(B \cap A')$.

8 In several ski resorts in Austria and Switzerland, the local sports authorities use high school students as 'ski instructors' to help deal with the surge in demand during vacations. However, to become an instructor, you have to pass a test and be a senior at your school. Here are the results of a survey of 120 students in a Swiss school who are training to become instructors. In this group, there are 70 boys and 50 girls. 74 students took the test, 32 boys and 16 girls passed the test, and the rest, including 12 girls, failed the test. 10 of the students, including 6 girls, were too young to take the ski test.

a) Copy and complete the table.

	Boys	Girls
Passed the ski test	32	16
Failed the ski test		12
Training, but did not take the test yet		
Too young to take the test		

b) Find the probability that

(i) a student chosen at random has taken the test

(ii) a girl chosen at random has taken the test

(iii) a randomly chosen boy and randomly chosen girl have both passed the ski test.

9 Two events A and B are such that $P(A) = \frac{9}{16}$, $P(B) = \frac{3}{8}$, and $P(A \mid B) = \frac{1}{4}$. Find the probability that

a) both events will happen

b) only one of the events will happen

c) neither event will happen.

10 Martina plays tennis. When she serves, she has a 60% chance of succeeding with her first serve and continuing the game. She has a 95% chance on the second serve. Of course if both serves are not successful, she loses the point.

a) Find the probability that she misses both serves.

If Martina succeeds with the first serve, her chances of gaining the point against Steffy is

75%. If she is only successful with the second serve, her chances against Steffy for that point go down to 50%.

b) Find the probability that Martina wins a point against Steffy.

11 For the events A and B, $P(A) = 0.6$, $P(B) = 0.8$ and $P(A \cup B) = 1$.

Find

a) $P(A \cap B)$

b) $P(A' \cup B')$.

12 In a survey, 100 students were asked, 'Do you prefer to watch television or play sport?' Of the 46 boys in the survey, 33 said they would choose sport, while 29 girls made this choice.

	Boys	Girls	Total
Television			
Sport	33	29	
Total	46		100

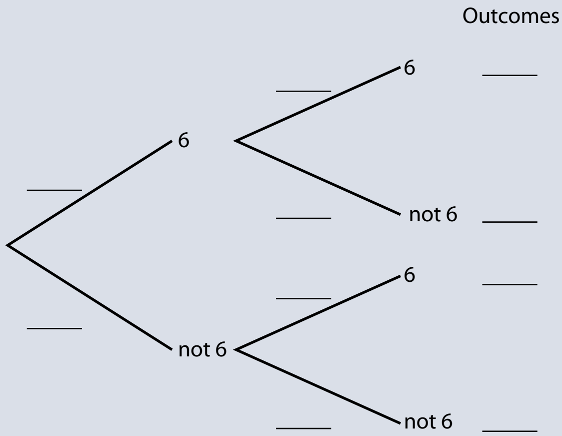
By completing this table or otherwise, find the probability that

a) a student selected at random prefers to watch television

b) a student prefers to watch television given that the student is a boy.

13 Two ordinary, six-sided dice are rolled and the total score is noted.

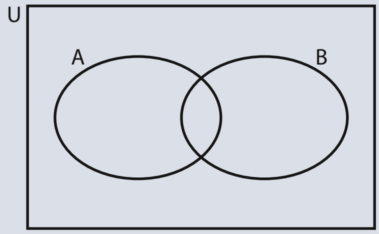
a) Complete the tree diagram by entering probabilities and listing outcomes.



b) Find the probability of getting one or more sixes.

14 The Venn diagram right shows a sample space U and events A and B.

$n(U) = 36$, $n(A) = 11$, $n(B) = 6$ and $n(A \cup B)' = 21$.



a) On the diagram, shade the region $(A \cup B)'$.

b) Find

(i) $n(A \cap B)$

(ii) $P(A \cap B)$.

c) Explain why events A and B are not mutually exclusive.

15 In a survey of 200 people, 90 of whom were female, it was found that 60 people were unemployed, including 20 males.

a) Using this information, complete the table below.

	Males	Females	Totals
Unemployed			
Employed			
Totals			200

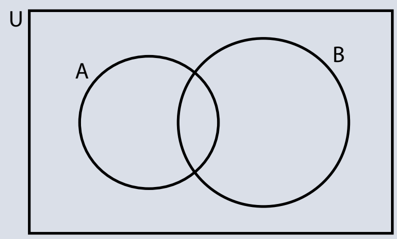
b) If a person is selected at random from this group of 200, find the probability that this person is

(i) an unemployed female

(ii) a male given that the person is employed.

16 A bag contains 10 red balls, 10 green balls and 6 white balls. Two balls are drawn at random from the bag without replacement. What is the probability that they are of different colours?

17 The Venn diagram right shows the universal set U and the sets A and B.



a) Shade the area in the diagram which represents the set $B \cap A'$.

$$n(U) = 100,\ n(A) = 30,\ n(B) = 50,\ n(A \cup B) = 65.$$

b) Find $n(B \cap A')$.

c) An element is selected at random from U. What is the probability that this element is in $B \cap A'$?

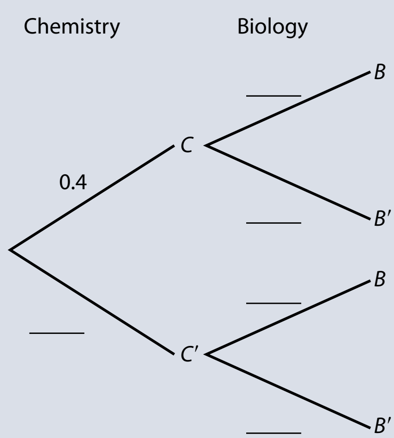
18 The events B and C are dependent, where C is the event 'a student takes chemistry', and B is the event 'a student takes biology'. It is known that

$$P(C) = 0.4,\ P(B|C) = 0.6,\ P(B|C') = 0.5.$$

a) Complete the following tree diagram.

b) Calculate the probability that a student takes biology.

c) Given that a student takes biology, what is the probability that the student takes chemistry?



19 Two fair dice are thrown and the number showing on each is noted. The sum of these two numbers is S. Find the probability that

a) S is less than 8

b) at least one die shows a 3

c) at least one die shows a 3 given that S is less than 8.

20 For events A and B, the probabilities are $P(A) = \frac{3}{11}$ and $P(B) = \frac{4}{11}$. Calculate the value of $P(A \cap B)$ if

a) $P(A \cup B) = \frac{6}{11}$

b) events A and B are independent.

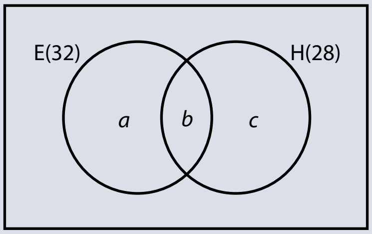
21 Consider events A and B such that $P(A) \neq 0$, $P(A) \neq 1$, $P(B) \neq 0$ and $P(B) \neq 1$. In each of the situations **a)**, **b)**, **c)** below, state whether A and B are mutually exclusive (M), independent (I), or neither (N).

a) $P(A|B) = P(A)$

b) $P(A \cap B) = 0$

c) $P(A \cap B) = P(A)$

22 In a school of 88 boys, 32 study economics (E), 28 study history (H) and 39 do not study either subject. This information is represented in the following Venn diagram.



a) Calculate the values a, b, c.

b) A student is selected at random.

  (i) Calculate the probability that he studies both economics and history.

  (ii) Given that he studies economics, calculate the probability that he does not study history.

c) A group of three students is selected at random from the school.

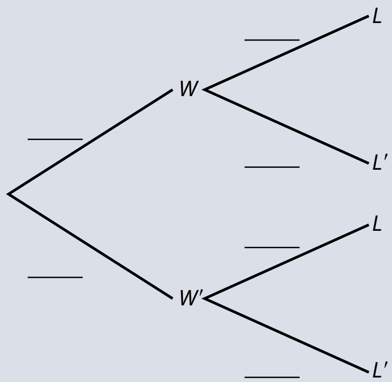
  (i) Calculate the probability that none of these students studies economics.

  (ii) Calculate the probability that at least one of these students studies economics.

23 A painter has 12 tins of paint. Seven tins are red and five tins are yellow. Two tins are chosen at random. Calculate the probability that both tins are the same colour.

24 Dumisani is a student at IB World College.
The probability that he will be woken by his alarm clock is $\frac{7}{8}$.
If he is woken by his alarm clock, the probability he will be late for school is $\frac{1}{4}$.
If he is not woken by his alarm clock, the probability he will be late for school is $\frac{3}{5}$.
Let W be the event 'Dumisani is woken by his alarm clock'.
Let L be the event 'Dumisani is late for school'.

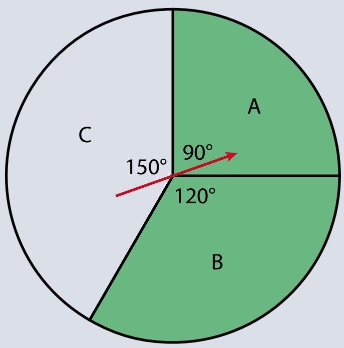
a) Copy and complete the tree diagram below.



b) Calculate the probability that Dumisani will be late for school.

c) Given that Dumisani is late for school, what is the probability that he was woken by his alarm clock?

25 The diagram shows a circle divided into three sectors A, B and C. The angles at the centre of the circle are 90°, 120° and 150°. Sectors A and B are shaded as shown.



The arrow is spun. It cannot land on the lines between the sectors. Let A, B, C and S be the events defined by

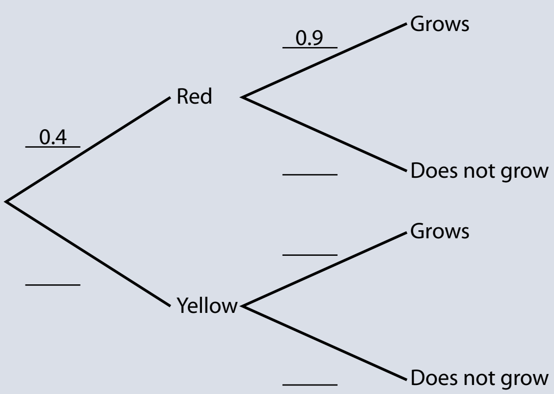
A : Arrow lands in sector A
B : Arrow lands in sector B
C : Arrow lands in sector C
S : Arrow lands in a shaded region.

Find

a) $P(B)$ **b)** $P(S)$ **c)** $P(A|S)$.

26 A packet of seeds contains 40% red seeds and 60% yellow seeds. The probability that a red seed grows is 0.9, and that a yellow seed grows is 0.8. A seed is chosen at random from the packet.

a) Complete the probability tree diagram below.



b) **(i)** Calculate the probability that the chosen seed is red and grows.
(ii) Calculate the probability that the chosen seed grows.
(iii) Given that the seed grows, calculate the probability that it is red.

27 Two unbiased six-sided dice are rolled, a red one and a black one. Let E and F be the events

E : the same number appears on both dice

F : the sum of the numbers is 10.

Find **a)** $P(E)$

b) $P(F)$

c) $P(E \cup F)$.

28 The table below shows the subjects studied by 210 students at a college.

	Year 1	Year 2	Totals
History	50	35	85
Science	15	30	45
Art	45	35	80
Totals	110	100	210

a) A student from the college is selected at random.

Let A be the event the student studies art.

Let B be the event the student is in year 2.

(i) Find $P(A)$.

(ii) Find the probability that the student is a year 2 art student.

(iii) Are the events A and B independent? Justify your answer.

b) Given that a history student is selected at random, calculate the probability that the student is in year 1.

c) Two students are selected at random from the college. Calculate the probability that one student is in year 1 and the other in year 2.

29 A bag contains 2 red balls, 3 blue balls and 4 green balls. A ball is chosen at random from the bag and is not replaced. A second ball is chosen. Find the probability of choosing one green ball and one blue ball in any order.

30 In a bilingual school there is a class of 21 pupils. In this class, 15 of the pupils speak Spanish as their first language and 12 of these 15 pupils are Argentinian. The other 6 pupils in the class speak English as their first language and 3 of these 6 pupils are Argentinian.

A pupil is selected at random from the class and is found to be Argentinian. Find the probability that the pupil speaks Spanish as his/her first language.

31 A new blood test has been shown to be effective in the early detection of a disease. The probability that the blood test correctly identifies someone with this disease is 0.99, and the probability that the blood test correctly identifies someone without that disease is 0.95. The incidence of this disease in the general population is 0.0001.

A doctor administered the blood test to a patient and the test result indicated that this patient had the disease. What is the probability that the patient has the disease?

32 The local Football Association consists of ten teams. Team A has a 40% chance of winning any game against a higher-ranked team, and a 75% chance of winning any game against a lower-ranked team. If A is currently in fourth position, find the probability that A wins its next game.

33 Given that events A and B are independent with $P(A \cap B) = 0.3$ and $P(A \cap B') = 0.3$, find $P(A \cup B)$.

34 A girl walks to school every day. If it is not raining, the probability that she is late is $\frac{1}{5}$. If it is raining, the probability that she is late is $\frac{2}{3}$. The probability that it rains on a particular day is $\frac{1}{4}$.

On one particular day the girl is late. Find the probability that it was raining on that day.

35 Given that $P(x) = \frac{2}{3}$, $P(y|x) = \frac{2}{5}$ and $P(y|x') = \frac{1}{4}$, find

a) $P(y')$ **b)** $P(x' \cup y')$.

36 The probability that a man leaves his umbrella in any shop he visits is $\frac{1}{3}$. After visiting two shops in succession, he finds he has left his umbrella in one of them. What is the probability that he left his umbrella in the second shop?

37 Two women, Ann and Bridget, play a game in which they take it in turns to throw an unbiased six-sided die. The first woman to throw a '6' wins the game. Ann is the first to throw.

a) Find the probability that

(i) Bridget wins on her first throw

(ii) Ann wins on her second throw

(iii) Ann wins on her nth throw.

b) Let p be the probability that Ann wins the game. Show that $p = \frac{1}{6} + \frac{25}{36}p$.

c) Find the probability that Bridget wins the game.

d) Suppose that the game is played six times. Find the probability that Ann wins more games than Bridget.

38 A box contains 22 red apples and 3 green apples. Three apples are selected at random, one after the other, without replacement.

a) The first two apples are green. What is the probability that the third apple is red?

b) What is the probability that exactly two of the three apples are red?

39 The probability that it rains during a summer's day in a certain town is 0.2. In this town, the probability that the daily maximum temperature exceeds 25 °C is 0.3 when it rains and 0.6 when it does not rain. Given that the maximum daily temperature exceeded 25 °C on a particular summer's day, find the probability that it rained on that day.

40 An integer is chosen at random from the first one thousand positive integers. Find the probability that the integer chosen is

a) a multiple of 4

b) a multiple of **both** 4 and 6.

41 The independent events A and B are such that $P(A) = 0.4$ and $P(A \cup B) = 0.88$. Find

a) $P(B)$

b) the probability that either A occurs or B occurs but **not both**.

42 Robert travels to work by train every weekday from Monday to Friday. The probability that he catches the 08:00 train on Monday is 0.66. The probability that he catches the 08:00 train on any other weekday is 0.75. A weekday is chosen at random.

a) Find the probability that he catches the train on that day.

b) Given that he catches the 08:00 train on that day, find the probability that the chosen day is Monday.

43 Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.

a) Write down the probability that Jack wins on his first throw.

b) Calculate the probability that Jill wins on her first throw.

c) Calculate the probability that Jack wins the game.

44 Bag A contains 2 red and 3 green balls.

a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen.

Bag B contains 4 red and n green balls.

b) Two balls are chosen without replacement from this bag. If the probability that two red balls are chosen is $\frac{2}{15}$, show that $n = 6$.

A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

c) Calculate the probability that two red balls are chosen.

d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die.

45 Given that $(A \cup B)' = \varnothing$, $\mathrm{P}(A'|B) = \frac{1}{3}$ and $\mathrm{P}(A) = \frac{6}{7}$, find $\mathrm{P}(B)$.

13 Differential Calculus I: Fundamentals

Assessment statements

6.1 Informal ideas of limit, continuity and convergence.

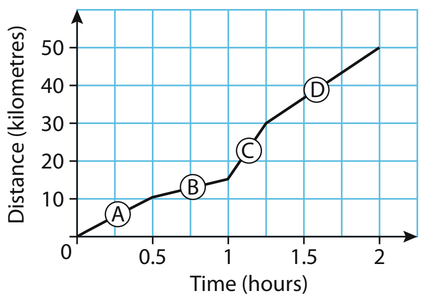
Definition of derivative from first principles: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

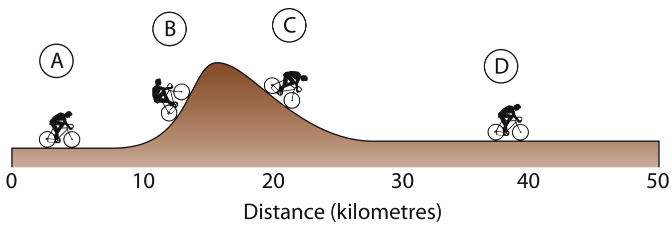
Derivative interpreted as gradient function and as rate of change.
Find equations of tangents and normals.
Identifying increasing and decreasing functions.
The second derivative.

6.2 Derivative of x^n.

6.3 Local maximum and minimum points.
Points of inflexion with zero and non-zero gradients.
Graphical behaviour of functions including the relationship between the graphs of f, f' and f''.

6.6 Kinematic problems involving displacement, s, velocity, v, and acceleration, a. (See also Chapter 15.)





A bicycle ride over a hill: The graph above left shows distance (km) versus time (hrs) for a 50-kilometre bicycle ride that included going up and then down a steep hill. There are four time intervals labelled A, B, C and D. During which interval is the bicyclist's speed the least? the greatest? During which two intervals is the bicyclist's speed about the same? How does the shape of the distance-time graph give information about the speed of the bicyclist during a certain interval? or at a particular moment (instant) during the ride?

Introduction

Calculus is the branch of mathematics that was developed to analyze and model change – such as velocity and acceleration. We can also apply it to study change in the context of slope, area, volume and a wide range of other real-life phenomena. Although mathematical techniques that you have studied previously deal with many of these concepts, the ability to model change was restricted. For example, consider the curve in Figure 13.1. This shows the motion of an object by indicating the distance (y metres) travelled after a certain amount of time (t seconds). Pre-calculus mathematics will only allow us to compute the **average velocity** between two different times (Figure 13.2). With calculus – specifically, techniques of differential calculus – we will be able to find the velocity of the object at a particular instant, known as its **instantaneous velocity** (Figure 13.3). The starting point for our study of calculus is the idea of a limit.

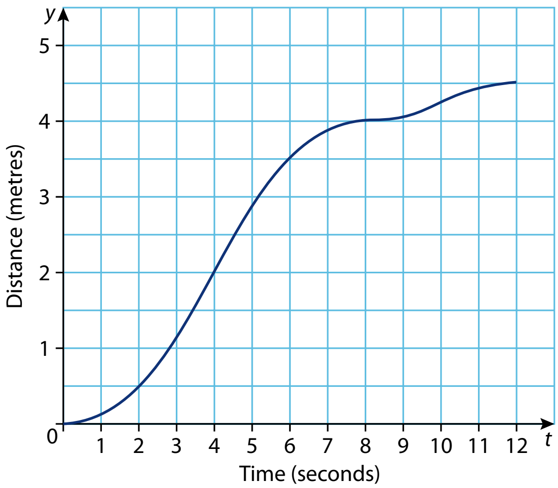


Figure 13.1 Distance–time graph for an object's motion.

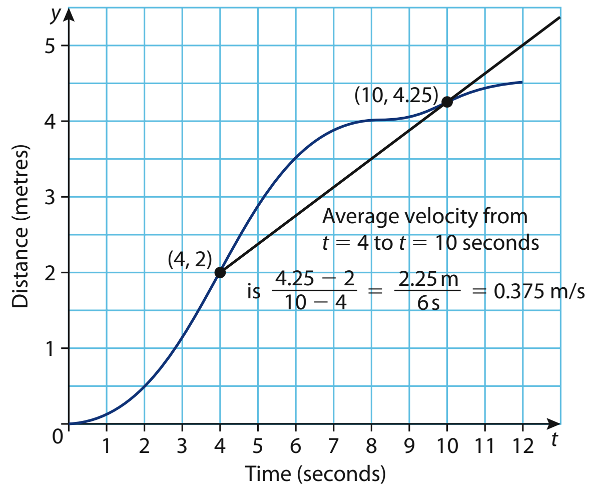


Figure 13.2 Computing average velocity from a distance–time graph.

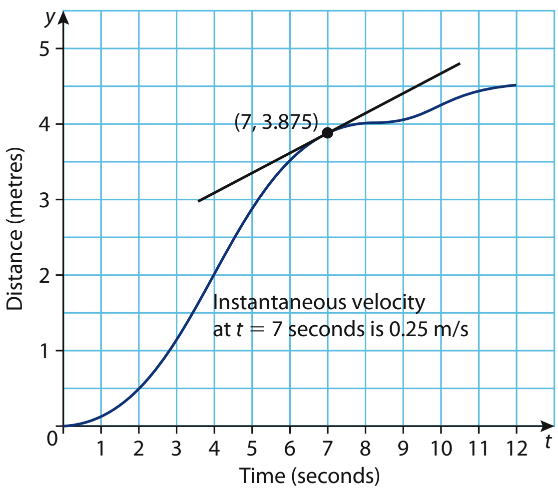


Figure 13.3 Instantaneous velocity from a distance–time graph.

13.1 Limits of functions

A **limit** is one of the ideas that distinguish calculus from algebra, geometry and trigonometry. The notion of a limit is a fundamental concept of calculus. Limits are not new to us. We often use the idea of a 'limit' in many non-mathematical situations. Mathematically speaking, we have encountered limits on at least two occasions previously in this book – finding the sum of an infinite geometric series (Section 4.4) and computing the irrational number e (Section 5.3).

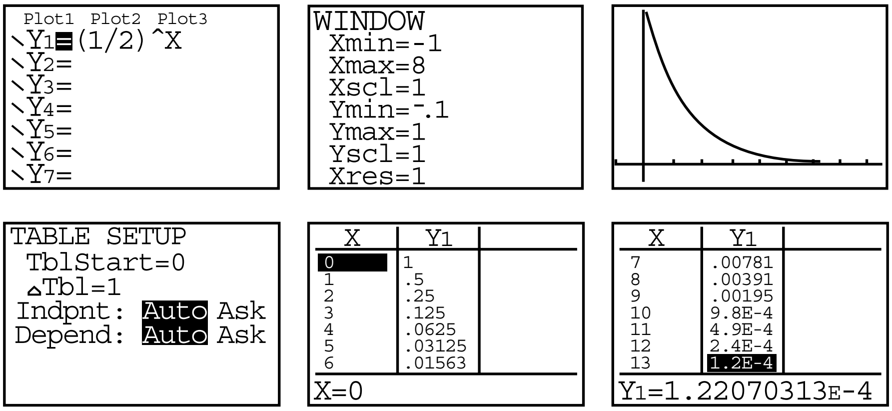
Recall from Section 4.4 that we established that if the sequence of partial sums for an infinite series **converges** to a finite number L we say that the infinite series has a 'sum' of L. Further on in that section, we used limits to algebraically confirm that the infinite series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ has a sum of 4. As part of the algebra for this, we reasoned that as the value of n increases in the positive direction without bound (i.e. $n \to +\infty$) the expression $\left(\frac{1}{2}\right)^n$ converges to zero – in other words, the **limit** of $\left(\frac{1}{2}\right)^n$ as n goes to positive infinity is zero. We express this result more efficiently using limit notation, as we did in Chapter 4, by writing $\lim\limits_{n \to \infty}\left(\frac{1}{2}\right)^n = 0$.

It is beyond the requirements of this course to establish a precise formal definition of a limit, but a closer look at justifying this limit and a couple of others can lead us to a useful informal definition.

Example 1

Evaluate $\lim_{n \to \infty} \left(\frac{1}{2}\right)^n$ by using your GDC to analyze the behaviour of the function $f(x) = \left(\frac{1}{2}\right)^x$ for large positive values of x.

Solution



The GDC screen images show the graph and table of values for $y = \left(\frac{1}{2}\right)^x$. Clearly, the larger the value of x, the closer that y gets to zero. Although there is no value of x that will produce a value of y equal to zero, we can get as close to zero as we wish. For example, if we wish to produce a value of y within 0.001 of zero, then we could choose $x = 10$ and $y = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$ $\approx 0.000\,976\,56$; and if we want a result within 0.000 0001 of zero, then we could choose $x = 24$ and $y = \left(\frac{1}{2}\right)^{24} = \frac{1}{16\,777\,216} \approx 0.000\,000\,059\,605$; and so on. Therefore, we can conclude that $\lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0$.

In calculus we are interested in limits of functions of real numbers. Although many of the limits of functions that we will encounter can only be approached and not actually reached (as in Example 1), this is not always the case. For example, if asked to evaluate the limit of the function $f(x) = \frac{x}{2} - 1$ as x approaches 6, we simply need to evaluate the function for $x = 6$. Since $f(6) = 2$, then $\lim_{x \to 6} \left(\frac{x}{2} - 1\right) = 2$.

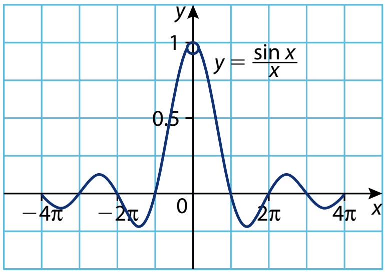
However, it is more common that we are unable to evaluate the limit of $f(x)$ as x approaches some number c because $f(c)$ does not exist.

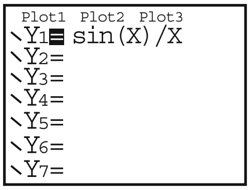
> The line $y = c$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$. For example, the line $y = 0$ (x-axis) is a horizontal asymptote of the graph of $y = \left(\frac{1}{2}\right)^x$ because $\lim_{x \to \infty} \left(\frac{1}{2}\right)^x = 0$.

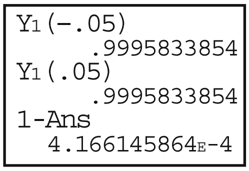
Example 2

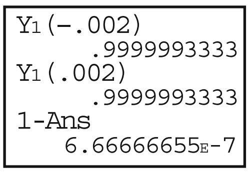
Find the following two limits using your GDC to analyze the behaviour of the relevant function.

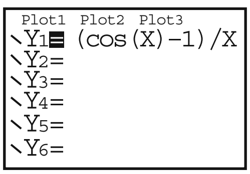
a) $\lim_{x \to 0} \frac{\sin x}{x}$ b) $\lim_{x \to 0} \frac{\cos x - 1}{x}$

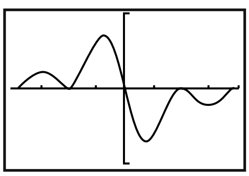


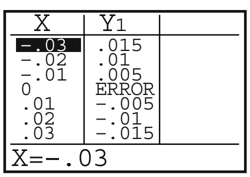












Solution

a) We are not able to evaluate this limit by direct substitution because when $x = 0$, $\frac{\sin x}{x} = \frac{0}{0}$ and is therefore undefined. Let's use our GDC again to analyze the behaviour of the function $f(x) = \frac{\sin x}{x}$ as x approaches zero from the right side and the left side.

Although there is no point on the graph of $y = \frac{\sin x}{x}$ corresponding to $x = 0$, it is clear from the graph that as x approaches zero (from either direction) the value of $\frac{\sin x}{x}$ converges to one. We can get the value of $\frac{\sin x}{x}$ arbitrarily close to 1 depending on our choice of x.

If we want $\frac{\sin x}{x}$ to be within 0.001 of 1, we choose $x = 0.05$ giving $\frac{\sin 0.05}{0.05} \approx 0.999\,583$ and $1 - 0.999\,583 = 0.000\,417 < 0.001$; and if we want $\frac{\sin x}{x}$ to be within 0.000 001 of 1, then we choose $x = 0.002$ giving $\frac{\sin 0.02}{0.02} \approx 0.999\,999\,3333$ and $1 - 0.999\,999\,3333$ $= 0.000\,000\,6667 < 0.000\,001$; and so on.

Therefore, $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

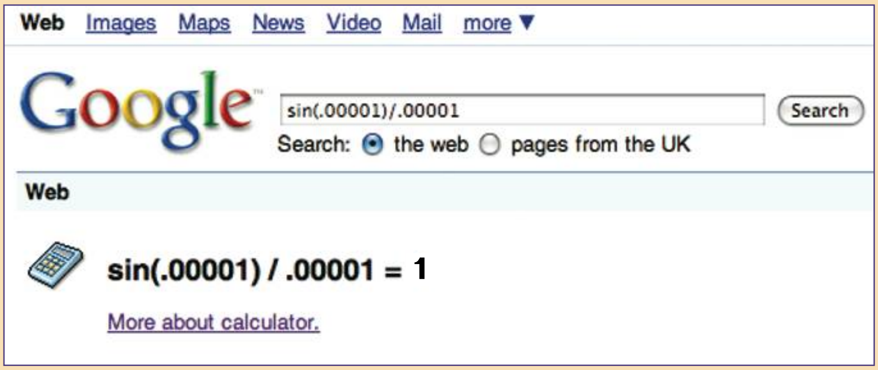
b) As with $y = \frac{\sin x}{x}$, substituting $x = 0$ into the function $y = \frac{\cos x - 1}{x}$ produces the meaningless fraction $\frac{0}{0}$. The graph of $y = \frac{\cos x - 1}{x}$ (GDC images right) reveals that the function values approach 0 as x goes to 0. A table produced on a GDC also shows that the function values approach zero from both directions.

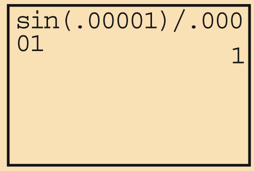
Therefore, $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

These two limits are confirmed analytically in the next section.

The analysis and result for Example 2 illustrates why it is preferred (and often necessary) that in calculus the argument of a trigonometric function be in radian measure rather than degrees. The limit of $\frac{\sin x}{x}$ as x goes to ∞ is not equal to 1 if x is in degrees. With your GDC in radian mode, duplicate the graph of $y = \frac{\sin x}{x}$ shown in the solution for Example 2. Now change the window dimensions on your GDC to Xmin $= -720$ and Xmax $= 720$ (equivalent to -4π and 4π) and graph $y = \frac{\sin x}{x}$ in degree mode. Explain why no graph appears.

It is interesting to note that if you ask your GDC to evaluate the function $y = \frac{\sin x}{x}$ for a sufficiently small value of x it will give a result of exactly 1. The function is undefined for $x = 0$ and can never give a result of exactly 1, so obviously the calculator is making an error. Due to memory restrictions the calculator has rounded off the result to 1. The GDC image below shows that for $x = 0.00001$ the result has been rounded to 1 when the actual value is $0.999\,999\,999\,98\overline{3}$ (digit 3 repeating). Even the Google calculator (see image below) gives an incorrect result.





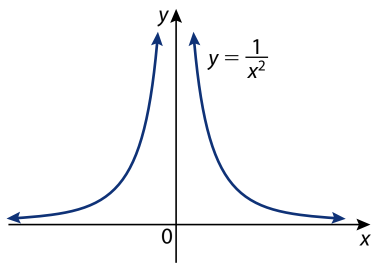
Functions do not necessarily converge to a finite value at every point – it's possible for a limit not to exist.

Example 3

Find $\lim\limits_{x \to 0} \frac{1}{x^2}$, if it exists.

Solution

As x approaches zero, the value of $\frac{1}{x^2}$ becomes increasingly large in the positive direction. The graph of the function (left) seems to indicate that we can make the values of $y = \frac{1}{x^2}$ arbitrarily large by choosing x close enough to zero. Therefore, the values of $y = \frac{1}{x^2}$ do not approach a finite number, so $\lim\limits_{x \to 0} \frac{1}{x^2}$ does not exist.



Although we can describe the behaviour of the function $y = \frac{1}{x^2}$ by writing $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$, this does not mean that we consider ∞ to represent a number – it does not. This notation is simply a convenient way to indicate in what manner the limit does not exist.

Limit of a function

If $f(x)$ becomes arbitrarily close to a unique finite number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L. The notation for indicating this is $\lim\limits_{x \to c} f(x) = L$.

When a function $f(x)$ becomes *arbitrarily close* to a finite number L, we say that $f(x)$ **converges** to L.

The line $x = c$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either $\lim\limits_{x \to c} f(x) = \infty$ or $\lim\limits_{x \to c} f(x) = -\infty$.
For example, the line $x = 0$ (y-axis) is a vertical asymptote of the graph of $y = \frac{1}{x^2}$ because $\lim\limits_{x \to 0} \frac{1}{x^2} = \infty$.

Often when trying to determine the limit of a quotient by direct substitution, we may get a meaningless fraction such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Such an expression is called an **indeterminate form** because we cannot use it to determine the desired limit. When confronted with an indeterminate form we need to perform some algebraic manipulation to the quotient to get it into a form so that the limit can be evaluated by direct substitution and/or applying known limits.

For our purposes in this course, it is also important to be able to apply some basic algebraic manipulation in order to evaluate the limits of some functions algebraically, rather than by conjecturing from a graph or table.

The following five properties of limits are also useful.

Properties of limits

Let a and b be real numbers, and let f and g be functions with the following limits.

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = K$$

1 Constant: $\lim_{x\to a} b = b$

2 Scalar multiple: $\lim_{x\to a} [b \cdot f(x)] = b \cdot L$

3 Sum or difference: $\lim_{x\to a} [f(x) \pm g(x)] = L \pm K$

4 Product: $\lim_{x\to a} [f(x) \cdot g(x)] = L \cdot K$

5 Quotient: $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K} \qquad K \neq 0$

Example 4

Evaluate each limit algebraically.

a) $\lim_{x\to\infty} \frac{5x-3}{x}$

b) $\lim_{p\to 0} (3x^2 - 4px + p^2)$

c) $\lim_{h\to 0} \frac{[(x+h)^2 - 6] - (x^2 - 6)}{h}$

d) $\lim_{x\to\infty} \frac{3x^2 + 5x - 1}{2x^2 + 1}$

Solution

a) $\lim_{x\to\infty} \frac{5x-3}{x} = \lim_{x\to\infty} \left(\frac{5x}{x} - \frac{3}{x}\right)$ Split the fraction into two terms and …

$= \lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{3}{x}$ … evaluate the limit of each term separately.

$= 5 - 0 = 5$ Therefore, $\lim_{x\to\infty} \frac{5x-3}{x} = 5$.

b) $\lim_{p\to 0} (3x^2 - 4px + p^2) = \lim_{p\to 0} 3x^2 - \lim_{p\to 0} 4px + \lim_{p\to 0} p^2$ Evaluate the limit of each term separately.

$= 3x^2 - 0 + 0 = 3x^2$ Therefore, $\lim_{p\to 0} (3x^2 - 4px + p^2) = 3x^2$.

c)
$$\lim_{h\to 0} \frac{[(x+h)^2 - 6] - (x^2 - 6)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 6 - x^2 + 6}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(2x + h)}{h}$$
$$= \lim_{h\to 0} 2x + \lim_{h\to 0} h$$
$$= 2x + 0 = 2x$$

Therefore, $\lim_{h\to 0} \frac{[(x+h)^2 - 6] - (x^2 - 6)}{h} = 2x$.

d) $\lim_{x\to\infty} \dfrac{3x^2 + 5x - 1}{2x^2 + 1}$

Dividing numerator and denominator by largest power of x. i.e. x^2.

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{5x}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 + \frac{5}{x} - \frac{1}{x^2}}{2 + \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{5}{x} - \lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{1}{x^2}}$$

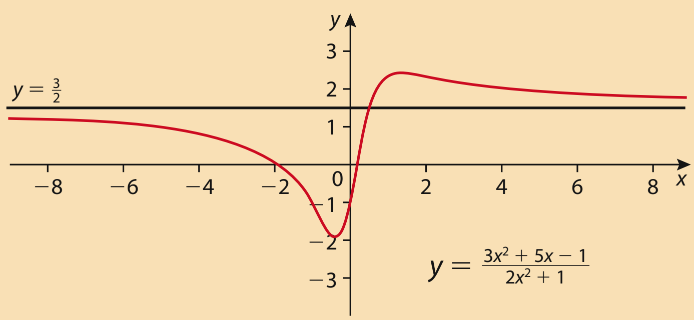
Applying $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ and $\lim_{x\to a} [f(x) \pm g(x)] = L \pm K$.

$$= \frac{3 + 0 - 0}{2 + 0}$$

Therefore, $\lim_{x\to\infty} \dfrac{3x^2 + 5x - 1}{2x^2 + 1} = \dfrac{3}{2}$.

The limits in parts b) and c) of Example 4 show that in some cases the limit of a function is itself a function.

Connect the limit in Example 4 part d) with the end behaviour of rational functions covered in Section 3.4. Since $\lim_{x\to\infty} \dfrac{3x^2 + 5x - 1}{2x^2 + 1} = \dfrac{3}{2}$, the rational function $y = \dfrac{3x^2 + 5x - 1}{2x^2 + 1}$ will have a horizontal asymptote of $y = \dfrac{3}{2}$. In other words, as $x \to \infty$ or $x \to -\infty$ the function will approach the value of $y = \dfrac{3}{2}$ as illustrated in the graph shown.



In this section we evaluated limits by guessing and checking with the help of our GDC. This process led us to conclude that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ and $\lim_{x\to 0} \cos \dfrac{x - 1}{x} = 0$.

It was reasonable to take this approach since it is not possible to perform algebraic manipulations on these expressions as we did with the expressions in Example 4. However, if possible we should always try to use analytic methods to evaluate a limit as illustrated in the next example.

Example 5

a) Estimate the value of $\lim_{x\to 0} \dfrac{\sqrt{x^2 + 4} - 2}{x^2}$ by evaluating the function $f(x) = \dfrac{\sqrt{x^2 + 4} - 2}{x^2}$ for $x = \pm 0.5, \pm 0.01, \pm 0.0001, \pm 0.000005, \pm 0.000001$.

b) Using algebra and properties of limits, evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}$.

c) Comment on the two results.

Solution

a) A GDC that displays results to an accuracy of ten significant figures gives the following results.

x	$f(x)=\dfrac{\sqrt{x^2+4}-2}{x^2}$
± 0.5	0.246 211 2512
± 0.01	0.249 998 438
± 0.0001	0.25
$\pm 0.000\,005$	0.248
$\pm 0.000\,003$	0.244 444 4444
$\pm 0.000\,001$	0

The GDC results in the table seem unusual. Initially as x approaches zero from either direction the function values appear to be approaching $\frac{1}{4}$, but then as the function is evaluated for values even closer to zero, the function values continue to decrease to zero.

Is $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}$ equal to $\frac{1}{4}$ or 0?

If we trust our GDC, we may be tempted to conclude that $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}=0$.

b) We cannot immediately apply the limit property for quotients, $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{L}{K}$ because we obtain the indeterminate form $\dfrac{0}{0}$.

We need to use the algebraic technique of multiplying numerator and denominator by the conjugate of the expression in the numerator. This will 'rationalize' the numerator – and may lead to an equivalent expression for which we can apply the quotient property for limits.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\cdot\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}\\
&= \lim_{x\to 0}\frac{\left(\sqrt{x^2+4}\right)^2-2^2}{x^2\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{x^2+4-4}{x^2\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}\\
&= \frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\sqrt{x^2+4}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}
\end{aligned}$$

Therefore, $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}=\dfrac{1}{4}$.

c) Because of memory limitations a GDC will sometimes give a false value. Because $\sqrt{x^2 + 4}$ is very close to 2 when x is very small, a GDC will eventually consider $\sqrt{x^2 + 4}$ to be equal to 2.000 000 00 … (to as many digits as the GDC is capable of computing) when x is sufficiently small. Your GDC is a very powerful tool, but like any tool it does have its limitations.

Exercise 13.1

In questions 1–4, evaluate each limit algebraically and then confirm your result by means of a table or graph on your GDC.

1 $\lim\limits_{n\to\infty} \dfrac{1 + 4n}{n}$

2 $\lim\limits_{h\to 0} (3x^2 + 2hx + h^2)$

3 $\lim\limits_{d\to 0} \dfrac{(x + d)^2 - x^2}{d}$

4 $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3}$

In questions 5–7, investigate the limit of the expression (if it exists) as $x \to \infty$ by evaluating the expression for the following values of x: 10, 50, 100, 1000, 10 000 and 1 000 000. Hence, make a conjecture for the value of each limit.

5 $\lim\limits_{x\to\infty} \dfrac{3x + 2}{x^2 - 3}$

6 $\lim\limits_{x\to\infty} \dfrac{5x - 6}{2x + 5}$

7 $\lim\limits_{x\to\infty} \dfrac{3x^2 + 2}{x - 3}$

In questions 8–13, find the limit, if it exists.

8 $\lim\limits_{x\to 4} \dfrac{x - 4}{x^2 - 16}$

9 $\lim\limits_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 1}$

10 $\lim\limits_{x\to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$ • **Hint:** multiply numerator and denominator by conjugate of numerator

11 $\lim\limits_{x\to\infty} \dfrac{x^3 - 1}{4x^3 - 3x + 1}$

12 $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$ • **Hint:** rewrite $\tan x$ as $\dfrac{\sin x}{\cos x}$ and apply property $\lim\limits_{x\to 0} [f(x) \cdot g(x)] = L \cdot K$

13 $\lim\limits_{\theta\to 0} \dfrac{\sin 3\theta}{\theta}$ • **Hint:** rewrite $\dfrac{\sin 3\theta}{\theta}$ as $\left(3\dfrac{\sin 3\theta}{3\theta}\right)$ and apply $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

14 Use the graphing or table capabilities of your GDC to investigate the values of the expression $\left(1 + \frac{1}{c}\right)^c$ as c increases without bound (i.e. $c \to \infty$). Explain the significance of the result.

15 If it is known that the line $y = 3$ is a horizontal asymptote for the function $f(x)$, state the value of each of the following two limits: $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

16 If it is known that the line $x = a$ is a vertical asymptote for the function $g(x)$ and $g(x) > 0$, what conclusion can be made about $\lim\limits_{x\to a} g(x)$?

17 State the equations of all horizontal and vertical asymptotes for the following functions. Confirm using your GDC.

a) $f(x) = \dfrac{3x - 1}{1 + x}$

b) $g(x) = \dfrac{1}{(x - 2)^2}$

c) $g(x) = \dfrac{1}{x - a} + b$

d) $R(x) = \dfrac{2x^2 - 3}{x^2 - 9}$

e) $d(x) = \dfrac{5 - 3x}{x^2 - 5x}$

f) $p(x) = \dfrac{x^2 - 4}{x - 4}$

For questions 18 and 19, a) use your GDC to estimate the limit, and b) use analytic methods to evaluate the limit.

18 $\lim\limits_{x \to 2} \dfrac{\sqrt{x^2 + 5} - 3}{x^2 - 2x}$

19 $\lim\limits_{x \to +\infty} \dfrac{4x - 1}{\sqrt{x^2 + 2}}$

20 Show that $\lim\limits_{h \to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h} = \dfrac{1}{2\sqrt{x}}$.

21 Show that $\lim\limits_{h \to 0} \dfrac{\frac{1}{x + h} - \frac{1}{x}}{h} = -\dfrac{1}{x^2}$.

13.2 The derivative of a function: definition and basic rules

Tangent lines and the slope (gradient) of a curve

In Section 1.6, we reviewed linear equations in two variables. And, later in Section 2.1, we established that any non-vertical line represents a function for which we typically assign the variables x and y for values in the domain and range of the function, respectively. Any linear function can be written in the form $y = mx + c$. This is the slope-intercept form for a linear equation, where m is the slope (or gradient) of the graph and c is the y-coordinate of the point at which the graph intersects the y-axis (i.e. the y-intercept). The value of the slope m, defined as $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{vertical change}}{\text{horizontal change}}$, will be the same for any pair of points, (x_1, y_1) and (x_2, y_2), on the line. An essential characteristic of the graph of a linear function is that it has a constant slope. This is not true for the graphs of non-linear functions.

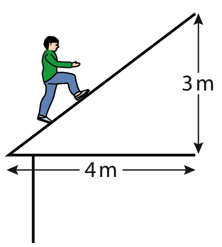
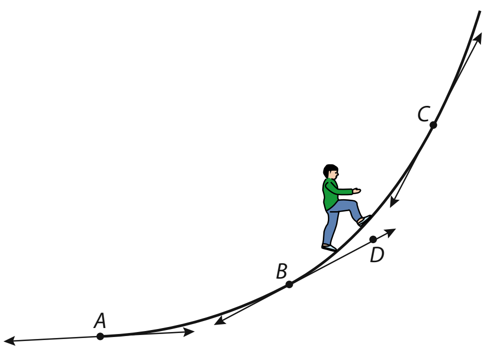


Figure 13.4 Slope of a straight line.

Consider a person walking up the side of a pitched roof as shown in Figure 13.4. At *any* point along the line segment PQ the person is experiencing a slope of $\frac{3}{4}$. Now consider someone walking up the curve shown in Figure 13.5, which passes through the three points A, B and C. As the person walks along the curve from A to C, he/she will experience a steadily increasing slope. The slope is continually changing from one point to the next along the curve. Therefore, it is incorrect to say that a non-linear function, whose graph is a curve, has ***a*** slope – it has *infinitely many* slopes. We need a means to determine the slope of a non-linear function *at a specific point* on its graph.

Figure 13.5 Slope of a curve.



Imagine if the slope of the curve in Figure 13.5 stopped increasing (remained constant) after point B. From that point on, a person walking up the curve would move along a line with a slope equal to the slope of the curve at point B. This line – containing point D in the diagram – only 'touches' the curve once at B. Line (BD) is **tangent** to the curve at point B. Therefore, finding the slope of the line that is tangent to a curve at a certain point will give us the slope of the curve at that point.

The slope (gradient) of a curve at a point is the slope of the line that is tangent to the curve at that point.

Finding the slope of a curve at a point – or better – finding a rule (function) that gives us the slope at any point on the curve is very useful information in many applications. The slope of a line, or of a curve at a point, is a measure of how fast variable y is changing as variable x changes. **The slope represents the rate of change of y with respect to x.** To find the slope of a tangent line, we first need to clarify what it means to say that a line is tangent to a curve at a point. Then we can establish a method to find the tangent line at a point.

● **Hint:** The word 'curve' can often mean the same as 'function', even if the function is linear.

The three graphs in Figure 13.6 show different configurations of tangent lines. A tangent line may cross or intersect the graph at one or more points.

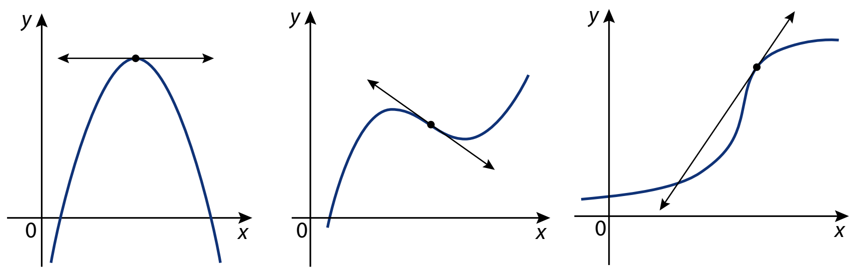
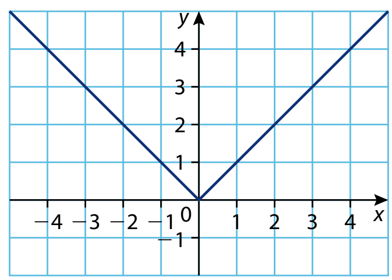


Figure 13.6 Different configurations of lines tangent to a curve.

Figure 13.7 $y = |x|$



For many functions, the graph has a tangent at *every* point. Informally, a function is said to be *smooth* if it has this property. Any linear function is certainly smooth, since the tangent at each point coincides with the original graph. However, some graphs are not smooth at every point. Consider the point $(0, 0)$ on the graph of the function $y = |x|$ (Figure 13.7). Zooming in on $(0, 0)$ will always produce a V-shape rather than smoothing out to appear more and more linear. Therefore, there is no tangent to the graph at this point.

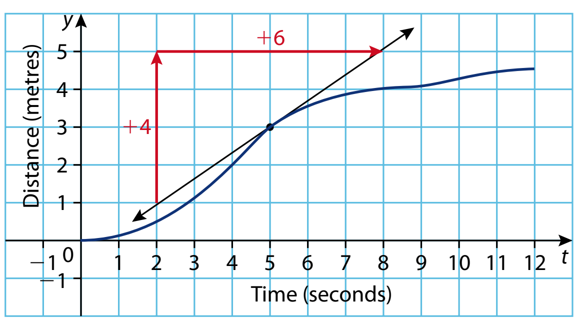


Figure 13.8 Estimating the slope of a tangent line.

One way to find the tangent line of a graph at a particular point is to make a visual estimate. Figure 13.8 reproduces the time-distance graph for an object's motion from the previous section (Figure 13.1). The slope at any point (t, y) on the curve will give us the rate of change of the distance y with respect to time t, in other words the object's **instantaneous velocity** at time t. In the figure, an estimate of the line tangent to the curve at $(5, 3)$ has been drawn. Reading from the graph, the slope appears to be $\frac{4}{6} = \frac{2}{3}$. Or, in other words, the object has a velocity of approximately 0.667 m/s at the instant when $t = 5$ seconds.

A more precise method of finding tangent lines makes use of a secant line and a limit process. Suppose that f is any smooth function, so the tangent to its graph exists at all points. A **secant line** (or chord) is drawn through the point for which we are trying to find a tangent to f and a second point on the graph of f, as shown in Figure 13.9a. If P is the point of tangency with coordinates $(x, f(x))$, choose a point Q to be horizontally some h units away. Hence, the coordinates of point Q are $(x + h, f(x + h))$. Then the slope of the secant line (PQ) is $m_{\text{sec}} = \dfrac{f(x+h) - f(x)}{(x+h) - x} = \dfrac{f(x+h) - f(x)}{h}$.
The right side of this equation is often referred to as a **difference quotient**. The numerator is the change in y, and the denominator h is the change in x. The limit process of achieving better and better approximations for the slope of the tangent at P consists of finding the slope of the secant (PQ) as Q moves ever closer to P, as shown in the graphs in Figure 13.9b and Figure 13.9c. In doing so, the value of h will approach zero.

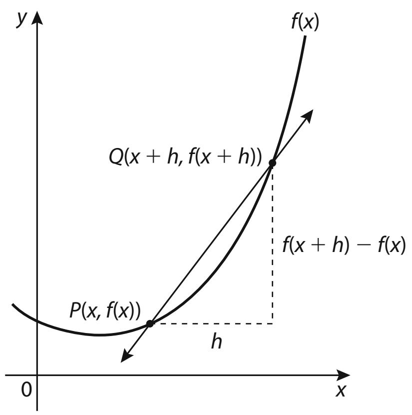


Figure 13.9a

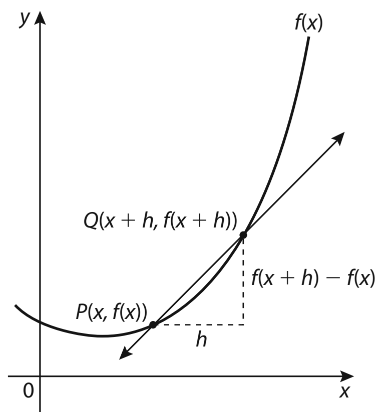


Figure 13.9b

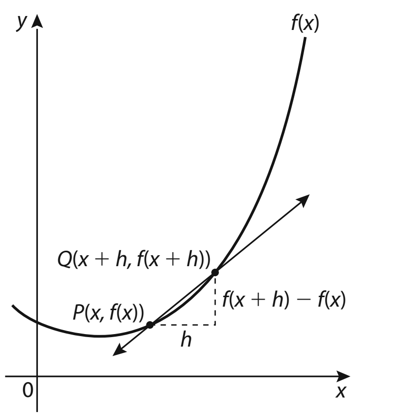


Figure 13.9c As h tends to zero, the secant line becomes a better approximation of the tangent line.

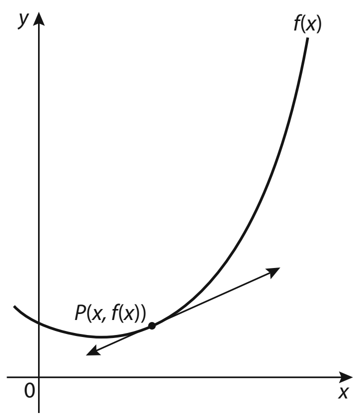


Figure 13.9d Tangent to f at point P.

By evaluating a limit of the slope of the secant lines as h approaches zero, we can find the exact slope of the tangent line at $P(x, f(x))$.

> The word 'secant', as applied to a line, comes from the Latin word *secare*, meaning to cut. The word 'tangent' comes from the Latin verb *tangere*, meaning to touch.

The slope (gradient) of a curve at a point

The slope of the curve $y = f(x)$ at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$m_{\tan} = \lim_{h \to 0} m_{\sec} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists.

Let's apply the definition of the slope of a curve at a point to find a rule, or function, for the slope of all of the tangent lines to a curve.

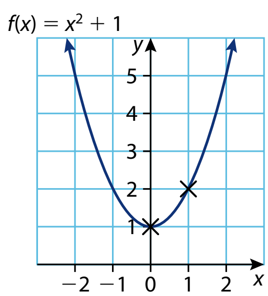
Example 6

Find a rule for the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$. Use this rule to find the exact slope of the curve at the point where $x = 0$ and at the point where $x = 1$.

Solution

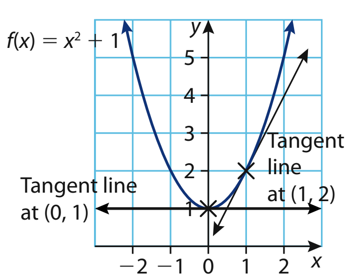
Let $(x, f(x))$ represent any point on the graph of f. By definition, the slope of the tangent line at $(x, f(x))$ is:

$$\begin{aligned}
m &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h} \\
&= \lim_{h \to 0} \frac{[x^2 + 2xh + h^2 + 1] - [x^2 + 1]}{h} \\
&= \lim_{h \to 0} \frac{x^2 - x^2 + 2xh + h^2 + 1 - 1}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h)}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{aligned}$$



Therefore, the slope at any point $(x, f(x))$ on the graph of f is $2x$.

At the point where $x = 0$, the slope is $2(0) = 0$. This makes visual sense because the point $(0, 1)$ is the vertex of the parabola $y = x^2 + 1$, and we expect that the tangent at this point is a horizontal line with a slope of zero. At the point where $x = 1$, the slope is $2(1) = 2$. This also makes visual sense because moving along the curve from $(0, 1)$ to $(1, 2)$ the slope is steadily increasing.



In Example 6, from the function $f(x) = x^2 + 1$ we used the limit process to derive another function with the rule $2x$. With this derived function we can compute the slope (gradient) of the graph of $f(x)$ at a point from simply inputting the x-coordinate of the point. This *derived* function is called the **derivative** of f at x. It is given the notation $f'(x)$, which is commonly read as 'f prime of x', or simply, 'the derivative of f of x.'

The derivative and differentiation

- The **derivative**, $f'(x)$, at a point x in the domain of f is the slope (gradient) of the graph of f at $(x, f(x))$, and is given by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
provided that this limit exists.
- If the derivative exists at each point of the domain of f, we say that f is **smooth**.
- The process of finding the derivative, $f'(x)$, is called **differentiation**.
- If $y = f(x)$, then $f'(x)$ is a formula for the instantaneous **rate of change** of y with respect to x.

If finding the derivative of a function indicated with the function notation $f(x)$, then – as shown already – the derivative is usually denoted as $f'(x)$. However, there are two other notations with which you should be familiar. Commonly, if a function is given as y in terms of x, then the derivative is denoted as y', read as 'y prime.' The notation $\frac{dy}{dx}$ is also often used to indicate a derivative, and is read as 'the derivative of y with respect to x.' Note: $\frac{dy}{dx}$ is not a fraction. If, for example, $y = x^2 + 1$, the derivative can be denoted by writing $\frac{d}{dx}(x^2 + 1) = 2x$. This is read as 'the derivative of $x^2 + 1$ with respect to x is $2x$.'

Differentiating from first principles

Depending on the particular purpose that you have in differentiating a function, you can consider the derivative as giving the slope of the graph of the function *or* the rate of change of the dependent variable (commonly y) with respect to the independent variable (commonly x). Both interpretations are useful and widely applied.

Using the limit definition directly to find the derivative of a function (as we did in Example 6) is often called 'differentiating from first principles'.

Example 7

Differentiating from first principles, find the derivative of $f(x) = x^3$.

Solution

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{(x+h)(x+h)^2 - x^3}{h} \\ &= \lim_{h \to 0} \frac{(x+h)(x^2 + 2hx + h^2) - x^3}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{h(3x^2 + 3hx + h^2)}{h} \\ &= \lim_{h \to 0} (3x^2 + 3hx + h^2) \\ &= 3x^2 \end{aligned}$$

Therefore, the derivative of $f(x) = x^3$ is $f'(x) = 3x^2$.

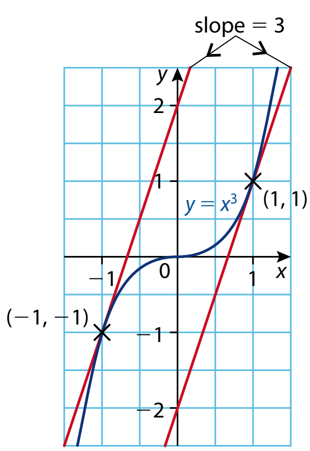


Figure 13.10 Two tangents to $y = x^3$ that are parallel.

As in Example 6, the result for Example 7 is a function that gives us the slope at any point on the graph of $y = x^3$. For example, the points $(1, 1)$ and $(-1, -1)$ both lie on $y = x^3$, and the slopes at these points are respectively $f'(1) = 3(1)^2 = 3$ and $f'(-1) = 3(-1)^2 = 3$. Hence, the tangents at these points will be parallel, as shown in Figure 13.10.

Let's examine the relationship between the slopes of tangents to the curve $f(x) = x^2 + 1$ (Example 6) and slopes of tangents to $g(x) = x^2$. Recall that we found the derivative of $f(x)$ to be $f'(x) = 2x$. It appears from the graphs of f and g, in Figure 13.11, that the slopes of tangents at points with the same x-coordinate are equal. For example, the tangent to g at the point $(1, 2)$ looks parallel to the tangent to f at $(1, 1)$, as shown in Figure 13.11. This implies that the derivatives of the two functions are equal. Rather than confirming this conjecture by finding the derivative of $g(x) = x^2$ by first principles (i.e. using the limit definition), let's use the graphical and computing power of our GDC. Any GDC model is capable of computing the slope of a curve at a point – either on the GDC's 'home' screen, or its graphing screen. The screen images below show computing derivative values for $y = x^2$ on the 'home' screen.

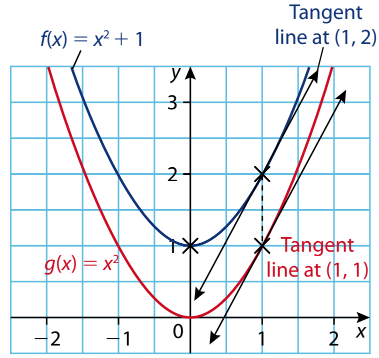
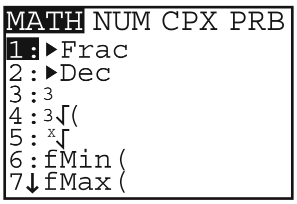
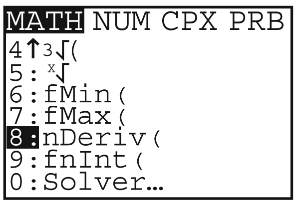


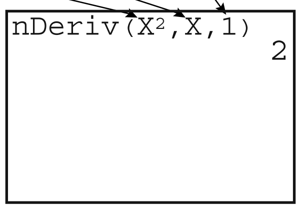
Figure 13.11

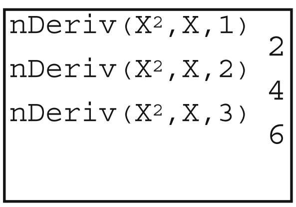
The exact command name and syntax for computing the value of a derivative at a point may vary from one GDC model to another.

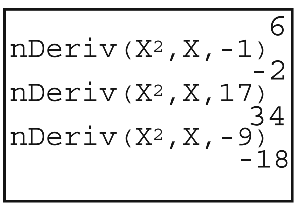
This command finds the value of the derivative of $\boldsymbol{y = x^2}$ in **terms of** $\boldsymbol{x}$, at the point $\boldsymbol{x = 1}$.











Our GDC results confirm our conjecture that the derivative of $g(x) = x^2$ is $g'(x) = 2x$.

Example 8

From first principles, find:

a) y' given $y = 3x^2 + 2x$ b) $\dfrac{dy}{dx}$ given $y = \dfrac{1}{x}$

Solution

We will apply the definition of the derivative, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, in both a) and b).

a) $$y' = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h)] - (3x^2 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{(3x^2 + 6hx + 3h^2 + 2x + 2h) - (3x^2 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{6hx + 3h^2 + 2h}{h}$$

$$= \lim_{h \to 0} (6x + 3h + 2) \quad \Rightarrow \quad y' = 6x + 2$$

b) $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h\to 0}\left(\frac{\frac{-h}{x(x+h)}}{\frac{h}{1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{-h}{x(x+h)}\cdot\frac{1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-1}{x^2 + hx}\right) \Rightarrow \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \text{ or } \frac{d}{dx}(x^{-1}) = -x^{-2}$$

Basic differentiation rules

We have now established the following results:

- If $f(x) = x^2$, then $f'(x) = 2x$.
- If $f(x) = x^2 + 1$, then $f'(x) = 2x$.
- If $f(x) = 3x^2 + 2x$, then $f'(x) = 6x + 2$.
- If $f(x) = x^3$, then $f'(x) = 3x^2$.
- If $f(x) = x^{-1}$, then $f'(x) = -x^{-2}$.

In addition, we know that if $f(x) = x$, then $f'(x) = 1$, since the line $y = x$ has a constant slope equal to 1; and that if $f(x) = 1$, then $f'(x) = 0$ because the line $y = 1$ is horizontal and thus has a constant slope equal to 0. Furthermore, the graph of any function $f(x) = c$, where c is a constant, is a horizontal line, confirming that if $f(x) = c$, $c \in \mathbb{R}$, then $f'(x) = 0$. In other words, the derivative of a constant is zero. This leads to our first basic rule of differentiation.

The constant rule

The derivative of a constant function is zero. That is, given c is a real number, and if $f(x) = c$, then $f'(x) = 0$.

These following results:

$$\begin{aligned} f(x) &= x^{-1} & &\Rightarrow & f'(x) &= -x^{-2} \\ f(x) &= x^0 = 1 & &\Rightarrow & f'(x) &= 0 \\ f(x) &= x^1 = x & &\Rightarrow & f'(x) &= 1 \\ f(x) &= x^2 & &\Rightarrow & f'(x) &= 2x \\ f(x) &= x^3 & &\Rightarrow & f'(x) &= 3x^2 \end{aligned}$$

can be summarized in the single statement:

$$\text{if } f(x) = x^n \text{ then } f'(x) = nx^{n-1} \text{ for } n = -1, 0, 1, 2, 3$$

In fact, this statement is true not just for these values but for *any* value of n that is a rational number ($n \in \mathbb{Q}$). This leads to our second basic rule of differentiation.

The derivative of x^n

Given n is a rational number, and if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Functions of the form $f(x) = x^n$ are called **power functions**, so the differentiation rule $\frac{d}{dx}(x^n) = nx^{n-1}$ gives the rule for differentiating power functions – and is often referred to as the **power rule**.

Recall from Chapter 4 the binomial theorem for positive integers

$$(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r.$$

Applying this to the limit definition of the derivative gives,

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^n + nx^{n-1}h + \frac{1}{2}n(n-1)x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \lim_{h \to 0} \frac{1}{2}n(n-1)x^{n-2}h + \ldots + \lim_{h \to 0} nxh^{n-2} + \lim_{h \to 0} h^{n-1}$$

$$= nx^{n-1} + 0 + \ldots + 0 + 0$$

$$= nx^{n-1}$$

Therefore, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Another basic rule of differentiation is suggested by our result that the derivative of $f(x) = x^2 + 1$ is $f'(x) = 2x$. The derivative of a sum of a number of terms is obtained by differentiating each term separately – i.e. differentiating 'term-by-term'. That is,

$$\frac{d}{dx}(x^2 + 1) = \frac{d}{dx}(x^2) + \frac{d}{dx}(1) = 2x + 0 = 2x.$$

The sum and difference rule

If $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$.

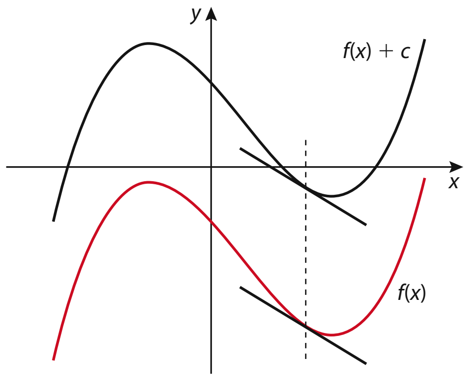
The sum rule for derivatives can help us give a very convincing justification of our first differentiation rule: the constant rule. The fact that the derivative of a constant must be zero can be verified by considering the transformation of the graph of a function (Section 2.4). The graph of the function $f(x) + c$, where $c \in \mathbb{R}$, is a vertical translation by c units of the graph of $f(x)$. As Figure 13.12 illustrates, when the graph of a function is translated vertically its shape is preserved. Hence, the slope of the tangent line to the graph of $f(x) + c$ will be the same as that for $f(x)$ at a particular value of x. This means that the derivatives for the two functions must be equal. That is,

$$\frac{d}{dx}[f(x) + c] = \frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[f(x)] + \frac{d}{dx}(c) = \frac{d}{dx}[f(x)]$$

This is only true if $\frac{d}{dx}(c) = 0$.

Figure 13.12 Translating the graph of a function vertically does not alter the slope of the tangent line at a particular value of x. Hence the derivatives of the two functions are equal.



A fourth basic rule of differentiation is illustrated by our result that the derivative of $f(x) = 3x^2 + 2x$ is $f'(x) = 6x + 2$. Using the sum rule, $f'(x) = \frac{d}{dx}(3x^2 + 2x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x) = 6x + 2$. The fact that $\frac{d}{dx}(3x^2) = 6x$ suggests that $3 \cdot \frac{d}{dx}(x^2) = 3 \cdot 2x = 6x$. In other words, the derivative of a function being multiplied by a constant is equal to the constant multiplying the derivative of the function.

The constant multiple rule

If $f(x) = c \cdot g(x)$ then $f'(x) = c \cdot g'(x)$.

As mentioned before, and as you have seen, there are different notations used for indicating a derivative or differentiation. These can be traced back to the fact that calculus was first developed by Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) independently of each other – and hence introduced different symbols for methods of calculus. The 'prime' notations y' and $f'(x)$ come from notations that Newton used for derivatives. The $\frac{dy}{dx}$ notation is similar to that used by Leibniz for indicating differentiation. Each has its advantages and disadvantages. For example, it is often easier to write our four basic rules of differentiation using Leibniz notation as shown below.

Constant rule: $\frac{d}{dx}(c) = 0, \ c \in \mathbb{R}$

Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}, \ n \in \mathbb{Q}$

Sum and difference rule: $\frac{d}{dx}[g(x) + h(x)] = \frac{d}{dx}[g(x)] + \frac{d}{dx}[h(x)]$

Constant multiple rule: $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)], c \in \mathbb{R}$

Example 9

For each function: (i) find the derivative using the basic differentiation rules; (ii) find the slope of the graph of the function at the indicated points; and (iii) use your GDC to confirm your answer for (ii).

Function | Points

a) $f(x) = x^3 + 2x^2 - 15x - 13$ — $(-3, 23), (3, -13)$

b) $f(x) = (2x - 7)^2$ — $(2, 9), (\frac{7}{2}, 0)$

c) $f(x) = 3\sqrt{x} - 6$ — $(4, 0), (9, 3)$

d) $f(x) = \dfrac{x^4}{4} - \dfrac{3x^3}{2} - 2x^2 + \dfrac{15x}{2} + \dfrac{3}{4}$ — $(5, -43), (0, 0)$

Solution

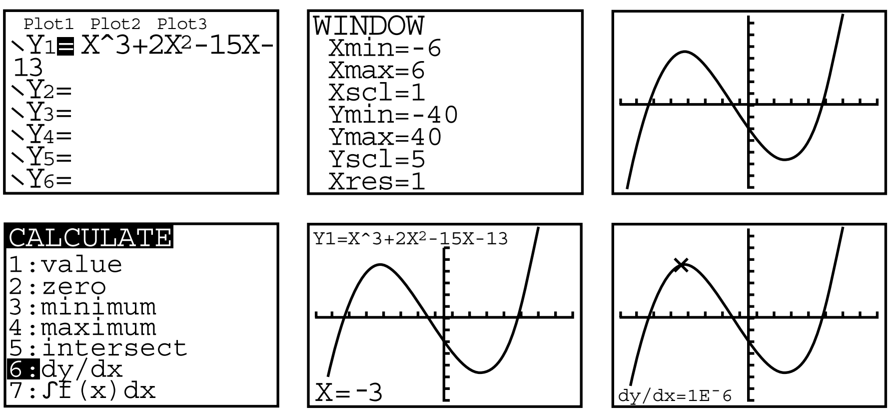
a) (i) $\dfrac{d}{dx}(x^3 + 2x^2 - 15x - 13) = \dfrac{d}{dx}(x^3) + 2\cdot\dfrac{d}{dx}(x^2) - 15\cdot\dfrac{d}{dx}(x) - \dfrac{d}{dx}(13)$

$$= 3x^2 + 2(2x) - 15(1) - 0$$
$$= 3x^2 + 4x - 15$$

Therefore, the derivative of $f(x) = x^3 + 2x^2 - 15x - 13$ is $f'(x) = 3x^2 + 4x - 15$.

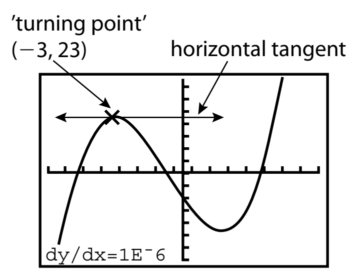
(ii) Slope of curve at $(-3, 23)$ is $f'(-3) = 3(-3)^2 + 4(-3) - 15$ $= 27 - 12 - 15 = 0$.
We should observe a horizontal tangent (slope $= 0$) to the curve at $(-3, 23)$.
Slope of curve at $(3, -13)$ is $f'(3) = 3(3)^2 + 4(3) - 15$ $= 27 + 12 - 15 = 24$.
We should observe a very steep tangent (slope $= 24$) to the curve at $(3, -13)$.

(iii) Not only can we use the GDC to compute the value of the derivative at a particular value of x on the 'home' screen, but we can also do it on the graph screen.



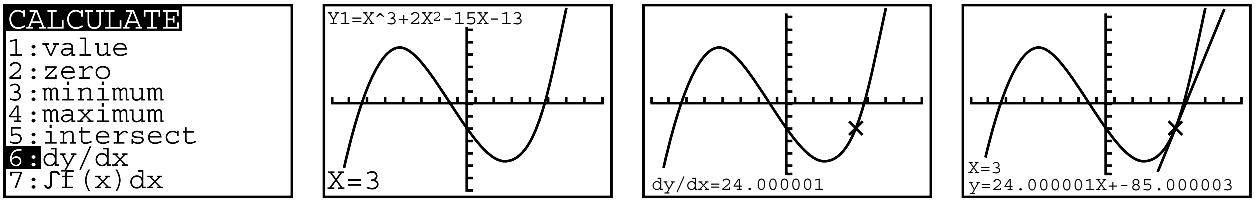
The GDC computes a slope of $1E^{-}6$ at the point $(-3, 23)$.
($1E^{-}6 = 1 \times 10^{-6} = 0.000\,001$)

Although the method the GDC uses is very accurate, sometimes there is a small amount of error in its calculation. This most commonly occurs when performing calculus computations (e.g. the value of the derivative at a point). $1E^{-}6 = 0.000\,001$ is very close to zero which is the exact value of the derivative. Observe that the graph of $y = x^3 + 2x^2 - 15x - 13$ appears to have a 'turning point' at $(-3, 23)$, confirming that a line tangent to the curve at that point would be horizontal.



Let's check on our GDC that the slope of the curve is 24 at $(3, -13)$. Again, the GDC exhibits a small amount of error in its result.

Most GDCs are also capable of drawing a tangent at a point and displaying its equation as shown in the final screen image below.



The equation of the tangent line at $(3, -13)$ is $y = 24x - 85$. We will look at finding the equations of tangent lines analytically in the last section of the chapter.

b) (i) $\dfrac{d}{dx}[(2x-7)^2] = \dfrac{d}{dx}[(2x-7)(2x-7)]$

Differentiate term-by-term after expanding.

$$= \frac{d}{dx}(4x^2 - 28x + 49)$$

$$= 4\frac{d}{dx}(x^2) - 28\frac{d}{dx}(x) + \frac{d}{dx}(49)$$

$$= 8x - 28 + 0$$

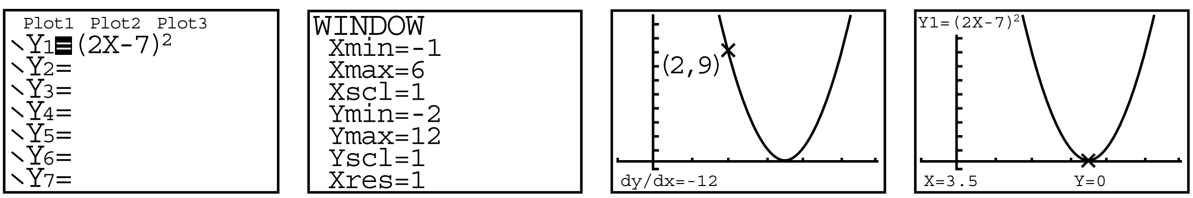
Therefore, the derivative of $f(x) = (2x-7)^2$ is $f'(x) = 8x - 28$.

(ii) Slope of curve at $(2, 9)$ is $f'(2) = 8(2) - 28 = -12$.

Slope of curve at $\left(\frac{7}{2}, 0\right)$ is $f'\left(\frac{7}{2}\right) = 8\left(\frac{7}{2}\right) - 28 = 0$.

Thus, we should observe a horizontal tangent to the curve at $\left(\frac{7}{2}, 0\right)$.

(iii)



There's no error this time in the GDC's computation of the slope at $(2, 9)$. The vertex of the parabola is at $\left(\frac{7}{2}, 0\right)$, confirming that it has a horizontal tangent at that point.

c) (i) $\dfrac{d}{dx}(3\sqrt{x} - 6) = 3\dfrac{d}{dx}(x^{\frac{1}{2}}) - \dfrac{d}{dx}(6)$

$$= 3\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - 0$$

$$= \frac{3}{2x^{\frac{1}{2}}}$$

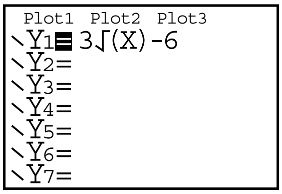
Therefore, the derivative of $f(x) = 3\sqrt{x} - 6$ is $f'(x) = \dfrac{3}{2x^{\frac{1}{2}}}$ or $f'(x) = \dfrac{3}{2\sqrt{x}}$.

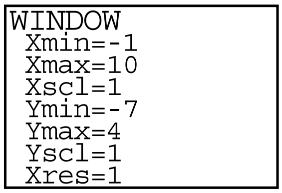
(ii) Slope of curve at $(4, 0)$ is $f'(4) = \frac{3}{2\sqrt{4}} = \frac{3}{4}$.

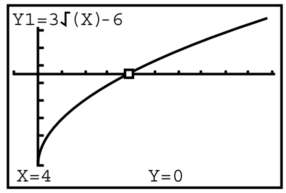
Slope of curve at $(9, 3)$ is $f'(9) = \frac{3}{2\sqrt{9}} = \frac{1}{2}$.

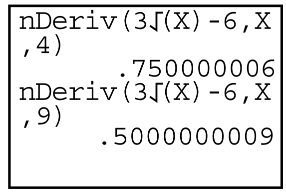
Thus, because the slope at $x = 9$ is less than that at $x = 4$, we should observe the graph of the equation becoming less steep as we move along the curve from $x = 4$ to $x = 9$.

(iii)









The slope of the graph of $y = 3\sqrt{x} - 6$ appears to steadily decrease as x increases. Let's check the results for (ii) by evaluating the derivative at a point on the 'home' screen. The GDC confirms the slopes for the curve when $x = 4$ and $x = 9$, but again the GDC computations have incorporated a small amount of error.

d) (i) $\frac{d}{dx}\left(\frac{x^4}{4} - \frac{3x^3}{2} - 2x^2 + \frac{15x}{2} + \frac{3}{4}\right)$

$$= \frac{1}{4}\frac{d}{dx}(x^4) - \frac{3}{2}\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + \frac{15}{2}\frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}(4x^3) - \frac{3}{2}(3x^2) - 2\frac{d}{dx}(2x) + \frac{15}{2}(1) + 0$$

$$= x^3 - \frac{9x^2}{2} - 4x + \frac{15}{2}$$

Therefore, the derivative of $f(x) = \frac{x^4}{4} - \frac{3x^3}{2} - 2x^2 + \frac{15x}{2} + \frac{3}{4}$

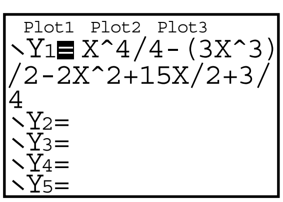
is $f'(x) = x^3 - \frac{9x^2}{2} - 4x + \frac{15}{2}$.

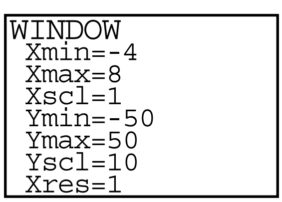
(ii) Slope of curve at $(5, -43)$ is $f'(5) = 5^3 - \frac{9(5)^2}{2} - 4(5) + \frac{15}{2} = 0$.

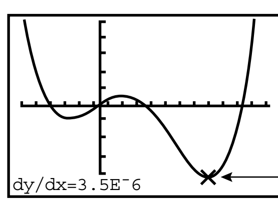
Thus, there should be a horizontal tangent to the curve at $(5, -43)$.

Slope of curve at $(0, 0)$ is $f'(0) = \frac{15}{2}$.

(iii) Your GDC is not capable of computing the derivative function – only the specific value of the derivative for a given value of x. However, we can have the GDC graph the values of the derivative over a given *interval* of x. We can then graph the derivative function found from differentiation rules (result from (i)) and see if the two graphs match.



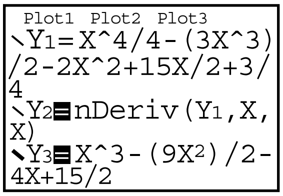




Horizontal tangent at $(5, -43)$

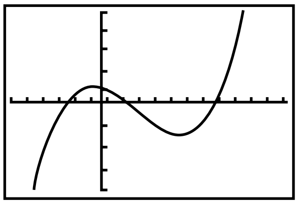
The command nDeriv(Y_1, X, X) computes the value of the

derivative of function Y_1 in terms of x for all x.

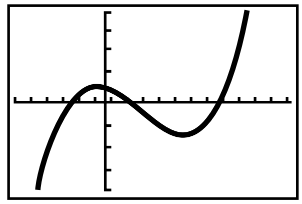


$Y_1 = \frac{x^4}{4} - \frac{3x^3}{2} - 2x^2 + \frac{15x}{2} + \frac{3}{4}$

Values of the derivative of $f(x)$ will be graphed as Y_2, and the derivative function, $f'(x) = x^3 - \frac{9x^2}{2} - 4x + \frac{15}{2}$, determined by manual application of differentiation rules (part (i)), will be graphed as Y_3. Note that the graph of Y_3 will be in bold style to distinguish it from Y_2, and that the equation Y_1 has been turned 'off.'



$Y_2 = \text{nDeriv}(Y_1, X, X)$



$Y_3 = x^3 - \frac{9x^2}{2} - 4x + \frac{15}{2}$

Since the two graphs match, this confirms that the derivative found in part (i) using differentiation rules is correct.

Example 10

The curve $y = ax^3 + 7x^2 - 8x - 5$ has a turning point at the point where $x = -2$. Determine the value of a.

Solution

There must be a horizontal tangent, and a slope of zero, at the point where the graph has a turning point.

$$\frac{dy}{dx} = \frac{d}{dx}(ax^3 + 7x^2 - 8x - 5)$$

$$= a\frac{d}{dx}(x^3) + 7\frac{d}{dx}(x^2) - 8\frac{d}{dx}(x) + \frac{d}{dx}(-5) = 3ax^2 + 14x - 8$$

$$\frac{dy}{dx} = 0 \text{ when } x = -2\text{: } 3a(-2)^2 + 14(-2) - 8 = 0$$

$$\Rightarrow\ 12a - 28 - 8 = 0 \ \Rightarrow\ 12a = 36 \ \Rightarrow\ a = 3$$

Recall that the derivative of a function is a formula for the **instantaneous rate of change** of the dependent variable (commonly y) with respect to the dependent variable (x). In other words, as illustrated earlier in this section, the slope of the tangent at a point gives the slope, or rate of change, of the curve at that point. The slope of a **secant line** (that crosses the curve at two points) gives the **average rate of change** between the two points.

Example 11

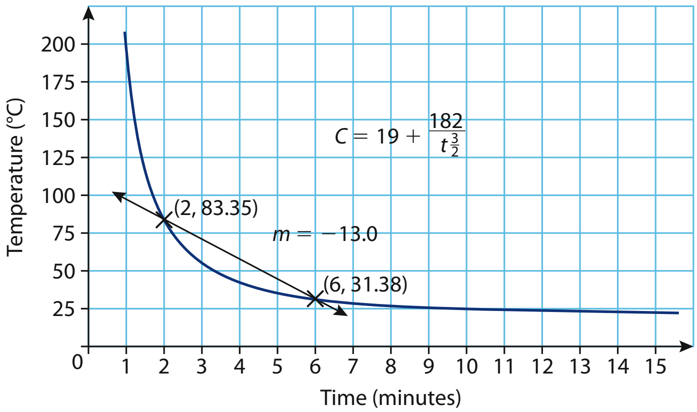
Boiling water is poured into a cup. The temperature of the water in degrees Celsius, C, after t minutes is given by $C = 19 + \frac{182}{t^{\frac{3}{2}}}$, for times $t \geqslant 1$ minute.

a) Find the average rate of change of the temperature from $t = 2$ to $t = 6$.

b) Find the rate of change of the temperature at the instant that $t = 4$.

Solution

a)



When $t = 2$, $C \approx 83.35°$ and when $t = 6$, $C \approx 31.38°$. The average rate of change from $t = 2$ to $t = 6$ is the slope of the line through the points (2, 83.35) and (6, 31.38).

Average rate of change $= \frac{83.35 - 31.38}{2 - 6} = \frac{51.97}{-4} = -12.9925.$

To an accuracy of 3 significant figures, the average rate of change from $t = 2$ to $t = 6$ is -13.0 °C per minute. During that period of time the water is, on average, becoming 13 degrees cooler every minute.

b) Let's compute the derivative $\frac{dC}{dt}$, i.e. the rate of change of degrees C with respect to time t, from which we can compute the rate the temperature is changing at the moment when $t = 4$.

$$\frac{dC}{dt} = \frac{d}{dt}\left(19 + \frac{182}{t^{\frac{3}{2}}}\right) = \frac{d}{dt}(19 + 182t^{-\frac{3}{2}}) = \frac{d}{dt}(19) + 182\frac{d}{dt}(t^{-\frac{3}{2}})$$

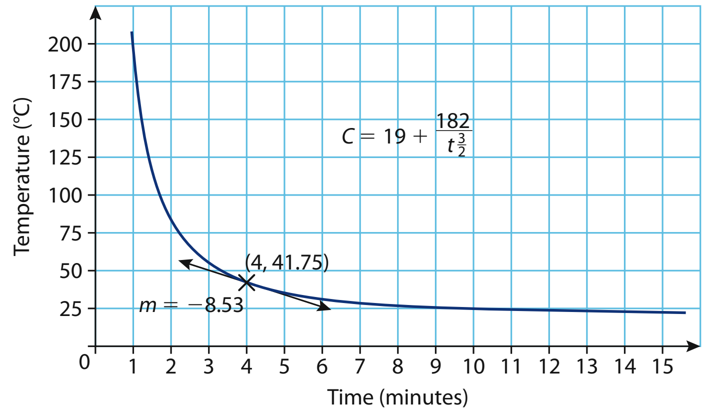
$$= 0 + 182\left(-\frac{3}{2}t^{-\frac{3}{2}-1}\right) = -273t^{-\frac{5}{2}}$$

$$\frac{dC}{dt} = -\frac{273}{t^{\frac{5}{2}}} = -\frac{273}{\sqrt{t^5}}$$

At $t = 4$:

$$\frac{dC}{dt} = -\frac{273}{\sqrt{4^5}} = -\frac{273}{32} \approx -8.53$$

Therefore, the temperature's instantaneous rate of change at $t = 4$ minutes is -8.53 °C per minute.



Differentiating sin x and cos x using limit definition for derivative

To add to our growing list of differentiation rules, we will now determine the derivatives for the sine and cosine functions. The results will help us determine the derivatives for the other trigonometric functions in Chapter 15.

The rigorous analytical method (applying limit definition of derivative) for finding these two derivatives requires two limit results that we found by decidedly non-rigorous methods in Example 2 in the previous section; namely that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$. We conjectured the value of these limits after exploring the behaviour of the expressions on our GDC. Example 5 illustrated that estimating limits by such informal methods is not foolproof. Hence, we will now put these two limit results on firmer ground through a more rigorous approach.

We first state, without proof, an important theorem in mathematics.

The squeeze theorem

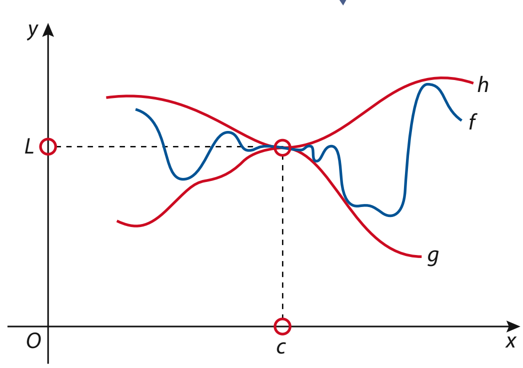
If $g(x) \leqslant f(x) \leqslant h(x)$ for all $x \neq c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

Figure 13.13 Squeezing f between g and h forces the limiting value of f to be between the limiting values of g and h.



The theorem describes a function f whose values are 'squeezed' between the values of two other functions, g and h. If g and h have the same limit as $x \to c$, then f has the same limit, as suggested by Figure 13.13.

Consider a sector of a circle with centre O, central angle θ (in radian measure) and radius 1 (see Figure 13.14). Further consider right triangle AOC, sector AOB and triangle AOB. We know that point B has coordinates $(\cos\theta, \sin\theta)$ and point C has coordinates $(1, \tan\theta)$. From Section 7.1, we also know that the area of a sector with central angle θ is $\frac{1}{2}r^2\theta$. It is clear that the area of sector AOB must be between the area of $\triangle AOC$ and the area of $\triangle AOB$, that is, the sector is 'squeezed' between the two triangles (Figure 13.15).

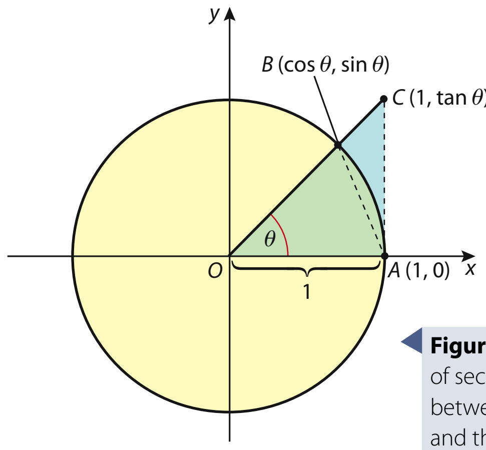


Figure 13.14 Area of sector AOB must be between the area of $\triangle AOC$ and the area of $\triangle AOB$.

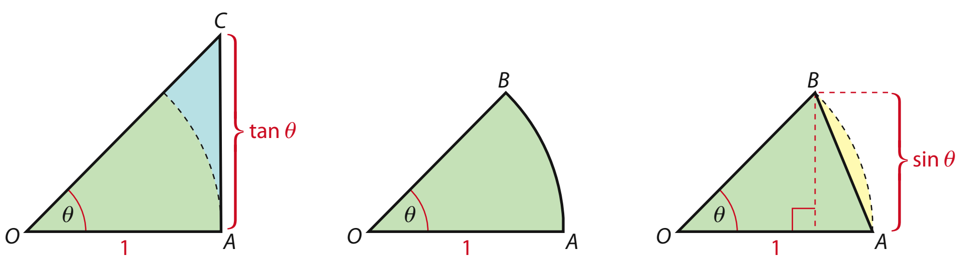


Figure 13.15 Area of sector AOB is squeezed between the two triangles.

Area of triangle $\geqslant$ Area of sector $\geqslant$ Area of triangle

$$\frac{\tan\theta}{2} \geqslant \frac{\theta}{2} \geqslant \frac{\sin\theta}{2}$$

Multiplying all the area expressions by $\dfrac{2}{\sin\theta}$ gives

$$\frac{1}{\cos\theta} \geqslant \frac{\theta}{\sin\theta} \geqslant 1.$$

Given the fact that if $\frac{a}{b} > \frac{c}{d}$, then $\frac{b}{a} < \frac{d}{c}$, we can write the reciprocals of the three expressions and reverse the inequality signs. This gives

$$\cos\theta \leqslant \frac{\sin\theta}{\theta} \leqslant 1.$$

It follows that

$$\lim_{\theta\to 0}\cos\theta \leqslant \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leqslant \lim_{\theta\to 0} 1.$$

From direct substitution, $\lim\limits_{\theta\to 0}\cos\theta = 1$. Thus,

$$1 \leqslant \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leqslant 1.$$

We can now apply the squeeze theorem and conclude that $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$.

Furthermore, because $\cos(-\theta) = \cos\theta$ and $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin(\theta)}{\theta}$, we can also conclude that this limit is true for all non-zero values of θ in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

The above result, $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, can be used to algebraically deduce that $\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x} = 0$. This is saved for you to do in Exercise 13.2, question 26.

Example 12

Differentiate from first principles:

a) $f(x) = \sin x$ b) $f(x) = \cos x$

Solution

For both of the derivatives we will need to make use of a compound angle identity and the limit results $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0}\frac{\cos x - 1}{x} = 0$.

a) We start by substituting into the limit definition for the derivative.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

Applying $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

Splitting argument into two fractions.

$$= \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right]$$

Factorizing common factors in each fraction.

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]$$

Applying $\lim\limits_{x\to a}[f(x)\cdot g(x)] = L\cdot K$.

$$= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

Applying $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0}\frac{\cos x - 1}{x} = 0$.

$$= \sin x\cdot 0 + \cos x\cdot 1$$

$$= \cos x$$

Thus, if $f(x) = \sin x$ then $f'(x) = \cos x$, or using Leibniz notation $\frac{d}{dx}(\sin x) = \cos x$.

b) Again, we start by substituting into the limit definition for the derivative.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

Applying $\cos(A + B) = \cos A\cos B - \sin A\sin B$.

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

Splitting argument into two fractions.

$$= \lim_{h\to 0}\left[\frac{\cos x\cos h - \cos x}{h} - \frac{\sin x\sin h}{h}\right]$$

Factorizing common factors in each fraction.

$$= \lim_{h\to 0}\left[\cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)\right]$$

Applying $\lim\limits_{x\to a}[f(x)\cdot g(x)] = L\cdot K$.

$$= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

Applying $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0}\frac{\cos x - 1}{x} = 0$.

$$= \cos x\cdot 0 - \sin x\cdot 1$$
$$= -\sin x$$

Thus, if $f(x) = \cos x$ then $f'(x) = -\sin x$, or using Leibniz notation $\frac{d}{dx}(\cos x) = -\sin x$.

We will confirm these two results graphically at the start of Chapter 15.

Exercise 13.2

In questions 1–4, find the derivative of the function by applying the limit definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

1 $f(x) = 1 - x^2$

2 $g(x) = x^3 + 2$

3 $h(x) = \sqrt{x}$

4 $r(x) = \frac{1}{x^2}$

5 Using your results from questions 1–4, find the slope of the graph of each function in 1–4 at the point where $x = 1$. Sketch each function and draw a line tangent to the graph at $x = 1$.

In questions 6–12, a) find the derivative of the function, and b) compute the slope of the graph of the function at the indicated point. Use a GDC to confirm your results.

6 $y = 3x^2 - 4x$ point $(0, 0)$

7 $y = 1 - 6x - x^2$ point $(-3, 10)$

8 $y = \frac{2}{x^3}$ point $(-1, 2)$

9 $y = x^5 - x^3 - x$ point $(1, -1)$

10 $y = (x + 2)(x - 6)$ point $(2, -16)$

11 $y = 2x + \frac{1}{x} - \frac{3}{x^3}$ point $(1, 0)$

12 $y = \frac{x^3 + 1}{x^2}$ point $(-1, 0)$

13 The slope of the curve $y = x^2 + ax + b$ at the point $(2, -4)$ is -1. Find the value of a and the value of b.

In questions 14–17, find the coordinates of any points on the graph of the function where the slope is equal to the given value.

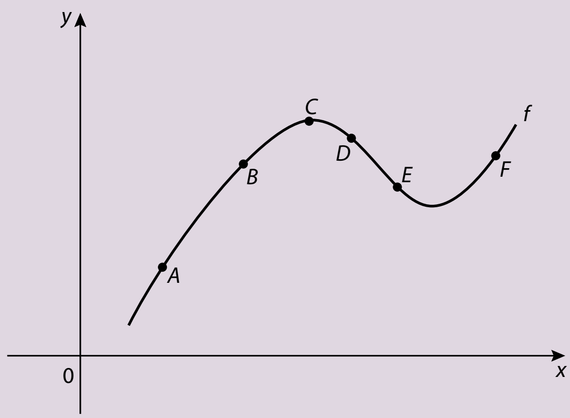
14 $y = x^2 + 3x$ slope $= 3$

15 $y = x^3$ slope $= 12$

16 $y = x^2 - 5x + 1$ slope $= 0$

17 $y = x^2 - 3x$ slope $= -1$

18 Use the graph of f to answer each of the following questions.



a) Between which two consecutive points is the average rate of change of the function greatest?

b) At what points is the instantaneous rate of change of f positive, negative and zero?

c) For which two pairs of points is the average rate of change approximately equal?

19 The slope of the curve $y = x^2 - 4x + 6$ at the point (3, 3) is equal to the slope of the curve $y = 8x - 3x^2$ at (a, b). Find the value of a and the value of b.

20 The graph of the equation $y = ax^3 - 2x^2 - x + 7$ has a slope of 3 at the point where $x = 2$. Find the value of a.

21 Find the coordinates of the point on the graph of $y = x^2 - x$ at which the tangent is parallel to the line $y = 5x$.

22 Let $f(x) = x^3 + 1$.

a) Evaluate $\dfrac{f(2+h) - f(2)}{h}$ for $h = 0.1$.

b) What number does $\dfrac{f(2+h) - f(2)}{h}$ approach as h approaches zero?

23 From first principles, find the derivative for the general quadratic function, $f(x) = ax^2 + bx + c$. Confirm your result by checking that it produces:
(i) the derivative of x^2 when $a = 1, b = 0, c = 0$
(ii) the derivative of $3x^2 - 4x + 2$ when $a = 3, b = -4, c = 2$.

24 A car is parked with the windows and doors closed for five hours. The temperature inside the car in degrees Celsius, C, is given by $C = 2\sqrt{t^3} + 17$ with t representing the number of hours since the car was first parked.

a) Find the average rate of change of the temperature from $t = 1$ to $t = 4$.

b) Find the function that gives the instantaneous rate of change of the temperature for any time t, $0 < t < 5$.

c) Find the time t at which the instantaneous rate of change of the temperature is equal to the average rate of change from $t = 1$ to $t = 4$.

25 A function f is even if $f(-x) = f(x)$ and a function g is odd if $g(-x) = -g(x)$.

a) If the function h is even, prove that the derivative of h is odd. In other words, if

$$h(-x) = h(x), \text{ then, } h'(-x) = -h'(x).$$

b) If the function p is odd, prove that the derivative of h is even. In other words, if

$$p(-x) = -p(x), \text{ then, } p'(-x) = p'(x).$$

26 Using algebraic manipulation and the proven result $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, prove that $\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x} = 0$.

In questions 27–30, find the indicated derivative by applying the limit definition of the derivative (i.e. by first principles). (See questions 20 and 21 in Exercise 13.1 for 27 and 28 below.)

27 $\dfrac{d}{dx}(\sqrt{x})$

28 $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$

29 $\dfrac{d}{dx}\left(\dfrac{2+x}{3-x}\right)$

30 $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x+2}}\right)$

31 Prove the constant rule by first principles. That is, prove that given a constant c, $c \in \mathbb{R}$, $\dfrac{d}{dx}(c) = 0$.

13.3 Maxima and minima – first and second derivatives

The relationship between a function and its derivative

The derivative, written in Newton notation as $f'(x)$ or in Leibniz notation as $\frac{dy}{dx}$, is a function derived from a function f that gives the slope of the graph of f at any x in the function's domain (given that the curve is 'smooth' at the value of x). The derivative is a slope, or rate of change, function. Knowing the slope of a function at different values in its domain tells us about properties of the function and the shape of its graph.

If the graph of a function is 'smooth' at a particular point, the function is considered to be *differentiable* at this point. In other words, a tangent line exists at this point. All functions that will be differentiated in this course will be differentiable at all values in the function's domain.

In the previous section, we observed that if a graph 'turns' at a particular point (for example, at the vertex of a parabola), then it has a horizontal tangent (slope $= 0$) at the point. Hence, the derivative will equal zero at a 'turning point'. In Section 3.2, we found the vertex of the graph of a quadratic function by using the technique of completing the square to write its equation in vertex form. We can also find the vertex by means of differentiation. As we look at the graph of a parabola moving from left to right (i.e. domain values increasing), it either turns from going down to going up (decreasing to increasing), or from going up to going down (increasing to decreasing) (Figure 13.16).

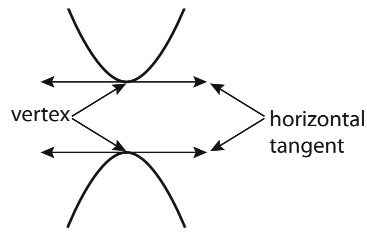


Figure 13.16

Example 13

Using differentiation, find the vertex of the parabola with the equation $y = x^2 - 8x + 14$.

Solution

Find the value of x for which the derivative, $\frac{dy}{dx}$, is zero.

$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 8x + 14) = 2x - 8 = 0 \Rightarrow x = 4$

Thus, the x-coordinate of the vertex is 4.

To find the y-coordinate of the vertex, we substitute $x = 4$ into the equation, giving $y = 4^2 - 8(4) + 14 = -2$. Therefore, the vertex has coordinates $(4, -2)$.

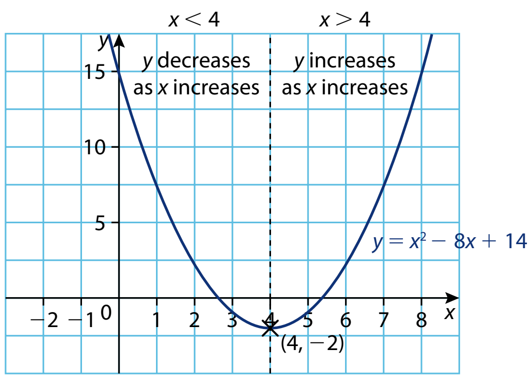


Figure 13.17 Slope changes from negative to positive as x increases.

We know that the parabola in Example 13 will 'open up' because the coefficient of the quadratic term, x^2, is positive. The parabola has a negative slope (decreasing) to the left of the vertex and a positive slope (increasing) to the right of the vertex (Figure 13.17). As the values of x increase, the derivative of $y = x^2 - 8x + 14$ will change from negative to zero to positive, accordingly.

$$\frac{dy}{dx} = 2x - 8 \Rightarrow \frac{dy}{dx} < 0 \text{ for } x < 4 \text{ and } \frac{dy}{dx} = 0 \text{ for } x = 4 \text{ and } \frac{dy}{dx} > 0 \text{ for } x > 4$$

In other words, the function $f(x) = x^2 - 8x + 14$ is decreasing for all $x < 4$; it is neither decreasing nor increasing at $x = 4$; and it is increasing for all $x > 4$. A point at which a function is neither increasing nor decreasing (i.e. there is a horizontal tangent) is called a **stationary point**. A convenient way to demonstrate where a function is increasing or decreasing and the location of any stationary points is with a **sign chart** for the function and its derivative, as shown in Figure 13.18 for $f(x) = x^2 - 8x + 14$. The derivative $f'(x) = 2x - 8$ is zero only at $x = 4$, thereby dividing the domain of f (i.e. $\mathbb{R}$) into two intervals: $x < 4$ and $x > 4$. $f'(x) = 2x - 8$ is a **continuous** function (i.e. no 'gaps' in the domain) so it is only necessary to test one point in each interval in order to determine the sign of all the values of the derivative in that interval. $f'(x)$ can only change sign at $x = 4$. For example, the fact that $f'(3) = 2(3) - 8 = -2 < 0$ means that $f'(x) < 0$ for all x when $x < 4$. Therefore, f is decreasing for all x in the open interval $(-\infty, 4)$.

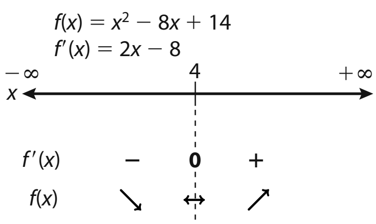


Figure 13.18 Sign chart for $f'(x)$ and $f(x)$.

Geometrically speaking, a function is **continuous** if there is no break in its graph; and a function is **differentiable** (i.e. a derivative exists) at any points where it is 'smooth'.

Increasing and decreasing functions and stationary points

If $f'(x) > 0$ for $a < x < b$, then $f(x)$ is **increasing** on the interval $a < x < b$.

If $f'(x) < 0$ for $a < x < b$, then $f(x)$ is **decreasing** on the interval $a < x < b$.

If $f'(x) = 0$ for $a < x < b$, then $f(x)$ is **constant** on the interval $a < x < b$.

If $f'(x) = 0$ for a single value $x = c$ on some interval $a < c < b$, then $f(x)$ has a **stationary point** at $x = c$. The corresponding point $(c, f(c))$ on the graph of f is called a stationary point.

It is at stationary points, or endpoints of the domain if the domain is not all real numbers, where a function may have a maximum or minimum value. These points at which extreme values of a function *may* occur are often referred to as **critical points**. Whether a function is increasing or decreasing on either side of a stationary point will indicate whether the stationary point is a maximum, minimum or neither.

Example 14

Consider the function $f(x) = 2x^3 + 3x^2 - 12x - 4,\ x \in \mathbb{R}$.

a) Find any stationary points of f.

b) Using the derivative of f, classify any stationary points as a maximum or minimum.

Solution

a) $f'(x) = 6x^2 + 6x - 12 = 0 \Rightarrow 6(x^2 + x - 2) = 0$

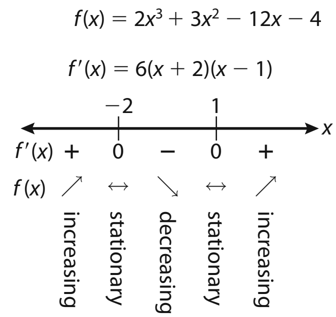
$\Rightarrow 6(x + 2)(x - 1) = 0 \Rightarrow x = -2$ or $x = 1$

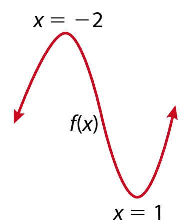
With a domain of all real numbers there are no domain endpoints that may be an extreme value. Thus, f has two critical points: one at $x = -2$ and the other at $x = 1$.

When $x = -2$: $y = 2(-2)^3 + 3(-2)^2 - 12(-2) - 4 = 16 \Rightarrow f$ has a stationary point at $(-2, 16)$.

When $x = 1$: $y = 2(1)^3 + 3(1)^2 - 12(1) - 4 = -11 \Rightarrow f$ has a stationary point at $(1, -11)$.

b) Construct a sign chart for $f'(x)$ and $f(x)$ (left) to show where f is increasing or decreasing. The derivative $f'(x)$ has two zeros, at $x = -2$ and $x = 1$, thereby dividing the domain of f into three intervals that need to be tested. Since $f'(-3) = 6(-1)(-4) = 24 > 0$, then $f'(x) > 0$ for all $x < -2$. Likewise, since $f'(2) = 6(4)(1) = 24 > 0$, then $f'(x) > 0$ for all $x > 1$. Thus, f is increasing on the open intervals $(-\infty, -2)$ and $(1, \infty)$. Since $f'(0) = -12 < 0$, then $f'(x) < 0$ for all x such that $-2 < x < 1$. Thus, f is decreasing on the open interval $(-2, 1)$, i.e. $-2 < x < 1$. From this information, we can visualize for increasing values of x that the graph of f is going up for all $x < -2$, then turning down at $x = -2$, then going down for values of x from -2 to 1, then turning up at $x = 1$, and then going up for all $x > 1$. The basic shape of the graph of f will look something like the rough sketch shown left. Clearly, the stationary point $(-2, 16)$ is a maximum and the stationary point $(1, -11)$ is a minimum.





The graph of $f(x) = 2x^3 + 3x^2 - 12x - 4$ from Example 14 (Figure 13.19) visually confirms the results acquired from analyzing the derivative of f.

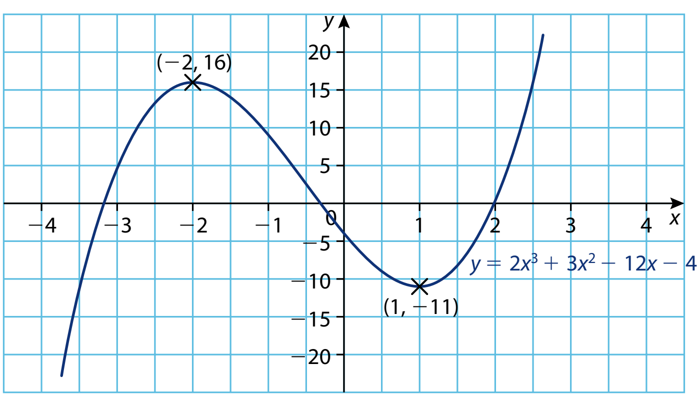


Figure 3.19

For Example 14, we can express the result for part b) most clearly by saying that $f(x)$ has a **relative maximum** value of 16 at $x = -2$, and $f(x)$ has a **relative minimum** value of -11 at $x = 1$. The reason that these *extreme* values are described as 'relative' (sometimes described as 'local') is because

The plural of 'maximum' is 'maxima', and the plural of 'minimum' is 'minima'. Maxima and minima are collectively referred to as 'extrema' – the plural of 'extremum' (extreme value). Extrema of a function that do not occur at domain endpoints will be 'turning points' of the graph of the function.

they are a maximum or minimum for the function in the immediate vicinity of the point, but not for the entire domain of the function. A point that is a maximum/minimum for the entire domain is called an **absolute**, or **global**, **maximum/minimum**.

The first derivative test

From Example 14, we can see that a function f has a maximum at some $x = c$ if $f'(c) = 0$ and f is *increasing* immediately to the left of $x = c$ and *decreasing* immediately to the right of $x = c$. Similarly, f has a minimum at some $x = c$ if $f'(c) = 0$ and f is *decreasing* immediately to the left of $x = c$ and *increasing* immediately to the right of $x = c$. It is important to understand, however, that not all stationary points are either a maximum or minimum.

Example 15

For the function $f(x) = x^4 - 2x^3$, find all stationary points and describe them completely.

Solution

$f'(x) = \frac{d}{dx}(x^4 - 2x^3) = 4x^3 - 6x^2 = 0 \Rightarrow 2x^2(2x - 3) = 0$

$\Rightarrow x = 0$ or $x = \frac{3}{2}$

The implied domain is all real numbers, so $x = 0$ and $x = \frac{3}{2}$ are the critical points of f.

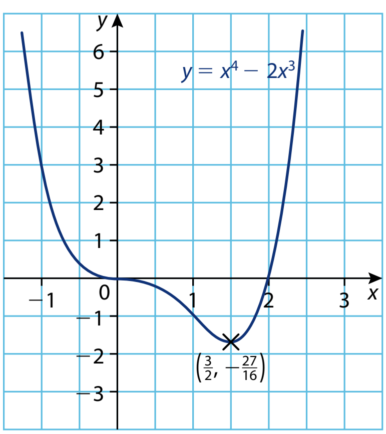
When $x = 0$, $y = f(0) = 0$.

When $x = \frac{3}{2}$, $y = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = \frac{81}{16} - \frac{54}{8} = -\frac{27}{16}$.

Therefore, f has stationary points at $(0, 0)$ and $\left(\frac{3}{2}, -\frac{27}{16}\right)$.

Because f has two stationary points, there are three intervals for which to test the sign of the derivative. We could use some form of a sign chart as shown previously, or we can use a more detailed table that summarizes the testing of the three intervals and the two critical points as shown below.

Interval/point	$x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x = \frac{3}{2}$	$x > \frac{3}{2}$
Test value	$x = -1$		$x = 1$		$x = 2$
Sign of $f'(x)$	$f'(-1) = -10 < 0$	0	$f'(1) = -2 < 0$	0	$f'(2) = 8 > 0$
Conclusion	f decreasing ↘	none	f decreasing ↘	abs. min.	f increasing ↗



On either side of $x = 0$, f does not change from either decreasing to increasing or from increasing to decreasing. Although there is a horizontal tangent at $(0, 0)$, it is *not* an extreme value (turning point). The function steadily decreases as x approaches zero, then at $x = 0$ the function has a rate of change (slope) of zero for an instant and then continues on decreasing. As x approaches $\frac{3}{2}$, f is decreasing and then switches to increasing at $x = \frac{3}{2}$.

Therefore, the stationary point $(0, 0)$ is neither a maximum nor a minimum; and the stationary point $\left(\frac{3}{2}, -\frac{27}{16}\right)$ is an absolute minimum. Or, in other words, f has an absolute (global) minimum value of $-\frac{27}{16}$ at $x = \frac{3}{2}$.

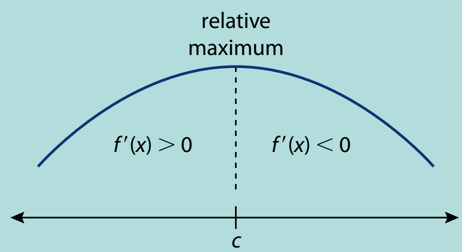
The reason that an *absolute*, rather than a *relative*, minimum value occurs at $x = \frac{3}{2}$ is because for all $x < \frac{3}{2}$ the function f is either decreasing or constant (at $x = 0$) and for all $x < \frac{3}{2}$ f is increasing.

First derivative test for maxima and minima of a function

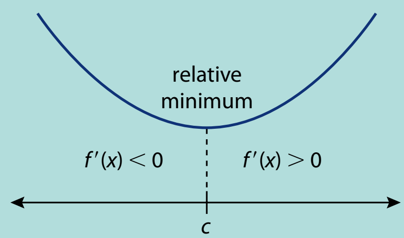
Suppose that $x = c$ is a critical point of a continuous and smooth function f. That is, $f(c) = 0$ and $x = c$ is a stationary point or $x = c$ is an endpoint of the domain.

I. At a stationary point $x = c$:

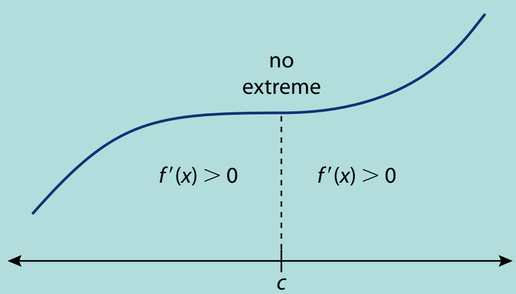
1. If $f'(x)$ changes sign from positive to negative as x increases through $x = c$, then f has a relative maximum at $x = c$.



2. If $f'(x)$ changes sign from negative to positive as x increases through $x = c$, then f has a relative minimum at $x = c$.

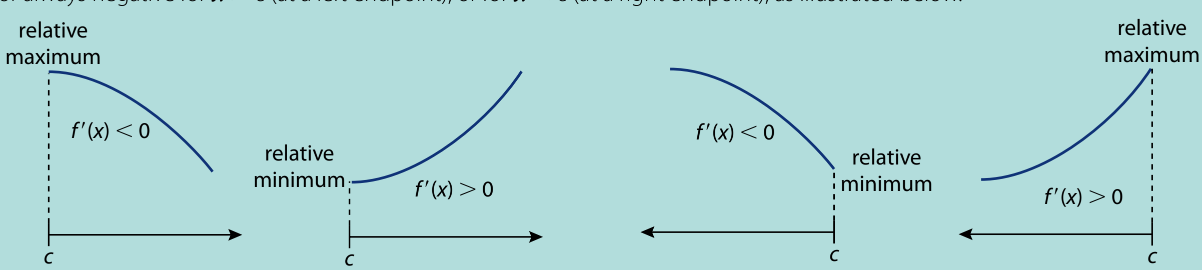


3. If $f'(x)$ does not change sign as x increases through $x = c$, then f has neither a relative maximum nor a relative minimum at $x = c$.



II. At a domain endpoint $x = c$:

If $x = c$ is an endpoint of the domain, then $x = c$ will be a relative maximum or minimum of f if the sign of $f'(x)$ is always positive or always negative for $x > c$ (at a left endpoint), or for $x < c$ (at a right endpoint), as illustrated below.



If it is possible to show that a relative maximum/minimum at $x = c$ is the greatest/least value for the entire domain of f, then it is classified as an absolute maximum/minimum.

Example 16

Apply the first derivative test to find any local extreme values for $f(x)$. Identify any absolute extrema.

$$f(x) = 4x^3 - 9x^2 - 120x + 25$$

Solution

$$f'(x) = \frac{d}{dx}(4x^3 - 9x^2 - 120x + 25) = 12x^2 - 18x - 120$$

$$f'(x) = 12x^2 - 18x - 120 = 0 \Rightarrow 6(2x^2 - 3x - 20) = 0$$
$$\Rightarrow 6(2x + 5)(x - 4) = 0$$

Thus, f has stationary points at $x = -\frac{5}{2}$ and $x = 4$.

To classify the stationary point at $x = -\frac{5}{2}$, we need to choose test points on either side of $-\frac{5}{2}$, for example, $x = -3$ (left) and $x = 0$ (right). Then we have

$$f'(-3) = 6(-1)(-7) = 42 > 0$$
$$f'(0) = 6(5)(-4) = -120 < 0$$

So f has a relative maximum at $x = -\frac{5}{2}$.

$$f\left(-\frac{5}{2}\right) = 4\left(-\frac{5}{2}\right)^3 - 9\left(-\frac{5}{2}\right)^2 - 120\left(-\frac{5}{2}\right) + 25 = 206.25$$

Therefore, f has a relative maximum value of 206.25 at $x = -\frac{5}{2}$.

To classify the stationary point at $x = 4$, we need to choose test points on either side of 4, for example, $x = 0$ (left) and $x = 5$ (right). Then we have

$$f'(0) = -120 < 0$$
$$f'(5) = 6(15)(1) = 90 > 0$$

So f has a relative minimum at $x = 4$.

$$f(4) = 4(4)^3 - 9(4)^2 - 120(4) + 25 = -343$$

Therefore, f has a relative minimum value of -343 at $x = 4$.

Change in displacement and velocity

Consider the motion of an object such that we know its position s relative to a reference point or line as a function of time t given by $s(t)$. The **displacement** of the object over the time interval from t_1 to t_2 is:

$$\text{change in } s = \text{displacement} = s(t_2) - s(t_1)$$

The **average velocity** of the object over the time interval is:

$$v_{\text{avg}} = \frac{\text{displacement}}{\text{change in time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

The object's **instantaneous velocity** at a particular time, t, is the value of the derivative of the position function, s, with respect to time at t.

$$\text{velocity} = \frac{ds}{dt} = s'(t)$$

Example 17

A rocket is launched upwards into the air. Its vertical position, s metres, above the ground at t seconds is given by
$s(t) = -5t^2 + 18t + 1$.

a) Find the average velocity over the time interval from $t = 1$ second to $t = 2$ seconds.

b) Find the instantaneous velocity at $t = 1$ second.

c) Find the maximum height reached by the rocket and the time at which this occurs.

Solution

a) $v_{\text{avg}} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{[-5(2)^2 + 18(2) + 1] - [-5 + 18 + 1]}{1}$

$= 3$ metres per second (or m s^{-1})

b) $s'(t) = -10t + 18 \Rightarrow s'(1) = -10 + 18 = 8\,\text{m s}^{-1}$

c) $s'(t) = -10t + 18 = 0 \Rightarrow t = 1.8$

Thus, s has a stationary point at $t = 1.8$. t must be positive and ranges from time of launch ($t = 0$) to when the rocket hits the ground, i.e. $h = 0$.

$$s(t) = -5t^2 + 18t + 1 = 0 \Rightarrow t = \frac{-18 \pm \sqrt{18^2 - 4(-5)(1)}}{2(-5)}$$

$$\Rightarrow t \approx \cancel{-0.5472} \text{ or } t \approx 3.655$$

So, the rocket hits the ground about 3.66 seconds after the time of launch. Hence, the domain for the position (s) and velocity (v) functions is $0 \leqslant t \leqslant 3.66$. Therefore, the function s has three critical points: $t = 0$, $t = 1.8$ and $t \approx 3.66$.

The maximum of the function, i.e. the maximum height, most likely occurs at the critical point $t = 1.8$. Let's confirm this.

Applying the first derivative test, we determine the sign of the derivative, $s'(t)$, for values on either side of $t = 1.8$, for example, $t = 0$ and $t = 2$. $s'(0) = 18 > 0$ and $s'(2) = -2 < 0$. Neither of the domain endpoints, $t = 0$ and $t \approx 3.66$, are at a maximum or minimum because the function is not constantly increasing or constantly decreasing before or after the endpoint. Since the function changes from increasing to decreasing at $t = 1.8$ and $s(1.8) = -5(1.8)^2 + 18(1.8) + 1 = 17.2$, then the rocket reaches a maximum height of 17.2 metres 1.8 seconds after it was launched.

The relationship between a function and its second derivative

You may have wondered why the strategy we are applying to locate and classify extrema for a function focuses on using the *first* derivative of the function. This implies that we are interested in using some other type of derivative, namely the *second* derivative. There is another useful test for the purpose of analyzing the stationary point of a function that makes use of the derivative of the derivative, i.e. the second derivative, of the function.

When we differentiate a function $y = f(x)$, we obtain the first derivative $f'(x)$ $\left(\text{also denoted as } \frac{dy}{dx}\right)$. Often this is a function that can also be differentiated. The result of doing so is the derivative of $f'(x)$, which is denoted in Newton notation as $f''(x)$ or in Leibniz notation as $\frac{d^2y}{dx^2}$ and called the second derivative of f with respect to x. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and $f''(x) = 6x$.

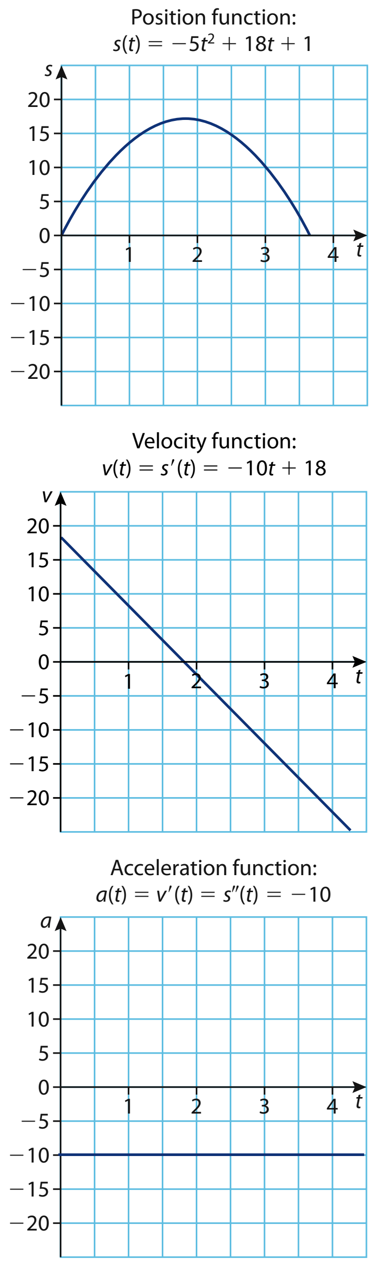


Figure 13.20 Position, velocity and acceleration functions for rocket.

Second derivatives, like first derivatives, occur often in methods of applying calculus. In Example 17, the function $s(t)$ gave the position, in metres above the ground, of a projectile (toy rocket) where t, in seconds, is the time since the projectile was launched. The function $s'(t)$, the first derivative of the position function, then gives the rate of change of the object's position, i.e. its velocity, in metres per second (m s^{-1}). Differentiation of this function gives the rate of change of the object's velocity, i.e. its *acceleration*, measured in metres per second per second (m s^{-2}).

The graphs of the position, velocity and acceleration functions for Example 17 aligned vertically (Figure 13.20) nicely illustrate the relationships between a function, its first derivative and its second derivative. The slope of the graph of $s(t)$ is initially a large positive value (graph is steep), but steadily decreases until it is zero (horizontal tangent) at $t = 1.8$ and then continues to decrease, becoming a large negative value (again, steep, but in the other direction). This corresponds to the real-life situation in which the rocket is launched with a high initial velocity ($v(0) = 18\text{ m s}^{-1}$) and then its velocity decreases steadily due to gravity. The rocket's velocity is zero for just an instant when it reaches its maximum height at $t = 1.8$ and then its velocity becomes more and more negative because it has changed direction and is moving back (negative direction) to the ground. The rate of change of the velocity, $v'(t)$, is constant and it is negative because the velocity is decreasing from positive values to zero to negative values. This is clear from the fact that the graph of the velocity function, $v(t)$, is a straight line with a negative slope. It follows then that the acceleration function – the rate of change of velocity – is a negative constant, $a = -10$ in this case, and its graph is a horizontal line.

In Example 17, it is not possible to have a negative function value for $s(t)$ because the rocket's position is always above, or at, ground level. In many motion problems in calculus, we consider a simplified version by limiting an object's motion to a line with its position given as its **displacement** from a fixed point (usually the origin). At a position left of the fixed point, the object's displacement is negative, and at a position right of the fixed point, the displacement is positive. Velocity can also be positive or negative depending on the direction of travel (i.e. the sign of the rate of change of the object's displacement). Likewise, acceleration is positive if velocity is increasing (i.e. rate of change of velocity is positive) and negative if velocity is decreasing.

A common misconception is that acceleration is positive for motion in the positive direction (usually 'right' or 'up') and negative for motion in the negative direction (usually 'left' or 'down'). Acceleration indicates how velocity is changing. Even though an object may be moving in a positive direction (e.g. to the right) if it is slowing down, then its acceleration is acting in the opposite direction and would be negative. In Example 17, the rocket was always accelerating in the negative direction, $-10\,\text{m}\,\text{s}^{-2}$, due to the force of gravity. Note: A more accurate value for the acceleration of a free-falling object due to gravity is $-9.8\,\text{m}\,\text{s}^{-2}$.

It would be incorrect to graph a function and its first and/or second derivative on the same axes. For example, the position $s(t)$, velocity $v(t)$ and acceleration $a(t)$ functions graphed on separate axes in Figure 13.20 will have different units on each vertical axis: metres for $s(t)$, metres per second for $v(t)$ and metres per second per second for $a(t)$.

Displacement can be negative, positive or zero. **Distance** is the absolute value of displacement. **Velocity** can be negative, positive or zero. **Speed** is the absolute value of velocity.

Motion along a line

If an object moves in a straight line such that at time t its displacement (position) from a fixed point is $s(t)$, then the first derivative $s'(t)$, also written as $\frac{ds}{dt}$, gives the velocity $v(t)$ at time t.

The second derivative $s''(t)$, also written as $\frac{d^2s}{dt^2}$, is the first derivative of $v(t)$. Hence, the second derivative of the displacement, or position, function is a measure of the rate at which the velocity is changing, i.e. it represents the acceleration of the object, which we express as

$$a(t) = v'(t) = s''(t) \quad \text{or} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Example 18

An object moves along a straight line so that after t seconds its displacement from the origin is s metres. Given that $s(t) = -2t^3 + 6t^2$, answer the following:

a) Find expressions for the (i) velocity and (ii) acceleration at time t seconds.

b) Find the (i) initial velocity and (ii) initial acceleration of the object (i.e. at time when $t = 0$).

c) Find the (i) maximum displacement and (ii) maximum velocity for the interval $0 \leqslant t \leqslant 3$.

Solution

a) (i) $v(t) = \frac{ds}{dt} = \frac{d}{dt}(-2t^3 + 6t^2) = -6t^2 + 12t$

(ii) $a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(-6t^2 + 12t) = -12t + 12$

b) (i) $v(0) = -6(0)^2 + 12(0) = 0 \quad \Rightarrow$ The object's initial velocity is $0\,\text{m}\,\text{s}^{-1}$.

(ii) $a(0) = -12(0) + 12 = 12 \quad \Rightarrow$ The object's initial acceleration is $12\,\text{m}\,\text{s}^{-2}$.

c) (i) To find the maximum displacement, we can apply the first derivative test to $s(t)$. Since the first derivative of displacement, $s(t)$, is velocity, $v(t)$, then the critical points of $s(t)$ are where the velocity is zero (stationary points) and domain endpoints.

$$s'(t) = v(t) = -6t^2 + 12t = 0 \Rightarrow 6t(-t + 2) = 0$$
$$\Rightarrow v(t) = 0 \text{ when } t = 0 \text{ or } t = 2$$

For the interval $0 \leqslant t \leqslant 3$, the critical points to be tested for finding the maximum displacement are at $t = 0$, $t = 2$ and $t = 3$. Check whether the velocity is increasing or decreasing on either side of the stationary point at $t = 2$ by finding the sign of $v(t)$ for $t = 1$ and $t = 2.5$.
$v(1) = -6(1)^2 + 12(1) = 6$ and $v(2.5) = -6(2.5)^2 + 12(2.5) = -7.5$
Hence, the displacement s is increasing for $0 < t < 2$ and decreasing for $2 < t < 3$. This indicates that the stationary point at $t = 2$ must be an absolute maximum for s in the interval $0 \leqslant t \leqslant 3$.

$s(2) = -2(2)^3 + 6(2)^2 = 8$

Therefore, the object has a maximum displacement of 8 metres at $t = 2$ seconds.

(ii) To find the maximum velocity, we can apply the first derivative test to $v(t)$. The first derivative of $v(t)$ is acceleration $a(t)$, which is the *second* derivative of $s(t)$. Hence, where $s''(t) = 0$ (acceleration is zero) indicates critical points for $v(t)$, i.e. where velocity may change from increasing to decreasing, or vice versa.

$$s''(t) = a(t) = \frac{d}{dt}(-6t^2 + 12t) = -12t + 12$$
$$\Rightarrow 12(-t + 1) = 0 \Rightarrow a(t) = 0 \text{ when } t = 1$$

For the interval $0 \leqslant t \leqslant 3$, the critical points to be tested for finding the maximum velocity are at $t = 0$, $t = 1$ and $t = 3$. Check whether the velocity is increasing or decreasing on either side of $t = 1$ by finding the sign of $a(t)$ for $t = 0.5$ and $t = 2$.
$a(0.5) = -12(0.5) + 12 = 6$ and $a(2) = -12(2) + 12 = -12$
Hence, the velocity v is increasing for $0 < t < 1$ and decreasing for $1 < t < 3$. This indicates that the point at $t = 1$ must be an absolute maximum for v in the interval $0 \leqslant t \leqslant 3$.

$v(1) = -6(1)^2 + 12(1) = 6$

Therefore, the object has a maximum velocity of 6 metres per second at $t = 1$ second.

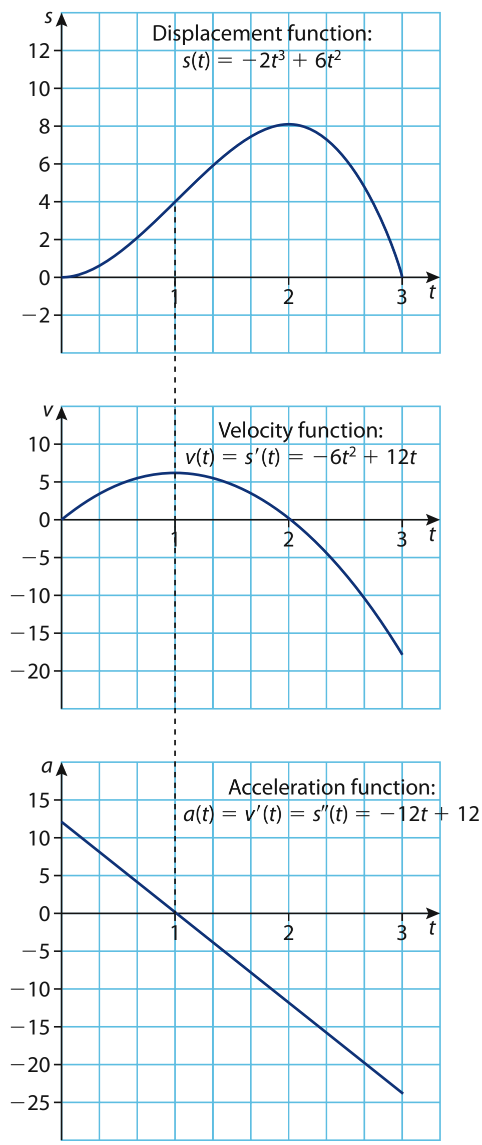
The second derivative of a function tells us how the first derivative of the function changes. From this we can use the second derivative, as we did the first derivative, to reveal information about the shape of the graph of a function. Note in Example 18 that the object's velocity changed from increasing to decreasing when the object's acceleration was zero at $t = 1$.

Let's examine graphically the significance of the point where acceleration is zero (i.e. velocity changing from increasing to decreasing) in connection to the displacement graph for Example 18. In other words, what can the second derivative of a function tell us about the shape of the function's graph?

Figure 13.21 shows the graphs of the displacement, velocity and acceleration functions for the motion of the object in Example 18. A dashed vertical line highlights the nature of the three graphs where $t = 1$. At this point, velocity has a maximum value and acceleration is zero. It is also where velocity changes from increasing to decreasing, which has a corresponding effect on the shape of the displacement function $s(t)$.

At the point where $t = 1$, the graph of $s(t)$ changes from curving 'upwards' (*concave up*) to curving 'downwards' (*concave down*) because its slope (corresponding to velocity) changes from increasing to decreasing. This can only occur when velocity (first derivative) has a maximum and hence where acceleration (second derivative) is zero. We can see from this illustration that for a general function $f(x)$, finding intervals where the first derivative $f'(x)$ is increasing (positive acceleration) or decreasing (negative acceleration) can be used to determine where the graph of $f(x)$ is curving upward or curving downward. A point at which a function's curvature (concavity) changes – as at $t = 1$ for the graph of $s(t)$ left – is called a **point of inflexion**.

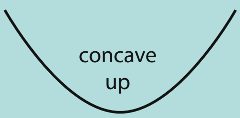
Figure 13.21



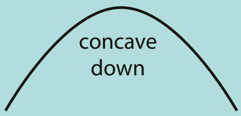
Concavity and the second derivative

The graph of $f(x)$ is **concave up** where $f'(x)$ is increasing and **concave down** where $f'(x)$ is decreasing. It follows that:

(i) if $f''(x) > 0$ for all x in some interval of the domain of f, the graph of f is concave up in the interval



(ii) if $f''(x) < 0$ for all x in some interval of the domain of f, the graph of f is concave down in the interval.



If $f(x)$ is a continuous function, its graph can only change concavity (up to down, or down to up) where $f''(x) = 0$. Hence, for a continuous function, an **inflexion point** may only occur where $f''(x) = 0$.

Note: Concavity is not defined for a line – it is neither concave up nor concave down.

Example 19

Determine the intervals on which the graph of $y = x^4 - 4x^3$ is concave up or concave down and identify any inflexion points.

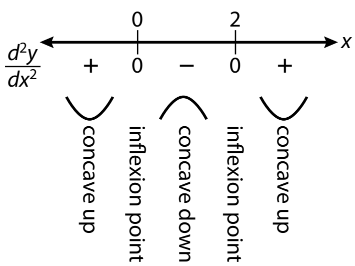
Solution

We first note that the function is continuous for its domain of all real numbers. To locate points of inflexion, we then find for what value(s) the second derivative is zero.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 4x^3) = 4x^3 - 12x^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3 - 12x^2) = 12x^2 - 24x = 12x(x - 2)$$

Setting $\frac{d^2y}{dx^2} = 0$, it follows that inflexion points may occur at $t = 0$ and $t = 2$. These two values divide the domain of the function into three intervals that we need to test. Let's choose $t = -1$, $t = 1$ and $t = 3$ as our test values. At $t = -1$, $\frac{d^2y}{dx^2} = 36 > 0$; at $t = 1$, $\frac{d^2y}{dx^2} = -12 < 0$; and at $t = 3$, $\frac{d^2y}{dx^2} = 36 > 0$. These results can be organized in a sign chart, illustrating that the graph of $y = x^4 - 4x^3$ is concave up for the open intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on the open interval $(0, 2)$.



At $t = 0$, $y = 0$ and at $t = 2$, $y = 2^4 - 4(2)^3 = -16$. Therefore, $(0, 0)$ and $(2, -16)$ are inflexion points because it is at these points the concavity of the graph changes.

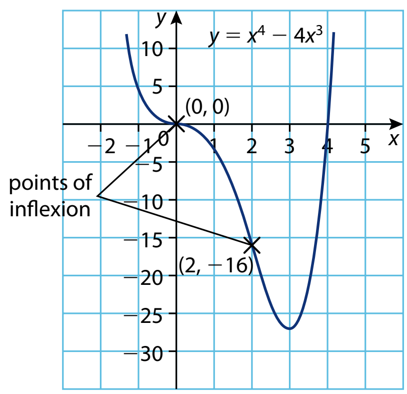


Figure 13.22 Inflexion points.

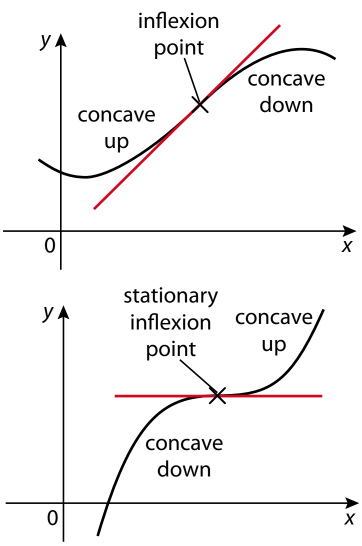


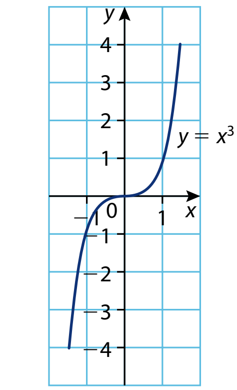
Figure 13.23 The concavity of a graph changes at a point of inflexion.

The graph of the function (Figure 13.22) from Example 19 reveals two different types of inflexion points. The slope of the curve at $(0, 0)$ is zero – i.e. it is a stationary point. The slope of the curve at the other inflexion point, $(2, -16)$, is negative.

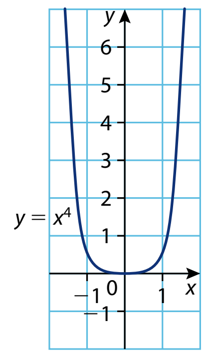
For either type of inflexion point, the graph crosses its tangent line at the point of inflexion, as shown in Figure 13.23.

The fact that the second derivative of a function is zero at a certain point does not guarantee that an inflexion point exists at the point.

The functions $y = x^3$ and $y = x^4$ will serve to illustrate that $\frac{d^2y}{dx^2} = 0$ is a necessary but not sufficient condition for the existence of an inflexion point.



- For $y = x^3$: $\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2 \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 6x \Rightarrow \frac{d^2y}{dx^2} = 0$ at $x = 0$. We can conclude from this that there may be an inflexion point at $x = 0$. We need to investigate further by checking to see if $\frac{d^2y}{dx^2}$ changes sign at $x = 0$. At $x = -1$, $\frac{d^2y}{dx^2} = -6$ and at $x = 1$, $\frac{d^2y}{dx^2} = 6$.
  Thus, there is an inflexion point at $x = 0$ (confirmed by graph) because the second derivative changes sign at $x = 0$.



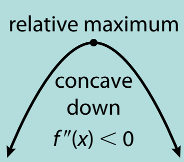
- For $y = x^4$: $\frac{dy}{dx} = \frac{d}{dx}(x^4) = 4x^3 \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3) = 12x^2 \Rightarrow \frac{d^2y}{dx^2} = 0$ at $x = 0$. Again, we need to see if $\frac{d^2y}{dx^2}$ changes sign at $x = 0$.
  At $x = -1$, $\frac{d^2y}{dx^2} = 12$ and at $x = 1$, $\frac{d^2y}{dx^2} = 12$. Thus, there is *no* inflexion point at $x = 0$ (confirmed by graph) because the second derivative does *not* change sign at $x = 0$.

The second derivative test

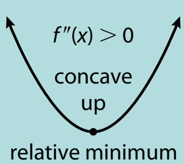
Earlier in this section, we developed the first derivative test for locating maxima and minima of a function. Instead of using the first derivative to check whether a function changes from increasing to decreasing (maximum) or decreasing to increasing (minimum) at a stationary point, we can simply evaluate the second derivative at the stationary point. If the graph is concave up at the stationary point then it will be a minimum, and if it is concave down then it will be a maximum. If the second derivative is zero at a stationary point (as for $y = x^3$ and $y = x^4$), no conclusion can be made and we need to go back to the first derivative test. Using the second derivative in this way is a very efficient method for telling us whether a stationary point is a relative maximum or minimum.

The second derivative test

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.



2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at $x = c$.



If $f''(c) = 0$, the test fails and the first derivative test should be applied.

Example 20

Find any relative extrema for $f(x) = 3x^5 - 25x^3 + 60x + 20$.

Solution

The implied domain of f is all real numbers. Solve $f'(x) = 0$ to obtain possible extrema.

$$\begin{aligned} f'(x) &= 15x^4 - 75x^2 + 60 = 0 \\ &15(x^4 - 5x^2 + 4) = 0 \\ &15(x^2 - 4)(x^2 - 1) = 0 \\ &15(x + 2)(x - 2)(x + 1)(x - 1) = 0 \end{aligned}$$

Therefore, f has four stationary points: $x = -2$, $x = -1$, $x = 1$ and $x = 2$.

Applying the second derivative test:

$$\begin{aligned} &f''(x) = 60x^3 - 150x = 30x(2x^2 - 5) \\ &f''(-2) = -180 < 0 \ \Rightarrow\ f \text{ has a relative maximum at } x = -2 \\ &f''(-1) = 90 > 0 \ \Rightarrow\ f \text{ has a relative minimum at } x = -1 \\ &f''(1) = -90 < 0 \ \Rightarrow\ f \text{ has a relative maximum at } x = 1 \\ &f''(2) = 180 > 0 \ \Rightarrow\ f \text{ has a relative minimum at } x = 2 \end{aligned}$$

Exercise 13.3

In questions 1–3, find the vertex of the parabola using differentiation.

1 $y = x^2 - 2x - 6$ **2** $y = 4x^2 + 12x + 17$ **3** $y = -x^2 + 6x - 7$

For questions 4–7, a) find the derivative, $f'(x)$, b) indicate the interval(s) for which $f(x)$ is increasing, and c) the interval(s) for which $f(x)$ is decreasing.

4 $y = x^2 - 5x + 6$ **5** $y = 7 - 4x - 3x^2$

6 $y = \frac{1}{3}x^3 - x$ **7** $y = x^4 - 4x^3$

For questions 8–13:
a) find the coordinates of any stationary points for the graph of the equation
b) state, with reasoning, whether each stationary point is a minimum, maximum or neither
c) sketch a graph of the equation and indicate the coordinates of each stationary point on the graph.

8 $y = 2x^3 + 3x^2 - 72x + 5$ **9** $y = \frac{1}{6}x^3 - 5$

10 $y = x(x - 3)^2$ **11** $y = x^4 - 2x^3 - 5x^2 + 6$

12 $y = x^3 - 2x^2 - 7x + 10$ **13** $y = x - \sqrt{x}$

14 An object moves along a line such that its displacement, s metres, from the origin O is given by $s(t) = t^3 - 4t^2 + t$.
a) Find expressions for the object's velocity and acceleration in terms of t.
b) For the interval $-1 \leqslant t \leqslant 3$, sketch the displacement-time, velocity-time, and acceleration-time graphs on separate sets of axes, vertically aligned as in Figure 13.21.
c) For the interval $-1 \leqslant t \leqslant 3$, find the time at which the displacement is a maximum and find its value.
d) For the interval $-1 \leqslant t \leqslant 3$, find the time at which the velocity is a minimum and find its value.
e) In words, accurately describe the motion of the object during the interval $-1 \leqslant t \leqslant 3$.

For each function $f(x)$ in questions 15–20, find any relative extrema and points of inflexion. State the coordinates of any such points. Use your GDC to assist you in sketching the function.

15 $f(x) = x^3 - 12x$

16 $f(x) = \frac{1}{4}x^4 - 2x^2$

17 $f(x) = x + \dfrac{4}{x}$

18 $y = x^2 - \dfrac{1}{x}$

19 $f(x) = -3x^5 + 5x^3$

20 $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

21 An object moves along a line such that its displacement, s metres, from a fixed point P is given by $s(t) = t(t - 3)(8t - 9)$.

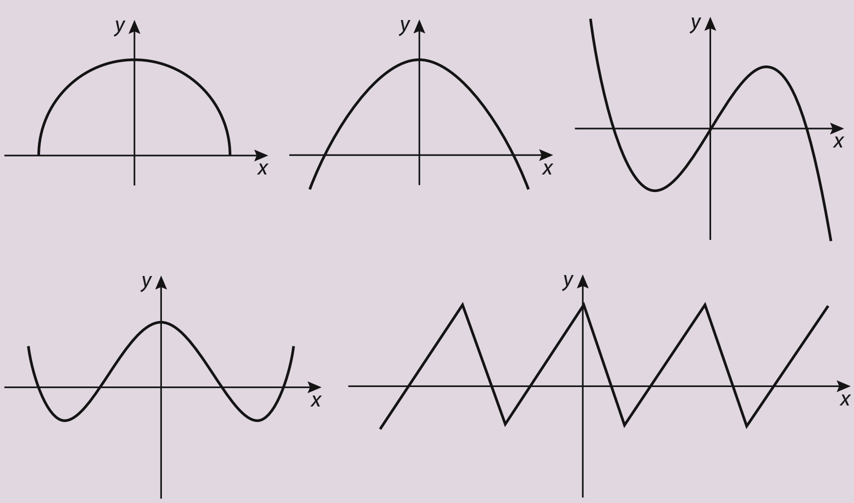
a) Find the initial velocity and initial acceleration of the object.
b) Find the velocity and acceleration of the object at $t = 3$ seconds.
c) Find for what values of t the object changes direction. What significance do these times have in connection to the displacement of the object?
d) Find for what value of t the object's velocity is a minimum. What significance does this time have in connection to the acceleration of the object?

22 The delivery cost per tonne of bananas, D (in thousands of dollars), when x tonnes of bananas are shipped is given by $D = 3x + \dfrac{100}{x}$, $x > 0$. Find the value of x for which the delivery cost per tonne of bananas is a minimum, and find the value of the minimum delivery cost. Explain why this cost is a minimum rather than a maximum.

23 The curve $y = x^4 + ax^2 + bx + c$ passes through the point $(-1, -8)$ and at that point $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} = 6$. Find the values of a, b and c and sketch the curve.

24 Find any maxima, minima or stationary points of inflexion of the function $f(x) = \dfrac{x^3 + 3x - 1}{x^2}$, stating, with explanation, the nature of each point.
Sketch the curve, indicating clearly what happens as $x \to \pm\infty$.

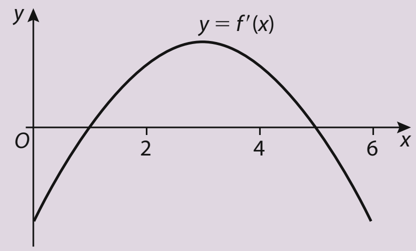
25 For each of the five functions graphed below sketch its derivative on a separate pair of axes. Do not use your GDC. It is helpful to use the result from question 25 in Exercise 13.2 – that the derivative of an even function is odd and the derivative of an odd function is even.



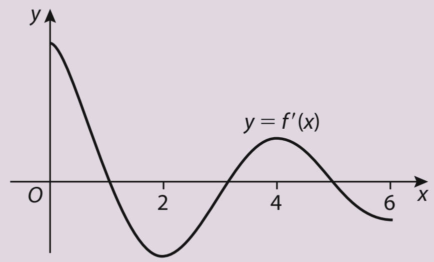
In questions 26 and 27, the graph of the **derivative** of a function *f* is shown.

a) On what intervals is *f* increasing or decreasing?
b) At what value(s) of x does *f* have a local maximum or minimum?

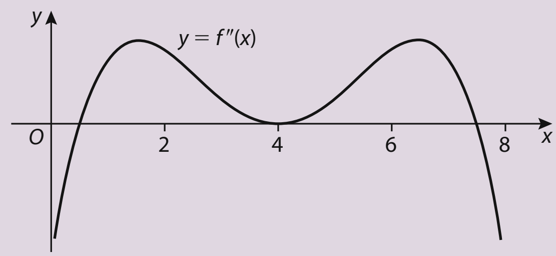
26



27



28 The graph of the **second derivative** f'' of a function *f* is shown. Approximate the x-coordinates of the inflexion points of *f*. Give reasons for your answers.



29 Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

$f(-2) = 8$ $\quad f(0) = 4$ $\quad f(2) = 0$ $\quad f'(2) = f'(-2) = 0$

$f'(x) > 0$ for $|x| > 2$ $\quad f'(x) < 0$ for $|x| < 2$ $\quad f''(x) < 0$ for $x < 0$ $\quad f''(x) > 0$ for $x > 0$

30 An object moves along a horizontal line such that its displacement, s metres, from its starting position at any time $t \geqslant 0$ is given by the function $s(t) = -2t^3 + 15t^2 - 24t$. The positive direction is to the right.

a) Find the intervals of time when the object is moving to the right, and the intervals when it is moving to the left.
b) Find the (i) initial velocity, and (ii) initial acceleration of the object.
c) Find the (i) maximum displacement, and (ii) maximum velocity for the interval $0 \leqslant t \leqslant 5$.
d) When is the object's acceleration equal to zero? Describe the motion of the object at this time.

31 a) Use your GDC to approximate to three significant figures the maximum and minimum values of the function $f(x) = x - \sqrt{2}\sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

b) Find $f'(x)$ and find the exact minimum and maximum values for $f(x)$ in the interval $0 \leqslant x \leqslant 2\pi$.

13.4 Tangents and normals

In many areas of mathematics and physics, it is useful to have an accurate description of a line that is tangent or normal (perpendicular) to a curve. The most complete mathematical description we can obtain is to find the algebraic equation of such lines. In this chapter, much of our work has been in connection to the slopes of tangent lines, so this will be our starting point.

Finding equations of tangents

We now make use of the basic differentiation rules that we established earlier to determine the equation of lines that are tangent to a curve at a point. The first example shows how we can approximate the square root of a number quite accurately without a calculator by making use of a tangent line.

Example 21

a) Find the equation of the line tangent to $y = \sqrt{x}$ at $x = 9$.
b) Use this tangent line to approximate $\sqrt{10}$.

Solution

a) We can find the equation of any line if we know its slope and a point it passes through. Since $y = 3$ when $x = 9$, the point of tangency is $(9, 3)$. We differentiate to find the slope of the curve at $x = 9$, thus giving us the slope of the tangent line.

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

At $x = 9$: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$ $\Rightarrow$ The slope of the curve and tangent line at $x = 9$ is $\frac{1}{6}$.

Now that we have a point and a slope for the line we can substitute in the point-slope form for the equation of a line.

$$y - 3 = \tfrac{1}{6}(x - 9) \quad \Rightarrow \quad y = \tfrac{1}{6}x + \tfrac{3}{2}$$

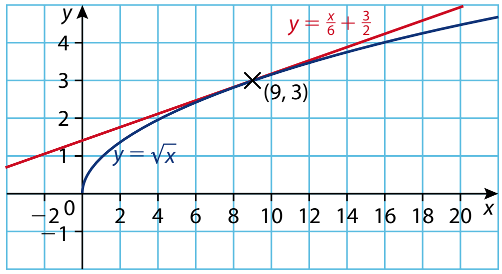
The equation of the line tangent to $y = \sqrt{x}$ at $x = 9$ is $y = \dfrac{x}{6} + \dfrac{3}{2}$.

b) For values of x near 9, $y = \sqrt{x} \approx \dfrac{x}{6} + \dfrac{3}{2}$.

$$\sqrt{10} \approx \frac{10}{6} + \frac{3}{2} = \frac{19}{6} \qquad \begin{array}{r} 3.1\overline{6} \\ 6\overline{)19.00} \end{array}$$

The actual value of $\sqrt{10}$ to 4 significant figures is 3.162. Our approximation expressed to 3 significant figures is 3.167. The percentage error is less than 0.2%.

Figure 13.24



Finding the tangent to a curve was a challenge that motivated many of the initial developments of calculus in the 17th century. In one of his books on mathematics, Descartes wrote the following about the problem of how to find a tangent to a curve:

> *And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know.*

The graphs of $y = \sqrt{x}$ and its tangent at $x = 9$, $y = \frac{x}{6} + \frac{3}{2}$, in Figure 13.24 illustrate that the tangent is a very good approximation to the curve in the interval $5 < x < 13$ centred on the point of tangency $(9, 3)$.

Example 22

Find the equation of the tangent to $f(x) = x + \frac{1}{x}$ at the point $\left(\frac{1}{2}, \frac{5}{2}\right)$.

Solution

$$f(x) = x + \frac{1}{x} = x + x^{-1}$$

$$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

When $x = \frac{1}{2}$, $f'\left(\frac{1}{2}\right) = 1 - \frac{1}{\left(\frac{1}{2}\right)^2} = -3$. Hence, the slope of the tangent is -3.

$$y - \frac{5}{2} = -3\left(x - \frac{1}{2}\right) \quad \Rightarrow \quad y = -3x + \frac{3}{2} + \frac{5}{2} \quad \Rightarrow \quad y = -3x + 4$$

The equation of the line tangent to $f(x) = x + \frac{1}{x}$ at $x = \frac{1}{2}$ is $y = -3x + 4$.

Example 23

Consider the function $g(x) = x^2(x - 1)$.

a) Find the two points on the graph of g at which the slope of the curve is 8.

b) Find the equations of the tangents at both of these points.

Solution

a) In order to differentiate by applying the power rule term-by-term, we first need to write the equation for g in expanded form:

$g(x) = x^2(x - 1) = x^3 - x^2$

$$g'(x) = \frac{d}{dx}(x^3 - x^2) = 3x^2 - 2x$$

$$g'(x) = 3x^2 - 2x = 8 \quad \Rightarrow \quad 3x^2 - 2x - 8 = 0$$

$$(3x + 4)(x - 2) = 0 \quad \Rightarrow \quad x = -\frac{4}{3} \text{ or } x = 2$$

$$g\left(-\frac{4}{3}\right) = \left(-\frac{4}{3}\right)^3 - \left(-\frac{4}{3}\right)^2 = -\frac{112}{27} \text{ and } g(2) = 2^3 - 2^2 = 4$$

Thus, the slope of the curve is equal to 8 at the points $\left(-\frac{4}{3}, -\frac{112}{27}\right)$ and $(2, 4)$.

b) Tangent at $\left(-\frac{4}{3}, -\frac{112}{27}\right)$:

$$y - \left(-\frac{112}{27}\right) = 8\left[x - \left(-\frac{4}{3}\right)\right] \quad \Rightarrow \quad y = 8x + \frac{32}{3} - \frac{112}{27}$$

$$\Rightarrow \quad y = 8x + \frac{176}{27}$$

Therefore, the equation of the tangent at $\left(-\frac{4}{3}, -\frac{112}{27}\right)$ is $y = 8x + \frac{176}{27}$.

Tangent at (2, 4):

$$y - 4 = 8(x - 2) \quad \Rightarrow \quad y = 8x - 16 + 4 \quad \Rightarrow \quad y = 8x - 12$$

Therefore, the equation of the tangent at (2, 4) is $y = 8x - 12$.

Figure 13.25 shows the results for Example 23 – the graph of the function g and the two tangent lines to the graph of the function that have a slope of 8. Note that the scales on the x- and y-axes are not equal which causes the slope of the tangent lines to appear less than 8 for this particular graph.

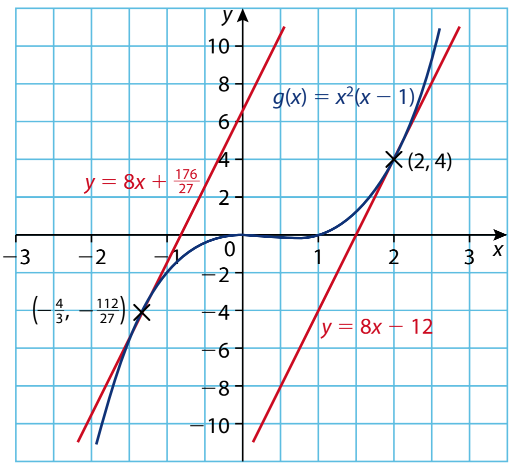
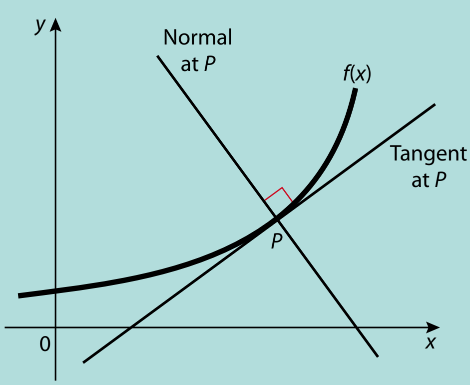


Figure 13.25

The normal to a curve at a point

Another line we often need to find is the line that is 'perpendicular' to a curve at a certain point, which we define to be the line that is perpendicular to the tangent at that point. In this particular context, we apply the adjective 'normal' rather than 'perpendicular' to denote that two lines are at right angles to one another.

A **normal** to a graph of a function at a point is the line through the point that is at a right angle to the tangent at the point. In other words, the tangent and normal to a curve at a certain point are perpendicular.



Recall that two perpendicular lines have slopes that are opposite reciprocals. If the slopes of two perpendicular lines are m_1 and m_2, then $m_1 = -\frac{1}{m_2}$ or $m_1 m_2 = -1$. The exception is if one of the lines is horizontal (slope is zero) and the other is vertical (slope is undefined).

Example 24

Find the equation of the normal to the graph of $y = 2x^2 - 6x + 3$ at the point $(1, -1)$.

Solution

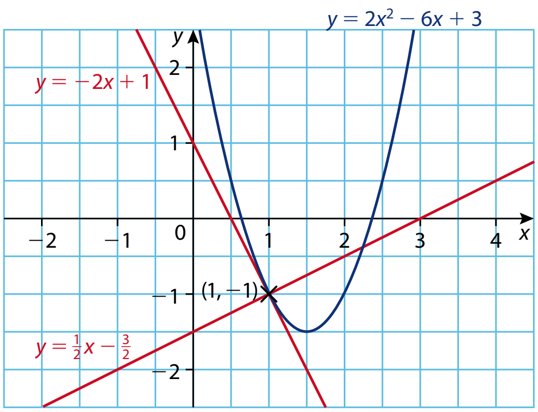
$$\frac{dy}{dx} = \frac{d}{dx}(2x^2 - 6x + 3) = 4x - 6$$

Slope of tangent at $(1, -1)$ is $4(1) - 6 = -2$. Hence, slope of normal is $+\frac{1}{2}$.

Equation of normal: $y - (-1) = \frac{1}{2}(x - 1) \quad \Rightarrow \quad y = \frac{1}{2}x - \frac{3}{2}$

Figure 13.26 shows the results for Example 24 with the curve at both its tangent and normal at the point $(1, -1)$. Please be aware that if you graph a function with its tangent and normal at a certain point, the normal will only appear perpendicular if the scales on both the x- and y-axes are equal. Regardless of whether the scales are equal or not, the tangent will always appear tangent to the curve.

Figure 13.26



Example 25

Consider the parabola with equation $y = \frac{1}{4}x^2$.

a) Find the equation of the normals at the points $(-2, 1)$ and $(-4, 4)$.

b) Show that the point of intersection of these two normals lies on the parabola.

Solution

a) $\frac{dy}{dx} = \frac{1}{2}x$

Slope of tangent at $(-2, 1)$ is $\frac{1}{2}(-2) = -1$, so the slope of the normal at that point is $+1$.

Then equation of normal at $(-2, 1)$ is: $y - 1 = x - (-2) \Rightarrow y = x + 3$

Slope of tangent at $(-4, 4)$ is $\frac{1}{2}(-4) = -2$, so the slope of the normal at that point is $\frac{1}{2}$.

Then equation of normal at $(-4, 4)$ is: $y - 4 = \frac{1}{2}[x - (-4)]$

$$\Rightarrow y = \frac{1}{2}x + 6$$

b) Set the equations of the two normals equal to each other to find their intersection.

$$x + 3 = \tfrac{1}{2}x + 6 \quad \Rightarrow \quad \tfrac{1}{2}x = 3 \quad \Rightarrow \quad x = 6 \ \text{ then } \ y = 9$$
$$\Rightarrow \quad \text{intersection point is } (6, 9)$$

Substitute the coordinates of the points into the equation for the parabola.

$$y = \tfrac{1}{4}x^2 \quad \Rightarrow \quad 9 = \tfrac{1}{4}(6)^2 \quad \Rightarrow \quad 9 = \tfrac{1}{4}\cdot 36 \quad \Rightarrow \quad 9 = 9$$

This confirms that the intersection point, $(6, 9)$, of the normals is also a point on the parabola.

Exercise 13.4

1 Find an equation of the tangent line to the graph of the equation at the indicated value of x.

a) $y = x^2 + 2x + 1 \qquad x = -3$

b) $y = x^3 + x^2 \qquad x = -\frac{2}{3}$

c) $y = 3x^2 - x + 1 \qquad x = 0$

d) $y = 2x + \frac{1}{x} \qquad x = \frac{1}{2}$

2 Find the equations of the normal to the functions in question 1 at the indicated value of x.

3 Find the equations of the lines tangent to the curve $y = x^3 - 3x^2 + 2x$ at any point where the curve intersects the x-axis.

4 Find the equation of the tangent to the curve $y = x^2 - 2x$ that is perpendicular to the line $x - 2y = 1$.

5 Using your GDC for assistance, make accurate sketches of the curves $y = x^2 - 6x + 20$ and $y = x^3 - 3x^2 - x$ on the same set of axes. The two curves have the same slope at an integer value for x somewhere in the interval $0 \leqslant x \leqslant \frac{3}{2}$.

a) Find this value of x.

b) Find the equation for the line tangent to each curve at this value of x.

6 Find the equation of the normal to the curve $y = x^2 + 4x - 2$ at the point where $x = -3$. Find the coordinates of the other point where this normal intersects the curve again.

7 Consider the function $g(x) = \dfrac{1 - x^3}{x^4}$. Find the equation of both the tangent and the normal to the graph of g at the point $(1, 0)$.

8 The normal to the curve $y = ax^{\frac{1}{2}} + bx$ at the point where $x = 1$ has a slope of 1 and intersects the y-axis at $(0, -4)$. Find the value of a and the value of b.

9 a) Find the equation of the tangent to the function $f(x) = x^3 + \frac{1}{2}x^2 + 1$ at the point $\left(-1, \frac{1}{2}\right)$.

b) Find the coordinates of another point on the graph of f where the tangent is parallel to the tangent found in a).

10 Find the equation of both the tangent and the normal to the curve $y = \sqrt{x}\,(1 - \sqrt{x})$ at the point where $x = 4$.

11 Consider the function $f(x) = (1 + x)^2(5 - x)$.

a) Show that the line tangent to the graph of f where $x = 1$ does not intersect the graph of the function again.

b) Also show that the tangent line at (0, 5) intersects the graph of f at a turning point.

c) Sketch the graph of f and the two tangents from a) and b).

12 Find equations of both lines through the point (2, −3) that are tangent to the parabola $y = x^2 + x$.

13 Find all tangent lines through the origin to the graph of $y = 1 + (x - 1)^2$.

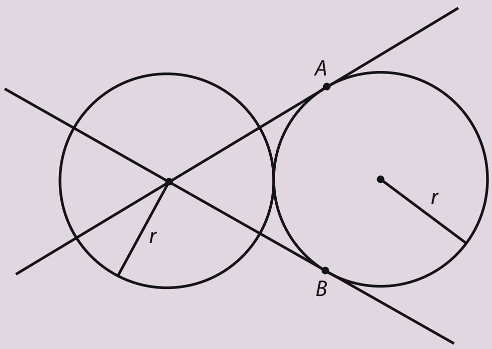
14 a) Find the equation of the tangent line to $y = \sqrt[3]{x}$ at $x = 8$.

b) Use the equation of this tangent line to approximate $\sqrt[3]{9}$ to three significant figures.

15 Find the equation of the tangent line for $f(x) = \dfrac{1}{\sqrt{x}}$ at $x = a$.

16 The tangent to the graph of $y = x^3$ at a point P intersects the curve again at another point Q.
Find the coordinates of Q in terms of the coordinates of P.

17 Two circles of radius r are tangent to each other. Two lines pass through the centre of one circle and are tangent to the other circle at points A and B as shown in the diagram. Find an expression for the distance between A and B.



18 Prove that there is no line through the point (1, 2) that is tangent to the curve $y = 4 - x^2$.

Practice questions

1 The function f is defined as $f(x) = x^2$.

a) Find the gradient (slope) of f at the point P, where $x = 1.5$.

b) Find an equation for the tangent to f at the point P.

c) Draw a diagram to show clearly the graph of f and the tangent at P.

d) The tangent of part **b)** intersects the x-axis at the point Q and the y-axis at the point R. Find the coordinates of Q and R.

e) Verify that Q is the midpoint of $[PR]$.

f) Find an equation, in terms of a, for the tangent to f at the point $S(a, a^2)$, $a \neq 0$.

g) The tangent of part **f)** intersects the x-axis at the point T and the y-axis at the point U. Find the coordinates of T and U.

h) Prove that, whatever the value of a, T is the midpoint of SU.

2 The curve with equation $y = Ax + B + \frac{C}{x}$, $x \in \mathbb{R}$, $x \neq 0$, has a minimum at $P(1, 4)$ and a maximum at $Q(-1, 0)$. Find the value of each of the constants A, B and C.

3 Differentiate:

a) $x^2(2 - 3x^3)$

b) $\frac{1}{x}$

4 Consider the function $f(x) = \frac{8}{x} + 2x,\ x > 0$.

a) Solve the equation $f'(x) = 0$. Show that the graph of f has a turning point at $(2, 8)$.

b) Find the equations of the asymptotes to the graph of f, and hence sketch the graph.

5 Find the coordinates of the stationary point on the curve with equation $y = 4x^2 + \frac{1}{x}$.

6 The curve $y = ax^3 - 2x^2 - x + 7$ has a gradient (slope) of 3 at the point where $x = 2$. Determine the value of a.

7 If $f(2) = 3$ and $f'(2) = 5$, find an equation of **a)** the line tangent to the graph of f at $x = 2$, and **b)** the line normal to the graph of f at $x = 2$.

8 The function $g(x)$ is defined for $-3 \leqslant x \leqslant 3$. The behaviour of $g'(x)$ and $g''(x)$ is given in the tables below.

x	$-3 < x < -2$	-2	$-2 < x < 1$	1	$1 < x < 3$
$g'(x)$	negative	0	positive	0	negative

x	$-3 < x < -\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2} < x < 3$
$g''(x)$	positive	0	negative

Use the information above to answer the following. In each case, justify your answer.

a) Write down the value of x for which g has a maximum.

b) On which intervals is the value of g decreasing?

c) Write down the value of x for which the graph of g has a point of inflexion.

d) Given that $g(-3) = 0$, sketch the graph of g. On the sketch, clearly indicate the position of the maximum point, the minimum point and the point of inflexion.

9 Given the function $f(x) = x^2 - 3bx + (c + 2)$, determine the values of b and c such that $f(1) = 0$ and $f'(3) = 0$.

10 **Figure 1** shows the graphs of the functions f_1, f_2, f_3, f_4. **Figure 2** includes the graphs of the derivatives of the functions shown in **Figure 1**.

Figure 1

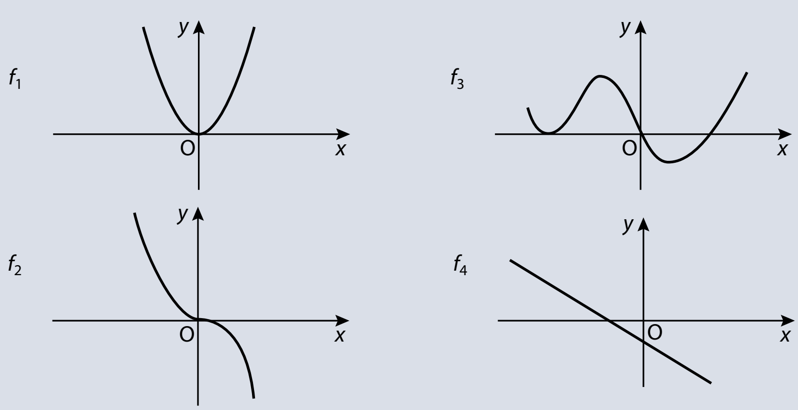
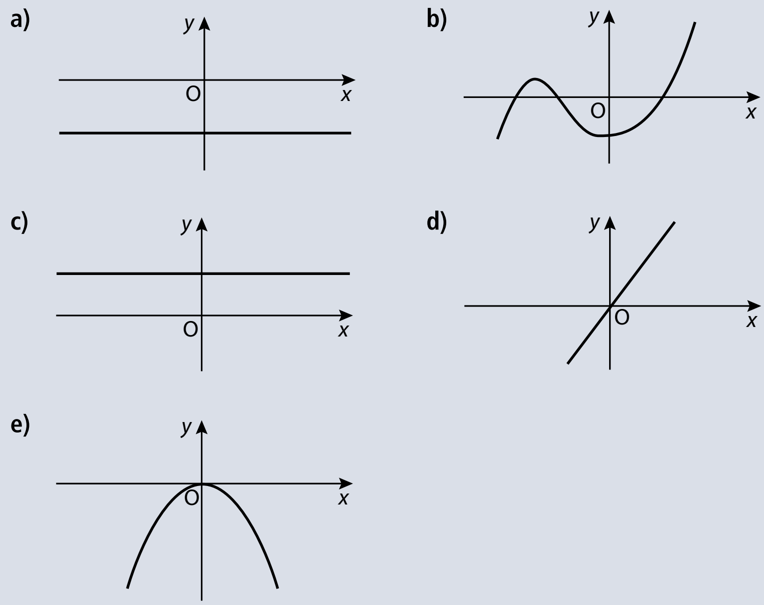


Figure 2



Complete the table below by matching each function with its derivative.

Function	Derivative diagram
f_1	
f_2	
f_3	
f_4	

11 Consider the function $f(x) = 1 + \sin x$.

a) Find the average rate of change of f from $x = 0$ to $x = \frac{\pi}{2}$.

b) Find the instantaneous rate of change of f at $x = \frac{\pi}{4}$.

c) At what value of x in the interval $0 < x < \frac{\pi}{2}$ is the instantaneous rate of change of f equal to the average rate of change of f from $x = 0$ to $x = \frac{\pi}{2}$ (answer to part **a)**)?

12 Consider the function $y = \frac{3x - 2}{x}$. The graph of this function has a vertical and a horizontal asymptote.

a) Write down the equation of

(i) the vertical asymptote

(ii) the horizontal asymptote.

b) Find $\frac{dy}{dx}$.

c) Indicate the intervals for which the curve is increasing or decreasing.

d) How many stationary points does the curve have? Explain using your result to **b)**.

13 Show that there are two points at which the function $h(x) = 2x^2 - x^4$ has a maximum value, and one point at which h has a minimum value. Find the coordinates of these three points, indicating whether it is a maximum or minimum.

14 The normal to the curve $y = x^{\frac{1}{2}} + x^{\frac{1}{3}}$ at the point (1, 2) meets the axes at (a, 0) and (0, b).
Find a and b.

15 The displacement, s metres, of a car, t seconds after leaving a fixed point A, is given by $s(t) = 10t - \frac{1}{2}t^2$.

a) Calculate the velocity when $t = 0$.
b) Calculate the value of t when the velocity is zero.
c) Calculate the displacement of the car from A when the velocity is zero.

16 A ball is thrown vertically upwards from ground level such that its height h metres at t seconds is given by $h = 14t - 4.9t^2$.

a) Write expressions for the ball's velocity and acceleration.
b) Find the maximum height the ball reaches and the time it takes to reach the maximum.
c) At the moment the ball reaches its maximum height, what is the ball's velocity and acceleration?

17 Find the exact coordinates of the inflexion point on the curve $y = x^3 + 12x^2 - x - 12$.

18 Consider the function $f(x) = 2\cos x - 3$. At the point on the curve where $x = \frac{\pi}{3}$, find:

a) the equation of the line tangent to f
b) the equation of the line normal to f.
Express both equations exactly.

19 A manufacturer produces closed cylindrical cans of radius r cm and height h cm. Each can has a total surface area of 54π cm^2.

a) Solve for h in terms of r, and hence find an expression for the volume, V cm^3, of each can in terms of r.
b) Find the value of r for which the cans have their maximum possible volume.

20 The curve $y = ax^2 + bx + c$ has a maximum point at (2, 18) and passes through the point (0, 10). Find a, b and c.

21 For the function $f(x) = \frac{1}{2}x^2 - 5x + 3$, find:

a) the equation of the tangent line at $x = -2$
b) the equation of the normal line at $x = -2$.

22 Consider the function $f(x) = x^4 - x^3$.

a) Find the coordinates of any maximum or minimum points. Identify each as relative or absolute.
b) State the domain and range of f.
c) Find the coordinates of any inflexion point(s).
d) Sketch the function clearly indicating any maximum, minimum or inflexion points.

23 Evaluate each limit.

a) $\displaystyle\lim_{x\to\infty} \frac{2 - 3x + 5x^2}{8 - 3x^2}$

b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4} - 2}{x}$

c) $\displaystyle\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$

d) $\displaystyle\lim_{h\to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}$

24 Find the derivative $f'(x)$ for each function.

a) $f(x) = \dfrac{x^2 - 4x}{\sqrt{x}}$ **b)** $f(x) = x^3 - 3\sin x$

c) $f(x) = \dfrac{1}{x} + \dfrac{x}{2}$ **d)** $f(x) = \dfrac{7}{3x^{13}}$

25 A point (p, q) is on the graph of $y = x^3 + x^2 - 9x - 9$, and the line tangent to the graph at (p, q) passes through the point $(4, -1)$. Find p and q.

26 For what values of c, such that $c \geqslant 0$, is the line $y = -\frac{1}{12}x + c$ normal to the graph of $y = x^3 + \frac{1}{3}$?

27 Find the points on the curve $y = \frac{1}{3}x^3 - x$ where the tangent line is parallel to the line $y = 3x$.

28 At what point does the line that is normal to the graph of $y = x - x^2$ at the point $(1, 0)$ intersect the graph of the curve a second time?

29 If $f(x) = \sqrt{x + 2}$, find $f'(x)$ by first principles.

30 An object moves along a line according to the position function $s(t) = t^3 - 9t^2 + 24t$. Find the positions of the object when

a) its velocity is zero

b) its acceleration is zero.

31 A particle moves along a straight line in the time interval $0 \leqslant t \leqslant 2\pi$ such that its displacement from the origin O is s metres given by the function $s = t + \sin t$.

a) Find the value(s) of t in the interval $0 \leqslant t \leqslant 2\pi$ when the particle's direction changes.

b) Show that the particle always remains on the same side of the origin O.

c) Find the value(s) of t in the interval $0 \leqslant t \leqslant 2\pi$ when the particle's acceleration is zero.

d) Sketch a graph of the particle's displacement from O for $0 \leqslant t \leqslant 2\pi$, and state the maximum value of s in this interval.

32 The curve whose equation is $y = ax^3 + bx^2 + cx + d$ has a point of inflexion at $(-1, 4)$, a turning point when $x = 2$, and it passes through the point $(3, -7)$. Find the values of a, b, c and d, and the y-coordinate of the turning point.

33 Find the stationary values of the function $f(x) = 1 - \dfrac{9}{x^2} + \dfrac{18}{x^4}$ and determine their nature.

34 a) Find the equation of the tangent to the curve $y = \dfrac{1}{x}$ at the point $(1, 1)$.

b) Find the equation of the tangent to the curve $y = \cos x$ at the point $\left(\dfrac{\pi}{2}, 0\right)$.

c) Deduce that $\dfrac{1}{x} > \cos x$ for $0 \leqslant x \leqslant \dfrac{\pi}{2}$.

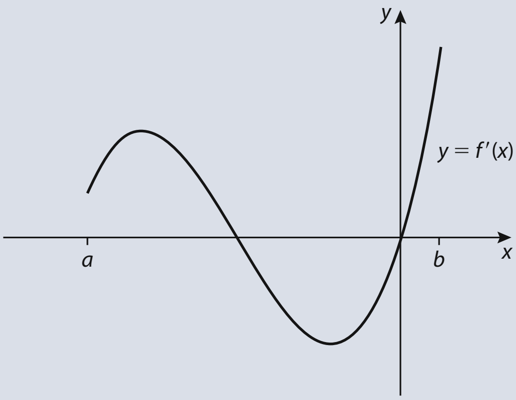
35 Show that there is just one tangent to the curve $y = x^3 - x + 2$ that passes through the origin.
Find its equation and the coordinates of the point of tangency.

36 The displacement, s metres, of a moving body B from a fixed point O, at time t seconds, is given by $s = 50t - 10t^2 + 1000$.

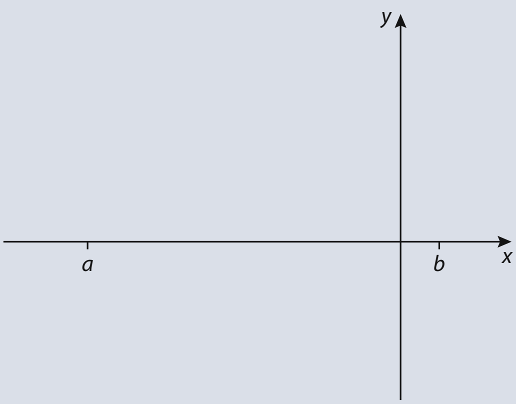
a) Find the velocity of B in $\mathrm{m\,s^{-1}}$.

b) Find its maximum displacement from O.

37 The diagram shows a sketch of the graph of $y = f'(x)$ for $a \leqslant x \leqslant b$.



On the grid below, which has the same scale on the x-axis, draw a sketch of the graph of $y = f(x)$ for $a \leqslant x \leqslant b$, given that $f(0) = 0$ and $f(x) \geqslant 0$ for all x. On your graph you should clearly indicate any minimum or maximum points, or points of inflexion.



14 Vectors, Lines and Planes

Assessment statements

4.1 Vectors as displacements in the plane and in three dimensions.
Components of a vector; column representation.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Algebraic and geometric approaches to the following topics:
the sum and difference of two vectors; the zero vector; the vector $-\mathbf{v}$;
multiplication by a scalar, $k\mathbf{v}$;
magnitude of a vector, $|\mathbf{v}|$;
unit vectors; base vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;
position vectors $\overrightarrow{OA} = \mathbf{a}$;
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

4.2 The scalar product of two vectors.
Perpendicular vectors; parallel vectors.
The angle between two vectors.

4.3 Vector equation of a line in two and three dimensions: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.
The angle between two lines.

4.4 Coincident, parallel, intersecting and skew lines, distinguishing between these cases.
Points of intersection.

4.5 The vector product of two vectors, $\mathbf{v} \times \mathbf{w}$.
Properties of the vector product.
Geometric interpretation of $|\mathbf{v} \times \mathbf{w}|$.

4.6 Vector equation of a plane $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.
Use of normal vector to obtain the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.
Cartesian equation of a plane $ax + by + cz = d$.

4.7 Intersections of: a line with a plane; two planes; three planes.
Angle between: a line and a plane; two planes.

Introduction

You have seen vectors in the plane in Chapter 9. We will limit our discussion to mainly three-dimensional space in this chapter. If you need to refresh your knowledge of the plane case, refer to Chapter 9.

Because we live in a three-dimensional world, it is essential that we study objects in three dimensions. To that end, we consider in this section a three-dimensional coordinate system in which points are determined by ordered triples. We construct the coordinate system in the following manner: Choose three mutually perpendicular axes, as shown in Figure 14.1, to serve as our reference. The orientation of the system is *right-handed* in the sense that if you hold your right hand so that the fingers curl from the positive x-axis towards the positive y-axis, your thumb points along the z-axis (see below). Looking at it in a different perspective, if you are looking straight at the system, the yz-plane is the plane facing you, and the xz-plane is perpendicular to it and extending out of the page towards you, and the xy-plane is the bottom part of that picture (Figure 14.2). The xy-, xz- and yz-planes are called the **coordinate planes**. Points in space are assigned coordinates in the same manner as in the plane. So, the point P (left) is assigned the ordered triple (x, y, z) to indicate that it is x, y and z units from the yz-, xz- and xy-planes.

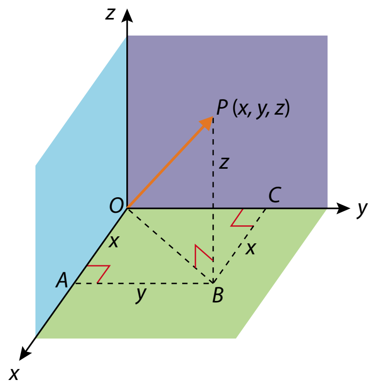
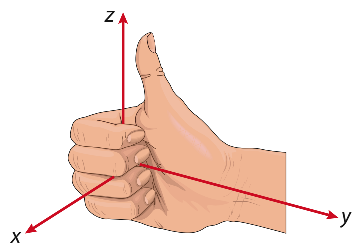


Figure 14.1

In this chapter, we will extend our study of vectors to space. The good news is that many of the rules you know from the plane also apply to vectors in space. So, we will only have to introduce a few new concepts. Some of the material will either be a repeat of what you have learned for two-dimensional space or an extension.



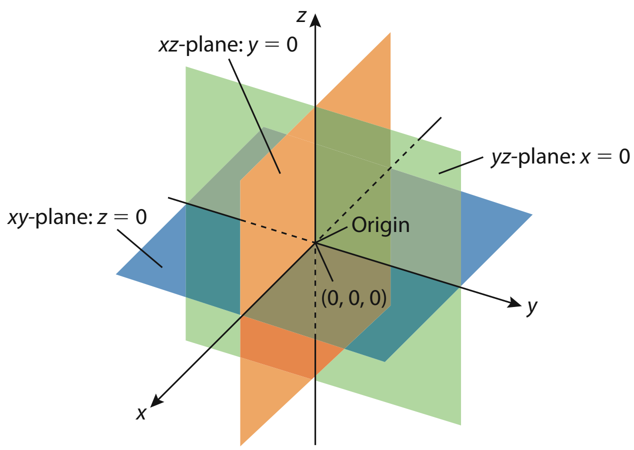


Figure 14.2 The coordinate planes divide space into 8 octants.

14.1 Vectors from a geometric viewpoint

Vectors can be represented geometrically by arrows in two- or three-dimensional space; the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude. The first point on the arrow is called the **initial point** of the vector and the tip is called the **terminal point**. We shall denote vectors in lower-case boldface type, such as $\mathbf{v}$, when using one letter to name the vector, and we will use $\overrightarrow{AB}$ to denote the vector from A to B. The handwritten notation will be the latter too.

If the initial point of a vector is at the origin, the vector is said to be in standard position. It is also called the **position vector** of point P. The terminal point will have coordinates of the form (x, y, z). We call these coordinates the **components** of $\mathbf{v}$ and we write $\mathbf{v} = (x, y, z)$ or $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The length (magnitude) of a vector $\mathbf{v}$ is also known as its **modulus** or its **norm** and it is written as $|\mathbf{v}|$.

Look back at Figure 14.1. Using Pythagoras' theorem, we can show that the magnitude of a vector $\mathbf{v}$, $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.

Let $\overrightarrow{OP} = \mathbf{v}$, then

$|\mathbf{v}| = |\overrightarrow{OP}| = \sqrt{OB^2 + BP^2}$, since the triangle OBP is right-angled at B. Now, consider triangle OAB, which is right-angled at A:

$OB^2 = OA^2 + AB^2 = x^2 + y^2$, and, therefore,

$$|\mathbf{v}| = \sqrt{OB^2 + BP^2} = \sqrt{(x^2 + y^2) + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

Two vectors like $\mathbf{v}$ and $\overrightarrow{AB}$ are equal (equivalent) if they have the same length (magnitude) and the same direction; we write $\mathbf{v} = \overrightarrow{AB}$. Geometrically, two vectors are equal if they are translations of one another as you see in Figures 14.3 and 14.4. Notice in Figure 14.4 that the four vectors are equal, even though they are in different positions.

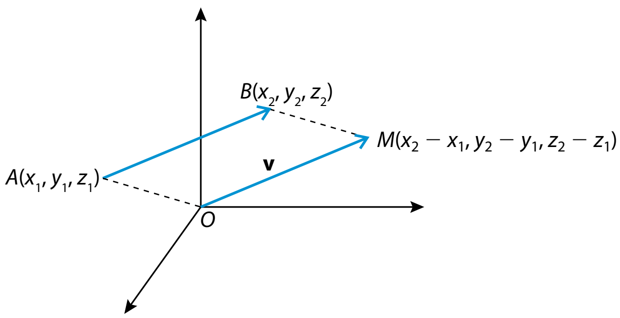


Figure 14.3

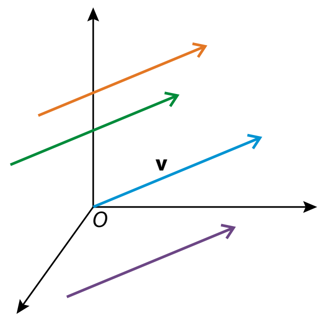


Figure 14.4

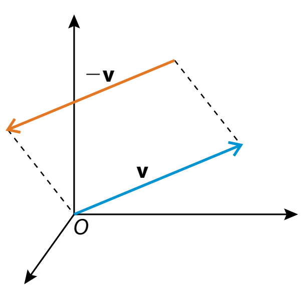


Figure 14.5

Because vectors are not affected by translation, the initial point of a vector $\mathbf{v}$ can be moved to any convenient position by making an appropriate translation.

Two vectors are said to be opposite if they have equal modulus but opposite direction (Figure 14.5).

If the initial and terminal points of a vector coincide, the vector has length zero; we call this the **zero vector** and denote it by $\mathbf{0}$.

The zero vector does not have a specific direction, so we will agree that it can be assigned any convenient direction in a specific problem.

Addition and subtraction of vectors

As you recall from Chapter 9, according to the **triangular rule**, if **u** and **v** are vectors, the sum **u** + **v** is the vector from the initial point of **u** to the terminal point of **v**, when the vectors are positioned so that the initial point of **v** is the terminal point of **u**, as shown in Figure 14.6.

Equivalently, **u** + **v** is also the diagonal of the parallelogram whose sides are **u** and **v**, as shown in Figure 14.7.

The difference of the two vectors **u** and **v** can be dealt with in the same manner. So, the vector **w** = **u** – **v** is a vector such that **u** = **v** + **w**.

In Figure 14.8, we can clearly see that the difference is along the diagonal joining the two terminal points of the vectors and in the direction from **v** to **u**.

If k is a real positive number, k**v** is a vector of magnitude $k|\mathbf{v}|$ and in the same direction as **v**. It follows that when k is negative, k**v** has magnitude $|k| \times |\mathbf{v}|$ and is in the opposite direction to **v** (Figure 14.9).

● **Hint:** When we discuss vectors, we will refer to real numbers as scalars.

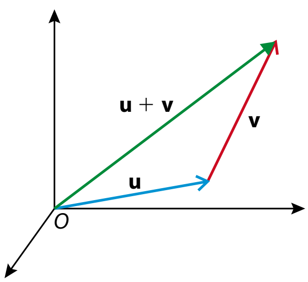


Figure 14.6

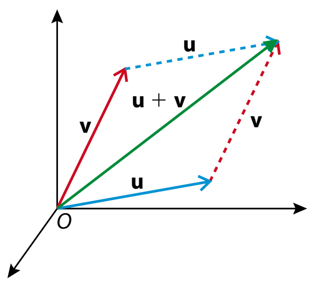


Figure 14.7

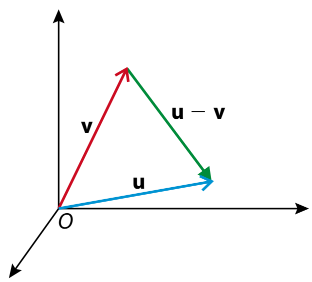


Figure 14.8

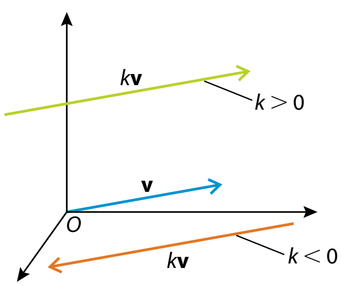


Figure 14.9

A result of the previous situation is the necessary and sufficient condition for two vectors to be parallel:

Two vectors are parallel if one of them is a scalar multiple of the other.
For example, the vector $(-3, 4, -2)$ is parallel to the vector $(4.5, -6, 3)$ since $(-3, 4, -2) = -\frac{2}{3}(4.5, -6, 3)$.

Components provide a simple way to algebraically perform several operations on vectors. First, by definition, we know that two vectors are equal if they have the same length and the same magnitude. So, if we choose to draw the two equal vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ from the origin, their terminal points must coincide, and hence $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$. So, we showed that equal vectors have the same components. The converse is obviously true, i.e. if $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$, the two vectors are equal. The following results are also obvious from the simple geometry of similar figures:

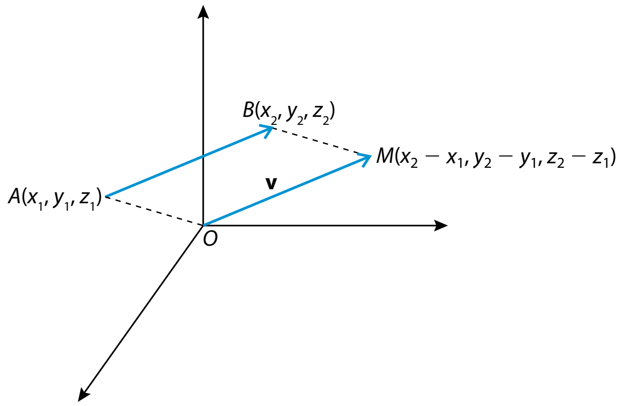
If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ and k is any real number, then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \text{ and } k\mathbf{u} = (ku_1, ku_2, ku_3).$$

If the initial point of the vector is not at the origin, the following theorem generalizes the previous notation to any position:

If $\overrightarrow{AB}$ is a vector with initial point $A(x_1, y_1, z_1)$ and terminal point $B(x_2, y_2, z_2)$, then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, as you see in Figure 14.10.

Figure 14.10



As illustrated in Figure 14.10, either by applying the distance formula or by using the equality of vectors **v** and $\overrightarrow{AB}$,

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Additionally, the following results can follow easily from properties of real numbers: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$; $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$; $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$; and the other obvious relationships.

Example 1

Given the points $A(-2, 3, 5)$ and $B(1, 0, -4)$,

a) find the components of vector $\overrightarrow{AB}$

b) find the components of vector $\overrightarrow{BA}$

c) find the components of vector $3\overrightarrow{AB}$

d) find the components of vector $\overrightarrow{OA} + \overrightarrow{OB}$

e) calculate $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$

f) calculate$|3\overrightarrow{AB}|$ and $|\overrightarrow{OA} + \overrightarrow{OB}|$.

Solution

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
$= (1 - (-2), 0 - 3, -4 - 5) = (3, -3, -9)$

b) Since $\overrightarrow{BA}$ is the opposite of $\overrightarrow{AB}$, then $\overrightarrow{BA} = (-3, 3, 9)$.

c) $3\overrightarrow{AB} = 3(3, -3, -9) = (9, -9, -27)$

d) $\overrightarrow{OA} + \overrightarrow{OB} = (-2 + 1, 3 + 0, 5 - 4) = (-1, 3, 1)$

e) $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{9 + 9 + 81} = 3\sqrt{11}$
$|\overrightarrow{BA}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{9 + 9 + 81} = 3\sqrt{11}$

f) $|3\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{81 + 81 + 729}$
$= \sqrt{891} = 9\sqrt{11}$

Obviously, $|3\overrightarrow{AB}| = 3|\overrightarrow{AB}|$!

$|\overrightarrow{OA} + \overrightarrow{OB}| = |(-1, 3, 1)| = \sqrt{1 + 9 + 1} = \sqrt{11}$

Notice that $|\overrightarrow{OA} + \overrightarrow{OB}| = \sqrt{11} \neq |\overrightarrow{OA}| + |\overrightarrow{OB}|$

$$= \sqrt{4 + 9 + 25} + \sqrt{1 + 0 + 16} = \sqrt{38} + \sqrt{17}.$$

In general, $|\lambda \mathbf{v}| = |\lambda|\,|\mathbf{v}|$, i.e. the magnitude of a multiple of a vector is equal to the absolute multiple of the magnitude of the vector. For example, $|-3\mathbf{v}| = 3|\mathbf{v}|$.

Example 2

Determine the relationship between the coordinates of point $M(x, y, z)$ so that the points M, $A(0, -1, 5)$ and $B(1, 2, 3)$ are collinear.

Solution

For the points to be collinear, it is enough to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. If the two vectors are parallel, then one of them is a scalar multiple of the other. Say $\overrightarrow{AM} = t\overrightarrow{AB}$.

$$\overrightarrow{AM} = (x, y + 1, z - 5) = t(1, 3, -2) = (t, 3t, -2t)$$

So, $x = t$, $y + 1 = 3t$, and $z - 5 = -2t$.

Unit vectors

A vector of length 1 is called a **unit vector**. So, in two-dimensional space, the vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are unit vectors along the x- and y-axes, and in three-dimensional space, the unit vectors along the axes are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are called the **base vectors** of the 3-space.

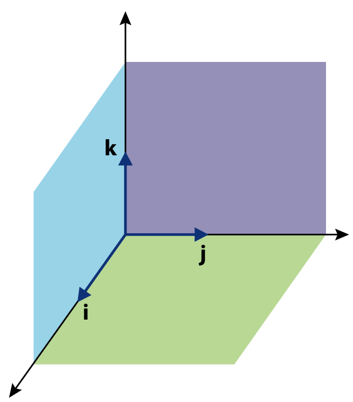


Figure 14.11

It follows immediately that each vector in 3-space can be expressed uniquely in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows:

$$\mathbf{u} = (x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z)$$
$$= x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

● **Hint:** The terms '2-space' and '3-space' are short forms for two-dimensional space and three-dimensional space respectively.

So, in Example 1, $\overrightarrow{AB} = (3, -3, -9) = 3\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}$.

Unit vectors can be found in any direction, not only in the direction of the axes. For example, if we want to find the unit vector in the same direction as $\mathbf{u}$, we need to find a vector parallel to $\mathbf{u}$, which has a magnitude of 1. Since $\mathbf{u}$ has a magnitude of $|\mathbf{u}|$, it is enough to multiply this vector by $1/|\mathbf{u}|$ to 'normalize' it. So, the unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

$\mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|}$. This is a unit vector since its length is 1. This is why:

Recall that $|\mathbf{u}|$ is a real number (scalar), and so is $1/|\mathbf{u}|$.

Let $1/|\mathbf{u}| = k \Rightarrow \mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u} = k\mathbf{u} \Rightarrow |\mathbf{v}| = |k\mathbf{u}| = k|\mathbf{u}| = \frac{1}{|\mathbf{u}|}\cdot|\mathbf{u}| = 1.$

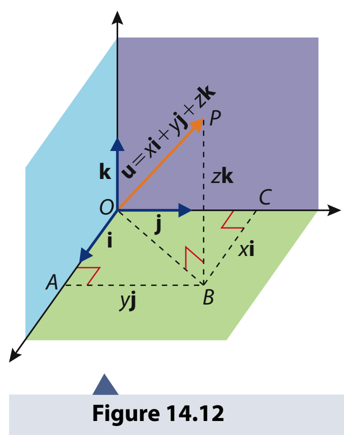


Figure 14.12

Example 3

Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Solution

The length of the vector $\mathbf{v}$ is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so the unit vector is

$$\frac{1}{\sqrt{14}}(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{\mathbf{i}}{\sqrt{14}} - \frac{2\mathbf{j}}{\sqrt{14}} + \frac{3\mathbf{k}}{\sqrt{14}}.$$

To verify that this is a unit vector, we find its length:

$$\sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = 1$$

The unit vector plays another important role: it determines the direction of the given vector.

Recall from Chapter 9 that, in 2-space, we can write the vector in a form that gives us its direction (in terms of the angle it makes with the horizontal axis, called the direction angle) and its magnitude.

In the diagram below, θ is the angle with the horizontal axis.

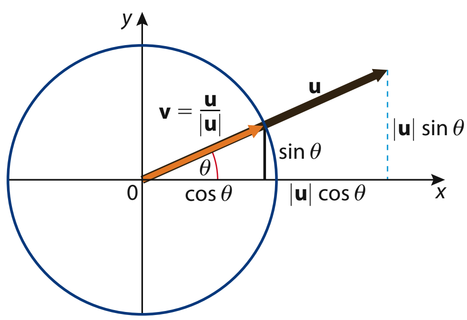
The unit vector $\mathbf{v}$, in the same direction as $\mathbf{u}$, is:

$$\mathbf{v} = 1\cos\theta\,\mathbf{i} + 1\sin\theta\,\mathbf{j}$$

and from the results above,

$$\mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u} \Rightarrow$$

$$\begin{aligned}\mathbf{u} &= |\mathbf{u}|\,(\mathbf{v})\\ &= |\mathbf{u}|\cos\theta\,\mathbf{i} + |\mathbf{u}|\sin\theta\,\mathbf{j}\\ &= |\mathbf{u}|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).\end{aligned}$$



Example 4

Find the vector with magnitude 2 that makes an angle of 60° with the positive x-axis.

Solution

$$\mathbf{v} = |\mathbf{v}|\,(\cos 60°\,\mathbf{i} + \sin 60°\,\mathbf{j}) = 2\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = \mathbf{i} + \sqrt{3}\mathbf{j}$$

Example 5

Find the direction and magnitude of the vector $\mathbf{v} = 2\sqrt{3}\mathbf{i} - 2\mathbf{j}$.

Solution

$$|\mathbf{v}| = \sqrt{(2\sqrt{3}\,)^2 + 4} = 4$$

$$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2},\ \sin\theta = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}$$

Example 6

a) Find the unit vector that has the same direction as $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

b) Find a vector of length 6 that is parallel to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Solution

a) The vector $\mathbf{v}$ has magnitude $|\mathbf{v}| = \sqrt{1 + 2^2 + (-2)^2} = 3$, so the unit vector $\boldsymbol{v}$ in the same direction as $\mathbf{v}$ is

$$\boldsymbol{v} = \frac{1}{3}\mathbf{v} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

b) Let $\mathbf{u}$ be the vector in question and $\boldsymbol{v}$ be the unit vector in the direction of $\mathbf{v}$.

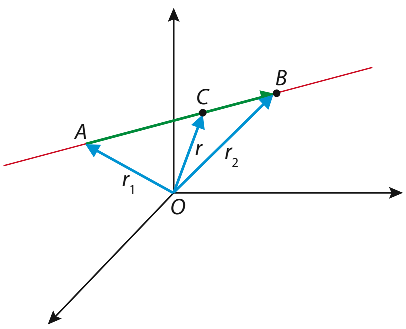
$$\mathbf{u} = 6 \cdot \boldsymbol{v} = 6 \times \frac{1}{\sqrt{14}}(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{6\mathbf{i}}{\sqrt{14}} - \frac{12\mathbf{j}}{\sqrt{14}} + \frac{18\mathbf{k}}{\sqrt{14}}$$

Example 7

Note: This problem introduces you to the vector equation of a line, as we will see in Section 14.4.

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points A and B in space, and λ is a real number, show that $\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$ is the position vector of a point C on the straight line joining A and B. Consider the cases where $\lambda = 0, 1, \frac{1}{2}, -1, 2$ and $\frac{2}{3}$.

Solution



Rewrite the equation:

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2 = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1)$$

Since $\mathbf{r}_2 - \mathbf{r}_1 = \overrightarrow{AB}$, then the position vector $\mathbf{r}$ of C, which is simply $\mathbf{r}_1 + \overrightarrow{AC}$ gives us $\mathbf{r} = \mathbf{r}_1 + \lambda\overrightarrow{AB} = \mathbf{r}_1 + \overrightarrow{AC}$, which in turn gives $\overrightarrow{AC} = \lambda\overrightarrow{AB}$. As you have seen before, this means that $\overrightarrow{AC}$ is parallel to $\overrightarrow{AB}$ and is a multiple of it.

If $\lambda = 0$, then $\mathbf{r} = \mathbf{r}_1$ and C is at A.

If $\lambda = 1$, then $\mathbf{r} = \mathbf{r}_1 + \overrightarrow{AB}$ and C is at B.

If $\lambda = \frac{1}{2}$, then $\mathbf{r} = \mathbf{r}_1 + \frac{1}{2}\overrightarrow{AB}$ and C is the midpoint of AB.

If $\lambda = -1$, then $\mathbf{r} = \mathbf{r}_1 + -\overrightarrow{AB}$ and A is the midpoint of CB.

If $\lambda = 2$, then $\mathbf{r} = \mathbf{r}_1 + 2\overrightarrow{AB}$ and B is the midpoint of AC.

If $\lambda = \frac{2}{3}$, then $\mathbf{r} = \mathbf{r}_1 + \frac{2}{3}\overrightarrow{AB}$ and C is $\frac{2}{3}$ the way between A and B.

Exercise 14.1

1 Write the vector $\overrightarrow{AB}$ in component form in each of the following cases.

a) $A\left(-\frac{3}{2}, -\frac{1}{2}, 1\right); B\left(1, -\frac{5}{2}, 1\right)$

b) $A\left(-2, -\sqrt{3}, -\frac{1}{2}\right); B\left(1, \sqrt{3}, -\frac{1}{2}\right)$

c) $A(2, -3, 5); B(1, -1, 3)$

d) $A(a, -a, 2a); B(-a, -2a, a)$

2 Given the coordinates of point P or Q and the components of $\overrightarrow{PQ}$, find the missing items.

a) $P\left(-\frac{3}{2}, -\frac{1}{2}, 1\right); \overrightarrow{PQ}\left(1, -\frac{5}{2}, 1\right)$

b) $\overrightarrow{PQ}\left(-\frac{3}{2}, -\frac{1}{2}, 1\right); Q\left(1, -\frac{5}{2}, 1\right)$

c) $P(a, -2a, 2a); \overrightarrow{PQ}(-a, -2a, a)$

3 Determine the relationship between the coordinates of point $M(x, y, z)$ so that the points M, A and B are collinear.

a) $A(0, 0, 5); B(1, 1, 0)$

b) $A(-1, 0, 1); B(3, 5, -2)$

c) $A(2, 3, 4); B(-2, -3, 5)$

4 Given the coordinates of the points A and B, find the symmetric image C of B with respect to A.

a) $A(3, -4, 0); B(-1, 0, 1)$

b) $A(-1, 3, 5); B\left(-1, \frac{1}{2}, \frac{1}{3}\right)$

c) $A(1, 2, -1); B(a, 2a, b)$

5 Given a triangle ABC and a point G such that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$, find the coordinates of G in each of the following cases.

a) $A(-1, -1, -1); B(-1, 2, -1); C(1, 2, 3)$

b) $A(2, -3, 1); B(1, -2, -5); C(0, 0, 1)$

c) $A(a, 2a, 3a); B(b, 2b, 3b); C(c, 2c, 3c)$

6 Determine the fourth vertex D of the parallelogram $ABCD$ having AB and BC as adjacent sides.

a) $A(\sqrt{3}, 2, -1); B(1, 3, 0); C(-\sqrt{3}, 2, -5)$

b) $A(\sqrt{2}, \sqrt{3}, \sqrt{5}); B(3\sqrt{2}, -\sqrt{3}, 5\sqrt{5}); C(-2\sqrt{2}, \sqrt{3}, -3\sqrt{5})$

c) $A\left(-\frac{1}{2}, \frac{1}{3}, 0\right); B\left(\frac{1}{2}, \frac{2}{3}, 5\right); C\left(\frac{7}{2}, -\frac{1}{3}, 1\right)$

7 Determine the values of m and n such that the vectors $\mathbf{v}(m - 2, m + n, -2m + n)$ and $\mathbf{w}(2, 4, -6)$ have the same direction.

8 Find a unit vector in the same direction as each vector.

a) $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

b) $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

c) $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

9 Find a vector with the given magnitude and in the same direction as the given vector.

a) Magnitude 2, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

b) Magnitude 4, $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

c) Magnitude 5, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

10 Let $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. Find

a) $|\mathbf{u} + \mathbf{v}|$

b) $|\mathbf{u}| + |\mathbf{v}|$

c) $|-3\mathbf{u}| + |3\mathbf{v}|$

d) $\frac{1}{|\mathbf{u}|}\mathbf{u}$

e) $\left|\frac{1}{|\mathbf{u}|}\mathbf{u}\right|$

11 Find the terminal points for each vector.

a) $\mathbf{w} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, given the initial point $(-1, 2, -3)$

b) $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, given the initial point $(-2, 1, 4)$

12 Find vectors that satisfy the stated conditions:

a) opposite direction of $\mathbf{u} = (-3, 4)$ and third the magnitude of $\mathbf{u}$

b) length of 12 and same direction as $\mathbf{w} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

c) of the form $x\mathbf{i} + y\mathbf{j} - 2\mathbf{k}$ and parallel to $\mathbf{w} = \mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

13 Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the vectors from each vertex of a triangle to the midpoint of the opposite side. Find the value of $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

14 Find the scalar t (or show that there is none) so that the vector $\mathbf{v} = t\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}$ is a unit vector.

15 Find the scalar t (or show that there is none) so that the vector $\mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}$ is a unit vector.

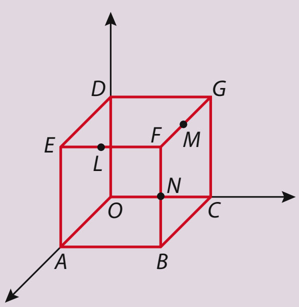
16 Find the scalar t (or show that there is none) so that the vector $\mathbf{v} = 0.5\mathbf{i} - t\mathbf{j} + 1.5t\mathbf{k}$ is a unit vector.

17 The diagram shows a cube of length 8 units.

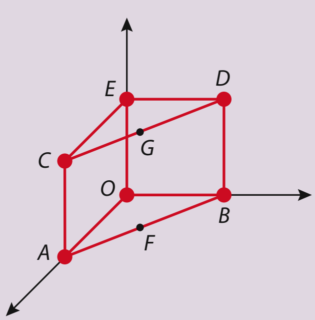
a) Find the position vectors of all the vertices.

b) L, M and N are the midpoints of the respective edges. Find the position vectors of L, M and N.

c) Show that $\overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{NL} = \vec{0}$.



18 A triangular prism is given with the lengths of the sides $OA = 8$, $OB = 10$ and $OE = 12$.



a) Find the position vectors of C and D.

b) F and G are the midpoints of the respective edges. Find their position vectors.

c) Find the vectors $\overrightarrow{AG}$ and $\overrightarrow{FD}$ and explain your results.

19 Find α such that $|\alpha\mathbf{i} + (\alpha - 1)\mathbf{j} + (\alpha + 1)\mathbf{k}| = 2$.

20 Let $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

Find the scalars α, β and μ (or show that they cannot exist) such that $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c} + \mu\mathbf{d}$.

21 Repeat question 20 for $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

22 Repeat question 20 for $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

23 Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors such that $|\mathbf{u} - \mathbf{v}| = |\mathbf{u} + \mathbf{v}|$.

a) What can you conclude about the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides?

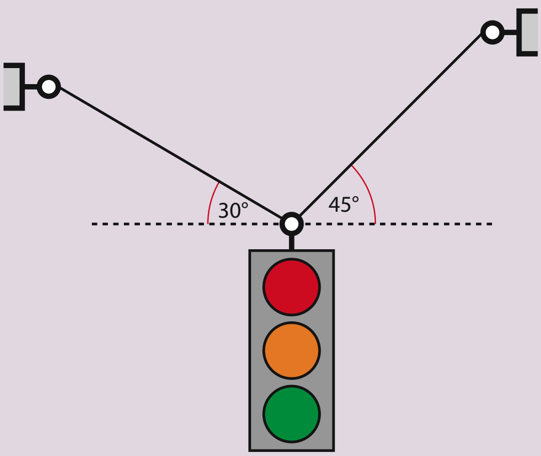
b) Show that if

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

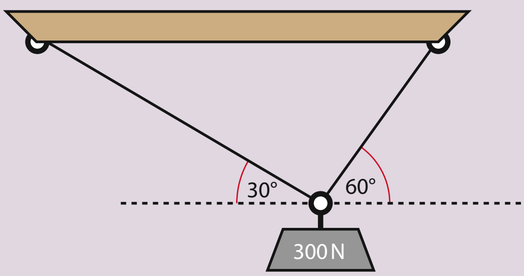
then

$$u_1v_1 + u_2v_2 + u_3v_3 = 0.$$

24 A 125 N traffic light is hanging from two flexible cables. The magnitude of the force that each cable applies to the 'eye ring' holding the lights is called the cable tension. Find the cable tensions if the light is in equilibrium.



25 Find the tension in the cables used to hold a weight of 300 N as shown in the diagram.



14.2 Scalar (dot) product

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors, the dot product (scalar) is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Result 1: $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u} = u_1 \cdot u_1 + u_2 \cdot u_2 + u_3 \cdot u_3 = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2$

From this definition, we can deduce another geometric 'definition' of the dot product:

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$, where θ is the angle between the two vectors.

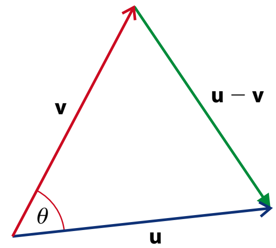


Figure 14.13

Proof:

Let **u** and **v** be drawn from the same point, as shown in Figure 14.13. Then

$$|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u}^2 + \mathbf{v}^2 - 2\mathbf{u} \cdot \mathbf{v}$$
$$= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}.$$

Also, using the law of cosines,

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta.$$

Conversely, using the law of cosines in the figure above gives

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - \mathbf{2}|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta,$$

which in turn will give

$$2|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2$$
$$= (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - [(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2]$$
$$= 2(u_1 v_1 + u_2 v_2 + u_3 v_3).$$

Thus, $|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta = u_1 v_1 + u_2 v_2 + u_3 v_3$ and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$.

By comparing the two results, we can conclude that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos \theta$.

From the geometric definition of the dot product, we can see that for vectors of a given magnitude, the dot product measures *the extent to which the vectors agree in direction*. As the difference in direction, from 0 to π *increases*, the dot product *decreases*:

If **u** and **v** have the same direction, then $\theta = 0$ and

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta = |\mathbf{u}||\mathbf{v}|.$$

This is the largest possible value for $\mathbf{u} \cdot \mathbf{v}$.

If **u** and **v** are at right angles, then $\theta = \frac{\pi}{2}$ and

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

If **u** and **v** have opposite directions, then $\theta = \pi$ and

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \pi = -|\mathbf{u}||\mathbf{v}|.$$

This is the least possible value for $\mathbf{u} \cdot \mathbf{v}$.

The scalar product can be used, among other things, to find angles between vectors:

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \Leftrightarrow \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}}{|\mathbf{u}|}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = u\cdot v$, where u and v are unit vectors in the direction of **u** and **v** respectively. That is, the cosine of the angle between two vectors is the dot product of the corresponding unit vectors.

Example 8

Find the angle between the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

Solution

From the previous results, we have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} = \frac{-3-12+4}{\sqrt{1+4+4}\sqrt{9+36+4}} = \frac{-11}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-11}{21}\right) \approx 2.12 \text{ radians}$$

Result 2: A direct conclusion of the previous definitions is that if two vectors are *perpendicular*, the dot product is *zero*.

This is so because when the two vectors are perpendicular the angle between them is $\pm 90°$ and, therefore,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 90° = |\mathbf{u}||\mathbf{v}|\cdot 0 = 0.$$

The base vectors of the coordinate system are obviously perpendicular: $\mathbf{i}\cdot\mathbf{j} = (1, 0, 0)\cdot(0, 1, 0) = 0$, and similarly, $\mathbf{i}\cdot\mathbf{k} = 0$ and $\mathbf{j}\cdot\mathbf{k} = 0$.

Result 3: If two vectors **u** and **v** are parallel, then $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.

Again, this is so because when the vectors are parallel the angle between them is either 0° or 180° and, therefore,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 0° = |\mathbf{u}||\mathbf{v}|\cdot 1 = |\mathbf{u}||\mathbf{v}|, \text{ or}$$

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 180° = |\mathbf{u}||\mathbf{v}|\cdot(-1) = -|\mathbf{u}||\mathbf{v}|.$$

Example 9

Determine which, if any, of the following vectors are orthogonal.

$$\mathbf{u} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},\ \mathbf{v} = -3\mathbf{i} + 5\mathbf{j} + 3\mathbf{k},\ \mathbf{w} = \mathbf{i} + \mathbf{k}$$

Solution

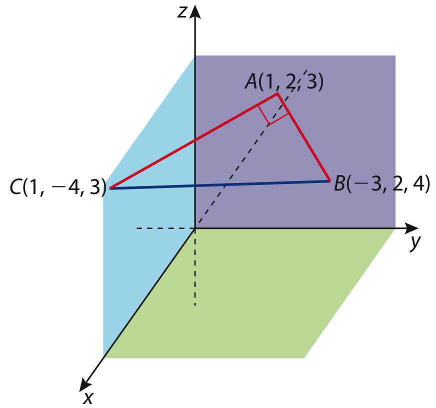
$\mathbf{u}\cdot\mathbf{v} = 7(-3) + 3\times 5 + 2\times 3 = 0$; orthogonal vectors

$\mathbf{u}\cdot\mathbf{w} = 7\times 1 + 3\times 0 + 2\times 1 = 9$; not orthogonal

$\mathbf{v}\cdot\mathbf{w} = -3\times 1 + 5\times 0 + 3\times 1 = 0$; orthogonal vectors

Example 10

$A(1, 2, 3)$, $B(-3, 2, 4)$ and $C(1, -4, 3)$ are the vertices of a triangle. Show that the triangle is right-angled and find its area.



Solution

$$\overrightarrow{AB} = (-3 - 1)\mathbf{i} + (2 - 2)\mathbf{j} + (4 - 3)\mathbf{k} = -4\mathbf{i} + \mathbf{k}$$

$$\overrightarrow{AC} = (1 - 1)\mathbf{i} + (-4 - 2)\mathbf{j} + (3 - 3)\mathbf{k} = -6\mathbf{j}$$

$$\overrightarrow{BC} = (1 - (-3))\mathbf{i} + (-4 - 2)\mathbf{j} + (3 - 4)\mathbf{k} = 4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$$

Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = -4 \times 0 + 0 \times -6 + 1 \times 0 = 0$, the vectors are perpendicular. So the triangle is right-angled at A.

The area of this right triangle is half the product of the legs.

$$\text{Area} = \tfrac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}| = \tfrac{1}{2}\cdot\sqrt{(-4)^2 + 1}\cdot 6 = 3\sqrt{17}$$

Theorem

a) $|\mathbf{u} \cdot \mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}|$

b) $|\mathbf{u} + \mathbf{v}| \leqslant |\mathbf{u}| + |\mathbf{v}|$

Proof

a) Since, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, then $|\mathbf{u} \cdot \mathbf{v}| = ||\mathbf{u}|\,|\mathbf{v}|\cos\theta| = |\mathbf{u}|\,|\mathbf{v}||\cos\theta|$, and as $|\cos\theta| \leqslant 1$, then $|\mathbf{u} \cdot \mathbf{v}| \leqslant |\mathbf{u}||\mathbf{v}|$.

b) $$\left.\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 \end{aligned}\right\} \Leftarrow \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

but, $\mathbf{u} \cdot \mathbf{v} \leqslant |\mathbf{u} \cdot \mathbf{v}|$, since $\mathbf{u} \cdot \mathbf{v}$ may also be negative while $|\mathbf{u} \cdot \mathbf{v}|$ is not.

Also, $|\mathbf{u} \cdot \mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}|$, therefore

$$\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 &\leqslant |\mathbf{u}|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + |\mathbf{v}|^2 \\ &\leqslant |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2. \end{aligned}$$

This then implies that $|\mathbf{u} + \mathbf{v}| \leqslant |\mathbf{u}| + |\mathbf{v}|$ (taking square roots).

Direction angles, direction cosines

Figure 14.14 shows a non-zero vector $\mathbf{v}$. The angles α, β and γ that the vector makes with the unit coordinate vectors are called the **direction angles** of $\mathbf{v}$, and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines**.

Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Considering the right triangles OAP, OCP and ODP, the hypotenuse in each of these triangles is OP, i.e. $|\mathbf{v}|$. From your trigonometry chapters, you know that the side adjacent to an angle θ in a right triangle is related to it by

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \Leftrightarrow \text{adjacent} = \text{hypotenuse} \cdot \cos\theta, \text{ so in this case}$$

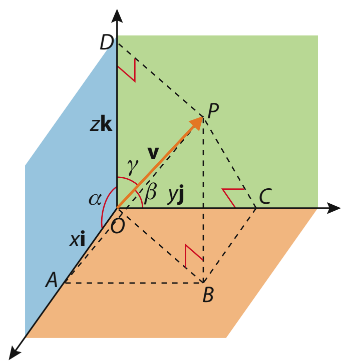
$x = |\mathbf{v}|\cos\alpha$, $y = |\mathbf{v}|\cos\beta$, $z = |\mathbf{v}|\cos\gamma$, and so

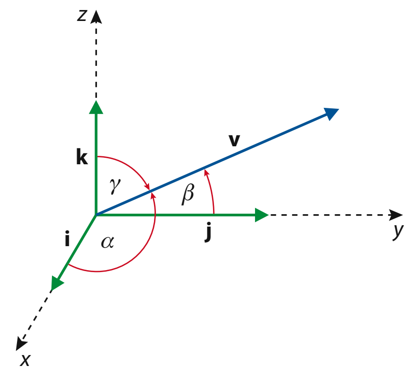
$\mathbf{v} = (|\mathbf{v}|\cos\alpha)\,\mathbf{i} + (|\mathbf{v}|\cos\beta)\,\mathbf{j} + (|\mathbf{v}|\cos\gamma)\,\mathbf{k} = |\mathbf{v}|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$.

Taking the magnitude of both sides,

$$|\mathbf{v}| = |\mathbf{v}|\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}.$$

Figure 14.14





It is also important that you remember that $\cos\alpha = \frac{x}{|\mathbf{v}|}$, $\cos\beta = \frac{y}{|\mathbf{v}|}$, $\cos\gamma = \frac{z}{|\mathbf{v}|}$.

Therefore,
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, i.e. the sum of the squares of the direction cosines is always 1.
For a unit vector, the expression will be of the form

$$\mathbf{u} = |\mathbf{u}|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}) = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}\ (|\mathbf{u}| = 1).$$

This means that for a unit vector its x-, y- and z-coordinates are its direction cosines.

Example 11

Find the direction cosines of the vector $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and then approximate the direction angles to the nearest degree.

Solution

$$|\mathbf{v}| = \sqrt{4^2 + (-2)^2 + 4^2} = 6 \Rightarrow \mathbf{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k},$$

$$\text{thus } \cos\alpha = \frac{2}{3}, \cos\beta = -\frac{1}{3}, \cos\gamma = \frac{2}{3}$$

From your GDC you will obtain:

$$\alpha = \cos^{-1}\left(\frac{2}{3}\right) \approx 48°, \beta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109°, \gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 48°$$

Example 12

Find the angle that a main diagonal of a cube with side a makes with the adjacent edges.

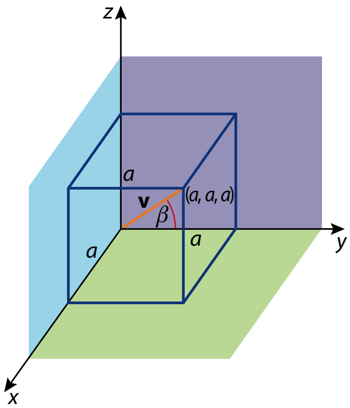
Solution

We can place the cube in a coordinate system such that three of its adjacent edges lie on the coordinate axes as shown (right). The diagonal, represented by the vector $\mathbf{v}$ has a terminal point (a, a, a). Hence,

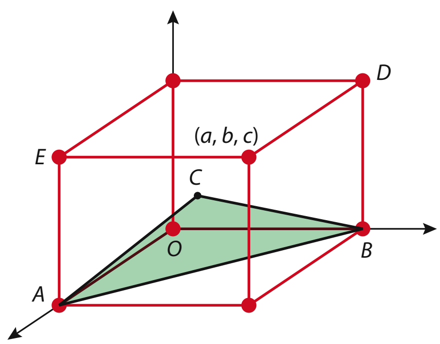
$|\mathbf{v}| = \sqrt{a^2 + a^2 + a^2} = a\sqrt{3}.$

Take angle β, for example:

$$\beta = \cos^{-1}\left(\frac{a}{a\sqrt{3}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$$



Example 13



The point C is at the centre of the rectangular box whose edges have measures a, b and c. Find the measure of angle $A\hat{C}B$ in terms of a, b and c.

Solution

The point diagonally opposite to A is $D(0, b, c)$. So, $C\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.
Consequently,

$$\overrightarrow{CA} = \left(a - \frac{a}{2}, 0 - \frac{b}{2}, 0 - \frac{c}{2}\right) = \left(\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right) \text{ and } \overrightarrow{CB}$$
$$= \left(0 - \frac{a}{2}, b - \frac{b}{2}, 0 - \frac{c}{2}\right) = \left(-\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}\right)$$

$$\cos A\hat{C}B = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} = \frac{-\frac{a^2}{4} - \frac{b^2}{4} + \frac{c^2}{4}}{\sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}\sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}}$$
$$= \frac{c^2 - a^2 - b^2}{a^2 + b^2 + c^2}$$

Exercise 14.2

1 Find the dot product and the angle between the vectors.

a) $\mathbf{u} = (3, -2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 6\mathbf{k}$

b) $\mathbf{u} = \begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$

c) $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$

d) $\mathbf{u} = \mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 5\mathbf{j} + 2\mathbf{k}$

e) $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{3}$

f) $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$

● **Hint:** Orthogonal means 'at right angles to each other'.

2 State whether the following vectors are orthogonal. If not orthogonal, is the angle acute?

a) $\mathbf{u} = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$

b) $\mathbf{u} = 3\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$

c) $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 6\mathbf{j} + 3\mathbf{k}$

3 a) Show that the vectors $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$ and $\mathbf{w} = y\mathbf{i} - x\mathbf{j}$ are both perpendicular to $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.

b) Find two unit vectors that are perpendicular to $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$. Plot the three vectors in the same coordinate system.

4 (i) Find the direction cosines of $\mathbf{v}$.

(ii) Show that they satisfy $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(iii) Approximate the direction angles to the nearest degree.

a) $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

b) $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

c) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

d) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{k}$

5 Find a unit vector with direction angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{2\pi}{3}$.

6 Find a vector with magnitude 3 and direction angles $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$.

7 Determine m so that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

a) $\mathbf{u} = (3, 5, 0)$; $\mathbf{v} = (m - 2, m + 3, 0)$

b) $\mathbf{u} = (2m, m - 1, m + 1)$; $\mathbf{v} = (m - 1, m, m - 1)$

8 Given the vectors $\mathbf{u} = (-3, 1, 2)$, $\mathbf{v} = (1, 2, 1)$, and $\mathbf{w} = \mathbf{u} + m\mathbf{v}$, determine the value of m so that the vectors $\mathbf{u}$ and $\mathbf{w}$ are orthogonal.

9 Given the vectors $\mathbf{u} = (-2, 5, 4)$ and $\mathbf{v} = (6, -3, 0)$, find, to the nearest degree, the measures of the angles between the following vectors.

a) $\mathbf{u}$ and $\mathbf{v}$

b) $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$

c) $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$

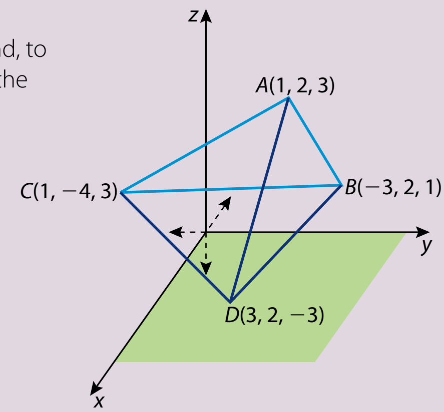
10 Consider the following three points: $A(1, 2, -3)$, $B(3, 5, -2)$ and $C(m, 1, -10m)$. Determine m so that

a) A, B and C are collinear

b) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.

11 Consider the triangle with vertices $A(4, -2, -1)$, $B(3, -5, -1)$ and $C(3, 1, 2)$. Find the vector equations of each of its medians and then find the coordinates of its centroid (i.e. where the medians meet).

12 Consider the tetrahedron $ABCD$ with vertices as shown in the diagram. Find, to the nearest degree, all the angles in the tetrahedron.



13 In question 12 above, use the angles you found to calculate the total surface area of the tetrahedron.

14 In question 12, what angles does $\overrightarrow{DC}$ make with each of the coordinate axes?

15 In question 12, find $(\overrightarrow{DA} - \overrightarrow{DB}) \cdot \overrightarrow{AC}$.

16 Find k such that the angle between the vectors $\begin{pmatrix} 3 \\ -k \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ k \end{pmatrix}$ is $\frac{\pi}{3}$.

17 Find k such that the angle between the vectors $\begin{pmatrix} k \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}$ is $\frac{\pi}{3}$.

18 Find x and y such that $\begin{pmatrix} 2 \\ x \\ y \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$.

19 Consider the vectors $\begin{pmatrix} 1 - x \\ 2x - 2 \\ 3 + x \end{pmatrix}$ and $\begin{pmatrix} 2 - x \\ 1 + x \\ 1 + x \end{pmatrix}$. Find the value(s) of x such that the two vectors are parallel.

20 In triangle ABC, $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$.

Find the measure of $A\hat{B}C$.
Find $\overrightarrow{AC}$ and use it to find the measure of $B\hat{A}C$.

21 Find the value(s) of b such that the vectors are orthogonal.

a) $(b, 3, 2)$ and $(1, b, 1)$

b) $(4, -2, 7)$ and $(b^2, b, 0)$

c) $\begin{pmatrix} b \\ 11 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2b \\ -b \\ -5 \end{pmatrix}$

d) $\begin{pmatrix} 2 \\ 5 \\ 2b \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ -b \end{pmatrix}$

22 If two vectors **p** and **q** are such that $|\mathbf{p}| = |\mathbf{q}|$, show that $\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - \mathbf{q}$ are perpendicular. (This proves that the diagonals of a rhombus are perpendicular to each other!)

23 Shortly after take-off, a plane is rising at a rate of 300 m/min. It is heading at an angle of 45° north-west with an airspeed of 200 km/h. Find the components of its velocity vector. The x-axis is in the east direction, the y-axis north and the z-axis is the elevation.

24 For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

25 For what value of t is the vector $t\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

26 For what value of t is the vector $4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$ perpendicular to the vector $t^2\mathbf{i} + t\mathbf{j}$?

27 Find the angle between the diagonal of a cube and a diagonal of one of the faces. Consider all possible cases!

28 Show that the vector $|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ bisects the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

29 Let $\mathbf{u} = \mathbf{i} + m\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + n\mathbf{k}$. Compute all values of m and n for which $\mathbf{u} \perp \mathbf{v}$ and $|\mathbf{u}| = |\mathbf{v}|$.

30 Show that $\frac{\pi}{4}, \frac{\pi}{6}, \frac{2\pi}{3}$ cannot be the direction angles of a vector.

31 If a vector has direction angles $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{4}$, find the third direction angle γ.

32 If a vector has all its direction angles equal, what is the measure of each angle?

33 If the direction angles of a vector $\mathbf{u}$ are α, β and γ, then what are the direction angles of $-\mathbf{u}$?

34 Find all possible values of a unit vector $\mathbf{u}$ that will be perpendicular to both $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

14.3 Vector (cross) product

In several applications of vectors there is a need to find a vector that is orthogonal to two given vectors. In this section we will discuss a new type of vector multiplication that can be used for this purpose.

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors, then the vector (cross) product is written as $\mathbf{u} \times \mathbf{v}$ and is defined as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k},$$

or, using the properties of determinants, we can observe that this definition is equivalent to

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 14

Given the vectors $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, find

a) $\mathbf{u} \times \mathbf{v}$ b) $\mathbf{v} \times \mathbf{u}$ c) $\mathbf{u} \times \mathbf{u}$

Solution

a) $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}\mathbf{k}$$

$$= 3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}.$$

You can also get the same result by simply evaluating the determinant using the short cut you learned in Chapter 6.

b) $$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} - 5\mathbf{j} - 9\mathbf{k}.$$

Observe here that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$!

c) $$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix}\mathbf{k} = 0.$$

Determinants have many useful applications when we are dealing with vector products. Here are some of the properties which we state without proof.

1 If two rows of a determinant are proportional, then the value of that determinant is zero.

So, for example, if $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = ma\mathbf{i} + mb\mathbf{j} + mc\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ ma & mb & mc \end{vmatrix} = 0.$$

This result leads to an important property of vector products:

Two non-zero vectors are parallel if their cross product is zero.

2 If two rows of a determinant are interchanged, then its value is multiplied by (-1).

So, for instance, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_2 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\mathbf{v} \times \mathbf{u}).$$

Properties

The following results are important in future work and are straightforward to prove. Most of the proofs are left as exercises.

1 $\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$

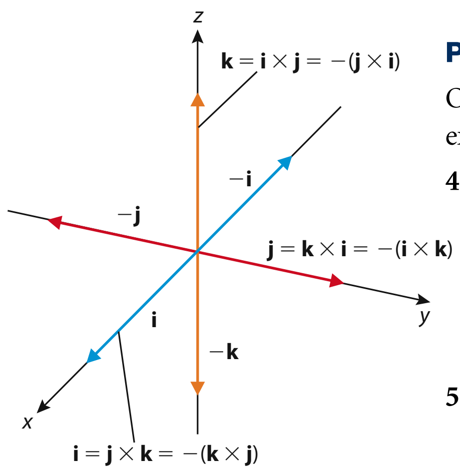
2 $\mathbf{u} \times 0 = 0$

3 $\mathbf{u} \times \mathbf{u} = 0$

4 $\mathbf{i} \times \mathbf{j} = \mathbf{k}; \mathbf{j} \times \mathbf{k} = \mathbf{i}; \mathbf{k} \times \mathbf{i} = \mathbf{j}$

5 $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ (i.e. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$.)

6 $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ (i.e. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$.)



Proofs

Only properties 4 and 5 will be proved here, the rest will be left as an exercise.

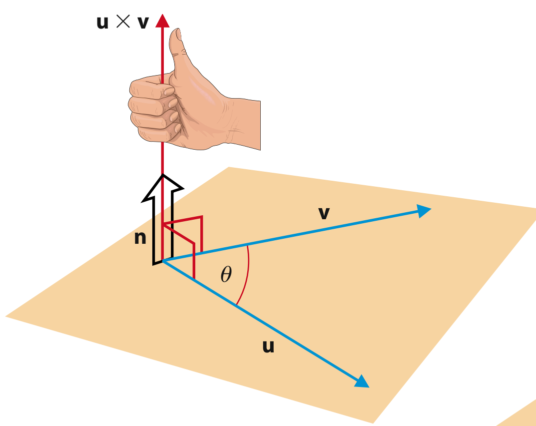
4 To prove the first result, we simply apply the definition.

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}, \text{ details are left as an exercise.}$$

5 $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (u_1, u_2, u_3) \cdot \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$, so that

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= u_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 + u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1 \\ &= 0 \end{aligned}$$

Properties 4 and 5 lead to an equivalent 'geometric' definition of the cross product:



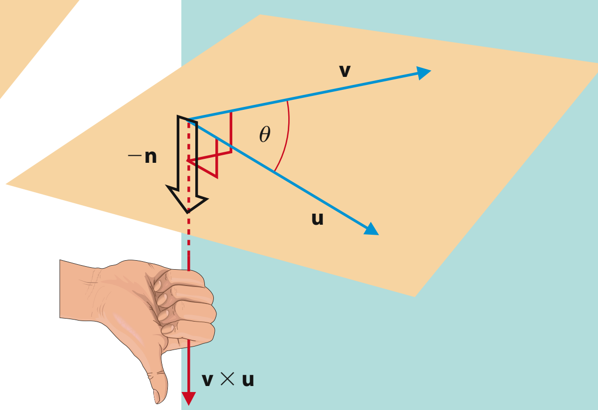
Definition of the cross product (Theorem):

$(\mathbf{u} \times \mathbf{v})$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, obeying the right-hand rule shown right, and has the magnitude:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$

Note: As we mentioned before, please do not forget that

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}).$$



Proof

The algebraic manipulation required for the proof is tremendous, so we will keep the details away from this discussion:

$$|\mathbf{u}\times\mathbf{v}| = \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2}$$
$$= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2}$$
$$= \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2} = \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}|\ |\mathbf{v}|\cos\theta)^2}$$
$$= |\mathbf{u}||\mathbf{v}|\sqrt{1-\cos^2\theta} = |\mathbf{u}||\mathbf{v}|\sin\theta$$

● **Hint:** The vector product gives you another method for finding the angle between two vectors.
$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

$\Leftrightarrow \sin\theta = \dfrac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$

Example 15

Find a unit vector orthogonal to both vectors $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

Solution

$\mathbf{u}\times\mathbf{v}$ is orthogonal to both vectors.

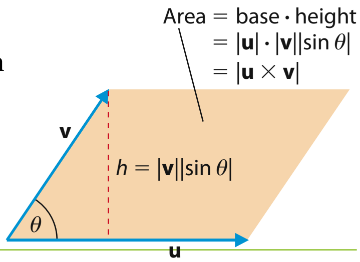
And $\mathbf{u}\times\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$. A unit vector in the same direction as $\mathbf{u}\times\mathbf{v}$ will also be orthogonal to both vectors. Remembering that a unit vector is equal to the vector itself multiplied by the reciprocal of its magnitude, as we have seen in Chapter 9 and in Section 14.1, we find the magnitude of $\mathbf{u}\times\mathbf{v}$ first.

$$|\mathbf{u}\times\mathbf{v}| = \sqrt{9 + 25 + 81} = \sqrt{115},$$

and the required unit vector is therefore

$$\frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|} = \frac{3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}}{\sqrt{115}}.$$

Corollary: The last result leads to the conclusion that the magnitude of the cross product is the area of the parallelogram that has **u** and **v** as adjacent sides.



Example 16

Show that the quadrilateral $ABCD$ with its vertices at the following points is a parallelogram and find its area.
$A(3, 0, 2)$, $B(6, 2, 5)$, $C(1, 2, 2)$, $D(4, 4, 5)$

Solution

$$\overrightarrow{AB} = \begin{pmatrix}3\\2\\3\end{pmatrix}, \overrightarrow{BD} = \begin{pmatrix}-2\\2\\0\end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix}3\\2\\3\end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix}-2\\2\\0\end{pmatrix}$$

This implies that $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{BD} = \overrightarrow{AC}$ which in turn means that the pairs of opposite sides of the quadrilateral are congruent and parallel. (You need only one pair.) Thus, $ABDC$ is a parallelogram with AB and BD as adjacent sides.

Furthermore, since

$$\overrightarrow{AB} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 3 \\ -2 & 2 & 0 \end{vmatrix} = -6\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

the area of the parallelogram is

$$|\overrightarrow{AB} \times \overrightarrow{BD}| = \sqrt{36 + 36 + 100} = \sqrt{172} = 2\sqrt{43}.$$

Example 17

Find the area of the triangle determined by the points $A(2, 2, 0)$, $B(-1, 0, 2)$ and $C(0, 4, 3)$.

Solution

The area of the triangle ABC is half the area of the parallelogram formed with AB and AC as its adjacent sides.

But $\overrightarrow{AB} = (-3, -2, 2)$ and $\overrightarrow{AC} = (-2, 2, 3)$, so $\overrightarrow{AB} \times \overrightarrow{AC} = (-10, 5, -10)$, and hence area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{15}{2}$.

The scalar triple product

(This product is very helpful in its geometric interpretation as it is of great help in finding the equation of a plane later in the chapter.)

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are three vectors, then the scalar triple product is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. The component expression of this product can be found by applying the above definition:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right) \\ &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

Example 18

Calculate the scalar triple product of the vectors:

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}, \mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}, \mathbf{w} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

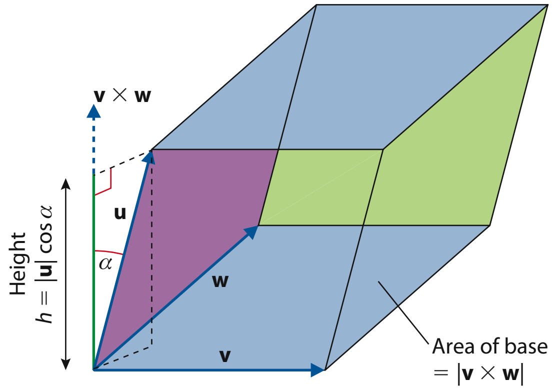
Solution

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & -1 & -5 \\ 2 & 5 & -5 \\ 1 & 4 & 3 \end{vmatrix} = 66$$

Geometric interpretation

$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped that has the three vectors as adjacent edges.

Proof



In the diagram above, $|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$, which is the base of the parallelepiped. Also,

$$\begin{aligned}|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| &= |\mathbf{u}||\mathbf{v} \times \mathbf{w}|\cos\alpha \\ &= |\mathbf{v} \times \mathbf{w}||\mathbf{u}|\cos\alpha.\end{aligned}$$

But, $|\mathbf{u}|\cos\alpha = h$, the height of the parallelepiped, and $|\mathbf{v} \times \mathbf{w}|$ is the area of the base; therefore, the triple product's absolute value is the volume of the parallelepiped.

A direct consequence of this theorem is that the volume of the parallelepiped is 0 if and only if the three vectors are coplanar. That is:

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ *and* $\mathbf{w} = (w_1, w_2, w_3)$ *are three vectors drawn from the same initial point, they lie in the same plane if:*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

> The parentheses in the scalar triple product is unnecessary, i.e. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$. Can you justify?

Example 19

Consider the three vectors

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k},\ 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{w} = m\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

a) Find the volume of the parallelepiped that has these vectors as sides.

b) Show that these vectors can never be on the same plane.

Solution

a) The volume of the parallelepiped is given by the absolute value of their scalar triple product:

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left\|\begin{matrix} 2 & 1 & m \\ 3 & 2 & 3 \\ m & 2 & 1 \end{matrix}\right\| = |-2m^2 + 9m - 11|$$

b) For the vectors to be coplanar, their scalar triple product must be zero. That is, $-2m^2 + 9m - 11 = 0$.

However, since this is a quadratic equation, it can have real roots if $b^2 - 4ac \geqslant 0$, but $b^2 - 4ac = 81 - 88 = -7 < 0$, and thus the equation does not admit any real roots and the three vectors can therefore never be coplanar.

Exercise 14.3

1 a) Find the cross product using the definition: $\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.
b) Compare your answer to $(\mathbf{i} \times \mathbf{i}) + (\mathbf{i} \times \mathbf{j}) + (\mathbf{i} \times \mathbf{k})$.

2 Repeat question 1 for $\mathbf{j} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

3 Repeat question 1 for $\mathbf{k} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

4 Use the definition of vector products to verify $\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$. (This is the distributive property of vector product over addition and subtraction.)

In questions 5–8, find $\mathbf{u} \times \mathbf{v}$ and check that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

5 $\mathbf{u} = (2, 3, -2)$, $\mathbf{v} = (-3, 2, 3)$

6 $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -2\mathbf{j} + 2\mathbf{k}$

7 $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$

8 $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + \mathbf{k}$

9 Consider the following vectors:

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k},\ \mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k},\ \mathbf{w} = m\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Find
a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
b) $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$
c) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$

10 Consider the following vectors:

$$\mathbf{u} = (3, 0, 4),\ \mathbf{v} = (1, 2, 8),\ \mathbf{w} = (2, 5, 6)$$

Find
a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
c) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$
d) $(\mathbf{v} \times \mathbf{w}) \times (\mathbf{u} \times \mathbf{v})$
e) $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
f) $(\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$

11 Find a unit vector that is orthogonal to both

$$\mathbf{u} = -6\mathbf{i} + 4\mathbf{j} + \mathbf{k} \text{ and } \mathbf{v} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

12 Find a unit vector that is normal (perpendicular) to the plane determined by the points $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$.

In questions 13–14, find the area of the parallelogram that has **u** and **v** as adjacent sides.

13 $\mathbf{u} = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

14 $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{j} - \mathbf{k}$

15 Verify that the points are the vertices of a parallelogram and find its area: (2, −1, 1), (5, 1, 4), (0, 1, 1) and (3, 3, 4).

16 Show that the points $P(1, -1, 2)$, $Q(2, 0, 1)$, $R(3, 2, 0)$ and $S(5, 4, -2)$ are coplanar.

17 For what value(s) of m are the following four points on the same plane? $A(m, 3, -2)$, $B(3, 4, m)$, $C(2, 0, -2)$ and $D(4, 8, 4)$.

In questions 18–19, find the area of the triangle with the given vertices.

18 $A(2, 6, -1)$, $B(1, 1, 1)$, $C(3, 5, 2)$

19 $A(3, 1, -2)$, $B(2, 5, 6)$, $C(6, 1, 8)$

In questions 20–22, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

20 $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

21 $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (1, 4, 3)$, $\mathbf{w} = (-3, 2, -2)$

22 $\mathbf{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

In questions 23–24, find the volume of the parallelepiped with **u**, **v** and **w** as adjacent edges.

23 $\mathbf{u} = (3, -5, 3)$, $\mathbf{v} = (1, 5, -1)$, $\mathbf{w} = (3, 2, -3)$

24 $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

In questions 25–26, determine whether the three given vectors are coplanar.

25 $\mathbf{u} = (2, -1, 2)$, $\mathbf{v} = (4, 1, -1)$, $\mathbf{w} = (6, -3, 1)$

26 $\mathbf{u} = (4, -2, -1)$, $\mathbf{v} = (9, -6, -1)$, $\mathbf{w} = (6, -6, 1)$

In questions 27–28, find m such that the following vectors are coplanar; otherwise, show that it is not possible.

27 $\mathbf{u} = (1, m, 1)$, $\mathbf{v} = (3, 0, m)$, $\mathbf{w} = (5, -4, 0)$

28 $\mathbf{u} = (2, -3, 2m)$, $\mathbf{v} = (m, -3, 1)$, $\mathbf{w} = (1, 3, -2)$

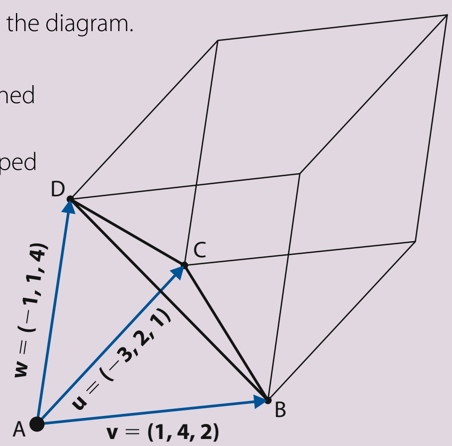
29 Consider the parallelepiped given in the diagram.

a) Find the volume.

b) Find the area of the face determined by **u** and **v**.

c) Find the height of the parallelepiped from vertex D to the base.

d) Find the angle that **w** makes with the plane determined by **u** and **v**.



30 a) From geometry, you know that the volume of a tetrahedron is $\frac{1}{3}$(base)(height). Use the results from the previous problem to find the volume of tetrahedron *ABCD*. Compare this volume to the volume of the parallelepiped and make a general conjecture.

b) Use the results you have from part a) to find the volume of the tetrahedron whose vertices are

$A(0, 3, 1)$, $B(3, 2, -2)$, $C(2, 1, 2)$ and $D(4, -1, 4)$.

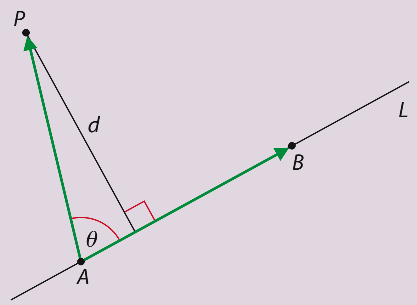
31 What can you conclude about the angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u} \times \mathbf{v}|$?

32 Show that $|\mathbf{u} \times \mathbf{v}| = \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2}$.

● **Hint:** Use right triangle trigonometry to find d in terms of θ first.

33 Use the diagram on the right to show that the distance from a point P in space to a line L through two points A and B can be expressed as

$$d = \frac{|\overrightarrow{AP} \times \overrightarrow{AB}|}{|\overrightarrow{AB}|}.$$



34 Use the result in the previous problem to find the distance from A to the line through the points B and C.

a) $A(-2, 2, 3)$, $B(2, 2, 1)$, $C(-1, 4, -3)$

b) $A(5, 4)$, $B(3, 2)$, $C(1, 3)$

c) $A(2, 0, 1)$, $B(1, -2, 2)$, $C(3, 0, 2)$

35 Express $(\mathbf{u} + \mathbf{v}) \times (\mathbf{v} - \mathbf{u})$ in terms of $(\mathbf{u} \times \mathbf{v})$.

36 Express $(2\mathbf{u} + 3\mathbf{v}) \times (4\mathbf{v} - 5\mathbf{u})$ in terms of $(\mathbf{u} \times \mathbf{v})$.

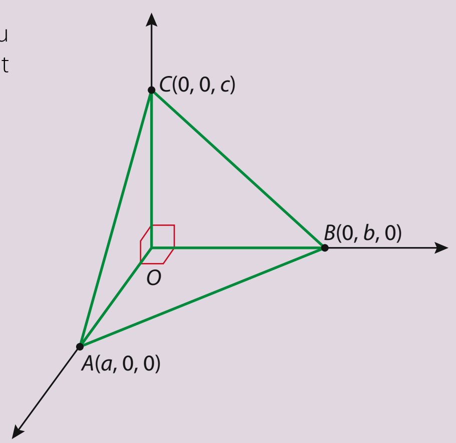
37 Express $(m\mathbf{u} + n\mathbf{v}) \times (p\mathbf{v} - q\mathbf{u})$ in terms of $(\mathbf{u} \times \mathbf{v})$, where m, n, p and q are scalars.

38 Refer to the diagram on the right. You are given a tetrahedron with vertex at the origin and base *ABC*.

a) Find the area of the base *ABC*. Call it o.

b) Find the area of each face of the tetrahedron and call them a, b and c.

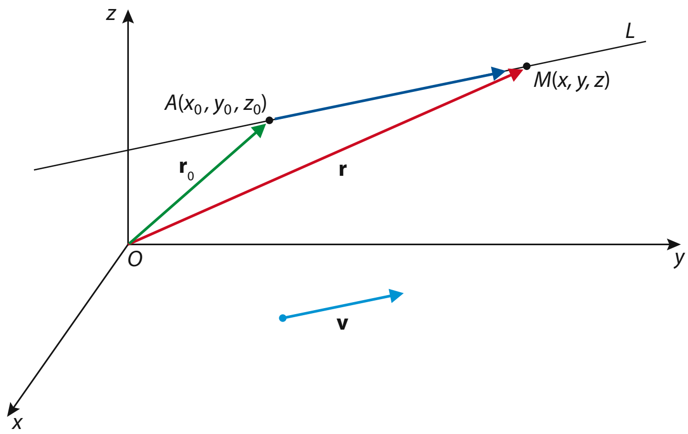
c) Show that $o^2 = a^2 + b^2 + c^2$. This is sometimes called the 3-D version of Pythagoras' theorem.



39 Find all vectors $\mathbf{v}$ such that $(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$; otherwise, show that it is not possible.

40 Find all vectors $\mathbf{v}$ such that $(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v} = \mathbf{i} + 5\mathbf{j}$; otherwise, show that it is not possible.

14.4 Lines in space



Similar to the plane, a straight line in space can be determined by any two points A and M that lie on it. Alternatively, the line can be determined by specifying a point on it and a direction given by a non-zero vector parallel to it. To investigate equations that describe lines in space, let us begin with a straight line L that passes through the point $A(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ as shown in the diagram. Now, if L is the line that passes through A and is parallel to the non-zero vector $\mathbf{v}$, then L consists of all the points $M(x, y, z)$ for which the vector $\overrightarrow{AM}$ is parallel to $\mathbf{v}$.

In Section 14.1 we established that:

Two vectors are parallel if one of them is a scalar multiple of the other. That is, $\mathbf{v}$ is parallel to $\mathbf{u}$ if and only if $\mathbf{v} = t\mathbf{u}$ for some real number t.

This means that for the point M to be on L, $\overrightarrow{AM}$ must be a scalar multiple of $\mathbf{v}$, i.e. $\overrightarrow{AM} = t\mathbf{v}$, where t is a scalar.

This equation can be written in coordinate form as

$$(x - x_0, y - y_0, z - z_0) = t(a, b, c) = (ta, tb, tc).$$

For two vectors to be equal, their components must be the same, then

$$x - x_0 = ta, y - y_0 = tb, z - z_0 = tc.$$

This leads to the result:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$$

The line that passes through the point $A(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = (a, b, c)$ has parametric equations:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

Example 20

a) Find parametric equations of the line through $A(1, -2, 3)$ and parallel to $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$.

b) Find parametric equations of the line through the points $A(1, -2, 3)$ and $B(2, 4, -2)$.

Solution

a) From the previous theorem, $x = 1 + 5t, y = -2 + 4t, z = 3 - 6t$.

b) We need to find a vector parallel to the given line. The vector $\overrightarrow{AB}$ provides a good choice: $\overrightarrow{AB} = (1, 6, -5)$. So the equations are

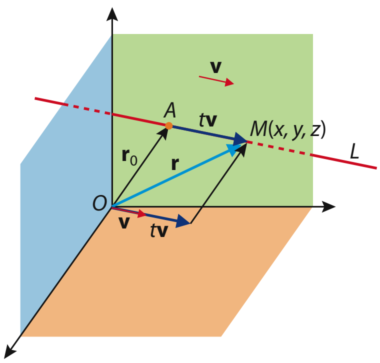
$$x = 1 + t, y = -2 + 6t, z = 3 - 5t.$$

Another set of equations could be

$$x = 2 + t, y = 4 + 6t, z = -2 - 5t.$$

Other sets are possible by considering any vector parallel to $\overrightarrow{AB}$.

Vector equation of a line



An alternative route to interpreting the equation

$$\overrightarrow{AM} = t\mathbf{v}$$

is to express it in terms of the position vectors $\mathbf{r}_0$ of the fixed point A, and $\mathbf{r}$, the position vector of M.

In Section 14.1, we discussed the difference of two vectors which can be of immediate use here.

$$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA}$$

$$\Rightarrow t\mathbf{v} = \mathbf{r} - \mathbf{r}_0$$

And hence we arrive at:

The **vector equation** of the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where $\mathbf{r}$ is the position vector of any point on the line, while $\mathbf{r}_0$ is the position vector of a fixed point (A in this case) on the line and $\mathbf{v}$ is the vector parallel to the given line.

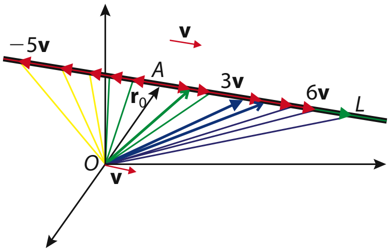


Figure 14.15 By observing Figure 14.15, you will notice, for example, that for each value of t you describe a point on the line. When $t > 0$, the points are in the same direction as $\mathbf{v}$. When $t < 0$, the points are in the opposite direction.

The two approaches are very closely related. We can even say that the parametric equations are a detailed form of the vector equation!

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$
$$\Leftrightarrow (x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$$
$$\Leftrightarrow (x, y, z) = (x_0, y_0, z_0) + (ta, tb, tc) = (x_0 + ta, y_0 + tb, z_0 + tc)$$

$$\Leftrightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

You can interpret vector equations in several ways. One of these has to do with displacement. That is, to reach point M from point O, you first arrive at A, and then go towards M along the line a multiple of $\mathbf{v}$, $t\mathbf{v}$.

Example 21

Find a vector equation of the line that contains $(-1, 3, 0)$ and is parallel to $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution

From the previous discussion,

$$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j}) + t(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

When $t = 0$, the equation gives the point $(-1, 3, 0)$. When $t = 1$, the equation yields

$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j}) + (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, a point shifted by $1\mathbf{v}$ down the line. Similarly, when $t = 3$,

$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j}) + 3(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 8\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, a point $3\mathbf{v}$ down the line, etc.

Alternatively, the equation can be written as

$$\mathbf{r} = (-1 + 3t)\mathbf{i} + (3 - 2t)\mathbf{j} + t\mathbf{k}.$$

This last form allows us to recognize the parametric equations of the line by simply reading the components of the vector on the right-hand side of the equation.

Example 22

Find a vector equation of the line passing through $A(2, 7)$ and $B(6, 2)$.

Solution

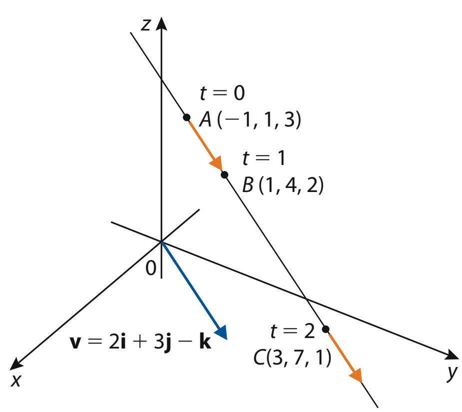
We let the vector $\overrightarrow{AB} = (6 - 2, 2 - 7) = (4, -5)$ be the vector giving the direction of the line, so

$$\mathbf{r} = (2, 7) + t(4, -5), \text{ or equivalently}$$

$$\mathbf{r} = 2\mathbf{i} + 7\mathbf{j} + t(4\mathbf{i} - 5\mathbf{j}).$$

Example 23

Find parametric equations for the line through $A(-1, 1, 3)$ and parallel to the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.



Solution

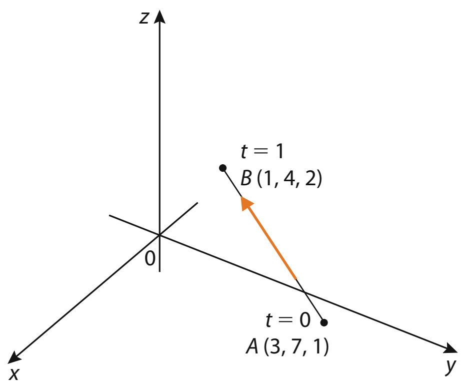
$$x = -1 + 2t, y = 1 + 3t, z = 3 - 1$$

If you select a few points with their parameter values you can see how the equation represents the line. For $t = 0$, as you expect, you are at point A; for $t = 1$, the point is B; and for $t = 2$, the point is C. The arrows show the direction of increasing values of t.

Line segments

Sometimes, we would like to 'parametrize' a line segment. That is, to write the equation so that it describes the points making up the segment. For example, to parametrize the line segment between $A(3, 7, 1)$ and $B(1, 4, 2)$, we first find the direction vector $AB = (-2, -3, 1)$, then we use point A as the fixed point on the line. Thus, the parametric equations are:

$$\begin{cases} x = 3 - 2t \\ y = 7 - 3t \\ z = 1 + t \end{cases}$$



Notice that when $t = 0$, the line starts at the point $A(3, 7, 1)$; when $t = 1$, the line is at $B(1, 4, 2)$. Therefore, to parametrize this segment we restrict the values of t to $0 \leqslant t \leqslant 1$. The new equations are then

$$x = 3 - 2t, y = 7 - 3t, z = 1 + t, 0 \leqslant t \leqslant 1.$$

In general, to parametrize a line segment AB so that we represent the points included between the endpoints only, we can use the vector equation

$$\mathbf{r}(t) = (1 - t)\overrightarrow{OA} + t\overrightarrow{OB}, 0 \leqslant t \leqslant 1.$$

In this parametrization, when $t = 0$, $\mathbf{r} = \overrightarrow{OA}$, and when $t = 1$, $\mathbf{r} = \overrightarrow{OB}$. This way $\mathbf{r}$ traces the segment AB from A to B for $0 \leqslant t \leqslant 1$.

Note: The parametrization with the vector equation can be expressed differently if we want to use parametric equation.

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are the endpoints of the segment, then

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)\overrightarrow{OA} + t\overrightarrow{OB} = \overrightarrow{OA} - t\overrightarrow{OA} + t\overrightarrow{OB} \\ &= \overrightarrow{OA} + t(\overrightarrow{OB} - \overrightarrow{OA}) \\ &= (x_1, y_1, z_1) + t(x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1), z_1 + t(z_2 - z_1)).\end{aligned}$$

So the parametric equations are

$$x = x_1 + t(x_2 - x_1), y = y_1 + t(y_2 - y_1), z = z_1 + t(z_2 - z_1),$$
$$0 \leqslant t \leqslant 1.$$

Example 24

Parametrize the segment through $A(2, -1, 5)$ and $B(4, 3, 2)$. Use the equation to find the midpoint of the segment.

Solution

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)\overrightarrow{OA} + t\overrightarrow{OB}, 0 \leqslant t \leqslant 1 \\ &= (1 - t)(2, -1, 5) + t(4, 3, 2) \\ &= (2 + 2t, -1 + 4t, 5 - 3t)\end{aligned}$$

For the midpoint, $t = \frac{1}{2}$, and hence its coordinates are

$$\mathbf{r}\left(\tfrac{1}{2}\right) = \left(2 + 2\left(\tfrac{1}{2}\right), -1 + 4\left(\tfrac{1}{2}\right), 5 - 3\left(\tfrac{1}{2}\right)\right) = \left(3, 1, \tfrac{7}{2}\right).$$

Note: This method can be used to find points that divide the segment in any ratio: $\frac{2}{3}$ the way from A to B, etc.

Equivalently, the parametric equations can be used.

$$x = 2 + t(4 - 2), y = -1 + t(3 + 1), z = 5 + t(2 - 5)$$
$$x = 2 + 2t, y = -1 + 4t, z = 5 - 3t, 0 \leqslant t \leqslant 1$$

Symmetric (Cartesian) equations of lines

Another set of equations for a line is obtained by eliminating the parameter from the parametric equation.

If $a \neq 0$, $b \neq 0$ and $c \neq 0$, then the set of parametric equations can be re-arranged to yield the set of Cartesian (symmetric) equations:

$$\left.\begin{aligned} x - x_0 = ta \Leftrightarrow \frac{x - x_0}{a} = t \\ y - y_0 = tb \Leftrightarrow \frac{y - y_0}{b} = t \\ z - z_0 = tc \Leftrightarrow \frac{z - z_0}{c} = t \end{aligned}\right\} \Leftrightarrow \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Notice that the coordinates (x_0, y_0, z_0) of the fixed point A on L appear in the numerators of the fractions, and that the components a, b and c of a direction vector appear in the denominators of these fractions.

Example 25

Find the Cartesian equations of the line through $A(3, -7, 4)$ and $B(1, -4, -1)$.

Solution

In order to use the Cartesian equation, we find the vector **v** parallel to the line. Since A and B are two points that lie on the line, the vector $\overrightarrow{AB}$ will suffice. Thus, we let

$$\mathbf{v} = \overrightarrow{AB} = (1 - 3)\mathbf{i} + (-4 + 7)\mathbf{j} + (-1 - 4)\mathbf{k} = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}.$$

If we use A as the fixed point, then the Cartesian equations are

$$\frac{x-3}{-2} = \frac{y+7}{3} = \frac{z-4}{-5}.$$

Similarly, if we use B as the fixed point, then

$$\frac{x-1}{-2} = \frac{y+4}{3} = \frac{z+1}{-5}.$$

Example 26

Let L be the line with Cartesian equations

$$\frac{x-2}{3} = \frac{y+1}{-2} = z - 4.$$

Find a set of parametric equations for L.

Solution

Since the numbers in the denominators are the components of a vector parallel to L, then

$$\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The point $(2, -1, 4)$ lies on L.

Thus, a set of parametric equations of L is

$$x = 2 + 3t, y = -1 - 2t, z = 4 + t.$$

A vector equation would be

$$\mathbf{r} = (2, -1, 4) + t(3, -2, 1).$$

Note: If any of the components a, b or c is zero, then the Cartesian equations are written in a mixed form. For example, if $c = 0$, then we write

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, z = z_0.$$

For example, the Cartesian set of equations for a line parallel to $2\mathbf{i} - 3\mathbf{j}$ through the point $(2, 1, -3)$ is

$$\frac{x-2}{2} = \frac{y-1}{-3}, z = -3.$$

Intersecting, parallel and skew straight lines

In the plane, lines can coincide, intersect or be parallel. This is not necessarily so in space. In addition to the three cases above, there is the case of skew straight lines. Although these lines are not parallel, they do not intersect either. They lie in different planes.

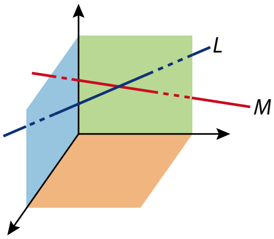


Figure 14.16a Two intersecting straight lines.

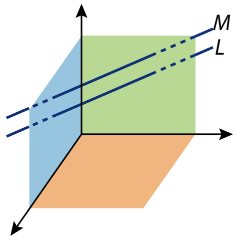


Figure 14.16b Two parallel lines.

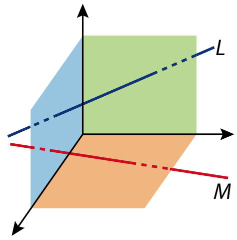


Figure 14.16c Two skew straight lines.

How do we know whether two lines are parallel?

If the 'direction' vectors are parallel, then the lines are. Check to see if one of the vectors is a scalar multiple of the other. Alternatively, you can find the angle between them, and if it is 0° or 180°, the lines are either parallel or coincident. The case for coincidence is always there, and you need to check it by examining a point on one of the lines to see whether it is also on the other line.

Example 27

Show that the following two lines are parallel.

$$L_1: x = 2 - 3t, y = t, z = -1 + 2t$$

$$L_2: x = 1 + 6s, y = 2 - 2s, z = 2 - 4s$$

Solution

Let $\mathbf{l}_1$ be the vector parallel to L_1 and $\mathbf{l}_2$ be the vector parallel to L_2.

$$\mathbf{l}_1 = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{l}_2 = 6\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}.$$

Now you can easily see that $\mathbf{l}_2 = -2\mathbf{l}_1$, and hence the vectors are parallel.

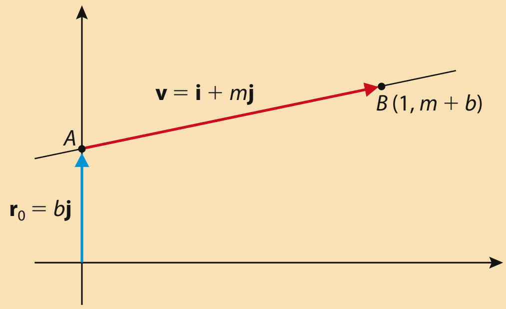
To check whether the lines coincide, we examine the point $(2, 0, -1)$, which is on the first line, and see whether it lies on the second line too.

If we choose $y = 0$, then $0 = 2 - 2s$, so $s = 1$; and when we substitute $s = 1$ into $x = 1 + 6s$ we find out that x must be 7 in order for the point $(2, 0, -1)$ to be on L_2. Therefore, the lines cannot intersect, and their 'direction' vectors are parallel, so they must be parallel.

Are the lines intersecting or skew?

If the direction vectors are not parallel, the lines either intersect or are skew. For the purposes of this course, the method starts by examining whether the lines intersect. If they do, we can find the coordinates of the point of intersection; if they do not intersect, we cannot find the coordinates of the point of intersection. Finding the coordinates of the point of intersection is a straightforward method that you already know: solving systems of equations. This can best be explained with an example.

The vector equation of a line has an interesting application in proving a result that you already know – the condition for two lines to be perpendicular.



Consider the slope-intercept form of the equation of the line is $y = mx + b$. We can think of the fixed point on the line to be its y-intercept, i.e. $\mathbf{r}_0 = b\mathbf{j}$, and another point $(1, m + b)$ on the line $\Rightarrow \mathbf{r} = \mathbf{i} + (m + b)\mathbf{j}$. So, the direction vector of the line is $\mathbf{v} = \mathbf{r} - \mathbf{r}_0 = (\mathbf{i} + (m + b)\mathbf{j}) - b\mathbf{j} = \mathbf{i} + m\mathbf{j}$.

Now, if you have two lines with slopes m_1 and m_2, their direction vectors can be written as $\mathbf{v}_1 = \mathbf{i} + m_1\mathbf{j}$ and $\mathbf{v}_2 = \mathbf{i} + m_2\mathbf{j}$. For the two lines to be perpendicular, their direction vectors will also be perpendicular, and hence

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{i} + m_1\mathbf{j}) \cdot (\mathbf{i} + m_2\mathbf{j}) = 1 + m_1m_2 = 0$$
$$\Rightarrow m_1m_2 = -1.$$

Example 28

The lines L_1 and L_2 have the following equations:

$$L_1 : x = 1 + 4t, y = 5 - 4t, z = -1 + 5t$$
$$L_2 : x = 2 + 8s, y = 4 - 3s, z = 5 + s$$

Show that the lines are skew.

Solution

We first examine whether the lines are parallel. Since the vector parallel to L_1 is $\mathbf{l}_1 = (4, -4, 5)$ and the vector parallel to L_2 is $\mathbf{l}_2 = (8, -3, 1)$, they are not scalar multiples of each other and the vectors and consequently the lines are not parallel.

For the lines to intersect, there should be some point $M(x_0, y_0, z_0)$ which satisfies the equations of both lines for some values of t and s. That is,

$$x_0 = 1 + 4t = 2 + 8s; y_0 = 5 - 4t = 4 - 3s; z_0 = -1 + 5t = 5 + s.$$

This leads to a set of three simultaneous equations in two unknowns: s and t.

By solving the first two equations:

$$\left.\begin{array}{l}1+4t=2+8s\\5-4t=4-3s\end{array}\right\}\Rightarrow 6=6+5s\Rightarrow s=0,\ t=\tfrac{1}{4}$$

For the system to be consistent, these values must satisfy the third equation, i.e. $-1+\frac{5}{4}=5+0$, which is false. Hence, the system is inconsistent and the lines are skew.

Example 29

The lines L_1 and L_2 have the following equations:

$$L_1: x=1+2t,\ y=3-4t,\ z=-2+4t$$
$$L_2: x=4+3s,\ y=4+s,\ z=-4-2s$$

Show that the lines intersect.

Solution

We first examine whether the lines are parallel. Since the vector parallel to L_1 is $\mathbf{l}_1=(2,-4,4)$ and the vector parallel to L_2 is $\mathbf{l}_2=(3,1,-2)$, they are not scalar multiples of each other and the vectors and consequently the lines are not parallel.

For the lines to intersect, there should be some point $M(x_0, y_0, z_0)$ which satisfies the equations of both lines for some values of t and s. That is,

$$x_0=1+2t=4+3s;\ y_0=3-4t=4+s;\ z_0=-2+4t=-4-2s.$$

This leads to a set of three simultaneous equations in two unknowns: s and t.

By solving the first two equations:

$$\left.\begin{array}{l}1+2t=4+3s\\3-4t=4+s\end{array}\right\}\Rightarrow 5=12+7s\Rightarrow s=-1,\ t=0$$

For the system to be consistent, these values must satisfy the third equation, i.e. $-2+4(0)=-4-2(-1)\Rightarrow -2=-2$, which is a correct statement. Hence, the two lines intersect.

The point of intersection can be found through substitution of the value of the parameter into the corresponding line equation:

$$L_1: (1,3,-2) \text{ and } L_2: (4-3, 4-1, -4-2(-1))=(1,3,-2)$$

Note: In vector form, finding the point of intersection, if it exists, follows a similar approach. For example, the vector equations of lines L_1 and L_2 are

L_1: $\mathbf{r}=(1,3,-2)+t(2,-4,4)$

L_2: $\mathbf{r}=(4,4,-4)+s(3,1,-2)$

The condition of intersection is therefore

$(1,3,-2)+t(2,-4,4)$
$=(4,4,-4)+s(3,1,-2)$,

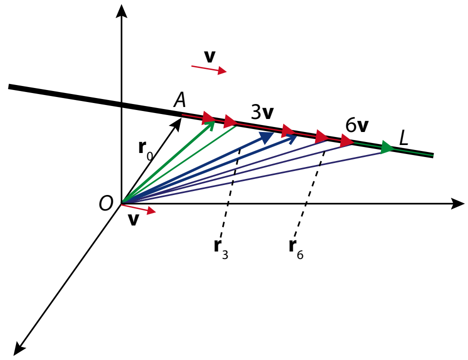
which leads to the same conclusion as in the case of parametric equations.

Application of lines to motion

The vector form of the equation of a line in space is more revealing when we think of the line as the path of an object, placed in an appropriate coordinate system and starting at position $A(x_0, y_0, z_0)$ and moving in the direction of $\mathbf{v}$.

Generally speaking, you find an object at an initial location A, represented by $\mathbf{r}_0$. The object moves on its path with a velocity vector $\mathbf{v} = (a, b, c)$. The object's position at any point in time after the start can then be described by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

Figure 14.17



Assuming the unit of time is seconds, the equation tells us that for every second, the object moves a units in the x direction, b in the y direction and c in the z direction. So, for example, after 2 seconds you find the object at $\mathbf{r} = \mathbf{r}_0 + 2\mathbf{v}$.

The speed of the object is then $|\mathbf{v}|$ in the $\mathbf{v}$ direction.

In general, we can write the vector equation in a slightly modified form.

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v}\\ &= \underbrace{\mathbf{r}_0}_{\text{initial position}} + \underbrace{t}_{\text{time}} \cdot \underbrace{|\mathbf{v}|}_{\text{speed}} \cdot \underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{direction}}\end{aligned}$$

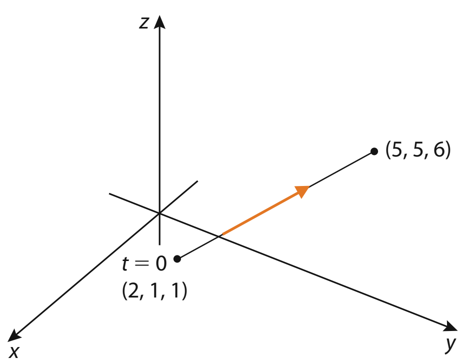
In other words, the position of an object at time t is the *initial position* plus its *rate* × *time* (distance moved) in the *direction* $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ of its straight-line motion.

Example 30

A model plane is to fly directly from a platform at a reference point (2, 1, 1) toward a point (5, 5, 6) at a speed of 60 m/min. What is the position of the plane (to the nearest metre) after 10 minutes?

Solution

The unit vector in the direction of the flight is $\mathbf{u} = \frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} + \frac{5}{5\sqrt{2}}\mathbf{k}$.



The position of the plane at any time t is

$$\begin{aligned}\mathbf{r}(t)_0 &= \mathbf{r}_0 + t(speed)(\mathbf{u})\\ &= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (10)(60)\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} + \frac{5}{5\sqrt{2}}\mathbf{k}\right)\\ &= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \left(\frac{360}{\sqrt{2}}\mathbf{i} + \frac{480}{\sqrt{2}}\mathbf{j} + \frac{600}{\sqrt{2}}\mathbf{k}\right).\end{aligned}$$

So, the plane is approximately at (257, 340, 425).

Example 31

An object is moving in the plane of an appropriately fitted coordinate system such that its position is given by

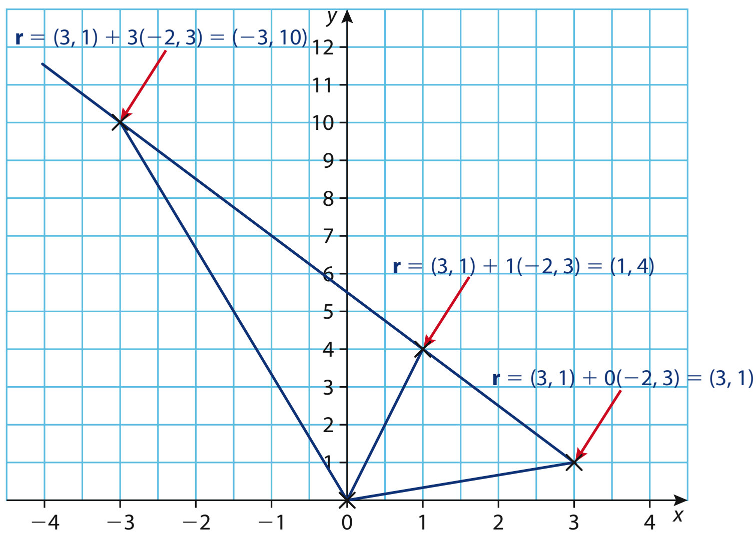
$$\mathbf{r} = (3, 1) + t(-2, 3),$$

where t stands for time in hours after start and distances are measured in km.

a) Find the initial position of the object.
b) Show the position of the object on a graph at start, 1 hour and 3 hours after start.
c) Find the velocity and speed of the object.

Solution

a) Initial position is when $t = 0$. This is the point $(3, 1)$.
b) See graph.



c) The velocity vector is $\mathbf{v} = (-2, 3)$, which means that every hour the object moves 2 units west and 3 units north.

The speed is $|\mathbf{v}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$ km/h.

We can also express the velocity as $\sqrt{13}$ km/h in the direction of $(-2, 3)$.

Note: We can also express the direction in terms of the unit vector in the direction of $\mathbf{v}$ instead. That is, we can say that the speed is $\sqrt{13}$ km/h in the direction of $\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$, or, equivalently, at an angle of $\cos^{-1}\left(\frac{-2}{\sqrt{13}}\right) \approx 124°$ to the positive x-direction.

Example 32

At 12:00 midday a plane A is passing in the vicinity of an airport at a height of 12 km and a speed of 800 km/h. The direction of the plane is $(4, 3, 0)$. [Consider that $(1, 0, 0)$ is a displacement of 1 km due east, $(0, 1, 0)$ due north, and $(0, 0, 1)$ is an altitude of 1 km.]

a) Using the airport as the origin, find the position vector **r** of the plane t hours after midday.
b) Find the position of the plane 1 hour after midday.
c) Another plane B is heading towards the airport with velocity vector $(-300, -400, 0)$ from a location $(600, 480, 12)$. Is there a danger of collision?

Solution

a) The position vector at midday is $(0, 0, 12)$. The direction of the velocity vector is given by the unit vector $\frac{1}{5}(4, 3, 0)$. So, the velocity vector of this plane is $800 \cdot \frac{1}{5}(4, 3, 0) = (640, 480, 0)$.
The position vector of the plane is $\mathbf{r} = (0, 0, 12) + t(640, 480, 0)$.
b) $\mathbf{r} = (0, 0, 12) + (640, 480, 0) = (640, 480, 12)$
c) A collision can happen if the two planes pass the same point at the same time.
The position vector for the second plane is $\mathbf{r} = (600, 480, 12) + t(-300, -400, 0)$.
If the two paths intersect, they may intersect at instances corresponding to t_1 and t_2 and they should have the same position, i.e.

$$(0, 0, 12) + t_1(640, 480, 0) = (600, 480, 12) + t_2(-300, -400, 0).$$

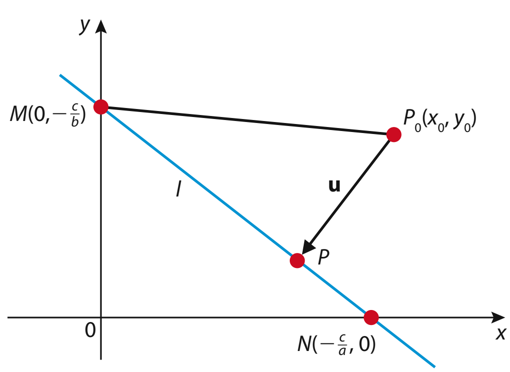
This gives rise to a set of three equations in two variables:

$$\left.\begin{aligned} 640t_1 &= 600 - 300t_2 \\ 480t_1 &= 480 - 400t_2 \\ 12 &= 12 \end{aligned}\right\}$$

Solving the system of equations simultaneously will give $t_1 = \frac{6}{7}$ and $t_2 = \frac{6}{35}$.
This means that the planes' paths will cross at $(548.57, 411.43, 12)$.
There is no collision though because plane A will pass that point at 12:51 while plane B will pass this point at 12.10!

Distance from a point to a line (optional)

2-space



Theorem: If the equation of a line l is written in the form $ax + by + c = 0$, then the distance from a point $P_0(x_0, y_0)$ to the line l is given by

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

There are several methods of proving this theorem. We will follow a vector approach, leaving some other interesting methods for the website.

The x- and y-intercepts of the line l are

$$N\left(-\frac{c}{a}, 0\right) \text{ and } M\left(0, -\frac{c}{b}\right).$$

So, a vector parallel to l can be any vector in the direction of

$$\overrightarrow{NM} = \left(\frac{c}{a}, -\frac{c}{b}\right).$$

For convenience we will consider the vector $\mathbf{L} = \left(\frac{1}{a}, -\frac{1}{b}\right)$.

Consider a vector in the direction of $\overrightarrow{P_0P}$ perpendicular to l. Consider vector $\mathbf{u} = (a, b)$ which is perpendicular to l because $\mathbf{d} \cdot \mathbf{L} = 0$ and hence parallel to $\overrightarrow{P_0P}$, then in triangle MPP_0, the distance $|\overrightarrow{P_0P}|$ is

$$|\overrightarrow{MP_0}|\cos(MP_0P).$$

$$\overrightarrow{MP_0} = \left(x_0, y_0 + \frac{c}{b}\right)$$

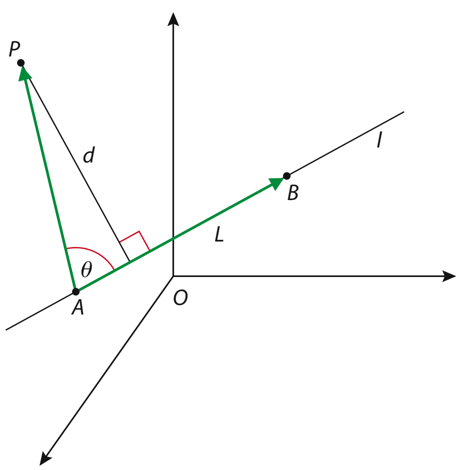
$$|\overrightarrow{P_0P}| = \left||\overrightarrow{MP_0}| \cdot \cos M\hat{P_0}P\right| = \left|\frac{|\overrightarrow{MP_0}| \cdot \overrightarrow{MP_0} \cdot \overrightarrow{P_0P}}{|\overrightarrow{MP_0}| \cdot |\overrightarrow{P_0P}|}\right| = \left|\frac{\overrightarrow{MP_0} \cdot \mathbf{u}}{|\mathbf{u}|}\right|$$

$$= \left|\frac{\left(x_0, y_0 + \frac{c}{b}\right) \cdot (a, b)}{\sqrt{a^2 + b^2}}\right| = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

3-space

$$d = |\overrightarrow{AP_0}| \cdot \sin(\theta) = |\overrightarrow{AP_0}| \cdot \frac{|\mathbf{L} \times \overrightarrow{AP_0}|}{|\mathbf{L}| \cdot |\overrightarrow{AP_0}|} = \frac{|\mathbf{L} \times \overrightarrow{AP_0}|}{|\mathbf{L}|}$$

where A is any point on line l and $\mathbf{L}$ is a vector parallel to l.



Example 33

Find the distance from the point $(1, 3)$ to the line with equation

$$2x - y = 7.$$

Solution

The equation can be written as $2x - y - 7 = 0$ and hence the distance is

$$d = \frac{|2(1) - 3 - 7|}{\sqrt{2^2 + 1}} = \frac{8}{\sqrt{5}}.$$

Example 34

Find the distance from the point $P(8, 1, -3)$ to the line containing $M(3, 0, 6)$ and $N(5, -2, 7)$.

Solution

We have:

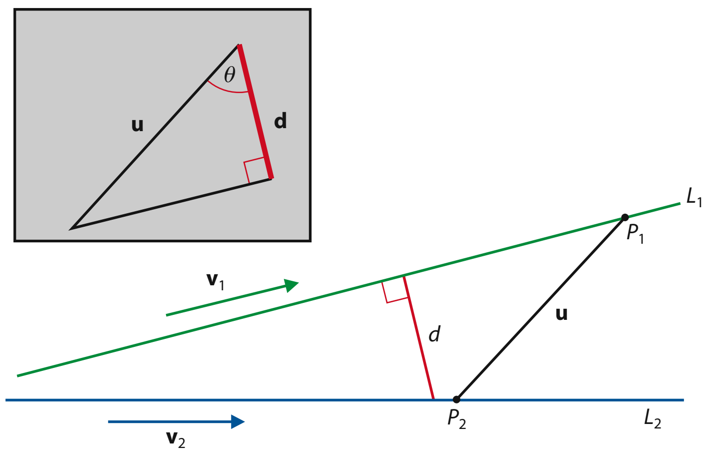
$$\overrightarrow{MN} = (2, -2, 1) \text{ and } \overrightarrow{MP} = (5, 1, -9) \Rightarrow$$

$$d = \frac{|(2, -2, 1) \times (5, 1, -9)|}{|(2, -2, 1)|} = \frac{|(17, 23, 12)|}{\sqrt{2^2 + (-2)^2 + 1}}$$

$$= \frac{\sqrt{17^2 + 23^2 + 12^2}}{3} = \frac{\sqrt{962}}{3}$$

Distance between two skew straight lines

In the diagram below, two skew straight lines L_1 and L_2 are given with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. We need to find the distance d, defined as the length of the 'common perpendicular', between them.



Consider any two fixed points P_1 on L_1 and P_2 on L_2. The distance d is the length of the orthogonal projection of vector $\mathbf{u} = \overrightarrow{P_1P_2}$ on a direction perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. This direction can be determined by $\mathbf{v}_1 \times \mathbf{v}_2$. So,

$$d = \left||\overrightarrow{P_1P_2}|\cos\theta\right| = \left||\overrightarrow{P_1P_2}|\frac{\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)}{|\overrightarrow{P_1P_2}||\mathbf{v}_1 \times \mathbf{v}_2|}\right| = \left|\frac{\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)}{|\mathbf{v}_1 \times \mathbf{v}_2|}\right|.$$

Example 35

Find the distance between the following skew lines:

$$L_1: \mathbf{r} = (2, 3, 1) + t(1, 2, -3);\ L_2: \mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + s(3\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Solution

The two fixed points could be taken as (2, 3, 1) and (4, 2, 0) while the vectors are $\mathbf{v}_1 = (1, 2, -3)$ and $\mathbf{v}_2 = (3, -1, 1)$.

$$d = \left| \frac{(2, -1, -1)(-1, -10, -7)}{|(-1, -10, -7)|} \right|$$

$$= \left| \frac{-2 + 10 + 7}{\sqrt{1 + 100 + 49}} \right| = \frac{15}{\sqrt{150}} = \frac{\sqrt{6}}{2}$$

Note: The minimum distance could be found using other methods too. One of them would be to consider the line going from any point on L_1 to any point on L_2. This will give a parametric equation in s and t. Then considering that this line will be perpendicular to both L_1 and L_2, i.e. $\mathbf{u} \cdot \mathbf{v}_1 = 0$, $\mathbf{u} \cdot \mathbf{v}_2 = 0$, enables us to set up a system of two equations that could be solved for s and t. Lastly, we get the distance between the points corresponding to the specific values we just established.

Exercise 14.4

1 Find a vector equation, a set of parametric equations and a set of Cartesian equations of the line containing the point A and parallel to the vector $\mathbf{u}$.
a) $A(-1, 0, 2)$, $\mathbf{u} = (1, 5, -4)$
b) $A(3, -1, 2)$, $\mathbf{u} = (2, 5, -1)$
c) $A(1, -2, 6)$, $\mathbf{u} = (3, 5, -11)$

2 Find all three forms of the equation of the line that passes through the points A and B.
a) $A(-1, 4, 2)$, $B(7, 5, 0)$
b) $A(4, 2, -3)$, $B(0, -2, 1)$
c) $A(1, 3, -3)$, $B(5, 1, 2)$

3 a) Write the equation of the line through the points (3, −2) and (5, 1) in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
b) Write the equation of the line through the points (0, −2) and (5, 0) in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

4 The equation of a line in 2-space is given by $\mathbf{r} = (2, 1) + t(3, -2)$.
Write the equation in the form $ax + by = c$.

5 Find the equation of a line through (2, −3) that is parallel to the line with equation $\mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$.

6 Find the equation of a line through (−2, 1, 4) and parallel to the vector $3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

7 In each of the following, find the point of intersection of the two given lines, and if they do not intersect, explain why.
a) L_1: $\mathbf{r} = (2, 2, 3) + t(1, 3, 1)$
L_2: $\mathbf{r} = (2, 3, 4) + t(1, 4, 2)$
b) L_1: $\mathbf{r} = (-1, 3, 1) + t(4, 1, 0)$
L_2: $\mathbf{r} = (-13, 1, 2) + t(12, 6, 3)$
c) L_1: $\mathbf{r} = (1, 3, 5) + t(7, 1, -3)$
L_2: $\mathbf{r} = (4, 6, 7) + t(-1, 0, 2)$

d) $L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$L_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$

8 Find the vector and parametric equations of each line:

a) through the points $(2, -1)$ and $(3, 2)$

b) through the point $(2, -1)$ and parallel to the vector $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

c) through the point $(2, -1)$ and perpendicular to the vector $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

d) with y-intercept $(0, 2)$ and in the direction of $2\mathbf{i} - 4\mathbf{j}$

9 Consider the line with equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

a) For what value of t does this line pass through the point $\left(0, \frac{11}{2}, \frac{9}{2}\right)$?

b) Does the point $(-1, 4, 6)$ lie on this line?

c) For what value of m does the point $\left(\frac{1-2m}{2}, 2m, 3\right)$ lie on the given line?

10 Consider the following equations representing the paths of cars after starting time $t \geqslant 0$, where distances are measured in km and time in hours. For each car, determine

(i) starting position

(ii) the velocity vector

(iii) the speed.

a) $\mathbf{r} = (3, -4) + t\begin{pmatrix} 7 \\ 24 \end{pmatrix}$

b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -12 \end{pmatrix}$

c) $(x, y) = (5, -2) + t(24, -7)$

11 Find the velocity vector of each of the following racing cars taking part in the Paris–Dakar rally:

a) direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ with a speed of 160 km/h

b) direction $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ with a speed of 170 km/h

12 After leaving an intersection of roads located at 3 km east and 2 km north of a city, a car is moving towards a traffic light 7 km east and 5 km north of the city at a speed of 30 km/h. (Consider the city as the origin for an appropriate coordinate system.)

a) What is the velocity vector of the car?

b) Write down the equation of the position of the car after t hours.

c) When will the car reach the traffic light?

13 Consider the vectors $\mathbf{u} = (1, a, b)$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

a) Find a and b so that $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

b) If O is the origin, P a point whose position vector is $\mathbf{v}$ and Q is with position vector $\mathbf{w}$, find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

c) Hence, find the sine of the angle and use it to find the area of the triangle OPQ.

14 The triangle ABC has vertices at the points $A(-1, 2, 3)$, $B(-1, 3, 5)$ and $C(0, -1, 1)$.

a) Find the size of the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence, or otherwise, find the area of triangle ABC.

Let L_1 be the line parallel to $\overrightarrow{AB}$ which passes through $D(2, -1, 0)$, and L_2 be the line parallel to $\overrightarrow{AC}$ which passes through $E(-1, 1, 1)$.

c) (i) Find the equations of the lines L_1 and L_2.

(ii) Hence, show that L_1 and L_2 do not intersect.

15 Consider the points $A(1, 3, -17)$ and $B(6, -7, 8)$ which lie on the line l.

a) Find an equation of line l, giving the answer in parametric form.

b) The point P is on l such that $\overrightarrow{OP}$ is perpendicular to l. Find the coordinates of P.

16 a) Starting with the equation of a line in the form $mx + ny = p$, find a vector equation of the line.

b) (i) Starting with a vector equation of a line where $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, with $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, find an equation of the line in the form $mx + ny = p$.

(ii) What is the relationship between the components of the direction vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and the slope of the line?

17 Find a parametrization for the line segment between points A and B in each of the following questions.

(i) $A(0, 0, 0)$, $B(1, 1, 3)$

(ii) $A(-1, 0, 1)$, $B(1, 1, -2)$

(iii) $A(1, 0, -1)$, $B(0, 3, 0)$

18 Find a vector equation and a set of parametric equations of the line through the point $(0, 2, 3)$ and parallel to the line $\mathbf{r} = (\mathbf{i} - 2\mathbf{j}) + 2t\mathbf{k}$.

19 Find a vector equation and a set of parametric equations of the line through the point $(1, 2, -1)$ and parallel to the line $\mathbf{r} = t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

20 Find a vector equation and a set of parametric equations of the line through the origin and the point $A(x_0, y_0, z_0)$.

21 Find a vector equation and a set of parametric equations of the line through $(3, 2, -3)$ and perpendicular to

a) the xz-plane

b) the yz-plane.

22 Write a set of symmetric equations for the line through the origin and the point $A(x_0, y_0, z_0)$, $x_0, y_0, z_0 \neq 0$.

In questions 23–29, determine whether the lines l_1 and l_2 are parallel, skew or intersecting. If they intersect, find the coordinates of the point of intersection.

23 $l_1\colon x - 3 = 1 - y = \dfrac{z-5}{2}$, $l_2\colon \mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{j} + \mathbf{k})$

24 $l_1\colon \begin{cases} x = -1 + s \\ y = 2 - 3s \\ z = 1 + 2s \end{cases}$ $\quad l_2\colon \begin{cases} x = 2 - 2m \\ y = -1 + 6m \\ z = -4m \end{cases}$

25 $l_1\colon \dfrac{x-3}{2} = \dfrac{1+y}{4} = 2 - z$, $l_2\colon \mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

26 $l_1\colon x - 1 = \dfrac{y-1}{3} = \dfrac{z+4}{2}$, $l_2\colon 1 - x = -1 - y = \dfrac{z}{2}$

27 l_1: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(-6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$, l_2: $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + m(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

28 $\frac{x-2}{5} = y - 1 = \frac{z-2}{3}$ and $\frac{x+4}{3} = \frac{7-y}{3} = \frac{10-z}{4}$

29 $x = 1 + t, y = 2 - 2t, z = t + 5$ and $x = 2 + 2t, y = 5 - 9t, z = 2 + 6t$

30 Find the point on the line

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(-3\mathbf{i} + \mathbf{j} + \mathbf{k})$$

that is closest to the origin. (Hint: use the parametric form and the distance formula and minimize the distance using derivatives!)

31 Find the point on the line

$$\mathbf{r} = 4\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

that is closest to the origin.

32 Find the point on the line

$$\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

that is closest to the point $(-1, 4, 1)$.

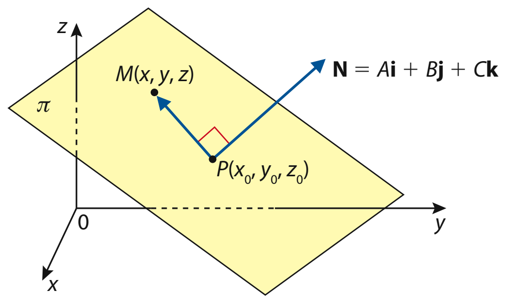
14.5 Planes

To define/specify a plane is to identify it in a way that makes it unique. One way is to set up an equation in a frame that will identify every point that belongs to the plane. There are several ways of specifying a plane but we will only mention four of them here. The rest will be cases that we address in some problems later. For more helpful geometric concepts please refer to the book's website.

A plane can be defined

- by three non-collinear points
- by two intersecting straight lines
- to be perpendicular to a certain direction and at a specific distance from the origin (for example)
- by being drawn through a given point and perpendicular to a given direction.

A direction, for our purposes, can be defined by a vector. In the case of a plane, the vector determining the direction is perpendicular to the plane and is said to be **normal to the plane**.



Equations of a plane

From the many ways of defining a plane above, the last two are mostly appropriate for deriving equations of a plane.

Cartesian (scalar) equation of a plane

Consider a plane π and a fixed point $P(x_0, y_0, z_0)$ on that plane. A vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, called the normal vector to the plane, is a vector perpendicular to the plane.

To find an equation for the plane, consider an arbitrary point $M(x, y, z)$ in space. Recalling that a line perpendicular to the plane is perpendicular to every line in the plane, we can conclude that for the point M to be on the plane, the vector $\mathbf{N}$ must be perpendicular to $\overrightarrow{PM}$.

Hence,

$$\mathbf{N} \cdot \overrightarrow{PM} = 0, \text{ but } \overrightarrow{PM} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}, \text{ and}$$

$$\mathbf{N} \cdot \overrightarrow{PM} = 0$$

$$\Leftrightarrow (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$$

Using the scalar product definition this can be simplified to

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

This is a Cartesian equation of a plane that passes through a point $P(x_0, y_0, z_0)$ and has a normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

Note: If $\mathbf{N}$ is normal to a given plane, then any vector parallel to $\mathbf{N}$ will be normal to the plane. Suppose we have chosen $3\mathbf{N}$ as our normal, then

$$3A(x - x_0) + 3B(y - y_0) + 3C(z - z_0) = 0$$

$$\Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0)$$

Specifically, the unit vector $\mathbf{n}$ in the same direction as $\mathbf{N}$ is of particular importance, as we will see soon.

Note: The above equation can be simplified further.
$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
$\Leftrightarrow Ax + By + Cz = Ax_0 + By_0 + Cz_0$, and setting $Ax_0 + By_0 + Cz_0 = D$ will give us a more concise form of the equation,

$$Ax + By + Cz = D$$

which is similar to the equation of a line in the plane, i.e. $Ax + By = C$.

Note: In many sources, the equation of the plane is given in the form

$$Ax + By + Cz + D = 0.$$

This is the case when we set the quantity $Ax_0 + By_0 + Cz_0 = -D$. Each form has some advantage in using it. We will adhere to the previous form for reasons that will be clear in the following discussion.

Example 36

Write an equation for the plane that contains $(2, -3, 5)$ and has normal $\mathbf{N} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

Solution

A Cartesian equation for the plane is of the form:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow 2(x - 2) + (y + 3) - 3(z - 5) = 0$$

$$\Rightarrow 2x + y - 3z = -14$$

Alternatively, since $\mathbf{N} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is the normal to the plane, then

$2x + y - 3z = D$, but the line contains the point $(2, -3, 5)$, and thus

$$2(2) - 3 - 3(5) = D \Rightarrow -14 = D.$$

And therefore

$$2x + y - 3z = -14 \text{ as before.}$$

Example 37

Show that every equation of the form $Ax + By + Cz = D$ with $A^2 + B^2 + C^2 \neq 0$ represents a plane in space.

Solution

The equation $Ax + By + Cz = D$ is a linear equation in 3 variables, x, y and z. This means that it has an infinite number of solutions, and hence we can be confident that there exist numbers x_0, y_0, z_0 such that $Ax_0 + By_0 + Cz_0 = D$.

Since the equation $Ax + By + Cz = D$ is also true, then

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D$$
$$\Leftrightarrow Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0$$
$$\Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

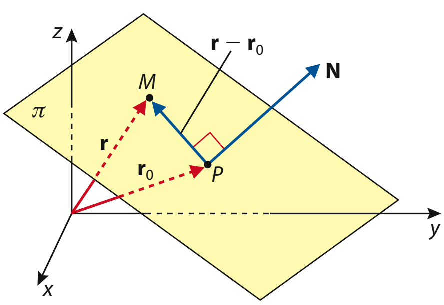
The last equation represents the equation of a plane through a fixed point $P(x_0, y_0, z_0)$ with a normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. The condition $A^2 + B^2 + C^2 \neq 0$ guarantees that $\mathbf{N} \neq \mathbf{0}$.

Vector equation of a plane

We can write the equation of the plane in vector notation. Using the same set up as before: the normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, a fixed point $P(x_0, y_0, z_0)$ with a position vector $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, and an arbitrary point $M(x, y, z)$ with a position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$.

The equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ can be interpreted as the scalar product $\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which you can also see in the diagram. The normal $\mathbf{N}$ is perpendicular to $PM = \mathbf{r} - \mathbf{r}_0$ and hence their dot product must be zero. Using the distributive property of the scalar product, we have

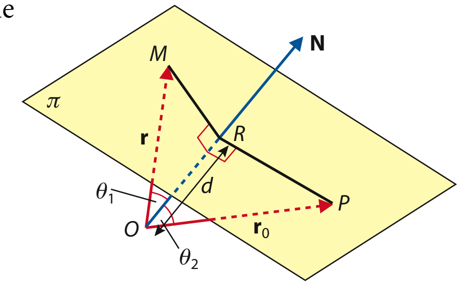
$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \Leftrightarrow \mathbf{N} \cdot \mathbf{r} - \mathbf{N} \cdot r_0 = 0$$
$$\Leftrightarrow \mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$$



This is one form of the vector equation of a plane that passes through a point with position vector $\mathbf{r}_0$ and has a normal $\mathbf{N}$.

Note: Notice here that $\mathbf{N} \cdot \mathbf{r} = Ax + By + Cz$ and $\mathbf{N} \cdot \mathbf{r}_0 = Ax_0 + By_0 + Cz_0 = D$, which shows that $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$ is another way of stating $Ax + By + Cz = D$.

Unit vector equation of a plane



The diagram shows vector $\mathbf{N}$ as drawn from the origin O, along with the position vectors $\mathbf{r}$ and $\mathbf{r}_0$.

The vector equation $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$ can be investigated further.

$$\mathbf{N} \cdot \mathbf{r} = |\mathbf{N}||\mathbf{r}|\cos\theta_1 = |\mathbf{N}|OR = |\mathbf{N}|d$$

where OR $= d$ is the distance from the origin to the plane.

Also

$$\mathbf{N} \cdot \mathbf{r}_0 = |\mathbf{N}||\mathbf{r}_0|\cos\theta_2 = |\mathbf{N}|d.$$

In both cases, the result is of course the same. Both sides of the equation $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$ are equal to the same value: the magnitude of the normal multiplied by the distance from the origin.

Furthermore, if we divide each side by $|\mathbf{N}|$, we get the distance from the origin to the plane.

That is,

$$\mathbf{N} \cdot \mathbf{r} = |\mathbf{N}|d \Rightarrow d = \frac{\mathbf{N} \cdot \mathbf{r}}{|\mathbf{N}|}$$

as well as

$$\mathbf{N} \cdot \mathbf{r}_0 = |\mathbf{N}|d \Rightarrow d = \frac{\mathbf{N} \cdot \mathbf{r}_0}{|\mathbf{N}|}.$$

This last result gives us the basis for forming a new vector equation of the plane in terms of a unit vector perpendicular to it.

Let us call the unit vector normal to the plane $\mathbf{n}$. So, using the results just established, we can write

$d = \dfrac{\mathbf{N} \cdot \mathbf{r}}{|\mathbf{N}|} = \dfrac{\mathbf{N} \cdot \mathbf{r}_0}{|\mathbf{N}|}$, which in turn can be simplified to

$\dfrac{\mathbf{N}}{|\mathbf{N}|} \cdot \mathbf{r} = \dfrac{\mathbf{N}}{|\mathbf{N}|} \cdot \mathbf{r}_0$, and since $\dfrac{\mathbf{N}}{|\mathbf{N}|} = \mathbf{n}$, then obviously

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 = d.$$

This equation is very practical when we need to find the distance from the origin to the plane. The distance from the origin to a plane is the scalar product between the unit normal and the position vector of any point on the plane.

This will be shown in the examples below.

Example 38

Write a vector equation for the plane that contains $(2, -3, 5)$ and has normal $\mathbf{N} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

Solution

We apply the results of the previous discussion:

$$\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$$

$$\Rightarrow (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot \mathbf{r} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$$

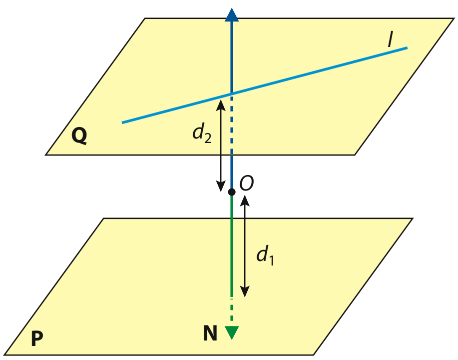
$$\Rightarrow (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot \mathbf{r} = -14$$

Notice that this result can easily transfer into Cartesian form by expanding the scalar product on the left.

$$(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot \mathbf{r} = -14$$
$$\Rightarrow (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -14$$
$$\Rightarrow 2x + y - 3z = -14$$

Example 39

Show that the line l with equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + k(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ is parallel to the plane P whose equation is $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ and find the distance between them.



Solution

Since $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ is the direction vector of the line l, and since this vector is perpendicular to $\mathbf{N} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, the normal to P, as $(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 6 - 4 - 2 = 0$, then the line l must be parallel to plane P.

To find the distance between the line and the plane P, we may find the distance between a point on l, $(2, -1, 5)$ for example, and plane P. One way would be to consider a plane Q containing the given point and parallel to P. The equation of plane Q can be found using the last result:

$$\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$$
$$\Rightarrow (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot \mathbf{r} = (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$
$$\Rightarrow (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot \mathbf{r} = 11$$

The distance from the origin to P is given by

$$d = \left|\frac{\mathbf{N} \cdot \mathbf{r_0}}{|\mathbf{N}|}\right| = \frac{3}{\sqrt{4 + 4 + 1}} = 1$$

while the distance from the origin to Q is

$$d = \left|\frac{\mathbf{N} \cdot \mathbf{r_0}}{|\mathbf{N}|}\right| = \frac{11}{\sqrt{4 + 4 + 1}} = \frac{11}{3}.$$

The distance between the two planes is the sum of these two distances since the planes are on opposite sides of the origin (see note), and hence the required distance is $\frac{14}{3}$.

Note: The vector equation for P is $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$, or $\mathbf{r} \cdot (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3$ and the equation for Q is $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot \mathbf{r} = 11$. The normals to the two planes are opposite, and hence they are on opposite sides of the origin. If the two normals are in the same direction, then the distance between them will be the difference of the two distances from the origin.

Example 40

Show that the plane with vector equation $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ contains the line with equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} + k(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

Solution

We have several methods available to us at this stage. One method is to check whether two points are common to the line and the plane. One point on the line is $(1, 3, 1)$. Since $(1, 3, 1) \cdot (2, -2, 1) = 2 - 6 + 1 = -3$, then the point is on the plane. Another point on the line can be found by choosing any value for k, say $k = 1$. Thus, another point has the position vector

$$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} + 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}.$$

Since $(4\mathbf{i} + 5\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 8 - 10 - 1 = -3$, this point will also lie on the plane and therefore the plane will contain the whole line.

Another method would be to check only one point and prove that the line is parallel to the plane as in Example 39 above.

Example 41

Find the vector equation of the line through $(1, 2, 3)$ that is perpendicular to the plane with vector equation $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ and find their point of intersection.

Solution

A vector parallel to the required line must be parallel to the normal vector to the plane. Hence, a vector equation of the line is
$\mathbf{r} = (1, 2, 3) + k(2, -2, 1)$.

To find the point of intersection, we consider any point on the line. Such a point would have the position vector $(1 + 2k, 2 - 2k, 3 + k)$. For this point to be on the plane, the following equation must be true:

$$(1 + 2k, 2 - 2k, 3 + k) \cdot (2, -2, 1) = -3;$$

i.e. $2 + 4k - 4 + 4k + 3 + k = -3$, so

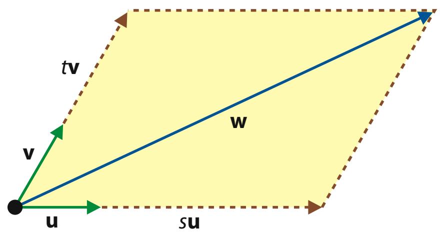
$9k = -4$, and $k = -\frac{4}{9}$ giving the point of intersection of the line and the plane as

$$\left(1 + 2 \times -\tfrac{4}{9}, 2 - 2 \times -\tfrac{4}{9}, 3 - \tfrac{4}{9}\right) = \left(\tfrac{1}{9}, \tfrac{26}{9}, \tfrac{23}{9}\right).$$

Parametric form for the equation of a plane

We start this section with an example that demonstrates the following theorem:

Three coplanar vectors $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are given. If* $\mathbf{u}$ *and* $\mathbf{v}$ *are not parallel, then* $\mathbf{w}$ *can always be expressed as a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$, *i.e. it is always possible to find two scalars s and t such that* $\mathbf{w} = s\mathbf{u} + t\mathbf{v}$. (The proof of the theorem is not included in this book.)



Using the diagram on the left, this means that it is always possible to construct a parallelogram whose diagonal is **w** and whose sides are the non-parallel vectors **u** and **v** or their multiples.

For example, given the two non-parallel vectors $\mathbf{u} = (1, 0, -2)$ and $\mathbf{v} = (3, 1, -9)$, then we can always find the scalars s and t so that vector $\mathbf{w} = (2, 1, -7)$ can be expressed as a linear combination of **u** and **v**. Thus,

$$(2, 1, -7) = s(1, 0, -2) + t(3, 1, -9).$$

To find s and t we solve the system of equations:

$$\begin{cases} s + 3t = 2 \\ t = 1 \\ -2s - 9t = -7 \end{cases}$$

This system is consistent and yields the solution $s = -1$ and $t = 1$. Thus,

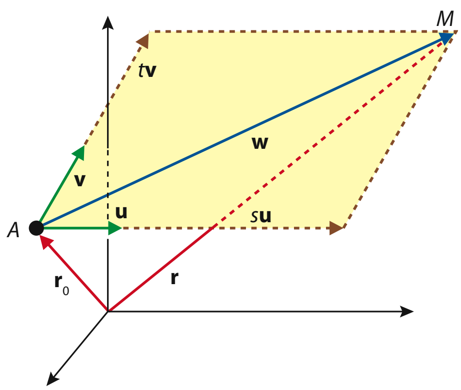
$$\mathbf{w} = -\mathbf{u} + \mathbf{v}.$$

Now, consider the plane which is parallel to vectors **u** and **v** and which contains the point A whose position vector is $\mathbf{r}_0$. As the figure below shows, the two vectors determine the direction of the plane and A 'fixes' it in space. So, if M is any point in this plane then, according to the previous theorem, $\overrightarrow{AM} = s\mathbf{u} + t\mathbf{v}$ where s and t are two scalars.

If **r** is the position vector of M, then

$$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{AM} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}.$$

Thus, any equation of the form $\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$, where s and t are independent scalars, represents the equation of a plane parallel to **u** and **v** and contains the point with position vector $\mathbf{r}_0$.



Note that this equation is not unique. This is because one can start at any other fixed point on the plane other than A and may choose any number of intersecting vectors in the plane other than **u** and **v**. The parametric form of the equation of the plane is seldom needed or used.

Example 42

Find an equation of a plane with normal $\mathbf{q} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and that contains the point $A(2, 1, 1)$. Use all forms you learned.

Solution

The Cartesian equation

Consider the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow 2(x - 2) - 3(y - 1) + (z - 1) = 0$$

The equation would be: $2x - 3y + z - 2 = 0$.

Or, start the equation:

$$Ax + By + Cz = D$$

$$\Rightarrow 2x - 3y + z = D$$

Since the plane contains the point $(2, 1, 1)$, then

$2(2) - 3(1) + 1 = D$, and thus $D = 2$.

Vector equations

Finding the vector equation can also be achieved by applying:

$$\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0 \Rightarrow (2, -3, 1) \cdot (x, y, z) = (2, -3, 1) \cdot (2, 1, 1)$$

$$\Rightarrow (2, -3, 1) \cdot (x, y, z) = 2$$

Note: It is easy to transform the vector equation into Cartesian form by simply performing the dot product. The opposite is also true.

Parametric equation

A parametric equation of this plane is not as straightforward and may not be the most efficient way of doing this problem. However, for the sake of giving an example we present a way of doing it.

The parametric form requires that we have two vectors parallel to the plane. We may find the two vectors by considering that they have to be perpendicular to $(2, -3, 1)$. So, take a vector $(1, 1, z)$ and find z so that this vector is perpendicular to $(2, -3, 1)$:

$$\Rightarrow 2 - 3 + z = 0 \text{ and } z = 1$$

Do the same with $(1, 0, z)$, i.e. $2 + 0 + z = 0$ and $z = -2$. Therefore, two vectors that are perpendicular to $(2, -3, 1)$ are $(1, 1, 1)$ and $(1, 0, -2)$, and a parametric equation of the plane is

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v} \Rightarrow \mathbf{r} = (2, 1, 1) + s(1, 1, 1) + t(1, 0, -2).$$

Observe that the choice of the vectors is arbitrary and hence the parametric form is not unique.

Example 43

Find the equation of the plane that contains the following three points:

$A(1, 3, 0)$, $B(-2, 1, 2)$ and $C(1, -2, -1)$.

Solution

Consider any point $M(x, y, z)$ on this plane. For this point to belong to the plane, the following vectors must be coplanar: $\overrightarrow{AM}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

This means that the parallelepiped with these vectors as edges is flat, i.e. with volume zero. Since we know that the volume of the parallelepiped is the absolute value of the scalar triple product, we equate that value to zero and get the equation. Here are the details:

$$\overrightarrow{AM} = (x - 1, y - 3, z), \overrightarrow{AB} = (-3, -2, 2), \overrightarrow{AC} = (0, -5, -1)$$

$$\Rightarrow \overrightarrow{AM} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} x-1 & y-3 & z \\ -3 & -2 & 2 \\ 0 & -5 & -1 \end{vmatrix} = 0$$

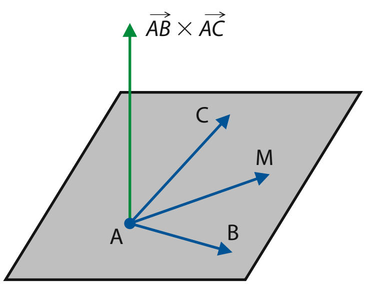
$$\Rightarrow 3(4x - y + 5z - 1) = 0$$

So, the Cartesian equation of the plane is $4x - y + 5z - 1 = 0$.

We can also deduce the vector equation for this plane by writing it in scalar product form:

$$(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1.$$

The vector form could also be achieved if we think of the problem as a plane containing a fixed point and normal to a given vector.



The normal can be found by computing the cross product of two vectors in the plane; in this case, we can take $\overrightarrow{AB}$ and $\overrightarrow{AC}$. So,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 2 \\ 0 & -5 & -1 \end{vmatrix} = 3(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}).$$

The equation of the plane is then

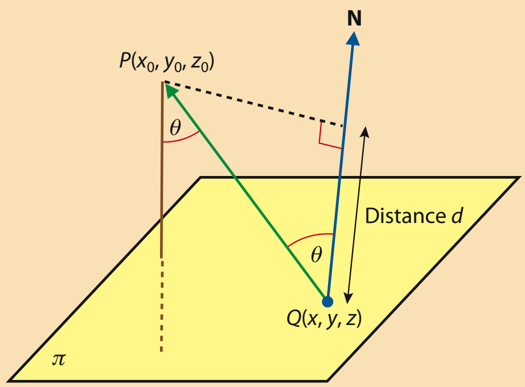
$$(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j})$$
$$(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1.$$

This is the same as above.

Distance between a point and a plane

The distance between a point $P(x_0, y_0, z_0)$ and a plane with equation $Ax + By + Cz = D$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$



In the diagram above, let $Q(x, y, z)$ be any point on the plane and $\mathbf{N}(A, B, C)$ be a normal to the plane. The distance we are looking for is d. Then,

$$d = \left| |\overrightarrow{QP}| \cos\theta \right| = \left| |\overrightarrow{QP}| \frac{\overrightarrow{QP} \cdot \mathbf{N}}{|\overrightarrow{QP}||\mathbf{N}|} \right|$$

$$= \left| \frac{\overrightarrow{QP} \cdot \mathbf{N}}{|\mathbf{N}|} \right| = \left| \frac{(A, B, C) \cdot (x_0 - x, y_0 - y, z_0 - z)}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \frac{|A(x_0 - x) + B(y_0 - y) + C(z_0 - z)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_0 + By_0 + Cz_0 - (Ax + Bx + Cz)|}{\sqrt{A^2 + B^2 + C^2}}$$

Since $Q(x, y, z)$ is on the plane, then $Ax + By + Cz = D$, so replacing this expression in the result above will yield

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

This formula is similar to the distance between a point and a line in 2-space.

Example 44

Show that the line l with equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + k(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ is parallel to the plane P whose equation is $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ and find the distance between them.

Solution

In a previous example we showed that the line is parallel to the plane because it is perpendicular to the normal of the plane. To find the distance, we used a relatively complex approach. At this moment we can utilize the distance formula just established to find the required distance.

A point on the line is $(2, -1, 5)$ and the Cartesian equation of the plane is simply

$$2x - 2y + z = -3.$$

Hence, the distance is

$$d = \frac{|2(2) - 2(-1) + 1(5) - (-3)|}{\sqrt{4 + 4 + 1}} = \frac{14}{3}.$$

Example 45

Find the distance between the two parallel planes: $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$.

Solution

It is enough to find the distance from one point on one of the planes to the other plane since all points are equidistant.

Take the point $(1, 1, z)$ on the first plane:

$$1 + 2 - 2z = 3, \text{ so } z = 0.$$

Thus, the point is $(1, 1, 0)$ and the distance between the planes is

$$d = \frac{|2(1) + 4(1) - 4(0) - 7|}{\sqrt{4 + 16 + 16}} = \frac{1}{6}.$$

Example 46

Find the distance between the two skew straight lines

$$L_1\colon \mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} \text{ and } L_2\colon \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}.$$

Solution

We can reduce this problem to the type in the previous example by creating two planes that contain the given lines and are parallel to each other.

For the two planes to be parallel, they must be perpendicular to the same vector. Hence, by finding the cross product of the direction vectors of the lines we would have found a vector perpendicular to both.

$$l_1 \times l_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

Considering the point $(2, 4, 5)$ on L_2, the plane containing this line will be

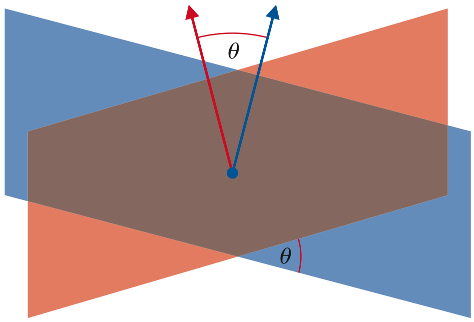
$$\begin{pmatrix} 11 \\ 36 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 36 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

or $11x + 36y + 20z = 266$ and the distance between $(1, 5, -1)$ on L_1 to this plane will be

$$d = \frac{|11(1) + 36(5) + 20(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}.$$

The angle between two planes

The angle between two planes is defined to be the *acute* angle between them as you see in the figure below.



Consider two planes **P** and **Q** with unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively. Their vector equations are of the form

$$\mathbf{r} \cdot \mathbf{n}_1 = d_1 \text{ and } \mathbf{r} \cdot \mathbf{n}_2 = d_2.$$

The angle between the planes is equal to the angle between the normal units $\mathbf{n}_1$ and $\mathbf{n}_2$ or the normals $\mathbf{N}_1$ and $\mathbf{N}_2$, and hence

$$\cos\theta = \mathbf{n}_1 \cdot \mathbf{n}_2 = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1|\,|\mathbf{N}_2|}.$$

For example, the angle between the planes with vector equations

$$\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3 \text{ and } \mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2$$

is given by

$$\cos\theta = \frac{(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})}{\sqrt{1+1+4} \cdot \sqrt{4+4+1}} = -\frac{2}{3\sqrt{6}}.$$

This is the cosine of the obtuse angle between the two planes. The acute angle between them is $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$.

Example 47

Find the angle between the planes with equations

$$2x - 3y = 0 \text{ and } 3x + y - z = 4.$$

Solution

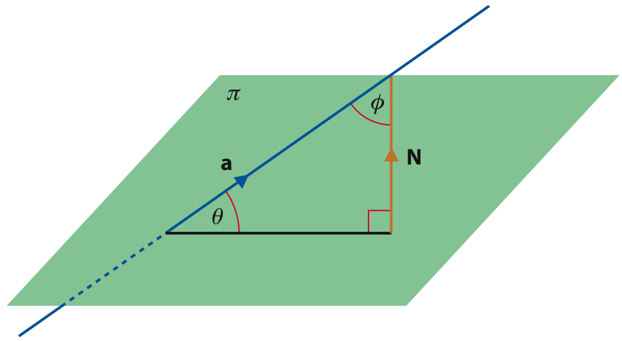
The two normals are $2\mathbf{i} - 3\mathbf{j}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$, and therefore the angle is given by

$$\cos\theta = \frac{(2\mathbf{i} - 3\mathbf{j}) \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{13}\sqrt{11}} = \frac{3}{\sqrt{143}}.$$

So, the angle between the planes is $\cos^{-1}\left(\frac{3}{\sqrt{143}}\right)$.

The angle between a line and a plane

The angle between a line and a plane can be defined as the angle θ formed by the line and its projection on the plane, as shown in the figure below.



Consider the line l with equation $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{a}$ and the plane with equation $\mathbf{r} \cdot \mathbf{N} = D$.

The acute angle ϕ between $\mathbf{N}$, the normal to the plane, and the line l can be found by using the law of cosines:

$$\cos\phi = \left|\frac{\mathbf{a} \cdot \mathbf{N}}{|\mathbf{a}|\,|\mathbf{N}|}\right|$$

If θ is the acute angle between the line and the plane then

$$\theta = \frac{\pi}{2} - \phi.$$

Therefore, to find the angle between the line and the plane, we

- either find the angle ϕ first and then find its complement, or
- since ϕ and θ are complements, then $\sin\theta = \cos\phi = \left|\frac{\mathbf{a} \cdot \mathbf{N}}{|\mathbf{a}|\,|\mathbf{N}|}\right|$.

For example: to find the angle between the line with equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

and the plane with equation

$$\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3$$

we find that

$$\sin\theta = \frac{(\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k})}{\sqrt{3} \cdot \sqrt{4}} = \frac{4}{3\sqrt{2}}$$

$$\Rightarrow \theta = \sin^{-1}\frac{4}{3\sqrt{2}}.$$

Example 48

Find the angle between the line with equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{2}, \text{ and the plane with equation}$$

$$2x - y - z = 7.$$

Solution

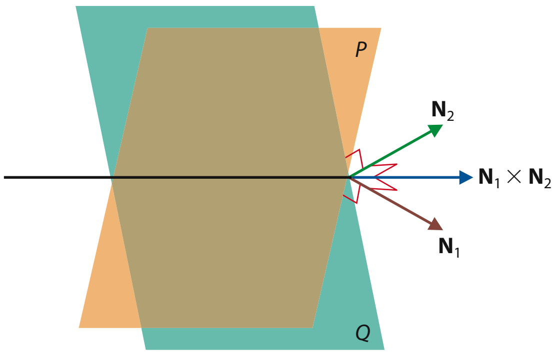
The direction of the line is given by $\mathbf{a} = (2, 3, 2)$ and the normal to the plane by

$\mathbf{N} = (2, -1, -1)$, and the angle is given by

$$\sin\theta = \left|\frac{(2, 3, 2)\cdot(2, -1, -1)}{\sqrt{17}\cdot\sqrt{6}}\right| = \frac{1}{\sqrt{102}}$$

$$\Rightarrow \theta = \sin^{-1}\frac{1}{\sqrt{102}}.$$

Line of intersection of two planes



Unless two planes are parallel they will intersect along a straight line. Consider two planes P and Q that have l as their line of intersection. Also let the planes have the following vector equations:

$$\mathbf{r}\cdot\mathbf{N}_1 = D_1 \text{ and } \mathbf{r}\cdot\mathbf{N}_2 = D_2.$$

Since the line l lies in plane P then $\mathbf{N}_1$, the normal to this plane, must be perpendicular to it. This is also true for $\mathbf{N}_2$. Therefore, the direction of line l is perpendicular to both $\mathbf{N}_1$ and $\mathbf{N}_2$.

To find the line of intersection, we will demonstrate two methods:

1 Use the cross product of $\mathbf{N}_1$ and $\mathbf{N}_2$ as the direction of l and a specific point on the line.

2 Use the fact that all points on l must satisfy the equations of both planes; i.e. we solve a system of equations.

These methods are best demonstrated when we apply them to a particular situation.

Let the planes P and Q have the equations:

$$\mathbf{r}\cdot(\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 6 \text{ and } \mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4.$$

1 To find a vector equation of the line of intersection, we need first to find the cross product of the two normals and then find a point on the line l.

$$(\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$$

To find a point on the line we use the fact that the points on that line must satisfy both equations. So, consider the points on both planes that have the x-coordinate zero; i.e.

$$(0, y, z) \cdot (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 6 \text{ and } (0, y, z) \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4$$

$$\Rightarrow \begin{cases} y - 3z = 6 \\ -y + z = 4 \end{cases} \Rightarrow z = -5 \text{ and } y = -9$$

So, the vector equation of the line is: $\mathbf{r} = (0, -9, -5) + t(2, 7, 3)$.

2 The second method uses a system of equations to find the equation. The equations of the planes in Cartesian form are:

$$x + y - 3z = 6 \text{ and } 2x - y + z = 4.$$

Since this system has to be solved simultaneously, and since there are two equations in three variables, we should consider one of these variables as a parameter and solve for the rest. So,

$$\begin{cases} x + y - 3z = 6 \\ 2x - y + z = 4 \end{cases} \Rightarrow 3x - 2z = 10 \Rightarrow z = -5 + \tfrac{3}{2}x;\ y = -9 + \tfrac{7}{2}x.$$

Therefore, we either consider x to be the parameter or, for convenience purposes, we replace it by another parameter such as the following:

$$x = 2\lambda,\ y = 7\lambda - 9,\ z = 3\lambda - 5$$

This equation is equivalent to the one found in part (1).

3 If the equations of the planes are in parametric form it may not be necessary to convert them into Cartesian form. However, from the example below, you may notice that it may be more straightforward to follow the Cartesian method.

Find the intersection between the two planes

$$P\colon \mathbf{r} = \mathrm{i} + \mathrm{j} + \lambda(2\mathbf{i} - \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k}), \text{ and}$$

$$Q\colon \mathbf{r} = 3\mathbf{i} - \mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k}).$$

A point on P will have the following coordinates: $(1 + 2\lambda + \mu, 1 - \mu, -\lambda + \mu)$, while a point on Q will have the coordinates $(3 + s + t, -s + 2t, -1 + 2s - t)$. For the collection of points on the intersection, we must have the coordinates satisfy both equations, and hence

$$1 + 2\lambda + \mu = 3 + s + t$$

$$1 - \mu = -s + 2t$$

$$-\lambda + \mu = -1 + 2s - t$$

This system of three equations in four unknowns must have an infinite number of solutions if the planes intersect – the set of points that belong to the line of intersection. We can solve this system best, after re-arranging terms, by Gaussian elimination:

$$\begin{pmatrix} 2 & 1 & -1 & -1 & 2 \\ 0 & -1 & 1 & -2 & -1 \\ -1 & 1 & -2 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{9}{2} & \frac{5}{2} \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

The last result expresses λ, μ and s in terms of t. To find the equation of the line we have to substitute these values in either of the two equations above. For example, it is easier to substitute for s in terms of t in the equation for Q. So, from the result above,

$$s = \frac{3}{2} - \frac{5}{2}t$$

and the line will have the vector equation

$$\begin{aligned}\mathbf{r} &= 3\mathbf{i} - \mathbf{k} + \left(\tfrac{3}{2} - \tfrac{5}{2}t\right)(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\\ &= \left(\tfrac{9}{2}\mathbf{i} - \tfrac{3}{2}\mathbf{j} + 2\mathbf{k}\right) + t\left(-\tfrac{3}{2}\mathbf{i} + \tfrac{9}{2}\mathbf{j} - 6\mathbf{k}\right).\end{aligned}$$

If $\lambda = \frac{1}{2} + \frac{3}{2}t$ and $\mu = \frac{5}{2} - \frac{9}{2}t$ are substituted into the equation for P we will get the same result. (Try it!)

As we notice from the above discussion, the process is long and complex, even with many steps that are 'hidden' to save space. Alternatively, the Cartesian solution may be more efficient.

P: point $(1, 1, 0)$ is on the plane, and $(2, 0, -1) \times (1, -1, 1) = (-1, -3, -2)$ is perpendicular to the plane, so the Cartesian equation is

$$(x - 1) + 3(y - 1) + 2z = 0, \text{ or } x + 3y + 2z = 4.$$

Q: point $(3, 0, -1)$ is on the plane, and $(1, -1, 2) \times (1, 2, -1) = (-3, 3, 3)$ is perpendicular to the plane, so the Cartesian equation is

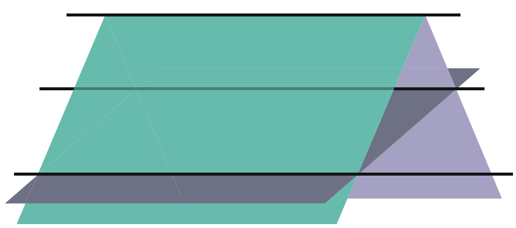
$$-3(x - 3) + 3y + 3(z + 1) = 0, \text{ or } x - y - z = 4.$$

The intersection between the planes is the result of solving the following system:

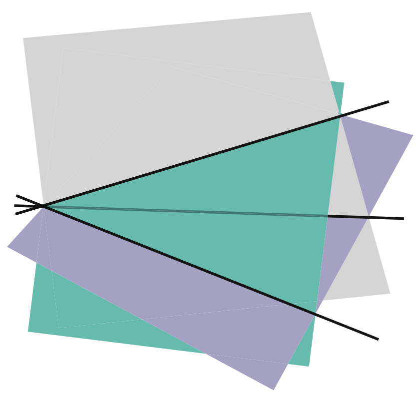
$$\begin{cases} x + 3y + 2z = 4 \\ x - y - z = 4 \end{cases} \Rightarrow x = 4 + m, y = -3m, z = 4m$$

This result compares to the previous one and appears to be more elegant!

Note:
Three planes can intersect in three lines as shown here.



The three lines of intersection are parallel. Hence, the system of equations they represent is inconsistent.



If the lines of intersection are not parallel, then the three planes meet at one point as shown. This system is consistent with a unique solution. The three planes can also all pass through one straight line. In that case, the system is consistent with an infinite number of solutions.

Exercise 14.5

1 Which of the points $A(3, -2, -1)$, $B(2, 1, -1)$, $C(1, 4, 0)$ lie in the plane

$$3x + 2y - 3z = 11?$$

2 Which of the points $A(3, 2, -3)$, $B(2, 1, -2)$, $C(1, 4, 0)$ lie in the plane

$$(\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -6?$$

In questions 3–16, find an equation for the plane satisfying the given conditions. Give two forms for each equation out of the three forms: Cartesian, vector or parametric.

3 Contains the point $(3, -2, 4)$ and perpendicular to $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

4 Contains the point $(-3, 2, 1)$ and perpendicular to $2\mathbf{i} + 3\mathbf{k}$

5 Contains the point $(0, 3, 1)$ and perpendicular to $3\mathbf{k}$

6 Contains the point $(3, -2, 4)$ and parallel to the plane $5x + y - 2z = 7$

7 Contains the point $(3, 0, 1)$ and parallel to the plane $y - 2z = 11$

8 Contains the point $(3, -2, 4)$ and the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

9 Contains the lines $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$

10 Contains the point $(1, -3, 2)$ and the line $x = 2t, y = 2 + t, z = -1 + 3t$

11 Contains the point $M(p, q, r)$ and perpendicular to the vector $\overrightarrow{OM}$

12 Contains the three points $(1, 2, 2)$, $(3, -1, 0)$ and $(7, 0, -2)$

13 Contains the three points $(2, -2, -2)$, $(3, -1, 3)$ and $(0, 1, 5)$

14 Contains the point $(1, -2, 3)$ and the line $x - 2 = y + 1 = \frac{z - 5}{3}$

15 Contains the two parallel lines

$$\mathbf{r} = (1, -1, 5) + t(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) \text{ and } \mathbf{r} = (-3, 4, 0) + t(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$$

16 Contains the point $(1, 1, 0)$ and parallel to the two lines

$$x = 2 + t, y = -1, z = t \text{ and } x = s, y = 2 - s, z = -1 + s$$

In questions 17–22, find the acute angle between the given lines or planes.

17 $3x + 4y - z = 1$ and $x - 2y = 3$

18 $4x - 7y + z = 3$ and $3x + 2y + 2z = 17$

19 $x = 4$ and $x + z = 4$

20 $x - 2y + 2z = 3$ and $x = 2 - 6t, y = 4 + 3t, z = 1 - 2t$

21 $(3\mathbf{i} - \mathbf{k}) \cdot (x, y, z) = 4$ and $\mathbf{r} = (2\mathbf{j} + 3\mathbf{k}) + \lambda(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

22 $x + y + z = 7$ and $z = 0$

In questions 23–26 find the points of intersection of the given line and plane.

23 $\mathbf{r} = 5\mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ and $(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -35$

24 $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$ and $4x - 2y + 3z - 30 = 0$

25 $x - 3 = \dfrac{y - 4}{5} = \dfrac{z - 6}{3}$ and $(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 5$

26 $x = t, y = 4 - \frac{1}{3}t, z = 5 - \frac{5}{3}t$ and $3x - y + 2z = 6$

In questions 27–30, find the line of intersection between the given planes.

27 $x = 10$ and $x + y + z = 3$

28 $2x - y + z = 5$ and $x + y - z = 4$

29 $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$ and $x - y - 2z = 5$

30 $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$ and $3x - y - z = 3$

31 Find a plane through $A(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$.

32 Find a plane through the points $A(1, 2, 3)$ and $B(3, 2, 1)$ and perpendicular to the plane $(4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 7$.

33 What point on the line through (1, 2, 5) and (3, 1, 1) is closest to the point $(2, -1, 5)$?

34 Find an equation of the plane that contains the line

$$\mathbf{r} = (-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

and is parallel to $\dfrac{x - 1}{3} = \dfrac{y + 2}{2} = \dfrac{z - 4}{4}$.

35 Find an equation of the plane that contains the line

$$x = 1 + 2t, y = 1 + 2t, z = 2 - t$$

and is parallel to $x - 1 = \dfrac{y - 2}{2} = z - 7$.

36 Show that the equation

$$\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$$

is the equation of a plane.

37 Find the equation of a plane that contains the point $(4, -3, -1)$ and is perpendicular to the planes $2x - 3y + 4z = 5$ and $4x - 3z = 5$.

38 Find the equation of a plane that contains the point (2, 3, 0) and is perpendicular to the plane $2x - 3y + 4z = 5$ and parallel to the line $\mathbf{r}(t) = (t - 3)\mathbf{i} + (4 - 2t)\mathbf{j} + (1 + t)\mathbf{k}$.

Review exercise

1 Briefly discuss how you test if two vectors are parallel or perpendicular. Use more than one approach.

2 Briefly discuss how you test if three vectors are coplanar.

3 Briefly discuss how you find the angle between two vectors.

4 Briefly discuss how you find the equation of a line.

5 Briefly discuss how you find the equation of a plane.

6 Briefly discuss how you find the angle between two planes.

7 Briefly discuss how you find the angle between a line and a plane.

Find vector, parametric and Cartesian equations for the lines in questions 8–15.

8 The line through the point $(4, -3, 0)$ parallel to the vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

9 The line through $A(-1, 1, 4)$ and $B(4, 6, -1)$.

10 The line through $A(2, 3, 0)$ and $B(0, 1, 2)$.

11 The line through the origin parallel to the vector $\mathbf{j} + 2\mathbf{k}$.

12 The line through the point $(4, -1, 2)$ parallel to the line $x = 2 + 3t, y = 3 - t, z = 4t$.

13 The line through $(1, 2, 2)$ parallel to the y-axis.

14 The line through $(3, 5, 6)$ perpendicular to the plane $4x - 8y + 7z = 23$.

15 The line through $(3, 5, 6)$ perpendicular to the vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$.

Find equations for the planes in questions 16–20.

16 The plane through $A(1, 3, 0)$ normal to the vector $\mathbf{n} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$.

17 The plane through $P(2, 0, 5)$ parallel to the plane $2x + 3y - z = 11$.

18 The plane through $(2, 1, -1)$, $(3, -1, 0)$ and $(1, -1, 2)$.

19 The plane through $B(2, 5, 4)$ perpendicular to the line $x = 2 + 3t, y = 3 - t, z = 4t$.

20 The plane through $P(2, -1, 2)$ perpendicular to the vector from the origin to P.

21 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $x = 2 + s$, $y = 4 + 2s, z = -4s - 1$.

22 Find the equation of the plane determined by the straight lines in the previous question.

23 Find the point of intersection of the lines $x = 2 - y = z - 1$ and $\dfrac{x-2}{2} = y - 3 = \dfrac{z-6}{5}$.

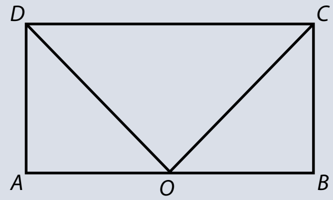
24 Find the equation of the plane determined by the straight lines in the previous question.

25 Find the equation of the plane through $M(1, -2, 1)$ and perpendicular to the vector from the origin to M.

Practice questions

1 $ABCD$ is a rectangle and O is the midpoint of $[AB]$. Express each of the following vectors in terms of $\overrightarrow{OC}$ and $\overrightarrow{OD}$

a) $\overrightarrow{CD}$ **b)** $\overrightarrow{OA}$ **c)** $\overrightarrow{AD}$



2 The vectors **i** and **j** are unit vectors along the x-axis and y-axis respectively. The vectors $\mathbf{u} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$ are given.

a) Find $\mathbf{u} + 2\mathbf{v}$ in terms of **i** and **j**.

A vector **w** has the same direction as $\mathbf{u} + 2\mathbf{v}$, and has a magnitude of 26.

b) Find **w** in terms of **i** and **j**.

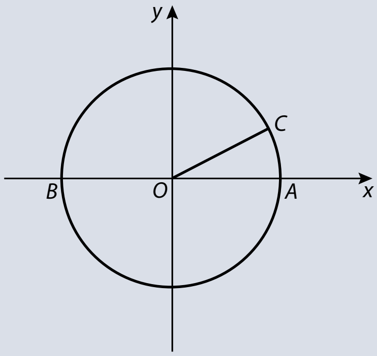
3 The circle shown has centre O and radius 6. $\overrightarrow{OA}$ is the vector $\begin{pmatrix}6\\0\end{pmatrix}$, $\overrightarrow{OB}$ is the vector $\begin{pmatrix}-6\\0\end{pmatrix}$ and $\overrightarrow{OC}$ is the vector $\begin{pmatrix}5\\\sqrt{11}\end{pmatrix}$.

a) Verify that A, B and C lie on the circle.

b) Find the vector $\overrightarrow{AC}$.

c) Using an appropriate scalar product, or otherwise, find the cosine of angle OAC.

d) Find the area of triangle ABC, giving your answer in the form $a\sqrt{11}$, where $a \in \mathbb{N}$.



4 The quadrilateral $OABC$ has vertices with coordinates $O(0, 0)$, $A(5, 1)$, $B(10, 5)$ and $C(2, 7)$.

a) Find the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

b) Find the angle between the diagonals of the quadrilateral $OABC$.

5 The vectors **u** and **v** are given by $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$. Find scalars a and b such that $a(\mathbf{u} + \mathbf{v}) = 8\mathbf{i} + (b - 2)\mathbf{j}$.

6 Find a vector equation of the line passing through $(-1, 4)$ and $(3, -1)$. Give your answer in the form $\mathbf{r} = \mathbf{p} + t\mathbf{d}$, where $t \in \mathbb{R}$.

7 In this question, the vector $\begin{pmatrix}1\\0\end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix}0\\1\end{pmatrix}$ a displacement due north. Distances are in kilometres and time in hours.

Two crews of workers are laying an underground cable in a north-south direction across a desert. At 06:00 each crew sets out from their base camp, which is situated at the origin (0, 0). One crew is in a Toyundai vehicle and the other in a Chryssault vehicle.

The Toyundai has velocity vector $\begin{pmatrix}18\\24\end{pmatrix}$ and the Chryssault has velocity vector $\begin{pmatrix}36\\-16\end{pmatrix}$.

a) Find the speed of each vehicle.

b) **(i)** Find the position vectors of each vehicle at 06:30.

(ii) Hence, or otherwise, find the distance between the vehicles at 06:30.

c) At this time (06:30) the Chryssault stops and its crew begin their day's work, laying cable in a northerly direction. The Toyundai continues travelling in the same direction, at the same speed, until it is exactly north of the Chryssault. The Toyundai crew then begin their day's work, laying cable in a southerly direction. At what time does the Toyundai crew begin laying cable?

d) Each crew lays an average of 800 m of cable in an hour. If they work non-stop until their lunch break at 11:30, what is the distance between them at this time?

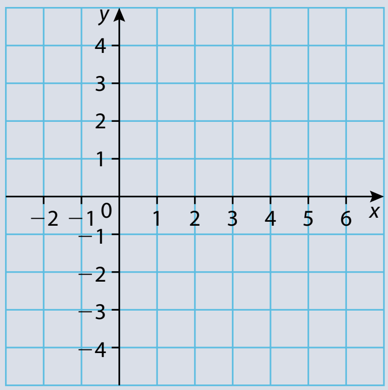
e) How long would the Toyundai take to return to base camp from its lunchtime position, assuming it travelled in a straight line and with the same average speed as on the morning journey? (Give your answer to the nearest minute.)

8 The line L passes through the origin and is parallel to the vector $2\mathbf{i} + 3\mathbf{j}$.
Write down a vector equation for L.

9 The triangle ABC is defined by the following information:

$\overrightarrow{OA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{AC}$ is parallel to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

a) On the grid below, draw an accurate diagram of triangle ABC.



b) Write down the vector $\overrightarrow{OC}$.

10 In this question, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement due north.
The point (0, 0) is the position of Shipple Airport. The position vector $\mathbf{r}_1$ of an aircraft, *Air One*, is given by

$$\mathbf{r}_1 = \begin{pmatrix} 16 \\ 12 \end{pmatrix} + t\begin{pmatrix} 12 \\ -5 \end{pmatrix},$$

where t is the time in minutes since 12:00.

a) Show that *Air One*

(i) is 20 km from Shipple Airport at 12:00

(ii) has a speed of 13 km/min.

b) Show that a Cartesian equation of the path of *Air One* is:

$$5x + 12y = 224.$$

The position vector $\mathbf{r}_2$ of an aircraft, *Air Two*, is given by

$$\mathbf{r}_2 = \begin{pmatrix} 23 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2.5 \\ 6 \end{pmatrix},$$

where t is the time in minutes since 12:00.

c) Find the angle between the paths of the two aircraft.

d) **(i)** Find a Cartesian equation for the path of *Air Two*.

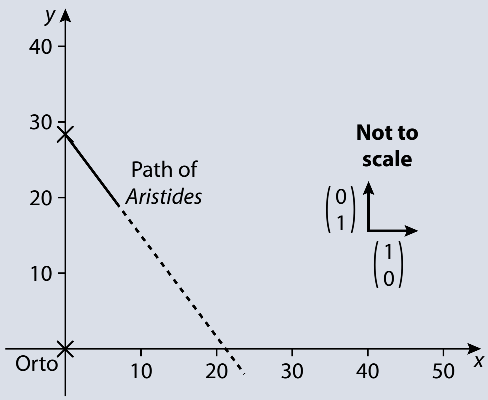
(ii) Hence, find the coordinates of the point where the two paths cross.

e) Given that the two aircraft are flying at the same height, show that they do not collide.

11 Find the size of the angle between the two vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$. Give your answer to the nearest degree.

12 A line passes through the point (4, -1) and its direction is perpendicular to the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find the equation of the line in the form $ax + by = p$, where a, b and p are integers to be determined.

13 In this question, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement due north. Distances are in kilometres.
The diagram shows the path of the oil tanker *Aristides* relative to the port of Orto, which is situated at the point (0, 0).



The position of the *Aristides* is given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + t\begin{pmatrix} 6 \\ -8 \end{pmatrix}$$

at a time t hours after 12:00.

a) Find the position of the *Aristides* at 13:00.

b) Find

(i) the velocity vector

(ii) the speed of the *Aristides*.

c) Find a Cartesian equation for the path of the *Aristides* in the form $ax + by = g$.

Another ship, the cargo vessel *Boadicea*, is stationary, with position vector $\begin{pmatrix} 18 \\ 4 \end{pmatrix}$.

d) Show that the two ships will collide, and find the time of collision.

To avoid collision, the *Boadicea* starts to move at 13:00 with velocity vector $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

e) Show that the position of the *Boadicea* for $t \geqslant 1$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}.$$

f) Find how far apart the two ships are at 15:00.

14 Find the angle between the following vectors **a** and **b**, giving your answer to the nearest degree.

$\mathbf{a} = -4\mathbf{i} - 2\mathbf{j}$
$\mathbf{b} = \mathbf{i} - 7\mathbf{j}$

15 In this question, a unit vector represents a displacement of 1 metre.
A miniature car moves in a straight line, starting at the point (2, 0). After t seconds, its position, (x, y), is given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0.7 \\ 1 \end{pmatrix}.$$

a) How far from the point (0, 0) is the car after 2 seconds?
b) Find the speed of the car.
c) Obtain the equation of the car's path in the form $ax + by = c$.
Another miniature vehicle, a motorcycle, starts at the point (0, 2) and travels in a straight line with constant speed. The equation of its path is

$$y = 0.6x + 2, \quad x \geqslant 0.$$

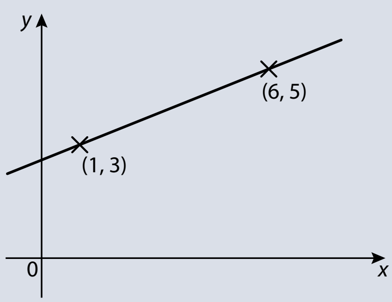
Eventually, the two miniature vehicles collide.
d) Find the coordinates of the collision point.
e) If the motorcycle left point (0, 2) at the same moment the car left point (2 , 0), find the speed of the motorcycle.

16 The diagram right shows a line passing through the points (1, 3) and (6, 5).

Find a vector equation for the line, giving your answer in the form

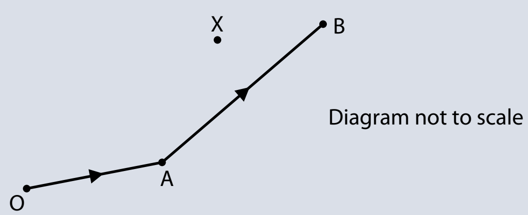
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + t\begin{pmatrix} c \\ d \end{pmatrix},$$

where t is any real number.



17 The vectors $\begin{pmatrix} 2x \\ x - 5 \end{pmatrix}$ and $\begin{pmatrix} x + 1 \\ 5 \end{pmatrix}$ are perpendicular for two values of x.
a) Write down the quadratic equation which the two values of x must satisfy.
b) Find the two values of x.

18 The diagram below shows the positions of towns O, A, B and X.



Town A is 240 km east and 70 km north of O.
Town B is 480 km east and 250 km north of O.
Town X is 339 km east and 238 km north of O.

A plane flies at a constant speed of 300 km h^{-1} from O towards A.

a) **(i)** Show that a unit vector in the direction of $\overrightarrow{OA}$ is $\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$.

(ii) Write down the velocity vector for the plane in the form $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

(iii) How long does it take for the plane to reach A?

At A the plane changes direction so it now flies towards B. The angle between the original direction and the new direction is θ, as shown in the following diagram. This diagram also shows the point Y, between A and B, where the plane comes closest to X.

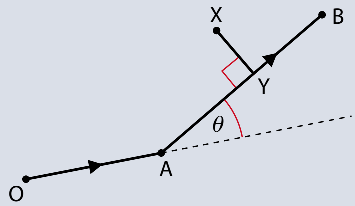


Diagram not to scale

b) Use the scalar product of two vectors to find the value of θ in degrees.

c) **(i)** Write down the vector $\overrightarrow{AX}$.

(ii) Show that the vector $\mathbf{n} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ is perpendicular to $\overrightarrow{AB}$.

(iii) By finding the projection of $\overrightarrow{AX}$ in the direction of $\mathbf{n}$, calculate the distance XY.

d) How far is the plane from A when it reaches Y?

19 A vector equation of a line is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $t \in \mathbb{R}$.
Find the equation of this line in the form $ax + by = c$, where a, b and $c \in \mathbb{Z}$.

20 Three of the coordinates of the parallelogram $STUV$ are $S(-2, -2)$, $T(7, 7)$ and $U(5, 15)$.

a) Find the vector $\overrightarrow{ST}$ and hence the coordinates of V.

b) Find a vector equation of the line (UV) in the form $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$, where $\lambda \in \mathbb{R}$.

c) Show that the point E with position vector $\begin{pmatrix} 1 \\ 11 \end{pmatrix}$ is on the line (UV), and find the value of λ for this point.

The point W has position vector $\begin{pmatrix} a \\ 17 \end{pmatrix}$, $a \in \mathbb{R}$.

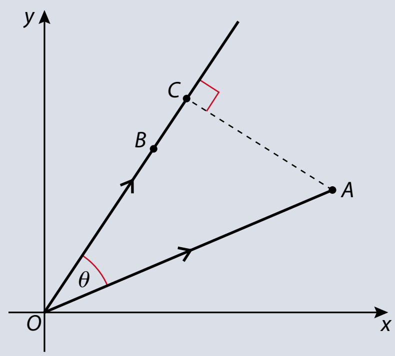
d) **(i)** If $\overrightarrow{EW} = 2\sqrt{13}$, show that one value of a is -3 and find the other possible value of a.

(ii) For $a = -3$, calculate the angle between $\overrightarrow{EW}$ and $\overrightarrow{ET}$.

21 Calculate the acute angle between the lines with equations

$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

22 The diagram on the right shows the point O with coordinates $(0, 0)$, the point A with position vector $\mathbf{a} = 12\mathbf{i} + 5\mathbf{j}$, and the point B with position vector $\mathbf{b} = 6\mathbf{i} + 8\mathbf{j}$. The angle between (OA) and (OB) is θ.



Find

a) $|\mathbf{a}|$

b) a unit vector in the direction of $\mathbf{b}$

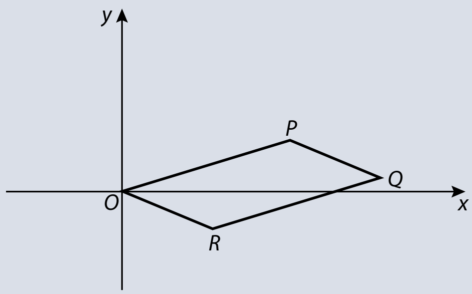
c) the exact value of $\cos\theta$ in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$.

23 The vector equations of two lines are given below.

$$\mathbf{r}_1 = \begin{pmatrix}5\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\end{pmatrix},\ \mathbf{r}_2 = \begin{pmatrix}-2\\2\end{pmatrix} + t\begin{pmatrix}4\\1\end{pmatrix}$$

The lines intersect at the point P. Find the position vector of P.

24 The diagram shows a parallelogram $OPQR$ in which $\overrightarrow{OP} = \begin{pmatrix}7\\3\end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix}10\\1\end{pmatrix}$.



a) Find the vector $\overrightarrow{OR}$.

b) Use the scalar product of two vectors to show that $\cos O\hat{P}Q = -\frac{15}{\sqrt{754}}$.

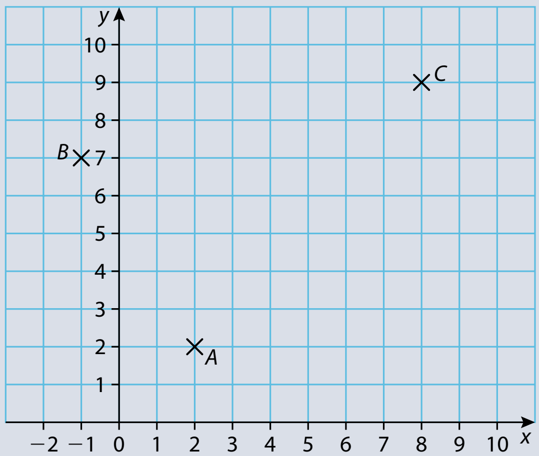
c) **(i)** Explain why $\cos P\hat{Q}R = -\cos O\hat{P}Q$.

(ii) Hence, show that $\sin P\hat{Q}R = \frac{23}{\sqrt{754}}$.

(iii) Calculate the area of the parallelogram $OPQR$, giving your answer as an integer.

25 The diagram shows points A, B and C, which are three vertices of a parallelogram $ABCD$.

The point A has position vector $\begin{pmatrix}2\\2\end{pmatrix}$.



a) Write down the position vector of B and C.

b) The position vector of point D is $\begin{pmatrix}d\\4\end{pmatrix}$. Find d.

c) Find $\overrightarrow{BD}$.

The line L passes through B and D.

d) **(i)** Write down a vector equation of L in the form $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}m\\n\end{pmatrix}$.

(ii) Find the value of t at point B.

e) Let P be the point (7, 5). By finding the value of t at P, show that P lies on the line L.

f) Show that $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{BD}$.

26 The points A and B have the position vectors $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ respectively.

a) **(i)** Find the vector $\overrightarrow{AB}$.

(ii) Find $|\overrightarrow{AB}|$.

The point D has position vector $\begin{pmatrix} d \\ 23 \end{pmatrix}$.

b) Find the vector $\overrightarrow{AD}$ in terms of d.

The angle $B\hat{A}D$ is 90°.

c) **(i)** Show that $d = 7$.

(ii) Write down the position vector of the point D.

The quadrilateral $ABCD$ is a rectangle.

d) Find the position vector of the point C.

e) Find the area of the rectangle $ABCD$.

27 Points A, B and C have position vectors $4\mathbf{i} + 2\mathbf{j}$, $\mathbf{i} - 3\mathbf{j}$ and $-5\mathbf{i} - 5\mathbf{j}$, respectively. Let D be a point on the x-axis such that $ABCD$ forms a parallelogram.

a) **(i)** Find $\overrightarrow{BC}$.

(ii) Find the position vector of D.

b) Find the angle between $\overrightarrow{BD}$ and $\overrightarrow{AC}$.

The line L_1 passes through A and is parallel to $\mathbf{i} + 4\mathbf{j}$. The line L_2 passes through B and is parallel to $2\mathbf{i} + 7\mathbf{j}$. A vector equation of L_1 is $\mathbf{r} = (4\mathbf{i} + 2\mathbf{j}) + s(\mathbf{i} + 4\mathbf{j})$.

c) Write down a vector equation of L_2 in the form $\mathbf{r} = \mathbf{b} + t\mathbf{q}$.

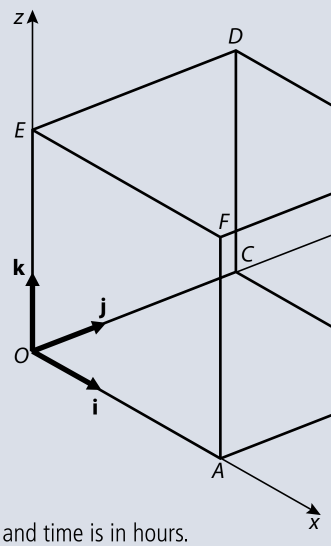
d) The lines L_1 and L_2 intersect at the point P. Find the position vector of P.

28 The diagram shows a cube, $OABCDEFG$, where the length of each edge is 5 cm. Express the following vectors in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

a) $\overrightarrow{OG}$

b) $\overrightarrow{BD}$

c) $\overrightarrow{EB}$



29 In this question, distance is in kilometres and time is in hours.

A balloon is moving at a constant height with a speed of 18 km h^{-1}, in the direction of the vector $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$.

At time $t = 0$, the balloon is at point B with coordinates (0, 0, 5).

a) Show that the position vector $\mathbf{b}$ of the balloon at time t is given by

$$\mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \frac{18t}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}.$$

At time $t = 0$, a helicopter goes to deliver a message to the balloon. The position vector **h** of the helicopter at time t is given by

$$\mathbf{h} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 49 \\ 32 \\ 0 \end{pmatrix} + t\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix}.$$

b) **(i)** Write down the coordinates of the starting position of the helicopter.

(ii) Find the speed of the helicopter.

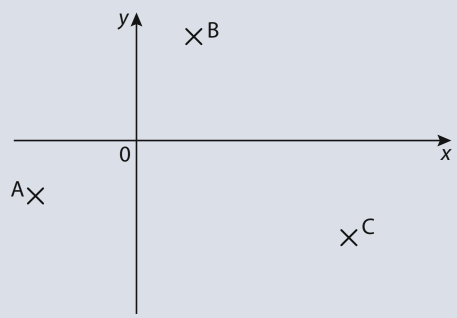
c) The helicopter reaches the balloon at point R.

(i) Find the time the helicopter takes to reach the balloon.

(ii) Find the coordinates of R.

30 In this question, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement of 1 km north.

The diagram below shows the positions of towns A, B and C in relation to an airport O, which is at the point (0, 0). An aircraft flies over the three towns at a constant speed of 250 km h^{-1}.



Town A is 600 km west and 200 km south of the airport.
Town B is 200 km east and 400 km north of the airport.
Town C is 1200 km east and 350 km south of the airport.

a) **(i)** Find $\overrightarrow{AB}$.

(ii) Show that the vector of length one unit in the direction of $\overrightarrow{AB}$ is $\begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$.

An aircraft flies over town A at 12:00, heading towards town B at 250 km h^{-1}.

Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be the velocity vector of the aircraft. Let t be the number of hours in flight after 12:00.

The position of the aircraft can be given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -600 \\ -200 \end{pmatrix} + t\begin{pmatrix} p \\ q \end{pmatrix}.$$

b) **(i)** Show that the velocity vector is $\begin{pmatrix} 200 \\ 150 \end{pmatrix}$.

(ii) Find the position of the aircraft at 13:00.

(iii) At what time is the aircraft flying over town B?

Over town B the aircraft changes direction so it now flies towards town C. It takes five hours to travel the 1250 km between B and C. Over town A the pilot noted that she had 17 000 litres of fuel left. The aircraft uses 1800 litres of fuel per hour when travelling at 250 km h^{-1}. When the fuel gets below 1000 litres a warning light comes on.

c) How far from town C will the aircraft be when the warning light comes on?

31 The coordinates of the points P, Q, R and S are $(4, 1, -1)$, $(3, 3, 5)$, $(1, 0, 2c)$ and $(1, 1, 2)$, respectively.

a) Find the value of c so that the vectors $\overrightarrow{OR}$ and $\overrightarrow{PR}$ are orthogonal.
For the remainder of the question, use the value of c found in part **a)** for the coordinate of the point R.

b) Evaluate $\overrightarrow{PS} \times \overrightarrow{PR}$.

c) Find an equation of the line l which passes through the point Q and is parallel to the vector PR.

d) Find an equation of the plane π which contains the line l and passes through the point S.

e) Find the shortest distance between the point P and the plane π.

32 Consider the points $A(1, 2, 1)$, $B(0, -1, 2)$, $C(1, 0, 2)$ and $D(2, -1, -6)$.

a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

b) Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$.

c) Hence, or otherwise, find the area of triangle ABC.

d) Find the equation of the plane P containing the points A, B and C.

e) Find a set of parametric equations for the line through the point D and perpendicular to the plane P.

f) Find the distance from the point D to the plane P.

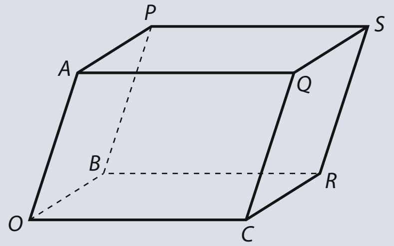
g) Find a unit vector which is perpendicular to the plane P.

h) The point E is a reflection of D in the plane P. Find the coordinates of E.

33 a) If $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, show that $\mathbf{u} \times \mathbf{v} = 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$.

b) Let $\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v}$ where λ and μ are scalars. Show that $\mathbf{w}$ is perpendicular to the line of intersection of the planes $x + 2y + 3z = 5$ and $2x - y + 2z = 7$ for all values of λ and μ.

34 Three points A, B and C have coordinates $(2, 1, -2)$, $(2, -1, -1)$ and $(1, 2, 2)$ respectively. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, where O is the origin, form three concurrent edges of a parallelepiped $OAPBCQSR$ as shown in the following diagram.



a) Find the coordinates of P, Q, R and S.

b) Find an equation for the plane $OAPB$.

c) Calculate the volume, V, of the parallelepiped given that $V = \overrightarrow{OA} \times \overrightarrow{OB} \cdot \overrightarrow{OC}$.

35 The triangle ABC has vertices at the points $A(-l, 2, 3)$, $B(-l, 3, 5)$ and $C(0, -1, 1)$.

a) Find the size of the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence, or otherwise, find the area of triangle ABC.

Let l_1 be the line parallel to $\overrightarrow{AB}$ which passes through $D(2, -1, 0)$ and l_2 be the line parallel to $\overrightarrow{AC}$ which passes through $E(-l, 1, 1)$.

c) (i) Find the equations of the lines l_1 and l_2.

(ii) Hence, show that l_1 and l_2 do not intersect.

d) Find the shortest distance between l_1 and l_2.

36 **a)** Solve the following system of linear equations:

$$x + 3y - 2z = -6$$
$$2x + y + 3z = 7$$
$$3x - y + z = 6$$

b) Find the vector $\mathbf{v} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$.

c) If $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{u} = m\mathbf{a} + n\mathbf{b}$ where m, n are scalars, and $\mathbf{u} \neq 0$, show that $\mathbf{v}$ is perpendicular to $\mathbf{u}$ for all m and n.

d) The line l lies in the plane $3x - y + z = 6$, passes through the point $(1, -1, 2)$ and is perpendicular to $\mathbf{v}$. Find the equation of l.

37 The points A, B, C, D have the following coordinates: $A(1, 3, 1)$, $B(1, 2, 4)$, $C(2, 3, 6)$, $D(5, -2, 1)$.

a) **(i)** Evaluate the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$, giving your answer in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

(ii) Find the area of the triangle ABC.

The plane containing the points A, B, C is denoted by Π and the line passing through D perpendicular to Π is denoted by L. The point of intersection of L and Π is denoted by P.

b) **(i)** Find the Cartesian equation of Π.

(ii) Find the Cartesian equation of L.

c) Determine the coordinates of P.

d) Find the perpendicular distance of D from Π.

38 The point $A(2, 5, -1)$ is on the line L, which is perpendicular to the plane with equation $x + y + z - 1 = 0$.

a) Find the Cartesian equation of the line L.

b) Find the point of intersection of the line L and the plane.

c) The point A is reflected in the plane. Find the coordinates of the image of A.

d) Calculate the distance from the point $B(2, 0, 6)$ to the line L.

39 **a)** The point $P(1, 2, 11)$ lies in the plane π_1. The vector $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ is perpendicular to π_1. Find the Cartesian equation of π_1.

b) The plane π_2 has equation $x + 3y - z = -4$.

(i) Show that the point P also lies in the plane π_2.

(ii) Find a vector equation of the line of intersection of π_1 and π_2.

c) Find the acute angle between π_1 and π_2.

40 A line l_1 has equation $\dfrac{x+2}{3} = \dfrac{y}{1} = \dfrac{z-9}{-2}$.

a) Let M be a point on l_1 with parameter μ. Express the coordinates of M in terms of μ.

b) The line l_2 is parallel to l_1 and passes through $P(4, 0, -3)$.

(i) Write down an equation for l_2.

(ii) Express $\overrightarrow{PM}$ in terms of μ.

c) The vector $\overrightarrow{PM}$ is perpendicular to l_1.

(i) Find the value of μ.

(ii) Find the distance between l_1 and l_2.

d) The plane π_1 contains l_1 and l_2. Find an equation for π_1, giving your answer in the form $Ax + By + Cz = D$.

e) The plane π_2 has equation $x - 5y - z = -11$. Verify that l_1 is the line of intersection of the planes π_1 and π_2.

41 a) Show that the lines $\dfrac{x-2}{1} = \dfrac{y-2}{3} = \dfrac{z-3}{1}$ and $\dfrac{x-2}{1} = \dfrac{y-3}{4} = \dfrac{z-4}{2}$ intersect and find the coordinates of P, the point of intersection.

b) Find the Cartesian equation of the plane π that contains the two lines.

c) The point $Q(3, 4, 3)$ lies on π. The line L passes through the midpoint of $[PQ]$. Point S is on L such that $|\overrightarrow{PS}| = |\overrightarrow{QS}| = 3$, and the triangle PQS is normal to the plane π. Given that there are two possible positions for S, find their coordinates.

42 a) The plane π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane π_2 has the equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(i) For points which lie in π_1 and π_2, show that $\lambda = \mu$.

(ii) Hence, or otherwise, find a vector equation of the line of intersection of π_1 and π_2.

b) The plane π_3 contains the line $\dfrac{2-x}{3} = \dfrac{y}{-4} = z + 1$ and is perpendicular to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Find the Cartesian equation of π_3.

c) Find the intersection of π_1, π_2 and π_3.

15 Differential Calculus II: Further Techniques and Applications

Assessment statements

6.2 Derivative of x^n ($n \in \mathbb{Q}$), sin x, cos x, tan x, e^x and ln x.
Differentiation of a sum and a real multiple of a function.
The chain rule for composite functions.
Implicit differentiation.
Related rates of change.
The product and quotient rules.
Derivatives of sec$x$, csc$x$, cot$x$, a^x, $\log_a x$, arcsin$x$, arccos$x$ and arctan$x$.

6.3 Optimization problems.

Introduction

The primary purpose of the earlier chapter on calculus, Chapter 13, was to establish some fundamental concepts and techniques of differential calculus. Chapter 13 also introduced some applications involving the differentiation of functions: finding maxima and minima of a function; kinematic problems involving displacement, velocity and acceleration; and finding equations of tangents and normals. The focus of this chapter is to expand our set of differentiation rules and techniques and to deepen and extend the applications introduced in Chapter 13 – particularly using methods of finding extrema in the context of finding an 'optimum' solution to a problem and solving problems involving more than one rate of change. We start by investigating the derivatives of some important functions.

It is not an exaggeration to consider Isaac Newton (1642–1727) the most influential person in the development of modern science and mathematics. Newton was educated at Cambridge University and later was a professor of mathematics there. When Newton entered Cambridge in 1661, he did not know much mathematics but he learned quickly by reading works of Euclid and Descartes and attending lectures of Isaac Barrow, the first professor of mathematics at Cambridge. Cambridge was closed in 1665 and 1666 because of the Great Plague that swept through London and other parts of England. Studying and thinking on his own during these two years (and still not yet 25 years old), Newton discovered that white light can be decomposed into rays of different colours, how to represent functions using infinite series (including the binomial theorem), formulated the law of universal gravitation, and developed differential and integral calculus (several years before its independent discovery by Leibniz – see page 707). These great discoveries were all published much later because of Newton's fear of criticism and controversy. In 1687, Newton published his *Principia Mathematica*, one of the greatest scientific works ever written, in which he presented his version of calculus and applied it to investigate and explain a wide range of physical phenomena.

Newton's intellectual interests were not restricted to physics and mathematics. He left behind many papers dealing with theology and alchemy (attempting to change ordinary metals into gold). He was also a successful Warden of the Royal Mint (overseeing the production of official coins) and held political office, representing Cambridge University in Parliament several times.

15.1 Derivatives of composite functions, products and quotients

Derivatives of composite functions: the chain rule

We know how to differentiate functions such as $f(x) = x^3 + 2x - 3$ and $g(x) = \sqrt{x}$, but how do we differentiate the composite function $f(g(x)) = \sqrt{x^3 + 2x - 3}$? The rule for computing the derivative of the composite of two functions, i.e. the 'function of a function', is called the **chain rule**. Because most functions that we encounter in applications are composites of other functions, it can be argued that the chain rule is the most important, and most widely used, rule of differentiation.

Below are some examples of functions that we can differentiate with the rules that we have learned thus far in Chapter 13, and further examples of functions which are best differentiated with the chain rule.

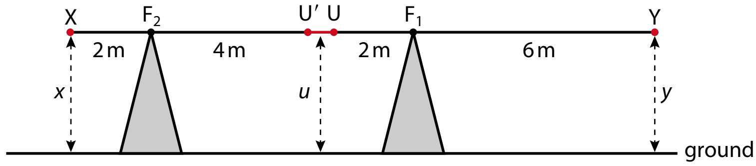
Differentiate *without* the chain rule	Differentiate *with* the chain rule
$y = \cos x$	$y = \cos 2x$
$y = 3x^2 + 5x$	$x = \sqrt{3x^2 + 5x}$
$y = \sin x$	$y = \sin^2 x$
$y = \frac{1}{3x^2}$	$y = \frac{1}{3x^2 + x}$

The chain rule says, in a very basic sense, that given two functions, the derivative of their composite is the product of their derivatives – remembering that a derivative is a rate of change of one quantity (variable) with respect to another quantity (variable). For example, the function $y = 8x + 6 = 2(4x + 3)$ is the composite of the functions $y = 2u$ and $u = 4x + 3$. Note that the function y is in terms of u, and the function u is in terms of x. How are the derivatives of these three functions related?

Clearly, $\frac{dy}{dx} = 8$, $\frac{dy}{du} = 2$ and $\frac{du}{dx} = 4$. Since $8 = 2 \cdot 4$, the derivatives relate such that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. In other words, rates of change multiply.

Again, if we think of derivatives as rates of change, the relationship $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ can be illustrated by a practical example. Consider the pair of levers in Figure 15.1 with lever endpoints U and U′ connected by a segment that can shrink and stretch but always remains horizontal. Hence, points U and U′ are always the same distance u from the ground.

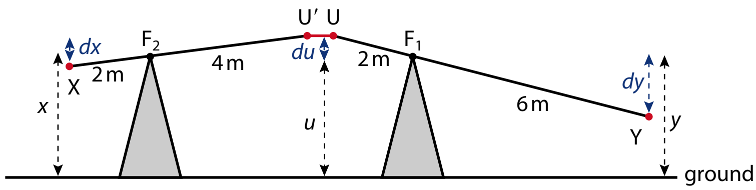
Figure 15.1 Two levers with horizontal connection between U′ and U.



As point Y moves down, points U and U′ move up, and point X moves down but at a rate different from that of Y. Let dy, du and dx represent the change in distance from the ground for the points Y, U and X, respectively. Because $YF_1 = 6$ and $UF_1 = 2$, if point Y moves such that $dy = 3$, then $du = 1$. Since $U'F_2 = 4$ and $XF_2 = 2$, if point U′ moves so that $du = 2$, then $dx = 1$.

Hence, $\frac{dy}{du} = 3$ and $\frac{du}{dx} = 2$.

Figure 15.2 *dx*, *du* and *dy* represent the change in distance from the ground for X, U and Y.



Combining these two results, we can see that for every 6 units that Y's distance changes, X's distance will change 1 unit. That is, $\frac{dy}{dx} = 6$.

Therefore, we can write $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$. In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

Example 1

The polynomial function $y = 16x^4 - 8x^2 + 1 = (4x^2 - 1)^2$ is the composite of $y = u^2$ and $u = 4x^2 - 1$. Use the chain rule to find $\frac{dy}{dx}$, the derivative of y with respect to x.

Solution

$$y = u^2 \;\Rightarrow\; \frac{dy}{du} = 2u$$

$$u = 4x^2 - 1 \;\Rightarrow\; \frac{du}{dx} = 8x$$

Applying the chain rule:
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 8x$$
$$= 2(4x^2 - 1) \cdot 8x$$
$$= 64x^3 - 16x$$

In this particular case, we could have differentiated the function in expanded form by differentiating term-by-term rather than differentiating the factored form by the chain rule. $\frac{dy}{dx} = \frac{d}{dx}(16x^4 - 8x^2 + 1) = 64x^3 - 16x$;

confirming the result above. It is not always easier to differentiate powers of polynomials by expanding and then differentiating term-by-term.
For example, it is far better to find the derivative of $y = (3x + 5)^8$ by the chain rule.

In Section 2.2, we often wrote composite functions using nested function notation. For example, the notation $f(g(x))$ denotes a function composed of functions f and g such that g is the 'inside' function and f is the 'outside' function. For the composite function $y = (4x^2 - 1)^2$ in Example 1, the 'inside' function is $g(x) = 4x^2 - 1$ and the 'outside' function is $f(u) = u^2$. Looking again at the solution for Example 1, we see that we can choose to express and work out the chain rule in function notation rather than Leibniz notation.

For $y = f(g(x)) = (4x^2 - 1)^2$ and $y = f(u) = u^2$, $u = g(x) = 4x^2 - 1$,

Leibniz notation	**Function notation**
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 8x$	$\frac{d}{dx}[f(g(x))] = f'(u) \cdot g'(x) = 2u \cdot 8x$
$= 2(4x^2 - 1) \cdot 8x$	$= f'(g(x)) \cdot g'(x) = 2(4x^2 - 1) \cdot 8x$
$= 64x^3 - 16x$	$= 64x^3 - 16x$

This leads us to formally state the chain rule in two different notations.

The chain rule

If $y = f(u)$ is a function in terms of u and $u = g(x)$ is a function in terms of x, the function $y = f(g(x))$ is differentiated as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{(Leibniz form)}$$

or, equivalently,

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \qquad \text{(function notation form)}$$

Let $\triangle u$ be the change in u corresponding to a change of $\triangle x$ in x, that is, $\triangle u = g(x + \triangle x) - g(x)$. Then the corresponding change in y is $\triangle y = f(u + \triangle u) - f(u)$. It would be tempting to try to **prove the chain rule** by writing $\frac{\triangle y}{\triangle x} = \frac{\triangle y}{\triangle u} \cdot \frac{\triangle u}{\triangle x}$, which is a true statement if none of the denominators are zero. Recognizing that the definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, is equivalent to $\frac{dy}{dx} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$, we could then proceed as follows:

$$\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \to 0} \left(\frac{\triangle y}{\triangle u} \cdot \frac{\triangle u}{\triangle x}\right) = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle u} \cdot \lim_{\triangle x \to 0} \frac{\triangle u}{\triangle x}$$

$$= \lim_{\triangle u \to 0} \frac{\triangle y}{\triangle u} \cdot \lim_{\triangle x \to 0} \frac{\triangle u}{\triangle x} \text{ because if } \triangle x \to 0 \text{ then } \triangle u \to 0$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

This would work as a proof if we knew that $\triangle u$, the change in u, was non-zero – but we do not know this. It is possible that a small change in x could produce no change in u. Nonetheless, this reasoning does provide an intuitive justification relating the chain rule to the limit definition of the derivative. A properly rigorous proof can be constructed with a different approach, but we will not present it here.

The chain rule needs to be applied carefully. Consider the function notation form for the chain rule $\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$. Although it is the product of two derivatives, it is important to point out that the first derivative involves the function f differentiated at $g(x)$ and the second is function g differentiated at x. The chain rule written in Leibniz form, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$, is easily remembered because it appears to be an obvious statement about fractions – but, they are *not* fractions. The expressions $\frac{dy}{dx}$, $\frac{dy}{du}$ and $\frac{du}{dx}$ are derivatives or, more precisely, limits and although du and dx essentially represent very small changes in the variables u and x, we cannot guarantee that they are non-zero.

The function notation form of the chain rule offers a very useful way of saying the rule 'in words', and, thus, a very useful structure for applying it.

f is 'outside' function $\qquad$ g is 'inside' function

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}\cdot g'(x)$$

derivative of 'outside' function with 'inside' function unchanged $\times$ derivative of 'inside' function

The chain rule in words:

$$\begin{pmatrix}\text{derivative of}\\ \text{composite}\end{pmatrix} = \begin{pmatrix}\text{derivative of 'outside' function}\\ \text{with 'inside' function unchanged}\end{pmatrix} \times \begin{pmatrix}\text{derivative of}\\ \text{'inside' function}\end{pmatrix}$$

Although this is taking some liberties with mathematical language, the mathematical interpretation of the phrase "with 'inside' function unchanged" is that the derivative of the 'outside' function f is evaluated at $g(x)$, the 'inside' function.

• **Hint:** The chain rule is our most important rule of differentiation. It is an indispensable tool in differential calculus. Forgetting to apply the chain rule when it needs to be applied, or by applying it improperly, is a common source of errors in calculus computations. It is important to understand it, practise it and master it.

The chain rule acquired its name because we use it to take derivatives of composites of functions by 'chaining' together their derivatives. A function could be the composite of more than two functions. If a function were the composite of three functions, we would take the product of three derivatives 'chained' together. For example, if $y = f(u)$, $u = g(v)$ and $v = h(x)$, the derivative of the function $y = f(g(h(x)))$ is $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$.

Example 2

Differentiate each function by applying the chain rule. Start by 'decomposing' the composite function into the 'outside' function and the 'inside' function.

a) $y = \cos 3x$ b) $y = \sqrt{3x^2 + 5x}$

c) $y = \dfrac{1}{3x^2 + x}$ d) $y = \sin^2 x$

e) $y = \sin x^2$ f) $y = \sqrt[3]{(7 - 5x)^2}$

Solution

a) $y = f(g(x)) = \cos 3x \Rightarrow$ 'outside' function is $f(u) = \cos u$

$\Rightarrow$ 'inside' function is $g(x) = 3x$

In Leibniz form: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = (-\sin u) \cdot 3 = -3\sin(3x)$

Or, alternatively, in function notation form:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = \underbrace{[-\sin(3x)]} \cdot 3 = -3\sin(3x)$$

derivative of 'outside' function with 'inside' function unchanged $\times$ derivative of 'inside' function

b) $y = f(g(x)) = \sqrt{3x^2 + 5x} \Rightarrow$

'outside' function is $f(u) = \sqrt{u} = u^{\frac{1}{2}}$

$f'(u) = \frac{1}{2}u^{-\frac{1}{2}} \Rightarrow$ 'inside' function is $g(x) = 3x^2 + 5x$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = \tfrac{1}{2}(3x^2 + 5x)^{-\frac{1}{2}} \cdot (6x + 5)$$

$$\frac{dy}{dx} = \frac{6x + 5}{2(3x^2 + 5x)^{\frac{1}{2}}} \text{ or } \frac{6x + 5}{2\sqrt{3x^2 + 5x}}$$

c) $y = f(g(x)) = \dfrac{1}{3x^2 + x} \Rightarrow$

'outside' function is $f(u) = \dfrac{1}{u} = u^{-1}$

$f'(u) = -u^{-2} \Rightarrow$ 'inside' function is $g(x) = 3x^2 + x$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = -(3x^2 + x)^{-2} \cdot (6x + 1)$$

$$\frac{dy}{dx} = -\frac{6x + 1}{(3x^2 + x)^2}$$

d) The expression $\sin^2 x$ is an abbreviated way of writing $(\sin x)^2$.

$y = f(g(x)) = \sin^2 x = (\sin x)^2 \Rightarrow$

'outside' function is $f(u) = u^2$

$f'(u) = 2u \Rightarrow$ 'inside' function is $g(x) = \sin x$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 2\sin x \cdot \cos x$$

$$\frac{dy}{dx} = 2\sin x \cos x$$

e) The expression $\sin x^2$ is equivalent to $\sin(x^2)$, and is **not** $(\sin x)^2$.

If $y = f(g(x)) = \sin(x^2)$, then the 'outside' function is $f(u) = \sin u$, and the 'inside' function is $g(x) = x^2$.

$$\text{By the chain rule, } \frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$
$$= \cos(x^2) \cdot 2x$$
$$\frac{dy}{dx} = 2x\cos(x^2)$$

f) First change from radical (surd) form to rational exponent form.

$$y = \sqrt[3]{(7-5x)^2} = (7-5x)^{\frac{2}{3}}$$

$$y = f(g(x)) = (7-5x)^{\frac{2}{3}} \Rightarrow \text{'outside' function } f(u) = u^{\frac{2}{3}}$$
$$\Rightarrow \text{'inside' function } g(x) = 7 - 5x$$

$$\text{By the chain rule, } \frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$
$$= \tfrac{2}{3}(7-5x)^{-\frac{1}{3}} \cdot (-5)$$
$$\frac{dy}{dx} = -\frac{10}{3(7-5x)^{\frac{1}{3}}} \text{ or } -\frac{10}{3(\sqrt[3]{7-5x})}$$

● **Hint:** Aim to write a function in a way that eliminates any confusion regarding the argument of the function. For example, write $\sin(x^2)$ rather than $\sin x^2$; $1 + \ln x$ rather than $\ln x + 1$; $5 + \sqrt{x}$ rather than $\sqrt{x} + 5$; $\ln(4 - x^2)$ rather than $\ln 4 - x^2$.

Example 3

Find the derivative of the function $y = (2x+3)^3$ by:

a) expanding the binomial and differentiating term-by-term

b) the chain rule.

Solution

a) $y = (2x+3)^3 = (2x+3)(2x+3)^2$

$$= (2x+3)(4x^2+12x+9)$$
$$= 8x^3 + 24x^2 + 18x + 12x^2 + 36x + 27$$
$$= 8x^3 + 36x^2 + 54x + 27$$
$$\frac{dy}{dx} = 24x^2 + 72x + 54$$

b) $y = f(g(x)) = (2x+3)^3 \Rightarrow y = f(u) = u^3;\ u = g(x) = 2x + 3$

$\Rightarrow f'(u) = 3u^2;\ g'(x) = 2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 2 = 6u^2$$
$$= 6(2x+3)^2$$
$$= 6(4x^2+12x+9)$$
$$= 24x^2 + 72x + 54$$

The product rule

With the differentiation rules that we have learned thus far we can differentiate some functions that are products. For example, we can differentiate the function $f(x) = (x^2 + 3x)(2x - 1)$ by expanding and then differentiating the polynomial term-by-term. In doing so, we are applying the sum and difference, constant multiple and power rules from Section 13.2.

$$f(x) = (x^2 + 3x)(2x - 1) = 2x^3 + 5x^2 - 3x$$

$$f'(x) = 2\frac{d}{dx}(x^3) + 5\frac{d}{dx}(x^2) - 3\frac{d}{dx}(x)$$

$$f'(x) = 6x^2 + 10x - 3$$

The sum and difference rule states that the derivative of a sum/difference of two functions is the sum/difference of their derivatives. Perhaps the derivative of the product of two functions is the product of their derivatives. Let's try this with the above example.

$$f(x) = (x^2 + 3x)(2x - 1)$$

$$f'(x) = \frac{d}{dx}(x^2 + 3x)\cdot\frac{d}{dx}(2x - 1)?$$

$$f'(x) = (2x + 3)\cdot 2?$$

$$f'(x) = 4x + 6? \quad \text{However, } 4x + 6 \neq 6x^2 + 10x - 3.$$

Thus, one important fact we have learned from this example is that the derivative of a product of two functions is *not* the product of their derivatives. However, there are many products, such as $y = (4x - 3)^3(x - 1)^4$ and $f(x) = x^2 \sin x$, for which it is either difficult or impossible to write the function as a polynomial. In order to differentiate functions like this, we need a '**product**' **rule**.

Gottfried Wilhelm Leibniz (1646–1716)

Leibniz was a German philosopher, mathematician, scientist and professional diplomat – and, although self-taught in mathematics, was a major contributor to the development of mathematics in the 17th century. He developed the elementary concepts of calculus independent of, but slightly after, Newton. Nevertheless, the notation that Leibniz created for differential and integral calculus is still in use today. Leibniz' approach to the development of calculus was more purely mathematical, whereas Newton's was more directly connected to solving problems in physics. Leibniz created the idea of differentials (infinitely small differences in length), which he used to define the slope of a tangent, before the modern concept of limits was fully developed. Thus, Leibniz considered the derivative $\frac{dy}{dx}$ as the quotient of two differentials, dy and dx. Though it caused some confusion and consternation in his time (and to some extent still), Leibniz manipulated differentials algebraically to establish many of the important differentiation rules – including the product rule.

The product rule

If y is a function in terms of x that can be expressed as the product of two functions u and v that are also in terms of x, the product $y = uv$ can be differentiated as follows:

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or, equivalently, if $y = f(x) \cdot g(x)$, then

$$\frac{dy}{dx} = \frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Proof of the product rule

Let $y = F(x) = f(x) \cdot g(x)$ where f and g are differentiable functions of x (i.e. derivative exists for all x) and their product is defined for all values of x in the domain.

We proceed by applying the limit definition of the derivative and properties of limits. Note that in the second line of the proof we have introduced the additional term, $f(x + h)g(x)$, and its opposite (thereby adding zero) in the numerator. The purpose of this is to allow us to analyze separately the changes in f and g as h goes to zero. Thus, in the fifth line we are eventually able to isolate limits that are the derivatives of f and g.

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right] \\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h}\right] + \lim_{h\to 0}\left[g(x)\frac{f(x+h) - f(x)}{h}\right] \\
&= \lim_{h\to 0}f(x+h) \cdot \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0}g(x) \cdot \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= f(x) \cdot g'(x) + g(x) \cdot f'(x)
\end{aligned}$$

A less formal but perhaps more intuitive justification can be provided by considering the product rule written in the form

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

and analyzing the relationship between the functions u, v and y when there is a small change in the variable x. Recall that the definition of the derivative (Section 13.2) is essentially the limit of $\frac{\text{change in } y}{\text{change in } x}$ as the 'change in x' goes to zero. Let δx (read 'delta x') and δy represent small changes in x and y, respectively. As $\delta x \to 0$, then $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, i.e. the derivative of y with respect to x.

Any small change in x, i.e. δx, will cause small changes, δu and δv, in the values of functions u and v respectively. Since $y = uv$, these changes will also cause a small change, δy, in the value of function y.

Now consider the rectangles in Figure 15.3. The area of the first smaller rectangle is $y = uv$. The values of u and v then increase by δu and δv respectively.

The area of the larger rectangle is $y + \delta y = uv + u\delta v + v\delta u + \delta u\delta v$.

The product uv changes by the amount $\delta y = u\delta v + v\delta u + \delta u\delta v$.

Dividing through by δx: $\dfrac{\delta y}{\delta x} = u\dfrac{\delta v}{\delta x} + v\dfrac{\delta u}{\delta x} + \delta u\dfrac{\delta v}{\delta x}$.

Let $\delta x \to 0$ and $\delta u \to 0$, then:

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\frac{\delta v}{\delta x} \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + 0 \cdot \frac{dv}{dx}$$

Giving $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$, the product rule.

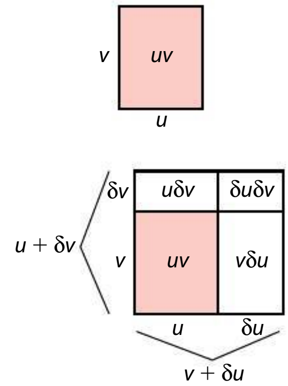


Figure 15.3

Example 4

Find the derivative of the function $y = (x^2 + 3x)(2x - 1)$ by:

a) expanding the binomial and differentiating term-by-term

b) the product rule.

Solution

a) Expanding gives $y = (x^2 + 3x)(2x - 1) = 2x^3 + 5x^2 - 3x$.

Therefore, $\dfrac{dy}{dx} = 6x^2 + 10x - 3$.

b) Let $u(x) = x^2 + 3x$ and $v(x) = 2x - 1$, then $y = u(x) \cdot v(x)$ or simply $y = uv$.

By the product rule (in Leibniz form),

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} &= (x^2 + 3x) \cdot 2 + (2x - 1) \cdot (2x + 3) \\ &= (2x^2 + 6x) + (4x^2 + 4x - 3) \\ &= 6x^2 + 10x - 3 \end{aligned}$$

This result agrees with the derivative we obtained earlier from differentiating the expanded polynomial.

Example 5

Given $y = x^2 \sin x$, find $\dfrac{dy}{dx}$.

Solution

Let $y = f(x) \cdot g(x) = x^2 \sin x \quad \Rightarrow \quad f(x) = x^2$ and $g(x) = \sin x$.

By the product rule (function notation form),

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}[f(x) \cdot g(x)] &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \\ &= x^2 \cdot \cos x + (\sin x) \cdot 2x \\ \frac{dy}{dx} &= x^2 \cos x + 2x \sin x \end{aligned}$$

As with the chain rule, it is very helpful to remember the structure of the product rule in words.

first factor second factor

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{[f(x)\cdot g(x)]}_{\substack{\text{product of}\\ \text{two functions,}\\ \text{i.e. factors}}} = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$= \text{first factor} \times \text{derivative of second factor} + \text{second factor} \times \text{derivative of first factor}$$

Example 6

Find an equation of the line tangent to the curve $y = \sin x\cos(2x)$ at the point where $x = \frac{\pi}{6}$.

Solution

To find the slope of the line tangent we need to find the derivative of $y = \sin x\cos(2x)$. To do this we will have to use more than one of the differentiation rules. Firstly, we need the product rule since the function consists of the two factors $\sin x$ and $\cos(2x)$. Secondly, the second factor is a composite of cosine and $2x$ so we need the chain rule. In essence the application of the chain rule will be 'nested' within the product rule.

$$\frac{dy}{dx} = \sin x\frac{d}{dx}(\cos(2x)) + \cos(2x)\frac{d}{dx}\sin x$$ Product rule applied to entire function.

$$\frac{dy}{dx} = \sin x(-2\sin(2x)) + \cos(2x)\cos x$$ Chain rule for $\frac{d}{dx}(\cos(2x))$.

$$\frac{dy}{dx} = -2\sin x\sin(2x) + \cos x\cos(2x)$$

At $x = \frac{\pi}{6}$, $\frac{dy}{dx} = -2\sin\left(\frac{\pi}{6}\right)\sin\left(2\cdot\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\cos\left(2\cdot\frac{\pi}{6}\right)$

$$= -2\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) = -2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = -\frac{\sqrt{3}}{4}.$$

Hence, slope of the tangent line is $-\frac{\sqrt{3}}{4}$.

Find the y-coordinate of the tangent point:

At $x = \frac{\pi}{6}$, $y = \sin\left(\frac{\pi}{6}\right)\cos\left(2\cdot\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow$ tangent point is $\left(\frac{\pi}{6}, \frac{1}{4}\right)$

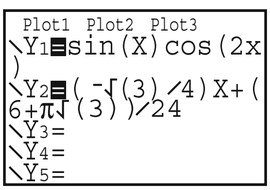
Using point-slope form for a linear equation, gives

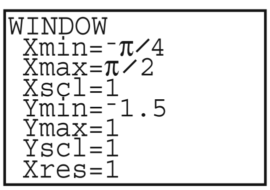
$$y - \frac{1}{4} = -\frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right) \Rightarrow y = -\frac{\sqrt{3}}{4}x + \frac{\pi\sqrt{3}}{24} + \frac{1}{4} \text{ or } y = -\frac{\sqrt{3}}{4}x + \frac{6+\pi\sqrt{3}}{24}.$$

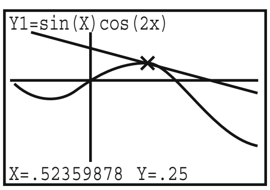
Therefore, an equation for the line tangent to $y = \sin x\cos(2x)$ at $x = \frac{\pi}{6}$ is

$$y = -\frac{\sqrt{3}}{4}x + \frac{6+\pi\sqrt{3}}{24}.$$

Our GDC can give a quick visual check for this result. $\left[\frac{\pi}{6} \approx 0.523\,598\,78\right]$







The quotient rule

Just as the derivative of the product of two functions is not the product of their derivatives, the derivative of a quotient of two functions is not the quotient of their derivatives. Let's derive a rule for the quotient of two functions by, once again, returning to the limit definition for the derivative.

Let $y = F(x) = \dfrac{f(x)}{g(x)}$ where f and g are differentiable functions of x and their quotient is defined for all values of x in the domain.

As with the proof of the product rule we introduce a term, $f(g)g(x)$ in this case, and its opposite (thereby adding zero) in the numerator (in the 3rd line below). This allows us (in the 5th line) to isolate limits that are the derivatives of f and g.

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x)g(x+h)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h \cdot g(x)g(x+h)} \\
&= \lim_{h\to 0}\frac{g(x)\dfrac{f(x+h)-f(x)}{h} - f(x)\dfrac{g(x+h)-g(x)}{h}}{g(x)g(x+h)} \\
&= \frac{\lim\limits_{h\to 0} g(x) \cdot \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0} f(x) \cdot \lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)g(x+h)} \\
&= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)g(x)} \\
&= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}
\end{aligned}
$$

The quotient rule

If y is a function in terms of x that can be expressed as the quotient of two functions u and v that are also in terms of x, the quotient $y = \frac{u}{v}$ can be differentiated as follows:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

or, equivalently, if $y = \dfrac{f(x)}{g(x)}$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

As with the chain rule and the product rule, it is helpful to recognize the structure of the quotient rule by remembering it in words:

$$\begin{pmatrix}\text{derivative}\\ \text{of quotient}\end{pmatrix} = \frac{(\text{denominator}) \times \begin{pmatrix}\text{derivative of}\\ \text{numerator}\end{pmatrix} - (\text{numerator})\begin{pmatrix}\text{derivative of}\\ \text{denominator}\end{pmatrix}}{(\text{denominator})^2}$$

● **Hint:** Since order is important in subtraction (subtraction is not commutative), be sure to set up the numerator of the quotient rule correctly.

● **Hint:** Note that we could have proved the quotient rule by writing the quotient $\frac{f(x)}{g(x)}$ as the product $f(x)\,[g(x)]^{-1}$ and apply the product rule and chain rule. As some of the examples here show, the derivative of a quotient can also be found by means of the product rule and/or the chain rule.

Example 7

For each function, find its derivative (i) by the quotient rule, and (ii) by another method.

a) $g(x) = \dfrac{5x-1}{3x^2}$ b) $h(x) = \dfrac{1}{2x-3}$ c) $f(x) = \dfrac{3x-2}{2x-5}$

Solution

a) (i) $g(x) = y = \dfrac{u}{v} = \dfrac{5x-1}{3x^2}$

$$g'(x) = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{3x^2 \cdot 5 - (5x-1)\cdot 6x}{(3x^2)^2}$$

$$= \frac{15x^2 - 30x^2 + 6x}{9x^4}$$

$$= \frac{3x(-5x+2)}{9x^4}$$

$$g'(x) = \frac{-5x+2}{3x^3}$$

(ii) Using algebra, 'split' the numerator:

$$g(x) = \frac{5x-1}{3x^2} = \frac{5x}{3x^2} - \frac{1}{3x^2} = \frac{5}{3x} - \frac{1}{3x^2} = \frac{5}{3}x^{-1} - \frac{1}{3}x^{-2}$$

Now, differentiate term-by-term using the power rule.

$$g'(x) = \frac{5}{3}\frac{d}{dx}(-x^{-1}) - \frac{1}{3}\frac{d}{dx}(x^{-3})$$

$$= \frac{5}{3}(-x^{-2}) - \frac{1}{3}(-2x^{-3})$$

$$g'(x) = -\frac{5}{3x^2} + \frac{2}{3x^3}$$

[Results for (i) and (ii) are equivalent:

$$-\frac{5}{3x^2} + \frac{2}{3x^3} = -\frac{5}{3x^2}\cdot\frac{x}{x} + \frac{2}{3x^3} = -\frac{5x}{3x^3} + \frac{2}{3x^3} = \frac{-5x+2}{3x^3}\Bigg]$$

b) (i) $y = \dfrac{f(x)}{g(x)} = \dfrac{1}{2x-3} \Rightarrow f(x) = 1$ and $g(x) = 2x - 3$

By the quotient rule (function notation form),

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

$$= \frac{(2x-3)\cdot 0 - 1\cdot(2)}{(2x-3)^2}$$

$$\frac{dy}{dx} = -\frac{2}{(2x-3)^2}$$

(ii) $y = f(g(x)) = \dfrac{1}{2x-3} = (2x-3)^{-1} \Rightarrow$ 'outside' function is $f(u) = u^{-1}$

$\Rightarrow f'(u) = -u^{-2}$

$\Rightarrow$ 'inside' function is $g(x) = 2x - 3$

By the chain rule (function notation form),

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x) = -(2x-3)^{-2}\cdot 2$$

$$\frac{dy}{dx} = -\frac{2}{(2x-3)^2}$$

c) (i) $f(x) = y = \frac{u}{v} = \frac{3x-2}{2x-5}$ $\quad f'(x) = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$= \frac{(2x-5)\cdot 3 - (3x-2)\cdot 2}{(2x-5)^2}$$

$$= \frac{6x - 15 - 6x + 4}{(2x-5)^2}$$

$$f'(x) = \frac{-11}{(2x-5)^2}$$

(ii) Rewrite $f(x)$ as a product and apply the product rule (with chain rule imbedded).

$f(x) = y = \frac{3x-2}{2x-5} = (3x-2)(2x-5)^{-1} \Rightarrow y = uv$, $u = 3x - 2$ and $v = (2x-5)^{-1}$

Note: $v = (2x-5)^{-1}$ is a composite function, so we'll need the chain rule to find $\frac{dv}{dx}$.

$$f'(x) = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (3x-2)\cdot\frac{d}{dx}[(2x-5)^{-1}] + (2x-5)^{-1}\cdot 3$$

$$= (3x-2)[-(2x-5)^{-2}\cdot 2] + 3(2x-5)^{-1}$$

Chain rule applied for $\frac{d}{dx}[(2x-5)^{-1}]$.

$$= (-6x+4)(2x-5)^{-2} + 3(2x-5)^{-1}$$

$$= (2x-5)^{-2}[(-6x+4) + 3(2x-5)]$$

Factorizing out GCF of $(2x-5)^2$.

$$= (2x-5)^{-2}[-6x + 4 + 6x - 15]$$

$$f'(x) = \frac{-11}{(2x-5)^2}$$

As Example 7 demonstrates, before differentiating a quotient it is worthwhile to consider if performing some algebra may allow other more efficient differentiation techniques to be used.

- **Hint:** The function $h(x) = \frac{3x^2}{5x-1}$ initially looks similar to the function g in Example 7, part a) (they're reciprocals). However, it is *not* possible to 'split' the denominator and express as two fractions. Recognize that $\frac{3x^2}{5x-1}$ is *not* equivalent to $\frac{3x^2}{5x} - \frac{3x^2}{1}$. Hence, in order to differentiate $h(x) = \frac{3x^2}{5x-1}$ we would apply either the quotient rule, or the product rule with the function rewritten as $h(x) = 3x^2(5x-1)^{-1}$ and using the chain rule to differentiate the factor $(5x-1)^{-1}$.

Higher derivatives

If $y = f(x)$ is a function of x then, in general, the derivative – expressed as either $\frac{dy}{dx}$ or $f'(x)$ – will be some other function of x. As we have learned the derivative indicates the rate of change of $f(x)$ with respect to x, as a function of x. In Section 13.3 we took the 'derivative of the derivative' of a function, that is, a function's second derivative, denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. The second derivative is an effective tool in verifying maximum, minimum and inflexion points on the graph of a function. In general, $\frac{d^2y}{dx^2}$ will also be a function of x and so may be differentiated to give the third derivative of y with respect to x, denoted by $\frac{d^3y}{dx^3}$. The nth derivative of y with respect to x is denoted by $\frac{d^ny}{dx^n}$. If the notation $f(x)$ is used, the first, second and third derivatives are written as $f'(x)$, $f''(x)$ and $f'''(x)$, respectively. The fourth derivative and higher is denoted using a superscript number rather than a 'prime' mark. For example, $f^{(4)}(x)$ represents the fourth derivative of the function f with respect to x.

The process of computing the nth derivative of a function can be very tedious and can only be achieved by computing the successive derivatives in turn. It is worthwhile to attempt to simplify the function $\frac{dy}{dx}$ before differentiating to find $\frac{d^2y}{dx^2}$, and in turn try to simplify this result before computing $\frac{d^3y}{dx^3}$, and so on.

Example 8

Given $y = \frac{1}{x}$, find a formula for the nth derivative $\frac{d^ny}{dx^n}$.

Solution

Let's take successive derivatives of the function until we can discern a pattern and then formulate a conjecture for the formula.

$$\begin{aligned}
y &= \frac{1}{x} = x^{-1}\\
\frac{dy}{dx} &= -x^{-2} = \frac{-1}{x^2}\\
\frac{d^2y}{dx^2} &= (-2)(-1)x^{-3} = \frac{2}{x^3}\\
\frac{d^3y}{dx^3} &= (-3)(2)(1)x^{-4} = \frac{-6}{x^4}\\
\frac{d^4y}{dx^4} &= (4)(3)(2)(1)x^{-5} = \frac{24}{x^5}\\
\frac{d^5y}{dx^5} &= (-5)(4)(3)(2)(1)x^{-6} = \frac{-120}{x^6}
\end{aligned}$$

We observe that the sign of the result alternates: negative when n is odd, and positive when n is even. Thus, we need to incorporate the expression $(-1)^n$ into our formula since the successive values of $(-1)^n$ are $-1, 1, -1, 1, \ldots$. Another factor needs to be $n!$ (n factorial) because $n! = n(n-1)(n-2)\cdot\cdot\cdot 2\cdot 1$. The last piece of the formula is that the power of x in the denominator is one more than the value of n.

$$\text{Therefore, } \frac{d^{(n)}y}{dx^{(n)}} = \frac{(-1)^n n!}{x^{n+1}}.$$

Exercise 15.1

1 Find the derivative of each function.

a) $y = (3x - 8)^4$ b) $y = \sqrt{1 - x}$ c) $y = \sin x \cos x$

d) $y = 2\sin\left(\frac{x}{2}\right)$ e) $y = (x^2 + 4)^{-2}$ f) $y = \frac{x + 1}{x - 1}$

g) $y = \frac{1}{\sqrt{x + 2}}$ h) $y = \cos^2 x$ i) $y = x\sqrt{1 - x}$

j) $y = \frac{1}{3x^2 - 5x + 7}$ k) $y = \sqrt[3]{2x + 5}$ l) $y = (2x - 1)^3(x^4 + 1)$

m) $y = \frac{\sin x}{x}$ n) $y = \frac{x^2}{x + 2}$ o) $y = \sqrt[3]{x^2}\cos x$

2 Find the equation of the line tangent to the given curve at the specified value of x. Express the equation exactly in the form $y = mx + c$.

a) $y = (2x^2 - 1)^3$ $x = -1$ b) $y = \sqrt{3x^2 - 2}$ $x = 3$

c) $y = \sin 2x$ $x = \pi$ d) $y = \frac{x^3 + 1}{2x}$ $x = 1$

3 An object moves along a line so that its position s relative to a starting point at any time $t \geqslant 0$ is given by $s(t) = \cos(t^2 - 1)$.

a) Find the velocity of the object as a function of t.

b) What is the object's velocity at $t = 0$?

c) In the interval $0 < t < 2.5$, find any times (values of t) for which the object is stationary.

d) Describe the object's motion during the interval $0 < t < 2.5$.

For questions 4–6, find the equation of a) the tangent, and b) the normal to the curve at the given point.

4 $y = \frac{2}{x^2 - 8}$ at (3, 2)

5 $y = \sqrt{1 + 4x}$ at (2, 3)

6 $y = \frac{x}{x + 1}$ at $(1, \frac{1}{2})$

7 Consider the trigonometric curve $y = \sin\left(2x - \frac{\pi}{2}\right)$.

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b) Find the exact coordinates of any inflexion points for the curve in the interval $0 < x < \pi$.

8 A curve has equation $y = x(x - 4)^2$.

a) For this curve, find

(i) the x-intercepts

(ii) the coordinates of the maximum point

(iii) the coordinates of the point of inflexion.

b) Use your answers to part a) to sketch a graph of the curve for $0 \leqslant x \leqslant 4$, clearly indicating the features you have found in part a).

9 Consider the function $f(x) = \frac{x^2 - 3x + 4}{(x + 1)^2}$.

a) Show that $f'(x) = \frac{5x - 11}{(x + 1)^3}$.

b) Show that $f''(x) = \frac{-10x + 38}{(x + 1)^4}$.

c) Does the graph of f have an inflexion point at $x = 3.8$? Explain.

10 Find the first and second derivatives of the function $f(x) = \dfrac{x - a}{x + a}$.

11 Given $y = \dfrac{1}{1 - x}$, find a formula for the nth derivative $\dfrac{d^n y}{dx^n}$.

12 The graph of the function $g(x) = \dfrac{8}{4 + x^2}$ is called the *witch of Agnesi*.

a) Find the exact coordinates of any extreme values or inflexion points.

b) Determine all values of x for which (i) $g(x) < 0$, (ii) $g(x) = 0$, and (iii) $g(x) > 0$.

c) Find (i) $\lim\limits_{x \to -\infty} g(x)$, and (ii) $\lim\limits_{x \to +\infty} g(x)$.

d) Sketch the graph of g.

13 Use the product rule to prove the constant multiple rule for differentiation. That is, show that $\dfrac{d}{dx}(c \cdot f(x)) = c \cdot \dfrac{d}{dx}(f(x))$ for any constant c.

14 If $y = x^4 - 6x^2$, show that y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are all negative on the interval $0 < x < 1$, but that $\dfrac{d^3y}{dx^3}$ is positive on the same interval.

15.2 Derivatives of trigonometric and exponential functions

Although it is important to provide formal justifications for any of our differentiation rules (as we did in the previous section), we should not forget that the derivative is a rule that gives us the slope of the line tangent to the graph of a function at a particular point. Thus, we can use a function's derivative to deduce the behaviour of its graph. Conversely, we can gain insight about the derivative of a function from the shape of its graph.

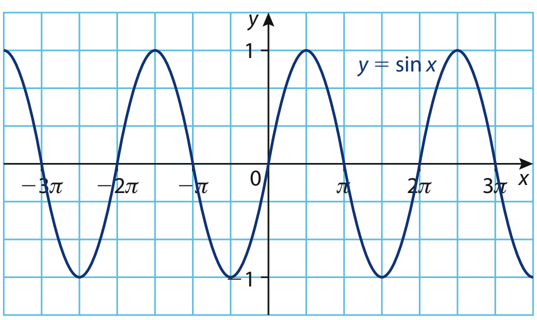


Figure 15.4

In Chapter 13, we formally determined that the derivative of $\sin x$ is $\cos x$ and that the derivative of $\cos x$ is $-\sin x$ by using the limit definition of the derivative. We could have made a very confident conjecture for the derivative of $\sin x$ by analyzing its graph as follows.

We start with the graph of $f(x) = \sin x$ (Figure 15.4). The graph of $y = \sin x$ is periodic, with period 2π, so the same will be true of its derivative that gives the slope at each point on the graph. Therefore, it's only necessary for us to consider the portion of the graph in the interval $0 \leqslant x \leqslant 2\pi$.

Figure 15.5 shows two pairs of axes having equal scales on the x- and y-axes and corresponding x-coordinates aligned vertically. On the top pair of axes $y = \sin x$ is graphed with tangent lines drawn at nine selected points. The points were chosen such that the slopes of the tangents at those points, in order, appear to be equal to $1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. The values of these slopes were then plotted in the bottom graph with the y-coordinate of each point indicating the slope of the curve for that particular x value. Hence, the points in the bottom pair of axes should be on the graph of the derivative of $y = \sin x$.

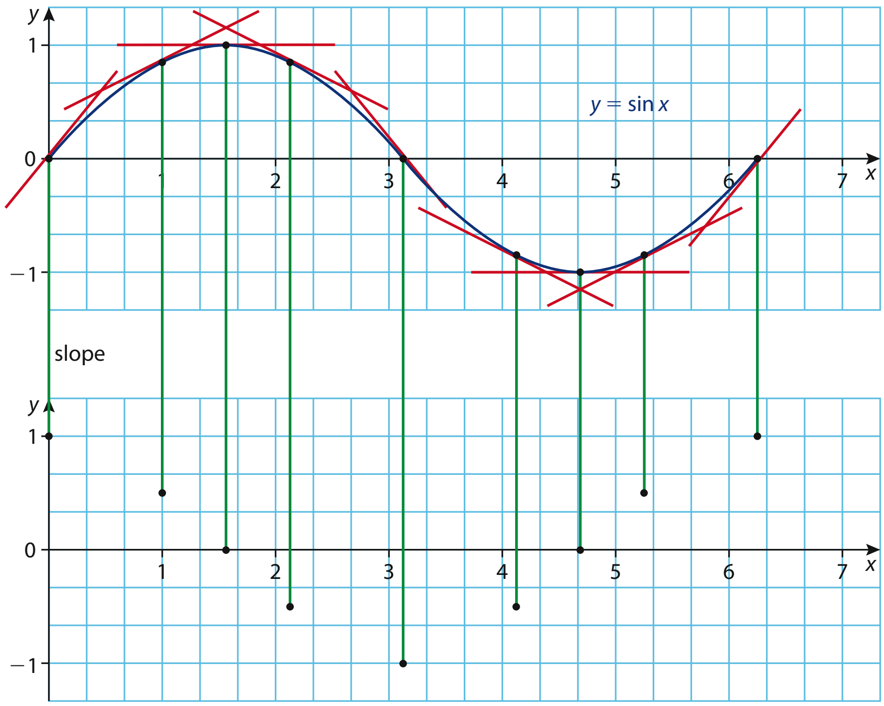


Figure 15.5: Analyzing the slope of tangents to the graph of $y = \sin x$.

Figure 15.6 is the same as Figure 15.5 except with the graph of $y = \sin x$, the grid lines and the lines connecting points between the two graphs removed.

● **Hint:** Note that the graphs in Figures 15.4, 15.5, 15.6 and 15.7 have x in radians. As mentioned previously, we must only use radian measure when trigonometric functions are involved in calculus.

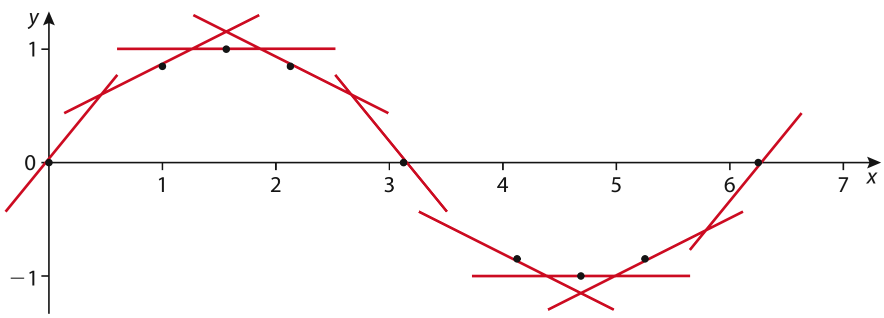


Figure 15.6

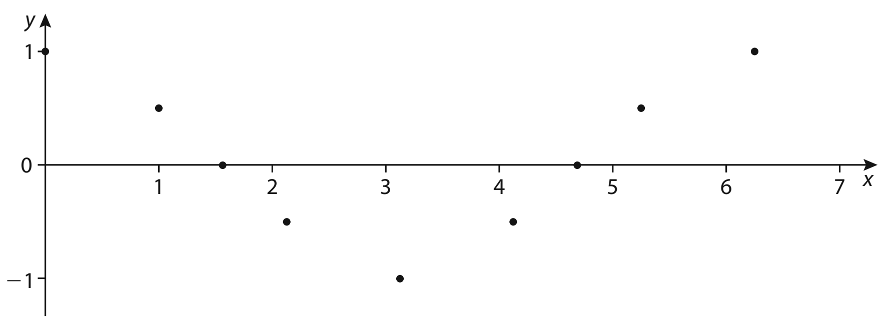
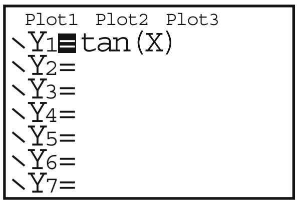


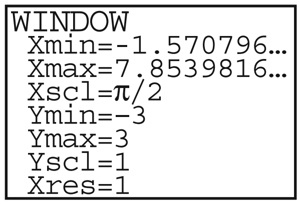
Figure 15.7

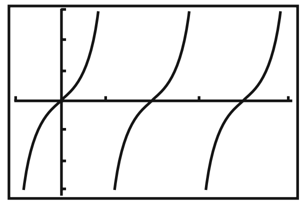
Clearly the points representing the slope of the tangents to $y = \sin x$ plotted in Figure 15.7 are tracing out the graph of $y = \cos x$.

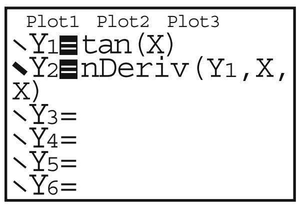
Although we will use this informal approach to conjecture the derivatives for $y = e^x$ and $y = \ln x$, it does not always work so smoothly. For example, let's analyze the graph of $y = \tan x$ in an attempt to guess its derivative.

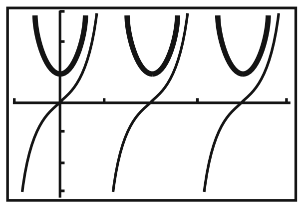
We can use our GDC command that evaluates the derivative of a function at a specified point to graph the value of the derivative at all points on a graph. We used this technique in Chapter 13 to confirm the result in Example 9 part d). The GDC screen images below show the graph of $y = \tan x$ and then the GDC graphing its derivative (in bold) on the same set of axes. Although, as pointed out in Section 13.3, in general it is incorrect to graph a function and its derivative on the same pair of axes (units on the vertical axis will not be the same), it is helpful in seeing the connection between the graph of a function and that of its derivative.











The graph of the derivative of $\tan x$ is always above the x-axis meaning that the derivative is always positive. This clearly agrees with the fact that the tangent function, except for where it is undefined, is always increasing (moving upwards) as the values of x increase. However, the shape of the graph does not bring to mind an easy conjecture for a rule for the derivative of $\tan x$.

Rather than use the limit definition for finding the derivative of $\tan x$ let's write $\tan x$ as $\frac{\sin x}{\cos x}$ and use the quotient rule.

$$\begin{aligned}
\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \qquad \text{Therefore, } \frac{d}{dx}(\tan x) = \sec^2 x.
\end{aligned}$$

Similarly, it can be shown that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

To find the derivative of sec x we can use the chain rule as follows.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}[(\cos x)^{-1}]$$

$$= -(\cos x)^{-2}(-\sin x) \qquad \text{Applying chain rule.}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

Therefore, $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Similarly, it can be shown that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

The table below lists the derivatives of the six trigonometric functions.

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Example 9

Find the derivative of each function.

a) $y = \cos(\sqrt{x})$

b) $y = \frac{x^3}{\sin x}$

c) $y = x^2 \tan(3x)$

d) $y = \sec^2(3x)$

Solution

a) $\frac{dy}{dx} = \frac{d}{dx}[\cos(\sqrt{x})] = -\sin(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x})$ Applying chain rule.

$$= -\sin(\sqrt{x}) \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= -\sin(\sqrt{x}) \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right) \qquad \text{Applying power rule.}$$

Therefore, $\frac{dy}{dx} = -\frac{\sin(\sqrt{x})}{2\sqrt{x}}$.

b) Method 1 (quotient rule):

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3}{\sin x}\right) = \frac{\sin x \cdot \frac{d}{dx}(x^3) - x^3 \cdot \frac{d}{dx}(\sin x)}{\sin^2 x} \qquad \text{Applying quotient rule.}$$

Therefore, $\frac{dy}{dx} = \frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}$.

Method 2 (product rule and chain rule):

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3}{\sin x}\right) = \frac{d}{dx}[x^3 \cdot (\sin x)^{-1}]$$ Rewriting as a product.

$$= x^3 \cdot \frac{d}{dx}[(\sin x)^{-1}] + (\sin x)^{-1} \cdot \frac{d}{dx}(x^3)$$ Applying product rule.

$$= x^3[-(\sin x)^{-2}\cos x] + (\sin x)^{-1}(3x^2)$$

$$= (\sin x)^{-2}[-x^3\cos x + 3x^2 \sin x]$$ Factor out common factor of $(\sin x)^{-2}$.

Therefore, $\dfrac{dy}{dx} = \dfrac{3x^2\sin x - x^3\cos x}{\sin^2 x}$.

c) $$\frac{dy}{dx} = \frac{d}{dx}[x^2\tan(3x)] = x^2 \cdot \frac{d}{dx}(\tan(3x)) + \tan(3x) \cdot \frac{d}{dx}(x^2)$$

$$= x^2 \cdot \frac{d}{dx}(\tan(3x)) + \tan(3x) \cdot \frac{d}{dx}(x^2)$$ Applying product rule.

$$= x^2(3\sec^2(3x)) + (\tan(3x))(2x)$$ Applying chain rule for $\frac{d}{dx}(\tan(3x))$.

$$= 3x^2\sec^2(3x) + 2x\tan(3x)$$

d) $$\frac{dy}{dx} = \frac{d}{dx}[\sec^2(3x)] = \frac{d}{dx}[(\sec(3x))^2]$$

$$= 2\sec(3x) \cdot \frac{d}{dx}(\sec(3x))$$ Applying chain rule 1st time.

$$= 2\sec(3x) \cdot \left(\sec(3x)\tan(3x) \cdot \frac{d}{dx}(3x)\right)$$ Applying chain rule 2nd time.

$$= 2\sec(3x) \cdot (\sec(3x)\tan(3x) \cdot 3)$$

$$= 6\sec^2(3x)\tan(3x)$$ Equivalent to $\frac{6\sin(3x)}{\cos^3(3x)}$.

Example 10

The motion of a particle moving along a straight line for the interval $0 < t < 12$ (t in seconds) is given by the function $s(t) = \sin\left(\frac{t}{2}\right) - \cos\left(\frac{t}{2}\right) + 1$, where s is the particle's displacement in centimetres from the origin O. The particle's displacement is negative when left of O and positive when right of O.

a) Find the exact time and displacement when the particle is (i) furthest to the right and (ii) furthest to the left during the interval $0 < t < 12$.

b) Find the particle's maximum speed to the right exactly and at what exact time it occurs.

Solution

For part a) displacement can only be a maximum or minimum when velocity is zero, i.e. $v(t) = 0$. Similarly for part b) velocity can only be a maximum or minimum when acceleration is zero, i.e. $a(t) = 0$. So we begin by finding the first and second derivatives of $s(t)$ giving us the velocity function, $v(t)$, and acceleration function, $a(t)$, respectively.

a) $v(t) = s'(t) = \frac{d}{dx}\left[\sin\left(\frac{t}{2}\right) - \cos\left(\frac{t}{2}\right) + 1\right] = \frac{1}{2}\cos\left(\frac{t}{2}\right) + \frac{1}{2}\sin\left(\frac{t}{2}\right)$

Solve $\frac{1}{2}\cos\left(\frac{t}{2}\right) + \frac{1}{2}\sin\left(\frac{t}{2}\right) = 0$:

$\sin\left(\frac{t}{2}\right) = -\cos\left(\frac{t}{2}\right)$

$\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)} = \tan\left(\frac{t}{2}\right) = -1$ Given that $\cos\left(\frac{t}{2}\right) \neq 0$.

$\tan\left(\frac{t}{2}\right) = -1$ when $\frac{t}{2} = \frac{3\pi}{4} + k \cdot \pi, k \in \mathbb{Z}$

Thus, $t = \frac{3\pi}{2} + k \cdot 2\pi, k \in \mathbb{Z}$. For $0 < t < 12$, $t = \frac{3\pi}{2}$ or $t = \frac{7\pi}{2}$.

(i) Checking the sign (direction) of the particle's velocity just before and after these two times will show if they are maximum or minimum values. Test values are $t = \pi$ and 2π for $t = \frac{3\pi}{2}$.

$v(\pi) = \frac{1}{2}\cos\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) = 0 + \frac{1}{2} \cdot 1 = \frac{1}{2} > 0 \Rightarrow$ displacement increasing before $t = \frac{3\pi}{2}$

$v(2\pi) = \frac{1}{2}\cos\left(\frac{2\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{2}\right) = \frac{1}{2}(-1) + 0 < 0 \Rightarrow$ displacement decreasing before $t = \frac{3\pi}{2}$

Hence, $s\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) + 1 = \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) + 1$
$= 1 + \sqrt{2}$ is a maximum.

Therefore, the particle is furthest to the right (maximum displacement) at $t = \frac{3\pi}{2}$ seconds when its displacement is $1 + \sqrt{2}$ cm.

(ii) Test values are $t = 3\pi$ and 4π for $t = \frac{7\pi}{2}$.

$v(3\pi) = \frac{1}{2}\cos\left(\frac{3\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{3\pi}{2}\right) = 0 + \frac{1}{2}(-1) < 0 \Rightarrow$ displacement decreasing before $t = \frac{7\pi}{2}$

$v(4\pi) = \frac{1}{2}\cos(2\pi) + \frac{1}{2}\sin(2\pi) = \frac{1}{2}(1) + 0 > 0 \Rightarrow$ displacement increasing after $t = \frac{7\pi}{2}$

Hence, $s\left(\frac{7\pi}{2}\right) = \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right) + 1 = -\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right) +$
$1 = 1 - \sqrt{2}$ is a minimum.

Therefore, the particle is furthest to the left (minimum displacement) at $t = \frac{7\pi}{2}$ seconds when its displacement is $1 - \sqrt{2}$ cm.

b) $a(t) = v'(t) = \frac{d}{dx}\left[\frac{1}{2}\cos\left(\frac{t}{2}\right) + \frac{1}{2}\sin\left(\frac{t}{2}\right)\right] = -\frac{1}{4}\sin\left(\frac{t}{2}\right) + \frac{1}{4}\cos\left(\frac{t}{2}\right)$

Solve $\frac{1}{4}\cos\left(\frac{t}{2}\right) - \frac{1}{4}\sin\left(\frac{t}{2}\right) = 0$:

$$\sin\left(\frac{t}{2}\right) = \cos\left(\frac{t}{2}\right)$$

$$\frac{\sin\left(\frac{t}{2}\right)}{\cos\left(\frac{t}{2}\right)} = \tan\left(\frac{t}{2}\right) = 1$$

Given that $\cos\left(\frac{t}{2}\right) \neq 0$.

$$\tan\left(\frac{t}{2}\right) = 1 \text{ when } \frac{t}{2} = \frac{\pi}{4} + k \cdot \pi, k \in \mathbb{Z}$$

Thus, $t = \frac{\pi}{2} + k \cdot 2\pi, k \in \mathbb{Z}$. For $0 < t < 12$, $t = \frac{\pi}{2}$ or $t = \frac{5\pi}{2}$.

To find maximum velocity (moving right, speed > 0), let's evaluate the velocity at all critical points, i.e. at endpoints for the time interval, $t = 0$ and $t = 12$, and where the acceleration is zero, $t = \frac{\pi}{2}$ and $t = \frac{5\pi}{2}$.

$$v(0) = \frac{1}{2}\cos(0) + \frac{1}{2}\sin(0) = \frac{1}{2}$$

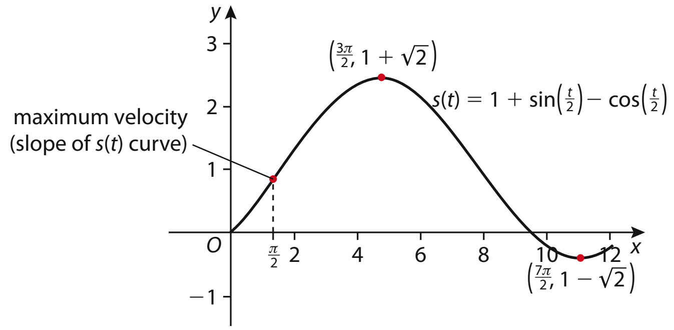
$$v\left(\frac{\pi}{2}\right) = \frac{1}{2}\cos\left(\frac{\pi}{4}\right) + \frac{1}{2}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{2} \approx 0.707$$

$$v\left(\frac{5\pi}{2}\right) = \frac{1}{2}\cos\left(\frac{5\pi}{4}\right) + \frac{1}{2}\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} = -\frac{\sqrt{2}}{2} \approx -0.707$$

$$v(12) = \frac{1}{2}\cos(6) + \frac{1}{2}\sin(6) \approx -0.424$$

Therefore, the particle has a maximum velocity of $\frac{\sqrt{2}}{2}$ cm/sec when $t = \frac{\pi}{2}$ seconds.

A graph of the displacement function $s(t) = \sin\left(\frac{t}{2}\right) - \cos\left(\frac{t}{2}\right) + 1$ gives a good visual confirmation of our results.



Derivatives of exponential functions

Let's review some important facts about exponential functions in general. An exponential function with base b is defined as $f(x) = b^x$, $b > 0$ and $b \neq 1$. The graph of f passes through $(0, 1)$, has the x-axis as a horizontal asymptote, and, depending on the value of the base of the exponential function b, will either be a continually increasing exponential growth curve (Figure 15.8) or a continually decreasing exponential decay curve (Figure 15.9).

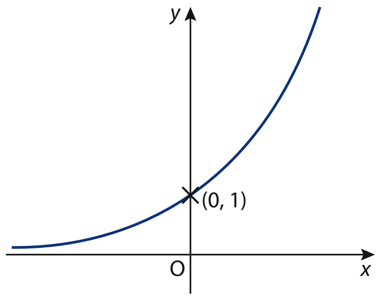


Figure 15.8

as $x \rightarrow \infty, f(x) \rightarrow \infty$

f is an increasing function

exponential growth curve

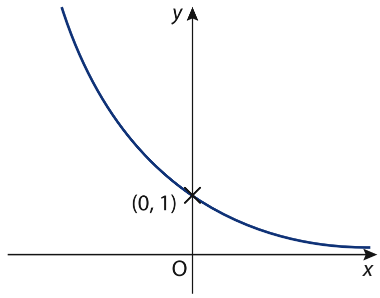


Figure 15.9

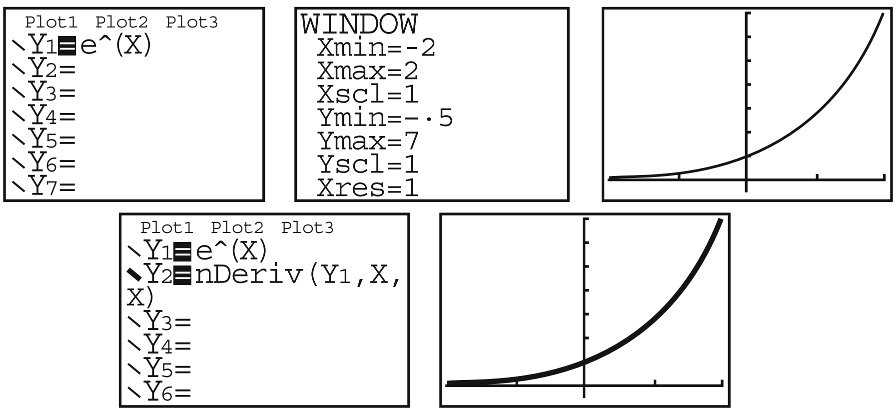
as $x \rightarrow \infty, f(x) \rightarrow 0$

f is a decreasing function

exponential decay curve

In Chapter 5 we learned that *the* exponential function e^x, sometimes written as 'exp x', is a particularly important function for modelling exponential growth and decay. The number e was defined in Section 5.3 as the limit of $\left(1 + \frac{1}{x}\right)^x$ as $x \rightarrow \infty$. Although the method was not successful in coming up with a conjecture for the derivative of the tangent function, let's try to guess the derivative of e^x by having our GDC graph its derivative.

• **Hint:** You may be tempted to find the derivative of e^x by applying the rule for differentiating powers $\frac{d}{dx}(x^n) = nx^{n-1}$ but this *only* applies if a variable is raised to a constant power. An exponential function, such as $y = e^x$, is a constant raised to a variable power, so the power rule does *not* apply.



The graph of the derivative of e^x appears to be identical to e^x itself! This is a very interesting result, but one which we will see fits in exactly with the nature of exponential growth/decay.

Let's try to apply the limit definition of the derivative to provide a formal justification.

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$ Applying limit definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$ Reverse of law of exponents: $a^m \cdot a^n = a^{m+n}$.

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$ Factorizing.

$$= \lim_{h \to 0} e^x \cdot \lim_{h \to 0} \frac{(e^h - 1)}{h}$$ Applying properties of limits.

$$= e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$ e^x is not affected by the value of h.

A closer look at the limit that is multiplying e^x reveals that it is equivalent to the slope of the graph of $y = e^x$ at $x = 0$: $\lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h}$. To finish our differentiation of e^x by first principles, we need to evaluate this limit. It is beyond the scope of this course to give a formal algebraic proof for the limit. Nevertheless, we can provide a convincing informal justification by evaluating the expression $\frac{e^h - 1}{h}$ for values of h approaching zero, as shown in the table.

h	$\frac{e^h - 1}{h}$
0.1	1.051 709 181
0.01	1.005 016 708
0.0001	1.000 050 002
0.000 001	1.000 000 005

Thus, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ and we can complete our algebraic work for the derivative of e^x.

$$\frac{d}{dx}(e^x) = e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

The derivative of the exponential function *is* the exponential function. More precisely, the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

The derivative of the exponential function

If $f(x) = e^x$, then $f'(x) = e^x$. Or, in Leibniz notation, $\frac{d}{dx}(e^x) = e^x$.

Example 11

Differentiate each of the following functions.

a) $y = e^{2x + \ln x}$ b) $y = \sqrt{x^2 + e^{4x}}$ c) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

a) Because $e^{2x + \ln x} = e^{2x}e^{\ln x}$ and $e^{\ln x} = x$, then $e^{2x + \ln x} = xe^{2x}$.

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x + \ln x}) = \frac{d}{dx}(xe^{2x})$$

$$= x \cdot \frac{d}{dx}(e^{2x}) + e^{2x} \cdot \frac{d}{dx}(x)$$ Applying the product rule.

Therefore, $\frac{dy}{dx} = 2xe^{2x} + e^{2x}$.

b) $$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x^2 + e^{4x}}) = \frac{d}{dx}\left[(x^2 + e^{4x})^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}(x^2 + e^{4x})^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2 + e^{4x})$$ Applying power rule and chain rule.

$$= \frac{2x + 4e^{4x}}{2\sqrt{x^2 + e^{4x}}}$$

Therefore, $\frac{dy}{dx} = \frac{x + 2e^{4x}}{\sqrt{x^2 + e^{4x}}}$.

c) $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

$$= \frac{(e^x + e^{-x}) \cdot \frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x}) \cdot \frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$ Quotient rule.

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + 2e^x e^{-x} + e^{-2x}) - (e^{2x} - 2e^x e^{-x} + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4e^x e^{-x}}{(e^x + e^{-x})^2}$$

Therefore, $\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}$.

What about exponential functions with bases other than e? We now differentiate the general exponential function $f(x) = b^x$, $b > 1$, $b \neq 0$, repeating the same steps we did with $f(x) = e^x$.

$$\frac{d}{dx}(b^x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$ Definition of derivative.

$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h}$$ Reverse of $a^m \cdot a^n = a^{m+n}$.

$$= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$ Factorizing.

$$= b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}$$ b^x is not affected by the value of h.

As with e^x, $\lim_{h \to 0} \frac{b^h - 1}{h}$ is equivalent to the slope of the graph of $f(x) = b^x$ at $x = 0$, i.e. $f'(0)$. Therefore, the derivative of the general exponential function $f(x) = b^x$ is $b^x \cdot f'(0)$. Although the value of $f'(0)$ will be a constant, it will depend on the value of the base b.

Application of the chain rule gives us the means to determine the value of $f'(0)$ in terms of b for the function $f(x) = b^x$. We can then state the rule for the derivative of the general exponential function $f(x) = b^x$.

We can use the laws of logarithms to write b^x in terms of e^x. Recall from Section 5.5 that $b^{\log_b x} = x$, and if $b = e$ then $e^{\ln x} = x$. Hence, $b^x = e^{x \ln b}$ because $e^{x \ln b} = e^{\ln(b^x)} = b^x$. We can now find the derivative of b^x by applying the chain rule to its equivalent expression $e^{x \ln b}$.

$y = f(g(x)) = e^{x \ln b}$ $\Rightarrow$ 'outside' function is $f(u) = e^u$

$f'(u) = e^u$ $\Rightarrow$ 'inside' function is $g(x) = x \ln b$

$g'(x) = \ln b$ [$\ln b$ is a constant]

$$\frac{dy}{dx} = f(g(x)) \cdot g'(x) = e^{x \ln b} \cdot \ln b$$

$$\frac{dy}{dx} = b^x \ln b$$

Therefore, $\frac{d}{dx}(b^x) = b^x \ln b$.

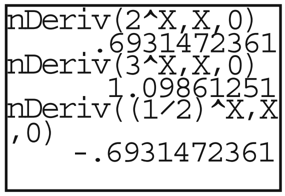
This result agrees with the fact that $\frac{d}{dx}(e^x) = e^x$. Using this 'new' general rule, $\frac{d}{dx}(b^x) = b^x \ln b$, then $\frac{d}{dx}(e^x) = e^x \ln e$. Since $\ln e = 1$ then $\frac{d}{dx}(e^x) = e^x$.

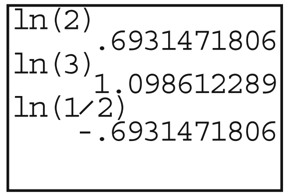
● **Hint:** Be careful to distinguish between the power rule, $\frac{d}{dx}(x^n) = nx^{n-1}$, where the base is a variable and the exponent is a constant, and the rule for differentiating exponential functions, $\frac{d}{dx}(b^x) = b^x \ln b$, where the base is a constant and the exponent is a variable.

The derivative of the general exponential function

For $b > 0$ and $b \neq 1$, if $f(x) = b^x$, then $f'(x) = b^x \ln b$. Or, in Leibniz notation, $\frac{d}{dx}(b^x) = b^x \ln b$.

Earlier we established that the derivative of the general exponential function $f(x) = b^x$ is $b^x \cdot f'(0)$, where $f'(0)$ is the slope of the graph at $x = 0$. From our result above, we can see that for a specific base b the slope of the curve $y = b^x$ when $x = 0$ is $\ln b$ because $b^0 \ln b = \ln b$. The first GDC screen image below shows the value of $f'(0)$ for $b = 2, 3$ and $\frac{1}{2}$. Evaluating $\ln 2$, $\ln 3$ and $\ln(\frac{1}{2})$ confirms that $f'(0)$ is equal to $\ln b$.





Example 12

Find the equation of the line tangent to the curve $y = 2^x$ at the point where $x = 3$. Express the equation of the line exactly in the form $y = mx + c$.

Solution

We first find the derivative of $y = 2^x$ and then evaluate it at $x = 3$ to get the slope of the tangent.

$$y' = \frac{d}{dx}(2^x) = 2^x(\ln 2) \Rightarrow y'(3) = 2^3(\ln 2) = 8 \ln 2 = \ln 2^8$$
$$= \ln 256 \Rightarrow m = \ln 256$$

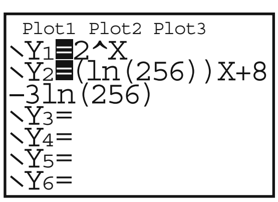
Finding the y-coordinate of the tangent point, $y(3) = 2^3 = 8 \Rightarrow$ point is $(3, 8)$

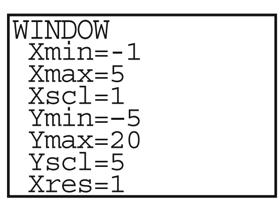
Substituting into the point-slope form for a linear equation, gives

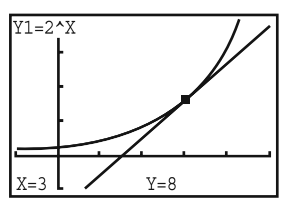
$$y - y_1 = m(x - x_1) \Rightarrow y - 8 = \ln 256(x - 3)$$

Therefore, the equation of the tangent line is $y = (\ln 256)x + 8 - 3 \ln 256$.

The GDC images below nicely confirm the result.







Example 13

Find the coordinates of the point P lying on the graph of $y = 5^x$ such that the line tangent to the curve at P passes through the origin.

Solution

Let $P = (x_0, y_0)$ be a point on the graph of $y = 5^x$. Since $\frac{dy}{dx} = 5^x(\ln 5)$ the slope of the tangent line to the curve at P is given by $\frac{dy}{dx} = 5^{x_0}(\ln 5)$.
Substituting into the point-slope form for a linear equation gives,

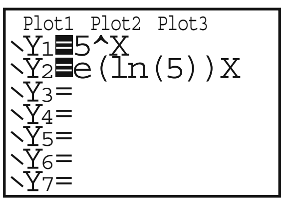
$$y - y_0 = 5^{x_0}(\ln 5)(x - x_0)$$

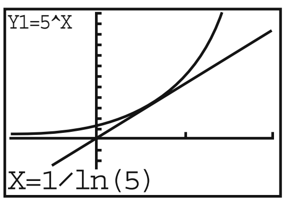
If the line passes through the origin then $(0, 0)$ must satisfy the equation.

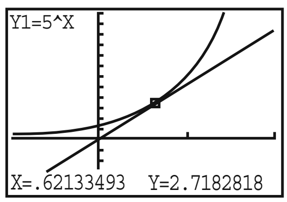
$$0 - y_0 = 5^{x_0}(\ln 5)(0 - x_0) \Rightarrow -y_0 = 5^{x_0}(\ln 5)(-x_0)$$

But $y_0 = 5^{x_0}$, so $-5^{x_0} = 5^{x_0}(\ln 5)(-x_0) \Rightarrow x_0 = \frac{5^{x0}}{5^{x0}\ln 5} = \frac{1}{\ln 5}$.

Then $y_0 = 5^{\frac{1}{\ln 5}} \Rightarrow (y_0)^{\ln 5} = \left(5^{\frac{1}{\ln 5}}\right)^{\ln 5} \Rightarrow (y_0)^{\ln 5} = 5 \Rightarrow y_0 = e$ because $e^{\ln x} = x$.
Therefore, the point P on the graph of $y = 5^x$ has coordinates $\left(\frac{1}{\ln 5}, e\right)$.
As a check let's find the equation of the tangent to $y = 5^x$ at this point.
Since $\frac{dy}{dx} = 5^{x_0}(\ln 5)$ the slope is $5^{\frac{1}{\ln 5}}(\ln 5)$, but we showed above that $5^{\frac{1}{\ln 5}} = e$. So the slope is equivalent to $e \ln 5$. Substituting in the point-slope form gives $y - e = e \ln 5\left(x - \frac{1}{\ln 5}\right) \Rightarrow y = e(\ln 5)x$. Clearly this line passes through $(0, 0)$.







If $f(x) = b^x$, then $f'(x) = b^x \cdot f'(0)$. The value of $f'(0)$ is the slope of the graph of $f(x) = b^x$ at the point $(0, 1)$. Hence, this will be a particular constant for each value of b ($b > 1$, $b \neq 0$). Therefore, if $f(x) = b^x$, then $f'(x) = kb^x$ where k is a constant dependent on the value of b. If the amount of a quantity y at a time t is given by $y = b^t$ then $\frac{dy}{dt} = kb^t = ky$. In other words, the rate of change of the quantity y at time t is proportional to the amount of y at time t. This is the essential behaviour of exponential growth/decay. It is because of this property that exponential functions have so many applications to real-life phenomena. Here are some good examples:

1 The rate of population growth for many living organisms is proportional to the size of the population p: $\frac{dp}{dt} = kp$.

2 The rate at which a radioactive substance decays is proportional to the amount A of the substance present: $\frac{dA}{dt} = kA$.

3 Newton's law of cooling states that if a substance is placed in cooler surroundings then its temperature decreases at a rate proportional to the temperature difference T between the temperature of the substance and the temperature of its surroundings: $\frac{dT}{dt} = kT$.

Exercise 15.2

1 Find the derivative of each function.

a) $y = x^2e^x$ b) $y = 8^x$ c) $y = \tan e^x$

d) $y = \dfrac{x}{1+\cos x}$ e) $y = \dfrac{e^x}{x}$ f) $y = \frac{1}{3}\sec^3 2x - \sec 2x$

g) $y = 4^{-x}$ h) $y = \cos x \tan x$ i) $y = \dfrac{x}{e^x - 1}$

j) $y = 4\cos(\sin 3x)$ k) $y = 2^{x+1}$ l) $y = \dfrac{1}{\csc x - \sec x}$

2 Find the equation of the line tangent to the given curve at the specified value of x. Express the equation exactly in the form $y = mx + c$.

a) $y = \sin x$ $x = \dfrac{\pi}{3}$

b) $y = x + e^x$ $x = \dfrac{\pi}{3}$

c) $y = 4\tan 2x$ $x = \dfrac{\pi}{8}$

3 Consider the function $g(x) = x + 2\cos x$. For the interval $0 \leqslant x \leqslant 2\pi$.

a) find the exact x-coordinates of any stationary points

b) determine whether each stationary point is a maximum, minimum or neither and give a brief explanation.

4 Find the coordinates of any stationary points on the curve $y = x - e^x$. Classify any such points as a maximum, minimum or neither and explain.

5 Find the coordinates of any stationary points for each function on the interval $0 \leqslant x \leqslant 2\pi$. Indicate whether a stationary point is a maximum, minimum or neither.

a) $f(x) = 4\sin x - \cos 2x$ b) $g(x) = \tan x(\tan x + 2)$

6 Find the equation of the normal line to the curve $y = 3 + \sin x$ at the point where $x = \dfrac{\pi}{2}$.

7 Consider the function $f(x) = e^x - x^3$.

a) Find $f'(x)$ and $f''(x)$.

b) Find the x-coordinates (accurate to three significant figures) for any points where $f'(x) = 0$.

c) Indicate the intervals for which $f(x)$ is increasing, and indicate the intervals for which $f(x)$ is decreasing.

d) For the values of x found in part b), state whether that point on the graph of f is a maximum, minimum or neither.

e) Find the x-coordinate of any inflexion point(s) for the graph of f.

f) Indicate the intervals for which $f(x)$ is concave up, and indicate the intervals for which $f(x)$ is concave down.

8 Show that the curves $y = e^{-x}$ and $y = e^{-x}\cos x$ are tangent at each point common to both curves. Sketch the two curves over the interval $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{3\pi}{2}$.

9 A particle moves in a straight line such that its displacement, s metres, is given by $s(t) = 4\cos t - \cos 2t$. If the particle comes to rest after T seconds, where $T > 0$, find:

a) the particle's acceleration at time T

b) the maximum speed of the particle for $0 < t < T$.

10 Find an equation for a line that is tangent to the graph of $y = e^x$ that passes through the origin.

11 Consider the exponential function $f(x) = 2^x$.

a) Find $f'(x)$.

b) Find the equation of the tangent to the graph of f at the point (0, 1).

c) Explain why the graph of f has no stationary points.

12 Consider the function $h(x) = \frac{x^2 - 3}{e^x}$.

a) Find the exact coordinates of any stationary points.

b) Determine whether each stationary point is a maximum, minimum or neither.

c) What do the function values approach as (i) $x \to \infty$ and (ii) $x \to -\infty$.

d) Write down the equation of any asymptotes for the graph of $h(x)$.

e) Make an accurate sketch of the curve indicating any extrema and points where the graph intersects the x- and y-axis.

13 Given $y = \sin x$, and $\frac{dy}{dx} = \sin(x + a)$, $\frac{d^2y}{dx^2} = \sin(x + b)$ and $\frac{d^3y}{dx^3} = \sin(x + c)$, find:

a) the values of a, b and c

b) a formula for $\frac{d^{(n)}y}{dx^{(n)}}$.

14 a) Find the first three derivatives of $y = xe^x$.

b) Suggest a formula for $\frac{d^{(n)}}{dx^{(n)}}(xe^x)$ that is true for all positive integers n.

c) Prove that your formula is true by using mathematical induction.

15.3 Implicit differentiation, logarithmic functions and inverse trigonometric functions

Implicit differentiation

An equation such as $3x - 2y - 8 = 0$ is said to define y as a function of x because it satisfies the definition of a function in that each value of x (domain) determines (corresponds to) a unique value of y (range). We can manipulate the equation in order to solve for y in terms of x, giving $y = \frac{3}{2}x - 4$. In this form, in which y is alone on one side of the equation, the equation is said to define y **explicitly** as a function of x. In the original form of the equation, $x - 2y - 8 = 0$, the function is said to define y **implicitly** as a function of x. If we wish to find the derivative of y with respect to x, $\frac{dy}{dx}$, from an equation in which y is defined implicitly as a function of x we can often solve for y and then differentiate using one of the rules that we have established. For example, if we were asked to find $\frac{dy}{dx}$ for the equation $xy = 1$ we can write y explicitly as a function of x and then differentiate.

$$xy = 1 \Rightarrow y = \frac{1}{x} = x^{-1} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$$

Most of the functions that we have encountered thus far can be described by expressing one variable explicitly in terms of another variable – for example, $y = \cos(2x)$ or $y = \sqrt{1 - x^2}$. But how do we find the derivative y for an equation where we are not able to solve for y explicitly? For example, if we have the equation

$$x^3 + y^3 - 9xy = 0 \text{ (Figure 15.10)}$$

we cannot solve for y in terms of x. However, there may exist one or more functions f such that if $y = f(x)$ then the equation

$$x^3 + [f(x)]^3 - 9x[f(x)] = 0$$

holds for all values of x in the domain of f. Hence, the function f is defined implicitly by the given equation.

With the assumption that the equation $x^3 + y^3 - 9xy = 0$ defines y as at least one differentiable function of x (see Figure 15.11), the derivative of y with respect to x, $\frac{dy}{dx}$, can be found by the technique of **implicit differentiation**.

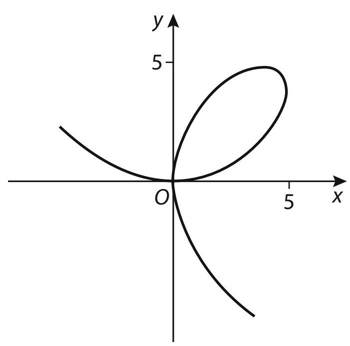


Figure 15.10 The graph of $x^3 + y^3 - 9xy = 0$ (called a *folium*, Latin for 'leaf'). This type of curve was first studied by Rene Descartes in 1638.

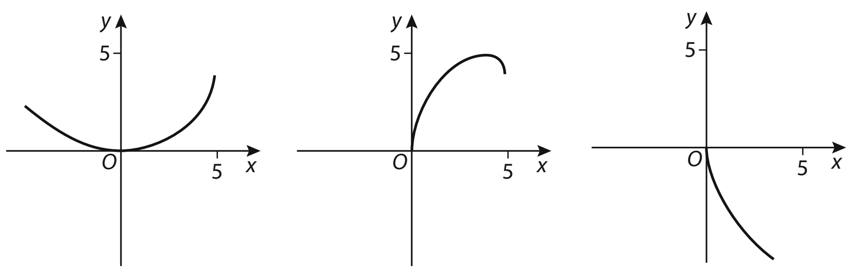


Figure 15.11 Although the equation $x^3 + y^3 - 9xy = 0$ is not a function, we can see that the graph of the equation can be separated into the graphs of three separate functions (they each pass the vertical line test for a function). This demonstrates that the equation implicitly defines y as three functions of x.

Initially we differentiate term-by-term, with respect to x, obtaining

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0).$$

The first and last terms are easily differentiated, and we can apply the constant rule to the third term, giving

$$3x^2 + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = 0.$$

Differentiating the second and third terms is a little more complicated requiring the use of the chain rule (and also product rule for the third term). If y is defined implicitly as a function of x, then y^3 is also a (composite) function of x. Thus, applying the appropriate rules, we have

$$3x^2 + 3y^2 \cdot \frac{d}{dx}(y) - 9\left(x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x)\right) = 0$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 9\left(x \cdot \frac{dy}{dx} + y\right) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9y = 0$$

Now we solve the equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx}(3y^2 - 9x) = -3x^2 + 9y \Rightarrow \frac{dy}{dx} = \frac{-3x^2 + 9y}{3y^2 - 9x}$$

Therefore, $\frac{dy}{dx} = \frac{-x^2 + 3y}{y^2 - 3x}$.

The process of implicit differentiation has given us a formula for $\frac{dy}{dx}$ that is the slope of the curve at any point (except where there is a vertical tangent and slope is undefined) and it is in terms of *both* x and y. This is not unexpected since we can see from the graph of the equation (Figure 15.10) that it is possible for two or three different points on the curve to have the same x-coordinate and the slope of the curve (given by $\frac{dy}{dx}$) will depend on the values of both x and y, and not only x as with functions where y is explicitly defined in terms of x.

In the examples and exercises of this section it is assumed that for any given equation y is implicitly defined as a differentiable function of x (or more than one differentiable function as in the above example) so that the technique of implicit differentiation can be applied.

Process of implicit differentiation

1. Differentiate, term-by-term, both sides of the equation **with respect to** x. The chain rule must be applied for any terms containing y.
2. Collect all terms containing $\frac{dy}{dx}$ on one side of the equation and all other terms on the other side.
3. Factor out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$ by dividing both sides by the factor multiplying $\frac{dy}{dx}$.
5. Simplify the result, if possible.

Example 14

Consider the equation for the unit circle $x^2 + y^2 = 1$ which is a relation (not a function).

a) Solve for y, and write all equations that express y as a function of x.
   Find $\frac{dy}{dx}$ for each of these functions.

b) Find $\frac{dy}{dx}$ by implicit differentiation.

c) Find the equation of the line tangent to the unit circle at the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Solution

a) Solving for y produces two equations, each defining y as a function of x.

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow y = \sqrt{1 - x^2} \text{ and } y = -\sqrt{1 - x^2}$$

Differentiating each of these with respect to x gives,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{1 - x^2}\right) = \frac{d}{dx}\left[(1 - x^2)^{\frac{1}{2}}\right] = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x) \Rightarrow$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1 - x^2}}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\sqrt{1 - x^2}\right) = \frac{d}{dx}\left[-(1 - x^2)^{\frac{1}{2}}\right]$$

$$= -\frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x) \Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}}$$

For the function $y = \sqrt{1 - x^2}$ we have $\frac{dy}{dx} = \frac{-x}{\sqrt{1 - x^2}}$.

Since $y = \sqrt{1 - x^2}$, then $\frac{dy}{dx} = -\frac{x}{y}$.

For the function $y = -\sqrt{1 - x^2}$ we have $\frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}}$.

Since $y = -\sqrt{1 - x^2}, \Rightarrow -y = \sqrt{1 - x^2}$, then $\frac{dy}{dx} = -\frac{x}{y}$.

b) $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$ Differentiating both sides term-by-term.

$2x + 2y\frac{dy}{dx} = 0$ Chain rule applied to differentiate y^2.

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$ Solving for $\frac{dy}{dx}$.

Therefore, $\frac{dy}{dx} = -\frac{x}{y}$.

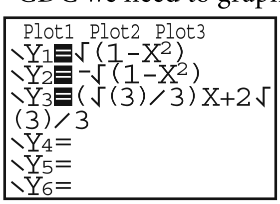
c) At the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ the slope of the tangent line is $\frac{dy}{dx} = -\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$

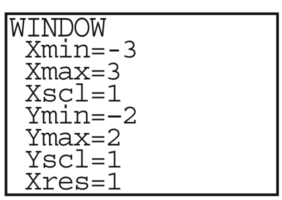
$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

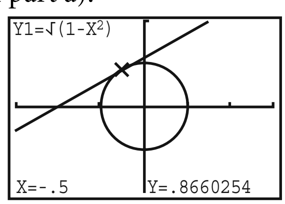
Substituting into the point-slope form gives,

$$y - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}\left(x + \frac{1}{2}\right) \Rightarrow y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{2} \Rightarrow y = \frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}$$

We can get a visual check by graphing the unit circle and the tangent line on our GDC. In order to graph the complete unit circle on our GDC we need to graph both functions found in part a).







● **Hint:** Example 14 illustrates that even when it is possible to solve an equation explicitly for y in terms of x, it may be more efficient to find $\frac{dy}{dx}$ by implicit differentiation.

Example 15

a) Find the points on the graph of $x^2 + 4xy + 13y^2 = 9$ at which the tangent is horizontal.

b) Determine whether each point is a maximum, minimum or neither.

Solution

a) We need to find $\frac{dy}{dx}$ which we do by implicit differentiation.

$$\frac{d}{dx}(x^2) + 4\frac{d}{dx}(xy) + 13\frac{d}{dx}(y^2) = \frac{d}{dx}(9)$$

Differentiating both sides term-by-term.

$$2x + 4\left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right) + 13\left(2y\frac{d}{dx}(y)\right) = 0$$

Applying chain and product rules.

$$2x + 4x\frac{dy}{dx} + 4y + 26y\frac{dy}{dx} = 0$$

Collecting terms containing $\frac{dy}{dx}$ on one side.

$$\frac{dy}{dx}(4x + 26y) = -2x - 4y$$

Factor out $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-2x - 4y}{4x + 26y} = \frac{-x - 2y}{2x + 13y}$$

Solving for $\frac{dy}{dx}$.

To find horizontal tangents, solve $\frac{dy}{dx} = 0$.

$$\frac{-x - 2y}{2x + 13y} = 0 \Rightarrow -x - 2y = 0 \Rightarrow y = -\frac{x}{2}$$

Of course, there are an infinite number of ordered pairs (x, y) that satisfy the equation $y = -\frac{x}{2}$. But the only ordered pairs that we want are ones that are on the curve $x^2 + 4xy + 13y^2 = 9$. So we substitute $-\frac{x}{2}$ for y and solve to find x-coordinates of points on the curve where $\frac{dy}{dx} = 0$.

$$x^2 + 4xy + 13y^2 = 9 \Rightarrow x^2 + 4x\left(-\frac{x}{2}\right) + 13\left(-\frac{x}{2}\right)^2 = 9$$

$$x^2 - 2x^2 + \frac{13}{4}x^2 = 9$$

$$4x^2 - 8x^2 + 13x^2 = 36$$

Multiplying both sides by 4.

$$9x^2 = 36$$

$$x^2 = 4 \Rightarrow x = 2 \text{ or } x = -2$$

y-coordinates: for $x = 2$, $y = -\frac{2}{2} = -1$; for $x = -2$, $y = -\left(\frac{-2}{2}\right) = 1$

Therefore, the tangents to the curve at $(2, -1)$ and $(-2, 1)$ are horizontal.

b) It is very difficult to determine the nature of the points $(2, -1)$ and $(-2, 1)$ by testing the sign of the derivative to either side of each point. Since $\frac{dy}{dx}$ is in terms of both x and y we need an explicit equation for y in terms of x to find the y-coordinate – but no explicit equation for y exists.

It is also impossible to graph the curve $x^2 + 4xy + 13y^2 = 9$ on our GDC to see its shape. Let's find the second derivative, $\frac{d^2y}{dx^2}$, and apply the second derivative test (Section 13.3).

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x-2y}{2x+13y}\right)$$

$$= \frac{(2x+13y)\left[\frac{d}{dx}(-x-2y)\right] - (-x-2y)[\frac{d}{dx}(2x+13y)]}{(2x+13y)^2}$$

Applying quotient rule.

$$= \frac{(2x+13y)\left(-1-2\frac{dy}{dx}\right) + (x+2y)\left(2+13\frac{dy}{dx}\right)}{(2x+13y)^2}$$

$$= (2x+13y)\left(-1-2\left(\frac{-x-2y}{2x+13y}\right)\right) + (x+2y)\left(2+13\left(\frac{-x-2y}{2x+13y}\right)\right)$$

Substituting for $\frac{dy}{dx}$.

$$= \frac{2x+13y}{2x+13y} \cdot \frac{(2x+13y)\left(-1+\frac{2x+4y}{2x+13y}\right) + (x+2y)\left(2-\frac{13x+26y}{2x+13y}\right)}{(2x+13y)^2}$$

$$= \frac{(2x+13y)(-2x-13y+2x+4y) + (x+2y)(4x+26y-13x-26y)}{(2x+13y)^3}$$

$$= \frac{(2x+13y)(-9y) + (x+2y)(-9x)}{(2x+13y)^3}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{9x^2+36xy+117y^2}{(2x+13y)^3} = \frac{-9(x^2+4xy+13y^2)}{(2x+13y)^3}$$

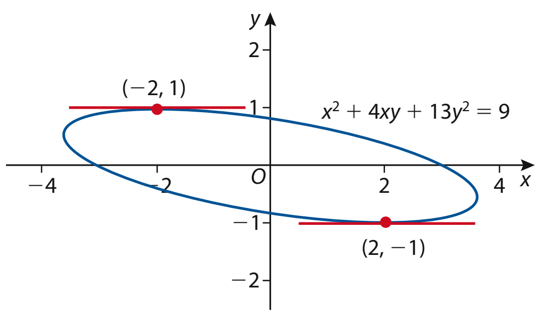
Now applying the second derivative test for both points where $\frac{dy}{dx} = 0$, we have

for $(2, -1)$, $\frac{d^2y}{dx^2} = \frac{-9(2^2+4(2)(-1)+13(-1)^2)}{(2(2)^2+13(-1))^3}$

$= \frac{81}{125} > 0 \Rightarrow (2, -1)$ is a minimum

for $(-2, 1)$, $\frac{d^2y}{dx^2} = \frac{-9(-2)^2+4(-2)(1)+13(1)^2)}{(2(-2)^2+13(1))^3}$

$= -\frac{3}{343} < 0 \Rightarrow (-2, 1)$ is a maximum



Even though it is not possible to graph the curve $x^2 + 4xy + 13y^2 = 9$ on our GDC, it is possible to find graphing software that can. The graph visually confirms our results for parts a) and b) of Example 15.

Previously we have established the rules for differentiating trigonometric functions and exponential functions. We still need to determine how to differentiate other important non-algebraic functions, namely logarithmic functions and inverse trigonometric functions.

Derivatives of logarithmic functions

At the start of the previous section we explored how we can often form a strong conjecture for the derivative of a function by analyzing the shape of the function's graph with the aid of some features of our GDC. Let's take this informal approach for finding the derivative for the natural logarithm function, $y = \ln x$, and then check our conjecture by deriving $\frac{d}{dx}(\ln x)$ by means of implicit differentiation.

The graph of $y = \ln x$ (Figure 15.12) is a particularly straightforward one. Its x-intercept is $(1, 0)$, and since its domain is all positive real numbers, it has no y-intercept. It is asymptotic to the y-axis, and the graph rises steadily, though less steeply as $x \to \infty$. There is neither an upper nor a lower bound, so its range is all real numbers.

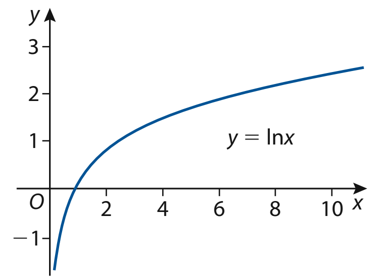
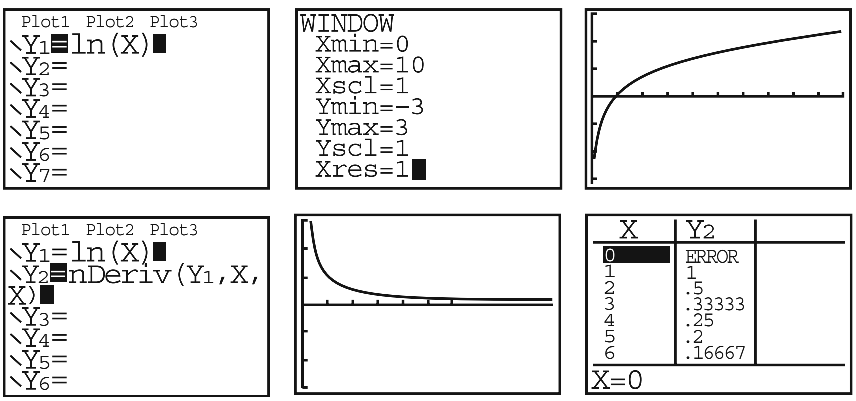


Figure 15.12

Let's cleverly use our GDC to view a graph of $y = \ln x$, a graph of its derivative, and to construct a table of ordered pairs with x and the value of the derivative at x (as computed by the GDC).



In the table, each value in the $\mathbf{Y}_2$ column is the slope of the curve (derivative) at the particular x value for $y = \ln x$. From the graph of the derivative and especially from the table, we conjecture that the derivative of $\ln x$ is $\frac{1}{x}$. This agrees with the fact that for $x > 0$, the slope of the graph of $y = \ln x$ is always positive and as x increases the slope decreases.

The inverse of $y = \ln x$ is $y = e^x$. Knowing this and that $\frac{d}{dx}(e^x) = e^x$, we can use implicit differentiation to confirm our conjecture.

$$y = \ln x$$

$$e^y = x \qquad \text{Inverse function relationship.}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \qquad \text{Differentiate implicitly.}$$

$$e^y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \text{Substituting } x \text{ for } e^y.$$

Therefore, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

The derivative of the natural logarithm function

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Or, in Leibniz notation, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

It is interesting to note that that the derivative of the **non-algebraic** function $f(x) = \ln x$ is the **algebraic** function $f'(x) = \frac{1}{x}$. Non-algebraic functions, such as trigonometric, exponential and logarithmic functions are often referred to as 'transcendental' functions. A **transcendental function** is a function that is not algebraic – in other words, it cannot be composed of a finite number of the elementary operations of addition, subtraction, multiplication, division and extracting a root. A **transcendental number** is a real number that is not a root of any polynomial equation with rational coefficients. For example, π and e are transcendental numbers.

What about the derivative of a logarithmic function with a base, b, other than e; that is, logarithmic functions other than the natural logarithmic function?

To find the derivative of $\log_b x$ with any base ($b > 0$, $b \neq 1$), we can use the change of base formula (Section 5.4) for logarithms to express $\log_b x$ in terms of the natural logarithm, $\ln x$, and then differentiate.

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{Applying change of base formula.}$$

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{d}{dx}\left(\frac{1}{\ln b} \cdot \ln x\right) \qquad \text{Differentiating both sides.}$$

$$= \frac{1}{\ln b} \cdot \frac{d}{dx}(\ln x) \qquad \frac{1}{\ln b} \text{ is a constant.}$$

$$= \frac{1}{\ln b} \cdot \frac{1}{x}$$

Therefore, $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$.

The derivative of the general logarithm function

If $f(x) = \log_b x$ ($b > 0$, $b \neq 1$), then $f'(x) = \frac{1}{x \ln b}$. Or, in Leibniz notation, $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$.

Example 16

a) Given $g(x) = \frac{1+x}{1-x}$, find $g'(x)$.

b) Hence, find $f'(x)$ for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$.

c) (i) Show that $f(x)$ is an odd function.
(ii) Show that $f(x)$ has no stationary points.
(iii) Show that $f(x)$ has one point of inflexion, and give its coordinates.

Solution

a) $g'(x) = \dfrac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$ Applying quotient rule.

$= \dfrac{1-x+1+x}{(1-x)^2}$

$\therefore g'(x) = \dfrac{2}{(1-x)^2}$

b) $f'(x) = \dfrac{d}{dx}\left[\ln\left(\dfrac{1+x}{1-x}\right)\right] = \dfrac{1}{\frac{1+x}{1-x}} \cdot \dfrac{d}{dx}\left(\dfrac{1+x}{1-x}\right)$ Applying $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and chain rule.

$= \left(\dfrac{1-x}{1+x}\right)\left(\dfrac{2}{(1-x)^2}\right)$ Substituting result from part a).

$= \dfrac{1}{1+x} \cdot \dfrac{2}{1-x}$

$\therefore f'(x) = \dfrac{2}{1-x^2}$

c) (i) In Section 7.3 we stated that a function f is odd if, for each x in the domain of f, $f(-x) = -f(x)$ with its graph symmetric about the origin. This symmetry leads to the fact (see question 25 in Exercise 13.2) that the graph of the derivative of an odd function is symmetric about the y-axis, i.e. an even function. A function f is even if $f(-x) = f(x)$. Thus, it will suffice to show that $f'(x)$ is even in order to show that $f(x)$ is odd.

$f'(-x) = \dfrac{2}{1-(-x)^2} = \dfrac{2}{1-x^2} = f(x)$

Therefore, $f'(x)$ is even and it follows that $f(x)$ is odd.

(ii) A stationary point for a function can only occur where its derivative is zero.

Clearly, $f'(x) = \dfrac{2}{1-x^2} \neq 0$ because a rational expression can only equal zero when its numerator is zero. Therefore, $f(x)$ has no stationary points.

(iii) To find any inflexion points we start by finding where the second derivative is zero.

$f''(x) = \dfrac{d}{dx}\left(\dfrac{2}{1-x^2}\right) = 2\dfrac{d}{dx}[(1-x^2)^{-1}]$ Power and chain rules instead of quotient rule.

$= 2[-(1-x^2)^{-2}(-2x)]$

$= f''(x) = \dfrac{4x}{(1-x^2)^2} = 0$ when $x = 0$

To confirm that an inflexion point does occur at $x = 0$ we need to show that the concavity of the graph of f changes at $x = 0$ ($f''(x)$ changes sign). Because $f(x)$ is defined only for $-1 < x < 1$, we choose $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ as test points.

$$f''\left(-\frac{1}{2}\right) = \frac{4\left(-\frac{1}{2}\right)}{\left(1 - \left(-\frac{1}{2}\right)^2\right)^2} = -\frac{32}{9} < 0 \text{ and}$$

$$f''\left(\frac{1}{2}\right) = \frac{4\left(\frac{1}{2}\right)}{\left(1 - \left(\frac{1}{2}\right)^2\right)^2} = \frac{32}{9} > 0$$

Since $f''(x)$ changes sign (and $f(x)$ changes concavity) at $x = 0$, f has an inflexion point there. $f(0) = \ln\left(\frac{1+0}{1-0}\right) = \ln(1) = 0$. Therefore, the inflexion point is at $(0, 0)$. (See GDC images below).

```
Plot1  Plot2  Plot3
\Y1■ln((1+X)/(1-
X))
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=-1.25
 Xmax=1.25
 Xscl=1
 Ymin=-6
 Ymax=6
 Yscl=1
 Xres=1
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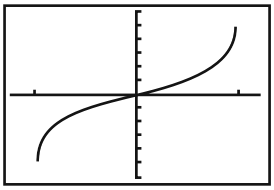


Example 17

Find the equation of the line tangent to the graph of $y = \log_{10}(x^3)$ at the point $x = 4$. Express the equation exactly with any logarithms being expressed as natural logarithms.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}[\log_{10}(x^3)] = \frac{1}{x^3 \ln 10} \cdot \frac{d}{dx}(x^3)$$ Applying $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$ and chain rule.

$$= \frac{1}{x^3 \ln 10} \cdot 3x^2$$

$$\frac{dy}{dx} = \frac{3}{x \ln 10}$$

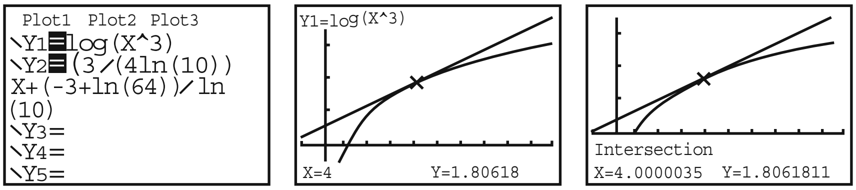
[Alternatively, we could have used laws of logarithms to write $y = \log_{10}(x^3) = 3\log_{10} x$ and then $\frac{dy}{dx} = 3\frac{d}{dx}(\log_{10} x) = \frac{3}{x \ln 10}$, avoiding use of the chain rule.]

When $x = 4$, $\frac{dy}{dx} = \frac{3}{4 \ln 10}$ and $y = \log_{10}(4^3) = \log_{10} 64 = \frac{\ln 64}{\ln 10}$ (using change of base formula). Thus, the tangent line intersects the curve at the point $\left(4, \frac{\ln 64}{\ln 10}\right)$ and has a slope of $\frac{3}{4 \ln 10}$. Substituting into the point-slope form for a linear equation gives:

$$y - \frac{\ln 64}{\ln 10} = \frac{3}{4 \ln 10}(x - 4) \quad \Rightarrow \quad y = \frac{3x}{4 \ln 10} - \frac{3}{\ln 10} + \frac{\ln 64}{\ln 10} \quad \Rightarrow$$

$$y = \frac{3x}{4 \ln 10} + \frac{-3 + \ln 64}{\ln 10}$$

Graphing the curve $y = \log_{10}(x^3)$ and the computed tangent line appears to give a good visual confirmation that the equation of the tangent line is correct.



Derivatives of inverse trigonometric functions

In the preceding pages, we established that the derivative of the *non-algebraic* (transcendental) function $f(x) = \ln x$ is the *algebraic* function $f'(x) = \frac{1}{x}$. The same is true for the inverse trigonometric functions – they are transcendental but their derivatives are algebraic. The inverse trigonometric functions were discussed in Section 7.6. We will now use implicit differentiation to find the derivatives of the inverse functions for sine, cosine, and tangent functions – which are usually referred to as arcsin x, arccos x and arctan x respectively. Their graphs are shown again in Figure 15.13.

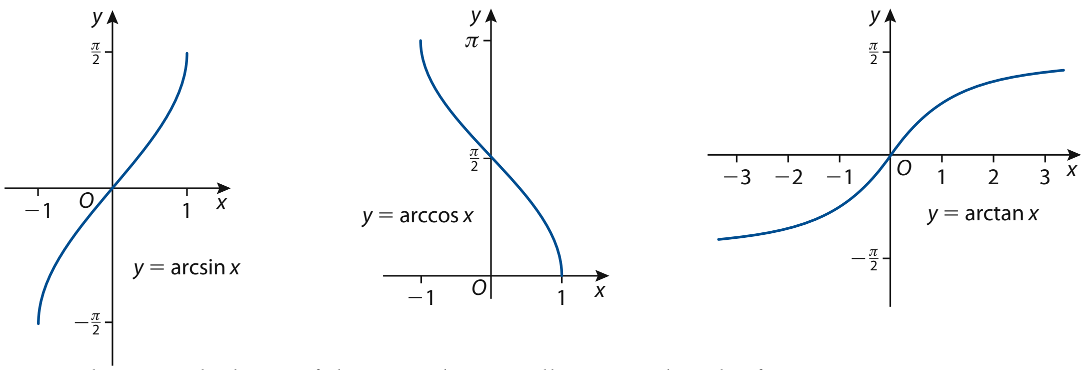


Figure 15.13

Given the smooth shape of their graphs we will assume that the functions $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$ are differentiable (i.e. the derivative exists) except where a vertical tangent exists. Since $y = \arcsin x$ and $y = \arccos x$ have vertical tangents at $x = -1$ and $x = 1$ they are differentiable throughout the interval $-1 < x < 1$. $y = \arctan x$ is differentiable for all real numbers.

• **Hint:** Recall from Chapter 7 that the notations $y = \arcsin x$ and $y = \sin^{-1} x$ are synonymous, but we will generally use $y = \arcsin x$.

Recall the definition of the arcsine function,

$$y = \arcsin x \Rightarrow \sin y = x \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

Differentiating $\sin y = x$ implicitly with respect to x gives:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \qquad \text{Differentiating both sides.}$$

$$(\cos y)\frac{dy}{dx} = 1 \qquad \text{Implicit differentiation.}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad \text{Dividing by } \cos y.$$

That is, $\frac{d}{dx}(\arcsin x) = \frac{1}{\cos y}$.

Dividing by $\cos y$ in the last step is allowed because $\cos y \neq 0$ for the interval in which $y = \arcsin x$ is differentiable, i.e. $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (quadrants I and IV). In fact, $\cos y > 0$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$. From the identity $\sin^2 x + \cos^2 x = 1$ we have $\cos x = \pm\sqrt{1 - \sin^2 x}$. Since $\cos y > 0$ we can replace $\cos y$ with $\sqrt{1 - \sin^2 y}$ and because $\sin y = x$ we get $\cos y = \sqrt{1 - x^2}$.

Therefore, $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$.

We can apply a similar process to find the derivative of the arcos x function, obtaining the result

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}.$$

Although the domain for the inverse sine and inverse cosine functions is the fairly narrow closed interval $-1 \leqslant x \leqslant 1$ and they are differentiable on the open interval $-1 < x < 1$, the inverse tangent function is defined and differentiable for all real numbers. To find $\frac{d}{dx}(\arctan x)$, we follow a similar procedure to that for $\frac{d}{dx}(\arcsin x)$.

The definition of the inverse tangent (arctan) function is

$$y = \arctan x \Rightarrow \tan y = x \text{ for } -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}.$$

Differentiating $\tan y = x$ implicitly with respect to x gives:

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$ Differentiating both sides.

$$(\sec^2 y)\frac{dy}{dx} = 1$$ Implicit differentiation.

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$ Dividing by $\sec^2 y$.

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}$$ Applying identity $1 + \tan^2 y = \sec^2 y$.

Therefore, $\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$. $\tan y = x$.

The derivatives for the inverse secant, inverse cosecant and inverse cotangent functions can also be found by means of implicit differentiation. They are included in the list below but are not necessary for this course.

Derivatives of the inverse trigonometric functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\text{arccsc}\, x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\text{arcsec}\, x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\text{arccot}\, x) = -\frac{1}{1 + x^2}$$

Example 18

Find the $\frac{dy}{dx}$ for each of the following.

a) $y = \cos^{-1}(e^{2x})$

b) $y = x\arcsin 2x + \frac{1}{2}\sqrt{1 - 4x^2}$

c) $\ln(x + y) = \arctan\left(\frac{x}{y}\right)$

Solution

a) $\dfrac{dy}{dx} = \dfrac{d}{dx}[\cos^{-1}(e^{2x})] = \dfrac{-1}{\sqrt{1-(e^{2x})^2}} \cdot \dfrac{d}{dx}(e^{2x})$ Chain rule and $\dfrac{d}{dx}(\arccos x) = \dfrac{-1}{\sqrt{1-x^2}}$.

$$= \frac{-1}{\sqrt{1-e^{4x}}} \cdot e^{2x} \cdot 2$$ Chain rule, again.

$$\frac{dy}{dx} = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

b) $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(x\arcsin 2x + \dfrac{1}{2}(1-4x^2)^{\frac{1}{2}}\right)$

$$= x\frac{d}{dx}(\arcsin 2x) + \arcsin 2x\frac{d}{dx}(x) + \frac{1}{2}\cdot\frac{1}{2}(1-4x^2)^{-\frac{1}{2}}\frac{d}{dx}(1-4x^2)$$

$$= x\left(\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right) + \arcsin 2x + \frac{-8x}{4\sqrt{1-4x^2}}$$

$$= \frac{2x}{\sqrt{1-4x^2}} + \arcsin 2x + \frac{-2x}{\sqrt{1-4x^2}}$$

$$\frac{dy}{dx} = \arcsin 2x$$

c) $\dfrac{d}{dx}[\ln(x+y)] = \dfrac{d}{dx}\left[\arctan\left(\dfrac{x}{y}\right)\right]$ Differentiating both sides implicitly.

$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$ Chain rule, $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$, quotient rule.

$$\frac{1+\frac{dy}{dx}}{x+y} = \frac{y - x\frac{dy}{dx}}{x^2+y^2}$$

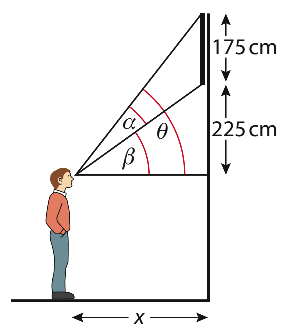
$$x^2 + y^2 + \frac{dy}{dx}x^2 + \frac{dy}{dx}y^2 = xy + y^2 - \frac{dy}{dx}x^2 - \frac{dy}{dx}xy$$

$$\frac{dy}{dx}(2x^2 + xy + y^2) = xy - x^2$$

$$\frac{dy}{dx} = \frac{xy - x^2}{2x^2 + xy + y^2}$$

Example 19

A painting that is 175 cm from top to bottom is hanging on the wall of a gallery such that it's base is 225 cm above the eye level of an observer. How far from the wall should the observer stand to get the best view of the painting, that is, so that the angle subtended at the observer's eye by the painting is a maximum? (This is similar to Example 34 in Section 7.6.)



Solution

Change all lengths from centimetres to metres.

$\tan\theta = \dfrac{4}{x}$ and $\tan\beta = \dfrac{\frac{9}{4}}{x}$

Because $0 < \theta < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$, we have

$\theta = \arctan\frac{4}{x}$ and $\beta = \arctan\frac{\frac{9}{4}}{x}$.

Substituting these values of θ and β into the equation $\alpha = \theta - \beta$ gives

$\alpha = \arctan\frac{4}{x} - \arctan\frac{\frac{9}{4}}{x}$.

Differentiating with respect to x gives:

$$\frac{d\alpha}{dx} = \frac{d}{dx}\left[\arctan(4x^{-1}) - \arctan\left(\frac{9}{4}x^{-1}\right)\right]$$

$$= \frac{1}{1+(4x^{-1})^2}(-4x^{-2}) - \frac{1}{1+\left(\frac{9}{4}x^{-1}\right)^2}\left(-\frac{9}{4}x^{-2}\right)$$

$$= \frac{-4}{x^2+16} + \frac{\frac{9}{4}}{x^2+\frac{81}{16}}$$

$$= \frac{-4}{x^2+16} + \frac{36}{16x^2+81}$$

Setting $\frac{d\alpha}{dx} = 0$, we get:

$$36(x^2+16) - 4(16x^2+81) = 0$$

$$-28x^2 + 252 = 0$$

$$x^2 = \tfrac{252}{28} = 9 \Rightarrow x = \pm 3, \text{ however } x \neq -3$$

We use the first derivative test to determine if the angle α is a maximum when $x = 3$, using test values of $x = 2$ and $x = 4$.

When $x = 2$, $\frac{d\alpha}{dx} = \frac{7}{145} > 0$ and when $x = 4$, $\frac{d\alpha}{dx} = -\frac{49}{2696} < 0$.

Hence, the angle α has an absolute maximum value at $x = 3$. Therefore, the observer should stand 3 metres away from the wall to get the 'best' view of the painting.

Summary of differentiation rules

Derivative of $f(x)$	$y = f(x)$	$\Rightarrow$	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
Derivative of x^n	$f(x) = x^n$	$\Rightarrow$	$f'(x) = nx^{n-1}$
Derivative of $\sin x$	$f(x) = \sin x$	$\Rightarrow$	$f'(x) = \cos x$
Derivative of $\cos x$	$f(x) = \cos x$	$\Rightarrow$	$f'(x) = -\sin x$
Derivative of $\tan x$	$f(x) = \tan x$	$\Rightarrow$	$f'(x) = \sec^2 x$
Derivative of $\sec x$	$f(x) = \sec x$	$\Rightarrow$	$f'(x) = \sec x \tan x$
Derivative of $\csc x$	$f(x) = \csc x$	$\Rightarrow$	$f'(x) = -\csc x \cot x$
Derivative of $\cot x$	$f(x) = \cot x$	$\Rightarrow$	$f'(x) = -\csc^2 x$

Note: derivative rules for trigonometric functions only apply if x is in radian measure.

Derivative of e^x	$f(x) = e^x$	$\Rightarrow$	$f'(x) = e^x$
Derivative of b^x	$f(x) = b^x$	$\Rightarrow$	$f'(x) = b^x \ln b$
Derivative of $\ln x$	$f(x) = \ln x$	$\Rightarrow$	$f'(x) = \frac{1}{x}$
Derivative of $\log_b x$	$f(x) = \log_b x$	$\Rightarrow$	$f'(x) = \frac{1}{x \ln b}$

Derivative of arcsin x $\quad f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

Derivative of arccos x $\quad f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$

Derivative of arctan x $\quad f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$

Derivative of arcsec x $\quad f(x) = \operatorname{arcsec} x \Rightarrow f'(x) = \frac{1}{x\sqrt{x^2-1}}$

Derivative of arccsc x $\quad f(x) = \operatorname{arccsc} x \Rightarrow f'(x) = -\frac{1}{x\sqrt{x^2-1}}$

Derivative of arccot x $\quad f(x) = \operatorname{arccot} x \Rightarrow f'(x) = -\frac{1}{1+x^2}$

Chain rule for composite functions: $\frac{dy}{dx} = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Product rule: $\frac{dy}{dx} = \frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

Quotient rule: $\frac{dy}{dx} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

Exercise 15.3

In questions 1–12, find the derivative of y with respect to x, $\frac{dy}{dx}$, by implicit differentiation.

1 $x^2 + y^2 = 16$

2 $x^2y + xy^2 = 6$

3 $x = \tan y$

4 $x^2 - 3xy^2 + y^3x - y^2 = 2$

5 $\frac{x}{y} - \frac{y}{x} = 1$

6 $xy\sqrt{x+y} = 1$

7 $x + \sin y = xy$

8 $x^2y^3 = x^4 - y^4$

9 $xy + e^y = 0$

10 $(x+2)^2 + (y+3)^2 = 25$

11 $x = \tan y$

12 $y + \sqrt{xy} = 3x^3$

In questions 13–16, find the lines that are a) tangent and b) normal to the curve at the given point.

13 $x^3 - xy - 3y^2 = 0, (2, -2)$

14 $16x^4 + y^4 = 32, (1, 2)$

15 $2xy + \pi \sin y = 2\pi, \left(1, \frac{\pi}{2}\right)$

16 $\sqrt[3]{xy} = 14x + y, (2, -32)$

17 For the circle $x^2 + y^2 = r^2$ show that the tangent line at any point (x_1, y_1) on the circle is perpendicular to the line that passes through (x_1, y_1) and the centre of the circle.

18 Consider the equation $x^2 + xy + y^2 = 7$.

a) Find the two points where the curve intersects the x-axis. Show that the tangents to the curve at these two points are parallel.

b) Find any points where the tangent to the curve is parallel to the x-axis.

c) Find any points where the tangent to the curve is parallel to the y-axis.

19 The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1) intersects the curve at what other point?

In questions 20 and 21, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the given equation.

20 $4x^2 + 9y^2 = 36$

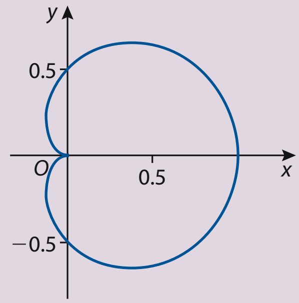
21 $xy = 2x - 3y$

22 Consider the equation $xy^3 = 1$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by two different methods.

a) Solve for y in terms of x and differentiate explicitly.

b) Differentiate implicitly.

23 The graph (shown right) of the equation $x^2 + y^2 = 2x^2 + 2y^2 - x^2$ is a type of curve called a *cardioid*. A cardioid is a heart-shaped curve generated by a fixed point on a circle as it rolls around another circle having the same radius. Find the equation of the line tangent to this particular cardioid at the point $(0, \frac{1}{2})$.



In questions 24–33, find the derivative of y with respect to x, $\frac{dy}{dx}$.

24 $y = \ln(x^3 + 1)$

25 $y = \ln(\sin x)$

26 $y = \log_5 \sqrt{x^2 - 1}$

27 $y = \ln\sqrt{\frac{1 + x}{1 - x}}$

28 $y = \sqrt{\log_{10} x}$

29 $y = \ln\left(\frac{a - x}{a + x}\right)$

30 $y = \ln(e^{\cos x})$

31 $y = \frac{1}{\log_3 x}$

32 $y = x\ln(x) - x$

33 $y = \ln(ax) - (\ln b)\log_b x$

34 Find the equation of the line tangent to the graph of $y = \log_2 x$ at the point $x = 8$. Express the equation exactly. Can you find a way to graph $y = \log_2 x$ on your GDC in order to check your answer?

35 Given $y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$ we could find $\frac{dy}{dx}$ by applying the chain rule and the quotient rule. However, it is much easier to first take the natural logarithm of both sides, use the properties of logarithms to simplify as much as possible, and then differentiate implicitly to find $\frac{dy}{dx}$. This technique is called *logarithmic differentiation*. Use this technique to show that $\frac{dy}{dx} = \frac{2x}{(x^2 - 1)^{\frac{1}{2}}(x^2 + 1)^{\frac{3}{2}}}$.

36 Find the x-coordinate, between 0 and 1, of the point of inflexion on the graph of the function $f(x) = x^2\ln(x^2)$. Express your answer exactly.

37 a) Given $g(x) = \frac{\ln x}{x}$, find expressions for $g'(x)$ and $g''(x)$.

b) Show that g has an absolute maximum at $x = e$, and state the maximum value of g.

In questions 38–41, find the derivative of y with respect to x, $\frac{dy}{dx}$.

38 $y = \arctan(x + 1)$

39 $y = \sin^{-1}\left(\frac{x}{\sqrt{1 + x^2}}\right)$

40 $y = \arccos\left(\frac{3}{x^2}\right)$

41 $\ln\sqrt{1 + x^2} = x\tan^{-1} x$

42 Given that $f(x) = \arcsin x + \arccos x$, find $f'(x)$. What can you conclude about the function f?

43 Show if a is a constant that

a) $\dfrac{d}{dx}\left[\arctan\left(\dfrac{x}{a}\right)\right] = \dfrac{a}{a^2 + x^2}$ b) $\dfrac{d}{dx}\left[\arcsin\left(\dfrac{x}{a}\right)\right] = \dfrac{1}{\sqrt{a^2 - x^2}}$

44 Find the equation of the line tangent to the curve $y = 4x \arctan 2x$ at the point on the curve where $x = \frac{1}{2}$. Express the equation exactly in the form $y = mx + c$, where m and c are constants.

45 Consider the function $f(x) = \arcsin(\cos x)$ with domain of $0 \leqslant x < \pi$.

a) Prove that f is a linear function.

b) Express the function exactly in the form $f(x) = ax + b$, where a and b are constants.

46 A 3-metre tall statue is on top of a column such that the bottom of the statue is 2 metres above the eye level of a person viewing the statue. How far from the base of the column should the person stand to get the best view of the statue, that is, so that the angle subtended at the observer's eye by the statue is a maximum?

47 A particle moves along the x-axis so that its displacement, s (in metres), from the origin at any time $t \geqslant 0$ (in seconds) is given by $s(t) = \arctan\sqrt{t}$.

a) Find the exact velocity of the particle at (i) $t = 1$ second, and at (ii) $t = 4$ seconds.

b) Find the exact acceleration of the particle at (i) $t = 1$ second, and at (ii) $t = 4$ seconds.

c) Describe the motion of the particle.

d) What is the limiting displacement of the particle as t approaches infinity?

15.4 Related rates

A claim was made in the first section of this chapter that 'the chain rule is the most important, and most widely used, rule of differentiation'. The chain rule has been repeatedly applied in all parts of this chapter thus far. Another important use of the chain rule is to find the rates of change of two or more variables that are changing with respect to time. Calculus provides us with the tools and techniques to solve problems where quantities (variables) are changing rather than static.

When a stone is thrown into a pond, a circular pattern of ripples is formed. In this situation we can observe an ever-widening circle moving across the water. As the circular ripple moves across the water, the radius r of the circle, its circumference C, and its area A all increase as a function of time t. Not only are these quantities (variables) functions of time, but their values at any particular time t are related to one another by familiar formulae such as $C = 2\pi r$ and $A = \pi r^2$. Thus their rates of change are also related to one another.

Example 20

A stone is thrown into a pond causing ripples in the form of concentric circles to move away from the point of impact at a rate of 20 cm per second. Find the following when a circular ripple has a radius of 50 cm and again when its radius is 100 cm.

a) the rate of change of the circle's circumference

b) the rate of change of the circle's area

Solution

In calculus, a derivative represents a rate of change of one variable with respect to another variable. If the circles are moving outward at a rate of 20 cm/sec, then the rate of change of the radius is 20 cm/sec, and in the notation of calculus we write

$$\frac{dr}{dt} = 20.$$

a) Knowing that the relationship between the radius, r, and the circumference, C, is $C = 2\pi r$, and that the rate of change of the radius with respect to time is $\frac{dr}{dt} = 20$, we can use the chain rule to find the rate of change of the circumference with respect to time, i.e. $\frac{dC}{dt}$.

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$$

We need to find $\frac{dC}{dr}$, the rate of change (derivative) of the circumference with respect to the radius. This rate can be derived from the relationship between the variables.

$$C = 2\pi r$$

$$\frac{d}{dr}(C) = \frac{d}{dr}(2\pi r)$$ Differentiate both sides with respect to r.

$$\frac{dC}{dr} = 2\pi$$ Implicit differentiation on the left side.

Since the circumference C is a linear function of the radius r ($C = 2\pi r$), the derivative $\frac{dC}{dr}$ is a constant.

We now substitute in for $\frac{dC}{dr}$ and $\frac{dr}{dt}$ to find the rate of change of the circumference with respect to time, $\frac{dC}{dt}$.

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt} \Rightarrow \frac{dC}{dt} = 2\pi \cdot 20 = 40\pi \text{ cm/sec}$$

The rate of change of a circular ripple's circumference is constant (40π). Therefore, the rate of change of the circumference is 40π cm/sec when the radius is 50 cm and also when its 100 cm.

b) Similarly, to find the rate of change of the area with respect to time, $\frac{dA}{dt}$, we can use the chain rule to write

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}.$$

Find $\frac{dA}{dr}$ from the formula, $A = \pi r^2$, that relates the variables A and r.

$$\frac{d}{dr}(A) = \frac{d}{dr}(\pi r^2)$$ Differentiate both sides with respect to r.

$$\frac{dA}{dr} = \pi(2r) = 2\pi r$$ Implicit differentiation on the left side.

● **Hint:** There is a slightly different method to determine $\frac{dC}{dt}$. We can find the rate by differentiating implicitly with respect to time, t, both sides of the equation, $C = 2\pi r$, that gives the relationship between the two changing quantities (variables).

$C = 2\pi r$

Differentiate both sides with respect to t:

$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$

Implicit differentiation:

$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$

Substitute $\frac{dr}{dt} = 20$:

$\frac{dC}{dt} = 2\pi \cdot 20 = 40\pi$ cm/sec

Since the area A is a non-linear function of the radius r ($A = \pi r^2$), the derivative $\frac{dA}{dr}$ is not a constant but has different values depending on the value of r.

We substitute in for $\frac{dA}{dr}$ and $\frac{dr}{dt}$ to find the rate of change of the area with respect to time, $\frac{dA}{dt}$.

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} \Rightarrow \frac{dA}{dt} = 2\pi r \cdot 20 = 40\pi r$$

Thus, the rate of change of the circle's area with respect to time, $\frac{dA}{dt}$, is a linear function in terms of the radius r.

When the radius is 50 cm, $\frac{dA}{dt} = 40\pi \cdot 50 = 2000\pi\,\text{cm}^2/\text{sec}$
$\approx 6280\,\text{cm}^2/\text{sec}$ [$\approx 0.628\,\text{m}^2/\text{sec}$].

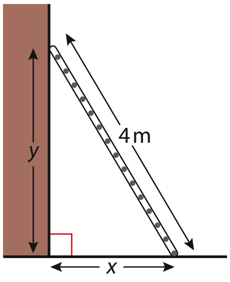
When the radius is 100 cm, $\frac{dA}{dt} = 40\pi \cdot 100 = 4000\pi\,\text{cm}^2/\text{sec}$
$\approx 12\,600\,\text{cm}^2/\text{sec}$ [$\approx 1.26\,\text{m}^2/\text{sec}$].

Note that when $r = 100$ cm the area is changing at twice the rate it was when $r = 50$ cm.

● **Hint:** It is important to include the appropriate units when giving a rate of change (derivative) answer. For example cm/sec, m^2/hour, litres/sec, etc.

Example 21

A 4-metre ladder stands upright against a vertical wall. If the foot of the ladder is pulled away from the wall at a constant rate of 0.75 m/sec, how fast is the top of the ladder coming down the wall at the instant it is (i) 3 metres above the ground, and (ii) 1 metre above the ground. Give answers approximate to three significant figures.



Solution

Let x and y represent the distances of the foot and top of the ladder, respectively, from the bottom of the wall. Then from Pythagoras' theorem, we have

$$x^2 + y^2 = 16.$$

Given that the ladder is being pulled away at a rate of 0.75 m/sec, then

$$\frac{dx}{dt} = 0.75 = \frac{3}{4}.$$

So we know the rate $\frac{dx}{dt}$, and we need to find $\frac{dy}{dt}$ when $y = 3$ and when $y = 1$.

Rather than starting with the chain rule and writing an equation relating the different rates, let's utilize the chain rule by differentiating implicitly with respect to time the equation relating the relevant variables x and y.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(16)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

(i) We know $\frac{dx}{dt} = \frac{3}{4}$, so to find $\frac{dy}{dt}$ when $y = 3$ m, we find the corresponding value for x.

$$x^2 + y^2 = 16 \Rightarrow x = \sqrt{16 - y^2}; \text{ for } y = 3\text{: } x = \sqrt{16 - 3^2} = \sqrt{7}$$

Hence, when $y = 3$: $\frac{dy}{dt} = -\frac{\sqrt{7}}{3} \cdot \frac{3}{4} = -\frac{\sqrt{7}}{4} \approx -0.661$ m/sec.

(ii) For $y = 1$: $x = \sqrt{16 - 1^2} = \sqrt{15}$

Hence, when $y = 1$: $\frac{dy}{dt} = -\frac{\sqrt{15}}{1} \cdot \frac{3}{4} = -\frac{3\sqrt{15}}{4} \approx -2.90$ m/sec.

It makes sense that $\frac{dy}{dt}$ is negative because the distance y decreases as the ladders slides down.

Example 22

In the preceding example, how fast is the angle between the ladder and the ground changing when $y = 2$ m?

Solution

We know $\frac{dx}{dt} = \frac{3}{4}$ and we seek to find $\frac{d\theta}{dt}$. We need a relationship, true at any instant, between the variables θ and x. Several trigonometric ratios could be used, but perhaps the most straightforward is

$$x = 4\cos\theta.$$

Now we differentiate implicitly with respect to t and solve for $\frac{d\theta}{dt}$.

$$\frac{d}{dt}(x) = \frac{d}{dt}(4\cos\theta)$$

$$\frac{dx}{dt} = -4\sin\theta\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{4\sin\theta}\frac{dx}{dt}$$

When $y = 2$ we find that $\sin\theta = \frac{1}{2}$. Substituting appropriately for $\sin\theta$ and $\frac{d\theta}{dt}$, we have

$$\frac{d\theta}{dt} = -\frac{1}{4\left(\frac{1}{2}\right)} \cdot \frac{3}{4} = -\frac{3}{8}.$$

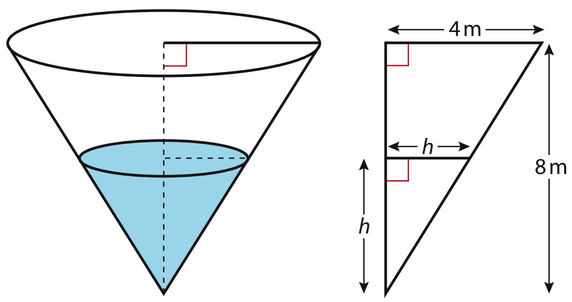
Therefore, the angle is decreasing at a rate of $\frac{3}{8}$ radians/sec (or approximately 21.5°/sec).

The solution strategy used in the preceding two examples is summarized below.

Solving problems involving related rates

1. Identify any rate(s) of change you know and the rate of change to be found.
2. Draw a diagram with all of the important information clearly labelled.
3. Write an equation relating the variables whose rates of change are either known or are to be found.
4. Using the chain rule, differentiate the equation implicitly with respect to time. Solve for the rate to be found.
5. Substitute in all known values for any variables and any rates of change. Compute the required rate of change. Be sure to include appropriate units with the result.

Example 23



Consider a conical tank as shown in the diagram. Its radius at the top is 4 metres and its height is 8 metres. The tank is being filled with water at a rate of 2 m^3/min. How fast is the water level rising when it is 5 metres high?

Solution

We know the rate of change of the volume with respect to time, that is, $\frac{dV}{dt} = 2$ m^3/min and we seek to find the rate of change of the height of the water level with respect to time, call it $\frac{dh}{dt}$.

Not including t, there are three variables involved in this problem: V, r and h. The formula for the volume of a cone will give us an equation that relates all of these variables.

$$V = \tfrac{1}{3}\pi r^2 h$$

If we differentiate this equation now we will get the rate $\frac{dr}{dt}$ in our result. We need to either find $\frac{dr}{dt}$ (which is possible) or eliminate r from the equation by solving for it in terms of one of the other variables and substitute. By using similar triangles we can write a proportion involving r and h.

$$\frac{r}{h} = \frac{4}{8} \Rightarrow r = \frac{h}{2}$$

Hence, $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \Rightarrow V = \frac{\pi}{12}h^3$.

Differentiating implicitly with respect to t and solving for $\frac{dh}{dt}$:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2\frac{dh}{dt} \Rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 5$ and $\frac{dV}{dt} = 2$ gives

$$\frac{dh}{dt} = \frac{4}{\pi(5)^2} \cdot 2 = \frac{8}{25\pi} \approx 0.102 \text{ m/min [or 10.2 cm/min]}.$$

Therefore, the water level is rising at a rate of 0.102 m/min when the water level is at 5 m.

● **Hint:** Be careful not to substitute in known quantities too early in the process of solving a related rates problem. Substitute the known values of any variables and any rates of change *after* differentiation. For example, in Example 23 h remained a variable (it is a quantity that is changing over time) until the last stage of the solution when we substituted $h = 5$. If we substituted earlier into $V = \frac{\pi}{12}h^3$, we would have obtained $\frac{dV}{dt} = 0$, which is obviously wrong.

The following example involves two rates of change.

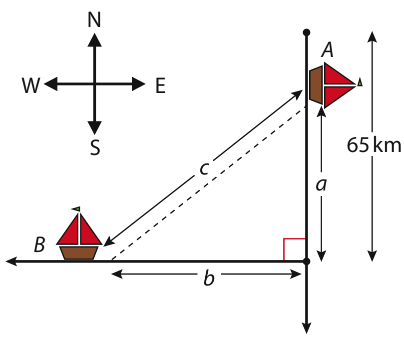
Example 24

At 12 noon ship A is 65 km due north of a second ship, B. Ship A sails south at a rate of 14 km/hr, and ship B sails west at a rate of 16 km/hr.

a) How fast are the two ships approaching each other $1\frac{1}{2}$ hours later at 1:30?

b) At what time do the two ships stop approaching and begin moving away from each other?

Solution



Let a and b be the distances that ships A and B, respectively, are from the intersection of the ships' paths (see diagram). Let c be the distance between the two ships. Since a is decreasing and b is increasing, we know that $\frac{da}{dt} = -14$ km/hr and $\frac{db}{dt} = 16$ km/hr.

a) The three variables are related by the equation

$$c^2 = a^2 + b^2.$$

Differentiating implicitly with respect to t gives

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}.$$

The rate at which the ships are approaching is $\frac{dc}{dt}$. Solving for $\frac{dc}{dt}$:

$$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c}$$

Substituting $\frac{da}{dt} = -14$ and $\frac{db}{dt} = 16$:

$$\frac{dc}{dt} = \frac{-14a + 16b}{c}$$

The distances a and b are both functions of time; thus, they can be written in terms of t as

$$a = 65 - 14t \text{ and } b = 16t.$$

Evaluating these expressions when $t = 1\frac{1}{2}$, gives $a = 44$, $b = 24$ and $c = \sqrt{44^2 + 24^2} \approx 50.12$. Substituting these values into the expression for $\frac{dc}{dt}$ gives

$$\frac{dc}{dt} \approx \frac{-14(44) + 16(24)}{50.12} \approx -4.629.$$

Therefore, at 1:30 the distance between the two ships is decreasing at a rate of approximately -4.63 km/hr.

b) The time at which the two ships will stop approaching each other and begin to move away is when the value of $\frac{dc}{dt}$ changes from negative to positive. So we need to find when $\frac{dc}{dt} = 0$.

$$\frac{dc}{dt} = \frac{-14a + 16b}{c} = 0 \Rightarrow -14a + 16b = 0$$

Substituting in $a = 65 - 14t$ and $b = 16t$ gives:

$$-14(65 - 14t) + (16t) = 0 \Rightarrow 452t - 910 = 0 \Rightarrow t = \frac{910}{452} \approx 2.013$$

Therefore, just moments after 2:00 the two ships will stop approaching and start moving away from each other.

Exercise 15.4

1 A water tank is in the shape of an inverted cone. Water is being drained from the tank at a constant rate of 2 m^3/min. (Since volume is decreasing, $\frac{dV}{dt}$ is negative.) The height of the tank is 8 m, and the diameter of the top of the tank is 6 m. When the height of the water is 5 m, find, in units of cm/min, the following:

a) the rate of change of the water level

b) the rate of change of the radius of the surface of the water.

2 A spherical balloon is being inflated at a constant rate of 240 cm^3/sec. [$V = \frac{4}{3}\pi r^3$]

a) At what rate is the radius increasing when the radius is equal to 8 cm?

b) At what rate is the radius increasing 5 seconds after the start of inflation?

3 Oil is dripping from a car engine on to a garage floor, making a growing circular stain. The radius, r, of the stain is increasing at a constant rate of 1 cm/hr. When the radius is 4 cm, find:

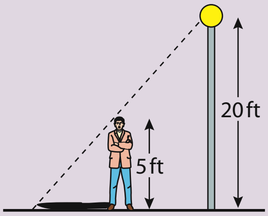
a) the rate of change of the circumference of the stain

b) the rate of change of the area of the stain.

4 A hot air balloon is rising straight up from a level field at a constant rate of 50 m/min. An observer is standing 150 m from the point on the ground where the balloon was launched. Let θ be the angle between the ground and the observer's line of sight to the balloon from the point at which the observer is standing (angle of elevation of the balloon). What is the rate of change of θ (in radians/min) when the height of the balloon is 250 m?

5 Jenny is flying a kite at a constant height above level ground of 72 m. The wind carries the kite away horizontally at a rate of 6 m/sec. How fast must Jenny let out the string at the moment when the kite is 120 m away from her?

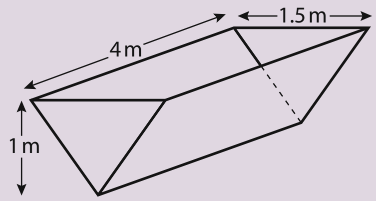
6 A 5-foot boy is walking toward a 20-foot lamp post at a constant rate of 6 ft/sec. The light from the lamp post causes the boy to cast a shadow. How fast is the tip of his shadow moving?



7 Two cars start from a point A at the same time. One travels west at 60 km/hr and the other travels north at 35 km/hr. How fast is the distance between them increasing 3 hours later?

8 A point moves along the curve $y = \sqrt{x^2 + 1}$ in such a way that $\frac{dx}{dt} = 4$. Find $\frac{dy}{dt}$ when $x = 3$.

9 A horizontal trough is 4 m long, 1.5 m wide and 1 m deep. Its cross-section is an isosceles triangle. Water is flowing into the trough at a constant rate of 0.03 m^3/sec. Find the rate at which the water level is rising 25 seconds after the water started flowing into the trough.



10 If the radius of a sphere is increasing at the constant rate of 3 mm/sec, how fast is the volume changing when the surface area is 10 mm^2? [Surface area $= 4\pi r^2$]

11 Two roads, A and B, intersect each other at an angle of 60°. Two cars, one on road A travelling at 40 km/hr and the other on road B travelling at 50 km/hr, are approaching the intersection. If, at a certain moment, the two cars are both 2 km from the intersection, how fast is the distance between them changing?

12 If the diagonal of a cube is increasing at a rate of 8 cm/sec, how fast is a side of the cube increasing?

13 A point P is moving along the circle with equation $x^2 + y^2 = 100$ at a constant rate of 3 units/sec. How fast is the projection of P on the x-axis moving when P is 5 units above the x-axis?

14 A jet is flying at a constant speed at an altitude of 10 000 m on a path that will take it directly over an observer on the ground. At a given instant the observer determines that the angle of elevation of the jet is $\frac{\pi}{3}$ radians and is increasing at a constant rate of $\frac{1}{60}$ radians/sec. Find the speed of the jet.

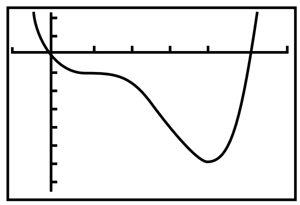
15 A television cameraman is filming an automobile race from a platform that is 40 metres from the racing track, following a car that is moving at 288 km/hr. How fast, in degrees per second, will the camera be turning when a) the car is directly in front of the camera and b) a half second later? Answer to the nearest whole degree.

16 A plane is flying due east at 640 km/hr and climbing vertically at a rate of 180 m/min. An airport tower is tracking it. Determine how fast the distance between the plane and the tower is changing when the plane is 5 km above the ground over a point exactly 6 km due west of the tower. Express the answer in km/hr.

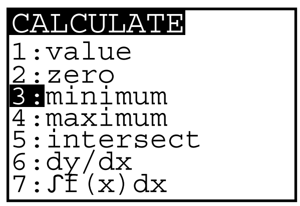
15.5 Optimization

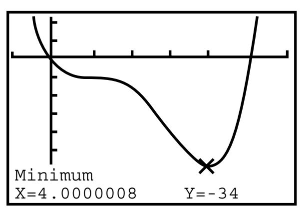
Many problems in science and mathematics involve finding the maximum or minimum value (**optimum** value) of a function over a specified or implied domain. The development of the calculus in the seventeenth century was motivated to a large extent by maxima and minima (**optimization**) problems. One such problem lead Pierre de Fermat (1601–1665) to develop his Principle of Least Time: a ray of light will follow the path that takes the least (or minimum) time. The solution to Fermat's principle lead to Snell's law, or law of refraction (see the investigation at the end of this section). The solution is found by applying techniques of differential calculus – which can also be used to solve other optimization problems involving ideas such as least cost, maximum profit, minimum surface area and greatest volume.

Previously, we learned the theory of how to use the derivative of a function to locate points where the function has a maximum or minimum (i.e. extreme) value. It is important to remember that if the derivative of a function is zero at a certain point it does not *necessarily* follow that the function has an extreme value (relative or absolute) at that point – it only ensures that the function has a horizontal tangent (stationary point) at that point. An extreme value *may* occur where the derivative is zero or at the endpoints of the function's domain.



The graph of $f(x) = x^4 - 8x^3 + 18x^2 - 16x - 2$ is shown left. The derivative of $f(x)$ is $f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2$. The function has horizontal tangents at both $x = 1$ and $x = 4$, since the derivative is zero at these points. However, an extreme value (absolute minimum) occurs only at $x = 4$. It is important to confirm – graphically (see GDC images) or algebraically – the precise nature of a point on a function where the derivative is zero. Some different algebraic methods for confirming that a value is a maximum or minimum will be illustrated in the examples that follow.



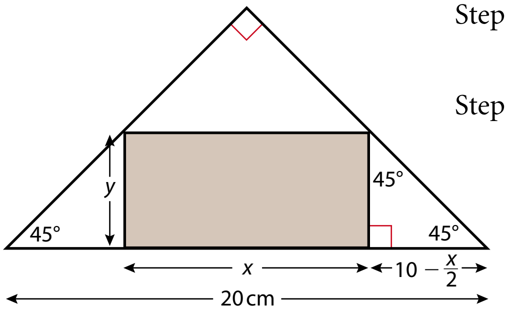


It is also useful to remember that one can often find extreme values (extrema) without calculus (e.g. using a 'minimum' command on a graphics calculator, as shown). Calculator or computer technology can be very helpful in modelling, solving or confirming solutions to optimization problems. However, it is important to learn how to apply algebraic methods of differentiation to optimization problems because it may be the only efficient way to obtain an accurate solution.

Let's start with a relatively straightforward example. We can use the steps in the solution to develop a general strategy that can be applied to more sophisticated problems.

Example 25 – Finding a maximum area (Developing a general strategy)

Find the maximum area of a rectangle inscribed in an isosceles right triangle whose hypotenuse is 20 cm long.



Solution

Step 1: Draw an accurate diagram. Let the base of the rectangle be x cm and the height y cm. Then the area of the rectangle is $A = xy\,\text{cm}^2$.

Step 2: Express area as a function in terms of only one variable.

It can be deduced from the diagram that $y = 10 - \frac{x}{2}$.

Therefore, $A(x) = x\left(10 - \frac{x}{2}\right) = 10x - \frac{x^2}{2}$.

x must be positive and from the diagram it is clear that x must be less than 20 (domain of A: $0 < x < 20$).

Step 3: Find the derivative of the area function and find for what value(s) of x it is zero.

$A'(x) = 10 - x \qquad A'(x) = 0$ when $x = 10$

Step 4: Analyze $A(x)$ at $x = 10$ and also at the endpoints of the domain, $x = 0$ and $x = 20$.

The second derivative test (Section 13.3) provides information about the concavity of a function. The second derivative is $A''(x) = -1$ and since $A''(x)$ is always negative then $A(x)$ is always concave down, indicating $A(x)$ has a maximum at $x = 10$.

$A(0) = 0$ and $A(20) = 0$, indicating $A(x)$ has an absolute maximum at $x = 10$.

Therefore, the rectangle has a maximum area equal to

$A(10) = 10\left(10 - \frac{10}{2}\right) = 50\,\text{cm}^2$.

General strategy for solving optimization problems

Step 1: Draw a diagram that accurately illustrates the problem. Label all known parts of the diagram. Using variables, label the important unknown quantity (or quantities) (for example, x for base and y for height in Example 25).

Step 2: For the quantity that is to be optimized (area in Example 25), express this quantity as a function in terms of a single variable. From the diagram and/or information provided, determine the domain of this function.

Step 3: Find the derivative of the function from Step 2, and determine where the derivative is zero. This value (or values) of the derivative, along with any domain endpoints, are the **critical values** ($x = 0$, $x = 10$ and $x = 20$ in Example 25) to be tested.

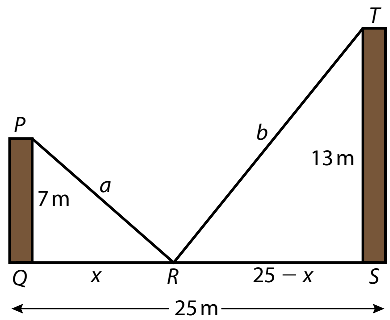
Step 4: Using algebraic (e.g. second derivative test) or graphical (e.g. GDC) methods, analyze the nature (maximum, minimum, neither) of the points at the critical values for the optimized function. Be sure to answer the precise question that was asked in the problem.

Example 26 – Finding a minimum length – two posts problem

Two vertical posts, with heights of 7 m and 13 m, are secured by a rope going from the top of one post to a point on the ground between the posts and then to the top of the other post. The distance between the two posts is 25 m. Where should the point at which the rope touches the ground be located so that the least amount of rope is used?

Solution

Step 1: An accurate diagram is drawn. The posts are drawn as line segments PQ and TS and the point where the rope touches the ground is labelled R. The optimum location of point R can be given as a distance from the base of the shorter post, QR, or from the taller post, SR. It is decided to give the answer as the distance from the shorter post – and this is labelled x. There are two other important unknown quantities: the lengths of the two portions of the rope, PR and TR. These are labelled a and b, respectively.



Step 2: The quantity to be minimized is the length L of the rope, which is the sum of a and b. From Pythagoras' theorem, $a = \sqrt{x^2 + 49}$ and $b = \sqrt{(25 - x)^2 + 169}$. Therefore, the function for length (L) can be expressed in terms of the single variable x as

$$L(x) = \sqrt{x^2 + 49} + \sqrt{(25 - x)^2 + 169}$$

$$= \sqrt{x^2 + 49} + \sqrt{x^2 - 50x + 625 + 169}$$

$$L(x) = \sqrt{x^2 + 49} + \sqrt{x^2 - 50x + 794}$$

From the given information and diagram, the domain of $L(x)$ is $0 \leqslant x \leqslant 25$.

Step 3: To facilitate differentiation, express $L(x)$ using fractional exponents:

$$L(x) = (x^2 + 49)^{\frac{1}{2}} + (x^2 - 50x + 794)^{\frac{1}{2}}$$

Then apply the chain rule for differentiation:

$$\frac{dL}{dx} = \frac{1}{2}(x^2 + 49)^{-\frac{1}{2}}(2x) + \frac{1}{2}(x^2 - 50x + 794)^{-\frac{1}{2}}(2x - 50) \Rightarrow$$

$$\frac{dL}{dx} = \frac{x}{\sqrt{x^2 + 49}} + \frac{x - 25}{\sqrt{x^2 - 50x + 794}}$$

By setting $\frac{dL}{dx} = 0$, we obtain

$$x\sqrt{x^2 - 50x + 794} = -(x - 25)\sqrt{x^2 + 49}$$

$$x^2(x^2 - 50x + 794) = (25 - x)^2(x^2 + 49)$$

$$x^4 - 50x^3 + 794x^2 = x^4 - 50x^3 + 674x^2 - 2450x + 30\,625$$

$$120x^2 + 2450x - 30\,625 = 0$$

$$5(4x - 35)(6x + 175) = 0$$

$$x = \frac{35}{4} \quad \text{or} \quad x = -\frac{175}{6}$$

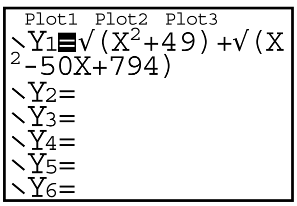
Step 4: Since $x = -\frac{175}{6}$ is not in the domain for $L(x)$, then the critical values are $x = 0$, $x = \frac{35}{4}$ and $x = 25$. Simply evaluate $L(x)$ for these critical values.

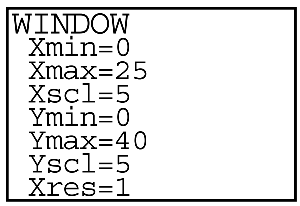
$L(0) = 7 + \sqrt{794} \approx 35.18, \quad L(25) = \sqrt{674} + 13 \approx 38.96,$

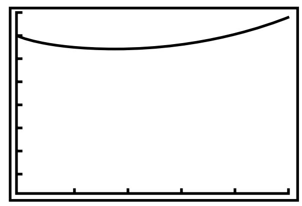
$L\left(\frac{35}{4}\right) = 5\sqrt{41} \approx 32.02$

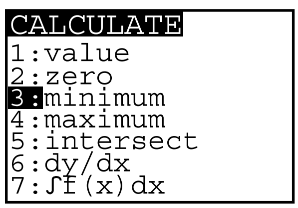
Therefore, the rope should touch the ground at a distance of $\frac{35}{4} = 8.75$ m from the base of the shorter post, to give a minimum rope length of approximately 32.02 m.

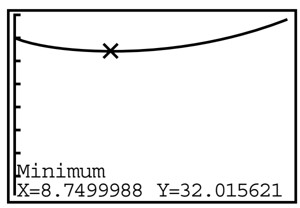
The minimum value could also be confirmed from the graph of $L(x)$, but it would be difficult to confirm using the second derivative test because of the algebra required. From this example, we can see that applied optimization problems can involve a high level of algebra. If you have access to suitable graphing technology, you could perform Steps 3 and 4 graphically rather than algebraically.











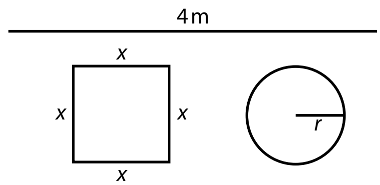
It is interesting to observe that the result for x produced by the calculator does not appear to be exact. Why is that? Algebraic techniques using differentiation give us the certainty of an exact solution while also allowing us to deal with the abstract nature of optimization problems involving parameters rather than fixed measurements (e.g. the heights of the posts).

In both Example 25 and 26, the extreme value occurred at a point where the derivative was zero. Although this often happens, an extreme value may occur at the endpoint of the domain.

Example 27 – An endpoint maximum

A supply of four metres of wire is to be used to form a square and a circle. How much of the wire should be used to make the square and how much should be used to make the circle in order to enclose the greatest amount of area? Guess the answer before looking at the following solution.

Solution



Step 1: Let $x =$ length of each edge of the square and $r =$ radius of the circle.

Step 2: The total area is given by $A = x^2 + \pi r^2$. The task is to write the area A as a function of a single variable. Therefore, it is necessary to express r in terms of x, or vice versa, and perform a substitution.

The perimeter of the square is $4x$ and the circumference of the circle is $2\pi r$. The total amount of wire is 4 m which gives

$$4 = 4x + 2\pi r \quad \Rightarrow \quad 2\pi r = 4 - 4x \quad \Rightarrow \quad r = \frac{2(1-x)}{\pi}$$

Substituting gives $A(x) = x^2 + \pi\left[\frac{2(1-x)}{\pi}\right]^2 = x^2 + \frac{4(1-x)^2}{\pi}$

$$= \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]$$

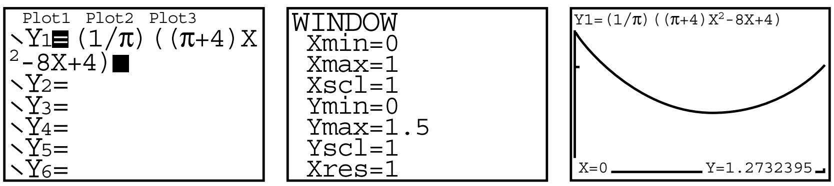
Because the square's perimeter is $4x$, then the domain for $A(x)$ is $0 \leqslant x \leqslant 1$.

Step 3: Differentiate the function $A(x)$, set equal to zero, and solve.

$$\frac{d}{dx}\left(\frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]\right) = \frac{1}{\pi}[2(\pi + 4)x - 8] = 0$$

$$2(\pi + 4)x - 8 = 0 \Rightarrow (\pi + 4)x = 4 \Rightarrow x = \frac{4}{\pi + 4} \approx 0.5601$$

The critical values are $x = 0$, $x \approx 0.5601$ and $x = 1$.

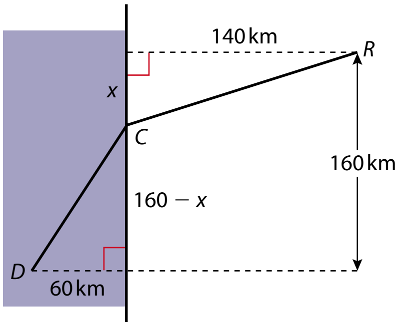


Step 4: Evaluating $A(x)$: $A(0) \approx 1.273$, $A(0.5601) \approx 0.5601$ and $A(1) = 1$. Therefore, the maximum area occurs when $x = 0$ which means <u>all</u> the wire is used for the circle.

What would the answer be if Example 27 asked for the dimensions of the square and circle to enclose the *least* total area?

Example 28 – Minimizing time

A pipeline needs to be constructed to link an offshore drilling rig to an onshore refinery depot. The oil rig is located at a distance (perpendicular to the coast) of 140 km from the coast. The depot is located inland at a distance (perpendicular) of 60 km from the coast. For modelling purposes, the coastline is assumed to follow a straight line. The point on the coastline nearest to the oil rig is 160 km from the point on the coastline nearest to the depot. The rate at which crude oil is pumped through the pipeline varies according to several variables, including pipe dimensions, materials, temperature, etc. On average, oil flows through the offshore section of the pipeline at a rate of 9 km per hour and 5 km per hour through the onshore section. Assume that both sections of pipeline can travel straight from one point to another. At what point should the pipeline intersect with the coastline in order for the oil to take a minimum amount of time to flow from the rig to the depot?



Solution

Step 1: The optimum location of the point, C, where the pipeline comes ashore will be designated by the distance, x, it is from the point on the coast that is a minimum distance (perpendicular) from the rig, R (140 km). The distance from R to C is $\sqrt{x^2 + 140^2}$ and the distance from D (depot) to C is $\sqrt{(160 - x)^2 + 60^2}$.

Step 2: The quantity to be minimized is time, so it is necessary to express the total time it takes the oil to flow from R to D in terms of a single variable.

$$\text{time} = \frac{\text{distance}}{\text{rate}} \quad \Rightarrow \quad \text{time (offshore)} = \frac{\sqrt{x^2 + 19\,600}\text{ km}}{9\text{ km/hr}};$$

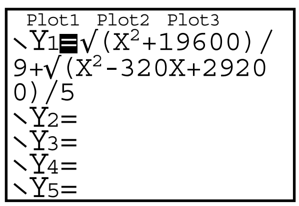
$$\text{time (onshore)} = \frac{\sqrt{x^2 - 320x + 29\,200}\text{ km}}{5\text{ km/hr}}$$

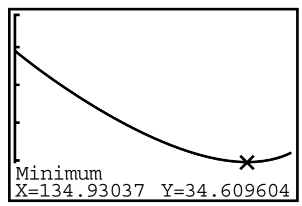
The function for time T in terms of x is:

$$T(x) = \frac{\sqrt{x^2 + 19\,600}}{9} + \frac{\sqrt{x^2 - 320x + 29\,200}}{5}$$

and the domain for $T(x)$ is $0 \leqslant x \leqslant 160$.

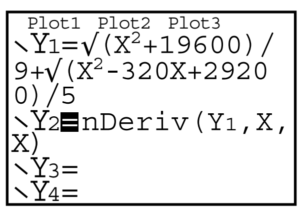
Steps 3/4: The algebra for finding the derivative of $T(x)$ is similar to that of Step 3 in Example 26. Let's use graphing technology to find the value of x that produces a minimum for $T(x)$.

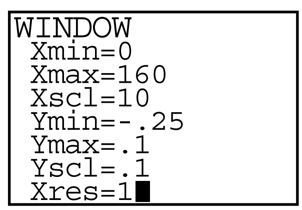


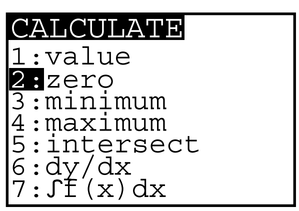


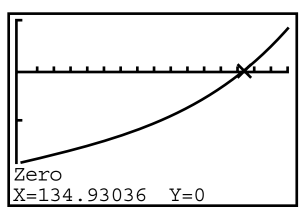
Therefore, the optimum point for the pipeline to intersect with the coast is approximately 134.9 km from the point on the coast nearest to the drilling rig.

The result could also be obtained by having a calculator or computer graph the derivative of $T(x)$ and compute any zeros for $T'(x)$ in the domain.





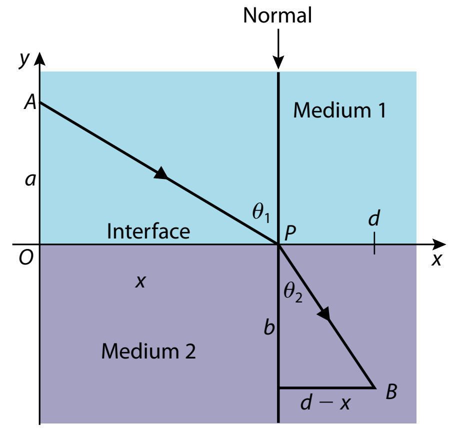




See the **Investigation** and how solving a problem similar to Example 28 derives Snell's law (or law of refraction).

Investigation – Snell's law

The speed of light depends on the medium through which light travels and is generally slower in denser media. The speed of light in a vacuum is an important physical constant and is exactly 299 792 458 m/s. A metre is defined to be the distance that light travels in a vacuum in $\frac{1}{299\,792\,458}$ of a second. Typically, the speed of light in a vacuum (denoted by the letter c) is given the approximate value of 3×10^8 m/s, but in the Earth's atmosphere light travels more slowly than that and even more slowly through glass and water.

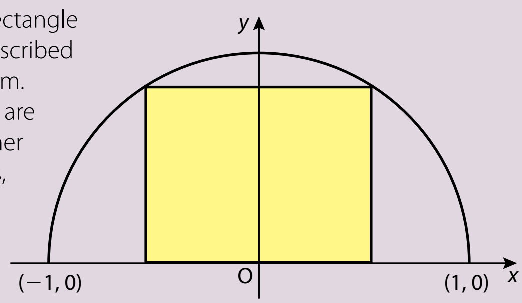


Fermat's principle in optics states that light travels from one point to another along a path for which time is a minimum. Investigate the path that a ray of light will follow in going from a point A in a transparent medium, where the speed of light is c_1, to a point B in a different transparent medium, where its speed is c_2, as illustrated in the diagram left. Using algebra and differentiation, prove that for time to be a minimum the following relationship must hold: $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$. This equation is known as Snell's law or the law of refraction. Why is a graphics calculator not helpful?

Assume that the two points, A and B, lie in the xy-plane and the x-axis (interface) separates the two media. A light ray is refracted (deflected) when it passes from one medium to another. θ_1 is the **angle of incidence** and θ_2 is the **angle of refraction** (both angles measured between ray and normal to the interface).

Exercise 15.5

1 Find the dimensions of the rectangle with maximum area that is inscribed in a semicircle with radius 1 cm. Two vertices of the rectangle are on the semicircle and the other two vertices are on the x-axis, as shown in the diagram.



2 A rectangular piece of aluminium is to be rolled to make a cylinder with open ends (a tube). Regardless of the dimensions of the rectangle, the perimeter of the rectangle must be 40 cm. Find the dimensions (length and width) of the rectangle that gives a maximum volume for the cylinder.

3 Find the minimum distance from the graph of the function $y = \sqrt{x}$ and the point $(\frac{3}{2}, 0)$.

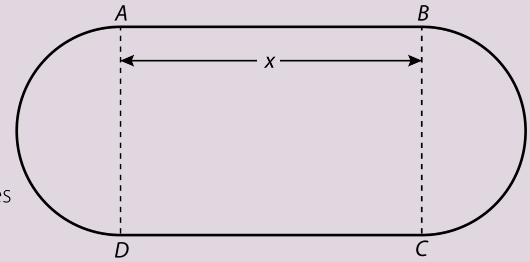
4 A rectangular box has height h cm, width x cm and length $2x$ cm. It is designed to have a volume equal to 1 litre (1000 cm^3).

a) Show that $h = \frac{500}{x^2}$ cm.

b) Find an expression for the total surface area, S cm^2, of the box in terms of x.

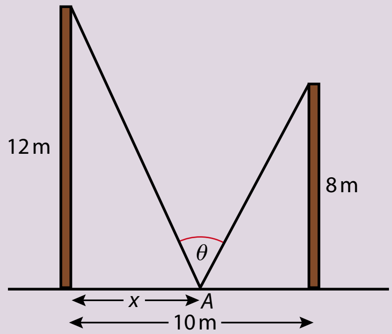
c) Find the dimensions of the box that produces a minimum surface area.

5 The figure right consists of a rectangle *ABCD* and two semicircles on either end. The rectangle has an area of 100 cm². If x represents the length of the rectangle *AB*, find the value of x that makes the perimeter of the entire figure a minimum.

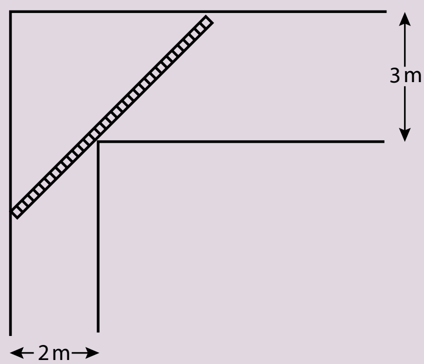


● **Hint:** Write an equation for θ in terms of x and find the value of x which makes θ a maximum by using your GDC.

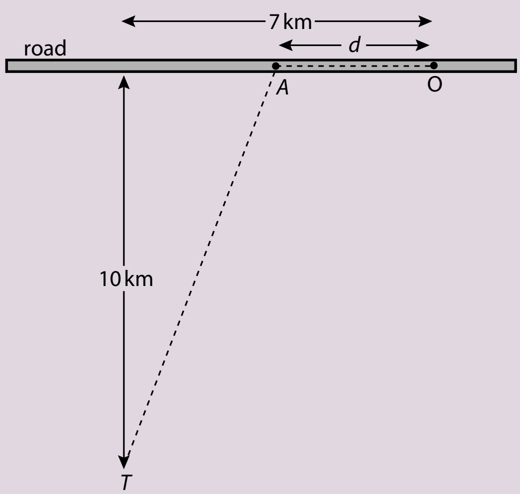
6 Two vertical posts, with heights 12 metres and 8 metres, are 10 metres apart on horizontal ground. A rope that stretches is attached to the top of both posts and is stretched down so that it touches the ground at point *A* between the two posts. The distance from the base of the taller post to point *A* is represented by x and the angle between the two sections of rope is θ. What value of x makes θ a maximum?



7 A ladder is to be carried horizontally down an L-shaped hallway. The first section of the hallway is 2 metres wide and then there is a right-angled turn into a 3-metre wide section. What is the longest ladder that can be carried around the corner?



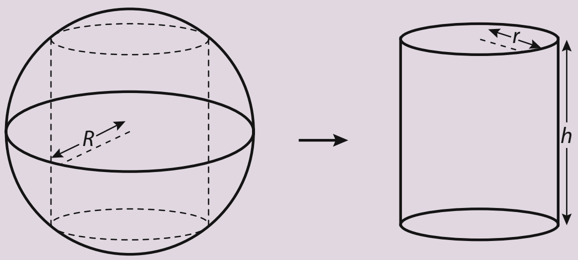
8 Charlie is walking from the wildlife observation tower (point *T*) to the Big Desert Park office (point *O*). The tower is 7 km due west and 10 km due south from the office. There is a road that goes to the office that Charlie can get to if she walks 10 km due north from the tower. Charlie can walk at a rate of 2 kilometres per hour (kph) through the sandy terrain of the park, but she can walk a faster rate of 5 kph on the road. To what point, *A*, on the road should Charlie walk to in order to take the least time to walk from the tower to the office? Find the value of *d* such that point *A* is *d* km from the office.



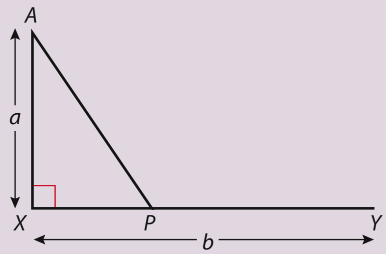
9 Two vertices of a rectangle are on the x-axis, and the other two vertices are on the curve $y = \dfrac{8}{x^2 + 4}$. (See Exercise 15.1, question 12.) Find the maximum area of the rectangle.

10 A ship sailing due south at 16 knots is 10 nautical miles north of a second ship going due west at 12 knots. Find the minimum distance between the two ships.

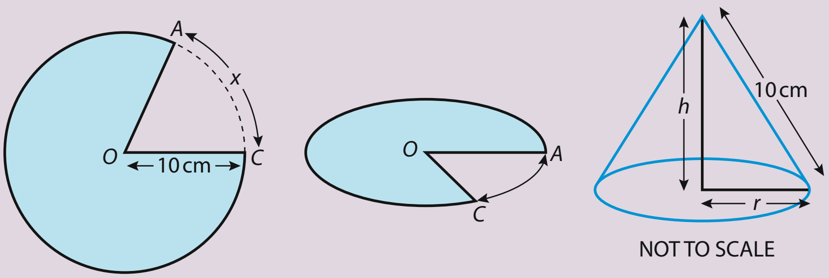
11 Find the height, h, and the base radius, r, of the largest right circular cylinder that can be made by cutting it away from a sphere with a radius of R.



12 Nadia is standing at point A that is a km away in the countryside from a straight road XY (see diagram). She wishes to reach the point Y where the distance from X to Y is b km. Her speed on the road is r km/hr and her speed travelling across the countryside is c km/hr, such that $r > c$. If she wishes to reach Y as quickly as possible, find the position of point P where she joins the road.

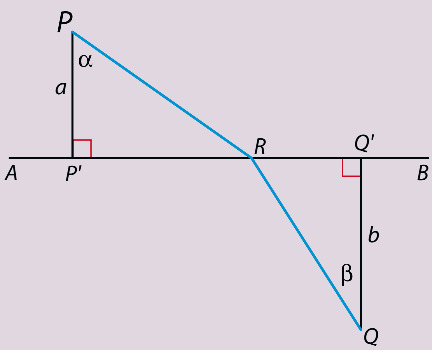


13 A cone of height h and radius r is constructed from a circle with radius 10 cm by removing a sector AOC of arc length x cm and then connecting the edges OA and OC. What arc length x will produce the cone of maximum volume, and what is the volume?



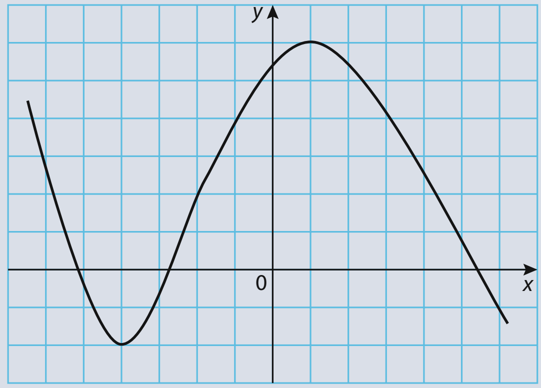
14 Point P is a units above the line AB, and point Q is b units below line AB (see diagram). The velocity of light is u units/second above AB and v units/second below AB, and $u > v$. The angles α and β are the angles that a ray of light makes with a perpendicular (normal) to line AB above and below AB, respectively. Show that the following relationship must hold true.

$$\frac{\sin\alpha}{\sin\beta} = \frac{u}{v}$$

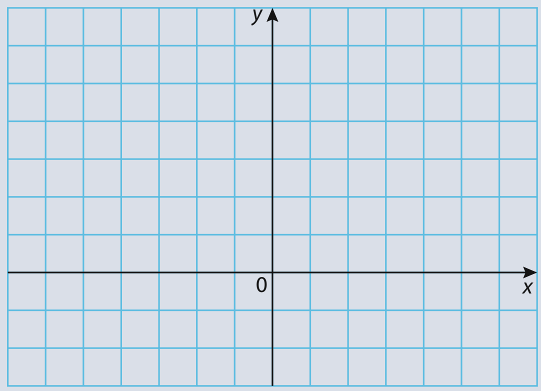


Practice questions

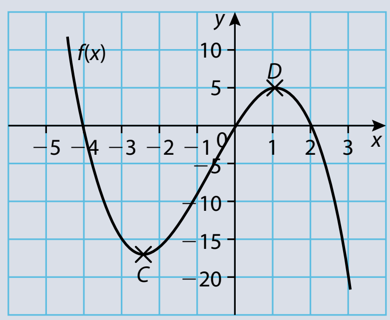
1 The diagram shows the graph of $y = f(x)$.



Copy the grid right and sketch the graph of $y = f'(x)$.



2 The diagram right shows part of the graph of the function $f: x \mapsto -x^3 - 2x^2 + 8x$.



The graph intersects the x-axis at $(-4, 0)$, $(0, 0)$ and $(2, 0)$. There is a minimum point at C and a maximum point at D.

a) The function may also be written in the form $f: x \mapsto -x(x - a)(x - b)$, where $a < b$. Write down the value of

(i) a **(ii)** b.

b) Find

(i) $f'(x)$

(ii) the exact values of x at which $f'(x) = 0$

(iii) the value of the function at D.

c) **(i)** Find the equation of the tangent to the graph of f at $(0, 0)$.

(ii) This tangent cuts the graph of f at another point. Give the x-coordinate of this point.

3 In a controlled experiment, a tennis ball is dropped from the uppermost observation deck (447 metres high) of the CN Tower in Toronto.

The tennis ball's velocity is given by

$$v(t) = 66 - 66e^{-0.15t}$$

where v is in metres per second and t is in seconds.

a) Find the value of v when

(i) $t = 0$ **(ii)** $t = 10$.

b) **(i)** Find an expression for the acceleration, a, as a function of t.

(ii) What is the value of a when $t = 0$?

c) **(i)** As t becomes large, what value does v approach?

(ii) As t becomes large, what value does a approach?

(iii) Explain the relationship between the answers to parts **c)(i)** and **(ii)**.

4 Given the function $f(x) = x^3 + 7x^2 + 8x - 3$,

a) identify any points as a relative maximum or minimum and find their exact coordinates

b) find the exact coordinates of any inflexion point(s).

5 Consider the function $g(x) = 2 + \frac{1}{e^{3x}}$.

a) **(i)** Find $g'(x)$.

(ii) Explain briefly how this shows that $g(x)$ is a decreasing function for all values of x (i.e. that $g(x)$ always decreases in value as x increases).

Let P be the point on the graph of g where $x = -\frac{1}{3}$.

b) Find an expression in terms of e for

(i) the y-coordinate of P

(ii) the gradient of the tangent to the curve at P.

c) Find the equation of the tangent to the curve at P, giving your answer in the form $y = mx + c$.

6 Consider the function f given by $f(x) = \frac{2x^2 - 13x + 20}{(x-1)^2}$, $x \neq 1$.

a) Show that $f'(x) = \frac{9x - 27}{(x-1)^3}$, $x \neq 1$.

The second derivative is given by $f''(x) = \frac{72 - 18x}{(x-1)^4}$, $x \neq 1$.

b) Using values of $f'(x)$ and $f''(x)$, explain why a minimum must occur at $x = 3$.

c) There is a point of inflexion on the graph of f. Write down the coordinates of this point.

7 Differentiate with respect to x:

a) $y = \frac{1}{(2x+3)^2}$

b) $y = e^{\sin 5x}$

c) $y = \tan^2(x^2)$

8 The curve with equation $y = Ax + B + \frac{C}{x}$, $x \in \mathbb{R}$, $x \neq 0$, has a minimum at $P(1, 4)$ and a maximum at $Q(-1, 0)$. Find the value of each of the constants A, B and C.

9 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point (1, 1) on the curve $x^3 + y^3 = 2$.

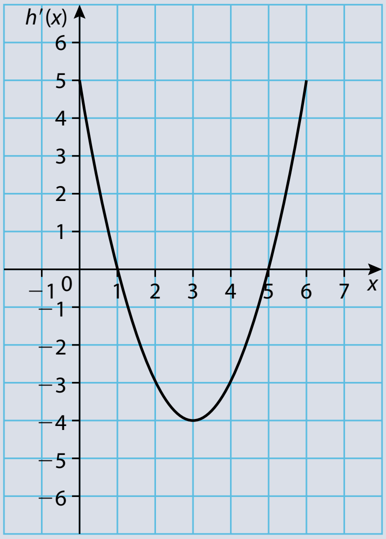
10 Differentiate with respect to x:

a) $y = \frac{x}{e^x - 1}$ **b)** $y = e^x \sin 2x$ **c)** $y = (x^2 - 1)\ln(3x)$

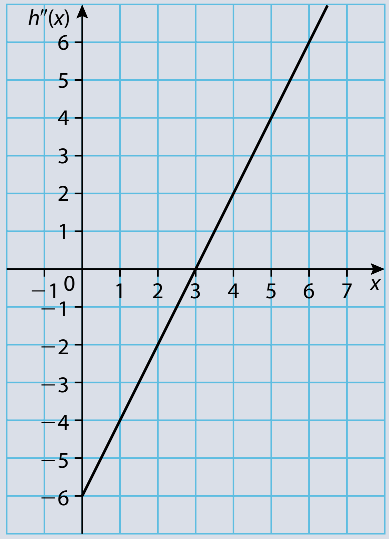
11 The normal to the curve $y = x^2 - 4x$ at the point $(3, -3)$ intersects the x-axis at point P and the y-axis at point Q. Find the equation of the normal and the coordinates of P and Q.

12 Let $y = h(x)$ be a function of x for $0 \leqslant x \leqslant 6$. The graph of h has an inflexion point at P and a maximum point at M.

Partial sketches of the curves of $h'(x)$ and $h''(x)$ are shown below.



$y = h'(x)$



$y = h''(x)$

Use the above information to answer the following.

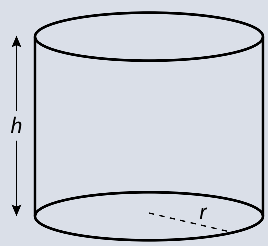
a) Write down the x-coordinate of P and justify your answer.

b) Write down the x-coordinate of M and justify your answer.

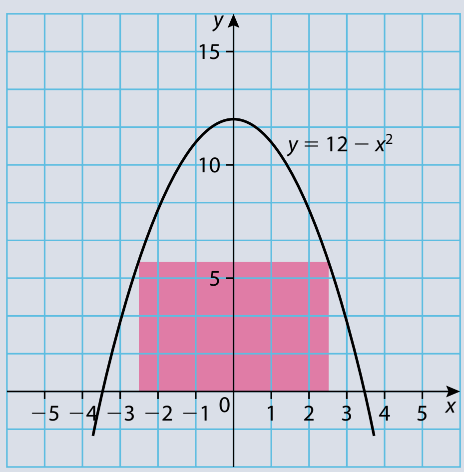
c) Given that $h(3) = 0$, sketch the graph of h. On the sketch, mark the points P and M.

13 Find the equation of the normal to the curve $x^2 + xy + y^2 - 3y = 10$ at the point (2, 3).

14 A cylinder is to be made with an exact volume of $128\pi\,\text{cm}^3$. What should be the height h and the radius r of the cylinder's base so that the cylinder's surface area is a minimum?



15 A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$, as shown in the diagram. What is the largest area that the rectangle can have, and what are its dimensions (i.e. length and width)?

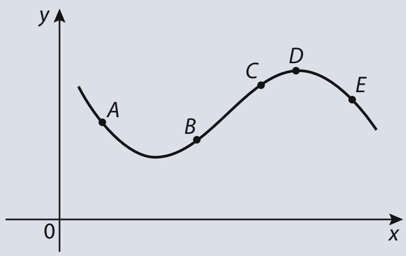


16 The figure below shows the graph of a function $y = f(x)$. At which one of the five points on the graph:

a) are $f'(x)$ and $f''(x)$ both negative?

b) is $f'(x)$ negative and $f''(x)$ positive?

c) is $f'(x)$ positive and $f''(x)$ negative?



17 Find the equation of the normal to the curve with equation $y = \frac{2x - 1}{x + 2}$ at the point $(-3, 7)$.

18 Find the equation of a) the tangent, and b) the normal to the curve $y = \ln(4x - 3)$ at the point $(1, 0)$.

19 Consider the function $f(x) = x^2 \ln x$.

a) Find the exact coordinates of any stationary points. Indicate whether it is a maximum or minimum (and absolute or relative).

b) Find the exact coordinates of any inflexion points.

20 a) Determine the constant a such that the function $f(x) = x^2 + \frac{a}{x}$ has (i) a local minimum at $x = 2$ and (ii) a local minimum at $x = -3$.

c) Show that the function cannot have a local maximum for any value of a.

21 A line passes through the point $(3, 2)$ and intersects both the x-axis and the y-axis, forming a triangular region in the first quadrant bounded by the x-axis, the y-axis and the line. Find the equation of such a line that creates a triangle of minimum area.

22 Find the equation of both the tangent and normal to the curve $y = x \tan x$ at the point where $x = \frac{\pi}{4}$.

23 A very important function in statistics is the equation for the **standard normal curve** (mean = 0, standard deviation = 1) given by $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{(2\pi)}}$.

a) Find the coordinates of any stationary points and of any inflexion points.

b) What happens when $x \to \infty$, and when $x \to -\infty$? Give the equation for any asymptotes.

c) Sketch a graph of $f(x)$ and indicate the location of any of the points found in part **a)**.

24 Let f be the function given by $f(x) = 2\ln(x^2 + 3) - x$.

a) Find the x-coordinate of each maximum and minimum point of f. Justify your answer(s).

b) Find the x-coordinate of each inflexion point of f. Justify your answer(s).

25 The rate at which cars on a road pass a certain point is known as the flow rate and is in units of cars per hour. The flow rate, F, of a certain road is given by $F(x) = \frac{2x}{18 + 0.015x^2}$ where x is the speed of the traffic in kilometres per hour. What speed will maximise the flow rate on the road?

26 If $2x^2 - 3y^2 = 2$, find the two values of $\frac{dy}{dx}$ when $x = 5$.

27 Differentiate $y = \arccos(1 - 2x^2)$ with respect to x, and simplify your answer.

28 For the function $f: x \mapsto x^2 \ln x$, $x > 0$, find the function f', the derivative of f with respect to x.

29 For the function $f: x \mapsto \frac{1}{2}\sin 2x + \cos x$, find the possible values of $\sin x$ for which $f'(x) = 0$.

30 Find the gradient of the tangent to the curve $3x^2 + 4y^2 = 7$ at the point where $x = 1$ and $y > 0$.

31 If $f(x) = \ln(2x - 1)$, $x > \frac{1}{2}$, find

a) $f'(x)$

b) the value of x where the gradient of $f(x)$ is equal to x.

32 Find the x-coordinate, between -2 and 0, of the point of inflexion on the graph of the function $f: x \mapsto x^2 e^x$. Give your answer to 3 decimal places.

33 A normal to the graph of $y = \arctan(x - 1)$, for $x > 0$, has equation $y = -2x + c$, where $c \in \mathbb{R}$. Find the value of c.

34 The function f is given by $f: x \mapsto e^{1+\sin \pi x}$, $x \geqslant 0$.

a) Find $f'(x)$.

Let x_n be the value of x where the $(n + 1)$th maximum or minimum point occurs, $n \in \{N\}$ (i.e. x_0 is the value of x where the first maximum or minimum occurs, x_1 is the value of x where the second maximum or minimum occurs, etc.).

b) Find x_n in terms of n.

35 Let $f(x) = x(\sqrt[3]{(x^2 - 1)^2})$, $-1.4 \leqslant x \leqslant 1.4$.

a) **Sketch** the graph of $f(x)$. (An exact scale diagram is **not** required.)

On your graph indicate the approximate position of

(i) each zero

(ii) each maximum point

(iii) each minimum point.

b) **(i)** Find $f'(x)$, clearly stating its domain.

(ii) Find the x-coordinates of the maximum and minimum points of $f(x)$, for $-1 < x < 1$.

c) Find the x-coordinate of the point of inflexion of $f(x)$, where $x > 0$, giving your answer correct to **four** decimal places.

36 The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point $(1, 7)$. Find the values of a and b.

37 Consider the function $y = \tan x - 8\sin x$.

a) Find $\frac{dy}{dx}$. **b)** Find the value of $\cos x$ for which $\frac{dy}{dx} = 0$.

38 Consider the tangent to the curve $y = x^3 + 4x^2 + x - 6$.

a) Find the equation of this tangent at the point where $x = -1$.

b) Find the coordinates of the point where this tangent meets the curve again.

39 Let $y = \sin(kx) - kx\cos(kx)$, where k is a constant.

Show that $\frac{dy}{dx} = k^2 x \sin(kx)$.

40 A curve has equation $xy^3 + 2x^2y = 3$. Find the equation of the tangent to this curve at the point (1, 1).

41 The function f is defined by

$$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}.$$

a) **(i)** Find an expression for $f'(x)$, simplifying your answer.

(ii) The tangents to the curve of $f(x)$ at points A and B are parallel to the x-axis. Find the coordinates of A and of B.

b) **(i)** Sketch the graph of $y = f'(x)$.

(ii) Find the x-coordinates of the three points of inflexion on the graph of f.

c) Find the range of

(i) f

(ii) the composite function $f \circ f$.

42 Air is pumped into a spherical ball which expands at a rate of 8 cm^3 per second (8 cm^3 s^{-1}). Find the **exact** rate of increase of the radius of the ball when the radius is 2 cm.

43 A curve has equation $x^3y^2 = 8$. Find the equation of the normal to the curve at the point (2, 1).

44 The function f is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.

a) **(i)** Show that

$$f'(x) = \frac{2x - x^2 \ln 2}{2^x}.$$

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

b) **(i)** Find the exact value of x satisfying the equation $f'(x) = 0$.

(ii) Show that this value gives a maximum value for $f(x)$.

c) Find the x-coordinates of the two points of inflexion on the graph of f.

45 Consider the function $f(t) = 3 \sec^2 t + 5t$.

a) Find $f'(t)$.

b) Find the **exact** values of

(i) $f(\pi)$

(ii) $f'(\pi)$.

46 Consider the equation $2xy^2 = x^2y + 3$.

a) Find y when $x = 1$ and $y < 0$.

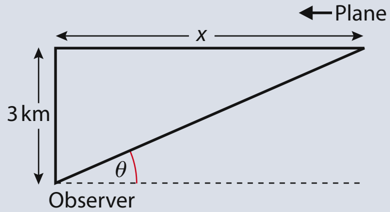
b) Find $\frac{dy}{dx}$ when $x = 1$ and $y < 0$.

47 Let $y = e^{3x} \sin(\pi x)$.

a) Find $\frac{dy}{dx}$.

b) Find the smallest positive value of x for which $\frac{dy}{dx} = 0$.

48 An airplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.



49 A curve has equation $f(x) = \dfrac{a}{b + e^{-cx}}$, $a \neq 0$, $b > 0$, $c > 0$.

a) Show that $f''(x) = \dfrac{ac^2 e^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3}$.

b) Find the coordinates of the point on the curve where $f''(x) = 0$.

c) Show that this is a point of inflexion.

50 The point $P(1, p)$, where $p > 0$, lies on the curve $2x^2y + 3y^2 = 16$.

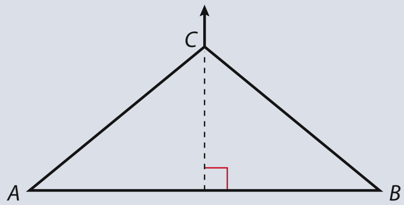
a) Calculate the value of p.

b) Calculate the gradient of the tangent to the curve at P.

51 The function f is defined by $f: x \mapsto 3^x$.

Find the solution of the equation $f''(x) = 2$.

52 The following diagram shows an isosceles triangle ABC with $AB = 10$ cm and $AC = BC$. The vertex C is moving in a direction perpendicular to (AB) with speed 2 cm per second.



Calculate the rate of increase of the angle $C\hat{A}B$ at the moment the triangle is equilateral.

53 If $y = \ln(2x - 1)$, find $\dfrac{d^2y}{dx^2}$.

54 Find the equation of the normal to the curve $x^3 + y^3 - 9xy = 0$ at the point (2, 4).

55 The function f' is given by $f'(x) = 2\sin\left(5x - \dfrac{\pi}{2}\right)$.

a) Write down $f''(x)$.

b) Given that $f\left(\dfrac{\pi}{2}\right) = 1$, find $f(x)$.

56 Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point $(1, -2)$.

57 The function f is given by $f(x) = \dfrac{x^5 + 2}{x}$, $x \neq 0$. There is a point of inflexion on the graph of f at the point P. Find the coordinates of P.

58 An experiment is carried out in which the number n of bacteria in a liquid is given by the formula $n = 650e^{kt}$, where t is the time in minutes after the beginning of the experiment and k is a constant. The number of bacteria doubles every 20 minutes. Find

a) the **exact** value of k

b) the rate at which the number of bacteria is increasing when $t = 90$.

59 Let f be a cubic polynomial function. Given that $f(0) = 2$, $f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$.

60 Let $f(x) = \cos^3(4x + 1)$, $0 \leqslant x \leqslant 1$.

a) Find $f'(x)$.

b) Find the **exact** values of the three roots of $f'(x) = 0$.

61 Given that $3^{x+y} = x^3 + 3y$, find $\frac{dy}{dx}$.

62 Let f be the function defined for $x > -\frac{1}{3}$ by $f(x) = \ln(3x + 1)$.

a) Find $f'(x)$.

b) Find the equation of the normal to the curve $y = f(x)$ at the point where $x = 2$.

Give your answer in the form $y = ax + b$ where $a, b \in \mathbb{R}$.

63 Let $y = x \arcsin x$, $x \in (-1, 1)$. Show that $\frac{d^2y}{dx^2} = \frac{2 - x^2}{(1 - x^2)^{\frac{3}{2}}}$.

64 Given that $e^{xy} - y^2 \ln x = e$ for $x \geqslant 1$, find $\frac{dy}{dx}$ at the point $(1, 1)$.

65 The function f is defined by $f(x) = \frac{2x}{x^2 + 6}$ for $x \geqslant b$ where $b \in \mathbb{R}$.

a) Show that $f'(x) = \frac{12 - 2x^2}{(x^2 + 6)^2}$.

b) Hence, find the smallest exact value of b for which the inverse function f^{-1} exists. Justify your answer.

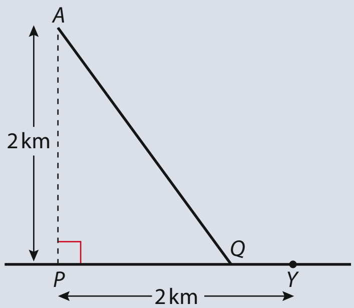
66 Consider the curve with equation $x^2 + xy + y^2 = 3$.

a) Find in terms of k, the gradient of the curve at the point $(-1, k)$.

b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k.

67 Find the gradient of the tangent to the curve $x^3y^2 = \cos(\pi y)$ at the point $(-1, 1)$.

68 André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

a) If PQ = x km, $0 \leqslant x \leqslant 2$, find an expression for the time T minutes taken by André to reach point Y.

b) Show that $\frac{dT}{dx} = \frac{5\sqrt{5}x}{\sqrt{x^2 + 4}} - 5$.

c) **(i)** Solve $\frac{dT}{dx} = 0$.

(ii) Use the value of x found in **part c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\frac{d^2T}{dx^2} = \frac{20\sqrt{5}}{(x^2 + 4)^{\frac{3}{2}}}$ and **hence** show that the time found in **part c)** is a minimum.

69 The function f is defined by $f(x) = xe^{2x}$.

It can be shown that $f^{(n)}(x) = (2^n x + n^{2n-1})e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the nth derivative of $f(x)$.

a) By considering $f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

b) Show that f has a point of inflexion Q at $x = -1$.

c) Determine the intervals on the domain of f where f is

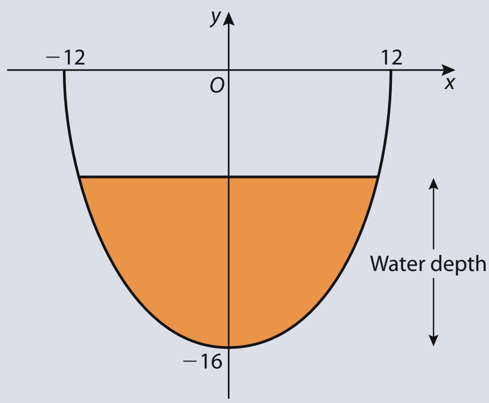
(i) concave up

(ii) concave down.

d) Sketch f', clearly showing any intercepts, asymptotes and the points P and Q.

e) Use mathematical induction to prove that $f^{(n)}(x) = (2^n x + n^{2n-1})e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the nth derivative of $f(x)$.

70 The diagram below shows the boundary of the cross-section of a water channel.



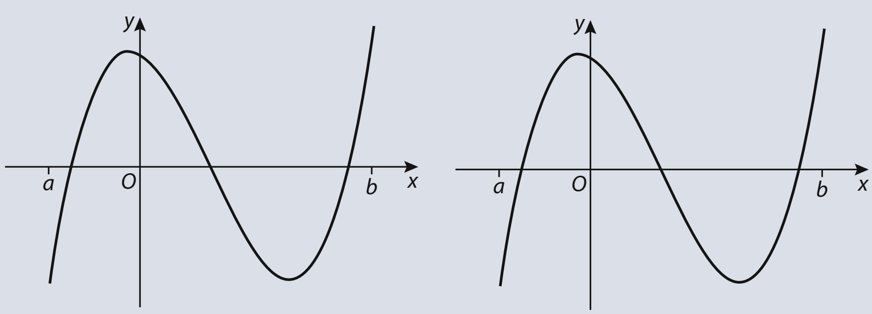
The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm.

The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm.
Find the width of the water surface in the channel when the water depth is 10 cm. Give your answer in the form $a \arccos b$ where $a, b \in \mathbb{R}$.

71 The graphs given below are those of the same function $y = f(x)$ for $a \leqslant x \leqslant b$.

Sketch, on the given axes, the graphs of **a)** $\frac{dy}{dx}$ and **b)** $\frac{d^2y}{dx^2}$.

Indicate clearly the positions of any asymptotes.



16 Integral Calculus

Assessment statements

6.4 Indefinite integration as anti-differentiation.
Indefinite integral of x^n, $\sin x$, $\cos x$, $1/x$ and e^x.
The composites of any of these with a linear function.

6.5 Anti-differentiation with a boundary condition to determine the constant term.
Definite integrals.
Area of the region enclosed by a curve and the x-axis or y-axis in a given interval.
Areas of regions enclosed by curves.
Volumes of revolution about the x-axis or y-axis.

6.6 Kinematic problems involving displacement s, velocity v and acceleration a.
Total distance travelled.

6.7 Further integration: integration by substitution; integration by parts.

Introduction

In Chapters 13 and 15 you learned about the process of differentiation. That is, given a function, how you can find its derivative. In this chapter, we will look at the reverse process. That is, given a function $f(x)$, how can we find a function $F(x)$ whose derivative is $f(x)$. This process is the opposite of differentiation and is therefore called **anti-differentiation**.

16.1 Anti-derivative

An **anti-derivative** of the function $f(x)$ is a function $F(x)$ such that

$$\frac{d}{dx}F(x) = F'(x) = f(x) \text{ wherever } f(x) \text{ is defined.}$$

For instance, let $f(x) = x^2$. It is not difficult to discover an anti-derivative of $f(x)$. Keep in mind that this is a power function. Since the power rule reduces the power of the function by 1, we examine the derivative of x^3:

$$\frac{d}{dx}(x^3) = 3x^2.$$

This derivative, however, is 3 times $f(x)$. To 'compensate' for the 'extra' 3, we have to multiply by $\frac{1}{3}$, so that the anti-derivative is now $\frac{1}{3}x^3$. Now,

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2.$$

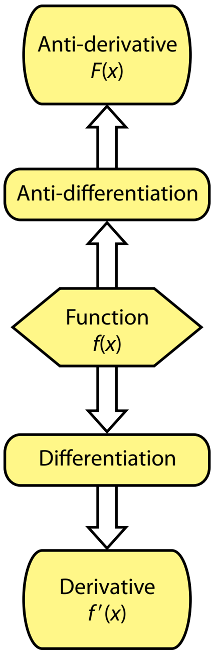
And, therefore, $\frac{1}{3}x^3$ is an anti-derivative of x^2.

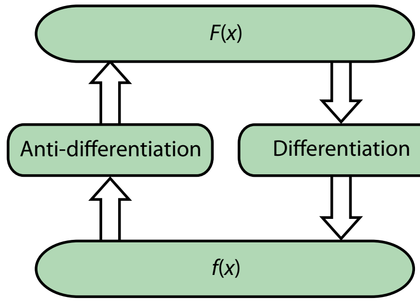
Table 16.1 shows some examples of functions, each paired with one of its anti-derivatives.

Table 16.1

Function $f(x)$	Anti-derivative $F(x)$
1	x
x	$\frac{x^2}{2}$
$3x^2$	x^3
x^4	$\frac{x^5}{5}$
$\cos x$	$\sin x$
$\cos 2x$	$\frac{1}{2}\sin 2x$
e^x	e^x
$\sin x$	$-\cos x$
$2x$	x^2

The diagrams below show the relationship between the derivative and the integral as opposite operations.





Example 1

Given the function $f(x) = 3x^2$, find an anti-derivative of $f(x)$.

Solution

$F_1(x) = x^3$ is such an anti-derivative because $\frac{d}{dx}(F_1(x)) = 3x^2$.

The following functions are also anti-derivatives because the derivative of each one of them is also $3x^2$.

$$H_1(x) = x^3 + 27,\ H_2(x) = x^3 - \pi, \text{ or } H_3(x) = x^3 + \sqrt{5}$$

Indeed, $F(x) = x^3 + c$ is an anti-derivative of $f(x) = 3x^2$ for any choice of the constant c.

This is so simply because

$$(F(x) + c)' = F'(x) + c' = F'(x) + 0 = f(x)!$$

Thus, we can say that any single function $f(x)$ has many anti-derivatives, whereas a function can have only one derivative.

If $F(x)$ is an anti-derivative of $f(x)$, then so is $F(x) + c$ for any choice of the constant c.

Stated slightly differently, this observation says:

If $F(x)$ is an anti-derivative of $f(x)$ over a certain interval I, then every anti-derivative of $f(x)$ on I is of the form $F(x) + c$.

This statement is an indirect conclusion of one of the results of the mean value theorem.

Two functions with the same derivative on an interval differ only by a constant on that interval.

We will state the mean value theorem here in order to establish the general rule for anti-derivatives.

Mean value theorem

A function $H(x)$, continuous over an interval $[a, b]$ and differentiable over $]a, b[$, satisfies

$$H(b) - H(a) = (b - a)H'(c) \text{ for some } c \in \,]a, b[.$$

Let $F(x)$ and $G(x)$ be any anti-derivatives of $f(x)$, i.e. $F'(x) = G'(x)$.

Take $H(x) = F(x) - G(x)$ and any two numbers x_1 and x_2 in the interval $[a, b]$ such that $x_1 < x_2$, then

$$\begin{aligned} H(x_2) - H(x_1) &= (x_2 - x_1)H'(c) = (x_2 - x_1)\cdot(F'(c) - G'(c)) \\ &= (x_2 - x_1)\cdot 0 = 0 \Rightarrow H(x_1) = H(x_2) \end{aligned}$$

which means that $H(x)$ is a constant function.
Hence, $H(x) = F(x) - G(x) =$ constant. That is, any two anti-derivatives of a function differ by a constant.

Notation:
The notation

$$\int f(x)\,dx = F(x) + c \qquad (1)$$

where c is an arbitrary constant, means that $F(x) + c$ is an anti-derivative of $f(x)$.

Equivalently, $F(x)$ satisfies the condition that

$$\frac{d}{dx}(F(x)) = F'(x) = f(x) \qquad (2)$$

for all x in the domain of $f(x)$.

It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2 dx = \tfrac{1}{3}x^3 + c \text{ is equivalent to } \frac{d}{dx}\left(\tfrac{1}{3}x^3\right) = x^2.$$

Note that if we differentiate an anti-derivative of $f(x)$, we obtain $f(x)$ back again.

Thus, $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$.

The expression $\int f(x)dx$ is called an **indefinite integral** of $f(x)$. The function $f(x)$ is called the **integrand** and the constant c is called the **constant of integration**.

The integral symbol $\int$ is made like an elongated capital S. It is, in fact, a medieval S, used by Leibniz as an abbreviation for the Latin word *summa*.

We think of the combination $\int [\,]\boldsymbol{dx}$ as a single symbol; we fill in the 'blank' with the formula of the function whose anti-derivative we seek. We may regard the differential $\boldsymbol{dx}$ as specifying the independent variable $\boldsymbol{x}$ both in the function $f(x)$ and in its anti-derivatives.

If an independent variable other than $\boldsymbol{x}$ is used, say t, the notation must be adjusted appropriately.

Thus, $\frac{d}{dt}\left(\int f(t)dt\right) = f(t)$ and $\int f(t)dt = F(t) + c$ are equivalent statements.

Derivative formula	Equivalent integration formula
$\frac{d}{dx}(x^3) = 3x^2$	$\int 3x^2 dx = x^3 + c$
$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\, dx = \sqrt{x} + c$
$\frac{d}{dt}(\tan t) = \sec^2 t$	$\int \sec^2 t\, dt = \tan t + c$
$\frac{d}{dv}(v^{\frac{3}{2}}) = \frac{3}{2}v^{\frac{1}{2}}$	$\int \frac{3}{2}v^{\frac{1}{2}}\, dv = v^{\frac{3}{2}} + c$

Note: The integral sign and differential serve as delimiters, adjoining the integrand on the left and right, respectively. In particular, we do not write $\int dx\, f(x)$ when we mean $\int f(x)\, dx$.

Basic integration formulae

Integration is essentially educated guesswork – given the derivative $f(x)$ of a function $F(x)$, we try to guess what the function $F(x)$ is. However, many basic integration formulae can be obtained directly from their companion differentiation formulae. Some of the most important are given in Table 16.2.

Table 16.2

	Differentiation formula	Integration formula
1	$\frac{d}{dx}(x) = 1$	$\int dx = x + c$
2	$\frac{d}{dx}(x^{n+1}) = (n+1)x^n, n \neq -1$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
3	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + c$
4	$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin v\, dv = -\cos v + c$
5	$\frac{d}{dt}(\tan t) = \sec^2 t$	$\int \sec^2 t\, dt = \tan t + c$
6	$\frac{d}{dv}(e^v) = e^v$	$\int e^v\, dv = e^v + c$
7	$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \ln\lvert x\rvert + c$
8	$\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = a^x$	$\int a^x dx = \frac{1}{\ln a}a^x + c$
9	$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$
10	$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + c$

Formula (7) is a special case of the 'power' rule formula (2), but needs some modification.

If we are given the task to integrate $\frac{1}{x}$, we may attempt to do it using the power rule:

$$\int \frac{1}{x}\, dx = \int x^{-1}\, dx = \frac{1}{(-1)+1}x^{(-1)+1} + c = \frac{1}{0}x^0 + c, \text{ which is undefined.}$$

However, the solution is clearly found by observing what you learned in Chapter 15.

In Section 15.3 you learned that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \ x > 0.$$

This implies

$$\int \frac{1}{x}\,dx = \ln x + c,\ x > 0.$$

However, the function $\frac{1}{x}$ is differentiable for $x < 0$ too. So, we must be able to find its integral.

The solution lies in the chain rule!

If $x < 0$, we can write $x = -u$ where $u > 0$. Then $dx = -du$, and

$\int \frac{1}{x}\,dx = \int \frac{1}{-u}(-du) = \int \frac{1}{u}\,du = \ln u + c,\ u > 0.$

But $u = -x$, therefore when $x < 0$

$\int \frac{1}{x}\,dx = \ln u + c = \ln(-x) + c$, and, combining the two results, we have

$\int \frac{1}{x}\,dx = \ln|x| + c,\ x \neq 0.$

Suppose that $f(x)$ and $g(x)$ are differentiable functions and k is a constant, then:

1. A constant factor can be moved through an integral sign, i.e.

   $\int kf(x)dx = k\int f(x)dx$

2. An anti-derivative of a sum (difference) is the sum (difference) of the anti-derivatives, i.e.

   $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$, or $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$

Example 2

Evaluate:

a) $\int 3\cos x\,dx$ b) $\int (x^3 + x^2)dx$

Solution

a) $\int 3\cos x\,dx = 3\int \cos x\,dx = 3\sin x + c$

b) $\int (x^3 + x^2)dx = \int x^3\,dx + \int x^2\,dx = \frac{x^4}{4} + \frac{x^3}{3} + c$

Sometimes it is useful to rewrite the integrand in a different form before performing the integration.

Example 3

Evaluate:

a) $\int \frac{t^3 - 3t^5}{t^5}\,dt$ b) $\int \frac{x + 5x^4}{x^2}\,dx$

Solution

a) $\int \frac{t^3 - 3t^5}{t^5}\,dt = \int \frac{t^3}{t^5}\,dt - \int \frac{3t^5}{t^5}\,dt = \int t^{-2}\,dt - \int 3\,dt = \frac{t^{-1}}{-1} - 3t + c$

$= \frac{-1}{t} - 3t + c$

b) $\int \frac{x + 5x^4}{x^2}\,dx = \int \frac{x}{x^2}\,dx + \int \frac{5x^4}{x^2}\,dx = \int \frac{1}{x}\,dx + \int 5x^2\,dx = \ln|x| + 5\cdot\frac{x^3}{3} + c$

Integration by simple substitution

In this section, we will study a technique called substitution that can often be used to transform complicated integration problems into simpler ones.

The method of substitution depends on our understanding of the chain rule as well as the use of variables in integration. Two facts to recall:

1. When we find an anti-derivative, we established earlier that the use of x is arbitrary. We can use any other variable as you have seen in several exercises and examples so far.

   So, $\int f(u)\,du = F(u) + c$, where u is a 'dummy' variable in the sense that it can be replaced by any other variable.

2. The chain rule enables us to say
   $$\frac{d}{dx}(F(u(x))) = F'(u(x))\cdot u'(x).$$
   This can be written in integral form as
   $$\int F'(u(x))\cdot u'(x)\,dx = F(u(x)) + c$$
   or, equivalently, since $F(x)$ is an anti-derivative of $f(x)$,
   $$\int f(u(x))\cdot u'(x)\,dx = F(u(x)) + c.$$

For our purposes, it will be useful and simpler to let $u(x) = u$ and to write $\frac{du}{dx} = u'(x)$ in its 'differential' form $du = u'(x)dx$, or, simply, $du = u'dx$.

With this notation, the integral can now be written as

$$\int f(u(x))\cdot \mathbf{u'(x)\,dx} = \int f(u)\,\mathbf{du} = F(u) + c.$$

The following example explains how the method works.

Example 4

Evaluate:

a) $\int (x^3 + 2)^{10}\cdot 3x^2\,dx$

b) $\int \tan x\,dx$

c) $\int \cos 5x\,dx$

d) $\int \cos x^2\cdot x\,dx$

e) $\int e^{3x+1}\,dx$

Solution

a) To integrate this function, it is simplest to make the following substitution.
   Let $u = x^3 + 2$, and so $du = 3x^2\,dx$. Now the integral can be written as
   $$\int (x^3 + 2)^{10}\cdot \mathbf{3x^2\,dx} = \int u^{10}\,\mathbf{du} = \frac{u^{11}}{11} + c = \frac{(x^3 + 2)^{11}}{11} + c.$$

b) This integrand has to be rewritten first and then we make the substitution.
   $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{\cos x}\cdot \sin x\,dx$$
   We now let $u = \cos x \Rightarrow du = -\sin x\,dx$, and
   $$\int \tan x\,dx = \int \frac{1}{\cos x}\cdot \mathbf{\sin x\,dx} = \int \frac{1}{u}\cdot(\mathbf{-du}) = -\int \frac{1}{u}\,du = -\ln|u| + c.$$

This last result can be then expressed in one of two ways:

$$\int \tan x\, dx = -\ln|\cos x| + c, \text{ or}$$

$$\int \tan x\, dx = -\ln|\cos x| + c = \ln|(\cos x)^{-1}| + c$$

$$= \ln\left|\frac{1}{(\cos x)}\right| + c = \ln|\sec x| + c$$

c) We let $u = 5x$, then $du = 5dx \Rightarrow dx = \frac{1}{5}du$, and so

$$\int \cos 5x\, dx = \int \cos u \cdot \tfrac{1}{5}du = \tfrac{1}{5}\int \cos u\, du = \tfrac{1}{5}\sin u + c$$

$$= \tfrac{1}{5}\sin 5x + c.$$

Another method can be applied here:

The substitution $u = 5x$ requires $du = 5dx$. As there is no factor of 5 in the integrand, and since 5 is a constant, we can multiply and divide by 5 so that we group the 5 and dx to form the du required by the substitution:

$$\int \cos 5x\, dx = \tfrac{1}{5}\int \cos x \cdot \mathbf{5dx} = \tfrac{1}{5}\int \cos u\, \mathbf{du} = \tfrac{1}{5}\sin u + c$$

$$= \tfrac{1}{5}\sin 5x + c$$

d) By letting $u = x^2$, $du = 2x\, dx$ and so

$$\int \cos x^2 \cdot x\, dx = \tfrac{1}{\mathbf{2}}\int \cos x^2 \cdot \mathbf{2}x\, dx = \tfrac{1}{2}\int \cos u\, du = \tfrac{1}{2}\sin u + c$$

$$= \tfrac{1}{2}\sin x^2 + c.$$

e) $\int e^{3x+1}\, dx = \frac{1}{3}\int e^{\mathbf{3x+1}}\mathbf{3dx} = \frac{1}{3}\int e^{\mathbf{u}}\, \mathbf{du} = \frac{1}{3}e^u + c = \frac{1}{3}e^{3x+1} + c$

In integration, multiplying by a constant 'inside' the integral and 'compensating' for that with the reciprocal 'outside' the integral depends on theorem 1 (page 775). That is,

$$\int kf(x)dx = k\int f(x)dx.$$

However, you ***cannot*** multiply with a variable. So, you ***cannot*** say, for example,

$$\int \cos x^2\, dx = \frac{1}{\mathbf{2x}}\int \cos x^2 \cdot \mathbf{2x}\, dx.$$

Note: The main challenge in using the substitution rule is to think of an appropriate substitution. You should try to select u to be a part of the integrand whose differential is also included (except for the constant). In Example 4a), we selected u to be $(x^3 + 2)$ knowing that $du = 3x^2dx$. Then we 'compensated' for the absence of 3! *Finding the right substitution is a bit of an art. You need to acquire it!* It is quite usual that your first guess may not work. Try another one!

Example 5

Evaluate each integral.

a) $\int e^{-3x}\, dx$

b) $\int \sin^2 x \cos x\, dx$

c) $\int 2\sin(3x - 5)\, dx$

d) $\int e^{mx+n}\, dx$

e) $\int x\sqrt{x}\, dx$, and $F(1) = 2$

Solution

a) Let $u = -3x$, then $du = -3dx$, and

$$\int e^{-3x}\, dx = -\tfrac{1}{3}\int e^{-3x}(\mathbf{-3dx}) = -\tfrac{1}{3}\int e^{\mathbf{u}}\, \mathbf{du} = -\tfrac{1}{3}e^u + c$$

$$= -\tfrac{1}{3}e^{-3x} + c.$$

b) Let $u = \sin x$, then $du = \cos x\, dx$, and

$$\int \mathbf{\sin^2 x \cos x\, dx} = \int \mathbf{u^2\, du} = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}\sin^3 x + c.$$

c) Let $u = 3x - 5$, then $du = 3dx$, and

$$\int 2\sin(\mathbf{3x-5})\,dx = 2\cdot\tfrac{1}{3}\int \sin(\mathbf{3x-5})\mathbf{3dx} = \tfrac{2}{3}\int \sin \boldsymbol{u}\,\mathbf{du}$$
$$= -\tfrac{2}{3}\cos u + c = -\tfrac{2}{3}\cos(3x-5) + c.$$

d) Let $u = mx + n$, then $du = m\,dx$, and

$$\int e^{mx+n}\,dx = \frac{1}{m}\int e^{\boldsymbol{mx+n}}\,\boldsymbol{m\,dx} = \frac{1}{m}\int e^{\boldsymbol{u}}\,\mathbf{du}$$
$$= \frac{1}{m}e^{u} + c = \frac{1}{m}e^{mx+n} + c.$$

e) $F(x) = \int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx = \dfrac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + c = \frac{2}{5}x^{\frac{5}{2}} + c$, but $F(1) = 2$

$F(1) = \frac{2}{5}1^{\frac{5}{2}} + c = \frac{2}{5} + c = 2 \Rightarrow c = \frac{8}{5}$

Therefore, $F(x) = \frac{2}{5}x^{\frac{5}{2}} + \frac{8}{5}$.

The previous discussion makes it clear that Table 16.2 is limited in scope, because we cannot use the integrals directly to evaluate composite integrals such as the ones in Examples 4 and 5 above. An adjusted table is therefore presented here.

Table 16.3

	Differentiation formula	Integration formula
1	$\frac{d}{dx}(u(x)) = u'(x) \Rightarrow du = u'(x)dx$	$\int du = u + c$
2	$\frac{d}{dx}\left(\frac{u^{n+1}}{(n+1)}\right) = u^n u'(x), n \neq -1 \Rightarrow d\left(\frac{u^{n+1}}{(n+1)}\right) = u^n u'(x)dx$	$\int u^n du = \frac{u^{n+1}}{n+1} + c, n \neq -1$
3	$\frac{d}{dx}(\sin(u)) = \cos(u)u'(x) \Rightarrow d(\sin(u)) = \cos(u)u'(x)dx$	$\int \cos u\, du = \sin u + c$
4	$\frac{d}{dx}(-\cos(u)) = \sin(u)u'(x) \Rightarrow d(-\cos(u)) = \sin(u)u'(x)dx$	$\int \sin u\, du = -\cos u + c$
5	$\frac{d}{dt}(\tan u) = \sec^2 u\, u'(t) \Rightarrow d(\tan u) = \sec^2 u\, u'(t)dt$	$\int \sec^2 u\, du = \tan u + c$
6	$\frac{d}{dx}(e^u) = e^u u'(x)dx \Rightarrow d(e^u) = e^u u'(x)dx$	$\int e^u du = e^u + c$
7	$\frac{d}{dx}(\ln\|u\|) = \frac{1}{u}u'(x) \Rightarrow d(\ln\|u\|) = \frac{1}{u}u'(x)dx$	$\int \frac{1}{u}du = \ln\|u\| + c$
8	$\frac{d}{dx}\left(\frac{a^u}{\ln a}\right) = a^u u'(x) \Rightarrow d\left(\frac{a^u}{\ln a}\right) = a^u u'(x)dx$	$\int a^u du = \frac{a^u}{\ln a} + c$
9	$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}u'(x) \Rightarrow d(\arcsin u) = \frac{1}{\sqrt{1-u^2}}u'(x)dx$	$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c$
10	$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}u'(x) \Rightarrow d(\arctan u) = \frac{1}{1+u^2}u'(x)dx$	$\int \frac{du}{1+u^2} = \arctan u + c$

Example 6

Evaluate each integral.

a) $\int\sqrt{6x+11}\,dx$

b) $\int(5x^3+2)^8x^2dx$

c) $\displaystyle\int\frac{x^3-2}{\sqrt[5]{x^4-8x+13}}dx$

d) $\int\sin^4(3x^2)\cos(3x^2)x\,dx$

Solution

a) We let $u = 6x + 11$ and calculate du:

$$u = 6x + 11 \Rightarrow du = 6dx$$

Since du contains the factor 6, the integral is still not in the proper form $\int f(u)du$. However, here we can use of two approaches:

(i) Introduce the factor 6, as we have done before, i.e.

$$\int\sqrt{6x+11}\,dx = \tfrac{1}{6}\int\sqrt{6x+11}\,\boxed{6dx}$$

$$= \tfrac{1}{6}\int\sqrt{u}\,\boxed{du} = \tfrac{1}{6}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{6}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{18}u^{\frac{3}{2}} + c$$

$$= \tfrac{1}{9}(6x+11)^{\frac{3}{2}} + c$$

Or,

(ii) Since $u = 6x + 11 \Rightarrow du = 6dx \Rightarrow dx = \dfrac{du}{6}$, then

$\int\sqrt{6x+11}\,dx = \int\sqrt{u}\dfrac{du}{6} = \tfrac{1}{6}\int u^{\frac{1}{2}}du$, then we follow the same steps as before.

b) We let $u = 5x^3 + 2$, then $du = 15x^2dx$. This means that we need to introduce the factor 15 into the integrand:

$$\int(5x^3+2)^8x^2dx = \frac{1}{15}\int(5x^3+2)^8 15x^2dx$$

$$= \frac{1}{15}\int u^8du = \frac{1}{15}\frac{u^9}{9} + c$$

$$= \frac{1}{135}(5x^3+2)^9 + c$$

c) We let $u = x^4 - 8x + 13 \Rightarrow du = (4x^3 - 8)dx = 4(x^3 - 2)dx$.

$$\int\frac{x^3-2}{\sqrt[5]{x^4-8x+13}}dx = \frac{1}{4}\int\frac{4(x^3-2)dx}{\sqrt[5]{x^4-8x+13}} = \frac{1}{4}\int\frac{du}{u^{\frac{1}{5}}}$$

$$= \frac{1}{4}u^{-\frac{1}{5}}du = \frac{1}{4}\frac{u^{\frac{4}{5}}}{\frac{4}{5}} + c$$

$$= \frac{5}{16}(x^4-8x+13)^{\frac{4}{5}} + c$$

d) We let $u = \sin(3x^2) \Rightarrow du = \cos(3x^2)6x\,dx$ using the chain rule!

$$\int \sin^4(3x^2)\cos(3x^2)x\,dx = \frac{1}{6}\int \sin^4(3x^2)\cos(3x^2)6x\,dx$$
$$= \frac{1}{6}\int u^4 du = \frac{1}{6}\frac{u^5}{5} + c$$
$$= \frac{1}{30}\sin^5(3x^2) + c$$

Exercise 16.1

In questions 1–15, find the most general anti-derivative of the function.

1 $f(x) = x + 2$

2 $f(t) = 3t^2 - 2t + 1$

3 $g(x) = \frac{1}{3} - \frac{2}{7}x^3$

4 $f(t) = (t - 1)(2t + 3)$

5 $g(u) = u^{\frac{2}{5}} - 4u^3$

6 $f(x) = 2\sqrt{x} - \dfrac{3}{2\sqrt{x}}$

7 $h(\theta) = 3\sin\theta + 4\cos\theta$

8 $f(t) = 3t^2 - 2\sin t$

9 $f(x) = \sqrt{x}(2x - 5)$

10 $g(\theta) = 3\cos\theta - 2\sec^2\theta$

11 $h(t) = e^{3t-1}$

12 $f(t) = \dfrac{2}{t}$

13 $h(t) = \dfrac{t}{3t^2 + 5}$

14 $h(\theta) = e^{\sin\theta}\cos\theta$

15 $f(x) = (3 + 2x)^2$

In questions 16–20, find f.

16 $f''(x) = 4x - 15x^2$

17 $f''(x) = 1 + 3x^2 - 4x^3$; $f'(0) = 2$, $f(1) = 2$

18 $f''(t) = 8t - \sin t$

19 $f'(x) = 12x^3 - 8x + 7$, $f(0) = 3$

20 $f'(\theta) = 2\cos\theta - \sin(2\theta)$

In questions 21–50, evaluate each integral.

21 $\int x(3x^2 + 7)^5 dx$

22 $\int \dfrac{x}{(3x^2 + 5)^4} dx$

23 $\int 2x^2\sqrt[4]{5x^3 + 2}\,dx$

24 $\int \dfrac{(3 + 2\sqrt{x})^5}{\sqrt{x}} dx$

25 $\int t^2\sqrt{2t^3 - 7}\,dt$

26 $\int \left(2 + \dfrac{3}{x}\right)^5 \left(\dfrac{1}{x^2}\right) dx$

27 $\int \sin(7x - 3)dx$

28 $\int \dfrac{\sin(2\theta - 1)}{\cos(2\theta - 1) + 3} d\theta$

29 $\int \sec^2(5\theta - 2)d\theta$

30 $\int \cos(\pi x + 3)dx$

31 $\int \sec 2t \tan 2t\,dt$

32 $\int xe^{x^2 + 1}dx$

33 $\int \sqrt{t}\,e^{2t\sqrt{t}}dt$

34 $\int \dfrac{2}{\theta}(\ln\theta)^2 d\theta$

35 $\int \dfrac{dz}{z\ln 2z}$

36 $\int t\sqrt{3 - 5t^2}\,dt$

37 $\int \theta^2\sec^2\theta^3 d\theta$

38 $\int \dfrac{\sin\sqrt{t}}{2\sqrt{t}} dt$

39 $\int \tan^5 2t\sec^2 2t\,dt$

40 $\int \dfrac{dx}{\sqrt{x}(\sqrt{x} + 2)}$

41 $\int \sec^5 2t \tan 2t\, dt$

42 $\int \frac{x+3}{x^2+6x+7} dx$

43 $\int \frac{k^3x^3}{\sqrt{a^2-a^4x^4}} dx$

44 $\int 3x\sqrt{x-1}\, dx$

45 $\int \csc^2 \pi t\, dt$

46 $\int \sqrt{1+\cos\theta} \sin\theta\, d\theta$

47 $\int t^2\sqrt{1-t}\, dt$

48 $\int \frac{r^2-1}{\sqrt{2r-1}} dr$

49 $\int \frac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}} x dx$

50 $\int \frac{t^2+2}{\sqrt{t-5}} dt$

16.2 Methods of integration: integration by parts

As far as this point, you will have noticed that while differentiation and integration are so strongly linked, finding derivatives is greatly different from finding integrals. With the derivative rules available, you are able to find the derivative of about any function you can think of. By contrast, you can compute anti-derivatives for a rather small number of functions. Thus far, we have developed a set of basic integration formulae, most of which followed directly from the related differentiation formulae that you saw in Table 16.2.

Using substitution, in some cases, helps us reduce the difficulty of evaluating some integrals by rendering them in familiar forms. However, there are far too many cases, where the simple substitution will not help. For example,

$$\int x \cos x\, dx$$

cannot be evaluated by the methods you have learned so far. We improve the situation in this section by introducing a powerful and yet simple tool called *integration by parts*.

Recall the product rule for differentiation:

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x),$$

which gives rise to the differential form

$d(u(x)v(x)) = v(x)d(u(x)) + u(x)d(v(x))$, and for convenience, we will write

$$d(uv) = vdu + udv.$$

If we integrate both sides of this equation, we get

$$\int d(uv) = \int vdu + \int udv \Leftrightarrow uv = \int vdu + \int udv.$$

Solving this equation for udv, we get

$$\int udv = uv - \int vdu.$$

This rule is the **integration by parts**.

The significance of this rule is not immediately apparent. We will see its great utility in a few examples.

Brook Taylor (1685–1731) is credited with devising integration by parts. Taylor is mostly known for his contributions to power series where his 'Taylor theorem' has several very important applications in mathematics and science.

Example 7

Evaluate $\int x \cos x \, dx$.

Solution

First, observe that you cannot evaluate this as it stands, i.e. it is not one of our basic integrals and no substitution can help either.

Notice how you need to make a clever choice of u and dv so that the integral on the right side is one that will ease your work ahead. We need to choose u (to differentiate) and dv (to integrate); thus we let

$$u = x, \text{ and } dv = \cos x \, dx.$$

Then $du = dx$, and $v = \sin x$. (We will introduce c at the end of the process.)

It is usually helpful to organize your work in a table form:

$$\begin{array}{ll} u = x & du = dx \\ dv = \cos x \, dx & v = \sin x \end{array}$$

This gives us:

$$\begin{aligned} \int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = \int u \, dv = & \; uv - \int v du \\ & = x \sin x - \int \sin x \, dx \\ & = x \sin x + \cos x + c \end{aligned}$$

To verify your result, simply differentiate the right-hand side.

$$\frac{d}{dx}(x \sin x + \cos x + c) = \sin x + x \cos x - \sin x + 0 = x \cos x$$

Note: What other choices can you make?

There are three other choices of u and dv in this problem:

1 If we let

$$\left.\begin{array}{ll} u = \cos x & du = -\sin x \, dx \\ dv = dx & v = \frac{x^2}{2} \end{array}\right\} \Rightarrow \int x \cos x \, dx = \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x \, dx$$

This new integral is worse than the one we started with!

2 If we let

$$\left.\begin{array}{ll} u = x \cos x & du = (\cos x - x \sin x) dx \\ dv = x dx & v = x \end{array}\right\} \Rightarrow \int x \cos x \, dx = x^2 \cos x - \int x(\cos x - x \sin x) dx$$

Again, this new integral is worse than the one we started with!

3 If we let

$$\begin{array}{ll} u = 1 & du = 0 \\ dv = x \cos x \, dx & v = ?? \end{array}$$

This is obviously a bad choice since we still do not know how to integrate $dv = x \cos x \, dx$.

The objective of integration by parts is to move from an integral $\int u dv$ (which we cannot see how to evaluate) to an integral $\int v du$ which we *can* integrate. So, keep in mind that integration by parts does not necessarily work all the time, and that we have to develop enough experience with such a process in order to make the 'correct' choice for u and vdu.

Example 8

Evaluate $\int xe^{-x}dx$.

Solution

We let

$$\left.\begin{array}{ll} u = x & du = dx \\ dv = e^{-x}\,dx & v = -e^{-x} \end{array}\right\} \Rightarrow \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + c$$

Example 9

Evaluate $\int \ln x\, dx$.

Solution

$$\left.\begin{array}{ll} u = \ln x & du = \dfrac{dx}{x} \\ dv = dx & v = x \end{array}\right\} \Rightarrow \int \ln x\, dx = x\ln x - \int \cancel{x}\frac{dx}{\cancel{x}}$$

$$= x\ln x - x + c$$

Example 10

Evaluate $\int x^2 \ln x\, dx$.

Solution

Since x^2 is easier to integrate than $\ln x$, and the derivative of $\ln x$ is also easier than $\ln x$ itself, we make the following substitution:

$$\left.\begin{array}{ll} u = \ln x & du = \dfrac{dx}{x} \\ dv = x^2\,dx & v = \dfrac{x^3}{3} \end{array}\right\} \Rightarrow \int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^{\cancel{3}2}}{3}\frac{dx}{\cancel{x}}$$

$$= \frac{x^3}{3}\ln x - \int \tfrac{1}{3}x^2 dx$$

$$= \frac{x^3}{3}\ln x - \tfrac{1}{9}x^3 + c$$

Example 11 – Repeated use of integration by parts

Evaluate $\int x^2 \sin x\, dx$.

Solution

Since $\sin x$ is equally easy to integrate or differentiate while x^2 is easier to differentiate, we make the following substitution:

$$\left.\begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = \sin x\,dx & v = -\cos x \end{array}\right\} \Rightarrow \int x^2 \sin x\, dx = -x^2\cos x + 2\int x\cos x\, dx$$

This first step simplified the original integral. However, the right-hand side still needs further integration. Here again, we use integration by parts.

$$\left.\begin{matrix} u = x^2 & du = 2x\,dx \\ dv = \cos x\,dx & v = \sin x \end{matrix}\right\} \Rightarrow \int 2x\cos x\,dx = 2x\sin x - 2\int \sin x\,dx$$
$$= 2x\sin x + 2\cos x + c$$

Combining the two results, we can now write

$$\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$
$$= x^2\cos x + 2x\sin x + 2\cos x + c.$$

Note: When making repeated applications of the integration by parts, you need to be careful not to change the 'nature' of the substitution in successive applications. For instance, in the previous example, the first substitution was $u = x^2$ and $dv = \sin x\,dx$. If in the second step, you had switched the substitution to $u = \cos x$ and $dv = 2x dx$, you would have obtained

$$\int x^2 \sin x\,dx = -x^2\cos x + x^2\cos x + \int x^2\sin x\,dx$$
$$= \int x^2 \sin x\,dx,$$

thus 'undoing' the previous integration and returning to the original integral.

Example 12

Evaluate $\int x^2 e^x dx$.

Solution

Since e^x is equally easy to integrate or differentiate while x^2 is easier to differentiate, we make the following substitution:

$$\left.\begin{matrix} u = x^2 & du = 2x\,dx \\ dv = e^x dx & v = e^x \end{matrix}\right\} \Rightarrow \int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

This first step simplified the original integral. However, the right-hand side still needs further integration. Here again, we use integration by parts.

$$\left.\begin{matrix} u = 2x & du = 2\,dx \\ dv = e^x dx & v = e^x \end{matrix}\right\} \Rightarrow \int 2xe^x dx = 2xe^x - 2\int e^x dx$$
$$= 2xe^x - 2e^x + c$$

Hence,

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$$
$$= x^2 e^x - 2xe^x + 2e^x + c.$$

Using integration by parts to find unknown integrals

Integrals like the one in the next example occur frequently in electricity problems. Their evaluation requires repeated applications of integration by parts followed by algebraic manipulation.

Example 13

Evaluate $\int \cos x\, e^x\, dx$.

Solution

Let

$$\left.\begin{array}{ll} u = e^x & du = e^x\, dx \\ dv = \cos x\, dx & v = \sin x \end{array}\right\} \Rightarrow \int \cos x\, e^x\, dx = e^x \sin x - \int \sin x\, e^x\, dx$$

The second integral is of the same nature, so we use integration by parts again.

$$\left.\begin{array}{ll} u = e^x & du = e^x\, dx \\ dv = \sin x\, dx & v = -\cos x \end{array}\right\} \Rightarrow \int \sin x\, e^x\, dx = -e^x \cos x + \int \cos x\, e^x\, dx$$

Hence,

$$\begin{aligned} \int \cos x\, e^x\, dx &= e^x \sin x - \int \sin x\, e^x\, dx \\ &= e^x \sin x - \left(-e^x \cos x + \int \cos x\, e^x\, dx\right) \\ &= e^x \sin x + e^x \cos x - \int \cos x\, e^x\, dx. \end{aligned}$$

Now, the unknown integral appears on both sides of the equation, thus

$$\begin{aligned} &\int \cos x\, e^x\, dx + \int \cos x\, e^x\, dx = e^x \sin x + e^x \cos x \\ &\Rightarrow 2\int \cos x\, e^x\, dx = e^x \sin x + e^x \cos x \\ &\Rightarrow \int \cos x\, e^x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + c. \end{aligned}$$

Example 14

Evaluate $\int x \ln x\, dx$.

Solution

$$\left.\begin{array}{ll} u = \ln x & du = \dfrac{dx}{x} \\ dv = x\, dx & v = \dfrac{x^2}{2} \end{array}\right\} \Rightarrow \int x^2 \ln x\, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{dx}{x}$$

$$= \frac{x^2}{2} \ln x - \int \frac{xdx}{2} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

Alternatively, we could have used a different substitution:

$$\left.\begin{array}{ll} u = x \ln x & du = (\ln x + 1)\, dx \\ dv = dx & v = x \end{array}\right\} \Rightarrow \int x \ln x\, dx = x^2 \ln x - \int x(\ln x + 1)\, dx$$

$$= x^2 \ln x - \int x \ln x\, dx - \int xdx$$

Adding $\int x \ln x\, dx$ to both sides and integrating $\int x dx$ we get

$$\int x \ln x\, dx + \int x \ln x\, dx = x^2 \ln x - \frac{x^2}{2} + c$$

$$\Rightarrow 2\int x \ln x\, dx = x^2 \ln x - \frac{x^2}{2} + c$$

$$\Rightarrow \int x \ln x\, dx = \frac{1}{2}\left(x^2 \ln x - \frac{x^2}{2} + c\right) = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

Note: The constant c is arbitrary, and hence it is unimportant that we use $c/2$ or C in our final answer.

Exercise 16.2

In questions 1–22, evaluate each integral.

1 $\int x^2 e^{-x^3} dx$

2 $\int x^2 e^{-x} dx$

3 $\int x^2 \cos 3x\, dx$

4 $\int x^2 \sin ax\, dx$

5 $\int \cos x \ln(\sin x) dx$

6 $\int x \ln x^2\, dx$

7 $\int x^2 \ln x\, dx$

8 $\int x^2 (e^x - 1) dx$

9 $\int x \cos \pi x\, dx$

10 $\int e^{3t} \cos 2t\, dt$

11 $\int \arcsin x\, dx$

12 $\int x^3 e^x dx$

13 $\int e^{-2x} \sin 2x\, dx$

14 $\int \sin(\ln x) dx$

15 $\int \cos(\ln x) dx$

16 $\int \ln(x + x^2) dx$

17 $\int e^{kx} \sin x\, dx$

18 $\int x \sec^2 x\, dx$

19 $\int \sin x \sin 2x\, dx$

20 $\int x \arctan x\, dx$

21 $\int \frac{\ln x}{\sqrt{x}} dx$

22 $\int t \sec^2 t\, dt$

23 In one scene of the movie *Stand and Deliver*, the teacher shows his students how to evaluate $\int x^2 \sin x\, dx$ by setting up a chart similar to the following.

	$\sin x$	
x^2	$-\cos x$	$+$
$2x$	$-\sin x$	$-$
2	$\cos x$	$+$

Multiply across each row and add the result. The integral is

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c.$$

Explain why the method works for this problem.

In questions 24–26, use the result of question 23 to evaluate each integral.

24 $\int x^4 \sin x\, dx$

25 $\int x^5 \cos x\, dx$

26 $\int x^4 e^x dx$

27 Show that the method used in question 23 will not work with

$$\int x^2 \ln x \, dx.$$

28 Show that $\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$, then use this *reduction formula* to show that $\int x^4 e^x dx = ax^4 e^x + bx^3 e^x + cx^2 e^x + dxe^x + fe^x + g$, where $a, b, c, \ldots, g$ are to be determined.

29 Show that $\int x^n \ln x \, dx = \dfrac{x^{n+1}}{n+1} \ln x - \dfrac{x^{n+1}}{(n+1)^2} + c.$

30 Show that $\int e^{mx} \cos nx \, dx = \dfrac{e^{mx}(m \cos nx + n \sin nx)}{m^2 + n^2} + c.$

31 Show that $\int e^{mx} \sin nx \, dx = \dfrac{e^{mx}(m \sin nx - n \cos nx)}{m^2 + n^2} + c.$

16.3 More methods of integration

In the previous section, we looked at a very powerful method for integration that has a wide range of applications. However, integration by parts does not work for all situations, and in some cases where it works, it may not be the most efficient of methods. We learned about substitution before. In this section we will consider a few trigonometric integrals and some substitutions related to trigonometric functions or their inverses.

This section is basically a set of examples that will show you how to deal with a variety of cases.

Some of the trigonometric identities you learned before will prove very helpful in this section. Key identities we will make use of are the following:

1 $\cos^2 \theta + \sin^2 \theta = 1$

2 $\sin^2 \theta = \dfrac{1 - \cos 2\theta}{2}$

3 $\cos^2 \theta = \dfrac{1 + \cos 2\theta}{2}$

4 $\sec^2 \theta = 1 + \tan^2 \theta$

Example 15

Evaluate $\int \sin^2 x \, dx$.

Solution

We can use identity (2) from the list above.

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2}\int (1 - \cos 2x) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

Example 16

Evaluate $\int \cos^4 \theta d\theta$.

Solution

Identity (3) will give us the following:

$$\int \cos^4 \theta d\theta = \int \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int (1 + 2\cos 2\theta + \cos^2 2\theta) d\theta$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{8}\left(2\theta + 2\sin 2\theta + \theta + \frac{1}{4}\sin 4\theta\right) + c$$

$$= \frac{1}{32}(12\theta + 8\sin 2\theta + \sin 4\theta) + c$$

Here is a list of a few cases and how to find the integral. There are a few more integrals that we did not list here. On exams, any non-standard cases will be accompanied by a recommended substitution.

Integral	How to find it
$\int \sin^m x \cos^n x\, dx$	If m is odd, then break $\sin^m x$ into $\sin x$ and $\sin^{m-1} x$, use the substitution $u = \cos x$ and change the integral into the form $\int \cos^p x \sin x\, dx = \int u^p du$. Similarly if n is odd.
$\int \tan^m x \sec^n x\, dx$	If m and n are odd, break off a term for $\sec x \tan x\, dx$ and express the integrand in terms of $\sec x$ since $d(\sec x) = \sec x \tan x\, dx$.
$\int \tan^n x\, dx$	Write the integrand as $\int \tan^{n-2} x \tan^2 x\, dx$, replace $\tan^2 x$ with $\sec^2 x - 1$ and then use $u = \tan x$.
$\int \sec^n x\, dx$	If n is even, factor a $\sec^2 x$ out and write the rest in terms of $\tan^2 x + 1$. If n is odd, factor a $\sec^3 x$ out. Here, integration by parts may be useful.

Example 17

Evaluate $\int \sec x\, dx$.

Solution

This integral is evaluated using a 'clever' multiplication by an atypical factor, then:

$$\int \sec x\, dx = \int \sec x \frac{\tan x + \sec x}{\tan x + \sec x} dx = \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} dx$$

Now use the substitution $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)$ dx; hence,

$$\int \sec x\, dx = \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} dx = \int \frac{du}{u}$$

$$= \ln|u| + c = \ln|\tan x + \sec x| + c.$$

Example 18

Evaluate $\int \sec^3 x\, dx$.

Solution

This can be evaluated using integration by parts and some of the results we have already established.

$$u = \sec x \qquad du = \sec x \tan x\, dx$$

$$dv = \sec^2 x dx \qquad v = \tan x$$

Hence,

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int \tan x \sec x \tan x\, dx \\ &= \sec x \tan x - \int \sec x \tan^2 x\, dx \\ &= \sec x \tan x - \int \sec x[\sec^2 x - 1]\, dx \\ &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx.\end{aligned}$$

Adding $\int \sec^3 x\, dx$ to both sides:

$$\begin{aligned}2\int \sec^3 x\, dx &= \sec x \tan x + \int \sec x\, dx \\ &= \sec x \tan x + \ln|\sec x + \tan x|\end{aligned}$$

And finally,

$$\int \sec^3 x\, dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + c.$$

Example 19

Evaluate $\int \sin^3 x \cos^3 x\, dx$.

Solution

This integral can be evaluated by separating either a cosine or a sine, then writing the rest of the expression in terms of sine or cosine.

We will separate a cosine here.

$$\begin{aligned}\int \sin^3 x \cos^3 x\, dx &= \int \sin^3 x \cos^2 x \cos x\, dx \\ &= \int \sin^3 x(1 - \sin^2 x)\cos x\, dx \\ &= \int (\sin^3 x - \sin^5 x)\cos x\, dx\end{aligned}$$

Now we let

$u = \sin x \Rightarrow du = \cos x\, dx$, and hence

$$\begin{aligned}\int \sin^3 x \cos^3 x\, dx &= \int (\sin^3 x - \sin^5 x)\cos x\, dx \\ &= \int (u^3 - u^5)\, du = \frac{u^4}{4} - \frac{u^6}{6} + c \\ &= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c.\end{aligned}$$

Some useful trigonometric substitutions

Evaluating integrals that involve $(a^2 - u^2)$, $(a^2 + u^2)$ or $(u^2 - a^2)$ may be rendered simpler by using some trigonometric substitution like the ones listed below.

Expression	Substitution	Simplified	du
$a^2 - u^2$	$u = a\sin\theta$	$a^2 - u^2 = a^2 - a^2\sin^2\theta$ $= a^2(1 - \sin^2\theta) = a^2\cos^2\theta$	$a\cos\theta d\theta$
$a^2 + u^2$	$u = a\tan\theta$	$a^2 + u^2 = a^2 + a^2\tan^2\theta$ $= a^2(1 + \tan^2\theta) = a^2\sec^2\theta$	$a\sec^2\theta d\theta$
$u^2 - a^2$	$u = a\sec\theta$	$u^2 - a^2 = a^2\sec^2 4\theta - a^2$ $= a^2(\sec^2\theta - 1) = a^2\tan^2\theta$	$a\sec\theta\tan\theta d\theta$

As you notice, this substitution is not the usual form. For convenience, we express the variable of integration in terms of the new variable. For example, rather than saying let $\theta = \arcsin\frac{u}{a}$, we say $u = a\sin\theta$. This allows us to easily find an expression for du. We will clarify the use of this type of substitution with a few examples. One important aspect of the process is how to revert back to the variable of integration. We will demonstrate that in the following examples.

Example 20

Evaluate $\int \frac{dx}{\sqrt{a^2 - x^2}}$.

Solution

This integrand is of the form involving $a^2 - u^2$, where $u = x$. We use the substitution $x = a\sin\theta$.

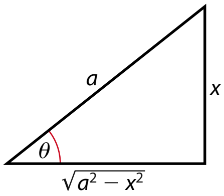
$\Rightarrow dx = a\cos\theta d\theta,$

$$\sqrt{a^2 - x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

Hence,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta d\theta}{a\cos\theta} = \int d\theta = \theta + c.$$

Now, consider the right triangle where $x = a\sin\theta \Leftrightarrow \sin\theta = \frac{x}{a}$.



$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \theta + c = \arcsin\frac{x}{a} + c.$$

Example 21

Evaluate $\int \frac{dt}{\sqrt{a^2 - t^2}}$.

Solution

This integrand is of the form involving $a^2 - u^2$, where $u = t$. We use the substitution $t = a\sin\theta$.

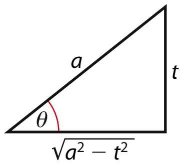
$$\Rightarrow dt = a\cos\theta d\theta,$$
$$a^2 - t^2 = a^2\cos^2\theta$$

And so

$$\int \frac{dt}{a^2 - t^2} = \int \frac{a\cos\theta d\theta}{a^2\cos^2\theta} = \frac{1}{a}\int \sec\theta d\theta = \frac{1}{a}\ln|\sec\theta + \tan\theta| + c.$$

Now, in the triangle right,

$$t = a\sin\theta \Leftrightarrow \sin\theta = \frac{t}{a},$$



$$\cos\theta = \frac{\sqrt{a^2 - t^2}}{a};$$

$$\tan\theta = \frac{t}{\sqrt{a^2 - t^2}}; \sec\theta = \frac{a}{\sqrt{a^2 - t^2}}$$

Consequently,

$$\int \frac{dt}{a^2 - t^2} = \frac{1}{a}\ln|\sec\theta + \tan\theta| + c = \frac{1}{a}\ln\left|\frac{a}{\sqrt{a^2 - t^2}} + \frac{t}{\sqrt{a^2 - t^2}}\right| + c$$

$$= \frac{1}{a}\ln\left|\frac{a + t}{\sqrt{a^2 - t^2}}\right| + c.$$

This is an acceptable answer. However, using the logarithmic properties you learned in Chapter 5, you can simplify further.

$$\int \frac{dt}{a^2 - t^2} = \frac{1}{a}\ln\left|\frac{a + t}{\sqrt{a^2 - t^2}}\right| + c = \frac{1}{a}\ln\sqrt{\frac{(a + t)^2}{a^2 - t^2}} + c$$

$$= \frac{1}{a}\ln\sqrt{\frac{(a + t)^2}{(a - t)(a + t)}} + c = \frac{1}{a}\ln\sqrt{\frac{(a + t)}{(a - t)}} + c$$

$$= \frac{1}{2a}\ln\left|\frac{(a + t)}{(a - t)}\right| + c$$

Example 22

Evaluate $\int \frac{dt}{a^2 + t^2}$.

Solution

This integrand is of the form involving $a^2 + u^2$, where $u = t$.
We use the substitution $t = a\tan\theta$.

$$\Rightarrow dt = a\sec^2\theta d\theta,$$

$$a^2 + t^2 = a^2(1 + \tan^2\theta) = a^2\sec^2\theta$$

And so

$$\int \frac{dt}{a^2 + t^2} = \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c.$$

Since $t = a\tan\theta$, then $\tan\theta = \frac{t}{a} \Rightarrow \theta = \arctan\frac{t}{a}$.

Consequently,

$$\int \frac{dt}{a^2 + t^2} = \frac{1}{a}\theta + c = \frac{1}{a}\arctan\frac{t}{a} + c.$$

Example 23

Evaluate $\int \sqrt{x^2 + 5}\, dx$.

Solution

This integrand is of the form involving $a^2 + u^2$, where $u = x$. We use the substitution $x = a\tan\theta = \sqrt{5}\tan\theta$.

$$\Rightarrow dx = \sqrt{5}\sec^2\theta d\theta,$$

$$5 + x^2 = 5(1 + \tan^2\theta) = 5\sec^2\theta d\theta$$

And so

$$\int \sqrt{x^2 + 5}\, dx = \int \sqrt{5\sec^2\theta}\sqrt{5}\sec^2\theta d\theta$$

$$= \int 5\sec^3\theta d\theta.$$

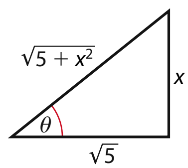
Now, earlier in Example 18, we have seen that

$$\int \sec^3 x\, dx = \frac{\sec x\tan x + \ln|\sec x + \tan x|}{2} + c.$$

And therefore

$$\int \sqrt{x^2 + 5}\, dx = 5\int \sec^3\theta d\theta = 5\left(\frac{\sec x\tan x + \ln|\sec x + \tan x|}{2}\right) + c.$$

Now, in the triangle right,



$$\tan\theta = \frac{x}{\sqrt{5}},$$

$$\sec\theta = \frac{\sqrt{5 + x^2}}{\sqrt{5}} = \sqrt{\frac{5 + x^2}{5}}, \text{ and so}$$

$$\int \sqrt{x^2 + 5}\, dx = 5\left(\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}\right) + c$$

$$= 5\left(\frac{\sqrt{\frac{5 + x^2}{5}} \cdot \frac{x}{\sqrt{5}} + \ln\left|\sqrt{\frac{5 + x^2}{5}} + \frac{x}{\sqrt{5}}\right|}{2}\right) + c$$

$$= \frac{\sqrt{5}}{2}\left(\sqrt{5 + x^2} \cdot x\right) + \frac{1}{2}\ln\left(\frac{\sqrt{5 + x^2} + x}{\sqrt{5}}\right) + c$$

$$= \frac{\sqrt{5}}{2}\left(\sqrt{5 + x^2} \cdot x\right) + \frac{1}{2}\left(\ln\left(\sqrt{5 + x^2} + x\right) - \ln\sqrt{5}\right) + c$$

$$= \frac{\sqrt{5}}{2}\left(x\sqrt{5 + x^2}\right) + \frac{1}{2}\left(\ln\left(\sqrt{5 + x^2} + x\right)\right) + C.$$

In the last step we set $-\frac{1}{2}\ln\sqrt{5} + c = C$.

Example 24

Evaluate $\int\sqrt{25 - 4x^2}\,dx$.

Solution

This integrand is of the form involving $a^2 - u^2$, where $u = 2x$. We use the substitution $2x = 5\sin\theta$.

$$2dx = 5\cos\theta d\theta \Rightarrow dx = \frac{5}{2}\cos\theta d\theta$$

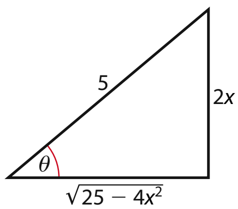
$$\sqrt{25 - 4x^2} = \sqrt{25 - 25\sin^2\theta} = 5\cos\theta$$

And so

$$\int\sqrt{25 - 4x^2}\,dx = \int 5\cos\theta\left(\frac{5}{2}\cos\theta d\theta\right) = \frac{25}{2}\int\cos^2\theta d\theta$$

$$= \frac{25}{2}\int\left(\frac{1 + \cos 2\theta}{2}\right)d\theta = \frac{25}{2}\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + c$$

$$= \frac{25}{8}(2\theta + \sin 2\theta) + c.$$



But, since $2x = 5\sin\theta$, then $\sin\theta = \frac{2x}{5} \Rightarrow \theta = \arcsin\frac{2x}{5}$, and since $\sin 2\theta = 2\sin\theta\cos\theta$, then

$$\int\sqrt{25 - 4x^2}\,dx = \frac{25}{8}(2\theta + \sin 2\theta) + c$$

$$= \frac{25}{8}\left(2\arcsin\frac{2x}{5} + 2\left(\frac{2x}{5}\right)\left(\frac{\sqrt{25 - 4x^2}}{5}\right)\right) + c$$

$$= \frac{25}{4}\arcsin\frac{2x}{5} + \frac{x\sqrt{25 - 4x^2}}{2} + c.$$

Exercise 16.3

In questions 1–44, evaluate each integral.

1 $\int\sin^3 t\cos^2 t\,dt$

2 $\int\sin^3 t\cos^3 t\,dt$

3 $\int\sin^3 3\theta\cos 3\theta\,d\theta$

4 $\int\frac{1}{t^2}\sin^5\left(\frac{1}{t}\right)\cos^2\left(\frac{1}{t}\right)dt$

5 $\int\frac{\sin^3 x}{\cos^2 x}dx$

6 $\int\tan^5 3x\sec^2 3x\,dx$

7 $\int\theta\tan^3\theta^2\sec^4\theta^2\,d\theta$

8 $\int\frac{1}{\sqrt{t}}\tan^3\sqrt{t}\sec^3\sqrt{t}\,dt$

9 $\int\tan^4(5t)dt$

10 $\int\frac{dt}{1 + \sin t}$

• **Hint:** multiply the integrand by $\frac{1 - \sin t}{1 - \sin t}$.

11 $\displaystyle\int \frac{d\theta}{1+\cos\theta}$

12 $\displaystyle\int \frac{1+\sin t}{\cos t}\,dt$

13 $\displaystyle\int \frac{\sin x - 5\cos x}{\sin x + \cos x}\,dx$

● **Hint:** find numbers a and b such that $\sin x - 5\cos x = a(\sin x + \cos x) + b(\cos x - \sin x)$.

14 $\displaystyle\int \frac{\sec\theta\tan\theta}{1+\sec^2\theta}\,d\theta$

15 $\displaystyle\int \frac{\arctan t}{1+t^2}\,dt$

16 $\displaystyle\int \frac{1}{(1+t^2)\arctan t}\,dt$

17 $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$

18 $\displaystyle\int \sin^3 x\,dx$

19 $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$

20 $\displaystyle\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx$

21 $\displaystyle\int \cos t\cos^3(\sin t)dt$

22 $\displaystyle\int \frac{\cos\theta + \sin 2\theta}{\sin\theta}\,d\theta$

23 $\displaystyle\int t\sec t\tan t\,dt$

24 $\displaystyle\int \frac{\cos x}{2-\sin x}\,dx$

25 $\displaystyle\int e^{-2x}\tan(e^{-2x})\,dx$

26 $\displaystyle\int \frac{\sec(\sqrt{t})}{\sqrt{t}}\,dt$

27 $\displaystyle\int \frac{dt}{1+\cos 2t}$

28 $\displaystyle\int \sqrt{1-9x^2}\,dx$

29 $\displaystyle\int \frac{dx}{(x^2+4)^{\frac{3}{2}}}$

30 $\displaystyle\int \sqrt{4+t^2}\,dt$

31 $\displaystyle\int \frac{3e^t dt}{4+e^{2t}}$

32 $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

33 $\displaystyle\int \frac{1}{\sqrt{4+9x^2}}\,dx$

34 $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$

35 $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

36 $\displaystyle\int \frac{x}{x^2+16}\,dx$

37 $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

38 $\displaystyle\int \frac{dx}{(9-x^2)^{\frac{3}{2}}}$

39 $\displaystyle\int x\sqrt{1+x^2}\,dx$

40 $\displaystyle\int e^{2x}\sqrt{1+e^{2x}}\,dx$

41 $\displaystyle\int e^{x}\sqrt{1-e^{2x}}\,dx$

42 $\displaystyle\int \frac{e^x dx}{\sqrt{e^{2x}+9}}$

43 $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx$

44 $\displaystyle\int \frac{x^3}{(x+2)^2}\,dx$

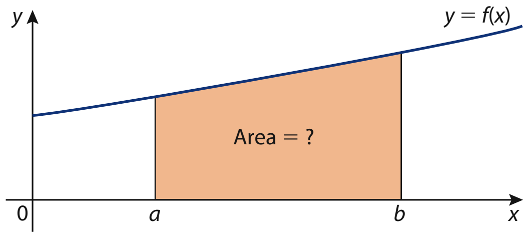
45 The integral $\displaystyle\int \frac{x}{x^2+9}\,dx$ can be evaluated either by trigonometric substitution or by direct substitution. Do it both ways and reconcile the results.

46 The integral $\displaystyle\int \frac{x^2}{x^2+9}\,dx$ can be evaluated either by trigonometric substitution or by rewriting the numerator as $(x^2+9)-9$. Do it both ways and reconcile the results.

16.4 Area and definite integral

The main goal of this section is to introduce you to the following major problem of calculus.

The area problem: Given a function $f(x)$ that is continuous and non-negative on an interval $[a, b]$, find the area between the graph of $f(x)$ and the interval $[a, b]$ on the x-axis.



● **Hint:** This is only an expository treatment that explains to you how the definite integral is developed. You will not be required to reproduce this calculation yourself.

We divide the base interval $[a, b]$ into n equal sub-intervals, and over each sub-interval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the sub-interval. The particular point does not matter – it can be above the centre, above one endpoint, or any other point in the sub-interval. In Figure 16.1 it is above the centre.

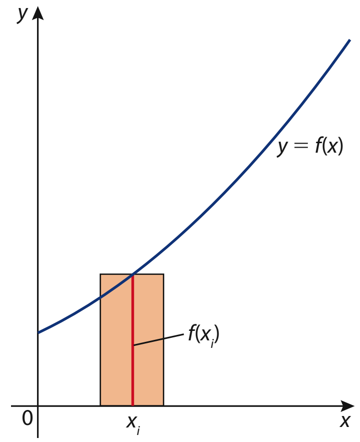


Figure 16.1

For each n, the total area of the rectangles can be viewed as an approximation to the exact area in question. Moreover, it is evident intuitively that as n increases, these approximations will get better and better and will eventually approach the exact area as a limit. See Figure 16.2.

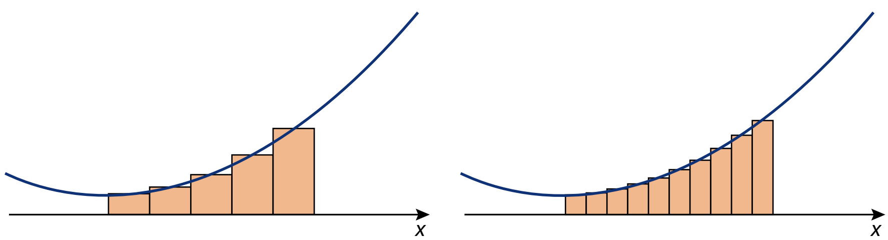


Figure 16.2

A traditional approach to this would be to study how the choice of where to erect the rectangular strip does not affect the approximation as the number of intervals increases. You can construct 'inscribed' rectangles, which, at the start, give you an underestimate of the area. On the other hand, you can construct 'circumscribed' rectangles that, at the start, overestimate the area. See Figure 16.3.

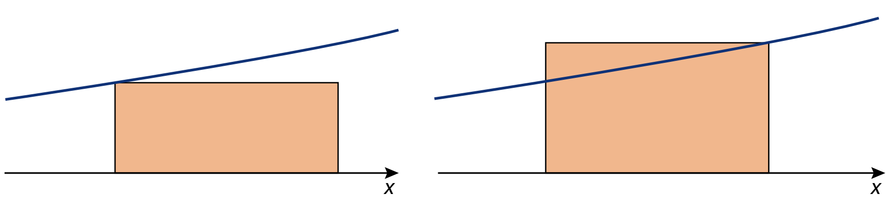


Figure 16.3

As the number of intervals increases, the difference between the overestimates and the underestimates will approach 0.

Figure 16.4

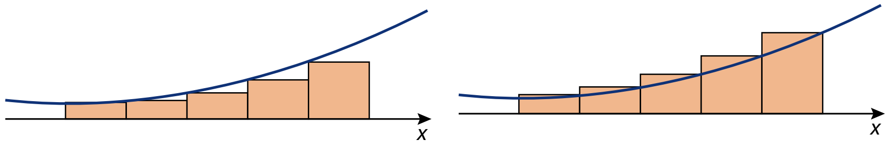
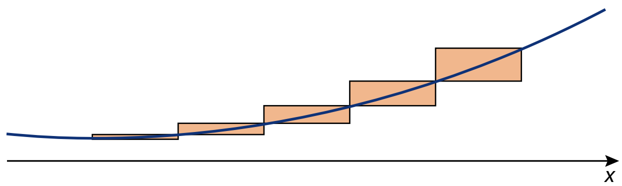


Figure 16.4 above shows n inscribed and subscribed rectangles and Figure 16.5 shows us the difference between the overestimates and the underestimates.

Figure 16.5



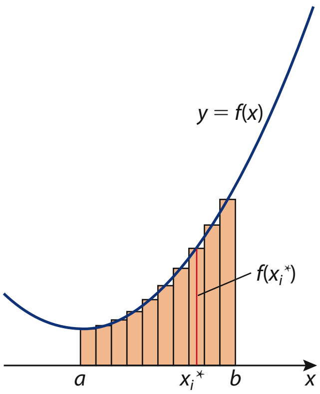


Figure 16.6

Figure 16.5 demonstrates that as the number n increases, the difference between the estimates will approach 0. Since, in Figure 16.1, we set up our rectangles by choosing a point inside the interval, the areas of the rectangles will lie between the overestimates and the underestimates, and hence as the difference between the extremes approaches zero, the rectangles we constructed will give the area of the region required.

If we consider the width of each interval to be Δx, the area of any rectangle is given as

$$A_i = f(x_i^*)\Delta x.$$

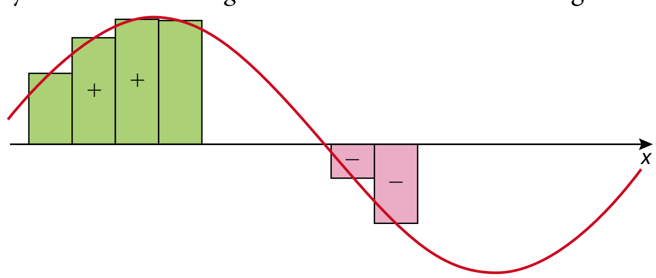
The total area of the rectangles so constructed is

$$A_n = \sum_{t=0}^{n} f(x_i^*)\Delta x$$

where x_i^* is an arbitrary point within any sub-interval $[x_{i-1}, x_i]$, $x_0 = a$ and $x_n = b$.

In the case of a function $f(x)$ that has both positive and negative values on $[a, b]$, it is necessary to consider the *signs* of the areas in the following sense.

Figure 16.7



On each sub-interval, we have a rectangle with width Δx and height $f(x^*)$. If $f(x^*) > 0$, this rectangle is above the x-axis; if $f(x^*) < 0$, this rectangle is below the x-axis. We will consider the sum defined above as the sum of the signed areas of these rectangles. That means the total area on the interval is the sum of the areas above the x-axis minus the sum of the areas of the rectangles below the x-axis.

We are now ready to look at a 'loose' definition of the definite integral.

If $f(x)$ is a continuous function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n sub-intervals of equal width $\Delta x = (b - a)/n$. We let $x_0 = a$ and $x_n = b$ and we choose $x_1^*, x_2^*, \ldots, x_n^*$ in these sub-intervals, so that x_i^* lies in the ith sub-interval $[x_{i-1}, x_i]$. Then the definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

For a list of recommended resources about definite integrals, visit www.pearsonhotlinks.com, enter the ISBN or title of this book and select weblink 2.

In the notation $\int_a^b f(x)\,dx$, in addition to the known integrand and differential, a and b are called the limits of integration: a is the lower limit and b is the upper limit.

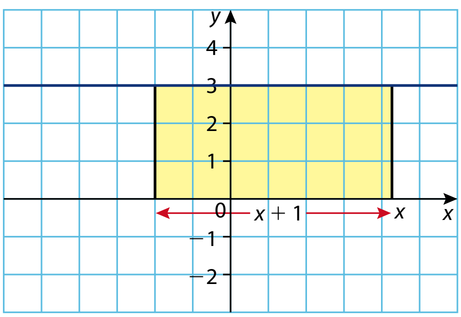
Note: Because we have assumed that $f(x)$ is continuous, it can be proved that the limit definition above always exists and gives the same value no matter how we choose the points x_i^*. If we take these points at the centre, at two-thirds the distance from the lower endpoint or at the upper endpoint, the value is the same. This is why we will state the definition of the integral from now on as

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Calling the area under the function an integral is no coincidence. To make the point, let us take the following example.

Example 25(I)

Find the area, $A(x)$, between the graph of the function $f(x) = 3$ and the interval $[-1, x]$, and find the derivative $A'(x)$ of this area function.



Solution

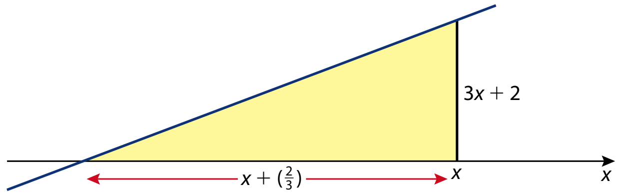
The area in question is

$$A(x) = 3(x - (-1)) = 3x + 3, \text{ and}$$

$$A'(x) = 3 = f(x).$$

Example 25(II)

Find the area, $A(x)$, between the graph of the function $f(x) = 3x + 2$ and the interval $[-2/3, x]$, and find the derivative $A'(x)$ of this area function.



Solution

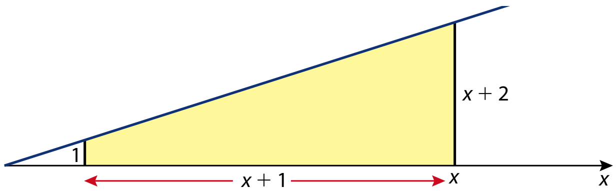
The area in question is

$$A(x) = \frac{1}{2}\left(x + \frac{2}{3}\right)(3x + 2) = \frac{1}{6}(3x + 2)^2, \text{ since this is the area of a triangle.}$$

$$\text{Hence, } A'(x) = \frac{1}{6} \times 2(3x + 2) \times 3 = 3x + 2 = f(x).$$

Example 25(III)

Find the area, $A(x)$, between the graph of the function $f(x) = x + 2$ and the interval $[-1, x]$, and find the derivative $A'(x)$ of this area function.



Solution

This is a trapezium, so the area is

$$A(x) = \frac{1}{2}(1 + (x + 2))(x + 1) = \frac{1}{2}(x^2 + 4x + 3), \text{ and}$$

$$A'(x) = \frac{1}{2} \times (2x + 4) = x + 2 = f(x).$$

Note that, in every case, $A'(x) = f(x)$.

The derivative of the area function $A(x)$ is the function whose graph forms the upper boundary of the region. It can be shown that this relation is true, not only for linear functions but for all continuous functions. Thus, to find the area function $A(x)$, we can look instead for a particular function whose derivative is $f(x)$. This is, of course, the anti-derivative of $f(x)$.

So, intuitively, as we have seen above, we define the area function as

$$A(x) = \int_a^x f(t)\,dt, \text{ that is, } A'(x) = f(x).$$

This is the trigger to the **fundamental theorem of calculus** which we will introduce in the following few pages. As we stressed at the outset, our intention here is to show you that this important theorem has its solid mathematical basis. However, examinations will not include questions requiring you to repeat the steps developed here. Just enjoy the discussion!

Before we begin the discussion, it is worth looking at some of the obvious properties of the definite integral.

Properties of the definite integral

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. When we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore, the result above follows.

2. $\int_a^a f(x)\,dx = 0$

   When $a = b$, then $\Delta x = 0$ and so the result above follows.

The following are a few straightforward properties:

3. $\int_a^b c\,dx = c(b - a)$

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant

6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   Property 6 can be demonstrated with a diagram (Figure 16.8) where the area from a to b is the sum of the two areas, i.e. $A(x) = A1 + A2$. Additionally, even if $c > b$ the relationship holds because the area from c to b in this case will be negative.

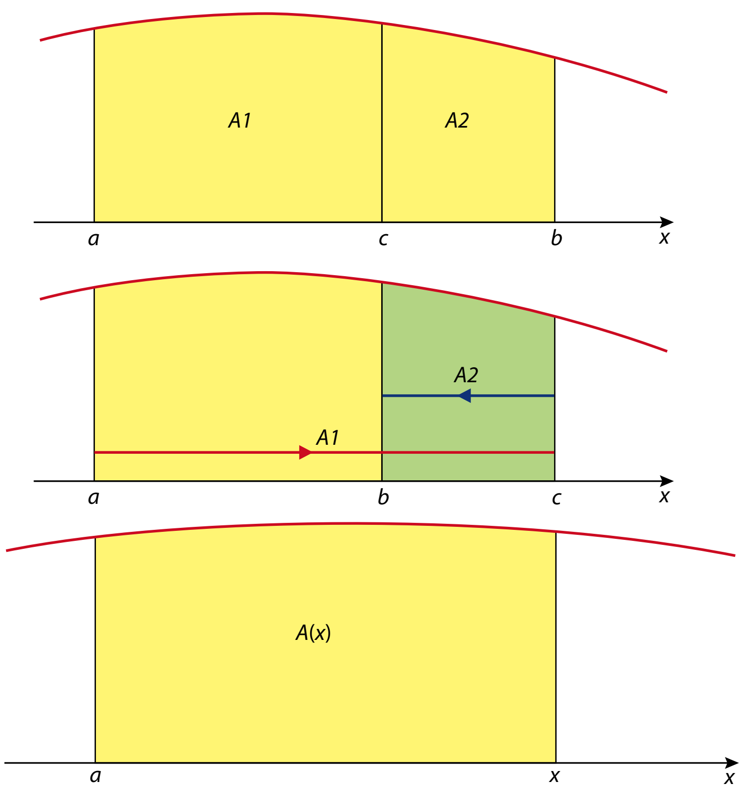
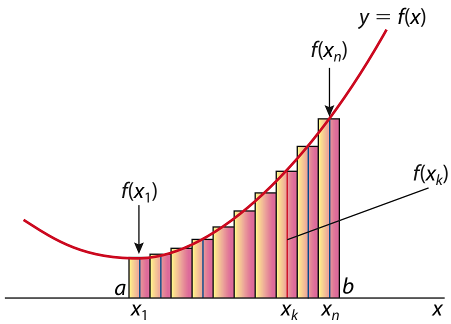


Figure 16.8

Average value of a function

As you recall from statistics, the average value of a variable is

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}.$$



We can also think of the average value of a function in the same manner. Consider a continuous function $f(x)$ defined over a closed interval $[a, b]$. We partition this interval into n sub-intervals of equal length in a fashion similar to the previous discussion. Each interval has a length

$$\triangle x = \frac{b-a}{n}.$$

Now, the average value of $f(x)$ can be defined as

$$av(f) = \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}.$$

Written in sigma notation:

$$av(f) = \frac{\sum_{k=1}^{n} f(x_k)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

However,

$$\Delta x = \frac{b-a}{n} \Rightarrow \frac{1}{n} = \frac{\triangle x}{b-a}; \text{ hence,}$$

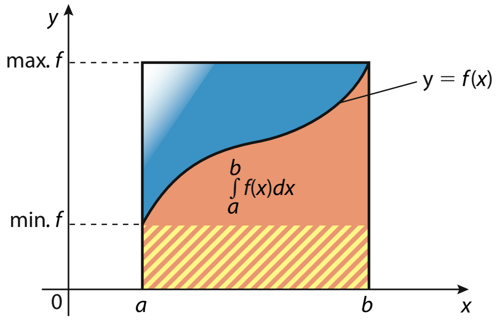
$$av(f) = \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{\triangle x}{b-a}\sum_{k=1}^{n} f(x_k) = \underbrace{\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\triangle x}_{\text{A Riemann sum for } f \text{ on } [a,\, b]}$$

This leads us to the following definition of the average value of a function $f(x)$ over an interval $[a, b]$.

The average (mean value) of an integrable function $f(x)$ over an interval $[a, b]$ is given by

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Max–min inequality



If *max. f* and *min. f* represent the maximum and minimum values of a non-negative continuous differentiable function $f(x)$ over an interval $[a, b]$, then the area under the curve lies between the area of the rectangle with base $[a, b]$ and the *min. f* as height and the rectangle with *max. f* as height.

That is,

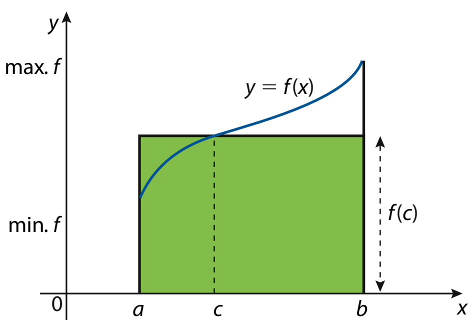
$$(b - a)\text{min.} f \leqslant \int_a^b f(x)dx \leqslant (b - a)\,max.\,f.$$

With the assumption that $b > a$, this in turn is equivalent to

$$\text{min.} f \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant max.\,f.$$

Now using the intermediate value theorem we can ascertain that there is at least one point $c \in [a, b]$ where $f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$

The value $f(c)$ in this theorem is in fact the average value of the function.



The first fundamental theorem of integral calculus

Our understanding of the definite integral as the area under the curve for $f(x)$ helps us establish the basis for the fundamental theorem of integral calculus.

In the definition of definite integral, let us make the upper limit a variable, say x. Then we will call the area between a and x, $A(x)$, i.e.

$$A(x) = \int_a^x f(t)dt.$$

Consequently,

$$A(x + h) = \int_a^{x+h} f(t)\,dt.$$

Now, if we want to find the derivative of $A(x)$, we evaluate.

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h}.$$

Using the properties of definite integrals discussed earlier, we have:

$$\begin{aligned} A(x + h) - A(x) &= \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt \\ &= \int_x^{a} f(t)\,dt + \int_a^{x+h} f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

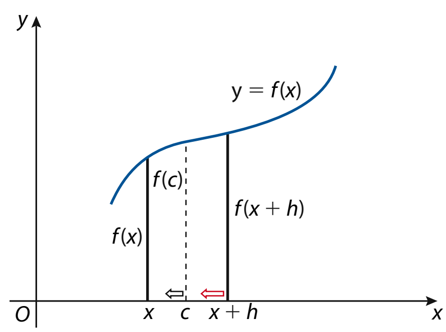
Therefore,

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

Looking at this result and what we established about the average value of $f(x)$ over the interval $[x, x + h]$ we can conclude that there is a point $c \in [x, x + h]$ such that

$$f(c) = \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

What happens to c as h approaches 0?



Answer: as h approaches 0, $x + h$ must approach x. This means, we are 'squeezing' c between x and a number approaching x. So, c must also approach x. That is,

$f(c) = f(x)$, and consequently

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(c) = f(x).$$

This last equation is stating that

$$\frac{d}{dx}(A(x)) = A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

This very powerful statement is called the first fundamental theorem of integral calculus. In essence it says that the processes of integration and derivation are inverses of one another.

Note: It is important to remember that $\int_a^x f(t)\,dt$ is a function of x!

Example 26

Find each of the following.

a) $\frac{d}{dx}\int_{-e}^{x} \sec^2 t\,dt$

b) $\frac{d}{dx}\int_{0}^{x} \frac{dt}{1+t^4}$

c) $\frac{d}{dx}\int_{x}^{\pi} \frac{1}{1+t^4}\,dt$

d) $\frac{d}{dx}\int_{0}^{2x+x^3} \frac{1}{1+t^4}\,dt$

e) $\frac{d}{dx}\int_{x}^{2x+x^3} \frac{1}{1+t^4}\,dt$

Solution

a) This is a direct application of the fundamental theorem.

$$\frac{d}{dx}\int_{-e}^{x} \sec^2 t\,dt = \sec^2 x$$

b) This is also straightforward.

$$\frac{d}{dx}\int_{0}^{x} \frac{dt}{1+t^4} = \frac{1}{1+x^4}$$

c) In this exercise, we need to rewrite the expression before we perform the calculation.

$$\frac{d}{dx}\int_{x}^{\pi} \frac{1}{1+t^4}\,dt = \frac{d}{dx}\int_{\pi}^{x} -\frac{1}{1+t^4}\,dt = -\frac{d}{dx}\int_{x}^{\pi} \frac{1}{1+t^4}\,dt = \frac{-1}{1+x^4}$$

d) Remembering that this is a function of x, and that the upper limit is a function of x, which makes $\int_{0}^{2x+x^3} \frac{1}{1+t^4}\,dt$ a composite of $\int_{0}^{u} \frac{1}{1+t^4}\,dt$ and $u = 2x + x^3$. So, we have to resort to the chain rule!

$$\frac{d}{dx}\int_{0}^{2x+x^3} \frac{1}{1+t^4}\,dt = \left(\frac{d}{du}\int_{0}^{u} \frac{1}{1+t^4}\right)\left(\frac{du}{dx}\right)$$

$$= \frac{1}{1+u^4} \cdot \frac{du}{dx}$$

$$= \frac{1}{1+(2x+x^3)^4} \cdot (2+3x^2)$$

$$= \frac{2+3x^2}{1+(2x+x^3)^4}$$

e) Again, here we need to rewrite the integral before evaluation.

$$\frac{d}{dx}\int_x^{2x+x^3}\frac{1}{1+t^4}\,dt = \frac{d}{dx}\left(\int_x^k \frac{1}{1+t^4}\,dt + \int_k^{2x+x^3}\frac{1}{1+t^4}\,dt\right)$$

$$= \frac{2+3x^2}{1+(2x+x^3)^4} - \frac{1}{1+x^4}$$

The second fundamental theorem of integral calculus

Recall that $A(x) = \int_a^x f(t)\,dt$. If $F(x)$ is any anti-derivatives of $f(x)$, then applying what we learned in earlier sections:

$$F(x) = A(x) + c \text{ where } c \text{ is an arbitrary constant.}$$

Now,

$$F(b) = A(b) + c = \int_a^b f(t)\,dt + c, \text{ and}$$

$$F(a) = A(a) + c = \int_a^a f(t)\,dt + c = 0 - c, \text{ and hence}$$

$$F(b) - F(a) = \int_a^b f(t)\,dt + c - c$$

$$= \int_a^b f(t)\,dt.$$

Second fundamental theorem of calculus

$$\int_a^b f(t)dt = F(b) - F(a)$$

The fundamental theorem is also referred to as the **evaluation theorem**. Also, since we know that $F'(x)$ is the rate of change in $F(x)$ with respect to x and that $F(b) - F(a)$ is the change in y when x changes from a to b, we can reformulate the theorem in words.

The integral of a rate of change is the **total change**.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Here are a few instances where this applies:

1. If $V'(t)$ is the rate at which a liquid flows into or out of a container at time t, then

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of liquid in the container between time t_1 and t_2.

2. If the rate of growth of a population is $n'(t)$, then

$$\int_{t_1}^{t_2} n'(t)dt = n(t_2) - n(t_1)$$

is the increase (decrease!) in population during the time period from t_1 to t_2.

3. Displacement situations are described separately later in the chapter.

This theorem has many other applications in calculus and several other fields. It is a very powerful tool to deal with problems of area, volume and work among other applications. In this book, we will apply it to finding areas between functions and volumes of revolution as well as displacement problems.

Notation:

We will use the following notation:

$$\int_a^b f(t)\,dt = F(x)\Big]_a^b = F(b) - F(a)$$

Example 27

a) Evaluate the integral $\int_{-1}^{3} x^5\,dx$.

b) Evaluate the integral $\int_0^4 \sqrt{x}\,dx$.

c) Evaluate the integral $\int_{\pi}^{2\pi} \cos\theta\,d\theta$.

d) Evaluate the integral $\int_1^2 \frac{4+u^2}{u^3}\,du$.

Solution

a) $\int_{-1}^{3} x^5\,dx = \frac{x^6}{6}\Big]_{-1}^{3} = \frac{3^6}{6} - \frac{1}{6} = \frac{364}{3}$

b) $\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big]_0^4 = \frac{2}{3}4^{\frac{3}{2}} - 0 = \frac{16}{3}$

c) $\int_{\pi}^{2\pi} \cos\theta\,d\theta = \sin\theta\Big]_{\pi}^{2\pi} = 0 - 0 = 0$

d) $$\begin{aligned}\int_1^2 \frac{4+u^2}{u^3}\,du &= \int_1^2 \left(\frac{4}{u^3} + \frac{1}{u}\right)du = \left[4\cdot\frac{u^{-2}}{-2} + \ln|u|\right]_1^2 \\ &= \left[-2u^{-2} + \ln u\right]_1^2 \\ &= (-2\cdot 2^{-2} + \ln 2) - (-2\cdot 1 + \ln 1) = -\frac{1}{2} + \ln 2 + 2 \\ &= \frac{3}{2} + \ln 2\end{aligned}$$

Using substitution with definite integral

In Section 16.1 we discussed the use of substitution to evaluate integrals in cases that are not easily recognized. We established the following rule:

$$\int f(u(x)) \cdot u'(x)dx = \int f(u)du = F(u(x)) + c = F(x) + c$$

When evaluating definite integrals by substitution, two methods are available.

1 Evaluate the indefinite integral first, revert to the original variable, and then use the fundamental theorem. For example, to evaluate

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\, dx,$$

we find the indefinite integral

$$\int \tan^5 x \sec^2 x\, dx = \int u^5 du = \tfrac{1}{6}u^6 = \tfrac{1}{6}\tan^6 x,$$

and then we use the fundamental theorem, i.e.

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\, dx = \tfrac{1}{6}\tan^6 x\Big]_0^{\frac{\pi}{3}} = \tfrac{1}{6}(\sqrt{3})^6 = \tfrac{27}{6} = \tfrac{9}{2}.$$

2 Use the following 'substitution rule' for definite integrals:

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Proof:
If $F(x)$ is an anti-derivative of $f(x)$, then by the fundamental theorem

$$\int_b^a f(u(x))u'(x)dx = F(u(x))\Big]_a^b = F(u(b)) - F(u(a)).$$

Also,

$$\int_{u(a)}^{u(b)} f(u)du = F(u)\Big]_{u(a)}^{u(b)} = F(u(b)) - F(u(a)).$$

Therefore, to evaluate

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\, dx,$$

letting $u = \tan x \Rightarrow u\left(\frac{\pi}{3}\right) = \sqrt{3}$, $u(0) = 0$, and so

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\, dx = \int_0^{\sqrt{3}} u^5 du = \tfrac{1}{6}u^6\Big]_0^{\sqrt{3}} = \frac{9}{2}.$$

Example 28

Evaluate $\int_2^6 \sqrt{4x + 1}\, dx$.

Solution

Let $u = 4x + 1$, then $du = 4dx$. The limits of integration are

$u(2) = 9$ and $u(6) = 25$, therefore

$$\int_2^6 \sqrt{4x + 1}\, dx = \frac{1}{4}\int_9^{25} \sqrt{u}\, du = \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big]_9^{25}$$

$$= \frac{1}{6}(125 - 27) = \frac{49}{3}.$$

Observe that using this method, we do not return to the original variable of integration. We simply evaluate the 'new' integral between the appropriate values of u.

Exercise 16.4

In questions 1–42, evaluate the integral.

1 $\displaystyle\int_{-2}^{1} (3x^2 - 4x^3)\,dx$

2 $\displaystyle\int_{2}^{7} 8\,dx$

3 $\displaystyle\int_{1}^{5} \frac{2}{t^3}\,dt$

4 $\displaystyle\int_{2}^{2} (\cos t - \tan t)\,dt$

5 $\displaystyle\int_{1}^{7} \frac{2x^2 - 3x + 5}{\sqrt{x}}\,dx$

6 $\displaystyle\int_{0}^{\pi} \cos\theta\,d\theta$

7 $\displaystyle\int_{0}^{\pi} \sin\theta\,d\theta$

8 $\displaystyle\int_{3}^{1} (5x^4 + 3x^2)\,dx$

9 $\displaystyle\int_{1}^{3} \frac{u^5 + 2}{u^2}\,du$

10 $\displaystyle\int_{1}^{e} \frac{2\,dx}{x}$

11 $\displaystyle\int_{1}^{3} \frac{2x}{x^2 + 2}\,dx$

12 $\displaystyle\int_{1}^{3} (2 - \sqrt{x})^2\,dx$

13 $\displaystyle\int_{0}^{\frac{\pi}{4}} 3\sec^2\theta\,d\theta$

14 $\displaystyle\int_{0}^{1} (8x^7 + \sqrt{\pi})\,dx$

15 a) $\displaystyle\int_{0}^{2} |3x|dx$ b) $\displaystyle\int_{-2}^{0} |3x|dx$ c) $\displaystyle\int_{-2}^{2} |3x|dx$

16 $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin 2x\,dx$

17 $\displaystyle\int_{1}^{9} \frac{1}{\sqrt{x}}\,dx$

18 $\displaystyle\int_{-2}^{2} (e^x - e^{-x})dx$

19 $\displaystyle\int_{-1}^{1} \frac{dx}{1 + x^2}$

20 $\displaystyle\int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}}$

21 $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{4 - x^2}}$

22 $\displaystyle\int_{-2}^{0} \frac{dx}{4 + x^2}$

23 $\displaystyle\int_{0}^{4} \frac{x^3\,dx}{\sqrt{x^2 + 1}}$

24 $\displaystyle\int_{1}^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}\,dx$

25 $\displaystyle\int_{e}^{e^2} \frac{dt}{t\ln t}$

26 $\displaystyle\int_{-1}^{2} 3x\sqrt{9 - x^2}\,dx$

27 $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\sqrt{3 + \cos x}}\,dx$

28 $\int_e^{e^2} \frac{\ln x}{x} dx$

29 $\int_1^{\sqrt{3}} \frac{\sqrt{\arctan x}}{1+x^2} dx$

30 $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln x)^2}}$

31 $\int_{-\ln 2}^{\ln 2} \frac{e^{2x}}{e^{2x}+9} dx$

32 $\int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-2x}dx}{\sqrt{1-e^{-4x}}}$

33 $\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sec^2 x\, dx$

34 $\int_0^{\sqrt{\pi}} 7x\cos x^2\, dx$

35 $\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

36 $\int_0^1 \frac{\sqrt{3}x}{\sqrt{4-3x^4}} dx$

37 $\int_0^{\frac{2}{\sqrt{3}}} \frac{dx}{9+4x^2}$

38 $\int_1^{\sqrt{2}} \frac{x\,dx}{3+x^4}$

39 $\int_0^{\frac{\pi}{6}} (1-\sin 3t)\cos 3t\, dt$

40 $\int_0^{\frac{\pi}{4}} e^{\sin 2\theta}\cos 2\theta\, d\theta$

41 $\int_0^{\frac{\pi}{8}} (3+e^{\tan 2t})\sec^2 2t\, dt$

42 $\int_0^{\sqrt{\ln \pi}} 4te^{t^2}\sin(e^{t^2})dt$

In questions 43–47, find the average value of the given function over the given interval.

43 x^4, $[1, 2]$

44 $\cos x$, $\left[0, \frac{\pi}{2}\right]$

45 $\sec^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$

46 e^{-2x}, $[0, 4]$

47 $\frac{e^{3x}}{1+e^{6x}}$, $\left[\frac{-\ln 3}{6}, 0\right]$

In questions 48–55, find the indicated derivative.

48 $\frac{d}{dx}\int_2^x \frac{\sin t}{t} dt$

49 $\frac{d}{dt}\int_t^3 \frac{\sin x}{x} dx$

50 $\frac{d}{dx}\int_{x^2}^0 \frac{\sin t}{t} dt$

51 $\frac{d}{dx}\int_0^{x^2} \frac{\sin u}{u} du$

52 $\frac{d}{dt}\int_{-\pi}^t \frac{\cos y}{1+y^2} dy$

53 $\frac{d}{dx}\int_{ax}^{bx} \frac{dt}{5+t^4}$

54 $\frac{d}{d\theta}\int_{\sin\theta}^{\cos\theta} \frac{1}{1-x^2} dx$

55 $\frac{d}{dx}\int_5^{x^{\frac{1}{4}}} e^{t^4+3t^2} dt$

56 Does the function $F(x) = \int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right)dt$ have an extreme value?

57 a) Find $\int_o^k \frac{dx}{3x+2}$, giving your answer in terms of k.

b) Given that $\int_o^k \frac{dx}{3x+2} = 1$, calculate the value of k.

58 Given that $p, q \in \mathbb{N}$, show that

$$\int_0^1 x^p(1-x)^q dx = \int_0^1 x^q(1-x)^p\, dx.$$

Do not attempt to evaluate the integrals.

59 Given that $k \in \mathbb{N}$, evaluate the integral.

a) $\int x(1-x)^k dx$ b) $\int_0^1 x(1-x)^k dx$

60 Let $F(x) = \int_3^x \sqrt{5t^2 + 2}\, dt$. Find

a) $F(3)$ b) $F'(3)$ c) $F''(3)$

61 Show that the function

$$f(x) = \int_x^{3x} \frac{dt}{t}$$

is constant over the set of positive real numbers.

16.5 Integration by method of partial fractions (Optional)

In this section, we will see how rational functions with polynomial denominators can be integrated. For example, if we were to find the indefinite integral $\int \frac{x+1}{x^2+5x+6} dx$, we first decompose the integrand into partial fractions and then the integration process would be straightforward.

$$\frac{x+1}{x^2+5x+6} \equiv \frac{a}{x+2} + \frac{b}{x+3}$$

(See Section 3.6 for details.)

After solving for a and b we can perform the integration:

$$\int \frac{x+1}{x^2+5x+6} \equiv \int\left(\frac{-1}{x+2} + \frac{2}{x+3}\right)dx = -\ln|x+2| + 2\ln|x+3| + c$$
$$= \ln\left|\frac{(x+3)^2}{x+2}\right| + c$$

Example 29

Find the indefinite integral $\int \frac{3x-1}{x^2+4x+4} dx$.

Solution

Using partial fractions will make the work simpler than otherwise.

From Example 42 of Section 3.6 we know:

$$\frac{3x-1}{x^2+4x+4} = \frac{3}{x+2} - \frac{7}{(x+2)^2}$$

Hence, the integral can be rewritten as:

$$\int \frac{3x-1}{x^2+4x+4} dx = \int \frac{3}{x+2} dx - \int \frac{7}{(x+2)^2} dx$$

These two integrals can be found by inspection, giving:

$$\int \frac{3x - 1}{x^2 + 4x + 4}dx = 3 \ln |x + 2| + \frac{7}{x + 2} + c$$

Example 30

Find the indefinite integral $\int \frac{2}{x^3 + 2x^2 + 2x}dx$.

Solution

Again, from Example 43 of Section 3.6, we have:

$$\frac{2}{x^3 + 2x^2 + 2x} = \frac{1}{x} - \frac{x + 2}{x^2 + 2x + 2}$$

Hence, we can write the integral as:

$$\int \frac{2}{x^3 + 2x^2 + 2x}dx = \int \frac{dx}{x} - \frac{x + 2}{x^2 + 2x + 2}dx = \int \frac{dx}{x} - \int \frac{x + 1 + 1}{x^2 + 2x + 2}dx$$

$$= \int \frac{dx}{x} - \frac{1}{2} \int \frac{2x + 4}{x^2 + 2x + 2}dx = \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x + 4}{(x + 1)^2 + 1}dx$$

$$= \ln |x| - \frac{1}{4} \ln(x^2 + 2x + 2) - \arctan (x + 1) + c$$

Example 31

Find the indefinite integral $\int \frac{5x^2 + 16x + 17}{2x^3 + 9x^2 + 7x - 6}dx$.

Solution

Again from Example 41 of Section 3.6 we have:

$$\int \frac{5x^2 + 16x + 17}{2x^3 + 9x^2 + 7x - 6}dx = \int \frac{3}{2x - 1}dx - \int \frac{1}{x + 2}dx + \int \frac{2}{x + 3}dx$$

$$= \frac{3}{2} \ln|2x - 1| - \ln|x + 2| + 2 \ln|x + 3| + c$$

Example 32

Evaluate $\int \frac{3x - 1}{x^3 + 8}dx$.

Solution

We first factorize the denominator:

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

Now, by using partial fractions we have:

$$\frac{3x - 1}{x^3 + 8} = \frac{a}{x + 2} + \frac{bx + c}{x^2 - 2x + 4}$$

Solving for a, b, and c will yield:

$$3x - 1 \equiv a(x^2 - 2x + 4) + (bx + c)(x + 2)$$
$$= (a + b)x^2 + (2b - 2a + c)x + 4a + 2c$$

This implies: $\begin{cases} a + b = 0 \\ 2b - 2a + c = 3 \\ 4a + 2c = -1 \end{cases}$

Solving this system of equations will yield:

$$a = \frac{-7}{12}, b = \frac{7}{12}, c = \frac{2}{3}$$

Therefore,

$$\int \frac{3x - 1}{x^3 + 8}dx = \int \frac{-\frac{7}{12}}{x + 2}dx + \int \frac{\frac{7}{12}x + \frac{2}{3}}{x^2 - 2x + 4}dx$$

Finally, using what you learned so far you can verify the answer to be:

$$\int \frac{3x - 1}{x^3 + 8}dx = -\frac{7}{12}\ln|x+2| + \frac{7}{24}\ln(x^2 - 2x + 4) - \frac{5\sqrt{3}}{12}\arctan\left(\frac{x - 1}{\sqrt{3}}\right) + c$$

Summary of procedures

In this book we will only consider five general cases. They are outlined below.

Possible cases for partial fractions

1 **Denominator is a quadratic** that factorises into two distinct linear factors, and numerator $p(x)$ is a constant or linear.

$$\frac{p(x)}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}$$

2 **Denominator is a quadratic** that factorises into two repeated linear factors, and numerator $p(x)$ is a constant or linear.

$$\frac{p(x)}{(ax + b)^2} = \frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

3 **Denominator is a cubic** that factorises into three repeated linear factors, and numerator $p(x)$ is a constant, linear or quadratic.

$$\frac{p(x)}{(ax + b)^3} = \frac{A}{ax + b} + \frac{B}{(ax + b)^2} + \frac{C}{(ax + b)^3}$$

4 **Denominator is a cubic** that factorises into one linear factor and one quadratic factor (that cannot be factorised), and numerator $p(x)$ is a constant, linear or quadratic.

$$\frac{p(x)}{(ax + b)(cx^2 + dx + e)} = \frac{A}{ax + b} + \frac{Bx + C}{cx^2 + dx + e}$$

5 **Denominator is a cubic** that factorises into three distinct linear factors, and numerator $p(x)$ is a constant, linear or quadratic.

$$\frac{p(x)}{(ax + b)(cx + d)(ex + f)} = \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{ex + f}$$

A consequence of the Fundamental Theorem of Algebra (see margin note in Section 3.3) guarantees that any polynomial with real coefficients can only have factors that are linear or quadratic.

Exercise 16.5

Evaluate each integral.

1 $\displaystyle\int \frac{5x+1}{x^2+x-2}dx$

2 $\displaystyle\int \frac{x+4}{x^2-2}dx$

3 $\displaystyle\int \frac{x+2}{x^2+4x+3}dx$

4 $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

5 $\displaystyle\int \frac{2x^2+x-12}{x^3+5x^2+6x}dx$

6 $\displaystyle\int \frac{4x^2+2x-1}{x^3+x^2}dx$

7 $\displaystyle\int \frac{3}{x^2+x-2}dx$

8 $\displaystyle\int \frac{5-x}{2x^2+x-1}dx$

9 $\displaystyle\int \frac{3x+4}{(x+2)^2}dx$

10 $\displaystyle\int \frac{12}{x^4-x^3-2x^2}dx$

11 $\displaystyle\int \frac{2}{x^3+x}dx$

12 $\displaystyle\int \frac{x+2}{x^3+3x}dx$

13 $\displaystyle\int \frac{3x+2}{x^3+6x}dx$

14 $\displaystyle\int \frac{2x+3}{x^3+8x}dx$

15 $\displaystyle\int \frac{x+5}{x^3-4x^2-5x}dx$

16.6 Areas

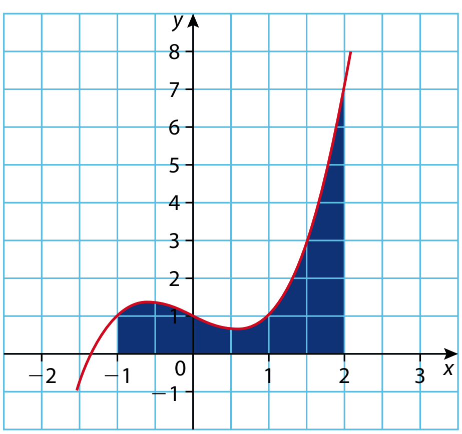
We have seen how the area between a curve, defined by $y = f(x)$, and the x-axis can be computed by the integral $\int_a^b f(x)dx$ on an interval $[a, b]$, where $f(x) \geqslant 0$. In this section, we shall find that integration can be used to find the area of more general regions between curves.

Areas between curves of functions of the form $y = f(x)$ and the x-axis

If the function $y = f(x)$ is always above the x-axis, finding the area is a straightforward computation of the integral $\int_a^b f(x)dx$.

Example 33

Find the area under the curve $f(x) = x^3 - x + 1$ and the x-axis over the interval $[-1, 2]$.

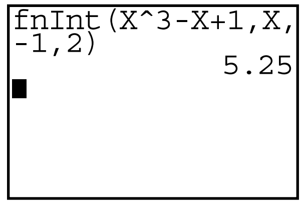


Solution

This area is simply

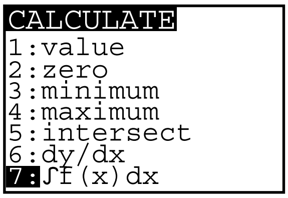
$$\int_{-1}^{2}(x^3 - x + 1)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2} + x\right]_{-1}^{2}$$

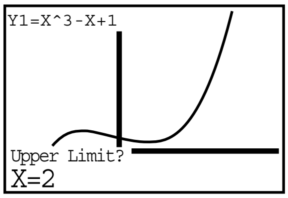
$$= (4 - 2 + 2) - \left(\frac{1}{4} - \frac{1}{2} - 1\right) = 5\frac{1}{4}.$$

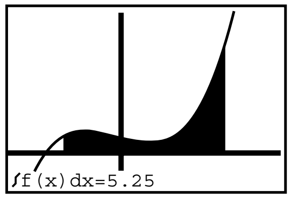


Using your GDC, this is done by simply choosing the 'MATH' menu, then the 'fnInt' menu item.

Or, you can type in your function and then go to the 'CALC' menu, where you choose '$\int f(x)\,dx$' and type in your integration limits. Here is what you see.



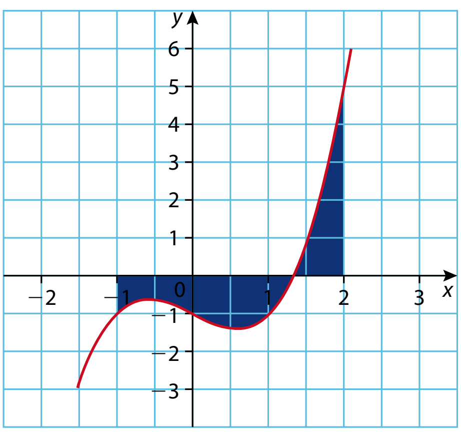




In some cases, you will have to adjust how you work. This is the case when the graph intersects the x-axis. Since you are interested in the area bounded by the curve and the interval $[a, b]$ on the x-axis, you do not want the 'signed' areas to cancel each other. This is why you have to split the process into different sub-intervals where you take the absolute values of the areas found and add them.

Example 34

Find the area under the curve $f(x) = x^3 - x - 1$ and the x-axis over the interval $[-1, 2]$.



Solution

As you see from the diagram, a part of the graph is below the x-axis, and its area will be negative. If you try to integrate this function without paying attention to the intersection with the x-axis, this is what you get:

$$\int_{-1}^{2}(x^3 - x - 1)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2} - x\right]_{-1}^{2}$$

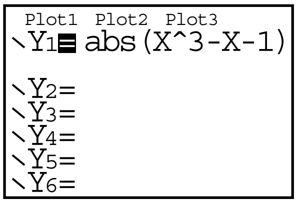
$$= (4 - 2 - 2) - \left(\frac{1}{4} - \frac{1}{2} + 1\right) = -\frac{3}{4}$$

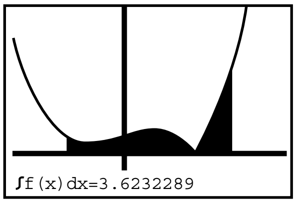
This integration has to be split before we start. However, this is a function where you cannot find the intersection point. So, we either use our GDC to find the intersection, or we just take the absolute values of the different parts of the region. This is done by integrating the absolute value of the function:

$$\text{Area} = \int_a^b |f(x)|\,dx$$

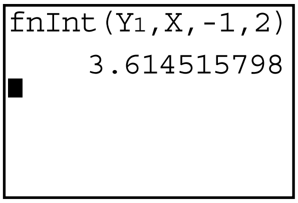
$$\text{Hence, area} = \int_{-1}^{2} |(x^3 - x - 1)|\,dx.$$

As we said earlier, this is not easy to find given the difficulty with the x-intercept. It is best if we make use of a GDC.





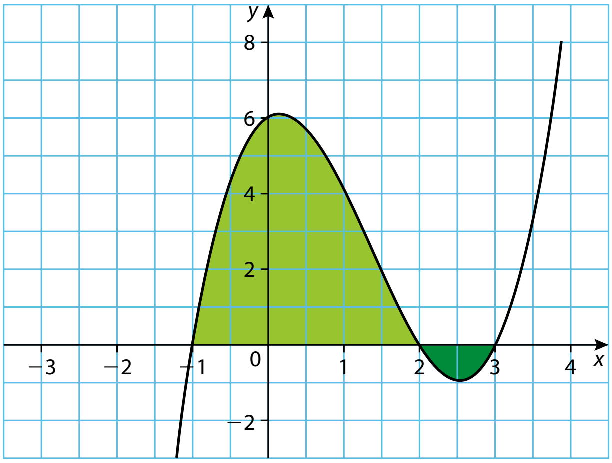
Or, using 'fnInt' directly:



The difference between them is that the latter is more of a rough approximation than the first.

Example 35

Find the area enclosed by the graph of the function $f(x) = x^3 - 4x^2 + x + 6$ and the x-axis.



Solution

This function intersects the x-axis at three points where $x = -1, 2$ and 3.
To find the area, we split it into two and then add the absolute values:

$$\text{Area} = \int_{-1}^{3} |f(x)|\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{3} (-f(x))\,dx$$

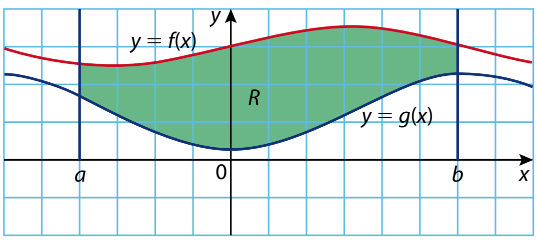
$$= \int_{-1}^{2} (x^3 - 4x^2 + x + 6)\,dx + \int_{2}^{3} (-x^3 + 4x^2 - x - 6)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + 6x\right]_{-1}^{2} + \left[-\frac{x^4}{4} + \frac{4x^3}{3} - \frac{x^2}{2} - 6x\right]_{2}^{3}$$

$$= \frac{45}{4} + \frac{7}{12} = \frac{71}{6}$$

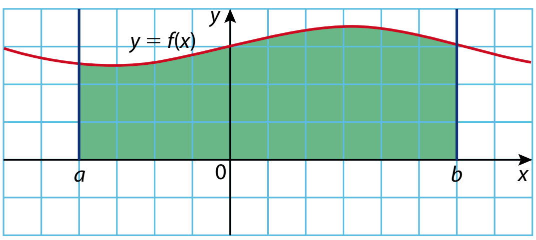
Area between curves

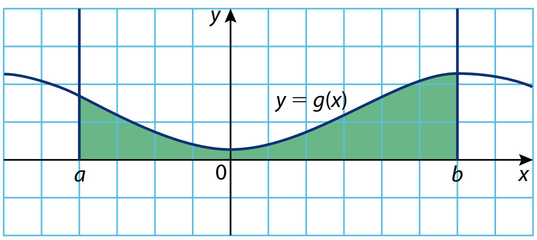
In some practical problems, you may have to compute the area between two curves. Suppose $f(x)$ and $g(x)$ are functions such that $f(x) \geqslant g(x)$ on the interval $[a, b]$, as shown in the diagram. Note that we do not insist that both functions are non-negative, but we begin by showing that case for demonstration purposes.



To find the area of the region R between the curves from $x = a$ to $x = b$, we subtract the area between the lower curve $g(x)$ and the x-axis from the area between the upper curve $f(x)$ and the x-axis; that is,

$$\text{Area of } R = \int_{a}^{b} f(x)\,dx - \int_{a}^{b} g(x)\,dx = \int_{a}^{b} [f(x) - g(x)]\,dx.$$

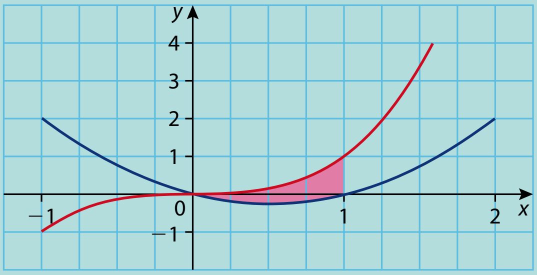




The fact just mentioned applies to all functions, not only positive functions. These facts are used to define the area between curves.

If $f(x)$ and $g(x)$ are functions such that $f(x) \geqslant g(x)$ on the interval $[a, b]$, then the area between the two curves is given by

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



Example 36

Find the area of the region between the curves $y = x^3$ and $y = x^2 - x$ on the interval $[0, 1]$. (See diagram above.)

Solution

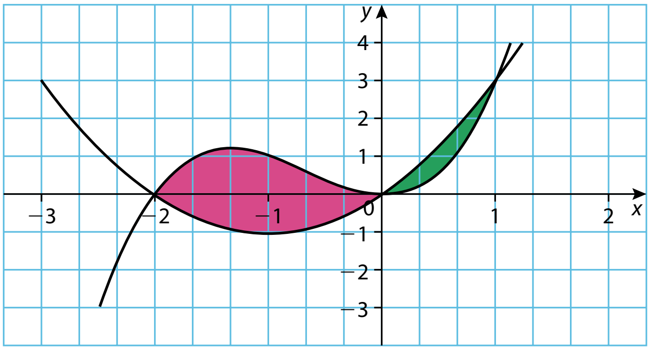
$y = x^3$ appears to be higher than $y = x^2 - x$ with one intersection at $x = 0$. Thus, the required area is

$$A = \int_0^1 [x^3 - (x^2 - x)]\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{12}.$$

In order to take all cases into consideration, we will present here another case where you must be very careful of how you calculate the area. This is the case where the two functions in question intersect at more than one point. We will clarify this with an example.

Example 37

Find the area of the region bounded by the curves $y = x^3 + 2x^2$ and $y = x^2 + 2x$.



Solution

The two curves intersect when

$$x^3 + 2x^2 = x^2 + 2x \Rightarrow x^3 + x^2 - 2x = 0 \Rightarrow x(x + 2)(x - 1) = 0,$$
i.e. when $x = -2, 0$ or 1.

The area is equal to

$$A = \int_{-2}^{0} [x^3 + 2x^2 - (x^2 + 2x)]\, dx + \int_{0}^{1} [x^2 + 2x - (x^3 + 2x^2)]\, dx$$

$$= \int_{-2}^{0} [x^3 + x^2 - 2x]\, dx + \int_{0}^{1} [-x^2 + 2x - x^3]\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right]_{0}^{1}$$

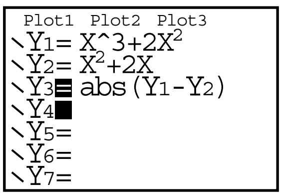
$$= 0 - \left[\frac{16}{4} - \frac{8}{3} - 4\right] + \left[-\frac{1}{4} - \frac{1}{3} + 1\right] - 0 = \frac{37}{12}.$$

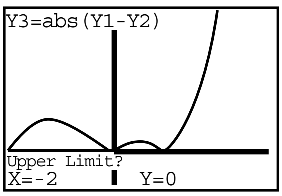
This discussion leads us to stating the general expression you should use in evaluating areas between curves.

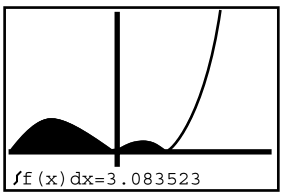
If $f(x)$ and $g(x)$ are continuous functions on the interval $[a, b]$, the area between the two curves is given by

$$A = \int_{a}^{b} |f(x) - g(x)| dx.$$

The above computation can be done with your GDC as follows:

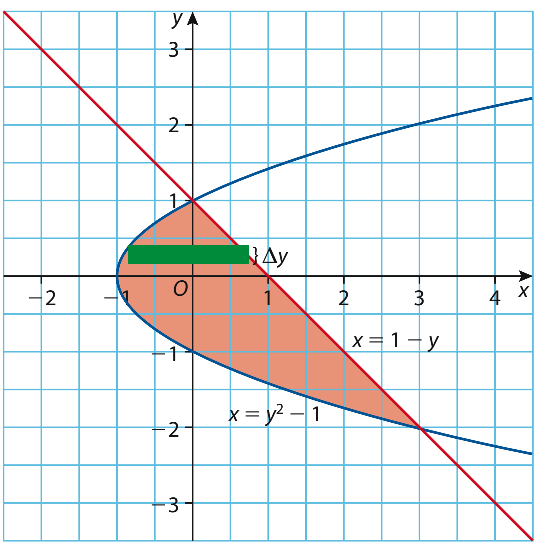






Areas along the y-axis

If we were to find the area enclosed by $y = 1 - x$ and $y^2 = x + 1$, it would be best to treat the region between them by regarding x as a function of y as you see in the graph here.



The area of the shaded region can be calculated using the following integral:

$$A(y) = \int_{-1}^{1} |(1 - y) - (y^2 - 1)|\,dy$$

$$= \int_{-2}^{1} |2 - y - y^2|\,dy = \left|2y - \frac{y^2}{2} - \frac{y^3}{3}\right|_{-2}^{1} = \frac{9}{2}$$

If we were to use y as a function of x, then the calculation would have involved calculating the area by dividing the interval into two: $[-1, 0]$ and $[0, 3]$.

In the first part the area is enclosed between $y = \sqrt{x+1}$ and $y = -\sqrt{x+1}$, and the area in the second part is enclosed by $y = 1 - x$ and $y = -\sqrt{x+1}$:

$$2\int_{-1}^{0} \sqrt{x+1}\,dx + \int_{0}^{3} \left((1 - x) - \left(-\sqrt{x+1}\right)\right)\,dx$$

(Calculation is left as an exercise.)

Exercise 16.6

In questions 1–22, sketch the region whose area you are asked for, and then compute the required area. In each question, find the area of the region bounded by the given curves.

1 $y = x + 1, y = 7 - x^2$

2 $y = \cos x, y = x - \frac{\pi}{2}, x = -\pi$

3 $y = 2x, y = x^2 - 2$

4 $y = x^3, y = x^2 - 2, x = 1$

5 $y = x^6, y = x^2$

6 $y = 5x - x^2, y = x^2$

7 $y = 2x - x^3, y = x - x^2$

8 $y = \sin x, y = 2 - \sin x$ (one period)

9 $y = \frac{x}{2}, y = \sqrt{x}, x = 9$

10 $y = \frac{x^4}{10}, y = 3x - x^3$

11 $y = \frac{1}{x}, y = \frac{1}{x^3}, x = 8$

12 $y = 2\sin x, y = \sqrt{3}\tan x, -\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{4}$

13 $y = x - 1$ and $y^2 = 2x + 6$

14 $x = 2y^2$ and $x = 4 + y^2$

15 $4x + y^2 = 12$ and $y = x$

16 $x - y = 7$ and $x = 2y^2 - y + 3$

17 $x = y^2$ and $x = 2y^2 - y - 2$

18 $y = x^3 + 2x^2, y = x^3 - 2x, x = -3$ and $x = 2$

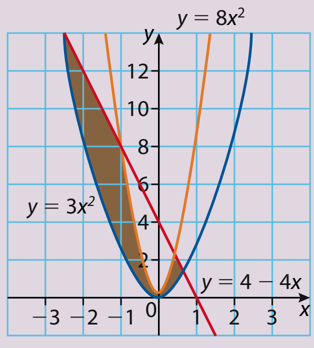
19 $y = \sec^2 x, y = \sec x \tan x, x = -\frac{\pi}{3}$ and $x = \frac{\pi}{6},$

20 $y = x^3 + 1$ and $y = (x + 1)^2$

21 $y = x^3 + x$ and $y = 3x^2 - x$

22 $y = 3 - \sqrt{x}$ and $y = \frac{2\sqrt{x} + 1}{2\sqrt{x}}$

23 Find the area of the shaded region.



24 Find the area of the region enclosed by $y = e^x$, $x = 0$ and the tangent to $y = e^x$ at $x = 1$.

25 Find the area of the region inside the 'loop' in the graph of the curve $y^2 = x^4(x + 3)$.

26 Find the area enclosed by the curve $y^2 = 2x^2 - 4x^4$.

27 Find the area of the region enclosed by $x = 3y^2$ and $x = 12y - y^2 - 5$.

28 Find the area of the region enclosed by $y = (x - 2)^2$ and $y = x(x - 4)^2$.

29 Find a value for $m > 0$ such that the area under the graph of $y = e^{2x}$ over the interval $[0, m]$ is 3 square units.

30 Find the area of the region bounded by $y = x^3 - 4x^2 + 3x$ and the x-axis.

16.7 Volumes with integrals

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, and then add the approximations and take the limit of the sum to produce an integral for the area. The same strategy can be used to find the volume of a solid.

● **Hint:** This is an introductory section that will not be examined. It is only used to give you an idea of why we use integrals to find volumes.

The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations and take the limit of the sum to produce an integral of the volume.

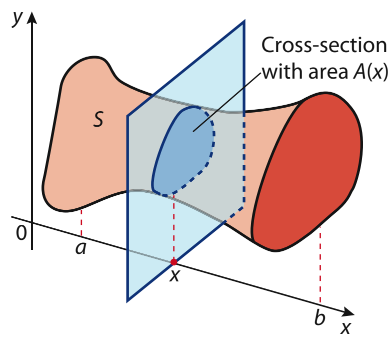
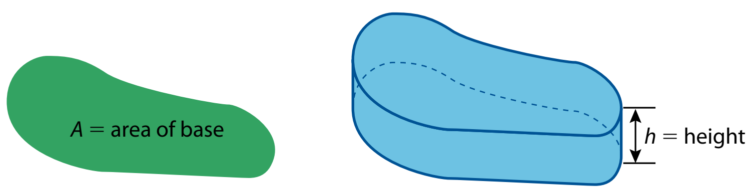


Figure 16.9

Given a solid whose volume is to be computed, we start by taking cross-sections perpendicular to the x-axis as shown in Figure 16.9. Each slab will be approximated by a cylindrical solid whose volume will be equal to the product of its base times its height.

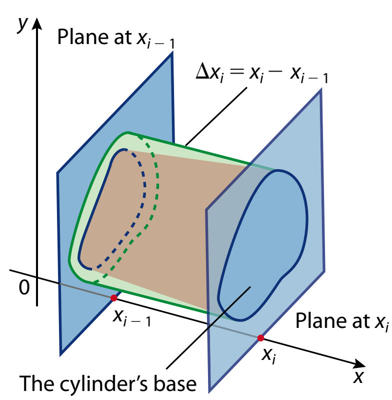


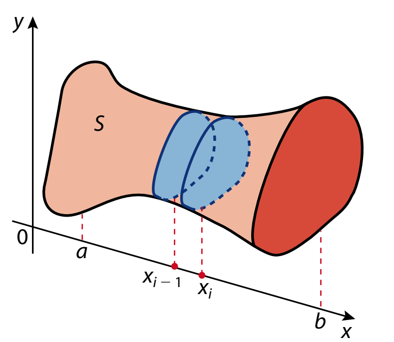
If we call the volume of the slab v_i and the area of its base $A(x)$, then

$$v_i = A(x_i)\cdot h = A(x_i)\cdot\Delta x_i.$$

Using this approximation, the volume of the whole solid can be found by

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x_i.$$





Taking the limit as n increases and the widths of the sub-intervals approach zero yields the definite integral:

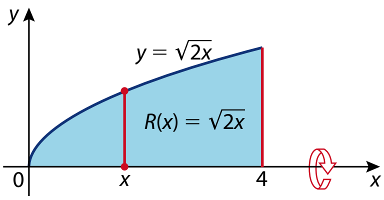
$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i)\Delta x_i = \int_a^b A(x)\,dx$$

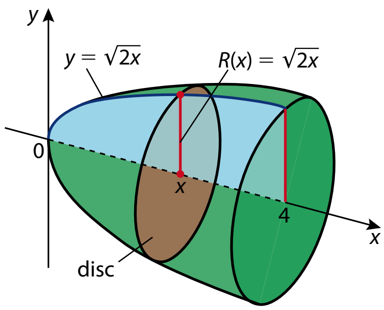
Note: If we place the solid along the y-axis and take the cross-sections perpendicular to that axis, we will arrive at a similar expression for the volume of the solid, i.e.

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(y_i)\triangle y_i = \int_a^b A(y)\,dy$$

Example 38

Find the volume of the solid formed when the graph of the parabola $y = \sqrt{2x}$ over $[0, 4]$ is rotated around the x-axis through an angle of 2π radians, as shown in the diagram.





Solution

The cross-section here is a circular disc whose radius is $y = \sqrt{2x}$. Therefore,

$$A(x) = \pi R^2 = \pi(\sqrt{2x})^2 = 2\pi x.$$

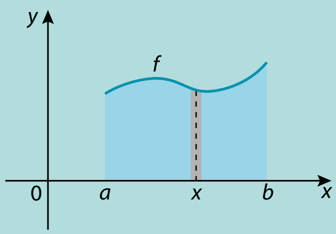
The volume is then

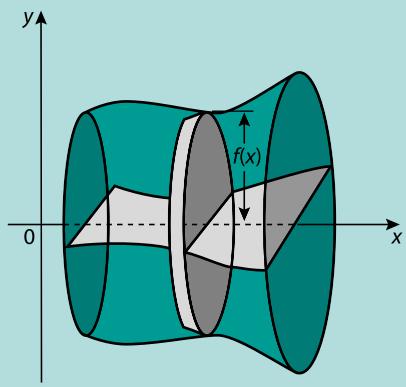
$$V = \int_0^4 A(x)\,dx = \int_0^4 2\pi x\,dx = \left[2\pi\frac{x^2}{2}\right]_0^4 = 16\pi \text{ cubic units.}$$

Example 38 above is a special case of the general process for finding volumes of the so-called 'solids of revolution'.

1 If a region is bounded by a closed interval $[a, b]$ on the x-axis and a function $f(x)$ is rotated about the x-*axis*, the volume of the resulting **solid of revolution** is given by

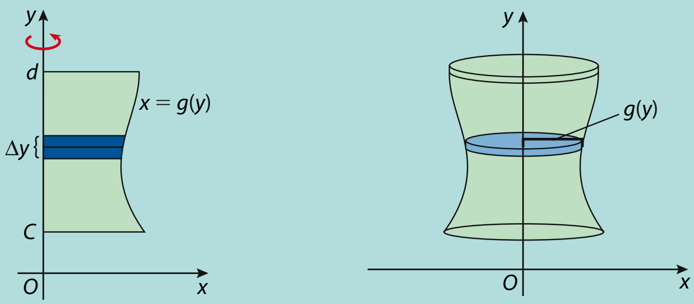
$$V = \int_a^b \pi(f(x))^2\,dx.$$





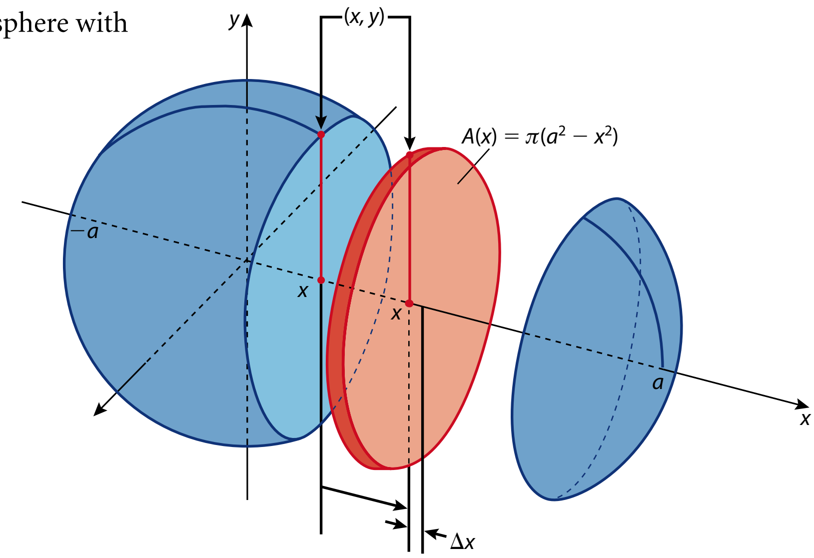
2 If the region bounded by a closed interval [*c*, *d*] on the *y*-axis and a function *g* (*y*) is rotated about the *y-axis*, the volume of the resulting solid of revolution is given by

$$V = \int_d^c \pi(g(y))^2 dy.$$



Example 39

Find the volume of a sphere with radius $R = a$.



Solution

If we place the sphere with its centre at the origin, the equation of the circle will be

$$x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}.$$

The cross-section of the sphere, perpendicular to the x-axis, is a circular disc with radius y, so the area is

$$A(x) = \pi R^2 = \pi y^2 = \pi(\sqrt{a^2 - x^2}) = \pi(a^2 - x^2).$$

So, the volume of the sphere is

$$V = \int_{-a}^{a} \pi(a^2 - x^2)\, dx = \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \pi\left(a^3 - \frac{a^3}{3}\right) - \pi\left(-a^3 + \frac{a^3}{3}\right)$$

$$= \pi\left(2a^3 - 2\frac{a^3}{3}\right) = \frac{4\pi a^3}{3}.$$

Note: If we want to rotate the right-hand region of the circle around the y-axis, then the cross-section of the sphere, perpendicular to the y-axis is a circular disc with radius x. Solving the equation for x instead:

$$x^2 + y^2 = a^2 \Rightarrow x = \pm\sqrt{a^2 - y^2}, \text{ and hence the area is}$$

$$A(y) = \pi R^2 = \pi x^2 = \pi\left(\sqrt{a^2 - y^2}\right)^2 = \pi(a^2 - y^2),$$

and the volume of the sphere is

$$V = \int_{-a}^{a} \pi(a^2 - y^2)\,dy = \pi\left[a^2 y - \frac{y^3}{3}\right]_{-a}^{a} = \pi\left(a^3 - \frac{a^3}{3}\right) - \pi\left(-a^3 + \frac{a^3}{3}\right)$$

$$= \pi\left(2a^3 - 2\frac{a^3}{3}\right) = \frac{4\pi a^3}{3}$$

This is the same result as above.

Example 40

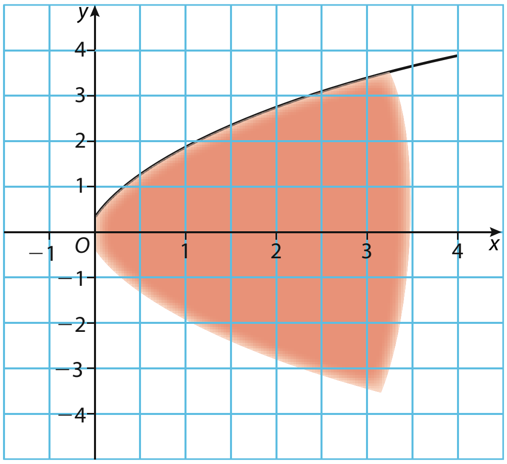
Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x}$, $x = 3$ and $y = 0$ is revolved about the x-axis.

Solution

$$V = \int_0^3 \pi(f(x))^2 dx$$

$$= \pi\int_0^3 (\sqrt{3x})^2 dx$$

$$= 3\pi\left[\frac{x^2}{2}\right]_0^3 = \frac{27\pi}{2}$$



Example 41

Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x}$, $y = 3$ and $x = 0$ is revolved about the y-axis.

Solution

Here, we first find x as a function of y.

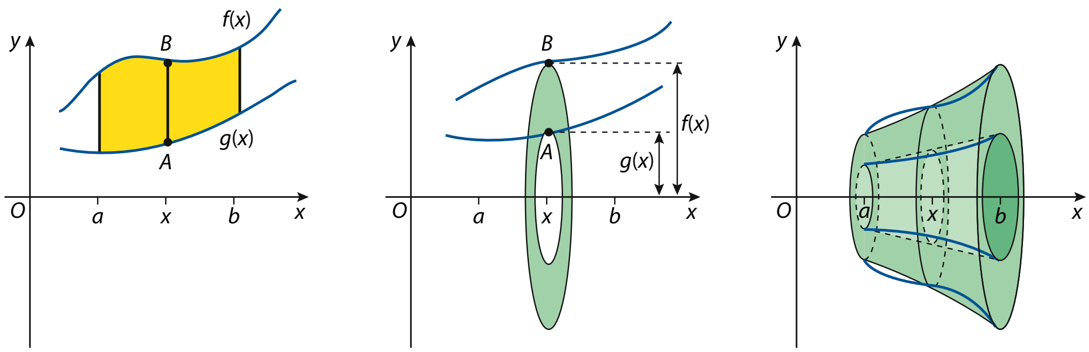
$$y = \sqrt{3x} \Rightarrow x = \frac{y^2}{3}, \text{ the interval on the } y\text{-axis is } [0, 3]$$

So, the volume required is

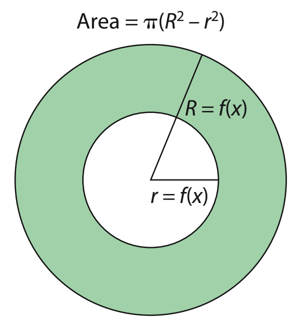
$$V = \int_0^3 \pi\left(\frac{y^2}{3}\right)^2 dy = \frac{\pi}{9}\int_0^3 y^4 dy = \frac{\pi}{9}\left[\frac{y^5}{5}\right]_0^3 = \frac{27\pi}{5}.$$

Washers

Consider the region R between two curves, $y = f(x)$ and $y = g(x)$, and from $x = a$ to $x = b$ where $f(x) > g(x)$. Rotating R about the x-axis generates a solid of revolution S. How do we find the volume of S?



Consider an arbitrary point x in the interval $[a, \mathrm{b}]$. The segment AB represents the difference $f(x) - g(x)$. When we rotate this slice, the cross-section perpendicular to the x-axis is going to look like a 'washer' whose area is



$$A = \pi(R^2 - r^2) = \pi((f(x))^2 - (g(x))^2).$$

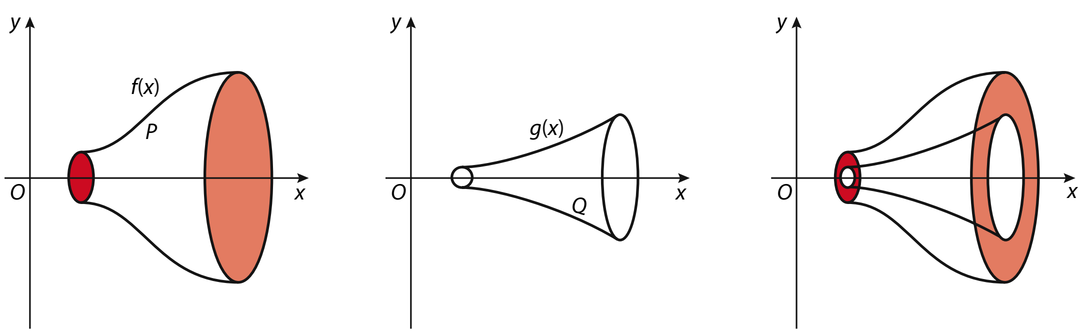
So, the volume of S is

$$V = \int_a^b A(x)dx = \pi\int_a^b ((f(x))^2 - (g(x))^2)dx.$$

Note: If you are rotating about the y-axis, a similar formula applies.

$$V = \pi\int_c^d ((p(y))^2 - (q(y))^2)dy$$

Note: To understand the washer more, you can think of it in the following manner: Let P be the solid generated by rotating the curve $y = f(x)$ and Q be the solid generated by rotating the curve $y = g(x)$. Then S can be found by removing the solid of revolution generated by $y = g(x)$ from the solid of revolution generated by $y = f(x)$, as shown.

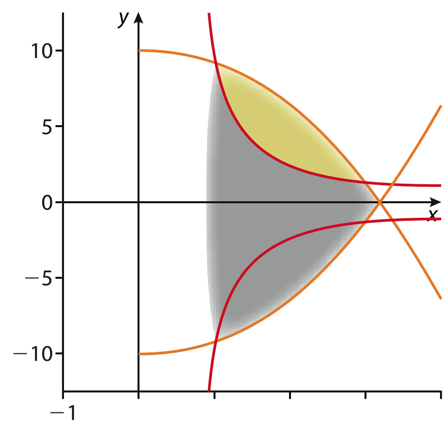


Therefore, volume of $S =$ volume of $P -$ volume of Q. And this justifies the formula:

$$V = \pi\int_a^b (f(x))^2dx - \pi\int_a^b (g(x))^2dx = \pi\int_a^b ((f(x))^2 - (g(x))^2)dx$$

Example 42

The region in the first quadrant between $f(x) = 6 - x^2$ and $h(x) = \dfrac{8}{x^2}$ is rotated about the x-axis. Find the volume of the generated solid.



Solution

The rotated region is shown in the diagram. $f(x)$ is larger than $h(x)$in this interval. Moreover, the two curves intersect at:

$$\frac{8}{x^2} = 6 - x^2 \Rightarrow x = \sqrt{2},\ x = 2$$

Hence, the volume of the solid of revolution is

$$V = \pi\int_{\sqrt{2}}^{2}\left((6 - x^2)^2 - \left(\frac{8}{x^2}\right)^2\right)dx$$

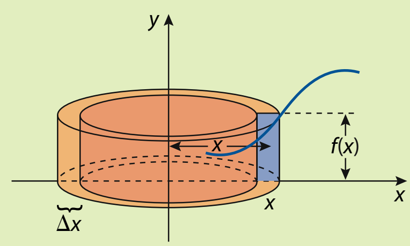
$$= \pi\int_{\sqrt{2}}^{2}\left(x^4 - 12x^2 + 36 - \frac{64}{x^4}\right)dx$$

$$= \pi\left[\frac{x^5}{5} - 4x^3 + 36x + \frac{64}{3x^3}\right]_{\sqrt{2}}^{2}$$

$$= \frac{736 - 512\sqrt{2}}{15}\pi.$$

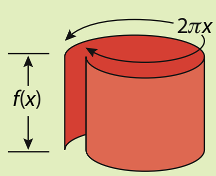
An alternative method: Volumes by cylindrical shells

Consider the region R under the curve $y = f(x)$. Rotate R about the y-axis. We divide R into vertical strips of width Δx each as shown. When we rotate a strip around the y-axis, we generate a cylindrical shell of Δx thickness and height $f(x)$. To understand how we get the volume, we can cut the shell vertically as shown and 'unfold' it. The resulting rectangular parallelpiped has length $2\pi x$, height $f(x)$ and thickness Δx.



So, the volume of this shell is

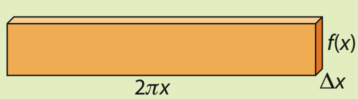
$$\Delta v_i = \text{length} \times \text{height} \times \text{thickness}$$
$$= (2\pi x) \times f(x) \times \Delta x.$$



The volume of the whole solid is the sum of the volumes of these shells as the number of shells increases, and consequently

$$V = \lim_{n\to\infty}\sum_{i=1}^{v}\Delta v_i = \lim_{\Delta x\to 0}\sum(2\pi x) \times f(x) \times \Delta x$$

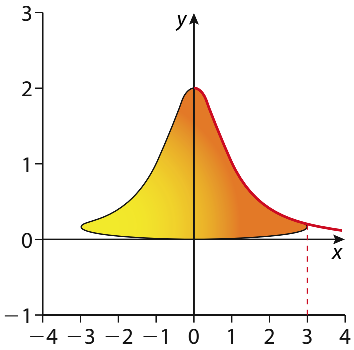
$$= 2\pi\int_a^b xf(x)dx.$$



In many problems involving rotation about the y-axis, this would be more accessible than the disc/washer method.

Example 43

Find the volume of the solid generated when we rotate the region under $f(x) = \dfrac{2}{1 + x^2}$, $x = 0$ and $x = 3$ around the y-axis.



Solution

Using the shell method, we have

$$V = 2\pi\int_0^3 x \times \frac{2}{1+x^2}dx$$
$$= 2\pi\int_0^3 \frac{2x}{1+x^2}dx = 2\pi\int_1^{10}\frac{du}{u}$$
$$= 2\pi[\ln u]_1^{10} = 2\pi\ln 10.$$

Exercise 16.7

In questions 1–19, find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis. Sketch the region, the solid and a typical disc.

1 $y = 3 - \frac{x}{3}, y = 0, x = 2, x = 3$

2 $y = 2 - x^2, y = 0$

3 $y = \sqrt{16 - x^2}, y = 0, x = 1, x = 3$

4 $y = \frac{3}{x}, y = 0, x = 1, x = 3$

5 $y = 3 - x, y = 0, x = 0$

6 $y = \sqrt{\sin x}, y = 0, 0 \leqslant x \leqslant \pi$

7 $y = \sqrt{\cos x}, y = 0, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{3}$

8 $y = 4 - x^2, y = 0$

9 $y = x^3 + 2x + 1, y = 0, x = 1$

10 $y = -4x - x^2, y = x^2$

11 $y = \sec x, x = \frac{\pi}{4}, x = \frac{\pi}{3}, y = 0$

12 $y = 1 - x^2, y = x^3 + 1$

13 $y = \sqrt{36 - x^2}, y = 4$

14 $x = \sqrt{y}, y = 2x$

15 $y = \sin x, y = \cos x, x = \frac{\pi}{4}, x = \frac{\pi}{2}$

16 $y = 2x^2 + 4, y = x, x = 1, x = 3$

17 $y = \sqrt{x^4 + 1}, y = 0, x = 1, x = 3$

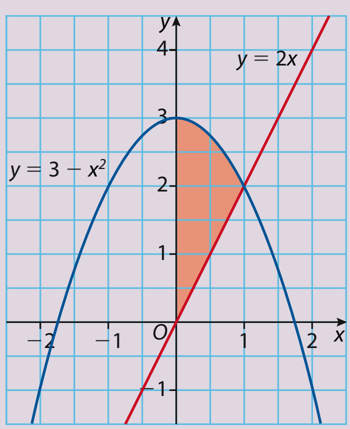
18 $y = 16 - x, y = 3x + 12, x = -1$

19 $y = \frac{1}{x}, y = \frac{5}{2} - x$

20 Find the volume resulting from a rotation of this region about

a) the x-axis

b) the y-axis.



In questions 21–31, find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis. Sketch the region, the solid and a typical disc/shell.

21 $y = x^2, y = 0, x = 1, x = 3$

22 $y = x, y = \sqrt{9 - x^2}, x = 0$

23 $y = x^3 - 4x^2 + 4x, y = 0$

24 $y = \sqrt{3x}, x = 5, x = 11, y = 0$

25 $y = x^2, y = \dfrac{2}{1 + x^2}$

26 $y = \sqrt{x^2 + 2}, x = 3, y = 0, x = 0$

27 $y = \dfrac{7x}{\sqrt{x^3 + 7}}, x = 3, y = 0$

28 $y = \sin x, y = \cos x, x = \dfrac{\pi}{4}, x = \dfrac{\pi}{2}$

29 $y = 2x^2 + 4, y = x, x = 1, x = 3$

30 $y = \sin(x^2), y = 0, x = 0, x = \sqrt{\pi}$

31 $y = 5 - x^3, y = 5 - 4x$

16.8 Modelling linear motion

In previous sections of this text, we have examined problems involving displacement, velocity and acceleration of a moving object. In different sections of Chapter 13, we applied the fact that a derivative is a rate of change to express velocity and acceleration as derivatives. Even though our earlier work on motion problems involved an object moving in one, two or even three dimensions, our mathematical models considered the object's motion occurring only along a straight line. For example, projectile motion (e.g. a ball being thrown) is often modelled by a position function that simply gives the height (displacement) of the object. In that way, we are modelling the motion as if it were restricted to a vertical line.

In this section, we will again analyze the motion of an object as if its motion takes place along a straight line in space. This can only make sense if the mass (and thus, size) of the object is not taken into account. Hence, the object is modelled by a particle whose mass is considered to be zero. This study of motion, without reference either to the forces that cause it or to the mass of the object, is known as **kinematics**.

Displacement and total distance travelled

Recall from Chapter 13 that given time t, displacement s, velocity v and acceleration a, we have the following:

$$v = \frac{ds}{dt}, a = \frac{dv}{dt}, \text{ and } a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

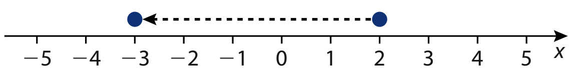
Let's review some of the essential terms we use to describe an object's motion.

Position, distance and displacement

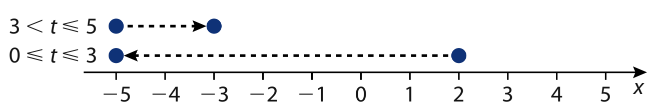
- The **position** $\boldsymbol{s}$ of a particle, with respect to a chosen axis, is a measure of how far it is from a fixed point (usually the origin) *and* of its direction relative to the fixed point.
- The **distance** $|\boldsymbol{s}|$ of a particle is a measure of how far it is from a fixed point (usually the origin) and does *not* indicate direction. Thus, distance is the magnitude of position and is always positive.
- The **displacement** is the *change* in position. The displacement of an object may be positive, negative or zero, depending on its motion.

It is important to understand the difference between displacement and distance travelled. Consider a couple of simple examples of an object moving along the x-axis.

1. In this first example, assume that the object does not change direction during the interval $0 \leqslant t \leqslant 5$. In other words, its velocity does not change from positive to negative or from negative to positive. If the position of the object at $t = 0$ is $x = 2$ and then the object moves so that at $t = 5$ its position is $x = -3$, its displacement, or change in position, is -5 because the object changed its position by 5 units in the negative direction. This can be calculated by (final position) $-$ (initial position) $= -3 - 2 = -5$. However, the distance travelled would be the absolute value of displacement, calculated by |final position $-$ initial position| $= |-3 - 2| = +5$.



2. In this example, the object's initial and final positions are the same as in the first example – that is, at $t = 0$ its position is $x = 2$ and at $t = 5$ its position is $x = -3$. However, the object changed direction in that it first travelled to the left (negative velocity) from $x = 2$ to $x = -5$ during the interval $0 \leqslant t \leqslant 3$, and then travelled to the right (positive velocity) from $x = -5$ to $x = -3$. The object's displacement is -5 – the same as in the first example because its net change in position is just the difference between final and initial positions. However, it's clear that the object has travelled further than in the first example. But we cannot calculate it in the same way as we did in the first example. We will have to make a separate calculation for each interval where the direction changed. Hence, total distance travelled $= |-5 - 2| + |-3 - (-5)| = 7 + 2 = 9$.



Velocity and speed

- The **velocity** $\boldsymbol{v} = \frac{ds}{dt}$ of a particle is a measure of how fast it is moving *and* of its direction of motion relative to a fixed point.
- The **speed** $|\boldsymbol{v}|$ of a particle is a measure of how fast it is moving and does *not* indicate direction. Thus, speed is the magnitude of velocity and is always positive.

Acceleration

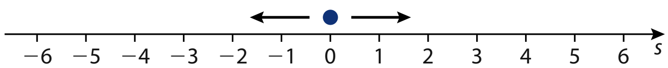
- The **acceleration** $\boldsymbol{a} = \frac{dv}{dt}$ of a particle is a measure of how fast its velocity is changing.

There is no separate word to describe the magnitude of acceleration, $|\boldsymbol{a}|$.

The definite integral is a mathematical tool that can be used in applications to calculate net change of a quantity (e.g. Δ position $\rightarrow$ displacement) and total accumulation (e.g. Σ area $\rightarrow$ volume).

Example 44

The displacement s of a particle on the x-axis, relative to the origin, is given by the position function $s(t) = -t^2 + 6t$, where s is in centimetres and t is in seconds.



a) Find a function for the particle's velocity $v(t)$ in terms of t. Graph the functions $s(t)$ and $v(t)$ on separate axes.

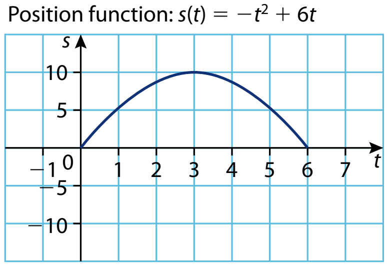
b) Find the particle's position at the following times: $t = 0, 1, 3$ and 6 seconds.

c) Find the particle's displacement for the following intervals: $0 \leqslant t \leqslant 1$, $1 \leqslant t \leqslant 3$, $3 \leqslant t \leqslant 6$ and $0 \leqslant t \leqslant 6$.

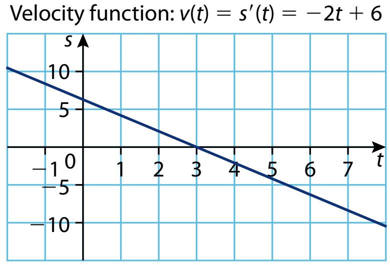
d) Find the particle's total distance travelled for the following intervals: $0 \leqslant t \leqslant 1$, $1 \leqslant t \leqslant 3$, $3 \leqslant t \leqslant 6$ and $0 \leqslant t \leqslant 6$.

Solution

Position function: $s(t) = -t^2 + 6t$



Velocity function: $v(t) = s'(t) = -2t + 6$



a) $v(t) = \frac{d}{dt}(-t^2 + 6t) = -2t + 6$

b) The particle's position at:
- $t = 0$ is $s(0) = -(0)^2 + 6(0) = 0$ cm
- $t = 1$ is $s(1) = -(1)^2 + 6(1) = 5$ cm
- $t = 3$ is $s(3) = -(3)^2 + 6(3) = 9$ cm
- $t = 6$ is $s(6) = -(6)^2 + 6(6) = 0$ cm

c) The particle's displacement for the interval:
- $0 \leqslant t \leqslant 1$ is Δposition $= s(1) - s(0) = 5 - 0 = 5$ cm
- $1 \leqslant t \leqslant 3$ is Δposition $= s(3) - s(1) = 9 - 5 = 4$ cm
- $3 \leqslant t \leqslant 6$ is Δposition $= s(6) - s(3) = 0 - 9 = -9$ cm
- $0 \leqslant t \leqslant 6$ is Δposition $= s(6) - s(0) = 0 - 0 = 0$ cm

This last result makes sense considering the particle moved to the right 9 cm then at $t = 3$ turned around and moved to the left 9 cm, ending where it started – thus, no change in net position.

d) The particle's total distance travelled for the interval:

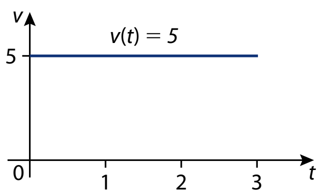
- $0 \leqslant t \leqslant 1$ is $|s(1) - s(0)| = |5 - 0| = 5$ cm
- $1 \leqslant t \leqslant 3$ is $|s(3) - s(1)| = |9 - 5| = 4$ cm
- $3 \leqslant t \leqslant 6$ is $|s(6) - s(3)| = |0 - 9| = |-9| = 9$ cm
- $0 \leqslant t \leqslant 6$: The object's motion changed direction (velocity $= 0$) at $t = 3$, so total distance is $|s(3) - s(0)| + |s(6) - s(3)|$
  $= |9 - 0| + |0 - 9| = 9 + 9 = 18$ cm

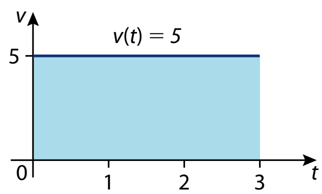
Since differentiation of the position function gives the velocity function $\left(\text{i.e. } v = \frac{ds}{dt}\right)$, we expect that the inverse of differentiation, integration, will lead us in the reverse direction – that is, from velocity to position. When velocity is constant, we can find the displacement with the formula:

$$\text{displacement} = \text{velocity} \times \Delta \text{ in time}$$

If we drove a car at a constant velocity of 50 km/h for 3 hours, our displacement (same as distance travelled in this case) is 150 km. If a particle travelled to the left on the x-axis at a constant rate of -4 units/sec for 5 seconds, the particle's displacement is -20 units.

The velocity–time graph below depicts an object's motion with a constant velocity of 5 cm/s for $0 \leqslant t \leqslant 3$. Clearly, the object's displacement is 5 cm/s $\times$ 3 sec $= 15$ cm for this interval.

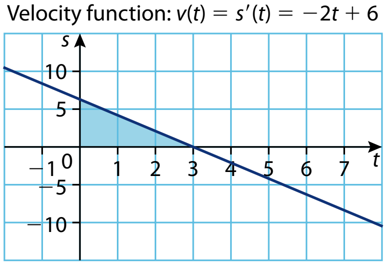


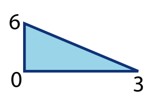


The rectangular area $(3 \times 5 = 15)$ under the velocity curve is equal to the object's displacement.

Looking back at Example 44, consider the area under the graph of $v(t)$ from $t = 0$ to $t = 3$.

Velocity function: $v(t) = s'(t) = -2t + 6$





Area $= \frac{1}{2} \times 3 \times 6 = 9$

Given the discussion above, we should not be surprised to see that the area under the velocity curve for a certain interval is equal to the displacement for that interval. We can argue that just as the total area can be found by summing the areas of narrow rectangular strips, the displacement can be found by summing small displacements ($v \cdot \Delta t$). Consider:

$$\text{displacement} = \text{velocity} \times \Delta \text{ in time} \Rightarrow s = v \cdot \Delta t \Rightarrow s = v \cdot dt$$

We learned earlier in this chapter that if $f(x) \geqslant 0$ then the definite integral $\int_a^b f(x)\,dx$ gives the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$. And if $f(x) \leqslant 0$ then $\int_a^b f(x)\,dx$ gives a number that is the opposite of the area between $y = f(x)$ and the x-axis from a to b.

Using integration to find displacement and total distance travelled

Given that $v(t)$ is the velocity function for a particle moving along a line, then:

$\int_a^b v(t)\,dt$ gives the displacement from $t = a$ to $t = b$.

$\left|\int_a^b v(t)\,dt\right|$ gives the total distance travelled from $t = a$ to $t = b$ if the particle does not change direction during the interval $a < t < b$.

If a particle changes direction at some $t = c$ for $a < c < b$, the total distance travelled for the particle is given by $\left|\int_a^c v(t)\,dt\right| + \left|\int_c^b v(t)\,dt\right|$.

In general, the total distance travelled by an object from time t_0 to t_1, with many switches in direction is given by $\int_{t_0}^{t_1} |v(t)|\,dt$.

Let's apply integration to find the displacement and distance travelled for the two intervals $3 \leqslant t \leqslant 6$ and $0 \leqslant t \leqslant 6$ in Example 40.

- For $3 \leqslant t \leqslant 6$:

$$\text{Displacement} = \int_3^6 (-2t + 6)\,dt = \left[-t^2 + 6t\right]_3^6$$
$$= [-(6)^2 + 6(6)] - [-(3)^2 + 6(3)] = 0 - 9 = -9$$

$$\text{Distance travelled} = \left|\int_3^6 (-2t + 6)\right| dt = \left|\left[-t^2 + 6t\right]_3^6\right|$$
$$= |[-(6)^2 + 6(6)] - [-(3)^2 + 6(3)]| = |0 - 9| = 9$$

- For $0 \leqslant t \leqslant 6$:

$$\text{Displacement} = \int_0^6 (-2t + 6)\,dt = \left[-t^2 + 6t\right]_0^6$$
$$= [-(6)^2 + 6(6)] - [0] = 0$$

$$\text{Distance travelled} = \left|\int_0^3 (-2t + 6)\,dt\right| + \left|\int_3^6 (-2t + 6)\,dt\right|$$

Particle changed direction at $t = 3$.

$$= \left|\left[-t^2 + 6t\right]_3^6\right| + \left|\left[-t^2 + 6t\right]_3^6\right|$$
$$= |(-9 + 18) - 0| + |0 - (-9 + 18)|$$
$$= |9| + |-9| = 9 + 9 = 18$$

Example 45

The function $v(t) = \sin(\pi t)$ gives the velocity in m/s of a particle moving along the x-axis.

a) Determine when the particle is moving to the right, to the left, and stopped. At any time it stops, determine if it changes direction at that time.

b) Find the particle's displacement for the time interval $0 \leqslant t \leqslant 3$.

c) Find the particle's total distance travelled for the time interval $0 \leqslant t \leqslant 3$.

Solution

a) $v(t) = \sin(\pi t) = 0 \Rightarrow \sin(k \cdot \pi) = 0$ for $k \in \mathbb{Z} \Rightarrow \pi t = k\pi \Rightarrow t = k, k \in \mathbb{Z}$ for $0 \leqslant t \leqslant 3$, $t = 0, 1, 2, 3$. Therefore, the particle is stopped at $t = 0, 1, 2, 3$.

Since $t = 0$ and $t = 3$ are endpoints of the interval, the particle can only change direction at $t = 1$ or $t = 2$.

$v(\frac{1}{2}) = \sin(\pi \cdot \frac{1}{2}) = 1$; $v(\frac{3}{2}) = \sin(\pi \cdot \frac{3}{2}) = -1 \Rightarrow$ direction changes at $t = 1$

$v(\frac{3}{2}) = \sin(\pi \cdot \frac{3}{2}) = -1$; $v(\frac{5}{2}) = \sin(\pi \cdot \frac{5}{2}) = 1 \Rightarrow$ direction changes again at $t = 2$

b) Displacement $= \displaystyle\int_0^3 \sin(\pi t)\, dt = \left[-\frac{1}{\pi}\cos(\pi t)\right]_0^3$

$$= -\frac{1}{\pi}\cos(3\pi) - \left(-\frac{1}{\pi}\cos(0)\right) = -\frac{1}{\pi}(-1) + \frac{1}{\pi}(1) = \frac{2}{\pi} \approx 0.637 \text{ metres}$$

c) Total distance travelled $= \left|\displaystyle\int_0^1 \sin(\pi t)\, dt\right| + \left|\displaystyle\int_1^2 \sin(\pi t)\, dt\right|$

$$+ \left|\int_2^3 \sin(\pi t)\, dt\right| = \left|\left[-\frac{1}{\pi}\cos(\pi t)\right]_0^1\right|$$

$$+ \left|\left[-\frac{1}{\pi}\cos(\pi t)\right]_1^2\right| + \left|\left[-\frac{1}{\pi}\cos(\pi t)\right]_2^3\right|$$

$$= \left|\frac{2}{\pi}\right| + \left|-\frac{2}{\pi}\right| + \left|\frac{2}{\pi}\right| = \frac{6}{\pi} \approx 1.91 \text{ metres}$$

Note that, in Example 45, the position function is not known precisely. The position function can be obtained by finding the anti-derivative of the velocity function.

$$s(t) = \int v(t)\, dt = \int \sin(\pi t)\, dt = -\frac{1}{\pi}\cos(\pi t) + C$$

We can only determine the constant of integration C if we know the particle's initial position (or position at any other specific time). However, the particle's initial position will not affect displacement or distance travelled for any interval.

Position and velocity from acceleration

If we can obtain position from velocity by applying integration then we can also obtain velocity from acceleration by integrating. Consider the following example.

Example 46

The motion of a falling parachutist is modelled as linear motion by considering that the parachutist is a particle moving along a line whose positive direction is vertically downwards. The parachute is opened at $t = 0$ at which time the parachutist's position is $s = 0$. According to the model, the acceleration function for the parachutist's motion for $t > 0$ is given by:

$$a(t) = -54e^{-1.5t}$$

a) At the moment the parachute opens, the parachutist has a velocity of 42 m/s. Find the velocity function of the parachutist for $t > 0$. What does the model say about the parachutist's velocity as $t \to \infty$?

b) Find the position function of the parachutist for $t > 0$.

Solution

a) $v(t) = \int a(t)\,dt = \int(-54e^{-1.5t})\,dt$

$$= -54\left(\frac{1}{-1.5}\right)e^{-1.5t} + C$$

$$= 36e^{-1.5t} + C$$

Since $v = 42$ when $t = 0$, then $42 = 36e^0 + C \Rightarrow 42 = 36 + C \Rightarrow C = 6$

Therefore, after the parachute opens ($t > 0$) the velocity function is $v(t) = 36e^{-1.5t} + 6$.

Since $\lim\limits_{t\to\infty} e^{-1.5t} = \lim\limits_{t\to\infty} \frac{1}{e^{1.5t}} = 0$, then as $t \to \infty$, $\lim\limits_{t\to\infty} v(t) = 6$ m/sec.

The limit of the velocity as $t \to \infty$, for a falling object, is called the **terminal velocity** of the object. While the limit $t \to \infty$ is never attained (the parachutist eventually lands on the ground), the velocity gets close to the terminal velocity very quickly. For example, after just 8 seconds, the velocity is $v(8) = 36e^{-1.5(8)} + 6 \approx 6.0002$ m/s.

b) $s(t) = \int v(t)\,dt = \int(36e^{-1.5t} + 6)\,dt$

$$= 36\left(\frac{1}{-1.5}\right)e^{-1.5t} + 6t + C$$

$$= -24e^{-1.5t} + 6t + C$$

Since $s = 0$ when $t = 0$, then $0 = -24e^0 + 6(0) + C$
$\Rightarrow 0 = -24 + C \Rightarrow C = 24$

Therefore, after the parachute opens ($t > 0$) the position function is $s(t) = -24e^{-1.5t} + 6t + 24$.

Uniformly accelerated motion

Motion under the effect of gravity in the vicinity of Earth (or other planets) is an important case of rectilinear motion. This is called uniformly accelerated motion.

If a particle moves with constant acceleration along the s-axis, and if we know the initial speed and position of the particle, then it is possible to have specific formulae for the position and speed at any time t. This is how:

Assume acceleration is constant, i.e. $a(t) = a$, $v(0) = v_0$ and $s(0) = s_0$.

$v(t) = \int a dt = at + c$, we know that $v(0) = v_0$, then

$v(0) = v_0 = a(0) + c \Rightarrow c = v_0$; hence $v(t) = at + v_0$

$s(t) = \int v(t) dt = \int (at + v_0) dt = \frac{1}{2}at^2 + v_0 t + c$, but $s(0) = s_0$, then

$s(0) = s_0 = \frac{1}{2}a(0^2) + v_0(0) + c \Rightarrow c = s_0$; hence

$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$

When this is applied to a free-fall model (s-axis vertical), then

$v(t) = -gt + v_0$, and

$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$, where $g = 9.8\ \text{m/s}^2$.

Example 47

A ball is hit, from a point 2 m above the ground, directly upward with initial velocity of 45 m/s. How high will the ball travel?

Solution

$v(t) = -9.8t + 45$

$s(t) = -\frac{1}{2}(9.8)t^2 + 45t + 2 = -4.9t^2 + 45t + 2$

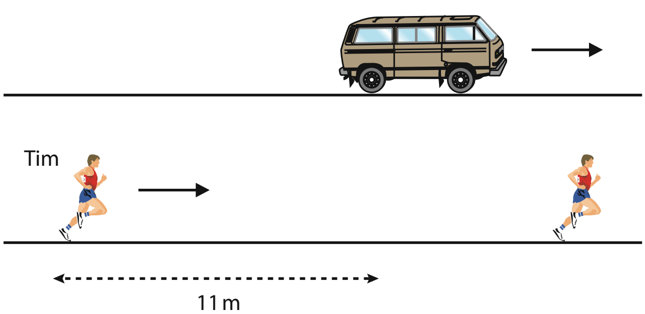
The ball will rise till $v(t) = 0, \Rightarrow 0 = -9.8t + 45, \Rightarrow t \approx 4.6\ \text{s}$

At this time,

$s(4.6) = -4.9(4.6)^2 + 45(4.6) + 2 \approx 105.32\text{m}.$

Example 48

Tim is running at a constant speed of 5 m/s to catch a bus that stopped at the station. The bus started as it was 11 m away with an acceleration of 1 m/s^2. How long will it take Tim to catch up with the bus?



Solution

To catch the bus at some time t, Tim will have to cover a distance s_T that is equal to 11 m plus s_b travelled by the bus.

$$s_T = 5t$$

$$s_b = \frac{1}{2}t^2$$

But $s_T = s_b + 11 = \frac{1}{2}t^2 + 11$, therefore

$$5t = \frac{1}{2}t^2 + 11 \Rightarrow t^2 - 10t + 22 = 0$$

So, $t \approx 3.3$ s, or $t \approx 6.7$ s.

Note: The reason we have two answers is that since Tim is travelling at a constant rate he may miss the door at first, and if he continues, the bus will catch up with him 6.7 s later!

Exercise 16.8

In questions 1–6, the velocity of a particle along a rectilinear path is given by the equation $v(t)$ in m/s. Find both the net distance and the total distance it travels between the times $t = a$ and $t = b$.

1 $v(t) = t^2 - 11t + 24, a = 0, b = 10$

2 $v(t) = t - \frac{1}{t^2}, a = 0.1, b = 1$

3 $v(t) = \sin 2t, a = 0, b = \frac{\pi}{2}$

4 $v(t) = \sin t + \cos t, a = 0, b = \pi$

5 $v(t) = t^3 - 8t^2 + 15t, a = 0, b = 6$

6 $v(t) = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right), a = 0, b = 1$

In questions 7–11, the acceleration of a particle along a rectilinear path is given by the equation $a(t)$ in m/s^2, and the initial velocity v_0 m/s is also given. Find the velocity of the particle as a function of t, and both the net distance and the total distance it travels between the times $t = a$ and $t = b$.

7 $a(t) = 3, v_0 = 0, a = 0, b = 2$

8 $a(t) = 2t - 4, v_0 = 3, a = 0, b = 3$

9 $a(t) = \sin t, v_0 = 0, a = 0, b = \frac{3\pi}{2}$

10 $a(t) = \frac{-1}{\sqrt{t + 1}}, v_0 = 2, a = 0, b = 4$

11 $a(t) = 6t - \frac{1}{(t + 1)^3}, v_0 = 2, a = 0, b = 2$

In each question 12–15, the velocity and initial position of an object moving along a coordinate line are given. Find the position of the object at time t.

12 $v = 9.8t + 5, s(0) = 10$

13 $v = 32t - 2, s(0.5) = 4$

14 $v = \sin \pi t, s(0) = 0$

15 $v = \frac{1}{t + 2}, t > -2, s(-1) = \frac{1}{2}$

In each question 16–19, the acceleration is given as well as the initial velocity and initial position of an object moving on a coordinate line. Find the position of the object at time t.

16 $a = e^t$, $v(0) = 20$, $s(0) = 5$

17 $a = 9.8$, $v(0) = -3$, $s(0) = 0$

18 $a = -4\sin 2t$, $v(0) = 2$, $s(0) = -3$

19 $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi}$, $v(0) = 0$, $s(0) = -1$

In questions 20–23, an object moves with a speed of $v(t)$ m/s along the s-axis. Find the displacement and the distance travelled by the object during the given time interval.

20 $v(t) = 2t - 4;\ 0 \leqslant t \leqslant 6$

21 $v(t) = |t - 3|;\ 0 \leqslant t \leqslant 5$

22 $v(t) = t^3 - 3t^2 + 2t;\ 0 \leqslant t \leqslant 3$

23 $v(t) = \sqrt{t} - 2;\ 0 \leqslant t \leqslant 3$

In questions 24–26, an object moves with an acceleration $a(t)$ m/s^2 along the s-axis. Find the displacement and the distance travelled by the object during the given time interval.

24 $a(t) = t - 2$, $v_0 = 0$, $1 \leqslant t \leqslant 5$

25 $a(t) = \dfrac{1}{\sqrt{5t+1}}$, $v_0 = 2$, $0 \leqslant t \leqslant 3$

26 $a(t) = -2$, $v_0 = 3$, $1 \leqslant t \leqslant 4$

27 The velocity of an object moving along the s-axis is

$$v = 9.8t - 3.$$

a) Find the object's displacement between $t = 1$ and $t = 3$ given that $s(0) = 5$.

b) Find the object's displacement between $t = 1$ and $t = 3$ given that $s(0) = -2$.

c) Find the object's displacement between $t = 1$ and $t = 3$ given that $s(0) = s_0$.

28 The displacement s metres of a moving object from a fixed point O at time t seconds is given by $s(t) = 50t - 10t^2 + 1000$.

a) Find the velocity of the object in m s^{-1}.

b) Find its maximum displacement from O.

29 A particle moves along a line so that its speed v at time t is given by

$$v(t) = \begin{cases} 5t, & 0 \leqslant t < 1 \\ 6\sqrt{t} - \dfrac{1}{t}, & t \geqslant 1 \end{cases}$$

where t is in seconds and v is in cm/s. Estimate the time(s) at which the particle is 4 cm from its starting position.

30 A projectile is fired vertically upward with an initial velocity of 49 m/s from a platform 150 m high.

a) How long will it take the projectile to reach its maximum height?

b) What is the maximum height?

c) How long will it take the projectile to pass its starting point on the way down?

d) What is the velocity when it passes the starting point on the way down?

e) How long will it take the projectile to hit the ground?

f) What will its speed be at impact?

16.9 Differential equations (Optional)

This section presents only an introduction to differential equations. More on differential equations can be found in the Options part: Calculus.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. A *first-order* differential equation is an equation that involves an unknown function and its *first derivative*. Examples of first-order differential equations are:

$$y' + 2xy = \sin x, \frac{dy}{dx} = y + 2x, \text{ and } \frac{dy}{dx} = -ky$$

In this part of the textbook we will consider only first-order differential equations that can be written in the form

$$\frac{dy}{dx} = f(x, y).$$

Here $f(x, y)$ is a function of two variables defined on a *region* in the xy-plane. By a solution to the differential equation, we mean the following.

Solution of a differential equation

We say that a *differentiable* function $y = y(x)$ is a solution to the differential equation

$$\frac{dy}{dx} = f(x, y)$$

on an interval of x-values (sometimes $\mathbb{R}$) when

$$\frac{d}{dx}y(x) = f(x, y(x)).$$

By $y(x)$, we mean 'y of x', i.e. y as a function of x, and not 'y times x'.

The initial condition $y(x_0) = y_0$ amounts to requiring the solution curve $y = y(x)$ to pass through the point (x_0, y_0).

Let us clarify these initial ideas by some examples.

Note: In algebra we usually seek the unknown variable values that satisfy an equation such as $3x^2 - 2x - 5 = 0$. By contrast, in solving a differential equation, we are looking for the unknown functions $y = y(x)$ for which an identity such as $y'(x) = 3x^2y(x)$ holds on some interval of real numbers. Usually, we will desire to find all solutions of the differential equation, if achievable.

Example 49

Verify that $y(x) = Ce^{x^3}$ is a solution to the differential equation

$$\frac{dy}{dx} = 3x^2y.$$

Solution

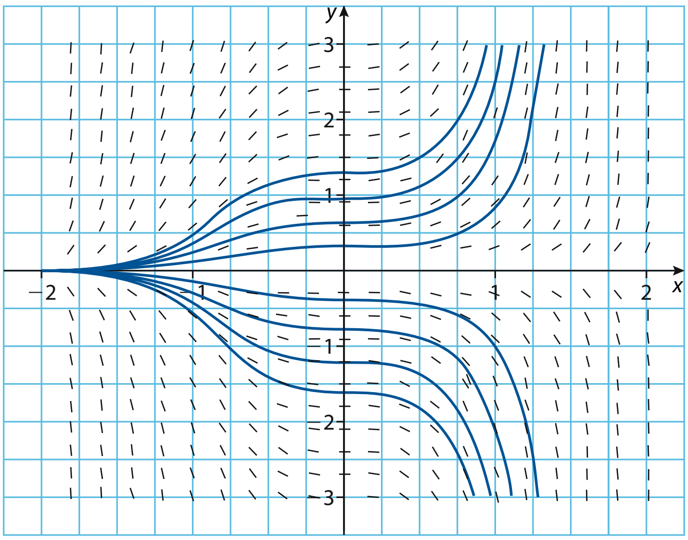
Since C is a constant in $y(x) = Ce^{x^3}$, then

$$\frac{dy}{dx} = C(3x^2e^{x^3}) = 3x^2(Ce^{x^3}) = 3x^2y.$$

Consequently every function $y(x)$ of the form $y(x) = Ce^{x^3}$ satisfies – and thus is a solution of – the differential equation

$$\frac{dy}{dx} = 3x^2y$$

for all real x. In fact $y(x) = Ce^{x^3}$ defines an infinite family of different solutions to this differential equation, one for each choice of the arbitrary constant C.



Example 50

Verify that

$$y(x) = -\frac{1}{2x^4 + 3}$$

is a solution to the differential equation

$$\frac{dy}{dx} = 8x^3y^2$$

over the interval $]-\infty, \infty[$.

Solution

Notice that the denominator in $y(x)$ is never zero and that $y(x)$ is differentiable everywhere. Furthermore, for all real numbers x,

$$\frac{d}{dx}y(x) = \frac{d}{dx}\left(-\frac{1}{2x^4 + 3}\right) = \frac{8x^3}{(2x^4 + 3)^2}$$

$$= 8x^3\left(-\frac{1}{2x^4 + 3}\right)^2 = 8x^3y^2$$

Thus,

$$y(x) = -\frac{1}{2x^4 + 3}$$

is a solution to the given differential equation.

Differential equations as mathematical models

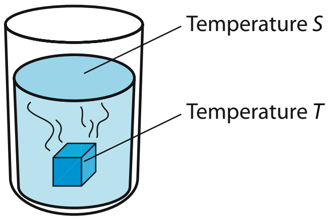
The following examples illustrate typical cases where scientific principles are translated into differential equations.

1 **Newton's law of cooling** states that the rate of change of the temperature T of an object is proportional to the difference between T and the temperature of the surrounding medium S.

That is,

$$\frac{dT}{dt} = k(T - S)$$

where k is a constant and S is usually considered constant.



2 **Population growth rate** in cases where the birth and death rates are not variable is proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP$$

where k is a constant.

Shortly, we will learn how to solve such problems.

Separable differential equations

In this section, we will limit our discussion to one basic type, the **separable differential equations**, also called **variables-separable differential equations**.

The first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called variable separable when the function $f(x, y)$can be factored into a product or quotient of two functions such as

$$\frac{dy}{dx} = g(x)h(y) \text{ or } \frac{dy}{dx} = \frac{p(x)}{q(y)}.$$

In such cases, the variables x and y can be separated by writing

$$\frac{dy}{h(y)} = g(x)dx \text{ or } q(y)dy = p(x)dx$$

and then simply integrating both sides with respect to x. That is,

$$\int \frac{dy}{h(y)} = \int g(x)dx + c \text{ or } \int q(y)dy = \int p(x)dx + c.$$

Note: You need to remember that $h(y)$ is a continuous function of y alone and $g(x)$ is a continuous function of x alone. The same goes for $q(y)$ and $p(x)$.

Note: We also may say that the method of solution is separation of variables.

Here are some examples of differential equations that are separable

Original differential equation	Rewritten with variables separated
$(x^2 + 4)y' = 3xy$	$\frac{dy}{y} = \frac{3x}{x^2+4}dx$
$\frac{3xe^y y'}{1+e^2y} = 5$	$\frac{3e^y}{1+e^{2y}}dy = \frac{5}{x}dx$
$\frac{dy}{dx} = xy + 4$	Not separable!
$3x^2 + y\frac{dy}{dx} = 7$	$y\,dy = (7 - 3x^2)dx$
$x^2\frac{dy}{dx} + y^2 = xy^2$	$\frac{1}{y^2}dy = \frac{(x-1)}{x^2}dx$
$y^2\frac{dy}{dx} + x^2 = xy^2$	Not separable!

We will end this section by looking at a few examples.

Example 51

Solve

$$y' - 9x^2y^2 = 5y^2.$$

Solution

We first factor the equation to separate the variables.

$$\frac{dy}{dx} = 5y^2 + 9x^2y^2 \Rightarrow \frac{dy}{dx} = y^2(5 + 9x^2)$$

$$\Rightarrow \frac{dy}{y^2} = (5 + 9x^2)dx$$

$$\Rightarrow -\frac{1}{y} = 5x + 3x^3 + c$$

$$\Rightarrow y = \frac{-1}{5x + 3x^3 + c}$$

This is a general solution for the differential equation. In this case we are able to express this function in explicit form.

Example 52

Solve

$$\frac{dy}{dx} = \frac{3x^2y}{1 + 4y^2}.$$

Solution

With very few steps, we can separate the variables:

$$\frac{1 + 4y^2}{y} = 3x^2dx$$

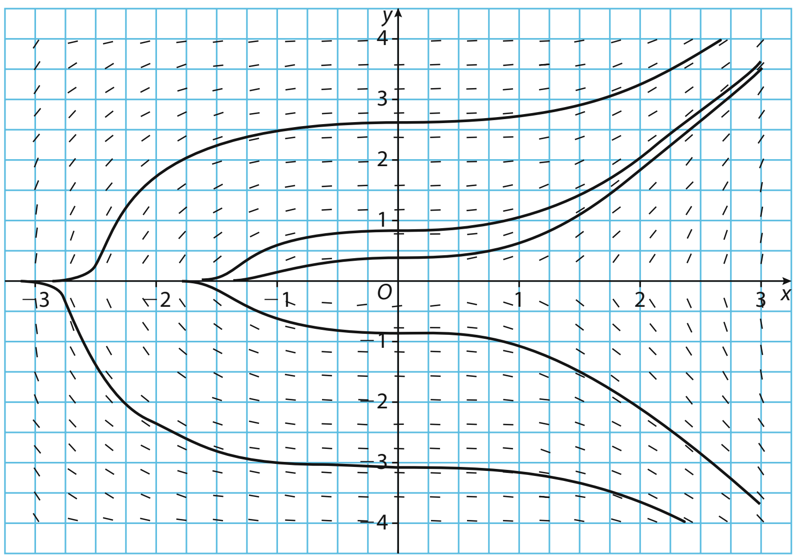
And now we can integrate both sides:

$$\int \frac{1 + 4y^2}{y}dy = \int 3x^2dx \Leftrightarrow \int \left(\frac{1}{y} + 4y\right)dy = \int 3x^2dx$$

$$\ln|y| + 2y^2 = x^3 + c$$

For every value of arbitrary constant c, this defines an exact but implicit solution $y(x)$ as it cannot be written in an explicit form $y = f(x)$.

Here are some of the solution curves for a few values of c.



Note: Here is a summary of *solving equations by separation of variables.*

1 Write the differential equation in the standard form $\frac{dy}{dx} = f(x, y)$.

2 Can you separate the variables, i.e. is $\frac{dy}{dx} = g(x)h(y)$ or $\frac{dy}{dx} = \frac{p(x)}{q(y)}$?

3 If so, separate the variables, to get $\frac{dy}{h(y)} = g(x)dx$ or $q(y)dy = p(x)dx$.

4 Integrate both parts to get $\int \frac{dy}{h(y)} = \int g(x)dx + c$ or $\int q(y)dy$
$= \int p(x)dx + c$.

5 Do the integrals if you can and don't forget the arbitrary constant. Even though we have two integrals, one on the left and one on the right, *it is enough to combine both arbitrary constants with one.*

6 If possible, resolve the resulting equation with respect to y, to get your equation in explicit form $y = f(x)$.

Example 53

Find the general solution of the population growth model

$$\frac{dP}{dt} = kP.$$

Solution

In this problem, we can easily separate the variables.

$$\frac{dP}{P} = kt$$

Now integrate both sides to get

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$$\ln|P| = kt + c$$

where c is an arbitrary constant. This last equation can be simplified to render an explicit expression for P:

$$\ln|P| = kt + c$$

$$\Rightarrow |P| = e^{kt+c} = e^{kt}e^{c} = Ae^{kt}$$

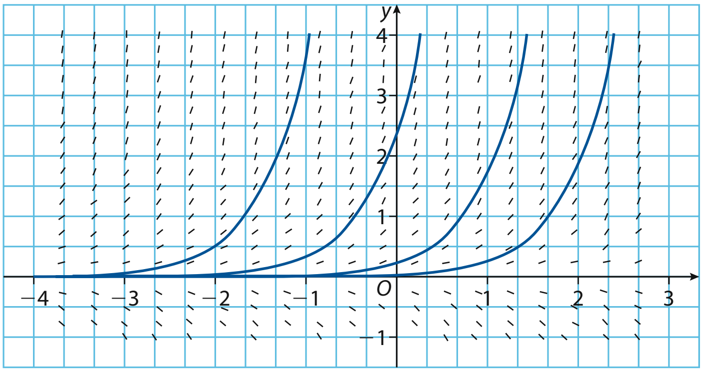
where we replaced e^c with A. Thus,

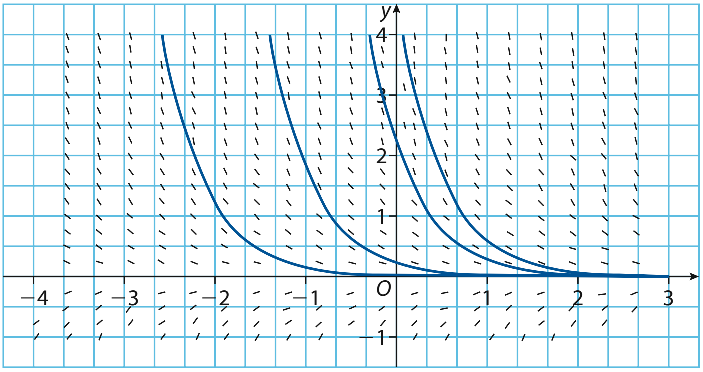
$$P = Ae^{kt} \text{ or } P = -Ae^{kt}.$$

This is the general solution and all solutions to this problem will be in this form.

If the constant k is positive, the model describes population growth; if it is negative, it is decay.

The first one corresponds to positive values of k and the second to negative values of k.





If the problem above had the additional 'initial value' that at t_0 the population is P_0, then this particular population satisfies

$$P = Ae^{kt}$$

and hence

$$P_0 = Ae^{kt_0} \Rightarrow A = \frac{P_0}{e^{kt_0}} = P_0e^{-kt_0}$$

and the solution to the initial value problem is

$$P = Ae^{kt} = P_0e^{-kt_0}e^{kt} = P_0e^{k(t-t_0)}.$$

There is a very important special case when $t_0 = 0$. The solution becomes

$$P = P_0e^{k(t-t_0)} = P_0e^{kt}$$

which is the usual growth model which starts at time $t = 0$ with initial population P_0.

Example 54

If a cold object is placed in warmer medium that is kept at a constant temperature S, then the rate of change of the temperature $T(t)$ with respect to time t is proportional to the difference between the surrounding medium and the object and hence it satisfies

$$\frac{dT}{dt} = k(S - T) \qquad T(0) = T_0$$

where $k > 0$ and $T_0 < S$, i.e. the initial temperature is less than the temperature of the surrounding medium. Find the solution to the initial value problem.

Solution

It is immediately apparent that this is a variables separable type of differential equations as:

$$\frac{dT}{dt} = k(S - T) \Leftrightarrow \frac{dT}{S - T} = k\,dt$$

We integrate and find the general solution first.

$$\int \frac{dT}{S - T} = \int k\,dt$$

$$-\ln|S - T| = kt + c_1$$

$$\ln|S - T| = -kt - c_1$$

where c_1 is an arbitrary constant. Now since we know that the temperature T is less than the surrounding temperature, then

$$\ln|S - T| = \ln(S - T).$$

The general solution then is:

$$\ln(S - T) = -kt - c_1$$

$$S - T = e^{-kt - c_1}$$

$$T = S - e^{-kt - c_1}$$

The initial condition implies:

$$T = S - e^{-kt - c1}$$
$$T_0 = S - e^{0 - c1}$$
$$e^{-c1} = S - T_0$$
$$-c_1 = \ln(S - T_0)$$
$$c_1 = -\ln(S - T_0)$$

Therefore, substituting this value in the general solution:

$$\begin{aligned}
\ln(S - T) &= -kt - c_1 \\
\ln(S - T) &= -kt + \ln(S - T_0) \\
\ln(S - T) - \ln(S - T_0) &= -kt \\
\ln\left(\frac{S - T}{S - T_0}\right) &= -kt \\
\frac{S - T}{S - T_0} &= e^{-kt} \\
S - T &= (S - T_0)e^{-kt} \\
T &= S - (S - T_0)e^{-kt}
\end{aligned}$$

This is an example of what is called 'limited growth'. This is so because the maximum value that T can achieve is S. For example, if a can of soda is left in a room with constant temperature of 21°, then the temperature of the soda will increase to reach the room temperature!

In fact, since $k > 0$ and S is a constant, then

$$T = S - (S - T_0)e^{-kt}$$

$$\frac{dT}{dt} = k(S - T_0)e^{-kt}.$$

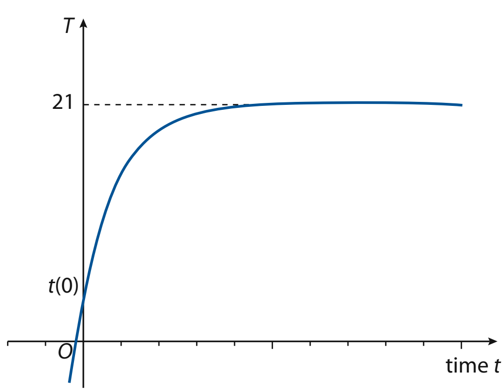
Also, since $T_0 < S$, then

$$\frac{dT}{dt} = k(S - T_0)e^{-kt} > 0.$$

The temperature will always increase. As time passes, i.e.

$$\lim_{t \to \infty} e^{-kt} = 0$$
$$\Rightarrow \lim_{t \to \infty} T = \lim_{t \to \infty}(S - (S - T_0)e^{-kt}) = S$$

The graph shows how the temperature climbs up to 21° but does not exceed it.



Example 55

Solve the initial value problem:

$$\frac{dy}{dx} = \frac{y}{x+1}; \; y(1) = 4$$

Solution

This is a variables separable type. We will separate the variables and integrate.

$$\frac{dy}{dx} = \frac{y}{x+1}$$

$$\frac{dy}{y} = \frac{dx}{x+1}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x+1}$$

$$\ln|y| = \ln|x+1| + c$$

$$|y| = e^{\ln|x+1|+c} = e^{\ln|x+1|}e^{c} = |x+1|e^{c}$$

Now, since c is an arbitrary constant, we can replace e^c with a constant C, and our solution becomes

$$|y| = C|x+1|.$$

Using the initial condition:

$4 = C|1+1| \Rightarrow C = 2$, and the particular solution

$|y| = 2|x+1|$, that is,

$y = \pm 2(x+1)$

Example 56

Solve the initial value problem:

$$\frac{dy}{dt} = e^{y-t}\frac{1+t^2}{\cos y}; \; y(0) = 0$$

Solution

This problem needs some work to get it separated.

$$\frac{dy}{dt} = e^{y}e^{-t}\frac{1+t^2}{\cos y}$$

$$e^{-y}\cos y\,dy = e^{-t}(1+t^2)\,dt$$

Both sides need integration by parts (left as an exercise for you).

$$\int e^{-y}\cos y\,dy = \int e^{-t}(1+t^2)\,dt$$

$$\tfrac{1}{2}e^{-y}(\cos y - \sin y) = e^{-t}(t^2 + 2t + 3) + c$$

With initial conditions applied:

$$\tfrac{1}{2}e^{-y}(\cos y - \sin y) = e^{-t}(t^2 + 2t + 3) + c$$
$$\tfrac{1}{2}e^{-0}(\cos 0 - \sin 0) = e^{-0}(0^2 + 2(0) + 3) + c$$
$$\tfrac{1}{2} = 3 + c \Rightarrow c = \tfrac{5}{2}$$

Therefore, our particular solution is:

$$\tfrac{1}{2}e^{-y}(\cos y - \sin y) = e^{-t}(t^2 + 2t + 3) + \tfrac{5}{2}$$
$$e^{-y}(\cos y - \sin y) = 2e^{-t}(t^2 + 2t + 3) + 5$$

Notice here that our solution cannot be expressed explicitly. In many cases, solutions to differential equations are given in implicit form.

Exercise 16.9

In questions 1–27, solve the given differential equation.

1 $x^{-3}dy = 4y\,dx,\ y(0) = 3$

2 $\dfrac{dy}{dx} = xy,\ y(0) = 1$

3 $y' - xy^2 = 0,\ y(1) = 2$

4 $y' - y^2 = 0,\ y(2) = 1$

5 $\dfrac{dy}{dx} - e^y = 0,\ y(0) = 1$

6 $y'e^{y-x} = 1$

7 $\dfrac{dy}{dx} = y^{-2}x + y^{-2},\ y(0) = 1$

8 $xdy - y^2\,dx = -dy,\ y(0) = 1$

9 $y^2dy - x\,dx = dx - dy,\ y(0) = 3$

10 $yy' = xy^2 + x,\ y(0) = 0$

11 $\dfrac{dy}{dx} = y^2x + x$

12 $y' = \dfrac{xy - y}{y + 1},\ y(2) = 1$

13 $e^{x-y}dy = x\,dx$

14 $y' = xy^2 - x - y^2 + 1$

15 $xy\ln xy' = (y + 1)^2$

16 $\dfrac{dy}{dx} = \dfrac{1 + 2y^2}{y\sin x}$

17 $\dfrac{dy}{dx} = x\sqrt{\dfrac{1 - y^2}{1 - x^2}},\ y(0) = 0$

18 $y'(1 + e^x) = e^{x-y},\ y(1) = 0$

19 $(y + 1)dy = (x^2y - y)dx,\ y(3) = 1$

20 $\cos y\,dx + (1 + e^{-x})\sin y\,dy = 0,\ y(0) = \dfrac{\pi}{4}$

21 $xy' - y = 2x^2y,\ y(1) = 1$

22 $xydx + e^{-x^2}(y^2 - 1)dy = 0,\ y(0) = 1$

23 $(1 + \tan y)y' = x^2 + 1$

24 $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{y^2 + 1}}$

25 $y\sec\theta\,dy = e^y\sin^2\theta\,d\theta$

26 $x\cos x = (2y + e^{3y})y',\ y(0) = 0$

27 $\dfrac{dy}{dx} = e^x - 2x,\ y(0) = 3$

28 The temperature T of a kettle in a room satisfies the differential equation

$$\frac{dT}{dt} = m(T - 21), \text{ where } t \text{ is in minutes and } m \text{ is a constant.}$$

a) Solve the differential equation showing that $T = Ce^{mt} + 21$, where C is an arbitrary constant.

b) Given that $T(0) = 99$ and $T(15) = 69$, find
 (i) the value of m and C
 (ii) t when $T = 39$.

Practice questions

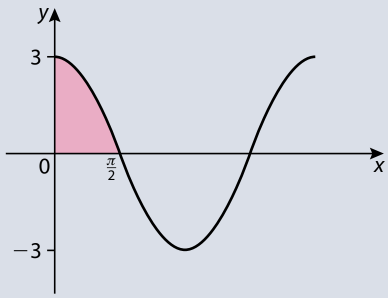
1 The graph represents the function

$f: x \mapsto p\cos x,\ p \in \mathbb{N}.$

Find

a) the value of p

b) the area of the shaded region.



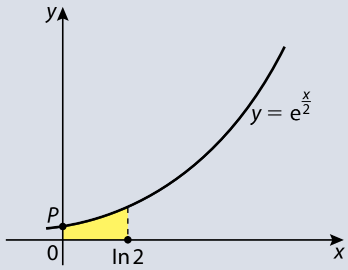
2 The diagram shows part of the graph of $y = e^{\frac{x}{2}}$.

a) Find the coordinates of the point P, where the graph meets the y-axis.

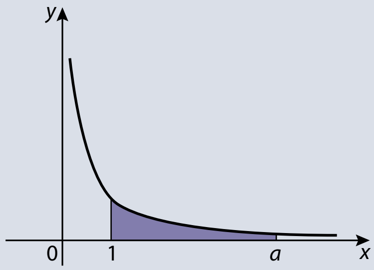
The shaded region between the graph and the x-axis, bounded by $x = 0$ and $x = \ln 2$, is rotated through 360° about the x-axis.

b) Write down an integral that represents the volume of the solid obtained.

c) Show that this volume is π cubic units.



3 The diagram shows part of the graph of $y = \frac{1}{x}$. The area of the shaded region is 2 units.

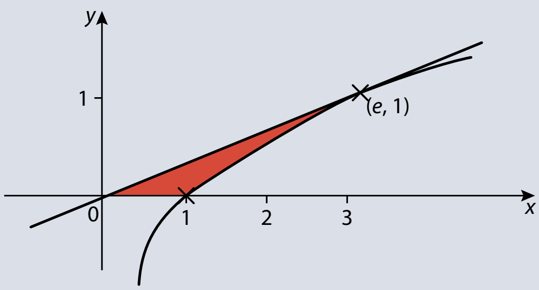


Find the exact value of a.

4 **a)** Find the equation of the tangent line to the curve $y = \ln x$ at the point $(e, 1)$, and verify that the origin is on this line.

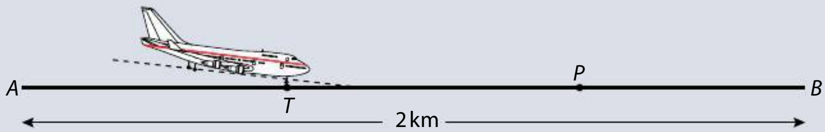
b) Show that $(x \ln x - x)' = \ln x$.

c) The diagram shows the region enclosed by the curve $y = \ln x$, the tangent line in part **a)**, and the line $y = 0$.



Use the result of part **b)** to show that the area of this region is $\frac{1}{2}e - 1$.

5 The main runway at Concordville airport is 2 km long. An aeroplane, landing at Concordville, touches down at point T, and immediately starts to slow down. The point A is at the southern end of the runway. A marker is located at point P on the runway.



Not to scale

As the aeroplane slows down, its distance, s, from A, is given by

$$s = c + 100t - 4t^2$$

where t is the time in seconds after touchdown and c metres is the distance of T from A.

a) The aeroplane touches down 800 m from A (i.e. $c = 800$).

(i) Find the distance travelled by the aeroplane in the first 5 seconds after touchdown.

(ii) Write down an expression for the velocity of the aeroplane at time t seconds after touchdown, and hence find the velocity after 5 seconds.

The aeroplane passes the marker at P with a velocity of 36 m s^{-1}. Find

(iii) how many seconds after touchdown it passes the marker

(iv) the distance from P to A.

b) Show that if the aeroplane touches down before reaching the point P, it can stop before reaching the northern end, B, of the runway.

6 **a)** Sketch the graph of $y = \pi \sin x - x$, $-3 \leqslant x \leqslant 3$, on millimetre square paper, using a scale of 2 cm per unit on each axis.
Label and number both axes and indicate clearly the approximate positions of the x-intercepts and the local maximum and minimum points.

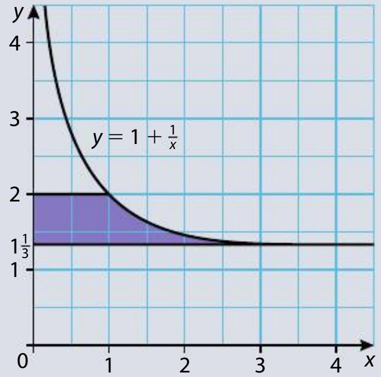
b) Find the solution of the equation $\pi \sin x - x = 0$, $x > 0$.

c) Find the indefinite integral

$$\int (\pi \sin x - x)\, dx$$

and hence, or otherwise, calculate the area of the region enclosed by the graph, the x-axis and the line $x = 1$.

7 The diagram shows the graph of the function $y = 1 + \frac{1}{x}$, $0 < x \leqslant 4$. Find the **exact** value of the area of the shaded region.



8 Note: Radians are used throughout this question.

a) (i) Sketch the graph of $y = x^2 \cos x$, for $0 \leqslant x \leqslant 2$, making clear the approximate positions of the positive intercept, the maximum point and the endpoints.

(ii) Write down the **approximate** coordinates of the positive x-intercept, the maximum point and the endpoints.

b) Find the **exact value** of the positive x-intercept for $0 \leqslant x \leqslant 2$.

Let R be the region in the first quadrant enclosed by the graph and the x-axis.

c) (i) Shade R on your diagram.

(ii) Write down an integral that represents the area of R.

d) Evaluate the integral in part **c) (ii)**, either by using a graphic display calculator, or by using the following information.

$$\frac{d}{dx}(x^2 \sin x + 2x \cos x - 2 \sin x) = x^2 \cos x$$

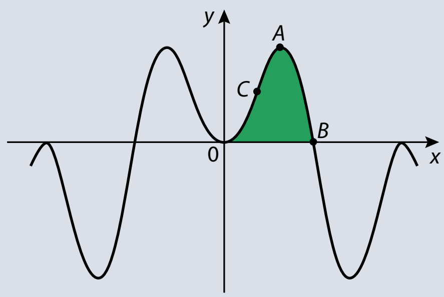
9 Note: Radians are used throughout this question.

The function f is given by

$$f(x) = (\sin x)^2 \cos x.$$

The diagram shows part of the graph of $y = f(x)$.

The point A is a maximum point, the point B lies on the x-axis, and the point C is a point of inflexion.



a) Give the period of f.

b) From consideration of the graph of $y = f(x)$, find, **to an accuracy of 1 significant figure**, the range of f.

c) (i) Find $f'(x)$.

(ii) Hence, show that at the point A $\cos x = \sqrt{\frac{1}{3}}$.

(iii) Find the exact maximum value.

d) Find the exact value of the x-coordinate at the point B.

e) (i) Find $\int f(x)\,dx$.

(ii) Find the area of the shaded region in the diagram.

f) Given that $f''(x) = 9(\cos x)^3 - 7 \cos x$, find the x-coordinate at the point C.

10 Note: Radians are used throughout this question.

a) Draw the graph of $y = \pi + x \cos x$, $0 \leqslant x \leqslant 5$, on millimetre square paper, using a scale of 2 cm per unit. Make clear

(i) the integer values of x and y on each axis

(ii) the approximate positions of the x-intercepts and the turning points.

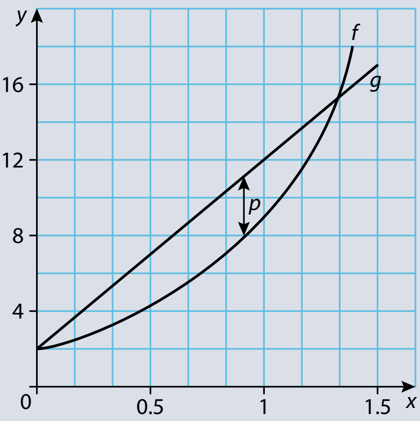
b) Without the use of a calculator, show that π is a solution of the equation $\pi + x \cos x = 0$.

c) Find another solution of the equation $\pi + x \cos x = 0$ for $0 \leqslant x \leqslant 5$, giving your answer to 6 significant figures.

d) Let R be the region enclosed by the graph and the axes for $0 \leqslant x \leqslant \pi$. Shade R on your diagram, and write down an integral which represents the area of R.

e) Evaluate the integral in part **d)** to an accuracy of **6** significant figures. (If you consider it necessary, you can make use of the result $\frac{d}{dx}(x \sin x + \cos x) = x \cos x$.)

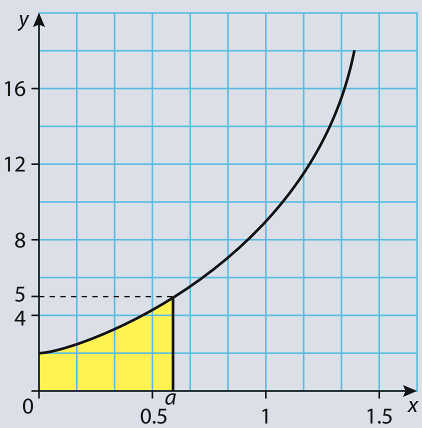
11 The diagram right shows the graphs of $f(x) = 1 + e^{2x}$ and $g(x) = 10x + 2, 0 \leqslant x \leqslant 1.5$.



a) **(i)** Write down an expression for the vertical distance p between the graphs of f and g.

(ii) Given that p has a maximum value for $0 \leqslant x \leqslant 1.5$, find the value of x at which this occurs.

The graph of $y = f(x)$ only is shown in the diagram right. When $x = a$, $y = 5$.



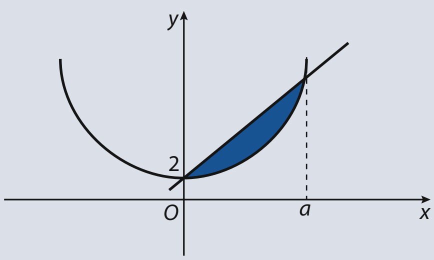
b) **(i)** Find $f^{-1}(x)$.

(ii) **Hence**, show that $a = \ln 2$.

c) The region shaded in the diagram is rotated through 360° about the x-axis. Write down an expression for the volume obtained.

12 The area of the enclosed region shown in the diagram is defined by

$$y \geqslant x^2 + 2, \, y \leqslant ax + 2, \text{ where } a > 0.$$



This region is rotated 360° about the x-axis to form a solid of revolution. Find, in terms of a, the volume of this solid of revolution.

13 Using the substitution $u = \frac{1}{2}x + 1$, or otherwise, find the integral $\int x\sqrt{\frac{1}{2}x + 1}\, dx$.

14 A particle moves along a straight line. When it is a distance s from a fixed point, where $s > 1$, the velocity v is given by $v = \dfrac{3s + 2}{2s - 1}$. Find the acceleration when $s = 2$.

15 The area between the graph of $y = e^x$ and the x-axis from $x = 0$ to $x = k$ $(k > 0)$ is rotated through 360° about the x-axis. Find, in terms of k and e, the volume of the solid generated.

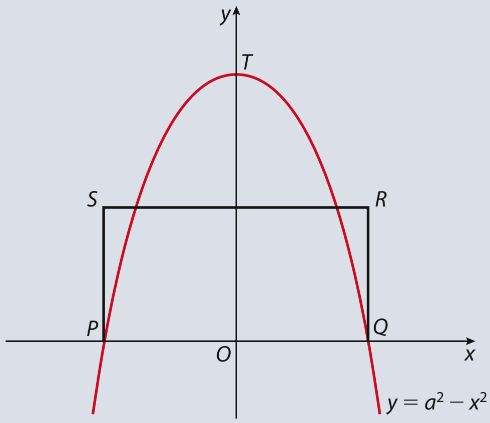
16 Find the real number $k > 1$ for which $\displaystyle\int_1^k \left(1 + \frac{1}{x^2}\right) dx = \frac{3}{2}$.

17 The acceleration, $a(t)\,\mathrm{m\,s^{-2}}$, of a fast train during the first 80 seconds of motion is given by

$$a(t) = -\frac{1}{20}t + 2$$

where t is the time in seconds. If the train starts from rest at $t = 0$, find the distance travelled by the train in the first minute.

18 In the diagram, PTQ is an arc of the parabola $y = a^2 - x^2$, where a is a positive constant, and $PQRS$ is a rectangle. The area of the rectangle $PQRS$ is equal to the area between the arc PTQ of the parabola and the x-axis.

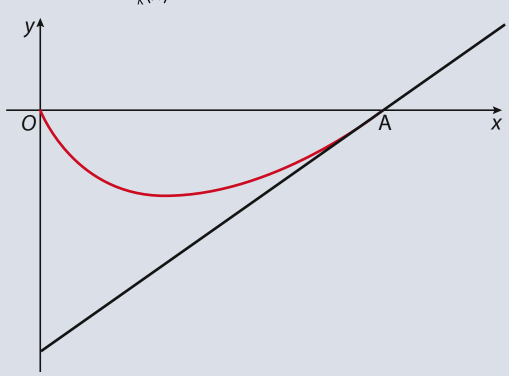


Find, in terms of a, the dimensions of the rectangle.

19 Consider the function $f_k(x) = \begin{cases} x\ln x - kx, & x > 0 \\ 0, & x = 0 \end{cases}$, where $k \in \mathbb{N}$

a) Find the derivative of $f_k(x)$, $x > 0$.
b) Find the interval over which $f(x)$ is increasing.

The graph of the function $f_k(x)$ is shown below.



c) **(i)** Show that the stationary point of $f_k(x)$ is at $x = e^{k-1}$.
(ii) One x-intercept is at (0, 0). Find the coordinates of the other x-intercept.
d) Find the area enclosed by the curve and the x-axis.
e) Find the equation of the tangent to the curve at A.
f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the x-axis.
g) Show that the x-intercepts of $f_k(x)$ for consecutive values of k form a geometric sequence.

20 Solve the differential equation $\frac{dy}{dx} = 1 + y^2$ given that $y = 0$ when $x = 2$.

21 The equation of motion of a particle with mass m, subjected to a force kx can be written as $kx = mv\frac{dv}{dx}$, where x is the displacement and v is the velocity. When $x = 0$, $v = v_0$. Find v, in terms of v_0, k and m, when $x = 2$.

22 **a)** Sketch and label the graphs of $f(x) = e^{-x^2}$ and $g(x) = e^{x^2} - 1$ for $0 \leqslant x \leqslant 1$, and shade the region A which is bounded by the graphs and the y-axis.

b) Let the x-coordinate of the point of intersection of the curves $y = g(x)$ and $y = g(x)$ be p.
Without finding the value of p, show that

$$\frac{p}{2} < \text{area of region } A < p.$$

c) Find the value of p correct to four decimal places.

d) Express the area of region A as a definite integral and calculate its value.

23 Let $f(x) = x\cos 3x$.

a) Use integration by parts to show that

$$\int f(x)dx = \tfrac{1}{3}x\sin 3x + \tfrac{1}{9}\cos 3x + c.$$

b) Use your answer to part **a)** to calculate the **exact** area enclosed by $f(x)$ and the x-axis in each of the following cases. **Give your answers in terms of π.**

(i) $\frac{\pi}{6} \leqslant x \leqslant \frac{3\pi}{6}$

(ii) $\frac{3\pi}{6} \leqslant x \leqslant \frac{5\pi}{6}$

(iii) $\frac{5\pi}{6} \leqslant x \leqslant \frac{7\pi}{6}$

c) Given that the above areas are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by $f(x)$ and the x-axis for

$$\frac{\pi}{6} \leqslant x \leqslant \frac{(2n+1)\pi}{6}, \text{ where } n \in \mathbb{Z}.$$

Give your answers in terms of n and π.

24 A particle is moving along a straight line so that t seconds after passing through a fixed point O on the line its velocity $v(t)\,\text{m s}^{-1}$ is given by

$$v(t) = t\sin\left(\frac{\pi}{3}t\right).$$

a) Find the values of t for which $v(t) = 0$, given that $0 \leqslant t \leqslant 6$.

b) **(i)** Write down a mathematical expression for the **total** distance travelled by the particle in the first six seconds after passing through O.

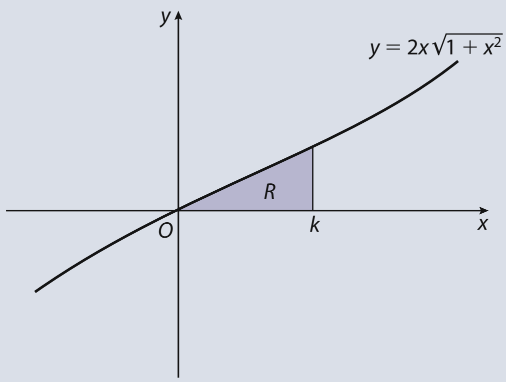
(ii) Find this distance.

25 A particle is projected along a straight-line path. After t seconds, its velocity v metres per second is given by $v = \frac{1}{2 + t^2}$.

a) Find the distance travelled in the first second.

b) Find an expression for the acceleration at time t.

26 The diagram below shows the shaded region R enclosed by the graph of $y = 2x\sqrt{1 + x^2}$, the x-axis, and the vertical line $x = k$.



a) Find $\frac{dy}{dx}$.

b) Using the substitution $u = 1 + x^2$ or otherwise, show that

$$\int 2x\sqrt{1 + x^2}\,dx = \tfrac{2}{3}(1 + x^2)^{\frac{3}{2}} + c.$$

c) Given that the area of R equals 1, find the value of k.

27 A particle moves in a straight line with velocity, in metres per second, at time t seconds, given by

$$v(t) = 6t^2 - 6t,\ t \geqslant 0.$$

Calculate the total distance travelled by the particle in the first two seconds of motion.

28 A particle moves in a straight line. Its velocity $v\,\mathrm{m\,s^{-1}}$ after t seconds is given by

$$v = e^{-\sqrt{t}} \sin t.$$

Find the total distance travelled in the time interval $[0, 2\pi]$.

29 The temperature T °C of an object in a room, after t minutes, satisfies the differential equation

$$\frac{dT}{dt} = k(T - 22), \text{ where } k \text{ is a constant.}$$

a) Solve the differential equation showing that $T = Te^{kt} + 22$, where A is a constant.

b) When $t = 0$, $T = 100$, and when $t = 15$, $T = 70$.

(i) Use this information to find the value of A and of k.

(ii) Hence, find the value of t when $T = 40$.

30 Solve the differential equation $x\frac{dy}{dx} - y^2 = 1$ given that $y = 0$ when $x = 2$. Give your answer in the form $y = f(x)$.

31 Use the substitution $u = x + 2$ to find $\int \frac{x^3}{(x + 2)^2}\,dx$.

32 a) On the same axes sketch the graphs of the functions, $f(x)$ and $g(x)$, where

$$f(x) = 4 - (1 - x)^2, \text{ for } -2 \leqslant x \leqslant 4,$$

$$g(x) = \ln(x + 3) - 2, \text{ for } -3 \leqslant x \leqslant 5.$$

b) **(i)** Write down the equation of any vertical asymptotes.

(ii) State the x-intercept and y-intercept of $g(x)$.

c) Find the values of x for which $f(x) = g(x)$.

d) Let A be the region where $f(x) \geqslant g(x)$ and $x \geqslant 0$.

(i) On your graph shade the region A.

(ii) Write down an integral that represents the area of A.

(iii) Evaluate this integral.

e) In the region A find the maximum vertical distance between $f(x)$ and $g(x)$.

33 Consider the differential equation $\frac{dy}{d\theta} = \frac{y}{e^{2\theta} + 1}$.

a) Use the substitution $x = e^{\theta}$ to show that

$$\int \frac{dy}{y} = \int \frac{dx}{x(x^2 + 1)}.$$

b) Find $\int \frac{dx}{x(x^2 + 1)}$.

c) Hence, find y in terms of θ, if $y = \sqrt{2}$ when $\theta = 0$.

17 Probability Distributions

5.5	Concept of discrete and continuous random variables and their probability distributions.
	Definition and use of probability density functions.
	Expected value (mean), mode, median, variance and standard deviation.
5.6	Binomial distribution, its mean and variance.
	Poisson distribution, its mean and variance.
5.7	Normal distribution.
	Properties of the normal distribution.
	Standardization of normal variables.

Introduction

Investing in securities, calculating premiums for insurance policies or overbooking policies used in the airline industry are only a few of the many applications of probability and statistics. Actuaries, for example, calculate the expected 'loss' or 'gain' that an insurance company will incur and decide on how high the premiums should be. These applications depend mainly on what we call probability distributions. A probability distribution describes the behaviour of a population in the sense that it lists the distribution of possible outcomes to an event, along with the probability of each potential outcome. This can be done by a table of values with their corresponding probabilities or by using a mathematical model.

In this chapter, you will get an understanding of the basic ideas of distributions and will study three specific ones: the binomial, Poisson and normal distributions.

17.1 Random variables

In Chapter 11, **variables** were defined as characteristics that change or vary over time and/or for different objects under consideration. A numerically valued variable x will vary or change depending on the outcome of the experiment we are performing. For example, suppose you are counting the number of mobile phones families in a certain city own. The variable of interest, x, can take any of the values 0, 1, 2, 3, etc. depending on the *random* outcome of the experiment. For this reason, we call the variable x a **random variable**.

> **Random variable**
> A **random variable** is a variable that takes on numerical values determined by the outcome of a random experiment.

When a probability experiment is performed, often we are not interested in all the details of the outcomes, but rather in the value of some numerical quantity determined by the result. For instance, in tossing two dice (used in plenty of games), often we care about their sum and not the values on the individual dice. Consider this specific experiment: A sample space for which the points are equally likely is given in Table 17.1 below. It consists of 36 ordered pairs (a, b) where a is the number on the first die and b is the number on the second die. For each sample point, we can let the *random* variable x stand for the sum of the numbers. The resulting values of x are also presented in Table 17.1.

$(1, 1); x = 2$	$(2, 1); x = 3$	$(3, 1); x = 4$	$(4, 1); x = 5$	$(5, 1); x = 6$	$(6, 1); x = 7$
$(1, 2); x = 3$	$(2, 2); x = 4$	$(3, 2); x = 5$	$(4, 2); x = 6$	$(5, 2); x = 7$	$(6, 2); x = 8$
$(1, 3); x = 4$	$(2, 3); x = 5$	$(3, 3); x = 6$	$(4, 3); x = 7$	$(5, 3); x = 8$	$(6, 3); x = 9$
$(1, 4); x = 5$	$(2, 4); x = 6$	$(3, 4); x = 7$	$(4, 4); x = 8$	$(5, 4); x = 9$	$(6, 4); x = 10$
$(1, 5); x = 6$	$(2, 5); x = 7$	$(3, 5); x = 8$	$(4, 5); x = 9$	$(5, 5); x = 10$	$(6, 5); x = 11$
$(1, 6); x = 7$	$(2, 6); x = 8$	$(3, 6); x = 9$	$(4, 6); x = 10$	$(5, 6); x = 11$	$(6, 6); x = 12$

Table 17.1 Sample space and the values of the random variable x in the two-dice experiment.

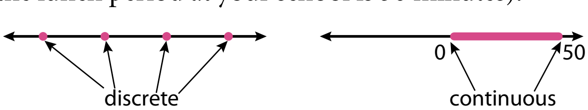
> Random variables are customarily denoted by upper-case letters, such as X and Y. Lower-case letters are used to represent particular values of the random variable. That is, if X represents the numbers resulting in the throw of a die, then $x = 2$ represents the case when the outcome is 2.

Notice that events can be more accurately and concisely defined in terms of the random variable x; for example, the event of tossing a sum at least equal to 5 but less than 9 can be replaced by $5 \leqslant x < 9$.

We can think of many examples of random variables:

- $X =$ the number of calls received by a household on a Friday night.
- $X =$ the number of free beds available at hotels in a large city.
- $X =$ the number of customers a sales person contacts on a working day.
- $X =$ the length of a metal bar produced by a certain machine.
- $X =$ the weight of newborn babies in a large hospital.

As you have seen in Chapter 11, these variables are classified as **discrete** or **continuous**, according to the values that x *can* assume. In the examples above, the first three are discrete and the last two are continuous. The random variable is discrete if its set of *possible* values is isolated points on the number line, i.e. there is a *countable* number of possible values for the variable. The variable is continuous if its set of *possible* values is an entire interval on the number line, i.e. it can take any value in an interval. Consider the number of times you toss a coin until the head side appears. The possible values are $x = 1, 2, 3, \ldots$. This is a discrete variable, even though the number of times may be infinite! On the other hand, consider the time it takes a student at your school to eat/have his/her lunch. This can be anywhere between zero and 50 minutes (given that the lunch period at your school is 50 minutes).



Example 1

State whether each of the following is a discrete or a continuous random variable.

1. The number of hairs on a Scottish Terrier
2. The height of a building
3. The amount of fat in a steak
4. A high school student's grade on a maths test
5. The number of fish in the Atlantic Ocean
6. The temperature of a wooden stove

Solution

1. Even though the number of hairs is 'almost' infinite, it is countable. So, it is a discrete random variable.
2. This can be any real number. Even when you say this building is 15 m high, the number could be 15.1 or 15.02, etc. Hence, it is continuous.
3. This is continuous, as the amount of fat could be zero or anything up to the maximum amount of fat that can be held in one piece.
4. Grades are discrete. No matter how detailed a score the teacher gives, the grades are isolated points on a scale.
5. This is almost infinite, but countable, and hence discrete.
6. This is continuous, as the temperature can take any value from room temperature to 100 degrees.

Discrete probability distribution

In Chapter 11, you learned how to work with the frequency distribution and relative or percentage frequency distribution for a set of numerical measurements on a variable X. The distribution gave the following information about X:

- The value of x that occurred.
- How often each value occurred.

You also learned how to use the mean and standard deviation to measure the centre and variability of the data set.

Here is an example of the frequency distribution of 25 families in Lower Austria that were polled in a marketing survey to list the number of litres of milk consumed during a particular week, reproduced on the next page. As you will observe, the table lists the number of litres consumed along with the relative frequency with which that number is observed. As you recall from Chapter 12, one of the interpretations of probability is that it is understood to be the long-term relative frequency of the event.

Number of litres	Relative frequency
0	0.08
1	0.20
2	0.36
3	0.20
4	0.12
5	0.04

Table 17.2

A table like this, where we replace the relative frequency with probability, is called a **probability distribution** of the random variable.

The **probability distribution** for a discrete random variable is a table, graph or formula that gives the possible values of X, and the probability $P(X = x)$ associated with each value of x.
This is also called the **probability mass function (pmf)** and in many sources it is called the **probability distribution function (pdf)**.

In other words, for every possible value x of the random variable X, the probability mass function specifies the probability of observing that value when the experiment is performed.

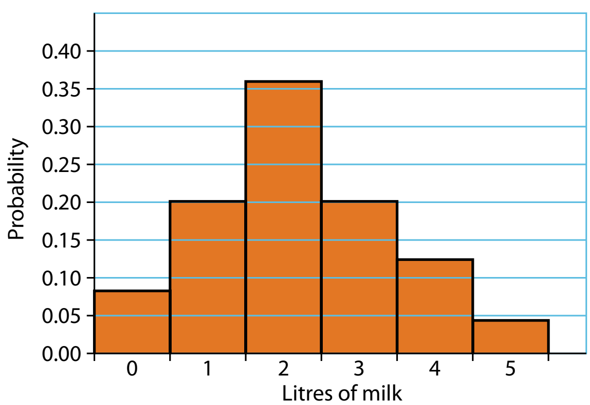
Note: we write $P(X = x)$ as $P(x)$ for convenience.

Letting x be the number of litres of milk consumed by a family above, the **probability distribution** of x would be as follows:

Table 17.3

x	0	1	2	3	4	5
$P(x)$	0.08	0.20	0.36	0.20	0.12	0.04

The other form of representing the probability distribution is with a histogram, as shown below. Every column corresponds to the probability of the associated value of x. The values of x naturally represent mutually exclusive events. Summing $P(x)$ over all values of x is equivalent to adding all probabilities of all simple events in the sample space, and hence the total is 1.



The result above can be generalized for all probability distributions:

Required properties of probability distribution functions of discrete random variables

Let X be a discrete random variable with probability distribution function, $P(x)$. Then:

- $0 \leqslant P(x) \leqslant 1$, for any value x.
- The individual probabilities sum to 1; that is, $\sum_x P(x) = 1$ where the notation indicates summation over all possible values x.

For some value x of the random variable X, we often wish to compute the probability that the observed value of X is at most x. This gives rise to the **cumulative distribution function (cdf)**.

Cumulative distribution function (cdf) (optional but very helpful)
The **cumulative distribution function** of a random variable X (also known as the 'cumulative probability function $F(x)$'), expresses the probability that X does not exceed the value x, as a function of x. That is,

$$F(x) = P(X \leqslant x) = \sum_{y:y \leqslant x} P(y)$$

The notation here indicates that summation is over all possible values of y that are less than or equal to x.

For example, in the milk consumption case, the cdf will look like the following table:

x	$F(x)$
0	0.08
1	0.28
2	0.64
3	0.84
4	0.96
5	1.00

So, $F(3) = 0.84$, stands for the probability of families that consume up to 3 litres of milk. This result of course can be achieved by adding the probabilities corresponding to $x = 0, 1, 2$ and 3.

In many cases, as we will see later, we use the cumulative distribution to find individual probabilities,

$$P(X = x) = P(X \leqslant x) - P(X < x).$$

For example, to find the probability that $x = 3$, we can use the cumulative distribution table.

$$P(x = 3) = P(x \leqslant 3) - P(x < 3) = 0.84 - 0.64 = 0.2$$

This property is of great value when studying the binomial and the Poisson distributions.

Example 2

Radon is a major cause of lung cancer. It is a radioactive gas produced by the natural decay of radium in the ground. Studies in areas rich with radium revealed that one-third of houses in these areas have dangerous levels of this gas. Suppose that two houses are randomly selected and we define the random variable X to be the number of houses with dangerous levels. Find the probability distribution of x by a table, a graph and a formula.

Solution

Since two houses are selected, the possible values of x are 0, 1 or 2. To find their probabilities, we utilize what we learned in Chapter 12. The assumption here is that we are choosing the houses randomly and independently of each other!

$$P(x = 2) = P(2) = P(\text{1st house with gas } \textit{and} \text{ 2nd house with gas})$$

$$= P(\text{1st house with gas}) \times P(\text{2nd house with gas}) = \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{1}{9}$$

$$P(x = 0) = P(0) = P(\text{1st house without gas } \textit{and} \text{ 2nd house without gas})$$

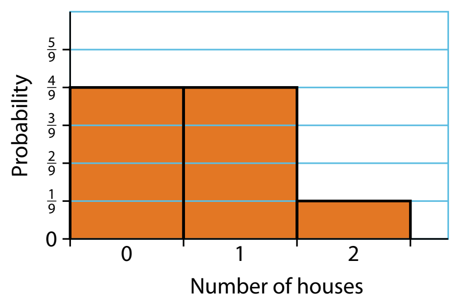
$$= P(\text{1st house without gas}) \times P(\text{2nd house without gas}) = \tfrac{2}{3} \times \tfrac{2}{3} = \tfrac{4}{9}$$

$$P(x = 1) = 1 - [P(0) + P(2)] = 1 - \left[\tfrac{4}{9} + \tfrac{1}{9}\right] = \tfrac{4}{9}$$

Table

x	0	1	2
$P(x)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Graph



Any type of graph can be used to give the probability distribution, as long as it shows the possible values of x and the corresponding probabilities. The probability here is graphically displayed as the height of a rectangle. Moreover, the rectangle corresponding to each value of x has an area equal to the probability $P(X = x)$. The histogram is the preferred tool due to its connection to the continuous distributions discussed later in the chapter.

Formula/rule

The probability distribution of X can also be given by the following rule. Don't be concerned now with how we came up with this formula, as we will discuss it later in the chapter. The only reason we are looking at it now is to illustrate the fact that a formula/rule can sometimes be used to give the probability distribution.

$$P(x) = \binom{2}{x} \cdot \left(\frac{1}{3}\right)^{x} \cdot \left(\frac{2}{3}\right)^{2-x}$$

where $\binom{2}{x}$ represents the binomial coefficient you saw in Chapter 4.

Notice that when x is replaced by 0, 1 or 2 we obtain the results we are looking for:

$$P(0) = \binom{2}{0} \cdot \left(\frac{1}{3}\right)^{0} \cdot \left(\frac{2}{3}\right)^{2-0} = 1 \cdot 1 \cdot \frac{4}{9} = \frac{4}{9}$$

$$P(1) = \binom{2}{1} \cdot \left(\frac{1}{3}\right)^{1} \cdot \left(\frac{2}{3}\right)^{2-1} = 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

$$P(2) = \binom{2}{2} \cdot \left(\frac{1}{3}\right)^{2} \cdot \left(\frac{2}{3}\right)^{2-2} = 1 \cdot \frac{1}{9} \cdot 1 = \frac{1}{9}$$

Example 3

Many universities have the policy of posting the grade distributions for their courses. Several of the universities have a grade-point average that codes the grades in the following manner: A = 4, B = 3, C = 2, D = 1 and F = 0. During the spring term at a certain large university, 13% of the students in an introductory statistics course received A's, 37% B's, 45% C's, 4% received D's and 1% received F's. The experiment here is to choose a student at random and mark down his/her grade. The student's grade on the 4-point scale is a random variable X.

Here is the probability distribution of X:

x	0	1	2	3	4
$P(x)$	0.01	0.04	0.45	0.37	0.13

Is this a probability distribution?

Solution

Yes, it is. Each probability is between 0 and 1, and the sum of all probabilities is 1.

What is the probability that a randomly chosen student receives a B or better?

$$P(x \geqslant 3) = P(x = 3) + P(x = 4) = 0.37 + 0.13 = 0.40$$

Example 4

In the codes example in Chapter 12, we saw the probability with which people choose the first digits for the codes for their cellphones. The probability distribution is copied below for reference.

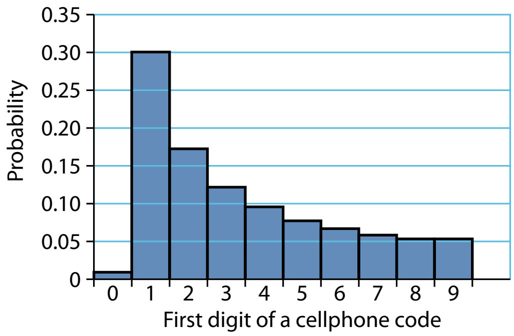
First digit	0	1	2	3	4	5	6	7	8	9
Probability	0.009	0.300	0.174	0.122	0.096	0.078	0.067	0.058	0.051	0.045

Here, X is the first digit chosen.

What is the probability that you pick a first digit and it is more than 5? Show a probability histogram for the distribution.

Solution

$$P(x > 5) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) = 0.221$$



Note that the height of each bar shows the probability of the outcome at its base. The heights add up to 1, of course. The bars in this histogram have the same width, namely 1. So, the areas also display the probability assignments of the outcomes. Think of such histograms (probability histograms) as idealized pictures of the results of very many repeated trials.

Expected values

The probability distribution for a random variable looks very similar to the relative frequency distribution discussed in Chapter 11. The difference is that the relative frequency distribution describes a *sample* of measurements, whereas the probability distribution is constructed as a *model* for the *entire population*. Just as the mean and standard deviation gave you measures for the centre and spread of the sample data, you can calculate similar measures to describe the centre and spread of the population.

The population mean, which measures the average value of X in the population, is also called the **expected value** of the random variable X. It is the value that you would *expect* to observe *on average* if you repeat the experiment an infinite number of times. The formula we use to determine the expected value can be simply understood with an example.

Let's revisit the milk consumption example. Let X be the number of litres consumed. Here is the table of probabilities again:

x	0	1	2	3	4	5
$P(x)$	0.08	0.20	0.36	0.20	0.12	0.04

Suppose we choose a large number of families, say 100 000. Intuitively, using the relative frequency concept of probability, you would expect to observe 8000 families consuming no milk, 20 000 consuming 1 litre, and the rest similarly done: 36 000, 20 000, 12 000 and 4000.

The average (mean) value of X, as defined in Chapter 11, would then be equal to

$$\frac{\text{sum of all measurements}}{n}$$

$$= \frac{0 \cdot 8000 + 1 \cdot 20\,000 + 2 \cdot 36\,000 + 3 \cdot 20\,000 + 4 \cdot 12\,000 + 5 \cdot 4000}{100\,000}$$

$$= \frac{0 \cdot 8000}{100\,000} + \frac{1 \cdot 20\,000}{100\,000} + \frac{2 \cdot 36\,000}{100\,000} + \frac{3 \cdot 20\,000}{100\,000} + \frac{4 \cdot 12\,000}{100\,000} + \frac{5 \cdot 4000}{100\,000}$$

$$= 0 \cdot 0.08 + 1 \cdot 0.20 + 2 \cdot 0.36 + 3 \cdot 0.20 + 4 \cdot 0.12 + 5 \cdot 0.04$$

$$= 0 \cdot \mathrm{P}(0) + 1 \cdot \mathrm{P}(1) + 2 \cdot \mathrm{P}(2) + 3 \cdot \mathrm{P}(3) + 4 \cdot \mathrm{P}(4) + 5 \cdot \mathrm{P}(5) = 2.2$$

That is, we expect to see families, *on average*, consuming 2.2 litres of milk! This does not mean that we know what a family *will* consume, but we can say what we *expect* to happen.

Let X be a discrete random variable with probability distribution $\mathrm{P}(x)$. The mean or **expected value** of X is given by

$$\mu = \mathrm{E}(X) = \sum x \cdot \mathrm{P}(x).$$

Insurance companies make extensive use of expected value calculations. Here is a simplified example.

An insurance company offers a policy that pays you €10 000 when you totally damage your car or €5000 for major damages (50%). They charge you €50 per year for this service. The question is, how can they make a profit?

To understand how they can afford this, suppose that the 'total damage' car accident rate, in any year, is 1 out of every 1000 cars, and that another 2 out of 1000 will have serious damages. Then we can display the probability model for this policy in a table like this:

Type of accident	Amount paid x	Probability $\mathrm{P}(X = x)$
Total damage	10 000	$\frac{1}{1000}$
Major damage	5000	$\frac{2}{1000}$
Minor or no damage	0	$\frac{997}{1000}$

The expected amount the insurance company pays is given by:

$$\mu = \mathrm{E}(X) = \sum x\mathrm{P}(x) = €10\,000\left(\frac{1}{1000}\right) + €5000\left(\frac{2}{1000}\right)$$
$$+ €0\left(\frac{997}{1000}\right) = €20$$

This means that the insurance company *expects* to pay, on average, an amount of €20 per insured car. Since it is charging people €50 for the policy, the company *expects* to make a profit of €30 per car. Thinking about the problem in a different perspective, suppose they insure 1000 cars, then the company would expect to pay €10 000 for 1 car and €5000 to each of two cars with major damage. This is a total of €20 000 for all cars, or an average of $\frac{20\,000}{1000}$ = €20 per car.

Of course, this expected value is not what actually happens to any *particular* policy. No individual policy actually costs the insurance company €20. We are dealing with random events, so a few car owners may require a payment of €10 000 or €5000, many others receive *nothing*!

Because of the need to anticipate such variability, the insurance company needs to know a measure of this variability, which is nothing but the **standard deviation**.

Variance and standard deviation

For data in Chapter 11, we calculated the variance by computing the deviation from the mean, $x - \mu$, and then squaring it. We do that with random variables as well.

We can use similar arguments to justify the formulae for the population variance σ^2 and, consequently, the population standard deviation σ. These measures describe the spread of the values of the random variable around the centre. We similarly use the idea of the 'average' or 'expected' value of the squared deviations of the x-values from the mean μ or $E(x)$.

Let X be a discrete random variable with probability distribution $P(x)$ and mean μ. The **variance of X** is given by

$$\sigma^2 = E((X - \mu)^2) = \sum (x - \mu)^2 \cdot P(x).$$

(This is sometimes called Var(X).)

Note: It can also be shown, similar to what you saw in Chapter 11, that you have another 'computation' formula for the variance:

$$\sigma^2 = \sum (x - \mu)^2 \cdot P(x) = \sum x^2 \cdot P(x) - \mu^2 = \sum x^2 \cdot P(x) - [E(x)]^2$$
$$= \sum x^2 \cdot P(x) - \left[\sum xP(x)\right]^2$$

The **standard deviation** σ of a random variable X is equal to the positive square root of its variance.

Let us go back to the milk consumption example. Recall that we calculated the expected value, mean, to be 2.2 litres. In order to calculate the variance, we can tabulate our work to make the manual calculation simple.

x	$P(x)$	Deviation $(x - \mu)$	Squared deviation $(x - \mu)^2$	$(x - \mu)^2 \cdot P(x)$
0	0.08	−2.2	4.84	0.3872
1	0.20	−1.2	1.44	0.2880
2	0.36	−0.2	0.04	0.0144
3	0.20	0.8	0.64	0.1280
4	0.12	1.8	3.24	0.3888
5	0.04	2.8	7.84	0.3136
		Total	$\sum (x - \mu)^2 \cdot P(x)$	1.52

So, the variance of the milk consumption is 1.52 litres^2, or the standard deviation is 1.233 litres.

GDC notes

The above calculations, along with the expected value calculation, can be easily done using your GDC.

First, store x and $\mathrm{P}(x)$ into L1 and L2.

```
L1      L2      L3     2
0       .08     --------
1       .2
2       .36
3       .2
4       .12
5       .04
------- -------
L2(1)=.08
```

Then, to find $x\,\mathrm{P}(x)$, we multiply L1 and L2 and store the result in L3.

```
L1*L2→L3
(0 .2 .72 .6 .4…
```

To find the expected value, you simply get the sum of the entries in L3, since they correspond to $\sum x \cdot \mathrm{P}(x)$.

```
L1*L2→L3
(0 .2 .72 .6 .4…
sum(L3)
                 2.2
```

To find the variance, we need to find the deviations from the mean; so we make L4 that deviation, i.e. we store L1 − 2.2 into L4. Then, to get the squared deviations multiplied by the corresponding probability, we set up L5 to be L4 squared multiplied by L2, the probability. Now, to find the variance, just add the terms of L5.

```
L1-2.2→L4
(-2.2  -1.2  -.2…
(L4)²*L2→L5
(.3872 .228 .01…
sum(L5)
                1.52
```

Software note

In the comfort of home/class, the above calculation can be performed on a computer with a simple spreadsheet like the following one:

x	$\mathrm{P}(x)$	$x\mathrm{P}(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2\mathrm{P}(x)$	
0	0.08	0	−2.2	4.84	0.3872	
1	0.2	0.2	−1.2	1.44	0.288	
2	0.36	0.72	−0.2	0.04	0.0144	
3	0.2	0.6	0.8	0.64	0.128	
4	0.12	0.48	1.8	3.24	0.3888	E4*B4
5	0.04	0.2	2.8	7.84	0.3136	
Totals	**1**	**2.2**			**1.52**	
				A3 − 2.2	E6^2	
	A2*B2		SUM(C2:C7)			

Example 5

A computer store sells a particular type of laptop. The daily demand for the laptops is given in the table below. X is the number of laptops in demand. They have only 4 laptops left in stock and would like to know how well they are prepared for all eventualities. Work out the expected value of the demand and the standard deviation.

x	0	1	2	3	4	5
$P(X = x)$	0.08	0.40	0.24	0.15	0.08	0.05

Solution

$$E(X) = \sum xP(x) = 0 \times 0.08 + 1 \times 0.40 + 2 \times 0.24 + 3 \times 0.15 + 4 \times 0.08 + 5 \times 0.05 = 1.90$$

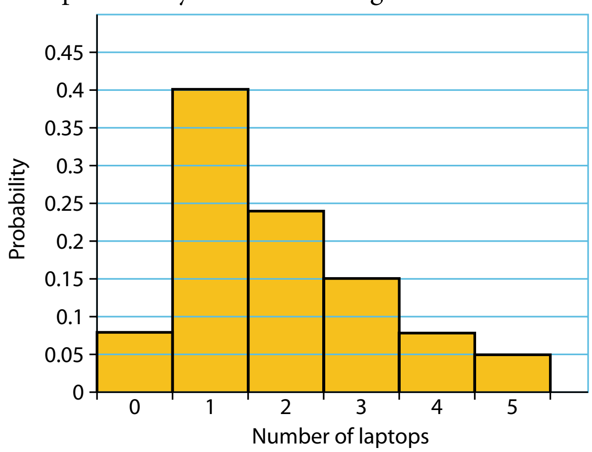
$$\begin{aligned}\mathrm{Var}(X) &= \sigma^2 = \sum (x - \mu)^2 P(x) \\ &= (0 - 1.9)^2 \cdot 0.08 + (1 - 1.9)^2 \cdot 0.40 + (2 - 1.9)^2 \cdot 0.24 \\ &\quad + (3 - 1.9)^2 \cdot 0.15 + (4 - 1.9)^2 \cdot 0.08 + (5 - 1.9)^2 \cdot 0.05 \\ &= 1.63\end{aligned}$$

$$\sigma = 1.28$$

Spreadsheet output is also given.

x	$P(x)$	$xP(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
0	0.08	0	−1.9	3.61	0.2888
1	0.4	0.4	−0.9	0.81	0.324
2	0.24	0.48	0.1	0.01	0.0024
3	0.15	0.45	1.1	1.21	0.1815
4	0.08	0.32	2.1	4.41	0.3528
5	0.05	0.25	3.1	9.61	0.4805
Totals	**1**	**1.9**			**1.63**

The graph of the probability distribution is given below.



As an approximation, we can use the *empirical rule* to see where most of the demand is expected to be. Recall that the empirical rule tells us that about 95% of the values would lie within 2 standard deviations from the mean. In this case $\mu \pm 2\sigma = 1.9 \pm 2 \times 1.28 \Rightarrow (-0.66, 4.46)$. This interval does not contain the 5 units of demand. We can say that it is unlikely that 5 or more customers of this shop will want to buy a laptop today.

```
sum(L1*L2)
               1.9
■
```

```
L1*L2→L3
(0 .4 .48 .45 ....
(L1-1.9)²*L2→L5
(.2888 .324 .00...
sum(L5)
              1.63
■
```

GDC

After entering the demand in L1 and the probabilities in L2, it is enough to find the sum of their product.

For the variance, we follow the same procedure as described in the previous example, see left.

Notice here that we combined several steps in one.

Exercise 17.1

1 Classify each of the following as discrete or continuous random variables.

a) The number of words spelled correctly by a student on a spelling test.
b) The amount of water flowing through the Niagara Falls per year.
c) The length of time a student is late to class.
d) The number of bacteria per cc of drinking water in Geneva.
e) The amount of CO produced per litre of unleaded gas.
f) The amount of a flu vaccine in a syringe.
g) The heart rate of a lab mouse.
h) The barometric pressure at Mount Everest.
i) The distance travelled by a taxi driver per day.
j) Total score of football teams in national leagues.
k) Height of ocean tides on the shores of Portugal.
l) Tensile breaking strength (in newtons per square metre) of a 5 cm diameter steel cable.
m) Number of overdue books in a public library.

2 A random variable Y has this probability distribution:

y	0	1	2	3	4	5
$\mathrm{P}(y)$	0.1	0.3		0.1	0.05	0.05

a) Find P(2).
b) Construct a probability histogram for this distribution.
c) Find μ and σ.
d) Locate the interval $\mu \pm \sigma$ as well as $\mu \pm 2\sigma$ on the histogram.
e) We create another random variable $Z = b + 1$. Find μ and σ of Z.
f) Compare your results for c) and e) and generalize for $Z = Y + b$, where b is a constant.

3 A discrete random variable X can assume five possible values: 12, 13, 15, 18 and 20. Its probability distribution is shown below.

x	12	13	15	18	20
$\mathrm{P}(x)$	0.14	0.11		0.26	0.23

a) What is P(15)?
b) What is the probability that x equals 12 or 20?
c) What is $\mathrm{P}(x \leqslant 18)$?
d) Find $\mathrm{E}(X)$.
e) Find $\mathrm{V}(X)$.
f) Let $Y = 0.5X - 4$. Find $\mathrm{E}(Y)$ and $\mathrm{V}(Y)$.
g) Compare your results in d), e) and f) and generalize for $Y = aX + b$, where a and b are constants.

4 Medical research has shown that a certain type of chemotherapy is successful 70% of the time when used to treat skin cancer. In a study to check the validity of such a claim, researchers chose different treatment centres and chose five of their patients at random. Here is the probability distribution of the number of successful treatments for groups of five:

x	0	1	2	3	4	5
P(x)	0.002	0.029	0.132	0.309	0.360	0.168

a) Find the probability that at least two patients would benefit from the treatment.
b) Find the probability that the majority of the group does not benefit from the treatment.
c) Find E(X) and interpret the result.
d) Show that $\sigma(X) = 1.02$.
e) Graph P(x). Locate μ, $\mu \pm \sigma$ and $\mu \pm 2\sigma$ on the graph. Use the empirical rule to approximate the probability that x falls in this interval. Compare this with the actual probability.

5 The probability function of a discrete random variable X is given by

$$P(X = x) = \frac{kx}{2}, \text{ for } x = 12, 14, 16, 18.$$

Set up the table showing the probability distribution and find the value of k.

6 X has probability distribution as shown in the table.

x	5	10	15	20	25
P(x)	$\frac{3}{20}$	$\frac{7}{30}$	k	$\frac{3}{10}$	$\frac{13}{60}$

a) Find the value of k.
b) Find $P(x > 10)$.
c) Find $P(5 < x \leqslant 20)$.
d) Find the expected value and the standard deviation.
e) Let $Y = \frac{1}{5}X - 1$. Find E(Y) and V(Y).

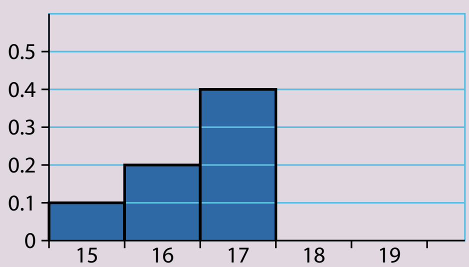
7 The discrete random variable Y has a probability density function

$$P(Y = y) = k(16 - y^2), \text{ for } y = 0, 1, 2, 3, 4.$$

a) Find the value of the constant k.
b) Draw a histogram to illustrate the distribution.
c) Find $P(1 \leqslant y \leqslant 3)$.
d) Find the mean and variance.

8 The probability distribution of students categorized by age that visit a certain movie house on weekends is given on the right. The probabilities for 18- and 19-year-olds are missing. We know that

$$P(x = 18) = 2P(x = 19).$$



a) Complete the histogram and describe the distribution.
b) Find the expected value and the variance.

9 In a small town, a computer store sells laptops to the local residents. However, due to low demand, they like to keep their stock at a manageable level. The data they have indicate that the weekly demand for the laptops they sell follows the distribution given in the table below.

X: number of laptops bought	0	1	2	3	4	5
$P(X = x)$	0.10	0.40	0.20	0.15	0.10	0.05

a) Find the mean and standard deviation of this distribution.

b) Use the empirical rule to find the approximate number of laptops that is sold about 95% of the time.

10 The discrete random variable X has probability function given by

$$P(x) = \begin{cases} \left(\frac{1}{4}\right)^{x-1} & x = 2, 3, 4, 5, 6 \\ k & x = 7 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Determine the value of k and the expected value of X.

11 The following is a probability distribution for a random variable Y.

y	0	1	2	3
$P(Y = y)$	0.1	0.11	k	$(k - 1)^2$

a) Find the value of k.

b) Find the expected value.

12 A closed box contains eight red balls and four white ones. A ball is taken out at random, its colour noted, and then returned. This is done three times. Let X represent the number of red balls drawn.

a) Set up a table to show the probability distribution of X.

b) What is the expected number of red balls in this experiment?

13 A discrete random variable Y has the following probability distribution function:

$P(Y = y) = k(4 - y)$, for $y = 0, 1, 2, 3$ and 4.

a) Find the value of k.

b) Find $P(1 \leqslant y < 3)$.

14 Airlines sometimes overbook flights. Suppose for a 50-seat plane, 55 tickets were sold. Let X be the number of ticketed passengers that show up for the flight. From records, the airline has the following **pmf** for this flight.

x	45	46	47	48	49	50	51	52	53	54	55
$P(x)$	0.05	0.08	0.12	0.15	0.25	0.20	0.05	0.04	0.03	0.02	0.01

a) Construct a **cdf** table for this distribution.

b) What is the probability that the flight will accommodate all ticket holders that show up?

c) What is the probability that not all ticket holders will have a seat on the flight?

d) Calculate the expected number of passengers who will show up.

e) Calculate the standard deviation of the passengers who will show up.

f) Calculate the probability that the number of passengers showing up will be within one standard deviation of the expected number.

15 A small internet provider has 6 telephone service lines operating 24-hours daily. Defining X as the number of lines in use at any specific 10-minute period of the day, the **pmf** of X is given in the following table.

x	0	1	2	3	4	5	6
P(x)	0.08	0.15	0.22	0.27	0.20	0.05	0.03

a) Construct a **cdf** table.
b) Calculate the probability that at most three lines are in use.
c) Calculate the probability that a customer calling for service will have a free line.
d) Calculate the expected number of lines in use.
e) Calculate the standard deviation of the number of lines in use.

16 Some flashlights use one AA-type battery. The voltage in any new battery is considered acceptable if it is at least 1.3 volts. 90% of the AA batteries from a specific supplier have an acceptable voltage. Batteries are usually tested till an acceptable one is found. Then it is installed in the flashlight. Let X be the number of batteries that must be tested.
a) What is P(1), i.e. P($x = 1$)?
b) What is P(2)?
c) What is P(3)?
d) To have $x = 5$, what must be true of the fourth battery tested? of the fifth one?
e) Use your observations above to obtain a general model for P(x).

17 Repeat question 16, but now consider the flashlight as needing two batteries.

18 A biased die with four faces is used in a game. A player pays 10 counters to roll the die. The table below shows the possible scores on the die, the probability of each score and the number of counters the player receives in return for each score.

Score	1	2	3	4
Probability	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
Number of counters player receives	4	5	15	n

Find the value of n in order for the player to get an expected return of 9 counters per roll.

19 Two children, Alan and Belle, each throw two fair cubical dice simultaneously. The score for each child is the sum of the two numbers shown on their respective dice.
a) (i) Calculate the probability that Alan obtains a score of 9.
(ii) Calculate the probability that Alan and Belle both obtain a score of 9.
b) (i) Calculate the probability that Alan and Belle obtain the same score.
(ii) Deduce the probability that Alan's score exceeds Belle's score.
c) Let X denote the largest number shown on the four dice.
(i) Show that for $P(X \leqslant x) = \left(\frac{x}{6}\right)^4$, for $x = 1, 2, \ldots, 6$.
(ii) Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$				$\frac{671}{1296}$

(iii) Calculate E(X).

20 Consider the 10 data items $x_1, x_2, \ldots, x_{10}$. Given that $\sum_{i=1}^{10} x^2_i = 1341$ and the standard deviation is 6.9, find the value of $\bar{x}$.

Questions 18–20, © International Baccalaureate Organization

17.2 The binomial distribution

Examples of discrete random variables are abundant in everyday situations. However, there are a few discrete probability distributions that are widely applied and serve as *models* for a great number of the applications. In this book, we will study two of them only: the **binomial distribution** and the Poisson distribution.

We will start with the basis of the binomial distribution.

Bernoulli distribution

If an experiment has two possible outcomes, '*success*' and '*failure*', and their probabilities are p and $1 - p$, respectively, then the number of successes, 0 or 1, has a **Bernoulli distribution**.

A discrete random variable X has a Bernoulli distribution if and only if it has two possible outcomes labelled by $x = 0$ and $x = 1$ in which $x = 1$ ('success') occurs with probability p and $x = 0$ ('failure') occurs with probability $1 - p$, where $0 < p < 1$. It therefore has probability function

$$p(x) = \begin{cases} 1 - p & \text{for } x = 0 \\ p & \text{for } x = 1, \end{cases}$$

which can also be written as

$$p(x) = p^x(1 - p)^{1-x},\ x = 0, 1.$$

The corresponding distribution function is

$$D(x) = \begin{cases} 1 - p & \text{for } x = 0 \\ 1 & \text{for } x = 1. \end{cases}$$

A sequence of **Bernoulli trials** occurs when a Bernoulli experiment is performed several independent times so that the probability of success, p, remains the same from trial to trial. In addition, we frequently use q to denote the probability of failure, i.e. $q = 1 - p$.

The distribution of heads and tails in coin tossing is an example of a Bernoulli distribution with $p = q = \frac{1}{2}$.

The Bernoulli distribution is one of the simplest discrete distributions, and it is the basis for other more complex discrete distributions. The definitions of a few types of distributions based on sequences of independent Bernoulli trials are summarized in the following table:

Distribution[1]	Definition
Binomial distribution	number of successes in n trials
Geometric distribution	number of failures before the first success
Negative binomial distribution	number of failures before the xth success

[1]These distributions will be discussed in more detail in the Options part.

In this part of the book, we will study the binomial distribution. The other two will be discussed in the options section.

Expected value and variance

The mean of a random variable X that has a Bernoulli distribution with parameter p is:

$$\begin{aligned} E(X) &= \sum_x xp(x) \\ &= 1(p) + 0(1 - p) = p \end{aligned}$$

The variance of X is:

$$\begin{aligned} Var(X) &= E(X^2) - (E(X))^2 \\ &= \sum_x x^2 p(x) - p^2 \\ &= 1^2 \cdot p + 0^2(1 - p) - p^2 \\ &= p - p^2 = p(1 - p) = pq \end{aligned}$$

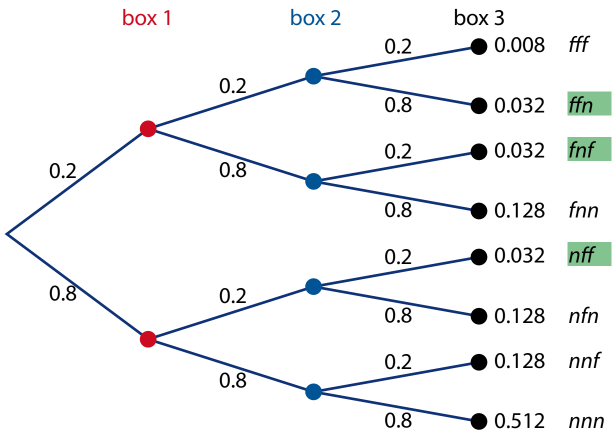
The binomial distribution

We will start our discussion of the binomial distribution with an example.

Suppose a cereal company puts miniature figures in boxes of cornflakes to make them attractive for children and thus boost sales. The manufacturer claims that 20% of the boxes contain a figure. You buy three boxes of this cereal. What is the probability that you'll get exactly three figures?

To get three figures means that the first box contains a figure (0.20 chance), as does the second (also 0.20), and the third (0.20). You want three figures; therefore, this is the intersection of three events and the probability is simply $0.20^3 = 0.008$.

If you want to calculate the probability of getting exactly two figures, the situation becomes more complicated. A tree diagram can help you visualize it better.



Let f stand for figure and n for no figure. There are three events of interest to us. Since we are interested in two figures, we want to see *ffn*, which has a probability of $0.2 \times 0.2 \times 0.8 = 0.2^2 \times 0.8 = 0.032$, and the other events of interest are *fnf* and *nff*, with probabilities $0.2 \times 0.8 \times 0.2 = 0.032$ and $0.8 \times 0.2 \times 0.2 = 0.032$.

Since the order of multiplication is not important, you see that three probabilities are the same. These three events are disjoint, as can be clearly seen from the tree diagram, and hence the probability of exactly two figures is the sum of the three numbers: $0.032 + 0.032 + 0.032$. Of course, you may realize by now that it would be much simpler if you wrote $3(0.032)$, since there are three events with the same probability.

What if you have five boxes?

The situation is similar, of course. However, a tree diagram would not be useful in this case, as there is too much information to assemble to see the solution. As you have seen above, no matter how you succeed in finding a figure, whether it is in the first box, the second or the third, it has the same probability, 0.2. So, to have two successes (finding figures) in the five boxes, you need the other three to be failures (no figures), with a probability of 0.8 for each failure. Therefore, the chance of having a case like *ffnnn* is $0.2^2 \times 0.8^3$. However, this can happen in several disjoint ways. How many? If you count them, you will find 10. This means the probability of having exactly two figures in five boxes is $10 \times 0.2^2 \times 0.8^3 = 0.2048$.
(Here are the 10 possibilities: *ffnnn*, *fnfnn*, *fnnfn*, *fnnnf*, *nffnn*, *nnffn*, *nnnff*, *nfnfn*, *nnfnf*, *nfnnf*.)

The number 10 is nothing but the *binomial* coefficient (Pascal's entry) you saw in Chapter 4. This is also the 'combination' of three events out of five.

The previous result can be written as $\binom{5}{2}0.2^2 \cdot 0.8^3$, where $\binom{5}{2}$ is the binomial coefficient.

You can find experiments like this one in many situations. Coin-tossing is only a simple example of this. Another very common example is opinion polls which are conducted before elections and used to predict voter preferences. Each sampled person can be compared to a coin – but a biased coin! A voter you sample in favour of your candidate can correspond to either a 'head' or a 'tail' on a coin. Such experiments all exhibit the typical characteristics of the **binomial experiment**.

A **binomial experiment** is one that has the following five characteristics:

1. The experiment consists of n identical trials.
2. Each trial has one of two outcomes. We call one of them success, S, and the other failure, F.
3. The probability of success on a single trial, p, is constant throughout the whole experiment. The probability of failure is $1 - p$, which is sometimes denoted by q. That is, $p + q = 1$.
4. The trials are independent.
5. We are interested in the number of successes x that are possible during the n trials. That is, $x = 0, 1, 2, \ldots, n$.

In the cereal company's example above, we started with $n = 3$ and $p = 0.2$ and asked for the probability of two successes, i.e. $x = 2$. In the second part, we have $n = 5$.

Let us imagine repeating a binomial experiment n times. If the probability of success is p, the probability of having x successes is $pppp\ldots$, x times (p^x), because the order is not important, as we saw before. However, in order to have exactly x successes, the rest, $(n - x)$ trials, must be failures, that is, with probability of $qqqq\ldots$, $(n - x)$ times (q^{n-x}). This is only one order (combination) where the successes happen the first x times and the rest are failures. In order to cater for 'all orders', we have to count the number of orders (combinations) possible. This is given by the binomial coefficient $\binom{n}{x}$.

We will state the following result without proof.

The binomial distribution

Suppose that a random experiment can result in two possible mutually exclusive and collectively exhaustive outcomes, 'success' and 'failure,' and that p is the probability of a success resulting in a single trial. If n independent trials are carried out, the distribution of the number of successes 'x' resulting is called the **binomial distribution**. Its probability distribution function for the binomial random variable X is:

$$\begin{aligned} \text{P}(x \text{ successes in } n \text{ independent trials}) &= \text{P}(x) = \binom{n}{x}p^x(1-p)^{n-x} \\ &= \binom{n}{x}p^x q^{n-x}, \text{ for } x = 0, 1, 2, \ldots, n. \end{aligned}$$

Notation:

The notation used to indicate that a variable has a binomial probability distribution with n trials and success probability of p is: $X \sim \text{B}(n, p)$.

Example 6

The computer shop orders its notebooks from a supplier, which like many suppliers has a rate of defective items of 10%. The shop usually takes a sample of 10 computers and checks them for defects. If they find two computers defective, they return the shipment. What is the probability that their random sample will contain two defective computers?

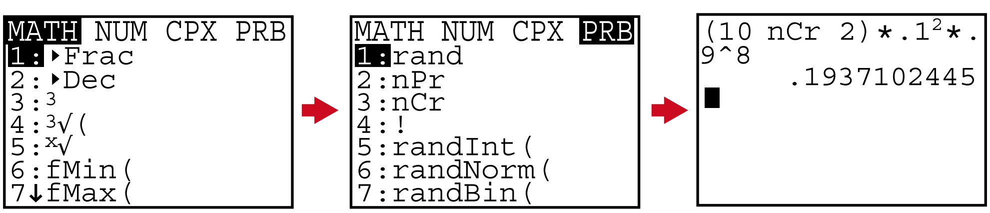
Solution

We will consider this to be a random sample and the shipment large enough to render the trials independent of each other. The probability of finding two defective computers in a sample of 10 is given by

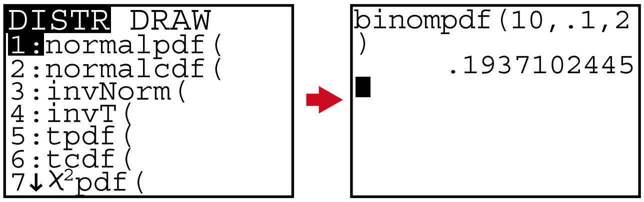
$$\text{P}(x = 2) = \binom{10}{2}0.1^2 0.9^{10-2} = 45 \times 0.01 \times 0.43047 = 0.194.$$

Of course, it is a daunting task to do all the calculations by hand. A GDC can do this calculation for you in two different ways.

The first possibility is to let the calculator do all the calculations in the formula above: Go to the math menu, then choose PRB, then go to #3.



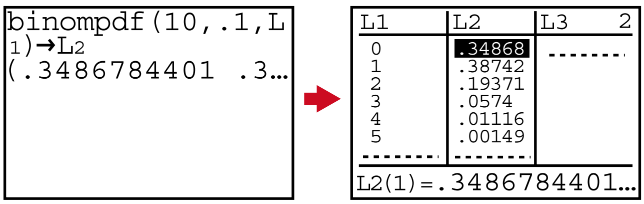
The second one is direct. We go to the 'DISTR' button, then scroll down to 'binompdf' and write down the two parameters followed by the number of successes:



Using a spreadsheet, you can also produce this result or even a set of probabilities covering all the possible values. The command used here for Excel is (BINOMDIST(B1:G1,10,0.1,FALSE)) which produced the table below:

x	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
P(x)	0.349	0.387	0.194	0.057	0.011	0.001	0.000	0.000	0.000	0.000	0.000

Similarly, the GDC can also give you a list of the probabilities:



Like other distributions, when you look at the binomial distribution, you want to look at its expected value and standard deviation.

Using the formula we developed for the expected value, $\sum x\text{P}(x)$, we can of course add $x\text{P}(x)$ for all the values involved in the experiment. The process would be long and tedious for something we can intuitively know. For example, in the defective items sample, if we know that the defective rate of the computer manufacturer is 10%, it is natural to *expect* to have $10 \times 0.1 = 1$ defective computer! If we have 100 computers with a defective rate of 10%, how many would you expect to be defective? Can you think of a reason why it would not be 10?

This is so simple that many people would not even consider it. The expected value of the successes in the binomial is actually nothing but the number of trials n multiplied by the probability of success, i.e. np!

The binomial probability model

n = number of trials

p = probability of success, probability of failure $q = 1 - p$

x = number of successes in n trials

$\text{P}(x) = \binom{n}{x}p^x(1-p)^{n-x} = \binom{n}{x}p^x q^{n-x}$, for $x = 0, 1, 2, \ldots, n$

Expected value $= \mu = np$

Variance $= \sigma^2 = npq$, $\sigma = \sqrt{npq}$

So, in the defective notebooks case, the expected number of defective items in the sample of 10 is $np = 10 \times 0.1 = 1$!

And the standard deviation is $\sigma = \sqrt{npq} = \sqrt{10 \times 0.1 \times 0.9} = 0.949$.

Question:
How do we know that the binomial distribution is a probability distribution?

Answer:
We can easily verify that the binomial distribution as developed satisfies the probability distribution conditions:

1. $0 \leqslant p(x) \leqslant 1$
2. $\sum_{x} p(x) = 1$

1. Since $p > 0$ by definition, then $p^x > 0$, for $x = 0, 1, 2, \ldots$. Similarly, $q^{n-x} > 0$. We also know that $\binom{n}{x} > 0$. Therefore,

$$p(x) = \binom{n}{x} p^x q^{n-x} > 0.$$

   $p(x) \leqslant 1$ will be a natural result of proving the second condition. If the sum of n positive parts is equal to 1, none of the parts can be greater than 1!

2. $\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$

   Recalling from Chapter 4, that the binomial theorem states

$$(p + q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} p(x).$$

   Since $p + q = 1$, then $(p + q)^n = 1$, and therefore

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p + q)^n = 1.$$

Expected value of the binomial (optional)

$$\mathrm{E}(X) = \sum_{x=0}^{n} x p(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

Notice that when $x = 0$, the first term in the summation equation is 0. Hence,

$$\mathrm{E}(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{(n-x)!x!} p^x q^{n-x} = \sum_{x=1}^{n} \frac{n!}{(n-x)!(x-1)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!(x-1)!} p p^{x-1} q^{n-x}$$

n and p are independent of x, so they can be factored out of the summation.

$$\mathrm{E}(X) = \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!(x-1)!}\, pp^{x-1}q^{n-x}$$

$$= np\sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!}\, p^{x-1}q^{n-x}$$

The term in the summation expression appears to be nothing but the probability of $(x-1)$ successes among $(n-1)$ trials.

$$\sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!}\, p^{x-1}q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{(n-1)!}{(n-1-(x-1)!(x-1))!}\, p^{x-1}q^{n-1-(x-1)}$$

If you replace $x-1$ by y and $n-1$ by m, then,

$$\sum_{x=1}^{n} \frac{(n-1)!}{(n-1-(x-1)!(x-1))!}\, p^{x-1}q^{n-1-(x-1)}$$

$$= \sum_{y=0}^{m} \frac{m!}{(m-y)!y!}\, p^{y}q^{m-y} = \sum_{y} p(y) = 1$$

This is nothing but the sum of all the probabilities of the random variable $Y = X - 1$ successes in $m = n - 1$ trials, and hence it is 1. Therefore,

$$\mu = \mathrm{E}(X) = np.$$

A slightly different manipulation of the summation rules will also be helpful to prove that

$$\sigma^2 = \mathrm{Var}(X) = npq.$$

The proof of both is optional and we will be content by providing you with the proof of the expected value only. Some of the references cited at the end of the book will contain detailed proofs of the variance formula.

Example 7

Among the studies carried out to examine the effectiveness of advertising methods, a study reported that 4 out of 10 web surfers remember advertisement banners after they have seen them.

a) If 20 web surfers are chosen at random and shown an ad, what is the expected number of surfers that would remember the ad?

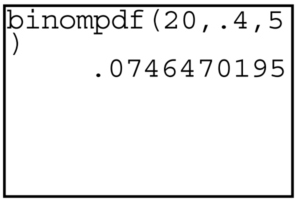
b) What is the chance that 5 of those 20 will remember the ad?

c) What is the probability that at most 1 surfer would remember the ad?

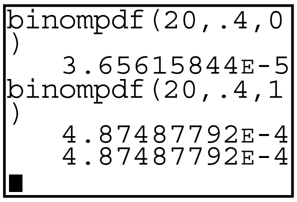
d) What is the chance that at least two surfers would remember the ad?

Solution

a) $X \sim (20, 0.4)$. The expected number is simply $20 \times 0.4 = 8$. We expect 8 of the surfers to remember the ad. Notice on the histogram below that the area in red corresponds to the expected value 8.

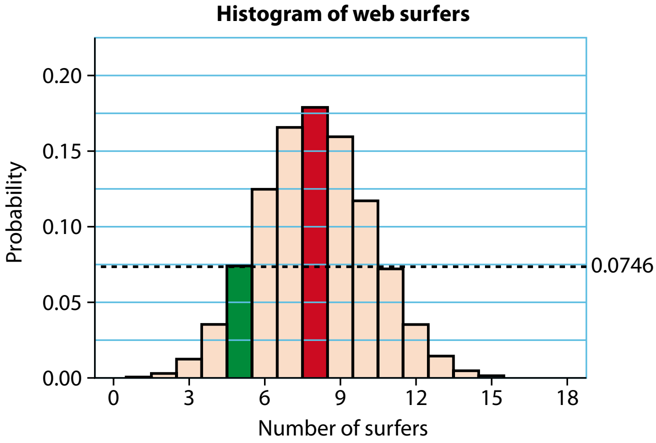


b) $P(5) = \binom{20}{5} 0.4^5 (0.6)^{15} = 0.0746$, or see the output from the GDC to the right. Graphically, this area is shown on the histogram as the green area.



c) $P(x \leqslant 1) = P(x = 0) + P(x = 1) = 0.000\,524$

d) $P(x \geqslant 2) = 1 - P(x \leqslant 1)$
$= 1 - 0.000\,524 = 0.999\,475$



The cumulative binomial distribution function

As you have seen in Section 17.1, the cumulative distribution function $F(x)$ of a random variable X expresses the probability that X does not exceed the value x. That is,

$$F(x) = P(X \leqslant x) = \sum_{y:y \leqslant x} p(y).$$

So, for the binomial distribution, the cumulative distribution function (cdf) is given by:

$$F(x) = P(X \leqslant x) = \sum_{y:y \leqslant x} p(y)$$

$$= \sum_{y:y \leqslant x} \binom{n}{y} p^y q^{n-y}$$

The cumulative distribution is very helpful when we need to find the probability that a binomial variable assumes values over a certain interval.

Example 8

A large shipment of light bulbs contains 4% defective bulbs. In a sample of 20 randomly selected bulbs from the shipment, what is the probability that

a) there are at most three defective bulbs?

b) there are at least 6 defective bulbs?

Solution

a) This can be considered as a binomial distribution with $n = 20$ and $p = 0.04$.

We need $P(x \leqslant 3)$, which we can calculate by either finding the probabilities for $x = 0, 1, 2$ and 3, and then add them, or by using the cumulative function. In both cases, we will use a GDC to produce the answers.

```
binompdf(20,0.04
,{0,1,2,3})
{.4420024339 .3…
Ans→L1
{.4420024339 .3…
sum(L1)
       .9925870629
```

```
binomcdf(20,0.04
,3)
      .9925870629
```

As you can see, using the cdf is a much more straightforward procedure.

b) Here we need $P(x \geqslant 6)$. The first approach is not feasible at all as we need to calculate 15 individual probabilities and add them. However, setting the problem as a complement and then using the cumulative distribution is much more efficient.

$$P(x \geqslant 6) = 1 - P(x < 6) = 1 - P(x \leqslant 5)$$

```
1-binomcdf(20,0.
04,5)
   9.765401703E-5
```

Exercise 17.2

1 Consider the following binomial distribution:

$$P(x) = \binom{5}{x}(0.6)^x(0.4)^{5-x},\ x = 0, 1, \ldots, 5$$

a) Make a table for this distribution.

b) Graph this distribution.

c) Find the mean and standard deviation in two ways:

(i) by formula

(ii) by using the table of values you created in part a).

d) Locate the mean μ and the two intervals $\mu \pm \sigma$ and $\mu \pm 2\sigma$ on the graph.

e) Find the actual probabilities for x to lie within each of the intervals $\mu \pm \sigma$ and $\mu \pm 2\sigma$ and compare them to the empirical rule.

2 A poll of 20 adults is taken in a large city. The purpose is to determine whether they support banning smoking in restaurants. It is known that approximately 60% of the population supports the decision. Let X represent the number of respondents in favour of the decision.

a) What is the probability that 5 respondents support the decision?

b) What is the probability that none of the 20 supports the decision?

c) What is the probability that at least 1 respondent supports the decision?

d) What is the probability that at least two respondents support the decision?

e) Find the mean and standard deviation of the distribution.

3 Consider the binomial random variable with $n = 6$ and $p = 0.3$.

a) Fill in the probabilities below.

k	0	1	2	3	4	5	6
$P(x \leqslant k)$							

b) Fill in the table below. Some cells have been filled for you to guide you.

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3				
At least 3				
More than 3	4, 5, 6	$P(x > 3)$	$1 - P(x \leqslant 3)$	0.070 47
Fewer than 3				
Between 3 and 5 (inclusive)				
Exactly 3				

4 Repeat question 3 with $n = 7$ and $p = 0.4$.

5 A box contains 8 balls: 5 are green and 3 are white, red and yellow. Three balls are chosen at random without replacement and the number of green balls Y is recorded.

a) Explain why Y is not a binomial random variable.

b) Explain why, when we repeat the experiment with replacement, then Y is a binomial.

c) Give the values of n and p and display the probability distribution in tabular form.

d) What is the probability that at most 2 green balls are drawn?

e) What is the expected number of green balls drawn?

f) What is the variance of the number of balls drawn?

g) What is the probability that some green balls will be drawn?

6 On a multiple choice test, there are 10 questions, each with 5 possible answers, one of which is correct. Nick is unaware of the content of the material and guesses on all questions.

a) Find the probability that Nick does not answer any question correctly.

b) Find the probability that Nick answers at most half of the questions correctly.

c) Find the probability that Nick answers at least one question correctly.

d) How many questions should Nick expect to answer correctly?

7 Houses in a large city are equipped with alarm systems to protect them from burglary. A company claims their system to be 98% reliable. That is, it will trigger an alarm in 98% of the cases. In a certain neighbourhood, 10 houses equipped with this system experience an attempted burglary.

a) Find the probability that all the alarms work properly.

b) Find the probability that at least half of the houses trigger an alarm.

c) Find the probability that at most 8 alarms will work properly.

8 Harry Potter books are purchased by readers of all ages! 40% of Harry Potter books were purchased by readers 30 years of age or older! 15 readers are chosen at random. Find the probability that

a) at least 10 of them are 30 or older

b) 10 of them are 30 or older

c) at most 10 of them are younger than 30.

9 A factory makes computer hard disks. Over a long period, 1.5% of them are found to be defective. A random sample of 50 hard disks is tested.

a) Write down the expected number of defective hard disks in the sample.

b) Find the probability that three hard disks are defective.

c) Find the probability that more than one hard disk is defective.

10 Car colour preferences change over time and according to the area the customer lives in and the car model he/she is interested in. In a certain city, a large dealer of BMW cars noticed that 10% of the cars he sells are 'metallic grey'. Twenty of his customers are selected at random, and their car orders are checked for colour. Find the probability that

a) at least five cars are 'metallic grey'

b) at most 6 cars are 'metallic grey'

c) more than 5 are 'metallic grey'

d) between 4 and 6 are 'metallic grey'

e) more than 15 are not 'metallic grey'.

In a sample of 100 customer records, find

f) the expected number of 'metallic grey' car orders

g) the standard deviation of 'metallic grey' car orders.

According to the empirical rule, 95% of the 'metallic grey' orders are between a and b.

h) Find a and b.

11 Dogs have health insurance too! Owners of dogs in many countries buy health insurance for their dogs. 3% of all dogs have health insurance. In a random sample of 100 dogs in a large city, find

a) the expected number of dogs with health insurance

b) the probability that 5 of the dogs have health insurance

c) the probability that more than 10 dogs have health insurance.

12 A balanced coin is tossed 5 times. Let X be the number of heads observed.

a) Using a table, construct the probability distribution of X.

b) What is the probability that no heads are observed?

c) What is the probability that all tosses are heads?

d) What is the probability that at least one head is observed?

e) What is the probability that at least one tail is observed?

f) Given that the coin is unbalanced in such a way that it shows 2 heads in every 10 tosses, answer the same questions above.

13 When John throws a stone at a target, the probability that he hits the target is 0.4. He throws a stone 6 times.

a) Find the probability that he hits the target exactly 4 times.

b) Find the probability that he hits the target for the first time on his third throw.

14 On a television channel the news is shown at the same time each day. The probability that Alice watches the news on a given day is 0.4. Calculate the probability that on five consecutive days, she watches the news on at most three days.

15 A satellite relies on solar cells for its power and will operate provided that at least one of the cells is working. Cells fail independently of each other, and the probability that an individual cell fails within one year is 0.8.

a) For a satellite with ten solar cells, find the probability that all ten cells fail within one year.

b) For a satellite with ten solar cells, find the probability that the satellite is still operating at the end of one year.

c) For a satellite with n solar cells, write down the probability that the satellite is still operating at the end of one year. Hence, find the smallest number of solar cells required so that the probability of the satellite still operating at the end of one year is at least 0.95.

17.3 Poisson distribution

The Poisson distribution arises when you count a number of events across time or over an area. You should think about the Poisson distribution for any situation that involves counting events. Some examples are:

- the number of emergency visits by an infant during the first year of life,
- the number of white blood cells found in a cubic centimetre of blood
- the number of sample defects on a car
- the number of typographical errors on a page
- the number of failures in a large computer system during a given day
- the number of delivery trucks arriving at a central warehouse in an hour
- the number of customers arriving for flights during each 15-minute time interval from 3:00 p.m. to 6:00 p.m. on weekdays
- the number of customers arriving at a checkout aisle in your local grocery store during a particular time interval.

Sometimes, you will see the count represented as a rate, such as

- the number of injuries per year due to horse kicks, or
- the number of defects per square metre.

So, in general, the Poisson distribution is used when measuring the number of occurrences of 'something' (number of successes) over an interval, or time period.

Four assumptions

Information about how the data was generated can help you decide whether the Poisson distribution fits. Assume that an interval is divided into a very large number of sub-intervals so that the probability of the occurrence of an event in any sub-interval is very small. The Poisson distribution is based on four assumptions. We will use the term 'interval' to refer to either a time interval or an area, depending on the context of the problem.

1. The probability of observing a single event over a small interval is approximately proportional to the size of that interval.
2. The probability of two events occurring in the same narrow interval is negligible.
3. The probability of an event within a certain interval does not change over different intervals.
4. The probability of an event in one interval is independent of the probability of an event in any other non-overlapping interval.

Notation
If a random variable follows a Poisson distribution, we write $X \sim P_0(\mu)$.

You should examine all of these assumptions carefully, but especially the last two. If either of these last two assumptions is violated, they can lead to extra variation, sometimes referred to as overdispersion.

Mathematical details

The Poisson distribution depends on a single parameter μ. The probability that the Poisson random variable equals x is

$$P(X = x) = e^{-\mu}\frac{\mu^x}{x!}$$

where μ is the average number of events observed over the specific interval, and x is the number of 'successes' we are interested in. μ can be any positive real number, while x has to be a positive integer. We will show that the parameter is actually the expected value below.

Note: IBO uses m instead of μ for the Poisson parameter.

Only for the curious!
The Poisson probability is related to the binomial probability. Here is a justification.

Suppose we want to find the probability distribution of the number of telephone calls to the front desk of a large company over a period of one hour. Think of the one hour as being split into n sub-intervals, each of which is small enough that at most one call could arrive within it with a probability p.

This means:

$$P(\text{one call}) = p$$

$$P(\text{no call}) = 1 - p, \text{ i.e. } P(\text{ more than one call}) = 0$$

This in turn will mean that the total number of calls received within one hour is equal to the total number of sub-intervals that contain one call! Hence, if we can consider the calls to be arriving independently of each other from one interval to the other, then the distribution of the number of calls per hour is a binomial distribution.

As n increases, p will become smaller. Since the binomial distribution has its expected value $\mu = np$, and considering the probability of x successes within one hour as n increases indefinitely, that is, $n \to \infty$, we have:

$$p(x) = \binom{n}{x}p^x(1-p)^{n-x}$$

$$\Rightarrow \lim_{n\to\infty}\binom{n}{x}p^x(1-p)^{n-x} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\ldots(n-x+1)}{x!}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^{n-x}$$

This is so, because if $\mu = np$, then $p = \frac{\mu}{n}$ and $1 - p = 1 - \frac{\mu}{n}$.

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-x+1)}{x!}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-x+1)}{x!}\frac{\mu^x}{n^x}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{\mu^x}{x!}\left(1-\frac{\mu}{n}\right)^n \frac{n(n-1)(n-2)\dots(n-x+1)}{n^x}\left(1-\frac{\mu}{n}\right)^{-x}$$

Now, since $\frac{\mu^x}{x!}$ is independent of n, then it can be factored out of the limit expression and we are left with:

$$\frac{\mu^x}{x!}\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n \frac{n(n-1)(n-2)\dots(n-x+1)}{n^x}\left(1-\frac{\mu}{n}\right)^{-x}$$

But, using the theorem that the limit of a product is the product of limits, then:

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n \frac{n(n-1)(n-2)\dots(n-x+1)}{n^x}\left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n \cdot \lim_{n\to\infty}\frac{n(n-1)(n-2)\dots(n-x+1)}{n^x} \cdot \lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x}$$

Also,

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = e^{-\mu};$$

$$\lim_{n\to\infty}\frac{n(n-1)(n-2)\dots(n-x+1)}{n^x} = \lim_{n\to\infty}\frac{n}{n}\cdot\frac{n-1}{n}\dots\frac{n-x+1}{n} = 1\cdot 1\dots 1 = 1$$

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^{-x} = 1^{-x} = 1$$

So, finally, we have

$$p(x) = \binom{n}{x}p^x(1-p)^{n-x} = \frac{\mu^x}{x!}\cdot e^{-\mu}\cdot 1\cdot 1 = e^{-\mu}\frac{\mu^x}{x!}.$$

Expected value of the Poisson distribution

Like any discrete distribution, the Poisson distribution has an expected value that can be found using the definition of expected value developed earlier.

$$\mathrm{E}(X) = \sum_{x=0}^{\infty} xp(x) = \sum_{x=0}^{\infty} xe^{-\mu}\frac{\mu^x}{x!}$$

Notice that when $x = 0$, the first term in the summation equation is 0. Hence

$$\mathrm{E}(X) = \sum_{x=1}^{\infty} xe^{-\mu}\frac{\mu^x}{x!} = \sum_{x=1}^{\infty} e^{-\mu}\frac{\mu\cdot\mu^{x-1}}{(x-1)!} = \mu\sum_{x=1}^{\infty} e^{-\mu}\frac{\mu^{x-1}}{(x-1)!}$$

We factored μ out of the summation as it is independent of x. Now notice that the expression in the summation is nothing but the total of all probabilities of the variable $x - 1$ over its domain. So, if we replace $x - 1$ by y, the expression in the summation formula becomes

$$\sum_{x=1}^{\infty} e^{-\mu}\frac{\mu^{x-1}}{(x-1)!} = \sum_{y=0}^{\infty} e^{-\mu}\frac{\mu^y}{y!} = 1$$

(since subtracting 1 from ∞ does not change anything.)

Therefore,

$$\mathrm{E}(X) = \mu \cdot \sum_{y=1}^{\infty} e^{-\mu} \frac{\mu^y}{y!} = \mu.$$

Also, since we considered the Poisson as a binomial when n tends to infinity, the variance can be verified to be μ in the following manner.

$$\mathrm{V}(X) = npq = n\left(\frac{\mu}{n}\right)\left(1 - \frac{\mu}{n}\right) = \mu\left(1 - \frac{\mu}{n}\right)$$

But as $n \to \infty, \frac{\mu}{\mathrm{n}} \to 0$; hence,

$$\mathrm{V}(X) = \mu\left(1 - \frac{\mu}{n}\right) = \mu.$$

Note: Observe that for a Poisson distribution, $\mathrm{E}(X) = \mathrm{V}(X) = \mu$.

Question:
How do we know that the Poisson distribution is a probability distribution?

Answer:
We can easily verify that Poisson as developed satisfies the probability distribution conditions:

1. $0 \leqslant p(x) \leqslant 1$
2. $\sum_x p(x) = 1$

1. Since $\mu > 0$, it is obvious that $p(x) > 0$, for $x = 0, 1, 2, \ldots$.
   $p(x) \leqslant 1$ will be a natural result of proving the second condition. If the sum of n positive parts is equal to 1, none of the parts can be greater than 1!
2. $$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} e^{-\mu} \frac{\mu^x}{x!} = e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$

But, $\sum_{x=0}^{\infty} \frac{\mu^x}{x!} = e^{\mu}$.

This is proved in Option 10: Series and differential equations.

This is a Taylor expansion of e^x: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$; hence,

$$\sum_{x=0}^{\infty} p(x) = e^{-\mu} e^{\mu} = 1.$$

Example 9

Police use speed cameras to record violations of speed limits. At a strategic spot in a city, the installed camera automatically turns itself on, on average once every 10 minutes. The pattern follows, approximately, a Poisson distribution.

a) Within a 10-minute interval, what is the chance that it is on
   (i) once
   (ii) twice
   (iii) at least once?

b) Within an hour, what is the chance that it is on

(i) once

(ii) twice

(iii) at least once?

Solution

a) For this part the interval is 10 minutes and $\mu = 1$. Then,

(i) $P(x = 1) = e^{-1}\frac{1^1}{1!} = e^{-1} = 0.368$

(ii) $P(x = 2) = e^{-1}\frac{1^2}{2!} = \frac{e^{-1}}{2} = 0.184$

(iii) $P(\text{at least once}) = 1 - P(\text{at most } 0) = 1 - P(x = 0)$

$= 1 - e^{-1}\frac{1^0}{0!} = 1 - e^{-1} = 0.632$

b) Here the expected value $\mu = 6$.

(i) $P(x = 1) = e^{-6}\frac{6^1}{1!} = 6e^{-6} = 0.0149$

(ii) $P(x = 2) = e^{-6}\frac{6^2}{2!} = e^{-6}\frac{36}{2} = 0.0446$

(iii) $P(\text{at least once}) = 1 - P(\text{at most } 0) = 1 - P(x = 0)$

$= 1 - e^{-6}\frac{6^0}{0!} = 1 - e^{-6} = 0.998$

The results above can of course be calculated directly using your GDC.

```
poissonpdf(1,1)
        .3678794412
poissonpdf(1,2)
        .1839397206
1-poissonpdf(1,0)
        .6321205588
```

```
poissonpdf(6,1)
        .0148725131
poissonpdf(6,2)
        .0446175392
1-poissonpdf(6,0)
        .9975212478
```

Cumulative Poisson distribution function

As you have seen in Sections 17.1 and 17.2, it is more practical in several situations to start with a cumulative distribution in order to calculate probabilities of several consecutive values. The cumulative Poisson distribution function plays the same role introduced in the previous sections.

The cumulative Poisson distribution function of a Poisson random variable X expresses the probability that X does not exceed a value x.

$$P(x \leqslant x) = \sum_{i=0}^{x} e^{-\mu}\frac{\mu^i}{i!}$$

Example 10

The number of radioactive particles released per minute from a meteoroid after it enters the atmosphere is recorded and the average is found to be 3.5 particles per minute. Find the probability that in any one minute there are at least 5 particles released.

Solution

This appears to be a Poisson model. We first set it up to make the calculation through the cumulative distribution function.

$$P(x \geqslant 5) = 1 - P(x \leqslant 4), \text{ and so}$$

```
1-Poissoncdf(3.5
,4)
        .274555047
```

That is, there is a probability of 27.5% that at least 5 particles are emitted.

Example 11

Small aircraft arrive at a certain airport according to a Poisson process with rate of 10 per hour.

a) What is the probability that during a 1-hour period
   (i) 8 small aircraft arrive?
   (ii) at most 8 small aircraft arrive?
   (iii) at least 9 small aircraft arrive?

b) What is the expected value and standard deviation of the number of small aircraft that arrive during a 90-minute period?

c) What is the probability that at least 1 small aircraft arrives during a 6-minute period?

d) What is the probability that 1 small aircraft arrives during two 6-minute separate periods?

e) What is the probability that 1 small aircraft arrives during a 12-minute period?

Solution

a) (i) $Po(x = 8|\mu = 10) = e^{-10}\dfrac{10^8}{8!} \approx 0.113$

```
PoissonPdf(10,8)
,4)
        .112599032
Poissoncdf(10,8)

        .332819679
```

(ii) $Po(x \leqslant 8|\mu = 10) = \sum_{x=0}^{8} e^{-10}\dfrac{10^x}{x!} \approx 0.333$

(iii)

```
1-Poissoncdf(10,
8)
        .667180321
```

$1 - Po(x \leqslant 8|\mu = 10) = 1 - \sum_{x=0}^{8} e^{-10}\dfrac{10^x}{x!} \approx 0.667$

b) A 90-minute period is 1.5 hours.
So, the expected value is $1.5 \times 10 = 15$ and the standard deviation is $\sqrt{15} = 3.87$. Recall that $V(X) = \mu$.

c) During a 6-minute period, the expected value is $\lambda = \frac{\mu}{10} = 1$, and

$$\begin{aligned} Po(x \geqslant 1 | \lambda = 1) &= 1 - Po(x = 0 | \lambda = 1) = 1 - e^{-1}\frac{1^0}{0!} \\ &= 1 - e^{-1} \approx 0.632. \end{aligned}$$

d) This event consists of two simple events: either 1 plane the first period and no plane the second, or no plane the first period and 1 plane the second.
Hence, [let $P(a, b)$ be the probability of a planes first and b planes second]

$$\begin{aligned} P(1 \text{ plane in two 6-minute periods}) &= P(1, 0)P(0, 1) + P(0, 1)P(1, 0) \\ &= e^{-1}\frac{1^1}{1!} \cdot e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^0}{0!} \cdot e^{-1}\frac{1^1}{1!} \\ &= 2e^{-2} \approx 0.271. \end{aligned}$$

e) Here the expected value is 2 aircraft, and hence

$$Po(x = 1 | \mu = 2) = e^{-2}\frac{2^1}{2!} \approx 0.135.$$

Exercise 17.3

1 Let X denote a random variable that has a Poisson distribution with mean $\mu = 3$. Find the following probabilities, both manually and with a GDC:

a) $P(x = 5)$

b) $P(x < 5)$

c) $P(x \geqslant 5)$

d) $P(x \geqslant 5 | x \geqslant 3)$

2 Let X denote a random variable that has a Poisson distribution with mean $\mu = 5$. Find the following probabilities, both manually and with a GDC:

a) $P(x = 5)$

b) $P(x < 4)$

c) $P(x \geqslant 4)$

d) $P(x \leqslant 6 | x \geqslant 4)$

3 The number of support phone calls coming into the central switchboard of a small computer company averages 6 per minute.

a) Find the probability that no calls will arrive in a given one-minute period.

b) Find the probability that at least two calls will arrive in a given one-minute period.

c) Find the probability that at least two calls will arrive in a given 2-minute period.

4 DVDs are tested by sending them through an analyzer that measures imbalance, using accepted industry standards. A brand of DVDs is known to have an error score of 0.1 per DVD, which is within the acceptable standards.

a) Find the probability that the next inspected DVD will have no error.

b) Find the probability that the next inspected DVD will have more than one error.

c) Find the probability that neither of the next two inspected DVDs will have any error.

5 In 2000, after an extensive study of road safety, Japan decided to set a maximum speed limit on their expressways of 100 km/h. In the study, it was reported that the number of deaths and serious injuries on expressways for regular passenger vehicles was 0.024 per million vehicle–kilometres.

a) (i) Find the probability that at most 15 serious incidents happen in a given block of 10^9 vehicle–kilometres.

(ii) Find the probability that at least 20 serious incidents happen in a given block of 10^9 vehicle–kilometres.

b) The rate for light motor vehicles was 0.036.

(i) Find the probability that at most 15 serious incidents happen in a given block of 10^9 vehicle–kilometres.

(ii) Find the probability that at least 20 serious incidents happen in a given block of 10^9 vehicle–kilometres.

6 Passengers arrive at a security checkpoint in a busy airport at the rate of 8 per 10-minute period. For the time between 8:00 and 8:10 on a specific day, find the probability that

a) 8 passengers arrive

b) no more than 5 passengers arrive

c) at least 4 passengers arrive.

7 In question 6 above, find each of the following probabilities.

a) The probability that three passengers arrive between 8:00 and 8:20.

b) The probability that three passengers arrive between 8:00 and 8:10 and 9:00 and 9:10.

8 A certain internet service website receives on average 0.2 hits per second. It is known that the number of hits on this site follows a Poisson distribution.

a) Find the probability that no hits are registered during the next second.

b) Find the probability that no hits are registered for the next 3 seconds.

9 The number of faults in the knit of a certain fabric has an average of 4.4 faults per square metre. It is also assumed to have a Poisson distribution.

a) Find the probability that a 1 m^2 piece of this fabric contains at least 1 fault.

b) Find the probability that a 3 m^2 piece of this fabric contains at least 1 fault.

c) Find the probability that three 1 m^2 pieces of this fabric contain 1 fault.

10 A supplier of copper wire looks for flaws before despatching it to customers. It is known that the number of flaws follows a Poisson probability distribution with a mean of 2.3 flaws per metre.

a) Determine the probability that there are exactly 2 flaws in 1 metre of the wire.

b) Determine the probability that there is at least one flaw in 2 metres of the wire.

11 a) Patients arrive at random at an emergency room in a hospital at the rate of 15 per hour throughout the day. Find the probability that 6 patients will arrive at the emergency room between 08:00 and 08:15.

b) The emergency room switchboard has two operators. One operator answers calls for doctors and the other deals with enquiries about patients. The first operator fails to answer 1% of her calls and the second operator fails to answer 3% of his calls. On a typical day, the first and second telephone operators receive 20 and 40 calls respectively during an afternoon session. Using the Poisson distribution find the probability that, between them, the two operators fail to answer two or more calls during an afternoon session.

12 The random variable X is Poisson distributed with mean μ and satisfies $P(x = 3) = P(x = 0) + P(x = 1)$.

a) Find the value of μ, correct to four decimal places.

b) For this value of μ evaluate $P(2 \leqslant x \leqslant 4)$.

13 Give all numerical answers to this question correct to three significant figures. Two typists were given a series of tests to complete. On average, Mr Brown made 2.7 mistakes per test while Mr Smith made 2.5 mistakes per test. Assume that the number of mistakes made by any typist follows a Poisson distribution.

a) Calculate the probability that, in a particular test,

(i) Mr Brown made two mistakes

(ii) Mr Smith made three mistakes

(iii) Mr Brown made two mistakes and Mr Smith made three mistakes.

b) In another test, Mr Brown and Mr Smith made a combined total of five mistakes. Calculate the probability that Mr Brown made fewer mistakes than Mr Smith.

17.4 Continuous distributions

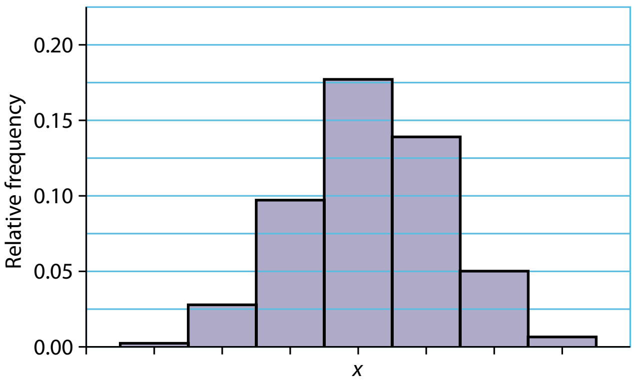
Continuous random variables

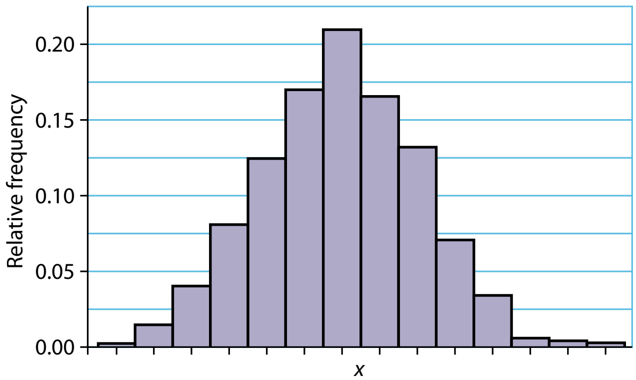
When a random variable X is discrete, you assign a positive probability to each value that X can take and get the probability distribution for X. The sum of all the probabilities associated with the different values of X is 1.

You have seen, in the discrete variable case, that we graphically represent the probabilities corresponding to the different values of the random variable X with a probability histogram (relative frequency histogram), where the area of each bar corresponds to the probability of the specific value it represents.

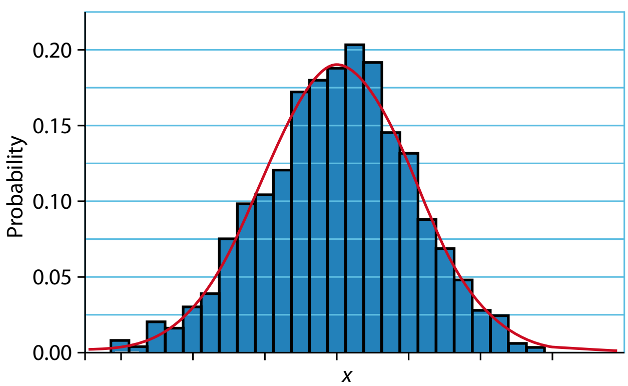
Consider now a continuous random variable X, such as height and weight, and length of life of a particular product – a TV set for example. Because it is continuous, the possible values of X are over an interval. Moreover, there are an infinite number of possible values of X. Hence, we cannot find a probability distribution function for X by listing all the possible values of X along with their probabilities, as you see in the histogram on the next page. If we try to assign probabilities to each of these uncountable values, the probabilities will no longer sum to 1, as is the case with discrete variables. Therefore, you must use a different approach to generate the probability distribution for such random variables.

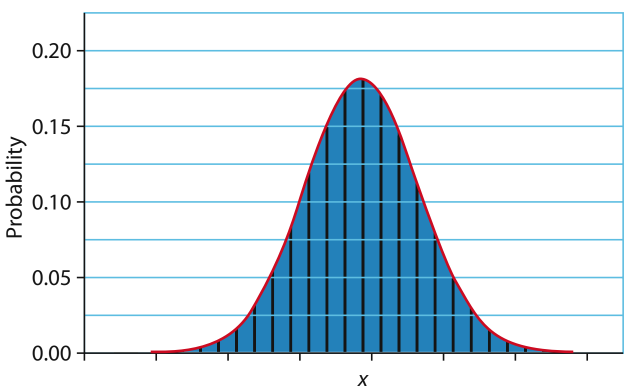
Suppose that you have a set of measurements on a continuous random variable, and you create a relative frequency histogram to describe their distribution. For a small number of measurements, you can use a small number of classes, but as more and more measurements are collected, you can use more classes and reduce the **class width**.





The histogram will slightly change as the class width becomes smaller and smaller, as shown in the diagrams below. As the number of measurements becomes very large and the class width becomes very narrow, the relative frequency histogram appears more and more like the smooth curve you see below. This is what happens in the continuous case, and the smooth curve describing the probability distribution of the continuous random variable becomes the **PDF** (**probability density function**) of X, represented by a curve $y = f(x)$. This curve is such that the entire area under the curve is 1 and the area between any two points is the probability that x falls between those two points.



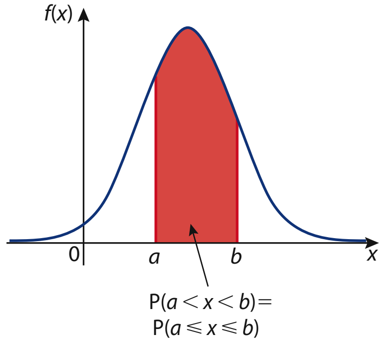


Probability density function

Let X be a continuous random variable. The probability density function, $f(x)$, of the random variable is a function with the following properties:

1. $f(x) > 0$ for all values of x.

2. The area under the probability density function $f(x)$ over all values of the random variable X is equal to 1.0, i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
3. Suppose this density function is graphed. Let a and b be two possible values of the random variable X, with $a < b$. Then the probability that x lies between a and b [$P(a < x < b)$] is the area under the density function between these points.



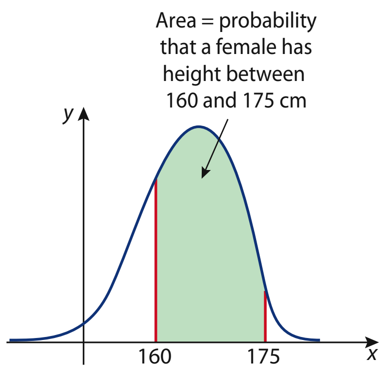
Notice that, based on this definition, the probability that x equals any point a is 0. This is so because the area above a value, say a, is a rectangle whose width is 0 or equivalently

$$P(X = a) = \int_a^a f(x)\,dx = 0.$$

So, for the continuous case, regardless of whether the endpoints a and b are themselves included, the area included between a and b is the same.

$$P(a < x < b) = P(a \leqslant x \leqslant b) = P(a \leqslant x < b) = P(a < x \leqslant b)$$

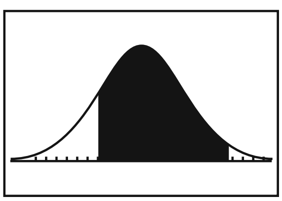
For example, the graph shows a model for the pdf f for a random variable X defined to be the height, in cm, of an adult female in Spain. The probability that the height of a female chosen at random from this population is between 160 and 175 is equal to the area under the curve between 160 and 175.

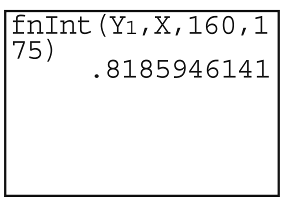


The function represented here is:

$$f(x) = \int_{160}^{175} \frac{e^{\frac{(x-165)^2}{50}}}{5\sqrt{2\pi}}\,dx$$

As you know from your integral calculus class, it is not an integral you can calculate exactly. We use a GDC to approximate it.





So, the chance of choosing a female at random with a height between 160 cm and 175 cm is approximately 81.9%.

Example 12

$f(x)$ as defined below describes a random variable X.

$$f(x) = \begin{cases} \frac{1}{512}(12x^2 - x^3 - 20x) & 2 \leqslant x \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

a) Verify that $f(x)$ is a probability density function.

b) Find $P(5 \leqslant x \leqslant 8)$.

Solution

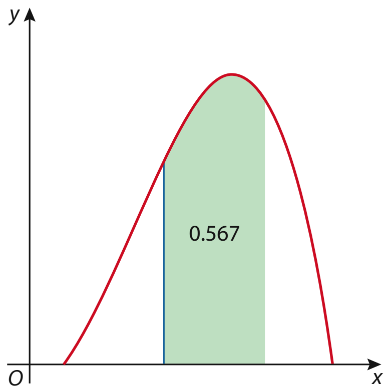
a) For $2 \leqslant x \leqslant 10$ we have $12x^2 - x^3 - 20x \geqslant 0$, so $f(x) \geqslant 0$.

We also need to check that $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_2^{10} \tfrac{1}{512}(12x^2 - x^3 - 20x)dx \\ &= \tfrac{1}{2048}(16x^3 - x^4 - 40x^2)]_2^{10} \\ &= \tfrac{1}{2048}(16\,000 - 100\,000 - 4000 - 128 + 16 + 160) = 1\end{aligned}$$

Therefore, $f(x)$ is a pdf.

b)



The probability that x lies between 5 and 8 is

$$\begin{aligned}\int_5^8 (12x^2 - x^3 - 20x)dx &= \tfrac{1}{2048}(16x^3 - x^4 - 40x^2)]_5^8 \\ &= \tfrac{1}{2048}(1536 - 375) \\ &= \tfrac{1161}{2048} \approx 0.567.\end{aligned}$$

Example 13

Find the value of k such that the following represents a probability density function of a random variable X.

$$f(x) = \begin{cases} kx^2(2 - x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution

For $f(x)$ to be a pdf, we need to satisfy both conditions.

a) For $0 \leqslant x \leqslant 2$ we must have $kx^2(2 - x) \geqslant 0$, and since $x^2 \geqslant 0$, then k must be positive.

b) For $\int_{-\infty}^{\infty} f(x)dx = 1$, then

$\int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2(2 - x)dx = 1$, and hence

$\int_0^2 kx^2(2 - x)dx = k\left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)\Big]_0^2 = 1$, and this in turn leads to

$k\left(\frac{16}{3} - \frac{16}{4}\right) = 1 \Rightarrow k\left(\frac{4}{3}\right) = 1$, and $k = \frac{3}{4}$.

Cumulative distribution functions

You have met the idea of the cumulative distribution functions for discrete random variables in Sections 17.1 to 17.3. In the same way, and using the fact that an integral is the limit of a sum, we have the following definition.

A **cumulative distribution function**, $F(x)$, of a random variable X with a density function $f(t)$ is defined by

$F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)dt$, where x is a value in the domain of the function $f(t)$.

$F(x)$ gives us the proportion of the population having values smaller than x.
Note here that $F(x)$ is an anti-derivative of $f(x)$, that is, $F'(x) = f(x)$.
Any distribution function has the following properties:

1. $F(x)$ is non-decreasing.
2. $\lim_{x \to \infty} F(x) = 0$; $\lim_{x \to -\infty} F(x) = 0$.
3. Since $P(a < x < b) = \int_a^b f(x)dx$, then $P(a < x < b) = \int_a^b f(x)dx = F(b) - F(a)$.

Note: The lower limit of integration is given as $-\infty$, but in essence, it is the smallest possible value of x.

Measures of centre, position and spread of a continuous distribution

Like discrete distributions, continuous distributions have their characteristics including **mean**, **median**, **mode**, **variance** and the **percentiles**. Next we will discuss each of them in more detail.

Mean

Recall that for a discrete random variable

$E(X) = \sum_x xp(x)$. Similarly, if we have a continuous random variable X with a *pdf* $f(x)$, then

$$E(X) = \int_x xf(x)dx.$$

$E(X)$ is called the expected value of X and it is also referred to as the mean μ.

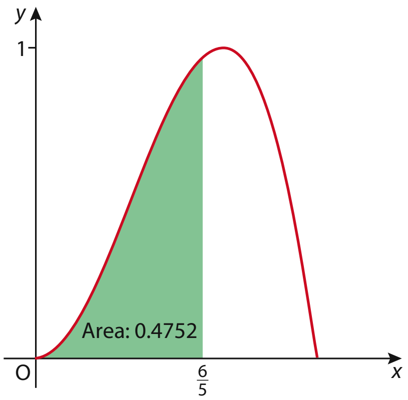
Example 14

The function $f(x)$ is a pdf for a random variable X.

$$f(x) = \begin{cases} \frac{3}{4}x^2(2 - x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find μ.

b) Find $P(x < \mu)$.



Solution

a) $\mu = \mathrm{E}(X) = \int_x xf(x)\,dx = \int_0^2 \frac{3}{4}x^3(2-x)\,dx = \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\cdot\frac{8}{5} = \frac{6}{5}$

b) $\mathrm{P}\left(x \leqslant \frac{6}{5}\right) = \int_0^{\frac{6}{5}} \frac{3}{4}x^2(2-x)\,dx = \frac{3}{4}\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^{\frac{6}{5}} = \frac{3}{4}\cdot\frac{386}{624} = 0.4752$

Mode

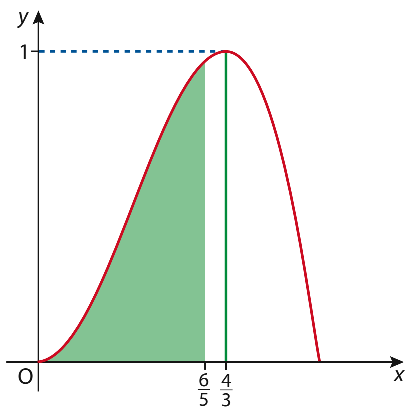
The mode, as you know, is the value of X for which $f(x)$ is the largest in the given domain of X.

To locate the mode, you may first draw a graph of $f(x)$ and then use the first or first and second derivative tests to find the maximum. Just recall that the maximum can happen at critical points.

So, in the previous example, it appears that the mode is slightly higher than the mean.

$$f(x) = \begin{cases} \frac{3}{4}x^2(2-x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow f'(x) = 3x - \frac{9}{4}x^2;\ f''(x) = 3 - \frac{9}{2}x$$

Now, $f'(x) = 0 \Rightarrow x = 0, x = \frac{4}{3}$, and at $x = 0, f(x) = 0$, while at $x = \frac{4}{3}, f''(x) = -3 < 0$, which means that $f(x)$ has a maximum.
So, the mode is at $x = \frac{4}{3}$.



Example 15

The random variable X has a pdf defined by

$$f(x) = \begin{cases} k(-x^2 + 2x + 15); & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k.

b) Determine the mean and the mode.

c) Find the value, m, of the random variable which is larger than 50% of the population.

Solution

a) The area under the curve must be equal to 1:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{5} k(-x^2 + 2x + 15)\,dx = 1$$

$$k\left[-\frac{x^3}{3} + x^2 + 15x\right]_0^5 = k\frac{175}{3} = 1 \Rightarrow k = \frac{3}{175}$$

b) $\mu = \displaystyle\int_0^5 x\left(\frac{3}{175}(-x^2 + 2x + 15)\right)dx = \frac{3}{175}\int_0^5 (-x^3 + 2x^2 + 15x)\,dx$

So, the mean is

$$\frac{3}{175}\left[-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{15x^2}{2}\right]_0^5 = \frac{55}{28}.$$

$f'(x) = 0 \Rightarrow -2x + 2 = 0$, and so, the mode is $x = 1$.

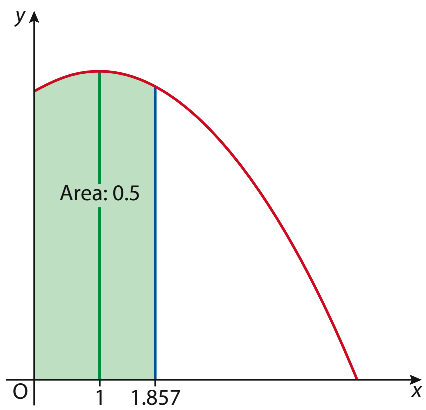
c) The value of m can be found by finding the value where the area to the left of it under the pdf is 0.5. Hence,

$$\int_{-\infty}^{m} f(x)\,dx = \int_0^m \frac{3}{175}(-x^2 + 2x + 15)\,dx = 0.5$$

$$\Rightarrow \frac{3}{175}\left[-\frac{x^3}{3} + x^2 + 15x\right]_0^m = \frac{1}{2}$$

$$\Rightarrow x \approx 1.857$$

Note: This is the method used next to find the median of the data.



Median and percentiles

The median for a random variable X that has a pdf $f(x)$ is a value m of the random variable such that 50% of the values of X are less than or equal to m. (Similarly, 50% of the values are larger than or equal to m). Thus, the median m satisfies

$$\int_{-\infty}^{m} f(x)\,dx = 0.5.$$

In other words, half of the area under the pdf lies to the left of m.

Example 16

The function $f(x)$ is a pdf for a random variable X.

$$f(x) = \begin{cases} \frac{3}{4}x^2(2 - x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the median m.

b) Find $P(x < m)$.

Solution

a) To find the median, we set up the integral as given by the definition,

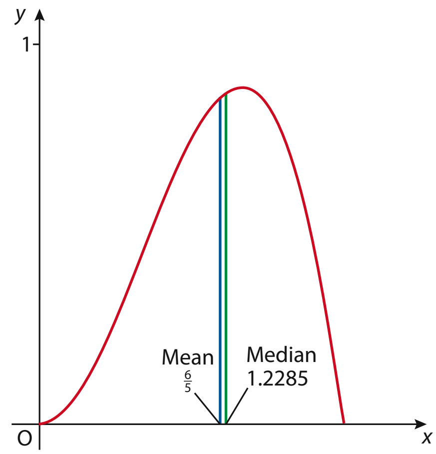
$$\int_{-\infty}^{m} f(x)\,dx = \int_{0}^{m} \frac{3}{4}x^2(2-x)\,dx = 0.5,$$ and then we solve for m,

$$\int_{0}^{m} \frac{3}{4}x^2(2-x)\,dx = \frac{3}{4}\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^m = \frac{3}{4}\left(\frac{2}{3}m^3 - \frac{m^4}{4}\right) = 0.5.$$

In the interval [0, 2], the solution is approx. 1.2285.

b) $P(x < 1.2285) = \int_{0}^{1.2285} \frac{3}{4}x^2(2-x)\,dx = 0.4999 \approx 0.5.$

This result confirms the calculation that 1.2285 is the median. Notice how the median is to the right of the mean as the distribution is slightly skewed to the left!



The **percentiles** can also be found in a similar manner. The kth percentile can be defined as the value n of the random variable such that *k% of the values of X are less than or equal to n*. Thus, the ***k*th percentile** satisfies

$$\int_{-\infty}^{n} f(x)\,dx = k\%.$$

Example 17

In the previous example,

a) find the first quartile

b) find the third quartile and the IQR.

Solution

a) The first quartile is the 25th percentile, and hence

$$\int_{0}^{n} \frac{3}{4}x^2(2-x)\,dx = \frac{3}{4}\left(\frac{2}{3}n^3 - \frac{n^4}{4}\right) = 0.25.$$

In the interval [0, 2], the solution is approx. **0.913**.

b) The third quartile is the 75th percentile, and hence

$$\int_{0}^{n} \frac{3}{4}x^2(2-x)\,dx = \frac{3}{4}\left(\frac{2}{3}n^3 - \frac{n^4}{4}\right) = 0.75.$$

In the interval [0, 2], the solution is approx. 1.514. The IQR is therefore IQR $= 1.514 - 0.913 =$ **0.601**.

Note: The median and the percentiles can be found using the **distribution function** $F(x)$. Recall that $F(x)$ gives the area under the pdf up to the value x of the random variable X. Using this fact then, to find the median, it is enough to solve the equation

$$F(x) = 0.5.$$

For the percentiles, a similar equation can be used, i.e.

$$F(x) = k\%.$$

Example 18

Use the distribution function above to find the median, first and third quartiles.

Solution

The distribution function is given by:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \frac{3}{4}t^2(2 - t)\,dt = \frac{3}{4}\left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)$$

Hence, to find the median we solve

$$F(m) = 0.5, \text{ i.e. } \frac{3}{4}\left(\frac{2}{3}m^3 - \frac{m^4}{4}\right) = 0.5,$$

which gives us the median as 1.2285.

To find the quartiles, we also solve

$$F(n) = 0.25, \text{ i.e. } \frac{3}{4}\left(\frac{2}{3}n^3 - \frac{n^4}{4}\right) = 0.25,$$

which gives us the first quartile as 0.913, and

$$F(n) = 0.75, \text{ i.e. } \frac{3}{4}\left(\frac{2}{3}n^3 - \frac{n^4}{4}\right) = 0.75,$$

which gives us the third quartile as 1.514.

Notice that $P(0.913 \leqslant x \leqslant 1.514) = F(1.514) - F(0.913) = 0.5$.

Variance and standard deviation

The variance of a random variable is defined as

$$\text{Var}(X) = \text{E}(X - \mu)^2.$$

This formula can be simplified, as in the discrete case, to

$$\text{Var}(X) = \text{E}(X)^2 - \mu^2 = \text{E}(X)^2 - (\text{E}(X))^2.$$

In the continuous case, if X has a pdf function $f(x)$, then

$$\text{Var}(X) = \int_x (x - \text{E}(X))^2 f(x)\,dx = \int_x x^2 f(x)\,dx - \mu^2$$
$$= \int_x x^2 f(x)\,dx - (\text{E}(X))^2.$$

The standard deviation is of course the square root of the variance,

$$\sigma = \sqrt{\text{Var (X)}}.$$

Example 19

The function $f(x)$ is a *pdf* for a random variable X.

$$f(x) = \begin{cases} \frac{3}{4}x^2(2-x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the variance and the standard deviation.

Solution

$\mathrm{Var}(X) = \int_x x^2 f(x)\,dx - (\mathrm{E}(X))^2 = \int_0^2 x^2 f(x)\,dx - (\mathrm{E}(X))^2$

Since we have already calculated $\mathrm{E}(X) = 6/5$, we only need to calculate $\int_0^2 x^2 f(x)\,dx$.

$$\int_0^2 x^2 f(x)\,dx = \int_0^2 x^2\left(\frac{3}{4}x^2(2-x)\right)dx = \left[\frac{3x^5}{10} - \frac{3x^6}{24}\right]_0^2 = 1.6\text{; hence,}$$

$$\mathrm{Var}(X) = 1.6 - (6/5)^2 = 0.16\text{, and } \sigma = \sqrt{0.16} = 0.4.$$

Exercise 17.4

1 The continuous random variable X has a pdf $f(x)$ where

$$f(x) = \begin{cases} kx^2 + \frac{3}{2} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k.
b) Find $P(x > 0.5)$.
c) Find $P(0 < x < 0.5)$.
d) Find the mean, median and standard deviation.

2 The continuous random variable X has a pdf $f(x)$ where

$$f(x) = \begin{cases} k(5-2x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k.
b) Find $P(x > 1.5)$.
c) Find $P(0.5 < x < 1.5)$.
d) Find the mean, median and standard deviation.

3 The continuous random variable X has a pdf $f(x)$ where

$$f(x) = \begin{cases} 2x - x^3 & 0 \leqslant x \leqslant k \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k.
b) Find $P(x > 0.5)$.
c) Find $P(0 < x < 0.5)$.
d) Find the mean, median and standard deviation.

4 The continuous random variable X has a pdf $f(x)$ where

$$f(x) = \begin{cases} k(x+1) & 0 \leqslant x \leqslant 1 \\ 2kx^2 & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k.
b) Find $P(x > 0.5)$
c) Find $P(1 < x < 1.5)$
d) Find the mean, median and standard deviation.

5 The continuous random variable X has a pdf $f(x)$ where

$$f(x) = \begin{cases} 2kx & 0 \leqslant x < 1 \\ 2kx^2 & 1 \leqslant x < 2 \\ k(8 - 2x) & 2 \leqslant x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch a graph for $f(x)$.
b) Find k.
c) Find the mean, median and standard deviation.
d) Find the IQR of this distribution.

6 The lifetime of flat batteries used in remote control units for one type of tv sets has a probability density function defined by

$$f(x) = \begin{cases} \frac{15}{76}(x^4 - 2x^2 + 2) & 0 \leqslant x \leqslant 1 \\ -\frac{15}{8056}(15x - 121) & 1 < x \leqslant 8\frac{1}{15} \\ 0 & \text{otherwise} \end{cases}$$

where x is measured in tens of hours.
a) Find the mean life of such batteries.
b) What is the probability that a battery will last at least 20 hours?
c) Each control unit is fitted with two batteries. The unit can only function if both batteries work. What is the probability that the unit will work for more than 20 hours?

7 The lifetime Y, in tens of hours, of light bulbs produced by a certain company has a probability density function defined by

$$f(y) = \begin{cases} \frac{3}{500}y(10 - y) & 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the mean life of a light bulb.
b) Find the median life of a light bulb.
c) Find the standard deviation of the life of a light bulb.
d) Find the probability that a light bulb will last more than 80 hours.
e) A lamp set is fitted with two such bulbs. For a new set, find
  (i) the probability that both bulbs will last more than 80 hours
  (ii) the probability that at least one of the bulbs has to be replaced before 80 hours.

8 The weekly amount of oil pumped out of an oil well, in hundreds of barrels, has density function $f(y)$ defined by

$$f(y) = \begin{cases} \frac{1}{8}y^2 & 0 \leqslant y < 2 \\ \frac{y}{8}(4 - y) & 2 \leqslant y \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the graph of the pdf.
b) Find the mean production per week of this well.
c) Find the IQR for the production per week.
d) When the production falls below 10% of the weekly production, some maintenance will have to be done in terms of replacing the pumps with more specialized ones. What level of production will warrant that?

9 A continuous random variable Y has a pdf $f(y)$ given by

$$f(y) = \begin{cases} \dfrac{c}{(1 - y)(y - 6)} & 2 \leqslant y \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $c = \dfrac{5}{4\ln 2}$.
b) Calculate the mean and standard deviation of Y.

10 A random variable Y has a pdf $f(y)$ defined as follows

$$f(y) = \begin{cases} a(by - y^2) & 0 \leqslant y \leqslant 5 \\ 0 & \text{otherwise}, \ a, b > 0 \end{cases}$$

a) Show that $b \geqslant 5$ and that $a = \dfrac{6}{25(3b - 10)}$.

b) If the expected value of Y is 2.5, find the values of a and b.

c) Find the variance of Y.

11 A random variable X has a pdf $f(x)$ defined over an interval $[a, b]$ where $b > a > 0$. The equation that defines $f(x)$ is $f(x) = k$.

a) Find k in terms of a and b.

b) Also, in terms of a and b, find

(i) the mean

(ii) the median

(iii) the variance.

Note: This is known as the **uniform distribution**.

12 The random variable X has the probability density function

$$f(x) = \begin{cases} \dfrac{5x^4}{31} & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find

(i) $P(1.2 < x < 1.7)$

(ii) the median

(iii) the value of k such that $P(x > k) = 0.25$.

b) Two independent observations from X are made. What is the probability that at least one of them is larger than 1.5?

13 (Optional) The distribution function $F(x)$ of a random variable X is

$$F(x) = \begin{cases} 0 & 0 \leqslant x < 5 \\ k(x^3 - 21x^2 + 147x - 335 & 5 \leqslant x \leqslant 7 \\ 1 & x > 7 \end{cases}$$

Find

a) the value of k

b) the probability density function $f(x)$

c) the median of X

d) $\text{Var}(X)$.

14 The random variable Y has pdf

$$f(y) = \begin{cases} 4y^k & 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise}, k > 0 \end{cases}$$

a) Find the value of k.

b) Find the mean of Y.

c) Find the value a for which $P(y > a) = 0.5$.

15 The time to first failure of an engine valve (in thousands of hours) is a random variable Y with a pdf

$$f(y) = \begin{cases} 2ye^{-y^2} & 0 \leqslant y \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $f(y)$ satisfies the requirements of a density function.

b) Find the probability that the valve will last at least 2000 hours before being serviced.

c) Find the mean of the random variable.

(Hint: You need to know that $\lim\limits_{x \to \infty} \displaystyle\int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$)

d) Find the median of Y.

e) Find the IQR.

f) An engine utilizes two such valves and needs servicing as soon as any of the two valves fail. Find the probability that the engine needs servicing before 200 hours of work.

16 The length of time required by students to complete a paper 2 HL exam is a random variable with a pdf

$$f(y) = \begin{cases} \frac{1}{2}\left(cy + \frac{y^2}{3}\right) & 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find c.
b) Find the probability that a randomly selected student will finish in less than 1 hour.
c) In a randomly selected group of 10 students, what is the probability that 3 students will finish the exam in less than 1 hour?
d) Given that Casper, who happened to be randomly selected, needs at least 1 hour to finish the exam, what is the probability that he will require at least 90 minutes?

17 The time, in months, in excess of one year to complete a building construction project is modelled by a continuous random variable Y months with a pdf

$$f(y) = \begin{cases} ky^2(5 - y) & 0 \leqslant y \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k = \frac{12}{625}$.
b) Find the mean, median and mode of this distribution (1 decimal place accuracy).
c) What proportion of the projects is completed in less than three months of excess time?
d) Find the standard deviation of the excess time.
e) What proportion of the projects is finished within one standard deviation of the mean excess time? Does your answer contradict the 'empirical rule'?

18 The delay time T for flights of a certain airline, in hours, has the pdf

$$f(t) = \begin{cases} \frac{4}{625}(5t^3 - t^4) & 0 \leqslant t \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

If a flight is delayed more than 5 hours, it is cancelled.
a) Find the mean and mode of the delay time.
b) Show that the median delay time is approximately 3.43 hours.
c) Find the probability that a randomly chosen flight will be delayed between 1 and 2 hours.
d) Find the standard deviation of the delay time.
e) Two flights are chosen at random. What is the probability that (assume flights delay times are independent of each other)
   (i) both are delayed more than 1 hour?
   (ii) at least one of them is delayed for more than 1 hour?
   (iii) only one of them is delayed more than 1 hour?

19 The probability density function $f(x)$ of the continuous random variable X is defined on the interval $[0, a]$ by

$$f(x) = \begin{cases} \frac{1}{8}x & 0 \leqslant x < 3 \\ \frac{27}{8x^2} & 3 < x \leqslant a \end{cases}$$

Find the value of a.

20 The probability density function $f(x)$, of a continuous random variable X is defined by

$$f(x) = \begin{cases} \frac{1}{4}x(4 - x^2) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the median value of X.

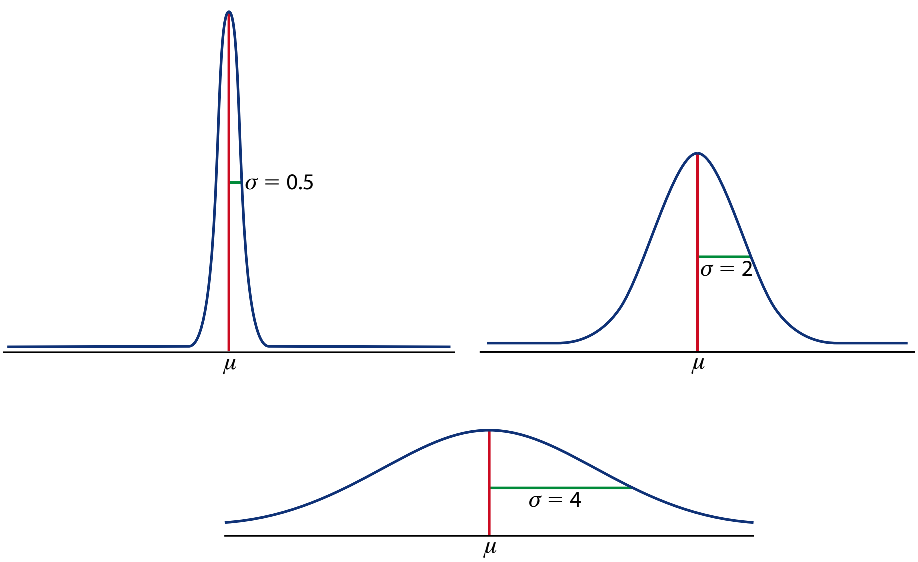
17.5 The normal distribution

Continuous probability distributions can assume a variety of shapes. However, for reasons of staying within (with some extensions) the boundaries of the IB syllabus, we will focus on one distribution. In fact, a large number of random variables observed in our surroundings possess a frequency distribution that is approximately bell-shaped. We call that distribution the **normal probability distribution**.

The most important type of continuous random variable is the *normal* random variable. The probability density function of a normal random variable X is determined by two parameters: the mean or expected value μ and the standard deviation σ of the variable.

The normal probability density function is a bell-shaped density curve that is symmetric about the mean μ. Its variability is measured by σ. The larger the value of σ the more variability there is in the curve. That is, the higher the probability of finding values of the random variable further away from the mean. Figure 17.1 represents three different normal density functions with the same mean but different standard deviations. Note how the curves 'flatten' as σ increases. This is so because the area under the curve has to stay equal to 1.

Figure 17.1



Probability density function of the normal distribution

The probability density function for a normally distributed random variable x is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ for } -\infty < x < \infty$$

where μ and σ^2 are any number such that $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$, and where e and π are the well-known constants $e = 2.718\,28\ldots$ and $\pi = 3.141\,59\ldots$.

Notation:
When a variable is normally distributed, we write $X \sim \mathrm{N}(\mu, \sigma^2)$.

Although we will not make direct use of the formula above, it is interesting to note its properties, because they help us understand how the normal distribution works. Notice that the equation is completely determined by the mean μ and the standard deviation σ.

The graph of a normal probability distribution is shown in Figure 17.2. As you notice, the mean or expected value locates the centre of the distribution, and the distribution is symmetric about this mean. Since the total area under the curve is 1, the symmetry of the curve implies that the area to the right of the mean and the area to the left are both equal to 0.5. The shape, or how 'flat' it is, is determined by σ, as we have seen in Figure 17.1. Large values of σ tend to reduce the height of the curve and increase the spread, and small values of σ increase the height to compensate for the narrowness of the distribution.

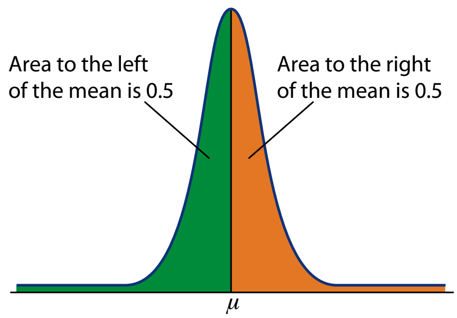


Figure 17.2

So, the normal distribution is fully determined by its mean, μ, and its standard deviation, σ. Changing μ without changing σ moves the normal curve along the horizontal axis without changing its spread. As you have seen above, the standard deviation σ controls the spread of the curve. You can also locate the standard deviation by eye on the curve. One σ to the right or left of the mean μ marks the point where the curvature of the curve changes. That is, as you move right from the mean, at the point where $x = \mu + \sigma$, the curve changes its curvature from downwards to upwards. Similarly, as you move one σ to the left from the mean the curve changes its curvature from downwards to upwards.

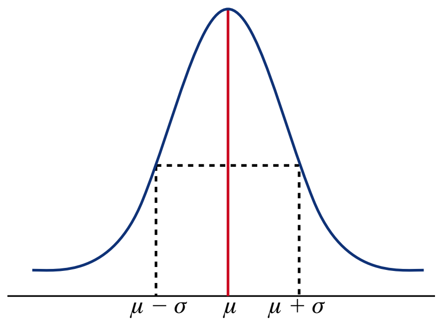


Figure 17.3

Although there are many normal curves, they all have common properties. Here is one important one that you have seen in Chapter 11:

The empirical rule – restated

In the normal distribution with mean μ and standard deviation σ.

- Approximately 68% of the observations fall within σ of the mean μ.
- Approximately 95% of the observations fall within 2σ of the mean μ.
- Approximately 99.7% of the observations fall within 3σ of the mean μ.

Figure 17.4 illustrates this rule. Later in this section, you will learn how to find these areas from a table or from your GDC.

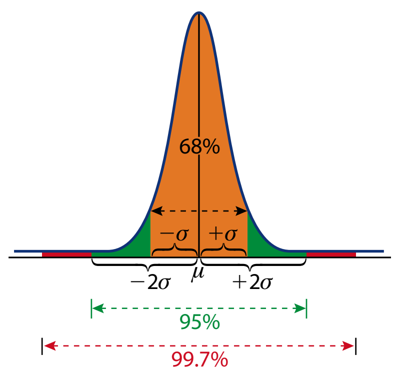


Figure 17.4

Example 20

Heights of young German men between 18 and 19 years of age follow a distribution that is approximately normal, with a mean of 181 cm and a standard deviation of 8 cm (approximately). Describe this population of young men.

Solution

According to the empirical rule, we find that approximately 68% of those young men have a height between 173 cm and 189 cm, 95% of them between 165 cm and 197 cm, and 99.7% between 157 cm and 205 cm. Looking further, you can say that only 0.15% are taller than 205 cm, or shorter than 157 cm.

As the empirical rule suggests, all normal distributions are the same if we measure in units of size σ about the mean μ as centre. Changing to these units is called *standardizing*. To standardize a value, measure how far it is from the mean and express that distance in terms of σ. This is how the calculation can be done:

Standardizing

If x is a value of a normal random variable, with mean μ and standard deviation σ, the standardized value of x is

$$z = \frac{x - \mu}{\sigma}.$$

A standardized value is also called the ***z*-score**.

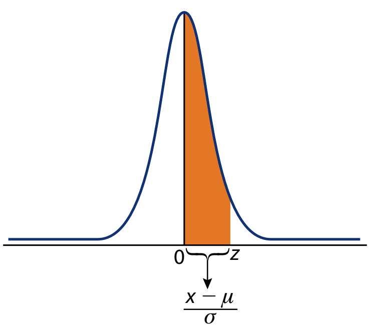


Figure 17.5

The quantity $x - \mu$ tells us how far our value is from the mean; dividing by σ then tells us how many standard deviations that distance is equal to.

The standardizing process, as you notice, is a transformation of the normal curve. For discussion purposes, assume the mean μ to be positive. The transformation $x - \mu$ shifts the graph back μ units. So, the new centre is shifted from μ back μ units. That is, the new centre is 0! Dividing by σ is going to 'scale' the distances from the mean and express everything in terms of σ. So, a point that is one standard deviation from the mean is going to be 1 unit above the new mean, i.e. it will be represented by $+1$. Now, if you look at the empirical rule we discussed earlier, points that are within one standard deviation from the mean will be within a distance of 1 in the new distribution. Instead of being at $\mu + \sigma$ and $\mu - \sigma$, they will be at $0 + 1$ and $0 - 1$ respectively, i.e. -1 and $+1$. (See Figure 17.6.)

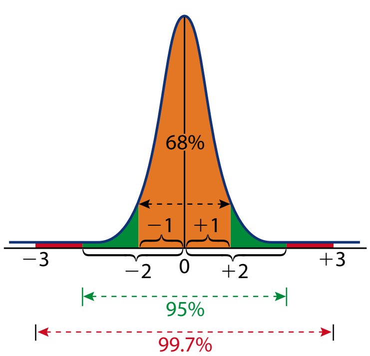


Figure 17.6

The new distribution we created by this transformation is called the **standard normal distribution**. It has a mean of 0 and a standard deviation of 1. It is a very helpful distribution because it will enable us to read the areas under any normal distribution through the standardization process, as will be demonstrated in the examples that follow.

Probability density function of the standard normal distribution

The probability density function for standard normal distribution is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \text{ for } -\infty < z < \infty.$$

Since linear transformations can transform all normal functions to standard, this becomes a very convenient and efficient way of finding the area under any normal distribution.

The proof that the mean and the variance of the standard normal variable are 0 and 1 respectively is straightforward.

Let $z = \dfrac{x - \mu}{\sigma}$ be the standard variable corresponding to a normal variable x.

$$\mathrm{E}(z) = \mathrm{E}\left(\frac{x-\mu}{\sigma}\right) = \mathrm{E}\left(\frac{1}{\sigma}(x-\mu)\right) = \frac{1}{\sigma}\mathrm{E}(x-\mu) = \frac{1}{\sigma}(\mu - \mu) = 0$$

$$\mathrm{V}(z) = \mathrm{V}\left(\frac{x-\mu}{\sigma}\right) = \mathrm{V}\left(\frac{1}{\sigma}(x-\mu)\right) = \frac{1}{\sigma^2}\mathrm{V}(x-\mu) = \frac{1}{\sigma^2}\mathrm{V}(x) = \frac{1}{\sigma^2}\sigma^2 = 1$$

Let us look at an example.

A young German man with a height of 192 cm has a z-score of

$$z = \frac{x - \mu}{\sigma} = \frac{192 - 181}{8} = 1.375$$

or 1.375 standard deviations above the mean. Similarly, a young man with a height of 175 cm is

$$z = \frac{x - \mu}{\sigma} = \frac{175 - 181}{8} = -0.75$$

or 0.75 standard deviations below the mean.

To find the probability that a normal variable x lies in the interval a to b, we need to find the area under the normal curve $N(\mu, \sigma^2)$ between the points a and b. However, there is an infinitely large number of normal curves – one for each mean and standard deviation. (See Figure 17.7.)

A separate table of areas for each of these curves is obviously not practical. Instead, we use one table for the standard normal distribution, which gives us the required areas. When you standardize a and b, you get two standard numbers z_1 and z_2 such that the area between z_1 and z_2 is the same as the area we need.

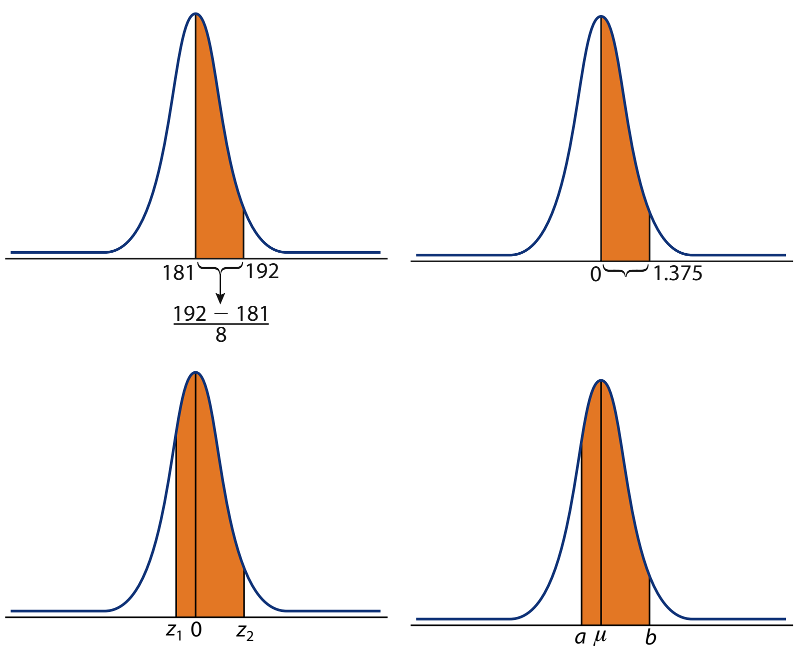


Figure 17.7

In the example above, if we are interested in the proportion of young German men whose height is between 175 cm and 192 cm, we calculate the z-scores for these numbers and then read the area from the table. (The normal distribution table is in the appendix). Here is an abbreviated version of the table and instructions on how to use it. (There are many tables of the areas under normal distributions. We will use a table constructed in a similar way to the one used on IB examinations.)

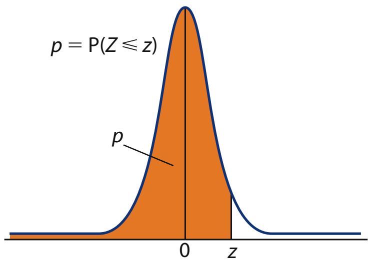


Figure 17.8

z	0.00	0.01	→	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040		0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438		0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832		0.5987	0.6026	0.6064	0.6103	0.6141
↓			→			↓		
↓			→			↓		
↓			→			↓		
1.3	0.9032	0.9049		0.9115	0.9131	0.9147	**0.9162**	0.9177
1.4	0.9192	0.9207		0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345		0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463		0.9505	0.9515	0.9525	0.9535	0.9545

The table, as constructed, gives you the areas under the normal distribution to the left of some value z, as you see in Figure 17.8.

The table starts at 0, and gives the areas till $z = 3.9$. To read an area to the left of a number z, say 1.37, you read the first column to find the first two digits of z. So, in the first column, we stop at the cell containing 1.3. To get the area for 1.37, we look at the first row and choose the column corresponding to 0.07. Where the row at 1.3 meets the column at 0.07 is the area under the normal distribution corresponding to 1.37, namely, 0.9147. That is, the probability of at most a height with $z = 1.37$ is 0.9147. Since the table does not go to 4 decimal places, our answers will not be very precise. So, to find the probability corresponding to a height of 192 cm, we need a z of 1.375, which is not in the table. We can use 1.37, 1.38, or take an average. If we want an average, we read the neighbouring area of 0.9162, and get the average to be 0.915 45.

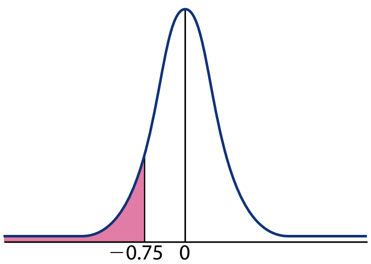


Figure 17.9

Unfortunately, due to limitations of space, this type of table does not cater for negative values of z. The good news is that, due to the symmetry of the distribution, the area to the left of a negative value of z is the same as the area to the right of its absolute value. So, if we are interested in the area to the left of −0.75, we look for the area to the right of 0.75, i.e. $1 - \mathrm{P}(z < 0.75) = 1 - 0.7734 = 0.2266$ (see Figure 17.10). So, in the example above, if we want to know the probability of a young German man, chosen at random, having a height between 175 cm and 192 cm, we look up the corresponding area under the standard normal distribution between −0.75 and 1.375. Since these two areas are cumulative, we need to subtract them, i.e. the required area is $0.915\,45 - 0.2266 = 0.6885$.

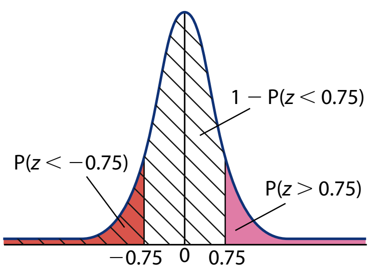
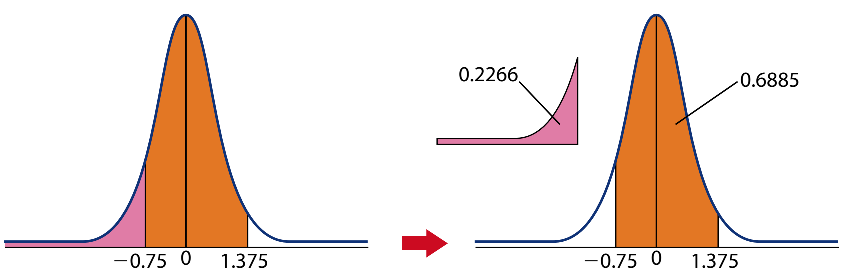


Figure 17.10
$\mathrm{P}(z > 0.75) = 1 - \mathrm{P}(z < 0.75)$.

Figure 17.11



What is the chance that a young German man is taller than 175 cm?

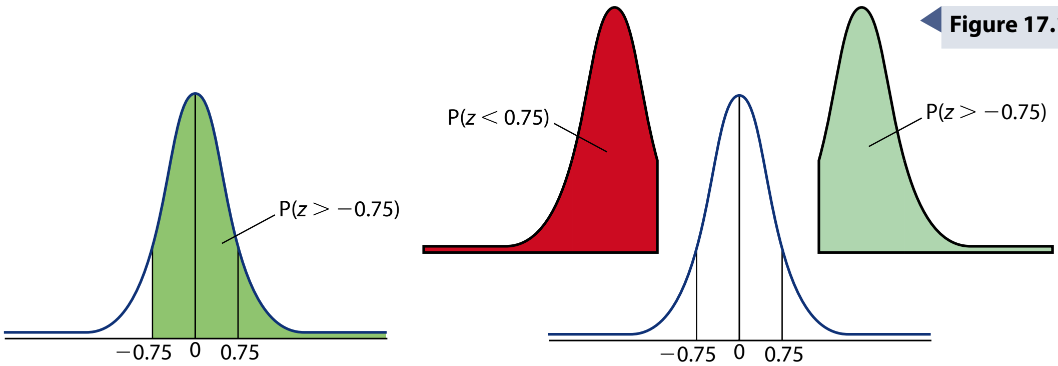


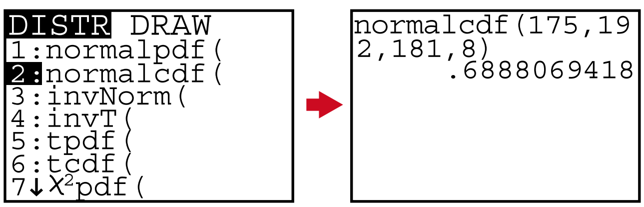
Figure 17.12

This means that we have to look at the area above −0.75. Due to symmetry, the area in question, which is to the right of −0.75, is equal to the area below 0.75, which in turn can be read directly from the table as 0.7734.

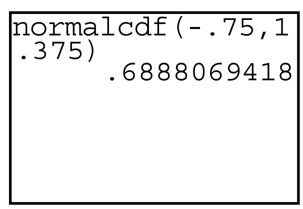
These calculations are much easier to calculate using a GDC, of course. Also, with the GDC, you do not need to standardize your variables either. However, because there are cases where you need to understand standardization and other cases where you are *required to use a table*, you need to know both methods.

Here is how your GDC can give you your answers.

You first go to the 'Distribution' menu and choose 'normalcdf'. Then you enter the numbers in the following order: *Lower limit, upper limit, mean, and standard deviation.* The result will be the area you need. See the screen images below.



If you want to use the standard normal, your commands will be the same, but you do not need to include the mean and standard deviation. They are the default.



If you need the probability that a young man is taller than 175 cm, you can also read it either by looking at the distribution with the original data or by standardizing.

Example 21

The age of graduate students in engineering programmes throughout the US is normally distributed with mean $\mu = 24.5$ and standard deviation $\sigma = 2.5$.

If a student is chosen at random,

a) what is the probability he/she is younger than 26 years old?

b) what proportion of students is older than 23.7 years?

c) what percentage of students is between 22 and 28 years old?

d) what percentage of the ages falls within 1 standard deviation of the mean? 2 standard deviations? 3 standard deviations?

Solution

If we let X = age of students, then $X \sim \mathrm{N}(\mu = 24.5, \sigma^2 = 6.25)$.

a) To answer this, we can either standardize and then read the table for the area left of 0.6:

$$\mathrm{P}\left(z < \frac{26 - 24.5}{2.5}\right) = \mathrm{P}(z < 0.6) = 0.7257\text{, or use a GDC:}$$

```
normalcdf(0,26,2
4.5,2.5)
        .7257469354
```

Notice here that we put 0 as a lower limit. You can put a number as a lower limit far enough from the mean to make sure you are receiving the correct cumulative distribution.

b) This can be done similarly:

$$\mathrm{P}(x > 23.7) = \mathrm{P}\left(z > \frac{23.7 - 24.5}{2.5} = -0.32\right)$$

So, by symmetry we know that

$$\mathrm{P}(z > -0.32) = \mathrm{P}(z < 0.32) = 0.6255.$$

With a GDC:

```
normalcdf(23.7,1
00,24.5,2.5
        .6255157701
█
```

Also, notice here that we wrote 100 as an upper limit, which is an arbitrary number far enough to the right to be sure we include the whole population.

c) $\mathrm{P}(22 < x < 28) = \mathrm{P}\left(\frac{22 - 24.5}{2.5} < z < \frac{28 - 24.5}{2.5}\right) = \mathrm{P}(-1 < z < 1.4)$

We find the area to the left of 1.4 and to the left of -1 and subtract them:

$$= 0.9192 - 0.1587 = 0.7606 = 76.06\%$$

With a GDC:

```
normalcdf(22,28,
24.5,2.5)
        .7605880293
```

d) This, as you know, is the empirical rule we talked about before. Let us see what percentage of the approximately normal data will lie within 1, 2 or 3 standard deviations.
We start with the traditional table:

$$P(-1 \leqslant z \leqslant 1) = P(z \leqslant 1) - P(z \leqslant -1) = 0.8413 - 0.1587$$
$$= \mathbf{0.6826}$$

This is the exact value corresponding to the empirical rule's 68%!

$$P(-2 \leqslant z \leqslant 2) = P(z \leqslant 2) - P(z \leqslant -2) = 0.9772 - 0.0228$$
$$= \mathbf{0.9544}$$

Again, this is the exact value corresponding to the empirical rule's 95%!

$$P(-3 \leqslant z \leqslant 3) = P(z \leqslant 3) - P(z \leqslant -3) = 0.9987 - 0.0013$$
$$= \mathbf{0.9973}$$

And again, this is the exact value corresponding to the empirical rule's 99.7%!

The inverse normal distribution

Another type of problem arises in situations similar to the one above when we are given a cumulative probability and would like to find the value in our data that has this cumulative probability. For example, what age marks the 95th percentile? That is, what age is higher or equal to 95% of the population? To answer this question, we need to reverse our steps. So far, we are given a value and then we look for the area corresponding to it. Now, we are given the area and we have to look for the number. That is why this is called the **inverse normal distribution**. Again, the approach is to find the *standard inverse normal number and then to 'de-standardize' it*. That is, to find the value from the original data that corresponds to the z-value at hand.

There is an inverse normal table available online. We will produce a part of the inverse normal table here for explanation.

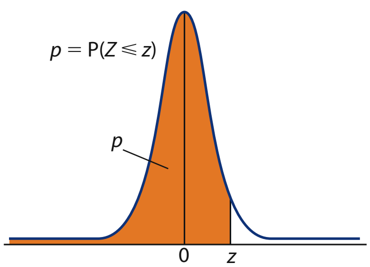


Figure 17.13

p	0.000	0.001	→	0.005	0.006	0.007	0.008	0.009
0.50	0.0000	0.0025		0.0125	0.0150	0.0175	0.0201	0.0226
0.51	0.0251	0.0276		0.0376	0.0401	0.0426	0.0451	0.0476
0.52	0.0502	0.0527		0.0627	0.0652	0.0677	0.0702	0.0728
0.53	0.0753	0.0778		0.0878	0.0904	0.0929	0.0954	0.0979
↓			→					
↓			→					
0.74	0.6433	0.6464		0.6588	0.6620	0.6651	0.6682	0.6713
0.75	0.6745	0.6776		0.6903	0.6935	0.6967	0.6999	0.7031
0.76	0.7063	0.7095		0.7225	0.7257	0.7290	0.7323	0.7356
0.77	0.7388	0.7421		0.7554	0.7588	0.7621	0.7655	0.7688
0.78	0.7722	0.7756		0.7892	0.7926	0.7961	0.7995	0.8030
0.79	0.8064	0.8099		0.8239	0.8274	0.8310	0.8345	0.8381

The table gives a selection of probabilities above the mean and the body of the table gives the z-value corresponding to that area. You know that 0 has a cumulative probability of 0.5. Look at the table and observe the intersection of the 0.50 row and the 0.000 column. It is 0, the mean of the standard normal distribution.

If we need to know what z-score the third quartile Q_3 is, for example, we need to look up 0.75. The z-score corresponding to Q_3 is 0.6745 as you see.

Suppose you want to find the z-score that leaves an area of 0.915 below it.

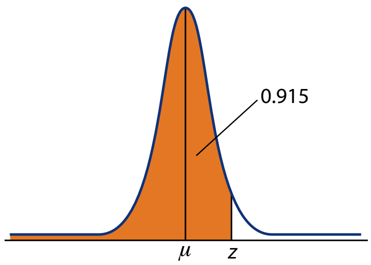


Figure 17.14

p	0.000	→	0.005
0.50	0.0000		0.0125
0.51	0.0251		0.0376
↓			
0.91	1.3408		1.3722

In the first column, we choose 0.91, then at the intersection of the row at 0.91 and the column at 0.005 the z-score corresponds to 0.915. So,

$$P(z < 1.3722) = 0.915.$$

The GDC can also be used in this case. The process is identical to the normal calculation. The difference is in choosing 'invNorm' instead.

```
invNorm(.5)
                0
invNorm(.915)
       1.37220381
invNorm(.75)
      .6744897495
```

In the young German men example, we would like to find what height leaves 95% of the population below it.

In this case, we look up the z-score corresponding to 0.95 and we find that it is $z = 1.6449$.

Now $z = 1.6449 = \dfrac{x - 181}{8} \Rightarrow x - 181 = 8 \times 1.6449$

$$\Rightarrow x = 181 + 8 \times 1.6449 = 194.16.$$

So, 95% of the young German men are shorter than 194.16 cm.

The GDC gives you this number with less effort:

```
invNorm(.95,181,
8)
       194.158829
```

Example 22

Since November 2007, the average time it takes fast trains (Eurostar) to travel between London and Paris is 2 hours 15 minutes, with a standard deviation of 4 minutes. Assume a normal distribution.

a) What is the probability that a randomly chosen trip will take longer than 2 hours and 20 minutes?

b) What is the probability that a randomly chosen trip will take less than 2 hours and 10 minutes?

c) What is the IQR of a trip on these trains?

Solution

We will do each problem using a table and a GDC to acquaint you with both methods.

a) The mean $\mu = 2.25$ and $\sigma = 0.067$.
2 hours 20 minutes = 2.33

$$P(x > 2.33) = P\left(z > \frac{2.33 - 2.25}{0.067}\right) = P(z > 1.25)$$

From the table: $P(z > 1.25) = 1 - P(z < 1.25) = 1 - 0.8944 = 0.1056$
Using your GDC:

```
normalcdf(2.3333
,100,2.25,.06667
)
           .10575261
```

The number 100 is arbitrary!

b) 2 hours 10 minutes = 2.167

$$P(x < 2.167) = P\left(z > \frac{2.167 - 2.25}{0.067}\right) = P(z < -1.25)$$

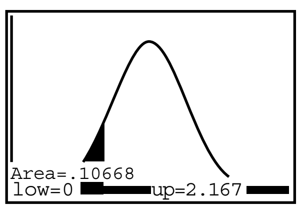
From the table, and by symmetry, this is the same as $P(z > 1.25)$, which we found in part a) above.

GDC

```
normalcdf(0,2.16
7,2.25,0.0667)
      .1066803378
```

or

```
ShadeNorm(0,2.16
7,2.25,0.0667)
```



c) To find the IQR, we need to find Q_1 and Q_3.

Q_1 is the number that leaves 25% of the data before it. So, we need to find the inverse normal variable that has an area of 0.25 before it.

From the table we can only do so using symmetry. So, we find the z-score that corresponds to 0.25 by finding its symmetrical number, which is the z-score with 0.75. So, we only need to find $z(0.75)$. The table of standard inverse normal gives us $z = 0.6745$.

So, Q_1 corresponds to -0.6745.

$$z = -0.6745 = \frac{x - 2.25}{0.067} \Rightarrow x - 2.25 = 0.067 \times (-0.6745)$$
$$\Rightarrow x = 2.25 - 0.045 = 2.205$$

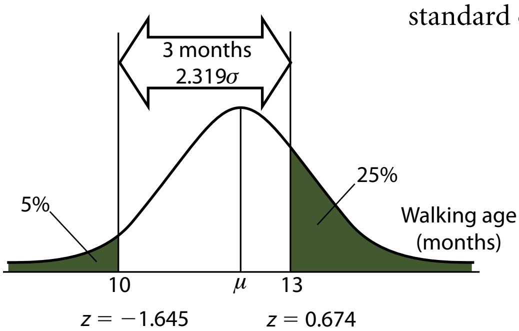
Q_3 corresponds to 0.6745.

$$z = 0.6745 = \frac{x - 2.25}{0.067} \Rightarrow x - 2.25 = 0.067 \times (0.6745)$$
$$\Rightarrow x = 2.25 + 0.045 = 2.295$$

IQR $= 2.295 - 2.205 = 0.090$ of an hour, i.e. 5.4 minutes.

Example 23

The age at which babies develop the ability to walk can be described by a normal model. It is known that 5% of babies learn how to walk by the age of 10 months and 25% need more than 13 months. Find the mean and standard deviation of the distribution.



Solution

Looking at the diagram at left will help you visualize the solution. We will show two approaches to this problem.

The first approach is to consider the distance between 10 and 13 months. In our data, that distance is 3 months, but how many standard deviations does that represent? Since we know that 10 months represents the lower 5%, and 13 months represents the upper 25%, we can obtain z-scores for those two data points without knowing the mean and standard deviation. Use the inverse table or a GDC:

```
invNorm(.05)
    -1.644853626
invNorm(.75)
     .6744897495
```

Therefore, the 3-month distance is equivalent to 0.674 – (–1.64) = 2.319 standard deviations or:

$$2.319\sigma = 3 \Rightarrow \sigma = 1.294$$

And finally we use either of the data points (10 months or 13 months) in our z-score formula to find the mean:

$$z = \frac{x - \mu}{\sigma} \Rightarrow 0.674 = \frac{13 - \mu}{1.294} \Rightarrow \mu = 12.128$$

Thus the mean age that babies begin to walk is 12.1 months with a standard deviation of 1.29 months.

The second approach uses a bit of algebra instead. After obtaining the z-scores from the inverse table or GDC (as above), we begin by writing two equations using the z-score formula:

$$-1.645 = \frac{10 - \mu}{\sigma} \Rightarrow \mu - 1.645\sigma = 10,$$

$$0.674 = \frac{13 - \mu}{\sigma} \Rightarrow \mu + 0.674\sigma = 13$$

Aha! We have two linear equations with two unknowns. Solve these to obtain:

$\mu = 12.128$
$\sigma = 1.294$

Again, we conclude the mean age that babies begin to walk is 12.1 months with a standard deviation of 1.29 months.

Exercise 17.5

1 The time it takes to change the batteries of your GDC is approximately normal with mean 50 hours and standard deviation of 7.5 hours.
Find the probability that your newly equipped GDC will last

a) at least 50 hours
b) between 50 and 75 hours
c) less than 42.5 hours
d) between 42.5 and 57.5 hours
e) more than 65 hours
f) 47.5 hours.

2 Find each of the following probabilities.

a) $P(|z| < 1.2)$
b) $P(|z| > 1.4)$
c) $P(x < 3.7)$, where $X \sim N(3, 3)$
d) $P(x > -3.7)$, where $X \sim N(3, 3)$

3 A car manufacturer introduces a new model that has an in-city mileage of 11.4 litres/100 kilometres. Tests show that this model has a standard deviation of 1.26. The distribution is assumed to be normal.
A car is chosen at random from this model.

a) What is the probability that it will have a consumption less than 8.4 litres/100 kilometres?
b) What is the probability that the consumption is between 8.4 and 14.4 litres/100 kilometres?

4 Find the value of z that will be exceeded only 10% of the time.

5 Find the value of $z = z_0$ such that 95% of the values of z lie between $-z_0$ and $+z_0$.

6 The scores on a public schools examination are normally distributed with a mean of 550 and a standard deviation of 100.

a) What is the probability that a randomly chosen student from this population scores below 400?
b) What is the probability that a student will score between 450 and 650?
c) What score should you have in order to be in the 90th percentile?
d) Find the IQR of this distribution.

7 A company producing and packaging sugar for home consumption put labels on their sugar bags noting the weight to be 500 g. Their machines are known to fill the bags with weights that are normally distributed with a standard deviation of 5.7 g. A bag that contains less than 500 g is considered to be underweight and is not appreciated by consumers.

a) If the company decides to set their machines to fill the bags with a mean of 512 g, what fraction will be underweight?

b) If they wish the percentage of underweight bags to be at most 4%, what mean setting must they have?

c) If they do not want to set the mean as high as 512 g, but instead at 510 g, what standard deviation gives them at most 4% underweight bags?

8 In a large school, heights of students who are 13 years old are normally distributed with a mean of 151 cm and a standard deviation of 8 cm.
Find the probability that a randomly chosen child is

a) shorter than 166 cm

b) within 6 cm of the average.

9 The time it takes Kevin to get to school every day is normally distributed with a mean of 12 minutes and standard deviation of 2 minutes. Estimate the number of days when he takes

a) longer than 17 minutes

b) less than 10 minutes

c) between 9 and 13 minutes.

There are 180 school days in Kevin's school.

10 X has a normal distribution with mean 16. Given that the probability that x is less than 16.56 is 64%, find the standard deviation, σ, of this distribution.

11 X has a normal distribution with mean 91. Given that the probability that x is larger than 104 is 24.6%, find the standard deviation σ of this distribution.

12 X has a normal distribution with variance of 9. Given that the probability that x is more than 36.5 is 2.9%, find the mean μ of this distribution.

13 X has a normal distribution with standard deviation of 32. Given that the probability that x is more than 63 is 87.8%, find the mean μ of this distribution.

14 X has a normal distribution with variance of 25. Given that the probability that x is less than 27.5 is 0.312, find the mean μ of this distribution.

15 X has a normal distribution such that the probability that x is larger than 14.6 is 93.5% and $P(x > 29.6) = 2.2\%$. Find the mean μ and the standard deviation σ of this distribution.

16 $X \sim N(\mu, \sigma^2)$. $P(x > 19.6) = 0.16$ and $P(x < 17.6) = 0.012$. Find μ and σ.

17 $X \sim N(\mu, \sigma^2)$. $P(x > 162) = 0.122$ and $P(x < 56) = 0.0276$. Find μ and σ.

18 Wooden poles used for electricity networks in rural areas are produced and have lengths that are normally distributed.
2% of the poles are rejected because they are considered too short, and 5% are rejected because they are too long.

a) Find the mean and standard deviation of these poles if the acceptable range is between 6.3 m and 7.5 m.

b) In a randomly selected sample of 20 poles, find the probability of finding 2 rejected poles.

19 Bottles of mineral water sold by a company are advertised to contain 1 litre of water. To guarantee customer satisfaction the company actually adjusts its filling process to fill the bottles with an average of 1012 ml. The process follows a normal distribution with standard deviation of 5 ml.

a) Find the probability that a randomly chosen bottle contains more than 1010 ml.

b) Find the probability that a bottle contains less than the advertised volume.

c) In a shipment of 10 000 bottles, what is the expected number of 'underfilled' bottles?

20 Cholesterol plays a major role in a person's heart health. High blood cholesterol is a major risk factor for coronary heart disease and stroke. The level of cholesterol in the blood is measured in milligrams per decilitre of blood (mg/dl). According to the WHO, in general, less than 200 mg/dl is a desirable level, 200 to 239 is borderline high, and above 240 is a high-risk level and a person with this level has more than twice the risk of heart disease as a person with less than a 200 level. In a certain country, it is known that the average cholesterol level of their adult population is 184 mg/dl with a standard deviation of 22 mg/dl. It can be modelled by a normal distribution.

a) What percentage do you expect to be borderline high?

b) What percentage do you consider are high risk?

c) Estimate the interquartile range of the cholesterol levels in this country.

d) Above what value are the highest 2% of adults' cholesterol levels in this country?

21 A manufacturer of car tyres claims that the treadlife of its winter tyres can be described by a normal model with an average life of 52 000 km and a standard deviation of 4000 km.

a) You buy a set of tyres from this manufacturer. Is it reasonable for you to hope they last more than 64 000 km?

b) What fraction of these tyres do you expect to last less than 48 000 km?

c) What fraction of these tyres do you expect to last between 48 000 km and 56 000 km?

d) What is the IQR of the treadlife of this type of tyre?

e) The company wants to guarantee a minimum life for these tyres. That is, they will refund customers whose tyres last less than a specific distance. What should their minimum life guarantee be so that they do not end up refunding more than 2% of their customers?

22 Chicken eggs are graded by size for the purpose of sales. In Europe, modern egg sizes are defined as follows: very large has a mass of 73 g or more, large is between 63 and 73 g, medium is between 53 and 63 g, and small is less than 53 g. The small size is usually considered as undesirable by consumers.

a) Mature hens (older than 1 year) produce eggs with an average mass of 67 g. 98% of the eggs produced by mature hens are above the minimum desirable weight. What is the standard deviation if the egg production can be modelled by a normal distribution?

b) Young hens produce eggs with a mean mass of 51 g. Only 28% of their eggs exceed the desired minimum. What is the standard deviation?

c) A farmer finds that 7% of his farm's eggs are 'underweight', and 12% are very large. Estimate the mean and standard deviation of this farmer's eggs.

23 A machine produces bearings with diameters that are normally distributed with mean 3.0005 cm and standard deviation 0.0010 cm. Specifications require the bearing diameters to lie in the interval 3.000 $\pm$ 0.0020 cm. Those outside the interval are considered scrap and must be disposed of. What fraction of the production will be scrap?

24 A soft-drink machine can be regulated so that it discharges an average μ cc per bottle. If the amount of fill is normally distributed with a standard deviation 9 cc, give the setting for μ so that 237 cc bottles will overflow only 1% of the time.

25 The machine described in the previous problem has standard deviation σ that can be adjusted to the required levels when needed. What is the largest value of σ that will allow the actual amount dispensed to fall within 30 cc of the mean with probability at least 95%?

26 The speeds of cars on a main highway are approximately normal. Data collected at a certain point show that 95% of the cars travel at a speed less than 140 km/h, and 10% travel at a speed less than 90 km/h.

a) Find the average speed and the standard deviation for the cars travelling that specific stretch of the highway.

b) Find the proportion of cars that travel more at speeds exceeding 110 km/h.

27 The random variable X is normally distributed and:

$P(x \leqslant 10) = 0.670$

$P(x \leqslant 12) = 0.937$

Find $E(X)$.

28 A machine is set to produce bags of salt, whose weights are distributed normally, with a mean of 110 g and standard deviation of 1.142 g. If the weight of a bag of salt is less than 108 g, the bag is rejected. With these settings, 4% of the bags are rejected.

The settings of the machine are altered and it is found that 7% of the bags are rejected.

a) (i) If the mean has not changed, find the new standard deviation, correct to three decimal places.

The machine is adjusted to operate with this new value of the standard deviation.

(ii) Find the value, correct to two decimal places, at which the mean should be set so that only 4% of the bags are rejected.

b) With the new settings from part a), it is found that 80% of the bags of salt have a weight which lies between A g and B g, where A and B are symmetric about the mean. Find the values of A and B, giving your answers correct to two decimal places.

Practice questions

1 Residents of a small town have savings which are normally distributed with a mean of \$3000 and a standard deviation of \$500.

a) What percentage of townspeople have savings greater than \$3200?

b) Two townspeople are chosen at random. What is the probability that both of them have savings between \$2300 and \$3300?

c) The percentage of townspeople with savings less than d dollars is 74.22%. Find the value of d.

2 A box contains 35 red discs and 5 black discs. A disc is selected at random and its colour noted. The disc is then replaced in the box.

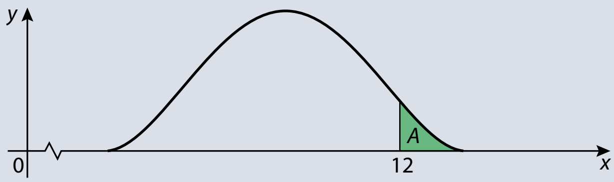
a) In eight such selections, what is the probability that a black disc is selected

(i) exactly once?

(ii) at least once?

b) The process of selecting and replacing is carried out 400 times. What is the expected number of black discs that would be drawn?

3 The graph shows a normal curve for the random variable X, with mean μ and standard deviation σ.



It is known that $P(x \geqslant 12) = 0.1$.

a) The shaded region A is the region under the curve where $x \geqslant 12$. Write down the area of the shaded region A.

It is also known that $P(x \leqslant 8) = 0.1$.

b) Find the value of μ, explaining your method in full.

c) Show that $\sigma = 1.56$ to an accuracy of 3 significant figures.

d) Find $P(x \leqslant 11)$.

4 A fair coin is tossed eight times. Calculate

a) the probability of obtaining exactly 4 heads

b) the probability of obtaining exactly 3 heads

c) the probability of obtaining 3, 4 or 5 heads.

5 The lifespan of a particular species of insect is normally distributed with a mean of 57 hours and a standard deviation of 4.4 hours.
The probability that the lifespan of an insect of this species lies between 55 and 60 hours is represented by the shaded area in the following diagram. This diagram represents the standard normal curve.

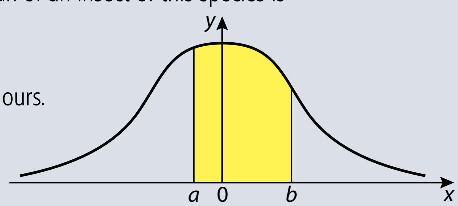
a) Write down the values of a and b.

b) Find the probability that the lifespan of an insect of this species is

(i) more than 55 hours

(ii) between 55 and 60 hours.

90% of the insects die after t hours.



c) **(i)** Represent this information on a standard normal curve diagram, similar to the one shown, indicating clearly the area representing 90%.

(ii) Find the value of t.

6 An urban highway has a speed limit of 50 km h^{-1}. It is known that the speeds of vehicles travelling on the highway are normally distributed, with a standard deviation of l0 km h^{-1}, and that 30% of the vehicles using the highway exceed the speed limit.

a) Show that the mean speed of the vehicles is approximately 44.8 km h^{-1}.

The police conduct a 'Safer Driving' campaign intended to encourage slower driving, and want to know whether the campaign has been effective. It is found that a sample of 25 vehicles has a mean speed of 41.3 km h^{-1}.

b) Given that the null hypothesis is
H0: the mean speed has been unaffected by the campaign state
state H1, the alternative hypothesis.

c) State whether a one-tailed or two-tailed test is appropriate for these hypotheses, and explain why.

d) Has the campaign had significant effect at the 5% level?

7 Intelligence quotient (IQ) in a certain population is normally distributed with a mean of 100 and a standard deviation of 15.

a) What percentage of the population has an IQ between 90 and 125?

b) If two persons are chosen at random from the population, what is the probability that both have an IQ greater than 125?

c) The mean IQ of a random group of 25 persons suffering from a certain brain disorder was found to be 95.2. Is this sufficient evidence, at the 0.05 level of significance, that people suffering from the disorder have, on average, a lower IQ than the entire population? State your null hypothesis and your alternative hypothesis, and explain your reasoning.

8 Bags of cement are labelled 25 kg. The bags are filled by machine and the actual weights are normally distributed with mean 25.7 kg and standard deviation 0.50 kg.

a) What is the probability a bag selected at random will weigh less than 25.0 kg?

In order to reduce the number of underweight bags (bags weighing less than 25 kg) to 2.5% of the total, the mean is increased without changing the standard deviation.

b) Show that the increased mean is 26.0 kg.

It is decided to purchase a more accurate machine for filling the bags. The requirements for this machine are that only 2.5% of bags be under 25 kg and that only 2.5% of bags be over 26 kg.

c) Calculate the mean and standard deviation that satisfy these requirements.

The cost of the new machine is $5000. Cement sells for $0.80 per kg.

d) Compared to the cost of operating with a 26 kg mean, how many bags must be filled in order to recover the cost of the new equipment?

9 The mass of packets of a breakfast cereal is normally distributed with a mean of 750 g and standard deviation of 25 g.

a) Find the probability that a packet chosen at random has mass

(i) less than 740 g

(ii) at least 780 g

(iii) between 740 g and 780 g.

b) Two packets are chosen at random. What is the probability that both packets have a mass that is less than 740 g?

c) The mass of 70% of the packets is more than x grams. Find the value of x.

10 In a country called Tallopia, the height of adults is normally distributed with a mean of 187.5 cm and a standard deviation of 9.5 cm.

a) What percentage of adults in Tallopia have a height greater than 197 cm?

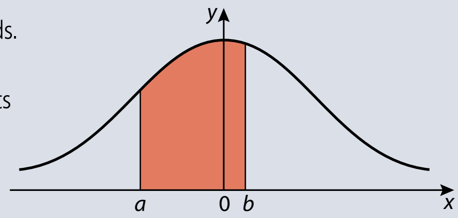
b) A standard doorway in Tallopia is designed so that 99% of adults have a space of at least 17 cm over their heads when going through a doorway. Find the height of a standard doorway in Tallopia. Give your answer to the nearest cm.

11 It is claimed that the masses of a population of lions are normally distributed with a mean mass of 310 kg and a standard deviation of 30 kg.

a) Calculate the probability that a lion selected at random will have a mass of 350 kg or more.

b) The probability that the mass of a lion lies between *a* and *b* is 0.95, where *a* and *b* are symmetric about the mean. Find the values of *a* and *b*.

12 Reaction times of human beings are normally distributed with a mean of 0.76 seconds and a standard deviation of 0.06 seconds. The graph right is that of the standard normal curve. The shaded area represents the probability that the reaction time of a person chosen at random is between 0.70 and 0.79 seconds.



a) Write down the values of *a* and *b*.

b) Calculate the probability that the reaction time of a person chosen at random is

(i) greater than 0.70 seconds

(ii) between 0.70 and 0.79 seconds.

Three per cent (3%) of the population have a reaction time less than *c* seconds.

c) **(i)** Represent this information on a diagram similar to the one above. Indicate clearly the area representing 3%.

(ii) Find *c*.

13 A factory makes calculators. Over a long period, 2% of them are found to be faulty. A random sample of 100 calculators is tested.

a) Write down the expected number of faulty calculators in the sample.

b) Find the probability that three calculators are faulty.

c) Find the probability that more than one calculator is faulty.

14 The speeds of cars at a certain point on a straight road are normally distributed with mean μ and standard deviation σ. 15% of the cars travelled at speeds greater than 90 km h^{-1} and 12% of them at speeds less than 40 km h^{-1}. Find μ and σ.

15 Bag A contains 2 red balls and 3 green balls. Two balls are chosen at random from the bag without replacement. Let *X* denote the number of red balls chosen. The following table shows the probability distribution for *X*.

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

a) Calculate E(*X*), the mean number of red balls chosen.

Bag B contains 4 red balls and 2 green balls. Two balls are chosen at random from bag B.

b) **(i)** Draw a tree diagram to represent the above information, including the probability of each event.

(ii) Hence, find the probability distribution for *Y*, where *Y* is the number of red balls chosen.

A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

c) Calculate the probability that two red balls are chosen.

d) Given that two red balls are obtained, find the conditional probability that a 1 or 6 was rolled on the die.

16 Ball bearings are used in engines in large quantities. A car manufacturer buys these bearings from a factory. They agree on the following terms: The car company chooses a sample of 50 ball bearings from the shipment. If they find more than 2 defective bearings, the shipment is rejected. It is a fact that the factory produces 4% defective bearings.

a) What is the probability that the sample is clear of defects?

b) What is the probability that the shipment is accepted?

c) What is the expected number of defective bearings in the sample of 50?

17 Each CD produced by a certain company is guaranteed to function properly with a probability of 98%. The company sells these CDs in packages of 10 and offers a money-back guarantee that all the CDs in a package will function.

a) What is the probability that a package is returned?

b) You buy three packages. What is the probability that exactly 1 of them must be returned?

18 The table below shows the probability distribution of a random variable X.

x	0	1	2	3
$P(X = x)$	$2k$	$2k^2$	$k^2 + k$	$2k^2 + k$

a) Calculate the value of k.

b) Find $E(X)$.

19 It is estimated that 2.3% of the cherry tomato fruits produced on a certain farm are considered to be small and cannot be sold for commercial purposes. The farmers have to separate such fruits and use them for domestic consumption instead.

a) 12 tomatoes are randomly selected from the produce. Calculate

(i) the probability that three are not fit for selling

(ii) the probability that at least four are not fit for selling.

b) It is known that the sizes of such tomatoes are normally distributed with a mean of 3 cm and a standard deviation of 0.5 cm. Tomatoes that are categorized as large will have to be larger than 2.5 cm. What proportion of the produce is large?

20 A factory has a machine designed to produce 1 kg bags of sugar. It is found that the average weight of sugar in the bags is 1.02 kg. Assuming that the weights of the bags are normally distributed, find the standard deviation if 1.7% of the bags weigh below 1 kg. Give your answer correct to the nearest 0.1 gram.

21 The continuous random variable X has probability density function $f(x)$ where

$$f_k(x) = \begin{cases} e - ke^{kx} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k = 1$.

b) What is the probability that the random variable X has a value that lies between $\frac{1}{4}$ and $\frac{1}{2}$? Give your answer in terms of e.

c) Find the mean and variance of the distribution. Give your answers *exactly* in terms of e.

The random variable X above represents the lifetime, in years, of a certain type of battery.

d) Find the probability that a battery lasts more than six months.

A calculator is fitted with three of these batteries. Each battery fails independently of the other two. Find the probability that at the end of six months

(i) none of the batteries has failed;

(ii) exactly one of the batteries has failed.

22 The lifetime of a particular component of a solar cell is Y years, where Y is a continuous random variable with probability density function

$$f(y) = \begin{cases} 0 & y < 0 \\ 0.5e^{-\frac{y}{2}} & y \geqslant 0 \end{cases}$$

a) Find the probability, correct to four significant figures, that a given component fails within six months.

Each solar cell has three components which work independently and the cell will continue to run if at least two of the components continue to work.

b) Find the probability that a solar cell fails within six months.

23 Ian and Karl have been chosen to represent their countries in the Olympic discus throw. Assume that the distance thrown by each athlete is normally distributed. The mean distance thrown by Ian in the past year was 60.33 m with a standard deviation of 1.95 m.

a) In the past year, 80% of Ian's throws have been longer than x metres.

Find x, correct to two decimal places.

b) In the past year, 80% of Karl's throws have been longer than 56.52 m. If the mean distance of his throws was 59.39 m, find the standard deviation of his throws, correct to two decimal places.

c) This year, Karl's throws have a mean of 59.50 m and a standard deviation of 3.00 m. Ian's throws still have a mean of 60.33 m and standard deviation 1.95 m. In a competition an athlete must have at least one throw of 65 m or more in the first round to qualify for the final round. Each athlete is allowed three throws in the first round.

(i) Determine which of these two athletes is more likely to qualify for the final on their first throw.

(ii) Find the probability that both athletes qualify for the final.

24 The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{6}x(1 + x^2) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the graph of f for $0 \leqslant x \leqslant 2$.

b) Write down the mode of X.

c) Find the mean of X.

d) Find the median of X.

25 A company buys 44% of its stock of bolts from manufacturer A and the rest from manufacturer B. The diameters of the bolts produced by each manufacturer follow a normal distribution with a standard deviation of 0.16 mm.
The mean diameter of the bolts produced by manufacturer A is 1.56 mm.
24.2% of the bolts produced by manufacturer B have a diameter less than 1.52 mm.

a) Find the mean diameter of the bolts produced by manufacturer B.

A bolt is chosen at random from the company's stock.

b) Show that the probability that the diameter is less than 1.52 mm is 0.312, to three significant figures.

c) The diameter of the bolt is found to be less than 1.52 mm. Find the probability that the bolt was produced by manufacturer B.

d) Manufacturer B makes 8000 bolts in one day. It makes a profit of \$1.50 on each bolt sold, on condition that its diameter measures between 1.52 mm and 1.83 mm.
Bolts whose diameters measure less than 1.52 mm must be discarded at a loss of \$0.85 per bolt.
Bolts whose diameters measure over 1.83 mm are sold at a reduced profit of \$0.50 per bolt.
Find the expected profit for manufacturer B.

18 The Mathematical Exploration – Internal Assessment

'If digressions can bring knowledge of new truths, why should they trouble us? … how do we know that we shall not discover curious things that are more interesting than the answers we originally sought?'

Galileo, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*, 1638

At the end of the Mathematics Higher Level course, you will take three written exams that will constitute the **External Assessment** component of the course: Paper 1 (covering the core syllabus – no GDC), Paper 2 (covering the core syllabus – GDC required), and Paper 3 (covering the option topic for which you registered). These written exams will contribute 80% to your final grade for the course that will be reported to you about six weeks after the exams finish.

Internal Assessment (IA) is another important component of the Mathematics Higher Level course and will contribute 20% to your final grade for the course. Thus, IA does comprise a significant part of the overall assessment for the course and should be taken seriously. It should also be pointed out that your work in completing the IA component differs in important ways from the written exams for the course.

- You do **not** perform IA work under strict time constraints as with written examinations.
- You have some freedom to help decide what mathematical topic you wish to explore.
- Your IA work involves writing about mathematics and not just doing mathematical procedures.
- Regular discussion with, and feedback from, your teacher will be essential.
- You should endeavour to explore a topic in which you have a genuine personal interest.
- You will be rewarded for evidence of creativity, curiosity and independent thinking.

The Mathematical Exploration

To satisfy the Internal Assessment component, you are required to write a report on a mathematical topic that you choose in consultation with your teacher. This report is formally referred to as the **Mathematical Exploration**. Throughout this chapter 'Mathematical Exploration' and 'report' refer to the same thing, i.e. the written piece of work that you submit for the Internal Assessment component of the course.

The Mathematical Exploration is aptly named because your primary objective in writing this report is to *explore* a topic in which you are genuinely interested and that is at an appropriate level for the course. Your teacher may provide you with a list of ideas (or 'stimuli') from which to choose a topic or which may help you to develop your own ideas for a topic to explore (see the **list of 200 ideas** printed later in this chapter). It is your responsibility to determine whether or not you are sufficiently interested in a particular topic – and it is your teacher's responsibility to determine if an exploration of the topic can be conducted at a level mathematically suitable for the course. Your teacher will help you determine if an exploration of a certain topic can potentially address the five assessment criteria satisfactorily. Your report should be approximately 6 to 12 pages long.

Internal Assessment Criteria

Your **Mathematical Exploration** report will be assessed by your teacher according to the following five criteria.

A **Communication**: This criterion assesses the **organisation** and **coherence** of the exploration. A well-organised exploration has an **introduction** and a **rationale** (which includes a brief explanation of why the topic was chosen). It describes the **aim of the exploration** and has a **conclusion**.

B **Mathematical presentation**: This criterion assesses to what extent you are able to:

- use appropriate mathematical language (notation, symbols and terminology);
- define key terms, where necessary;
- use multiple forms of mathematical representation such as formulae, diagrams, tables, charts, graphs and models.

C **Personal engagement**: This criterion assesses the extent to which you engage with the exploration, and present it in such a way that clearly shows **your own personal approach**. Personal engagement may be recognised in different attributes and skills. These include thinking independently and/or creatively, addressing personal interest, presenting mathematical ideas in your own way, using simple language to describe complex ideas, and applying unfamiliar mathematics.

D <u>**Reflection**</u>: This criterion assesses how well you **review**, **analyse** and **evaluate** the exploration. Although reflection may be seen in the conclusion to the exploration, you should also give evidence of reflective thought throughout the exploration. Reflection may be demonstrated by consideration of limitations and/or extensions and by relating mathematical ideas to your own previous knowledge.

E <u>**Use of mathematics**</u>: This criterion assesses to what extent and **how well you use mathematics** in your exploration. The mathematics that is explored in your report needs to be **sufficiently sophisticated**. The chosen topic should involve mathematics either in the Mathematics Higher Level syllabus, at a similar level, or beyond the level of the syllabus. Sophistication in mathematics may include understanding and use of challenging mathematical concepts, looking at a problem from different perspectives, or seeing underlying structures to link different areas of mathematics.

Your report will earn a numerical score out of a total of 20 possible marks. The five criteria do not contribute equally to the overall score for your Mathematical Exploration. For example, criterion E (Use of mathematics) is 30% of the overall score, whereas criteria B (Mathematical presentation) and D (Reflection) contribute 15% each.

It is very important that you familiarize yourself with the assessment criteria and refer to them while you are writing your report. The scoring levels for each criteria and associated descriptors are as follows.

A	**Communication**
0	The exploration does not reach the standard described by the descriptors below.
1	The exploration has some coherence.
2	The exploration has some coherence and shows some organisation.
3	The exploration is coherent and well organised.
4	The exploration is coherent, well organised, concise and complete.

B	**Mathematical presentation**
0	The exploration does not reach the standard described by the descriptors below.
1	There is some appropriate mathematical presentation.
2	The mathematical presentation is mostly appropriate.
3	The mathematical presentation is appropriate throughout.

C Personal engagement	
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of limited or superficial personal engagement.
2	There is evidence of some personal engagement.
3	There is evidence of significant personal engagement.
4	There is abundant evidence of outstanding personal engagement.

D Reflection	
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of limited or superficial reflection.
2	There is evidence of meaningful reflection.
3	There is substantial evidence of critical reflection.

E Use of Mathematics	
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used. Limited understanding is demonstrated.
2	Some relevant mathematics is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Good knowledge and understanding are demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and reflects the sophistication expected. Good knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and reflects the sophistication and rigour expected. Thorough knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is precise and reflects the sophistication and rigour expected. Thorough knowledge and understanding are demonstrated.

Guidance

Conducting an in-depth individual exploration into the mathematics of a particular topic can be an interesting and very rewarding experience. It is important to take all stages of your work on the Mathematical Exploration seriously – not only because it is worth 20% of your final grade for the course, but also because of the opportunity to pursue your own personal interests without the pressure of examination conditions. The Mathematical Exploration will require a significant amount of time and energy to complete successfully. It should *not* be approached as simply an extended homework assignment. The task of writing the report will demand a considerable amount of research, analysis, reading, consultation (with your teacher only), thinking, writing, editing, mathematical work, problem solving and proofreading. Hopefully, it will also be enjoyable, thought provoking and satisfying, and give you the opportunity to gain a deeper appreciation for the beauty, power and usefulness of mathematics.

Although it is required that your Mathematical Exploration be completely your own work, you should be consulting with your teacher on a regular basis throughout the time given to you to research and write your report. Your teacher should provide support and advice during the planning and writing stages of your report. Both you and your teacher will need to sign the internal assessment coversheet verifying the authenticity of your Mathematical Exploration.

All of the work connected with the exploration must be your own. Your Mathematical Exploration must reflect intellectual honesty in research practices and must provide the reader with the exact sources of quotations, ideas and points of view with a complete and accurate bibliography. There are a number of acceptable bibliographic styles. Whatever style is chosen, it must include all relevant source information and be applied consistently. Group work is not allowed with the Mathematical Exploration. Also, if you are writing an Extended Essay for mathematics, you are not allowed to submit the same piece of work for the Mathematical Exploration – and you are strongly advised not to write about the same mathematical topic for both.

In organizing a successful Mathematical Exploration, consider the following suggestions.

1 Select a topic in which you are **genuinely interested**. Include a brief explanation in the early part of your report about why you chose your topic – including why you find it interesting.

2 Consult with your teacher that the topic is at the **appropriate level of mathematics**, i.e. that it is at the same level of mathematics in the HL syllabus, or beyond.

3 Find as much **information** about the topic as possible. Although information found on internet websites can be very helpful, try to also find information from books, journals, textbooks and other print material.

4 Prepare and organize your material into a **thorough and interesting report**. Although there is no requirement that you present your report to your class, it should be written so that your fellow classmates can follow it without trouble. Your report needs to be **logically organized** and use appropriate mathematical terminology and notation.

5 The most important aspects of your report should be about **mathematical communication and using mathematics**. Although other aspects of your topic (e.g. historical, personal, cultural etc.) can be discussed, be careful not to lose focus on the mathematical features.

6 Two of the assessment criteria – personal engagement and reflection – are about **what you think about the topic** you are exploring. Don't hesitate to pose your own relevant and insightful questions as part of your report, and then to address these questions using mathematics at a suitably sophisticated level along with sufficient written commentary.

7 Although your teacher will expect and require you to work independently, you are allowed to **consult with your teacher** – and your teacher is allowed to give you advice and feedback to a certain extent while you are working on your report. It is especially important to check with your teacher that any **mathematics in your report is correct**. Your teacher will not give mathematical answers or corrections, but can indicate where any errors have been made or where improvement is needed.

Mathematical Exploration HL – Student Checklist

Is your report written entirely by yourself – and trying to avoid simply replicating work and ideas from sources you found during your research?	☐	Yes	☐	No
Have you strived to: apply your personal interest; develop your own ideas; and use critical thinking skills during your exploration and demonstrate these in your report?	☐	Yes	☐	No
Have you referred to the five assessment criteria while writing your report?	☐	Yes	☐	No
Does your report focus on good mathematical communication – and does it read like an article for a mathematical journal?	☐	Yes	☐	No
Does your report have a clearly identified introduction and conclusion?	☐	Yes	☐	No
Have you documented all of your source material in a detailed bibliography in line with the IB academic honesty policy?	☐	Yes	☐	No
Not including the bibliography, is your report 6 to 12 pages?	☐	Yes	☐	No
Are graphs, tables and diagrams sufficiently described and labelled?	☐	Yes	☐	No
To the best of your knowledge, have you used and demonstrated mathematics that is at the same level, or above, of that studied in IB Mathematics HL?	☐	Yes	☐	No
Have you attempted to discuss mathematical ideas, and use mathematics, with a sufficient level of sophistication and rigour?	☐	Yes	☐	No
Are formulae, graphs, tables and diagrams in the main body of text? (preferably no full-page graphs; and no separate appendices)	☐	Yes	☐	No
Have you used technology – such as a GDC, spreadsheet, mathematics software, drawing and word-processing software – to enhance mathematical communication?	☐	Yes	☐	No
Have you used appropriate mathematical language (notation, symbols, terminology) and defined key terms?	☐	Yes	☐	No
Is the mathematics in your report performed precisely and accurately?	☐	Yes	☐	No
Has calculator/computer notation and terminology **not** been used? ($y = x^2$, not $y = x$^2; $\approx$, not = for approximate values; π, not pi; $\|x\|$, not abs(x); etc)	☐	Yes	☐	No
At suitable places in your report – especially in the conclusion – have you included reflective and explanatory comments about the mathematical topic being explored?	☐	Yes	☐	No

List of 200 ideas/topics for a Mathematical Exploration

The topics listed here range from fairly broad to quite narrow in scope. It is possible that some of these 200 could be the title or focus of a **Mathematical Exploration**, while others will require you to investigate further to identify a narrower focus to explore. Do not restrict yourself only to the topics listed below. This list is only the 'tip of the iceberg' with regard to potential topics for your Mathematical Exploration. Reading through this list may stimulate you to think of some other topic in which you would be interested in exploring. Many of the items listed below may be unfamiliar to you. A quick search on the internet should give you a better idea what each is about and help you determine if you're interested enough to investigate further – and see if it might be a suitable topic for your Mathematical Exploration.

	Algebra and number theory	
Modular arithmetic	Goldbach's conjecture	Probabilistic number theory
Applications of complex numbers	Diophantine equations	Continued fractions
General solution of a cubic equation	Applications of logarithms	Polar equations
Patterns in Pascal's triangle	Finding prime numbers	Random numbers
Pythagorean triples	Mersenne primes	Magic squares and cubes
Loci and complex numbers	Matrices and Cramer's rule	Divisibility tests
Egyptian fractions	Complex numbers and transformations	Euler's identity: $e^{i\pi} + 1 = 0$
Chinese remainder theorem	Fermat's last theorem	Natural logarithms of complex numbers
Twin primes problem	Hypercomplex numbers	Diophantine application: Cole numbers
Odd perfect numbers	Euclidean algorithm for GCF	Palindrome numbers
Factorable sets of integers of the form $ak + b$	Algebraic congruences	Inequalities related to Fibonacci numbers
Combinatorics – art of counting	Boolean algebra	Graphical representation of roots of complex numbers
Roots of unity	Fermat's little theorem	Prime number sieves
Recurrence expressions for phi (golden ratio)		
	Geometry	
Non-Euclidean geometries	Cavalieri's principle	Packing 2D and 3D shapes
Ptolemy's theorem	Hexaflexagons	Heron's formula
Geodesic domes	Proofs of Pythagorean theorem	Minimal surfaces and soap bubbles
Tesseract – a 4D cube	Map projections	Tiling the plane – tessellations
Penrose tiles	Morley's theorem	Cycloid curve

Geometry (continued)		
Symmetries of spider webs	Fractal tilings	Euler line of a triangle
Fermat point for polygons and polyhedra	Pick's theorem and lattices	Properties of a regular pentagon
Conic sections	Nine-point circle	Geometry of the catenary curve
Regular polyhedra	Euler's formula for polyhedra	Eratosthenes – measuring earth's circumference
Stacking cannon balls	Ceva's theorem for triangles	Constructing a cone from a circle
Conic sections as loci of points	Consecutive integral triangles	Area of an ellipse
Mandelbrot set and fractal shapes	Curves of constant width	Sierpinksi triangle
Squaring the circle	Polyominoes	Reuleaux triangle
Architecture and trigonometry	Spherical geometry	Gyroid – a minimal surface
Geometric structure of the universe	Rigid and non-rigid geometric structures	Tangrams
Calculus/analysis and functions		
Mean value theorem	Torricelli's trumpet (Gabriel's horn)	Integrating to infinity
Applications of power series	Newton's law of cooling	Fundamental theorem of calculus
Brachistochrone (minimum time) problem	Second order differential equations	L'Hôpital's rule and evaluating limits
Hyperbolic functions	The harmonic series	Torus – solid of revolution
Projectile motion	Why ***e*** is base of natural logarithm function	
Statistics and modelling		
Traffic flow	Logistic function and constrained growth	Modelling growth of tumours
Modelling epidemics/spread of a virus	Modelling the shape of a bird's egg	Correlation coefficients
Central limit theorem	Modelling change in record performances for a sport	Hypothesis testing
Modelling radioactive decay	Least squares regression	Modelling the carrying capacity of the earth
Regression to the mean	Modelling growth of computer power past few decades	
Probability and probability distributions		
The Monty Hall problem	Monte Carlo simulations	Random walks
Insurance and calculating risks	Poisson distribution and queues	Determination of π by probability
Lotteries	Bayes' theorem	Birthday paradox
Normal distribution and natural phenomena	Medical tests and probability	Probability and expectation

Games and game theory		
The prisoner's dilemma	Sudoku	Gambler's fallacy
Poker and other card games	Knight's tour in chess	Billiards and snooker
Zero sum games		
Topology and networks		
Knots	Steiner problem	Chinese postman problem
Travelling salesman problem	Königsberg bridge problem	Handshake problem
Möbius strip	Klein bottle	
Logic and sets		
Codes and ciphers	Set theory and different 'size' infinities	Mathematical induction (strong)
Proof by contradiction	Zeno's paradox of Achilles and the tortoise	Four colour map theorem
Numerical analysis		
Linear programming	Fixed-point iteration	Methods of approximating π
Applications of iteration	Newton's method	Estimating size of large crowds
Generating the number e	Descartes' rule of signs	Methods for solving differential equations
Physical, biological and social sciences		
Radiocarbon dating	Gravity, orbits and escape velocity	Mathematical methods in economics
Biostatistics	Genetics	Crystallography
Computing centres of mass	Elliptical orbits	Logarithmic scales – decibel, Richter, etc.
Fibonacci sequence and spirals in nature	Predicting an eclipse	Change in a person's BMI over time
Concepts of equilibrium in economics	Mathematics of the 'credit crunch'	Branching patterns of plants
Column buckling – Euler theory		
Miscellaneous		
Paper folding	Designing bridges	Mathematics of rotating gears
Mathematical card tricks	Curry's paradox – 'missing' square	Bar codes
Applications of parabolas	Music – notes, pitches, scales…	Voting systems
Flatland by Edwin Abbott	Terminal velocity	Towers of Hanoi puzzle
Photography	Art of M.C. Escher	Harmonic mean
Sundials	Navigational systems	The abacus
Construction of calendars	Slide rules	Different number systems
Mathematics of juggling	Global positioning system (GPS)	Optical illusions
Origami	Napier's bones	Celtic designs/knotwork
Design of product packaging	Mathematics of weaving	

Website support

Further guidance and information concerning Internal Assessment is available from the authors' website at www.wazir-garry-math.org. You are encouraged to register with our site. Along with a considerable amount of support for other aspects of the IB Mathematics Higher Level course, there will be a section on our website devoted specifically to the Mathematical Exploration. We will be regularly updating our site so that you will have access to thorough and useful advice, materials and updates regarding how to get the most out of your Mathematical Exploration.

19 Sample Examination Papers

Paper 1 Sample A

Paper 1 is a non-calculator paper. Your exam paper will have instructions on the first page, some of which are reproduced here.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

It is important that you remember to show work because examiners will award marks for correct work leading to the final solution. Also, if your final answer is incorrect, you will not end up losing all the marks.

Section A

Answer all the questions in the spaces provided. Working may be continued below the lines, if necessary.

Section B

Answer all the questions on the answer sheets provided. Please start each question on a new page.

Section A

1 [*Maximum mark:* 6]
Given that $\log_b a = 0.74$ and $\log_b(a - 1) = 0.65$, find the value of the following expression:

$$\log_b(a^4 - 1) - 2\log_b(a^2 + 1) + \log_b(a^3 + a) - \log_b(a + 1)$$

Give your answer to 2 decimal places.

2 [*Maximum mark:* 6]
Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2 + 1}$$

from 0 to $\sqrt{e^3 - 1}$ about the y-axis.

3 [*Maximum mark:* 6]
Find out where the normal line to the curve $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersects the curve a second time.

4 [*Maximum mark:* 8]
Flaws appear randomly in a roll of textile at an average of 2 per metre length.
a) Find the probability that more than 3 flaws appear in a randomly chosen metre from this material.
b) Find the probability that more than 3 flaws appear in a randomly chosen 2-metre piece from this fabric.
c) Two pieces of 1 metre each are chosen at random. Find the probability the total number of flaws is 3.

5 [*Maximum mark:* 5]
If $\frac{3\pi}{2} < t < 2\pi$ and $\cos t = \frac{3}{\sqrt{10}}$, find the value of $\operatorname{cosec} t + \cos 2t$.

6 [*Maximum mark:* 6]
When $P(x) = ax^5 + 3x^2 - 2x + b$ is divided by $(x + 2)$ the remainder is -47, while if it is divided by $(x - 1)$ the remainder is 4. Find the values of a and b.

7 [*Maximum mark:* 4]
Simplify the following expression and write your answer in the form $a + bi$ where a and b are real numbers.

$$\frac{(2 + 3i)^2}{3 - 2i}$$

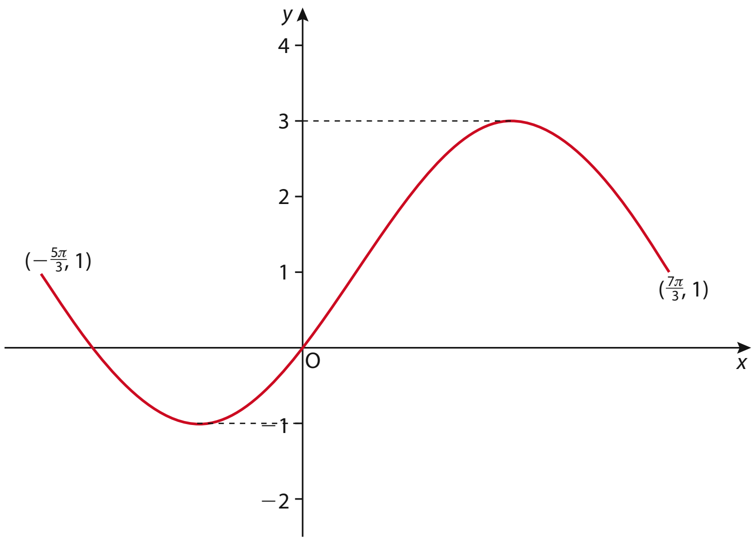
8 [*Maximum mark:* 5]
Find the value of k such that the following is a convergent geometric series whose sum to infinity is 12.

$$\sum_{i=1}^{n} 4k^{i-1}$$

9 [*Maximum mark:* 6]
The following graph is the graph of a function of the form

$$y = a\sin(b(x - c)) + d.$$

Determine the values of a, b, c and d.



10 [*Maximum mark:* 8]
Graph the rational function

$$y = \frac{2x^2 + 3x}{x^2 + x - 2}.$$

Show clearly all x- and y-intercepts and asymptotes.

Section B

11 [*Maximum mark:* 13]
The position vectors of the points A, B and C are

$$\mathbf{a} = 4\mathbf{i} - 9\mathbf{j} - \mathbf{k},\ \mathbf{b} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \text{ and } \mathbf{c} = 4\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}.$$

a) Find the unit vector parallel to $\overrightarrow{AB}$. [*3 marks*]
b) Find the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [*3 marks*]
c) Find a set of parametric equations for plane ABC. [*3 marks*]
d) Find the area of triangle ABC. [*4 marks*]

12 [*Maximum mark:* 21]
a) A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{6}$, $P(B/A) = \frac{2}{5}$.
Calculate the probability that
(i) both A and B occur
(ii) either A or B occurs, but not both
(iii) A occurs, knowing that B has occurred. [*10 marks*]

b) An industrial company has f female and m male employees. The employees arrive in the morning at random, and we will assume that the probability that the first employee to arrive any day will be a female is

$$\frac{f}{f + m}.$$

Mr Guard plans to watch the employees arrive on four consecutive days. If females arrive first every day on all four days, he will conclude that $f > m$; if males arrive first every day of the four days, he will conclude that $f < m$; otherwise, he will conclude that $f = m$. Ms Reception on the other hand wants to watch for 7 days. If females arrive first on 6 or 7 days, she will conclude that $f > m$; if males arrive first on 6 or 7 days, she will conclude that $f < m$; otherwise, she will conclude that $f = m$.
(i) If $f = m$, who, if any, is more likely to be wrong?
(ii) If $f = \frac{m}{2}$, what is the probability that Ms Reception will wrongly conclude that $f = m$? [*11 marks*]

13 [*Maximum mark:* 14]
a) A sequence of real numbers $\{u_n\}$ is defined by

$$u_{n+1} = 2u_n \cos\theta - u_{n-1};\ n > 1,\ u_0 = 1,\ u_1 = \cos\theta.$$

Prove by mathematical induction that $u_n = \cos n\theta$. [*8 marks*]

b) The kth term v_k of the series $12 + 30 + 58 + \ldots$ is given by

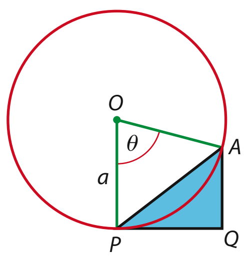
$$v_k = 5k^2 + 3k + 4.$$

Find $\sum_{k=1}^{n} v_k$, and express your answer in terms of n. [*6 marks*]

Hint: use the fact that $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

[IBO, 1978]

14 [*Maximum mark:* 12]
From a fixed point A on a circle with centre O and radius a, a perpendicular is dropped to the tangent at P to the circle.



a) Given the central angle θ as shown, prove that the area of triangle APQ is

$$\tfrac{1}{2}a^2|\sin\theta|(1 - \cos\theta).$$ [*7 marks*]

b) As P moves along the circle, find the maximum value of the area of triangle APQ. [*5 marks*]

Paper 1 Sample B

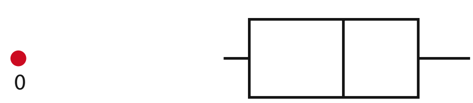
Section A

1 [*Maximum mark:* 6]
12 and $-\frac{4}{9}$ are the second and fifth terms of a geometric sequence.

a) Find the sum of the first n terms of this sequence.

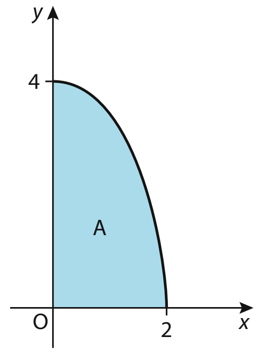
b) Find the sum to infinity of this sequence.

2 [*Maximum mark:* 4]
The data $\{0, 11, 12, 12, 14, 16, 17, 18, 19, 20, 21\}$ are represented by the box-plot below.



0 is considered an outlier because it is more than 1.5 IQR (interquartile range) below the first quartile. Show that this is true.

3 [*Maximum mark:* 8]
The diagram below shows the shaded region A, in the first quadrant that is enclosed by the curve $y^2 = 8(2 - x)$. Find the ratio of the volume of the solid formed when A is rotated through 2π radians around the x-axis to the volume of the solid formed when A is rotated through 2π radians around the y-axis.



4 [*Maximum mark:* 6]
$P(x) = x^3 + bx^2 + x + c$ is divisible by $(x - 2)$ but leaves a remainder of -35 when divided by $(x + 3)$. Find b and c.

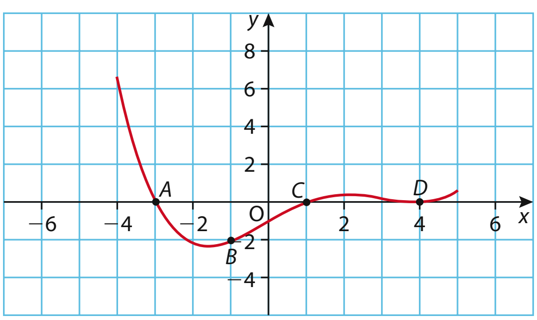
5 [*Maximum mark:* 6]
Consider the function

$$f(x) = e^{2x - x^2}.$$

a) Find the maximum value of this function.

b) Find the x-coordinate of the points of inflexion.

6 [*Maximum mark:* 6]
The diagram shows the graph of a function $y = f(x)$, which passes through the points $A(-3, 0)$, $B(-1, -2)$, $C(1, 0)$ and $D(4, 0)$.



a) Graph the function $g(x) = f(x + 2)$ clearly indicating the coordinates of the images of A, B, C and D.

b) Graph the function $h(x) = f(2x + 2)$ clearly indicating the coordinates of the images of A, B, C and D.

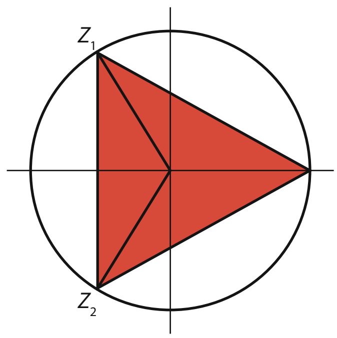
7 [*Maximum mark:* 6]

The lines L_1 and L_2 have the following equations:

$$L_1\text{: } r = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix};\ L_2\text{: } \frac{x-5}{2} = \frac{y+1}{-3} = z+5$$

a) Find the point of intersection between these lines.
b) Find the cosine of the acute angle between the lines.

8 [*Maximum mark:* 6]



The cube roots of 1 are 1, $z_1 = a + bi$, and $z_2 = a - bi$.
a) Find the values of a and b.
b) In the Argand diagram, join the points corresponding to the roots. Find the area of the triangle formed by the points as vertices.

9 [*Maximum mark:* 6]
Find all solutions to the equation

$$\cos 4\theta + \sin^2 2\theta = \frac{1}{4}$$

in the interval $[0, 2\pi]$.

10 [*Maximum mark:* 6]

The sum of the first n terms of a sequence is $\frac{3n^2 + 4n}{2}$, $n \in \mathbb{Z}^+$.

a) Find the first three terms of the sequence.
b) Find an expression for the nth term of the sequence, giving your answer in terms of n.

Section B

11 [*Maximum mark:* 20]
Consider the plane P with equation $2x + 3y - z = 11$ and the line L with equation

$$\frac{x-2}{-1} = \frac{y-1}{2} = \frac{z-3}{5}.$$

a) Show that the point $A(1, 3, 8)$ lies on the line L. [*3 marks*]
b) Find the coordinates of point B, the intersection between line L and plane P. [*3 marks*]

c) Find an equation of a line M containing A and perpendicular to P. [4 *marks*]

d) Find the coordinates of the point C, the intersection of line M and P. [2 *marks*]

e) Hence or otherwise, find the point D symmetric to A about plane P. [4 *marks*]

f) Find a set of parametric equations of the line through B and D. [4 *marks*]

12 [*Maximum mark:* 20]
Consider the complex number $z = \cos\theta + i\sin\theta$.

a) Show that $z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2i\sin\theta$. [2 *marks*]

b) Show that $z^n + \frac{1}{z^n} = 2\cos(n\theta)$, and find a similar expression for $z^n - \frac{1}{z^n}$. [4 *marks*]

c) Hence, show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$, and find a similar expression for $\cos^5\theta$. [8 *marks*]

d) Hence, find $\int\left(4\sin^2\left(\frac{3\theta}{2}\right) - 32\sin^5\theta\right)d\theta$. [6 *marks*]

13 [*Maximum mark: 20*]
A function f is defined by

$$f(x) = \frac{x^2 + x - 2}{x^2 - 2x - 3}.$$

a) What is the largest possible domain of f? Find its derivative $f'(x)$. Show that $f'(x)$ has a constant sign over its domain. [6 *marks*]

b) Write down the equations of the asymptotes of the curve $y = f(x)$. [3 *marks*]

c) Use the information developed so far to sketch the graph of $f(x)$. [3 *marks*]

d) Find the real numbers P, Q and R such that the following is true for all values of x in the domain of f:

$$f(x) = P + \frac{Q}{x+1} + \frac{R}{x-3}.$$ [4 *marks*]

e) Use the expression above to find

$$\int f(x)dx.$$ [4 *marks*]

Paper 1 Sample C
Section B

1 [*Maximum mark:* 5]

When the function $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x + 1)$, the remainder is -20.

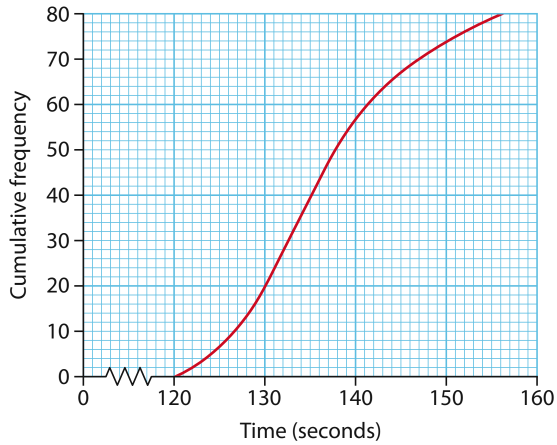
Find the value of a.

2 [*Maximum mark:* 5]

A bag contains 2 red balls, 3 blue balls and 4 green balls. A ball is chosen at random from the bag and is not replaced. A second ball is chosen. Find the probability of choosing one green ball and one blue ball in any order.

3 [*Maximum mark:* 6]

The 80 applicants for a Sports Science course were required to run 800 metres and their times were recorded. The results were used to produce the following cumulative frequency graph.



Estimate

a) the median [2 *marks*]

b) the interquartile range. [4 *marks*]

4 [*Maximum mark:* 6]

Find the coordinates of the point where the line with the vector equation

$\boldsymbol{r} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ intersects the plane with the equation

$2x + 3y - z = 2$.

5 [*Maximum mark:* 7]

a) Express the complex number $8i$ in polar form. [3 *marks*]

b) The cube root of $8i$ which lies in the first quadrant is denoted by z. Express z

(i) in polar form [2 *marks*]

(ii) in Cartesian form. [2 *marks*]

6 [*Maximum mark:* 7]

Find the equation of the line that is tangent to the curve $3x^2 + 4y^2 = 7$ where $x = 1$ and $y > 0$.

7 [*Maximum mark:* 6]

Find the value of x satisfying the equation

$(3^x)(4^{2x+1}) = 6^{x+2}$

Give your answer in the form $\dfrac{\ln a}{\ln b}$ where $a, b \in \mathbb{Z}$.

8 [*Maximum mark:* 6]

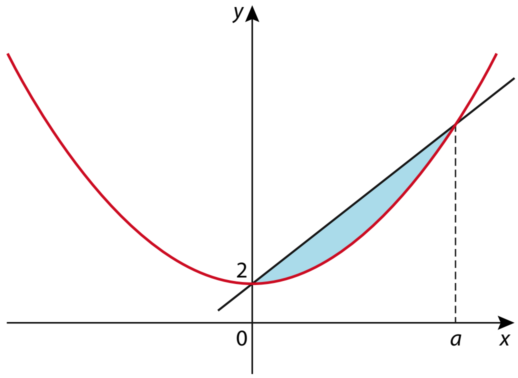
a) The independent events A and B are such that $P(A) = 0.4$ and $P(A \cup B) = 0.88$. Find $P(B)$. [4 *marks*]

b) Find the probability that either A occurs or B occurs, but **not** both. [2 *marks*]

9 [*Maximum mark:* 6]

The area of the enclosed region shown in the diagram is defined by

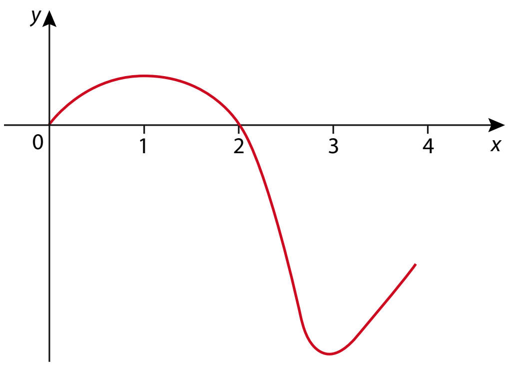
$y \geqslant x^2 + 2$, $y \leqslant ax + 2$, where $a > 0$



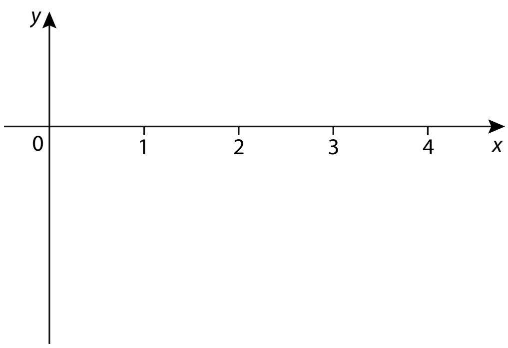
The region is rotated 360° about the x-axis to form a solid of revolution. Find, in terms of a, the volume of this solid of revolution.

10 [*Maximum mark:* 6]

The diagram below shows the graph of equation $y_1 = f(x)$, $0 \leqslant x \leqslant 4$.



Copy the axes below and sketch the graph of $y_2 = \int_0^x f(t)\, dt$, marking clearly the points of inflexion.



Section B

11 [*Maximum mark:* 17]

The probability density function of the random variable X is given by

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{4 - x^2}}, & \text{for } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of the constant k. [5 *marks*]

b) Show that $E(X) = \dfrac{6\left(2 - \sqrt{3}\right)}{\pi}$ [7 *marks*]

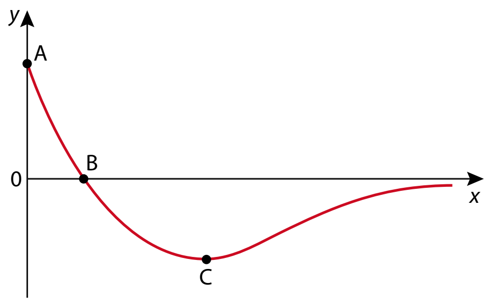
c) Find the median of X. [5 *marks*]

12 [*Maximum mark:* 16]

a) Find the root of the equation $e^{2-2x} = 2e^{-x}$ giving the answer as a logarithm. [4 *marks*]

b) The curve $y = e^{2-2x} - 2e^{-x}$ has a minimum point. Find the coordinates of this minimum. [7 *marks*]

c) The curve $y = e^{2-2x} - 2e^{-x}$ is shown below.



Write down the coordinates of the points A, B and C. [3 *marks*]

d) Hence state the set of values of k for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct roots. [2 *marks*]

13 [*Maximum mark:* 13]

a) Show that the following system of equations will have a unique solution when $a \neq -1$.

$$\begin{aligned} x + 3y - z &= 0 \\ 3x + 5y - z &= 0 \\ x - 5y + (2 - a)z &= 9 - a^2 \end{aligned}$$

[5 *marks*]

b) Given that $a \neq -1$, state the solution in terms of a. [6 *marks*]

c) Hence, solve

$$\begin{aligned} x + 3y - z &= 0 \\ 3x + 5y - z &= 0 \\ x - 5y + z &= 8 \end{aligned}$$

[2 *marks*]

14 [*Maximum mark:* 14]

a) Using mathematical induction, prove that

$$\sum_{r=1}^{n}(r+1)2^{r-1} = n(2^n).$$ [7 *marks*]

b) The first three terms of a geometric sequence are also the first, eleventh and sixteenth terms of an arithmetic sequence.

The terms of the geometric sequence are all different.

The sum to infinity of the geometric sequence is 18.

(i) Find the common ratio of the geometric sequence, clearly showing all working. [4 *marks*]

(ii) Find the common difference of the arithmetic sequence. [3 *marks*]

Paper 2 Sample A

Paper 2 is a GDC paper. Your exam paper will have some instructions on the first page, some of which are reproduced here.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

It is important that you remember to show work because examiners will award marks for correct work leading to the final solution. Also, if your final answer is incorrect, you will not end up losing all the marks.

Specific to GDC papers: If you use a GDC to arrive at your conclusion, you need to show work leading to what you entered into your GDC. For example, if you are to find the area of a certain region under a curve between two points a and b, then you set up the integral leading to the solution but not necessarily the symbolic manipulation required.

Example
Find the area enclosed by the curve $f(x) = 2x^3 - 9x^2 + x + 12$ and the x-axis.

Suggested answer
To find the area of this region, we observe that the function intersects the x-axis at three different points: at $x = -1$, $x = 1.5$ and $x = 4$.

Therefore, the area of the region is $\int_{-1}^{4} |2x^3 - 9x^2 + x + 12|\,dx \approx 39.1$.

Section A

1 [*Maximum mark:* 5]

For what values of x is the following inequation true?

$$-7x^2 - 27x + 4 \geqslant 0$$

2 [*Maximum mark:* 6]
In triangle ABC, $BC = 6$, $AC = 7$ and $\angle A = 30°$. Find all possible values of AB.

3 [*Maximum mark:* 6]
An experiment can result in one or both of events A or B with the following probabilities:

	A	A'
B	0.34	0.46
B'	0.15	0.05

Find:

a) $P(A \cup B)$ b) $P(A|B)$.

c) Are A and B independent? Justify.

4 [*Maximum mark:* 4]
In a binomial experiment with n trials, the probability of success $p = 0.6$ and $P(x < 2) = 0.1792$. Find the value of n.

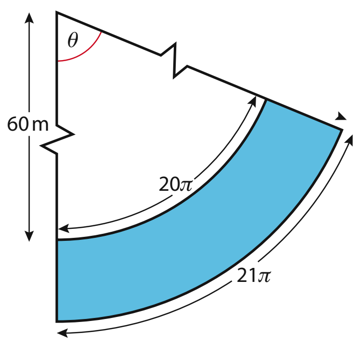
5 [*Maximum mark:* 7]

Consider the function $f(x) = \dfrac{2e^x}{1 + 3e^x}$.

a) Find $f^{-1}(x)$.

b) Find the exact domain of $f^{-1}(x)$.

6 [*Maximum mark:* 5]
A part of a track is shown in the diagram. The radius of the inner circle is 60 m and the width of the track is 3 m. The length of the inner arc is 20π and the outer arc is 21π. Find the area of the track.



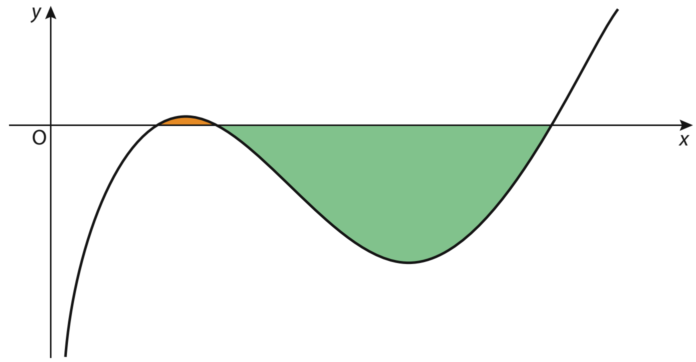
7 [*Maximum mark:* 6]
Consider the complex number $z = 5\sqrt{3} - 5i$.

a) Express z in the form $re^{i\theta}$, presenting your answer in exact form.

b) Find the fifth roots of the complex number and sketch them in an Argand diagram.

8 [*Maximum mark:* 7]
The figure below is that of the function

$$f(x) = \cos x \ln x,\ 0 \leqslant x \leqslant 2\pi.$$



Find the ratio of the shaded area below the x-axis to the shaded area above the x-axis.

9 [*Maximum mark:* 9]
The continuous random variable has the following pdf:

$$f(x) = \begin{cases} mx\sqrt{4 - x^2} & x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of m.
b) Find the ratio of the area between the mean and median to that between the mean and mode.

10 [*Maximum mark:* 5]
Solve the initial value problem

$$e^x \sin 2y \frac{dy}{dx} = \cos y(e^{2x} - x), y(0) = 0.$$

Section B

11 [*Maximum mark:* 18]
A function is defined by

$$f(x) = \tfrac{1}{3}x^3 - x^2 - 3x + 9, x \in \{\mathbb{R}\}.$$

a) Find the points where the graph of this function intersects the x-axis. [*3 marks*]
b) Find the point in the first quadrant where the normal to the curve at $(0, 9)$ meets the curve again. [*4 marks*]
c) Find the local maximum and minimum of the function in the interval $[-2, 5]$. [*3 marks*]
d) Sketch the graph of the function. [*3 marks*]
e) Find the area enclosed by the function and the line connecting the maximum to the minimum points. [*5 marks*]

12 [*Maximum mark:* 20]
a) Write down the expanded form of $(1 + x)^n$ using binomial coefficients. Include the term containing x^r where $0 \leqslant r \leqslant n$. [*4 marks*]
b) Calculate the coefficient of x^5 in the expansion of $(1 + x)^7(1 + x)^{11}$. [*4 marks*]

c) Calculate the coefficient of x^r in the expansion of the identity $(1 + x)^m(1 + x)^n = (1 + x)^{m+n}$ where $0 \leqslant r \leqslant n$, and $0 \leqslant r \leqslant m$. [*4 marks*]

d) Hence, show that

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \dots + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}.$$

[*4 marks*]

e) By considering $n = m = r$, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}.$$

[*4 marks*]

13 [*Maximum mark:* 22]

The amount of salt extracted in a large salt mine is modelled by a normal distribution with a mean of 1.5 cubic metres per hour of production time, and a standard deviation of 0.375 cubic metres.

a) (i) Find the probability that in a randomly chosen hour of production, the output is between 1.2 cubic metres and 1.875 cubic metres. [*3 marks*]

(ii) In 10% of the production hours, the output is considered low. How many cubic metres are considered low? [*3 marks*]

b) The production process can be adjusted to meet production demand. So, the mean and the standard deviation can be altered. The management would like to see that the production exceeds 2 cubic metres at most 10% of the time, and falls short of 0.5 cubic metres 5% of the time. Find the values of the required mean and standard deviation. [*8 marks*]

Because of the hard nature of the extraction process, the machines used in the process occasionally stop and have to be restarted. The number of stoppages per hour of production is modelled by a Poisson distribution with a mean of 3 stoppages.

c) (i) Find the probability that the machines stop at least 4 times in each of three successive hours of production. [*5 marks*]

(ii) Find the probability that the machines stop 20 times during a randomly chosen 8-hour shift. [*3 marks*]

Paper 2 Sample B

Section A

1 [*Maximum mark:* 7]

A pizza producer packs half-baked pizzas in boxes, freezes them and distributes them to consumers. For a pizza to fit in a box, the diameter must not exceed 30 centimetres. All pizzas with diameter larger than 30 cm have to be re-done. To comply with the label, pizzas must not be smaller than 27 cm in diameter. It is found that 4% of the pizzas are too large while 1% are too small. Assuming the diameters of these pizzas to be normally distributed, find the mean and standard deviation.

2 [*Maximum mark:* 5]
An infinite geometric series converges to 24. The sum of the first three terms is 208/9.
Find the sum of the first 6 terms.

3 [*Maximum mark:* 6]
Consider the function

$$f(x) = 3^{3x - x^2}.$$

a) Find the maximum value of this function.

b) Find the coordinates of the points of inflexion.

4 [*Maximum mark:* 5]
Solve the following inequation:

$$\frac{|x-1|+3}{|x+1|-2} < 2$$

5 [*Maximum mark:* 6]
Consider the function

$$f(x) = \sin\left(\sqrt{4 - x^2}\right).$$

a) Find the domain and range of the function.

b) For what values of x does this function have an extreme value?

6 [*Maximum mark:* 7]
The probability density function of a random variable is

$$f(x) = \begin{cases} k(2 - \log_3(4x^2 + 1)) & -a \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $a = \sqrt{2}$.
b) Find the value of k correct to 3 decimal places.

7 [*Maximum mark:* 7]
The number of defects per square metre of fabric is known to follow a Poisson distribution. It is discovered that

$$\mathrm{P}(x \leqslant 3) = 0.2381.$$

a) Find the average number of defects per square metre, to the nearest integer.
b) You randomly pick 1 square metre of this fabric for inspection. Find the probability of observing at least 3 defects.

8 [*Maximum mark:* 6]
You invest an amount of \$1000 at an interest rate of 6% compounded semi-annually.
How much money will you have in 20 years?
If you were offered to invest the money at continuous compounding, how long will it take you to earn the same amount?

9 [*Maximum mark:* 5]

Find the equation of the tangent line to the curve defined by

$$\ln(xy) = 2x$$

at the point $(1, e^2)$.

10 [*Maximum mark:* 6]

Solve the differential equation

$$\frac{dy}{dx} = \frac{e^{(y^2 + \sin x)}}{y \sec x}; \; y(0) = \sqrt{\ln 2}.$$

Section B

11 [*Maximum mark:* 21]

a) $5^x = e^{kx}$ for all real numbers x. Find the value of k. [*4 marks*]

b) Use the value of k found in a) to find the derivative of $f(x) = 5^x$. [*3 marks*]

c) A random variable X has the following probability density function:

$$f(x) = \begin{cases} 5^x & 0 \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of a. [*4 marks*]

(ii) Find the expected value of X. [*4 marks*]

d) Three values of this random variable are chosen at random. What is the probability that:

(i) at least one of the values is larger than 0.5? [*3 marks*]

(ii) at most two of the values are less than 0.5? [*3 marks*]

12 [*Maximum mark:* 18]

Let A be the matrix $\begin{pmatrix} -2 & 2 & 3 \\ -4 & 5 & 5 \\ 2 & 1 & -3 \end{pmatrix}$ and I be the identity matrix of order 3.

a) Show that: $\det(A - kI) = -k^3 + 22k - 6$. [*4 marks*]

b) With an appropriate choice of k, find the determinant of A. [*3 marks*]

c) You are given that the matrix A satisfies the equation

$$-6I + 22A - A^3 = 0.$$

(i) Express the matrix A^{-1} in terms of A. [*5 marks*]

(ii) Hence, show that the three planes

$$20x - 9y + 5z = 2,$$
$$2x + 2z = 3,$$
$$14x - 6y + 2z = 5$$

intersect at one point. Find the coordinates of that point. [*6 marks*]

13 [*Maximum mark:* 21]
Consider the complex number $z = \cos\theta + i\sin\theta$.

a) Use DeMoivre's theorem to find z^5. [*3 marks*]

b) Hence, show that $\cos 5\theta = 11\cos^5\theta - 10\cos^3\theta + 5\sin^4\theta\cos\theta$, and $\sin 5\theta = 15\sin\theta\cos^4\theta - 10\sin\theta\cos^2\theta + \sin^5\theta$. [*6 marks*]

c) Hence, show that $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$. [*6 marks*]

d) Hence, find the solutions to the equation
$t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0$,
expressing your answer correct to 3 d.p. [*6 marks*]

Paper 2 Sample C

Section A

1 [*Maximum mark:* 6]

Triangle ABC has $\hat{C} = 42°$, BC $= 1.74$ cm, and area 1.19 cm^2.

a) Find AC. [*2 marks*]

b) Find AB. [*4 marks*]

2 [*Maximum mark:* 5]

Find the values of a and b, where a and b are real, given that
$(a + bi)(2 - i) = 5 - i$.

3 [*Maximum mark:* 6]

The function f is defined as $f(x) = \dfrac{3x - 4}{x + 2}$, $x \neq -2$.

a) Find an expression for $f^{-1}(x)$. [*5 marks*]

b) Write down the domain of f^{-1}. [*1 marks*]

4 [*Maximum mark:* 6]

The function f is defined as $f(x) = \sin x \ln x$ for $x \in [0.5, 3.5]$.

a) Write down the x-intercepts. [*2 marks*]

b) The area above the x-axis is A and the **total** area below the x-axis is B.

If $A = kB$, find k. [*4 marks*]

5 [*Maximum mark:* 6]

The weights in grams of bread loaves sold at a supermarket are normally distributed with mean 200 grams. The weights of 88% of the loaves are less than 200 grams. Find the standard deviation.

6 [*Maximum mark:* 6]

Find $\int e^{2x} \sin x \, dx$.

7 [*Maximum mark:* 6]

The number of car accidents occurring per day on a highway follows a Poisson distribution with mean 1.5.

a) Find the probability that more than two accidents will occur on a given day. [*2 marks*]

b) Given that at least one accident occurs on another day, find the probability that more than two accidents occur on that day. [*4 marks*]

8 [*Maximum mark:* 6]

There are 10 seats in a row in a waiting room. There are six people in the room.

a) In how many different ways can they be seated? [*2 marks*]

b) In the group of six people, there are three sisters who must sit next to each other.
In how many different ways can the group be seated? [*4 marks*]

9 [*Maximum mark:* 6]

Solve the differential equation given that $y = 1$ when $x = -1$.

$$(x + 2)^2 \frac{dy}{dx} = 4xy \quad (x > -2)$$

10 [*Maximum mark:* 6]

The radius and height of a cylinder are both equal to x cm. The curved surface area of the cylinder is increasing at a constant rate of 10 cm^2/sec. When $x = 2$, find the rate of change of

a) the radius of the cylinder [*4 marks*]

b) the volume of the cylinder. [*2 marks*]

Section B

11 [*Maximum mark:* 12]

A machine is set to produce bags of salt, whose weights are distributed normally, with a mean of 110 grams and standard deviation of 1.142 grams. If the weight of a bag of salt is less than 108 grams, the bag is rejected. With these settings, 4% of the bags are rejected.

The settings of the machine are altered and it is found that 7% of the bags are rejected.

a) (i) If the mean has not changed, find the new standard deviation, **correct to three decimal places.** [*4 marks*]

The mean is adjusted to operate with this new value of the standard deviation.

(ii) Find the value, **correct to two decimal places**, at which the mean should be set so that only 4% of the bags are rejected. [*4 marks*]

b) With the new settings from part (a), it is found that 80% of the bags of salt have a weight which lies between A grams and B grams, where A and B are symmetric about the mean.

Find the values of A and B, giving your answers **correct to two decimal places**. [*4 marks*]

12 [*Total mark:* 22]

Part A [*Maximum mark:* 12]

A bag contains a very large number of ribbons. One quarter of the ribbons are yellow and the rest are blue. Ten ribbons are selected at random from the bag.

a) Find the expected number of yellow ribbons selected. [*2 marks*]

b) Find the probability that exactly six of these ribbons are yellow. [*2 marks*]

c) Find the probability that at least two of these ribbons are yellow. [*3 marks*]

d) Find the most likely number of yellow ribbons selected. [*4 marks*]

e) What assumption have you made about the probability of selecting a yellow ribbon? [*1 mark*]

Part B [*Maximum mark:* 10]

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{x}{1+x^2}, & \text{for } 0 \leqslant x \leqslant k \\ 0, & \text{otherwise} \end{cases}$$

a) Find the exact value of k. [*5 marks*]

b) Find the mode of X. [*2 marks*]

c) Calculate $\mathrm{P}(1 \leqslant X \leqslant 2)$. [*3 marks*]

13 [*Total mark:* 26]

Part A [*Maximum mark:* 14]

a) The line L_1 passes through the point A(0, 1, 2) and is perpendicular to the plane $x - 4y - 3z = 0$. Find a Cartesian equation of L_1. [*2 marks*]

b) The line L_2 is parallel to L_1 and passes through the point P(3, −8, −11). Find the vector equation of the line L_2. [*2 marks*]

c) (i) The point Q is on the line L_1 such that $\overrightarrow{PQ}$ is perpendicular to L_1 and L_2.

Find the coordinates of Q.

(ii) Hence find the distance between L_1 and L_2. [*10 marks*]

Part B [*Maximum mark:* 12]

Consider this system of equations.

$$\begin{aligned} x + 2y + kz &= 0 \\ x + 3y + z &= 3 \\ kx + 8y + 5z &= 6 \end{aligned}$$

a) Find the set of values of k for which this system of equations has a **unique** solution. [*6 marks*]

b) For each value of k that results in a non-unique solution, find the solution set. [*6 marks*]

20 Theory of Knowledge

What is TOK?

Theory of knowledge is concerned with how we know what we claim to know. As an IB diploma student you take classes in a number of areas of knowledge corresponding to the IB hexagon. While we call what we learn in each of these subjects 'knowledge', each seems to go about the process of getting this knowledge in a different way. Theory of knowledge examines these different ways of knowing and asks a number of questions about what sort of things can be considered facts, knowledge, good evidence and truth in each of the IB subjects.

Mathematics is rather puzzling as an area of knowledge. Most other subjects that we study in the IB base their knowledge claims upon observations of the world. Mathematics does not. Yet mathematics has profoundly practical applications in the world. How can this be? Knowledge claims in the sciences – while often fairly secure – are nonetheless provisional in some sense. Science allows the possibility that it is wrong – that some new observation or discovery will overturn previously held beliefs. The statements of mathematics, on the other hand, are certain. $1 + 1 = 2$ is not just probably true. It is certain. **It cannot be otherwise.** This is because $1 + 1 = 2$ can be proved. These features give mathematics a special place in TOK.

Explain why probability theory is certain even though it deals in probabilities.

Think of your favourite topic in the HL course. In this topic, can you identify (1) a mathematical transformation and (2) an invariant under this transformation? If you get stuck, ask your maths teacher. (Hint: when studying a function $f(x)$ defined on the real numbers, the function itself is a transformation of the whole real number line, and the set of points that are unmoved by the function, i.e. for which $f(x) = x$ are the invariants. This set is the fixed point set of the function. These points would be represented graphically as the points where the graph of $y = f(x)$ intersected the 45 degree line $y = x$.)

What is mathematics?

It is remarkably difficult to pin down exactly what mathematics is about. A first attempt might be: 'mathematics is the study of numbers'. Certainly, much of our school mathematics is concerned with operations on numbers and the relations between them. This is what is called **arithmetic**. But there is much more to mathematics than numbers, and mathematicians do not take kindly to being thought of as simply good at adding up the bill in the restaurant (actually many of them are not). One of the oldest fields of mathematical thought is **geometry**. When we study geometrical objects such as points, lines, planes, triangles, circles and ellipses we are not studying numbers as such. Rather we are studying the structure of space itself – in particular those aspects that stay unchanged under various types of geometrical transformation. These aspects we call **invariants**. Modern mathematics takes this idea further and studies structures, which are far removed from numbers or even our everyday intuitions about space and time. We could do far worse than define mathematics as the study of transformations and invariants.

What are the foundations of mathematics?

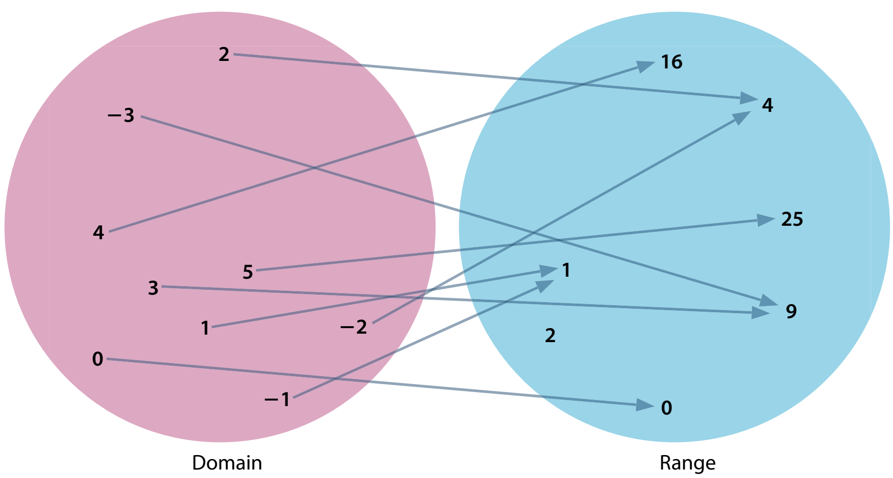
Sets

Modern mathematicians build up the raw materials of their subject from quite humble beginnings. Let us look at how they do this. They start off with some basic concepts about sets. A set, as you know, is just a collection of elements placed inside curly brackets. For example, we could consider a set $A = \{1, 2, 3, 4\}$. We can say that 1 belongs to A: $1 \in A$, but that 5 does not belong to A: $5 \notin A$. The notions of what it is to be a set and to belong to a set are **primitive**. This means that they cannot be explained in terms of more simple notions. If you keep on asking the question 'why?' (as some small children do), the questions stop when you get to a primitive concept (you find yourself answering: 'it just is'). Aristotle was aware of this when he stated that any explanation has to end somewhere. We can now answer him that explanations end in primitive concepts.

Think about an explanation in one of your IB subjects. Keep on asking the question 'why?' until you can go no further. What you are left with is a primitive notion. Are the primitive notions in physics different from those in history?

Mappings

We also need the idea of a mapping between sets. A mapping from A to B is a rule that assigns an element of B to each element of A. The functions that you study in your Higher Level Mathematics course are examples of mappings between the set of real numbers and itself.



An example of a mapping between two sets.

Notice that for a mapping to be well defined, every member of the domain set has to have an arrow (and only one arrow) pointing from it. But some members of the range set can have more than one arrow pointing to them (and the number 2 has no arrow pointing to it). This is an example of a many-to-one mapping. What is the mapping represented here?

Bertrand Russell and A. N. Whitehead, in their monumental book *Principia Mathematica* (1913), reduced the whole of mathematics to these simple notions. With a bit of work and a great deal of care and patience we can establish basic truths of arithmetic, such as $1 + 1 = 2$. (Proving this takes about four pages of quite sophisticated mathematical argument; this is a surprise to many students who think that 2 is *defined* to be $1 + 1$.)

Because we can build the whole of mathematics out of these primitive ideas of sets and mappings, does this mean that this is what mathematics is about?

Russell's paradox[1]

In constructing mathematics from set theory, we must be careful that we do not allow sets to be members of themselves. Consider the collection $D = \{d: d$ is a set **and** d contains more than 1 element$\}$. By this definition, D actually belongs to itself, since D contains more than one element. There is something rather strange about this, which might make us suspicious. The self-reference involved in thinking about sets that are members of themselves leads to a famous paradox discovered by the English philosopher and mathematician Bertrand Russell. He considered the set that is defined as follows: $S = \{s: s$ is a set **and** s does not belong to itself$\}$. The question he then asked was: Does S belong to itself or not? If the answer is yes – S does belong to itself – then, by the definition of S, S does not belong to itself. If S does not belong to itself, then, by the definition of S, S does belong to itself. Either way we get a contradiction. Russell realized that certain large collections (such as that of all sets) were actually too big to be a set. A collection like this is called a **proper class**.

[1]Bertrand Russell *The Principles of Mathematics* (1903) Cambridge

The barber of Seville

Russell's paradox is similar to the story of the barber of Seville. There was a man who lived in Seville who was a barber. He had a monopoly on the shaving industry in Seville. He shaved every man in the town who did not shave himself. What is contradictory about this?

Mathematics and the real world

1 + 1 = 2?

The objects of mathematics, such as the number systems that we use, are built up from elementary ideas about sets. In this sense, mathematics can be seen as a rather elaborate abstract game, which seems to be about nothing in particular. Bertrand Russell wrote: 'Mathematics is a subject where we do not know what we are talking about, nor whether what we are saying is true.' A possible response to this could be: 'We don't need to establish formally that $1 + 1 = 2$. It is easy to prove. Here I have one apple and there another apple. I put them together and I have two apples!' What is wrong with this approach? Think carefully about what abstractions we are making from the real world in order that this argument works. Does it still work with two glasses of water poured together, or two piles of sand pushed together, or two rabbits (male and female) left together for a suitable length of time? These are all examples of the rather curious and sometimes awkward connection between the world of mathematics and the real world.

The Platonist view of mathematics

Plato was aware of the tension between the world of perfect geometrical objects – points with no area, lines with no width, perfect circles and triangles – and the messy physical world. There are no perfectly thin lines, infinitely small points and perfect circles in the real world. But he thought there was a world of perfect mathematical objects underlying the imperfect physical world of our everyday experiences. This mathematical world existed independently of human beings. There would still be nine planets in the solar system long after human beings have ceased to exist on Earth (well, eight actually!). Plato's thinking can help explain the usefulness of mathematics. After all, mathematics is often described as the language of the natural sciences – it is almost impossible to do biology, chemistry or physics without it. But increasingly, mathematics is becoming the *lingua franca* of the social sciences. For example, cutting-edge research in economics is highly mathematical. Governments use highly complex mathematical models to make predictions about future inflation, unemployment and growth rates. This makes a lot of sense, if we grant that mathematics is 'out there' as part of the structure of our physical and social world, as Plato thought it was. That would explain why mathematical methods are so effective in solving real world problems. This is called the **Platonist** view of mathematics.

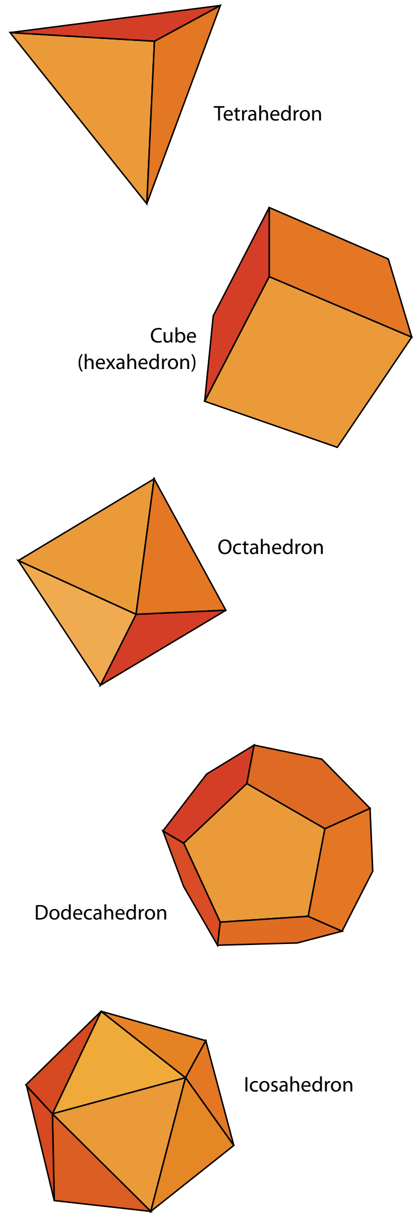
Formalist and constructivist mathematics

There are two responses to Plato's view that mathematics is 'out there'. One emphasizes the game-like nature of mathematics and the other the fact that it is played by human beings. The **formalist** approach treats mathematics as an abstract formal game. The game proceeds using an agreed set of rules from agreed starting points, or **axioms**. The individual symbolic statements of mathematics mean nothing outside the game, just as 'checkmate' is meaningless outside chess and 'fifteen-love' is meaningless outside the game of tennis.

The formalist must concede that any use mathematics has in the outside world is largely a coincidence. Is this a point against the formalist view of mathematics?

The **constructivist** sees mathematics as a human activity. To this way of thinking, when there are no more humans there will be no more mathematics. Mathematics is produced by individuals or societies in much the same way as literature and other cultural artefacts. Again, the constructivist has the problem of explaining the success of mathematics in describing, understanding and predicting the outside world. How can a man-made system fit the non-human world so well?

There is another problem with the constructivist view of mathematics. It seems that we are accountable to the truths of mathematics. Mathematicians speak of mathematics as having an independent existence – maths is there to be discovered rather than being man-made. It is certainly true, as we shall see later, that maths can throw up quite unexpected results. Is this compatible with the description of mathematics as being built up out of a few basic and abstract raw materials? In order to try to answer this question, we need to look a little more closely at what constitutes mathematical truth.



Plato thought that underlying the messy real world was the perfect world of mathematics. The five regular polyhedra shown are often called the Platonic solids.

Think about the question of whether mathematics is 'out there' in the world or whether it is an invention of human beings. Does this question occur in other areas of knowledge? Does it make sense to ask if English literature is out there in the world? Does it make sense to ask if chemistry is invented by humans?

What is truth in mathematics?

Let us look again at what we mean by **mathematical truth**. Mathematical statements are true if they can be proved. Before it is proved, a mathematical statement is called a **conjecture**. Once it is proved it is called a **theorem**. So, theorems are mathematical truths.

The idea of proof in mathematics is very old. In around 300 BC, the geometer Euclid of Alexandria formalized the notion of proof in his book *The Elements*. He proved a number of truths about geometrical figures. A proof is a list of statements. Each statement is derived from the preceding statement in the list using only the rules of logic. This is called **chain reasoning**. But what starts the chain in the first place? The first statement in the chain must be, in some sense, either true by definition or self-evident in some way. These self-evident truths are called **axioms**. They are considered to be basic or primitive mathematical truths. By definition, they cannot be proved. A mathematical proof builds a chain of reasoning from the axioms to final mathematical results – theorems.

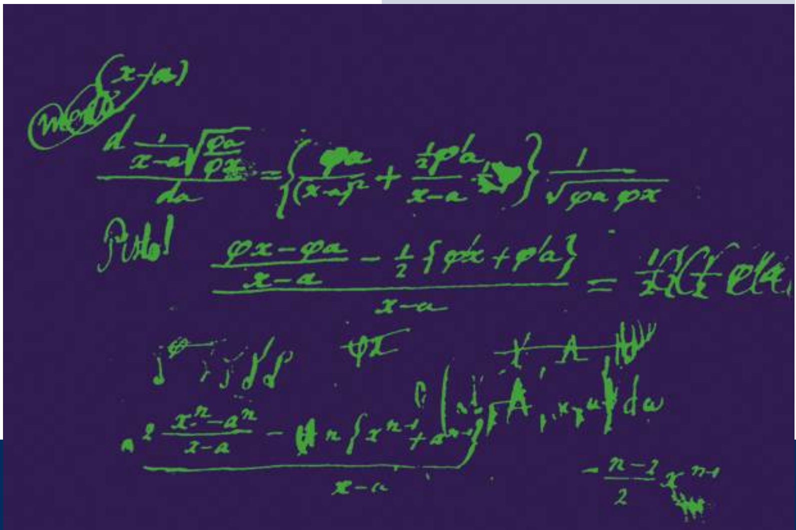
They are very special from a TOK perspective because it seems that a theorem is an example of knowledge that is certain. A mathematical theorem is not just probably true. It is true in the sense that, given the definitions of the terms it uses and the axiom system used to prove it, **it cannot be otherwise**. In TOK, we rarely meet truths that are certain in this absolute sense.

Theorem, theory and proof

Be careful that you do not confuse the word 'theorem' with the similar-sounding 'theory'. In mathematics, a theory is an established piece of mathematical work that might contain many theorems. In other words, mathematical theories are pieces of true mathematics. In science, the word is more problematic. It might apply to an established piece of science that has been tested and found to yield accurate predictions and to give good explanations of phenomena in the physical world. But the term can also refer to a more tentative idea that has not yet been thoroughly tested. It is a common mistake in TOK essays to make a statement such as: 'The theory of evolution is only a theory so it cannot be considered knowledge'. Evolution theory belongs to the first type of theory – it is as well supported by evidence as the fact that water is H_2O – but the essay treats it as belonging to the second.

A word of warning is also needed about the word **proof**. Strictly speaking, proof is the mathematical process outlined above – where a mathematical statement is derived from axioms in a step-by-step manner. Proof implies absolute certainty. Be careful applying this word outside mathematics.

Part of a manuscript by the French mathematician Evariste Galois.



Absolute certainty is generally not achievable in science for a number of reasons that you may have discovered in your TOK course. Scientific results are not proved in this strict sense, it is better to describe them as being 'secure' or 'well supported by the evidence'.

To see how mathematical proof works, let us prove a simple theorem.

Theorem: Let x and y be odd integers. Then $x + y$ is an even integer.

Proof: The definition of an odd number is that it is an even number plus 1. An even number is a number in the 2× table.

So, write $x = 2m + 1$ for some integer m. In a similar fashion, $y = 2n + 1$ for some integer n.

$x + y = (2m + 1) + (2n + 1) = 2m + 2n + 2$

We can take out the common factor of 2 to give: $x + y = 2(m + n + 1)$

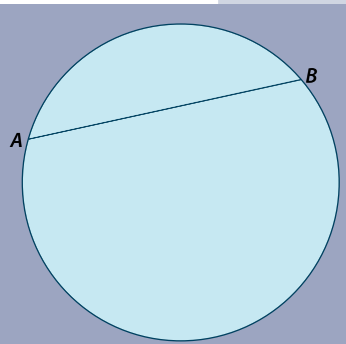
Since m, n and 1 are integers, it follows that $m + n + 1$ is also an integer.[2]

Hence, $x + y$ is 2× an integer and so must be even. QED

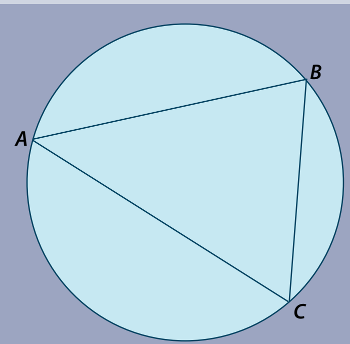
We write QED (*Quad Erat Demonstrandum* – meaning 'which was to be shown') at the end, to show that the proof is finished.

Let us take a closer look at some features of this method of proof. First notice that we have in effect proved an infinite sequence of statements including: 3 + 5 is even, 3 + 7 is even, 5 + 7 is even, and so on. We could have attempted to do a sort of mathematical experiment by checking whether the result holds for some randomly chosen odd numbers: 3 + 5 = 8, which is even; 3 + 7 = 10, which is even; 5 + 7 = 12, which is even; and so on. But this is not a proof. There is always an infinite number of examples that we have not tried and for which the result might not hold. This is what mathematicians call a **semantic** method. But, as you have probably learned from studying the natural sciences in TOK, it takes a single counter-example to disprove a conjecture. The same is true in mathematics. Why not try Moser's circle problem (shown right) to see what we mean? A proof is a **syntactic** method. It does not look at particular examples of odd numbers but rather depends on features that all odd numbers have in common (namely their oddness!). We have been able to do this by using algebra. We have substituted letters for numbers to allow us to talk generally about odd numbers rather than specific examples. This is typical of a mathematical

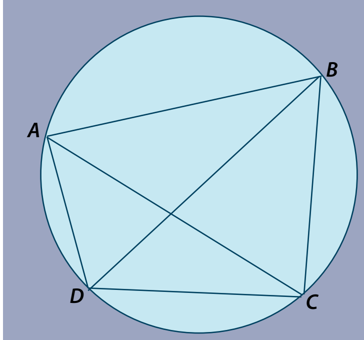
Moser's circle problem illustrates the difference between an experimental approach to mathematics – a semantic method (trying out a conjecture to see if it works) – and proving it – a syntactic method.



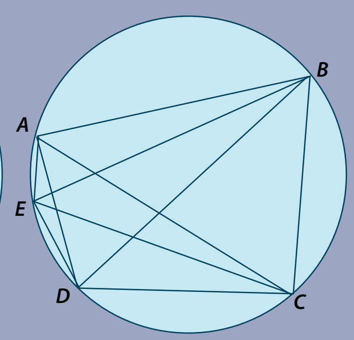
Draw a circle. Label 2 points on its circumference. Draw a line between them. This line divides the circle into 2 regions.



Add third point *C*. Draw lines between *C* and the other points. There are now 4 regions.



4 points, 8 regions



5 points, 16 regions

The question is: Can you predict how many regions there will be when you add a sixth point? Can you prove why this is so?

[2] There is a further subtlety here in the statement that $n + m + 1$ is an integer because m and n are. This is because the integers are closed under addition because $\mathbb{Z}^+$ is a group – closure is a property of groups. Groups are structures that underlie most mathematics.

proof. Once the conjecture is proved we can state categorically that it is true, now and for all time. It does not depend on culture, nationality, personal points of view, language or gender. It does not matter who proved it. It could be a university professor of mathematics or it could be an eight-year-old. It simply does not matter. In mathematics, proof means truth and that is the end of the story.

Axioms

A mathematical statement is true if it could be derived from the basic axioms of set theory by using only the rules of logic. In the HL course, the rules of logic are packaged in a convenient way to help us solve problems. We call this package 'the rules of algebra'. These are rules such as: You can add the same number to both sides of an equality and it remains equal (if $y = x$ then $y + 5 = x + 5$).

We can use these rules of algebra to solve mathematical problems. Each problem we solve is a little theorem. An example is: If $x + 5 = 10$ then $x = 5$. This is rather a simple theorem, but it is a theorem nevertheless. If you write any of your standard maths problems in the form '**If** … (problem to be solved) **then** (solution)' you get a theorem. (This assumes that you have solved the problem correctly!) But there is one additional set of assumptions that we do not explicitly mention when we solve these problems (or prove these theorems). That is, the assumptions that the axioms of set theory on which we base all our mathematics (and without which none of our mathematics would mean anything) are true. But how do we choose which axioms to use? How do we know that we have chosen a good set of axioms? These questions are not easy to answer. We shall examine them using a concrete example.

Euclid's postulates

What axioms did Euclid propose for doing plane geometry?
Here are Euclid's axioms. He called them 'postulates'.

Euclid's postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the given line segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. If two lines are drawn, which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side, if extended far enough.

In some sense, Euclid's axioms express mathematical intuitions about the nature of geometrical objects. What is clear in any case is that they are not established using observation of the external world. Objects such as points, lines, circles and planes do not exist in the real world with the perfect qualities they possess in mathematics.

Euclidian geometry

Let us try to use Euclid's postulates to do some geometry (see right).

Let us now examine the construction and see which postulates were used.

Step 1: drawing the arcs is allowed by postulate 3 (twice).

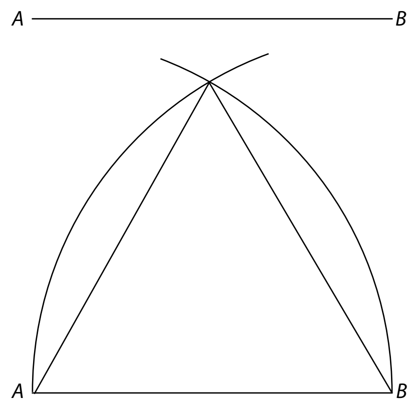
Step 2: drawing the line segments *AC* and *BC* is allowed by postulate 1 (twice).

It follows from step 1 that the line segments *AC* and *BC* are both equal to *AB*. Therefore, they must be equal (this is sometimes quoted as a separate axiom – that two line segments equal to the same line segment must be of equal length).

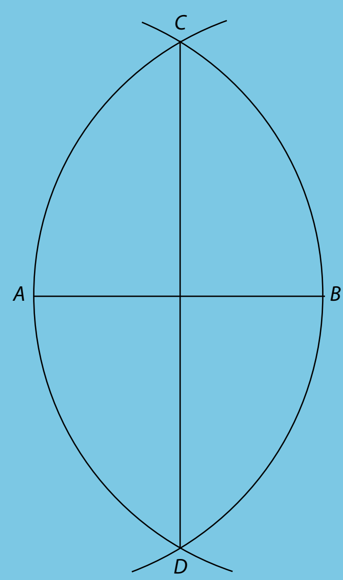
Non-Euclidian geometry

Take a look at Euclid's postulate 5. This cannot be proved as a theorem from the other axioms (that this is impossible can itself be proved!) although many people have attempted this. Euclid himself only used the first four axioms in the first 28 propositions of the *Elements*, but he was forced to use the fifth axiom, so-called 'parallel postulate', in the 29th proposition. The independence of the parallel postulate means that we can choose whether we accept it or not. If we accept it, parallel lines do not meet. The geometry we get is the familiar geometry of the plane. This is the geometry that we can use to construct buildings and other physical objects. In 1823, Janos Bolyai and Nicolai Lobachevsky independently realized that entirely self-consistent non-Euclidian geometries could be envisaged in which the fifth axiom did not hold. There are two quite different geometries in this case: those in which parallel lines meet at some point – elliptical geometry – and those in which parallel lines diverge – hyperbolic geometry. An example of elliptical geometry is the geometry of long distance travel on the Earth's surface. The shortest path between two points (say the most efficient route of a jet airliner) is a curve called a great circle.[3] The parallel lines of longitude are great circles that meet at the poles. If parallel lines diverge, we get so-called hyperbolic geometry. An example of doing hyperbolic geometry would be to draw lines on a saddle.

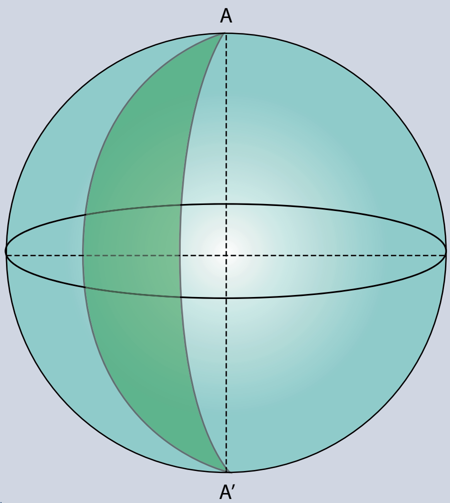
Problem: To construct an equilateral triangle on a given line segment using Euclid's axioms.



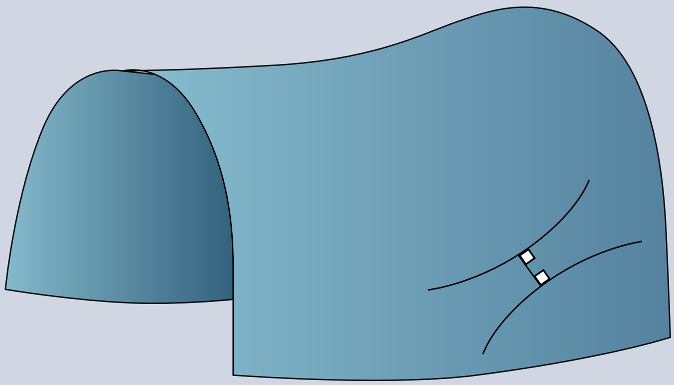
Why not try out a construction yourself, using the postulates of Euclidian geometry? Extend the arcs below the segment *AB* to meet again at *D*. Join *CD* with a line segment. The task is to prove that *CD* is the perpendicular bisector of *AB* using Euclid's postulates. (The perpendicular bisector of *AB* is a line segment *CD* that cuts *AB* exactly in half and the angle it makes with *AB* is exactly a right angle.)



[3] A 'great circle' is the largest circle that can be drawn on a sphere, and is the intersection between the surface of a sphere and a plane passing through the centre of the sphere. The shortest path between two points on a sphere follows a great circle.



In elliptical geometry, parallel lines converge.



In hyperbolic geometry, parallel lines diverge.

Consistency and completeness

We saw that postulate 5 is independent of the other four - that it could not be derived from them. More generally, there are two questions that can be asked of any set of axioms:
(1) Is the set **consistent**? In other words, is it impossible to derive a contradiction from them (to derive both the statements 'P is true' and 'not P is true')?
(2) Is the set **complete**? That is, any (semantically) true statement can be derived from them.

Are Euclid's axioms complete? Surprisingly, the German mathematician David Hilbert[4] showed that Euclid needs another 15 axioms to have a complete set to do what we now call Euclidian geometry.

In 1931, the Austrian logician Kurt Gödel[5] proved the devastating result that you could not prove the consistency and the completeness of the axioms for set theory that were used in *Principia Mathematica*. This famous incompleteness theorem proves, by an ingenious argument, that consistency and completeness is unprovable in any system rich enough to include the laws of arithmetic. So, it could be that mathematics is based upon rather shaky foundations. This might mean that there is a true statement of mathematics lurking somewhere in the recesses of the subject, which is not provable within the system. More serious, from a mathematical point of view, is the possibility that we can derive a contradiction of the form 'P is true' and 'not P is true' from the axioms using the rules of logic. Producing a contradiction means instant death for any area of knowledge. If you believe that 'P is true' and that 'not P is true' then one of your beliefs has to be false. This makes the combined belief 'P is

[4]David Hilbert *Foundations of Geometry* (1902) Gottingen

[5]Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I. *Monatshefte für Mathematik und Physik* 38, 173-98 (1931)

Do you hold any contradictory beliefs? If so, what are the implications for what you consider to be knowledge?

true and not P is true' false under all circumstances. So, if an area of knowledge throws up a contradiction, it simply cannot ever be true. It is condemned to being false whatever the actual state of the world. Since knowledge can be thought of (at least as a first approximation) as justified true belief, a statement that is forever false cannot be knowledge.

To what extent is mathematics really a language?

Beautiful equations

Einstein suggested that the most incomprehensible thing about the universe was that it was comprehensible. From a TOK point of view, the most incomprehensible thing about the universe is that it is comprehensible in the language of mathematics. Galileo wrote: 'Philosophy is written in this grand book, the universe … It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures….'[6]

[6]Galileo, *Il Saggiatore* (1623) Rome

What is perhaps most puzzling is not just that we can describe the universe in mathematical terms, but the mathematics we need to do this is mostly simple, elegant and even beautiful.

To illustrate this, let us look at some of the famous equations of physics. Most of you will be familiar with at least some of the following:

Relation between force and acceleration: $F = ma$ (more generally this is $F = \frac{d}{dt}(mv)$)

Gravitational force between two bodies: $F = \frac{Gm_1m_2}{r^2}$

Energy of rest mass: $E = mc^2$

Kinetic energy of a moving body: $E = \frac{1}{2}mv^2$

Electrostatic force between two charges: $F = \frac{kq_1q_2}{r^2}$

Maxwell's equations: $\nabla \times \mathbf{B} - \frac{d\mathbf{E}}{dt} = 4\pi\mathbf{J} \quad \nabla \times \mathbf{E} + \frac{d\mathbf{B}}{dt} = 0 \quad \nabla \cdot \mathbf{B} = 0 \quad \nabla \cdot \mathbf{E} = 4\pi\rho$

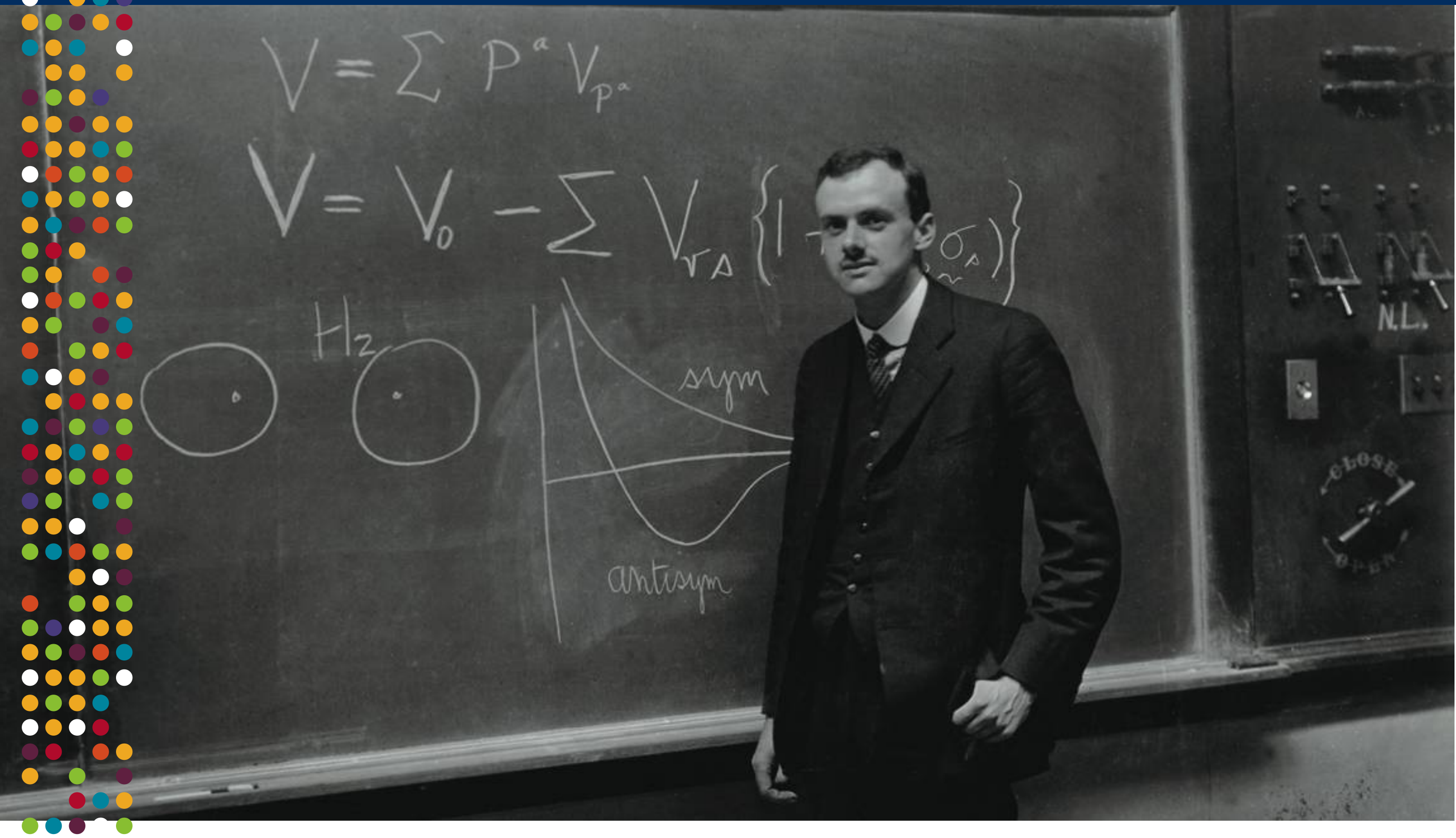
Einstein's field equation for general relativity: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 8\pi T_{\mu\nu}$

Similar equations can be found in the other natural sciences. Can you think of any?

Is there a sense in which these equations are elegant or beautiful?

I must admit that I find it perplexing that the whole crazy complex universe can be described by such simple, elegant and even beautiful equations. It seems that our mathematics fits the universe rather well. It is difficult to believe that maths is just a mind game that we humans have invented.

But the argument for simplicity and beauty goes further. Symmetry in the underlying algebra led mathematical physicists to propose the existence of new fundamental particles, which were subsequently discovered. In some cases, beauty and elegance of the mathematical description have even been used as evidence of its truth. The physicist Paul Dirac said: 'It seems that if one is working from the point of view of getting beauty in one's equations, and if one has really a sound insight, one is on a sure line of progress.'



'God used beautiful mathematics in creating the world.'
Paul Dirac

Dirac's own equation for the electron must qualify for being one of the most profoundly beautiful of all. Its beauty lies in the extraordinary neatness of the underlying mathematics – it all seems to fit so perfectly together:

$$\left(\beta mc^2 + \sum_{k=1}^{3} \alpha_k p_k c\right) \psi(x, t) = ih\frac{d\psi}{dt}(x, t)$$

The physicist and mathematician Palle Jorgensen[7] has written: '[Dirac] … liked to use his equation for the electron as an example, stressing that he was led to it by paying attention to the beauty of the math, more than to the physics experiments.'

[7]Palle Jorgensen *Operator Commutation Relations* (1984) New York

I shall leave the last word on this subject to Dirac himself, writing in *Scientific American* in 1963:

'I think that there is a moral to this story, namely that it is more important to have beauty in one's equations than to have them fit experiment.'

By any standards, this is an extraordinary statement for a mathematical physicist to make.

How good are your mathematical intuitions?

Mathematics can sometimes surprise us. Our mathematical intuitions can sometimes let us down, badly. In this section, we shall try out two basic scenarios upon our unsuspecting intuitions and see how they fare.

Scenario 1: The rare genetic disease

Consider the following. There is a very rare genetic disease amongst the population. Very few people have the disease. As a precaution, a test has been developed to check in a particular case whether a person has the disease or not. Although the test is quite good, it is not perfect – it is only 99% accurate. A person X takes the test and it shows positive. The question for your intuition: What is the probability that X actually has the disease?

Think about this for a moment before we go on with the analysis.

Many of the students and teachers that I have worked with in the past have given the same answer: The probability that X actually has the disease, given a positive test result, is around 99%. Did you say the same?

If you did, your mathematical intuition let you down – very badly.

Let us put some numbers into this problem. For the sake of simplicity, let us assume that the country in which the test takes place has a population of 10 million. We are told that the disease is very rare. Let us assume that only 100 people have it in the whole country.

We are told that the test is 99% accurate so that of the 100 cases of the disease the test would show positive in 99 cases and negative in 1 case. So far, so good.

Now let us look at the 9 999 900 people who do not have the disease. In 99% of these cases the test does its job and records a negative result. But in 1% of these cases the test records a positive result. 1% of 9 999 900 is 99 999. This means that if the whole population were tested 99 999 + 99 = 100 098 test results would be positive. Of these, only 99 people have the disease. Therefore, the probability of having the disease, given a positive test result, is 99/100 098 = 0.0989% – in other words, about a tenth of a per cent or one in one thousand. This is a bit different from the 99% that most people guess. How well did you do?

What went wrong with the intuition here?

The important factor in this problem is not just the accuracy of the test *but the accuracy of the test relative to the incidence of the disease*. Because the disease is so rare, the actual number of people with the disease is overwhelmed by the false positive results of the test – the 1% or so of the population who do not have the disease, but the test shows positive anyway. If more people had the disease and if the test was more accurate, the test scoring positive would be a better predictor of X actually having the disease.

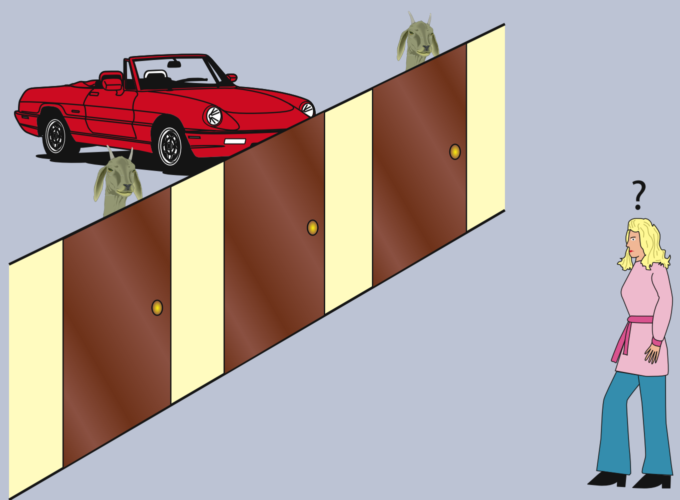
Try this problem out with some other numbers to check how the test could be made more useful.

Scenario 2: The Monty Hall game

The second scenario is also based on probability theory. The problem refers to a TV game show, which is loosely based on the actual show *Let's Make a Deal*[8]. A contestant in the show is shown three doors and told (truthfully), by the game show host Monty Hall, that behind one of the doors is a luxury sports car and behind the other two doors are goats. The contestant is told that she must pick a door. She will be allowed to take home whatever is behind the door she picks. We shall assume at this stage that she prefers to win the car. So she goes ahead and picks a door. At this point, Monty Hall opens another door to reveal a goat. (Whenever this game is played, Monty Hall chooses a door concealing a goat.) He then asks the competitor: 'Do you want to switch to the other closed door?'

[8]A widely known statement of the problem was published in Marilyn vos Savant's *Ask Marilyn* column in *Parade* (1990).

What does your intuition tell you? Should the contestant switch or should she stick to her original choice?



The Monty Hall problem: should the contestant switch?

Take a little time to think this through. You might like to try this game with a friend to see experimentally what the best strategy is.

Clearly, because there is one car and two goats, the probability of picking the car if the competitor does not switch doors is 1/3.

If she does switch, what is the probability of winning the car? Let us ask a related question. If she does switch, under what circumstances can she lose the car? Clearly, the only way she can lose the car is if her original choice was right. In other words, she has a 1/3 probability of losing. This must mean that by switching, her probability of winning the car is 2/3.

In other words, by switching she doubles her probability of winning.

Does this make sense? Even after this explanation many of the students and teachers that attend my workshops are not convinced. They argue that they cannot see how an asymmetry has been introduced into the situation.

The crucial point is that Monty Hall knows where the car is. He always opens a door to reveal a goat. It is this act that produces the required asymmetry.

Consider an extreme version of the Monty Hall problem. Imagine 100 closed doors containing 1 car and 99 goats. Let us suppose, for the sake of the argument, that our contestant chooses door number 1. Monty Hall then opens 98 doors to reveal goats. The contestant would be foolish not to switch to the one remaining door (and multiply her probability of winning by a factor of 99).

Try this problem out on your friends and relatives. Are their mathematical intuitions letting them down?

Is the fact that mathematics can surprise us and go against our intuitions evidence that mathematics exists independently of us?

[9] John R. Searle *The Construction of Social Reality* (1995) London

What is a social fact?

The philosopher John Searle[9] points out that many of the facts in our lives are actually socially constructed. He uses money as his central example. Money is money because we believe it to be money. There is something rather strange about this. Normally speaking, when we define a term X, we do not expect the definition to refer to X. Did we not learn in TOK that it was bad to define X in terms of itself? Was this not the reason why our TOK teacher advised us to keep clear of dictionary definitions: 'knowledge – that which is known'. Searle thinks that this sort of circularity is characteristic of what he calls a **socially constructed fact**. He asserts that the social agreements that we make collectively that something should be money makes it such. So 'X functions as money in society S' is a socially constructed fact. As such, statements about it are objective and capable of being evaluated as true or false. Our socially constructed reality includes the concepts of wife, girlfriend, driving licence, bank account, traffic lights, rules of etiquette, nationality, legality, country, nationality, debt, honour, and so on. Many of the physical objects around us are defined in terms of their function, and hence in terms of our intentions, and hence are socially constructed. The concept of a chair or a knife is socially constructed. This is what makes them so difficult to define without using the words 'function' or 'intention'.

Try to define a chair without making reference to human intentions.

Is mathematics a social fact?

Reuben Hersch[10] argues that numbers (and any other mathematical entities) are social constructions. If we acknowledge that they are not just out there in the world independent of human beings and they are not just thoughts in people's heads (our intuitions can be wrong after all) then what are they? There is a third possibility. Mathematics is a construction of human society.

Hersch proposes that mathematics is itself a whole interconnected web of socially constructed reality. Here he is in an interview with John Brockman on the Edge website:[11]

[10] Reuben Hersch *What Is Mathematics, Really?* (1997) Oxford

[11] http://www.edge.org/3rd_culture/hersh/hersh_p1.html (accessed Feb 2008)

'Mathematics is neither physical nor mental, it's social. It's part of culture, it's part of history, it's like law, like religion, like money, like all those very real things, which are real only as part of collective human consciousness. Being part of society and culture, it's both internal and external. Internal to society and culture as a whole, external to the individual, who has to learn it from books and in school. That's what math is.'

When asked what he called his theory of mathematics, Hersch replied that he calls it humanism 'because it's saying that math is something human. There's no math without people. Many people think that ellipses and numbers and so on are there whether or not any people know about them; I think that's a confusion.'

Hersch points out that we do use numbers to describe physical reality and that this seems to contradict the idea that numbers are a social construction. It is important to note here that we use numbers in two distinct ways: as nouns and adjectives. When we say nine apples, nine is an adjective. 'If it's an objective fact that there are nine apples on the table, that's just as objective as the fact that the apples are red, or that they're ripe, or anything else about them, that's a fact'. The problem occurs when we make a subconscious switch to 'nine' as an abstract noun in the sort of problems we deal with in maths class. Hersch thinks that this is not really the same nine. They are connected, but the number nine is an abstract object as part of a number system. It is a human creation.

Politics and maths learning

Hersch sees both a political and a pedagogic dimension to his thinking about mathematics. He thinks that a humanistic vision of mathematics chimes in with more progressive politics. How can politics enter mathematics? As soon as we think of mathematics as a social construction then the exact arrangements by which this construction comes about – the institutions that build and maintain it – become important. These arrangements are political. Particularly interesting for us here is how a different view of maths can bring about changes in maths teaching and learning. Let us return to Hersch:

'Let me state three possible philosophical attitudes towards mathematics. Platonism says mathematics is about some abstract entities, which are independent of humanity. Formalism says mathematics is nothing but calculations. There's no meaning to it at all. You just come out with the right answer by following the rules. Humanism sees mathematics as part of human culture and human history. It's hard to come to rigorous conclusions about this kind of thing, but I feel it's almost obvious that Platonism and Formalism are anti-educational, and interfere with understanding, and Humanism at least doesn't hurt and could be beneficial. Formalism is connected with rote, the traditional method, which is still common in many parts of the world. Here's an algorithm; practise it for a while; now here's another one. That's certainly what makes a lot of people hate mathematics. (I don't mean that mathematicians who are formalists advocate teaching by rote. But the formalist conception of mathematics fits naturally with the rote method of instruction.) There are various kinds of Platonists. Some are good teachers, some are bad. But the Platonist idea, that, as my friend Phil Davis puts it, Pi is in the sky, helps to make mathematics intimidating and remote. It can be an excuse for a pupil's failure to learn, or for a teacher's saying, "Some people just don't get it." The humanistic philosophy brings mathematics down to earth, makes it accessible psychologically, and increases the likelihood that someone can learn it, because it's just one of the things that people do.'

Do you agree with Reuben Hersch's humanist picture of mathematics – that mathematics is a social construction? Do you think he is right in his association of formalism with rote learning of maths and Platonism with the idea of maths being something remote that some people simply 'do not get'?

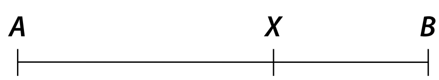
Are you really only intelligent if you can do maths?

There is a possibility that the arguments explored in this section might cast light on an aspect of mathematics learning which has seemed puzzling – why it is that mathematical ability is seen to be closely correlated with a certain type of intelligence. Mathematics has, moreover, seemed to polarize society into two distinct groups: those that can do it and those that cannot. Those that cannot do it often feel the stigma of failure. Is Hersch right in attributing this to a formalistic or platonic view? Is he right to suggest that if maths is just a meaningless set of formal exercises, then it will not be valued in the main by society? If maths is out there to be discovered, it does seem reasonable to imagine that a particular individual who does not make the discovery might experience a sense of failure. The interesting question in this case is: What practical consequences in the classroom would follow from a humanist view of mathematics?

The golden ratio

There are some intriguing links between mathematics and the arts. One link that seems to fascinate many students of mathematics is the ancient idea of the **golden ratio**. Consider a line segment AB. The Greek mathematicians were interested in dividing AB by placing a point X in such a way that the ratio of the smaller piece to the longer piece was equal to the ratio of the longer piece to the whole line.

In other words: $XB/AX = AX/AB$



Let us rescale our units so that $AB = 1$ unit. Let $AX = x$. Then $XB = 1 - x$.

The equation above gives us: $\dfrac{1-x}{x} = \dfrac{x}{1}$

Rearranging gives us: $1 - x = x^2$

This gives the quadratic equation: $x^2 + x - 1 = 0$

Solving this equation using the quadratic formula gives:

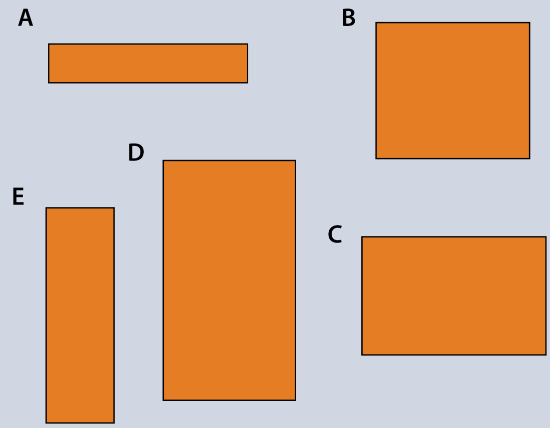
$x = \dfrac{-1+\sqrt{5}}{2}$ and $x = \dfrac{-1-\sqrt{5}}{2}$ or $x = 0.618\,033\,988\,75\ldots$ or $-1.618\,033\,988\,75\ldots$

The first of these solutions is known as the **golden ratio**. Because of the special symmetry of the relationship between the different parts of the line segment above, this ratio was thought to be special or perfect in some way. Rectangles in which the ratio of the shorter to the longer side is equal to the golden ratio were thought to be especially beautiful. Try this out yourself in the rectangle beauty contest. Choose the rectangle that is most pleasing to you. Measure the sides and calculate the ratio between the shorter and the longer side. How close are you to the golden ratio?

A4 paper has dimensions of 210 mm × 297 mm. 210/297 = 0.707, which is a little high. A4 paper is a little too 'fat' to be a golden rectangle.

Measure some rectangles in your school or home environment – for example, credit cards, postcards, books, tables. How close are they to golden rectangles?

Rectangle beauty contest



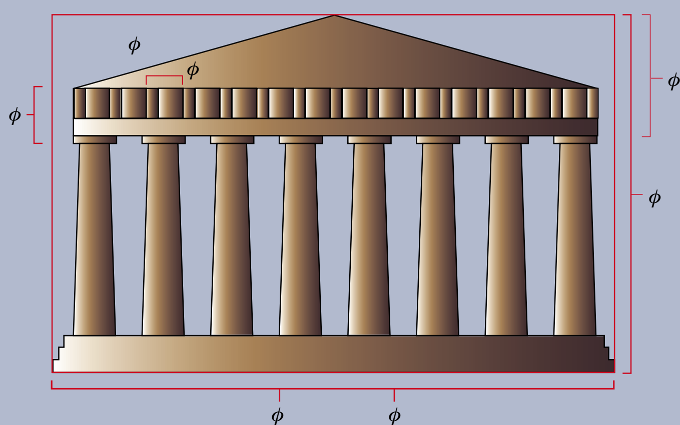
Which rectangle do you find the most pleasing?

A2, A3 and A5 paper are also all a little too fat to be golden rectangles. Why is this?

The golden ratio and the arts

There are many studies of the occurrence of the golden ratio in the natural and human worlds. It occurs in nature in connection with spirals and the Fibonacci sequence. The golden ratio has also been exploited by human beings in art, architecture and music. For example, the golden ratio was exploited by the ancient Greeks in their designs for temples and other buildings. The Parthenon in Athens is constructed using the golden section at key points.

The Greek letter ϕ is often used for the golden section.



Golden ratios have been consciously used in the structure of some musical compositions. The French composer Debussy is known to have used this ratio in his orchestral piece *La Mer*, for example. The 55 bar introduction to 'Dialogue du vent et de la mer' breaks down into five sections of 21, 8, 8, 5 and 13 bars in length, which are numbers in the Fibonacci sequence. The golden ratio point of bar 34 in this passage is signalled by the entry of the trombones and percussion. More generally, we can ask ourselves how many pieces of music (or films or plays or dance performances) have some sort of structurally significant event roughly two-thirds of the way through the piece?

Think of a film you have seen recently. At what point in the film did the moment of highest tension occur? How far into the film did this happen? Calculate this as a proportion. Is it close to 62%?

The Fibonacci sequence

The golden ratio is linked closely to the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

What are the next two terms in the sequence?

If we divide successive terms in the sequence:

$\frac{1}{1} = 1,\ \frac{1}{2} = 0.5,\ \frac{2}{3} = 0.6667,\ \frac{3}{5} = 0.6,\ \frac{5}{8} = 0.625,$
$\frac{8}{13} = 0.6154,\ \frac{13}{21} = 0.6190,\ \frac{21}{34} = 0.6176, \ldots$

What is going on here?

Much has been written about how this sequence occurs in nature. It is naturally associated with certain types of growth. Ian Stewart, in his book *Nature's Numbers,* describes how these numbers are naturally associated with the spiral growth of many types of shell, for example. There is nothing mystical about this link. But it is tempting to think again about the Platonists and their view of mathematics as somehow embedded in the outside world.

The golden ratio suggests a strong link between mathematics and the arts. In theory of knowledge, it also raises a set of interesting questions about the nature of beauty. If we find certain rectangles pleasing because of the golden ratio, we might also find certain faces beautiful because of the ratios between the features, and find certain paintings or pieces of music beautiful, because of their proportions. Beauty would not be entirely in the eye of the beholder – it would be in the mathematics.

Answers

Chapter 1

Exercise 1.1

1 $\mathbb{Z} \subset \mathbb{Q}$ **2** $\mathbb{N} \subset \mathbb{Q}$ **3** $\mathbb{R} \subset \mathbb{C}$

4 $\mathbb{N} \subset \mathbb{Z}$ **5** $\mathbb{Z}^+ \subset \mathbb{Z}$ **6** $\mathbb{N} \subset \mathbb{R}$

7 $\frac{71}{33}$ **8** $\frac{1787}{150}$ **9** $\frac{61}{7}$

10 $\{1, 3, 5, 7\}$ **11** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

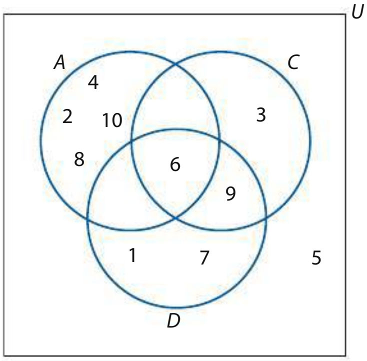
12 $\varnothing$ (empty set) **13** $\{1, 2, 3, 4, 5, 6, 7, 8\}$

14 $\{2, 4, 6\}$ **15** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

16 a) A is set of all even numbers between 2 and 10 inclusive; B is set of all odd numbers between 1 and 9 inclusive; C is set of all multiples of 3 between 3 and 9 inclusive

b) (i) $\varnothing$ (ii) U (iii) B
(iv) A (v) $\{6\}$ (vi) $\{3, 9\}$
(vii) $\{9\}$ (viii) $\{2, 4, 5, 8, 10\}$ (ix) $\{2, 4, 8, 10\}$

c)



17 a) $\{x \mid 0 < x \leqslant 1\}$ b) $\{x \mid -2 \leqslant x < 3\}$
c) $\{x \mid 1 < x \leqslant 4\}$ d) $\{x \mid -2 \leqslant x \leqslant 0,\ 3 \leqslant x \leqslant 4\}$
e) $\{x \mid 3 \leqslant x \leqslant 4\}$ f) $\{x \mid -2 \leqslant x \leqslant 0,\ 1 < x \leqslant 4\}$

18 $x > -10$ **19** $x \leqslant -3$ **20** $x > \frac{10}{3}$

21 $x > -\frac{3}{8}$ **22** $\frac{3}{2} \leqslant x < \frac{7}{4}$ **23** $-3 \leqslant x < 1$

24 False, $x = -1$ **25** True **26** False, $x = 0$

27 False, $x = \frac{1}{2}$ **28** True **29** True

30 False, $x = -1$ **31** False, $x = \frac{1}{2}$ **32** 14.5

33 9 **34** 8.2 **35** $\pi - 3$

36 $\frac{11\pi}{3}$ **37** $\frac{832}{77}$

38 $-5 \leqslant x \leqslant 3$, closed, bounded

39 $-10 < x \leqslant -2$, half-open, bounded

40 $x \geqslant 1$, half-open, unbounded

41 $x < 4$, open, unbounded

42 $0 \leqslant x < 2\pi$, half-open, bounded

43 $a \leqslant x \leqslant b$, closed, bounded

44 $]-3, \infty[$ **45** $]-4, 6[$ **46** $]-\infty, 10]$

47 $[0, 12[$ **48** $]-\infty, \pi[$ **49** $[-3, 3]$

50 $x \geqslant 6$ $[6, \infty[$ **51** $4 \leqslant x < 10$ $[4, 10[$

52 $x < 0$ $]-\infty, 0[$ **53** $0 < x < 25$ $]0, 25[$

54 $|x| < 6$ **55** $|x| \geqslant 4$ **56** $|x| \leqslant \pi$ **57** $|x| > 1$

58 13 **59** 4 **60** -25 **61** -5

62 $3 - \sqrt{3}$ **63** -1

64 $x = -5$ or $x = 5$ **65** $x = -1$ or $x = 7$

66 $x = -4$ or $x = 16$ **67** No solution; false for all x

68 $x = -2$ or $x = -\frac{4}{3}$ **69** $x = \frac{32}{3}$ or $x = -\frac{28}{3}$

70 $x = -\frac{42}{5}$ or $x = \frac{72}{5}$ **71** $x = 0$ or $x = -4$

72 a) $x = 2, y = -2$ b) $x = 2, y = -2$

Exercise 1.2

1 h^2 **2** 3 **3** $6\sqrt{5}$ **4** $\frac{2\sqrt{7}}{7}$

5 4 **6** $\frac{\sqrt{3}}{2}$ **7** $3\sqrt{5} + 20$ **8** -2

9 $7\sqrt{2}$ **10** $40\sqrt{10}$ **11** $2\sqrt[3]{6}$ **12** $2|xy|\sqrt{3xy}$

13 m **14** $\frac{3\sqrt{2}}{2}$ **15** $x^8|1+x|$ **16** $3\sqrt{7}$

17 $6\sqrt{2} + 4\sqrt{3}$ **18** $17\sqrt{5}$ **19** $\frac{\sqrt{5}}{5}$ **20** $\frac{\sqrt{2}}{5}$

21 $2\sqrt{21}$ **22** $\frac{\sqrt{2}}{2}$ **23** $\frac{\sqrt{5} - 1}{2}$ **24** $\frac{2\sqrt{5} - 3}{11}$

25 $3 + 2\sqrt{3}$ **26** $\frac{4\sqrt{5} - 4\sqrt{2}}{3}$ **27** $\sqrt{x} - \sqrt{y}$ **28** $5 + 3\sqrt{3}$

29 $\frac{\sqrt{1 - x^2}}{x}$ **30** $\sqrt{x + h} + \sqrt{x}$ **31** $\frac{1}{\sqrt{a} + 3}$ **32** $\frac{1}{\sqrt{x} + \sqrt{y}}$

33 $\frac{m - 7}{(7 - x)\left(\sqrt{m} + \sqrt{7}\right)}$

Exercise 1.3

1 2 **2** 27 **3** 16 **4** 16

5 8 **6** 8 **7** $\frac{4}{9}$ **8** $\frac{3}{4}$

9 $\frac{125}{8}$ **10** $\frac{1}{9}$ **11** 1 **12** $\frac{16}{3}$

13 $\frac{-64}{27}$ **14** x^2y^6 **15** $-x^2y^6$ **16** $-8x^3y^9$

17 $\frac{32y^7}{x}$ **18** $\frac{1}{64m^6}$ **19** $\frac{p^2}{3k^3}$ **20** -8

21 25 **22** $x^{\frac{7}{6}}$ **23** $\frac{1}{4a^2}$ **24** x

25 $2(a - b)$ **26** $\frac{(x + 4y)^{\frac{3}{2}}}{2}$ **27** $\sqrt{p^2 + q^2}$ **28** 5^{3x-1}

29 $\frac{1}{x^{\frac{1}{6}}} + \frac{1}{x^{\frac{1}{4}}}$ **30** $\frac{26}{9}(3^n)$ **31** 16 **32** $2\sqrt{3}x^2y^4$

33 $\sqrt{1 + n^2}$ **34** $\sqrt{x}$

Exercise 1.4

1 2.54×10^2 **2** 7.81×10^{-3} **3** 7.41×10^6

4 1.04×10^{-6} **5** 4.98 **6** 1.99×10^{-3}

7 1.49×10^8 **8** 8.99×10^{-5} **9** 1.50×10^8

10 9.11×10^{-31} **11** 0.0027 **12** 50 000 000

13 0.000 000 090 35 **14** 4 180 000 000 000 **15** 2.5×10^3

16 2×10^4 **17** 8.2×10^{-5} **18** 5.6×10^{18}

19 1.8×10^5 20 5×10^1 21 8.2×10^{-5}
22 5.56×10^1

Exercise 1.5

1 $x^2 + x - 20$ 2 $6h^2 - 11h + 3$ 3 $y^2 - 81$
4 $16x^2 + 16x + 4$ 5 $4n^2 - 10n + 25$
6 $4y^2 - 20y + 25$ 7 $36a^2 - 49b^2$
8 $4x^2 + 12x + 9 - y^2$ 9 $a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3$
10 $a^4x^4 + 4a^3bx^3 + 6a^2b^2x^2 + 4ab^3x + b^4$
11 $4 - 5x^2$ 12 $8x^3 - 1$
13 $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ 14 $x^2 + y^2$
15 $9 - m^2$ 16 $x^2 - 2\sqrt{x^2+1} + 2$
17 $12(x+2)(x-2)$ 18 $x^2(x-6)$
19 $(x+4)(x-3)$ 20 $-(m-1)(m+7)$
21 $(x-8)(x-2)$ 22 $(y+1)(y+6)$
23 $3(n-5)(n-2)$ 24 $2x(x+1)(x+9)$
25 $(a+4)(a-4)$ 26 $(3y+1)(y-5)$
27 $(5n^2+2)(5n^2-2)$ 28 $a(x+3)^2$
29 $(m+1)^2(2n-1)$ 30 $(x+1)(x-1)(x^2+1)$
31 $y(6-y)$ 32 $2y^2(2y^2-5y-48)$
33 $(2x-5)^2$ 34 $(2x+3)^{-3}(4x+3) = \dfrac{4x+3}{(2x+3)^3}$
35 $(n-2)^3(1-n)$ 36 $m\left(m-\dfrac{2}{3}\right)^2$ 37 $\dfrac{1}{x+1}$
38 $\dfrac{1}{2n}$ 39 $\dfrac{a+b}{5}$ 40 $x+2$
41 -1 42 $4x$ 43 $\dfrac{3x+2}{x+1}$
44 $\dfrac{3y-1}{y+2}$ 45 -1 46 $\dfrac{(2x-1)(x-1)}{x(x-2)}$
47 $\dfrac{1-n}{n}$ 48 $\dfrac{6-8x}{2x-1}$ 49 $\dfrac{-2x+5}{15}$
50 $\dfrac{b-a}{ab}$ 51 $\dfrac{10-3x}{(x-3)^2}$ 52 $\dfrac{x^2+x+3}{x^2+3x}$
53 $\dfrac{2x}{x^2-y^2}$ 54 $\dfrac{-2}{x-2}$ 55 6
56 $\dfrac{2}{7x-21}$ 57 $\dfrac{1}{ab-b^2}$ 58 $-\dfrac{5}{2}x(x+1)$
59 $\dfrac{3y-10}{(y+2)(y-5)}$ 60 $\dfrac{(x-3)(x+2)}{23x^2}$ 61 $\dfrac{x+\sqrt{2}}{x^2-2}$
62 $\dfrac{10-5x\sqrt{3}}{4-3x^2}$ 63 $\dfrac{x+2\sqrt{xy}+y}{x-y}$ 64 $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

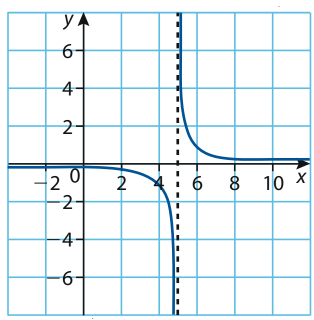
Exercise 1.6

1 $x = h - \dfrac{n}{m}$ 2 $a = \dfrac{v^2+t}{b}$ 3 $b_1 = \dfrac{2A}{h} - b_2$
4 $r = \sqrt{\dfrac{2A}{\theta}}$ 5 $k = \dfrac{gh}{f}$ 6 $t = \dfrac{x}{a+b}$
7 $r = \sqrt[3]{\dfrac{3V}{\pi h}}$ 8 $k = \dfrac{g}{F(m_1+m_2)}$ 9 $y = -\dfrac{2}{3}x - 5$
10 $y = -4$ 11 $y = \dfrac{5}{4}x + 6$ 12 $x = \dfrac{7}{3}$
13 $y = -4x + 11$ 14 $y = -\dfrac{5}{2}x - 7$
15 a) 17 b) $\left(0, \dfrac{5}{2}\right)$ 16 a) $\sqrt{40}$ b) $(2, 3)$
17 a) $\dfrac{\sqrt{82}}{3}$ b) $\left(-1, \dfrac{7}{6}\right)$ 18 a) $\sqrt{533}$ b) $\left(1, \dfrac{11}{2}\right)$
19 $k = 1$ or 9 20 $k = -11$ or -3
21 $(\sqrt{5})^2 + (\sqrt{45})^2 = (\sqrt{50})^2$ 22 Sides are: $\sqrt{29}, \sqrt{29}, \sqrt{58}$
23 Sides are: $\sqrt{45}, \sqrt{10}, \sqrt{45}, \sqrt{10}$ 24 $(5, 1)$
25 $\left(4, \dfrac{1}{2}\right)$ 26 $(3, -4)$ 27 $(3.8, -1.6)$
28 No solution 29 $(-1, 2)$ 30 $(-1, 3)$
31 $(-3, -8)$
32 Lines are coincident; solution set is all points on the line $y = -\dfrac{1}{4}x - \dfrac{3}{4}$
33 $\left(\dfrac{20}{3}, \dfrac{40}{3}\right)$ 34 $\left(\dfrac{1}{2}, 3\right)$ 35 $(-5, 10)$
36 $(5, -3)$ 37 $(14.1, 10.4)$ 38 $\left(\dfrac{11}{19}, -\dfrac{18}{19}\right)$

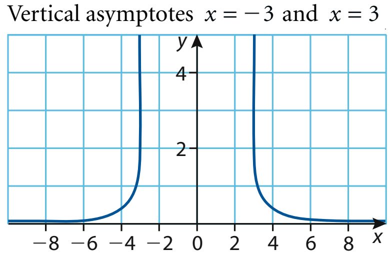
Chapter 2

Exercise 2.1

1 G 2 L 3 H 4 K
5 J 6 C 7 A 8 I
9 F 10 $A = \dfrac{C^2}{4\pi}$ 11 $A = \dfrac{l^2\sqrt{3}}{4}$ 12 $A = 4x^2 + 60x$
13 $h = x\sqrt{2}$
14 a) 9.4 b) $V = \dfrac{3525}{P}$
15 a) $F = kx$ b) 6.25 c) 37.5 N
16 $\{-6.2, -1.5, 0.7, 3.2, 3.8\}$ 17 $r > 0$
18 $\mathbb{R}$ 19 $\mathbb{R}$ 20 $t \leqslant 3$ 21 $\mathbb{R}$
22 $x \neq \pm 3$ 23 $-1 \leqslant x \leqslant 1$ and $x \neq 0$
24 No, $x = c$ is a vertical line
25 a) (i) $\sqrt{17}$ (ii) 7 (iii) 0
b) $x < 4$ c) Domain: $x \geqslant 4$, range: $h(x) \geqslant 0$
26 a) Domain $\{x : x \in \mathbb{R}, x \neq 5\}$, range $\{y : y \in \mathbb{R}, y \neq 0\}$
b) y-intercept $\left(0, -\dfrac{1}{5}\right)$, vertical asymptote $x = 5$, horizontal asymptote $y = 0$

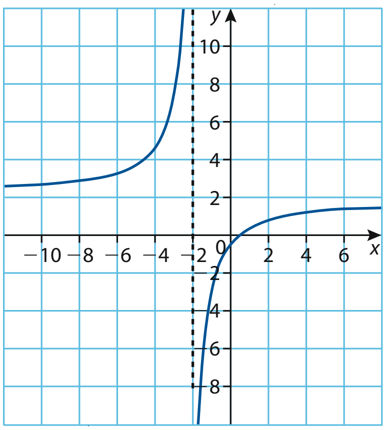


27 a) Domain $\{x : x < -3, x > 3\}$, range $\{y : y > 0\}$
b) Vertical asymptotes $x = -3$ and $x = 3$



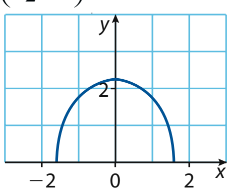
28 a) Domain $\{x : x \in \mathbb{R}, x \neq -2\}$, range $\{y : y \in \mathbb{R}, y \neq 2\}$

b) y-intercept $\left(0, -\frac{1}{2}\right)$, vertical asymptote $x = -2$, horizontal asymptote $y = 2$



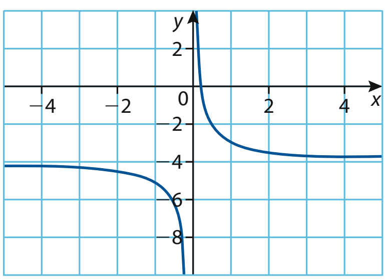
29 a) Domain $\left\{x : -\frac{\sqrt{10}}{2} \leqslant x \leqslant \frac{\sqrt{10}}{2}\right\}$, range $\left\{y : 0 \leqslant y \leqslant \sqrt{5}\right\}$

b) y-intercept $\left(0, \sqrt{5}\right)$, x-intercepts $\left(-\frac{\sqrt{10}}{2}, 0\right)$ and $\left(\frac{\sqrt{10}}{2}, 0\right)$



30 a) Domain $\{x : x \in \mathbb{R},\ x \neq 0\}$, range $\{y : y \in \mathbb{R},\ y \neq -4\}$

b) Vertical asymptote $x = 0$, horizontal asymptote $y = -4$



Exercise 2.2

1 a) $(f \circ g)(5) = 1$, $(g \circ f)(5) = \frac{1}{7}$

b) $(f \circ g)(x) = \frac{2}{x-3}$, $(g \circ f)(x) = \frac{1}{2x-3}$

2 a) 1 b) -7 c) 7
d) -47 e) -1 f) -79
g) $1 - 2x^2$ h) $-4x^2 + 12x - 7$ i) $4x - 9$
j) $-x^4 + 4x^2 - 2$

3 $(f \circ g)(x) = 12x + 7$, domain: $x \in \mathbb{R}$; $(g \circ f)(x) = 12x - 1$, domain: $x \in \mathbb{R}$

4 $(f \circ g)(x) = 4x^2 + 1$, domain: $x \in \mathbb{R}$; $(g \circ f)(x) = -2x^2 - 2$, domain: $x \in \mathbb{R}$

5 $(f \circ g)(x) = \sqrt{x^2 + 2}$, domain: $x \in \mathbb{R}$; $(g \circ f)(x) = x + 2$, domain: $x \geqslant -1$

6 $(f \circ g)(x) = \frac{2}{x+3}$, domain: $x \in \mathbb{R}$, $x \neq -3$;
$(g \circ f)(x) = -\frac{x+2}{x+4}$, domain: $x \in \mathbb{R}$, $x \neq -4$

7 $(f \circ g)(x) = x$, domain: $x \in \mathbb{R}$; $(g \circ f)(x) = x$, domain: $x \in \mathbb{R}$

8 $(f \circ g)(x) = x^4$, domain: $x \in \mathbb{R}$;
$(g \circ f)(x) = -x^4 + 4x^3 - 6x^2 + 4x$, domain: $x \in \mathbb{R}$

9 $(f \circ g)(x) = \frac{2}{4x^2 - 1}$, domain: $x \neq 0, x \neq \pm\frac{1}{2}$;
$(g \circ f)(x) = \frac{(4-x)^2}{4x^2}$, domain: $x \neq 0, x \neq 4$

10 $(f \circ g)(x) = 1 + x^2$, domain: $-1 \leqslant x \leqslant 1$;
$(g \circ f)(x) = \sqrt[3]{-x^6 + 4x^3 - 3}$, domain: $x \in \mathbb{R}$

11 $(f \circ g)(x) = x$, domain: $x \neq -3$; $(g \circ f)(x) = x$, domain: $x \neq -3$

12 $(f \circ g)(x) = \frac{x^2 - 1}{x^2 - 2}$, domain: $x \neq \pm\sqrt{2}$;
$(g \circ f)(x) = \frac{2x - 1}{(x-1)^2}$, domain: $x \neq 1$

13 a) $(g \circ h)(x) = \sqrt{9 - x^2}$, domain: $-3 \leqslant x \leqslant 3$, range: $y \geqslant 0$
b) $(h \circ g)(x) = -x + 11$, domain: $x \geqslant 1$, range: $y \leqslant 10$

14 a) $(f \circ g)(x) = \frac{1}{10 - x^2}$, domain: $x \neq \pm\sqrt{10}$, range: $y \neq 0$
b) $(g \circ f)(x) = 10 - \frac{1}{x^2}$, domain: $x \neq 0$, range: $y < 10$

15 $h(x) = x + 3$, $g(x) = x^2$ **16** $h(x) = x - 5$, $g(x) = \sqrt{x}$

17 $h(x) = \sqrt{x}$, $g(x) = 7 - x$ **18** $h(x) = x + 3$, $g(x) = \frac{1}{x}$

19 $h(x) = x + 1$, $g(x) = 10^x$ **20** $h(x) = x - 9$, $g(x) = \sqrt[3]{x}$

21 $h(x) = x^2 - 9$, $g(x) = |x|$ **22** $h(x) = \sqrt{x - 5}$, $g(x) = \frac{1}{x}$

23 a) Domain of f: $x \geqslant 0$ b) Domain of g: $x \in \mathbb{R}$
c) $(f \circ g)(x) = \sqrt{x^2 + 1}$, domain $x \in \mathbb{R}$

24 a) Domain of f: $x \neq 0$ b) Domain of g: $x \in \mathbb{R}$
c) $(f \circ g)(x) = \frac{1}{x+3}$, domain $x \neq -3$

25 a) Domain of f: $x \neq \pm 1$ b) Domain of g: $x \in \mathbb{R}$
c) $(f \circ g)(x) = \frac{3}{x^2 + 2x}$, domain $x \neq 0, -3$

26 a) Domain of f: $x \in \mathbb{R}$ b) Domain of g: $x \in \mathbb{R}$
c) $(f \circ g)(x) = x + 3$, domain $x \in \mathbb{R}$

Exercise 2.3

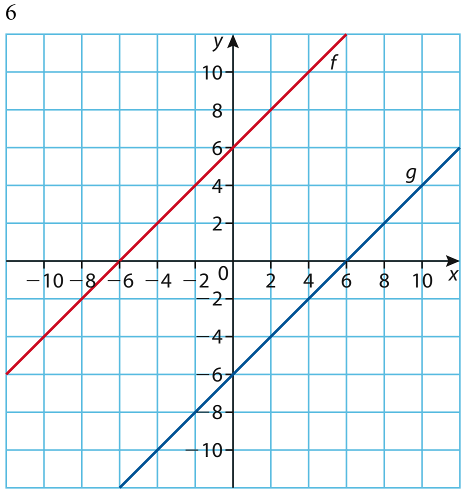
1 a) 2 b) 6

2 a) -1 b) b

3 4

4 6

5



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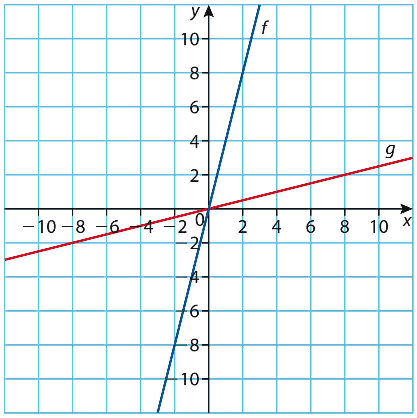

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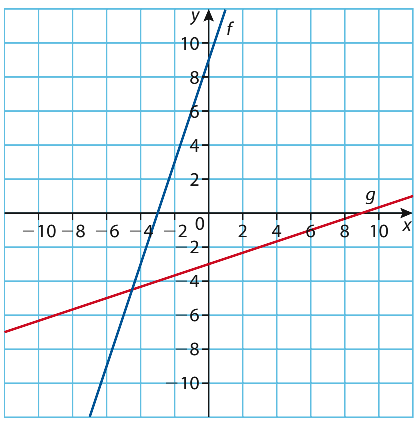

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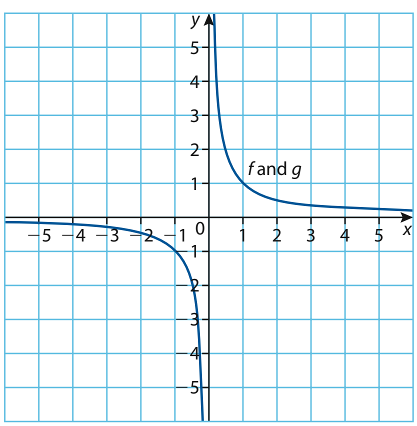

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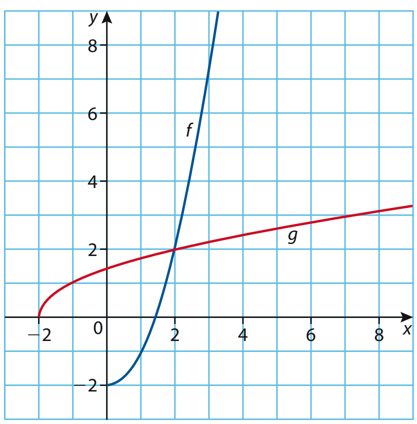

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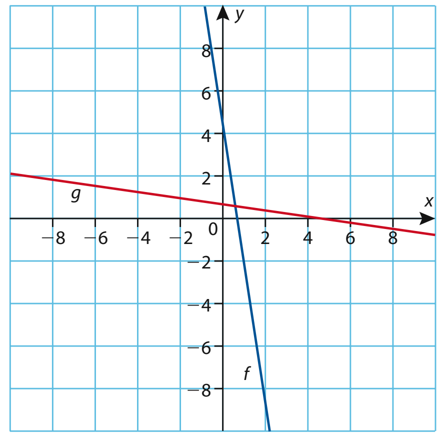

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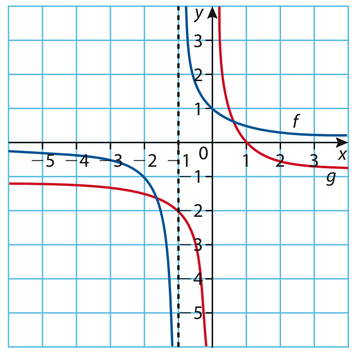

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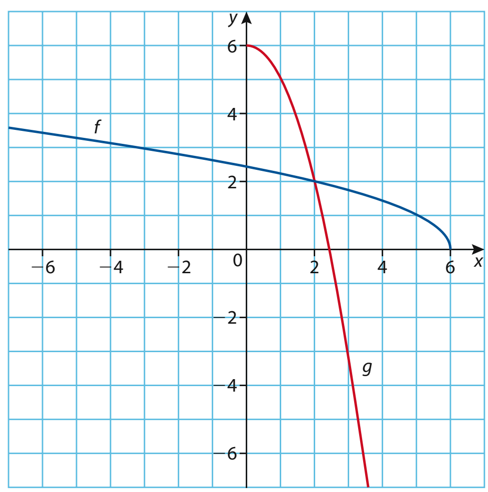

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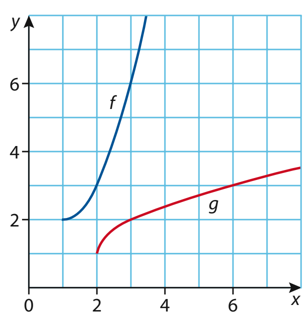

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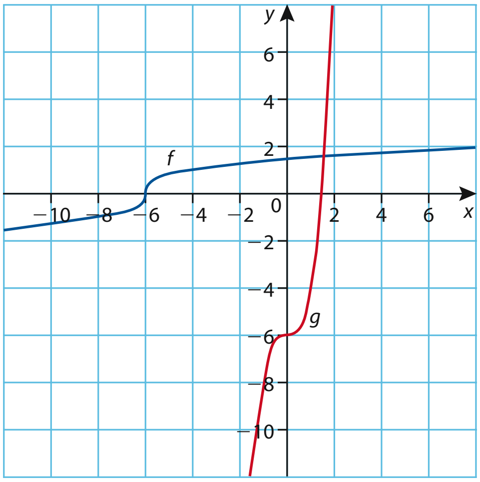


15 $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}, x \in \mathbb{R}$

16 $f^{-1}(x) = 4x - 7, x \in \mathbb{R}$

17 $f^{-1}(x) = x^2, x \geqslant 0$

18 $f^{-1}(x) = \frac{1}{x} - 2, x \in \mathbb{R}, x \neq 0$

19 $f^{-1}(x) = \sqrt{4 - x}, x \leqslant 4$

20 $f^{-1}(x) = x^2 + 5, x \geqslant 0$

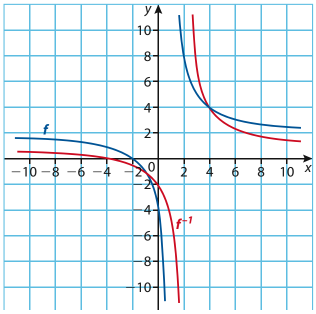
21 $f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}, x \in \mathbb{R}$

22 $f^{-1}(x) = 1 + \sqrt{x+1}, x \geqslant -1$

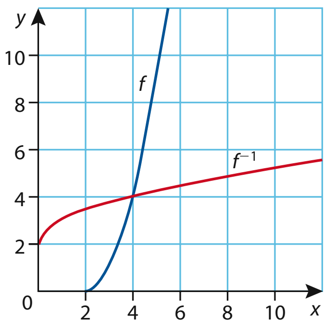
23 $f^{-1}(x) = \sqrt{\frac{1+x}{1-x}}, -1 \leqslant x \leqslant 1$

24 $f^{-1}(x) = \sqrt[3]{x-1}, x \in \mathbb{R}$

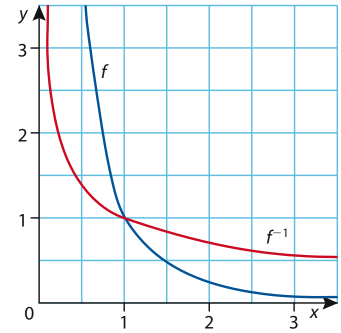
25 $f^{-1}(x) = \frac{x+3}{x-2}$



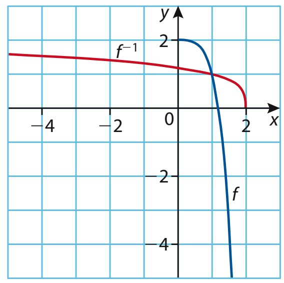
26 $x \geqslant 2,\ f^{-1}(x) = \sqrt{x} + 2$



27 $x > 0,\ f^{-1}(x) = \sqrt{\frac{1}{x}}$



28 $x > 0,\ f^{-1}(x) = \sqrt[4]{2 - x}$



29 $x < -1,\ -1 \leqslant x \leqslant 1,\ x > 1$

30 $\frac{3}{2}$

31 5

32 -4

33 $\frac{7}{2}$

34 $g^{-1} \circ h^{-1} = \frac{1}{2}x - 3$

35 $h^{-1} \circ g^{-1} = \frac{1}{2}x - \frac{3}{2}$

36 $(g \circ h)^{-1} = \frac{1}{2}x + \frac{1}{2}$

37 $(h \circ g)^{-1} = 2x + 2$

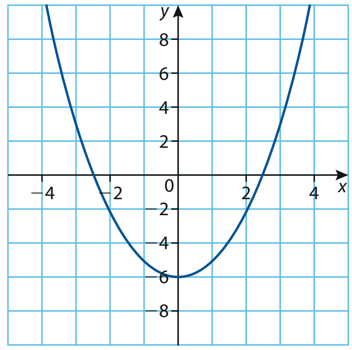
38 $f(f(x)) = f\left(\frac{a}{x+b} - b\right) = \frac{a}{\frac{a}{x+b} - b + b} - b$

$$= \frac{a}{\frac{a}{x+b}} - b = \frac{a}{1} \cdot \frac{x+b}{a} - b = x + b - b = x$$

Since $f(f(x)) = x$, then the function f is its own inverse.

Exercise 2.4

1



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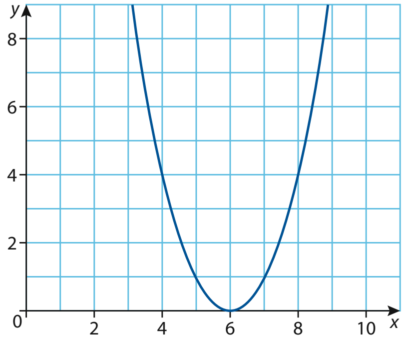

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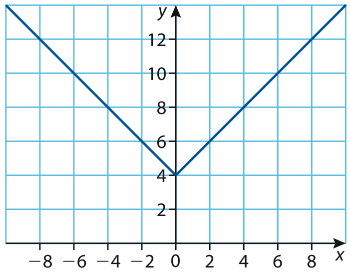

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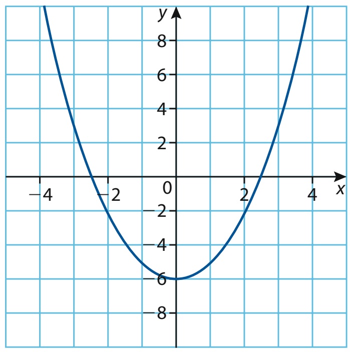

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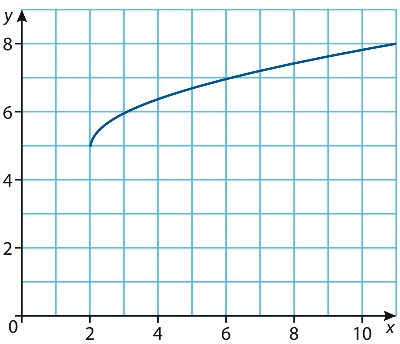

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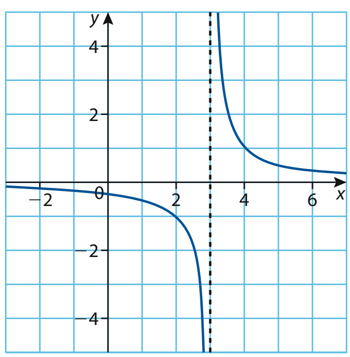

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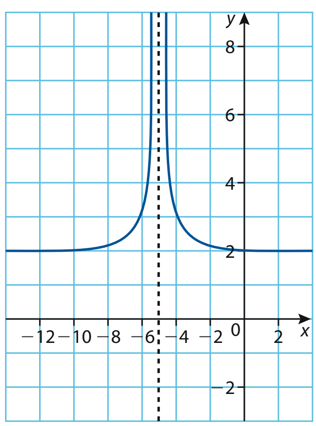

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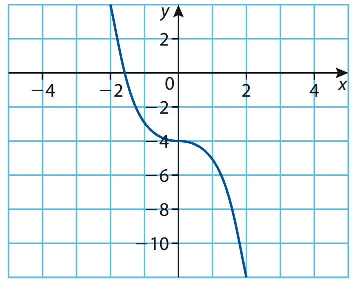

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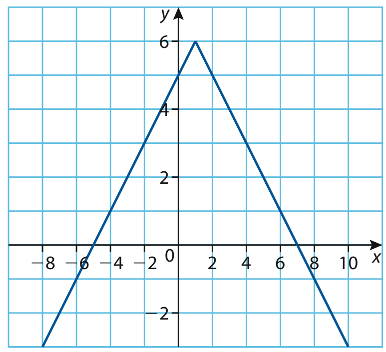

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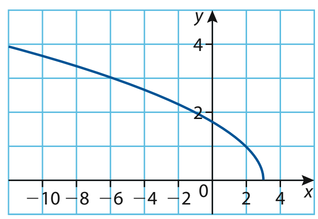

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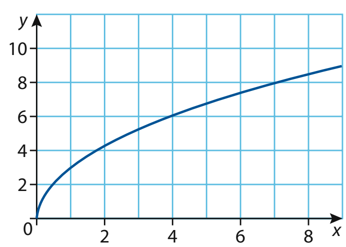

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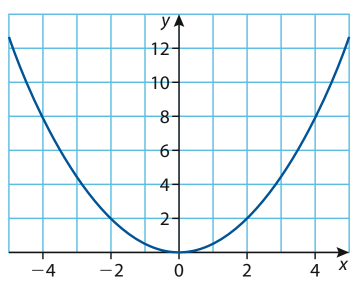

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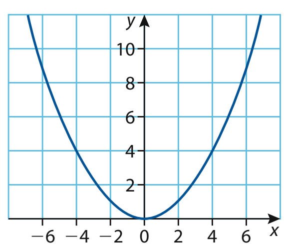

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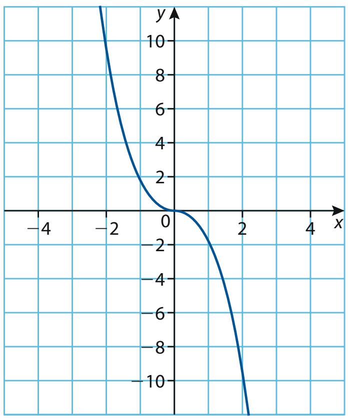


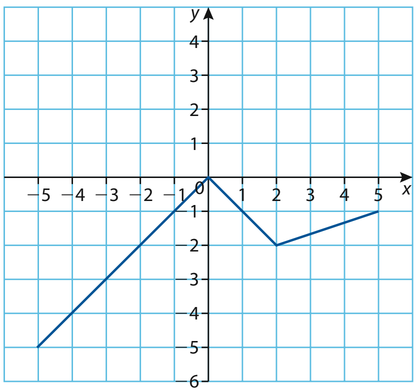
15 $y = -x^2 + 5$

16 $y = \sqrt{-x}$

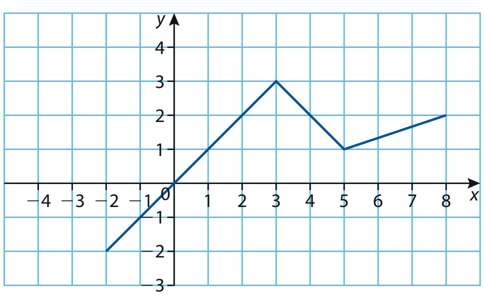
17 $y = -|x + 1|$

18 $y = \dfrac{1}{x - 2} - 2$

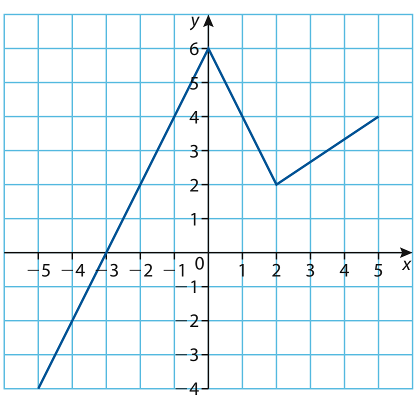
19 a)



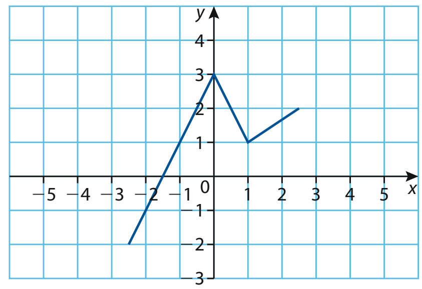
b)



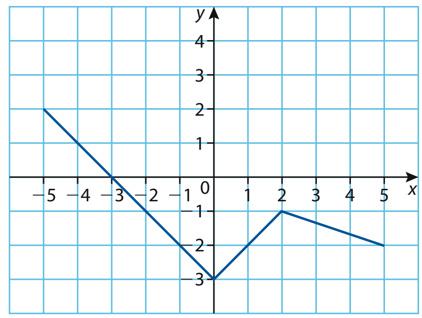
c)



d)



e)



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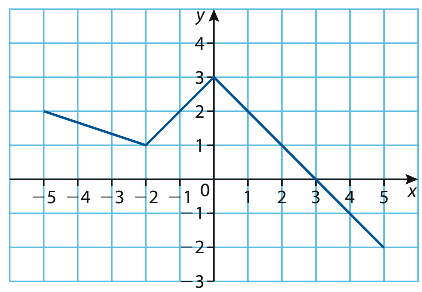
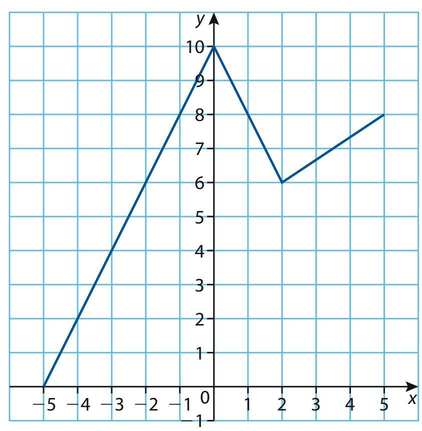


g)



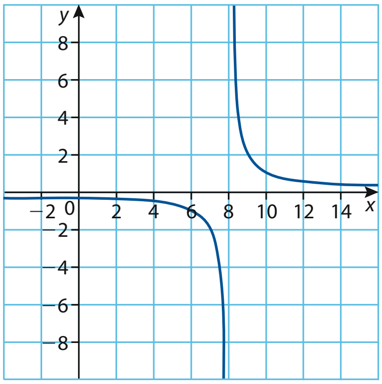
20 Horizontal translation 3 units right; vertical translation 5 units up (or reverse order).

21 Reflect over the x-axis; vertical translation 2 units up (or reverse order).

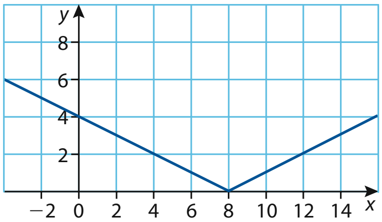
22 Horizontal translation 4 units left; vertical shrink by factor $\frac{1}{2}$ (or reverse order).

23 Horizontal translation 1 unit right; horizontal shrink by factor $\frac{1}{3}$; vertical translation 6 units down.

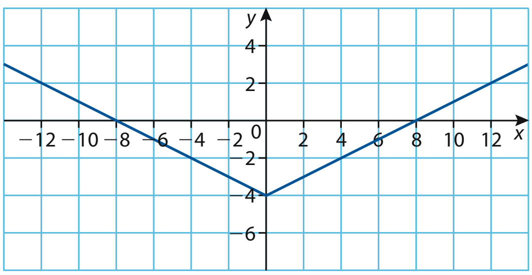
24 a)



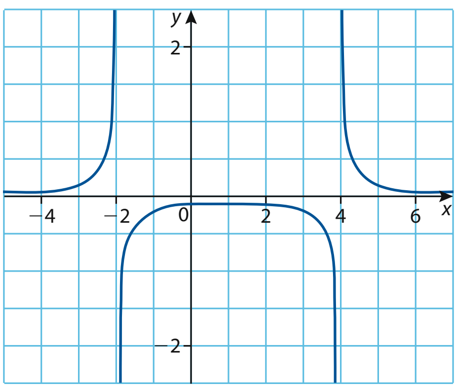
b)



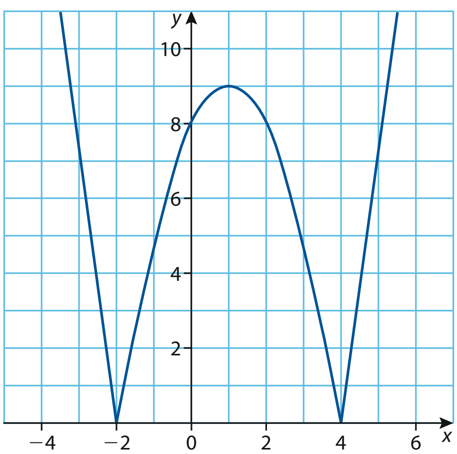
c)



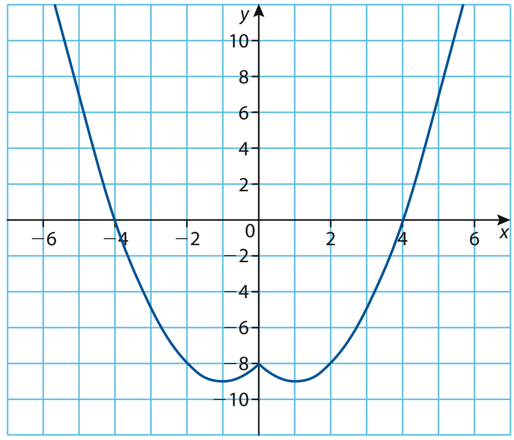
25 a)



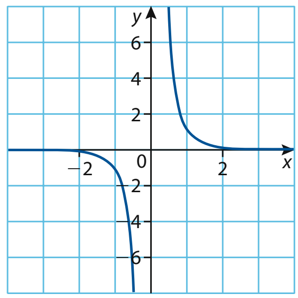
b)



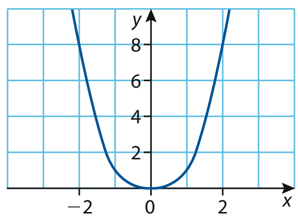
c)



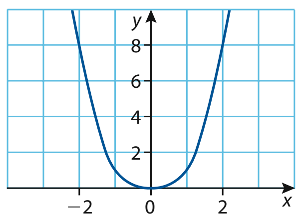
26 a)



b)



c)



Practice questions

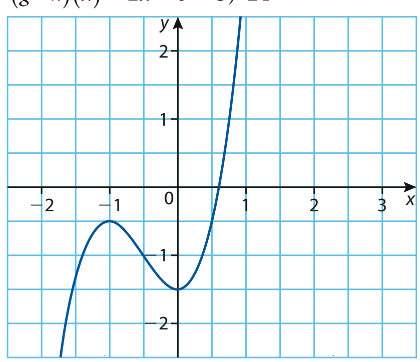
1 a) $a = -3, b = 1$ b) range: $y \geqslant 0$

2 a) 5 b) 3

3 a) $g^{-1}(x) = -3x + 4$ b) $x = \frac{2}{3}$

4 a) $(g \circ h)(x) = 2x - 3$ b) 24

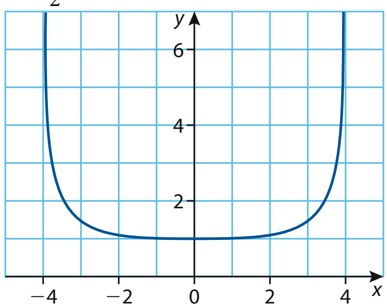
5 a)



maximum at $\left(-1, -\frac{1}{2}\right)$; minimum at $\left(0, -\frac{3}{2}\right)$

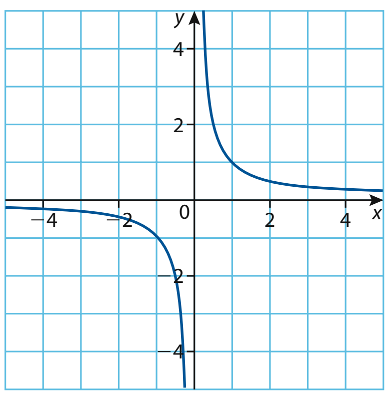
6 a) $k = \frac{1}{2}$ b) $p = -5$ c) $q = 3$

7 a)



b) $x = 4,\ x = -4$ c) range: $y \geqslant 1$

8 a)

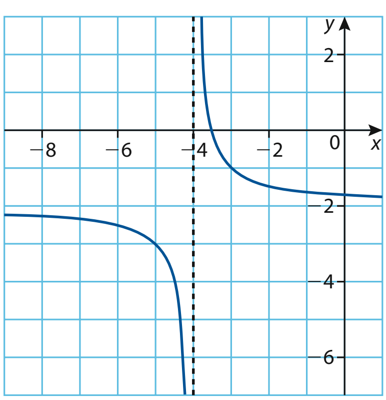


b) $h(x) = \dfrac{1}{x+4} - 2$

c) (i) x-intercept: $\left(-\frac{7}{2}, 0\right)$; y-intercept: $\left(0, -\frac{7}{4}\right)$

(ii) Vertical asymptote: $x = -4$; horizontal asymptote: $y = -2$

(iii)



9 a) (i) $\sqrt{11}$ (ii) 7 (iii) 0

b) $x < -3$

c) $(g \circ f)(x) = x - 2$

10 a) 4 b) $(g^{-1} \circ h)(x) = 2x^2 + 6$ c) $x = \pm 2\sqrt{2}$

11 a) $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$

b) $(f \circ g)(x) = \dfrac{12}{x} - 1$

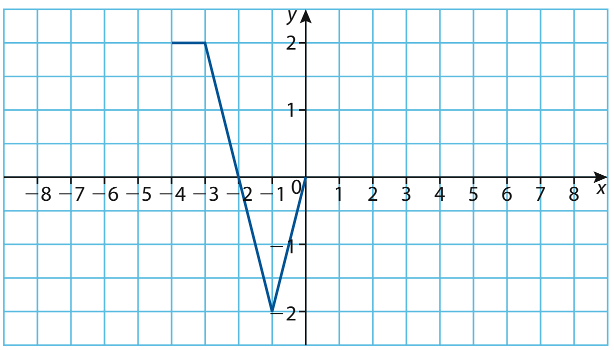
c) $(f \circ g)^{-1}(x) = \dfrac{12}{x+1}$

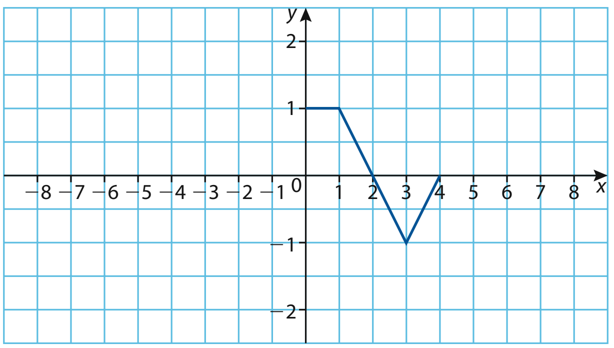
d) $(g \circ g)(x) = x$

12 a) (i) $a = 8$ (ii) $b = -3$

b) Reflection over x-axis

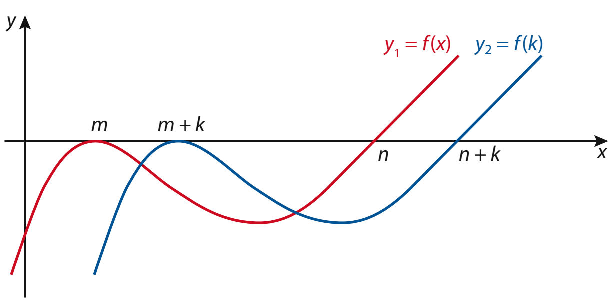
13 a)





b) $A'(-3, -2)$

14



15 $(f \circ g^{-1})(x) = \sqrt[3]{x} + 1$

16 a) $g(x) = \dfrac{x}{2x+1}$ b) $\dfrac{2}{9}$

17 a) $-\dfrac{\sqrt{2}}{2} \leqslant x \leqslant \dfrac{\sqrt{2}}{2},\ x \neq 0$

b) $f(x) \geqslant 0$

18 $f^{-1}(x) = \dfrac{x+1}{x-2},\ x \neq 2$

19 a) $-\frac{1}{2} < A < 2$ b) $f^{-1}(x) = \dfrac{-2x-1}{x-2}$

20 a) $g(x) = \sqrt[3]{x+1}$ b) $g(x) = \sqrt[3]{x} + 1$

21 a) $S = \{x : -\sqrt{3} < x < \sqrt{3}\}$ b) $f(x) \geqslant \dfrac{\sqrt{3}}{3}$

22 $\dfrac{x}{4}$

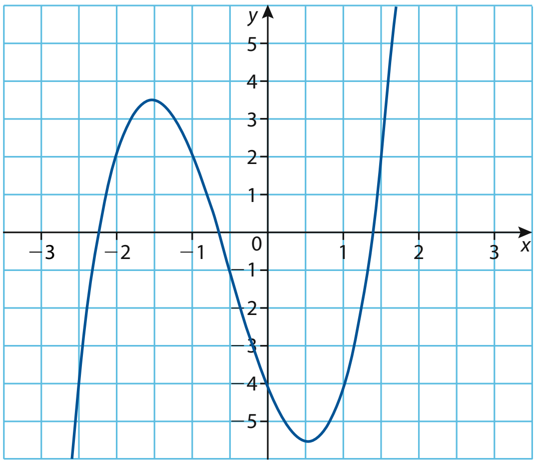
23 a) $A(1, 25),\ B(4, 0),\ C(7, -35),\ D(10, 0)$

b) $A(-1, -25),\ B(0, 0),\ C(1, 35),\ D(2, 0)$

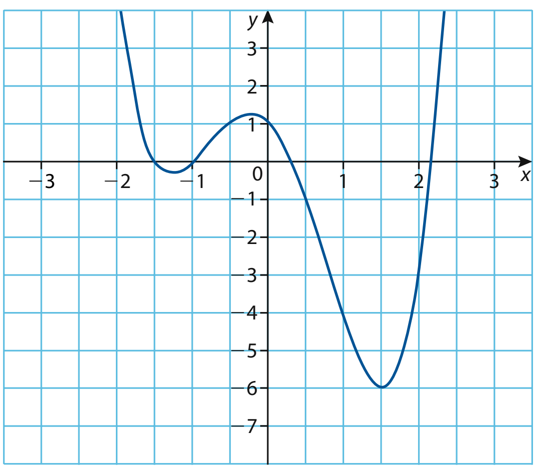
Chapter 3

Exercise 3.1

1 $-8; -8$
2 $0; 33$
3 $29; 2375$
4 $0; -3c+6$
5 $k = 2$
6 $k = 2$
7 a) $-16, 2, 2, -4, -4, 14, 62$
b) 3
c)



8 a) $52, 5, 0, 1, -4, -3, 40$
b) 4
c)



9 $a = \frac{12}{11}$

10 $b = -\sqrt{3}$

11 a) (i) (↖, ↗) (ii) (↙, ↘)
(iii) (↖, ↘) (iv) (↙, ↗)
(v) (↖, ↗) (vi) (↙, ↘)
(vii) (↙, ↗) (viii) (↖, ↘)
b) If leading term has positive coefficient and even exponent, then (↖, ↗).
If leading term has negative coefficient and even exponent, then (↙, ↘).
If leading term has positive coefficient and odd exponent, then (↙, ↗).
If leading term has negative coefficient and odd exponent, then (↖, ↘).

Exercise 3.2

1 a) $f(x) = (x-5)^2 + 7$
b) Horizontal translation 5 units right; vertical translation 7 units up.
c) Minimum: $(5, 7)$

2 a) $f(x) = (x+3)^2 - 1$
b) Horizontal translation 3 units left; vertical translation 1 unit down.
c) Minimum: $(-3, -1)$

3 a) $f(x) = -2(x+1)^2 + 12$
b) Horizontal translation 1 unit left; reflection over x-axis; vertical stretch by factor 2; vertical translation 12 units up.
c) Maximum: $(-1, 12)$

4 a) $f(x) = 4\left(x - \frac{1}{2}\right)^2 + 8$
b) Horizontal translation $\frac{1}{2}$ unit right; vertical stretch by factor 4; vertical translation 8 units up.
c) Maximum: $\left(\frac{1}{2}, 8\right)$

5 a) $f(x) = \frac{1}{2}(x+7)^2 + \frac{3}{2}$
b) Horizontal translation 7 units left; vertical shrink by factor $\frac{1}{2}$; vertical translation $\frac{3}{2}$ units up.
c) Minimum: $\left(-7, \frac{3}{2}\right)$

6 $x = 2, x = -4$
7 $x = 5, x = -2$
8 $x = \frac{3}{2}, x = 0$
9 $x = 6, x = -1$
10 $x = 3$
11 $x = \frac{1}{3}, x = -4$
12 $x = 3, x = 2$
13 $x = 2, x = \frac{1}{4}$
14 $x = -2 \pm \sqrt{7}$
15 $x = 5, x = -1$
16 No real solution
17 $x = -4 \pm \sqrt{13}$
18 $x = 2, x = -4$
19 $x = \frac{2 \pm \sqrt{22}}{2}$
20 a) $x = 2 \pm \sqrt{5}$
b) Axis of symmetry: $x = 2$
c) Minimum value of f is -5
21 Two real solutions
22 No real solutions
23 Two real solutions
24 No real solutions
25 $p = \pm 2\sqrt{2}$
26 $k < 4$
27 $k < -1,\ k > 1$
28 $m < -3,\ m > 3$
29 $k > 12$
30 $x - 2 - x^2 \Rightarrow -(x^2 - x + 2) \Rightarrow -\left(x^2 - x + \frac{1}{4}\right) - \frac{7}{4}$
$\Rightarrow -\left(x - \frac{1}{2}\right)^2 - \frac{7}{4} \leqslant -\frac{7}{4}$ for all x
31 $y = -2x^2 + 6x + 8$
32 $y = x^2 - \frac{7}{2}x - \frac{1}{2}$
33 $-1 < k < 15$
34 $m < -2\sqrt{10}$ or $m > 2\sqrt{10}$
35 $f(x) = 3x^2 + 5x - 2$
36 $f(2) = 8$
37 $x < 1$ or $x > 3$
38 $\Delta = (2-t)^2 - 4(2)(t^2+3) > 0 \Rightarrow -7t^2 - 4t - 20 > 0$; because $\Delta_1 = -544$ for $-7t^2 - 4t - 20$ and leading coefficient is negative, then graph of $y = -7t^2 - 4t - 20$ is a parabola opening down and always below x-axis; hence, Δ for original equation is always negative; thus, no real roots

39 $x = \frac{-(-a^2-1) \pm \sqrt{(-a^2-1)^2 - 4a(a)}}{2a} = \frac{a^2 + 1 \pm \sqrt{a^4 - 2a^2 + 1}}{2a}$

$= \frac{a^2 + 1 \pm \sqrt{(a^2-1)^2}}{2a} = \frac{a^2 + 1 \pm (a^2 - 1)}{2a} \Rightarrow x = \frac{2a^2}{2a}$

$= a$ or $x = \frac{2}{2a} = \frac{1}{a}$

40 a) sum $= -3$, product $= -\frac{5}{2}$

b) sum $= -3$, product $= -1$

c) sum $= 0$, product $= -\frac{3}{2}$

d) sum $= a$, product $= -2a$

e) sum $= 6$, product $= -4$

f) sum $= \frac{1}{3}$, product $= -\frac{2}{3}$

41 $4x^2 + 5x + 4 = 0$

42 a) $\frac{1}{9}$ b) $\frac{1}{12}$ c) $\frac{55}{27}$

43 a) -2 and -6 b) $k = 12$

44 a) $-\frac{1}{4}$ b) $4x^2 + x + 1 = 0$

45 a) $x^2 - 19x + 25 = 0$ b) $25x^2 + 72x - 5 = 0$

Exercise 3.3

1 $3x^2 + 5x - 5 = (x+3)(3x-4) + 7$

2 $3x^4 - 8x^3 + 9x + 5 = (x-2)(3x^3 - 2x^2 - 4x + 1) + 7$

3 $x^3 - 5x^2 + 3x - 7 = (x-4)(x^2 - x - 1) - 11$

4 $9x^3 + 12x^2 - 5x + 1 = (3x-1)(3x^2 + 5x) + 1$

5 $x^5 + x^4 - 8x^3 + x + 2 = (x^2 + x - 7)(x^3 - x + 1) + (-7x + 9)$

6 $(x-7)(x-1)(2x-1)$ **7** $(x-2)(2x+1)(3x+2)$

8 $(x-2)^2(x+4)(3x+2)$ **9** $Q(x) = x - 2$, $R = -2$

10 $Q(x) = x^2 + 2$, $R = -3$ **11** $Q(x) = 3$, $R(x) = 20x + 5$

12 $Q(x) = x^4 + x^3 + 4x^2 + 4x + 4$, $R = -2$

13 $P(2) = 5$ **14** $P(-1) = -17$

15 $P(-7) = -483$ **16** $P\left(\frac{1}{4}\right) = \frac{49}{64}$

17 $x = 2 + i$ or $x = 2 - i$ **18** $x = \frac{1+\sqrt{5}}{2}$ or $x = \frac{1-\sqrt{5}}{2}$

19 $k = \sqrt{1 - x\sqrt{3}}$ or $k = -\sqrt{1 - x\sqrt{3}}$

20 $a = 5$, $b = 12$

21 $x^3 - 3x^2 - 6x + 8$ **22** $x^4 - 3x^3 - 7x^2 + 15x + 18$

23 $x^3 - 6x^2 + 12x - 8$ **24** $x^3 - x^2 + 2$

25 $x^4 + 2x^3 + x^2 + 18x - 72$ **26** $x^4 - 8x^3 + 27x^2 - 50x + 50$

27 $x = 2 + 3i$, $x = 3$

28 a) $a = -1$, $b = -2$ b) $3x + 2$

29 $a = \frac{4}{3}$, $b = \frac{1}{3}$

30 $x = 3$, $x = -1$, $x = -\frac{1}{4} + \frac{\sqrt{3}}{4}i$, $x = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$

31 $a = -1$, $b = -4$, $c = 4$ **32** $p = -5$, $q = 23$, $r = -51$

33 $a = -5$ **34** $m = -2$, $n = -6$

35 $b = 18$ **36** b) $R = 3$

37 a) sum $= \frac{2}{3}$, product $= 5$ b) sum $= 1$, product $= 7$

c) sum $= \frac{1}{3}$, product $= -\frac{1}{2}$

39 $-9, 3, 6$

40 $2, -4, 8$

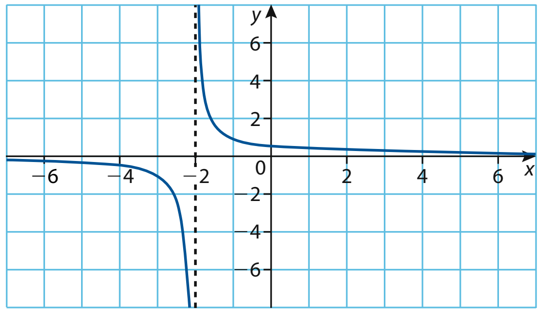
41 $3 + 2i, 2 + i, 2 - i$

42 $k = 3$

43 $k = -8$

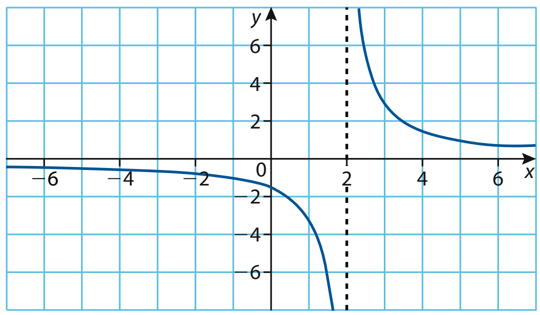
Exercise 3.4

1



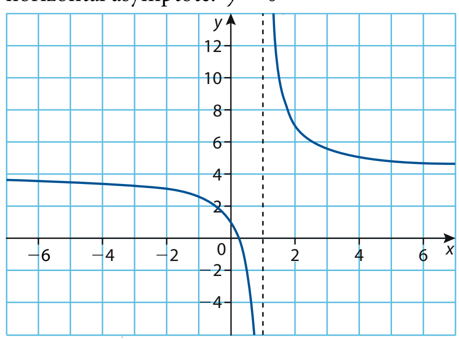
vertical asymptote: $x = -2$
horizontal asymptote: $y = 0$

2



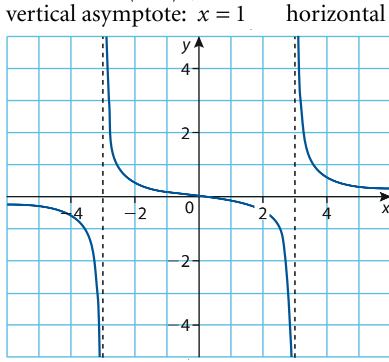
vertical asymptote: $x = 2$
horizontal asymptote: $y = 0$

3



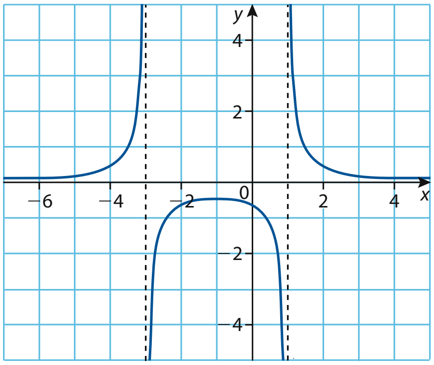
x-intercept: $\left(\frac{1}{4}, 0\right)$, y-intercept: $(0, 1)$
vertical asymptote: $x = 1$ horizontal asymptote: $y = 4$

4



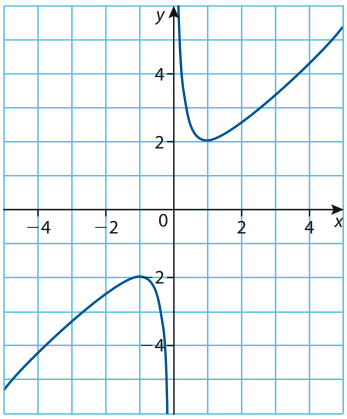
x- and y-intercept: $(0, 0)$
vertical asymptotes: $x = -3, x = 3$
horizontal asymptote: $y = 0$

5



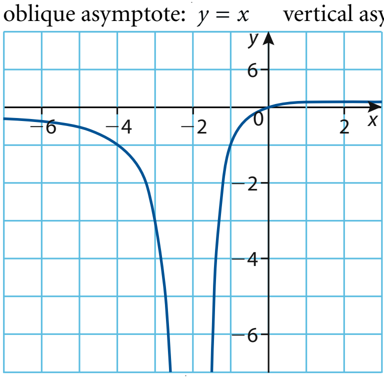
x-intercept: none, y-intercept: $\left(0, -\frac{2}{3}\right)$
vertical asymptotes: $x = -3$, $x = 1$
horizontal asymptote: $y = 0$

6



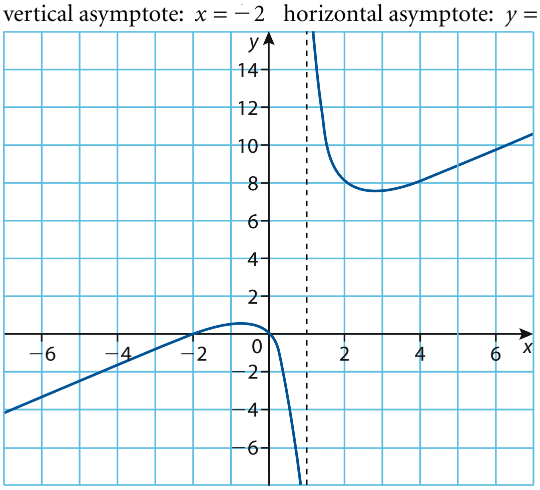
oblique asymptote: $y = x$ vertical asymptote: $x = 0$

7



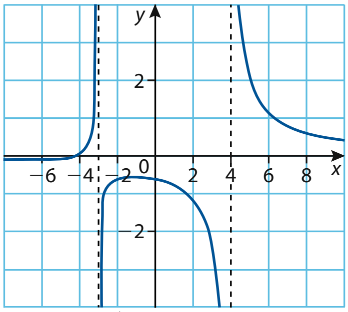
x- and y-intercept: $(0, 0)$
vertical asymptote: $x = -2$ horizontal asymptote: $y = 0$

8



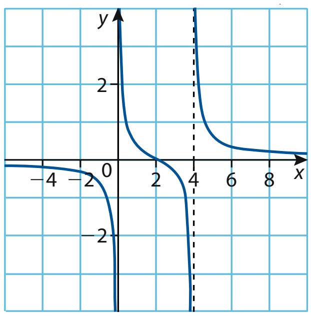
x- and y-intercept: $(0, 0)$
vertical asymptote: $x = 1$ oblique asymptote: $y = x + 3$

9



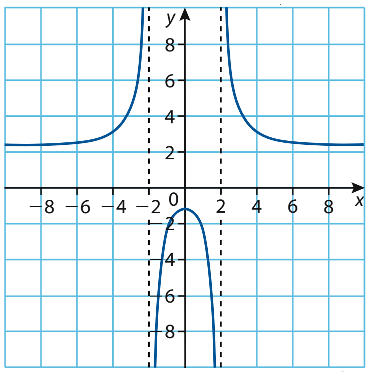
x-intercept: $(-4, 0)$ y-intercept: $\left(0, -\frac{2}{3}\right)$
vertical asymptotes: $x = -3$ and $x = 4$
horizontal asymptote: $y = 0$

10



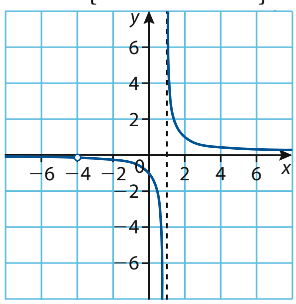
x-intercept: $(2, 0)$ y-intercept: none
vertical asymptotes: $x = 0$ and $x = 4$
horizontal asymptote: $y = 0$

11



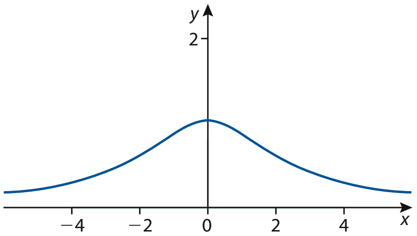
domain $\{x : x \in \mathbb{R},\ x \neq \pm 2\}$ range $\left\{y : y \leqslant -\frac{5}{4} \text{ or } y > 2\right\}$

12



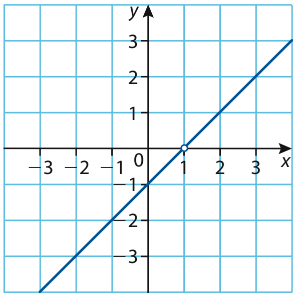
domain $\{x : x \in \mathbb{R},\ x \neq -4, 1\}$ range $\{y : y \in \mathbb{R},\ y \neq 0\}$

13



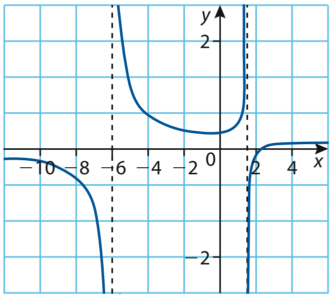
domain $\{x : x \in \mathbb{R}\}$ range $\{y : 0 < y \leqslant 1\}$

14



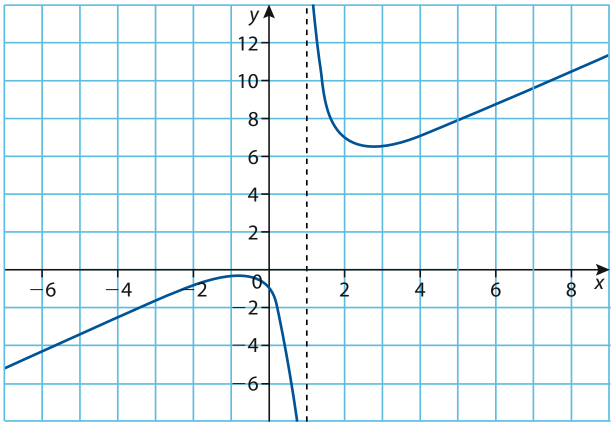
domain $\{x : x \in \mathbb{R},\ x \neq 1\}$
range $\{y : y \in \mathbb{R},\ y \neq 0\}$

15



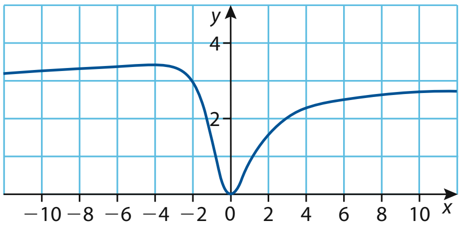
x-intercept: $\left(\frac{5}{2}, 0\right)$ y-intercept: $\left(0, \frac{5}{18}\right)$
vertical asymptotes: $x = -6$ and $x = \frac{3}{2}$
horizontal asymptote: $y = 0$

16



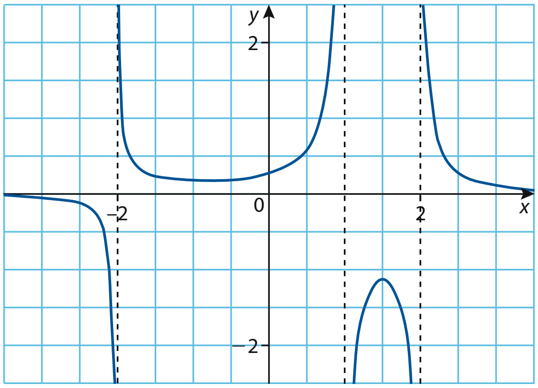
x-intercept: none
y-intercept: $(0, -1)$
vertical asymptote: $x = 1$
oblique asymptote: $y = x + 2$

17



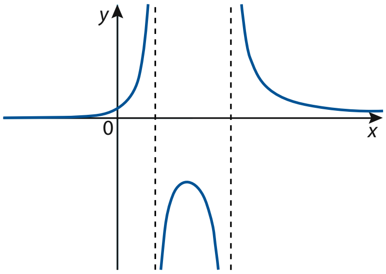
x- and y-intercept: $(0, 0)$
horizontal asymptote: $y = 3$

18

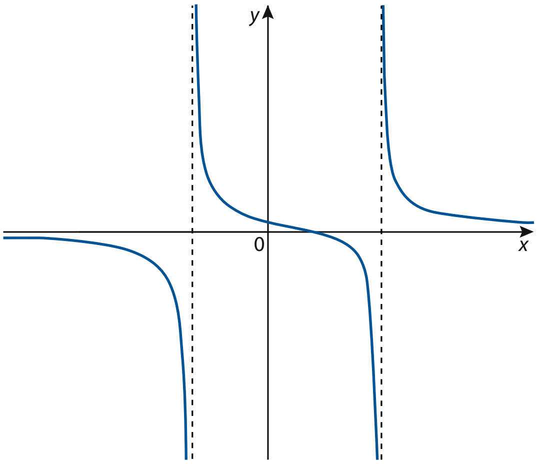


x-intercept: none y-intercept: $\left(0, \frac{1}{4}\right)$
vertical asymptotes: $x = -2$, $x = 1$ and $x = 2$
horizontal asymptote: $y = 0$

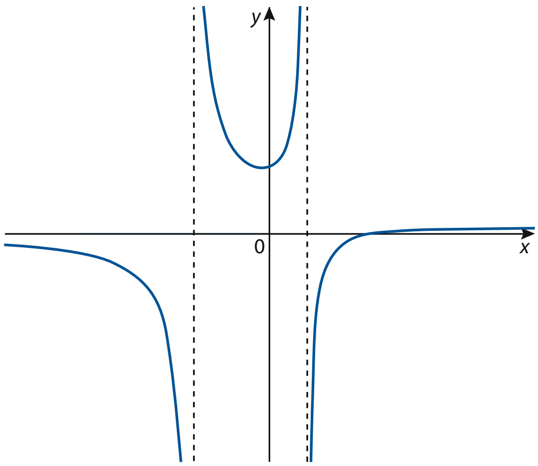
19 a)



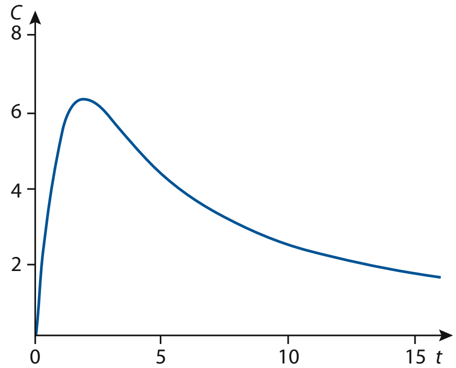
b)



c)



20 a)



b) At $t = 2$ minutes, concentration is 6.25 mg/l.

c) It continues to decrease and approaches zero as amount of time increases.

d) 50 minutes (49 minutes 55 seconds)

Exercise 3.5

1 $x = 3$ **2** $x = 9$

3 $x = 5$ or $x = -2$ **4** $x = 11$ or $x = 3$

5 $x = -5$ **6** $x = 1$ or $x = -2$

7 $x = 2$ or $x = -2$ **8** $x = \frac{1}{2}$

9 $x = \pm\sqrt{5}$ **10** $x = 27$ or $x = -\frac{125}{8}$

11 $x = 3$ or $x = 2$ **12** $x = \frac{1 \pm \sqrt{41}}{4}$

13 $x = \frac{4}{3}$ or $x = -4$ **14** $x = 15$ or $x = \frac{9}{2}$

15 No solution **16** $x = 2$ or $x = -1$

17 $x = 2$ or $x = \frac{1}{2}$ **18** $x = 9$

19 $x = -5$ **20** $x = \frac{\pm 2\sqrt{5}}{5}$ or $x = \pm 1$

21 $x = \frac{1 + \sqrt{41}}{2}$ **22** $x = \frac{49}{4}$ or $x = \frac{64}{9}$

23 $-\frac{2}{3} < x < 2$ **24** $x < -2, x \geqslant 3$

25 $-10 \leqslant x \leqslant 6$ **26** $x < \frac{3}{2}, x > 2$

27 $x > \frac{17}{2}$ **28** $-4 \leqslant x \leqslant -1, 1 \leqslant x \leqslant 4$

29 $x < -1, x > 2$ **30** $x < -1, -\frac{2}{3} < x < 3, x > 4$

31 a) $p = \frac{9}{4}$ b) $p < \frac{9}{4}$ c) $p > \frac{9}{4}$

32 $x < -1, x > \frac{1}{3}$

33 a) $m + \frac{1}{n} > 2 \Rightarrow mn + 1 > 2n \Rightarrow mn - 2n + 1 > 0$; since $m > n \Rightarrow mn > n^2$ it follows that $mn - 2n + 1 > n^2 - 2n + 1$ and since $n^2 - 2n + 1 = (n-1)^2 > 0$ then $mn - 2n + 1 > 0$ $\Rightarrow m + \frac{1}{n} > 2$

b) $(m+n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4 \Rightarrow (m+n)\left(\frac{1}{m} + \frac{1}{n}\right)mn > 4mn \Rightarrow$ $(m+n)(n+m) > 4mn \Rightarrow m^2 + 2mn + n^2 > 4mn \Rightarrow$ $m^2 - 2mn + n^2 > 0 \Rightarrow (m-n)^2 > 0$ which is true for all x and is equivalent to original inequality – thus, $(m+n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4$ is true for all x.

34 $x = \frac{-1 \pm \sqrt{13}}{2}$, $x = 1$ or $x = -2$

35 $(a + b + c)^2 < 3(a^2 + b^2 + c^2)$
$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2ac + 2bc < 3a^2 + 3b^2 + 3c^2$
$\Rightarrow 0 < 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc$
$\Rightarrow a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + a^2 - 2ac + c^2 > 0$
$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 > 0.$
Since all the numbers are unequal, the squares of their differences are strictly larger than zero therefore their sum too is strictly larger than zero.

36 a) $1 < x < 3$ b) $x < -2, -1 < x < 1, x > 3$

37 If a and b have the same sign, then $|a + b| = |a| + |b|$; and if a and b are of opposite sign, then $|a + b| < |a| + |b|$.

Practice questions

1 $x = a$ or $x = 3b$ **2** $x \leqslant 4$

3 $c = 5$ **4** $a = -\frac{1}{2}, b = 4, c = -2$

5 $\omega = -2, p = 2, q = -8$

6 a) $m > -2$ b) $-2 < m < 0$

7 $a = 2, b = -1, c = -2$

8 $x < 5, x > \frac{15}{2}$

9 $-1 < k < 15$

10 a) $f(x) = 2 - \frac{3}{(x+2)^2 + 1}$

b) (i) $\lim_{x \to +\infty} f(x) = 2$ (ii) $\lim_{x \to -\infty} f(x) = 2$

c) $(-2, -1)$

11 $k \in \mathbb{R}$ **12** $a = -1$

13 $a = \frac{7}{4}, b = -\frac{1}{4}$ **14** $a = -6$

15 $a = 4$ **16** $a = -2, b = 6$

17 $a = 1$ **18** $k = 6$

19 $k = 6$ **20** $-2.80 < k < 0.803$ (3 s.f.)

21 $-3 \leqslant k \leqslant 4.5$ **22** $-4 \leqslant m \leqslant 0$

23 $1 \leqslant x \leqslant 3$ **24** $-2.30 < x < 0$ or $1 < x < 1.30$

25 $-3 \leqslant x \leqslant \frac{1}{3}$ **26** $x < -1$ or $4 < x \leqslant 14$

27 $x \leqslant 3$ or $x \geqslant 27$ **28** $x = 2 - i$ and $x = 2$

29 $x < \frac{1}{3}$

Chapter 4

Exercise 4.1

1 $-1, 1, 3, 5, 7$ **2** $-1, 1, 5, 13, 29$

3 $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}$ **4** $5, 8, 11, 14, 17$

5 $1, 7, -5, 19, -29$ **6** $3, 7, 13, 21, 31$

7 $-1, 1, 3, 5, 7, 97$

8 $2, 6, 18, 54, 162, 4.786 \times 10^{23}$

9 $\frac{2}{3}, -\frac{2}{3}, \frac{6}{11}, -\frac{4}{9}, \frac{10}{27}, \frac{50}{1251}$

10 1, 2, 9, 64, 625, 1.776×10^{83}

11 3, 11, 27, 59, 123, 4.50×10^{15}

12 0, 3, $\frac{3}{7}, \frac{21}{13}, \frac{39}{55}$, approx. 1

13 2, 6, 18, 54, 162, 4.786×10^{23}

14 $-1, 1, 3, 5, 7, 97$

15 $u_n = \frac{1}{4}u_{n-1}, u_1 = \frac{1}{3}$

16 $u_n = \frac{4a^2}{3}u_{n-1}, u_1 = \frac{1}{2}a$

17 $u_n = u_{n-1} + a - k, u_1 = a - 5k$

18 $u_n = n^2 + 3$

19 $u_n = 3n - 1$

20 $u_n = \frac{2n-1}{n^2}$

21 $u_n = \frac{2n-1}{n+3}$

22 a) $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}$

23 a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

Exercise 4.2

1 $3, \frac{19}{5}, \frac{23}{5}, \frac{27}{5}, \frac{31}{5}, 7$

2 a) Arithmetic, $d = 2, a_{50} = 97$
b) Arithmetic, $d = 1, a_{50} = 52$
c) Arithmetic, $d = 2, a_{50} = 97$
d) Not arithmetic, *no common difference*
e) Not arithmetic, *no common difference*
f) Arithmetic, $d = -7, a_{50} = -341$

3 a) 26
b) $a_n = -2 + 4(n-1)$
c) $a_1 = -2, a_n = a_{n-1} + 4$ for $n > 1$

4 a) 1
b) $a_n = 29 - 4(n-1)$
c) $a_1 = 29, a_n = a_{n-1} - 4$ for $n > 1$

5 a) 57
b) $a_n = -6 + 9(n-1)$
c) $a_1 = -6, a_n = a_{n-1} + 9$ for $n > 1$

6 a) 9.23
b) $a_n = 10.07 - 0.12(n-1)$
c) $a_1 = 10.07, a_n = a_{n-1} - 0.12$ for $n > 1$

7 a) 79
b) $a_n = 100 - 3(n-1)$
c) $a_1 = 100, a_n = a_{n-1} - 3$ for $n > 1$

8 a) $-\frac{27}{4}$
b) $a_n = 2 - \frac{5}{4}(n-1)$
c) $a_1 = 2, a_n = a_{n-1} - \frac{5}{4}$ for $n > 1$

9 13, 7, 1, $-5, -11, -17, -23$

10 299, $299\frac{1}{4}, 299\frac{1}{2}, 299\frac{3}{4}$, 300

11 $a_n = -10 + 4(n-1) = 4n - 14$

12 $a_n = -\frac{142}{3} + \frac{11}{3}(n-1) = -51 + \frac{11}{3}n$

13 88

14 36

15 11

16 16

17 11

18 9, 3, $-3, -9, -15$

19 99.25, 99.50, 99.75

20 $a_n = 4n - 1$

21 $a_n = \frac{19n - 277}{3}$

22 $a_n = 4n + 27$

23 Yes, 3271th term

24 Yes, 1385th term

25 No

Exercise 4.3

1 Geometric, $r = 3^a, g_{10} = 3^{9a+1}$

2 Arithmetic, $d = 3, a_{10} = 27$

3 Geometric, $r = 2, b_{10} = 4096$

4 Neither, not geometric, $r = 2, c_{10} = -1534$

5 Geometric, $r = 3, u_{10} = 78\,732$

6 Geometric, $r = 2.5, a_{10} = 7629.394\,531\,25$

7 Geometric, $r = -2.5, a_{10} = -7629.394\,531\,25$

8 Arithmetic, $d = 0.75, a_{10} = 8.75$

9 Geometric, $r = -\frac{2}{3}, a_{10} = -\frac{1024}{2187}$

10 Arithmetic, $d = 3$

11 Geometric, $r = -3$

12 Geometric, $r = 2$

13 Neither

14 Neither

15 Arithmetic, $d = 1.3$

16 a) 32 b) $-3 + 5(n-1)$
c) $a_1 = -3, a_n = a_{n-1} + 5$ for $n > 1$

17 a) -9 b) $19 - 4(n-1)$
c) $a_1 = 19, a_n = a_{n-1} - 4$ for $n > 1$

18 a) 69 b) $-8 + 11(n-1)$
c) $a_1 = -8, a_n = a_{n-1} + 11$ for $n > 1$

19 a) 9.35 b) $10.05 - 0.1(n-1)$
c) $a_1 = 10.05, a_n = a_{n-1} - 0.1$ for $n > 1$

20 a) 93 b) $100 - (n-1)$
c) $a_1 = 100, a_n = a_{n-1} - 1$ for $n > 1$

21 a) $-\frac{17}{2}$ b) $2 - 1.5(n-1)$
c) $a_1 = 2, a_n = a_{n-1} - 1.5$ for $n > 1$

22 a) 384 b) $3 \times 2^{n-1}$
c) $a_1 = 3, a_n = 2a_{n-1}$ for $n > 1$

23 a) 8748 b) $4 \times 3^{n-1}$
c) $a_1 = 4, a_n = 3a_{n-1}$ for $n > 1$

24 a) -5 b) $5 \times (-1)^{n-1}$
c) $a_1 = 5, a_n = -a_{n-1}$ for $n > 1$

25 a) -384 b) $3 \times (-2)^{n-1}$
c) $a_1 = 3, a_n = -2a_{n-1}$ for $n > 1$

26 a) $-\frac{4}{9}$ b) $972 \times (-\frac{1}{3})^{n-1}$
c) $a_1 = 972, a_n = (-\frac{1}{3})a_{n-1}$ for $n > 1$

27 a) $\frac{2187}{64}$ b) $a_n = -2\left(-\frac{3}{2}\right)^{n-1}$
c) $a_1 = -2, a_n = -\frac{3}{2}a_{n-1}, n > 1$

28 a) $\frac{390\,625}{117\,649}$ b) $a_n = 35\left(\frac{5}{7}\right)^{n-1}$
c) $a_1 = 35, a_n = \frac{5}{7}a_{n-1}, n > 1$

29 a) $-\frac{3}{64}$ b) $a_n = -6\left(\frac{1}{2}\right)^{n-1}$
c) $a_n = -6, a_n = \frac{1}{2}a_{n-1}, n > 1$

30 a) 1216 b) $9.5 \times 2^{n-1}$
c) $a_1 = 9.5, a_n = 2a_{n-1}, n > 1$

31 a) $69.833\,729\,609\,375 = \frac{893\,871\,739}{12\,800\,000}$
b) $a_n = 100\left(\frac{19}{20}\right)^{n-1}$
c) $a_1 = 100, a_n = \frac{19}{20}a_{n-1}, n > 1$

32 a) $0.002\,085\,685\,73 = \frac{2187}{1\,048\,576}$
b) $a_n = 2\left(\frac{3}{8}\right)^{n-1}$ c) $a_1 = 2, a_n = \frac{3}{8}a_{n-1}, n > 1$

33 6, 12, 24, 48

34 35, 175, 875

35 36

36 21, 63, 189, 567

37 $-24, 24$

38 1.5, $a_n = 24\left(\frac{1}{2}\right)^{n-1}$

39 $a_4 = \pm 3, r = \pm\frac{1}{2}, a_n = 24\left(\pm\frac{1}{2}\right)^{n-1}$

40 $\frac{49}{3}$

41 10th term

42 Yes, 10th term

43 Yes, 10th term

44 2228.92

45 £945.23

46 €2968.79

47 7745 thousands

48 $\frac{98}{9}$

49 10th term

50 €3714.87

51 £2921.16

Exercise 4.4

1 11 280

2 $-\dfrac{10\,5469}{1024}$

3 0.7

4 $\dfrac{10}{7}$

5 $\dfrac{16+4\sqrt{3}}{39}$

6 a) $\dfrac{52}{99}$ b) $\dfrac{449}{990}$ c) $\dfrac{7459}{2475}$

7 13 026.135 (£13 026.14)

8 940

9 6578

10 42 625

11 $\dfrac{n(7+3n)}{2}$

12 17 terms

13 85 terms

14 $d = 4$

15 a) 250, 125 250, b) 83 501

16 $a = 1, d = 5$

17 2890

18 0.290

19 -2.065

20 11 400

21 1.191

22 49.2

23 $\dfrac{6}{5}$

24 $\dfrac{3+\sqrt{6}}{2}$

25 $3, \dfrac{18}{5}, \dfrac{93}{25}, \dfrac{468}{125}, \dfrac{15}{4}\left(1-\dfrac{1}{5^n}\right)$

26 $\dfrac{1}{6}, \dfrac{1}{4}, \dfrac{3}{10}, \dfrac{1}{3}; \dfrac{n}{2n+4}$

27 $\sqrt{2}-1, \sqrt{3}-1, 1, \sqrt{5}-1; \sqrt{n+1}-1$

28 1.945, 152.42

29 127, 128

30 $\dfrac{819}{128}, \dfrac{32}{5}$

31 11 866

32 763 517

33 14 348 906

34 ≈ 150

Exercise 4.5

1 a) 120 b) 120 c) 20 d) 336

2 a) 1 b) 1 c) 120 d) 120

3 a) 70 b) 70 c) 330 d) 330

4 a) 0 b) 39 916 800 c) 0 d) 10

5 a) F b) F c) T

6 24

7 72

8 312

9 16 777 216

10 262 144

11 1 757 600 000

12 81 000

13 a) 40 320 b) 384

14 a) 40 320 b) 720

15 JANE, JAEN, JNAE, JNEA, JEAN, JENA, AJNE, AJEN, ANJE, ANEJ, AEJN, AENJ, NJAE, NJEA, NEJA, NEAJ, NAJE, NAEJ, EJAN, EJNA, EAJN, EANJ, ENJA, ENAJ

16 Mag, Mga, Mai, …(60 of them)

17 a) 175 760 000 b) 174 790 000

18 a) 4080 b) 1680 c) 1050 d) 1980 e) 3150

19 a) 296 b) 1460 c) 504

20 a) 125 000 b) 117 600 c) 61 250 d) 176 400

21 768

22 a) 36 b) 256

23 a) 5985 b) 2376 c) 2475

24 a) 2280 b) 748 c) 770

25 a) 1 192 052 400 b) 4560, 0.000 38% c) 265 004 096, 22.2%

26 a) 74 613 b) 7560

27 54 867 456 000

Exercise 4.6

1 a) $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

b) $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

c) $x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

d) $16 - 32x^3 + 24x^6 - 8x^9 + x^{12}$

e) $x^7 - 21bx^6 + 189b^2x^5 - 945b^3x^4 + 2835b^4x^3 - 5103b^5x^2 + 5103b^6x - 2187b^7$

f) $64n^6 + 192n^3 + 240 + \dfrac{160}{n^3} + \dfrac{60}{n^6} + \dfrac{12}{n^9} + \dfrac{1}{n^{12}}$

g) $\dfrac{81}{x^4} - \dfrac{216}{x^2\sqrt{x}} + \dfrac{216}{x} - 96\sqrt{x} + 16x^2$

2 a) 56 b) 0 c) 1225 d) 32 e) 0

3 a) $x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + 448xy^6 + 128y^7$

b) $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

c) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

d) $x^{18} - 12x^{15} + 60x^{12} - 160x^9 + 240x^6 - 192x^3 + 64$

e) $x^7 - 21bx^6 + 189b^2x^5 - 945b^3x^4 + 2835b^4x^3 - 5103b^5x^2 + 5103b^6x - 2187b^7$

f) $64n^6 + 192n^3 + 240 + \dfrac{160}{n^3} + \dfrac{60}{n^6} + \dfrac{12}{n^9} + \dfrac{1}{n^{12}}$

g) $\dfrac{81}{x^4} - \dfrac{216}{x^2\sqrt{x}} + \dfrac{216}{x} - 96\sqrt{x} + 16x^2$

h) 112 i) $1792\sqrt{3}$

j) 16 k) $-23 + 10i\sqrt{2}$

4 a) $x^{45} - 90x^{43} + 3960x^{41}$

b) Does not exist as the powers of x decrease by 2's starting at 45. There is no chance for any expression to have zero exponent.

c) $\binom{45}{43}x^2\left(\dfrac{-2}{x}\right)^{43} + \binom{45}{44}x\left(\dfrac{-2}{x}\right)^{44} + \left(\dfrac{-2}{x}\right)^{45} = -\binom{45}{43}\dfrac{2^{43}}{x^{41}} + \binom{45}{44}\dfrac{2^{44}}{x^{43}} - \dfrac{2^{45}}{x^{45}}$

d) $\binom{45}{21}x^{24}\left(\dfrac{-2}{x}\right)^{21} = -\binom{45}{21}\cdot 2^{21}x^3$

5 $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n!}{(n-k)!k!} = \dfrac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$

6 $(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

$2^n = 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} \Rightarrow 2^n - 1 = \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}$

7 Answers vary

8 $\left(\frac{1}{3} + \frac{2}{3}\right)^6 = 1$

9 $\left(\frac{2}{5} + \frac{3}{5}\right)^8 = 1$

10 $\left(\frac{1}{7} + \frac{6}{7}\right)^n = 1$

11 15

12 90 720

13 16 128

14 1.1045, 0.9045

15 Proof

16 a) $\dfrac{7}{9}$ b) $\dfrac{38}{110}$ c) $\dfrac{31\,808}{9900}$

17 $-145\,152$

18 $35a^3$

19 96 096

20 $243n^5 - 810n^4m + 1080n^3m^2 - 720n^2m^3 + 240nm^4 - 32m^5$

21 7 838 208

Exercise 4.7

1 $2 + 4 + 6 + \ldots + 2n = n(n+1)$

2–20 All proofs

Practice questions

1 $D = 5, n = 20$

2 €2098.63

3 a) Nick: 20 Charlotte: 17.6
b) Nick: 390 Charlotte: 381.3
c) Charlotte will exceed the 40 hours during week 14.
d) In week 12 Charlotte will catch up with Nick and exceed him.

4 a) Loss for the second month = 1060 g
Loss for the third month = 1123.6 g
b) Plan A loss = 1880 g
Plan B loss = 1898.3 g
c) (i) Loss due to plan A in all 12 months = 17 280 g
(ii) Loss due to Plan B in all 12 months = 16 869.9 g

5 a) €895.42 b) €6985.82

6 a) 142.5 b) 19 003.5

7 $1, \sqrt[3]{7}, 1, 1, \sqrt[3]{7}, 1, \ldots; 2, 0, 2, 0, 2, \ldots$

8 a) On the 37th day b) 407 km

9 a) 1.5 b) 207 595
c) 2009 d) 619 583
e) Market saturation

10 $-4, 3006$

11 a) $\sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2}$ b) $\frac{1}{2}$
c) (i) $\frac{1}{4}$ (ii) $\frac{1}{2}$ d) (i) $\frac{1}{512}$ (ii) 2

12 a) 1220 b) 36 920

13 a) Area A = 1, Area B = $\frac{1}{9}$ b) $\frac{1}{81}$
c) $1 + \frac{8}{9}, 1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2$ d) 0

14 a) Neither, geometric converging, arithmetic, geometric diverging
b) 6

15 a) (i) Kell: 18 400, 18 800; YBO: 18 190, 19 463.3
(ii) Kell: 198 000; YBO: 234 879.62
(iii) Kell: 21 600; YBO: 31 253.81
b) (i) After the second year
(ii) 4th year

16 a) 62 b) 936

17 a) $7000(1 + 0.0525)^t$ b) 7 years
c) Yes, since 10 084.7 > 10 015.0

18 a) 11 b) 2 c) 15

19 15, −8 **20** −2, −7 **21** 10 300

22 Proof

23 a) $a_n = 8n - 3$ b) 50

24 2 099 520

25 $6n - 5$ **26** 72 **27** 559

28 −3, 3 **29** 9 **30** 62

31 $-\frac{36}{5}$

32 a) 4 b) $16(4^n - 1)$

33 a) $|x| < 1.5$ b) 5

34 3168

35 a) $\frac{n(3n+1)}{2}$ b) 30

36 −7

37 1275 ln 2

38 a) 4, 8, 16
b) (i) $u_n = 2^n$ (ii) proof

39 a) $\frac{2}{3}$ b) 9

40 2, −3 **41** 55 **42** −2, 4

43 $\frac{\theta}{1 - \cos\theta}$

44 a) 1, 5, 9 b) $4n - 3$

45 a) $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
a) 32.808 040 1001

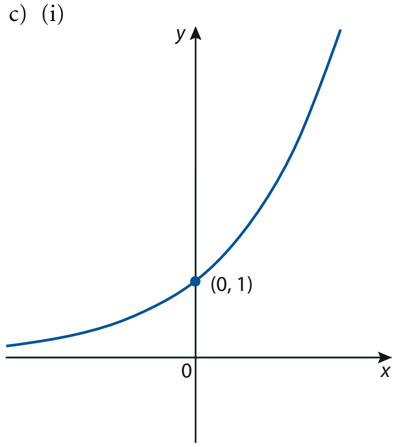
46 a) $5000(1.063)^n$ b) 6786.35
c) (i) $5000(1.063)^n > 1000$ (ii) 12

47 Proof **48** 7

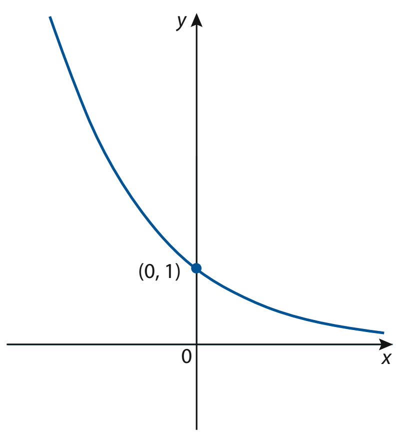
Chapter 5

Exercise 5.1 and 5.2

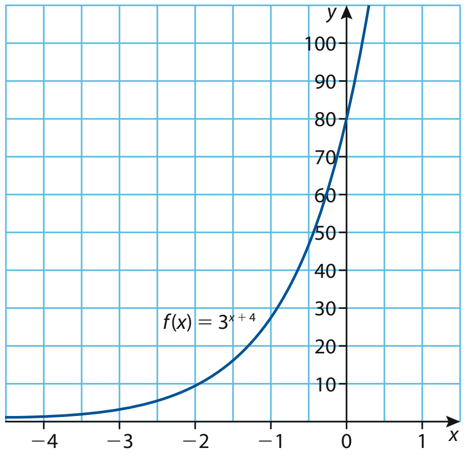
1 a) $y = b^x$
b) Domain $\{x : x \in \mathbb{R}\}$, range $\{y : y > 0\}$
c) (i)



(ii)



2



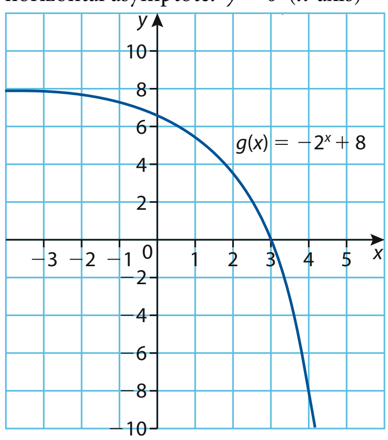
domain: $x \in \mathbb{R}$

range: $y > 0$

y-intercept: $(0, 81)$

horizontal asymptote: $y = 0$ (x-axis)

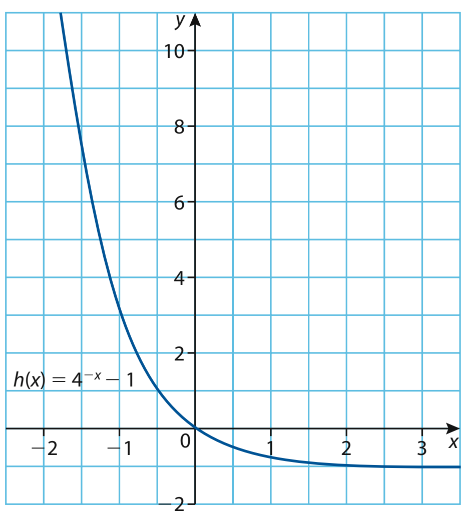
3



domain: $x \in \mathbb{R}$ range: $y < 8$

y-intercept: $(0, 7)$ horizontal asymptote: $y = 8$

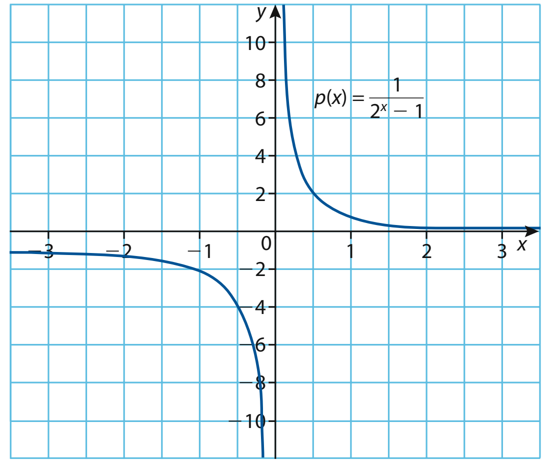
4



domain: $x \in \mathbb{R}$ range: $y > -1$

y-intercept: $(0, 0)$ horizontal asymptote: $y = -1$

5

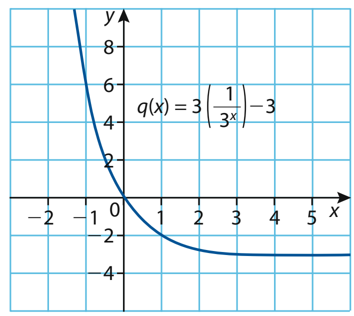


domain: $x \in \mathbb{R}$, $x \neq 0$ range: $y < -1$ or $y > 0$

y-intercept: none

horizontal asymptotes: $y = 0$ and $y = -1$

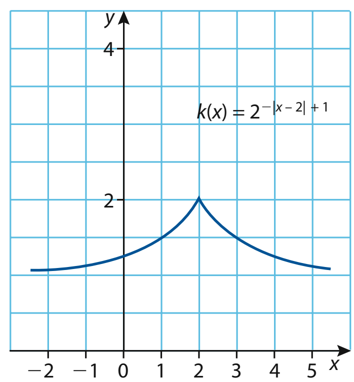
6



domain: $x \in \mathbb{R}$ range: $y > -3$

y-intercept: $(0, 0)$ horizontal asymptote: $y = -3$

7



domain: $x \in \mathbb{R}$ range: $y > 1$

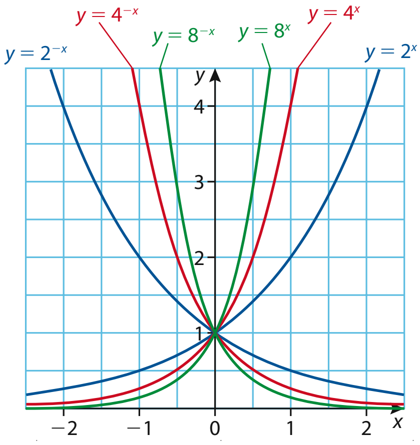
y-intercept: $\left(0, \frac{5}{4}\right)$ horizontal asymptote: $y = 1$

8 Domain: $x \in \mathbb{R}$

range: if $a > 0 \Rightarrow y > d$, if $a < 0 \Rightarrow y < d$

y-intercept: $\left(0, a(b)^{-c} + d\right)$ horizontal asymptote: $y = d$

9



10 a) $y = \left(\frac{1}{2}\right)^x$ b) $y = \left(\frac{1}{4}\right)^x$ c) $y = \left(\frac{1}{8}\right)^x$

11 $y = b^x$ is steeper

12 $P(t) = 100\,000(3)^{\frac{t}{25}}$ where t is number of years

a) 900 000 b) 2 167 402 c) 8 100 000

13 $N(t) = 10^4(2)^{\frac{t}{3}}$

a) 20 000 b) 80 000

c) 5 120 000 d) 10 485 760 000

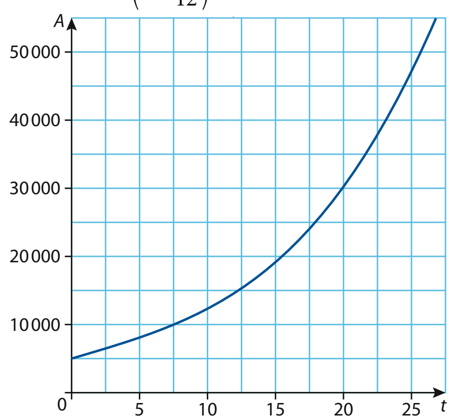
14 a) $A(t) = A_0(2)^{\frac{t}{10}}$ b) 7.18%

15 a) \$17 204.28 b) \$29 598.74

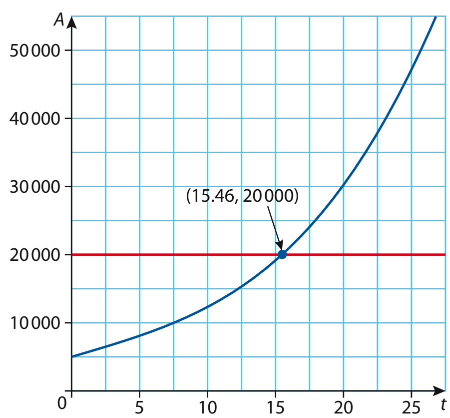
c) \$50 922.51

16 a) $A(t) = 5000\left(1 + \frac{.09}{12}\right)^{12t}$

b)



c)



minimum number of years is 16

17 a) \$16 850.58 b) \$17 289.16

c) \$17 331.09 d) \$17 332.47

18 a) \$2 b) \$2.61 c) \$2.71 d) \$2.72 e) \$2.72

19 a) 240 310 b) 192 759

20 8.90%

21 $0.0992A_0$ (or 9.92% of A_0 remains)

22 a) $A(w) = 1000(0.7)^w$ b) About 20 weeks

23 $b > 0$ because if $b = 0$ then the result is always zero, and if $b < 0$ then b^x gives a positive result when x is an even integer and a negative result when x is an odd integer.

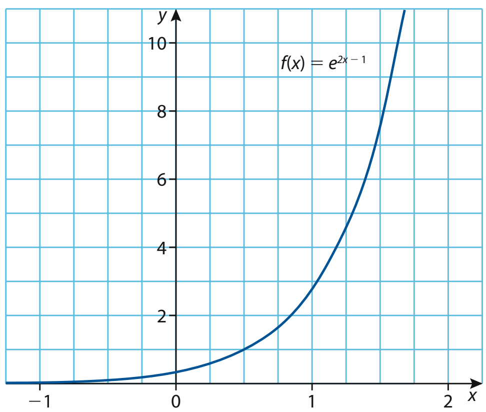
24 Payment plan I: \$465; payment plan II: \$10 737 418.23

25 a) $a = 2, k = 3$ b) $a = \frac{1}{3}, k = 2$

c) $a = 3, k = -4$ d) $a = 10, k = \frac{3}{2}$

Exercise 5.3

1

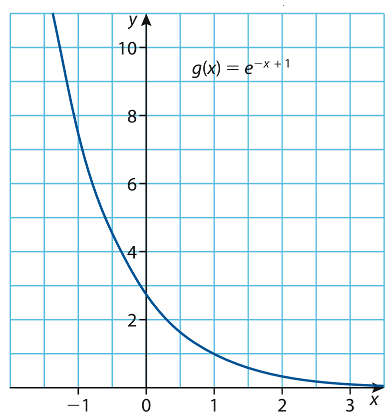


a) Domain: $x \in \mathbb{R}$, range: $y > 0$

b) x-intercept: none, y-intercept: $\left(0, \frac{1}{e}\right)$

c) Horizontal asymptote: $y = 0$

2

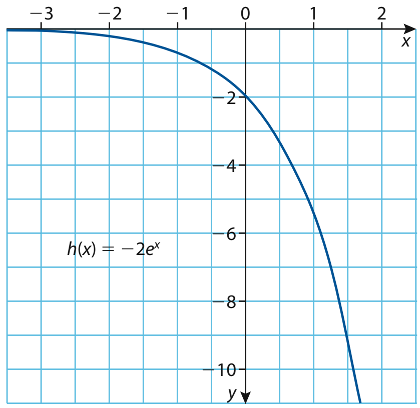


a) Domain: $x \in \mathbb{R}$, range: $y < 0$

b) x-intercept: none, y-intercept: $(0, e)$

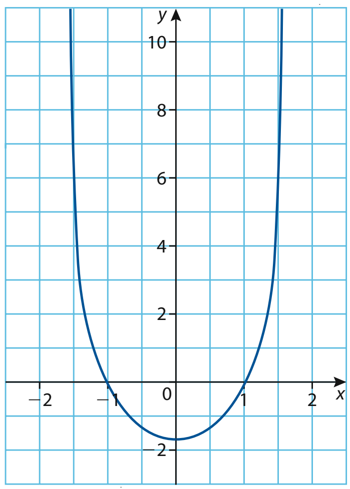
c) Horizontal asymptote: $y = 0$

3



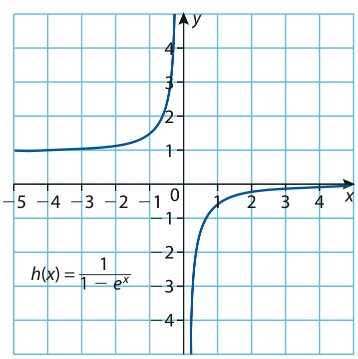
a) Domain: $x \in \mathbb{R}$, range: $y < 0$
b) x-intercept: none, y-intercept: $(0, -2)$
c) Horizontal asymptote: $y = 0$

4



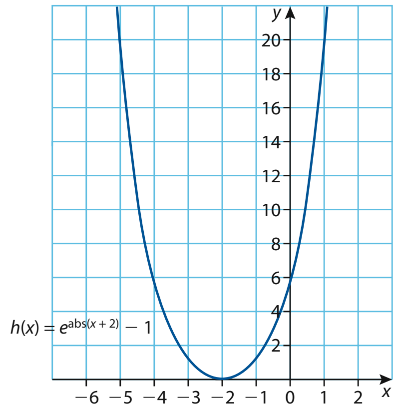
a) Domain: $x \in \mathbb{R}$, range: $y \geqslant 1 - e$
b) x-intercept: $(-1, 0)$ and $(1, 0)$, y-intercept: $(0, 1 - e)$
c) No asymptotes

5



a) Domain: $x \in \mathbb{R}, x \neq 0$, range: $y < 0, y > 1$
b) x-intercept: none, y-intercept: none
c) Horizontal asymptotes: $y = 0$ and $y = 0$

6



a) Domain: $x \in \mathbb{R}$, range: $y \geqslant 0$
b) x-intercept: $(-2, 0)$, y-intercept: $(0, e^2 - 1)$
c) No asymptotes

7 a) $e = \lim\limits_{x \to \infty} \left(1 + \frac{1}{n}\right)^n$
b) 0.366 032 3413, 0.367 861 0464, 0.367 879 2572
c) 0.367 88; reciprocal of e, $\frac{1}{e} \approx 0.367\,879\,4412$

8 $y = \left(x + \frac{1}{x}\right)^x$ will not intersect $y = 2.72$ because

$\lim\limits_{x \to \infty} \left(x + \frac{1}{x}\right)^x = e \approx 2.718\,281\,828\ldots < 2.72$

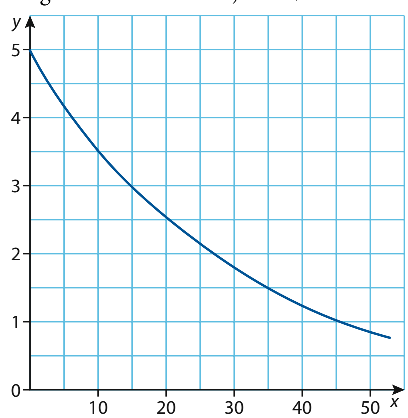
9 Bank A: earn 608.79 euros in interest.
Bank B: earn 609.16 euros in interest.
Bank B account earns 0.37 euros more in interest.

10 Blue Star has greater total of \$1358.42 which is \$11.93 more than the Red Star.

11 a) 0.976 kg b) 0.787 kg c) 0.0916 kg d) 0.002 54 kg

12 a) 5 kg b) 71.7%

c)



d) 20 days

13 a) $8\frac{1}{2}\%$ compounded semi-annually is the better investment.

14 a) $r \approx 1.070\,37$ (6 s.f.) b) 7.037% (4 s.f.)

15 a) Less than 1 b) Less than 1
c) Greater than 1 d) Greater than 1

16 a) £1568.31, £2459.60
b) 15.4 years
c) 15.4 years
d) Same; doubling time is independent of initial amount

Exercise 5.4

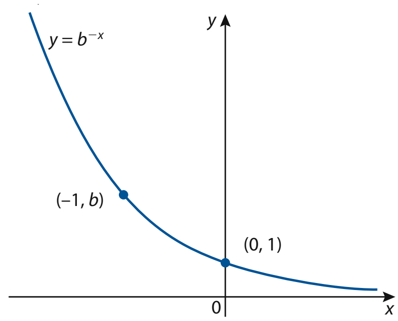
1 $2^4 = 16$ 2 $e^0 = 1$ 3 $10^2 = 100$
4 $10^{-2} = 0.01$ 5 $7^3 = 343$ 6 $e^{-1} = \frac{1}{e}$
7 $10^y = 50$ 8 $e^{12} = x$ 9 $e^3 = x + 2$
10 $\log_2 1024 = 10$ 11 $\log_{10} 0.0001 = -4$ 12 $\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$
13 $\log_3 81 = 4$ 14 $\log_{10} 1 = 0$ 15 $\ln 5 = x$
16 $\log_2 0.125 = -3$ 17 $\ln y = 4$
18 $\log_{10} y = x + 1$ 19 6 20 3
21 -3 22 5 23 $\frac{3}{4}$ 24 $\frac{1}{3}$
25 -3 26 13 27 0 28 6
29 -3 30 $\sqrt{2}$ 31 3 32 $\frac{1}{2}$
33 -2 34 88 35 $\frac{1}{2}$ 36 18
37 $\frac{1}{3}$ 38 π 39 1.6990 40 0.2386
41 3.912 42 0.5493 43 1.398 44 0.2090
45 4.605 46 13.82 47 $x > 2$ 48 $x \in \mathbb{R}$
49 $x > 0$ 50 $x < \frac{8}{5}$ 51 $-2 \leqslant x < 3$ 52 $x < 0$
53 Domain $\{x : x > 0,\ x \neq 1\}$, range $\{y : y \in \mathbb{R},\ y \neq 0\}$
54 Domain $\{x : x > 1\}$, range $\{y : y \geqslant 0\}$
55 Domain: $x > 0, x \neq 1$, range: $y < 0$
56 $f(x) = \log_4 x$ 57 $f(x) = \log_2 x$
58 $f(x) = \log_{10} x$ 59 $f(x) = \log_3 x$
60 $\log_2 2 + \log_2 m = 1 + \log_2 m$ 61 $\log 9 - \log x$
62 $\frac{1}{5}\ln x$ 63 $\log a + 3\log b$
64 $\log 10x + \log(1+r)^t = \log 10 + \log x + t\log(1+r)$
65 $3\ln m - \ln n$ 66 $\log_b p + \log_b q + \log_b r$
67 $2\log_b p + 3\log_b q - \log_b r$ 68 $\frac{\log_b p}{4} + \frac{\log_b q}{4}$
69 $\frac{\log_b q}{2} + \frac{\log_b r}{2} - \frac{\log_b p}{2}$ 70 $\log_b p + \frac{1}{2}\log_b q - \log_b r$
71 $3\log_b p + 3\log_b q - \frac{1}{2}\log_b r$ 72 $\log x$
73 $\log_3 72$ 74 $\ln\left(\frac{y^4}{4}\right)$ 75 $\log_b 4$ 76 $\log\left(\frac{p}{qr}\right)$
77 $\ln\left(\frac{36}{e}\right)$ 78 9.97 79 -5.32 80 2.06
81 -0.179 82 4.32 83 1.86
84 $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$ 85 $\log e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10}$
86 $dB = 10\log\left(\frac{I}{10^{-16}}\right) = 10(\log I - \log 10^{-16}) = 10(\log I + 16)$
$= 10\log 10^{-4} + 160 = 10(-4) + 160 = 120$ decibels

Exercise 5.5

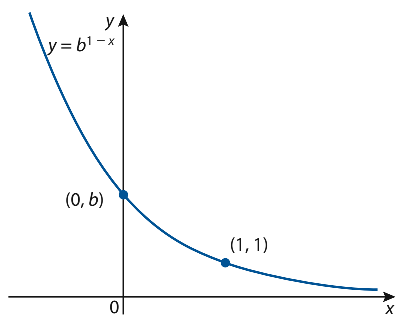
1 0.699 2 2.5 3 7.99 4 3.64
5 -1.92 6 2.71 7 0.434 8 2.12
9 4.42 10 0.225 11 0.642 12 22.0
13 3 14 0 or -1 15 $\frac{\ln\left(\frac{3}{2}\right)}{\ln 6}$ or $\frac{\ln\left(\frac{4}{3}\right)}{\ln 6}$
16 1 or -1
17 a) \$6248.58 b) $9\frac{1}{4}$ years
18 12.9 years
19 20 hours (≈ 19.93)
20 a) 24 years (≈ 23.45) b) 12 years (≈ 11.9)
c) 9 years (≈ 8.04)
21 6 years
22 a) 99.7% b) 139 000 years
23 a) 37 dogs b) 9 years
24 a) 458 litres b) 8.89 minutes $\approx$ 8 min. 53 seconds
c) 39 minutes
25 a) 5 kg b) 17.7 days
26 $x = \frac{20}{3}$ 27 $x = 104$ 28 $x = \frac{1}{e^3}$
29 $x = 4$ 30 $x = 98$ 31 $x = \pm\sqrt{e^{16}} \approx \pm 2980.96$
32 $x = 2$ or $x = 4$ 33 $x = 9$ 34 $x = \frac{13}{5}$
35 $x = 3$ 36 $x = 1$ or $x = 100$
37 $x > \frac{1}{\sqrt[5]{100}}$ 38 $x < 2$ 39 $0 < x < \ln 6$
40 $0.161 < x < 1.14$ (approx. to 3 s.f.)

Practice questions

1 a) $(8, 0)$ b) $(0, 2)$ c) $\left(-\frac{2}{3}, 3\right)$
2 a) 183 g (3 s.f.) b) 154 years (3 s.f.)
3 a) $a_n = \ln(y^n)$, $S_n = \frac{n(n+1)}{2}\ln y$
b) $a_n = \ln(xy^n)$, $S_n = n\ln x + \frac{n(n+1)}{2}\ln y$
4 $x = 2$ 5 $y = 16$ 6 $x = 0, \ln\left(\frac{1}{2}\right)$ or $-\ln$
7 $x = e^{-4e}$ or e^{2e}
8 a) $x = 3$ b) $x = 6$
9 a) $\log\left(\frac{a^2b^3}{c}\right)$ b) $\ln\left(\frac{ex^3}{\sqrt{y}}\right)$
10 1900 years
11 $c = 22$
12 a)



b)



13 a) $k \approx 0.000\,4332$ b) 17.7% (3 s.f.)
14 $x \approx 1.28$
15 $1.52 < x < 1.79 \cup 17.6 < x < 19.1$
16 $-1 < x < -0.800 \cup x > 1$
17 a) $x = -\frac{1}{2}$ or $x = 0$
b) $x = \frac{1}{\ln a - 2}$ or $x = \frac{\log_a e}{1 - 2\log_a e}$
c) $a = e^2$
18 $a = -2, b = 3$

19 $x = \sqrt{e}, x = e$

20 a) $V = \$265.33$ b) 235 months

21 $x = 5^{\frac{5}{3}}$ or $x = 5^{\frac{-5}{3}}$

22 $x = e - 3$ or $x = \frac{1}{e} - 3$

23 $x = -2.50, -1.51$ or 0.440 (3 s.f.)

24 $k = \frac{\ln 2}{20}$

25 a) $f(x) = \ln\left(\frac{x}{x+2}\right)$ b) $f^{-1}(x) = -\frac{2e^x}{e^x - 1}$ or $\frac{2e^x}{1 - e^x}$

26 a) (i) Minimum value of f is 0.

(ii) From part (i) $f(x) \geqslant 0 \Rightarrow e^x - 1 - x \geqslant 0 \Rightarrow e^x \geqslant 1 + x$

d) $n > e^{100}$

Chapter 6

Exercise 6.1 and 6.2

1 a) (i) $\begin{pmatrix} x-1 & x-3 \\ y+3 & y+1 \end{pmatrix}$ (ii) $\begin{pmatrix} -x-7 & 3x+3 \\ 3y-7 & 11-y \end{pmatrix}$

b) $x = -3, y = 5$ c) $x = 3, y = -3$

d) $AB = \begin{pmatrix} 2x-2 & xy-2x+6 \\ xy-x+y+11 & -3 \end{pmatrix}$;

$BA = \begin{pmatrix} -2x-3y+1 & x^2+x-9 \\ y^2-3y-6 & 4x+3y-6 \end{pmatrix}$

2 a) $x = 2, y = -10$ b) $p = 2, q = -4$

3 a) $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 3 & 1 & 2 & 3 & 2 & 0 \\ 3 & 5 & 2 & 3 & 3 & 3 & 2 \\ 1 & 2 & 9 & 1 & 3 & 1 & 0 \\ 2 & 3 & 1 & 6 & 1 & 2 & 4 \\ 3 & 3 & 3 & 1 & 4 & 3 & 0 \\ 2 & 3 & 1 & 2 & 3 & 6 & 0 \\ 0 & 2 & 0 & 4 & 0 & 0 & 4 \end{bmatrix}$

Matrix signifies the number of routes between each pair that go via one other city.

4 a) $A + C = \begin{pmatrix} x+1 & 10 & y+1 \\ 0 & -x-3 & y+3 \\ 2x+y+7 & x-3y & -x+2y-1 \end{pmatrix}$

b) $\begin{pmatrix} 17m+2 & -6 \\ 4-9m & 9 \\ 7m-2 & -17 \end{pmatrix}$

c) Not possible d) $x = 3, y = 1$

e) Not possible f) $m = 3$

5 $a = -3, b = 3, c = 2$

6 $x = 4, y = -3$

7 $m = 2, n = 3$

8 Shop A: €18.77

9 a) $\begin{pmatrix} 2 & 4 \\ -2 & 12 \end{pmatrix}$ b) associative

c) $\begin{pmatrix} -22 & 16 \\ 60 & -7 \end{pmatrix}$ d) associative

10 $AB = [88\ 142]$, which represents total profit.

11 $r = 3, s = -2$

12 a) (i) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1^n & n \\ 0 & 3^n \end{pmatrix}$

b) (i) $\begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix}$

(iii) $\begin{pmatrix} 81 & 324 \\ 0 & 81 \end{pmatrix}$ (iv) $\begin{pmatrix} 3^n & 3^{n+1} \\ 0 & 3^n \end{pmatrix}$

13 $\left(\frac{11}{3}, \frac{8}{3}\right)$ **14** $(1, -4)$

15 5 **16** $(5, 1)$

Exercise 6.3

1 a) $\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$ b) $M = \begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$

c) $\begin{pmatrix} -39 & -44 \\ 17 & 19 \end{pmatrix}$

d) (i) $N = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$ (ii) $N = \begin{pmatrix} -14 & -11 \\ -7 & -6 \end{pmatrix}$

e) If $AB = C$ then $B = A^{-1}C$, while if $BA = C$, then $B = CA^{-1}$. Also, $A^{-1}C \neq CA^{-1}$.

2 $\begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 0 \end{pmatrix}$

3 a) $|A| = -5 \neq 0$ b) $\begin{pmatrix} \frac{9}{5} & \frac{11}{5} & -\frac{8}{5} \\ \frac{6}{5} & \frac{9}{5} & -\frac{7}{5} \\ 1 & 1 & -1 \end{pmatrix}$ c) $\begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{5} \end{pmatrix}$

4 a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ b) $\begin{pmatrix} \frac{3}{a}+1 & -1 \\ -a-2 & a \end{pmatrix}$

5 $x = 2$ or $x = 3$

6 $n = 0.5$

7 a) $X = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{4} & -\frac{7}{6} \end{pmatrix}$ b) $Y = \begin{pmatrix} 1 & \frac{13}{12} \\ -1 & -\frac{5}{3} \end{pmatrix}$

c) $X \neq Y$ – not commutative

8 a) $PQ = \begin{pmatrix} 5 & -4 & 3 \\ 33 & 5 & -1 \\ 2 & -3 & 2 \end{pmatrix}$, $QP = \begin{pmatrix} 4 & -5 & -8 \\ 8 & 0 & -4 \\ 7 & 10 & 8 \end{pmatrix}$

b) $P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -\frac{7}{5} & \frac{1}{5} & \frac{11}{5} \\ 1 & 0 & -2 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -1 & 1 \\ 2 & -\frac{7}{4} & 1 \end{pmatrix}$

$P^{-1}Q^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$

$Q^{-1}P^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$

$(PQ)^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$

$(QP)^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$

9 a) $\begin{pmatrix} -7 \\ 3 \\ -2 \end{pmatrix}$ b) $\begin{pmatrix} -7 \\ 3 \\ -2 \end{pmatrix}$

10 $x = -1$ **11** $x = 1, y = 2$
12 $(0, 1)$ **13** $(-3, -29), (0, 1)$
14 $17x - 8y + 37 = 0$; $y + 2 = 0$; $x + 5 = 0$ **15** 165; 80; 136
16 $x = \frac{89}{2}$ or $x = \frac{129}{8}$; $x = -4$ or $x = -2$ or $x = -3 \pm\sqrt{21}$
17 -3; 3
18 a) -25
b) $x^2 - 7x - 25$, constant $= \det(A)$
c) $-(a + d)$
d) $f(A) = 0$
e) $ad - bc$; $x^2 - (a + d)x + (ad - bc)$,
constant $= \det(A)$; $f(A) = 0$
19 a) -22
b) $x^3 - x^2 - 22x + 22$, constant $= -\det(A)$
c) Opposite of the sum of the main diagonal
d) $f(A) = 0$

Exercise 6.4

1 $\mathbf{m} = 2$ or $\mathbf{m} = 3$
2 a) $a = 7, b = 2$ b) $(-1, 2, -1)$
3 $\mathbf{m} = 2$
4 a) $(-1, 3, 2)$ b) $(5, 8, -2)$
c) $\left(\frac{13}{16} + \frac{5}{16}t, \frac{11}{16} + \frac{19}{16}t, t\right)$ d) $(-7, 3, -2)$
e) $(-1 + 2t, 2 - 3t, t)$ f) inconsistent
g) $(-2, 4, 3)$ h) $(4, -2, 1)$
5 a) $k \neq \frac{-1 \pm \sqrt{33}}{4}$ b) $k = 1$
c) $\begin{pmatrix} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{pmatrix}$
6 a) $\frac{71 \pm i\sqrt{251}}{42}$ b) $k = 2$
c) $\begin{pmatrix} 1 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & \frac{4}{5} & \frac{-3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{6}{5} & -1 \end{pmatrix}$
7 $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 0 & 2 & \frac{-16}{13} & \frac{-19}{13} \\ 0 & 1 & 0 & 1 & \frac{-11}{13} & \frac{-9}{13} \\ 0 & 0 & 1 & -1 & \frac{12}{13} & \frac{11}{13} \end{pmatrix}$
B is the inverse of A
8 a) $f(x) = 4x^2 - 6x - 5$
b) $f(x) = \frac{1}{2}(m - 27)x^2 + \frac{3}{2}(17 - m)x + m,\ m \in \mathbb{R}$
c) $f(x) = 3x^3 - 2x^2 - 7x + 3$
d) $f(x) = \frac{1}{6}(4 - m)x^3 + \frac{1}{3}(4 - m)x^2 - \frac{5}{6}(4 - m)x + m,\ m \in \mathbb{R}$
9 $m = 2$, $\begin{pmatrix} -t - \frac{3}{5} \\ -t - \frac{19}{5} \\ 5t \end{pmatrix}$ **10** $m = -1$, $\begin{pmatrix} 7t - \frac{9}{5} \\ \frac{3}{5} - 11t \\ 5t \end{pmatrix}$

11 a) 3 b) $\begin{pmatrix} 3 & -4 & -6 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$ c) 3
d) -1672 e) $\begin{pmatrix} 2 & 1 & -3 & 5 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & 36 & -184 \\ 0 & 0 & 0 & -\frac{209}{9} \end{pmatrix}$ f) -1672

Practice questions

1 $x = -7$ or $x = 1$
2 a) $\begin{pmatrix} a^2 + 4 & 2a - 2 \\ 2a - 2 & 5 \end{pmatrix}$
b) $a = -1$; $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
3 $B = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$
4 $a = \frac{28}{33}$; $b = \frac{59}{33}$; $c = \frac{20}{33}$; $d = \frac{28}{33}$
5 a) $A^{-1} = \begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$
b) (i) $X = (C - B)A^{-1}$ (ii) $X = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$
6 a) $A + B = \begin{pmatrix} a + 1 & b + 2 \\ c + d & 1 + c \end{pmatrix}$
b) $AB = \begin{pmatrix} a + bd & 2a + bc \\ c + d & 3c \end{pmatrix}$
7 a) $\begin{pmatrix} 0.1 & 0.4 & 0.1 \\ -0.7 & 0.2 & 0.3 \\ -1.2 & 0.2 & 0.8 \end{pmatrix}$
b) $x = 1.2, y = 0.6, z = 1.6$
8 a) $Q = \begin{pmatrix} -3 & 2 \\ 1 & \frac{14 - a}{3} \end{pmatrix}$
b) $CD = \begin{pmatrix} -14 & -4 + 4a \\ -2 & 2 + 7a \end{pmatrix}$
c) $D^{-1} = \frac{1}{5a + 2}\begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$
9 a) $(7, 2)$ b) $(-1, 2, -1)$
10 a) $B = A^{-1}C$ b) $DA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
c) $(1, -1, 2)$
11 a) Det = 0 b) $\lambda = 5$ c) $(2 - 3t, 1 + t, t)$
12 No answer required – proof

Chapter 7

Exercise 7.1

1 $\frac{\pi}{3}$ **2** $\frac{5\pi}{6}$ **3** $-\frac{3\pi}{2}$ **4** $\frac{\pi}{5}$
5 $\frac{3\pi}{4}$ **6** $\frac{5\pi}{18}$ **7** $-\frac{\pi}{4}$ **8** $\frac{20\pi}{9}$
9 $-\frac{8\pi}{3}$
10 135° **11** -630° **12** 115° **13** 210°
14 -143° **15** 300° **16** 115° **17** $89.95^\circ \approx 90^\circ$

18 480° **19** $390^\circ, -330^\circ$ **20** $\frac{7\pi}{2}, -\frac{\pi}{2}$ **21** $535^\circ, -185^\circ$

22 $\frac{11\pi}{6}, -\frac{13\pi}{6}$ **23** $\frac{11\pi}{3}, -\frac{\pi}{3}$

24 $3.25 + 2\pi \approx 9.5,\ 3.25 - 2\pi \approx -3.03$

25 12.6 cm **26** 14.7 cm

27 1.5 radians, or approx. 85.9° **28** $r \approx 7.16$

29 Area $\approx 13.96 \approx 14.0\ \text{cm}^2$ **30** Area $\approx 131\ \text{cm}^2$

31 $\alpha = 3$ (radian measure), or $\alpha = 172^\circ$ **32** 32 cm

33 6.77 cm

34 a) 3π radians/second b) 11.9 km/hr

35 19.8 radians/second **36** $v = \omega \times r$

37 28.3 cm **38** 20 944 sq metres

39 a) $r \approx 30.6$ cm b) $r \approx 0.0771$ cm

40 $150\sqrt{3}\ \text{cm}^2$ **41** Area of circle $= \left(\frac{\pi - 2}{4\pi}\right)A$

Exercise 7.2

1 a) $t = \frac{\pi}{6} : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right);\ t = \frac{\pi}{3} : \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

2 0.6 **3** 1.0 **4** 0.5 **5** 0.5

6 2.7 **7** 0.1 **8** 0.3 **9** 1.6

10 a) I b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

11 a) IV b) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

12 a) IV b) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

13 a) Negative x-axis b) $(0, -1)$

14 a) II b) $(-0.416, 0.909)$

15 a) I b) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

16 a) IV b) $(0.540, 0.841)$

17 a) II b) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

18 a) III b) $(-0.929, -0.369)$

19 $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3} = \frac{1}{2},\ \tan\frac{\pi}{3} = \sqrt{3}$

20 $\sin\frac{5\pi}{6} = \frac{1}{2},\ \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2},\ \tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$

21 $\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \tan\left(-\frac{3\pi}{4}\right) = 1$

22 $\sin\frac{\pi}{2} = \frac{1}{2},\ \cos\frac{\pi}{2} = 0,\ \tan\frac{\pi}{2}$ is undefined

23 $\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \cos\left(-\frac{4\pi}{3}\right) = -\frac{1}{2},\ \tan\left(-\frac{4\pi}{3}\right) = -\sqrt{3}$

24 $\sin 3\pi = 0,\ \cos 3\pi = -1,\ \tan 3\pi = 0$

25 $\sin\frac{3\pi}{2} = -1,\ \cos\frac{3\pi}{2} = 0,\ \tan\frac{3\pi}{2}$ is undefined

26 $\sin\left(-\frac{7\pi}{6}\right) = \frac{1}{2},\ \cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2},\ \tan\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

27 $\sin(1.25\pi) = -\frac{\sqrt{2}}{2},\ \cos(1.25\pi) = -\frac{\sqrt{2}}{2},\ \tan(1.25\pi) = 1$

28 $\sin\frac{13\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2};\ \cos\frac{13\pi}{6} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

29 $\sin\frac{10\pi}{3} = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2};\ \cos\frac{10\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$

30 $\sin\frac{15\pi}{4} = \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2};\ \cos\frac{15\pi}{4} = \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$

31 $\sin\frac{17\pi}{6} = \sin\frac{5\pi}{6} = \frac{1}{2};\ \cos\frac{17\pi}{6} = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

32 a) $-\frac{\sqrt{3}}{2}$ b) $-\frac{\sqrt{2}}{2}$ c) undefined

d) 2 e) $-\frac{2\sqrt{3}}{3}$

33 a) 0.598 b) $-\frac{\sqrt{3}}{3}$ c) $\frac{1}{2}$ d) 1.04 e) 0

34 I, II **35** II

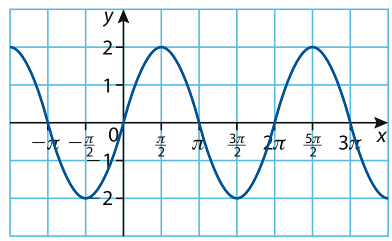
36 III **37** II

38 I, IV **39** I

40 IV **41** II, IV

Exercise 7.3

1



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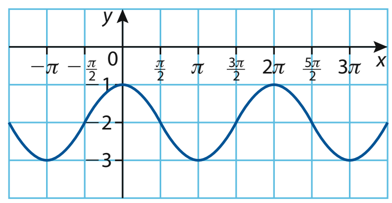

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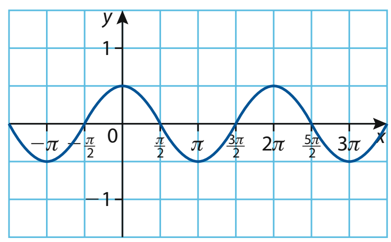

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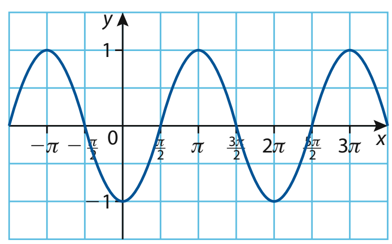

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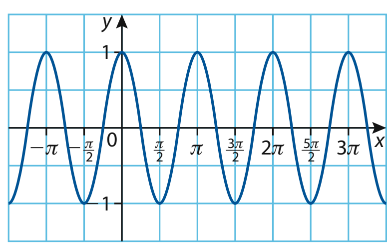

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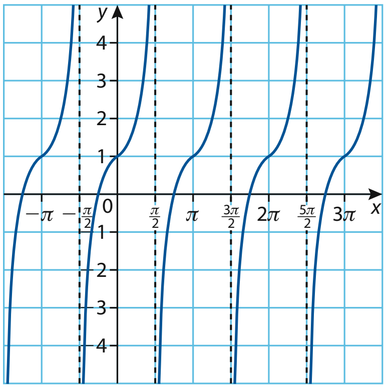

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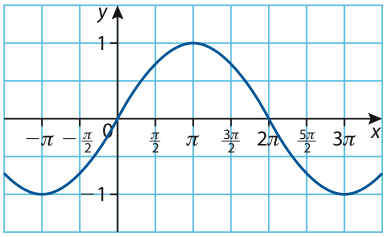

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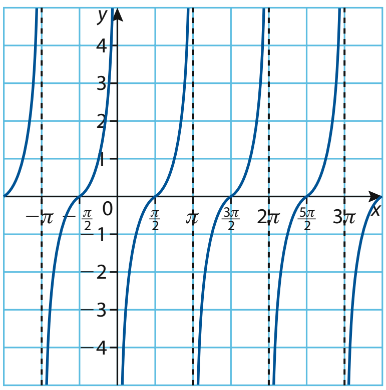

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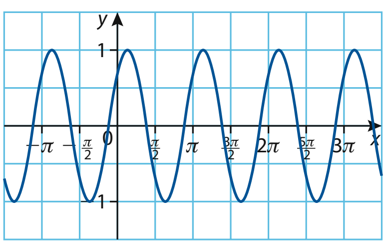
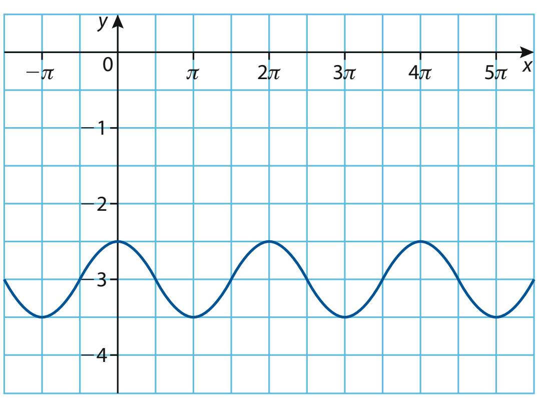


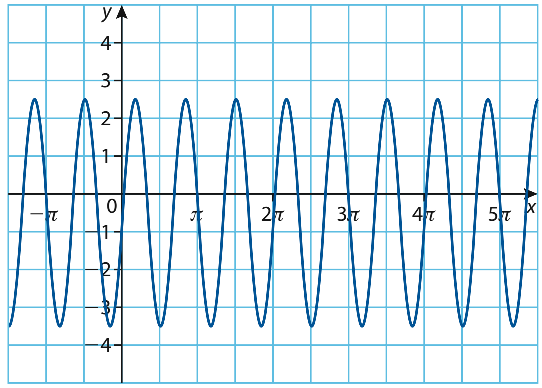
10 a)



amplitude $= \frac{1}{2}$, period $= 2\pi$

b) Domain: $x \in \mathbb{R}$, range: $-3.5 \leqslant y \leqslant -2.$

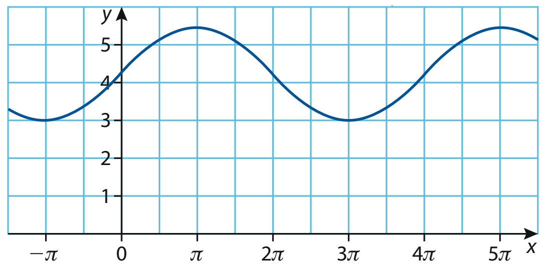
11 a)



amplitude $= 3$, period $= \frac{2\pi}{3}$

b) Domain: $x \in \mathbb{R}$, range: $-3.5 \leqslant y \leqslant 2.5$

12 a)



amplitude 1.2, period $= 4\pi$

b) Domain: $x \in \mathbb{R}$, range: $3.1 \leqslant y \leqslant 5.5$

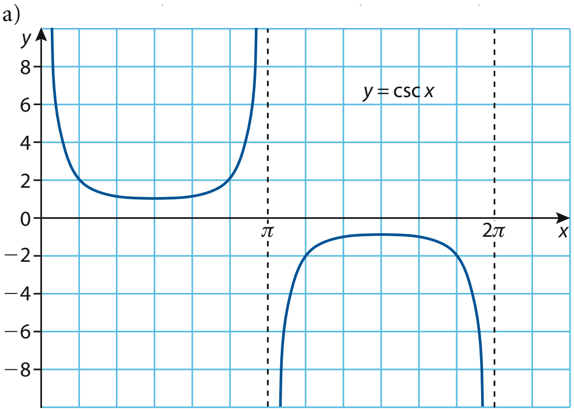
13 $A = 3$, $B = 7$

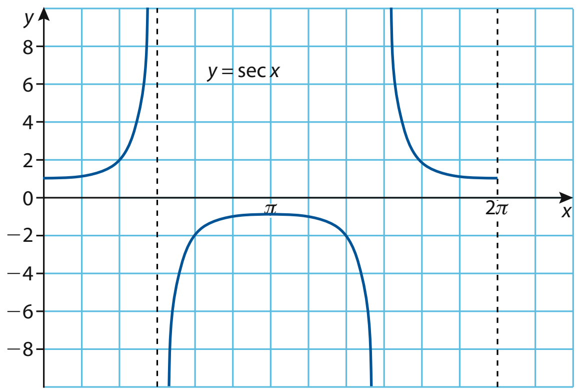
14 $A = 2.7$, $B = 5.9$

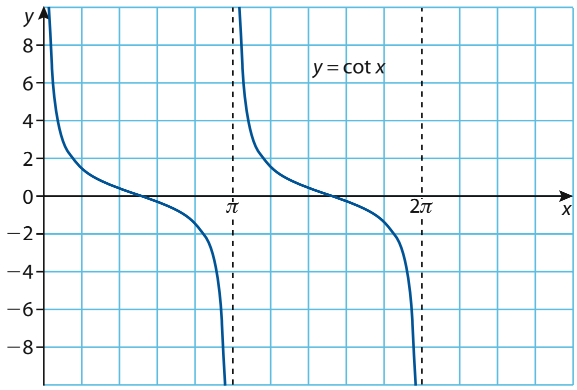
15 $A = 1.9$, $B = 4.3$

16 a) $p = 8$ b) $q = 6$

17 a)







b) $y = \sec x$, range: $y \geqslant 1$, $y \leqslant -1$;
$y = \csc x$, range: $y \geqslant 1$, $y \leqslant -1$; $y = \cot x$, range: $y \in \mathbb{R}$

18 a) $a = 2, b = 3, c = -1$ b) $\dfrac{5\pi}{18}$

19 $a = 3, b = -\dfrac{\pi}{4}, c = -1$

Exercise 7.4

1 $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

2 $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

3 $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

4 $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

5 $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

6 $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

7 $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

8 $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

9 $x = 0, \dfrac{3\pi}{4}, \pi, \dfrac{7\pi}{4}, 2\pi$

10 $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

11 $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

12 $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

13 $x \approx 0.412, 2.73$

14 $x \approx 1.91, 4.37$

15 $x \approx 1.11, 4.25$

16 $x \approx 5.64, 3.78, 2.50, 0.639$

17 $x \approx 2.96, 5.32$

18 $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

19 $x \approx 5.85, 5.01, 2.71, 1.86$

20 $x \approx 3.43, 0.291, 2.71, 1.86$

21 $\dfrac{5\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{2}, -\dfrac{\pi}{2}, -\dfrac{3\pi}{2}, -\dfrac{5\pi}{2}$

22 $\dfrac{\pi}{6}, -\dfrac{11\pi}{6}$

23 $\dfrac{7\pi}{12}, \dfrac{19\pi}{12}$

24 $0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}, 2\pi$

25 $x = \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

26 $\theta = -\dfrac{3\pi}{4}, \dfrac{\pi}{4}$

27 $x = 30^\circ, 60^\circ, 210^\circ, 240^\circ$

28 $\alpha = -\dfrac{\pi}{6}, \dfrac{\pi}{6}$

29 $\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

30 $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

31 $x = 225^\circ, 315^\circ$

32 $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

33 $t \approx 1.5$ hours

34 a) 80th day (March 21) and approximately 263rd day (September 20)
b) 105th day (April 15) and approximately 238th day (August 26)
c) 94 days – from 125th day to 218th day

35 $x = \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{2}, \dfrac{4\pi}{3}$

36 $\theta = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

37 $x = -45^\circ, 63.4^\circ$

38 $x \approx -1.87, 1.87$

39 $x \approx 56.3^\circ$

40 $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

41 No solution

42 $x \approx 0^\circ, 71.6^\circ, 180^\circ, 252^\circ$

Exercise 7.5

1 $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

2 $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

3 $2-\sqrt{3}$

4 $\dfrac{-\sqrt{6}-\sqrt{2}}{4}$

5 $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

6 $2-\sqrt{3}$

7 a) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ b) $\sqrt{\dfrac{\sqrt{6}+\sqrt{2}+4}{8}}$

8 $\tan\left(\dfrac{\pi}{2}-\theta\right) = \dfrac{\sin\left(\dfrac{\pi}{2}-\theta\right)}{\cos\left(\dfrac{\pi}{2}-\theta\right)} = \dfrac{\sin\dfrac{\pi}{2}\cos\theta - \cos\dfrac{\pi}{2}\sin\theta}{\cos\dfrac{\pi}{2}\cos\theta + \sin\dfrac{\pi}{2}\sin\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

9 $\sin\left(\dfrac{\pi}{2}-\theta\right) = \sin\dfrac{\pi}{2}\cos\theta - \cos\dfrac{\pi}{2}\sin\theta = \cos\theta$

10 $\csc\left(\dfrac{\pi}{2}-\theta\right) = \dfrac{1}{\sin\left(\dfrac{\pi}{2}-\theta\right)} = \dfrac{1}{\sin\dfrac{\pi}{2}\cos\theta - \cos\dfrac{\pi}{2}\sin\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

11 a) $\dfrac{4}{5}$ b) $\dfrac{7}{25}$ c) $\dfrac{24}{25}$

12 a) $\dfrac{\sqrt{5}}{3}$ b) $-\dfrac{4\sqrt{5}}{9}$ c) $-\dfrac{1}{9}$

13 $\sin 2\theta = -\dfrac{4\sqrt{5}}{9}, \cos 2\theta = \dfrac{1}{9}, \tan 2\theta = -4\sqrt{5}$

14 $\sin 2\theta = \dfrac{24}{25}, \cos 2\theta = \dfrac{7}{25}, \tan 2\theta = \dfrac{24}{7}$

15 $\sin 2\theta = \dfrac{4}{5}, \cos 2\theta = -\dfrac{3}{5}, \tan 2\theta = -\dfrac{4}{3}$

16 $\sin 2\theta = -\dfrac{2\sqrt{15}}{16}\left(\text{or } \sin 2\theta = -\dfrac{\sqrt{15}}{8}\right), \cos 2\theta = -\dfrac{7}{8}, \tan 2\theta = \dfrac{\sqrt{15}}{7}$

17 $-\cos x$

18 $-\cos x$

19 $\tan x$

20 $-\sin x$

21 $\dfrac{1+\sin\theta\cos\theta}{\cos\theta}$

22 $\dfrac{1}{\sin^3\theta}$

23 $\dfrac{\sin\theta+\cos\theta}{\sin 2\theta}$

24 $\dfrac{1+\sin^2\theta}{\cos^2 x}$

25 $\cos^3\theta$

26 1

27 $\cos^2\theta$

28 $2\tan^2\theta$

29 $2\sin\alpha\cos\beta$

30 $\cos^2 A$

31 $2\cos\alpha\cos\beta$

32 1

33–46 No answers required (proofs)

47 $\tan\theta = \dfrac{5x}{x^2+14}$

48 $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

49 $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

50 $x = 90^\circ$ and -90°

51 $x \approx 0.375, 2.77$

52 $x \approx 0.615, 2.53, 3.76, 5.67$

53 $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

54 $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

55 $x = 0, \dfrac{\pi}{4}, \pi, \dfrac{5\pi}{4}$

56 $x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

57 $x = 30^\circ, 90^\circ, 105^\circ, 150^\circ, 165^\circ$

58 $3\sin x - 4\sin^3 x$

59 b) $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Exercise 7.6

1 $\frac{\pi}{2}$ **2** $\frac{\pi}{4}$ **3** $-\frac{\pi}{3}$ **4** $\frac{2\pi}{3}$

5 0 **6** $-\frac{\pi}{3}$ **7** $\frac{\pi}{3}$ **8** $\frac{3}{2}$

9 12 **10** Not possible **11** $\frac{\pi}{4}$ **12** Not possible

13 $\frac{3}{5}$ **14** $\frac{24}{25}$ **15** Not possible **16** $\frac{\pi}{3}$

17 $\frac{2\sqrt{5}}{5}$ **18** $\frac{4}{5}$ **19** $\frac{63}{65}$

20 $\frac{2\sqrt{20}-3\sqrt{10}}{30}\left(\text{or } \frac{4\sqrt{5}-3\sqrt{10}}{30}\right)$

21 $\sqrt{1-x^2}$ **22** $\frac{\sqrt{1-x^2}}{x}$ **23** $\frac{1}{\sqrt{x^2+1}}$

24 $2x\sqrt{1-x^2}$ **25** $\sqrt{\frac{1-x}{1+x}}$

26 $\frac{-x^3+x+2x\sqrt{1-x^2}}{x^2+1}$

27 $\cos\left(\arcsin\frac{4}{5}+\arcsin\frac{5}{13}\right)=\cos\left(\arccos\frac{16}{65}\right)$

$\cos\left(\arcsin\frac{4}{5}\right)\cos\left(\arcsin\frac{5}{13}\right)-\sin\left(\arcsin\frac{4}{5}\right)\sin\left(\arcsin\frac{5}{13}\right)=\frac{16}{65}$

$\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}=\frac{36}{65}-\frac{20}{65}=\frac{16}{65}$ Q.E.D

28 $\sin\left(\arctan\frac{1}{2}+\arcsin\frac{1}{3}\right)=\sin\left(\frac{\pi}{4}\right)$

$\sin\left(\arctan\frac{1}{2}\right)\cos\left(\arctan\frac{1}{3}\right)+\cos\left(\arctan\frac{1}{2}\right)\sin\left(\arctan\frac{1}{3}\right)=\frac{\sqrt{2}}{2}$

$\frac{\sqrt{5}}{5}\cdot\frac{3\sqrt{10}}{10}+\frac{2\sqrt{5}}{5}\cdot\frac{\sqrt{10}}{10}=\frac{3\sqrt{50}}{50}+\frac{2\sqrt{50}}{50}=\frac{25\sqrt{2}}{50}=\frac{\sqrt{2}}{2}$ Q.E.D

29 $x=\frac{1}{2}$ **30** $x\approx 0.580,\ 2.56$

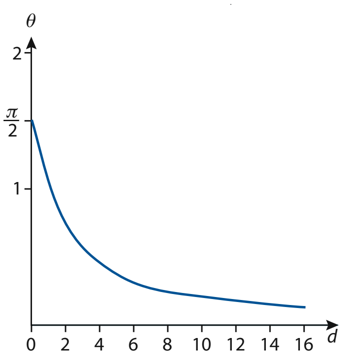
31 $x\approx 2.21$ **32** $x\approx 1.11,\ 4.25$

33 $x=\frac{\pi}{4},\frac{5\pi}{4}$; $x\approx 2.82,\ 5.96$ **34** $x=\frac{\pi}{4}$; $x\approx 0.464$

35 $x\approx 1.37,\ 4.91$

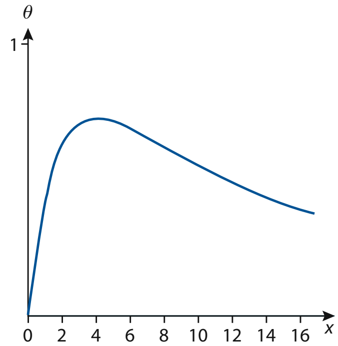
36 $x=\pi,\ 2\pi$; $x\approx 0.912,\ 2.23,\ 4.05,\ 5.37$

37 $x=0,\ \pi$; $x\approx 1.89,\ 5.03$ **38** $\theta=\arctan\left(\frac{2}{d}\right)$



39 a) (ii) $\theta=\arctan\left(\frac{7x}{x^2+15.84}\right)$

(iii)

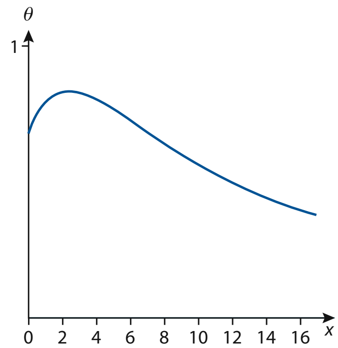


(iv) 3.98 m; sit in the 2nd row

b) (ii) $\theta=\arctan\left(\frac{7\left(x\cos\frac{\pi}{9}+2.5\right)}{\left(x\cos\frac{\pi}{9}+2.5\right)^2+\left(8.8-x\sin\frac{\pi}{9}\right)\left(1.8-x\sin\frac{\pi}{9}\right)}\right)$

$\left[\text{note: } 20^\circ=\frac{\pi}{9}\right]$

(iii)



(iv) 2.5 m; sit in the 3rd row

Practice questions

1 a) 135 cm b) 85 cm
c) $t=0.5$ sec d) 1 sec

2 $x=0,\ 2\pi$

3 $\theta\approx 2.12$ (radian measure)

4 a) (i) -1 (ii) 4π
b) four

5 a) $p=35$ b) $q=29$ c) $m=\frac{1}{2}$

6 $x=0,\ 1.06,\ 2.05$

7 a) $x=\frac{2\pi}{3},\frac{4\pi}{3}$ b) $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}$

8 a) $\sin x=\frac{1}{3}$ b) $\cos 2x=\frac{7}{9}$
c) $\sin 2x=-\frac{4\sqrt{2}}{9}$

9 a) $1.6\sin\left(\frac{2\pi}{11}\left(x-\frac{9}{4}\right)\right)+4.2$
b) Approximately 3.15 metres
c) Approximately 12:27 p.m. to 7:33 p.m.

10 $x\approx 0.785,\ 1.89$

11 a) 15 cm
b) area $\approx 239\text{ cm}^2$

12 $k > 2.5, k < -2.5$

13 $k = 1, a = -2$

14 $\sec\theta = -\frac{3}{2}$

15 a) $\frac{84}{85}$ b) $-\frac{13}{85}$ c) $-\frac{84}{13}$

16 $\sin 2p^\circ = \frac{4}{5}, \sin 3p^\circ = \frac{11\sqrt{5}}{25}$

17 a) $-\frac{5}{13}$ b) $\frac{12}{13}$ c) $-\frac{120}{169}$ d) $\frac{119}{169}$

18 $\tan\theta = \frac{1}{3}$ or -3

19 $\tan x = \frac{-(k+1)}{k-1}\tan\alpha \left(\text{or } \tan x = \frac{\tan\alpha(k+1)}{1-k}\right)$

20 $\theta = \pm\frac{3\pi}{8}, \pm\frac{\pi}{8}$

21 b) $x \approx 0.412$

c) $\cos(2) \leqslant g(x) \leqslant 1$

22 24.1°

23 $\frac{72}{\pi}\arccos\frac{8}{13}$ cm

Chapter 8

Exercise 8.1

1 b) $\cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}, \cot\theta = \frac{4}{3}, \sec\theta = \frac{5}{4}, \csc\theta = \frac{5}{3}$

c) $\theta \approx 36.9^\circ; 53.1^\circ$

2 b) $\sin\theta = \frac{\sqrt{39}}{8}, \tan\theta = \frac{\sqrt{39}}{5}, \cot\theta = \frac{5\sqrt{39}}{39}, \sec\theta = \frac{8}{5}$,

$\csc\theta = \frac{8\sqrt{39}}{39}$

c) $\theta \approx 51.3^\circ; 38.7^\circ$

3 b) $\sin\theta = \frac{2\sqrt{5}}{5}, \cos\theta = \frac{\sqrt{5}}{5}, \cot\theta = \frac{1}{2}, \sec\theta = \sqrt{5}$,

$\csc\theta = \frac{\sqrt{5}}{2}$

c) $\theta \approx 63.4^\circ; 26.6^\circ$

4 b) $\sin\theta = \frac{\sqrt{51}}{10}, \tan\theta = \frac{\sqrt{51}}{7}, \cot\theta = \frac{7\sqrt{51}}{51}, \sec\theta = \frac{10}{7}$,

$\csc\theta = \frac{10\sqrt{51}}{51}$

c) $\theta \approx 45.6^\circ; 44.4^\circ$

5 b) $\sin\theta = \frac{3\sqrt{10}}{10}, \cos\theta = \frac{\sqrt{10}}{10}, \tan\theta = 3, \sec\theta = \sqrt{10}$,

$\csc\theta = \frac{\sqrt{10}}{3}$

c) $\theta \approx 71.6^\circ; 18.4^\circ$

6 b) $\cos\theta = \frac{3}{4}, \tan\theta = \frac{\sqrt{7}}{3}, \cot\theta = \frac{3\sqrt{7}}{7}, \sec\theta = \frac{4}{3}$,

$\csc\theta = \frac{4\sqrt{7}}{7}$

c) $\theta \approx 41.4^\circ; 48.6^\circ$

7 b) $\sin\theta = \frac{\sqrt{60}}{11}, \cos\theta = \frac{\sqrt{61}}{11}, \tan\theta = \frac{2\sqrt{915}}{61}$,

$\cot\theta = \frac{\sqrt{915}}{30}, \csc\theta = \frac{11\sqrt{60}}{60}$

c) $\theta \approx 44.8^\circ; 45.2^\circ$

8 b) $\sin\theta = \frac{9\sqrt{181}}{181}, \cos\theta = \frac{10\sqrt{181}}{181}, \cot\theta = \frac{10}{9}, \sec\theta = \frac{\sqrt{181}}{10}$,

$\csc\theta = \frac{\sqrt{181}}{9}$

c) $\theta \approx 42.0^\circ; 48.0^\circ$

9 b) $\sin\theta = \frac{7\sqrt{65}}{65}, \tan\theta = \frac{7}{4}, \cot\theta = \frac{4}{7}, \sec\theta = \frac{\sqrt{65}}{4}$,

$\csc\theta = \frac{\sqrt{65}}{7}$

c) $\theta \approx 60.3^\circ; 29.7^\circ$

10 $\theta = 60^\circ, \frac{\pi}{3}$

11 $\theta = 45^\circ, \frac{\pi}{4}$

12 $\theta = 60^\circ, \frac{\pi}{3}$

13 $\theta = 60^\circ, \frac{\pi}{3}$

14 $\theta = 45^\circ, \frac{\pi}{4}$

15 $\theta = 30^\circ, \frac{\pi}{6}$

16 $x \approx 86.6$

17 $x \approx 8.60$

18 $x \approx 20.6$

19 $x \approx 374$

20 $x = 18$

21 $x = 200$

22 $\alpha = 30^\circ, \beta = 60^\circ$

23 $\alpha \approx 67.4^\circ, \beta \approx 22.6^\circ$

24 $\alpha = 30^\circ, \beta = 60^\circ$

25 $\alpha \approx 20.0^\circ, \beta \approx 70.0^\circ$

26 114 metres

27 67.4°

28 4.05 metres

29 4105 m

30 $44^\circ, 68^\circ, 68^\circ$

31 5.76 km/hr

32 69.5 m

33 28.7 m

34 151 m

35 59.2 m

36 $3\sqrt{5}$

37 -0.6

38 $\frac{|ap + bq + c|}{\sqrt{a^2 + b^2}}$

39 Verify

40 14°

Exercise 8.2

1 $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$

2 $\sin\theta = \frac{12}{37}, \cos\theta = -\frac{35}{37}, \tan\theta = -\frac{12}{35}$

3 $\sin\theta = -\frac{\sqrt{2}}{2}, \cos\theta = \frac{\sqrt{2}}{2}, \tan\theta = -1$

4 $\sin\theta = -\frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = \frac{\sqrt{3}}{3}$

5 a) $\sin 120^\circ = \frac{\sqrt{3}}{2}, \cos 120^\circ = -\frac{1}{2}, \tan 120^\circ = -\sqrt{3}, \cot 120^\circ = -\frac{\sqrt{3}}{3}, \sec 120^\circ = -2, \csc 120^\circ = \frac{2\sqrt{3}}{3}$

b) $\sin 135^\circ = \frac{\sqrt{2}}{2}, \cos 135^\circ = -\frac{\sqrt{2}}{2}, \tan 135^\circ = -1, \cot 135^\circ = -1, \sec 135^\circ = -\sqrt{2}, \csc 135^\circ = \sqrt{2}$

c) $\sin 330^\circ = -\frac{1}{2}, \cos 330^\circ = \frac{\sqrt{3}}{2}, \tan 330^\circ = -\frac{1}{2}, \cot 330^\circ = -2, \sec 330^\circ = \frac{2\sqrt{3}}{3}, \csc 330^\circ = -2$

d) $\sin 270^\circ = -1, \cos 270^\circ = 0, \tan 270^\circ = \text{undef.}, \cot 270^\circ = 0, \sec 270^\circ = \text{undef.}, \csc 270^\circ = -1$

e) $\sin 240^\circ = -\frac{\sqrt{3}}{2}, \cos 240^\circ = -\frac{1}{2}, \tan 240^\circ = \sqrt{3}, \cot 240^\circ = \frac{\sqrt{3}}{3}, \sec 240^\circ = -2, \csc 240^\circ = -\frac{2\sqrt{3}}{3}$

f) $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}, \cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}, \tan \frac{5\pi}{4} = 1, \cot \frac{5\pi}{4} = 1, \sec \frac{5\pi}{4} = -\sqrt{2}, \csc \frac{5\pi}{4} = -\sqrt{2}$

g) $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}, \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}, \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}, \sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}, \csc\left(-\frac{\pi}{6}\right) = -2$

h) $\sin\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{2}, \cos\left(\frac{7\pi}{6}\right) = -\frac{1}{2}, \tan\left(\frac{7\pi}{6}\right) = -\sqrt{3}, \cot\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{3}, \sec\left(\frac{7\pi}{6}\right) = -2, \csc\left(\frac{7\pi}{6}\right) = \frac{2\sqrt{3}}{3}$

i) $\sin(-60^\circ) = -\frac{\sqrt{3}}{2}, \cos(-60^\circ) = \frac{1}{2}, \tan(-60^\circ) = -\sqrt{3}, \cot(-60^\circ) = -\frac{\sqrt{3}}{3}, \sec(-60^\circ) = 2, \csc(-60^\circ) = -\frac{2\sqrt{3}}{3}$

j) $\sin\left(-\frac{3\pi}{2}\right) = 1, \cos\left(-\frac{3\pi}{2}\right) = 0, \tan\left(-\frac{3\pi}{2}\right) = \text{undef.}, \cot\left(-\frac{3\pi}{2}\right) = 0, \sec\left(-\frac{3\pi}{2}\right) = \text{undef.}, \csc\left(-\frac{\pi}{6}\right) = 1$

k) $\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2}, \cos\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \tan\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{3}, \cot\left(\frac{5\pi}{3}\right) = -\sqrt{3}, \sec\left(\frac{5\pi}{3}\right) = -\frac{2\sqrt{3}}{3}, \csc\left(\frac{5\pi}{3}\right) = 2$

l) $\sin(-210^\circ) = -\frac{1}{2}, \cos(-210^\circ) = -\frac{\sqrt{3}}{2}, \tan(-210^\circ) = \frac{\sqrt{3}}{3}, \cot(-210^\circ) = \sqrt{3}, \sec(-210^\circ) = -\frac{2\sqrt{3}}{3}, \csc(-210^\circ) = -2$

m) $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(-\frac{\pi}{4}\right) = -1, \cot\left(-\frac{\pi}{4}\right) = -1, \sec\left(-\frac{\pi}{4}\right) = \sqrt{2}, \csc\left(-\frac{\pi}{4}\right) = -\sqrt{2}$

n) $\sin \pi = 0, \cos \pi = -1, \tan \pi = 0, \cot \pi = \text{undef.}, \sec \pi = -1, \csc \pi = \text{undef.}$

o) $\sin 4.25\pi = \frac{\sqrt{2}}{2}, \cos 4.25\pi = \frac{\sqrt{2}}{2}, \tan 4.25\pi = 1, \cot 4.25\pi = 1, \sec 4.25\pi = \sqrt{2}, \csc 4.25\pi = \sqrt{2}$

6 $\sin\theta = \frac{15}{17}, \tan\theta = \frac{15}{8}, \cot\theta = \frac{8}{15}, \sec\theta = \frac{17}{8}, \csc\theta = \frac{17}{15}$

7 $\sin\theta = -\frac{6\sqrt{61}}{61}, \cos\theta = \frac{5\sqrt{61}}{61}$

8 $\cos\theta = -1, \tan\theta = 0, \cot\theta = \text{undef.}, \sec\theta = -1, \csc\theta = \text{undef.}$

9 $\sin\theta = -\frac{\sqrt{3}}{2}, \cos\theta = \frac{1}{2}, \tan\theta = -\sqrt{3}, \cot\theta = -\frac{\sqrt{3}}{3}, \csc\theta = -\frac{2\sqrt{3}}{3}$

10 a) (i) 30° (ii) 85°
b) (i) 45° (ii) 7°
c) (i) 60° (ii) 20°

11 a) $6\sqrt{3}$ b) 87.5 c) $675\sqrt{2}$

12 28.5°

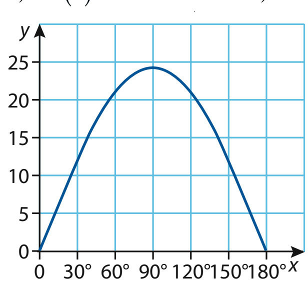
13 a) 236 b) 97.4

14 a) 9.06 b) 119

15 $ab\sin\theta$

16 $x\sqrt{3}$ **17** $\frac{2hf\cos\theta}{h+f}$ **18** Verify

19 a) $A(x) = 24\sin x$ b) $0^\circ < x < 180^\circ$



c) $(90^\circ, 24)$

20 c) 7.02 m

21 1740 km

22 a) $\sec\theta = \frac{1}{\sqrt{1-x^2}}, 0 \leqslant x < \frac{\pi}{2}$

b) $\sin\beta = \frac{y\sqrt{1+y^2}}{1+y^2}$

23 $\cos\theta = OA, \tan\theta = PB, \cot\theta = CP, \sec\theta = OB, \csc\theta = OC$

Exercise 8.3 and 8.4

1 Infinite triangles **2** One triangle **3** One triangle

4 One triangle **5** Two triangles **6** One triangle

7 $BC \approx 17.9, AC \approx 27.0, A\widehat{C}B = 115^\circ$

8 $AB \approx 18.1, BC \approx 22.5, B\widehat{A}C = 65^\circ$

9 $AB \approx 3.91, BC \approx 1.56, A\widehat{B}C = 111^\circ$

10 $AB \approx 326, AC \approx 149, B\widehat{A}C = 43^\circ$

11 $AB \approx 74.1, B\widehat{A}C \approx 60.2^\circ, A\widehat{B}C \approx 48.8^\circ$

12 $B\widehat{A}C \approx 75.5^\circ, A\widehat{B}C \approx 57.9^\circ, A\widehat{C}B \approx 46.6^\circ$

13 $B\widehat{A}C \approx 81.6^\circ, A\widehat{B}C \approx 60.6^\circ, A\widehat{C}B \approx 37.8^\circ$

14 Two possible triangles:

(1) $B\widehat{A}C \approx 55.9^\circ, A\widehat{C}B \approx 81.1^\circ, AB \approx 40.6$

(2) $B\widehat{A}C \approx 124.1^\circ, A\widehat{C}B \approx 12.9^\circ, AB \approx 9.17$

15 Two possible triangles:
(1) $A\hat{B}C \approx 72.2^\circ$, $A\hat{C}B \approx 45.8^\circ$, $AB \approx 0.414$
(2) $A\hat{B}C \approx 107.8^\circ$, $A\hat{C}B \approx 10.2^\circ$, $AB \approx 0.102$

16 10.8 cm and 30.4 cm
17 $51.3^\circ, 51.3^\circ, 77.4^\circ$
18 71.6° or 22.4°
19 Distance $\approx$ 743 metres
20 20.7°
21 Area $\approx 151.2\text{ cm}^2$
22 a) $BC = 5\sin 36^\circ$ or $BC \geqslant 5$ b) $5\sin 36^\circ < BC < 5$
c) $BC < 5\sin 36^\circ$
23 a) $BC = 5\sqrt{3}$ or $BC \geqslant 10$ b) $5\sqrt{3} < BC < 10$
c) $BC < 5\sqrt{3}$
24 $x \approx 64.9$ m, $y \approx 56.9$ m
25 a) $x = 5$ c) $\dfrac{15\sqrt{3}}{14}$
26 $\dfrac{21\sqrt{15}}{4}$
27 a) Obtuse triangle b) acute triangle
28 21.1
29 a) 14 b) $\cos\theta = \dfrac{3}{5}$, $WY = 2\sqrt{65}$
c) $2\sqrt{5}$ d) 13.9°
30 51.3°
31–32 Verify

Exercise 8.5

1 a) $\tan 70^\circ \approx 2.75$ b) $y = x\tan 70^\circ$
2 a) $\tan(-20^\circ) \approx -0.364$ b) $y = x\tan(-20^\circ)$
3 a) 1 b) $y = -x + 2$
4 a) $\tan 22^\circ \approx 0.404$ b) $y = x\tan 22^\circ - \dfrac{3}{2}$
5 45° 6 33.7° 7 60.3° 8 71.6° 9 45°
10 a) $y = \dfrac{\sqrt{3}}{3}x$ b) 56.6°
11 $AB \approx 19.3$ cm
12 $P\hat{R}O \approx 71.8^\circ$, $S\hat{R}O \approx 51.3^\circ$, area $\approx 20.9\text{ cm}^2$
13 406.1 metres
14 2.70 metres
15 a) 1291.8 km b) 42.8°
16 59.5 cm
17 $\Delta ABC = 72\text{ cm}^2$, $\Delta ABD = 24\sqrt{3} \approx 41.6\text{ cm}^2$,
$\Delta BCD \approx 34.6\text{ cm}^2$, $\Delta ACD \approx 69.3\text{ cm}^2$
18 $D\hat{E}F \approx 41.9^\circ$ 19 43.0 metres 20 95.9°

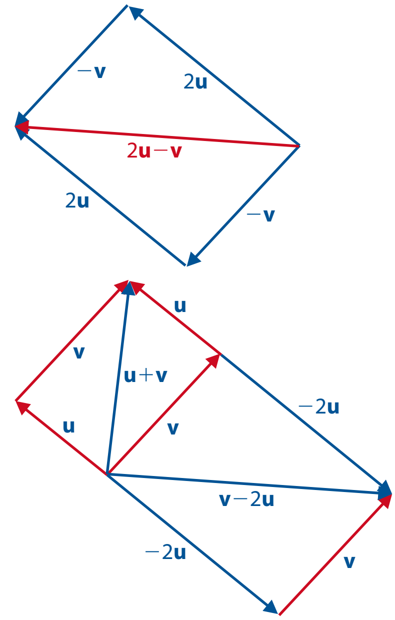
Practice questions

1 $\sin A\hat{O}B = \dfrac{24}{25}$
2 $\sin 2\theta = \dfrac{21}{29}$, $\cos 2\theta = \dfrac{20}{29}$
3 101.5°
4 $\sin 2A = \dfrac{120}{169}$
5 a) 29.1 m b) 41.9 m
6 $C\hat{A}B \approx 86.4^\circ$
7 a) 38.2° b) 17.3 cm^2
8 a) $A\hat{C}B \approx 116^\circ$ b) 155 cm^2
9 78.5 km
10 $J\hat{K}L \approx 31^\circ$
11 a) 3.26 cm b) 7.07 cm^2
12 70.5°
13 a) 91 m b) $1690\sqrt{3}$
c) (ii) $A_2 = 26x$ (iii) $x = 40\sqrt{3}$
d) (i) Supplementary angles have equal sines.
14 a) $2\sqrt{2} + 4$ b) $2\sqrt{6} + 3\sqrt{3} + 2\sqrt{2} + 3$
15 Proof
16 a) $0 < \theta < 120^\circ$ b) verify c) 60°
17 a) 120 cm^2 b) 2.16 c) 161 cm^2
18 Verify
19 $\cos\theta = \dfrac{b}{2a}$

Chapter 9

Exercise 9.1 and 9.2

1



2 a) $\sqrt{41}$ b) $\mathbf{u} = (4, -5)$
c) $\mathbf{v} = \left(\dfrac{4}{\sqrt{41}}, \dfrac{-5}{\sqrt{41}}\right)$ d) 1
3 a) $\sqrt{53}$ b) $\mathbf{u} = (7, -2)$
c) $\mathbf{v} = \left(\dfrac{7}{\sqrt{53}}, \dfrac{-2}{\sqrt{53}}\right)$ d) 1
4 a) 3 b) $(-3, 0)$ c) $(-1, 0)$ d) 1
5 a) 5 b) $(0, 5)$ c) $(0, 1)$ d) 1
6 a) $\overrightarrow{PQ} = (5, -6)$ b) $\sqrt{61}$ d) $(4, -5)$
7 a) $\overrightarrow{PQ} = (4, 6)$ b) $2\sqrt{13}$ d) $(3, 7)$
8 a) $\overrightarrow{PQ} = (5, 5)$ b) $5\sqrt{2}$ d) $(4, 6)$
9 a) $\overrightarrow{PQ} = (4, 6)$ b) $2\sqrt{13}$ d) $(3, 7)$
10 **a, c**
11 $(1, -1)$
12 $(8, -1)$
13 $(4, 8)$
14 $(-5, -5)$
15 a) $\mathbf{u} + \mathbf{v} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{u} - \mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$, $2\mathbf{u} + 3\mathbf{v} = 3\mathbf{i} + 7\mathbf{j}$,
$2\mathbf{u} - 3\mathbf{v} = 9\mathbf{i} - 11\mathbf{j}$
b) $|\mathbf{u} + \mathbf{v}| = 2\sqrt{2}$, $|\mathbf{u} - \mathbf{v}| = 4\sqrt{2}$, $|\mathbf{u}| + |\mathbf{v}| = 2\sqrt{10}$,
$|\mathbf{u}| - |\mathbf{v}| = 0$
c) $|2\mathbf{u} + 3\mathbf{v}| = \sqrt{58}$, $|2\mathbf{u} - 3\mathbf{v}| = \sqrt{202}$, $2|\mathbf{u}| + 3|\mathbf{v}| = 5\sqrt{10}$,
$2|\mathbf{u}| - 3|\mathbf{v}| = -\sqrt{10}$
16 $\left(\dfrac{11}{8}, -\dfrac{1}{4}\right)$
17 $\mathbf{u} = \frac{8}{5}\mathbf{i} - \frac{7}{5}\mathbf{j}$; $\mathbf{v} = -\frac{1}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
18 $\sqrt{13}, \sqrt{17}$
19 a) $\mathbf{v} + \mathbf{u}$ b) $\mathbf{v} + 0.5\mathbf{u}$ c) $\mathbf{v} - \mathbf{u}$ d) $0.5(\mathbf{v} - \mathbf{u})$
20 $(6, 8)$
21 $x = 3, y = 5$
22 $(6, 2)$
23 $\dfrac{5}{2}(2, 3) - \dfrac{1}{2}(2, 1)$
24 $r(1, -1) + (r - 5)(-1, 1)$
25 $2(2, 5) - 5(3, 2)$
26 $\begin{pmatrix} x \\ y \end{pmatrix} = \left(\dfrac{x+y}{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\dfrac{y-x}{2}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Exercise 9.3

1 a) $0°$ b) $90°$ c) $180°$ d) $56.31°$ e) $135°$

2 a) $\sqrt{13}, 33.69°$ b) $\sqrt{13}, 213.69°$
c) $2\sqrt{13}, 33.69°$ d) $3\sqrt{13}, 213.69°$
e) $5\sqrt{13}, 213.69°$ f) $\sqrt{13}, 33.69°$

3 a) $\sqrt{65}, \tan^{-1}\left(-\frac{7}{4}\right)+\pi$ b) $\sqrt{29}, \tan^{-1}\left(\frac{5}{2}\right)$
c) $3\sqrt{65}, \tan^{-1}\left(-\frac{7}{4}\right)+\pi$ d) $2\sqrt{29}, \tan^{-1}\left(\frac{5}{2}\right)+\pi$
e) $5\sqrt{41}, \tan^{-1}\left(-\frac{31}{8}\right)+\pi$ f) $2\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right)+\pi$

4 a) $(145.54, 273.71)$ b) $(40.70, 14.49)$
c) $(-6\sqrt{2}, 6\sqrt{2})$ d) $(120, -120\sqrt{3})$

5 $(0, 4)$

6 a) $\left(\frac{3}{5}, \frac{4}{5}\right)$ b) $\frac{2}{\sqrt{29}}\mathbf{i} - \frac{5}{\sqrt{29}}\mathbf{j}$

7 $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right); \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

8 $\frac{21}{5}\mathbf{i} - \frac{28}{5}\mathbf{j}$

9 $\pm\frac{3}{\sqrt{13}}(2\mathbf{i}+3\mathbf{j})$

10 $\pm\frac{7}{5}(4\mathbf{i}+3\mathbf{j})$

11 $\pm\frac{3}{\sqrt{13}}(3\mathbf{i}-2\mathbf{j})$

12 a) $\vec{P} = (840\cos 80°, -840\sin 80°)$;
$\vec{W} = (60\cos 30°, -60\sin 30°)$
b) $\vec{V} = (840\cos 80° + 60\cos 30°, -840\sin 80° - 60\sin 30°)$
$= (197.83, -857.24)$
c) Speed $= 879.77$ km/h, bearing $167°$

13 a) $\vec{P} = (520\cos 110)\vec{i} + (520\sin 110)\vec{j}$
$= -177.85\vec{i} + 488.64\vec{j}$
$\vec{W} = (64\cos 160)\vec{i} + (64\sin 160)\vec{j} = -60.14\vec{i} + 21.89\vec{j}$
b) Speed $= 580.6$ km/h, bearing $337.8°$

14 24.15, 6.47

15 200 m east of the initial point.

16 Force $= 8176.152$ N at an angle of $-10.85°$ to the x-axis.

17 Water $= 12.36$, boat $= 38.04$

18 T $= 35.89$, S $= 41.57$

19 35.9 km/h at N 12.88° W

20 At N 11.54° W

21 P $= (10, 6)$

22 N 11.54° E, 293.9 km/h

23 a) $(4, 6)$ b) $(0, -2)$ and $(20, 6)$

24 No answer required – proof

25 No answer required – proof

26 No answer required – proof

27 a) 50 m b) 5 minutes
c) N 19.47° W, 5.3 minutes

28 a) $\mathbf{p} = (220, 200\sqrt{3})$ b) speed = 410.37, N 32.42° E

29 66.6 N, S 28.5° E (or N 151.5° E)

Exercise 9.4

1 a) $0, 90°$ b) $13, 54°$ c) $11, 42°$ d) $2\sqrt{3}, 30°$
e) $4, 90°$ f) $3\sqrt{3}, 30°$ g) $-12\sqrt{3}, 150°$ h) $-16, 180°$

2 a) -1 b) -1 c) $(57, -38)$
d) $(-12, -15)$ e) -6 f) 3
g) Scalar multiplication is distributive over addition of vectors. Multiplication is not associative.

3 Neither, perpendicular, perpendicular

4 a) 2000 b) 6450 c) 155

5 a) 26.6, 63.4, 90 b) 41.4, 74.5, 64.1
c) 41.6, 116.6, 21.8

6 a) $(5t, -3t)$ b) $(3t, 2t)$

7 a) $(x-1)(x-3) + (y-2)(y-4) = 0$
b) $(x-3)(x+1) + (y-4)(y+7) = 0$

8 No

9 $t = \frac{21}{5}$

10 $b = \sqrt{6}$ or $b = -\sqrt{6}$

11 $\left(\frac{4\sqrt{3}+3}{10}, \frac{4-3\sqrt{3}}{10}\right)$ or $\left(\frac{3-4\sqrt{3}}{10}, \frac{4+3\sqrt{3}}{10}\right)$

12 $t = 0$

13 Sides of rhombus: $\vec{a}$ and $\vec{b}$ with $|\vec{a}| = |\vec{b}|$, diagonals are $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b} \Rightarrow (\vec{a}+\vec{b})(\vec{a}-\vec{b}) = (\vec{a})^2 - \vec{a}\vec{b} + \vec{a}\vec{b} - (\vec{b})^2 = 0$

14 a) 5.6 b) $\frac{8}{\sqrt{17}}$

15 $440\sqrt{2}$

16 a) 1 b) 0 c) $\frac{21}{\sqrt{34}}$

17 No answer required – proof

18 $\frac{48 \pm 25\sqrt{3}}{39}$

19 $\alpha = 63.4°, \beta = 71.6°, \theta = 45°$

20 No answer required – proof

Practice questions

1 a) $\mathbf{v} - \mathbf{u}$
b) $(\frac{1}{2})(\mathbf{v} - \mathbf{u})$
c) $(\frac{1}{2})(\mathbf{u} + \mathbf{v})$
d) $(\frac{3}{2})\mathbf{v} - (\frac{1}{2})\mathbf{u}$

2 a) $(6, -1)$ b) $\frac{6}{\sqrt{37}}(6, -1)$

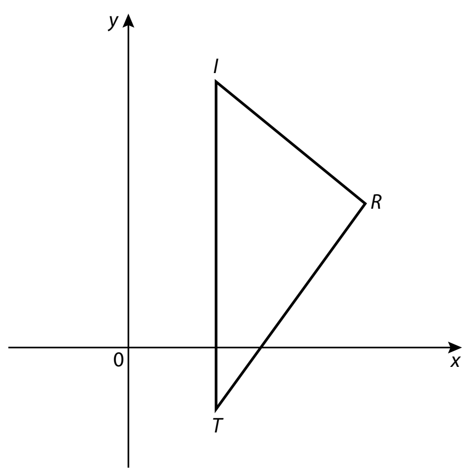
3 a) $OR = 15$ b) $\begin{pmatrix}-5\\5\sqrt{5}\end{pmatrix}$ c) $\frac{1}{\sqrt{6}}$ d) $75\sqrt{5}$

4 a) $\overrightarrow{MR} = \begin{pmatrix}11\\4\end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix}-3\\6\end{pmatrix}$
b) $83.4°$
c) $\mathbf{u} = \frac{1}{2}\overrightarrow{MR}, \mathbf{v} = -\frac{1}{2}\overrightarrow{MR} \Rightarrow \mathbf{u} \parallel \mathbf{v}$ and $|\mathbf{u}| = |\mathbf{v}|$

5 $m = \frac{63}{46}, n = \frac{37}{46}$

6 a) 15 km/h, 19.7 km/h b) $\begin{pmatrix}4.5\\6\end{pmatrix}; \begin{pmatrix}9\\-4\end{pmatrix}$
c) 11.4 km d) At 8 a.m.
e) 12.2 km f) 54 minutes

7 a)



b) $\overrightarrow{IR} = \begin{pmatrix} 5 \\ -\frac{25}{6} \end{pmatrix}$

8 a) $\begin{pmatrix} 745 \\ 1000 \end{pmatrix}$ b) 600 km/h c) at 1.5 hrs

d) $\begin{pmatrix} 325 \\ 940 \end{pmatrix}$ e) 451 km

9 $2n^2 - n + 12 = 0$ does not have real solutions, so it is not possible.

10 $\alpha = \frac{\pi}{2} - 2\theta$ **11** 0

Chapter 10

Exercise 10.1

1 $5 + 2i$ **2** $7 - \sqrt{7}i$ **3** $-6 + 0i$

4 $-7 + 0i$ **5** $0 + 9i$ **6** $0 - \frac{5}{4}i$

7 $-1 - i$ **8** $-5 + 9i$ **9** $14 + 23i$

10 $-2 + 7i$ **11** $34 - 13i$ **12** $5 - i$

13 $\frac{16}{29} - \frac{11}{29}i$ **14** $\frac{4}{13} - \frac{7}{13}i$ **15** 1

16 $\frac{25}{36}$ **17** $-\frac{1}{13} - \frac{18}{13}i$ **18** $8 - i$

19 $-7 - 3i$ **20** $4 + 10i$ **21** $\frac{5}{13} + \frac{12}{13}i$

22 $\frac{48}{25} + \frac{36}{25}i$ **23** $2 + 9i$ **24** 68

25 $\frac{8}{13} - \frac{63}{26}i$ **26** $\frac{7}{65} + \frac{4}{65}i$ **27** $\frac{5}{169} + \frac{12}{169}i$

28 $\frac{12}{25} + \frac{8}{25}i$ **29** $\frac{498}{169} + \frac{553}{169}i$ **30** $-\frac{33}{25} - \frac{56}{25}i$

31 $\frac{17}{13} - \frac{19}{13}i$ **32** $x = -\frac{1}{2}, y = -2$; and $x = 1, y = 1$

33 a) -8 c) 2^{48}

34 a) $-4i$ c) 2^{46}

35 $x^2 + y^2 = 4$ **36** $\frac{9 - \sqrt{2}}{3} + \frac{2}{3}i$

37 $x = -\frac{2}{65}, y = \frac{29}{65}$ **38** $\frac{1}{2}(1 + i)$

39 $5 + 12i$ **40** $(x, y) = (2, -1)$ or $(x, y) = (-2, 1)$

41 a) $(x, y) = (1, 3)$ or $(x, y) = (-1, -3)$

b) $2i, -1 - i$

42 $\left\{-3i, \frac{3(\sqrt{3} + i)}{2}, \frac{3(-\sqrt{3} + i)}{2}\right\}$

43 $\frac{1}{2} - 2i, 3$ **44** $f(x) = 2x^4 - 11x^3 + 15x^2 + 17x - 11$

45 $f(x) = x^4 + 2x^3 + 8x + 16$

46 $5 - 2i, -3$ **47** $1 + i\sqrt{3}, -\frac{2}{3}$ **48** Verify

49 a) $k = 0 \pm 1$ b) $k = \pm\sqrt{3 \pm 2\sqrt{2}}$

50 $z_1 = 1 + i, z_2 = 2 - i$ **51** $z_1 = \frac{7 - 4i}{3}, z_2 = \frac{1 + 6i}{3}$

Exercise 10.2

1 $2\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$ **2** $2\text{ cis}\left(\frac{\pi}{6}\right)$ **3** $2\sqrt{2}\text{ cis}\left(\frac{7\pi}{4}\right)$

4 $2\sqrt{2}\text{ cis}\left(\frac{11\pi}{6}\right)$ **5** $4\text{ cis}\frac{5\pi}{3}$ **6** $3\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)$

7 $4\text{ cis}\left(\frac{\pi}{2}\right)$ **8** $6\text{ cis}\left(\frac{7\pi}{6}\right)$ **9** $\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$

10 $15\text{ cis }\pi$ **11** $\frac{1}{5}\text{ cis }(5.64)$ **12** $3\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)$

13 $\pi\text{ cis }(0)$ **14** $e\text{ cis}\left(\frac{\pi}{2}\right)$ **15** $\frac{-\sqrt{3}}{2} + \frac{i}{2}, \frac{\sqrt{3}}{2} + \frac{i}{2}$

16 $1, \frac{1}{2} - \frac{\sqrt{3}\,i}{2}$ **17** $\frac{-\sqrt{3}}{2} + \frac{i}{2}, -i$ **18** $-i, \frac{-1}{2} + \frac{\sqrt{3}\,i}{2}$

19 $\frac{\sqrt{6} + \sqrt{2}}{2} + i\frac{\sqrt{6} - \sqrt{2}}{2}, \frac{9(-\sqrt{6} + \sqrt{2})}{8} - i\frac{9(\sqrt{6} + \sqrt{2})}{8}$

20 $-3\sqrt{3} - 3 + i(3\sqrt{3} - 3), \frac{3\sqrt{3} - 3}{4} - \frac{i(3\sqrt{3} + 3)}{4}$

21 $\frac{-\sqrt{2}}{2}(1 + i), \frac{\sqrt{2}}{2}(1 + i)$

22 $6, \frac{-3}{4} - \frac{3\sqrt{3}i}{4}$

23 $\frac{5\sqrt{6} - 15\sqrt{2}}{48} - i\frac{5\sqrt{6} + 15\sqrt{2}}{48}, \frac{-5\sqrt{6} - 15\sqrt{2}}{64} + i\frac{5\sqrt{6} - 15\sqrt{2}}{64}$

24 $-3\sqrt{3} + 3 + i(3\sqrt{3} + 3), \frac{3\sqrt{3} + 3}{4} + \frac{i(3\sqrt{3} - 3)}{4}$

25 $z_1 = 2\text{ cis}\frac{\pi}{6}, z_2 = 4\text{ cis}\frac{-\pi}{3}, \frac{1}{z_1} = \frac{1}{2}\text{ cis} -\frac{\pi}{6}, \frac{1}{z_2} = \frac{1}{4}\text{ cis}\frac{\pi}{3},$

$z_1z_2 = 8\text{ cis}\frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{1}{2}\text{ cis}\frac{\pi}{2}$

26 $z_1 = 2\sqrt{2}\text{ cis}\frac{\pi}{6}, z_2 = 4\sqrt{3}\text{ cis}\frac{-\pi}{3}, \frac{1}{z_1} = \frac{\sqrt{2}}{4}\text{ cis}\frac{\pi}{6}, \frac{1}{z_2} = \frac{\sqrt{3}}{12}\text{ cis}\frac{\pi}{3}$

,

$z_1z_2 = 8\sqrt{6}\text{ cis}\frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{\sqrt{6}}{6}\text{ cis}\frac{\pi}{2}$

27 $z_1 = 8\text{ cis}\frac{\pi}{6}, z_2 = 3\sqrt{2}\text{ cis}\frac{-3\pi}{4}, \frac{1}{z_1} = \frac{1}{8}\text{ cis}\frac{-\pi}{6}, \frac{1}{z_2} = \frac{\sqrt{2}}{6}\text{ cis}\frac{3\pi}{4},$

$z_1z_2 = 24\sqrt{2}\text{ cis}\frac{-7\pi}{12}, \frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}\text{ cis}\frac{11\pi}{12}$

28 $z_1 = \sqrt{3}\text{ cis}\frac{\pi}{2}, z_2 = 2\sqrt{2}\text{ cis}\frac{-2\pi}{3}, \frac{1}{z_1} = \frac{\sqrt{3}}{3}\text{ cis}\frac{-\pi}{2},$

$\frac{1}{z_2} = \frac{\sqrt{2}}{8}\text{ cis}\frac{2\pi}{3}, z_1z_2 = 2\sqrt{6}\text{ cis}\frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{\sqrt{6}}{4}\text{ cis}\frac{-5\pi}{6}$

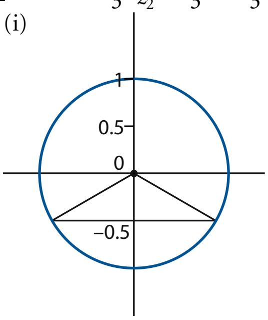
29 $z_1 = \sqrt{10}\text{ cis}\frac{\pi}{4}, z_2 = 2\sqrt{2}\text{ cis}\frac{\pi}{2}, \frac{1}{z_1} = \frac{\sqrt{10}}{10}\text{ cis}\frac{\pi}{4}, \frac{1}{z_2} = \frac{\sqrt{2}}{8}\text{ cis}\frac{\pi}{2}$

$, z_1z_2 = 4\sqrt{5}\text{ cis}\frac{3\pi}{4}, \frac{z_1}{z_2} = \frac{\sqrt{5}}{2}\text{ cis}\frac{-\pi}{4}$

30 $z_1 = 2\text{ cis}\frac{\pi}{3}, z_2 = 2\sqrt{3}\text{ cis }0, \frac{1}{z_1} = \frac{1}{2}\text{ cis}\frac{-\pi}{3}, \frac{1}{z_2} = \frac{\sqrt{3}}{6}\text{ cis }0,$

$z_1z_2 = 4\sqrt{3}\text{ cis}\frac{\pi}{3}, \frac{z_1}{z_2} = \frac{\sqrt{3}}{3}\text{ cis}\frac{\pi}{3}$

31 b) (i)



(ii) $\arg(z_1) = \frac{-\pi}{6}, \arg(z_2) = \frac{-5\pi}{6}$

32 Verify

33 a) $\frac{\sqrt{3}}{2} - \frac{3i}{2}$ b) $\frac{-2\sqrt{3}}{3}$ c) $\sqrt{3}i$

34 $|z_1| = 4, \arg(z_1) = \frac{-\pi}{6}, |z_2| = 2\sqrt{2}, \arg(z_2) = \frac{\pi}{4}, |z_3| = 8\sqrt{2},$

$\arg(z_3) = \frac{\pi}{12}$

35 $22 - 2\sqrt{3} \approx 18.5$

36 a) $\{(x, y): x^2 + y^2 = 9\}$, the circle centre (0, 0) radius 3

b) $\{(x, y): x = 0\}$, the y-axis

c) $\{(x, y): x = 4\}$, the line $x = 4$

d) $\{(x, y): (x - 3)^2 + y^2 = 4\}$, the circle centre (3, 0) radius 2

e) $\{(x, y): 1 - x + 3$ and $y = 0\}$, the line segment between (1, 0) and (3, 0)

37 a) $\{(x, y): x^2 + y^2 \leqslant 9\}$, the disk centre $(0, 0)$ radius 3
b) $\{(x, y): x^2\}\ (y + 3)^2 - 4|$, all points excluding the interior of the disk centre $(0, 3)$ radius 2

Exercise 10.3

1 $-2 - 2i\sqrt{3}$ **2** 3 **3** $3i$

4 $\sqrt{2} - \sqrt{6} + i\left(\sqrt{2} + \sqrt{6}\right)$ **5** $\frac{13}{2} + \frac{13i\sqrt{3}}{2}$

6 $\frac{3e}{2} + \frac{3ei\sqrt{3}}{2}$ **7** $2\sqrt{2}e^{i\frac{\pi}{4}}$ **8** $2e^{i\frac{\pi}{6}}$

9 $2\sqrt{2}e^{-i\frac{\pi}{6}}$ **10** $4e^{-i\frac{\pi}{3}}$ **11** $3\sqrt{2}e^{i\frac{3\pi}{4}}$

12 $4e^{i\frac{\pi}{2}}$ **13** $6e^{i\frac{7\pi}{6}}$ **14** $3\sqrt{2}e^{i\frac{3\pi}{4}}$

15 $\pi e^{2\pi i}$ or simply π **16** $e^{1+i\frac{\pi}{2}}$ **17** $32i$

18 -64 **19** $-10\,077\,696$ **20** $-262\,144$

21 1296 **22** $17\,496(-1 - i)$ **23** $\frac{1}{1296}$

24 $\frac{1}{559\,872}\left(\sqrt{3} - i\right)$

25 $-128\sqrt{3} - 128i$

26 $\sqrt{6} + i\sqrt{2}, -\sqrt{6} - i\sqrt{2}$

27 $2e^{i\frac{\pi}{9}}; 2e^{i\frac{7\pi}{9}}; 2e^{i\frac{13\pi}{9}}$

28 $\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$

29 $\left(-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right) + i\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right); \left(-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right) + i\left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right)$;

$\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right) + i\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right); \left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right) + i\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)$;

$-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

30 $\sqrt[5]{18}e^{i\frac{4\pi}{15}}; \sqrt[5]{18}e^{i\frac{10\pi}{15}}; \sqrt[5]{18}e^{i\frac{16\pi}{15}}; \sqrt[5]{18}e^{i\frac{22\pi}{15}}; \sqrt[5]{18}e^{i\frac{28\pi}{15}}$

31 $2; 2e^{i\frac{2\pi}{5}}; 2e^{i\frac{4\pi}{5}}; 2e^{i\frac{6\pi}{5}}; 2e^{i\frac{8\pi}{5}}$

32 $e^{i\left(-\frac{\pi}{16}\right)}; e^{i\frac{3\pi}{16}}; e^{i\frac{7\pi}{16}}; e^{i\frac{11\pi}{16}}; \ldots; e^{i\frac{27\pi}{16}}$

33 $2e^{i\frac{5\pi}{18}}; 2e^{i\frac{17\pi}{18}}; 2e^{i\frac{29\pi}{18}}$

34 $\pm 2, \pm 2i$

35 $\sqrt{8}e^{i\left(\frac{3\pi}{20}\right)}; \sqrt{8}e^{i\left(\frac{11\pi}{20}\right)}; \ldots; \sqrt{8}e^{i\left(\frac{35\pi}{20}\right)}$

36 $\left(-\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\right); \left(-\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\right)$;

$\left(\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\right) + i\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\right); -\sqrt{2} + i\sqrt{2}; \sqrt{2} - i\sqrt{2}$

37 $\cos(4\beta) + i\sin(4\beta)$ **38** $\cos(7\beta) + i\sin(7\beta)$

39 $\cos(3\beta) + i\sin(3\beta)$ **40** $\cos(2\beta) + i\sin(2\beta)$

41 Proof **42–43** Verify

45 b) $2\cos 2n\alpha = z^n + \frac{1}{z^n}; 2i\sin 2n\alpha = z^n - \frac{1}{z^n}$

46 7 **47** b) $1 - i$ **48** b) $\frac{3 + \sqrt{5}}{2}$

49 524 288 **50** $\frac{3}{2}$

Practice questions

1 $x = 2, y = -1$

2 a) 0 b) $x^2 + y^2 - xy$

3 a) $2i$ c) 65 536

4 a) $z_1 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right); z_2 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

c) $\frac{z_1}{z_2} = \frac{\sqrt{6} + \sqrt{2}}{4} + i\frac{\sqrt{6} - \sqrt{2}}{4}; \cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$;

$\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$

5 $\left(\frac{z_1}{z_2}\right)^3 = \left(\frac{a^3\sqrt{2}}{2b^3}\right) - i\left(\frac{a^3\sqrt{2}}{2b^3}\right)$

6 $|z| = 4$ **7** $a = \frac{11}{5}, b = \frac{3}{5}$

8 $b = \sqrt{3}$ **9** $a = 0, b = -1$

10 a) $z^5 - 1 = (z - 1)\left(z^4 + z^3 + z^2 + z + 1\right)$

b) $\operatorname{cis}\left(\pm\frac{2\pi}{5}\right); \operatorname{cis}\left(\pm\frac{4\pi}{5}\right)$

c) $\left(z^2 - \left(2\cos\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - \left(2\cos\frac{4\pi}{5}\right)z + 1\right)$

11 a) $8i = 8\operatorname{cis}\frac{\pi}{2}$

b) (i) $z = 2\operatorname{cis}\frac{\pi}{6}$

(ii) $z = \sqrt{3} + i$

12 a) $|z| = 1; \arg(z) = \frac{2\pi}{3}$

c) $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

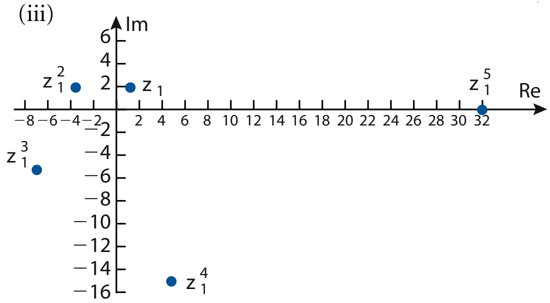
13 $-5 - 12i$

14 c) $z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

15 $p = -\frac{2}{5}; q = \frac{6}{5}$

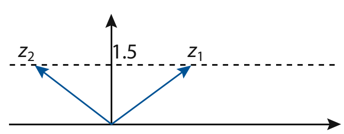
16 (ii) $z_1^2 = 4\operatorname{cis}\frac{4\pi}{5}; z_1^3 = 8\operatorname{cis}\frac{6\pi}{5}; z_1^4 = 16\operatorname{cis}\frac{\pi}{5}; z_1^5 = 32$

(iii)



(iv) Enlargement scale factor of 2 with $(0, 0)$ as centre, and a rotation of $\frac{2\pi}{5}$.

17 b) (i)



(ii) $\frac{5\pi}{6}$

c) $k = 4$

18 $a = 3, b = 1$

19 No answers required – proofs

20 a) $z = \frac{1}{2}e^{i\theta}$

c) $S_\infty = \frac{e^{i\theta}}{1 - \frac{1}{2}e^{i\theta}}$

d) (i) $S_\infty = \frac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$

21 $a = 8$; $b = 25$; $c = 26$

22 $z = 2 + 4i$

23 $z_1 = 1 + 4i; z_2 = \frac{7}{2} - \frac{1}{2}i$

24 a) $z_1 = 2 + 2i; z_2 = 2 - 2i$ b) $\frac{z_1^4}{z_2^2} = -8i$

d) 0 e) $n = 4k$, where $k \in \mathbb{Z}$

Chapter 11

Exercise 11.1

Note: Some answers may differ from one person to another due to different graph accuracies.

1 a) Student, all students in a community, random sample of few students, qualitative

b) Exam, 10th-grade students in a country, a sample from a few schools, quantitative

c) Newborns, heights of newborns in a city, sample from a few hospitals, quantitative

d) Children, eye colour of children in a city, sample of children at schools, qualitative

e) Working persons, commuters in a city, sample of few districts, quantitative

f) Country leaders, sample of few presidents, qualitative

g) Students, origin countries of a group of international school students, qualitative

2 Answers are not unique!

a) Skewed to the right as few players score very high

b) Symmetric

c) Skewed to the right

d) Unimodal, or bi-modal, symmetric or skewed, etc.

3 a) b) Quantitative

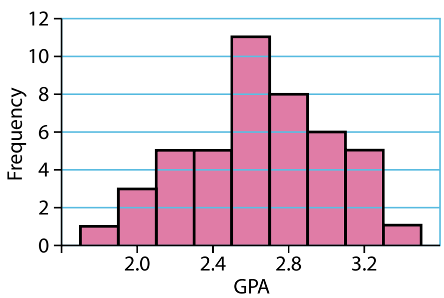
c) d) Qualitative

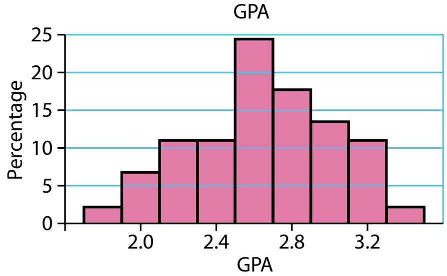
4 a) Discrete b) Continuous

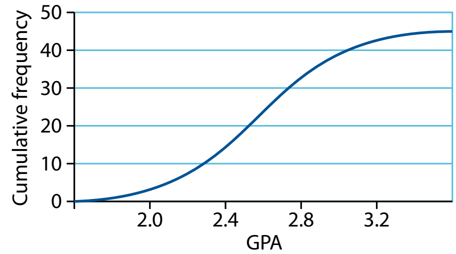
c) Continuous d) Discrete

e) Continuous f) Discrete (debatable!)

5

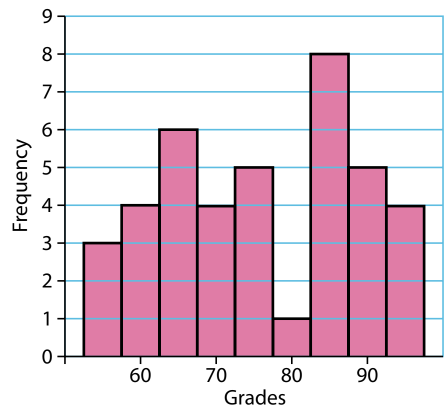






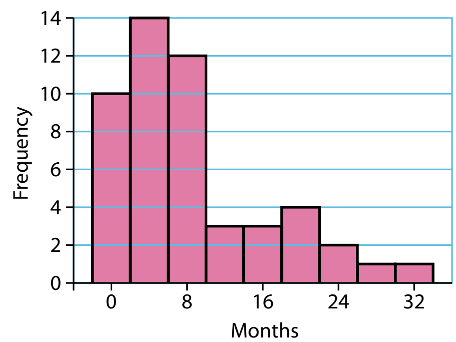
Relatively symmetric; no outliers

6



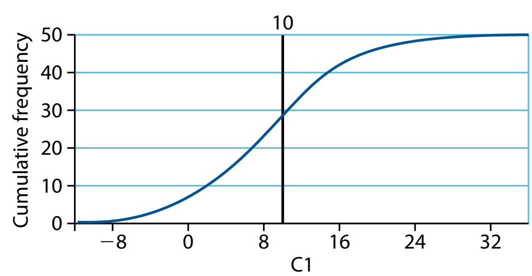
The grades appear to be divided into two groups, one with mode around 65 and the other around 85. No outliers are detected.

7 a)



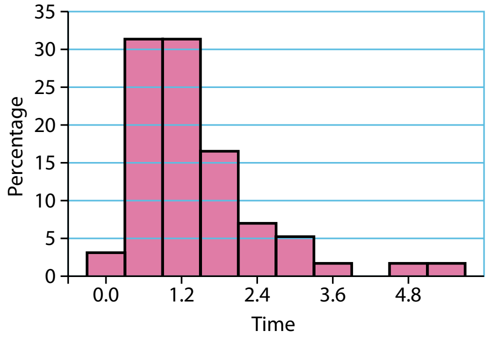
b) The data is skewed to the right.

c)

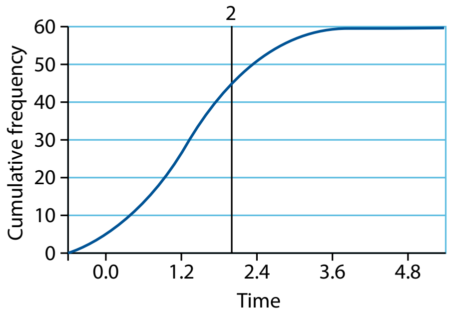


Apparently, more than 35 out of the 50 will lose the licence, about 70%.

8 a)



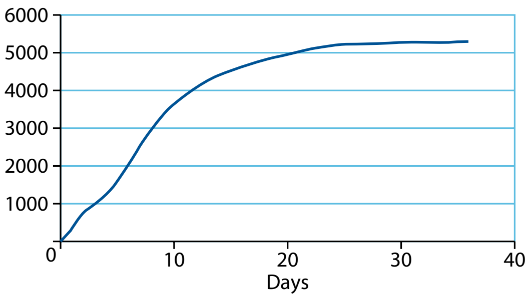
b)



Apparently, about 15 customers have to wait more than 2 minutes.

9 a) Skewed to the right, there is a mode at about 7 days stay, and a few extremes that stayed more than 20 days. A good proportion stayed for about 3 days.

b)

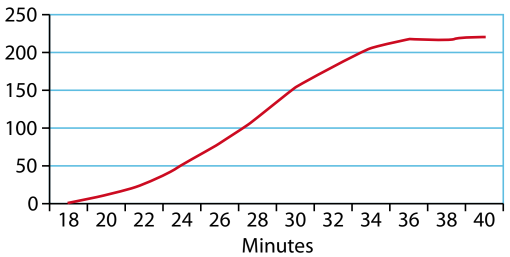


c) Approximately 35% of the patients

10 a) 40 minutes

b) Approximately 30%

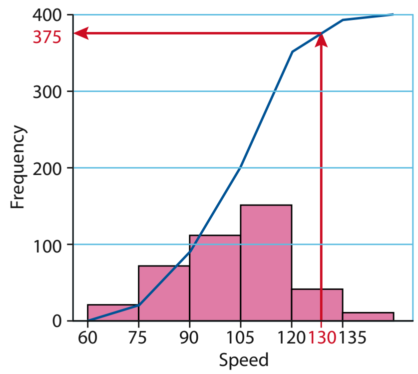
c)



11 a)

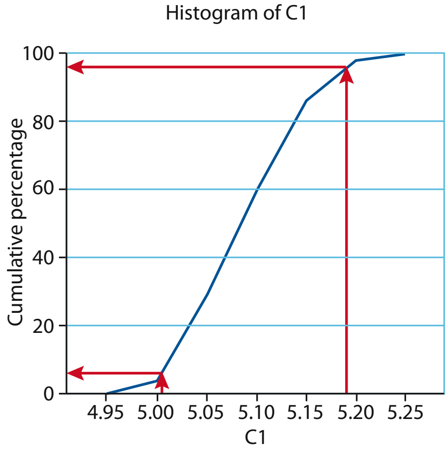
Speed	Frequency
60–75	20
75–90	70
90–105	110
105–120	150
120–135	40
135– …	10

b) c)



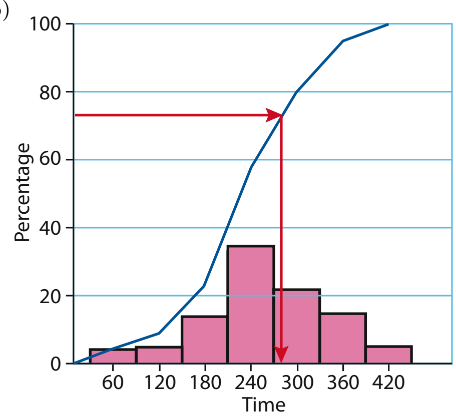
d) As you see from the diagram, $\frac{25}{400} = 6.25\%$.

12 a)



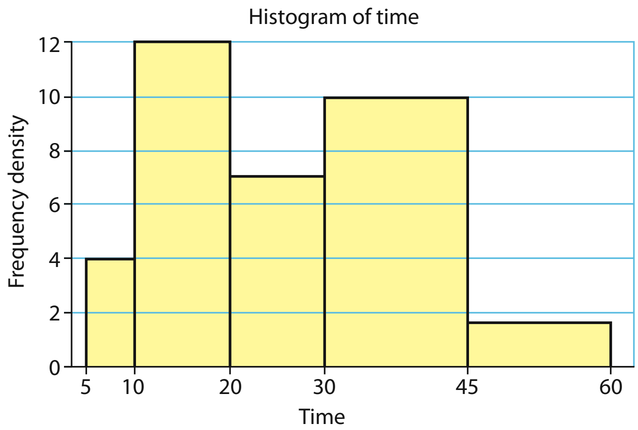
b) About 5% at the lower end and also about 5% at the upper end.

13 a) b)

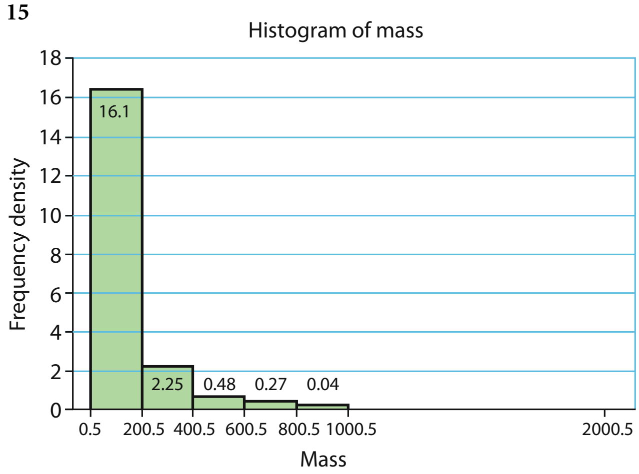


c) As you see from the diagram, about 250 seconds.

14



15



16

x	$0 \leqslant x < 10$	$10 \leqslant x < 20$	$20 \leqslant x < 30$	$30 \leqslant x < 60$	$60 \leqslant x < 80$	$80 \leqslant x < 120$
Frequency	120	80	60	60	40	40

17

x	$0 \leqslant x < 1$	$1 \leqslant x < 3$	$3 \leqslant x < 6$	$6 \leqslant x < 10$	$10 \leqslant x < 15$	$15 \leqslant x < 20$	$20 \leqslant x < 30$
Frequency	6	10	20	30	50	20	18

18

x	$0 \leqslant x < 3$	$3 \leqslant x < 8$	$8 \leqslant x < 12$	$12 \leqslant x < 16$	$16 \leqslant x < 24$	$24 \leqslant x < 30$	$30 \leqslant x < 36$
Frequency	10	10	18	20	30	20	10

19 $m = 10$, $n = 45$, $p = 2.5$, $q = 33$

Exercise 11.2

1 a) 6 b) 6

c) It appears to be symmetric as the mean and median are the same. A histogram supports this view.

2 a) 7.8 b) 7.5 c) 7 or 8

3 Average = 1.16, median = 1. Median is more appropriate as the data is skewed to the right.

4 Mean = 7494.7, median = 837.5. There are extreme values and hence the median is more appropriate.

5 Mean = median = 430. It appears to be symmetric and hence either measure would be fine.

6 a) 49.56 b) 49.93

7 2.052

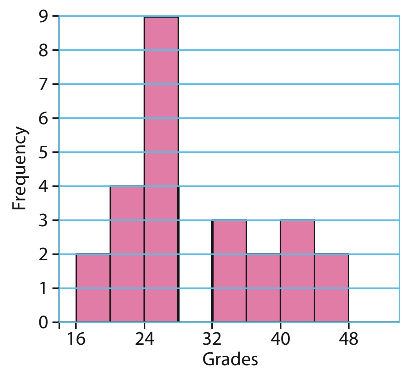
8 a) 29.96

b)

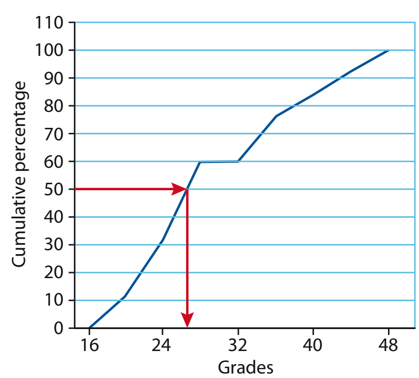
1	89
2	0223344
2	5666777
3	34
3	568
4	022
4	66

Median is 27

c)

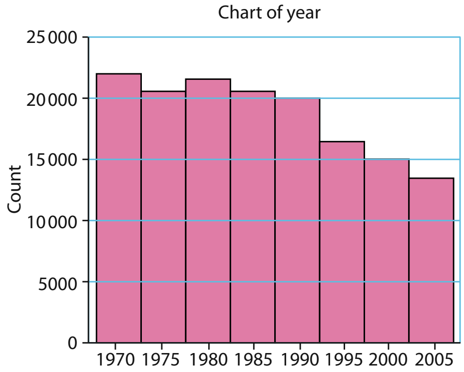


d)



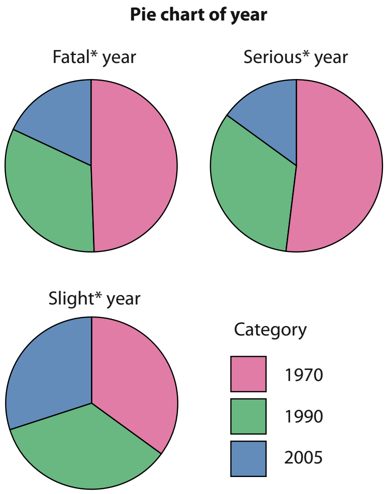
The median ≈ 27

9 a)

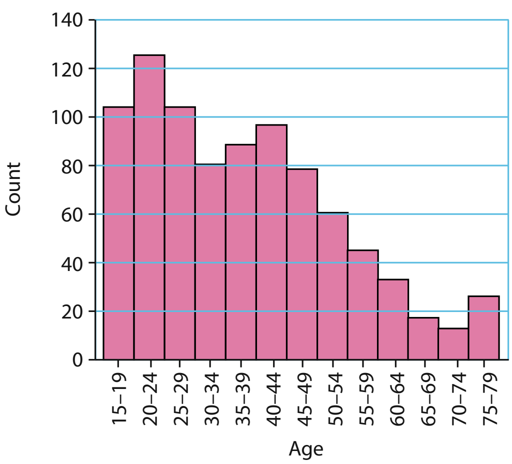


There appears to be a decline in the total number of injuries.

b)

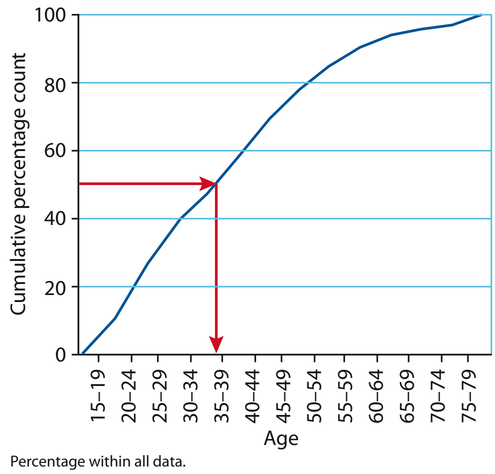


10 a)



b) 37.6

c)



From the graph, the median is approx. at 36.

11 Median ≈ 8 days; mean = 9.5 days

12 Median ≈ 28 minutes; mean = 28.7 minutes

13 Median ≈ 105; mean = 103 km/h

14 Median ≈ 5.075; mean = 5.09

15 Median ≈ 210; mean = 228.6

16 a) 41.6

b) 61.6

17 a) 61.4

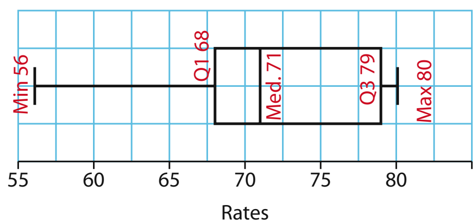
b) 63.8

Exercise 11.3

Where S_{n-1} is given, please multiply the answer by $\sqrt{\frac{n-1}{n}}$ to get the answer to S_n.

1 a) Mean = 71.47, $S_{n-1} = 7.29$

b)



c) No outliers

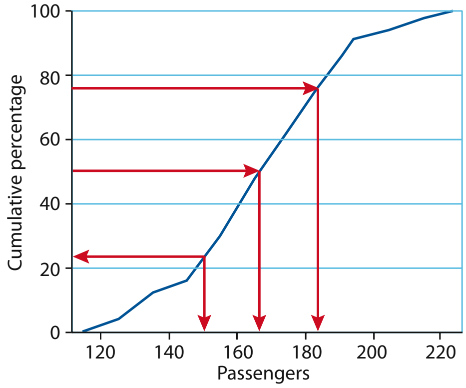
2 a) Mean = 162.6, $S_{n-1} = 23.35$

b)

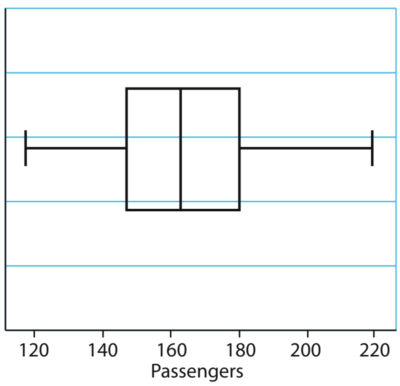
11	79
12	567
13	089
14	123679
15	033445689
16	02334568
17	1344789
18	02255779
19	8
20	9
21	08

Median = 162.5

c)



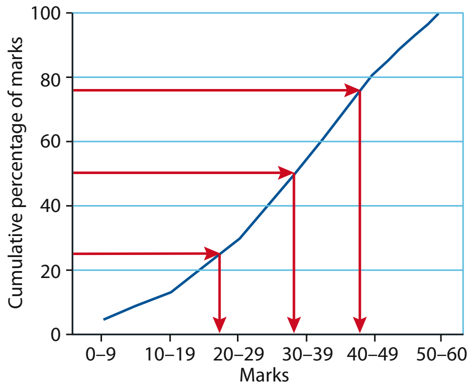
Q1 ≈150, median ≈ 165, Q3 ≈ 182



d) Real Q1 = 146.75, Q3 = 179.25, IQR = 32.5. No outliers

e) $\bar{x} \pm 3s_{n-1} = (92.55, 232.65)$. No outliers

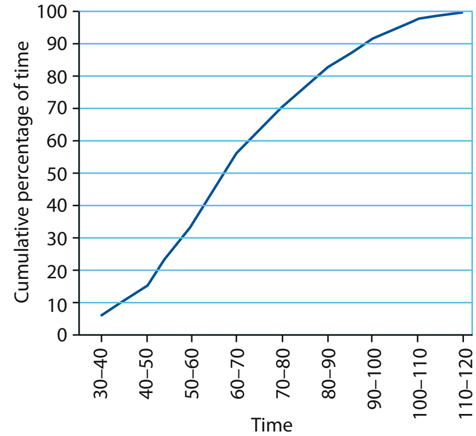
3 a) and b)



Percentage within all data.

Q1 ≈ 18, median ≈ 29, Q3 ≈ 39

4 a)



Percentage within all data.

b) Median = 63, IQR = 27

c) About 68

5 29.6

6 a) Mean = 72.1, $S_{n-1} = 6.1$

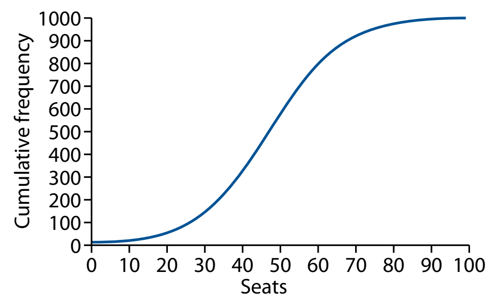
b) New mean = 85.1, S will not change.

7 a)

$x \leqslant 10$	$x \leqslant 20$	$x \leqslant 30$	$x \leqslant 40$	$x \leqslant 50$
15	65	165	335	595

$x \leqslant 60$	$x \leqslant 70$	$x \leqslant 80$	$x \leqslant 90$	$x \leqslant 100$
815	905	950	980	1000

b)



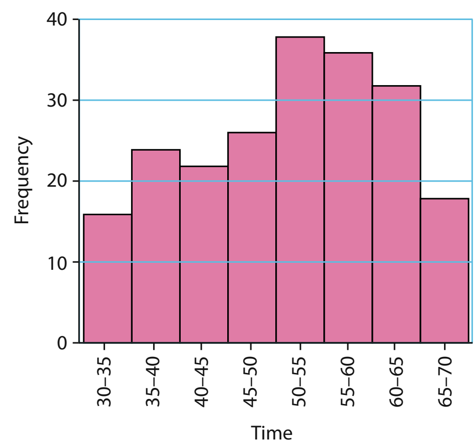
c) (i) Around 50

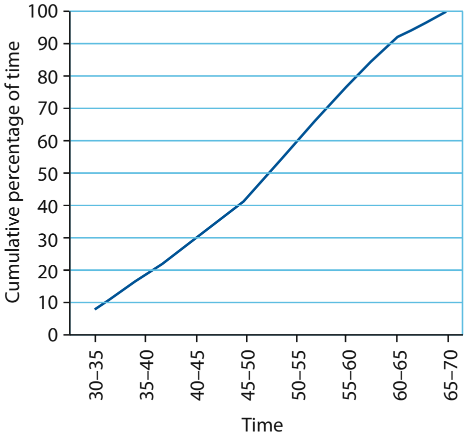
(ii) Q1 = 40, Q3 = 60, IQR = 20

(iii) About 170 days

(iv) Approximately 70 seats

8 a)



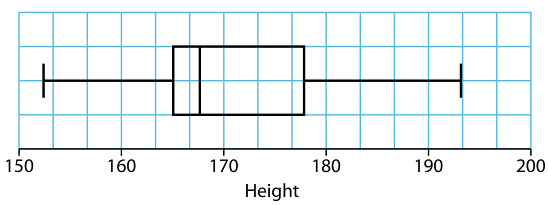
b) 

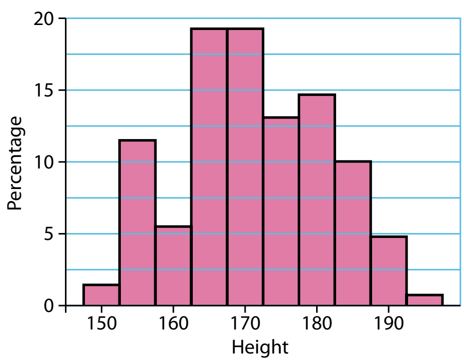
Percentage within all data.

Median = 53, IQR = 15

c) Mean = 51.3 and $S_{n-1} = 34.8$

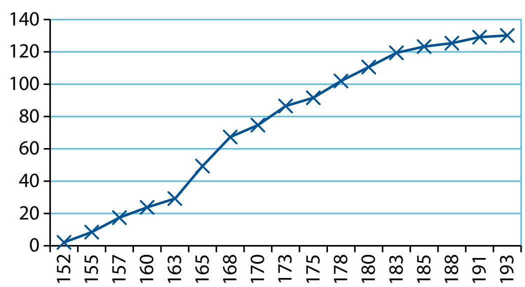
9 a) Q1 = 165.1, median = 167.64, Q3 = 177.8, minimum = 152, maximum = 193

b) 



c) Mean = 170.5, standard deviation = 9.61

d) The heights are widely spread from very short to very tall players. Heights are slightly skewed to the right, bimodal at 165 and 170, no apparent outliers. The heights between the first quartile and the median are closer together than the rest of the data.

e) 

Approximately 183 cm tall

f) 171.3

10 a) 12 b) 12 c) 111

11 a) 31 b) Increase

12 36.7

13 $x = 6, y = 11$

14 Mean = 11.12, variance = 24.6 (calculating $\sigma^2 = 23.6$)

15 Standard deviation = 6.1, IQR ≈ 6

16 Standard deviation ≈ 4.5, IQR ≈ 6

17 Standard deviation ≈ 16.7, IQR ≈ 15

18 Standard deviation ≈ 0.056, IQR ≈ 0.05

19 Standard deviation ≈ 82.3, IQR ≈ 60

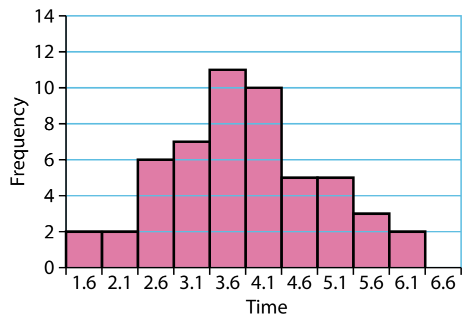
Practice questions

1 a) 12 b) $\sqrt{30.83}$

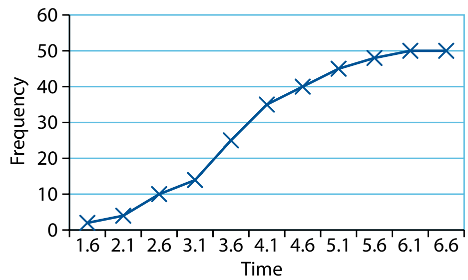
2 4

3 a)

Time	1.6	2.1	2.6	3.1	3.6	4.1	4.6	5.1	5.6	6.1	6.6
Frequency	2	2	6	4	11	10	5	5	3	2	0



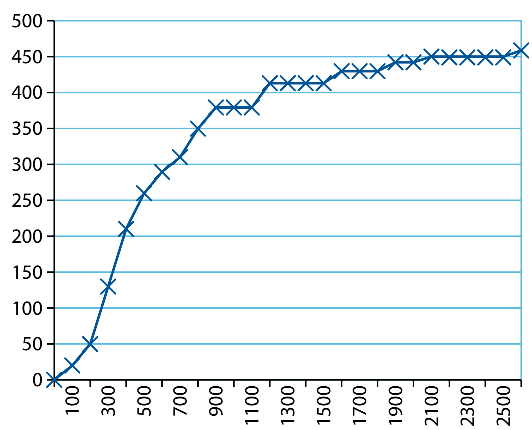
b) 86% c) approx. 4 d) 3.86, 1.1

e) 

f) Minimum = 1.6, Q1 = 3, median = 4, Q3 = 4.5, maximum = 6.2

4 a) Median and IQR as the data is skewed with outliers.

b) Mean = 682.6, standard deviation = 536.2

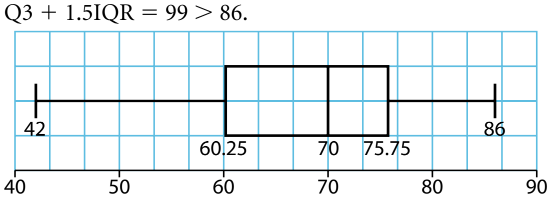
c) 

d) Q1 = 300, median = 500, Q3 = 800, IQR = 500
e) There are a few outliers on the right side. Outliers lie above Q3 + 1.5IQR = 1550.
f) Data is skewed to the right, with several outliers from 1600 onwards. It is bimodal at 300–400.

5 a) Spain, Spain b) France
c) On average, it appears that France produces the more expensive wines as 50% of its wines are more expensive than most of the wines from the other countries. Italy's prices seem to be symmetric while France's prices are skewed to the left. Spain has the widest range of prices.

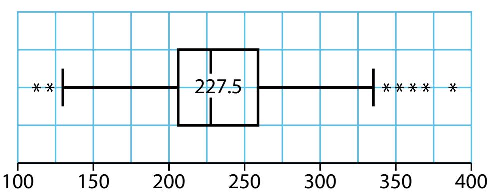
6 a) Mean = 52.65, standard deviation = 7.66
b) Median = 51.34, IQR = 2.65
c) Apparently, the data is skewed to the right with a clear outlier of 112.72! This outlier pulled the value of the mean to the right and increased the spread of the data. The median and IQR are not influenced by the extreme value.

7 a) The distribution does not appear to be symmetric as the mean is less than the median, the lower whisker is longer than the upper one and the distance between Q1 and the median is larger than the distance between the median and Q3. Left skewed.
b) There are no outliers as Q1 − 1.5IQR = 37 < 42 and Q3 + 1.5IQR = 99 > 86.
c)



d) See a)

8 a) 225
b) Q1=205, Q3 = 255, 90th percentile=300, 10th percentile = 190
c) IQR = 50, since Q1 − 1.5IQR = 130 > minimum and Q3 + 1.5IQR = 330 < 400 then there are outliers on both sides.
d)

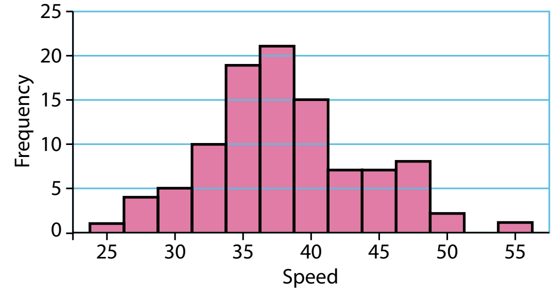


e) The distribution has many outliers. Apparently skewed to the right with more outliers there. The middle 50% seem to be very close together while the whiskers appear to be quite spread.

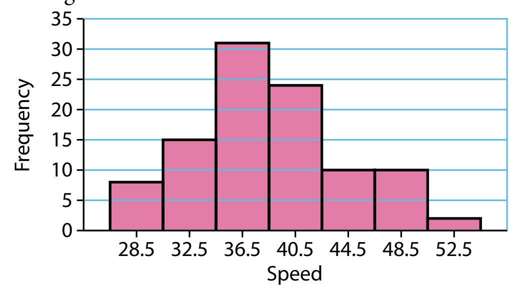
9 a)

Speed	Frequency
26–30	8
31–34	15
35–38	31
39–42	24
43–46	10
47–50	10
51–54	2

b)



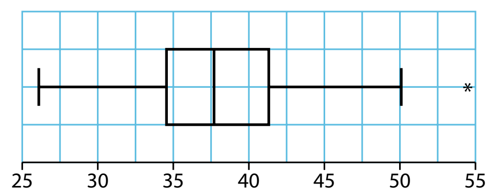
Data is relatively symmetric with possible outlier at 55.
The mode is approximately 37.
Histogram created from table:



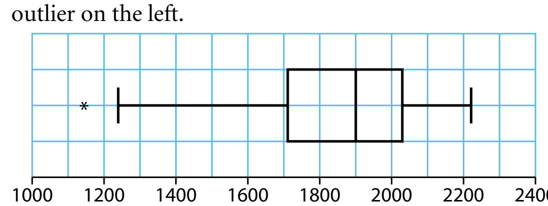
c) Mean = 38.2, standard deviation = 5.7
d)

Speed	Cu. frequency
26–30	8
31–34	23
35–38	54
39–42	78
43–46	88
47–50	98
51–54	100

e) Median = 37.6, Q1 = 34.5, Q3 = 41.3, IQR = 6.8
f) There are outliers on the right since Q3 + 1.5IQR = 51.5 < maximum = 54.



10 a) Mean = 1846.9, median = 1898.6, standard deviation = 233.8, Q1 = 1711.8, Q3 = 2031.3, IQR = 319.5
b) Q1 − 1.5IQR = 1232.55 > minimum, so there is an outlier on the left.
c)



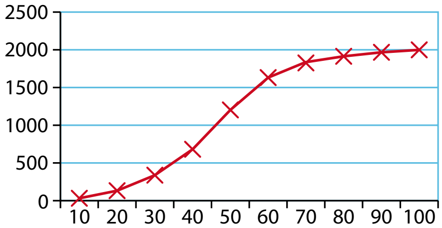
d)]1613, 2081[
e) The mean and standard deviation will get larger. The rest will not change much.

11 a) 49.6 minutes b) 48.9 minutes

12 a)

⩽10	⩽20	⩽30	⩽40	⩽50	⩽60	⩽70	⩽80	⩽90	⩽100
30	130	330	670	1190	1630	1810	1900	1960	2000

b)



c) (i) 47 (ii) About 500 (iii) Above 60

13 1.74

14 a) $m = 12$ b) Standard deviation = 5

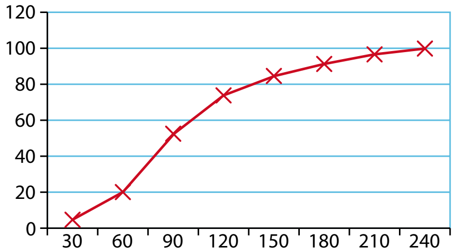
15 $k = 4$

16 a) 97.2

b)

30	60	90	120	150	180	210	240
5	20	53	74	85	92	97	100

c)



d) Median = 88
Q1 = 66
Q3 = 124

17 a) (i) 10 (ii) 24
b) Mean = 63, standard deviation = 20.5
c) Skew to the left d) 65

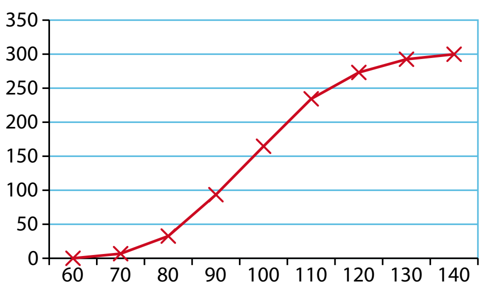
18 a) 7.41

b)

Weight	Number of packets
$w \leqslant 85$	5
$w \leqslant 90$	15
$w \leqslant 95$	30
$w \leqslant 100$	56
$w \leqslant 105$	69
$w \leqslant 110$	76
$w \leqslant 115$	80

c) (i) Median = 97 (ii) Q3 = 101 d) 0 e) 0.282

19 a) 98.2
b) (i) $a = 165, b = 275$
(ii)

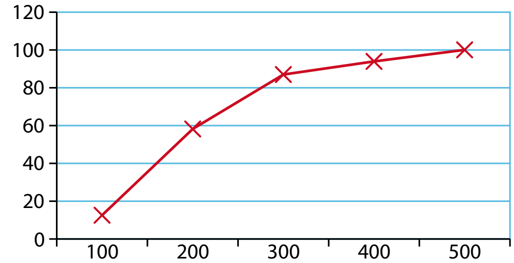


c) (i) 34% (ii) 115

20 a) (i) 24 (ii) 158 b) 40 c) 7%

21 $a = 3$

22 a)



b) IQR = 110 c) $a = 7, b = 6$
d) 199 e) (i) 9 (ii) $\frac{15}{28}$

23 a) (i) 20 (ii) 24 b) 10

24 a)

Mark	[0, 20[	[20, 40[	[40, 60[	[60, 80[	[80, 100[
Number of students	22	50	66	42	20

b) Pass mark = 43%

25 a) 183 b) 14

26 $a = 3, b = 7, c = 11, d = 11$

27 a) 100 b) $a = 55, b = 75$

28 $x = 4, y = 10$

Chapter 12

Exercise 12.1 and 12.2

Note: Some answers may differ from one person to another due to different graph accuracies.

1 a) {left-handed, right-handed}
b) All real numbers from (say) 50 cm to 210 cm.
c) All real numbers from 0 to 720 (say).

2 {(1, h), (2, h), …, (1, t), …, (6, t)}

3 a) {(1, hearts), …, (king, hearts), (1, spades), …}
b) {[(1, hearts), (king, diamonds)], …,[(1, spades), (10, diamonds)],…}
c) a: 52, b: 1326

4 a) 0.47 b) Anywhere from 0 to 20
c) 10 000

5 a) {(1, 1), (1, 2), …, (4, 4)} b) {3, 4, …, 9}

6 a) {(b, b), (b, g), (b, y), (g, b), (g, g), (g, y), (y, b), (y, g), (y, y)}
b) {(y, y), (y, b), (y, g)}
c) {(b, b), (g, g), (y, y)}

7 a) {(b, g), (b, y), (g, b), (g, y), (y, b), (y, g)}
b) {(y, b), (y, g)} c) $\varnothing$

8 a) {(t, t, t), (t, t, h), (t, h, t), (h, t, t), (h, t, h), (h, h, t), (t, h, h), (h, h, h)}
b) {(h, t, h), (h, h, t), (t, h, h), (h, h, h)}

9 {(I, fly), (I, dr), (I, tr), (H, dr), (H, b)}
{(I, fly)}

10 a) {(1, g), (1, f), …, (0, c)}
b) {(0, c), (0, s)}
c) {(1, g), (1, f), (0, g), (0, f)}
d) {(1, g), (1, f), (1, s), (1, c)}

11 a) $\{(G_1, K_1, M_1), (G_1, K_2, M_1), (G_1, K_1, M_2), \ldots\}$
b) A = all triplets containing G_2; B = all triplets not containing K_1; C = all triplets containing M_2.

c) $A \cup B$ = all males or persons who drink; $A \cap C$ = all single males; C' = all non–single persons; $A \cap B \cap C$ = all single males who drink; $A' \cap B$ = all females who drink.

12 a) {(R, L, L, S), (L, R, L, R), …}, 81
b) {(R, R, R, R), (L, L, L, L), (S, S, S, S)}
c) {(R, R, L, L), (R, L, R, S), …}
d) {(R, L, R, S), (S, S, R, L), …}

13 a) $\{(T, SY, O), (C, SN, O), \ldots\}$
b) $\{(T, SY, O), (T, SY, F), (B, SY, O), \ldots\}$
c) $\{(C, SY, O), (C, SN, O), (C, SY, F), \ldots\}$
d) $C \cap SY = \{(C, SY, O), (C, SY, F)\}$
$C' = \{(T, \ldots, \ldots), (B, \ldots, \ldots)\}$
$C \cup SY$ = all triplets containing C or SY.

14 a) {(1, 1, 1), (1, 1, 0), (0, 1, 0), …}
b) X = {(1, 1, 0), (1, 0, 1), (0, 1, 1)}
c) Y = {(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1)}
d) Z = {(1, 1, 1), (1, 1, 0), (1, 0, 1)}
e) Z' = {(0, 1, 1), (0, 1, 0), (0, 0, 1), (0, 0, 0), (1, 0, 0)}
X ∪ Z = {(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)}
X ∩ Z = { (1, 1, 0), (1, 0, 1)}
Y ∪ Z = {(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)}
Y ∩ Z = { (1, 1, 1), (1, 1, 0), (1, 0, 1)}

15 a) {1, 2, 31, 32, 41, 42, 51, 52, 341, 342, …, 3452}
b) {31, 32, 41, 42, 51, 52}
c) All except {1, 2}
d) {1, 31, 41, 51, 341, 351, 431, 451, 531, 541, 3451, 4351, …}

Exercise 12.3

1 a) $\frac{3}{10}$ b) $\frac{3}{4}$

2 a) 0.63 b) 1

3 a) (i) $\frac{1}{52}$ (ii) $\frac{7}{26}$ (iii) $\frac{4}{13}$ (iv) $\frac{10}{13}$
b) (i) $\frac{1}{51}$ (ii) $\frac{13}{17}$
c) (i) $\frac{1}{52}$ (ii) $\frac{10}{13}$

4 a) $\frac{4}{5}$ b) $\frac{11}{30}$ c) 1

5 a) $\frac{1}{2}$ b) $\frac{1}{12}$

6 a) $\frac{1}{7}$ b) $\frac{4}{7}$

7 a) (i) {(1, 1), (1, 2), …, (6, 6)}
(ii) $\frac{1}{6}$ (iii) $\frac{2}{9}$ (iv) $\frac{5}{6}$
b) (i) 0 (ii) $\frac{1}{9}$ (iii) $\frac{5}{36}$ (iv) 0

8 a) 0.04 b) 0.55 c) 0.1548
d) 0.060 372 e) 0.104 022

9 a) Yes b) no c) no

10 a) 0.06 b) 0.42 c) 0.3364 d) 0.412

11 a) 0.183 b) 0.69

12 $x > \frac{n-1}{2}$ **13** a) $n = 20$ b) $n = 12$

14 a) $\frac{5}{18}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$

15 a) $\frac{1}{3}$ b) $\frac{5}{14}$

16 a) $\frac{3243}{10\,829} \approx 0.299$ b) $\frac{143}{39\,984} \approx 0.0036$

17 a) $\frac{144}{3553} \approx 0.0405$ b) $\frac{943}{2261} \approx 0.417$

18 a) $\frac{1}{91}$ b) $\frac{4}{91}$

19 a) $\frac{78}{253} \approx 0.308$ b) $\frac{576}{1265} \approx 0.455$

20 a) 593 775 b) $\frac{608}{2639} \approx 0.230$
c) $\frac{2426}{7917} \approx 0.306$ d) $\frac{1045}{1131} \approx 0.924$

21 a) $\frac{10\,005}{29\,900\,492} \approx 0.000\,33$ b) $\frac{269\,265}{777\,412\,792} \approx 0.000\,35$
c) $\frac{777\,143\,527}{777\,412\,792} \approx 0.9997$ d) $\frac{85\,266\,221}{777\,412\,792} \approx 0.1097$

22 a) $\frac{3}{28} \approx 0.107$ b) $\frac{17}{190} \approx 0.0895$
c) $\frac{153}{190} \approx 0.805$

23 $\frac{7}{16} \approx 0.4375$ **24** $\frac{111}{400} \approx 0.2775$

25 a) $\frac{264}{1885} \approx 0.140$ b) $\frac{166}{84\,825} \approx 0.001\,96$
c) $\frac{2584}{395\,85} \approx 0.0653$

26 a) 0.096 b) 0.008 c) 0.512

Exercise 12.4

1 $\frac{7}{20}$

2 a) $\frac{5}{10}$ b) $\frac{4}{10}$ c) $\frac{2}{10}$ d) $\frac{1}{10}$ e) $\frac{2}{5}$

3 $P(A \cap B) = \frac{1}{9} \neq 0 \neq P(A)P(B)$

4 $\frac{29}{35}$

5 0.90

6 a) 92%
b) (i) 0.64% (ii) 15.36% (iii) 14.72%
c) 48.68%

7 a) 10 000 b) $\frac{9}{10}$ c) 0.3439 d) $\frac{1000}{3439}$

8 a) $\frac{15}{16}$ b) $\frac{4}{5}$ c) $\frac{1}{5}$

9 a) {(1, 1), (1, 2), …, (6, 6)}
b)

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

c) $\frac{11}{36}$ d) $\frac{11}{12}$ e) $\frac{1}{3}$ f) $\frac{2}{3}$

10 a) $\frac{7}{15}$ b) $\frac{11}{75}$ c) $\frac{9}{35}$
d) $\frac{46}{75}$ e) $\frac{11}{20}$
f) No: P(female) ≠ P(female/grade 12) − for example

11 a) (i) 0.56 (ii) 0.15
b) $\frac{15}{56}$ c) no

12

P(A)	P(B)	Conditions for events A and B	P($A \cap B$)	P($A \cup B$)	P($A\|B$)
0.3	0.4	Mutually exclusive	**0.00**	**0.7**	**0.00**
0.3	0.4	Independent	**0.12**	**0.58**	**0.30**
0.1	0.5	Mutually exclusive	**0.00**	0.60	**0.00**
0.2	0.5	Independent	0.10	**0.60**	**0.20**

13 a) 0.30 b) yes

14 a) 65% b) 35% c) 52%

15 a) 0.56 b) 0.10

16 a) $\frac{1}{216}$ b) $\frac{91}{216}$ c) $\frac{75}{216}$

17 a) 0.21 b) 0.441 c) 0.657

18 a) $\frac{23}{144}$ b) $\frac{11}{144}$ c) $\frac{15}{144}\left(\text{or } \frac{5}{48}\right)$ d) $\frac{9}{23}$

19 a) $A \cap B = \{(10,5),(10,4),\ldots,(10,1),(1,10),\ldots,(5,10)\}$, P = 0.069

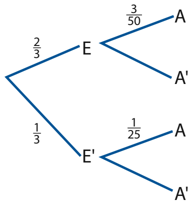
b) $A \cup B = \{(1,12), ..., (1,1), (2,12), ..., (3,12), ..., (4,11), ..., (5,10), ...\}$,
P = 0.778

c) list, P = 0.931 d) list, P = 0.222
e) same as c) f) same as d)
g) This is $(A \cup B) - (A \cap B)$; P = 0.709

20 Proof
21 a) $\frac{1}{36}$ b) $\frac{1}{28}$ c) $\frac{91}{216}$ d) 0.5
22 $\frac{2}{3}$
23 a) 0.103 b) 0.0887 c) 0.537
24 a) 0.10 b) 0.000 01
25 a) $\frac{21}{22} \approx 0.955$ b) $\frac{23}{66} \approx 0.348$ c) $\frac{13}{63} \approx 0.206$
26 a) 0.36 b) 0.64 c) 0.75 d) 0.17
e) 0.0455 f) 0.682
27 a) 0.8805 b) 0.0471

Exercise 12.5

1 a)



b) $\frac{4}{75}$ c) $\frac{3}{4}$
2 a) 0.060 1872 b) 0.003 37
3 a) 0.52 b) 0.692
4 a) $\frac{1}{3}$ b) $\frac{1}{2}$
5 a) 0.055 b) 0.444
6 0.875
7 a) 85.5% b) 10.5%
8 a) 0.64 b) 0.703
9 a) 0.7 b) 0.50
10 0.915
11 a) 3.6% b) 66.7% c) 5.37% d) 4.24%
12 0.382 **13** Antonio
14 a) 0.1445 b) 0.000 58
15 a) 0.425 b) 0.176
16 a) 0.93 b) 0.108
17 a) P(F) = 0.4, P(F ∩ T) = 0.224, P(F ∪ A') = 0.944,
P(F'|A) = 0.35
b) M is mutually exclusive with F. T is independent as
P(F ∩ T) = P(T) P(F), …
c) (i) 0.716 (ii) 0.704
18 a) 0.012 b) 0.030 c) 0.40

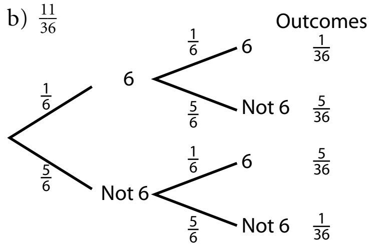
Practice questions

1 a) 0.30 b) 0.72 c) 0.70
2 a) 0.0004 b) 0.9996 c) 0.0004
3 0.999 98
4 a) (i) 0.85 (ii) 0.80 (iii) 0.15 b) 0.083
5 a) (i) 0.3405 (ii) 0.0108 (iii) 0.9622 (iv) 0.30
b) Yes
6 a) 0.63 b) 0.971
7 a) 0.60 b) yes, $P(B|A) = P(B) = 0.60$ c) 0.42

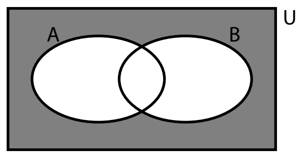
8 a)

	Boys	Girls
Passed the ski test	32	16
Failed the ski test	**14**	12
Training, but did not take the test yet	**20**	**16**
Too young to take the test	**4**	**6**

b) (i) 0.6167 (ii) 0.56 (iii) 0.1463
9 a) $\frac{3}{32}$ b) $\frac{3}{4}$ c) $\frac{5}{32}$
10 a) 0.02 b) 0.64
11 a) 0.4 b) 0.6
12 a) 0.38 b) 0.283
13 b) $\frac{11}{36}$



14 a)

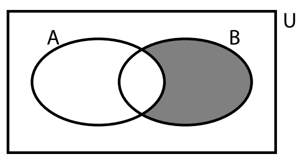


b) (i) 2 (ii) $\frac{1}{18}$ c) $n(A \cap B) \neq 0$

15 a)

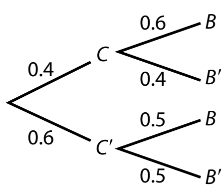
	Male	Female	Total
Unemployed	20	40	60
Employed	90	50	140
Total	110	90	200

b) (i) $\frac{1}{5}$ (ii) $\frac{9}{14}$
16 $\frac{44}{65}$
17 a)



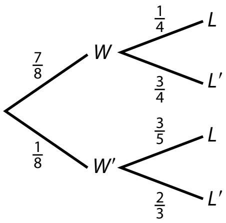
b) 35 c) 0.35

18 a)



b) 0.54 c) 0.444
19 a) $\frac{7}{12}$ b) $\frac{11}{36}$ c) $\frac{1}{3}$
20 a) $\frac{1}{11}$ b) $\frac{12}{121}$
21 a) Independent b) M c) N
22 a) $a = 21, b = 11, c = 17$
b) (i) $\frac{1}{8}$ (ii) $\frac{21}{32}$
c) (i) 0.253 (ii) 0.747
23 $\frac{31}{66}$

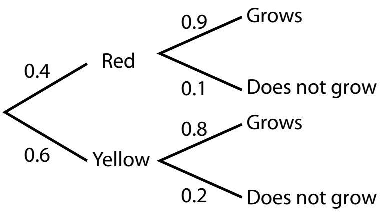
24 a)



b) $\frac{47}{160}$ c) $\frac{35}{47}$

25 a) $\frac{1}{3}$ b) $\frac{7}{12}$ c) $\frac{3}{7}$

26 a)



b) (i) 0.36 (ii) 0.84 (iii) 0.429

27 a) $\frac{1}{6}$ b) $\frac{1}{12}$ c) $\frac{2}{9}$

28 a) (i) $\frac{8}{21}$ (ii) $\frac{1}{6}$ (iii) no, $P(A \cap B) \neq P(A)P(B)$

b) $\frac{10}{17}$ c) $\frac{200}{399}$

29 $\frac{1}{3}$ **30** $\frac{4}{5}$ **31** 0.001 98 **32** $\frac{19}{30}$

33 0.80 **34** $\frac{10}{19}$ **35** a) $\frac{13}{20}$ b) $\frac{11}{15}$

36 $\frac{2}{5}$

37 a) (i) $\frac{5}{36}$ (ii) $\frac{25}{216}$ (iii) $\frac{1}{6}\left(\frac{5}{6}\right)^{2n-2}$

b) No answer required – proof

c) $\frac{5}{11}$ d) 0.432

38 a) 0.957 b) 0.301

39 $\frac{1}{9}$

40 a) 0.25 b) 0.083

41 a) 0.80 b) 0.56

42 a) 0.732 b) $\frac{11}{61}$

43 a) $\frac{2}{3}$ b) $\frac{2}{9}$ c) $\frac{3}{4}$

44 a) $\frac{1}{10}$ b) proof

c) $\frac{11}{90}$ d) $\frac{3}{11}$

45 $\frac{3}{7}$

Chapter 13

Exercise 13.1

1 4 **2** $3x^2$ **3** $2x$ **4** 6

5 0 **6** $\frac{5}{2}$ **7** d.n.e. (increases without bound)

8 $\frac{1}{8}$ **9** $\frac{3}{2}$ **10** $\frac{\sqrt{2}}{4}$ **11** $\frac{1}{4}$

12 1 **13** 3 **14** $\frac{1}{e}$

15 $\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$

16 As $x \to a$, $g(x) \to +\infty$

17 a) Horizontal: $y = 3$; vertical: $x = -1$

b) Horizontal: $y = 0$; vertical: $x = 2$

c) Horizontal: $y = b$; vertical: $x = a$

d) Horizontal: $y = 2$; vertical: $x = \pm 3$

e) Horizontal: $y = 0$; vertical: $x = 0$, $x = 5$

f) Horizontal: none; vertical: $x = 4$

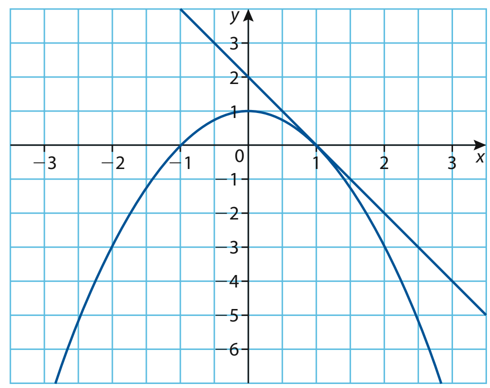
18 $\frac{1}{3}$ **19** 4

Exercise 13.2

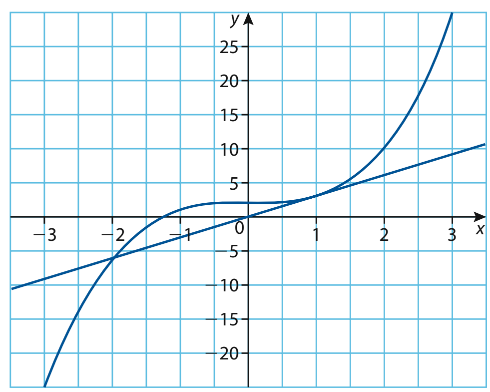
1 $f'(x) = -2x$ **2** $g'(x) = 3x^2$

3 $h'(x) = \frac{1}{2\sqrt{x}}$ **4** $r'(x) = -\frac{2}{x^3}$

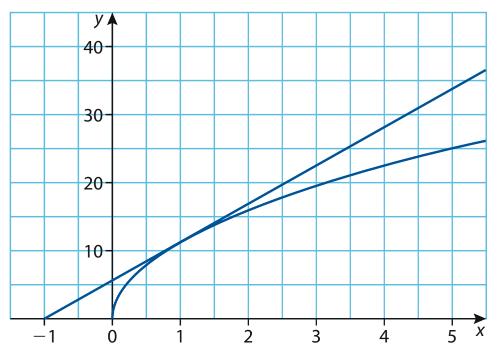
5 (i)



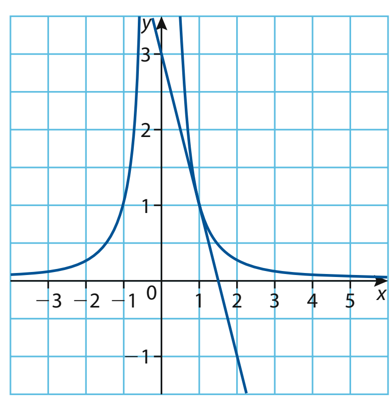
(ii)



(iii)



(iv)



6 a) $y' = 6x - 4$ b) -4

7 a) $y' = -2x - 6$ b) 0

8 a) $y' = -\frac{6}{x^4}$ b) -6

9 a) $y' = 5x^4 - 3x^2 - 1$ b) 1

10 a) $y' = 2x - 4$ b) 0

11 a) $y' = 2 - \frac{1}{x^2} + \frac{9}{x^4}$ b) 10

12 a) $y' = 1 - \frac{2}{x^3}$ b) 3

13 $a = -5, b = 2$ **14** $(0, 0)$

15 $(2, 8)$ and $(-2, -8)$ **16** $\left(\frac{5}{2}, -\frac{21}{4}\right)$

17 $(1, -2)$

18 a) Between A and B

b) Rate of change is positive at A, B and F;
rate of change is negative at D and E;
rate of change is zero at C

c) Pair B and D, and pair E and F

19 $a = 1, b = 5$ **20** $a = 1$ **21** $(3, 6)$

22 a) 12.61 b) 12 **23** $f'(x) = 2ax + b$

24 a) $4.\overline{6}$ degrees Celsius per hour

b) $C'(t) = 3\sqrt{t}$

c) $t = \frac{196}{81} \approx 2.42$ hours

25–26 Proof

27 $\frac{1}{2\sqrt{x}}$ **28** $-\frac{1}{x^2}$

29 $\frac{5}{(3-x)^2}$ $\left[\text{or } \frac{5}{(x-3)^2}\right]$ **30** $-\frac{1}{2\sqrt{(x+2)^3}}$

Exercise 13.3

1 $(1, -7)$ **2** $\left(-\frac{3}{2}, 8\right)$ **3** $(3, 2)$

4 a) $y' = 2x - 5$ b) increasing for $x > \frac{5}{2}$

c) decreasing for $x < \frac{5}{2}$

5 a) $y' = -6x - 4$ b) increasing for $x < -\frac{2}{3}$

c) decreasing for $x > -\frac{2}{3}$

6 a) $y' = x^2 - 1$ b) increasing for $x > 1, x < -1$

c) decreasing for $-1 < x < 1$

7 a) $y' = 4x^3 - 12x^2$ b) increasing for $x > 3$

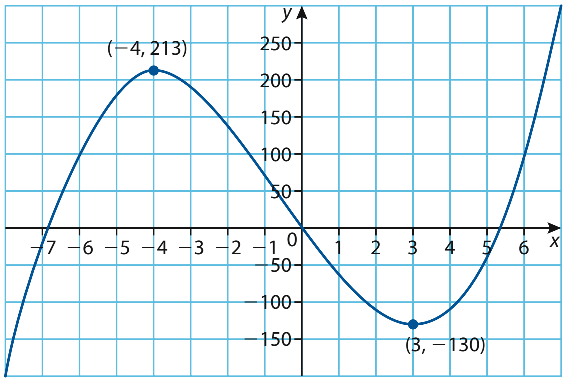
c) decreasing for $x < 0, 0 < x < 3$

8 a) $(3, -130), (-4, 213)$

b) $(3, -130)$ minimum because 2nd derivative is positive at $x = 3$

$(-4, 213)$ maximum because 2nd derivative is negative at $x = -4$

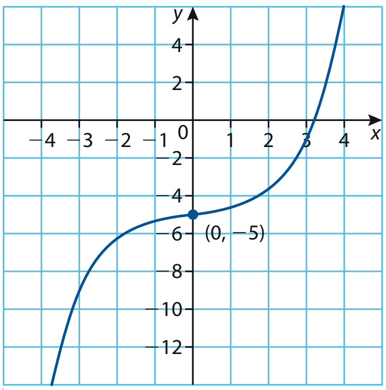
c)



9 a) $(0, -5)$

b) Stationary point is neither a maximum nor minimum because 1st derivative is always positive.

c)

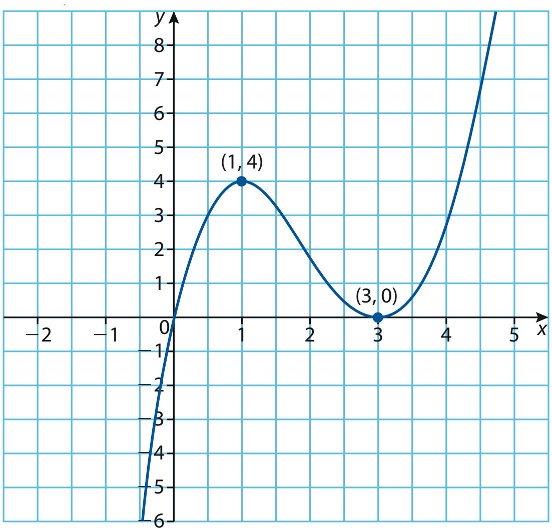


10 a) $(1, 4), (3, 0)$

b) $(1, 4)$ maximum because 2nd derivative is negative at $x = 1$

$(3, 0)$ minimum because 2nd derivative is positive at $x = 3$

c)

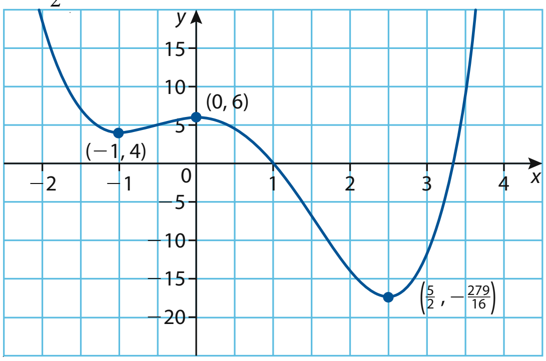


11 a) $(-1, 4), (0, 6), \left(\frac{5}{2}, -\frac{279}{16}\right)$

b) $(-1, 4)$ minimum because 2nd derivative is positive at $x = -1$

$(0, 6)$ maximum because 2nd derivative is negative at $x = 0$

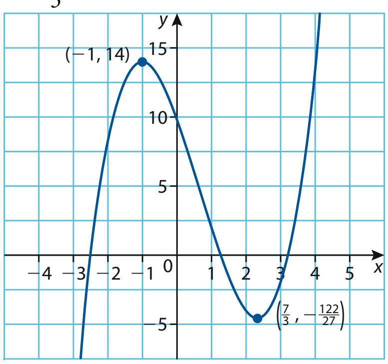
at $\left(\frac{5}{2}, -\frac{279}{16}\right)$ minimum because 2nd derivative is positive at $x = \frac{5}{2}$

c) 

12 a) $(-1, 14), \left(\frac{7}{3}, -\frac{122}{27}\right)$

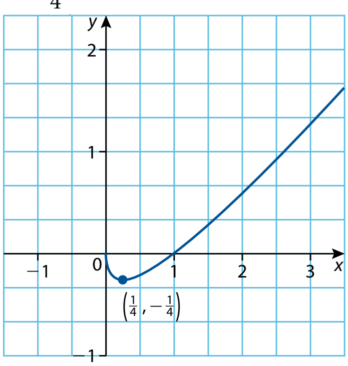
b) $(-1, 14)$ maximum because 2nd derivative is negative at $x = -1$

$\left(\frac{7}{3}, -\frac{122}{27}\right)$ minimum because 2nd derivative is positive at $x = \frac{7}{3}$

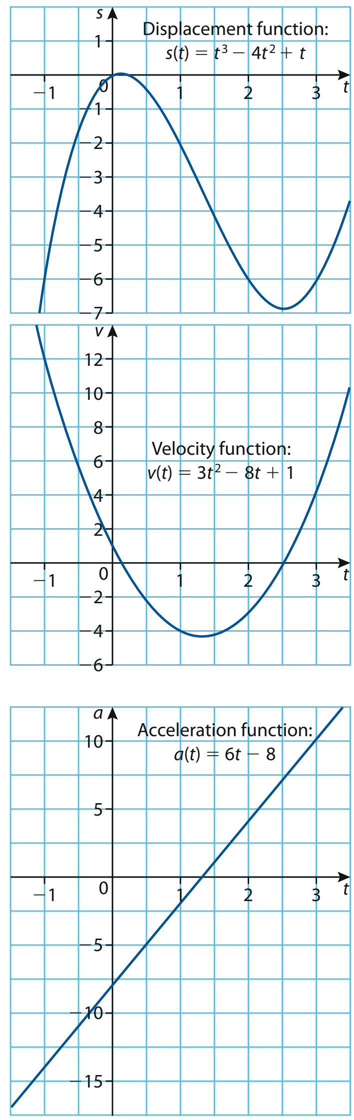
c) 

13 a) $\left(\frac{1}{4}, -\frac{1}{4}\right)$

b) $\left(\frac{1}{4}, -\frac{1}{4}\right)$ minimum because 2nd derivative is positive at $x = \frac{1}{4}$

c) 

14 a) $v(t) = 3t^2 - 8t + 1; \quad a(t) = 6t - 8$

b) 

c) $t \approx 0.131$, displacement ≈ 0.0646

d) $t = 1.\overline{3}$, displacement $= -4.\overline{3}$

e) Object moves right at a decreasing velocity then turns left with increasing velocity then slowing down and turning right with increasing velocity.

15 Relative maximum at $(-2, 16)$; relative minimum at $(2, 16)$; inflexion point at $(0, 0)$

16 Absolute minima at $(-2, -4)$ and $(2, -4)$; relative maximum at $(0, 0)$; inflexion points at $\left(-\frac{2\sqrt{3}}{3}, -\frac{20}{9}\right)$ and $\left(\frac{2\sqrt{3}}{3}, -\frac{20}{9}\right)$

17 Relative maximum at $(-2, -4)$; relative minimum at $(2, 4)$; no inflexion points

18 Relative minimum at $\left(-\frac{\sqrt[3]{4}}{2}, \frac{3\sqrt[3]{2}}{2}\right)$; inflexion point at $(1, 0)$

19 Relative minimum at $(-1, -2)$; relative maximum at $(1, 2)$; inflexion points at $\left(-\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{8}\right)$, $(0, 0)$ and $\left(\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{8}\right)$

20 Relative minimum at $(-1, 0)$; absolute minimum at $(2, -27)$; relative maximum at $(0, 5)$; inflexion points at $(1.22, -13.4)$ and $(-0.549, 2.32)$

21 a) $v(0) = 27 \text{ m s}^{-1}$, $a(0) = -66 \text{ m s}^{-2}$

b) $v(3) = 45 \text{ m s}^{-1}$, $a(3) = 78 \text{ m s}^{-2}$

c) $t = \frac{1}{2}$ and $t = 2\frac{1}{4}$; where displacement has a relative maximum or minimum

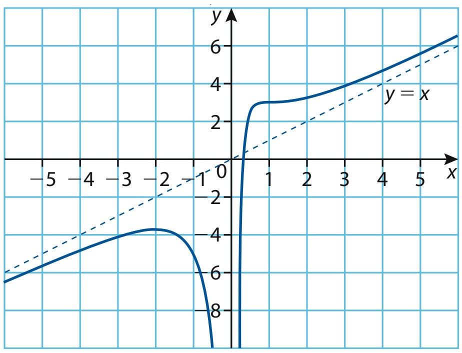
d) $t = \frac{11}{8} = 1.375$; where acceleration is zero

22 $x \approx 5.77$ tonnes; $D \approx 34.6$ (\$34 600); this cost is a minimum because cost decreases to this value then increases

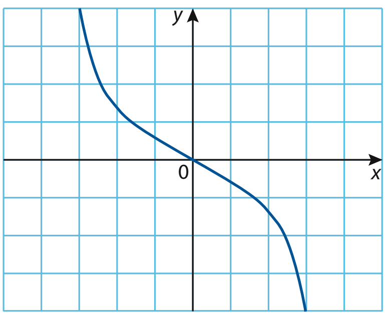
23 $a - 3,\ b = 4,\ c = -2$

24 Relative maximum at $\left(-2, -\frac{15}{4}\right)$, stationary inflexion point at $(1, 3)$

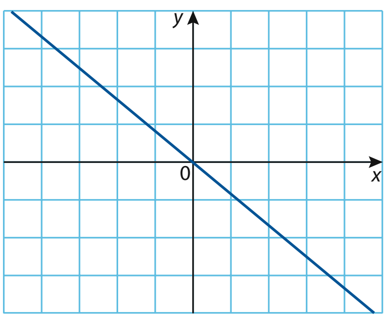
$f(x) \to x$ as $x \to \pm\infty$



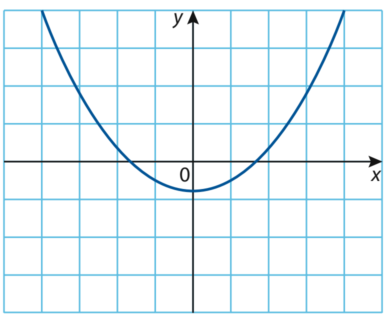
25 a)



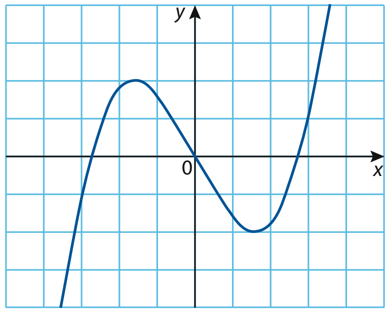
b)



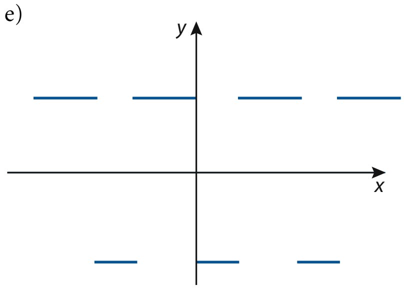
c)



d)



e)



26 a) Increasing on $1 < x < 5$; decreasing on $x < 1,\ x > 5$

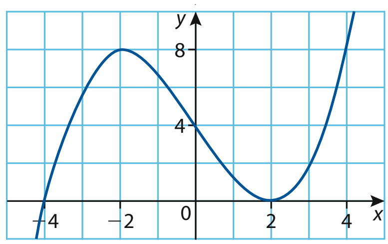
b) Minimum at $x = 1$; maximum at $x = 5$

27 a) Increasing on $0 \leqslant x < 1,\ 3 < x < 5$; decreasing on $1 < x < 3,\ x > 5$

b) Minimum at $x = 3$; maximum at $x = 1$ and $x = 5$

28 $x \approx 0.5$ and $x \approx 7.5$

29



30 a) Right $1 < t < 4$; left $t < 1,\ t > 4$

b) $v_0 = -24,\ a_0 = 30$

c) $d_{max} = 16$ at $t = 4$, $v_{max} = 13.5$ at $t = 2.5$

d) Velocity is maximum at $t = 2.5$

31 a) Maximum at $x \approx 6.50$, minimum at $x \approx -0.215$

b) Maximum is $\frac{7\pi}{4} + 1$, minimum is $\frac{\pi}{4} - 1$

Exercise 13.4

1 a) $y = -4x - 8$ b) $y = \frac{4}{27}$

c) $y = -x + 1$ d) $y = -2x + 4$

2 a) $y = \frac{1}{4}x + \frac{19}{4}$ b) $x = -\frac{2}{3}$

c) $y = x + 1$ d) $y = \frac{1}{2}x + \frac{11}{4}$

3 At $(0, 0)$: $y = 2x$; at $(1, 0)$: $y = -x + 1$; at $(2, 0)$: $y = 2x - 4$

4 $y = -2x$

5 a) $x = 1$

b) For $y = x^2 - 6x + 20$, eq. of tangent is $y = -4x + 19$

For $y = x^3 - 3x^2 - x$, eq. of tangent is $y = -4x + 1$

6 Normal: $y = \frac{1}{2}x - \frac{7}{2}$; intersection pt: $\left(-\frac{1}{2}, -\frac{15}{4}\right)$

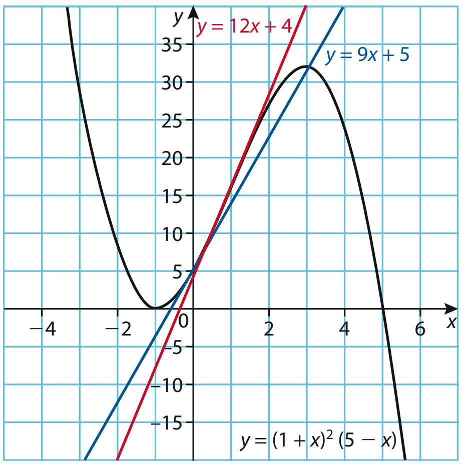
7 Eq. of tangent: $y = -3x + 3$; eq. of normal: $y = \frac{1}{3}x - \frac{1}{3}$

8 $a = 4,\ b = -7$

9 a) $y = 2x + \frac{5}{2}$ b) $\left(\frac{2}{3}, \frac{41}{27}\right)$

10 Eq. of tangent: $y = -\frac{3}{4}x + 1$; eq. of normal: $y = \frac{4}{3}x - \frac{22}{3}$

11



12 $y = 11x - 25$ and $y = -x - 1$

13 $y = \left(2\sqrt{2} - 2\right)x$ and $y = -\left(2\sqrt{2} + 2\right)x$

14 a) $y = \frac{1}{12}x + \frac{4}{3}$ b) $\sqrt[3]{9} \approx 2.08$

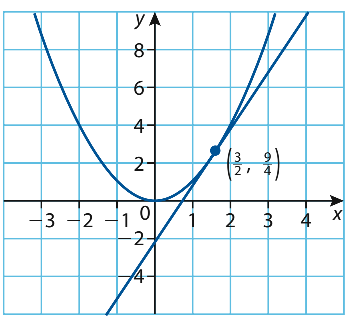
15 $y = -\frac{1}{2\sqrt{a^3}}x + \frac{3}{2\sqrt{a}}$

16 $x_Q = -2x_P,\ y_Q = -8y_P$

Practice questions

1 a) Gradient = 3 b) $y = 3x - \frac{9}{4}$

c)



d) $Q\left(\frac{3}{4}, 0\right)$, $R\left(0, -\frac{9}{4}\right)$

f) $y = 2ax - a^2$

g) $T\left(\frac{a}{2}, 0\right)$, $U\left(0, -a^2\right)$

h) x-coord.: $\frac{a+0}{2} = \frac{a}{2}$; y-coord.: $\frac{a^2 - a^2}{2} = 0$

2 $A = 1,\ B = 2,\ C = 1$

3 a) $4x - 15x^4$

b) $-\frac{1}{x^2}$

4 a) $x = 2$ or -2; $f'(1) = -6 < 0$ (decreasing) and $f'(3) = \frac{10}{9} > 0$ (increasing) $\therefore f(2)$ is a turning point

b) vertical asymptote: $x = 0$ (y-axis); oblique asymptote: $y = 2x$

5 $\left(\frac{1}{2}, 3\right)$

6 $a = 1$

7 a) $y = 5x - 7$

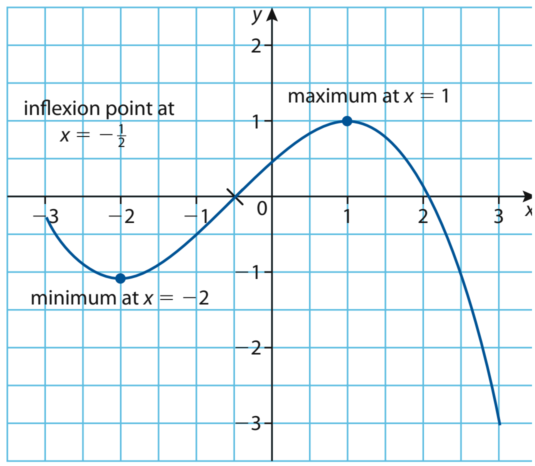
b) $y = -\frac{1}{5}x + \frac{17}{5}$

8 a) $x = 1$

b) $-3 < x < -2,\ 1 < x < 3$

c) $x = -\frac{1}{2}$

d)



9 $b = 2,\ c = 3$

10

function	diagram
f_1	d
f_2	e
f_3	b
f_4	a

11 a) $\frac{2}{\pi}$ b) $\frac{\sqrt{2}}{2}$ c) $x \approx 0.881$

12 a) (i) $x = 0$ (ii) $y = 3$

b) $\frac{dy}{dx} = \frac{2}{x^2}$

c) Increasing for all x, except $x = 0$

d) No stationary points because $\frac{dy}{dx} = \frac{2}{x^2} \neq 0$

13 Maximum at $(-1, 1)$, minimum at $(0, 0)$, maximum at $(1, 1)$

14 $a = \frac{8}{3},\ b = \frac{16}{5}$

15 a) 10 m s^{-1} b) 10 sec c) 50 metres

16 a) $v = 14 - 9.8t$

b) $t \approx 1.43$ sec

c) Velocity = 0, acceleration = -9.8 m s^{-2}

17 $(-4, 120)$

18 a) $y = \left(-\sqrt{3}\right)x + \frac{\pi\sqrt{3}}{3} - 2$

b) $y = \left(\frac{\sqrt{3}}{3}\right)x - \frac{\pi\sqrt{3}}{9} - 2$

19 a) $h = \frac{27 - r^2}{r}$; $V = \pi r\left(27 - r^2\right)$

b) $r = 3$

20 $a=-2$, $b=8$, $c=10$

21 a) $y=-7x+1$

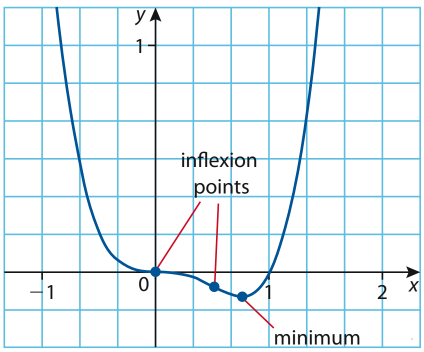
b) $y=\dfrac{x}{7}+\dfrac{107}{7}$

22 a) Absolute minimum at $\left(\dfrac{3}{4},-\dfrac{27}{256}\right)$

b) Domain: $x\in\mathbb{R}$, range: $y\geqslant-\dfrac{27}{256}$

c) Inflexion points at $(0,0)$ and $\left(\dfrac{1}{2},-\dfrac{1}{16}\right)$

d)



23 a) $-\dfrac{5}{3}$ b) $\dfrac{1}{4}$ c) 3 d) $\dfrac{1}{2\sqrt{x+2}}$

24 a) $f'(x)=\dfrac{3x-4}{2\sqrt{x}}$

b) $f'(x)=3x^2-3\cos x$

c) $f'(x)=-\dfrac{1}{x^2}+\dfrac{1}{2}$

d) $f'(x)=-\dfrac{91}{3x^{14}}$

25 3 solutions: $\left(\dfrac{11}{2},\dfrac{1105}{8}\right)$, $(2,-15)$, and $(-2,5)$

26 $\dfrac{17}{2}$

27 $\left(2,\dfrac{2}{3}\right)$, $\left(-2,-\dfrac{2}{3}\right)$

28 $(-1,-2)$

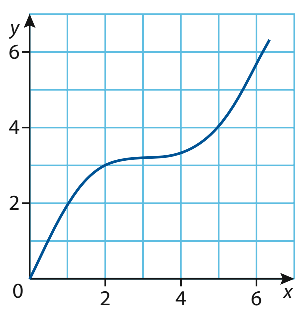
30 $(2,20)$, $(4,16)$

31 a) particle does not change direction for $0\leqslant t\leqslant 2\pi$

b) $v=1+\cos t\geqslant 0$ for $0\leqslant t\leqslant 2\pi$

c) $t=0,\ \pi,\ 2\pi$

d) Maximum value of s is 2π



32 $a=\dfrac{1}{4}$, $b=\dfrac{3}{4}$, $c=-6$, $d=-\dfrac{5}{2}$; y-coord. is $-\dfrac{19}{2}$

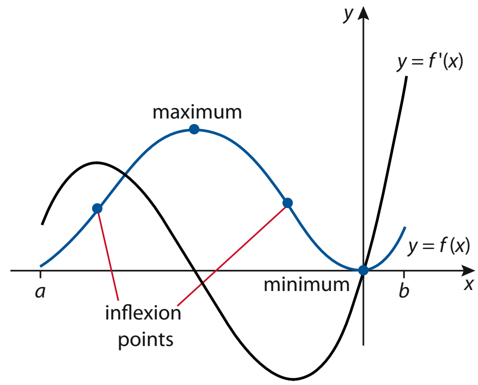
33 Absolute minimum points at $\left(-2,-\dfrac{1}{8}\right)$ and $\left(2,-\dfrac{1}{8}\right)$

34 a) $y=-x+2$ b) $y=-x+\dfrac{\pi}{2}$

35 b) $y=2x$, $(1,2)$

36 a) $v=50-20t$ b) $s=1062.5$ m

37



Chapter 14

Exercise 14.1

1 a) $\left(\frac{5}{2},-2,0\right)$

b) $(3,2\sqrt{3},0)$

c) $(-1,2,-2)$

d) $(a,-4a,-a)$

2 a) $Q\left(-\frac{1}{2},-3,2\right)$

b) $P\left(\frac{5}{2},-2,0\right)$

c) $Q(0,-4a,3a)$

3 a) $(x,y,z)=(t,t,5-5t)$, or $(x,y,z)=(1+t,1+t,-5t)$

b) $(x,y,z)=(-1+4t,5t,1-3t)$

c) $(x,y,z)=(2-4t,3-6t,4+t)$

4 a) $C(7,-8,-1)$

b) $C\left(-1,\frac{11}{2},\frac{29}{3}\right)$

c) $C(2-a,4-2a,-b-2)$

5 a) $\left(-\frac{1}{3},1,\frac{1}{3}\right)$

b) $\left(1,-\frac{5}{3},-1\right)$

c) $\left(\dfrac{a+b+c}{3},\dfrac{2a+2b+2c}{3},a+b+c\right)$

6 a) $D(-1,1,-6)$

b) $D(-2\sqrt{2},2\sqrt{3},1-4\sqrt{5})$

c) $D\left(\frac{5}{2},-\frac{2}{3},-4\right)$

7 $m=5$, $n=1$

8 a) $\mathbf{v}=\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$

b) $\mathbf{v}=\dfrac{3}{\sqrt{14}}\mathbf{i}-\dfrac{2}{\sqrt{14}}\mathbf{j}+\dfrac{1}{\sqrt{14}}\mathbf{k}$

c) $\mathbf{v}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$

9 a) $\frac{2}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

b) $\dfrac{2}{\sqrt{14}}(6\mathbf{i}-4\mathbf{j}+2\mathbf{k})$

c) $\dfrac{5}{3}(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$

10 a) $|\mathbf{u}+\mathbf{v}|=\sqrt{29}$

b) $|\mathbf{u}|+|\mathbf{v}|=\sqrt{14}+\sqrt{5}$

c) $|-3\mathbf{u}|+|3\mathbf{v}|=3\sqrt{14}+3\sqrt{5}$

d) $\dfrac{1}{|\mathbf{u}|}\mathbf{u}=\dfrac{\mathbf{i}}{\sqrt{14}}+\dfrac{3\mathbf{j}}{\sqrt{14}}-\dfrac{2\mathbf{k}}{\sqrt{14}}$

e) $\left|\dfrac{1}{|\mathbf{u}|}\mathbf{u}\right|=1$

11 a) $(3,4,-5)$ b) $(0,-2,5)$

12 a) $(1, -\frac{4}{3})$ b) $\sqrt{6}(4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ c) $-\frac{2}{3}\mathbf{i} + \frac{8}{3}\mathbf{j} - 2\mathbf{k}$

13 0 14 $\pm\frac{\sqrt{14}}{14}$ 15 None 16 None

17 a) $\mathbf{a} = (8, 0, 0)$, $\mathbf{b} = (8, 8, 0)$, $\mathbf{c} = (0, 8, 0)$, $\mathbf{d} = (0, 0, 8)$, $\mathbf{e} = (8, 0, 8)$, $\mathbf{f} = (8, 8, 8)$
b) $\mathbf{l} = (8, 4, 8)$, $\mathbf{m} = (4, 8, 8)$, $\mathbf{n} = (8, 8, 4)$
c) proof

18 a) $\mathbf{c} = (8, 0, 12)$, $\mathbf{d} = (0, 10, 12)$
b) $\mathbf{f} = (4, 5, 0)$, $\mathbf{g} = (4, 5, 12)$
c) $\overrightarrow{AG} = (-4, 5, 12) = \overrightarrow{FD}$

19 $\pm\frac{\sqrt{6}}{3}$

20 $(\alpha, \beta, \mu) = \left(\frac{31}{7}, -\frac{15}{7}, \frac{6}{7}\right)$ 21 $(\alpha, \beta, \mu) = (2, -1, 3)$

22 Not possible 23 Rectangle

24 $T_1 = 125\left(\sqrt{3} - 1\right)$ N; $T_2 = 175\left(\frac{3\sqrt{2} - \sqrt{6}}{2}\right)$ N

25 $T_1 = 150$ N; $T_2 = 150\sqrt{3}$ N

Exercise 14.2

1 a) $-16, 117.65°$ b) $-20, 64.68°$ c) $13, 40.24°$
d) $-15, 151.74°$ e) $6, 60°$ f) $-6, 120°$

2 a) Orthogonal b) acute c) orthogonal

3 a) $\mathbf{v} \cdot \mathbf{u} = 0 = \mathbf{wu}$ b) $\frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{j}, \frac{-3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$

4 a) (i) $\cos\alpha = \frac{2}{\sqrt{14}}, \cos\beta = \frac{-3}{\sqrt{14}}, \cos\gamma = \frac{1}{\sqrt{14}}$
(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{2^2}{14} + \frac{(-3)^2}{14} + \frac{1^2}{14} = 1$
(iii) $\alpha \approx 58°, \beta \approx 143°, \gamma \approx 74°$
b) (i) $\cos a = \frac{1}{\sqrt{6}}, \cos\beta = \frac{-2}{\sqrt{6}}, \cos\gamma = \frac{1}{\sqrt{6}}$
(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1^2}{6} + \frac{2^2}{6} + \frac{1^2}{6} = 1$
(iii) $\alpha \approx 66°, \beta \approx 145°, \gamma \approx 66°$
c) (i) $\cos\alpha = \frac{3}{\sqrt{14}}, \cos\beta = \frac{-2}{\sqrt{14}}, \cos\gamma = \frac{1}{\sqrt{14}}$
(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3^2}{14} + \frac{(-2)^2}{14} + \frac{1^2}{14}$ $= 1$
(iii) $\alpha \approx 37°, \beta \approx 122°, \gamma \approx 74°$
d) (i) $\cos\alpha = \frac{3}{5}, \cos\beta = 0, \cos\gamma = \frac{-4}{5}$
(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3^2}{25} + \frac{0^2}{25} + \frac{4^2}{25} = 1$
(iii) $\alpha \approx 53°, \beta \approx 90°, \gamma \approx 143°$

5 $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix}$ 6 $\begin{pmatrix} \frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \\ 0 \end{pmatrix}$

7 a) $m = -\frac{9}{8}$ b) $m = 1$ or $-\frac{1}{4}$

8 $m = -14$

9 a) 127° b) 63° c) 73°

10 a) $m = \frac{1}{3}$ b) $m = -\frac{1}{4}$

11 m_A: $\mathbf{r} = (4, -2, -1) + m(-1, 0, \frac{3}{2})$;
m_B: $\mathbf{r} = (3, -5, -1) + n(\frac{1}{2}, \frac{9}{2}, \frac{3}{2})$
m_C: $\mathbf{r} = (3, 1, 2) + k(\frac{1}{2}, -\frac{9}{2}, -3)$; centroid $(\frac{10}{3}, -2, 0)$

12 90, 90, 82, 74, 60, 54, 53, 52, 47, 43, 38, 37

13 68.22

14 103.3°, 133.5°, 46.5°

15 0

16 $k = 2$

17 $k = 0$ or $k = 4$

18 $x = -20, y = -14$

19 $x = 5$

20 117°, $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$, 33°

21 a) $b = -\frac{1}{2}$ b) $b = 0$ or $b = \frac{1}{2}$
c) $b = \frac{5}{2}$ or $b = 3$ d) $b = \pm 4$

22 a) $b = -\frac{1}{2}$ b) $b = \frac{1}{2}$

23 $(-140.8, 140.8, 18)$ 24 $t = 2$

25 $t = -\frac{1}{2}$ 26 $t = 0$ or $t = \frac{1}{2}$

27 90° or $\cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$ 28 Proof

29 $m = \frac{7}{4}, n = -\frac{1}{4}$ 30 Proof

31 $\frac{\pi}{3}, -\frac{2\pi}{3}$ 32 $\cos^{-1}\left(\pm\frac{\sqrt{3}}{3}\right)$

33 $\pi - \alpha, \pi - \beta, \pi - y$ 34 $\mathbf{k}(8\mathbf{i} + \mathbf{j} - 10\mathbf{k})$

Exercise 14.3

1 a) $\mathbf{k} - \mathbf{j}$ b) same

2 a) $\mathbf{i} - \mathbf{k}$ b) same

3 a) $\mathbf{j} - \mathbf{i}$ b) same

4 Proof 5 (13, 0, 13) 6 $6\mathbf{i} - 8\mathbf{j} - 8\mathbf{k}$

7 $\begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix}$ 8 $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

9 a) $-2m^2 + 9m - 11$ b) $-2m^2 + 9m - 11$
c) $-2m^2 + 9m - 11$

10 a) $(-40, -115, 30)$ b) $(-150, 60, 0)$
c) $(-80, -160, -640)$
d) $(80, 160, 640)$ e) $(-40, -115, 30)$
f) $(-150, 60, 0)$

11 $\frac{\sqrt{1774}}{1774}\begin{pmatrix} 19 \\ 33 \\ -18 \end{pmatrix}$ 12 $\frac{\sqrt{6}}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ 13 $\sqrt{209}$

14 $\sqrt{139}$ 15 $2\sqrt{43}$ 16 Proof

17 $m = 1$ or $m = \frac{11}{4}$ 18 $\frac{\sqrt{374}}{2}$ 19 $5\sqrt{29}$

20 128 21 21 22 1 23 78

24 63 25 No 26 Yes 27 $-2, \frac{6}{5}$

28 Not possible

29 a) 49 b) $7\sqrt{5}$ c) $\frac{7\sqrt{5}}{5}$ d) $\cos^{-1}\left(\frac{7\sqrt{10}}{30}\right)$

30 a) $\frac{49}{3}$, V(tetrahedron) $= \frac{1}{3}$ V(parallelepiped) b) $\frac{4}{3}$

31 45° 32 Proof 33 Proof

34 a) $\sqrt{\frac{564}{29}}$ b) $\frac{6\sqrt{5}}{5}$ c) $\sqrt{\frac{3}{2}}$

35 $2(\mathbf{u} \times \mathbf{v})$ 36 $23(\mathbf{u} \times \mathbf{v})$ 37 $(mp + nq)(\mathbf{u} \times \mathbf{v})$

38 a) $o = \frac{1}{2}\left(\sqrt{(ab)^2 + (ac)^2 + (bc)^2}\right)$
b) $a = \frac{1}{2}ab; b = \frac{1}{2}bc; c = \frac{1}{2}ac$
c) result obvious

39 $\begin{pmatrix} 5t - \frac{1}{3} \\ -t + \frac{2}{3} \\ 3t \end{pmatrix}$

40 Not possible

Exercise 14.4

1 a) $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1+t \\ 5t \\ 2-4t \end{pmatrix}$

b) $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+2t \\ -1+5t \\ 2-t \end{pmatrix}$

c) $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+3t \\ -2+5t \\ 6-11t \end{pmatrix}$

2 a) $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$ b) $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$

c) $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$

3 a) $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ b) $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

4 $2x + 3y = 7$

5 $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$

6 $\mathbf{r} = (-2, 1, 4) + t(3, -4, 7)$

7 a) $(1, -1, 2)$ b) $(-17, -1, 1)$
c) No d) No

8 a) $\mathbf{r} = (2, -1) + t(1, 3)$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+t \\ -1+3t \end{pmatrix}$

b) $\mathbf{r} = (2, -1) + t(-3, 7)$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-3t \\ -1+7t \end{pmatrix}$

c) $\mathbf{r} = (2, -1) + t(7, 3)$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+7t \\ -1+3t \end{pmatrix}$

d) $\mathbf{r} = (0, 2) + t(2, -4)$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ 2-4t \end{pmatrix}$

9 a) $t = \frac{3}{2}$ b) no c) $m = \frac{7}{2}$

10 a) (i) $(3, -4)$ (ii) $(7, 24)$ (iii) 25
b) (i) $(-3, 1)$ (ii) $(5, -12)$ (iii) 13
c) (i) $(5, -2)$ (ii) $(24, -7)$ (iii) 25

11 a) $(-96, 128)$ b) $\left(\frac{2040}{13}, -\frac{850}{13}\right)$

12 a) $(24, 18)$
b) $\mathbf{r} = (3, 2) + t(24, 18)$
c) In 10 minutes

13 a) $a = -3, b = -5$
b) $-\frac{\sqrt{21}}{6}$
c) $\frac{\sqrt{15}}{6}, \frac{\sqrt{35}}{2}$

14 a) 146.8° b) 3.87
c) (i) L_1: $\mathbf{r} = (2, -1, 0) + t(0, 1, 2)$; L_2: $\mathbf{r} = (-1, 1, 1) + t(1, -3, -2)$

15 a) $(x, y, z) = (1 + t, 3 - 2t, -17 + 5t)$
b) $(4, -3, -2)$

16 a) $\mathbf{r} = \left(\frac{p}{m}, 0\right) + t(n, -m)$
b) (i) $bx - ay = bx_0 - ay_0$ (ii) slope $= \frac{b}{a}$

17 (i) $\mathbf{r} = (t, t, 3t), 0 \leqslant t \leqslant 1$
(ii) $\mathbf{r} = (2t - 1, t, 1 - 3t), 0 \leqslant t \leqslant 1$
(iii) $\mathbf{r} = (1 - t, 3t, t - 1), 0 \leqslant t \leqslant 1$

18 $\mathbf{r} = (2\mathbf{j} + 3\mathbf{k}) + 2t\mathbf{k}$
$\begin{cases} x = 0 \\ y = 2 \\ z = 3 + 2t \end{cases}$

19 $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$
$\begin{cases} 1 + 2t \\ 2 - 3t \\ -1 + t \end{cases}$

20 $\mathbf{r} = t(x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k})$
$\begin{cases} tx_0 \\ ty_0 \\ tz_0 \end{cases}$

21 a) $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t\mathbf{j}$
$\begin{cases} 3 \\ 2 + t \\ -3 \end{cases}$
b) $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t\mathbf{i}$
$\begin{cases} 3 + t \\ 2 \\ -3 \end{cases}$

22 $\frac{x - x_0}{x_0} = \frac{y - y_0}{y_0} = \frac{z - z_0}{z_0}$

23 Intersect at $(1, 3, 1)$

24 Parallel

25 Skew lines

26 Skew lines

27 Parallel

28 Skew lines

29 $(4, -4, 8)$

30 $\left(\frac{16}{11}, \frac{35}{11}, \frac{13}{11}\right)$ **31** $\left(\frac{17}{11}, -\frac{7}{11}, \frac{72}{11}\right)$ **32** $\left(\frac{43}{11}, \frac{58}{11}, -\frac{1}{11}\right)$

Exercise 14.5

1 B and C

2 A

3 $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 26$; $2x - 4y + 3z - 26 = 0$

4 $\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -3$; $2x + 3z + 3 = 0$

5 $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3$; $3z - 3 = 0$; $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

6 $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5$; $5x + y - 2z - 5 = 0$

7 $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2$; $y - 2z + 2 = 0$

8 $\begin{pmatrix}1\\-6\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 23;\ r = \begin{pmatrix}3\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\0\\-1\end{pmatrix}$

9 $\begin{pmatrix}-2\\2\\1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = -1;\ -2x + 2y + z = -1$

10 $\begin{pmatrix}18\\-3\\-11\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 5;\ 18x - 3y - 11z = 5$

11 $\begin{pmatrix}p\\q\\r\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = p^2 + q^2 + r^2;\ px + qy + rz = p^2 + q^2 + r^2$

12 $4x - 2y + 7z = 14;\ \begin{pmatrix}4\\-2\\7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 14;$

$\mathbf{r} = \begin{pmatrix}1\\2\\2\end{pmatrix} + m\begin{pmatrix}2\\-3\\-2\end{pmatrix} + n\begin{pmatrix}4\\1\\-2\end{pmatrix}$

13 $8x + 17y - 5z + 8 = 0;\ \begin{pmatrix}8\\17\\-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = -8;$

$\mathbf{r} = \begin{pmatrix}2\\-2\\-2\end{pmatrix} + s\begin{pmatrix}1\\1\\5\end{pmatrix} + t\begin{pmatrix}-3\\2\\2\end{pmatrix}$

14 $\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 3;\ x - y = 3$

15 $\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = -86;\ 30x + y - 23z + 86 = 0$

16 $\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 1;\ x - z = 1;\ r = \begin{pmatrix}1\\1\\0\end{pmatrix} + m\begin{pmatrix}1\\0\\1\end{pmatrix} + n\begin{pmatrix}1\\-1\\1\end{pmatrix}$

Note: All answers for 17–22 are to the nearest degree.

17 64° **18** 90° **19** 45° **20** 50°

21 24° **22** 55° **23** $(3, 6, -10)$ **24** $(2, -2, 6)$

25 No intersection **26** Plane contains line

27 $\mathbf{r} = \begin{pmatrix}10\\-7\\0\end{pmatrix} + t\begin{pmatrix}0\\-1\\1\end{pmatrix}$ **28** $\mathbf{r} = \begin{pmatrix}3\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}$

29 No intersection

30 $\mathbf{r} = \begin{pmatrix}\\ \\ \\\end{pmatrix} + t\begin{pmatrix}0\\-1\\1\end{pmatrix}$ **31** $\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

32 $x + 6y + z = 16$ **33** $\left(\frac{31}{21}, \frac{37}{21}, \frac{85}{21}\right)$

34 $\begin{pmatrix}10\\1\\-8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = -32;\ 10x + y - 8z + 32 = 0$

35 $\begin{pmatrix}4\\-3\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 5;\ 4x - 3y + 2z - 5 = 0$

36 $(BC)x + (AC)y + (AB)z = ABC$

37 $\mathbf{r} = \begin{pmatrix}4\\-3\\-1\end{pmatrix} + \mathbf{r}\begin{pmatrix}2\\-3\\4\end{pmatrix} + \mathrm{s}\begin{pmatrix}4\\0\\-3\end{pmatrix}$

38 $\mathbf{r} = \begin{pmatrix}2\\3\\0\end{pmatrix} + m\begin{pmatrix}2\\-3\\4\end{pmatrix} + n\begin{pmatrix}1\\-2\\1\end{pmatrix}$

Practice questions

1 a) $\overrightarrow{OD} - \overrightarrow{OC}$ b) $\frac{1}{2}(\overrightarrow{OD} - \overrightarrow{OC})$ c) $\frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC})$

2 a) $5\mathbf{i} + 12\mathbf{j}$ b) $10\mathbf{i} + 24\mathbf{j}$

3 a) $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = 6$

b) $\overrightarrow{AC} = \begin{pmatrix}-1\\\sqrt{11}\end{pmatrix}$ c) $\frac{1}{\sqrt{12}}$ d) $6\sqrt{11}$

4 a) $(10, 5)$ b) $(-3, 6)$; 90°

5 $a = 2, b = 8$

6 $\mathbf{r} = (3, -1) + t(4, -5)$

7 a) 39.4 b) (i) $(9, 12), (18, -8)$ (ii) $\sqrt{481}$

c) 7 a.m. d) 24.4 km e) 54 minutes

8 $\mathbf{r} = t(2\mathbf{i} + 3\mathbf{j})$

9 b) $(2, 3.25)$

10 c) 90°

d) (i) $12x - 5y = 301$ (ii) $(28, 7)$

11 117°

12 $2x + 3y = 5$

13 a) $(6, 20)$ b) (i) $(6, -8)$ (ii) 10

c) $4x + 3y = 84$ d) collide at 15:00

f) 26 km

14 72°

15 a) 3.94 m b) 1.22 m/s

c) $x - 0.7y = 2$ d) $\left(\frac{170}{29}, \frac{160}{29}\right)$

e) Speed = 1.24 m/s

16 $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + t\begin{pmatrix}5\\2\end{pmatrix}$

17 $2x^2 + 7x - 15 = 0,\ x = \frac{3}{2},\ x = -5$

18 a) (ii) $(288, 84)$ (iii) 50 minutes b) 20.6°

c) (i) $(99, 168)$ (iii) $XY = 75$ d) 180 km

19 $3x + 2y = 7$

20 a) $\overrightarrow{ST} = \begin{pmatrix}9\\9\end{pmatrix},\ V(-4, 6)$ b) $\mathbf{r} = (-4, 6) + \lambda(1, 1)$

c) $\lambda = 5$ d) (i) $a = 5$ (ii) 157°

21 81.9°

22 a) 13 b) $\frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$ c) $\frac{56}{65}$

23 $(2, 3)$

24 a) $(3, -2)$ c) (iii) 23 square units

25 a) $\overrightarrow{OB} = \begin{pmatrix}-1\\7\end{pmatrix};\ \overrightarrow{OC} = \begin{pmatrix}8\\9\end{pmatrix}$ b) $d = 11$

c) $\overrightarrow{BD} = \begin{pmatrix}12\\-3\end{pmatrix}$ d) (i) $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}12\\-3\end{pmatrix}$ (ii) $t = 0$

26 a) (i) $\overrightarrow{AB} = \begin{pmatrix}-5\\1\end{pmatrix}$ (ii) $AB = \sqrt{26}$ b) $\overrightarrow{AD} = \begin{pmatrix}d-2\\25\end{pmatrix}$

c) (ii) $\overrightarrow{OD} = \begin{pmatrix}7\\23\end{pmatrix}$ d) $\overrightarrow{OC} = \begin{pmatrix}2\\24\end{pmatrix}$ e) 130

27 a) (i) $\overrightarrow{BC} = -6\mathbf{i} - 2\mathbf{j}$ (ii) $\overrightarrow{OD} = -2\mathbf{i}$ b) 82.9°

c) $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + t(2\mathbf{i} + 7\mathbf{j})$ d) $15\mathbf{i} + 46\mathbf{j}$

28 a) $(5, 5, -5)$ b) $(-5, 0, 5)$ c) $(5, 5, -5)$

29 b) (i) $(49, 32, 0)$ (ii) 54 km/h

c) (i) $\frac{5}{6}$ hours (ii) $(9, 12, 5)$

30 a) (i) $\overrightarrow{AB} = \begin{pmatrix}800\\600\end{pmatrix}$

b) (ii) $\begin{pmatrix}-400\\-50\end{pmatrix}$ (iii) at 16:00 hours

c) 27.8 km

31 a) $c = 1$ b) $3\mathbf{i} + 3\mathbf{k}$

c) $\mathbf{r} = 3(1 - t)\mathbf{i} + (3 - t)\mathbf{j} + (5 + 3t)\mathbf{k}$

d) $9x - 15y + 4z - 2 = 0$ e) $\frac{15}{\sqrt{322}}$

32 a) $\overrightarrow{AB} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$; $\overrightarrow{BC} = \mathbf{i} + \mathbf{j}$

b) $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ c) $\frac{\sqrt{6}}{2}$

d) $-x + y + 2z = 3$

e) $\begin{cases} 2 - t \\ -1 + t \\ -6 + 2t \end{cases}$ f) $3\sqrt{6}$

g) $\frac{1}{\sqrt{6}}(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ h) $E(-4, 5, 6)$

33 Proof

34 a) $P(4, 0, -3)$, $Q(3, 3, 0)$, $R(3, 1, 1)$, $S(5, 2, 1)$

b) $3x + 2y + 4z = 0$

c) 0

35 a) $147°$ b) 2.29

c) (i) $L_1 : \begin{cases} 2 \\ -1 + \lambda \\ 2\lambda \end{cases}$; $L_2 : \begin{cases} -1 + \mu \\ 1 - 3\mu \\ 1 - 2\mu \end{cases}$ (ii) no solution

d) $\frac{9}{\sqrt{21}}$

36 a) $(1, -1, 2)$

b) $11\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$

c) $\mathbf{v}.\mathbf{u} = 0$

d) $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 6 \\ 13 \\ -5 \end{pmatrix}$

37 a) (i) $-5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ (ii) $\frac{\sqrt{35}}{2}$

b) (i) $-5x + 3y + z = 5$

(ii) $\frac{x - 5}{-5} = \frac{y + 2}{3} = z - 1$

c) $(0, 1, 2)$ d) $\sqrt{35}$

38 a) $x - 2 = y - 5 = z + 1$ b) $\left(\frac{1}{3}, \frac{10}{3}, -\frac{8}{3}\right)$

c) $A'\left(-\frac{4}{3}, \frac{5}{3}, -\frac{13}{3}\right)$ d) $\frac{\sqrt{654}}{3}$

39 a) $3x - 4y + z = 6$

b) (ii) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 13 \end{pmatrix}$ c) $53.7°$

40 a) $(3\mu - 2, \mu, 9 - 2\mu)$

b) (i) $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

(ii) $\overrightarrow{PM} = \begin{pmatrix} 3\mu - 6 \\ \mu \\ 12 - 2\mu \end{pmatrix}$

c) (i) $\mu = 3$ (ii) $3\sqrt{6}$

d) $2x - 4y + z = 5$ e) verify

41 a) $(1, -1, 2)$

b) $2x - y + z = 5$

c) $(3, 1, 3)$ and $(1, 2, 2)$

42 a) (i) $\lambda = \mu$

(ii) $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

b) $3x - 2y + z = 5$

c) $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

Chapter 15

Exercise 15.1

1 a) $y' = 12(3x - 8)^3$ b) $y' = -\frac{1}{2\sqrt{1 - x}}$

c) $y' = \cos^2 x - \sin^2 x$

d) $y' = \cos\left(\frac{x}{2}\right)$ e) $y' = -\frac{4x}{(x^2 + 4)^3}$

f) $y' = \frac{-2}{(x - 1)^2}$

g) $y' = \frac{-1}{2\sqrt{(x + 2)^3}}$ $\left[\text{or } \frac{-1}{(2x + 4)\sqrt{x + 2}}\right]$

h) $y' = -2\sin x\cos x$

i) $y' = \frac{-x + 2}{2\sqrt{(1 - x)^3}}$ $\left[\text{or } \frac{-x + 2}{(2 - 2x)\sqrt{1 - x}}\right]$

j) $y' = \frac{-6x + 5}{(3x^2 - 5x + 7)^2}$ k) $y' = \frac{2}{3\sqrt[3]{(2x + 5)^2}}$

l) $y' = 2(2x - 1)^2(7x^4 - 2x^3 + 3)$

2 a) $y = -12x - 11$ b) $y = \frac{9}{5}x - \frac{2}{5}$

c) $y = 2x - 2\pi$ d) $y = \frac{1}{2}x + \frac{1}{2}$

3 a) $v(t) = -2t\sin(t^2 - 1)$ b) velocity $= 0$

c) $t = \sqrt{\pi + 1} \approx 2.04$, $t = 1$

d) Accelerating to the right then slowing down, turning around, accelerating to the left, slowing down, turning around again, then accelerating to the right.

4 a) $y = -12x + 38$ b) $y = \frac{1}{12}x + \frac{7}{4}$

5 a) $y = \frac{2}{3}x + \frac{5}{3}$ b) $y = -\frac{3}{2}x + 6$

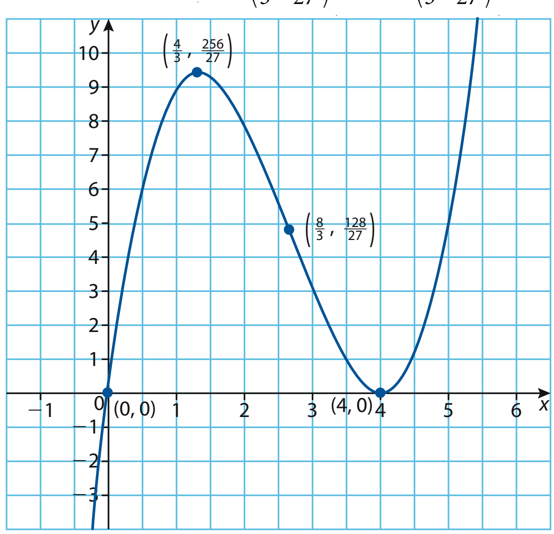
6 a) $y = \frac{1}{4}x + \frac{1}{4}$ b) $y = -4x + \frac{9}{2}$

7 a) $\frac{dy}{dx} = 2\sin(2x)$; $\frac{d^2y}{dx^2} = 4\cos(2x)$

b) $\left(\frac{\pi}{4}, 0\right)$ and $\left(\frac{3\pi}{4}, 0\right)$

8 a) (i) $(0, 0)$ and $(4, 0)$ (ii) $\left(\frac{4}{3}, \frac{256}{27}\right)$ (iii) $\left(\frac{8}{3}, \frac{128}{27}\right)$

b)



9 c) $f''(3.8)=0$ and $f''(3)=\frac{1}{3}>0,\ f''(4)=-\frac{2}{625}<0$, therefore graph of f changes concavity from up to down at $x=3.8$ verifying that graph of f does have an inflexion point at $x=3.8$

10 $\frac{dy}{dx}=\frac{2a}{(x+a)^2};\ \frac{d^2y}{dx^2}=\frac{-4a}{(x+a)^3}$

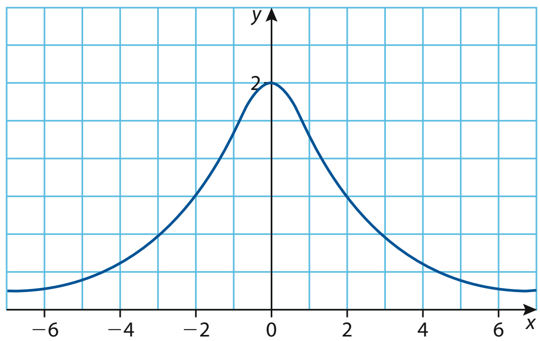
11 $\frac{d^ny}{dx^n}=\frac{(-1)^{n+1}n!}{(x-1)^{n+1}}\left(\text{or } \frac{n!}{(1-x)^{n+1}}\right)$

12 a) Max. at $(0,2)$; inflexion pts at or $(-2,1)$ and $(2,1)$

b) (i) None (ii) none (iii) all $x\in\mathbb{R}$

c) (i) $\lim_{x\to\infty} g(x)=0$ (ii) $\lim_{x\to-\infty} g(x)=0$

d)



13 $\frac{d}{dx}(c\cdot f(x))=\frac{d}{dx}(c)\cdot f(x)+c\cdot\frac{d}{dx}(f(x))$

$=0\cdot f(x)+c\cdot\frac{d}{dx}(f(x))=c\cdot\frac{d}{dx}(f(x))$

14 $y=x^2(x^2-6)=0$ when $x=0$ and $x=\pm\sqrt{6}$;

$y\left(\frac{1}{2}\right)=-\frac{23}{16}<0$, so $y<0$ for $0<x<1$

$\frac{dy}{dx}=4x(x^2-3)=0$ when $x=0,\ x=\pm\sqrt{3}$; when

$x=\frac{1}{2},\ \frac{dy}{dx}=-\frac{11}{2}<0$, so $\frac{dy}{dx}<0$ for $0<x<1$

$\frac{d^2y}{dx^2}=12(x^2-1)=0$ when $x=0,\ x=\pm1$; when

$x=\frac{1}{2},\ \frac{d^2y}{dx^2}=-9<0$, so $\frac{d^2y}{dx^2}<0$ for $0<x<1$

$\frac{d^3y}{dx^3}=24x>0$ for $0<x<1$

Exercise 15.2

1 a) $y'=x^2e^x+2xe^x$ b) $y'=8^x\ln 8$

c) $y'=e^x\sec^2(e^x)$ d) $y'=\frac{\cos x+x\sin x+1}{(1+\cos x)^2}$

e) $y'=\frac{xe^x-e^x}{x^2}$ f) $y'=2\tan^3(2x)\sec(2x)$

g) $y'=\left(\frac{1}{4}\right)^x\ln\left(\frac{1}{4}\right)$ h) $y'=\cos x$

i) $y'=\frac{-xe^x+e^x-1}{(e^x-1)^2}$ j) $y'=-12\cos(3x)\sin(\sin(3x))$

k) $y'=2\ln 2(2^x)$ l) $y'=\frac{\cos^3 x-\sin^3 x}{(\cos x-\sin x)^2}$

2 a) $y=\frac{1}{2}x+\frac{3\sqrt{3}-\pi}{6}$

b) $y=2x+1$

c) $y=16x+4-2\pi$

3 a) $x=\frac{\pi}{6},\ x=\frac{5\pi}{6}$

b) Maximum at $\frac{\pi}{6}$, minimum at $\frac{5\pi}{6}$

4 $(0,-1)$ is an absolute maximum

5 a) Maximum at $\left(\frac{\pi}{2},5\right)$; minimum at $\left(\frac{3\pi}{2},-3\right)$

b) Minimum at $\left(\frac{3\pi}{4},-1\right)$ and $\left(\frac{7\pi}{4},-1\right)$

6 $x=\frac{\pi}{2}$

7 a) $f'(x)=e^x-3x^2;\ f''(x)=e^x-6x$

b) $x\approx3.73$ or $x\approx0.910$ or $x\approx-0.459$

c) Decreasing on $(-\infty,-0.459)$ and $(0.910, 3.73)$; increasing on $(-0.459, 0.910)$ and $(3.73,\infty)$

d) $x\approx-0.459$ (minimum); $x\approx0.910$ (maximum); $x\approx3.73$ (minimum)

e) $x\approx0.204$ or $x\approx2.83$

f) Concave up on $(-\infty, 0.204)$ and $(2.83,\infty)$; concave down on $(0.204, 2.83)$

8 The two functions intersect for all x such that $\cos x=1$, i.e. $x=k\cdot2\pi,\ k\in\mathbb{Z}$. The derivatives for the two functions are $y'=-e^{-x}$ and $y'=-e^{-x}(\cos x+\sin x)$. The derivatives are equal whenever $x=k\cdot2\pi,\ k\in\mathbb{Z}$. Therefore, the functions are tangent at all of the intersection points.

9 a) $8\ \mathrm{m\,s^{-2}}$ b) $2.09\ \mathrm{m\,s^{-1}}$

10 $y=ex$

11 a) $f'(x)=2^x\ln 2$

b) $y=x\ln2+1$

c) $f'(x)=2^x\ln2\neq0$ for any x

12 a) $(-1,-2e)$ and $\left(3,\frac{6}{e^3}\right)$

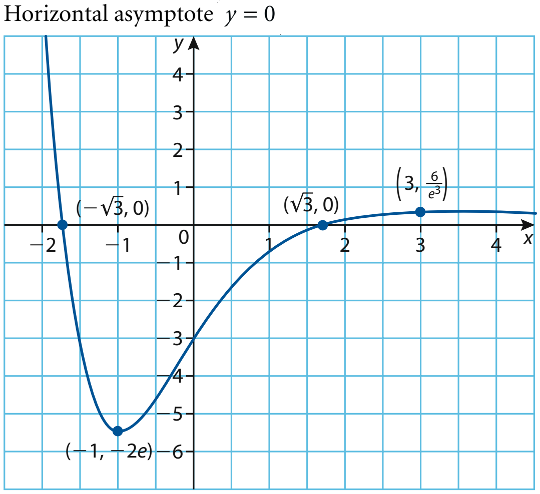
b) $(-1,-2e)$ is a minimum; $\left(3,\frac{6}{e^3}\right)$ is a maximum

c) (i) $\lim_{x\to\infty} h(x)=0$

(ii) as $x\to-\infty$, $h(x)$ increases without bound

d) Horizontal asymptote $y=0$

e)



13 a) $a = \frac{\pi}{2}, b = \pi, c = \frac{3\pi}{2}$

b) $\frac{d^{(n)}}{dx^{(n)}}(\sin x) = \sin\left(x + n \cdot \frac{\pi}{2}\right), \quad n \in \mathbb{Z}^+$

14 a) $\frac{d}{dx}\left(xe^x\right) = xe^x + e^x$; $\frac{d^2}{dx^2}\left(xe^x\right) = xe^x + 2e^x$;

$\frac{d^3}{dx^3}\left(xe^x\right) = xe^x + 3e^x$

b) $\frac{d^{(n)}}{dx^{(n)}}\left(xe^x\right) = xe^x + ne^x$

Exercise 15.3

1 $\frac{dy}{dx} = -\frac{x}{y}$

2 $\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$

3 $\frac{dy}{dx} = \cos^2 y \quad \left[\text{or } \frac{dy}{dx} = \frac{1}{1+x^2}\right]$

4 $\frac{dy}{dx} = \frac{-2x + 3y^2 - y^3}{-6xy + 3xy^2 - 2y}$

5 $\frac{dy}{dx} = \frac{x^2y + y^3}{x^3 + xy^2}$

6 $\frac{dy}{dx} = \frac{-2xy - 2y^2 - xy}{2x^2 + 2xy + xy}$

7 $\frac{dy}{dx} = \frac{y-1}{\cos y - x}$

8 $\frac{dy}{dx} = \frac{4x^3 - 2xy^3}{3x^2y^2 + 4y^3}$

9 $\frac{dy}{dx} = \frac{-y}{x + e^y}$

10 $\frac{dy}{dx} = \frac{x+2}{y+3}$

11 $\frac{dy}{dx} = -\sin^2(x+y) \quad \left[\text{or } \frac{dy}{dx} = -\frac{x^2}{x^2+1}\right]$

12 $\frac{dy}{dx} = \frac{18x^2\sqrt{xy} - y}{x + 2\sqrt{xy}}$

13 $y = -\frac{7}{5}x + \frac{4}{5}$; $y = \frac{5}{7}x - \frac{24}{7}$

14 $y = -2x + 4$; $y = \frac{1}{2}x + \frac{3}{2}$

15 $y = -\frac{\pi}{2}x + \pi$; $y = \frac{2}{\pi}x + \frac{\pi^2 - 4}{2\pi}$

16 $y = -\frac{352}{23}x - \frac{32}{23}$; $y = \frac{23}{352}x - \frac{5655}{176}$

17 $x^2 + y^2 = r^2 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$; at point (x_1, y_1), $m = -\frac{x_1}{y_1}$; centre of circle is $(0,0)$; slope of line through (x_1, y_1) and $(0,0)$ is $\frac{y_1}{x_1}$; because $-\frac{x_1}{y_1} \times \frac{y_1}{x_1} = -1$, the tangent to the circle at (x_1, y_1) and the line through (x_1, y_1) and $(0,0)$ are perpendicular

18 a) $\left(\sqrt{7}, 0\right), \left(-\sqrt{7}, 0\right)$; $\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$, at both points $\frac{dy}{dx} = -2$

b) $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$ and $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$

c) $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$ and $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$

19 $(0,0)$

20 $\frac{dy}{dx} = -\frac{4x}{9y}, \frac{d^2y}{dx^2} = \frac{-36y^2 - 16x^2}{81y^3}$

21 $\frac{dy}{dx} = \frac{2-y}{(x+3)^2}, \frac{d^2y}{dx^2} = \frac{2y - 4}{(x+3)^2}$

22 a) $\frac{dy}{dx} = \frac{-1}{3x^{\frac{4}{3}}}, \frac{d^2y}{dx^2} = \frac{4}{9x^{\frac{7}{3}}}$

b) $\frac{dy}{dx} = -\frac{y}{3x}, \frac{d^2y}{dx^2} = \frac{4y}{9x^2}$

23 $y = x + \frac{1}{2}$

24 $\frac{dy}{dx} = \frac{3x^2}{x^3 + 1}$

25 $\frac{dy}{dx} = \cot x$

26 $\frac{dy}{dx} = \frac{x}{\left(x^2 - 1\right)\ln 5}$

27 $\frac{dy}{dx} = \frac{-1}{x^2 - 1}$

28 $\frac{dy}{dx} = \frac{1}{2x \ln 10 \sqrt{\log x}}$

29 $\frac{dy}{dx} = \frac{2a}{x^2 - a}$

30 $\frac{dy}{dx} = -\sin x$

31 $\frac{dy}{dx} = \frac{-1}{x \ln 3 \left(\log_3 x\right)^2}$

32 $\frac{dy}{dx} = \ln x$

33 0

34 $y = \left(\frac{1}{8\ln 2}\right)x - \frac{1}{\ln 2} + 3$

35 Verify

36 $x = \frac{1}{e^{\frac{3}{2}}}$

37 a) $g'(x) = \frac{1 - \ln x}{x^2}, \; g''(x) = \frac{-3 + 2\ln x}{x^3}$

b) $g'(x) = 0$ only at $x = e$; $g''(e) = -\frac{1}{e^3} < 0$, $\therefore$ abs. max. at $x = e$, max. value of g is $\frac{1}{e}$

38 $\frac{dy}{dx} = \frac{1}{x^2 + 2x + 2}$

39 $\frac{dy}{dx} = \frac{1}{x^2 + 1}$

40 $\frac{dy}{dx} = \frac{6}{x\sqrt{x^4 - 9}}$

41 $\frac{dy}{dx} = \left(\tan^{-1} x + \frac{x}{x^2 + 1}\right)e^{x \tan^{-1} x}$

42 $f'(x) = 0$; the graph of $f(x)$ is horizontal

43 Verify

44 $y = \left(\frac{\pi + 4}{2}\right)x + \frac{\pi - 4}{4}$

45 a) For $0 \leqslant x < \pi$, $f'(x) = -1$, therefore $f(x)$ is linear

b) $y = -x + \frac{\pi}{2}$

46 $\sqrt{10} \approx 3.16$ m

47 a) $\frac{1}{4}\,\text{m s}^{-1}, \frac{1}{20}\,\text{m s}^{-1}$

b) $-\frac{1}{4}\,\text{m s}^{-2}, -\frac{13}{800}\,\text{m s}^{-2}$

c) The particle initially is moving very fast to the right and then gradually slows down while continuing to move to the right.

d) $\lim_{t \to \infty} s(t) = \frac{\pi}{2}$ m

Exercise 15.4

1 a) -18.1 cm/min b) -6.79 cm/min

2 a) 0.298 cm/sec b) 0.439 cm/sec

3 a) 2π cm/hr b) 8π cm/hr

4 $\frac{d\theta}{dt} = \frac{3}{34} \approx 0.0882$ radians/min

5 26.4 m/sec

6 2 ft/sec

7 69.6 km/hr

8 $\frac{dy}{dt} = \frac{12}{\sqrt{10}} \approx 3.79$

9 0.01 m/sec

10 30 mm^3/sec

11 45 km/hr

12 $\frac{8\sqrt{3}}{3} \approx 4.62$ cm/sec

13 1.5 units/sec

14 $222.\dot{2}$ m/sec = 800 km/hr

15 a) 115 degrees/sec b) 57 degrees/sec

16 -485 km/hr

Exercise 15.5

1 $\sqrt{2}$ by $\frac{\sqrt{2}}{2}$

2 $13\frac{1}{3}$ cm by $6\frac{2}{3}$ cm

3 $\dfrac{\sqrt{5}}{2}$

4 b) $S = 4x^2 + \dfrac{3000}{x}$ c) 7.21 cm × 14.4 cm × 9.61 cm

5 $x = 5\sqrt{2\pi} \approx 12.5$ cm

6 $x \approx 3.62$ m

7 Longest ladder ≈ 7.02 m

8 $d \approx 2.64$ km

9 $\dfrac{8}{5}$ units^2

10 6 nautical miles

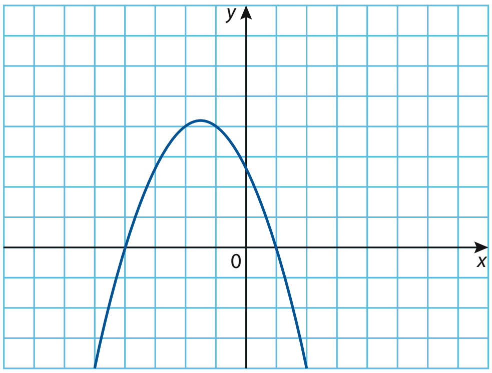
11 $h = R\sqrt{2}$, $r = \dfrac{R\sqrt{2}}{2}$

12 Distance of point P from point X is $\dfrac{ac}{\sqrt{r^2 - c^2}}$

13 $x \approx 51.3$ cm, maximum volume ≈ 403 cm^3

Practice questions

1



2 a) (i) $a = -4$ (ii) $b = 2$

(i) $f'(x) = -3x^2 - 4x + 8$

(ii) $\dfrac{-2 + 2\sqrt{7}}{3}$, $\dfrac{-2 - 2\sqrt{7}}{3}$

(iii) $f(1) = 5$

c) (i) $y = 8x$ (ii) $x = -2$

3 a) (i) $v(0) = 0$ (ii) $v(10) \approx 51.3$

b) (i) $a(t) = 0.99e^{-0.15t}$ (ii) $a(0) = 0.99$

c) (i) 66 (ii) 0

(iii) As object falls it approaches terminal velocity

4 a) $\left(-\dfrac{2}{3}, -\dfrac{149}{27}\right)$ is a minimum, $(-4, 13)$ is a maximum

b) $\left(-\dfrac{7}{3}, \dfrac{101}{27}\right)$ is an inflexion point

5 a) (i) $g'(x) = -\dfrac{3}{e^{3x}}$

(ii) $e^{3x} > 0$ for all x, hence $-\dfrac{3}{e^{3x}} < 0$ for all x; therefore, $f(x)$ is decreasing for all x

b) (i) $e + 2$

(ii) $g'(-\frac{1}{3}) = -3e$

c) $y = -3ex + 2$

6 b) $f'(3) = 0$ and $f''(3) > 0 \Rightarrow$ stationary point at $x = 3$ and graph of f is concave up at $x = 3$, so $f(3)$ is a minimum

c) $(4, 0)$

7 a) $-\dfrac{4}{(2x+3)^3}$

b) $5\cos(5x)e^{\sin(5x)}$

8 $A = 1$, $B = 2$, $C = 1$

9 $\dfrac{dy}{dx} = -1$, $\dfrac{d^2y}{dx^2} = -4$

10 a) $\dfrac{dy}{dx} = \dfrac{-xe^x + e^x - 1}{(e^x - 1)^2}$

b) $\dfrac{dy}{dx} = 2e^x\cos(2x) + e^x\sin(2x)$

c) $\dfrac{dy}{dx} = 2x\ln x + 2x\ln 3 + x - \dfrac{1}{x}$

11 $y = -\dfrac{1}{2}x - \dfrac{3}{2}$, $P(-3, 0)$, $Q\left(0, -\dfrac{3}{2}\right)$

12 a) $x = 3$; sign of $h''(x)$ changes from negative (concave down) to positive (concave up) at $x = 3$

b) $x = 1$; $h'(x)$ changes from positive (h increasing) to negative (h decreasing) at $x = 1$

13 $y = \dfrac{5}{7}x + \dfrac{11}{7}$

14 $h = 8$ cm, $r = 4$ cm

15 Maximum area is 32 square units; dimensions are 4 by 8

16 a) E b) A c) C

17 $y = -\dfrac{1}{5}x + \dfrac{32}{5}$

18 a) $y = 4x - 4$

b) $y = -\dfrac{1}{4}x + \dfrac{1}{4}$

19 a) Absolute minimum at $\left(\dfrac{1}{\sqrt{e}}, -\dfrac{1}{2e}\right)$

b) Inflexion point at $\left(\dfrac{1}{\sqrt{e^3}}, -\dfrac{3}{2e^3}\right)$

20 a) (i) $a = 16$ (ii) $a = 54$

b) $f'(x) = 2x - \dfrac{a}{x^2} = 0 \Rightarrow x = \sqrt[3]{\dfrac{a}{2}}$;

$f''(x) = 2 + \dfrac{2a}{x^3} \Rightarrow f''\left(\sqrt[3]{\dfrac{a}{2}}\right) = 4 > 0$; hence, f is concave up at any critical point, so it cannot be a maximum

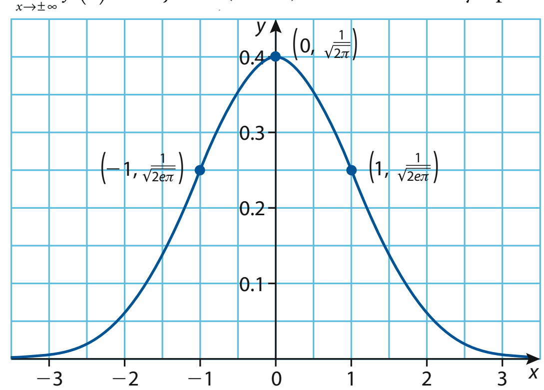
21 $y = -\dfrac{2}{3}x + 4$

22 $y = \left(\dfrac{\pi + 2}{2}\right)x - \dfrac{\pi^2}{8}$; $y = \left(\dfrac{-2}{\pi + 2}\right)x + \dfrac{\pi}{2\pi + 4} + \dfrac{\pi}{4}$

23 a) Maximum at $\left(0, \dfrac{1}{\sqrt{2\pi}}\right)$, inflexion points at $\left(-1, \dfrac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \dfrac{1}{\sqrt{2e\pi}}\right)$

b) $\lim\limits_{x \to \pm\infty} f(x) = 0$; $y = 0$ (x-axis) is a horizontal asymptote

c)



24 a) Min. at $x = 1$ because $f''(1) = \dfrac{1}{2} > 0$; max. at $x = 3$ because $f''(3) = -\dfrac{1}{6} < 0$

b) Inflexion points at $x = -\sqrt{3}$ and $x = \sqrt{3}$ because $f''(x)$ changes sign at both values

25 $x = 20\sqrt{3} \approx 34.6$ km/hr

26 $\frac{dy}{dx} = \frac{5}{6}$ or $\frac{dy}{dx} = -\frac{5}{6}$

27 $\frac{dy}{dx} = \frac{-2x}{2x^4 - 2x^2 + 1}$

28 $\frac{dy}{dx} = 2x\ln x + x$

29 $\sin x = \frac{1}{2}$, $\sin x = -1$

30 $-\frac{3}{4}$

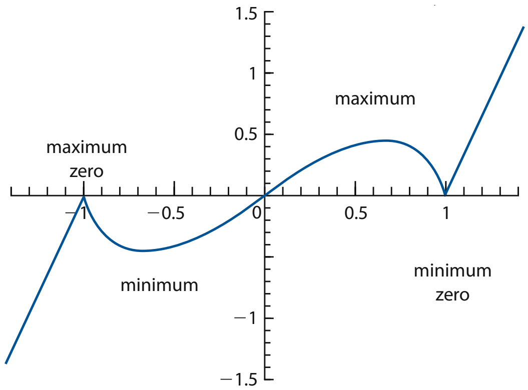
31 a) $f'(x) = \frac{2}{2x-1}$ b) $x = \frac{1+\sqrt{17}}{4}$

32 $x \approx -0.586$

33 $c = 4 + \frac{\pi}{4}$

34 a) $f'(x) = \pi\cos(\pi x)e^{1+\sin\pi x}$ b) $x_n = \frac{2n+1}{2}$

35 a)



b) (i) $f'(x) = \frac{7x^2 - 3}{3(x^2-1)^{\frac{1}{3}}}$, domain: $-1.4 \leqslant x \leqslant 1.4, x \neq \pm 1$

(ii) Maximum at $x = \sqrt{\frac{3}{7}}$, minimum at $x = -\sqrt{\frac{3}{7}}$

c) $x \approx 1.1339$

36 $a = -4, b = 18$

37 a) $\frac{dy}{dx} = \sec^2 x - 8\cos x$ b) $\cos x = \frac{1}{2}$

38 a) $y = -4x - 8$ b) $(-2, 0)$

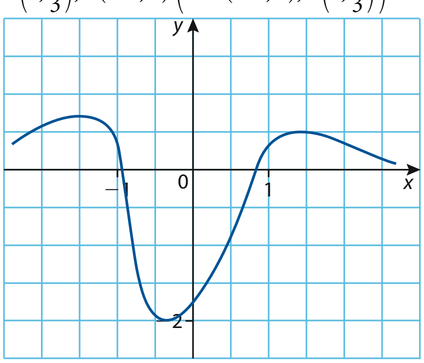
39 Proof

40 $y = -x + 2$

41 a) (i) $f'(x) = \frac{2(x^2-1)}{(x^2+x+1)^2}$

(ii) $A\left(1, \frac{1}{3}\right), B(-1, 3)$ $\left(\text{or } A(-1,3), B\left(1, \frac{1}{3}\right)\right)$

b) (i)



(ii) $x \approx -0.347, 1.53, 1.88$

c) (i) Range of f: $\left[\frac{1}{3}, 3\right]$ (ii) range of $f \circ f$: $\left[\frac{1}{3}, \frac{7}{13}\right]$

42 $\frac{1}{2\pi}$ cm/s

43 $y = \frac{4}{3}x - \frac{5}{3}$

44 a) (ii) $f''(x) = \frac{x^2(\ln 2)^2 - 4x\ln 2 + 2}{2^x}$

b) (i) $x = \frac{2}{\ln 2}$

(ii) $f''\left(\frac{2}{\ln 2}\right) < 0$; therefore, a maximum

c) $x = \frac{2+\sqrt{2}}{\ln 2} \approx 4.93$, $x = \frac{2-\sqrt{2}}{\ln 2} \approx 0.845$

45 a) $f'(t) = 6\sec^2 t\tan t + 5$ $\left[\text{or } f'(t) = \frac{6\sin t}{\cos^3 t} + 5\right]$

b) (i) $3 + 5\pi$ (ii) 5

46 a) $y = -1$ b) $\frac{dy}{dx} = \frac{4}{5}$

47 a) $\frac{dy}{dx} = 3e^{3x}\sin(\pi x) + \pi e^{3x}\cos(\pi x)$ b) $x \approx 0.743$

48 240 km/hr

49 b) $\left(-\frac{1}{c}\ln b, \frac{a}{2b}\right)$

50 a) $p = 2$ b) $-\frac{4}{7}$

51 $x \approx 0.460$

52 $\frac{1}{10}$ radians/sec

53 $\frac{d^2y}{dx^2} = \frac{-4}{(2x-1)^2}$

54 $y = -\frac{5}{4}x + \frac{13}{2}$

55 a) $f''(x) = 10\cos\left(5x - \frac{\pi}{2}\right)$

b) $f(x) = -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}$

56 $\frac{5}{4}$

57 $(-0.803, -2.08)$

58 a) $k = \frac{\ln 2}{20}$ b) 510 bacteria per minute

59 $f(x) = -\frac{1}{5}x^3 + \frac{12}{5}x^2 - 3x + 2$

60 a) $f'(x) = -12\cos^2(4x+1)\sin(4x+1)$

b) $x = \frac{\pi - 2}{8}$, $x = \frac{3\pi - 2}{8}$, $x = \frac{\pi - 1}{4}$

61 $\frac{dy}{dx} = \frac{3x^2 - (\ln 3)3^{x+y}}{(\ln 3)3^{x+y} - 3}$

62 a) $f'(x) = \frac{3}{3x+1}$ b) $y = -\frac{7}{3}x + \frac{14}{3} + \ln 7$

63 Verify

64 $\frac{dy}{dx} = \frac{1-e}{e}$

65 b) $b = \sqrt{6}$

66 a) $\frac{dy}{dx} = \frac{2-k}{2k-1}$ b) $k = 2$

67 $\frac{3}{2}$

68 a) $5\sqrt{5}\sqrt{x^2+4} + 5(2-x)$ minutes

c) (i) $x = 1$ (ii) 30 minutes

(iii) $\frac{d^2T}{dx^2} > 0$ for $x = 1$; therefore, it's a minimum

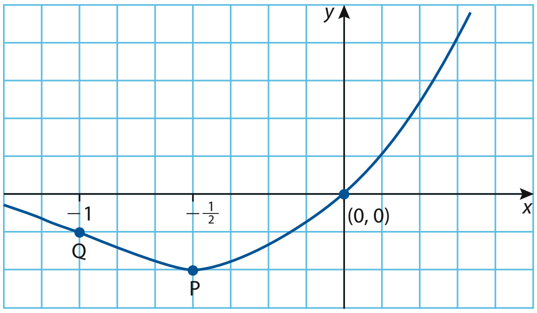
69 a) $P\left(-\frac{1}{2}, -\frac{1}{2e}\right)$

b) $f''(x) = 4x + 4 = 0$ at $x = -1$, and $f''(x)$ changes sign at $x = -1$

c) (i) Concave up for $x > -1$

(ii) Concave down for $x < -1$

d)



e) Show true for $n = 1$:

$f'(x) = e^{2x} + 2xe^{2x}$
$= e^{2x}(1 + 2x) = (2x + 2^0)\,e^{2x}$

Assume true for $n = k$, i.e. $f^{(k)}(x)$
$= (2^k x + k \times 2^{k-1})\,e^{2x},\ k \geqslant 1$

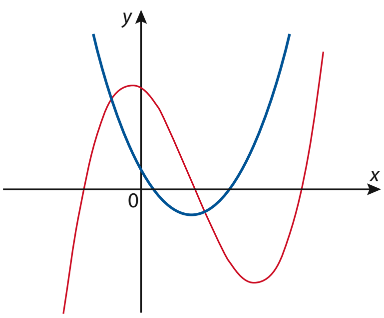
Consider $n = k + 1$, *i.e.* an attempt to find $\frac{d}{dx}(f^{(k)}(x))$

$f^{(k+1)}(x) = 2^k e^{2x} + 2e^{2x}(2^k x + k \times 2^{k-1})$
$= (2^k + 2(2^k x + k \times 2^{k-1}))\,e^{2x}$
$= (2 \times 2^k x + 2^k + k \times 2 \times 2^{k-1})\,e^{2x}$
$= (2^{k+1} x + 2^k + k \times 2^k)\,e^{2x}$
$= (2^{k+1} x + (k+1)2^k)\,e^{2x}$

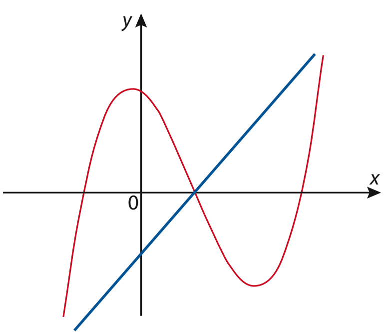
$P(n)$ is true for $k \Rightarrow P(n)$ is true for $k + 1$, and since true for $n = 1$, result proved by mathematical induction $\forall n \in \mathbb{Z}^+$

70 $\frac{72}{\pi}\arccos\frac{8}{13}$ cm

71 a)



b)



Chapter 16

Exercise 16.1

1 $\frac{x^2}{2} + 2x + c$ **2** $t^3 - t^2 + t + c$

3 $\frac{x}{3} - \frac{x^4}{14} + c$ **4** $\frac{2t^3}{3} + \frac{t^2}{2} - 3t + c$

5 $\frac{5u^{\frac{7}{5}}}{7} - u^4 + c$ **6** $\frac{4x\sqrt{x}}{3} - 3\sqrt{x} + c$

7 $-3\cos\theta + 4\sin\theta + c$ **8** $t^3 + 2\cos t + c$

9 $\frac{4x^2\sqrt{x}}{5} - \frac{10x\sqrt{x}}{3} + c$ **10** $3\sin\theta - 2\tan\theta + c$

11 $\frac{1}{3}e^{3t-1} + c$ **12** $2\ln|t| + c$

13 $\frac{1}{6}\ln(3t^2 + 5) + c$ **14** $e^{\sin\theta} + c$

15 $\frac{(2x+3)^3}{6} + c$ **16** $-\frac{5x^4}{4} + \frac{2x^3}{3} + cx + k$

17 $-\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^2}{2} + 2x - \frac{11}{20}$ **18** $\frac{4t^3}{3} + \sin t + ct + k$

19 $3x^4 - 4x^2 + 7x + 3$ **20** $2\sin\theta + \frac{1}{2}\cos 2\theta + c$

21 $\frac{(3x^2+7)^6}{36} + c$ **22** $-\frac{1}{18(3x^2+5)^3} + c$

23 $\frac{8\sqrt[4]{(5x^3+2)^5}}{75} + c$ **24** $\frac{(2\sqrt{x}+3)^6}{6} + c$

25 $\frac{\sqrt{(2t^3-7)^3}}{9} + c$ **26** $-\frac{(2x+3)^6}{18x^6} + c$

27 $-\frac{\cos(7x-3)}{7} + c$ **28** $-\frac{1}{2}\ln(\cos(2\theta-1)+3) + c$

29 $\frac{1}{5}\tan(5\theta-2) + c$ **30** $\frac{1}{\pi}\sin(\pi x + 3) + c$

31 $\frac{1}{2}\sec 2t + c$ **32** $\frac{1}{2}e^{x^2+1} + c$

33 $\frac{1}{3}e^{2t\sqrt{t}} + c$ **34** $\frac{2}{3}(\ln\theta)^3 + c$

35 $\ln|\ln 2z| + c$ **36** $-\frac{1}{15}\sqrt{(3-5t^2)^3} + c$

37 $\frac{1}{3}\tan\theta^3 + c$ **38** $-\cos\sqrt{t} + c$

39 $\frac{1}{12}\tan^6 2t + c$ **40** $2\ln(\sqrt{x}+2) + c$

41 $\frac{1}{10}\sec^5 2t + c$ **42** $\frac{1}{2}\ln|x^2 + 6x + 7| + c$

43 $-\frac{k^3}{2a^4}\sqrt{a^2 - a^4x^4} + c = -\frac{k^3}{2|a|^3}\sqrt{1 - a^2x^4} + c$

44 $\frac{2}{5}(3x^2 - x - 2)\sqrt{x-1} + c$ **45** $-\frac{1}{\pi}\cot\pi t + c$

46 $-\frac{2}{3}\sqrt{(1+\cos\theta)^3} + c$

47 $\frac{2}{105}(15t^3 - 3t^2 - 4t - 8)\sqrt{1-t} + c$

48 $\frac{1}{15}(3r^2 + 2r - 13)\sqrt{2r-1} + c$ **49** $\frac{1}{2}\ln\left(e^{x^2} + e^{-x^2}\right) + c$

50 $\frac{2}{15}(3t^2 + 20t + 230)\sqrt{t-5} + c$

Exercise 16.2

1 $-\frac{1}{3}e^{-x^3} + c$ **2** $-e^{-x}(x^2 + 2x + 2) + c$

3 $\frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + \frac{1}{3}x^2\sin 3x + c$

4 $\frac{1}{a^3}(2\cos ax - a^2x^2\cos ax + 2ax\sin ax) + c$

5 $\sin x(\ln(\sin x) - 1) + c$ **6** $\frac{1}{2}x^2(\ln x^2 - 1) + c$

7 $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c$ **8** $2e^x + x^2e^x - 2xe^x - \frac{1}{3}x^3 + c$

9 $\frac{1}{\pi^2}(\cos\pi x + \pi x\sin\pi x) + c$ **10** $\frac{3}{13}\cos 2t\,e^{3t} + \frac{2}{13}e^{3t}\sin 2t + c$

11 $\sqrt{1-x^2} + x\arcsin x + c$ 12 $e^x(x^3 - 3x^2 + 6x - 6) + c$

13 $-\frac{1}{4}e^{-2x}(\cos 2x + \sin 2x) + c$

14 $\frac{1}{2}x\big(\sin(\ln x) - \cos(\ln x)\big) + c$

15 $\frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + c$

16 $\ln|x+1| - 2x + x\ln|x^2 + x| + c$

17 $\frac{e^{kx}(k\sin x - \cos x) + c}{k^2+1}$ 18 $x\tan x + \ln|\cos x|$

19 $\frac{2}{3}\sin^3 x$ 20 $\frac{1}{2}\arctan x(1 + x^2) - \frac{1}{2}x + c$

21 $2\sqrt{x}\,(\ln x - 2) + c$ 22 $t\tan t + \ln|\cos x| + c$

23 Verification

24 $-x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c$

25 $x^5\sin x + 5x^4\cos x - 20x^3\sin x - 60x^2\cos x + 120x\sin x + 120\cos x + c$

26 $e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + c$

27 Proof 28 Proof 29 Proof

30 Proof 31 Proof

Exercise 16.3

1 $\frac{1}{80}\cos 5t - \frac{1}{48}\cos 3t - \frac{1}{8}\cos t; c\,\frac{\cos^5 t}{5} - \frac{\cos^3 t}{3} + c$

2 $\frac{\cos^6 t}{6} - \frac{\cos^4 t}{4} + c$

3 $\frac{\cos^4 3\theta}{12} + c$

4 $\frac{1}{3}\cos^3\left(\frac{1}{t}\right) - \frac{2}{5}\cos^5\left(\frac{1}{t}\right) + \frac{1}{7}\cos^7\left(\frac{1}{t}\right) + c$

5 $\sec x + \cos x + c$ 6 $\frac{1}{18}\tan^6 3x + c$

7 $\frac{1}{24}\left(3\tan^4\theta^2 + 2\tan^6\theta^2\right) + c$

8 $\frac{2}{5}\sec^5\sqrt{t} - \frac{2}{3}\sec^3\sqrt{t} + c$

9 $\frac{1}{15}\left(\tan^3 5t - 3\tan 5t + 15t\right) + c$

10 $\tan t - \sec t + c$ 11 $\csc t - \cot t + c$

12 $-\ln|1 - \sin t| + c$ 13 $-2x - 3\ln|\sin x + \cos x| + c$

14 $\arctan(\sec\theta) + c$ 15 $\frac{1}{2}(\arctan t)^2 + c$

16 $\ln|\arctan t| + c$ 17 $\arcsin(\ln x) + c$

18 $\frac{-\cos x}{3}\left(\sin^2 x + 2\right) + c$ 19 $\frac{2}{5}\left(\cos^2 x\sqrt{\cos x} - 5\sqrt{\cos x}\right) + c$

20 $\frac{-\cos\sqrt{x}}{3}\left(2\sin^2\sqrt{x} + 4\right) + c$

21 $\frac{\sin(\sin t)}{3}\left(\cos^2(\sin t) + 2\right) + c$

22 $\ln|\sin\theta| + 2\sin\theta + c$ 23 $t\sec t - \ln|\sec t + \tan t| + c$

24 $-\ln(2 - \sin x) + c$ 25 $\frac{1}{2}\ln\left|\cos\left(e^{-2x}\right)\right| + c$

26 $2\ln\left|\sec\sqrt{x} + \tan\sqrt{x}\right| + c$ 27 $\frac{1}{2}\tan x + c$

28 $\frac{1}{6}\left(\arcsin 3x + 3x\sqrt{1 - 9x^2}\right) + c$

29 $\frac{x}{4\sqrt{x^2+4}} + c$

30 $2\ln\left|t + \sqrt{t^2+4}\right| + \frac{1}{2}t\sqrt{t^2+4} + c$

31 $\frac{3}{2}\arctan\left(\frac{1}{2}e^t\right) + c$ 32 $\frac{1}{2}\arcsin\left(\frac{2}{3}x\right) + c$

33 $\frac{1}{3}\ln\left|\frac{3}{2}x + \frac{1}{2}\sqrt{9x^2+4}\right| + c$ 34 $\ln\left|\sqrt{1 + \sin 2x} + \sin x\right| + c$

35 $-\sqrt{4 - x^2} + c$ 36 $\frac{1}{2}\ln\left(x^2 + 16\right) + c$

37 $-\arcsin\left(\frac{x}{2}\right) - \frac{\sqrt{4-x^2}}{x} + c$ 38 $\frac{1}{9}\frac{x}{\sqrt{9-x^2}} + c$

39 $\frac{(x^2+1)^{\left(\frac{3}{2}\right)}}{3} + c$ 40 $\frac{(e^{2x}+1)^{\left(\frac{3}{2}\right)}}{3} + c$

41 $\frac{1}{2}\left(\arcsin\left(e^x\right) + e^x\sqrt{1 - e^{2x}}\right) + c$ 42 $\ln\left(\frac{1}{3}e^x + \frac{1}{3}\sqrt{e^{2x} + 9}\right) + c$

43 $2\sqrt{x}\,(\ln x - 2) + c$

44 $12\ln(x+2) + \frac{8}{x+2} + \frac{x^2}{2} - 4x + c$

45 $\frac{1}{2}\ln\left(x^2+9\right) + c_1$; $x = 3\tan\theta$ yields $\ln\left(\frac{\sqrt{x^2+9}}{3}\right) + c_2$; they differ by a constant

46 $x - 3\arctan\left(\frac{x}{3}\right) + c_1$; $x = 3\tan\theta$ yields $3(\tan\theta - \theta) + c_2 = 3\left(\frac{x}{3} - \arctan\frac{x}{3}\right) + c_2$

Exercise 16.4

1 24 2 40

3 $\frac{24}{25}$ 4 0

5 $\frac{176\sqrt{7} - 44}{5}$ 6 0

7 2 8 -268

9 $\frac{64}{3}$ 10 2

11 $\ln\left(\frac{11}{3}\right)$ 12 $\frac{44}{3} - 8\sqrt{3}$

13 3 14 $\sqrt{\pi} + 1$

15 a) 6 b) 6 c) 12 16 1

17 4 18 0

19 $\frac{\pi}{2}$ 20 $\frac{\pi}{6}$

21 $\frac{\pi}{3}$ 22 $\frac{\pi}{8}$

23 $\frac{14\sqrt{17} + 2}{3}$ 24 $\frac{1}{\pi}$

25 $\ln(2)$ 26 $16\sqrt{2} - 5\sqrt{5}$

27 $\sqrt{14} - \sqrt{10}$ 28 $\frac{3}{2}$

29 $\pi^{\frac{3}{2}}\left(\frac{2\sqrt{3}}{27} - \frac{1}{12}\right)$ 30 $\frac{\pi}{6}$

31 $-\frac{1}{2}\ln\left(\frac{37}{52}\right)$

32 $-\arctan\left(\frac{\sqrt{15} - \sqrt{7}}{4}\right)$ or $\frac{1}{2}\left(\arcsin\left(\frac{1}{4}\right) - \arcsin\left(\frac{3}{4}\right)\right)$

33 $\frac{2}{3}$ 34 0

35 -4 36 $\frac{\pi}{6}$

37 $\frac{1}{6}\arctan\left(\frac{4\sqrt{3}}{9}\right)$ 38 $\frac{\pi\sqrt{3} - 3\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2}\right)}{18}$

39 $\frac{1}{6}$ 40 $\frac{e-1}{2}$

41 $1 + \frac{e}{2}$ 42 $2\cos(1) + 2$

43 $\frac{31}{5}$ 44 $\frac{2}{\pi}$

45 $\frac{12 - 4\sqrt{3}}{\pi}$ 46 $\frac{e^8 - 1}{8e^8}$

47 $\frac{\pi}{6\ln 3}$ 48 $\frac{\sin x}{x}$

49 $-\dfrac{\sin t}{t}$

50 $-2x\dfrac{\sin x^2}{x^2}$

51 $2x\dfrac{\sin x^2}{x^2}$

52 $\dfrac{\cos t}{1+t^2}$

53 $\dfrac{b-a}{5+x^4}$

54 $-\csc\theta - \sec\theta$

55 $\dfrac{1}{4x^{\frac{3}{4}}}\left(e^{x+3x^{\frac{1}{2}}}\right)$

56 Yes

57 a) $\dfrac{1}{3}\ln\left(\dfrac{3k+2}{2}\right)$ b) $k = \dfrac{2\left(e^3-1\right)}{3}$

58 Proof

59 $-(1-x)^{k+1}\left(\dfrac{1}{k+1}+\dfrac{1-x}{k+2}\right)$

60 a) 0 b) $\sqrt{47}$ c) $\dfrac{15\sqrt{47}}{47}$

61 Proof

Exercise 16.5

1 $\frac{1}{2}((1+2\sqrt{2})\ln|x-\sqrt{2}| + (1-2\sqrt{2})\ln|x+\sqrt{2}|)$

2 $3\ln|x-2| - 2\ln|x| + c$

3 $\frac{1}{2}\ln|x^2+4x+3| + c$

4 $-\ln|x+1| + 6\ln|x| - \dfrac{9}{x+1} + c$

5 $\ln|x+3| + 3\ln|x+2| - 2\ln|x| + c$

6 $\ln|x+1| + 3\ln|x| + \dfrac{1}{x} + c$

7 $-\ln|x+2| + \ln|x-1| + c$

8 $\dfrac{3\ln|2x-1|}{2} - 2\ln|x+1| + c$

9 $3\ln|x+2| + \dfrac{2}{x+2} + c$

10 $\ln|x-2| - 4\ln|x+1| + 3\ln|x| + \dfrac{6}{x} + c$

11 $-\ln|x^2+1| + 2\ln|x| + c$

12 $\dfrac{\sqrt{3}}{3}\arctan\left(\dfrac{\sqrt{3}x}{3}\right) - \dfrac{\ln|x^2+3|}{3} + \dfrac{2\ln|x|}{3} + c$

13 $\dfrac{\sqrt{3}}{2}\arctan\left(\dfrac{x}{\sqrt{6}}\right) - \dfrac{\ln|x^2+6|}{6} + \dfrac{\ln|x|}{3} + c$

14 $\dfrac{\sqrt{2}}{2}\arctan\left(\dfrac{\sqrt{2}x}{4}\right) - \dfrac{3}{16}\ln|x^2+8| + \dfrac{3}{8}\ln|x| + c$

15 $\dfrac{\ln|x-5|}{3} + \dfrac{2\ln|x+1|}{3} - \ln|x| + c$

Exercise 16.6

1 $\dfrac{125}{6}$

2 $\dfrac{9\pi^2}{8}+1$

3 $4\sqrt{3}$

4 $\frac{10}{3}$

5 $\frac{8}{21}$

6 $\frac{125}{24}$

7 $\frac{13}{12}$

8 4π

9 $\frac{59}{12}$

10 Approx. 361.95 (4 points of intersection!)

11 $3\ln 2 - \dfrac{63}{128}$

12 Between $-\dfrac{\pi}{6}$ and $\dfrac{\pi}{6}$, $\sqrt{3}\ln\left(\dfrac{3}{4}\right) - 2\sqrt{3} + 4$

13 18

14 $\frac{32}{3}$

15 $\frac{64}{3}$

16 9

17 $\frac{9}{2}$

18 19

19 $\dfrac{2\sqrt{3}}{3}+2$

20 $\frac{37}{12}$

21 $\frac{1}{2}$

22 $\dfrac{2\sqrt{2}}{3}$

23 $\frac{269}{54}$

24 $\dfrac{e}{2}-1$

25 $\dfrac{288\sqrt{3}}{35}$

26 $\dfrac{2\sqrt{2}}{3}$

27 $\dfrac{16}{3}$

28 25.36

29 $m = 0.973$

30 $\dfrac{37}{12}$

Exercise 16.7

1 $\dfrac{127\pi}{27}$

2 $\dfrac{64\sqrt{2}\pi}{15}$

3 $\dfrac{70\pi}{3}$

4 6π

5 9π

6 2π

7 $\left(\dfrac{\sqrt{3}}{2}+1\right)\pi$

8 $\dfrac{512\pi}{15}$

9 Approx. 5.937π

10 $\dfrac{32\pi}{3}$

11 $\pi\left(\sqrt{3}-1\right)$

12 $\dfrac{23\pi}{210}$

13 $288\pi - \dfrac{160\pi\sqrt{5}}{3}$

14 $\dfrac{64}{15}\pi$

15 $\pi\left(\dfrac{1}{2}-\dfrac{1}{4}\sqrt{3}\right)$

16 $\dfrac{1778}{5}\pi$

17 $\dfrac{252}{5}\pi$

18 1419π

19 $\dfrac{9}{8}\pi$

20 a) $\dfrac{88}{15}\pi$ b) $\dfrac{7}{6}\pi$

21 40π

22 $9\pi\left(2-\sqrt{2}\right)$

23 $\dfrac{32}{15}\pi$

24 $\dfrac{4}{5}\pi\left(121\sqrt{33}-25\sqrt{15}\right)$

25 $2\pi\left(\ln 2 - \dfrac{1}{4}\right)$

26 $2\pi\left(\dfrac{11}{3}\sqrt{11}-\dfrac{2}{3}\sqrt{2}\right)$

27 $\dfrac{28}{3}\pi\left(\sqrt{34}-\sqrt{7}\right)$

28 $\pi\left(\dfrac{1}{2}\sqrt{2}\pi - \pi + 2\right)$

29 $\dfrac{284}{3}\pi$

30 2π

31 $\dfrac{256}{15}\pi$

Exercise 16.8

1 $\frac{70}{3}$ m, 65 m

2 8.5 m to the left, 8.5 m

3 1 m, 1 m

4 2 m, $2\sqrt{2}$ m

5 18 m, 28.67 m

6 $\dfrac{4}{\pi}$ m, $\dfrac{4}{\pi}$ m

7 $3t$, 6 m, 6 m

8 $t^2 - 4t + 3$, 0, 2.67 m

9 $1-\cos t$, $\left(\dfrac{3\pi}{2}+1\right)$ m, $\left(\dfrac{3\pi}{2}+1\right)$ m

10 $4 - 2\sqrt{t+1}$, 2.43 m, 2.91 m

11 $3t^2 + \dfrac{1}{2(1+t)^2} + \dfrac{3}{2}$, 11.3 m, 11.3 m

12 $4.9t^2 + 5t + 10$

13 $16t^2 - 2t + 1$

14 $\dfrac{1}{\pi} - \dfrac{\cos\pi t}{\pi}$

15 $\ln\left(t+2\right) + \dfrac{1}{2}$

16 $e^t + 19t + 4$

17 $4.9t^2 - 3t$

18 $\sin(2t) - 3$

19 $-\cos\left(\dfrac{3t}{\pi}\right)$

20 12; 20

21 $\dfrac{13}{2}; \dfrac{13}{2}$

22 $\dfrac{9}{4}; \dfrac{11}{4}$

23 $2\sqrt{3}-6$; $6-2\sqrt{3}$

24 $-\dfrac{10}{3}; \dfrac{17}{3}$

25 $\dfrac{204}{25}$

26 $-6; \dfrac{13}{2}$

27 $\dfrac{166}{5}; \dfrac{166}{5}; \dfrac{166}{5}$

28 a) $50 - 20t$ b) 1062.5

29 1.27 s

30 a) 5 s b) 272.5 m c) 10 s d) -49 m/s e) 12.46 s f) -73.08 m/s

Exercise 16.9

1 $y = \pm 10e^{x^4}$

2 $y = \pm e^{\frac{1}{2}x^2}$

3 $y = \dfrac{2}{2 - x^2}$

4 $y = \dfrac{1}{3 - x}$

5 $y = \ln\left(\dfrac{e}{1 - ex}\right)$

6 $y = \ln\left(e^x - C\right)$

7 $y^3 = \dfrac{3(x+1)^2}{2} - \dfrac{1}{2}$

8 $y = \dfrac{1}{\ln|x+1| + 1}$

9 $2y^3 + 6y = 3x^2 + 6x + 72$

10 $y^2 = e^{x^2} - 1$

11 $\arctan y = \dfrac{x^2}{2} + c$

12 $y + \ln|y| = \dfrac{x^2}{2} - x + 1$

13 $x + \ln\dfrac{1}{x + Ce^x + 1}$

14 $\dfrac{y-1}{y+1} = e^{(x-1)^2} + c$

15 $(y+1)\ln|y+1| + 1 = (y+1)\left(\ln|\ln x|\right) + c$

16 $1 + 2y^2 = c\tan^4\dfrac{x}{2}$

17 $\arcsin y = 1 - \sqrt{1 - x^2}$

18 $y = \ln\left(\ln\dfrac{e\left(e^x + 1\right)}{1 + e}\right)$

19 $y + |\ln y| = \dfrac{x^3}{3} - x - 5$

20 $\cos y = \dfrac{\sqrt{2}}{4}\left(e^x + 1\right)$

21 $|y| = |x|\,e^{x^2 - 1}$

22 $2\ln|y| - y^2 = e^{x^2} - 2$

23 $y + \ln|\sec y| = \dfrac{1}{3}x^3 + x + c$

24 $\sqrt{\left(y^2 + 1\right)^3} = 3e^t\left(t - 1\right) + c$

25 $e^{-y}\left(y + 1\right) = -\dfrac{1}{3}\sin^3\theta + c$

26 $e^{3y} + 3y^2 = 3\left(\cos x + x\sin x\right) - 2$

27 $y = e^x - x^2 + 2$

28 b) $C = 78; m = \dfrac{1}{15}\ln\dfrac{8}{13}$; 45.3 minutes

Practice questions

1 a) $p = 3$ b) 3 square units

2 a) $(0, 1)$ b) $V = \displaystyle\int_0^{\ln 2}\left(e^{\frac{x}{2}}\right)^2 dx$

3 $a = e^2$

4 a) $y = \dfrac{x}{e}$

5 a) (i) 400 m (ii) $v = 100 - 8t$, 60 m/s
(iii) 8 s (iv) 1344 m
b) Distance needed 625

6 b) 2.31 c) $-\pi\cos x - \dfrac{x^2}{2} + c$, 0.944

7 $\ln 3$

8 a) (ii) $(1.57, 0)$; $(1.1, 0.55)$; $(0, 0)$, $(2, -1.66)$
b) $x = \dfrac{\pi}{2}$ c) (ii) $\displaystyle\int_0^{\frac{\pi}{2}} x^2\cos x\,dx$ d) $\dfrac{\pi^2}{2} - 2$

9 a) 2π
b) Range: $\{y \mid -0.4 < y < 0.4\}$
c) (i) $-3\sin^3 x + 2\sin x$ (iii) $\dfrac{2\sqrt{3}}{9}$
d) $\dfrac{\pi}{2}$
e) (i) $\frac{1}{3}\sin^3 x + c$ (ii) $\frac{1}{3}$
f) $\arccos\dfrac{\sqrt{7}}{3} \approx 0.491$

10 c) 3.696 72 d) $\displaystyle\int_0^{\pi}(\pi + x\cos x)dx$
e) $\pi^2 - 2 \approx 7.869\,60$

11 a) (i) $10x + 1 - e^{2x}$ (ii) $\dfrac{\ln 5}{2} \approx 0.805$
b) (i) $f^{-1}(x) = \dfrac{\ln(x - 1)}{2}$
c) $v = \pi\displaystyle\int_0^{\ln 2}(1 + e^{2x})^2\,dx$

12 $\pi\left(\dfrac{2}{15}a^5 + \dfrac{2}{3}a^3\right)$

13 $4\left(\dfrac{2}{5}\left(\dfrac{1}{2}x + 1\right)^{\frac{5}{2}} - \dfrac{2}{3}\left(\dfrac{1}{2}x + 1\right)^{\frac{3}{2}}\right) + c$

14 $a = -\dfrac{56}{27}$

15 $\dfrac{\pi}{2}\left(e^{2k} - 1\right)$

16 $k = 2$

17 1800 m

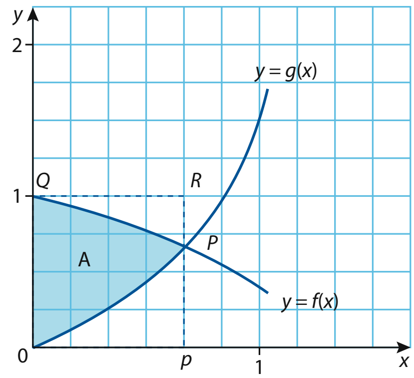
18 $2a$ by $\dfrac{2}{3}a^2$

19 a) $\ln x + 1 - k$ b) $x > \dfrac{1}{e}$
c) (ii) $\left(e^k, 0\right)$ d) $\dfrac{e^{2k}}{4}$
e) $y = x - e^k$ f) Verify
g) Common ratio $= e$

20 $x^2 - 4y^2 = 4$

21 $v = \sqrt{v_0^2 + \dfrac{4k}{m}}$

22 a)



b) Proof c) 0.6937
d) $\displaystyle\int_0^p\left(e^{-x^2} - \left(e^{-x^2} - 1\right)\right)dx \approx 0.467$

23 a) Verify
b) $\dfrac{2\pi}{9}; \dfrac{4\pi}{9}; \dfrac{6\pi}{9}$
c) $\dfrac{n\pi}{9}(n + 1)$

24 a) $t = 0$, 3, or 6
b) (i) $\displaystyle\int_0^6\left|t\sin\left(\dfrac{\pi}{3}\right)t\,dt\right|$ (ii) 11.5 m

25 a) 0.435 b) $\dfrac{-2t}{\left(2 + t^2\right)^2}$

26 a) $\dfrac{dy}{dx} = \dfrac{2x^2}{\sqrt{1 + x^2}} + 2\sqrt{1 + x^2}$
b) Verify c) $k = 0.918$

27 6 m

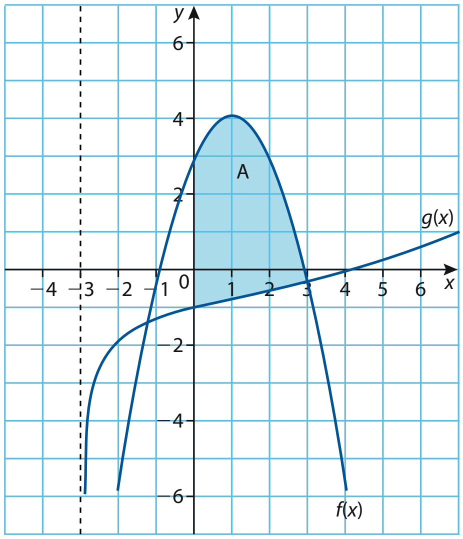
28 0.852

29 a) Verify
a) (i) $A = 78$; $k = \dfrac{1}{15}\ln\dfrac{48}{78}$ (ii) 45.3

30 $y = \tan\left(\ln\dfrac{x}{2}\right)$

31 $\dfrac{(x+2)^2}{2} - 6(x + 2) + 12\ln|x + 2| + \dfrac{8}{x + 2} + c$

32 a)



b) (i) $x = -3$;

(ii) $x - \text{int} = e^2 - 3$; $y - \text{int} = \ln 3 - 2$

c) -1.34; 3.05

d) (ii) $\int_0^{3.05} \left(4 - (1-x)^2 - (\ln(x+3) - 2)\right) dx$

(iii) 10.6

e) 4.63

33 a) Verify

b) $\ln x - \frac{1}{2}\ln(x^2+1) + c$

c) $y = \dfrac{2e^\theta}{\sqrt{e^{2\theta}+1}}$

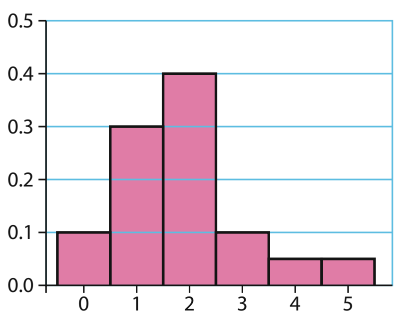
Chapter 17

Exercise 17.1

1 a) Discrete b) Continuous c) Continuous
d) Discrete e) Continuous f) Continuous
g) Discrete h) Continuous i) Continuous
j) Discrete k) Continuous l) Continuous
m) Discrete

2 a) 0.4

b)



c) 1.85, 1.19 e) 2.85, 1.19

f) $E(X + b) = E(X) + b$; $V(X + b) = V(X)$

3 a) 0.26 b) 0.37 c) 0.77
d) 16.29 e) 8.126 f) 4.125; 2.013 25
g) $E(aX + b) = aE(X) + b$; $V(aX + b) = a^2V(X)$

4 a) 0.969 b) 0.163 c) 3.5

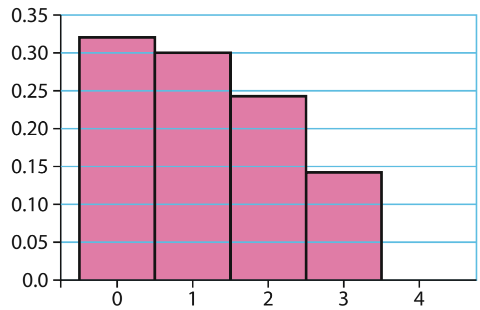
5 $k = \frac{1}{30}$

x	12	14	16	18
$P(X = x)$	$6k$	$7k$	$8k$	$9k$

6 a) $k = \frac{1}{10}$ b) $\frac{37}{60}$ c) $\frac{19}{30}$
d) $E(X) = 16$, SD $= 7$ e) $E(Y) = \frac{11}{5}$; SD $= \frac{7}{5}$

7 a) $\frac{1}{50}$

b)



c) $\frac{17}{25}$

d) $\mu = 1.2$; var $= 0.9$

8 a) $P(x = 18) = 0.2$, $P(x = 19) = 0.1$, symmetric distribution.
b) $\mu = 17$, SD $= 1.095$

9 a) $\mu = 1.9$, SD $= 1.34$
b) Between 0 and 5

10 $k = 0.667$, $E(X) = 5.44$

11 a) $k = 0.3$ or 0.7
b) for $k = 0.3$: $E(X) = 2.18$; for $k = 0.7$: $E(X) = 1.78$

12 a)

y	0	1	2	3
$P(Y = y)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

b) 2

13 a) $k = \frac{1}{10}$ b) $\frac{1}{2}$

14 a) <See table below>
b) 0.85 c) 0.15 d) 48.87
e) 2.057 f) 0.77

x	45	46	47	48	49	50	51	52	53	54	55
CDF	0.05	0.13	0.25	0.4	0.65	0.85	0.9	0.94	0.97	0.99	1

15 a)

x	0	1	2	3	4	5	6
CDF	0.08	0.23	0.45	0.72	0.92	0.97	1

b) 0.72 c) 0.97
d) 2.63 e) 1.44

16 a) 0.90 b) 0.09 c) 0.009
d) Unacceptable, acceptable e) $p(x) = (0.1^{x-1}) \times 0.90$

17 a) 0 b) 0.81 c) 0.162
d) Either acceptable or unacceptable, acceptable
e) $(x - 1)(0.1^{x-2}) \times 0.90^2$, $x > 1$

18 $n = 30$

19 a) (i) $\frac{1}{9}$ (ii) $\frac{1}{81}$
b) (i) $\frac{73}{648}$ (ii) $\frac{575}{1296}$

c) (ii)

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$	$\frac{65}{1296}$	$\frac{175}{1296}$	$\frac{369}{1296}$	$\frac{671}{1296}$

(iii) $\frac{6797}{1296}$

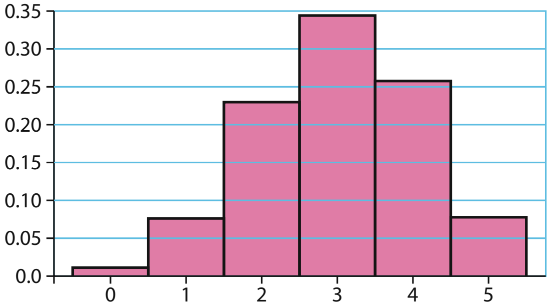
20 9.3

Exercise 17.2

1 a)

x	0	1	2	3	4	5
$P(X = x)$	0.010 24	0.0768	0.2304	0.3456	0.2592	0.077 76

b)



c) (i) Mean = 3, SD = 1.095

(ii) Mean = 3, SD = 1.095

d) Between 2 and 4, and between 1 and 5

e) 0.8352, 0.990. Slightly more than the empirical rule.

2 a) 0.001 294 494 b) 0.000 000 011

c) 0.999 999 99 d) 0.999 999 66

e) Mean = 12, SD = 2.19

3 a)

k	0	1	2	3	4	5	6
$P(x \leqslant k)$	0.117 65	0.420 17	0.744 31	0.929 53	0.989 07	0.999 27	1

b)

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3	0, 1, 2, 3	$P(x \leqslant 3)$	$P(x \leqslant 3)$	0.929 53
At least 3	3, 4, 5, 6	$P(x \geqslant 3)$	$1 - P(x \leqslant 2)$	0.255 69
More than 3	4, 5, 6	$P(x > 3)$	$1 - P(x \leqslant 3)$	0.070 47
Fewer than 3	0, 1, 2	$P(x \leqslant 2)$	$P(x \leqslant 2)$	0.744 31
Between 3 and 5 (inclusive)	3, 4, 5	$P(3 \leqslant x \leqslant 5)$	$P(x \leqslant 5) - P(x \leqslant 2)$	0.254 96
Exactly 3	3	$P(x = 3)$	$P(x = 3)$	0.185 22

4 a)

k	0	1	2	3	4	5	6	7
$P(x \leqslant k)$	0.027 99	0.158 63	0.419 90	0.710 21	0.903 74	0.981 16	0.998 36	1

b)

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3	0, 1, 2, 3	$P(x \leqslant 3)$	$P(x \leqslant 3)$	0.710 21
At least 3	3, 4, 5, 6, 7	$P(x \geqslant 3)$	$1 - P(x \leqslant 2)$	0.580 10
More than 3	4, 5, 6, 7	$P(x > 3)$	$1 - P(x \leqslant 3)$	0.289 79
Fewer than 3	0, 1, 2	$P(x \leqslant 2)$	$P(x \leqslant 2)$	0.419 90
Between 3 and 5 (inclusive)	3, 4, 5	$P(3 \leqslant x \leqslant 5)$	$P(x \leqslant 5) - P(x \leqslant 2)$	0.561 26
Exactly 3	3	$P(x = 3)$	$P(x = 3)$	0.290 304

5 a) p is not constant, trials are not independent.

b) p becomes constant.

c) $n = 3, p = \frac{5}{8}$

y	0	1	2	3
$P(Y = y)$	0.052 73	0.263 672	0.439 453	0.244 141

d) 0.755 86 e) 1.875 f) 0.703 125 g) 0.947 27

6 a) 0.107 374 b) 0.993 63 c) 0.892 63 d) 2

7 a) 0.817 073 b) 1 c) 0.016 1776

8 a) 0.033 833 b) 0.024 486 c) 0.782 722

9 a) 0.75 b) 0.032 5112 c) 0.172 678

10 a) 0.043 1745 b) 0.997 614 c) 0.011 2531

d) 0.130 567 e) 0.956 826 f) 10

g) 3 h) 4, 16

11 a) 3 b) 0.101 308 c) 0.000 214 925

12 a)

x	0	1	2	3	4	5
$P(x)$	0.031 25	0.156 25	0.312 50	0.312 50	0.156 25	0.031 25

b) 0.031 25 c) 0.031 25 d) 0.968 75 e) 0.968 75

f) a)

x	0	1	2	3	4	5
$P(x)$	0.327 68	0.409 60	0.204 80	0.051 20	0.006 40	0.000 32

b) 0.327 68 c) 0.000 32 d) 0.672 32 e) 0.999 68

13 a) 0.138 b) 0.144

14 0.912 96

15 a) 0.107 b) 0.893 c) $n = 14$

Exercise 17.3

Note: most answers are rounded.

1 a) 0.100 82 b) 0.8153 c) 0.1847 d) 0.3203

2 a) 0.1755 b) 0.2650 c) 0.7350 d) 0.6764

3 a) 0.0025 b) 0.9826 c) 0.9999

4 a) 0.9048 b) 0.0047 c) 0.8187

5 a) (i) 0.0344 (ii) 0.8197

b) (i) 0.0001 (ii) 0.9986

6 a) 0.1396 b) 0.1912 c) 0.9576

7 a) 0.000 0768 b) 0.000 076 824

8 a) 0.8187 b) 0.5488

9 a) 0.9877 b) 0.999 998 c) 0.000 0244

10 a) 0.265 b) 0.990

11 a) 0.0908 b) 0.408

12 a) 2.8473 b) 0.617

13 a) 0.245, 0.214, 0.0524 b) 0.464

Exercise 17.4

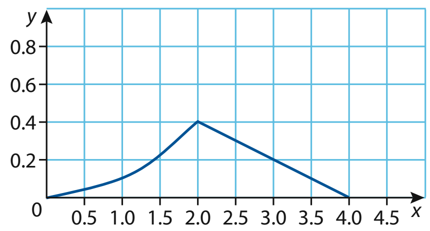
1 a) $k = -\frac{3}{2}$ b) 0.3125 c) 0.6875
d) 0.375, 0.3473, 0.2437

2 a) $\frac{1}{6}$ b) $\frac{1}{8}$ c) $\frac{1}{2}$
d) $\frac{7}{9}$, 0.697, 0.533

3 a) $k = \sqrt{2}$ b) 0.766 c) 0.234
d) 0.754, 0.765, 0.3127

4 a) $\frac{6}{37}$ b) $\frac{133}{148}$ c) $\frac{19}{74}$
d) $\frac{50}{37}$, 1.5, 0.528

5 a)

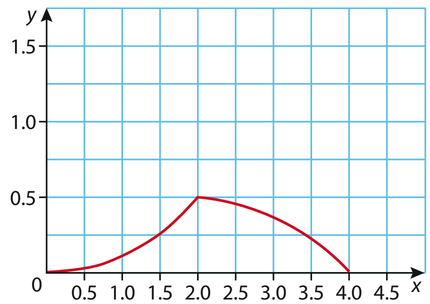


b) $\frac{3}{29}$ c) $\frac{113}{58}$, 1.89, 0.757 d) 0.983

6 a) 24.7 hours b) 0.514 c) 0.264

7 a) 50 hours b) 50 hours c) 22.4 hours
d) 0.104 e) (i) 0.010 82 (ii) 0.9892

8 a)



b) $\frac{7}{3}$ c) 0.694 d) 134 barrels

9 b) $\frac{7}{2}$, 0.916

10 b) $a = \frac{6}{125}$; $b = 5$ c) 1.25

11 a) $k = \frac{1}{(b-a)}$
b) mean = median = $\frac{(a+b)}{2}$; variance = $\frac{(a-b)^2}{12}$

12 a) (i) 0.378 (ii) 1.752 (iii) 1.892
b) 0.955

13 a) $\frac{1}{8}$

b) $$f(x) = \begin{cases} 0 & 0 \leqslant x < 5 \\ \dfrac{3(x-7)^2}{8} & 5 \leqslant x \leqslant 7 \\ 0 & x > 7 \end{cases}$$

c) 5.4126 d) 0.15

14 a) $k = 3$ b) $\frac{4}{5}$ c) 0.8409

15 b) 0.0183 c) $\frac{\sqrt{\pi}}{2}$ d) 0.8326 e) 0.641 f) 0.0769

16 a) $\frac{5}{9}$ b) 0.1944 c) 0.1941 d) 0.6207

17 b) 3, 3.1, 3.3 c) 0.475 d) 1 e) 0.64, no

18 a) $\frac{10}{3}, \frac{15}{4}$ c) 0.0803 d) 0.891
e) (i) 0.987 (ii) 0.9999 (iii) 0.9996

19 $\frac{54}{11}$ **20** 1.08

Exercise 17.5

Note: some answers are rounded.

1 a) 0.5 b) 0.499 571 c) 0.158 655
d) 0.682 690 e) 0.022 750 f) 0

2 a) 0.769 86 b) 0.161 514 c) 0.656 947
d) 0.999 944

3 a) 0.008 634 b) 0.982 732

4 1.28

5 1.96

6 a) 0.066 807 b) 0.682 69 c) 678.16
d) 134.898

7 a) 1.8% b) 509.975 c) 5.71

8 a) 0.9696 b) 0.546 746

9 a) 1 day b) 29 days c) 112 days

10 1.56 **11** 18.95

12 30.81 **13** 100.28

14 29.95

15 $\mu = 21.037, \sigma = 4.252$

16 $\mu = 18.988, \sigma = 0.615$

17 $\mu = 121.936, \sigma = 34.39$

18 a) $\mu = 6.966, \sigma = 0.324$ b) 0.252

19 a) 0.655 422 b) 0.008 198 c) 82 bottles

20 a) 0.227 319 b) 0.55% c) 29.678
d) 229.182

21 a) Not likely: chance is 0.14% b) 15.87%
c) 68.27% d) 5396 e) 43 785

22 a) 6.817 b) 3.4315
c) $\mu = 64.14, \sigma = 7.545$

23 7.3% **24** 216.06 **25** 15.31

26 a) $\mu = 111.89, \sigma = 17.9$ **27** 0.919

28 a) (i) $\sigma = 1.355$ (ii) $\mu = 110.37$
b) A = 108.63; B = 112.11

Practice questions

1 a) 34.5% b) 0.416 c) 3325

2 a) (i) 0.393 (ii) 0.656 b) 50

3 a) 0.1 b) 10 d) 0.739

4 a) $\frac{35}{128}$ b) $\frac{7}{32}$ c) $\frac{91}{128}$

5 a) $a = -0.455, b = 0.682$
b) (i) 0.675
(ii) 0.428
c) (ii) $t = 62.6$

6 a) $\mu = 50 - 10(0.522\,44) \approx 44.8$
b) H1: the mean speed has been affected by the campaign.
c) One-tailed test, as we are interested in a decrease in the mean only (not also an increase).

7 a) 70.1% b) 0.002 26 c) p-value = 5.48%

8 a) 0.0808 c) $\mu = 25.5, \sigma = 0.255$ d) 12 500

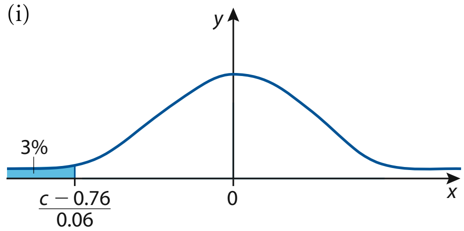
9 a) (i) 0.345 (ii) 0.115 (iii) 0.540
b) 0.119 c) 737

10 a) 15.9% b) 210

11 a) 0.0912 b) $a = 251, b = 369.$

12 a) $a = -1, b = 0.5$ b) (i) 0.841 (ii) 0.533

c) (i)



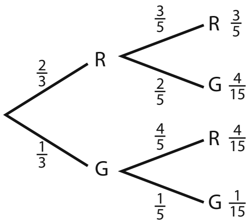
(ii) 0.647

13 a) 2 b) 0.182 c) 0.597

14 $\mu = 66.6, \sigma = 22.6$

15 a) 0.8

b) (i)



(ii)

x	0	1	2
$P(X = x)$	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{2}{5}$

c) $\frac{3}{10}$ d) $\frac{1}{9}$

16 a) 0.129 886 b) 0.676 714 c) 2

17 a) 0.1829 b) 0.3664

18 a) $\frac{1}{5}$ b) $\frac{7}{5}$

19 a) (i) 0.217% (ii) 0.012%

b) 84.13%

20 $\sigma = 0.009\,43 \text{ kg} \approx 9.4 \text{ g}$

21 b) $\frac{e}{4} - \sqrt{e} + \sqrt[4]{e}$

c) $\mu = \frac{e}{2} - 1; \quad \sigma^2 = 1 + \frac{e}{3} - \frac{e^2}{4}$

d) $\sqrt{e} - \frac{e}{2}$

e) $\left(\sqrt{e} - \frac{e}{2}\right)^3$

f) $\binom{3}{2}\left(1 - \sqrt{e} + \frac{e}{2}\right)\left(\sqrt{e} - \frac{e}{2}\right)^2$

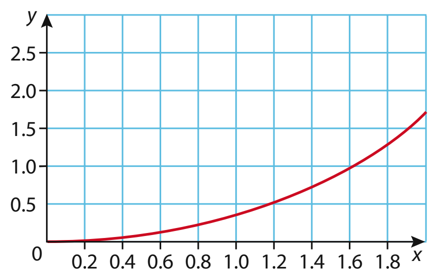
22 a) 0.2212 b) 0.125

23 a) $x = 58.69$ b) $\sigma = 3.41$

c) (i) Karl

(ii) 0.002 39

24 a)



b) 2 c) 1.51 d) 1.61

25 a) $\mu = 1.63$ c) 0.434 d) \$6605.28

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Statistics and Probability

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Statistics and Probability

Assessment statements

7.1 Cumulative distribution functions for both discrete and continuous distributions.
Geometric distribution. Negative binomial distribution.
Probability generating functions for discrete random variables.
Using probability generating functions to find the mean, variance and distribution of the sum of n independent random variables.

7.2 Linear transformation of a single random variable.
Mean of linear combinations of n random variables.
Variance of linear combinations of n independent random variables.
Expectation of the product of independent random variables.

7.3 Unbiased estimators and estimates.
Comparison of unbiased estimators based on variances.
$\overline{X}$ as an unbiased estimator for μ.
S^2 as an unbiased estimator for σ^2.

7.4 The normal distribution of linear combinations of independent normal random variables. In particular, $X \sim \mathrm{N}(\mu, \sigma^2) \Rightarrow \overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.
The central limit theorem.

7.5 Finding confidence intervals for the mean of a normal population.

7.6 Null and alternative hypotheses, H_0 and H_1.
Significance level.
Critical regions, critical values, p-values, one-tailed and two-tailed tests.
Type I and II errors, including calculations of their probabilities.
Testing hypotheses for the mean of a normal population.

7.7 Introduction to bivariate distributions.
Covariance and (population) product moment correlation coefficient ρ.
Proof that $\rho = 0$ in the case of independence and ±1 in the case of a linear relationship between X and Y.
Definition of the (sample) product moment correlation coefficient R in terms of n paired observations on X and Y. Its application to the estimation of ρ.
Informal interpretation of r, the observed value of R. Scatter diagrams.
The following topics are based on the assumption of bivariate normality.
Use of the t-statistic to test the null hypothesis $\rho = 0$.
Knowledge of the facts that the regression of X on Y $(E(X)|Y = y)$ and Y on X $(E(Y)|X = x)$ are linear.
Least-squares estimates of these regression lines (proof not required).
The use of these regression lines to predict the value of one of the variables given the value of the other.

1 Introductory Expectation Algebra

Review

Before starting to work on this option, it may be helpful to look at the chapters in the book relating to the subject – namely, Chapters 11, 12, and 17.

We defined a random variable as a variable that takes on numerical values determined by the outcome of a random experiment.

We also distinguish between two types of variables:

Discrete random variable, if it can take on no more than a countable number of values; and

Continuous random variable, if it can take any value in an interval.

1.1 The expected value of X

We defined the expected value of a random variable as

$\mathrm{E}(X) = \sum_{\text{all } x} xp(x)$ when X is discrete, and

$\mathrm{E}(X) = \int_{\text{all } x} xp(x)\,dx$ when X is continuous.

The expected value of a linear function of X

We start our discussion of the algebra of expectations of random variables with a very simple example.

Example 1

You have a large box containing an equal number of chips with the numbers 0 and 1 on them.

You draw one chip and record the number. Find the expected value and variance of the number you record.

Solution

Since there is an equal chance of drawing 0 or 1, then

$$\mathrm{E}(X) = \sum xp(x) = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{1}{2}$$

$$\mathrm{Var}(X) = \sum x^2 p(x) - \left(\mathrm{E}(X)\right)^2 = 0^2\cdot\frac{1}{2} + 1^2\cdot\frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Example 2

You have a large box containing an equal number of chips with the numbers 0 and 2 on them.

You draw one chip and record the number. Find the expected value and variance of the number you record.

Solution

Since there is an equal chance of drawing 0 or 2, then

$$\mathrm{E}(Y) = \sum yp(y) = 0\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = 1$$

$$\mathrm{Var}(Y) = \sum y^2 p(y) - \left(\mathrm{E}(Y)\right)^2 = 0^2\cdot\frac{1}{2} + 2^2\cdot\frac{1}{2} - (1)^2 = 1$$

Notice here that $Y = 2X$, and $\mathrm{E}(Y) = 2\mathrm{E}(X)$, while $\mathrm{Var}(Y) = 4\mathrm{Var}(X)$.

Example 3

You have a large box containing an equal number of chips with the numbers 0 and 3 on them.

You draw one chip and record the number. Find the expected value and variance of the number you record.

Solution

Since there is an equal chance of drawing 0 or 3, then

$$\mathrm{E}(Y) = \sum yp(y) = 0\cdot\frac{1}{2} + 3\cdot\frac{1}{2} = 3\cdot\mathrm{E}(X)$$

$$\mathrm{Var}(Y) = \sum y^2 p(y) - \left(\mathrm{E}(Y)\right)^2 = 0^2\cdot\frac{1}{2} + 3^2\cdot\frac{1}{2} - \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 9\cdot\mathrm{Var}(X)$$

Theorem

$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$, with $a, b \in \mathbb{R}$

Proof

Discrete case:

$$\begin{aligned} \mathrm{E}(aX+b) &= \sum(ax+b)p(x) = \sum(axp(x)+bp(x)) \\ &= \sum axp(x) + \sum bp(x) = a\sum xp(x) + b\sum p(x) \\ &= a\mathrm{E}(X) + b(1) = a\mathrm{E}(X) + b \end{aligned}$$

Continuous case:

$$\begin{aligned} \mathrm{E}(aX+b) &= \int(ax+b)p(x)dx = \int(axp(x)+bp(x))\,dx \\ &= \int axp(x)dx + \int bp(x)dx = a\int xp(x)\,dx + b\int p(x)\,dx \\ &= a\mathrm{E}(X) + b(1) = a\mathrm{E}(X) + b \end{aligned}$$

1.2 Variance

We defined the variance of a random variable X as

For the discrete case:

$\sigma^2 = \mathrm{E}((X-\mu)^2) = \sum(x-\mu)^2 \cdot p(x)$. We will call it $\mathrm{Var}(X)$.

We also found a short-cut formula for the variance.

$$\sigma^2 = \sum(x-\mu)^2 \cdot p(x) = \sum x^2 \cdot p(x) - \mu^2 = \sum x^2 \cdot p(x) - [\mathrm{E}(X)]^2$$

For the continuous case:

$\sigma^2 = \mathrm{E}((X-\mu)^2) = \int(x-\mu)^2 \cdot p(x)\,dx$, and the short cut is

$$\sigma^2 = \int(x-\mu)^2 \cdot p(x)dx = \int x^2 \cdot p(x)dx - \mu^2 = \int x^2 \cdot p(x)dx - [\mathrm{E}(X)]^2$$

The variance of a linear function of X:
$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$

Proof

Discrete case:

Let $Y = aX + b$, which means that the random variable Y takes values $y = ax + b$ with the same probability as $p(x)$ since a and b are constants.

$$\begin{aligned} \mathrm{Var}(aX+b) = \mathrm{Var}(Y) &= \sum(Y-\mathrm{E}(Y))^2 p(y) \\ &= \sum(aX+b-a\mathrm{E}(X)-b)^2 p(x) \\ &= \sum(a(X-\mathrm{E}(X)))^2 p(x) \\ &= a^2\sum(X-\mathrm{E}(X))^2 p(x) = a^2\mathrm{Var}(X) \end{aligned}$$

The continuous case is left for you to verify as it runs in a parallel manner to the discrete case.

Linear combinations of random variables

In this section, we present some results whose proofs go beyond the scope of the HL course and this publication. Let us start with an example.

Example 4

You have a large box containing an equal number of chips with the numbers 0 and 1 on them.

You draw one chip, record the number and return it to the box, then draw another chip and record the number. Find the expected value and variance of the sum of the numbers you record.

Solution

Since there is an equal chance of drawing 0 or 1, then the probability that the chip number is 0 or 1 is $\frac{1}{2}$.

The random variable in question is the sum of the two numbers, $Z = X_1 + X_2$. The values and their probabilities are summarized below.

z	sample points	$p(z)$
0	(0, 0)	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
1	(1, 0), (0, 1)	$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$
2	(1, 1)	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

$$\mathrm{E}(Z) = \sum zp(z) = 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} = 1$$

$$\mathrm{Var}(Z) = \sum z^2 p(z) - (\mathrm{E}(Z))^2 = 0^2\cdot\frac{1}{4} + 1^2\cdot\frac{1}{2} + 2^2\cdot\frac{1}{4} - 1^2 = \frac{1}{2}$$

Example 5

You have a large box containing an equal number of chips with the numbers 0 and 1 on them.

You draw one chip, record the number and return it to the box, then draw another chip and record the number. Find the expected value and variance of the difference of the numbers you record.

Solution

Since there is an equal chance of drawing 0 or 1, then the probability that the chip number is 0 or 1 is $\frac{1}{2}$.

The random variable in question is the difference of the two numbers, $Z = X_1 - X_2$. The values and their probabilities are summarized below.

z	sample points	$p(z)$
0	(0, 0), (1, 1)	$\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$
1	(1, 0)	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
−1	(0, 1)	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

$$\mathrm{E}(Z) = \sum zp(z) = 0\cdot\frac{1}{2}+1\cdot\frac{1}{4}-1\cdot\frac{1}{4}=0$$

$$\mathrm{Var}(Z) = \sum z^2 p(z) - (\mathrm{E}(Z))^2 = 0^2\cdot\frac{1}{2}+1^2\cdot\frac{1}{4}+(-1)^2\cdot\frac{1}{4}-0^2=\frac{1}{2}$$

Theorem

Let X and Y be *any* two random variables, then $E(aX \pm bY) = aE(X) \pm bE(Y)$, and if the two variables are **independent**, then $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$.

What follows is an outline of a proof of the expected value equation for independent variables X and Y when $a = b = 1$. The proof for any two variables includes material beyond the scope of this course. This proof can be omitted if you wish.

Let X be a random variable that can assume values x_i, where $i = 1, 2, \ldots, n$, and Y be an independent random variable that can assume values y_j, where $j = 1, 2, \ldots, m$.

Hence, since X and Y are independent, $p(x_i + y_j) = p(x_i)p(y_j)$.

And if we are interested in $(X + Y)$, then we need to consider all possible values $(x_i + y_j)$. Here is a table summarizing the values and their corresponding probabilities.

	x_1	x_2	…	x_n	
y_1	$(x_1+y_1)p(x_1)p(y_1)$	$(x_2+y_1)p(x_2)p(y_1)$		$(x_n+y_1)p(x_n)p(y_1)$	$y_1p(y_1)+\mathrm{E}(X)p(y_1)$
y_2	$(x_1+y_2)p(x_1)p(y_2)$			$(x_n+y_2)p(x_n)p(y_2)$	$y_2p(y_2)+\mathrm{E}(X)p(y_2)$
⋮					
y_m	$(x_1+y_m)p(x_1)p(y_m)$			$(x_n+y_m)p(x_n)p(y_m)$	$y_mp(y_m)+\mathrm{E}(X)p(y_m)$

We are interested in E($X + Y$), and therefore we need to calculate the sum

$$\mathrm{E}\left(X+Y\right)=\sum\left(x_i+y_j\right)p(x_i)p(y_j).$$

Now, consider the first row of the table:

$$\begin{aligned}
&\left(x_1+y_1\right)p(x_1)p(y_1)+\left(x_2+y_1\right)p(x_2)p(y_1)+\ldots+\left(x_n+y_1\right)p(x_n)p(y_1)\\
&=p(y_1)\left[\left(x_1+y_1\right)p(x_1)+\left(x_2+y_1\right)p(x_2)+\ldots+\left(x_n+y_1\right)p(x_n)\right]\\
&=p(y_1)\left[\left(x_1p(x_1)+x_2p(x_2)+\ldots+x_np(x_n)\right)+y_1p(x_1)+y_1p(x_2)+\ldots+y_1p(x_n)\right]\\
&=p(y_1)\left[\mathrm{E}(X)+y_1\left(p(x_1)+p(x_2)+\ldots+p(x_n)\right)\right]=p(y_1)\left[\mathrm{E}(X)+y_1\right]\\
&=\mathrm{E}(X)p(y_1)+y_1p(y_1)
\end{aligned}$$

Now, taking the last column of the table into consideration, add all its terms:

$$\begin{aligned}
&y_1p(y_1)+\mathrm{E}(X)p(y_1)+y_2p(y_2)+\mathrm{E}(X)p(y_2)+\ldots+y_mp(y_m)+\mathrm{E}(X)p(y_m)\\
&=\left(y_1p(y_1)+y_2p(y_2)+\ldots+y_mp(y_m)\right)+\left(\mathrm{E}(X)p(y_1)+\mathrm{E}(X)p(y_2)+\ldots+\mathrm{E}(X)p(y_m)\right)\\
&=\mathrm{E}(Y)+\mathrm{E}(X)\left(p(y_1)+p(y_2)+\ldots+p(y_m)\right)=\mathrm{E}(Y)+\mathrm{E}(X)
\end{aligned}$$

The proof for the non-independent case can be run in a similar manner but will require more involvement in the 'joint' distribution of the variables concepts, which are beyond our scope at the moment.

Using the linear functions concept developed earlier, we can easily verify that $\mathrm{E}\left(aX+bY\right)=a\mathrm{E}\left(X\right)+b\mathrm{E}\left(Y\right)$ because

$$\mathrm{E}\left(aX+bY\right)=\mathrm{E}\left(\left(aX\right)+\left(bY\right)\right)=\mathrm{E}\left(aX\right)+\mathrm{E}\left(bY\right)=a\mathrm{E}\left(X\right)+b\mathrm{E}\left(Y\right).$$

The case for $\mathrm{E}\left(aX-bY\right)=a\mathrm{E}\left(X\right)-b\mathrm{E}\left(Y\right)$ is similarly carried out.

The proof for the case of n independent variables is an exercise in mathematical induction, which will be left to the exercises and can also be omitted if the teacher wishes. Here is the result, stated without proof.

$$\mathrm{E}\left(a_1X_1\pm a_2X_2\pm\ldots\pm a_nX_n\right)=a_1\mathrm{E}\left(X_1\right)\pm a_2\mathrm{E}\left(X_2\right)\pm\ldots\pm a_n\mathrm{E}\left(X_n\right)$$

The case for variances will require, in addition to the above, more work with the concept of covariance and hence again we will leave the proof out and accept the result without proof.

$$\mathrm{Var}\left(aX\pm bY\right)=a^2\mathrm{Var}\left(X\right)+b^2\mathrm{Var}\left(Y\right)$$

Please notice here that we add the variances regardless of whether the variables are added or subtracted.

Again the result can be extended to n **independent** variables.

$$\mathrm{Var}\left(a_1X_1\pm a_2X_2\pm\ldots\pm a_nX_n\right)=a_1^2\mathrm{Var}\left(X_1\right)+a_2^2\mathrm{Var}\left(X_2\right)+\ldots+a_n^2\mathrm{Var}\left(X_n\right)$$

Example 6

Looking back at the two previous examples, you can notice that:

$E(X) = E(Y) = \frac{1}{2}$, and $E(X+Y) = 1 = \frac{1}{2} + \frac{1}{2} = E(X) + E(Y)$, and also

$E(X-Y) = 0 = \frac{1}{2} - \frac{1}{2} = E(X) - E(Y)$, and additionally

$\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{1}{4}$, and $\mathrm{Var}(X \pm Y) = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

This example demonstrates the theorem as applied to the case where $a = b = 1$.

Example 7

To demonstrate the theorem above in more detail, let us consider the two random variables X and Y where X is the number showing when we roll a tetrahedral die and Y is the number showing when we roll a cubical die. Here are their probability distributions:

x	1	2	3	4
$p(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

x	1	2	3	4	5	6
$p(y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$x+y$	2	3	4	5	6	7	8	9	10
	(1, 1)	(1, 2) (2, 1)	(1, 3) (3, 1) (2, 2)	(1, 4) (4, 1) (2, 3) (3, 2)	(1, 5) (2, 4) (4, 2) (3, 3)	(1, 6) (2, 5) (3, 4) (4, 3)	(2, 6) (3, 5) (4, 4)	(3, 6) (4, 5)	(4, 6)
$p(x+y)$	$\frac{1}{4}\cdot\frac{1}{6}$	$2\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$3\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$4\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$4\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$4\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$3\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$2\left(\frac{1}{4}\cdot\frac{1}{6}\right)$	$\left(\frac{1}{4}\cdot\frac{1}{6}\right)$

This can be summarized as:

$x+y$	2	3	4	5	6	7	8	9	10
$p(x+y)$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{3}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$

$E(X) = \frac{5}{2}, V(X) = \frac{5}{4}$

$E(Y) = \frac{7}{2}, V(Y) = \frac{35}{12}$

$E(X+Y) = 6, V(X+Y) = \frac{25}{6}$

Interesting application I

If several observations of the same random variable are examined, then the results above have to be applied with great care.

1. If $X_1, \ldots, X_n$ are observations of the same random variable X, then

$$\begin{aligned}\mathrm{E}(a_1X_1 \pm a_2X_2 \pm \ldots \pm a_nX_n) &= a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2) \pm \ldots \pm a_n\mathrm{E}(X_n)\\ &= a_1\mathrm{E}(X) \pm a_2\mathrm{E}(X) \pm \ldots \pm a_n\mathrm{E}(X)\\ &= (a_1 \pm a_2 \pm \ldots \pm a_n)\mathrm{E}(X).\end{aligned}$$

   Special cases:

   $\mathrm{E}(X_1 + X_2) = \mathrm{E}(X) + \mathrm{E}(X) = 2\mathrm{E}(X)$

   Also, $\mathrm{E}(2X) = \mathrm{E}(X + X) = \mathrm{E}(X) + \mathrm{E}(X) = 2\mathrm{E}(X)$ which is a special case of $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$, when $a = 2$ and $b = 0$.

   This result can be generalized to

   $\mathrm{E}(X_1 + X_2 + \ldots + X_n) = n\mathrm{E}(X)$, and

   $$\mathrm{E}(nX) = n\mathrm{E}(X).$$

2. If $X_1, \ldots, X_n$ are **independent** observations of the same random variable X, then

$$\begin{aligned}\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \ldots \pm a_nX_n) &= a_1^2\mathrm{Var}(X) + a_2^2\mathrm{Var}(X) + \ldots + a_n^2\mathrm{Var}(X)\\ &= (a_1^2 + a_2^2 + \ldots + a_n^2)\mathrm{Var}(X)\end{aligned}$$

   Special cases:

   $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X) + \mathrm{Var}(X) = 2\mathrm{Var}(X)$

   However,

   $\mathrm{Var}(2X) = \mathrm{Var}(X + X) \neq \mathrm{Var}(X) + \mathrm{Var}(X) = 2\mathrm{Var}(X)$

   because X and X are NOT independent!

   $\mathrm{Var}(2X) = 2^2\mathrm{Var}(X) = 4\mathrm{Var}(X)$ which is a special case of

   $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$, when $a = 2$ and $b = 0$.

   Here too, the results can be generalized.

   $\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = n\mathrm{Var}(X)$, while

   $\mathrm{Var}(nX) = n^2\mathrm{Var}(X)$.

Example 8

Part I: Throw an unbiased cubical die and define the random variable as the number on the upper side of the die. Compute the expected value and variance of this random variable.

Part II: Throw two unbiased cubical dice and define the random variable as the sum of the numbers on the upper side of each dice. Compute the expected value and variance of this random variable.

Part III: Throw one die and define the random variable as twice the number on the upper side of the die. Compute the expected value and variance of this random variable.

Solution

Here are the probability distributions of the related random variables.

Part I

X	1	2	3	4	5	6	Σ
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$E(X)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{2}$
$E(X^2)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{25}{6}$	$\frac{36}{6}$	$\frac{91}{6}$
$Var(X)$			$\frac{91}{6}$	$-$	$\left(\frac{7}{2}\right)^2$	$=$	$\frac{35}{12}$

Part II

$Y=X_1+X_2$	2	3	4	5	6	7	8	9	10	11	12	Σ
$P(Y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1
$E(Y)$	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{20}{36}$	$\frac{30}{36}$	$\frac{42}{36}$	$\frac{40}{36}$	$\frac{36}{36}$	$\frac{30}{36}$	$\frac{22}{36}$	$\frac{12}{36}$	$\frac{252}{36}=7$
$E(Y^2)$	$\frac{4}{36}$	$\frac{18}{36}$	$\frac{48}{36}$	$\frac{100}{36}$	$\frac{180}{36}$	$\frac{294}{36}$	$\frac{320}{36}$	$\frac{324}{36}$	$\frac{300}{36}$	$\frac{242}{36}$	$\frac{144}{36}$	$\frac{1974}{36}$
$Var(Y)$								$\frac{1974}{36}$	$-$	$(7)^2$	$=$	$\frac{35}{6}$

Part III

$Y=2X$	2	4	6	8	10	12	Σ
$P(Y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$E(Y)$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{6}{6}$	$\frac{8}{6}$	$\frac{10}{6}$	$\frac{12}{6}$	$\frac{42}{6}=7$
$E(Y^2)$	$\frac{4}{6}$	$\frac{16}{6}$	$\frac{36}{6}$	$\frac{64}{6}$	$\frac{100}{6}$	$\frac{144}{6}$	$\frac{364}{6}$
$Var(Y)$			$\frac{364}{6}$	$-$	$(7)^2$	$=$	$\frac{70}{6}$

Notice the following:

$$E(X_1+X_2)=E(2X)=7=2\times\frac{7}{2}=2E(X),$$

$$Var(X_1+X_2)=\frac{35}{6}=2\left(\frac{35}{12}\right)=2Var(X), \text{ while}$$

$$Var(2X)=\frac{70}{6}=\frac{35}{3}=4\left(\frac{35}{12}\right)=4Var(X)$$

Example 9

A multiple choice quiz of 10 questions offers four choices, one of which is correct. A student is guessing on all questions.

a) Find the expected value and variance of the number of questions answered correctly by the student.

b) Set up a table showing the probability distribution of the number of questions answered correctly by the student.

c) Use the table to calculate the expected number and variance of the number of questions answered correctly by the student.

d) The teacher will give a score of 3 marks for each question answered correctly and will not penalize wrong answers. Find the expected score and variance of the scores of the guessing student.

e) Set up a table for the distribution of scores of the student and use it to calculate the expected value and variance of the scores.

Solution

a) This is a binomial distribution with $n = 10$ and probability of success $p = 0.25$.

$\mathrm{E}(X) = np = 10(0.25) = 2.5$

$\mathrm{Var}(X) = npq = 10(0.25)(0.75) = 1.875$

b), c)

x	0	1	2	3	4	5	6	7	8	9	10	Total
$p(x)$	0.056	0.188	0.282	0.25	0.146	0.058	0.016	0.003	4E-04	3E-05	1E-06	1
$xp(x)$	0	0.188	0.563	0.751	0.584	0.292	0.097	0.022	0.003	3E-04	1E-05	2.5
$x^2p(x)$	0	0.188	1.126	2.253	2.336	1.46	0.584	0.151	0.025	0.002	1E-04	8.125
											$\mathrm{Var}(X) = \Sigma x^2p(x) - (\mathrm{E}(X))^2$	1.875

Observe that the expected value and variance agree completely with the theoretical values found in a).

d) Let $Y = 3X$ be the variable representing the score for each question, then:

$\mathrm{E}(Y) = 3\mathrm{E}(X) = 7.5$

$\mathrm{Var}(Y) = 9\mathrm{Var}(X) = 16.875$

e)

$y = 3x$	0	3	6	9	12	15	18	21	24	27	30	Total
$p(y)$	0.056	0.188	0.282	0.25	0.146	0.058	0.016	0.003	4E-04	3E-05	1E-06	1
$yp(y)$	0	0.563	1.689	2.253	1.752	0.876	0.292	0.065	0.009	8E-04	3E-05	7.5
$y^2p(y)$	0	1.689	10.14	20.27	21.02	13.14	5.256	1.363	0.222	0.021	9E-04	73.125
											$\mathrm{Var}(Y) = \Sigma y^2p(y) - (\mathrm{E}(Y))^2$	16.875

Observe that the expected value and variance agree completely with the theoretical values found in d).

Interesting application II

Suppose we repeatedly take samples of size n from a population with mean μ and variance σ^2. Each time we calculate the mean $\bar{X}$ and the variance $\text{Var}(X)$ of the n observations. This way, $\bar{X}$ becomes a random variable itself. Thus we can use what we developed earlier to find $\text{E}(\bar{X})$ and $\text{Var}(\bar{X})$.

Since $\bar{X} = \dfrac{\sum X_i}{n} = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$, and since $\text{E}(X_1) = \text{E}(X_2) = \cdots = \text{E}(X_n) = \mu$, then

$$\begin{aligned}
\text{E}(\bar{X}) &= \text{E}\left(\frac{\sum X_i}{n}\right) = \text{E}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) \\
&= \text{E}\left(\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right) = \frac{1}{n}\text{E}(X_1 + X_2 + \ldots + X_n) \\
&= \frac{1}{n}\left(\text{E}(X_1) + \text{E}(X_2) + \ldots + \text{E}(X_n)\right) = \frac{1}{n}(\mu + \mu + \ldots + \mu) \\
&= \frac{1}{n} \cdot n\mu = \mu.
\end{aligned}$$

Also, since $\text{Var}(X_1) = \text{Var}(X_2) = \cdots = \text{Var}(X_n) = \sigma^2$, then

$$\begin{aligned}
\text{Var}(\bar{X}) &= \text{Var}\left(\frac{\sum X_i}{n}\right) = \text{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) \\
&= \text{Var}\left(\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right) = \frac{1}{n^2}\text{Var}(X_1 + X_2 + \ldots + X_n) \\
&= \frac{1}{n^2}\left(\text{Var}(X_1) + \text{Var}(X_2) + \ldots + \text{Var}(X_n)\right) = \frac{1}{n^2}(\sigma^2 + \sigma^2 + \ldots + \sigma^2) \\
&= \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}.
\end{aligned}$$

These two results are of great importance in later chapters dealing with sampling distributions, confidence intervals and hypothesis testing.

Example 10

A multiple choice quiz of 10 questions offers four choices, one of which is correct. A correct answer is worth 3 marks. A randomly selected group of 36 students who are not familiar with the topic are all guessing on all questions.

a) Find the expected mean score of this group.

b) Find the variance of the mean scores of such groups.

c) You are told that the distribution of scores is normal. What is the probability that a student in this group scores at least 9 marks?

d) Under the same conditions as above, what is the probability that this group's mean is at least 9 marks?

Solution

a) As we proved before, $E(\bar{X}) = \mu = 7.5$.

b) Similarly, $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n} = \dfrac{16.875}{36} = 0.46875$.

c) This is an individual observation probability under a normal distribution with mean 7.5 and variance 16.875.

$P(x \geqslant 9) = 0.3575$ – This is the area under N(7.5, 16.875).

d) This is an average value. The probability uses a normal distribution with mean 7.5 and variance 0.468 75.

$P(\bar{x} \geqslant 9) = 0.000687$ – This is the area under N(7.5, 0.468 75).

1.3 Linear combinations of random variables with known distributions

Normal

A very significant property of normally distributed random variables is that a linear function of one of them or a linear combination of several is also normally distributed.

In particular:

If X is normally distributed with a mean μ and a variance σ^2, i.e. $X \sim (\mu, \sigma^2)$, then $Y = aX + b$ is also normally distributed such that $Y \sim N(a\mu + b, a^2\sigma^2)$.

If X and Y are two normally distributed random variables, then $Z = aX \pm bY$ is also normally distributed with the following results:

$$X \sim (\mu_x, \sigma_x^2), Y \sim (\mu_y, \sigma_y^2) \Rightarrow Z \sim (\mu_x \pm \mu_y, \sigma_x^2 + \sigma_y^2)$$

Example 11

Test scores in a HL class are to be 'curved' as follows: every student will receive 5 marks which are then added to twice the score on the test itself. Given that the test scores are normally distributed with an average of 35 and a standard deviation of 7 marks, find

a) the mean and standard deviation of the 'curved' score

b) the probability that a student receives a score of at least 65 after curving.

Solution

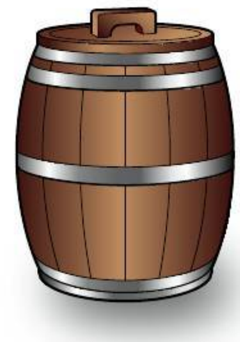
a) Let X be the raw score on the test, and hence $Y = 2X + 5$ will be the curved score.

$\mathrm{E}(Y) = 2 \times 35 + 5 = 75$

$\sigma = \sqrt{\mathrm{Var}(2X+5)} = \sqrt{2^2\,\mathrm{Var}(X)} = 2\sqrt{49} = 14$

b) $\mathrm{P}(Y \geqslant 65) = 0.7625$

Example 12



Wooden barrels are traditionally used to store pickled cucumber in some European countries. To hold the wood together, steel rims are fixed around them. To keep the steel tight around the wood, the rims are slightly smaller in diameter, so that when they are to be fitted, they are heated, to expand slightly, and then fitted over the wood and allowed to cool.

Diameters of one type of these barrels are known to have a normal distribution with mean of 56 cm and a standard deviation of 0.20 cm. The rims, without heating, are constructed so that they yield a diameter that is also normally distributed with a mean of 55.70 cm and a standard deviation of 0.30 cm. The rims are heated so that the diameter increases by 1.5%.

a) What is the probability that a randomly chosen rim will fit around a randomly chosen barrel without heating?

b) What is the probability that a randomly chosen rim will fit around a randomly chosen barrel with heating?

Solution

Let the barrel diameter be B and the rim diameter be R. Therefore, $B \sim \mathrm{N}(56,\ 0.04)$ and $R \sim \mathrm{N}(55.7,\ 0.09)$.

a) Before heating, for a rim to fit around a barrel, the rim's diameter must be larger than the barrel's diameter, i.e. $R - B > 0$.

Hence, if we want to find the probability, we need to consider the distribution of the random variable $(R - B)$. Since R and B are randomly chosen, they are independent random variables and the new variable (call it $Y = R - B$) will also be normal.

$\mathrm{E}(Y) = \mathrm{E}(R) - \mathrm{E}(B) = 55.90 - 56 = -0.03$, and

$\mathrm{Var}(Y) = \mathrm{Var}(R) + \mathrm{Var}(B) = 0.09 + 0.04 = 0.13.$

Therefore, $Y \sim \mathrm{N}(-0.03,\ 0.13)$, and hence

$\mathrm{P}(R - B > 0) = 0.409.$

b) After heating, the diameter of the rim becomes $1.015R$. For a heated rim to fit around a barrel, the rim's diameter must be larger than the barrel's diameter, i.e. $1.015R - B > 0$.

Hence, if we want to find the probability, we need to consider the distribution of the random variable $(1.015R - B)$. Since R and B are randomly chosen, they are independent random variables and the new variable (call it $H = 1.015R - B$) will also be normal.

$\mathrm{E}(H) = 1.015\mathrm{E}(R) - \mathrm{E}(B) = 56.5355 - 56 = .5355$, and

$\mathrm{Var}(H) = 1.015^2\mathrm{Var}(R) + \mathrm{Var}(B) = 0.0927 + 0.04 = 0.133$.

Therefore, $H \sim \mathrm{N}(0.5355, 0.133)$, and hence

$\mathrm{P}(1.015R - B > 0) = 0.929$.

Poisson – sum of two independent Poisson variables (Optional)

The Poisson case is somewhat different from the normal variables. The basic property of the Poisson where $\mathrm{E}(X) = \mathrm{Var}(X)$ limit the cases where you can combine the variables and still have that property hold. It only holds for the sum.

If we have two independent Poisson variables, X and Y, such that $X \sim Po(\lambda)$ and $Y \sim Po(\mu)$.

$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y) = \lambda + \mu$, and since X and Y are independent, then

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \lambda + \mu$, and therefore

$\mathrm{E}(X+Y) = \mathrm{Var}(X+Y)$ which leads us to consider the random variable $X + Y$ also as a Poisson variable. (A formal proof is omitted.)

Notice that this conclusion is not true for all linear combinations of Poisson variables. For example, the random variable $aX + bY$, where a and b are not both equal to 1, cannot be a Poisson variable because

$\mathrm{E}(aX + bY) = a\lambda + b\mu$, while

$\mathrm{Var}(aX + bY) = a^2\lambda + b^2\mu$, and since $a^2 \neq a$ and $b^2 \neq b$, then

$\mathrm{E}(aX + bY) \neq \mathrm{Var}(aX + bY)$ and the variable cannot be a Poisson variable.

Note: In general, aside from the normal variables and Poisson variables, the linear combinations of variables of the same type do not necessarily follow the same type of distribution. So, you cannot say that the sum of two binomial or geometric variables is normal or geometric.

Example 13

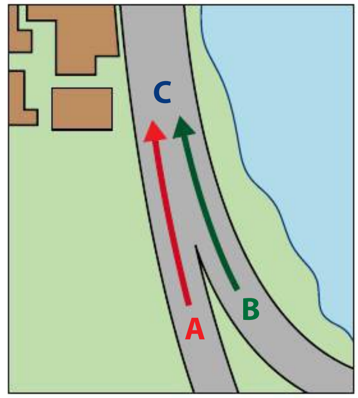
The two streets shown merge into one street at the entrance of a large city. During morning rush hours, the number of cars arriving at the junction through street A is known to be a Poisson variable with mean of 9 cars per minute. The number of cars arriving through B is also Poisson distributed with mean of 12 cars per minute.

a) What is the probability that in any minute there are more than 10 cars arriving from A?

b) What is the probability that in any minute there are more than 10 cars arriving from B?

c) What is the probability that in any minute more than 20 cars join into street C?

d) The capacity of street C is a maximum of 30 cars per minute. If that limit is exceeded then a traffic jam will develop. What is the probability that any minute in the morning rush hour a traffic jam develops?



Solution

a) This is a Poisson cumulative probability calculation where the mean of the distribution is 9: $\mathrm{P}(x > 10) = 1 - \mathrm{P}(x \leqslant 10) = 0.294$. (See right for GDC output.)

b) This is a Poisson cumulative probability calculation where the mean of the distribution is 12: $\mathrm{P}(x > 10) = 1 - \mathrm{P}(x \leqslant 10) = 0.653$. (See right for GDC output.)

```
1-poissoncdf(9,10)
          .2940116791
1-poissoncdf(12,10)
          .6527705824
```

c) This is also a Poisson with mean of 9 + 12 = 21.

$\mathrm{P}(x > 20) = 1 - \mathrm{P}(x \leqslant 20) = 0.529$

d) $\mathrm{P}(x > 30) = 1 - \mathrm{P}(x \leqslant 30) = 0.0242$

```
1-poissoncdf(21,
20)
          .5290256358
1-poissoncdf(21,
30)
          .0241529605
```

1.4 Summary of formulae

Formula	Note
$\mathrm{E}(X) = \sum_{\text{all } x} xp(x)$	discrete
$\mathrm{E}(X) = \int_{\text{all } x} xp(x)\,dx$	continuous
$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$, with $a, b \in \mathbb{R}$	
$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$	
$\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$	
$\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$	independent
$\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$	independent
$\mathrm{E}(a_1X_1 \pm a_2X_2 \pm \ldots \pm a_nX_n) = a_1\mathrm{E}(X_1) \pm a_2\mathrm{E}(X_2) \pm \ldots \pm a_n\mathrm{E}(X_n)$	
$\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \ldots \pm a_nX_n) = a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots + a_n^2\mathrm{Var}(X_n)$	independent
$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = n\mathrm{E}(X)$, and $\mathrm{E}(nX) = n\mathrm{E}(X)$	
$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = n\mathrm{Var}(X)$	
$\mathrm{Var}(nX) = n^2\mathrm{Var}(X)$	

Exercise 1

1 A discrete random variable X has the following probability distribution.

x	0	1	2	3	4
$P(X = x)$	0.1296	0.3456	0.3456	0.1536	0.0256

a Find $P(x \geqslant 2)$ and $P(1 \leqslant x \leqslant 3)$.

b Calculate $E(X)$ and $Var(X)$.

c Let $Y = 9 - 2X$. Calculate $E(Y)$ and $Var(Y)$.

2 A random variable X has the following probability distribution.

x	11	12	13	14	15
$P(X = x)$	0.25	0.2	0.35	k	0.07

a Find the value of k and draw a histogram to represent the distribution.

b Find $P(12 < x \leqslant 14)$ and $P(x \geqslant 14)$.

c Find $E(X)$ and $Var(X)$.

d If $Y = 2X$, find $E(Y)$ and $Var(Y)$ in two ways:

i Using what you learned in this chapter
ii Creating a table for all possible values of Z and then performing the calculations.

e If $Z = X_1 + X_2$, where X_1 and X_2 are randomly chosen independent values of X, find $E(Z)$ and $Var(Z)$ in two ways:

i Using what you learned in this chapter
ii Creating a table for all possible values of Z and then performing the calculations.

3 Two unbiased dice, one cubical and one tetrahedral, are tossed together. The number that each die lands on is the score.

a Set up the probability distribution tables for the scores on each die.

b Calculate the mean and variance of each of the two variables.

c Set up the probability distribution table for the sum of scores on both dice.

d Calculate the mean and variance of the sum of scores in two ways:

i Using the table you created in **c**
ii Using what you learned in this chapter.

4 We run an experiment where 36 cubical unbiased dice are thrown simultaneously and the average of the score is calculated. Supposing the experiment is repeated a large number of times (infinite?), calculate the expected value of the average score of 36 dice and their standard deviation.

5 The probability distribution for a random variable M is given below.

m	1	2	3	4	5
$P(M = m)$	$50k + k^2 - 5$	$35k - 2k^2 - 3$	$6k^2 + 10k - 1$	$32k - 3$	$5k^2 + 12k - 1$

Calculate

a k **b** $E(M)$ **c** $Var(M)$

If $N = 2M_1 + 3M_2$, where M_1 and M_2 are randomly chosen values of M, find

d $E(N)$ **e** $Var(N)$

6 Two independent random variables X and Y are given with the following properties:

$E(X) = 3$, $Var(X) = 2$; $E(Y) = 7$, $Var(Y) = 1$.

Calculate

a $E(X + Y)$, $Var(X + Y)$ **b** $E(X - Y)$, $Var(X - Y)$

c $E(2X + 3Y)$, $Var(2X + 3Y)$ **d** $E(2X - 3Y)$, $Var(2X - 3Y)$

7 Two independent random variables X and Y are given with the following properties:

$E(X^2) = 9$, $Var(X) = 2$; $E(Y^2) = 16$, $Var(Y) = 3$.

Calculate

a $E(X + Y)$, $Var(X + Y)$ **b** $E(X - Y)$, $Var(X - Y)$

c $E(2X + 3Y)$, $Var(2X + 3Y)$ **d** $E(2X - 3Y)$, $Var(2X - 3Y)$

8 Two independent random variables X and Y are given with the following properties:

$E(X^2) = 12$, $Var(X) = 5$; $E(Y^2) = 6$, $Var(Y) = 2$.

Calculate

a $E(2X + Y)$, $Var(2X + Y)$ **b** $E(X - 3Y)$, $Var(X - 3Y)$

c $E(2X + 3Y)$, $Var(2X + 3Y)$ **d** $E(2X - 3Y)$, $Var(2X - 3Y)$

9 Aluminum pipes are produced for an industrial process by two machines. One machine produces 60% of the pipes, each with length 1.05 m, and the second machine produces 40% of the pipes, each with length 0.95 m. All pipes are collected in a central storage place.

a Find the expected length and variance of a pipe.

b An instrument uses two of these pipes joined together in its production. Construct a table showing all possible lengths of the joined pipes and use the table to find the expected length and variance of the joined pipes.

Use the theorems you learned in this chapter to consolidate your results.

c Another instrument uses three of these pipes. Repeat the calculations for **b**. To help you out with the table here is a part of it:

l = *length*	2.85		3.05	3.15
$P(l)$	0.064	0.288		

10 Juice dispensers use juice concentrate to give out the final juice you drink. A machine that dispenses apple juice uses, on average, 40 cm^3 of juice concentrate and 260 cm^3 of water mixed with sugar and other ingredients to give a 'promised' glass of 300 cm^3 of apple juice. The volume of concentrate from this machine has a normal distribution with mean of 40 cm^3 and a standard deviation of 5 cm^3, and the volume of water has a mean of 260 cm^3 and a standard deviation of 8 cm^3.

a What is the probability that a glass from this dispenser will contain more than 305 cm^3?

b You can get a 'double glass' from this machine. The machine will deal with the order as if it is two glasses. So, it produces two glasses successively. What is the probability that the amount you receive is less than 590 cm^3?

c A different dispenser deals with the double glass differently. It will simply double the amount of concentrate and the amount of water. What is the probability that the amount you receive is less than 590 cm^3?

11 A ballpoint pen has an internal chamber filled with ink that is dispensed at the tip during use by the rolling action of a small metal sphere. Some pens have a small sphere with diameter 0.9 mm. The sphere must be held in place by a metal container as shown in the figure to the left. The metal spheres are produced by a machine and their diameters have a normal distribution with mean 0.9 mm and standard deviation of 0.05 mm. The containers are produced by different machines. The diameter of the opening of the container is normally distributed with a mean of 0.8 mm and standard deviation of 0.006 mm. The containers that are too large cannot hold the spheres and those that are too small do not allow enough ink.

Technically, the difference in diameters must not be smaller than 0.003 and not larger than 0.008. One sphere and one container are usually chosen at random to assemble into a pen. What is the probability that they will match?

12 The average number of customers who can be served at the main cash counter in a local supermarket is known to follow a Poisson distribution with 3 customers every 2 minutes. During peak time, a secondary counter must be opened.
The new counter also has a Poisson distribution and can serve 1 customer per minute.
What is the probability that if both counters are open, 5 or more customers can be served every 2 minutes?

Practice questions 1

1 Roger uses public transport to go to school each morning. The time he waits each morning for the transport is normally distributed with a mean of 15 minutes and a standard deviation of 3 minutes.

a On a specific morning, what is the probability that Roger waits more than 12 minutes?

b During a particular week (Monday–Friday), what is the probability that

i his total waiting time does not exceed 65 minutes?
ii he waits less than 12 minutes on at least three days of the week?
iii his average daily waiting time is more than 13 minutes?

2 The weights of male nurses in a hospital are known to be normally distributed with mean $\mu = 72$ kg and standard deviation $\sigma = 7.5$ kg. The hospital has a lift (elevator) with a maximum recommended load of 450 kg. Six male nurses enter the lift. Calculate the probability p that their combined weight exceeds the maximum recommended load.

3 Let X be a random variable with a Poisson distribution such that $\text{Var}(X) = (\text{E}(X))^2 - 6$.

a Show that the mean of the distribution is 3.

b Find $\text{P}(x \leqslant 3)$.

Let Y be another random variable, independent of X, with a Poisson distribution such that $\text{E}(Y) = 2$.

c Find $\text{P}(x + y < 4)$.

d Let $U = X + 2Y$.

i Find the mean and variance of U.
ii State with a reason whether or not U has a Poisson distribution.

4 Let X and Y be two independent variables with $\text{E}(X) = 5$, $\text{Var}(X) = 3$, $\text{E}(Y) = 4$, $\text{Var}(Y) = 2$. Find

a $\text{E}(2X)$ **b** $\text{Var}(2X)$

c $\text{E}(3X - 2Y)$ **d** $\text{Var}(3X - 2Y)$

5 a The independent variables U and V are such that $U \sim \text{N}(66, 5)$ and $V \sim \text{N}(19, 3)$. Calculate the probability that a randomly selected observation from U is more than three times a randomly selected observation from V.

b Let X be a random variable. By expanding the expression $\text{E}(X - \text{E}(X))^2$ show that $\text{E}(X^2) \geqslant (\text{E}(X))^2$.

6 The weights, X kg, of male birds of a certain species are normally distributed with mean 4.5 kg and standard deviation 0.2 kg. The weights, Y kg, of female birds of this species are normally distributed with mean 2.5 kg and standard deviation 0.15 kg.

a i Find the mean and variance of $2Y - X$.
ii Find the probability that the weight of a randomly chosen male bird is more than twice the weight of a randomly chosen female bird.

b Two randomly chosen male birds and three randomly chosen female birds are placed together on a weighing machine for which the recommended maximum weight is 16 kg. Find the probability that this maximum weight is exceeded.

7 A shop sells apples and pears. The weights, in grams, of the apples may be assumed to have a $\text{N}(200, 15^2)$ distribution and the weights of the pears, in grams, may be assumed to have a $\text{N}(120, 10^2)$ distribution.

a Find the probability that the weight of a randomly chosen apple is more than double the weight of a randomly chosen pear.

b A shopper buys 3 apples and 4 pears. Find the probability that the total weight is greater than 1000 grams.

8 **a** The random variable Y is such that $\mathrm{E}(2Y + 3) = 6$ and $\mathrm{Var}(2 - 3Y) = 11$.

Calculate

i $\mathrm{E}(Y)$

ii $\mathrm{Var}(Y)$

iii $\mathrm{E}(Y^2)$

b Independent random variables R and S are such that

$R \sim \mathrm{N}(5, 1)$ and $S \sim \mathrm{N}(8, 2)$.

The random variable V is defined by $V = 3S - 4R$.

Calculate $\mathrm{P}(V > 5)$.

2 Some Discrete Probability Distributions

In the book you have seen several probability distributions. The discrete distributions we studied are: Bernoulli, binomial, and Poisson. In this publication we will examine a few other distributions.

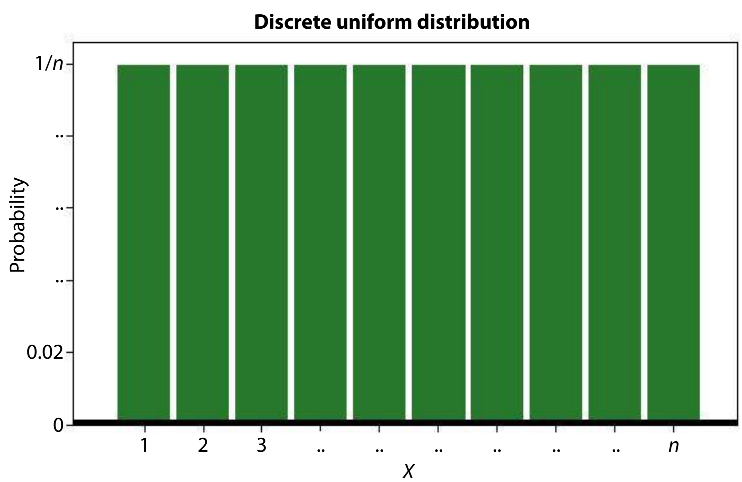
2.1 The discrete uniform distribution (Optional – will not be examined)

The simplest of the discrete distributions is the **uniform distribution**. This distribution describes the several situations where the outcomes of an experiment are all equally likely. In general, if an experiment has n possible outcomes, each of which are equally likely, then each outcome must have the same probability p. Since $\sum_{all\ x} p = 1 \Rightarrow np = 1$, then we can define the distribution of X as follows.

If the random variable X assumes the values $x_1, x_2, \ldots, x_n$ with equal probabilities, then the **discrete uniform distribution** is defined as

$$X \sim \mathrm{DU}(n),\ \mathrm{P}(X = x) = p(x) = \frac{1}{n},\ x = x_1, x_2, \ldots, x_n$$

The bar graph representing a uniform distribution is given below.



Example 1

The most familiar example is the throwing of an unbiased cubical die. X is the number showing on the top face.

a) Find the mean value of X.

b) Find the variance of X.

c) Find an expression for the cumulative function $F(x)$ and hence find $P(x < 5)$.

Solution

a) $\mu = \sum xp(x) = \sum x \cdot \frac{1}{6} = \frac{1}{6}\sum x = \frac{1}{6}(1 + 2 + \ldots + 6) = \frac{7}{2}$

b) $\text{Var}(X) = \text{E}\left(X^2\right) - (\text{E}(X))^2 = \frac{1}{6}(1 + 4 + 9 + \ldots + 36) - \left(\frac{7}{2}\right)^2$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

c) Recall from Section 17.1 that the cumulative distribution function is defined as

$$F(x) = \text{P}\left(X \leqslant x\right) = \sum_{y:y \leqslant x} \text{P}(y).$$

In this particular case:

$$F(x) = \text{P}\left(X \leqslant x\right) = \sum_{y=1}^{x} \text{P}\left(y\right) = \sum_{y=1}^{x} \frac{1}{6} = \frac{x}{6}$$

$$\text{P}\left(x < 5\right) = \text{P}\left(X \leqslant 4\right) = \frac{4}{6} = \frac{2}{3}$$

Expected value

Since $p\left(x\right) = \frac{1}{n}$, $x = x_1, x_2, \ldots, x_n$, then the expected value can easily be found using the established rules.

$$\text{E}\left(X\right) = \sum xp(x) = \sum x \cdot \frac{1}{n} = \frac{1}{n}\sum x$$

In the most used model, where $x_i \in \mathbb{Z}^+$, we have

$$\text{E}\left(X\right) = \frac{1}{n}\sum x = \frac{1}{n}(1 + 2 + \ldots + n) = \frac{1}{n} \cdot \frac{n\left(n+1\right)}{2} = \frac{n+1}{2}.$$

Notice that in the cubical die example, the expected value is $\frac{6+1}{2} = \frac{7}{2}$ as shown earlier.

Variance

Also here, we utilize the 'computation' formula for evaluating the variance.

$$\text{Var}(X) = \text{E}\left(X^2\right) - (\text{E}(X))^2 = \frac{1}{n}\sum x^2 - \left(\frac{1}{n}\sum x\right)^2$$

Again, in the most used model, where $x_i \in \mathbb{Z}^+$, we have

$$\mathrm{Var}(X) = \frac{1}{n}\sum x^2 - \left(\frac{1}{n}\sum x\right)^2 = \frac{1}{n}\left(1^2 + 2^2 + \ldots + n^2\right) - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} = \frac{n^2 - 1}{12}.$$

Notice that in the die example the variance is $\frac{6^2 - 1}{12} = \frac{35}{12}$ as shown earlier.

Cumulative distribution function

As we have seen in the die example, the cumulative distribution function is given by

$$F(x) = \mathrm{P}(X \leqslant x) = \sum_{y=1}^{x}\mathrm{P}(y) = \sum_{y=1}^{x}\frac{1}{n} = \frac{1}{n}\sum_{y=1}^{x}1 = \frac{x}{n}.$$

Exercise 2.1

1 X is a uniformly distributed random variable with values 2, 4, 6, 8, 10, 12, 14.

a Find the mean of X.

b Find the variance of X.

c Establish $F(x)$ and hence find $\mathrm{P}(x < 12)$.

2 The pmf of a random variable X is given in the table below.

x	11	13	15	17	19
$\mathrm{P}(X = x)$	k	k	k	k	k

a Find the value of k.

b Find the mean and standard deviation of X.

c Find $\mathrm{P}(X < \mathrm{E}(X))$.

3 Calculators have built-in functions that produce random numbers. A simple one would be the one that produces random digits, 0 to 9, such that each of them has an equal chance of being chosen.

a What is the probability that any digit is chosen?

b Find the expected value and variance of a random digit appearing.

c If you produce a 3-digit random number, what is the probability it is the number 123?

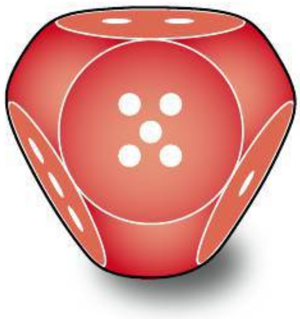
d What is the probability that a 3-digit random number is even?

4 Consider the dodecahedral die. This die has 12 faces that can represent integers between 1 and 12, inclusive.

a Find the expected value if we throw it a large number of times and mark the number on the top face.

b Find the variance.

c Find the probability that a sum of 12 will result if we throw two such dice.



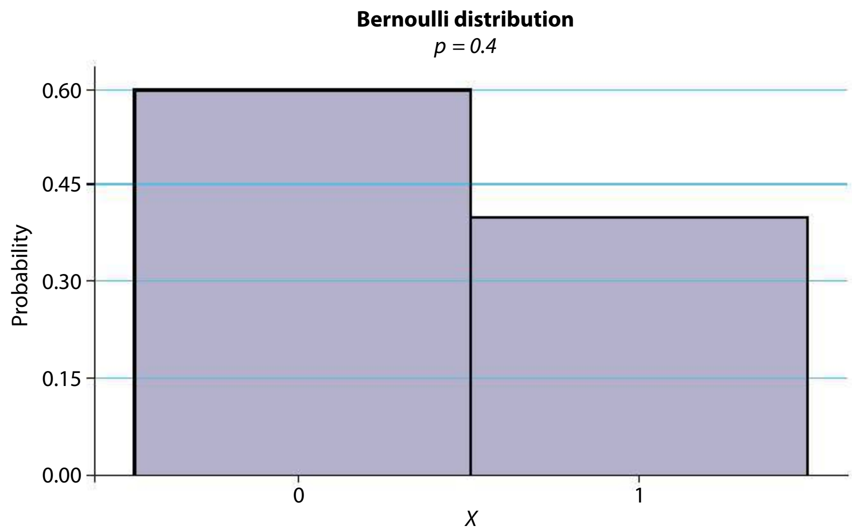
5 Consider the octahedral die. This die has 8 faces that can represent integers between 1 and 8, inclusive.

a Find the expected value if we throw it a large number of times and mark the number on the top face.

b Find the variance.

c Find the probability that a sum of 12 will result if we throw two such dice.

2.2 Bernoulli distribution (important reading – it will not be examined)



The Bernoulli[1] distribution is a discrete distribution having two possible outcomes labelled $x = 0$ and $x = 1$ in which $x = 1$ ('success') occurs with probability p and $x = 0$ ('failure') occurs with probability $1 - p$, where $0 < p < 1$. It therefore has probability function

$$p(x) = \begin{cases} 1-p & \text{for } x = 0 \\ p & \text{for } x = 1 \end{cases}$$

This can also be written

$$P(X = x) = p(x) = p^x(1-p)^{1-x}; x \in \{0,1\}.$$

The **distribution function** for the Bernoulli is

$$F(x) = P(X \leqslant x) = \sum_{y=0}^{x} P(y);$$ when $y = 0$, $F(x) = 1 - p$, and when $y = 1$,

$F(x) = 1 - p + p = 1$. Hence, the distribution function can be written as

$$F(x) = P(X \leqslant x) = \begin{cases} 1-p & \text{for } x = 0 \\ 1 & \text{for } x = 1 \end{cases}$$

[1]Treated in Section 17.2 of the book.

Note: It is a practice to call the probability of failure q, i.e. $q = 1 - p$. We will follow this practice in the rest of the chapter. So, the mass function as well as the distribution functions will be

$$p(x) = \begin{cases} q \text{ for } x = 0 \\ p \text{ for } x = 1 \end{cases}, \text{ and}$$

$$F(x) = \mathrm{P}\left(X \leqslant x\right) = \begin{cases} q & \text{for } x = 0 \\ 1 & \text{for } x = 1 \end{cases}$$

Expected value and variance

Since the Bernoulli experiment has two possible outcomes, this can be summarized in a table.

x	0	1	
$\mathrm{P}(X = x)$	q	p	1
$\mathrm{E}(X)$	$0q = 0$	$1 \cdot p = p$	p
$\mathrm{E}(X^2)$	$0^2q = 0$	$1^2 \cdot p$	p
$\mathrm{Var}(X)$	$\mathrm{E}(X^2) - (\mathrm{E}(X))^2 =$	$p - p^2 =$	$p(1 - p) = pq$

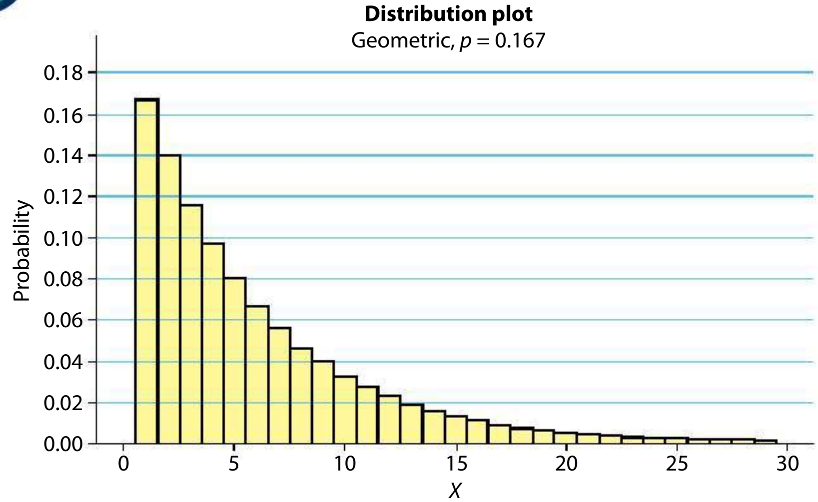
The Bernoulli experiment

A Bernoulli variable is the basic variable in several discrete probability distributions as we discussed in Section 17.2. The Bernoulli experiment, which gives rise to such distributions as the binomial, is characterized by the following:

- the experiment consists of n repeated trials
- the outcome of each trial may be classified in two ways: success or failure
- the probability of success, which we call p, is constant from trial to trial
- the repeated trials are statistically independent.

For example, if we are interested in the distribution of X, the number of successes in a specified number n of Bernoulli trials, then the distribution is the **binomial distribution**. If we are interested in the number of failures till the first success happens, we have the **geometric distribution**, and if we are interested in the number of trials till the rth success happens, then we have the **negative binomial distribution**.

2.3 Geometric distribution



X = total number of trials

Note: Another interpretation of the geometric distribution is that it is the number of failures until the first success. In that respect $x = 0, 1, 2, 3, \ldots$ With this interpretation $\mathrm{E}(X) = \frac{q}{p}$, the variance is the same as before, i.e. $\frac{q}{p^2}$

Consider a Bernoulli experiment where successive trials are performed with a probability of success p as usual. If we consider X to be the number of trials until a success first occurs, then X is said to follow a **geometric** distribution.

As defined above, X is a discrete distribution with domain 1, 2, 3, …, and the event $\{X = x\}$ means that we have a sequence of $x - 1$ failures followed by a success.

$$\underbrace{F \quad F \quad \ldots \quad F}_{x-1} \; S$$

Since this is a Bernoulli experiment and hence the trials are independent, then the probability that this sequence of trials happens is

$$\underbrace{q \cdot q \cdot \cdots \cdot q}_{(x-1)\ times} \cdot p = pq^{x-1}$$

Now we can state the definition.

If X has *pmf*

$$\mathrm{P}(X = x) = \mathrm{P}(x) = pq^{x-1}, \quad x = 1, 2, 3, \ldots$$

then X is said to have a **geometric distribution** with parameter p, and we write $X \sim \mathrm{Geo}(p)$.

(p is also called the probability of success here.)

The above function describes a probability distribution because:

- $\mathrm{P}(x) \geqslant 0$ since $0 \leqslant p \leqslant 1 \Leftrightarrow 0 \leqslant q \leqslant 1 \Rightarrow pq^{x-1} \geqslant 0$, and
- $\sum_{\text{all } x} \mathrm{P}(x; p) = \sum_{x=1}^{\infty} pq^{x-1} = p\sum_{x=1}^{\infty} q^{x-1}$. The term under the summation $\sum_{x=1}^{\infty} q^{x-1}$ can be interpreted as an infinite geometric series with first term 1,

and a common ratio r = q; and since $|r| = |q| < 1$, then

$$\sum_{x=1}^{\infty} q^{x-1} = \frac{1}{1-q} = \frac{1}{p}, \text{ and therefore } \sum_{\text{all } x} \mathrm{P}(x; p) = p\sum_{x=1}^{\infty} q^{x-1} = p \cdot \frac{1}{p} = 1.$$

Example 2

The proportion of left-handed people in a certain area is 7% of the population. In this population, we pick people at random and see whether they are left-handed. What is the probability that

a) the second person you ask is the first left-handed person you pick?

b) the fifth person you ask is the first left-handed person?

```
DISTR DRAW
ø↑Fcdf(
A:binompdf(
B:binomcdf(
C:poissonpdf(
D:poissoncdf(
E:geometpdf(
F:geometcdf(
```

Solution

a) $\mathrm{P}(x = 2) = 0.07 \cdot 0.93 = 0.0651$

b) $\mathrm{P}(x = 5) = 0.07 \cdot 0.93^4 = 0.0524$

Using your GDC, you can also get the same results.

```
geometpdf(.07,2)
               .0651
geometpdf(.07,5)
          .0523636407
```

Cumulative distribution function (The distribution function)

By now you recall the definition of the distribution function $F(x)$.

$$F(x) = \mathrm{P}(X \leqslant x) = \sum_{y=1}^{x} \mathrm{P}(y)$$

Applied to the geometric model, we have

$$F(x) = \mathrm{P}(X \leqslant x) = \sum_{y=1}^{x} \mathrm{P}(y) = \sum_{y=1}^{x} pq^{y-1} = p\sum_{y=1}^{x} q^{y-1}$$

Again, $\sum_{y=1}^{x} q^{y-1}$ is a geometric series with 1 as first term and q as common ratio; hence,

$$\sum_{y=1}^{x} q^{y-1} = 1 \cdot \frac{1-q^x}{1-q} = \frac{1-q^x}{p}, \text{ and therefore}$$

$$F(x) = p\sum_{y=1}^{x} q^{y-1} = p \cdot \frac{1-q^x}{p} = 1 - q^x.$$

Note: There is a quicker way of dealing with the distribution function.

If we consider the probability of having more than x trials for the first success to happen, that is, $\mathrm{P}(X > x)$ which means that we need x successive failures before the success happens, thus

$$\mathrm{P}(X > x) = q^x.$$

Then, the distribution function

$$F(x) = \mathrm{P}(X \leqslant x) = 1 - \mathrm{P}(X > x) = 1 - q^x.$$

Note: In many instances in this book, and in several other books, we choose to call the probability of failure q, i.e. $1 - p = q$. The geometric distribution function is then

$$F(x) = 1 - q^x = 1 - (1 - p)^x.$$

We can also express the geometric model itself as

$$\mathrm{P}(X = x) = p(x) = p(1 - p)^{x-1}, \quad x = 1, 2, 3, \ldots$$

Example 3

```
DISTR DRAW
ø↑Fcdf(
A:binompdf(
B:binomcdf(
C:poissonpdf(
D:poissoncdf(
E:geometpdf(
F:geometcdf(
```

```
geometcdf(1/6,4)
       .5177469136
1-geometcdf(1/6,4)
       .4822530864
```

An unbiased die is thrown repeatedly until 1 shows on its top face.

a) Find the probability that it takes at most 4 throws to get a 1.

b) Find the probability that it takes more than 4 throws to get a 1.

Solution

a) The probability of success $p = \frac{1}{6}$ and failure $q = \frac{5}{6}$, and hence

$$\mathrm{P}(x \leqslant 4) = 1 - \left(\frac{5}{6}\right)^4 = \frac{671}{1296} \approx 0.518$$

b) $\mathrm{P}(x > 4) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296} \approx 0.482$

Expected value

$$\mathrm{E}(X) = \sum_{\text{all } x} x\mathrm{P}(x) = \sum xpq^{x-1} = p\sum xq^{x-1},$$

but

$$\sum xq^{x-1} = 1 + 2q + 3q^2 + 4q^3 + \ldots + nq^{n-1} + \ldots$$

Now, multiply both sides of this equation by q, rearrange and subtract the two equations. (This is a technique we used in Chapter 17.)

$$\begin{aligned} \sum xq^{x-1} &= 1 + 2q + 3q^2 + 4q^3 + \ldots + nq^{n-1}\ldots \\ q \cdot \sum xq^{x-1} &= \qquad q + 2q^2 + 3q^3 + 4q^4 + \ldots \\ \hline \left(\sum xq^{x-1}\right)(1 - q) &= 1 + q + q^2 + q^3 + \ldots \end{aligned}$$

The right-hand side of the equation is an infinite geometric series.

$1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q} = \frac{1}{p}$, and therefore

$$\left(\sum xq^{x-1}\right)(1 - q) = \left(\sum xq^{x-1}\right)p = 1 + q + q^2 + q^3 + \ldots = \frac{1}{p}$$

$$\Rightarrow \sum xq^{x-1} = \frac{1}{p^2}$$

Finally,

$$\mathrm{E}(X) = p\sum xq^{x-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Remember that if we consider X to be the number of failures until the first success, then $\mathrm{E}(X) = \frac{q}{p}$.

Variance (optional)

Also,

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2 = \sum x^2 pq^{x-1} - \left(\frac{1}{p}\right)^2 = p\sum x^2 q^{x-1} - \left(\frac{1}{p}\right)^2$$

However,

$$\begin{aligned} \sum x^2 q^{x-1} &= 1 + 4q + 9q^2 + 16q^3 + \ldots + n^2 q^{n-1} \\ q \cdot \sum x^2 q^{x-1} &= \quad\;\; q + 4q^2 + 9q^3 + 16q^4 + \ldots \\ \hline \left(\sum x^2 q^{x-1}\right)(1-q) &= 1 + 3q + 5q^2 + 7q^3 + \ldots \end{aligned}$$

But,

$$1 + 3q + 5q^2 + 7q^3 + \ldots = 2\left(1 + 2q + 3q^2 + 4q^3 + \ldots\right) - \left(1 + q + q^2 + q^3 + \ldots\right)$$

However, we proved above that

$1 + q + q^2 + q^3 + \ldots = \frac{1}{p}$, and $1 + 2q + 3q^2 + 4q^3 + \ldots = \frac{1}{p^2}$; therefore,

$1 + 3q + 5q^2 + 7q^3 + \ldots = 2\frac{1}{p^2} - \frac{1}{p} = \frac{2-p}{p^2} = \frac{1+1-p}{p^2} = \frac{1+q}{p^2}$, and finally

$$\left(\sum x^2 q^{x-1}\right)(1-q) = 1 + 3q + 5q^2 + 7q^3 + \ldots = \frac{1+q}{p^2}$$

$$\left(\sum x^2 q^{x-1}\right) = \frac{1+q}{p^2(1-q)} = \frac{1+q}{p^3}$$

Thus the variance

$$\mathrm{Var}(X) = p\sum x^2 q^{x-1} - \left(\frac{1}{p}\right)^2 = p\frac{1+q}{p^3} - \frac{1}{p^2} = \frac{q}{p^2}.$$

Summary

If $X \sim \mathrm{Geo}(p)$, then

$$\mathrm{E}(X) = \frac{1}{p} \text{ and } \mathrm{Var}(X) = \frac{q}{p^2}.$$

Example 4

In some countries there is a lottery called '6 out of 45', where the player chooses 6 numbers out of the first 45 positive integers. To win the main prize, the numbers chosen must match the numbers drawn by the lottery company at random. A person plays this lottery once a week.

a) What is the probability of winning the lottery in any week?

b) What is the probability of winning the lottery in week 52?

c) What is the probability of winning within the first 52 weeks? (After you win, you will not play again!)

d) How long should one expect to wait to win the lottery?

e) Find the standard deviation for the number of weeks to win.

Solution

a) Since the lottery company draws the numbers at random, there are $\binom{45}{6}$ ways of drawing these numbers, and the probability of winning is

$$p = \frac{1}{\binom{45}{6}} = \frac{1}{8145060} \approx 0.0000001227738.$$

b) This is a geometric distribution, as winning in one week is independent of the others and the probability of winning is constant.

$$P(x = 52) = \frac{1}{8145060}\left(\frac{8145059}{8145060}\right)^{51} \approx 0.0000001227730.$$

```
geometpdf(1/(45
nCr 6),52)
  1.227730353E-7
geometcdf(1/(45
nCr 6),52)
    6.38421758E-6
█
```

c) This is a cumulative probability calculation:

$$P(X \leqslant x) = 1 - q^x \Rightarrow P(X \leqslant 52) = 1 - \left(\frac{8145059}{8145060}\right)^{52} = 0.00000638$$

d) $E(X) = \dfrac{1}{p} = \dfrac{1}{\dfrac{1}{8145060}} = 8145060$ weeks,

that is about 156 636 years or 1740 life cycles if one manages to live 90 years every time!

e) $\mathrm{Var}(X) = \dfrac{q}{p^2} = \dfrac{\dfrac{8145059}{8145060}}{\left(\dfrac{1}{8145060}\right)^2} = \dfrac{8145060}{8145059} \approx 1,$

and the standard deviation is about 1 week!

Example 5

In a large factory for producing coffee cups, they have a production line that is known to produce chipped cups at a rate of 5%. The quality control person in charge picks items at random from the production line, and checks if they are defective.

a) Find the probability that the first chipped cup
   (i) does not appear in the first 10 selected
   (ii) appears in the first 5 selected.

b) What is the smallest number of items, x, to be checked so that the probability of finding a chipped cup on or before it is at least 95%?

Solution

```
1-geometcdf(.05,
10)
      .5987369392
geometcdf(.05,5)
      .2262190625
```

a) (i) $P(x > 10) = 0.95^{10} = 0.5987$

   (ii) $P(x \leqslant 5) = 1 - 0.95^5 = 0.2262$

b) If the first chipped cup is to appear on the xth draw or before is to be at least 95%, then for that cup to be drawn after the xth draw is at most 5%:

$$\mathrm{P}\left(X > x\right) = q^x \Rightarrow \mathrm{P}\left(X > x\right) = 0.95^x, \text{ so}$$

$$0.95^x \leqslant 0.05 \Rightarrow \ln\left(0.95^x\right) \leqslant \ln 0.05$$

$$\Rightarrow x\ln 0.95 \leqslant \ln 0.05 \Rightarrow x \geqslant \frac{\ln 0.05}{\ln 0.95} = 58.4$$

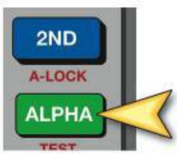
Why did we change the direction of the inequality?

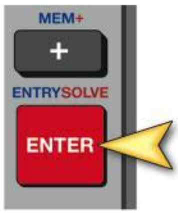
Therefore, we will need to draw at least 59 cups.

```
EQUATION SOLVER
eqn:0=█eometcdf(
.05,X)-0.95
```

To use your GDC for this, first go to the solver and type in your equation as shown right.

After pressing  put the cursor next to the ‘X’, then press the following two buttons in succession.






```
geometcdf(.05…=0
■X=58.999999999…
 bound={-1E99,1 …
■left-rt=-.00105
```

Exercise 2.2 and 2.3

In each of the following questions, please check the assumptions that need to be made before you apply the probability distribution in question.

1 A random variable $X \sim \mathrm{Geo}(0.32)$. Calculate

a $\mathrm{P}(x = 3)$ **b** $\mathrm{P}(x < 3)$

c $\mathrm{P}(x \leqslant 3)$ **d** $\mathrm{E}(X)$

2 An ice cream company starts a game of awarding \$100 in their ice creams. In every 500 ice creams they put 4 such awards. We buy ice creams until we get an award. Find the probability that we will buy

a exactly three ice creams to get an award

b at most three ice creams to get an award

c at least three ice creams to get an award.

3 Find the expected number of ice creams in question 2 we would need to buy in order to get one \$100 award. If the price for one such ice cream is \$1.50, will the company lose money? What is the standard deviation of the number of sold ice creams?

4 A die is biased in a way that the probability that any number shows up is proportional to that number, i.e. $\mathrm{P}(x = 3) = \frac{3}{12}$ for example.

The random variable, X, in this experiment is the number of throws up to and including the first 6. Find

a $\mathrm{P}(x = 3)$ **b** $\mathrm{P}(x \leqslant 3)$

c $\mathrm{P}(x > 5)$ **d** $\mathrm{E}(X)$

e $\mathrm{Var}(X)$ **f** the most likely number of throws until a six appears.

5 There is a tram stop next to Roberto's house. This stop is served by many trams that go to different destinations in the city. He takes tram number 43. 30% of the trams arriving at this stop are number 43. Roberto passes time by counting the number of trams that stop till he catches number 43 to his school.

a What is the number of trams he expects to wait for?

b What is the most likely number of trams he may end up waiting for?

c What is the probability that he will count at most 3 trams?

6 Your GDC generates random numbers by using a random digit generator that selects digits between 0 and 9 randomly in a way that each one of these digits is equally likely to be generated.

a Let X be the digit generated in the process. Find

i $P(x \leqslant 6)$ **ii** $P(x > 3)$
iii $E(X)$ **iv** $Var(X)$

b Let X be the number of digits generated before we get a 6.

i Find the probability that the first occurrence of a 6 is at the sixth digit generated.
ii Find the most likely number of digits to be generated to obtain a 6.
iii Find the expected number of digits generated to achieve a 6.

7 We are given a random variable X, such that $X \sim \text{Geo}(p)$. We also know that $P(x \leqslant 3) = 0.488$.

a Find P. **b** Find $E(X)$ and $Var(X)$. **c** Find $P(x > 5)$.

8 Marko plays tennis relatively well. However, his serves need some improvement. In practice he misses 25% of his serves. Let us call X the number of serves he makes, up to and including the first unsuccessful serve.

a Find the probability that his first unsuccessful serve happens on his third serve.

b Find the probability that he will have at least three successes before he hits a wrong serve.

c Find the expected number of unsuccessful serves and the standard deviation.

9 A lightbulb factory has a defective rate of 8%. Quality control engineers select bulbs at random from the production belt and verify their quality.

a Find the probability that the first defective bulb is found on the sixth pick.

b Find the probability that the first defective bulb is found in fewer than 5 picks.

10 It is widely accepted that about 13% of the population are left-handed. A researcher needs some left-handed people for an experiment. The researcher starts 'recruiting' participants for the experiment from a large group of volunteers.

a On average, how many participants must the researcher check to find a left-handed person?

b What is the probability that the researcher will not find any left-handed person among the first four volunteers she checks?

c What is the probability that the first left-handed person is the sixth volunteer to be checked?

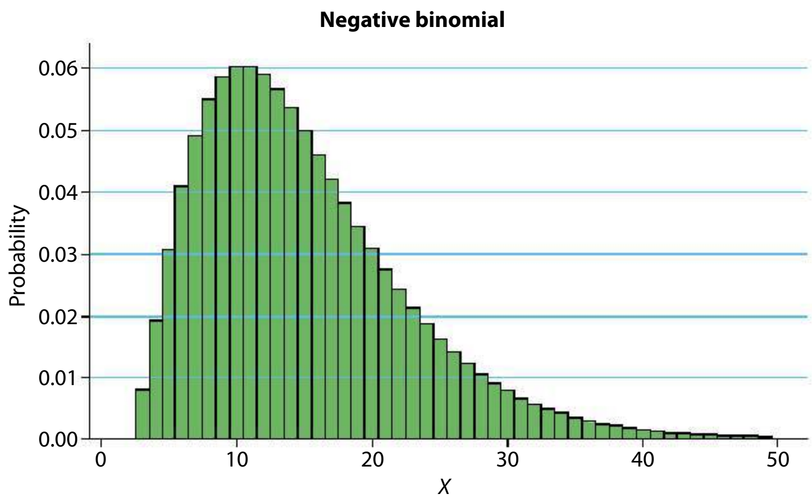
d What is the probability that she finds a left-handed person before checking the 10th volunteer?

11 An oil company is digging for oil in a certain area in the desert. They will drill holes until they find a productive well. The probability they will hit a productive well is 20% at any attempt.

a What is the probability that the third hole drilled is the first to give a productive well?

b It is usually not feasible to drill more than 10 holes to find a productive one. What is the probability that they will fail in this area?

2.4 The negative binomial distribution



X = total number of trials

The geometric distribution introduced in the previous section can be generalized into a more general model. Consider looking for the rth success rather than the first success in a sequence of independent trials of the Bernoulli type. How can we determine the probability that this rth success occurs on the xth trial?

We need the rth success to occur on the xth trial; hence, $r-1$ successes should occur within the $x-1$ trials before the xth one. The rest of the trials, $x-1-(r-1)=x-r$, are failures that occur within the first $x-1$ trials.

$$\underbrace{\overbrace{\overbrace{\underset{1}{S}\quad \underset{2}{F}\quad \underset{3}{S}\quad \underset{4}{S}\quad \cdots\quad \cdots\quad \underset{x-1}{F}}^{(r-1)\,\textit{successes and}\,(x-r)\,\textit{failures}}}^{(x-1)\,\textit{trials}}\quad \overbrace{\underset{x\textit{th trial}}{S}}^{r\textit{th success}}}_{x\textit{ trials}}$$

This means that we have

$\binom{x-1}{r-1}$ ways of getting $r-1$ successes, whose probability according to the binomial theorem is $\binom{x-1}{r-1}p^{r-1}q^{x-r}$.

As the xth trial will be a success with probability p, the probability of observing the rth success to occur on the xth trial is

$$\binom{x-1}{r-1}p^{r-1}q^{x-r}\cdot p=\binom{x-1}{r-1}p^{r}q^{x-r}.$$

The **negative binomial distribution** is used when the number of successes is fixed and we are interested in the number of failures before reaching the fixed number of successes. An experiment which follows a negative binomial distribution will satisfy the following requirements.

1. The experiment consists of a sequence of independent trials.
2. Each trial has two possible outcomes, S or F.
3. The probability of success, p, is constant from one trial to another.
4. The experiment continues until a total of r successes are observed, where r is fixed in advance.

A random variable X which follows a negative binomial distribution is denoted $X \sim \text{NB}(r, p)$.
The probability mass function of the negative binomial is given by

$$\text{P}(X=x)=\binom{x-1}{r-1}p^{r}q^{x-r},\ \text{where}$$

$$0\leqslant p\leqslant 1; r\in\{0,1,2,\ldots\}; x\in\{r,r+1,r+2,\ldots\}.$$

In several cases, statisticians describe the negative binomial distribution in a slightly different manner. X is considered as the number of failures before the rth success. The possible values of X here are $\{0, 1, 2, \ldots\}$. The statement $\text{P}(X = x)$ is equivalent to the probability of $r-1$ successes in the first $x + r - 1$ trials and a success on the $(x + r)$th trial. The pmf of the negative binomial is then of the form

$$\text{P}(X=x)=\binom{x+r-1}{r-1}p^{r}q^{x}.$$

In Example 6, $\text{P}(x = 10)$

$$=\binom{10+3-1}{3-1}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{10}$$

$= 0.0493$.

Example 6

In training, a volleyball player misses his serve one out of every six serves. Find the probability that there are 10 good serves before he misses his serve for the third time.

Solution

Let X be the number of trials before his third miss. So, $X \sim \text{NB}\left(3,\frac{1}{6}\right)$.

Here we are considering the miss to be a success. The number of trials is

10 failures + 2 successes + 1 = 13 (1 is the third success).

Hence,

$$\text{P}(x=13)=\binom{12}{2}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{10}=0.0493.$$

Expected value and variance

A random variable X, having a negative binomial distribution with parameters r and p, is the sum of r independent random variables, each one *geometrically* distributed with parameter p. Intuitively, X is the number of trials needed for the first success, plus the number of trials needed for the second success, ..., plus the number of trials needed for the rth success. Thus, the mean and variance of a random variable X, with parameters r and p, are derived as follows:

$$X = G_1 + G_2 + \ldots + G_r \quad \text{with } G_1, G_2, \ldots, G_r \text{ geometrically distributed with parameter } p$$

$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}(G_1) + \mathrm{E}(G_2) + \ldots + \mathrm{E}(G_r) \\ &= \frac{1}{p} + \frac{1}{p} + \ldots + \frac{1}{p} \text{ (added together } r \text{ times)} \\ &= \frac{r}{p} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{V}(G_1) + \mathrm{V}(G_2) + \ldots + \mathrm{V}(G_r) \\ &= \frac{q}{p^2} + \frac{q}{p^2} + \ldots + \frac{q}{p^2} = \frac{rq}{p^2} \end{aligned}$$

Example 7

Find the expected number of serves the player in the previous example makes before missing for the third time. Also, find the variance.

Solution

$$\mathrm{E}(X) = \frac{r}{p} = \frac{3}{\frac{1}{6}} = 18$$

$$\mathrm{Var}(X) = \frac{rq}{p^2} = \frac{3 \cdot \frac{5}{6}}{\frac{1}{6^2}} = 90$$

Example 8

At a storage space used to collect used cars for a large car dealership, 20% of the cars are usually in need of repairs before they are put on sale again. The lead mechanic has three repair teams at his disposal. He selects the cars at random and checks them one at a time. If the car works, he sends it to be put on display. If the car has defects, he contracts one of the teams to refurbish it. Suppose it takes 2 hours to test a car in good condition and 6 hours to test and refurbish a defective car. Find the mean and standard deviation of the total time it takes the mechanic to use all his three teams.

Solution

Let X be the number of the test on which the third team has to be used. X has a negative binomial distribution with $p = 0.20$. Thus, the expected value is

$$E(X) = \frac{r}{p} = \frac{3}{0.2} = 15, \text{ and the variance is}$$

$$\text{Var}(X) = \frac{rq}{p^2} = \frac{3(0.8)}{0.2^2} = 60.$$

Now, since it takes 4 extra hours to repair a defective car, the total time necessary to contract all three teams is

$$T = 2X + 3(4).$$

Hence,

$$E(T) = 2E(X) + 12 = 2(15) + 12 = 42 \text{ hours, and}$$

$$\text{Var}(T) = 2^2\text{Var}(X) = 4(60) = 240.$$

Thus, the total time needed to use all three teams has a mean of 42 hours and a standard deviation of $\sqrt{240} = 15.5$ hours.

Exercise 2.4

1 The probability that a student believes a rumour about the school closing the next day is 0.75. Find

a the probability that the 8th person to hear the rumour will be the 5th to believe it.

b the probability that the 15th person to hear the rumour will be the 10th to believe it.

c the expected number of students necessary to have 10 believers. Find the standard deviation.

2 If we accept that the probability of having a male or female child is 0.50, find

a the probability that a pair's 3rd child is their first son.

b the probability that a family's 5th child is their second daughter.

c the probability that a family's 6th child is their fifth or sixth son.

3 Actors often forget their lines when taping films. A certain actor misses his lines 30% of the time.

a What is the probability that this actor will get his lines right for the first time on the 5th take?

b What is the probability that this actor will get his lines right for the second time on the 5th take?

4 To raise money for charity, Anna is selling greeting cards in her neighbourhood. She is a nice and polite young lady and sells a card with a probability of 0.45. She needs to sell 6 cards a day. Find the probability that

a she will visit 12 houses

b she will visit at most 8 houses

c she will visit at least 10 houses.

5 Find the expected number of houses in question 4 that Anna will visit in a day.

6 Farmers in northern Austria have water wells on their property. Geological studies indicate that well drilling is successful 40% of the time. Find the probability that a farmer will succeed in having his third water well on the fifth attempt at drilling.

7 30% of the students in a large school have indications of 'Math Anxiety'. The school uses a testing program that identifies students with anxiety. They will involve three students with positive signs of anxiety in a program designed to help them minimize their fear of the subject.

a Find the probability that 10 students have to be tested in order to find the three that will go through the improvement program.

b If each test requires 2 hours to be completed, find the expected value and standard deviation of the total time necessary to identify the three students.

8 10% of the laptops manufactured on an assembly line at a computer company have defective screens. For quality control purposes, laptops are randomly selected, one at a time, and tested.

a What is the probability that the *first* non-defective laptop will be found on the second test?

b What is the probability that the *third* non-defective laptop will be found on the fifth test?

c What is the probability that the *third* non-defective laptop will be found on or before the fifth test?

d Find the mean and standard deviation of the number of the test on which

i the first non-defective laptop is found
ii the third non-defective laptop is found.

9 Telephone lines to my internet provider are all busy 60% of the time.

a If I am calling this provider, what is the probability that I will get through on **i** the first attempt and **ii** the third attempt?

b If I need to call twice, what is the probability that I will complete my calls on **i** the second attempt and **ii** the fifth attempt?

c If you and I need to call this provider (independently), what is the probability that a total of four attempts will be necessary for both of us to get through?

d Find the expected number of calls that I need to attempt in order to get through my first call. Find the standard deviation.

e Find the expected number of calls that I need to attempt in order to get through my third call. Find the standard deviation.

10 Geological research in the North Sea indicates that exploratory oil wells in the area close to the shore are successful 10% of the time. A company has the rights to drill for oil in a certain area near the shores of Norway.

a What is the probability that the first successful oil well is found on the third hole drilled?

b What is the probability that the third successful oil well is found on the seventh hole drilled?

c Find the expected number of wells that must be drilled if this company has to set up three successful oil wells. Find the standard deviation.

d If the cost of drilling a hole in the North Sea is on average 45 million euros and the cost of drilling and setting up a successful well is 139 million euros, find the expected cost and standard deviation of setting up three wells for this company.

2.5 The hypergeometric distribution (Optional – will not be examined)

The hypergeometric distribution models the total number of successes, X, in a fixed size sample n drawn without replacement from a finite population of size N.

The distribution is discrete, existing only for non-negative integers less than the number of samples or the number of possible successes, whichever is greater. The hypergeometric distribution differs from the binomial only in that the population is finite and the sampling from the population is without replacement.

The hypergeometric distribution has three parameters that have direct physical interpretations. N is the size of the population. M is the number of items with the desired characteristic in the population. n is the number of elements in the drawn samples. Sampling 'without replacement' means that once a particular sample is chosen, it is removed from the relevant population for all subsequent selections.

The hypergeometric probability mass function (pmf) is

$$\mathrm{P}\left(X = x\right) = \frac{\begin{pmatrix} M \\ x \end{pmatrix}\begin{pmatrix} N - M \\ n - x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}}, x \in \left\{0, 1, \ldots, n\right\}.$$

We say that the distribution of X is hypergeometric and we write $X \sim \mathrm{Hyp}(n, M, N)$.

One of the most common ways of understanding the hypergeometric distribution is through specific examples.

Example 9

In a set of 50 batteries, there are 10 defective batteries. We select, at random, a set of 5 batteries without replacement. What is the probability that the sample we choose contains 3 defective batteries?

Solution

There are $\binom{50}{5}$ ways of selecting a sample of 5 from these 50 batteries. The number of ways of choosing 3 defective batteries from among the 10 defective ones is $\binom{10}{3}$. However, each time we select 3 defective batteries, the other 2 batteries have to be chosen from among the 'good' batteries, and there are $50 - 10 = 40$ of them. For each choice of 3 defective batteries there are $\binom{40}{2}$ ways of choosing a 'good' battery, and hence there are $\binom{10}{3}\binom{40}{2}$ ways of choosing a sample with 3 defective batteries and 2 'good' ones. Therefore, the required probability is

$$\mathrm{P}(x=3)=\frac{\binom{10}{3}\binom{40}{2}}{\binom{50}{5}}=\frac{93\,600}{2118\,760}=\frac{2340}{52\,969}\approx 0.0442.$$

Note: It turns out that $M = 10$, $N = 50$, and $n = 5$.

The problem of finding the probability of such a 'picking' problem is sometimes called the **urn problem**, since it asks for the probability that x out of n balls drawn are 'good' from an urn that contains M 'good' balls and $N - M$ 'bad' balls.

Expected value and variance

The expected value and variance will be given without proof.

$$\mathrm{E}(X)=n\cdot\frac{M}{N}$$

$$\mathrm{Var}(X)=n\frac{M}{N}\left(1-\frac{M}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Letting $p=\frac{M}{N}$, the above parameters will be

$\mathrm{E}(X)=n\cdot p$, and

$$\mathrm{Var}(X)=np(1-p)\left(\frac{N-n}{N-1}\right)=npq\left(\frac{N-n}{N-1}\right).$$

Notice here that the expected values of the binomial and the hypergeometric are the same, whereas the variances differ by the factor

$$\frac{N-n}{N-1}.$$

This is often called the **finite population correction factor**. Since $N - n \leqslant N - 1$, the correction factor is less than 1, giving the hypergeometric random variable less variance than the binomial random variable. However, when n is small relative to N, this number approaches 1, and makes the two distributions almost identical.

Example 10

In a few neighbouring states of Austria you can still find brown bears. The bears are thought to be near extinction in this area. Environmentalists capture 5 of these bears before they hibernate, tag them and then release them to mix with the rest of the population in an effort to learn more about this population. A year later, a sample of 10 such bears are captured. It is believed that there are 40 of them in this area.

a) Find the expected number of tagged bears in the sample of 10.

b) Find $\mathrm{P}(x = 2)$.

c) Find $\mathrm{P}(x \leqslant 2)$.

Solution

a) $\mathrm{E}(X) = n \cdot p = 10 \cdot \dfrac{5}{40} = 1.25$

b) $\mathrm{P}(x = 2) = \dfrac{\binom{5}{2}\binom{35}{7}}{\binom{40}{10}} = 0.278$

c) $\mathrm{P}(x \leqslant 2) = \mathrm{P}(x = 0, 1, \text{or } 2) = \dfrac{\binom{5}{0}\binom{35}{10} + \binom{5}{1}\binom{35}{9} + \binom{5}{2}\binom{35}{8}}{\binom{40}{10}} = 0.911$

Exercise 2.5

1 There are 13 blue and 7 red marbles in a jar. We randomly draw 4 marbles from the jar. Find the probability that

a all four will be blue

b at most two will be blue

c at least two will be blue.

2 Find the expected number and the standard deviation of the number of blue marbles that will appear in question 1.

3 From a batch of 24 batteries, 6 are selected to be tested. The batch contains 4 defective batteries. What is the probability that

a all 6 will be non-defective?

b only 2 are defective?

c at most 2 are defective?

d at least 2 are defective?

4 In question 3, how many defective batteries might we expect to be included in the 6 that are selected?

5 An insurance company bought 100 laptop computers, 10 of which are Macintosh. 20 laptops are chosen at random to be sent to their location in a major city.

   a What is the probability that no Macs are included in the package?

   b What is the probability that 3 Macs will be included?

   c What is the probability that at most 3 Macs will be included?

   d What is the expected number of Macs to be included in any package?

6 At a flea market, a stand owner selling chocolate boxes included money prizes inside 3 of the 20 boxes he is trying to sell.

   a If a customer buys 4 boxes, what is the probability that none of the purchased boxes contains any prize?

   b If a customer buys 4 boxes, what is the probability that two of the purchased boxes contain prizes?

   c If a customer buys 4 boxes, what is the probability that at least two of the purchased boxes contain prizes?

7 An urn contains 15 marbles, of which 7 are green, 5 are blue and 3 are red. 4 marbles are drawn simultaneously from the urn.

   a What is the probability that

      i all 4 are green?
      ii 2 are green and one blue and one red?
      iii at least 2 are green?
      iv they have all colours?

   b Find the expected number of green marbles in the draw.

8 A large print shop has 14 printing machines. 6 of these machines do colour printing. Every week, 5 of the machines are randomly chosen for inspection. If they are defective, they have to be serviced.

   a What is the probability that

      i 2 of the machines are colour printers?
      ii none of the machines are colour printers?
      iii at most 3 machines are colour printers?

   b i What is the expected number of colour printers that will be inspected?
      ii Inspection of a black-and-white printer takes 2 hours, while a colour printer requires 4 hours of testing. Find the expected number of hours per week spent checking machines.

9 A shipment of 20 iPhones includes 3 that are defective. What is the minimum number of iPhones that we must select to make sure that the probability of selecting at least one defective iPhone is at least 80%.

10 In a large company that claims affirmative action (no bias according to gender or race) a 6-member board of directors has to be chosen from among 20 qualified employees. 8 of the employees were non-natives and 10 were female. The selection is supposed to be at random.

   a If the board contained only 1 non-native, do you have any reason to doubt the randomness of the selection?

   b If the board contained 2 females, do you have any reason to doubt the randomness of the selection?

c If the selection is really random, what is the expected number of non-natives to be selected and what is the standard deviation?

d If the selection is really random, what is the expected number of males to be selected and what is the standard deviation?

11 A quality control engineer inspects a random sample of 3 GDCs from each incoming lot of size 20, and accepts the lot if all are in working condition; otherwise the whole lot is inspected and the cost is passed on to the supplier.

a What is the probability that the lot will be accepted without any further inspection if it really contained four GDCs that are defective?

b What is the probability that the lot will be inspected if the number of defective GDCs is only one?

12 20 microprocessor chips are in stock. Three have etching errors that cannot be detected by the naked eye. Five chips are selected at random and installed in a piece of equipment.

a Set up a table for the pmf for X, the number of chips selected and have etching errors.

b Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

c Find the probability that at least one chip with an etching error will be selected.

Practice questions 2

1 Let $X_1, X_2, \ldots, X_{20}$ be independent random variables each having a geometric distribution with probability of success p equal to 0.6.

Let $Y = \sum_{i=1}^{20} X_i$.

a Explain why the random variable Y has a negative binomial distribution.

b Find the mean and variance of Y.

c Calculate $\mathrm{P}(Y = 30)$.

2 **a** The random variable X has a geometric distribution with parameter $p = \frac{1}{4}$.

What is the value of $\mathrm{P}(x \leqslant 4)$?

b A magazine publisher promotes his magazine by putting a concert ticket at random in one out of every four magazines. If you need 8 tickets to take friends to the concert, what is the probability that you will find your last ticket when you buy the 20th magazine?

c How are the two distributions in parts **a** and **b** related?

3 Probability Generating Functions

3.1 Generating functions

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable, say x, in a formal *power series*.

> A power series is a series of the form:
>
> $$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
>
> where x is a variable and the a_ns are constants called the **coefficients** of the series. For each fixed x, the series above is a series of constants that we can test for convergence or divergence. A series may converge for some values of x and diverge for other values of x. [This topic is beyond the scope of this option and it is a part of the *calculus* option (Topic 9).]
>
> The sum of the series is a function:
>
> $$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$
>
> whose domain is the set of all x for which the series converges. Notice that the function f resembles polynomials with one difference, that it has infinitely many terms.
>
> For example, if we take $a_n = 1$ for all n, then the power series is the usual geometric series
>
> $$1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$
>
> which converges for $|x| < 1$ and diverges for $|x| \geqslant 1$

Generating functions are widely used in mathematics, and play an important role in probability theory. For example, consider a sequence $\{a_i = 0, 1, 2, \ldots\}$ of real numbers. The numbers can be 'bundled up' in several kinds of 'generating functions'. The 'typical' *generating function* of the series corresponding to this sequence is the function defined as:

$$G(t) = \sum_{n=0}^{\infty} a_i t^i$$

for those values of the parameter t for which the sum converges. For a given series, there exists a *radius of convergence* $R \geqslant 0$ such that the series converges absolutely if $|t| < R$ and diverges if $|t| > R$.

> Issues of convergence always arise in dealing with infinite series. In this chapter certain operations on series, such as rearrangement and term-by-term differentiation, are only justified when the series satisfies, convergence conditions (Topic 9). For the purpose of this option, although you should realize when your solutions depend on assumptions about convergence, you do not need to worry about the details. You can assume that, unless stated otherwise, all the necessary conditions hold.

$G(t)$ may be differentiated or integrated term by term any number of times when $|t| < R$. For well-defined series, $G(t)$ can be written in closed form, and the individual numbers in the sequence can be recovered either by series expansion or by taking derivatives.

In this chapter, we will discuss the concept of a probability generating function. When you have completed it you should be able to

- understand the concept of a probability generating function and be able to construct and use the probability generating function for specific distributions such as Binomial, negative Binomial, Geometric and Poisson
- use formulae for the mean and variance of a discrete random variable in terms of its probability generating function, and to use these formulae to calculate the mean and variance of probability distributions
- use the result that the probability generating function of the sum of independent variables is the product of the individual probability generating functions of those variables.

We will start with the definition of a generating function for a sequence.

Definition 1

The *generating function* for the sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of real numbers is the infinite series:

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_nx^n.$$

Example

The generating function for the sequence with general term $a_n = 4$ is $\sum_{n=0}^{\infty} 4x^n$.

The generating function for the sequence $a_n = n + 3$ is $\sum_{n=0}^{\infty}(n+3)x^n$, and that for $a_n = 5^n$ is $\sum_{n=0}^{\infty} 5^n x^n$.

Example 1

Find the generating function for the sequence 1, 1, 1, 1, 1, 1, 1, 1, 1.

Solution

The generating function for the sequence 1, 1, 1, 1, 1, 1, 1, 1, 1 is:

$$1 + x + x^2 + \ldots + x^8.$$

This is a geometric series with 9 terms.

Hence

$$1 + x + x^2 + \ldots + x^8 = \frac{1 - x^9}{1 - x} = \frac{x^9 - 1}{x - 1}.$$

Thus, the generating function for the sequence 1, 1, 1, 1, 1, 1, 1, 1, 1 is

$$G(x) = \frac{x^9 - 1}{x - 1}.$$

Example 2

Find the generating function for the sequence $a_i = \binom{n}{i}$, where n is a certain positive integer and $i = 0, 1, 2, \ldots n$.

Solution

The generating function for this sequence is:

$$G(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n.$$

By the Binomial Theorem, this is obviously:

$$G(x) = (1 + x)^n.$$

Example 3

Find the generating function for 1, 5, 5^2, 5^3, ….

Solution

Since we know from Example 2 that this is the sequence $a_n = 5^n$, its generating function is:

$$G(x) = \sum_{n=0}^{\infty} 5^n x^n = 1 + 5x + 5^2x^2 + \ldots$$

This is clearly an infinite geometric series that converges if $|5x| < 1$, i.e., $-\frac{1}{5} < x < \frac{1}{5}$, and therefore, its limit is $G(x) = 1 + 5x + 5^2x^2 + \ldots = \frac{1}{1 - 5x}$.

3.2 Probability generating function

Consider a discrete random variable, X that takes non-negative values, x_i. Let the probabilities, p_i, associated with these values, that is

$$p_i = P(X = x_i),\ i = 0, 1, 2, \ldots$$

(If X takes a finite number of values, we simply assign zero probabilities to those values that cannot occur.)

The *probability generating function*, often abbreviated as PGF, which is usually denoted by $G_X(t)$, is defined as:

$$G(t) = \sum_{i=0}^{\infty} p_i t^{x_i} = p_0 t^{x_0} + p_1 t^{x_1} + p_2 t^{x_2} + \ldots$$

$$= \sum_{\text{all} x} p_x t^x = E(t^X)$$

In this book we will use the IBO notation $G(t)$ instead of $G_X(t)$. (Except in some cases where we need to distinguish between two random variables. So for example, when we discuss the PGFs of X and Y, we will use $G_X(t)$ and $G_Y(t)$). Also, remember: $G(t)$ only exists if this series converges.

It is important to remember that in this power series expansion

$$G(t) = \sum_{i=0}^{\infty} p_i t^{x_i} = p_0 t^{x_0} + p_1 t^{x_1} + p_2 t^{x_2} + \ldots$$

the coefficient p_i of t^{x_i} is the probability $P(X = x_i)$. As a convention, when it is clear from context that the random variable describes nonnegative integers, we will use x and x_i or i interchangeably.

We will start with a simple example.

Example

Consider the discrete random variable X, which has the probability distribution shown below.

x	1	3	5	7	9
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{20}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{1}{10}$

In this case, the PGF of X is given by

$$G(t) = \sum_{\text{all} x} p_i t^{x_i} = \frac{1}{5}t^1 + \frac{3}{20}t^3 + \frac{1}{4}t^5 + \frac{3}{10}t^7 + \frac{1}{10}t^9$$

Our first reaction is that it does not seem to be a very handy definition, particularly as t does not have any clear meaning. Nonetheless, you will realize in the course of this chapter that it provides a powerful tool for finding the mean and variance of certain probability distributions.

Recalling the definition given earlier, you can see that $G(t)$ is created by multiplying each value of t^X by the associated probability and then adding. Thus, $G(t)$ is the expected value of t^X. This is:

$$G(t) = E(t^X)$$

If we substitute $t = 1$ in the equation above, we get:

$$G(1) = E(1^X) = E(1) = 1$$

This is so, because according to the definition:

$$G(1) = \frac{1}{5}1^1 + \frac{3}{20}1^3 + \frac{1}{4}1^5 + \frac{3}{10}1^7 + \frac{1}{10}1^9 = \frac{1}{5} + \frac{3}{20} + \frac{1}{4} + \frac{3}{10} + \frac{1}{10} = \frac{20}{20} = 1$$

Generally, this is true because if you substitute $t = 1$ into the general definition of $G(t)$, you have:

$$G(1) = \sum_{\text{all } x} p_i 1^{x_i} = \sum_{\text{all } x} p_i = p_1 + p_2 + \ldots = 1$$

Since the sum of probabilities in a PGF is 1.

Example 4

Throw two unbiased dice and add the numbers on the upper faces. Let X be the number of throws till you get the first 6. Find the PGF for this variable and verify that $G(1) = 1$.

Solution

Since we have 36 possibilities, of which 5 give a sum of 6, then the probability of throwing a sum of 6 is $\frac{5}{36}$ and the probability of not throwing a sum of 6 is $\frac{31}{36}$.

Now, X has a geometric distribution where:

$$P(X = x) = \frac{5}{36}\left(\frac{31}{36}\right)^{x-1} \text{ where } x = 1, 2, 3, \ldots$$

Hence

$$G(t) = \frac{5}{36}t^1 + \frac{5}{36}\left(\frac{31}{36}\right)t^2 + \frac{5}{36}\left(\frac{31}{36}\right)^2 t^3 + \frac{5}{36}\left(\frac{31}{36}\right)^3 t^4 + \ldots$$

This is nothing but an infinite geometric series with first term $\frac{5}{36}t$ and a common ratio of $\left(\frac{31}{36}\right)t$.

As you recall from Chapter 4, an infinite geometric series converges to $\frac{a}{1-r}$ provided that $|r| < 1$.

Therefore, in this case, provided that $\left|\left(\frac{31}{36}\right)t\right| < 1$, the series converges to

$$G(t) = \frac{\frac{5}{36}t}{1 - \frac{31}{36}t} = \frac{5t}{36 - 31t}.$$

Now substitute $t = 1$. This gives $G(1) = \frac{5}{36 - 31} = \frac{5}{5} = 1$.

Note: $t = 1$ is possible in this case because $r = \frac{31}{36}t = \frac{31}{36} < 1$.

Properties of generating functions

Consider $G(t)$ together with its first and second derivatives $G'(t)$ and $G''(t)$. (The differentiation is with respect to t.)

$$G(t) = \sum_{all\,x} p_i t^{x_i} = p_0 t^{x_0} + p_1 t^{x_1} + p_2 t^{x_2} + \dots$$

$$G'(t) = x_0 p_0 t^{x_0 - 1} + x_1 p_1 t^{x_1 - 1} + x_2 p_2 t^{x_2 - 1} + \dots$$

$$G''(t) = x_0(x_0 - 1) p_0 t^{x_0 - 2} + x_1(x_1 - 1) p_1 t^{x_1 - 2} + x_2(x_2 - 1) p_2 t^{x_2 - 2} + \dots$$

Now consider the values of these functions at $t = 1$:

$$G(1) = \sum_{all\,x} p_i 1^{x_i} = p_0 1^{x_0} + p_1 1^{x_1} + p_2 1^{x_2} + \dots = \sum_{all\,x} p_i$$

$$\begin{aligned} G'(1) &= x_0 p_0 1^{x_0 - 1} + x_1 p_1 1^{x_1 - 1} + x_2 p_2 1^{x_2 - 1} + \dots \\ &= x_0 p_0 + x_1 p_1 + x_2 p_2 + \dots \end{aligned}$$

$$\begin{aligned} G''(1) &= x_0(x_0 - 1) p_0 1^{x_0 - 2} + x_1(x_1 - 1) p_1 1^{x_1 - 2} + x_2(x_2 - 1) p_2 1^{x_2 - 2} + \dots \\ &= x_0(x_0 - 1) p_0 + x_1(x_1 - 1) p_1 + x_2(x_2 - 1) p_2 + \dots \\ &= x_0^2 p_0 + x_1^2 p_1 + x_2^2 p_2 + \dots - (x_0 p_0 + x_1 p_1 + x_2 p_2 + \dots) \end{aligned}$$

Now, recall that $E(f(X)) = \sum_{all\,x} f(x) p(X = x)$ and look at the results above.

We can deduce the following properties:

Property 1

$$G(1) = \sum_{all\,x} p_i = 1$$

This is so because the sum of all probabilities must be equal to 1. This property helps us decide whether a generating function represents a probability distribution.

Property 2

$$G'(1) = x_0 p_0 + x_1 p_1 + x_2 p_2 + \dots = \sum_{all\,x} x P(X = x) = E(X)$$

This is the basic definition of the expected value of a random variable. Thus, the first derivative of the PGF evaluated at $t = 1$ is nothing but the expected value of the random variable.

Property 3

$$\begin{aligned} G''(1) &= x_0(x_0 - 1) p_0 + x_1(x_1 - 1) p_1 + x_2(x_2 - 1) p_2 + \dots \\ &= \sum_{all\,x} x(x - 1) P(X = x) = E(X(X - 1)) \end{aligned}$$

Using the results above, we can deduce the following theorem.

Theorem 1

The variance of a probability distribution can be expressed as:

$$V(X) = G''(1) + G'(1) - (G'(1))^2$$

Proof

Recall that the variance of a random variable is given by:

$V(X) = E(X^2) - (E(X))^2$

From Property 3 above we have:

$G''(1) = E(X(X-1)) = E(X^2 - X) = E(X^2) - E(X)$

Now, with simple algebraic manipulation we have:

$G''(1) = E(X^2) - G'(1) \Rightarrow E(X^2) = G''(1) + G'(1)$

Thus

$$\begin{aligned} V(X) &= E(X^2) - (E(X))^2 \\ &= G''(1) + G'(1) - (G'(1))^2 \end{aligned}$$

3.3 PGFs of known probability distributions

Here are the PGFs of some of the common distributions:

1 Bernoulli (Not required on exams)

Since $X = 0$, or $X = 1$, and $P(X = 1) = p$, then

$p_1 = p$, $p_0 = 1 - p = q$ and $p_x = 0$ for $x \neq 0$ or 1.

Thus, $G(t) = E(t^X) = \sum_{\text{all } x} p_x t^x = p_0 t^0 + p_1 t^1 = q + pt$

Also, $G'(t) = p$ and $G''(t) = 0$

Thus, $E(X) = G'(1) = p$ as expected, also as expected:

$V(X) = G''(1) + G'(1) - (G'(1))^2 = 0 + p - p^2 = p(1-p) = pq.$

2 Geometric

With probability of success as p and failure as q, then

$p_x = pq^{x-1}$, $x = 1, 2, 3, \ldots$, and consequently

$$\begin{aligned} G(t) = E(t^X) &= \sum_{\text{all } x} p_x t^x = p_1 t^1 + p_2 t^2 + p_3 t^3 + \ldots \\ &= pt + pqt^2 + pq^2t^3 + \ldots = pt + pt(qt) + pt(qt)^2 + \ldots \end{aligned}$$

This series is an infinite geometric series with first term pt and common ratio qt that converges if $|qt| < 1 \Rightarrow |t| < \dfrac{1}{q}$.

Thus

$$G(t) = pt + pt(qt) + pt(qt)^2 + \ldots = \frac{pt}{1 - qt}.$$

Now,

$$G(1) = \frac{p}{1-q} = \frac{p}{p} = 1$$

$$G'(t) = \frac{p(1-qt)+qpt}{(1-qt)^2} = \frac{p}{(1-qt)^2}$$

$$\Rightarrow G'(1) = E(X) = \frac{p}{p^2} = \frac{1}{p}$$

Finally

$$G''(t) = \frac{2pq(1-qt)}{(1-qt)^4} = \frac{2pq}{(1-qt)^3}$$

$$\Rightarrow G''(1) = \frac{2q}{p^2}$$

Thus, the variance is

$$V(X) = G''(1) + G'(1) - (G'(1))^2$$

$$= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} = \frac{2q-q}{p^2} = \frac{q}{p^2}$$

Note here the simplicity with which we were able to calculate the expected value and variance in comparison to what we did on pages 1074–1075.

Remember that our discussion of the geometric distribution considered x as the number of *trials* until the first success, and hence $x = 1, 2, 3, \ldots$. However, if we consider x to be the number of failures till the first success, then $p_i = pq^i$, $i = 0, 1, 2, 3, \ldots$ and hence

$$G(t) = E(t^x) = \sum_{\text{all } x} p_i t^{x_i} = p_0 t^0 + p_1 t^1 + p_2 t^2 + p_3 t^3 + \ldots$$

$$= p + pqt + pq^2t^2 + pq^3t^3 + \ldots = p(1 + qt + (qt)^2 + (qt)^3 + \ldots)$$

$$= p\left(\frac{1}{1-qt}\right) = \frac{p}{1-qt}$$

$$G'(t) = \frac{pq}{(1-qt)^2}$$

$$\Rightarrow G'(1) = E(X) = \frac{pq}{p^2} = \frac{q}{p}$$

and

$$G''(t) = \frac{2pq^2}{(1-qt)^3}$$

$$\Rightarrow G''(1) = \frac{2q^2}{p^2}$$

So the variance is

$$V(X) = G''(1) + G'(1) - (G'(1))^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2+pq}{p^2} = \frac{q(q+p)}{p^2} = \frac{q}{p^2}$$

3 Binomial

Let $X \sim \mathrm{B}(n, p)$, then

$$p(X = x) = \binom{n}{x} p^x q^{n-x} \text{ with } x = 0,\ 1,\ 2, \ldots$$

$$G(t) = E(t^X) = \sum_{\text{all } x} p_x t^x = p_0 t^0 + p_1 t^1 + p_2 t^2 + \ldots$$

$$= \binom{n}{0} p^0 q^{n-0} t^0 + \binom{n}{1} p^1 q^{n-1} t^1 + \binom{n}{2} p^2 q^{n-2} t^2 + \binom{n}{3} p^3 q^{n-3} t^3 + \ldots$$

$$= \binom{n}{0} p^0 t^0 q^{n-0} + \binom{n}{1} p^1 t^1 q^{n-1} + \binom{n}{2} p^2 t^2 q^{n-2} + \binom{n}{3} p^3 t^3 q^{n-3} + \ldots$$

$$= \binom{n}{0} (pt)^0 q^{n-0} + \binom{n}{1} (pt)^1 q^{n-1} + \binom{n}{2} (pt)^2 q^{n-2} + \binom{n}{3} (pt)^3 q^{n-3} + \ldots$$

Using the Binomial theorem, the last expression is the expansion of

$(pt + q)^n$

Thus $G(t) = (q + pt)^n$.

The first two derivatives are:

$G'(t) = np(q + pt)^{n-1}$

$G''(t) = n(n-1)p^2(q + pt)^{n-2}$

Thus

$G'(1) = np(q + p)^{n-1} = np$

$G''(1) = n(n-1)\, p^2(q + p)^{n-2} = n(n-1)p^2$

And so

$E(X) = G'(1) = np$, and

$V(X) = G''(1) + G'(1) - (G'(1))^2$

$= n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p) = npq$

4 Poisson

Let $X \sim \text{Po}(\lambda)$, then

$$p(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}, x = 0, 1, 2 \ldots$$

Now,

$$G(t) = \sum_{\text{all } x} p_x t^x = p_0 t^0 + p_1 t^1 + p_2 t^2 + \ldots = e^{-\lambda} \frac{\lambda^0}{0!} t^0 + e^{-\lambda} \frac{\lambda^1}{1!} t^1 + e^{-\lambda} \frac{\lambda^2}{2!} t^2 + \ldots$$

$$= e^{-\lambda} \left(\frac{\lambda^0}{0!} t^0 + \frac{\lambda^1}{1!} t^1 + \frac{\lambda^2}{2!} t^2 + \ldots \right) = e^{-\lambda} \left(\frac{(\lambda t)^0}{0!} + \frac{(\lambda t)^1}{1!} + \frac{(\lambda t)^2}{2!} + \ldots \right)$$

The expression in brackets is a power series expansion for $e^{\lambda t}$. Thus, the generating function for Poisson is:

$$G(t) = \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} t^x = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}$$

Notation

This is Topic 3 material, which is mentioned here for your information only.

Consider the first two derivatives of $G(t)$

$$G'(t) = \lambda e^{\lambda(t-1)} \Rightarrow G'(1) = E(X) = \lambda e^{\lambda(1-1)} = \lambda$$

and

$$G''(t) = \lambda^2 e^{\lambda(t-1)} \Rightarrow G''(1) = \lambda^2 e^{\lambda(1-1)} = \lambda^2$$

Thus

$$V(X) = G''(1) + G'(1) - (G'(1))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

5 Negative Binomial (stated without proof)

Let $X \sim \text{NegBin}(n, p)$, then

$$G(t) = \sum_{i=0}^{\infty} \binom{i-1}{n-1} p^n q^{i-n} t^i = \left(\frac{pt}{1-qt}\right)^n, |t| < \frac{1}{q}$$

Example 5

Find the probability mass function (PMF) for the distribution with PGF :

$$G(t) = \frac{1}{2-t}.$$

Solution

$$G(t) = \frac{1}{2-t} = \frac{1}{2}\left(\frac{1}{1-\frac{t}{2}}\right) = \frac{1}{2}\left(1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \ldots\right)$$

The result in brackets is due to the fact that the sum of an infinite geometric series with first term 1 and common ratio r is $\frac{1}{1-r}$.

Thus

$$G(t) = \frac{1}{2}\left(1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \ldots\right) = \frac{1}{2}\sum_{x=0}^{\infty}\left(\frac{t}{2}\right)^x = \sum_{x=0}^{\infty}\left(\frac{1}{2}\right)^{x+1} t^x$$

Now, since by definition

$$G(t) = \sum_{i=0}^{\infty} p_i t^{x_i} = p_o t^{x_o} + p_1 t^{x_1} + \ldots$$

$$= \sum_{all\, x} p_x t^x$$

Then it is clear from the expression above that the PMF for this distribution is

$$p(x) = \left(\frac{1}{2}\right)^{x+1}, x = 0, 1, 2, \ldots$$

Example 6

If the random variable X has the PGF

$$G(t) = \frac{t+2}{(2-t^2)(4-t)},$$

find $P(X = 3)$.

Solution

We can rearrange the function as follows

$$G(t) = (t+2)\cdot\frac{1}{2\left(1-\frac{t^2}{2}\right)}\cdot\frac{1}{4\left(1-\frac{t}{4}\right)} = \frac{(t+2)}{8}\cdot\frac{1}{\left(1-\frac{t^2}{2}\right)}\cdot\frac{1}{\left(1-\frac{t}{4}\right)}$$

Remembering the sum of infinite geometric series we know that

$$\frac{1}{\left(1-\frac{t^2}{2}\right)} = 1+\frac{t^2}{2}+\left(\frac{t^2}{2}\right)^2+\left(\frac{t^2}{2}\right)^3+\ldots, \text{ and}$$

$$\frac{1}{\left(1-\frac{t}{4}\right)} = 1+\frac{t}{4}+\left(\frac{t}{4}\right)^2+\ldots$$

Thus

$$G(t) = \left(\frac{t}{8}+\frac{1}{4}\right)\cdot\left(1+\frac{t^2}{2}+\left(\frac{t^2}{2}\right)^2+\left(\frac{t^2}{2}\right)^3+\ldots\right)\cdot\left(1+\frac{t}{4}+\left(\frac{t}{4}\right)^2+\left(\frac{t}{4}\right)^3\ldots\right)$$

Now, the coefficient of t^3 is equal to $P(X=3)$. Therefore, by considering the terms that will contain t^3 in the product above we have:

$$P(X=3) = \frac{1}{8}\cdot\frac{1}{2}\cdot 1+\frac{1}{8}\cdot 1\cdot\frac{1}{16}\cdot+\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot 1\cdot\frac{1}{64} = \frac{27}{256}$$

Theorem 2

Let X be a discrete random variable, whose possible values are all non-negative integers. The following statements are true:

- $G(0) = P(X=0)$
- $G'(0) = P(X=1)$
- $G''(0) = 2P(X=2), \Rightarrow P(X=2) = \frac{G''(0)}{2}$
- In general: $G^{(i)}(0) = i!P(X=i) \Rightarrow P(X=i) = \frac{G^{(i)}(0)}{i!}$ where $G^{(i)}(t)$ is the ith derivative of $G(t)$.

Proof

Because the possible values are all non-negative integers, $i = 0, 1, 2, \ldots$, we can write $G(t)$ in the form

$$G(t) = \sum_{\text{all } x} P(X=x)t^x = P(X=0)t^0 + P(X=1)t^1 + P(X=2)t^2 + P(X=3)t^3 + \ldots$$

$$= P(X=0) + P(X=1)t + P(X=2)t^2 + P(X=3)t^3 + \ldots$$

$$\Rightarrow G(0) = P(X=0)$$

Also,

$$G'(t) = P(X = 1) + 2P(X = 2)t + 3P(X = 3)t^2 + \ldots \Rightarrow G'(0) = P(X = 1)$$

$$G''(t) = 2P(X = 2) + (3\cdot 2)P(X = 3)t + \ldots \Rightarrow G''(0) = 2P(X = 2)$$

Continuing this way, we obtain the general formula

$$G^{(i)}(0) = i!\,P(X = i) \Rightarrow P(X = i) = \frac{G^{(i)}(0)}{i!}$$

We can see its application in the example below.

Example 7

Consider some PGFs of known distributions given previously.

Consider the Geometric distribution.

$$G(t) = \frac{pt}{1 - qt} \Rightarrow G(0) = 0 \text{ which is obviously } p(X = 0).$$

$$G'(t) = \frac{p}{(1 - qt)^2} \Rightarrow G'(0) = p = p(X = 1), \text{ as well as}$$

$$G''(t) = \frac{2pq}{(1 - qt)^3} \Rightarrow G''(0) = 2pq \Rightarrow p(X = 2) = \frac{2pq}{2} = pq.$$

Try the other models yourself and you will see that once you have the PGF of a distributions, then you can completely define the distribution (see Exercise 4).

3.4 Probability generating function of the sum of independent random variables

Note: It is important to observe that if two random variables X and Y are independent then $E(XY) = E(X)E(Y)$.

Example

Throw a 4-sided fair die and a 6-sided die. The numbers at the bottom side for each are recorded. The random number we consider here is the product of the numbers on each die.

Here is a table of the 'joint' events. Let X represent the 6-sided die and Y represent the 4-sided one.

Y \ X	1	2	3	4	5	6
1	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
2	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$

Y \ X	1	2	3	4	5	6
3	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
4	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$

Here is the PMF for the product

Y \ X	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24
p	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$

Now, the expected values of the different random variables are given below.

$$E(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

$$E(Y) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{4} = \frac{10}{4} = \frac{5}{2}$$

$$\begin{aligned}E(XY) &= 1\cdot\frac{1}{24} + 2\cdot\frac{2}{24} + 3\cdot\frac{2}{24} + 4\cdot\frac{3}{24} + 5\cdot\frac{1}{24} + 6\cdot\frac{3}{24} + 8\cdot\frac{2}{24}\\ &\quad + 9\cdot\frac{1}{24} + 10\cdot\frac{1}{24} + 12\cdot\frac{3}{24} + 15\cdot\frac{1}{24} + 16\cdot\frac{1}{24} + 18\cdot\frac{1}{24}\\ &\quad + 20\cdot\frac{1}{24} + 24\cdot\frac{1}{24} = \frac{210}{24} = \frac{35}{4}\end{aligned}$$

Notice that $E(XY) = E(X)E(Y)$.

Theorem 3

If X and Y are independent non-negative integer-valued random variables, with generating functions $G_X(t)$ and $G_Y(t)$ respectively, then the generating function $G_{X+Y}(t)$ of $X + Y$ is given by:

$$G_{X+Y}(t) = G_X(t)G_Y(t).$$

Proof outline

Because X and Y are independent, so are t^X and t^Y. Hence, we know that $E(t^X t^Y) = E(t^X)\,E(t^Y)$. In order to see the result, note that

$$G_{X+Y}(t) = E(t^{X+Y}) = E(t^X t^Y) = E(t^X)E(t^Y) = G_X(t)G_Y(t).$$

Note: The result in Theorem 3 can be generalized to include more than two independent random variables.

Example

Using the fact that a Binomial distribution is a repeated Bernoulli distribution, we can find the PGF of a binomial very easily by using Theorem 3.

Let $X \sim B(n, p)$

Recall that the PGF for a Bernoulli experiment is

$G(t) = q + pt$

Since the Binomial is the sum of the Bernoulli trials, let Y represent a Bernoulli trial, then

$X = Y_1 + Y_2 + \ldots + Y_n$, and hence

$G_X(t) = G_{(Y_1+Y_2+\ldots+Y_n)}(t) = (G_Y(t))^n = (q + pt)^n$.

Example 8

a) A fair die is thrown and the random variable is the number on the upper face is marked. Find the generating function for the distribution of the random variable.

b) Two fair dice are thrown and the sum of the two numbers is considered. Find the generating function for the random variable.

Solution

a) Since the probability for each face is the same, then

$$G(t) = 0t^0 + \frac{1}{6}t^1 + \frac{1}{6}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{6}t^5 + \frac{1}{6}t^6$$

b) Since we are adding the two variables, then

$$\begin{aligned} G_{X+Y}(t) &= G_X(t)G_Y(t) = (G(t))^2 \\ &= \left(\frac{1}{6}(t + t^2 + t^3 + t^4 + t^5 + t^6)\right)^2 \\ &= \frac{1}{36}(t^2 + 2t^3 + 3t^4 + 4t^5 + 5t^6 + 6t^7 + 5t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12}) \end{aligned}$$

The last expression enables us to directly read the probability of any possible sum. For example, $P(X + Y = 6) = \frac{5}{36}, P(X + Y = 10) = \frac{3}{36}$, etc.

Exercise 3

1 Find a formula for a generating function for 1, –1, 1, –1, 1, –1, 1, –1, 1, –1, …

2 Find a formula for a generating function for:

a 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, …

b 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, …

3 $X \sim B(80, 0.25)$ and $Y \sim \text{Po}(1.5)$ are independent. Find E(XY).

4 Verify Theorem 2, using $B(n, p)$.

5 Verify Theorem 2, using Po(λ).

6 Let X be a discrete random variable with $x = 1, 2, 3, \dots$ and a PMF $P(x) = \dfrac{1}{2^x}$.

a Find the PGF, $G(t)$, of this random variable.

b Verify that $G'(0) = P(X = 1)$ and $G''(0) = 2P(X = 2)$.

7 The discrete random variable X is the number of times we throw a fair die to get a 5. Find the PGF, as well as the expected number of throws and the variance.

8 The discrete random variable X is the number of times we throw a pair of fair dice to get a sum of 5. Find the PGF, as well as the expected number of throws and the variance.

9 A discrete random variable, Y, has the PGF $G(s) = \dfrac{k}{(5-s)^2}$.

a Find the value of k.

b Find $\mathrm{E}(Y)$.

c Find $P(Y = 2)$.

10 Consider the random variable X defined over the set of non-negative integers. The probability generating function for X is

$$G(s) = \frac{m}{(7-4s)^3},$$

where m is a certain real number.

a Find the value of m.

b Find $\mathrm{E}(X)$ and $\mathrm{V}(X)$.

c Find $P(X = 0)$, $P(X = 1)$, and $P(X = 2)$.

11 Kat and Won review for their exams by randomly choosing questions from a huge question bank. They compete in a way that the one that solves the problem correctly first (without looking at the solutions) gets a free dinner. They stop as soon as one of the solutions is correct. When Won tries, she manages to get the correct solution one third of the times, while Kat gets the correct solution one fourth of the times. (We will assume that repeated attempts are independent.) Let X be the total number of questions attempted. Won starts.

a Find the probability generating function for X.

b Find $\mathrm{E}(X)$ and $\mathrm{V}(X)$.

12 The discrete random variable X has the following PMF

$$G(t) = \left(\frac{p}{1-qt}\right)^2$$

where $0 < p < 1$ and $q = 1 - p$, and $x = 0, 1, 2, \dots$

a Find $\mathrm{E}(X)$ and $\mathrm{V}(X)$.

b Find $P(x = 1)$ and $P(x = 2)$.

13 Find a 'closed' form for the PGF of throwing a fair die and observing the number on the upper face.

14 Find a closed form for the PGF of a random variable X for which $P(X = x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots, n$. Find the expected value.

15 A random variable X defined over the set of non-negative integers has a PMF

$$P(X = x) = \frac{k}{e^x}.$$

Find the value of k, the PGF, the expected value and the variance.

4 Sampling and Sampling Distributions

In statistical study, we will be studying populations and trying to make inferences about these populations based on sample information. We know that when we use sample information to say things about the population there will be some random error. These concepts will be discussed. For now, we want to know how to obtain this sample information and what we can say about this sample information in a probability setting. To get this sample information, we often take from the population a **simple random sample**.

4.1 Simple random sample

Suppose that we want to pick a sample of n items from a population of N items. A **simple random sample** is selected such that *every item has an equal probability* of being selected and the items are selected independently – the selection of one item does not change the probability of selecting any other items.

A simple random sample is the ideal sample. In a number of real-world sampling studies, analysts develop alternative sampling procedures to lower the costs of sampling. But the basis for determining if these strategies are acceptable is to determine how closely the results approximate to a simple random sample.

To understand this, let's look at a very simple example.

Consider the population of the number of hours 6 students spent on homework on one night:

2 1 2 0 3 4

We pick one student at random from this population and define the 'random variable' X to be the number of hours a student spends on homework: X = number of hours, and so $x = 0, 1, 2, 3, 4$.

Recall that the following is the probability distribution of the discrete random variable X along with the mean and standard deviation of the population.

Table 1

x	$P(X = x)$	
0	$\frac{1}{6}$	$\mu_x = \sum x P(x) = 0 \cdot \frac{1}{6} + \ldots + 4 \cdot \frac{1}{6} = 2$
1	$\frac{1}{6}$	$\sigma_x^2 = \sum (x - \mu_x)^2 P(x)$
2	$\frac{2}{6}$	$= (0-2)^2 \frac{1}{6} + \cdots + (4-2)^2 \frac{1}{6}$
3	$\frac{1}{6}$	$= \frac{10}{6} \approx 1.667$
4	$\frac{1}{6}$	$\sigma_x = \sqrt{\frac{10}{6}} \approx 1.29$

Now let us create a new population by taking every possible sample of size $n = 2$ and obtain the sample mean, $\bar{x}$, of each sample.

Table 2

x_1	x_2	$\bar{x}$	x_1	x_2	$\bar{x}$	x_1	x_2	$\bar{x}$
0	1	0.5	0	2	1	0	2	1
0	3	1.5	0	4	2	1	2	1.5
1	2	1.5	1	3	2	1	4	2.5
2	2	2	2	3	2.5	2	3	2.5
2	4	3	2	4	3	3	4	3.5

Table 2 lists every possible sample of size $n = 2$ which can be obtained when sampled from the population of size $N = 6$ $\left[\binom{6}{2} = 15\right]$. Every possible sample of size 2 has the same chance of being chosen from the population. For each sample of size 2, the sample mean, $\bar{x}$, was calculated. Moreover, if we were to pick two items from the population, we must end up with one of these samples of size 2, and so, we must end up with one of these sample means.

So our (theoretical) population of sample means consists of the following 15 values: $\bar{x}$ = 0.5, 1, 1, 1.5, 1.5, 1.5, 2, 2, 2, 2.5, 2.5, 2.5, 3, 3, 3.5.

The following is the probability distribution of $\overline{X}$.

Table 3

$\overline{x}$	$P(\overline{X} = \overline{x})$
0.5	$\frac{1}{15}$
1	$\frac{2}{15}$
1.5	$\frac{3}{15}$
2	$\frac{3}{15}$
2.5	$\frac{3}{15}$
3	$\frac{2}{15}$
3.5	$\frac{1}{15}$

Using the same method as above, we have:

$$\mu_{\overline{x}} = \sum \overline{x}P(\overline{x}) = 0.5 \cdot \frac{1}{15} + \ldots + 3.5 \cdot \frac{1}{15} = 2$$

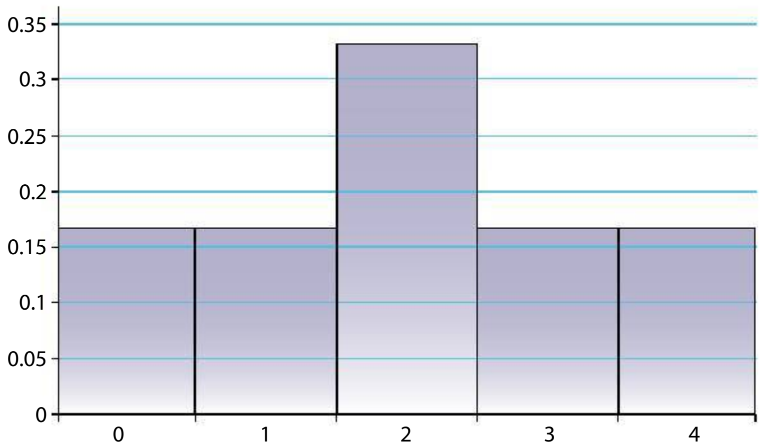
$$\sigma_{\overline{x}}^2 = \sum \left(\overline{x} - \mu_{\overline{x}}\right)^2 P(\overline{x})$$

$$= (0.5 - 2)^2 \frac{1}{15} + \cdots + (3.5 - 2)^2 \frac{1}{15}$$

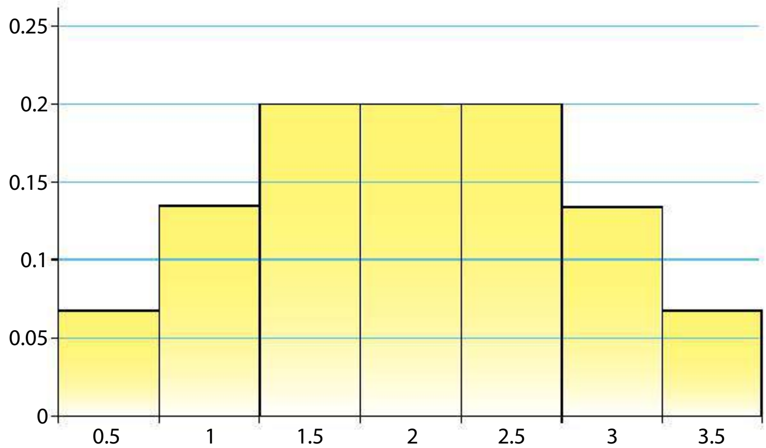
$$= \frac{10}{15} \approx 0.667$$

$$\sigma_{\overline{x}} = \sqrt{\frac{10}{15}} \approx 0.816$$

Notice, the original population was symmetric and the population of $\overline{X}$ is also symmetric.



We call this probability distribution the **sampling distribution of the mean.**



4.2 Sampling distributions

Consider a random sample selected from a population to make an inference about some population characteristic, such as the population mean, by using a sample statistic such as the sample mean, $\overline{X}$. The inference is based on the realization that every random sample would have a different number for $\overline{X}$ and thus $\overline{X}$ is a random variable. The sampling distribution of this statistic is the probability distribution of the values it could take over all possible samples of the same number of observations drawn from the population.

Recall that the original population is the one we wish to study and exists in real life. The sampling distribution of the mean, and for that matter, any sampling distribution, is a theoretical distribution that we mathematically derive. However, through these sampling distributions, we will be able to make inferences about the population we sample from.

Sampling distribution of the sample mean

Notice that in Table 1 (page 1106), we not only derived the sampling distribution of the mean, but we also computed the mean of that new population and it was the same as the mean of the original population. The variance and standard deviation were smaller than the variance and standard deviation of the original population.

We can generalize this situation as follows:

Let $\overline{X}$ denote the sample mean of a random sample of n observations from a large population with mean μ and variance σ^2. Then:

1. The sampling distribution of $\overline{X}$ has a mean $\mathrm{E}\left(\overline{X}\right) = \mu$.
2. The sampling distribution of $\overline{X}$ has a standard deviation $\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$.

Before we justify these results, let us look at some examples.

Let us consider choosing a sample of size 2 from the sample space of {1, 2, 3, 4, 5, 6} which are equally likely to be chosen and then calculate their average. We can simulate that by throwing two dice and calculating the average of the two numbers. The beginning of a table summarizing a large number of outcomes is shown left.

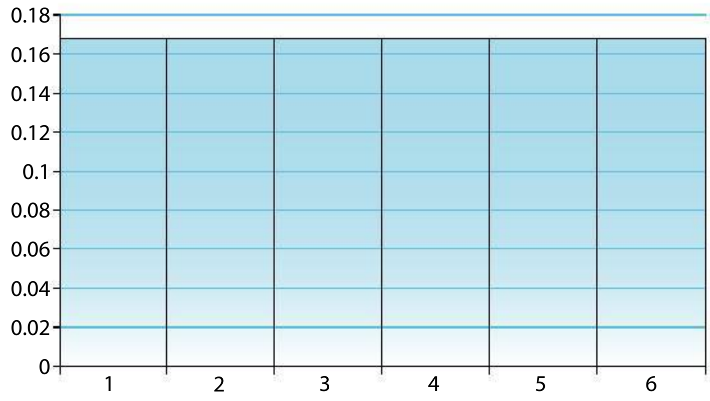
x_1	x_2	$\overline{x}$
3	2	2.5
6	6	6.0
4	3	3.5
4	1	2.5
5	6	5.5
5	4	4.5
5	6	5.5
4	4	4.0
3	4	3.5
3	1	2.0
5	4	4.5

Table 4

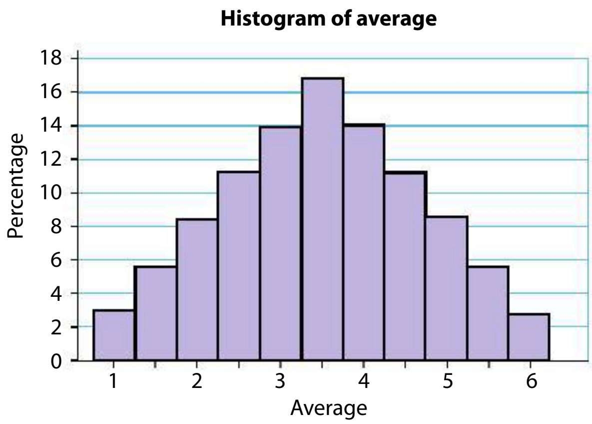
The distribution of values in the sample space is given by the table below. It is left for you to verify that the mean $\mu = 3.5$ and standard deviation $\sigma = 1.87$.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The histogram representing the population is shown below.



The histogram representing the average is as shown.



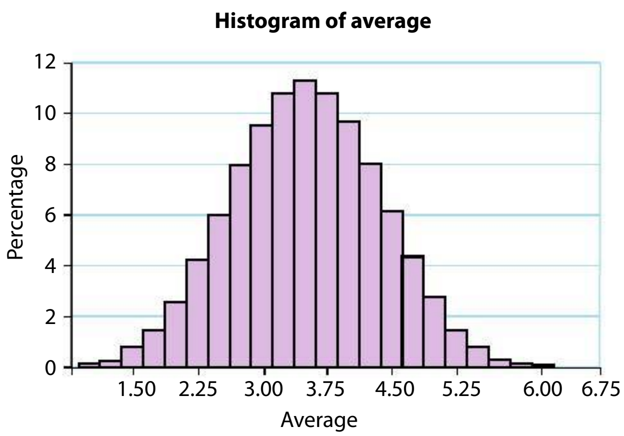
The mean and standard deviation of the sampling distribution is given by the software used for the simulation as

$\mu_{\bar{X}} = 3.5$ and standard deviation $\sigma_{\bar{X}} = 1.21$.

If we simulate throwing four dice and calculating the average, the result will be again

$\mu_{\bar{X}} = 3.5$ and $\sigma_{\bar{X}} = 0.8555$.

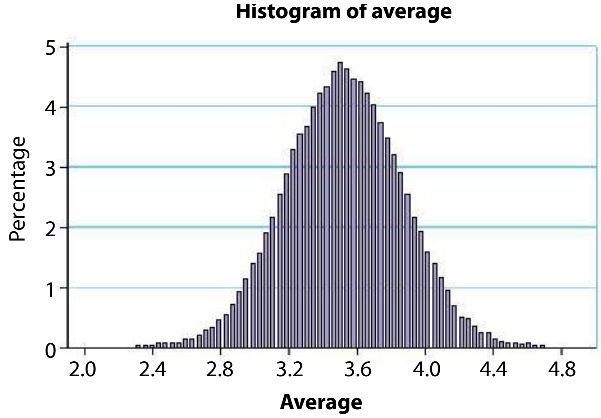
Here is the histogram.



If we sample 25 every time, the result will be again

$\mu_{\bar{X}} = 3.5$ and $\sigma_{\bar{X}} = 0.3415$.

The histogram will be as shown.



n	$\mu_{\bar{X}}$	$\sigma_{\bar{X}}$	$\frac{\sigma}{\sqrt{n}}$
Original data	3.5	1.71	1.71
2	3.5	1.21	1.209
4	3.5	0.8555	0.855
25	3.5	0.3415	0.342

Table 5

You should have noticed the dramatic changes in the shape of the histogram: The larger the sample size, the closer the histogram to the 'normal' shape. Also, you will have noticed that the centre of each of these sampling distribution simulations is 3.5, the mean of the population.

Remarkable too, is how close the standard deviation is to $\frac{\sigma}{\sqrt{n}}$.

This phenomenon is the result of the important theorem called the **central limit theorem**.

Central limit theorem

If random samples of size n are drawn from a population whose mean is μ and standard deviation σ, when n is large, then the sampling distribution of the mean $\bar{X}$, is approximately normally distributed, with mean μ and

standard deviation $\frac{\sigma}{\sqrt{n}}$.

The larger n becomes the more accurate is this approximation.

Note: How do we decide when the sample size is large enough?

1. If the sample population is **normal**, then the sampling distribution of the mean will be normal, no matter what sample size we choose.
2. If the population is approximately symmetric, the sampling distribution of the mean will become normal for relatively small sample sizes. Recall how the shape of the distribution changed for even a size of 2 in the dice case earlier.
3. If the population is not symmetric, the sample size n must be at least 30, for the sampling distribution of $\bar{X}$ to become approximately normal.

Note: The central limit theorem can be 'adjusted' to apply to the **sum of the sample measurements** $\sum x_i$. The distribution of the sum, when n becomes large, is approximately normal with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$.

Example 1

A soft-drink vending machine is set so that the amount of drink dispensed is a normal random variable with a mean of 100 cm^3 and a standard deviation of 7.5 cm^3.

a) What is the probability that at a randomly chosen time the machine dispenses at least 102 cm^3?

b) What is the probability that the average amount dispensed of a randomly chosen sample of 36 is at least 102 cm^3?

Solution

a) This is a normal probability calculation which can be read directly from your GDC.

So, $\mathrm{P}(x > 102) = 0.395$.

```
normalcdf(102,10
00,100,7.5)
        .394862968
```

b) This is a sampling distribution of the mean with $\mu_{\bar{X}} = 100$ and standard deviation $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{7.5}{6} = 1.25$.

So, $\mathrm{P}(\bar{x} > 102) = 0.0548$.

```
normalcdf(102,10
00,100,1.25)
       .0547992894
```

This example demonstrates the difference between probability calculations for an observation from the population to that of an average of a sample.

Justification of the central limit theorem

There are many versions of the CLT, the proof of which is beyond the scope of this course. We will present here an outline of justification for the case of random sampling with replacement from large or infinite populations.

Consider the random variable X which has a distribution with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

If we take n independent observations $X_1, X_2, \ldots, X_n$ from X, then

$$\mathrm{E}(X_1) = \mathrm{E}(X_2) = \cdots = \mathrm{E}(X_n) = \mu, \text{ and}$$
$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \cdots = \mathrm{Var}(X_n) = \sigma^2.$$

Now,

$$\bar{X} = \frac{\sum_i X_i}{n} = \frac{1}{n}\sum_i X_i.$$

Hence,

$$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{1}{n}\sum_i X_i\right) = \frac{1}{n}\mathrm{E}\left(\sum_i X_i\right)$$
$$= \frac{1}{n}\sum_i \mathrm{E}(X_i) = \frac{1}{n}\cdot n\mu = \mu.$$

Also,

$$\mathrm{Var}(\overline{X}) = \mathrm{Var}\left(\frac{1}{n}\sum_i X_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_i X_i\right)$$
$$= \frac{1}{n^2}\sum_i \mathrm{Var}(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

and therefore

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}.$$

Standardized distribution for means

Using the previous discussion, we can say that when we standardize $\overline{X}$, the resulting variable

$$z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$$

is approximately normal with a mean of 0 and standard deviation of 1. That is, z is approximately N(0, 1). This distribution is of much use for estimation and hypothesis testing as we will see in the next chapter.

Standardized distribution for means, when σ is not known: the *t*-distribution

Probably a more common and realistic case than that discussed in the previous section is that we do not know the population standard deviation σ. When this is the case we cannot simply substitute s for σ in the equation

$$z = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}}$$

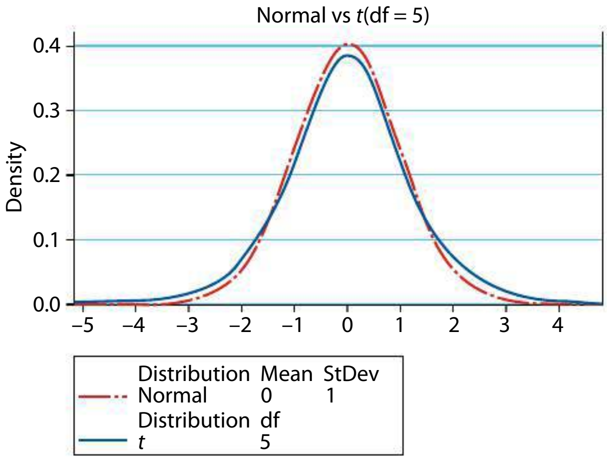
and assume that the variable is normally distributed. When the parent population is approximately normal, the distribution of the variable

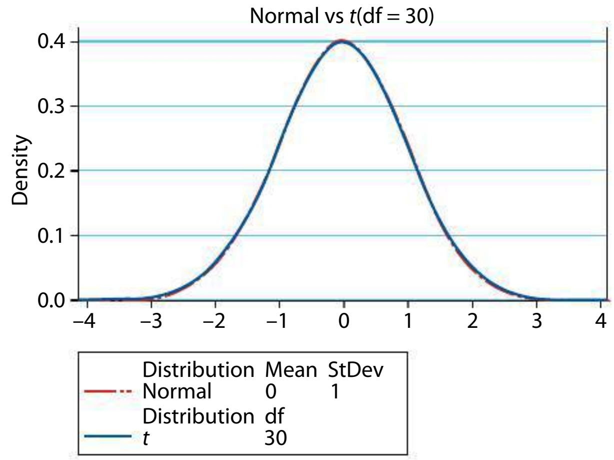
$$t = \frac{\overline{x} - \mu}{s / \sqrt{n}}$$

is the widely used t-distribution.

The t-distribution is actually a family of symmetrical density functions, with a single parameter υ that determines the particular member of the family. This parameter is known as the number of **degrees of freedom**, and in this case is $\upsilon = n - 1$. It can be shown that as n increases, the t-distribution becomes more and more *normal.*

The graph below shows how, as the number of degrees of freedom increases, the t-distribution gets very close to normal.





Reading areas under the *t*-distribution curve is delegated to the GDC. It can also be read from specialized tables.

For example, if we need to find the *t*-value that corresponds to a cumulative probability of 0.95 with 10 degrees of freedom, then we read down the first column till we reach the cell for $v = 10$, and then we move horizontally till we reach the column corresponding to the probability of 0.95. In this case we find that $t = 1.812\,461$, i.e. 1.812 461 is the number such that $\mathrm{P}(x < 1) = 0.95$.

p	0.9	0.95	0.975	0.99	0.995	0.9995
$\upsilon = 1$	3.077684	6.313752	12.7062	31.82052	63.65674	636.6192
2	1.885618		4.302653	6.964557	9.924843	31.59905
3	1.637744		3.182446	4.540703	5.840909	12.92398
4	1.533206		2.776445	3.746947	4.604095	8.610302
5	1.475884		2.570582	3.36493	4.032143	6.868827
6	1.439756		2.446912	3.142668	3.707428	5.958816
7	1.414924		2.364624	2.997952	3.499483	5.407883
8	1.396815		2.306004	2.896459	3.355387	5.041305
9	1.383029		2.262157	2.821438	3.249836	4.780913
10		1.812461	2.228139	2.763769	3.169273	4.586894

Using your GDC, first open the 'DISTR' menu, scroll down to 'invT' and then enter the cumulative probability followed by the number of degrees of freedom.

```
DISTR DRAW
1:normalpdf(
2:normalcdf(
3:invNorm(
4:invT(
5:tpdf(
6:tcdf(
7↓χ²pdf(
```

```
invT(.95,10)
       1.812461102
```

Sampling distribution of the sample proportion (Optional)

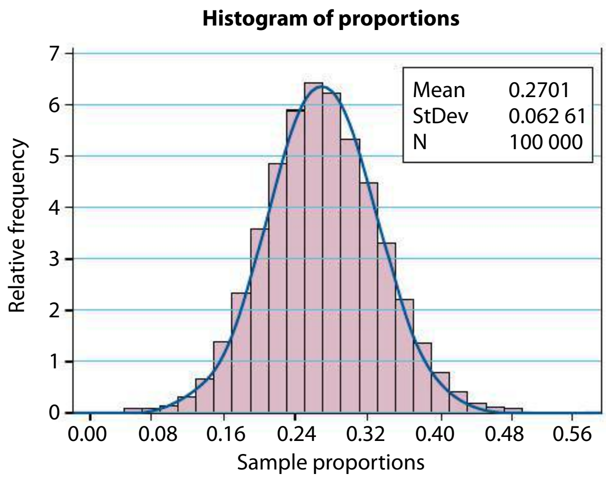
There are many practical examples of the binomial random variable X. One common application involves, for example, voter preferences in upcoming elections. We usually use a random sample of n people to estimate the proportion p of people in the population who have a specific characteristic. If x of the sampled people possess this characteristic, then the sample proportion is $\hat{p} = \frac{x}{n}$. This value can be used to estimate the population proportion p.

We will use simulation to study the distribution of the proportion of interest from repeated sampling. For example, the 'Greens' party in a certain country receives about 27% of the votes in national elections. To see how we can study this, we will simulate drawing random samples of size 50 from this large population and see how the distribution of proportions from these samples will look.

The table below shows the outcome (a part) of this simulation. When a cell contains 1, then this is a success; when it 0, it is a failure. The column 'Sum' contains the sum of all the '1's in the sample, and the last column contains the quotient of this number and 50, i.e. the proportion of success.

C1	C2	C3	C4	C5	C6	...	Sum	Proportion
0	0	0	0	0	0	...	16	0.32
0	0	0	0	0	0	...	9	0.18
0	0	1	0	1	1	...	13	0.26
0	0	0	0	0	0	...	10	0.20
0	0	0	0	0	0	...	12	0.24

Here is a histogram of the results.



As you see from the histogram, the distribution of proportions $\hat{p}$ is approximately normal with a mean 0.27, which is the proportion of the population itself and a standard deviation of 0.0626.

As you will see below, 0.0626 is approximately equal to

$$\sqrt{\frac{0.27(1-0.27)}{50}} = 0.0628.$$

For large samples, the distribution of the sample proportions is an extension to the central limit theorem.

As you recall from the core material, the binomial random variable X has a probability distribution $p(x) = \binom{n}{x} p^x q^{n-x}$, has a mean $E(X) = np$ and a variance $V(X) = npq$.

Now, since $\hat{p} = \frac{x}{n}$, and since $E(X) = np$, then

$E(\hat{p}) = E\left(\frac{X}{n}\right) = E\left(\frac{1}{n}X\right) = \frac{1}{n} \cdot np = p$ using properties of expected value.

Also,

$$V(\hat{p}) = V\left(\frac{1}{n}X\right) = \frac{1}{n^2}V(X) = \frac{1}{n^2}npq = \frac{pq}{n}.$$

Therefore, we can now state the following:

> If a random sample of n is selected from a binomial population with success probability p, then the sampling distribution of the sample proportion $\hat{p} = \frac{x}{n}$ will have a mean equal to p and standard deviation
>
> $$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}}, \text{ with } q = 1 - p.$$
>
> When the sample size is large enough, then the sampling distribution will be approximately normal.

Note: The sample size is considered large if $np > 5$ and $nq > 5$.

Example 2

In a certain country, the 'Green' voters are truly 27% of the voter population. If you take a random sample of 50 potential voters, what is the probability that the percentage of 'Green' voters in the sample is larger than or equal to 30%?

Solution

The sampling distribution of the 'Green' voter proportions is approximately normal with mean 0.27 and standard deviation 0.0628 as shown above.

Thus,

$$P(\hat{p} \geqslant 0.30) = P\left(z \geqslant \frac{0.30 - 0.27}{\sqrt{\frac{0.27 \cdot 0.73}{50}}}\right) = P(z \geqslant 0.478) = 0.316.$$

Or, using your GDC as shown right.

```
normalcdf(0.30,1
 000,0.27,0.0628
          .316429384
```

Exercise 4

1 Some kinds of ketchup are distributed in bottles of 875 cm^3 volume. The volumes in these bottles are normally distributed with a standard deviation of 12 cm^3.

A sample of 15 bottles is taken and the mean volume is found.

a What is the probability that a bottle contains less than 870 cm^3 of ketchup?

b What is the probability that the mean volume of the 15 bottles is less than 870 cm^3 of ketchup?

2 In a national examination, the scores during one session was 67.2 and the standard deviation was 5 marks.

A sample of 40 tests was taken and the average of the 40 calculated. Find

a $P(X > 75)$, where X represents the grade of an individual student

b $P(\bar{X} > 75)$, where $\bar{X}$ is the average of the 40 tests

c $P(65 < \bar{X} < 75)$.

3 The duration of human pregnancies is assumed to follow a normal distribution with a mean of 38 weeks and a standard deviation of 2 weeks.

a What percentage of pregnancies last between 38 and 40 weeks?

b How many weeks would the shortest 25% of the pregnancies last?

c A medical team in one of the major hospitals is collecting data about pregnant women. They have selected 120 women to take part in the study. Among the data collected is the duration of the pregnancy. Describe in detail the distribution of the mean length of their pregnancies.

d What is the probability that the mean duration of the pregnancies of these patients is less than 37 weeks?

e If you are told that the distribution of the pregnancies' duration is not normal but slightly skewed to the left, would any of your answers to **a**, **b**, **c**, or **d** change? Justify each one.

4 Customer purchases from a chain of supermarkets around Europe as collected from company records for the last three years show a right-skewed distribution with mean of €27 and a standard deviation of €19.

a Can you determine the probability that the next customer will spend more than €32? Justify your response.

b Can you determine the probability that the next 5 customers will spend on average more than €32? Justify your response.

c Can you determine the probability that the next 45 customers will spend on average more than €32? Justify your response.

5 $X \sim N(\mu, 9)$. A random sample is selected from this population.

a Find $P(|\bar{X} - \mu| < 3)$ if the sample size is 25.

b If $P(|\bar{X} - \mu| < 3) = 0.9$, what should the sample size n approximately be?

6 A random variable X has a mean μ and a standard deviation σ. The distribution of means of samples with 64 observations has a variance of 0.4. Find the value of σ.

7 A company claims a defective rate of 4% in the batteries they manufacture.

a If a sample of 100 batteries is checked, what is the probability that the defective rate could be higher than 4.5%, assuming the company's claim to be correct?

b You take a sample of 100 batteries. You find 6 defective batteries. What can you conclude about the company's claim and why?

8 Batteries from a large manufacturer are known to have lifetimes that are exponentially distributed with a mean of 20 working hours.

a Find the probability that a battery survives 30 hours.

b Find the probability that the average lifetime of 100 randomly selected batteries exceeds 20.2 hours.

c Find the probability that the average lifetime of 2 randomly selected batteries exceeds 20.2 hours.

9 A juice bottling machine discharges an average of μ cm^3 of juice per bottle. The volume dispensed by this machine is known to be normally distributed with $\sigma = 30$ cm^3.

a If a sample of n bottles is randomly selected, find

i $p\left(\left|\bar{X} - \mu\right| < 10\right)$, if $n = 9$

ii $p\left(\left|\bar{X} - \mu\right| < 10\right)$, if $n = 25$

iii $p\left(\left|\bar{X} - \mu\right| < 10\right)$, if $n = 64$.

b Do you see a pattern relating the probability to the size of the sample? Explain.

10 Tempered glass strength is measured in thousands of psi or bar (1 bar $\approx$ 14.5 psi). One brand of tempered glass has an average strength of 960 bar with a standard deviation of 138 bar.

a What is the probability that the average strength of 100 randomly chosen pieces of this glass exceeds 1000 bar?

b Find an interval that includes, with a probability of 0.95, the average strength of 100 randomly selected pieces of this glass.

(Hint: try to find k such that $p\left(\left|\bar{X} - \mu\right| < k\right) \approx 0.95$)

11 The tip percentage at a restaurant has a mean value of 18% and a standard deviation of 6%.

a What is the approximate probability that the sample mean tip percentage for a random sample of 40 customers is between 16% and 19%?

b If the sample size has been 10 rather than 40, could the requested probability have been calculated using the given information? Explain.

12 A juice factory buys apples from a large contractor. They have an agreement that the apples provided should meet certain standards in terms of size, bruises, yellowing and other defects. From every shipment a random sample of 180 apples is selected and examined. The whole shipment will be rejected if more than 5% of the sample is not to standard. Suppose 7% of the apples are substandard. What is the probability that the shipment will be accepted nonetheless?

13 The assembly line that produces an electronic component for a video system has historically resulted in a 3% defective rate. A random sample of 400 units is selected.

a What is the probability that the proportion of defective components in the sample is greater than 5%?

b Suppose that in fact the 400-units sample resulted in a 5% defective rate. What does that suggest about the defective rate on the assembly line? Explain.

14 The manufacturer of a painkiller pill claims that the proportion of headache sufferers who get relief by taking one of their pills is 63%. A random sample of 1000 headache sufferers is selected and given the pill.

a What is the probability that less than 59% obtain some relief?

b Suppose that the sample of 1000 resulted in a 59% success rate. What does this suggest about the manufacturer's claim?

<aw044>

<X>Exercise 4.1

15 A continuous random variable X has a mean 10 and a variance 9. A random sample of 2250 observations is taken on X. Find the probability that the sample mean exceeds 11.

16 A discrete random variable X has a probability distribution given in the table below.

x_i	0	1	2	3
p_i	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Determine an approximation to the probability that a random sample of 800 observations on X will have a total less than 1100, giving your answer to the nearest percentage point.

17 Bags of brown sugar are marked as containing 1 kg of sugar. In reality, the mean mass of sugar per bags is 1.04 kg. The mass of sugar varies from bag to bag, and has a standard deviation of 25 g. Making a suitable assumption, estimate the proportion of bags that contain less than 1 kg of sugar.

18 A builder orders 200 planks of oak and 150 planks of mahogany. The mean and standard deviation of the masses (given in kg) of oak planks are 25 and 1.3 respectively. The corresponding figures for the pine planks are 20 and 1.1 respectively. Assuming that the planks delivered to the builder are random samples from the population of planks, determine the probability that the wood delivered has a total mass that is

a less than 7.5 tons **b** between 7.8 and 8.3 tons.

19 The mean weight of trout in a fish farm is 980 g and the standard deviation is 100 g. What is the probability that a catch of 15 trout will have a mean weight per fish more than 1050g?

20 The girls of the age of 15 in a large town have a mean height of 166 cm and standard deviation of 6 cm.

a In one school there is a mathematics group with 5 girls. What is the probability that the mean height of this group is between 162 cm and 170 cm?

b In another school there is an English group with 8 girls. What is the probability that the mean height of the English group is between 162 cm and 170 cm?

21 In a potato chips factory, chips are packed in bags whose masses are distributed normally with a mean of 100 g and standard deviation of 1.3 g. Find the probability that the mass of 25 bags selected at random will be within 5 g of the expected mass.

22 The distribution of lengths of rods produced by a machine is normal with mean 100 cm and standard deviation 15 cm.

a What is the probability that a randomly chosen rod has a length of 105 cm or more?

b What is the probability that the average length of a randomly chosen set of 60 rods of this type is 105 cm or more?

23 The daily rainfall in a holiday resort follows a normal distribution with mean p mm and standard deviation σ mm. The rainfall each day is independent of the rainfall on other days.

On a randomly chosen day, there is a probability of 0.05 that the rainfall is greater than 10.2 mm.

In a randomly chosen 7-day week, there is a probability of 0.025 that the **mean** daily rainfall is less than 6.1 mm.

Find the value of p and of σ.

5 Confidence Intervals

5.1 Point estimators

We are now ready to use what we have learned about statistics in the previous sections to do statistical inference. The simplest case to begin with is point estimation. If we have a population with some unknown parameter, we will use sample information to say something about the parameter. What is a point estimator?

Estimator and estimate

Point estimation is analogous, in many respects, to shooting at a target in sports. The estimator, which generates the estimates, is analogous to the tool used (revolver, arrow, darts). A particular estimate is comparable to one attempt; and the parameter of interest is the centre of the target (in many cases it is called the 'bull's eye'). Drawing a single sample from the population and using it to compute an estimate of the parameter is similar to shooting once at the centre.

Suppose an athlete shoots once at a target and hits the centre. Do we conclude that this athlete is an excellent one? Certainly not. We would not decide on the quality of the athlete based on such small evidence. On the other hand, if the athlete manages to hit the target 50 times in a row then we may consider him/her as an expert.

The point here is that we cannot judge the goodness of a point estimation method on the basis of a single estimate. Instead, we must see the results when the method is implemented several times. Because the estimates are numbers, we assess the goodness of the point estimator by creating a frequency distribution of the values of the estimates gained in repeated sampling and note how closely this distribution masses about the target parameter.

Definition

An **estimator** of a population parameter is a random variable that depends on the sample information and whose value provides approximations to this unknown parameter. A specific value of that random variable is called an **estimate**.

So, a statistic that is used as an estimator of a particular unknown parameter is a **point estimator**. Note that a point estimator does not depend on any unknown parameter.

In terms of establishing a general notation, we have the following definition for a point estimator.

Point estimator and point estimate

Let θ represent a population parameter (such as the population mean μ or the population proportion p or π). A **point estimator**, $\hat{\theta}$ of a population parameter, θ, is a measure calculated from the sample information that yields a single number called a **point estimate**. For example, the sample mean $\overline{X}$ is a point estimator of the population mean μ, and the value that it assumes for a given set of data is called the **point estimate**.

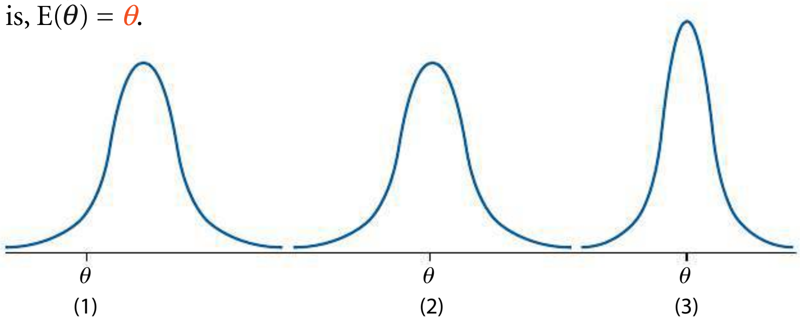
Unbiasedness

Suppose we wish to specify a point estimate for a population parameter, θ. With the shooting at a target example in mind, it is highly desirable for the distribution of estimates (the sampling distribution of the estimator) to cluster about the target parameter as shown in panels (2) and (3) in the diagram below. Point estimators similar to these cases are called **unbiased** estimators. An unbiased estimator is one whose expected value is the parameter it is trying to estimate. So the long-term average of the statistic is the parameter of the population $E(\hat{\theta}) = \theta$.

The sampling distribution in panel (1) below represents a biased estimator for which $E(\hat{\theta}) > \theta$.

Definition: unbiased estimator

The point estimator $\hat{\theta}$ is said to be an unbiased estimator of the parameter θ if the expected value, or mean, of the sampling distribution of $\hat{\theta}$ is θ; that is, $E(\hat{\theta}) = \theta$.



The figure above demonstrates the difference between a biased and an unbiased estimator. The figure shows the sampling distributions of three different statistics that are used to estimate a population parameter θ.

Note: The distribution in the first panel is not likely to yield an estimate close to the real value. Its centre is to the right of the true value, making it very likely that an estimate will be substantially larger than the true value. That is, if this statistic is used to make an estimate for θ based on data from one sample, and another estimate from a second sample and another from a third sample, and so on, the long-run average of these estimates will far exceed the true value of θ.

The distributions in the second and third panels are centred at the true value θ. Thus, while some estimates will be smaller than θ and some will be larger, the long-run average will not tend to overestimate or underestimate the true value of θ. Each statistic in (2) and (3) is unbiased. However, since the standard deviation in the third panel is relatively smaller than the one in the second panel, estimates using this statistic will nearly always be closer to the true value of θ than estimates using the statistic in panel (2). So we can make the following generalization.

Given a choice between several unbiased statistics that could be used to estimate a population parameter, the best statistic to use is the one with the smallest standard deviation. Such estimates are known to be most *efficient*.

An efficient estimator reflects the reliability of the estimator in terms of its tendency to have a smaller standard error for the same sample size when compared other estimators.

The median, for instance is an unbiased estimator of μ when the sample distribution is normally distributed. However, the standard error is 1.25 greater than that of the sample mean, so the sample mean is a more efficient estimator than the median.

The sample mean, sample variance, and sample proportion are unbiased estimators of their corresponding population parameters.

1. The **sample mean** $\overline{X}$ is an unbiased estimator of μ $[\mathrm{E}(\overline{X}) = \mu]$.
2. The **sample variance** s^2_{n-1} is an unbiased estimator of σ^2 $\left[\mathrm{E}\left(s^2_{n-1}\right) = \sigma^2\right]$.
3. The **sample proportion** $\overline{p}$ is an unbiased estimator of p $\left[\mathrm{E}\left(\overline{p}\right) = p\right]$.

Consistent estimators

A statistics is a **consistent estimator** of a parameter if its probability that it will be close to the parameter's true value approaches 1 with increasing sample size.

The standard error of a consistent estimator becomes smaller as the sample size gets larger.

The sample mean and sample proportion are consistent estimators: from their formulas, as n gets larger, the standard errors get smaller. Recall that the standard error for the mean is $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ and for the proportion it is $\sigma_p = \sqrt{\dfrac{pq}{n}}$.

Examples

1 If $\overline{X}$ is the mean of a random sample from an infinite population, then $E(\overline{X}) = \mu$, i.e. it is an unbiased estimate of μ.

Using properties of expected values, we have:

$$\overline{X} = \sum \frac{x_i}{n} = \frac{1}{n}\sum x_i \Rightarrow E(\overline{X}) = \frac{1}{n}E\left(\sum x_i\right) = \frac{1}{n}\sum E(x_i) = \frac{1}{n}\cdot n\mu = \mu$$

2 Also, $\overline{p} = \dfrac{x}{n}$ is an unbiased estimator of p in a Binomial distribution with parameters n and p.

This is so since $E(x) = np \Rightarrow E\left(\dfrac{x}{n}\right) = \dfrac{1}{n}E(x) = \dfrac{1}{n}\cdot np = p$.

Why is $s^2 = \sum \frac{(x_i - \bar{X})^2}{n-1}$ an unbiased estimate of σ^2?

Theorem

$s^2 = \sum \frac{(x_i - \bar{X})^2}{n-1}$ is an unbiased estimate of σ^2.

Proof

$$E(s^2) = E\left(\sum_1^n \frac{(x_i - \bar{X})^2}{n-1}\right) = \frac{1}{n-1}E\left(\sum_1^n (x_i - \bar{X})^2\right)$$

$$= \frac{1}{n-1}E\left(\sum_1^n ([x_i - \mu] - [\bar{X} - \mu])^2\right)$$

The last line can be simplified to

$$E(s^2) = \frac{1}{n-1}\left(\sum_1^n E[x_i - \mu]^2 - n \cdot E[\bar{X} - \mu]^2\right)$$

Then, since $E(x_i - \mu)^2 = \sigma^2$ and $E[\bar{X} - \mu]^2 = \frac{\sigma^2}{n}$, so

$$E(s^2) = \frac{1}{n-1}\left(\sum_1^n \sigma^2 - n \cdot \frac{\sigma^2}{n}\right) = \frac{1}{n-1}(n\sigma^2 - \sigma^2) = \sigma^2.$$

Now we can say also why s_n^2 is not unbiased

$$E(s_n^2) = \frac{1}{n}\left(\sum_1^n \sigma^2 - n \cdot \frac{\sigma^2}{n}\right) = \frac{1}{n}(n\sigma^2 - \sigma^2) = \frac{n-1}{n}\sigma^2.$$

This last statement justifies why the 'common sense' explanation used in most non-mathematical statistics books is correct because it shows that s_n^2 tends to underestimate σ^2 as $E(s_n^2) = \frac{n-1}{n}\sigma^2 < \sigma^2$.

5.2 Confidence interval for the mean, μ, of a population

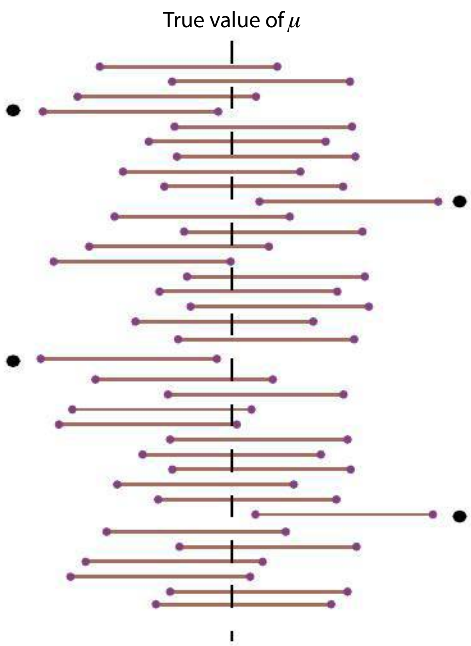
Consider the problem of estimating the mean monthly salaries of teachers in public schools in Austria. Suppose that, due to the large number of teachers involved, the distribution of salaries is normal with a known standard deviation of €350. In order to get some idea of the mean, μ, we must take a sample and obtain either a point estimate or an interval estimate of μ. We take a sample of 25 teachers and find that the mean of the sample is €1370. This is a point estimate of the mean monthly salaries of the Austrian teachers.

Does this mean that the average income of Austrian teachers is €1370? Certainly not, since if we take another sample, we may get €1300 or €1400, etc. It would be nice if we can find a statistic that can give us a point estimate that exactly reveals the true value of the parameter in question. However, the estimate we obtain depends on which sample we pick.

Different samples will nearly always provide different estimates due to sampling variability. In practice, you hardly ever select a sample that will provide you with an estimate exactly equal to the value of the population parameter. Thus, the point estimate we obtain from a sample for the mean μ, for example, says nothing about how close our estimate might be to μ.This variability of the estimates brings about the importance of indicating how accurately we are estimating the population parameter. An alternative to reporting a single sensible value for the parameter being estimated is to calculate an entire interval of plausible values – an **interval estimate** or **confidence interval**.

Confidence interval estimator
A **confidence interval estimator** for a population parameter θ is a rule for determining (based on sample information) a range, or interval, that is likely to include the parameter. The corresponding estimate is called a **confidence interval estimate**.

A confidence interval is always calculated by first selecting a **confidence level**, which is a measure of the degree of reliability of the interval. A confidence interval with a 95% confidence level for the true average salary might have a lower limit of €1232.8 and an upper limit of €1507.2. Then we would consider, at the 95% confidence level, any value of μ between €1232.8 and €1507.2 to be plausible. A 95% confidence level implies that 95% of *all* samples of this type would render an interval that includes μ (or any parameter θ that we are estimating), while 5% of such intervals might present an incorrect interval. The figure right illustrates this idea by showing several of the confidence intervals; 95% of them 'capture' the mean, while 5% miss it.



Stated differently, if we take repeated samples from the population and use the mean of each sample every time to construct a 95% confidence interval, in the long run, roughly 95% of these intervals will succeed to contain the mean μ. (Remember that we really do not know where μ is!)

Constructing a confidence interval for the mean μ of a population

The primary model and properties of confidence intervals are easily understood by first focusing on a simple, although rather unrealistic, problem situation. We will start with estimating the mean μ of a population under the following conditions.

The population distribution is normal.

The value of the population standard deviation σ is known.

In this publication, we will also assume that n is large enough for the central limit theorem to apply. That is, the sampling distribution of $\bar{x}$ is approximately normal with expected value $\mu_{\bar{x}} = \mu$ and a standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$. That is,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

follows a standard normal distribution.

Normality of the population distribution is often a sound hypothesis. However, if we do not know the mean μ, it is not likely that we would know σ. In later sections we will discuss less restrictive models.

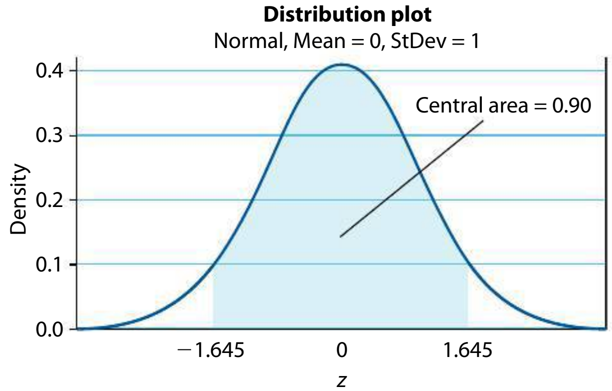
To better understand the development of the confidence interval, we will begin by considering a specific confidence level, say 90%.

Let us start by finding the two numbers Z and $-Z$ that include the central area of 0.90 under the standard normal curve. Either using a GDC or from the table we find that $z = 1.645$ is the number we are looking for. This means that

$$P(-1.645 \leqslant z \leqslant 1.645) = 0.90.$$

Applying this to the standardized value of the mean we have

$$P\left(-1.645 \leqslant \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leqslant 1.645\right) = 0.90.$$



Generalizing this result to the sampling distribution of the mean and simplifying the inequality inside the parenthesis we get the following result.

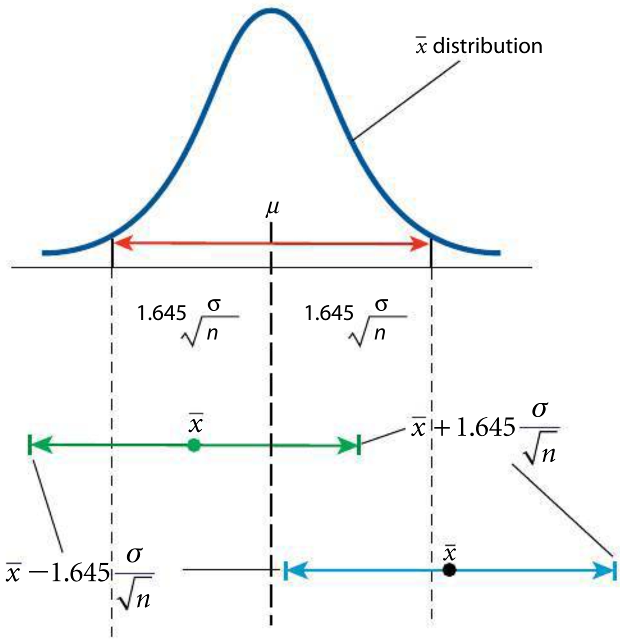
$$-1.645 \leqslant \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leqslant 1.645 \Leftrightarrow \mu - 1.645\frac{\sigma}{\sqrt{n}} \leqslant \bar{x} \leqslant \mu + 1.645\frac{\sigma}{\sqrt{n}}$$

Approximately 90% of the samples will result in an $\bar{x}$ value that is within 1.645 standard deviations of the true population mean. Observe the figure below and notice the following:

If $\bar{x}$ is within $1.645\frac{\sigma}{\sqrt{n}}$ of μ, then the interval

$$\bar{x} - 1.645\frac{\sigma}{\sqrt{n}} \text{ to } \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}$$

will definitely contain the mean μ. (This will happen for 90% of all possible samples.) On the other hand, if $\bar{x}$ is further away from μ than $1.645\frac{\sigma}{\sqrt{n}}$, which will happen for about 10% of the samples, the interval will not contain the true value of μ.



Now we can summarize the result as follows.

When n is large and σ is known, a 90% confidence interval for the population mean μ is

$$\left(\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}\right).$$

Hint

This result can be developed algebraically with a few steps.

Since the area under the standard normal curve between -1.645 and 1.645 is 0.90,

$$P\left(-1.645 \leqslant \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leqslant 1.645\right) = 0.90, \text{ as we have seen above.}$$

Now, for the inequality inside the parenthesis, some algebraic manipulation will have to be done in order to create an interval that includes the mean μ. (You may want to provide some missing steps!)

Multiply through with $\frac{\sigma}{\sqrt{n}}$.

$$-1.645 \leqslant \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leqslant 1.645 \Leftrightarrow -1.645\frac{\sigma}{\sqrt{n}} \leqslant \bar{x} - \mu \leqslant 1.645\frac{\sigma}{\sqrt{n}}$$

Subtract $\bar{x}$ from each term to obtain

$$-\bar{x} - 1.645\frac{\sigma}{\sqrt{n}} \leqslant -\mu \leqslant -\bar{x} + 1.645\frac{\sigma}{\sqrt{n}}.$$

Multiply through with -1 and rearrange to obtain

$$\bar{x} - 1.645\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}$$

which is the desired inequality.

Because each inequality in the sequence above is equivalent to the original one, the probability associated with each is 0.90. In particular,

$$P\left(\bar{x} - 1.645\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}\right) = 0.90.$$

However, much care must be taken in interpreting this statement. We usually express probability statements such that the random variable is usually in the middle, such as

$$P(-1.645 \leqslant z \leqslant 1.645) = 0.90.$$

In this case, remembering that μ is a fixed unknown constant, the variable is in the interval itself! Thus, we have a random interval having left endpoint

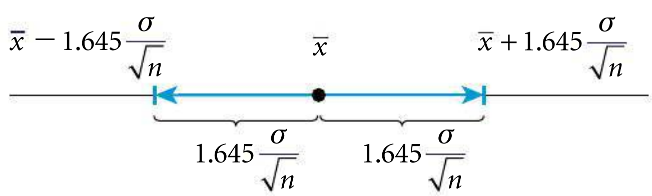
$\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}$ and right endpoint $\bar{x} + 1.645\frac{\sigma}{\sqrt{n}}$, which in interval notation is

$$\left(\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}\right).$$

The interval above is random because its endpoints are random variables. Note that the interval is centred at the sample mean $\bar{x}$ and extends

$1.645\frac{\sigma}{\sqrt{n}}$ to each side of $\bar{x}$. Thus, the interval's width is $2\left(1.645\frac{\sigma}{\sqrt{n}}\right)$,

which is not random! Only the location of the interval (its midpoint $\bar{x}$) is random.



Warning: We ***cannot*** say 'the probability that the population mean μ lies between $\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}$ and $\bar{x} + 1.645\frac{\sigma}{\sqrt{n}}$ is 0.90'. The mean μ is not a random variable, it is constant.

So, now we can either say that we are 90% confident that the interval

$\left(\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}\right)$ contains the true mean of the population,

or that the probability that the random interval $\left(\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}\right)$ includes the true mean of the population is 90%.

Calculating a confidence interval

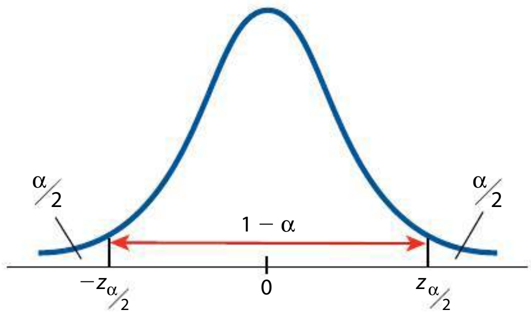
In the Austrian teachers' example, a 90% confidence interval is calculated as follows.

$$\left(\bar{x} - 1.645\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.645\frac{\sigma}{\sqrt{n}}\right) = \left(1370 - 1.645\frac{350}{\sqrt{25}}, 1370 + 1.645\frac{350}{\sqrt{25}}\right)$$
$$= (1254.85, 1485.15)$$

While if we want a 95% confidence interval, then we use $z = 1.96$ since a central area under the standard normal distribution lies between -1.96 and 1.96 and hence a 95% confidence interval is

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right) = \left(1370 - 1.96\frac{350}{\sqrt{25}}, 1370 + 1.96\frac{350}{\sqrt{25}}\right) = (1232.8, 1507.2)$$

The formulae we just developed for a 90% or 95% confidence interval suggest that any level of confidence can be achieved by replacing 1.645 or 1.96 by the appropriate standard normal critical value. As shown in the figure below, a probability of $1 - \alpha$ is achieved by using $z_{\alpha/2}$ in place of 1.645.



A $100(1 - \alpha)$% confidence interval for the mean μ of a normal population when σ is known is given by

$$\left(\bar{x} \pm z_{\alpha/2}\sigma_{\bar{x}}\right) = \left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Example 1

A sample of 36 100-watt light bulbs is tested for the length of their lifetime. The sample gave a mean of 985 hours. These light bulbs are known to have a standard deviation of 100 hours. Calculate a 99% confidence interval for the mean lifetime of all such light bulbs.

Solution

The critical value for this interval is $z_{\alpha/2} = z_{0.01/2} = z_{0.005} = 2.58$. Hence, the 99% confidence interval is

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = \left(985 - 2.58\frac{100}{\sqrt{36}}, 985 + 2.58\frac{100}{\sqrt{36}}\right) = (942, 1028)$$

5.3 Precision vs confidence

A frequently asked question is: Why don't we always get a large confidence interval? For example, why settle for a 90% confidence when a 99% is available?

Remember that a 90% interval extends $1.645\frac{\sigma}{\sqrt{n}}$ to each side of $\bar{x}$, while a 99% confidence interval extends $2.58\frac{\sigma}{\sqrt{n}}$ to each side of $\bar{x}$. This means that the more confidence we require, the wider the interval. The wider the interval, the less precise our estimate will be. For example, a 100% confidence interval for μ is simply $(-\infty, \infty)$. How much information do we get from such an interval? In fact, there is no need for any confidence interval development here; we knew without any work that such an interval would contain μ.

If you think of the length of the interval as a measure of its accuracy or precision, then for the same sample size, there is a trade-off between accuracy and confidence. An alternative strategy is given below. However, note that this is not required in your HL examination.

When there is interest in both a specific confidence level and specific precision, an appropriate sample size can be calculated.

Suppose that we are interested in a sample that ensures a level of precision defined by its width w and that we are interested in a level of confidence defined by the critical number $z_{\alpha/2}$. What sample size should we settle for?

The width of a confidence interval can, at most, be equal to w:

$$2\left(z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \leqslant w$$

Since all values are given except n, then we solve that inequality for n.

$\sqrt{n} \geqslant \dfrac{2z_{\alpha/2}\sigma}{w}$, and since all variables are positive, then squaring both sides will yield

$$n \geqslant \left(\frac{2z_{\alpha/2}\sigma}{w}\right)^2.$$

Example 2

What sample size is needed if, in the Austrian teachers' example, we are interested in having the estimate accurate to €100 with a confidence of 95%?

Solution

$$n \geqslant \left(\frac{2z_{\alpha/2}\sigma}{w}\right)^2 \Rightarrow n \geqslant \left(\frac{2 \times 1.96 \times 350}{100}\right)^2 \Rightarrow n \geqslant 188.24$$

Since n must be a whole number, then we choose $n = 189$ teachers.

The half-interval width $z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$ is sometimes called the **bound on the error of estimation** or **the margin of error** associated with a $100(1-\alpha)\%$ confidence level; i.e. with $100(1-\alpha)\%$ confidence (90% for example), the point estimate $\bar{x}$ will not be further than $z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$ from μ.

Note: Sometimes, the maximum 'acceptable' margin of error is used in determining the sample size. If we call this maximum value for the margin of error ε, then $w = 2\varepsilon$ and the sample size is determined by

$$n \geqslant \left(\frac{2z_{\alpha/2}\sigma}{w}\right)^2 = \left(\frac{2z_{\alpha/2}\sigma}{2\varepsilon}\right)^2 = \left(\frac{z_{\alpha/2}\sigma}{\varepsilon}\right)^2.$$

Example 3

50 measurements from a population whose standard deviation is known to be 23.4 resulted in a confidence interval for the population mean of (120.38, 130.67). Find the level of confidence used.

Solution

The width of the interval is 130.67 − 120.38 = 10.29. Thus,

$$width = 2\left(z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 10.29 \Rightarrow 2\left(z_{\alpha/2}\frac{23.4}{\sqrt{50}}\right) = 10.29 \Rightarrow z_{\alpha/2} = 1.555$$

```
normalcdf(-100,-
1.555)
       .059973046
```

Looking up 1.555 in tables, or using a GDC/computer, we get that this value of z corresponds to $\frac{\alpha}{2} = 0.05997 \approx 0.06 \Rightarrow \alpha = 0.12$ and hence the level of confidence is $1 - 0.12 = 0.88$ or 88%.

5.4 A confidence interval for μ when σ is unknown

The confidence intervals we discussed earlier have a major hitch: in order to be able to set up the interval, you need to know σ. As you have seen in the previous chapter, this is rarely, if ever, the case.

For populations that are approximately normal, if σ is not known, then we must use the unbiased estimate s calculated from the sample data.

However, the variable $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ which constituted the basis for our interval will now become $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$, which is not a standard normal variable. In the previous chapter we introduced this variable as t-variable with $n - 1$ degrees of freedom.

So, in this book (and in the IB), when the standard deviation of the population is not known, the t-distribution will be used. The basic

> The discussion of cases where the population cannot be assumed normal is beyond the IB/HL syllabus and this book. Also, when the population is large, many statisticians use the normal distribution. So the confidence interval would be
> $$\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$
> The argument here is that since the sample size is large, s will be very close to σ and introducing it into the equation will not bring any significant variation. However, the IB HL syllabus asks for the t-distribution to be used whenever σ is unknown, a wise decision which we will follow in the book.
> Important: From this point on, when we write s in any formula that is intended to estimate σ, we mean the unbiased estimate s_{n-1}.

structure of the confidence interval will stay the same. That is, so far we have the interval in the form

estimate ± (critical value) (standard error of the estimate)

and in the previous case, where we knew σ,

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

will become

$$\bar{x} \pm t_{(n-1),\,\alpha/2}\frac{s}{\sqrt{n}}.$$

Example 4

Back injuries may result from packing more than you can carry when going on mountain hikes. A study about the weight carried by hikers in a mountainous region chose a random sample of 20 hikers and checked the weight of their backpacks. The sample has an average of 10.2 kg and a standard deviation 3.8 kg. Develop a 95% confidence interval for the mean μ of all the weight carried by hikers in that region.

Solution

A 95% confidence level requires a critical t-value of 2.09 (19 degrees of freedom). The confidence interval is then

$$\bar{x} \pm t_{(n-1),\,\alpha/2}\frac{s}{\sqrt{n}} = 10.2 \pm 2.09 \cdot \frac{3.8}{\sqrt{20}} = (8.42, 11.98).$$

Or, using a GDC:

```
EDIT CALC TESTS
2↑T-Test
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8↓TInterval...
```

```
TInterval
 Inpt:Data Stats
 x:10.2
 Sx:3.8
 n:20
 C-Level:.95
```

```
TInterval
 (8.4215,11.978)
 x=10.2
 Sx=3.8
 n=20
```

Interpretation: We can be 95% confident that the true average weight carried by the hikers is between 8.42 kg and 11.98 kg. Or, equivalently, we are 95% confident that if we use 10.2 as an estimate of the true average, then the error in this estimate will not exceed 1.78 kg (half of the interval width).

Example 5

A producer of dairy products claims that the content of the 'light' yogurt they produce contains only 1% fat. To check this claim, we randomly collect 24 bottles of this product and check their fat content. Here are the results.

0.95	0.92	0.93	1.00	0.85	0.78	0.93	0.93	1.06	0.81	1.05	1.00
0.85	0.95	0.86	0.92	0.81	0.93	1.05	1.06	0.96	1.05	1.02	0.96

Set up a 99% confidence interval.

Solution

Here are the required statistics.

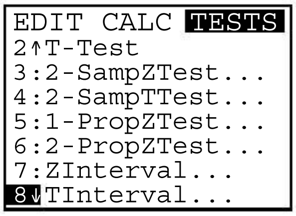
$\bar{x} = 0.9429$, $s = 0.0841$

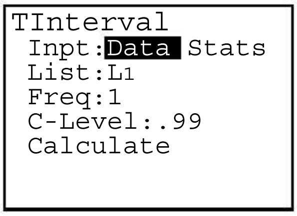
The critical t-value for 99% confidence level with 23 degrees of freedom is 2.807.

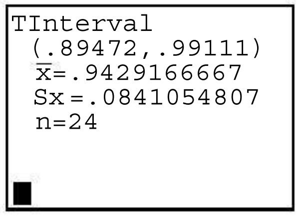
The confidence interval is then

$$\bar{x} \pm t_{(n-1),\, \alpha/2} \frac{s}{\sqrt{n}} = 0.9429 \pm 2.807 \cdot \frac{0.0841}{\sqrt{23}} = (0.895, 0.991).$$

Or, using a GDC, first enter the data into a list, and then calculate the interval.







Example 6

Workers in heavy industry, such as metalwork, drilling, and stone cutting are at risk of suffering from hearing loss because of exposure to high levels of noise. 49 workers in such industries are tested for hearing loss. Hearing level is usually measured in dBHL, where dB stands for decibels and HL for hearing level. This is the level where the subject starts to recognize noise. The higher the level, the more the hearing loss. A subject without exposure to high levels of noise has a hearing level up to 19 dBHL.

Here are the test results: $n = 49$, $\bar{x} = 35.0$ dBHL, $s = 19.0$ dBHL.

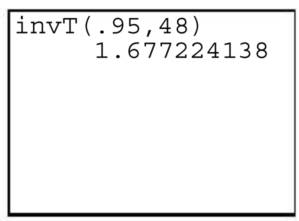
Find a 90% confidence interval for the average hearing level of all workers in heavy industry.

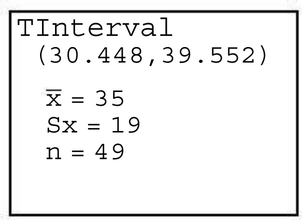
Solution

A 90% confidence interval is given by

$$\bar{x} \pm t_{48,0.05} \frac{s}{\sqrt{n}} = 35.0 \pm 1.677 \cdot \frac{19.0}{\sqrt{49}} = (30.4, 39.6).$$

(Screenshots for the calculations are shown right and below.)





This shows you that, even with 90% confidence level, there is so much information you get by calculating confidence intervals. In this case, since 19 dBHL is way below our 90% confidence interval, there is little doubt that these workers suffer extensive damage to their hearing.

Example 7 – Data given in a frequency table

The time taken to finish an entrance exam to a major university is recorded for a random selection of 300 students. Calculate a 95% confidence interval for the time it takes a student sitting for such an exam.

Time (min.)	70–75	75–80	80–85	85–90	90–95	95–100	100–105	105–110
Frequency	8	18	30	61	98	53	24	8

Solution

In order to do the work, manually or by GDC, we need to prepare the data so that we can find estimates of the mean and standard deviation. To that end, we represent each class by its mid-value. So, for the 70–75 class we use 72.5, and so on. Here is our adjusted table.

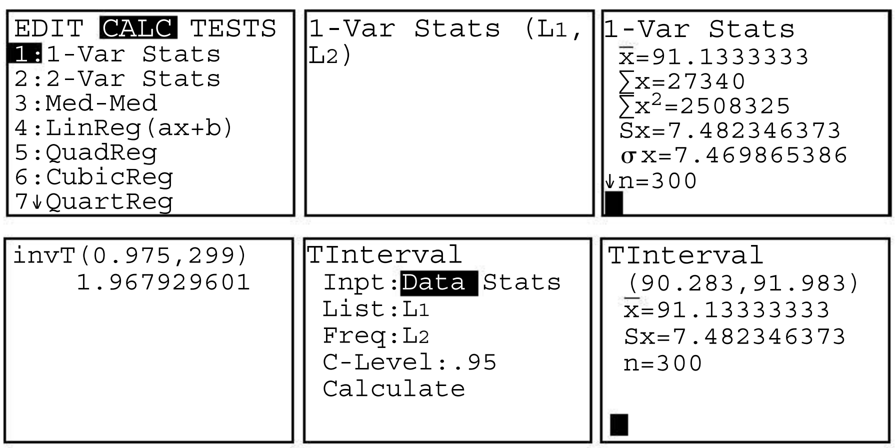
Time (min.)	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
Frequency	8	18	30	61	98	53	24	8

Here are the statistics we need: $n = 300$, $\overline{x} = 91.13$ min, $s = 7.48$ min.

A 95% confidence interval is given by

$$\overline{x} \pm t_{299,0.025}\frac{s}{\sqrt{n}} = 91.13 \pm 1.968 \cdot \frac{7.48}{\sqrt{300}} = (90.28, 91.98).$$

(Screenshots for the calculations are shown below.)



5.5 Confidence intervals for paired observations

Are automobiles equipped with ABS safer to drive than those without ABS? ABS (from the German **A**nti**b**lockier**s**ystem) is a safety system preventing the wheels from locking while braking. In an effort to check the safety of cars with ABS, two identical cars, one with and the other without ABS, were driven. The speeds and the time (in seconds) it took each to stop (on a dry surface) were recorded. The shorter the time, the safer the car is. Here are the results.

Speed (km/h)	20	30	40	50	60	70	80	90	100	120
ABS	3.7	4.6	5.8	6.5	7.1	7.3	7.7	8.2	8.4	8.9
Without ABS	3.6	4.5	5.9	6.8	7.4	7.8	8.0	8.6	9.0	9.4

To investigate the situation, a measure for the difference in the stopping times is required.

The experiment here is designed in such a way that each observation in one sample is *matched* with an observation in the other sample. Thus it is logical to compare the performance of each car under the two different situations – with and without ABS. This type of experiment is called **matched pairs** experiment.

To find a confidence interval, say 95%, for the differences, we *create* a new variable which we will call D and which measures the difference in stopping time, $D = [\text{without ABS}] - [\text{ABS}]$. When this difference is positive, then the stopping time of the ABS equipped car is better, and vice versa.

The table of values for D is given below.

Speed (km/h)	20	30	40	50	60	70	80	90	100	120
D	−0.1	−0.1	0.1	0.3	0.3	0.5	0.3	0.4	0.6	0.5

The calculation of the confidence interval is identical to the t-intervals developed earlier except for the notation! Thus, the 95% confidence interval is

$$\bar{d} \pm t_{9,0.025}\frac{s_D}{\sqrt{10}} = 0.28 \pm 20262 \cdot \frac{0.244}{\sqrt{10}} = (0.105, 0.455).$$

The critical t-value, $\bar{d}$, and s_D are calculated using the GDC.

```
1-Var Stats
 x̄=.28
 Σx=2.8
 Σx²=1.32
 Sx=.2440400696
 σx=.2315167381
↓n=10
```

```
invT(.975,9)
     2.262157158
```

Notation

We will use μ_D to represent the parameter, the true mean difference of stopping times of all cars without ABS and those with ABS. We will use the d to represent the calculated value for D, s_D^2 the variance, and $\bar{d}$ the estimated average from the sample.

With your GDC, you can also achieve the same results.

```
TInterval
 Inpt:Data Stats
 List:L1
 Freq:1
 C-Level:.95
 Calculate
```

```
TInterval
 (.10542,.45458)
 x̄=.28
 Sx=.2440400696
 n=10
```

In general, a $100(1-\alpha)\%$ confidence interval for the mean μ_D, the mean difference of the means of two matched-pairs samples is given by

$$\bar{d} \pm t_{(n-1),\,\alpha/2}\frac{s_D}{\sqrt{n}}$$

provided that the differences are approximately normal.

Example 8

A training program is designed to help people lose weight without going through harsh dieting. To check the effectiveness of this program, the weights of 12 randomly chosen participants were recorded when they joined the program and then two months later. Here are the data.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
Weight before	97	70	91	87	77	86	92	83	94	121	80	92
Weight after	96	71	88	84	75	84	92	84	93	119	79	91

Find a 90% confidence interval for the mean difference in weight loss due to the program.

Solution

Since the observations are taken in pairs, this will be a matched pairs interval. Hence, we need to set up a row consisting of the weight loss in the subjects. This is simply the difference between 'Weight before' and 'Weight after'.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
Weight loss	1	−1	3	3	2	2	0	−1	1	2	1	1

The confidence interval is then

$$\bar{d} \pm t_{(n-1),\,\alpha/2}\frac{s_D}{\sqrt{n}} = 1.167 \pm t_{11,\,0.05}\frac{1.34}{\sqrt{12}} = 1.167 \pm 1.796\cdot\frac{1.34}{\sqrt{12}} = (0.472, 1.86).$$

Or using a GDC:

```
1-Var Stats
 x̄=1.166666667
 Σx=14
 Σx²=36
 Sx=1.337115847
 σx=1.280190958
↓n=12
```

5.6 Confidence interval for a population proportion (large samples) (Optional)

Often we wish to make an inference about the proportion of individuals or objects in a population that possesses a particular property of interest.

For example, in the 'Green' party case. Suppose we would like to estimate the proportion of voters that will end up voting 'Green'. We select a random sample of 120 voters and discover that 31 of them claim to vote 'Green'. We need to calculate a 95% confidence interval for the true population proportion.

As we have seen in the previous chapter, the distribution of the sample proportion is approximately normal with a mean of p and a standard deviation $\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}}$. However, since p is unknown, then we use $\sqrt{\frac{\hat{p}\hat{q}}{n}}$ instead. The interval for the proportion follows the same structure as that for the mean, i.e. $\bar{x} \pm z_{\alpha/2}\sigma_{\bar{X}}$ will become $\hat{p} \pm z_{\alpha/2}\sigma_{\hat{p}}$.

$\hat{p} = \frac{x}{n}$ is the sample proportion where x is the number of 'successes' in the sample.
$\hat{q} = 1 - \hat{p}$.

The sample size must be large enough for this interval to be acceptable, namely, $np \geqslant 5$ and $nq \geqslant 5$.

A $100(1-\alpha)$% confidence interval for the proportion p of a normal population when the sample is large is given by

$$\left(\hat{p} \pm z_{\alpha/2}\sigma_{\hat{p}}\right) = \left(\hat{p} - z_{\alpha/2}\sigma_{\hat{p}},\ \hat{p} + z_{\alpha/2}\sigma_{\hat{p}}\right) = \left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}\right).$$

Thus, the interval is

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}\right) = \left(\frac{31}{120} - 1.96\sqrt{\frac{\frac{31}{120}\cdot\frac{89}{120}}{120}},\ \frac{31}{120} + 1.96\sqrt{\frac{\frac{31}{120}\cdot\frac{89}{120}}{120}}\right)$$

$$= (0.180, 0.337).$$

Using a GDC:

```
EDIT CALC TESTS
5↑1-PropZTest...
6:2-PropZTest...
7:ZInterval...
8:TInterval...
9:2-SampZInt...
0:2-SampTInt...
A↓1-PropZInt...
```

```
1-PropZInt
 x:31
 n:120
 C-Level:.95
 Calculate
```

```
1-PropZInt
 (.18002,.33665)
 p̂=.2583333333
 n=120
```

Example 9

A national airline claim that their flights are 'mostly' on time. We chose a random sample of 165 flights completed this year so far and found that 153 of them were actually on time. Find a 95% confidence interval for the true percentage of on-time flights for this airline.

Solution

$$\hat{p} = \frac{x}{n} = \frac{153}{165} = 0.927$$

$$\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}\right) = \left(0.927 - 1.96\sqrt{\frac{0.927 \times 0.073}{165}},\ 0.927 + 1.96\sqrt{\frac{0.927 \times 0.073}{165}}\right)$$
$$= (0.887,\ 0.967)$$

We are 95% confident that the true proportion of on-time flights can be between 88.7% and 96.7%.

Using a GDC:

```
1-PropZInt
 x:153
 n:165
 C-Level:.95
 Calculate
```

```
1-PropZInt
 (.88765,.9669)
 p̂=.9272727273
 n=165
```

5.7 Sample size determination (Optional)

Again, in estimating the population proportion, like the population mean, when there is interest in both a specific confidence level and specific precision, an appropriate sample size can be calculated.

Suppose that we are interested in a sample that ensures a level of precision defined by its width w and that we are interested in level of confidence defined by the critical number $z_{\alpha/2}$. What sample size should we settle for?

The width of a confidence interval can, at most, be equal to w:

$$2\left(z_{\alpha/2}\sqrt{\frac{pq}{n}}\right) \leqslant w$$

Since all values are given except n, then we solve that inequality for n.

$$2\left(z_{\alpha/2}\sqrt{\frac{pq}{n}}\right) \leqslant w \Rightarrow \sqrt{n} \geqslant \frac{2z_{\alpha/2}\sqrt{pq}}{w}$$

Since all variables are positive, then squaring both sides will yield

$$n \geqslant \left(\frac{2z_{\alpha/2}\sqrt{pq}}{w}\right)^2 \Rightarrow n \geqslant \left(\frac{2z_{\alpha/2}}{w}\right)^2 pq.$$

As is the case with the mean, the half-interval width $z_{\alpha/2}\sqrt{\frac{pq}{n}}$ is called the **bound on the error of estimation** or **the margin of error** associated with a $100(1-\alpha)\%$ confidence level; i.e. with $100(1-\alpha)\%$ confidence (90% for example), the point estimate $\hat{p}$ will not be further than $z_{\alpha/2}\sqrt{\frac{pq}{n}}$ from p.

Again, if we call this maximum value for the margin of error ε, then $w = 2\varepsilon$ and the sample size is determined by

$$n \geqslant \left(\frac{2z_{\alpha/2}}{2\varepsilon}\right)^2 pq = \left(\frac{z_{\alpha/2}}{\varepsilon}\right)^2 pq.$$

A problem with using this formula is that it depends on p, which we do not know. Statisticians have resorted to one of two solutions:

- use an estimate of p either from previous knowledge or from a convenient sample, or
- since in the formula above, $z_{\alpha/2}$ and ε are determined by the levels of confidence and precision, we can use the maximum value of pq which is $\frac{1}{4}$.

In this case $n \geqslant \frac{1}{4}\left(\frac{z_{\alpha/2}}{\varepsilon}\right)^2$.

Note: Since p and q add up to 1, then the product $pq = p(1-p)$ is a quadratic function in p. The maximum of this function is at $p = \frac{1}{2}$.

Example 10

In an upcoming local election, a party wants to estimate the percentage of voters planning on voting for them. They would like the error in the estimate not to exceed 3% at 95% confidence. What should the sample size be to achieve that objective?

Solution

Since we have no prior information about the population proportion here, we can use the formula developed earlier.

$$n \geqslant \frac{1}{4}\left(\frac{z_{\alpha/2}}{\varepsilon}\right)^2 \Rightarrow n \geqslant \frac{1}{4}\left(\frac{1.96}{0.03}\right)^2 = 1067.11$$

So, our sample size would be 1068 potential voters.

Exercise 5

1 Each of the following pieces of information are from random samples taken from populations that have two unknown parameters, the mean μ and the standard deviation σ. Find the unbiased estimates for μ and σ in each case.

a 64, 67, 75, 76, 80, 82, 84, 92, 94

b 0.685, 0.690, 0.687, −0.684, 0.693, −0.681, 0.688, 0.698, 0.678, −0.690

c

x	40	50	60	70	80	90	100
Frequency	2	5	11	25	18	10	4

d $\sum x = 611, \sum x^2 = 33267, n = 13$

e $\sum x = 867, \sum x^2 = 74135, n = 13$

2 The volume of mineral water supplied in small plastic bottles has a normal distribution and claimed to contain 500 cm^3 of water with a standard deviation of 2 cm^3. A sample of 10 bottles produced the following amounts.

502.0, 500.5, 498.0, 499.2, 501.0, 498.7, 499.6, 495.4, 501.2, 499.8

a Find unbiased estimates for the mean and standard deviation.

b Find a 90% confidence interval for the true mean of the population of all such bottles.

c Given the result above, what can you conclude about the company's claim?

d Find a 98% confidence interval for the true mean of the population of all such bottles.

e Calculate the width of each interval you found in **b** and in **d**. Compare the widths and make a comment.

f Suppose the standard deviation is not known. Do the calculation for part **c** without using the given standard deviation. Compare the two results.

3 A factory is acquiring a new measuring tool claimed to have very high accuracy. To make the decision, they tried it on measuring a piece of equipment that has a length of 50 cm. Here are the readings of the new tool in a sample of 36 trials.

50.027	49.912	50.135	49.993	49.829	49.696	50.049	49.864	50.191
50.009	50.139	50.044	50.082	50.096	49.919	50.061	50.018	50.330
50.133	50.147	49.769	50.137	49.876	50.179	50.173	49.850	50.044
50.014	49.680	50.116	50.016	50.163	50.371	49.714	49.767	50.017

Find a 95% confidence for the error this tool makes in measuring the 50 cm equipment.

4 My doctor asked me to keep track of my blood sugar level. So, I record the sugar level every day in the morning. I took a sample of 36 measurements and found out the sample estimate of the standard deviation to be 11.49. The confidence interval for the sugar level is (112.60, 119.63). What level of confidence did I use?

5 A sample of 100 bags of sugar are taken from a production line of Berto Packaging Company and found to have an average mass of $\bar{x} = 998$ g and a standard deviation $s_{n-1} = 10.2$ g.

a Calculate a 96% confidence interval for the mean mass of a sugar bag produced on this production line.

b The label on each bag reads '1 kg'. What fraction of the time do you believe the company is open to complaint by customers?

6 A sample of 40 flour bags from a production line of Berto Company produced a 95% confidence interval for the mass of a flour bag equal to (1008.3, 1066.7).

a Find a 98% confidence interval for the mean mass of such a bag.

b Suppose you take 60 random samples of 40 bags each and you calculate a 95% confidence interval for the mean mass μ of such a bag. Find the expected number of such intervals that may contain μ.

7 A quality control engineer has to decide on a sampling procedure for an assembly line. The assembly line assembles different plastic pipes of a certain length that are used in the production of refrigerators. She needs to be 95% confident that the sample mean will not differ from the true mean by more than 1.5 mm. The 'historical' standard deviation of this process is known to be 4 mm. How large should the sample be?

8 The age of smokers when they smoked their first cigarette is the subject of a study done at a medical school. The table below shows the data for a random sample of 500 smokers.

Age	14–	16–	18–	20–	22–	24–	26–	28–	30–	32–	34–
Number	28	72	84	114	96	52	34	14	4	0	2

Calculate a 95% interval for the average age of all the smokers in this city with their first cigarette.

9 A diet program aims to help people lose weight within 4 weeks of starting the program. In order for the promoters to be specific in their advertisement, they want to know if the program really helps people lose weight, and by how much. A sample of 12, relatively 'weighty', volunteers took part in the experiment. The experiment was to weigh each participant before they participated in the program, following all instructions, for a month. At the end of the month they were weighed again. Here are the results.

Participant	1	2	3	4	5	6	7	8	9	10	11	12
Weight before	95	87	102	92	85	86	97	105	112	110	90	96
Weight after	93	86	99	92	86	84	95	102	107	109	90	95

Calculate a 90% confidence interval for the weight loss owing to the program by all who participated.

10 A random sample of 300 households in a large city were asked whether they own a computer. 207 households own at least one computer. Find a 90% confidence interval for the proportion of households that own at least one computer in this city.

11 A car service shop knows from previous experience that the time needed to change oil on a car is normally distributed with a standard deviation of 5 minutes. However, as new cars are being made more efficient, he is interested in knowing the average time it takes to do the task. He chose 16 oil changes at random and recorded their time, which is listed below. Compute a 99% confidence interval for the mean of all oil changes in this shop.

16, 10, 12, 11, 18, 15, 12, 24, 25, 20, 18, 24, 13, 18, 21, 16

12 A statistician wants to estimate the average weight loss of people who are on an improved diet plan. In the previous version of the plan, she knows that the standard deviation is 5 kg. How large should a sample be to estimate the mean weight loss to within 1 kg, with 90% confidence?

13 In many countries in Europe the law requires drivers to have headlights on during the day. A newspaper report in one of these countries stated that only 25% of the drivers follow this law. The police departments in that country, in order to get more information about the issue, randomly chose 2000 cars and counted the number that have their headlights on. The number was 410. Construct a 95% confidence interval for the proportion of cars following that law.

14 The political environment in a certain country took a sharp turn away from the ruling party. A pollster, appointed by one of the opposition parties, would like to collect data in an effort to predict this party's share of the vote in the upcoming elections. The party insists that the estimate should not be more than 0.03 points off the actual proportion. They would also like to be 95% confident. What sample size should the pollster have?

15 Screws are produced with a mean length of 4 cm and a variance of 0.04 cm^2. How large a sample should be taken to be 95% certain that the mean of the sample will be within 0.1 cm of the population mean length?

16 A packaging machine produces packets of margarine with a mean of 250 g and standard deviation 4 g.

a If 20 packets are chosen at random and weighed, what is the probability that they will have a mean of more than 247 g?

b What size sample must be taken to be 90% certain that the mean of the sample will be between 246 g and 254 g?

17 A large consignment of apples is examined by selecting a random sample of 50 boxes. It is found that 12 contain at least one bad apple. Assuming that these boxes may be regarded as being a random sample from the boxes in the consignment, obtain an approximate 99% confidence interval for the proportion of boxes containing at least one bad apple, giving your confidence interval correct to three decimal places.

18 Suppose we have two unbiased estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ of a parameter θ. Let $V\left(\hat{\theta}_1\right) = \sigma_1^2$ and $V\left(\hat{\theta}_2\right) = \sigma_2^2$.

a Show that $T = k\hat{\theta}_1 + (1-k)\hat{\theta}_2$ is an unbiased estimate of θ.

b If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, find the value of k that will minimize the variance of T.

19 A random sample $\{X_1, X_2, X_3\}$ is chosen from a population whose density function is

$$f(x) = \begin{cases} \dfrac{1}{\lambda} e^{-\frac{x}{\lambda}}, & x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the following estimators of λ:

$$\hat{\theta}_1 = X_1,\ \hat{\theta}_2 = \frac{X_1 + X_2}{2},\ \hat{\theta}_3 = \frac{X_1 + 2X_2}{3},\ \hat{\theta}_4 = \bar{X}.$$

In your work, you may use the following two facts about the function f:

$E(X) = \lambda; V(X) = \lambda^2$.

a Which of the estimators is unbiased?

b Among these estimates, which is the most efficient?

20 In Example 1, we showed that if $X \sim B(n, p)$ then $\bar{p} = \dfrac{X}{n}$ is an unbiased estimator of p. If we want to estimate V(X) we sometimes use the estimate $v = n\overline{pq}$.

a Show that v is a biased estimator of $V(X)$.

b Modify v to get an unbiased estimator for $V(X)$.

Practice questions 5

1 A market research company has been asked to find an estimate of the mean hourly wage rate for a group of skilled workers. It is known that the population standard deviation of the hourly wage of workers is \$4.00. Using a confidence interval for the mean, determine how large a sample is required to yield a probability of 95% that the estimate of the mean hourly wage is within \$0.25 of the actual mean.

2 *Give your answers to* ***four*** *significant figures.*

The following is a random sample of 16 measurements of the density of aluminium. Assume that the measurements are normally distributed.

2.704	2.709	2.711	2.706
2.708	2.705	2.709	2.701
2.705	2.707	2.710	2.700
2.703	2.699	2.702	2.701

Construct a 95% confidence interval for the density of aluminium, showing all steps clearly.

3 *Give all numerical answers to this question correct to* ***two*** *decimal places.*

A radar device records the speed, v kilometres per hour, of cars on a road. The speed of these cars is normally distributed. The results for 1000 cars are recorded in the following table.

Speed	Number of cars
$40 \leqslant v < 50$	9
$50 \leqslant v < 60$	35
$60 \leqslant v < 70$	93
$70 \leqslant v < 80$	139
$80 \leqslant v < 90$	261
$90 \leqslant v < 100$	295
$100 \leqslant v < 110$	131
$110 \leqslant v < 120$	26
$120 \leqslant v < 130$	11

a For the cars on the road, calculate

i an unbiased estimate of the mean speed
ii an unbiased estimate of the variance of the speed.

b For the cars on the road, calculate

i a 95% confidence interval for the mean speed
ii a 90% confidence interval for the mean speed.

c Explain why one of the intervals found in part **b** is a subset of the other.

4 Carlos drives to work every morning. He records the times taken, in minutes, to complete the journey over a 10-day period. The times are as follows:

32.6 30.9 35.8 34.3 36.3 31.9 33.2 32.7 31.3 32.8

Assuming that these times form a random sample from a normal population, calculate

a unbiased estimates of the mean and variance of this population

b a 90% confidence interval for the mean.

5 A chicken farmer wishes to find a confidence interval for the mean weight of his chickens. He therefore randomly selects n chickens and weighs them. Based on his results, he obtains the following 95% confidence interval.

[2148 grams, 2188 grams]

The weights of the chickens are known to be normally distributed with a standard deviation of 100 grams.

a Find the value of n.

b Assuming that the same confidence interval had been obtained from weighing 166 chickens, what would be its level of confidence?

6 In an opinion poll, 540 out of 1200 people interviewed stated that they support government policy on taxation.

a **i** Calculate an unbiased estimate of the proportion, p, of the whole population supporting this policy.
ii Calculate the standard error of your estimate.
iii Calculate a 95% confidence interval for p.

b State an assumption required to find this interval.

7 The random variable X is normally distributed with mean μ. A random sample of 12 observations is taken on X, and it is found that $\sum_{i=1}^{12}(x_i - \bar{x})^2 = 99$.

a Determine a 95% confidence interval for μ.

b Another confidence interval [60.31, 65.69] is calculated for this sample. Find the confidence level for this interval.

8 The random variable X is normally distributed with mean μ and standard deviation 2.5.

A random sample of 25 observations of X gave the result $\sum x = 315$.

a Find a 90% confidence interval for μ.

b It is believed that $P(x \leqslant 14) = 0.55$. Determine whether or not this is consistent with your confidence interval for μ.

9 **a** In a random sample of 1100 people in Switzerland, it was found that 580 of them had a connection to the internet. Calculate the 95% confidence interval for the proportion of people in Switzerland having a connection to the internet.

b How large should the sample have been to make the width of the 95% confidence interval less than 0.02?

10 When a scientist measures the concentration of a solution, the measurement obtained may be assumed to be a normally distributed random variable with mean μ and standard deviation 1.6.

a He makes 5 independent measurements of the concentration of a particular solution and correctly calculates the following confidence interval for μ.

[22.7, 26.1]

Determine the confidence level of this interval.

b He is now given a different solution and is asked to determine a 95% confidence interval for its concentration. The confidence interval is required to have a width less than 2. Find the minimum number of independent measurements required.

6 Hypothesis Testing

6.1 Concepts of hypothesis testing

We will start this section with an example.

In an assembly plant for personal computers, the time technicians spend putting together systems is of extreme importance. Supervisors need to carry out regular inspections of the time each process takes. At a production cluster, a supervisor believes that the average time it takes his technicians to complete a certain task has changed. Previously, it took on average 2 minutes ($\mu = 2$) to complete this task with a standard deviation of $\sigma = 0.2$. So, now the supervisor believes that $\mu \neq 2$. How might we decide if, based on a sample of 100 observations, there is evidence to show the mean is significantly different from 2 minutes?

We begin by calculating the mean of this sample and find that it is $\bar{x} = 2.2$ minutes. By calculating a confidence interval (95% for example), we have the following result.

$$\left(\bar{x} \pm z_{\alpha/2}\sigma_{\bar{x}}\right) = \left(2.2 - 1.96\frac{0.2}{\sqrt{100}},\ 2.2 - 1.96\frac{0.2}{\sqrt{100}}\right) = (2.161,\ 2.239)$$

So, we are 95% confident that the true mean could lie between 2.161 minutes and 2.239 minutes. That is, since 2 minutes is not included in this interval, we are confident that we have evidence from the collected data to show that the mean time is different from 2 minutes. This means that the claim that $\mu = 2$ is rejected on the basis of collected evidence.

What we just did here is a test of the hypothesis $\mu = 2$.

So, what is a statistical hypothesis?

A statistical hypothesis is a claim (assertion) about a population parameter. It is often a claim about the value of the parameter.

In the previous example, the hypothesis is the claim that the mean time $\mu = 2$ minutes.

In hypothesis-testing problems, we have two contradictory hypotheses under consideration. One hypothesis in the example above is that $\mu = 2$ and the other is $\mu \neq 2$. If it were possible to collect data from the entire population, we would know which of the two hypotheses is true. Usually this is not possible, or very difficult and inefficient. As a result, we must decide which hypothesis we think is true by using information from a well-randomized sample.

In many countries, court trials are familiar cases where a choice between two opposing arguments must be made. The person charged of an offence must be judged as innocent or guilty. Initially, in these countries, the accused is assumed innocent. Only strong evidence to the contrary will be the basis for the innocence claim to be rejected and the guilty claim to be assumed instead. The burden of proof is then on the prosecution to prove the guilty claim. In other countries, the opposite is true. That is, once the prosecution puts someone on trial (with sufficient evidence, of course), the accused is assumed guilty. The burden of proof falls on the defence to establish innocence.

In every hypothesis testing problem, we will have a null hypothesis that we denote by H_0 (in the previous example $\mu = 2$) and an alternative hypothesis that we denote by H_1(in the previous example $\mu \neq 2$). The test is a method for deciding which of the two hypotheses is correct. We shall assume the null hypothesis H_0 is the correct one. After carrying out the test, this hypothesis will only be rejected in favour of the alternative one, H_1, if sample evidence is incompatible with H_0. If we fail to provide evidence against H_0, then the claim is that '*we fail to reject* H_0'.

Thus, in general we have:

The null hypothesis, H_0,

- states the assumption to be tested
  e.g. the mean time technicians take to complete the task is 2 minutes (H_0: $\mu = 2$)
- begins with the assumption that the null hypothesis is true
- refers to the status quo
- may or may not be rejected.

The alternative hypothesis, H_1,

- is the opposite of the null hypothesis, e.g. the true mean time is significantly different from 2 minutes (H_1: $\mu \neq 2$)
- challenges the status quo
- is generally the hypothesis that is believed (or needed to be proven) to be true by the researcher.

We will explore the ideas of hypothesis testing with a further example.

Example 1

A factory produces light bulbs that are known to last 8000 hours on average, with a standard deviation of 1000 hours. A new type has been developed and is claimed to last more than the present ones. The management is concerned that the cost of the process producing the new type is more expensive than the traditional method. They can justify shifting to the new process, only if they have evidence that the new light bulbs do actually have a longer life than 8000 hours. For that purpose, a random sample of 25 new bulbs was tested by simulating the use for more than 8000 hours and gave an average life of 8600 hours. Is this enough evidence to justify the shift to the new process?

Solution

Since we are attempting to show that the population mean exceeds 8000 hours, it seems reasonable to base our decision on the value of the sample mean. If $\overline{x}$ is 'much larger' (or significantly more) than 8000 hours, then we have compelling evidence that the mean $\mu > 8000$. To determine whether the sample mean of 8600 exceeds 8000 by an amount that would be considered 'unlikely' to occur *by chance*, we will either

- calculate the chance that a sample of mean 8600 or larger, can happen by chance from a population of mean 8000. In this case we call $\overline{x}$ the **test statistic**.

or

- calculate the z-value for $\overline{x}$ and see what chance such a z-value has. In this case, the z-value is called the **test statistic**.

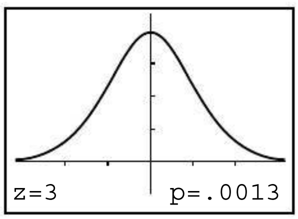
Using a GDC or computer software, we find the following:

$$\sigma_{\overline{x}} = \frac{1000}{\sqrt{25}} = 200, \text{ and hence}$$

$$P\left(\overline{x} \geqslant 8600 \middle| \mu = 8000, \sigma_{\overline{x}} = 200\right) = 0.00135.$$

```
Z-Test
 Inpt:Data Stats
 μ0:8000
 σ:1000
 x̄:8600
 n:25
 μ:≠μ0 <μ0 >μ0
 Calculate Draw
```

```
Z-Test
 μ>8000
 z=3
 p=.0013499672
 x̄=8600
 n=25
```



This is a very small probability that a population whose mean is 8000 could yield a sample with 8600 by mere chance. We conclude to *reject* the null hypothesis in favour of the alternative. That is, we have enough evidence to reject the claim that the new light bulbs are the same as the old ones.

Using the z-value approach will lead us to the same conclusion.

$$z = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}} = \frac{8600 - 8000}{1000 / \sqrt{25}} = 3.0$$

That is, the z-value of $\overline{x}$ is 3 standard deviations away from the mean of 8000. This is sufficiently large to consider it a rare event if the mean of the population were 8000.

6.2 The hypothesis testing procedure

Distributions with known variance

In the light bulbs example above, how can we be confident about our decision that the new light bulbs are better than the old ones?

We start with the **null hypothesis**, which we designate as H_0, and we specify a population parameter, μ in this case, and we suggest a value for that parameter, 8000 here. We usually write down a null hypothesis about a mean, for example, as

H_0: $\mu = \mu_0$ (H_0: $\mu = 8000$)

This is a short way of indicating the two items we need most: the nature of the parameter we hope to learn about (the true mean) and a particular assumed value for that parameter (8000 in this case). We need the particular value so we can judge our observed *statistic* against it.

The **alternative hypothesis**, H_1 (sometimes called H_a), contains the value(s) of the parameter that we regard as reasonable in case the null hypothesis is rejected. In the light bulbs example, the alternative is the life of the bulbs being more than 8000 hours. We also write it as

H_1: $\mu > \mu_0$ (H_1: $\mu > 8000$)

Note: In the light bulbs example, we were interested in an alternative:

H_1: $\mu > \mu_0$, which is called an **upper-tail test**.

But in other cases we could also be interested in

H_1: $\mu < \mu_0$, which is called a **lower-tail test**, or

H_1: $\mu \neq \mu_0$, which is called a **two-tail test** (like the example of 2-minute completion time).

What persuades us to believe that the light bulbs have a life more than 8000 hours? Does a sample mean of 8200 provide us with the evidence that the mean is really more than 8000? What about 8400 hours? or 8600?

We should not expect to have a sample mean exactly equal to 8000 as observations vary from one sample to the other. We base our decision on how significantly surprising our sample result is under the assumption that the true mean is 8000 in this example. That is, do we consider 8200 to be a surprising result? If not, is 8400 or 8600 surprising?

To answer the question, we have to remember that the distribution of sample means, according to the CLT, is normal with a mean of 8000, and a standard *error* $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{1000}{\sqrt{25}} = 200$.

So, how surprising is 8200?

To answer this question we find

$$P(\bar{x} > 8200) = P\left(z > \frac{8200 - 8000}{200}\right) = P(z > 1) = 0.159.$$

```
normalcdf(1,1000
,0,1)
      .1586552596
```

So, if the mean life of these bulbs were 8000, the chance of randomly getting a sample mean of 8200 or above is about 16%. In other words, there is a good chance that a population with mean 8000 can give out a sample of average 8200.

How surprising is 8400?

To answer this question we find

$$P(\bar{x} > 8400) = P\left(z > \frac{8400 - 8000}{200}\right) = P(z > 2) = 0.0228.$$

```
normalcdf(2,1000
,0,1)
       .022750062
normalcdf(3,1000
,0,1)
      .0013499672
```

Thus, if the true mean life were 8000, the chance that we can get a sample with average 8400 or more is less than 2.3%. You may think that this result is significantly 'surprising' and you conclude that the mean lifetime has to be larger than 8000 in order to produce such a sample by chance.

How surprising is the 8600?

As we have seen above,

$$P(\bar{x} > 8600) = P\left(z > \frac{8600 - 8000}{200}\right) = P(z > 3) = 0.00135.$$

In this case, the event of finding a random sample with a mean of 8600 or above from a population with mean of 8000 is extremely unlikely, and we find ourselves convinced that the population must have a higher mean than 8000 in order to render a random sample with a mean of 8600 or more by mere chance.

As you notice from the previous discussion, the fundamental step in our analysis is the question: 'are the sample data unexpected, given the null hypothesis?' The key calculation is to determine how likely the sample data we observed would be if the null hypothesis were the true model of the world. That is why we need a *probability*. We would like to find the probability of observing sample data like these *given* the null hypothesis. This probability is the value we base our decision on. This probability is called the ***p*-value**.

A small *p*-value indicates that the sample data we see would be very unlikely had our null hypothesis been true. That is, we start with a model in mind, we collect the data, and then the model tells us that this data we have is unlikely to have happened. That is surprising. The model and data are not compatible and hence *we have to make a decision.* Either the model, the null hypothesis, is true and we have been unlucky to get such a remarkably unexpected sample, or the null hypothesis is at fault – that is, we were not correct to use it as a basis for calculating our *p*-value. Given that the sample data is 'tangible' and real, while the model (null hypothesis) is an assumption, we are tempted to reject the model.

When the p-value is large (or just not small enough), what do we conclude? In that case, we have not found anything unlikely or surprising or unexpected. So, we have no reason to reject the null hypothesis. In this case, it does not mean that we 'proved' the null hypothesis. It only means that it 'does not appear that the hypothesis is false'. Formally, we say that 'we fail to reject the null hypothesis'. All we were able to establish is that the sample data we have at hand is consistent with the model. We did not and could not collect 'all' the evidence to support the null hypothesis. Unfortunately, the decision to reject it is more appealing usually as we have a contradicting example that proves it wrong!

Each hypothesis testing problem will involve a null hypothesis H_0 and an alternative hypothesis H_1.

For example, for the claim than an IB candidate has less than 7 hours of sleep per day:

H_0: $\mu = 7$, H_1: $\mu < 7$.

The null hypothesis, H_0,

- states the assumption to be tested (e.g. the mean daily time an IB candidate sleeps is 7 hours; H_0: $\mu = 7$)
- is about a population parameter, not about a sample statistic (μ and not $\bar{x}$)
- starts with the assumption that the null hypothesis is true
- is analogous to the concept of innocent until proven guilty in court cases
- refers to the status quo
- may or may not be rejected.

The alternative hypothesis, H_1,

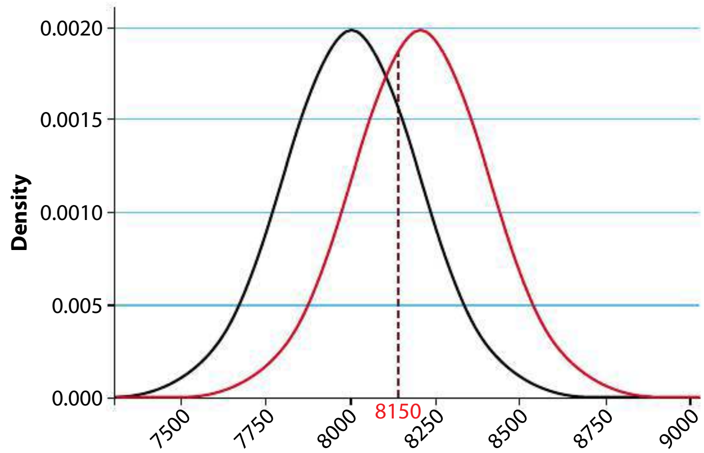
- is contradictory to the null hypothesis (e.g. the true mean time is significantly less than 7 hours; H_1: $\mu < 7$)
- disputes the status quo
- is usually the hypothesis that is suspected (or wanted to be verified) to be true by the investigator.

When performing the hypothesis test, we make our decision according to a **decision rule** (also called **critical region**), which tells us when to reject the null hypothesis. We have a (100α)% error rate of making the incorrect decision of rejecting the null hypothesis when it is true. We call this the **level of significance of the test** α.

How small must the *p*-value be?

To answer this question, we need to investigate the ramifications of our decision. So, as we discussed earlier, our decision is to reject or not to reject the null hypothesis. Like any situation, where a decision has to be made, we are open to make a mistake. If I reject the null hypothesis based on sample data, it could well be that this data was so unrepresentative that I was misled to reject the hypothesis. If I fail to reject the hypothesis, it could be that the sample belongs to a population whose mean is close to 8000, for example, but not 8000. To demonstrate this, see the figure right. We receive a sample mean of 8150. The probability that a sample of mean 8150 or more when the population has a mean of 8000 is given by

$$\begin{aligned} P(\bar{x} > 8150) &= P\left(z > \frac{8150 - 8000}{200}\right) \\ &= P(z > 1.5) = 0.0668. \end{aligned}$$



The sample belongs to a population whose mean is 8200, but the chance to have a sample with this mean from a population having our hypothesized mean is 6.7%. This could well lead us to conclude that the sample is consistent with the model and we end up making the error of not rejecting the null hypothesis.

So, what types of errors may we end up committing?

When we perform a hypothesis test, we can make mistakes in two ways:

- the null hypothesis is true, but we end up rejecting it, or
- the null hypothesis is false, but we fail to reject it.

These two types of errors are known as **Type I** and **Type II** errors. Type I is associated with rejecting the null hypothesis when it is true, and Type II for not rejecting it when it is false.

Here is an illustration that helps us keep track of our decision.

		The truth	
		H_0 true	**H_0 false**
Our decision	**Do not reject H_0**	Correct decision	Type II error
	Reject H_0	Type I error	Correct decision

So, in general, the decision of how small we want the *p*-value to be depends on how high the probability of Type I error is desired. In the example of the light bulbs, committing a Type I error means that the life of the bulbs is actually 8000 hours but we end up saying that it is higher. The price of our decision would be to cause the company to spend more money to produce a new line which is only as good as the old one. Management, of course, wants to minimize the chance of this happening.

A Type II error in this example is to conclude that you don't have evidence to say that the new light bulbs have a longer life, when they actually do. The consequence for this decision is to deprive the company from benefitting from the new innovation.

Hypothesis testing – analogy to court verdicts

There is a stark resemblance between court verdicts and hypothesis tests that is summarized in the following table.

	The truth		Decision	The truth	
Verdict ↓	Innocent	Guilty	Decision ↓	H_0 true	H_0 false
Innocent	Correct decision	Error	Do not reject H_0	Correct decision	Type II error
Guilty	Error	Correct decision	Reject H_0	Type I error	Correct decision

When the p-value is small, it indicates that our sample data are ***unusual given*** H_0. If our data are '*unusual enough*', then we cannot assume that this could have occurred only by chance. Since the data *did* occur, then something must be incorrect. All we can do is to reject the null hypothesis.

A **statistically significant** result in hypothesis testing can be interpreted as a significantly rare event that will convince us to reject H_0.

But how unusual is 'unusual'? How small must the *p*-value be?

We can define *unusual events* arbitrarily by setting a limit for our p-value. If our p-value falls below that point, we will reject H_0. We will call such results **statistically significant**.

The α-level is also called the **significance level** or **level of significance**. When we reject a hypothesis, we say that it was rejected at the 'k%' level of significance, where $k = 1, 5, 10$, or any other number.

The limit is called an **alpha level** (α-level). Common α-levels are 0.01, 0.05, and 0.10. A statistician has to consider the alpha level carefully, dependent on the situation. For example, if you are testing a hypothesis about the safety of a brake system in cars, you may want the α-level extremely low. If you are testing whether students use the school bus or not, you might be content with $\alpha = 0.10$. The level used mostly is $\alpha = 0.05$.

Sir **Ronald Aylmer Fisher** (1890–1962) was a statistician, evolutionary biologist, and geneticist. He is accredited with creating the foundations for modern statistics. Among his contributions is the discussion of the amount of evidence needed to reject a null hypothesis. He wrote that it was *situation dependent*, but remarked that for many applications, 1 out of 20, i.e. 5%, might be a reasonable value.

When the p-value is not smaller than the α-level, then we say that '*we have insufficient evidence to reject* H_0', or '*we fail to reject* H_0'. We do not say '*we accept* H_0'. By failing to find evidence against it, we have not proven it, as it was assumed in the first place.

$\alpha = \mathrm{P}(\textit{Type I error}) =$ P(H_0 is rejected when H_0 is true), and $\beta = \mathrm{P}(\textit{Type II error}) =$ P(H_0 is **not** rejected when H_0 is **not** true).

Note: From the preceding discourse we can say that $\mathrm{P}(\textit{Type I error}) = \alpha$. It should not be a surprise then if we call the probability of Type II error β.

Example 2

From 1998 to 2004, the amount of nicotine that could be inhaled from cigarettes increased by an average of 10 per cent. Nicotine is the chemical that causes cigarettes to be addictive, and studies found higher levels in all classes of cigarettes, including those branded 'light.' There is some suspicion that local cigarette companies boosted their cigarettes' nicotine content to maintain or increase present addictive levels. The last recorded level of nicotine content is 1.8 mgc (milligram per cigarette) with a standard deviation of 0.2 mgc. To investigate whether the present level has really been increased we analyze a random sample of 100 cigarettes for nicotine content. The average content of the 100 cigarettes is 1.84. Is there evidence, at the 5% level of significance, to conclude that our suspicion is justified? Also, interpret Type I and Type II errors in this case.

Solution

Here we are testing

H_0: $\mu = 1.8$

against

H_1: $\mu > 1.8$.

To find the p-value, we calculate:

$$\mathrm{P}\left(\bar{x} > 1.84 \middle| \mu = 1.8, \sigma_{\bar{X}} = \frac{0.2}{\sqrt{100}}\right) \approx 0.0228$$

It seems that this event is quite unlikely to happen merely by chance from a population whose mean is 1.8 and hence we reject the null hypothesis. Our suspicion that the cigarette companies boosted the nicotine content of their cigarettes is justified.

Type I error in this case would be to claim that the nicotine content in cigarettes has been 'boosted' when it actually has been unchanged. In this case, and since we decided to reject the null hypothesis, we are open to this type of error.

Type II error in this case would be to conclude that there is no evidence of an increase in nicotine levels, when the companies had truly increased them.

Note: Calculating the p-value can be done using your GDC, as shown on the screenshot right.

```
normalcdf(1.84,1
000,1.8,0.020)
       .022750062
```

Additionally, the whole hypothesis test can be performed by your GDC.

First you open the STAT menu and open the TESTS submenu.

```
EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7↓ZInterval...
```

Now choose the 'Z-Test', and fill the data in.

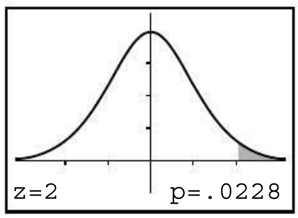
```
Z-Test
 Inpt:Data Stats
 μ0:1.8
 σ:.2
 x̄:1.84
 n:100
 μ:≠μ0 <μ0 >μ0
 Calculate Draw
```

You can now choose either 'Calculate' or 'Draw'.

'Calculate' will give you the following display.

```
Z-Test
 μ>1.8
 z=2
 p=.022750062
 x̄=1.84
 n=100
```

'Draw' will give this display.



In both cases, you will be able to read the p-value of 0.0228.

Example 3

The drying time for a type of car paint is known to be normally distributed with mean of 75 minutes and standard deviation of 9 minutes. Car painters for an automobile company have discovered an additive which shortens the drying time. However, if the company approves the use of this additive, the cost of painting a car will naturally increase. They will not approve unless they have strong evidence that the additive does reduce the drying time. A test on 49 new cars gave a mean drying time of 72 minutes.

a) What do you recommend to the company? Use 5% level of significance.

b) Discuss Type I and Type II errors.

Solution

a) In this problem, we are testing

H_0: $\mu = 75$ against

H_1: $\mu < 75$.

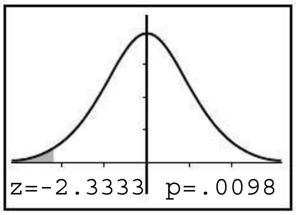
To find the p-value, we calculate:

$$P\left(\bar{x} < 72 \middle| \mu = 75, \sigma_{\bar{x}} = \frac{9}{\sqrt{49}}\right) \approx 0.0098$$

Since this value is less than 5%, we reject the null hypothesis and conclude that we have enough evidence that the average drying time is less than 75 minutes. So, the company may go ahead and start using the additive.

```
Z-Test
 Inpt:Data Stats
 μ0:75
 σ:9
 x̄:72
 n:49
 μ:≠μ0 <μ0 >μ0
 Calculate Draw
```

```
Z-Test
 μ<75
 z=-2.333333333
 p=.0098153068
 x̄=72
 n=49
```



b) **Type I error:** Conclude that the mean drying time is less than 75 minutes when it actually is still 75 minutes. Here again, we are open to this type of error since we rejected H_0. Ramifications of this error are to invest in using the additive when it is essentially worthless.

Type II error: Conclude that we do not have evidence to say the mean drying time is less than 75 minutes, when it actually is less than 75 minutes. Ramifications of this error would be to deprive the company from the benefit of having shorter drying time.

Example 4

A manufacturer of sprinklers used for fire prevention in buildings claims that the set-off temperature for their system is 55° with a standard deviation of 1.5°. It is very important for the set-off temperature to be accurate. If it is lower than the threshold mark, it may start sprinkling water without a fire, causing unnecessary panic and damage. If it starts late, then the fire will cause the damage. To test the claim, 16 sprinklers were randomly selected and tested, and yielded a sample mean of 56.02°. Does the data contradict the manufacturer's claim at 2% level of significance?

Solution

Our parameter of interest is μ, the true average set-off temperature.

The null hypothesis is H_0: $\mu = 55°$

The alternative is H_1: $\mu \neq 55°$

The p-value is the probability that a 55°-population yields a sample of 16 with a mean as extreme as 56.02° or further on both sides of the mean! Which means, the probability of receiving a sample with $\bar{x} \geqslant 56.02°$ or $\bar{x} \leqslant 53.98°$. This p-value can be calculated directly.

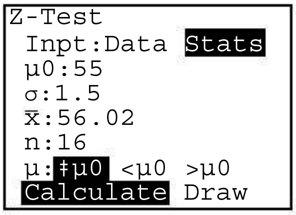
$$2 \times P\left(\bar{x} \leqslant 53.98 \middle| \mu = 55, \sigma_{\bar{x}} = \frac{1.5}{\sqrt{16}}\right) = 0.0065$$

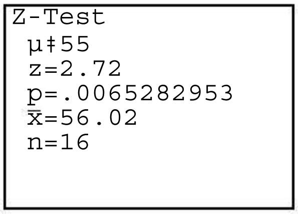
It can also be calculated through standardizing the sample mean.

$$2 \times P\left(z \leqslant \frac{53.98 - 55}{\frac{1.5}{\sqrt{16}}} \middle| \mu = 0, \sigma = 1\right) = 0.00653$$

as you can see from the GDC screen right.

```
2*normalcdf(-100
,53.98,55,.375)
      .0065282953
2*normalcdf(-100
,-2.72)
      .0065282953
```





Finally, it can be calculated by using the built-in test of the GDC.

The p-value $0.005 < 0.02$. We reject the null hypothesis. We have enough evidence to claim that the sprinklers of this company may not conform to the claimed 55° claim.

Note: As you noticed in the previous example, this is a two-tailed test.

6.3 Hypothesis testing using critical values

You recall from the discussion of confidence intervals that, in order to construct a confidence interval, we need to find the critical value $z_{\alpha/2}$ corresponding to the level of significance at hand. Such critical values can also be used in hypothesis testing procedures.

Such an approach was a common practice before the advent of computers and calculators. Calculating p-values, as you may have observed so far, is not an easy task without the use of technology. Instead, we can choose a few familiar α-levels and remember the corresponding critical value. Rather than calculating the probability corresponding to our test statistic (mean μ, or z-value for the mean, for example) we would instead find how far from the hypothesized mean our test statistic should be in order to be rejected. That is, finding a limit or a threshold beyond which we reject the null hypothesis. If our test statistic lies beyond that threshold, then it is considered rare and we reject the null hypothesis. The process can best be explained by using a specific example.

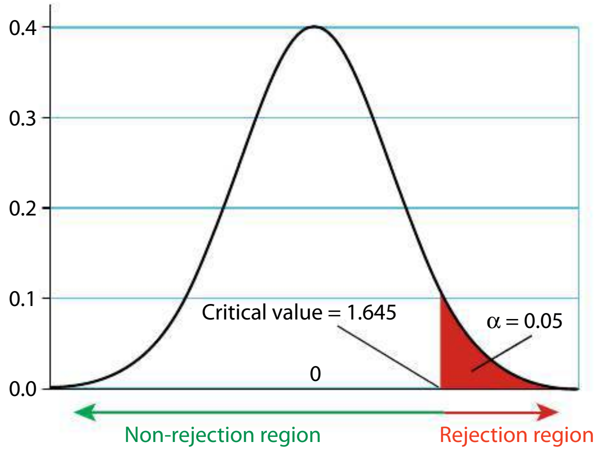
Example 5

Look again at the light bulbs factory example. Remember that a random sample of 25 new bulbs was tested and gave an average life of 8600 hours. Is this enough evidence at the 5% *level of significance*, to justify the shift to the new process?

Solution

To answer this question without calculation of the p-value, we can set up our normal distribution model to distinguish between what we consider rare and what we do not consider rare. This can be done in one of two ways: using the standard normal distribution or the original normal distribution with mean of 8000 and standard deviation of 1000.

Using the standard normal:



Since the probability of Type I error is specified as 5%, the critical number is $z = 1.645$ which we can find either from the table or from a GDC (see below). The area to the right of 1.645 is considered the 'rare' events area and hence the **rejection region** and the area to the left of 1.645 is the **non-rejection region**.

```
DISTR DRAW
1:normalpdf(
2:normalcdf(
3:invNorm(
4:invT(
5:tpdf(
6:tcdf(
7↓χ²pdf(
```

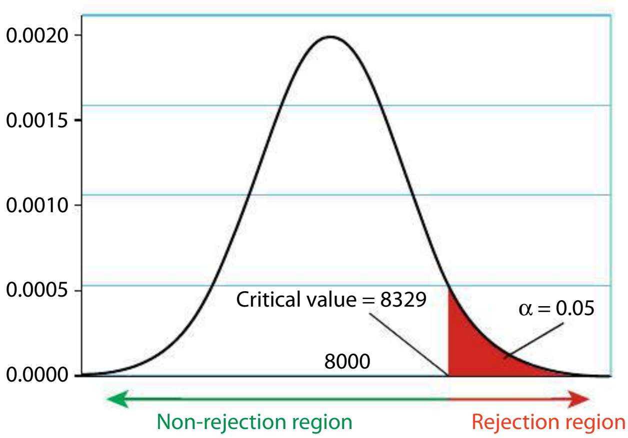
```
invNorm(0.95)
     1.644853626
```

Next we calculate our test statistic, the z-value corresponding to the sample mean.

$$z = \frac{8600 - 8000}{\frac{1000}{\sqrt{25}}} = 3.0$$

Since this value of the observed test statistic is to the right of the critical number 1.645, i.e. it lies in the rejection region, we reject the null hypothesis.

Using the original distribution:



The rejection region is also called the **critical region**.

The critical value using the raw data can be found in a similar manner to the one above (see below), i.e. $\bar{x}^* = 8329$. The area to the right of 8329 is considered the 'rare' events area and hence the rejection region, and the area to the left of 8329 is the non-rejection region. In this case, we do not need to do extra calculations for the test statistics as it is the mean of the sample 8600 which is in the rejection region.

```
invNorm(0.95,800
0,200)
     8328.970725
```

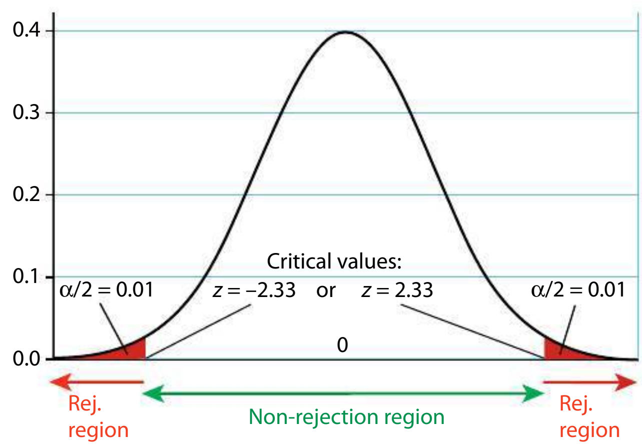
You notice that the results of the tests using the p-value approach or the critical values approach do not contradict each other. In most cases, it is advisable that you use the p-value approach, since you can read the results directly from your technology tool and then draw your conclusion. The other advantage is that when you reject or fail to reject a hypothesis, you know your exact 'observed' significance. Whereas if you apply a critical value approach, you must be prepared to settle for a flat reject/fail to reject decision.

Example 6

Using the sprinklers example (Example 4), we want to test the claim that the set-off temperature for the system is 55° with a standard deviation of 1.5°. Random sample data for 16 sprinklers gave a sample mean of 56.02°. Does the data contradict the manufacturer's claim at 2% level of significance?

Solution

Using the standard normal:



Notice here that since this is a two-tail test, then the level of significance is the total sum of the two tails.

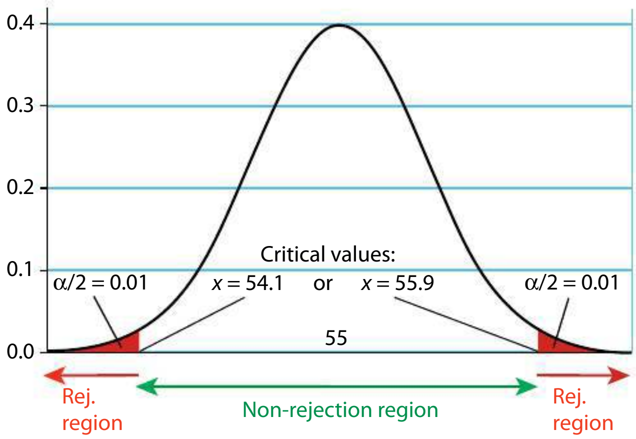
Next we calculate the test statistic as before.

$$z = \frac{56.02 - 55}{\frac{1.5}{\sqrt{16}}} = 2.72$$

This value is in the rejection region since it is larger than the critical value $z = 2.33$. So we reject the null hypothesis.

```
invNorm(0.99)
     2.326347877
invNorm(0.01)
    -2.326347877
█
```

Using the original distribution:




```
invNorm(0.01,55,
.375)
     54.12761955
invNorm(0.99,55,
.375)
     55.87238045
█
```

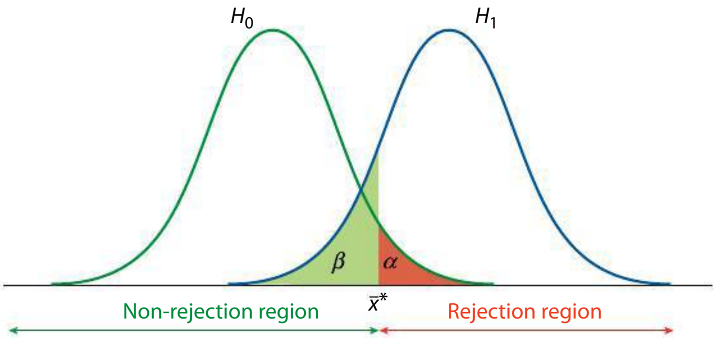
The critical numbers are found in a similar manner to the previous ones (see below). Now, we compare the sample mean to the critical value.

Since 56.02 is larger than the critical value 55.9, we reject the null hypothesis.

6.4 Type I and Type II errors revisited

We mentioned earlier that the two types are related. In this section we will investigate, graphically, how these two errors are related.

Suppose we are testing a hypothesis $H_0 = \mu_0$ against an alternative $H_1 = \mu_1$ where $\mu_1 > \mu_0$. The test will be an upper-tail test. If the level of significance is α, then we can find a critical value $\overline{x}^*$ that separates the rejection and non-rejection regions as shown below.



Notice that since α is the probability of rejecting H_0 when it is true, then it corresponds to the area in the upper tail of H_0. Also, since β is the probability of not rejecting H_0 when it is false, it is the area in the lower tail of the alternative H_1. If we decrease α, $\bar{x}^*$ will move to the right, making β larger. On the other hand, if we increase α, then β will decrease. Hence, for the same standard deviation, there is a trade-off between α and β; increasing one will decrease the other and vice versa. This is why statisticians do not have a free hand in making the errors as small as they wish. A way to reduce both error levels is to reduce the standard error, which can happen by increasing sample size. However, it is neither practical nor feasible in many cases to increase the sample size. Also, you may notice here that, in order to calculate β, you will need a specific alternative hypothesis.

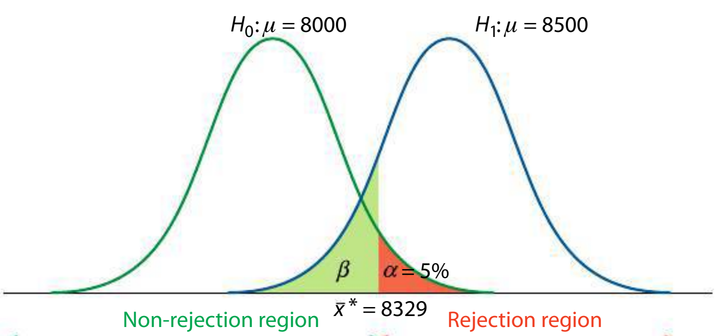
Example 7

Look at the light bulbs factory example again. Remember that a random sample of 25 new bulbs was tested and we are interested in knowing, at the 5% ***level of significance***, whether the new process is more efficient than the old one. Suppose that it is suggested that the new process yields light bulbs with average life of 8500 hours. Calculate the probability of Type II error.

Solution

As you recall we have calculated $\bar{x}^*$ and found it to be equal to 8329. So, finding β in this case is simply finding the area under H_1 to the left of 8329. This area, as you see in the GDC output below, is 19.6%. This is the area in the non-rejection region, i.e. we may not reject H_0 when it is false.

```
normalcdf(-10000
,8329,8500,200)
     .1962755241
```



Note: The unshaded area under H_1 to the right of $\bar{x}^*$ is equal to $1 - \beta$ and is called the **power** of the test. This is so because it corresponds to rejecting the null hypothesis when it is false.

Summary of hypothesis-testing terminology

Null hypothesis (H_0): A maintained hypothesis that is held to be true unless sufficient evidence to the contrary is obtained.

Alternative hypothesis (H_1): A hypothesis against which the null hypothesis is tested and which will be held to be true if the null is held false.

One-sided alternative: An alternative hypothesis involving all possible values of a population parameter on either one side or the other of (that is, either greater than or less than) the value specified by a simple null hypothesis.

Two-sided alternative: An alternative hypothesis involving all possible values of a population parameter other than the value specified by a simple null hypothesis.

Hypothesis test decisions: A decision rule is formulated, leading the investigator to either accept or reject the null hypothesis on the basis of sample evidence. (*Decisions or decision rules are often called the critical region of the test and tell you when to reject a null hypothesis.*)

Type I error: The rejection of a true null hypothesis.

Type II error: The acceptance of a false null hypothesis.

Significance level: The probability of rejecting a null hypothesis that is true. This probability is sometimes expressed as a percentage, so a test of significance level α is referred to as a 100α%-level test.

Example 8

A company claims that the average age of their new staff is 26 years. Members of the board of directors believe that the average age of the new staff is higher than 26. The human resource department wants to investigate the issue and collects data using a sample of randomly chosen new staff of 25 and finds that $\bar{x} = 27$ and $s_{n-1} = 6$.

a) Find the appropriate critical regions (using raw data) corresponding to a significance level of

(i) 0.05 (ii) 0.01.

State your conclusion in each case and state what type of error may happen in this case.

b) Given that the true population mean is 30, calculate the probability of making a Type II error when the level of significance is

(i) 0.05 (ii) 0.01.

c) How is the change in the probability of a Type I error related to the change in the probability of a Type II error?

```
invT(0.95,24)
     1.710882023
invT(0.99,24)
     2.492159469
```

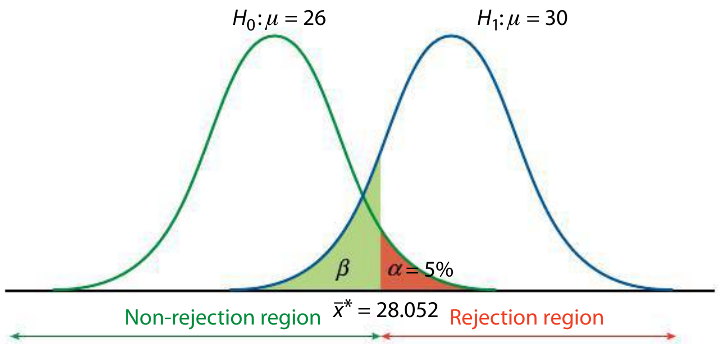
Solution

a) For both parts, this is an upper-tail test where we need to use the t-distribution since the variance of the population is unknown. Thus, we need to test the hypotheses:

$H_0: \mu = 26, H_1: \mu > 26$

(i) To find the critical value, which we will call $\bar{x}^*$ here, we first need to find that value under the t-distribution that leaves an area of 5% above it, or 95% below it.

$$1.71 = \frac{\bar{x}^* - 26}{6/\sqrt{25}} \Rightarrow \bar{x}^* = 26 + 1.71 \cdot \frac{6}{5} = 28.052$$



Since the mean of the sample is in the non-rejection region, we fail to reject the null hypothesis; thus, at the 5% level of significance, we cannot reject the hypothesis that the mean age of the new staff is 26. Type II error may happen here since we are not rejecting a hypothesis that could be false!

(ii) Since we failed to reject H_0 at 5%, we will not be able to reject it at 1%.

However, here are the calculations.

$$2.49 = \frac{\bar{x}^* - 26}{6/\sqrt{25}} \Rightarrow \bar{x}^* = 26 + 2.49 \cdot \frac{6}{5} = 28.988$$

Of course, 27 is also in the non-rejection region. The conclusion here is the same as in (i).

b) (i) To find β, the Type II error probability, we need to find the area corresponding to the region left of 28.052 under a distribution with mean 30.

$\beta = \mathrm{P}\left(t < \dfrac{28.052 - 30}{6/\sqrt{25}}\right) = \mathrm{P}(t < -1.623)$. So, we look under the t-distribution with 24 degrees of freedom for the required probability.

Therefore, $\beta = 0.058$.

```
tcdf(-5,-1.623,2
4)
      .0588058696
 tcdf(-5,-0.843,2
 4)   .2037551141
```

(ii) Similar to (i), we need the area to the left of 28.988.

$$\beta = \mathrm{P}\left(t < \frac{28.988 - 30}{6/\sqrt{25}}\right) = \mathrm{P}(t < -0.843)$$

Therefore, $\beta = 0.204$.

c) Notice here that, with the sample size unchanged, as the probability of Type I error decreases, the probability of Type II error increases. This is typical of the two errors; keeping the sample size the same, there is a trade-off between the two probabilities.

To find the probability of Type II error, you need to

1. Decide on the hypotheses you are testing. Your alternative hypothesis must be specific – it is not enough to say H_1: $\mu > \mu_0$, it has to be a specific number that is larger than μ_0.
2. Find the critical number, using raw data, for rejecting H_0.
3. Set up your rejection and non-rejection regions, using raw data!
4. Find the area corresponding to the non-rejection region under the mean hypothesized by the alternative.

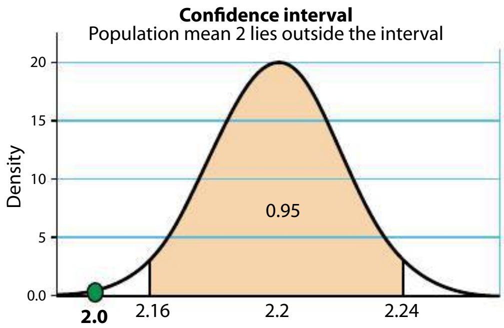
Tip: It is extremely helpful if you sketch a diagram similar to what you have seen above. Your β is always the area corresponding to the non-rejection region under the 'alternative' curve!

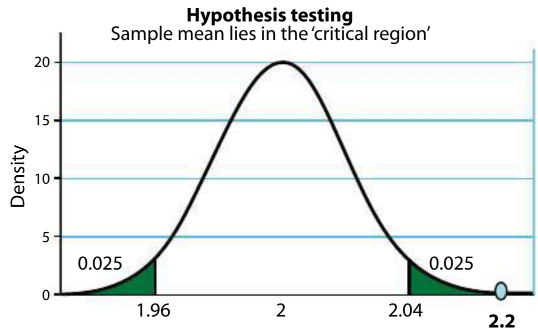
6.5 Confidence intervals and hypothesis tests

Confidence intervals and hypothesis tests are constructed from the same computations. They have identical assumptions and settings. As you have seen at the start of this chapter with the assembly plant for personal computers (where we tested the hypothesis that the time finishing a task with 2 minutes has been rejected on the basis of a 95% confidence interval), we can utilize the same procedure for other tests of hypothesis. The difference between confidence intervals and hypothesis tests is that confidence intervals refer to the confidence level, say 95%, while hypothesis tests employ the level of significance, 5% in this case. When using confidence intervals, the approach is opposite to that of the hypothesis test in the following sense. In a test, we claim a value for the parameter in the null hypothesis, $\mu = 2$ for example, then we check whether the sample value is consistent with that value (recall the interpretation of the p-value). In using confidence intervals, we examine whether the hypothesized value of the parameter is consistent with the confidence interval.

For example, in the task time situation, the hypothesis testing procedure is to set up a critical region and reject the null hypothesis if the sample data is inconsistent with the null hypothesis value. On the other hand, in the confidence interval approach, we construct the confidence interval from the sample data and reject the null hypothesis if its value is inconsistent with the sample data. See page 1154.

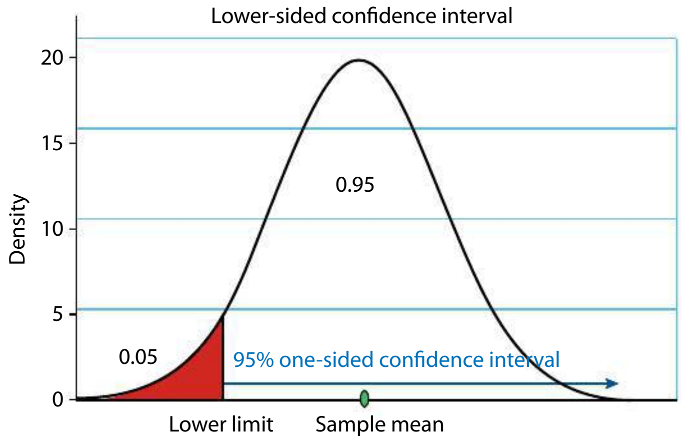
To generalize, we can say that for a two-tail hypothesis test with α level of significance, a confidence interval of $1 - \alpha$ renders the same decision in rejecting or failing to reject a certain null hypothesis.





What about one-tail tests?

In fact, one-sided confidence intervals are possible to construct. For a 5% level of significance lower-tail test for example, a one-sided 95% confidence interval can be constructed by leaving a 5% lower tail as shown below.
A one-sided confidence interval leaves one side unbounded. One-sided confidence intervals are becoming more and more common in statistical practice.



One-sided confidence intervals need some extra calculation that you may want to postpone till after the exam. We will give it to you here for reference purposes only.
A lower-sided interval is of the following form:

$\left[\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right[$, and an

upper-sided interval is of the form: $\left]-\infty, \bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right]$.

Notice here that we use z_α rather than $z_{\alpha/2}$.

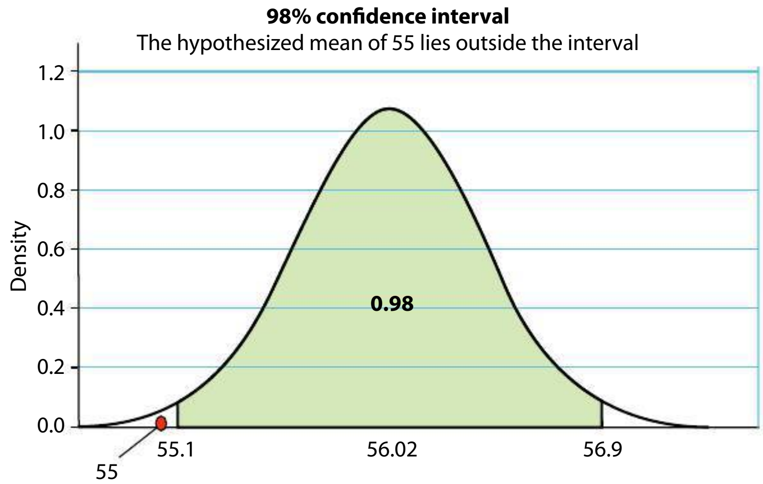
Example 9

Using the sprinklers example again, we want to test the claim that the set-off temperature for the system is 55° with a standard deviation of 1.5°. Random sample data for 16 sprinklers gave a sample mean of 56.02°. Does the data contradict the manufacturer's claim at 2% level of significance?

Solution

Using the hypothesis testing procedure was done earlier. Let us take the confidence interval approach.

We need to set up a 98% confidence interval here. The result is shown below.



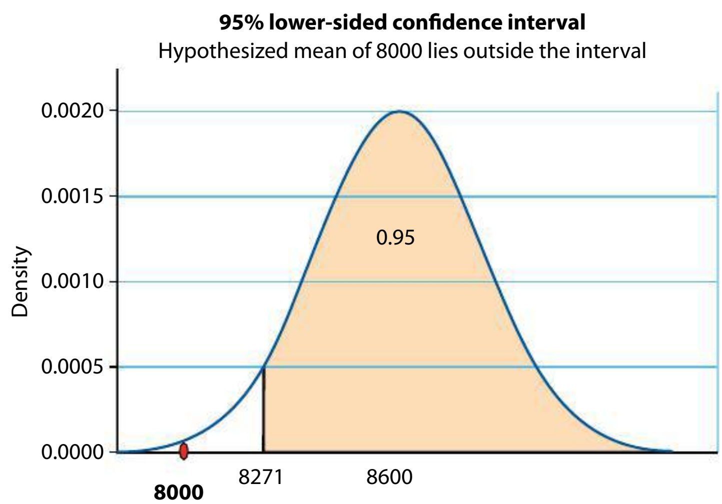
As you observe here, since the mean hypothesized by the null hypothesis (55), is below the lower limit of the interval, we reject the null hypothesis as we did earlier. Finding a confidence interval along with a hypothesis test gives statisticians an additional idea of how different the hypothesized mean is from the collected data. They can then judge whether the difference is a meaningful one for the problem at hand.

Example 10

Look at the light bulbs factory example again. Remember that a random sample of 25 new bulbs was tested and gave an average life of 8600 hours. Is this enough evidence at the 5% ***level of significance*** to justify the shift to the new process?

Solution

This is an upper-tail test, and hence the confidence interval is lower sided.



Notice here that the 8000 mean of the population is lower than the lower endpoint of the interval, and hence we reject the null hypothesis as we did before. The additional information we get here is that not only might the new system be better than the old one, but we are 95% confident that the true mean life of these bulbs lies above 8271 hours.

6.6 Distributions with unknown variance

Using the t-model needs more assumptions than normality. The first is the **randomization assumption**, which means that our sample is a simple random sample from the population in question. The second condition is that the **sample size** must not exceed 10% of the population size. In exams, these assumptions are assumed to be true.

If the population we want to study is approximately normal, then as we did in confidence intervals, the hypothesis testing will be done using the t-distribution.

The construction of the hypothesis test using the t-distribution is similar to the procedure outlined before with one difference. We use the t-distribution with $n - 1$ degrees of freedom rather than the normal distribution.

Let us use the same examples above, but with a difference. The difference is that the standard deviation is estimated from the sample rather than given.

Example 11

The light bulbs are known to last 8000 hours on average. The random sample of 25 new bulbs gave an average life of 8600 hours and a standard deviation of 1000 hours. Is this enough evidence at the 5% level of significance to justify the shift to the new process?

Solution

p-value approach

We are testing:

H_0: $\mu = 8000$

H_1: $\mu > 8000$

The standard error of the mean can now be replaced by $\dfrac{s}{\sqrt{n}} = \dfrac{1000}{\sqrt{25}} = 200$.

$$\mathrm{P}\left(\bar{x} \geqslant 8600 \middle| \mu = 8000, \frac{s}{\sqrt{n}} = 200\right) = \mathrm{P}\left(t \geqslant \frac{8600 - 8000}{200}\right) = \mathrm{P}(t \geqslant 3) = 0.0031$$

```
tcdf(3,100,24)
     .0031028683
```

The t-distribution here has 24 degrees of freedom.

Since 0.0031 is very small, we will reject the null hypothesis.

Notice here that even though our decision did not change, the p-value has more than doubled! This is so because the tails of the t-distribution are ‘fatter’ than those for the standard normal distribution.

Critical value approach

Since we need a 5% level of significance, we find our critical t-value to be 1.71, and hence our critical region is to the right of this number.

```
invT(.95,24)
      1.710882023
```

Our test statistic is $t = \dfrac{8600 - 8000}{200} = 3.0.$

Our test statistic, 3.0, is to the right of 1.71 and hence we reject the null hypothesis.

Notice here that our critical value of 1.71 is larger than $z_{0.05} = 1.645$.

Using your GDC, here are the results.

```
EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7↓ZInterval...
```

```
T-Test
 Inpt:Data Stats
 μ0:8000
 x̄:8600
 Sx:1000
 n:25
 μ:≠μ0 <μ0 >μ0
 Calculate Draw
```

```
T-Test
 μ>8000
 t=3
 p=.0031028683
 x̄=8600
 Sx=1000
 n=25
```

From the screenshots above, you can see that the p-value is 0.0031 and that the test statistic is 3.0 as we found before.

Example 12

Consider the amount of nicotine example where we need to test the mean of 1.8 mgc against an alternative of higher content. The random sample of 100 cigarettes gave an average content of 1.84 mgc and a standard deviation of 0.2 mgc. Is there evidence, at the 5% level of significance, to conclude that our suspicion of an increase is justified?

Solution

Since we do not know the standard deviation, then it is wiser to use the t-distribution even though the sample size is large.

***p*-value approach**

We are testing:

H_0: $\mu = 1.8$

H_1: $\mu > 1.8$

The standard error of the mean can now be replaced by $\dfrac{s}{\sqrt{n}} = \dfrac{0.2}{\sqrt{100}} = 0.02.$

$$\mathrm{P}\left(\bar{x} \geqslant 1.84 \middle| \mu = 1.8, \frac{s}{\sqrt{n}} = 0.02\right) = \mathrm{P}\left(t \geqslant \frac{1.84 - 1.80}{0.02}\right) = \mathrm{P}(t \geqslant 2) = 0.024$$

```
tcdf(2,100,99)
      .0241198442
```

The t-distribution here has 99 degrees of freedom.

Since 0.024 is very small, we will reject the null hypothesis.

Notice here that our decision did not change, and the p-value of 0.024 is not much larger than the 0.0228 that we had before. This is so because the sample size is large enough to make the t-distribution very close to the standard normal distribution.

Critical value approach

```
invT(.95,99)
    1.660391096
```

Since we need a 5% level of significance, we find our critical t-value to be 1.66, and hence our critical region is to the right of this number.

Our test statistic is $t = \dfrac{1.84 - 1.80}{0.02} = 2.0.$

Our test statistic, 2.0, is to the right of 1.66 and hence we reject the null hypothesis.

Notice here that our critical value of 1.66 is slightly larger than $z_{0.05} = 1.645.$

Using your GDC, here are the results.

```
T-Test
 Inpt:Data Stats
 μ0:1.8
 x̄:1.84
 Sx:.2
 n:100
 μ:≠μ0 <μ0 >μ0
 Calculate Draw
```

```
T-Test
 μ>1.8
 t=2
 p=.0241198442
 x̄=1.84
 Sx=.2
 n=100
```

Again, these are the same results as above.

6.7 Large-sample hypothesis test for a population proportion (Optional)

$\hat{p} = \dfrac{x}{n}$ is the sample proportion where x is the number of 'successes' in the sample.
$\hat{q} = 1 - \hat{p}$.
The sample size must be large enough for this interval to be acceptable, namely, $np \geqslant 5$ and $nq \geqslant 5$. Some statisticians consider 'large' to be $np \geqslant 10$ and $nq \geqslant 10$. **We will use 5**.

As you have seen in the confidence interval discussion, when the sample size is large enough, the sampling distribution of the sample proportion, $\hat{p}$, is approximately normal with an expected value p, the population proportion, and a variance of $\dfrac{pq}{n}$ (i.e. standard deviation of $\sqrt{\dfrac{pq}{n}}$).

Testing a hypothesis about the proportion is not different from testing a hypothesis about the mean.

There will be a null hypothesis H_0 and an alternative hypothesis H_1.

H_0: $p = p_0$ (the hypothesized value)

H_1:

H_1: $p > p_0$, an upper-tail test, or

H_1: $p < p_0$, a lower-tail test, or

H_1: $p \neq p_0$, a two-tail test.

In running a hypothesis test for the proportion, here again we can use the p-value approach as well as the critical region approach. We will demonstrate this with some examples.

Example 13

In many countries the sale of new cars that use leaded petrol is banned. In 2004, one country started to phase out older cars that use leaded petrol. In 2004, 27% of the cars used leaded petrol. In 2006, in a study to investigate the effectiveness of the efforts, a random sample of 120 cars found that 22 of them still use leaded petrol. Is there evidence, at the 5% level of significance, that the proportion of leaded-petrol cars has been reduced?

Solution

p-value approach

We are testing:

H_0: $p = 0.27$

H_1: $p < 0.27$

The standard error of the proportion is

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0 q_0}{n}} = \sqrt{\frac{0.27 \times 0.73}{120}} = 0.040528.$$

$$\hat{p} = \frac{22}{120} = 0.18333$$

$$\mathrm{P}\left(\hat{p} < 0.18333 \middle| \mu = p = 0.27, \sigma = 0.040528\right) = \mathrm{P}\left(z < \frac{0.18333 - 0.27}{0.040528}\right) = \mathrm{P}\left(z < -2.138\right) = 0.0163$$

Since 0.0163 is very small, we will reject the null hypothesis. That is, there is enough evidence to conclude that the proportion of leaded-petrol cars has been reduced.

Critical value approach

Since we need a 5% level of significance, we find our critical value to be -1.645, and hence our critical region is to the left of this number.

```
invNorm(.05)
   -1.644853626
```

Our test statistic is $z = \frac{0.18333 - 0.27}{0.0405}28 = -2.138.$

Our test statistic, -2.138, is to the left of -1.645 and hence we reject the null hypothesis.

Using your GDC gives you the same results.

```
EDIT CALC TESTS
1:Z-Test...
2:T-Test...
3:2-SampZTest...
4:2-SampTTest...
5:1-PropZTest...
6:2-PropZTest...
7↓ZInterval...
```

```
1-PropZTest
 p0:.27
 x:22
 n:120
 prop≠p0 <p0 >p0
 Calculate Draw
```

```
1-PropZTest
 prop<.27
 z=-2.138451496
 p=.01624
 p̂=.1833333333
 n=120
```

- **Examiner's hint:** In exams, you will have instructions to give your answer to a certain accuracy (3 s.f. for example). However, the instructions apply to your final answer only. It is good practice **not** to round your *intermediate* answers.

Example 14

The following news on 23 May 2009, created concern among passengers.

'Police in London have arrested a drunken ABC Airlines pilot before he could fly a plane with 204 passengers from Heathrow Airport to Chicago, USA.

The pilot, whose name was not revealed, was arrested after he failed a breathalyzer test, ABC Airlines said in a statement issued Thursday. The alcohol test was conducted by police called in by airport security staff who noticed the pilot was drunk. "Employees at all levels of the company are not allowed to be on duty while under the influence of drugs or alcohol, and regular screening is carried out," ABC Airlines said, according to XYZ News. The flight was delayed by 75 minutes as the airline looked for a substitute pilot for the Boeing 777 plane.'

This news is the latest in a sequence of such news. In April 2009, also at Heathrow, another pilot was arrested and earlier a pilot at Amsterdam's Schiphol airport was also arrested for the same reasons. The impression in the airline industry is that about 20% of pilots are usually under the influence of alcohol. To combat such a problem there is a rule in the industry where airline pilots are banned from drinking alcohol less than eight hours before flying – the so-called 'bottle to throttle' rule. They are deemed unfit to fly if they are a quarter of the drink-drive limit – 20 mg per 100 ml of blood.

To check whether there is any change in the pilots' behaviour, a random sample of 220 pilots were tested and 43 of them had levels of alcohol above the acceptable mark. Does this provide any evidence of a change in the proportion of 'drunken' pilots?

Solution

p-value approach

We are testing:

H_0: $p = 0.20$

H_1: $p \neq 0.27$

The standard error of the proportion is

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0 q_0}{n}} = \sqrt{\frac{0.20 \times 0.80}{220}} = 0.02697.$$

$$\hat{p} = \frac{43}{220} = 0.1955$$

$$\mathrm{P}\left(\hat{p} < 0.1955 \middle| \mu = p = 0.20, \sigma = 0.02697\right)$$

$$= \mathrm{P}\left(z < \frac{0.1955 - 0.20}{0.02697}\right) = \mathrm{P}\left(z < -0.1669\right) = 0.4337$$

Since 0.4337 is very large, we fail to reject the null hypothesis. That is, there is no evidence to conclude that the proportion of drunken pilots has changed.

Critical value approach

Since we need a 5% level of significance, we find our critical value to be -1.645, and hence our critical region is to the left of this number.

```
invNorm(.05)
     -1.644853626
```

Our test statistic is $z = \dfrac{0.1955 - 0.20}{0.02697} = -0.1669$.

Our test statistic, -0.1669, is to the right of -1.645 and hence we do not reject the null hypothesis.

Using your GDC gives you the same results as above.

```
1-PropZTest
 p0:.2
 x:43
 n:220
 prop≠p0 <p0 >p0
 Calculate Draw
```

```
1-PropZTest
 prop≠.2
 z=-.1685499656
 p=.8661506317
 p̂=.1954545455
 n=220
```

Exercise 6

1 The lengths of metal bars produced by a particular machine are normally distributed with mean length 310 cm and the standard deviation 10 cm. The machine is serviced, after which a sample of 50 bars is given a mean length of 313 cm. Is there evidence, at the 5% level, of a change in the mean length of the bars produced by the machine, assuming that the standard deviation remains the same?

2 Experience has shown that the scores obtained in a particular test are normally distributed with mean score 72 and variance 40. When the test is taken by a random sample of 30 students, the mean score is 69.5. Is there sufficient evidence, at the 1% level, that these students have not performed as well as expected?

3 A sample size of 100 is taken from a population with the mean value 4.4 and a standard deviation of 0.18. Given that the mean value of the sample was 4.35, test at the 2% significance level whether the mean value of the sample is smaller than the population mean.

4 A random sample of eight observations of a normal variable gave $\bar{x} = 6.35$ and the sample variance 0.0625. Test, at the 3% level, whether the mean of the distribution is 6.5.

5 Electrical resistivity is a measure of how strongly a certain type of material resists the flow of electric current. The SI unit of electrical resistivity is the ohm meter, Ωm. Five readings of the resistivity, in $10^{-8}\,\Omega$m, of a piece of wire gave the following results: 2.41, 2.49, 2.39, 2.52 and 2.48. If the wire is pure gold, the resistivity would be $2.44 \times 10^{-8}\,\Omega$m. If the wire were impure, the resistance would be decreased. Test, at the 5% level, the hypothesis that the wire is pure gold.

6 Packs of prosciutto slices are sold in 1kg packs. A sample of 15 packs was selected at random and the masses, measured in kg, noted. The following results were obtained: $\sum x = 14.4$, $\sum x^2 = 17.83$. Assuming that the masses of the packs follow a normal distribution with the variance σ^2, test at the 5% level whether the packs are significantly underweight if

a σ^2 is unknown **b** $\sigma^2 = 0.0064$.

7 In an area where high-voltage electricity cables are close to residential areas, there was suspicion the number of cancer cases in infants has increased. The national level of infant cancer cases is known to be 3%. To test this claim, an independent group collected data from a randomly selected group of 412 infants in the area. Among these infants they found 13 cases of cancer.

a Set up and test an appropriate hypothesis at the 5% level of significance.

b What type of error may occur in this case?

c If the real rate in this area is indeed at 3.8%, find the probability of Type II error.

8 In a certain European country there is a suspicion among government officials that the cost of the national health system has increased due to an unusual increase in hospital stays of patients considered to have minor illnesses which do not need hospitalization. The previous proportion of such cases was 30%. A random sample of 4184 patients' records was investigated and it was found that 1360 of these patients had minor illnesses that did not need hospitalization.

a Set up and test an appropriate hypothesis at the 1% level of significance.

b What type of error may occur in this case? Explain its repercussions.

c If, in fact, the real percentage is 32%, calculate the Type II error probability and explain its ramifications.

9 Before the 2009 economic crisis, consumer confidence in one of the EU countries was at 54%. That is, 54% of the people asked thought the country would be doing better in the next 12 months. In June 2009, a randomly chosen sample of 1518 people produced 782 positive responses.

a Set up an appropriate hypothesis and test it at the 5% level of significance.

b If the true proportion is 50%, calculate the probability of Type II error.

10 A large clothing company has several branches in different parts of the city. In two main commercial areas, they even have two shops. Recently, they started selling their products using online orders. They need to know whether their online endeavour decreased sales in their stores. Over a period of 10 months after the introduction of the online sales, they randomly selected 15 monthly shop sales to compare to their historical average of 3.2 (million euros). Data from the sample yielded $\overline{X} = 3.1$ and $s_{n-1} = 0.214$.

a Test the hypothesis at 5% level of significance. Set up your rejection and non-rejection regions, find the critical numbers, and state your *p*-value.

b If the sales are indeed 3.15, calculate and interpret the probability of Type II error.

11 Buying online is changing fast, in terms of the number of people using the service and in terms of the nature of the 'online' buyer.

Until 2008, the average age of online consumers was 24.1 years. Since most families now have computers, the age appears to have increased. An online sale company wants to investigate the increase in the age of the consumer and collected information from a randomly chosen sample of 80 consumers. The average age of this group is 25.0 years. The standard deviation s is 4.7 years.

a Test the hypothesis at 5% level of significance. Set up your rejection and non-rejection regions, find the critical numbers, and state your *p*-value.

b If the sales are indeed 24.8 years, calculate and interpret the probability of Type II error.

12 A company with a large number of cars would like to keep fuel costs at a minimum. Last year, they changed many of the cars into more economical cars. This year they are evaluating the process to see if it is effective. Fuel consumption till last year was, on average, 11.1 km/l (kilometres per litre). They sampled 60 company trips with different cars. The mean consumption for this sample was 11.9 km/l with a standard deviation s of 2.06 km/l.

Is there any evidence at the 5% level of significance that their efforts are being successful?

13 In question 12, a different procedure was followed in order to be more precise in how the company will save on fuel consumption. They have offers from two car manufacturers, Fiat and Renault, who produce cars with company specifications. A random sample of 12 employees was selected. These employees used two cars to travel the same trips: one trip with a Fiat and the second with a Renault. The table below shows the average fuel consumption of these trips using both types of cars.

Employee	1	2	3	4	5	6	7	8	9	10	11	12
Fiat	10.9	10.8	11.1	11	9.7	12	11.4	10.9	11.2	11.3	10.9	11.0
Renault	10.9	11.0	11.2	11.2	10.4	11.8	11.7	10.8	11.4	11.7	11.1	12.0

Test at the 5% level of significance whether there is a difference in fuel consumption between the two car types.

14 Wine makers have to look at the alcohol fermentation in their wines in order to determine the 'maturity' of their wine. The must-wine density is a good indicator of the maturity point. There are 2 instruments used. The hydrometer is a mechanical instrument that is very simple, but time consuming. The hydrostatic instrument, which is digital, is more expensive but fast. A large winery is interested in knowing whether there is any significant difference between the readings of the two instruments. If the difference is not more than 0.003, then they will go ahead and purchase the hydrostatic one. 20 samples of wine were tested using both instruments under the same conditions of heat and humidity. The table below gives the readings of both.

Sample	Hydrometer	Hydrostatic	Sample	Hydrometer	Hydrostatic
1	1.001 953 749	1.009 005 55	11	1.000 174 634	1.010 908 346
2	0.995 239 266	1.011 043 866	12	1.000 472 892	1.009 957 204
3	0.999 517 477	1.010 817 529	13	1.002 212 579	1.010 606 275
4	0.999 797 889	1.009 032 132	14	0.999 625 691	1.010 158 302
5	0.998 780 33	1.010 315 304	15	0.999 977 748	1.010 285 715
6	0.997 086 023	1.012 601 204	16	0.995 736 68	1.012 364 667
7	0.998 186 679	1.010 897 463	17	0.997 117 977	1.009 553 711
8	0.997 878 224	1.012 589 215	18	1.000 683 835	1.008 243 634
9	0.998 350 181	1.014 861 438	19	1.000 713 597	1.005 254 222
10	0.998 562 702	1.010 255 724	20	0.998 244 044	1.011 645 023

At the 5% level of significance, decide whether there is evidence to convince the large winery to buy the new equipment. Interpret, but do not calculate, what Type I and Type II errors mean here and state their consequences.

15 The common knowledge in car accidents is that the passenger next to the driver has the worst seat in the car. To study this phenomenon, an engineering group ran an experiment where they replaced humans with dummies and the cars were steered by remote control. The cars were new models of several medium- sized cars. The cars were driven at a speed of 60 km/h into a head-on collision with a fixed barrier. The damage to the chest of the driver and passenger were recorded. The higher the score the more damage had happened. The data in the table is from a randomly selected sample of 28 such events in the experiment.

Car	Driver	Passenger	Car	Driver	Passenger	Car	Driver	Passenger	Car	Driver	Passenger
1	48	49	8	42	46	15	46	52	22	53	49
2	45	42	9	48	46	16	43	58	23	48	49
3	45	45	10	50	54	17	47	52	24	47	47
4	48	53	11	46	52	18	45	43	25	55	53
5	40	42	12	46	52	19	44	46	26	43	58
6	36	37	13	46	42	20	43	58	27	50	47
7	50	47	14	53	53	21	44	46	28	36	37

a Is there any evidence to conclude that the passenger is more at risk than the driver? Test an appropriate hypothesis at the 5% level of significance.

b If, in fact, it is known that the average score difference for chest injury is known to be 1.5, calculate the Type II error probability and interpret what it means.

16 The masses of bags of salt packaged by a certain company are normally distributed with a standard deviation $\sigma = 30$ g. The label on each bag says 750 g. There is a concern in the company that their machines are 'overfilling' the bags, which is costly for the company. 16 bags are randomly selected and you need to run an appropriate test at the 5% level of significance.

a Show that if your sample mean is larger than 762.34, then you reject the null hypothesis.

b Your sample has a mean of 765, which is obviously larger than 762.34. You need to run the test using the *p*-value approach and consolidate your conclusion in both cases.

c If the actual average weight of the packaging process is 770 g, find the probability of making a Type II error.

17 A law firm claims that among their strengths is the time they spend with their clients. They claim that client interview time is at least 60 minutes. A random sample of 15 clients gave the following results. If t is the interview length, then $\sum t = 896$ and $\sum t^2 = 40\,172$. Assuming the interview time to be normal, test whether the law firm is overstating the length of clients' time at the 5% level of significance.

Practice questions 6

1 A computer manufacturing company buys large quantities of hard disks from several suppliers. Hard-disk quality is checked by a process called RTT which gives results on a continuous scale from 0 to 100. Based on previous experience the company assumes that the results are normally distributed with a mean of 68 and standard deviation of 3.

Shipments arrive from suppliers on a daily basis. A sample of 10 hard disks is taken from each shipment at random and tested. If the mean of the sample is more than 67, the shipment is accepted, otherwise it is rejected.

a What is the probability that a hard disk selected at random has a result less than 67?

b Find the probability that a shipment is rejected.

c There is a $1000 penalty each day that a shipment is rejected. A particular supplier's hard disks have a mean of 67.5 and a standard deviation of 2.8.

i What is the probability that this supplier's shipment is accepted?
ii What is the expected penalty per 6-day week for this supplier?

d The company's own production of hard disks has a mean of 68 and a standard deviation of 3. However, to keep the production within the acceptable limits, the company samples 10 hard disks every hour and examines whether the sample is accepted or rejected. During a particular hour, the following results were recorded for a sample that was randomly chosen for testing.

68.1747, 68.0473, 66.3189, 66.2735, 66.957, 66.9738,
66.1438, 67.0744, 66.1875, 67.8804

At the 5% level of significance, determine whether the sample meets the company's standard for acceptance.

e Every week, the company randomly selects the test results of 1000 hard disks and checks if these results come from a normal distribution (with a mean of 68 and standard deviation of 3) or not. The following table gives the results, R, for one such test.

Results	Frequency
$56 \leqslant R < 59$	5
$59 \leqslant R < 62$	17
$62 \leqslant R < 65$	146
$65 \leqslant R < 68$	333
$68 \leqslant R < 71$	360
$71 \leqslant R < 74$	113
$R \geqslant 74$	26

Check, at the 5% level of significance, whether the above data comes from a normal population with a mean of 68 and standard deviation of 3.

2 The 10 children in a class are given two jigsaw puzzles to complete. The time taken by each child to solve the puzzles was recorded as follows.

<X>Exercise

<PQ>Practice questions

Child	A	B	C	D	E	F	G	H	I	J
Time to solve Puzzle 1 (min)	10.2	12.3	9.6	13.8	14.3	11.6	10.5	8.3	9.3	9.9
Time to solve Puzzle 2 (min)	11.7	12.9	9.9	13.6	16.3	12.2	12.0	8.4	9.8	9.5

a For each child, calculate the time taken to solve Puzzle 2 minus the time taken to solve Puzzle 1.

b The teacher believes that Puzzle 2 takes longer, on average, to solve than Puzzle 1.

i State hypotheses to test this belief.
ii Carry out an appropriate t-test at the 1% significance level and state your conclusion in the context of the problem.

3 Sarah cycles to work and she believes that the mean time taken to complete her journey is 30 minutes. To test her belief, she records the times (in minutes) taken to complete her journey over a 10-day period as follows:

30.1 32.3 33.6 29.8 28.9 30.6 31.1 30.2 32.1 29.4

You may assume that the journey times are normally distributed with mean μ minutes.

a State suitable hypotheses.

b Test Sarah's belief, at the 5% significance level.

c Justify your choice of test.

4 Anne tosses a coin which has probability p of giving a head. Anne thinks that it is a fair coin for which $p = 0.5$. However, Anne's friend thinks that $p > 0.5$. In order to investigate the value of p, Anne decides to toss the coin 15 times.

a State appropriate null and alternative hypotheses.

Let X denote the number of heads obtained. Anne decides to reject the null hypothesis if $X \geqslant 11$.

b **i** What name is given to the region $X \geqslant 11$?
ii Explain what is meant by the significance level and find its value in this case.

It is known that $p = 0.6$.

c Find the probability of a Type II error.

d When Anne tosses the coin 15 times, she obtains 10 heads.

i What type of error does she commit?
ii Explain briefly the consequences of this error.

5 Doctor Tosco claims to have found a diet that will reduce a person's weight, on average, by 5 kg in a month. Doctor Crocci claims that the average weight loss is less than this. Ten people use this diet for a month. Their weights before and after are shown below.

Person	A	B	C	D	E	F	G	H	I	J
Weight before (kg)	82.6	78.8	83.1	69.9	74.2	79.5	80.3	76.2	77.8	84.1
Weight after (kg)	75.8	74.1	79.2	65.6	72.2	73.6	76.7	72.9	75.0	79.9

a State suitable hypotheses to test the doctors' claims.

b Use an appropriate test to analyse these data. State your conclusion at

i the 1% significance level
ii the 10% significance level.

c What assumption do you have to make about the data?

6 The ten children in a class were each given two puzzles and the times taken, in seconds, to solve them were recorded as follows.

Child	A	B	C	D	E	F	G	H	I	J
Puzzle 1	66.3	71.9	62.8	69.8	64.6	74.9	68.8	72.6	70.4	74.2
Puzzle 2	64.8	71.6	59.9	68.1	66.0	72.4	67.7	70.9	69.8	74.6

It is claimed that, on average, a child takes the same time to solve each puzzle. Treating the data as matched pairs, use a two-tailed test at the 5% significance level to determine whether or not this claim is justified.

7 Competitors at the World's Strongest Man contest have to hold an extremely heavy weight, with their arms held out straight, for as long as possible. It is claimed that a particular training schedule will improve the time (i.e. increase it) that a competitor can hold the weight for. Competitors are tested before and after the training schedule.

The times, in seconds, before and after training are shown in the table below.

Competitor	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Time before training	80	62	45	73	65	53	61	48	81	50	50	29	52	33	71
Time after training	85	74	60	67	69	55	68	46	89	60	64	26	61	33	72

Stating the null and alternative hypotheses, carry out an appropriate test at the 1% significance level to decide if the claim is justified.

8 Juan plays a quiz game. The scores he achieves on the separate topics may be modelled by independent normal distributions.

a On the topic of sport, the scores have the distribution $N(75, 12^2)$.

Find the probability that Juan scores less than 57 points on the topic of sport.

b On the topic of literature, Juan's scores have a mean of 45, and 30% of his scores are greater than 50.

Find the standard deviation of his scores on the topic of literature.

c Juan claims that he scores better in current affairs than in sport. He achieves the following scores on current affairs in 10 separate quizzes.

91 84 75 92 88 71 83 90 85 78

Perform a hypothesis test at the 5% significance level to decide whether there is evidence to support his claim.

9 **a** It was found that x people in a sample of 225 supported a smoking ban in public places. If the 95% confidence interval for the proportion of people supporting the ban in the population from which the sample was taken is [0.2297, 0.3481] calculate the value of x.

b A coin is thought to be biased.

To test the coin for bias, Amanda suggests that it should be tossed three times. If all three tosses are heads or all three tosses are tails, then we conclude that the coin is biased.

Roger suggests that it should be tossed eight times. If at least six tosses are heads or at least six tosses are tails, then we conclude that the coin is biased.

i Determine which of the two methods has the smaller probability of making a Type I error.

ii Determine the probability that Roger will make a Type II error when the probability of a head is actually 0.6.

10 A teacher wants to determine whether practice sessions improve the ability to memorize digits.

He tests a group of 12 children to discover how many digits of a twelve-digit number could be repeated from memory after hearing them once. He gives them test 1, and following a series of practice sessions, he gives them test 2 one week later. The results are shown in the table below.

Child	A	B	C	D	E	F	G	H	I	J	K	L
No. of digits remembered on test 1	4	6	4	7	8	5	6	7	6	8	4	7
No. of digits remembered on test 2	7	8	5	5	10	7	7	10	8	6	3	9

a State appropriate null and alternative hypotheses.

b Test at the 5% significance level whether or not practice sessions improve ability to memorize digits, justifying your choice of test.

7 Linear Regression

7.1 Correlation and covariance

Scatter plot

The total time you devote getting ready for an exam impacts on the score you obtain in that exam.

In general, the foot size of an adult is related to the height of that adult.

Smoking increases the chances of a heart attack.

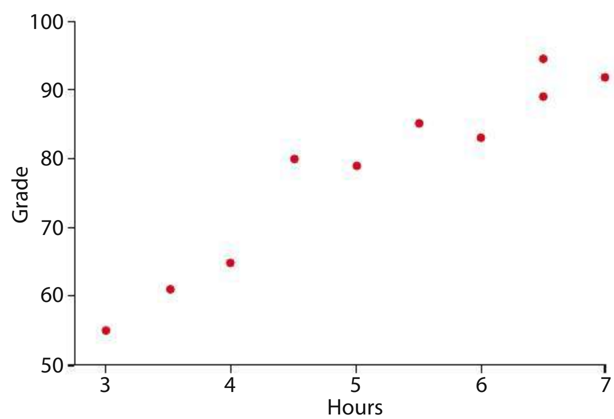
Such statements as those above concern the relationship between two variables. So far you have considered how to describe the characteristics of one variable. In this section, you will look at relationships between two variables. This is why we call this study *bivariate statistics.*

To study the relationship between two variables, we measure both variables on the same subjects. For example, if we are interested in the relationship between height and foot size, then for a group of individuals we record each person's height and foot size. This way we know which foot size goes with which height. Similarly, we record the grades of each individual in the study along with his/her time preparing for the exam. So, our data are sets of ordered pairs. These data allow us to study the link (association) between height and foot size or time and grade. In fact, taller people tend to have larger foot sizes. And the more you prepare for an exam the higher your grade is. We say that pairs of variables like these are *associated.*

Here are the grades of 10 students in an IB Mathematics SL class. The table gives the time they spent preparing for their math test and the score they achieved.

Student	Tim	Joon	S-youn	Kevin	Steve	Niki	Henry	Anton	Cindy	Lukas
Hours	4	4.5	6	3.5	3	5	5.5	6.5	7	6.5
Grade	65	80	83	61	55	79	85	89	92	95

Here is a graph (scatter plot) of the data given in the table.



The horizontal axis shows the number of hours spent studying and on the vertical axis shows the scores received. As you will notice, it appears that the more hours spent studying the higher the grade. We say that the grades on tests and the time preparing for them are **associated**. We call the time the **explanatory variable** (independent) and the grade the **response variable** (dependent). The students whose time and grades are recorded are the **subjects** of the **experiment/study**.

Definition
Two variables measured on the same subjects are **associated**, if specific values of one variable tend to occur in connection with particular values of the other variable.

For instance, larger values for the foot size of an individual tend to occur in connection with taller individuals. Or, a higher rate of serious road accidents happens in connection with drivers that have a high level of alcohol concentration in their blood. We claim that height and foot size are positively associated as well as alcohol level and involvement in serious road accidents. We can also claim that there is a negative association between time spent watching TV and scores on weekly tests for teenagers.

In our effort to study the nature of the relationship between two variables we try to look into how changes in the values of one variable help explain the variation in the other variable. For instance, we look at how the increase in a person's height can explain the increase in his/her foot size. As discussed above, we call the first variable **explanatory** and the second the **response** variable. These are traditionally called **independent** and **dependent** variables.

Definition
A **response variable** measures an outcome of a study. An **explanatory variable** explains the changes in the response variable. If the study is to determine the relationship between weight and blood pressure, then weight is the explanatory variable and blood pressure is the response variable. If the study is to investigate the relationship between the level of fertilizer and the crop volume during an agricultural season, then the level of fertilizer is explanatory, the crop is the response.

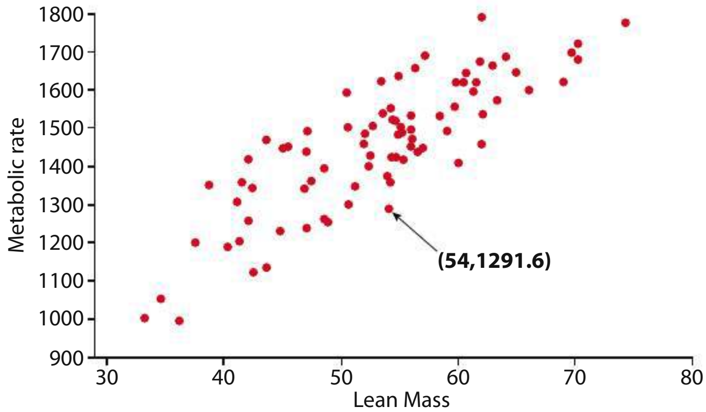
The principles that guide our work on data are:

- Start with graphical display, and then explore numerical summaries.
- Look for overall patterns and deviations from those patterns.
- When the overall pattern is quite regular, use a mathematical model to describe it.

Graphical displays associated with one variable include histograms, box plots and others. In bivariate statistics the graphical tool we use is the **scatter plot**, or **scatter diagram**. In a scatter plot, each observation is represented by a point on a grid. The horizontal component represents the explanatory variable and the vertical component represents the response variable.

Example 1

The data presented below is for 80 adults in a dieting program. The researchers believe that the metabolic rate (Calories burnt per 24 hours) is influenced by the lean body mass (in kg without fat).



Does the scatter plot show that there is an association between the metabolic rate and lean mass?

You will observe that there is a positive association between these two variables, i.e. the greater the weight, the higher is the metabolic rate.

What to look for in a scatter plot?

As a rule of thumb, when we examine a scatter plot, we may look at the following characteristics:

- Overall pattern (form, direction and strength)
- Striking deviations from pattern (outliers)

In this example, the *form* is roughly linear. That is, the points appear to cluster around a straight line. The *direction*, as mentioned earlier appears to be a positive association. The strength is determined by how closely the points follow the form (will be revisited later), even though some points stray away from the line. In this case it does not appear that there are any outliers.

An outlier is an observation whose values fall outside the overall pattern of the relationship.

Example 2

The table below lists the fuel consumption of 34 small cars in km/litre during city driving and highway driving. Make a scatter plot of the data and comment on any patterns you observe.

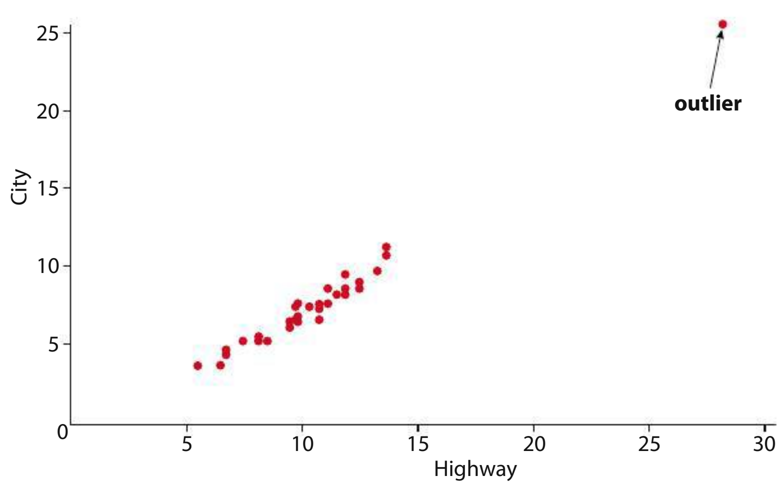
City	Highway
7.3	10.2
8.5	11.9
8.5	11.9
7.3	10.7
7.7	10.7
5.1	8.5
4.7	6.8
4.3	6.8

(Table continues overleaf)

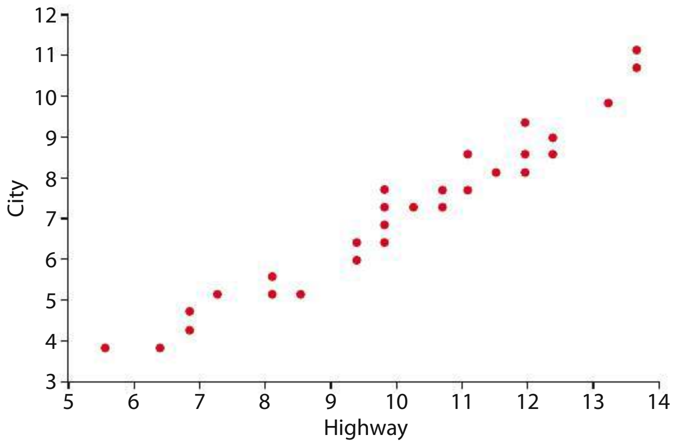
City	Highway
7.3	9.8
3.8	6.4
3.8	5.5
6.4	9.4
5.1	7.3
9.4	11.9
6.8	9.8
5.5	8.1
8.5	11.1
8.5	12.4
6.4	9.8
11.1	13.7
5.1	8.1
9.0	12.4
8.1	11.5
8.1	11.9
6.8	9.8
7.7	11.1
6.8	9.8
7.7	9.8
10.7	13.7
9.8	13.2
8.5	12.4
7.7	11.1
6.0	9.4
25.6	28.2

Solution

Here is a scatter plot of the data.



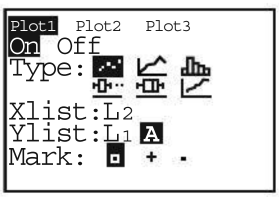
The data indicate that the fuel consumption in highway driving and city driving, as expected, are positively associated. The relationship appears to be strong as the data are tightly clustered around a positively sloped line. However, we can see that there is one observation that is positioned quite far from the rest of the data. This observation is an outlier. Outliers in statistics are important. Sometimes they indicate a problem in the data being observed and sometimes they may have a special significance. In our case, the data corresponds to a ‘hybrid’ car, which uses battery power in addition to fuel and hence the high performance. In that sense, this observation is not typical of the study and must be removed in order to get a clear indication of the nature of the relationship between the two variables. Here is an adjusted scatter plot without the hybrid car.

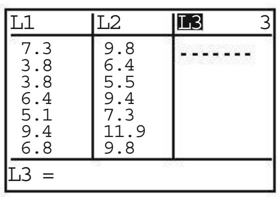


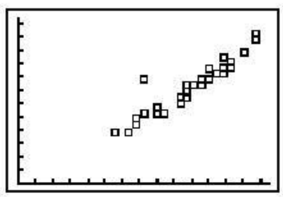
You can use either Excel or your GDC to produce scatter plot.

Here are the instructions for a TI-84:

- First enter the data into two lists L1 and L2 in this case.
- Then go to STAT PLOT
- Choose Plot1.
- Then choose the scatter plot and the correct lists as shown.
- Graph.

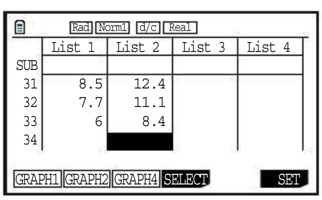


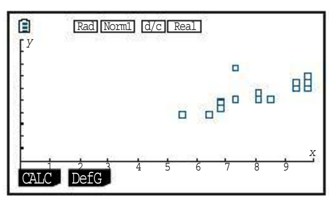




For a CASIO fx-CG20, you do the following:

- Go to Menu and choose Statistics.
- Press EXE and then fill in the lists as shown.
- Choose GRAPH1 (or 2).

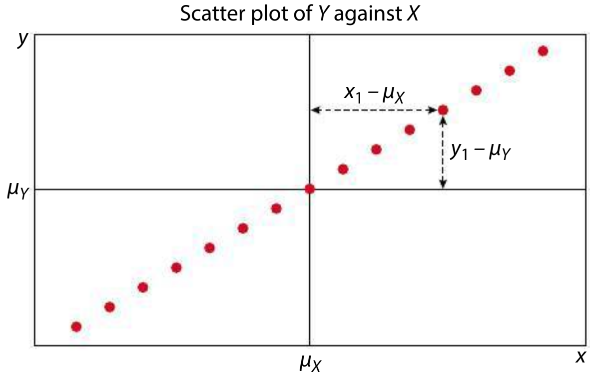




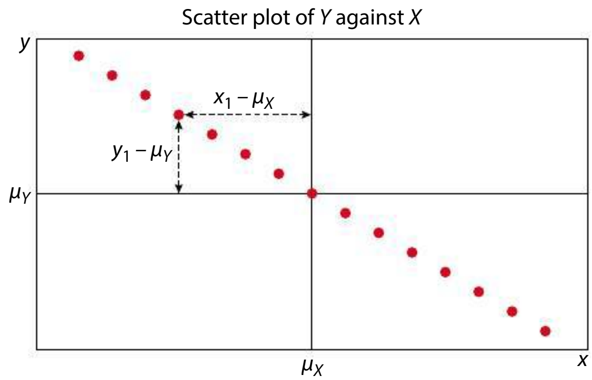
Covariance

Intuitively, we think of the dependence of two variables X and Y as implying that one variable, Y for example, either increases or decreases as X changes. In this book, we will confine our discussion to two measures of dependence: the ***covariance*** between two random variables and their ***correlation coefficient***.

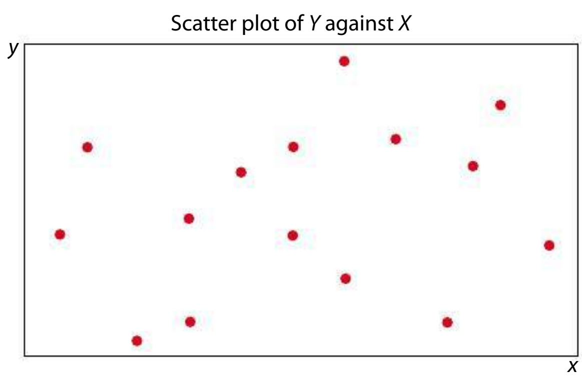
In the scatter plot below, we give plots of variables X and Y, for samples of size 15.



All points fall on a straight line. Obviously X and Y are dependent in this case. Suppose we know $E(X) = \mu_X$ and $E(Y) = \mu_Y$. Locate the point with coordinates (μ_X, μ_Y) and then locate any point (x_1, y_1) for example and measure the deviations $(x_1 - \mu_X)$ and $(y_1 - \mu_Y)$. If the point is in the upper right corner, then both deviations are positive. Similarly, if the point is in the lower left corner, both deviations are negative. The product of the deviations $(x_1 - \mu_X)(y_1 - \mu_Y)$ is positive. This is a typical and extreme case of positive association. When the line representing the pattern in the data is positively sloped, the product of deviations of the mean is on average positive, that is $E((X - \mu_X)(Y - \mu_Y)) > 0$.



In the scatter plot above, the data follow a negatively sloped pattern. If the point is in the upper left corner, then the X-deviations are negative while the Y-deviations are positive. Similarly, if the point is in the lower right corner, the X-deviations are positive while the Y-deviations are negative. The product of the deviations $(x_1 - \mu_X)(y_1 - \mu_Y)$ is negative. These situations do not occur for the diagram below where little dependence (if any) exists between the variables.



The deviations $(x_1 - \mu_X)$ and $(y_1 - \mu_Y)$ sometimes assume the same algebraic sign and sometimes opposite signs. Thus, the product $(x_1 - \mu_X)(y_1 - \mu_Y)$ will be positive sometimes and negative other times and average may be close to zero.

The discussion above indicates that the average $E((X-\mu_X)(Y-\mu_Y))$ provides a measure of the linear dependence between X and Y. This quantity is called the **covariance** of X and Y.

Definition

If X and Y are random variables with means μ_X and μ_Y the **covariance** of X and Y is

$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

The larger the absolute value of the covariance of X and Y, the greater the linear dependence between X and Y. Positive values indicate that Y increases as X increases and negative values indicate that Y decreases as X increases. A zero value of the covariance indicates that the variables are linearly *uncorrelated* and that there is no linear association between X and Y.

Some facts worth knowing about covariance

1 A short-cut calculation formula can be helpful if you were to do the calculations without using built in functions in your GDC or software:

$$\begin{aligned}\text{cov}(X,Y) &= E\left(\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right)\\ &= E\left(XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y\right)\\ &= E(XY) - E\left(X\mu_Y\right) - E\left(\mu_X Y\right) + E\left(\mu_X\mu_Y\right)\\ &= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y\\ &= E(XY) - \mu_Y\mu_X - \mu_X\mu_Y + \mu_X\mu_Y = E(XY) - \mu_X\mu_Y\end{aligned}$$

2 In fact, the above result leads to

$$\text{cov}(X,X) = E(XX) - \mu_X\mu_X = E\left(X^2\right) - \mu_X^2 = V(X)$$

3 If X and Y are **not independent**, then

$V(X+Y) = V(X) + 2\text{cov}(X, Y) + V(Y)$

4 If X and Y are **independent**, then

$$\text{cov}(X,Y) = \text{E}(XY) - \mu_X\mu_Y = \text{E}(X)\text{E}(Y) - \mu_X\mu_Y = 0$$

Consequently,

$V(X+Y) = V(X) + V(Y)$

Note that the converse of the theorem above is not true: if $\text{cov}(X, Y) = 0$, then X and Y are not necessarily independent.

Unfortunately, it is difficult to employ the covariance of X and Y as an absolute measure of association between variables because its value depends on the scales used.

In Example 2, the covariance of the data expressed as km/litre is 3.8. However, if we change the scale from km/litre to mile/litre, then the

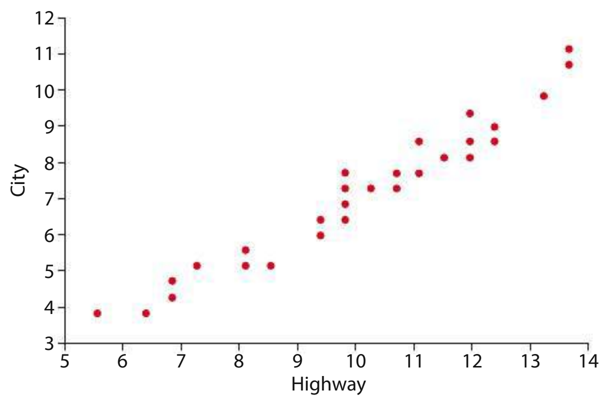
covariance will be 1.49 even though the scatter plot does not indicate any change in the form nor the strength of association between the two variables.
This problem with covariance can be eliminated by 'standardizing' its value and using the *correlation coefficient*, ρ instead.

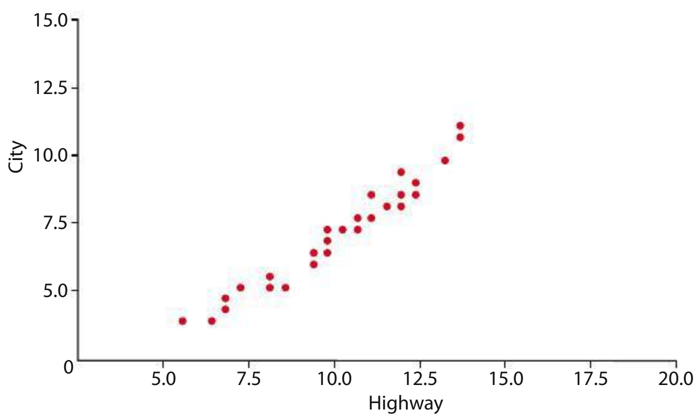
$$\rho_{XY} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$$

Since σ_X and σ_Y are both positive, the sign of the correlation coefficient is the same as that of the covariance.
Note: All models discussed concerning correlation and regression assume that data are samples that come from normal populations.

Correlation

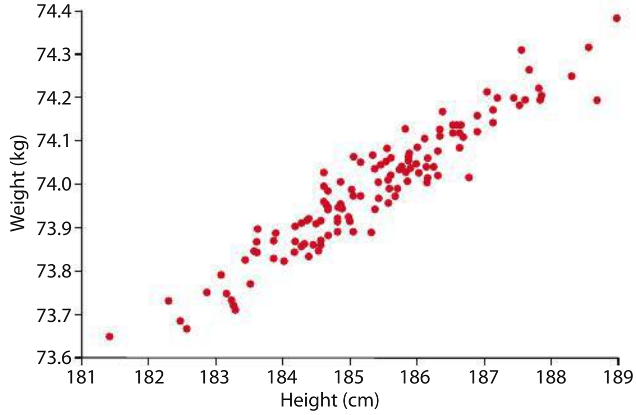
A scatter plot is a good device that reveals the form, trend and strength of the association between two quantitative variables. At this level, we are only interested in linear relations. As mentioned earlier, we say that a linear relationship is strong if the data are tightly packed around the line, and weak if they are widely dispersed around the line. Our judgment using our eyes only may be misleading though. Look at the two scatter plots.





The graph on the left is a copy of the second graph in Example 6. The graph gives the impression that the association is stronger than it is in the other graph. This is due to the change in scale on the vertical axis. However, both scatter plots represent the same situation. We will need a more robust measure to support our first graphical impressions. This measure is the **correlation coefficient**.

Let us consider height and weight data collected from 130 19-year-olds. The measurements were made in metric units. Here is the scatter plot.



Not surprisingly, the association between the two variables is strong. To measure the strength of this association, we use the correlation coefficient given by the following formula.

Definition
The correlation coefficient measures the strength and direction of the linear relationship between two quantitative variables when it exists.

For a set of data (x_i, y_i) of size n, the correlation coefficient is

$$R = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{S_x}\right)\left(\frac{y_i - \bar{y}}{S_y}\right)$$

where $\bar{x}$ and $\bar{y}$ are the means of the variables and S_x and S_y are the standard deviations. Specific values of R are denoted by r.

R is also called the Pearson product-moment correlation coefficient. In fact, R is an unbiased estimate of the population coefficient, which is given by:

$$\rho = \frac{cov(X, Y)}{\sigma_x \sigma_y}$$
$$= \frac{1}{n}\sum\left(\frac{x_i - \mu_x}{\sigma_x}\right)\left(\frac{y_i - \mu_y}{\sigma_y}\right)$$

The GDCs use r.

In exams, you will not be asked to calculate the coefficient by hand but to interpret the GDC result. There are several equivalent forms for the equation but it is not necessary at this stage to calculate any of them!

This formula is somewhat complex to calculate. However, it helps us see what correlation is instead. In practice, you will read the result from your calculator or computer output.

If we look at the formula, we see that the first component $\frac{x_i - \bar{x}}{S_x}$ is nothing but the standardized value for x_i. Similarly, the second component $\frac{y_i - \bar{y}}{S_y}$ is the standardized value for y_i. So, the correlation coefficient can be written as $R = \frac{\sum z_x z_y}{n-1}$. That is, the correlation coefficient is an average of the products of the standardized values of the two variables.

Note: Whether we use the definition of r or ρ, it can be shown that they are equivalent. Hence, using your GDC will give you the correct value. If you are interested in seeing how to show their equivalence, here is one method.

Starting with ρ:

$$\rho = \frac{1}{n}\sum\left(\frac{x_i - \mu_y}{\sigma_y}\right)\left(\frac{y_i - \mu_y}{\sigma_y}\right) = \frac{1}{n}\sum\left(\frac{x_i - \mu_x}{\sqrt{\frac{\sum(x_i - \mu_x)^2}{n}}}\right)\left(\frac{y_i - \mu_y}{\sqrt{\frac{\sum(y_i - \mu_y)^2}{n}}}\right)$$

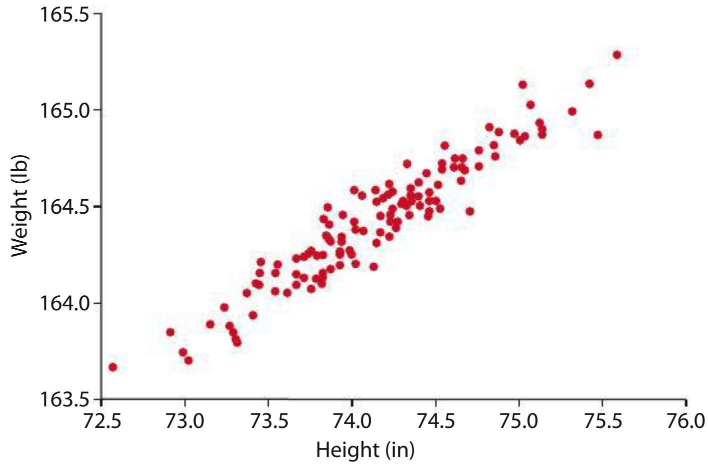
$$= \frac{1}{n}\sum\left(\frac{x_i - \mu_x}{\frac{1}{n}\sqrt{\sum(x_i - \mu_x)^2}}\right)\left(\frac{y_i - \mu_y}{\sqrt{\sum(y_i - \mu_y)^2}}\right) = \sum\frac{(x_i - \mu_x)(y_i - \mu_y)}{\sqrt{\sum(x_i - \mu_x)^2}\sqrt{\sum(y_i - \mu_y)^2}}$$

Starting with r:

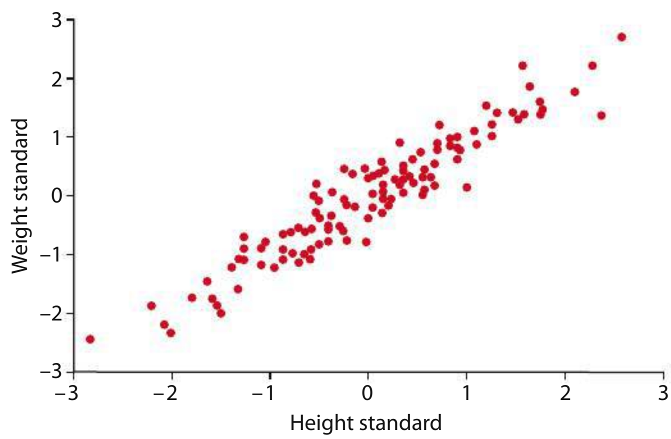
$$R = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{S_x}\right)\left(\frac{y_i - \bar{y}}{S_y}\right) = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{\sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}}}\right)\left(\frac{y_i - \bar{y}}{\sqrt{\frac{\sum(y_i - \bar{y})^2}{n-1}}}\right)$$

$$= \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{\frac{1}{n-1}\sqrt{\sum(x_i - \bar{x})^2}}\right)\left(\frac{y_i - \bar{y}}{\sqrt{\sum(y_i - \bar{y})^2}}\right) = \sum\frac{(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2}\sqrt{\sum(y_i - \bar{y})^2}}$$

Let us take the weight–height data and express it in pounds and inches instead. Here is what we get:



As you notice, other than the scale on the axes being inches and pounds, the plot has the same form and direction and strength as the original one. Similarly, when you standardize the variables, you are subtracting a constant from each value and dividing by another constant. If you plot the standardized variables, here is what you get:



As you will notice, other than the centre of the data being at the origin, the form, direction and strength appear to be the same.

This fact is verified by calculating the correlation coefficient for all three forms of the data. The result is always the same, 0.95 (software use).

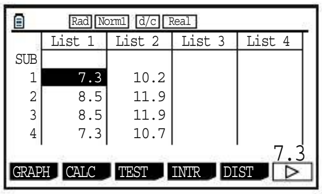
For **Example 2**, the correlation can be read from TI-84's regression output below. You first need to enable the Diagnostics on your GDC, then run LinReg from the Stats menu.

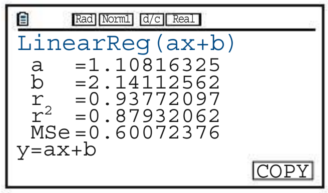
```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7↓QuartReg
```

```
LinReg(ax+b) L1,
L2
```

```
LinReg
 y=ax+b
 a=1.108163254
 b=2.141125622
 r²=.8793206242
 r=.9377209735
```

On CASIO fx-CG20, from the list, choose CALC, REG, X, ax + b (or a + bx).





You may have observed in the technology output that r^2 is also reported. *This measure is not required for your exam*. However, it is an extremely useful and powerful tool. r^2 is known as the **coefficient of determination**. It reports the portion of variation in the response variable that can be explained by the variation in the explanatory variable. As such, r^2 can be expressed as a percentage. Using the data from Example 2, $r^2 = 0.879$, which can be interpreted as 'if all else is equal, then 88% of the variation in city consumption can be explained by variation in the highway consumption', i.e. on average, for cars with the same characteristics, if there is a 1 km/l change in City consumption, we expect that 88% of this change can be explained by changes in the Highway consumption. Using the data from Example 1, $r = 0.84$ and $r^2 = 0.7056$, which means that approximately 70.6% of the changes in the metabolic rate can be explained by changes in the lean mass. Finally, using the data from Example 3 below, $r^2 = 0.9025$ which means that, all else equal, approximately 90% of the variation in weight could be explained by variation in the height of those teenagers.

Properties of the correlation coefficient

- The correlation coefficient is a measure of the *strength* of the *linear* association between two *quantitative* variables.
  - Do not apply correlation to non-quantitative data!
  - The coefficient makes sense only if there is a linear relationship. It does not prove a linear relationship. If there is a linear association, the coefficient will describe its strength.
- The outliers can distort the correlation. Special attention must be paid to such outliers.
- The correlation is always a number between -1 and $+1$. Values of R near 0 indicate a weak relationship. Values close to $+1$ or -1 indicate strong association.

When there is no association, $\text{cov}(X, Y) = 0$.

Hence, $\rho = \dfrac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \dfrac{0}{\sigma_X \sigma_Y} = 0$.

A proof for the values ± 1 is beyond the scope of this book.

- R does not change as we change the units of measurement.
- R has no units and is not a percentage! Don't express a correlation of 0.85 as 85% for example.
- Correlation between two variables means that there is some association between them. It does NOT mean that one of them *causes* the other.

So, correlation does not mean causation, i.e. two variables can have a strong correlation without one of them being the cause of the changes in the other. For example, there may be a strong correlation between the amount of crude oil imported by country X and the rate of birth in country Y. That does not necessarily mean that the increase of oil imports causes an increase in birth rate. However, in some cases, there may be a causal relationship. For example, the increase in level of income in a certain country and the decrease of unemployment can have a strong negative correlation. This association is also causal. However, the task of proving the causal relationship comes with economics.

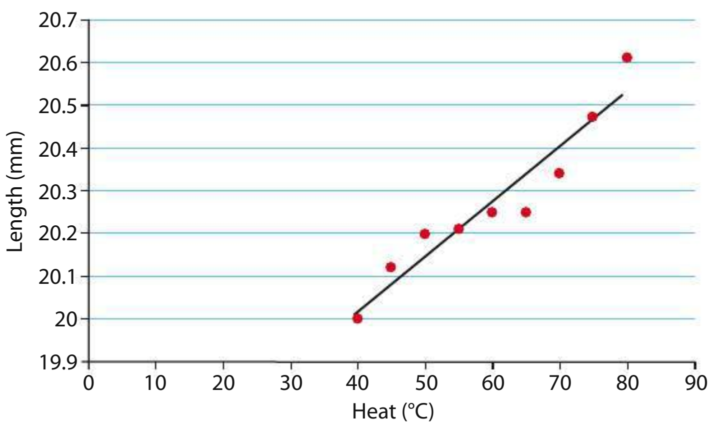
Example 3

The table below gives you the data for a lab experiment involving the length (mm) of a metal alloy bar used in electronic equipment when it is exposed to heat (°C).

Heat (°C)	40	45	50	55	60	65	70	75	80
Length (mm)	20	20.12	20.20	20.21	20.25	20.25	20.34	20.47	20.61

Draw a scatter plot. Comment on the strength of the relationship. Use both r and r^2.

Solution



Here is the scatter plot.

It appears that we have a relatively strong relationship where the points are tightly spread around the trend line.

This is confirmed by calculating the correlation coefficient. In this case, regardless of which formula we use(r or ρ), the correlation is approximately 0.95521. Using the $r^2 = 91.2\%$ implies that 91.2% of the variation in the length can be explained by variation in the temperature.

Exercise 7.1

1 The following table lists the values of a response variable x against an explanatory variable y. Draw a scatter plot and comment on the strength of the relationship.

x	12	6	12	11	16	13	11	12	11	12	12	12	15	16	14	13	13	8	10	11
y	8	10	9	6	14	10	10	9	15	14	10	6	12	8	13	11	11	9	9	6

2 The data below represents the outcome of an experiment on a small car, relating fuel consumption to speed.

Speed km/h	60	65	70	75	80	85	90	95	100	105	110	120	130	140	150
Fuel consumption km/L	16.9	16.8	15.9	15.9	14.4	14.3	13.2	14.3	12.1	12.0	10.2	9.8	9.0	8.0	7.1

a) Make a scatter plot.

b) Describe the relationship and justify your choice of which variable is the explanatory and which is the response.

c) Is the relationship strong? Explain your answer.

3 The following data is from World Bank statistics relating the Gross National Income per Capita (GNI/Cap) to Purchasing Power Parity (PPP) for a few developed countries. (The exchange rate adjusts so that an identical product in two different countries has the same price when expressed in the same currency.) For example, a chocolate bar that sells for C$1.50 in a Canadian city should cost US$1.00 in a U.S. city, when the exchange rate between Canada and the U.S. is 1.50 USD/CDN. (Both chocolate bars cost US$1.00.)

Country	GNI/Cap	PPP
NOR	85380	57130.0
CH	70350	49180.0
DK	58980	40140.0
SWE	49930	39600.0
NL	49720	42590.0
FIN	47170	37180.0
USA	47140	47020.0
AUT	46710	39410.0
BEL	45420	37840.0
D	43330	38170.0
F	42390	34440.0
JPN	42150	34790.0
SGP	40920	54700.0

a) Make a scatter plot.

b) Describe the relationship and justify your choice of which variable is the explanatory and which is the response.

c) Is the relationship strong? Explain your answer.

4 In hotel management, it is necessary to estimate the electricity consumption in relation to number of visitors. Here is the data for a large hotel.

Visitors	232	311	321	334	352	375	412	447	456	472	480	495	512
Consumption	237	278	270	303	298	328	387	390	376	402	431	430	432

a) Make a scatter plot.

b) Describe the relationship and justify your choice of which variable is the explanatory and which is the response.

c) Is the relationship strong? Explain your answer.

7.2 Least squares regression

You have seen above that correlation measures the strength and direction of a linear relationship between two quantitative variables. So, if we suspect from a scatter plot that the relationship is linear, then we need to summarize this linear behaviour, i.e. we need to find an equation of a straight line that *best fits* the trend in the data. In this sub-section, we will discuss how to find a **line of best fit** that describes the linear relationship between an explanatory and response variable when it exists.

Finding a line of best fit means finding a line that comes as close as possible to the points in the data set. Obviously, there is no *straight* line that *contains* all the points in the set.

Regression line

A **regression line** is a straight line that describes how a response variable changes with changes in an explanatory variable.

Let Y be the response variable and X be the explanatory variable. Since for the same value of the explanatory variable X we can expect several values of the response variable Y, our linear model enables us, on average, to predict the value of Y given a value of $X = x$, and hence we write the equation of the linear regression line in the form

$$E(Y) = \alpha + \beta x$$

This is to say, given a specific value of x, the expected value of Y is equal to $\alpha + \beta x$ where α is the value corresponding to $x = 0$, and β is the slope representing the rate with which the response variable changes with every change of one unit in the explanatory variable (gradient).

Note: The regression model can be stated "formally" as $E(Y|X = x) = \alpha + \beta x$

In cases like this, our data are only samples from a population and consequently, we can only estimate the regression equation.

From sample data we estimate the regression equation and we write our estimate as

$$y = bx + a$$

where b, the slope of the line, is an estimate of β and reflects how the response variable, Y, changes according to changes in the explanatory variable X. a is an estimate of α and is the value of the response variable corresponding to a zero value in X.

In the example of height—weight, the equation is

$$Weight\ (kg) = 56.1 + 0.0966\ Height\ (cm)$$

That is $b = 0.0966$ and $a = 56.1$.

This means that ***on average***, for every increase (decrease) of 1 cm in height, we predict an increase (decrease) of 0.0966 kg in weight. The interpretation of a is peculiar. As you know from algebra, a stands for the value of y (which is *Weight* in this case) corresponding to a zero value of x (which is height in this case). However, for this problem the interpretation is not ideal! It corresponds to a height of zero. The general rule in this is that if 0 is not included in the domain of the explanatory variable, then trying to interpret the intercept is pointless.

This issue has to do with what we call **extrapolation**. Extrapolation is the use of the regression line for predicting values far off the range of values of the explanatory variable x used to find the equation of that line. *Such predictions are often inaccurate.*

Why the least-squares regression line?

Let us take a simple example. The graph below represents a few points in a data set. The green line is the line of best fit. Take for example the point (x_1, y_1). The point on the line $(x_1, \hat{y}_1)$ is the point whose y-coordinate $\hat{y}_1$ *predicts* the real y-coordinate, using the line of best fit. The distance $y_1 - \hat{y}_1$ is the error in this prediction. Similarly is $y_2 - \hat{y}_2$ and all other $y_i - \hat{y}_i$. The line of best fit is the line that minimizes the sum of all these errors. However, like the variance, some of these errors are positive and some are negative and may eventually cancel each other out. To avoid this, like we did with the variance, we try to minimize the squares of these errors. That is, the line of best fit is the line that minimizes the sum $\sum(y_i - \hat{y}_i)^2$. Hence, it has the name of the **least-squares** line of regression $\hat{y} = bx + a$.

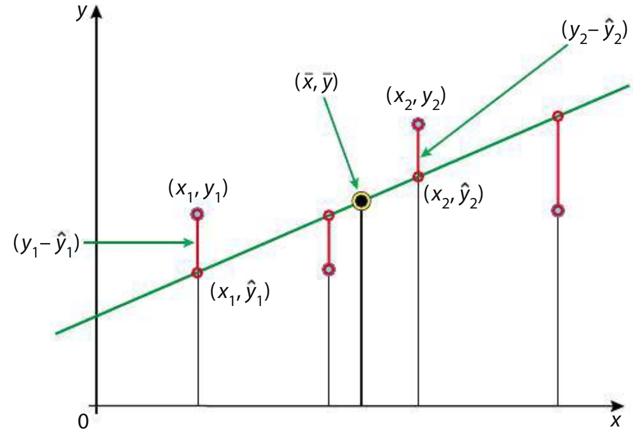
The process of finding the slope of such a line is beyond the scope of this book. Here are some of the forms of the many forms of the resulting formulas for the slope and intercept

$$b = \frac{\operatorname{cov}(X,Y)}{V(X)} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{\sum x_i y_i - n\overline{xy}}{\sum x_i^2 - n\bar{x}^2} = r\frac{s_y}{s_x}$$

Here, r is the correlation coefficient, $\bar{x}$, $\bar{y}$, s_x, and s_y are the means and standard deviations of the explanatory and response variables. The last form demonstrates the close relationship between the slope of the regression line and the correlation coefficient. One conclusion you can draw from this formula is that along a line of regression with slope b, a change of 1 standard deviation in the x direction will result in a change of r standard deviations in the y direction.

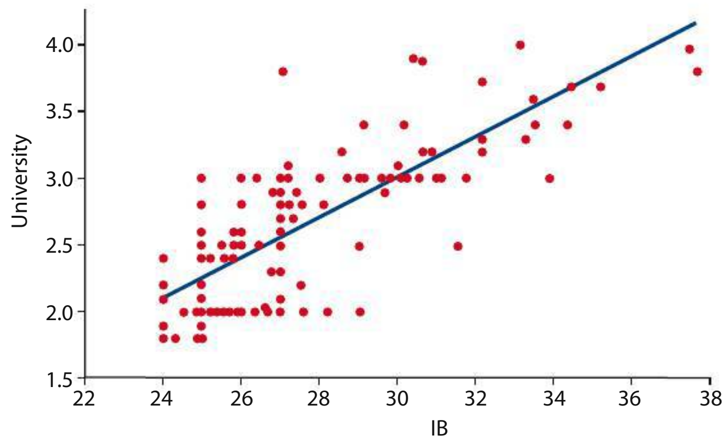
After estimating the slope, and using the fact that the line has to contain the point with coordinates $(\bar{x}, \bar{y})$, the intercept, a, can be found using $a = \bar{y} - b\bar{x}$.

As you will notice from the equations, every regression line should contain the point $(\bar{x}, \bar{y})$ with the averages of the variables as coordinates.



Example 4

The following scatter plot represents a random sample of IB students who went through four years of university and a comparison of their scores on the IB exams they took and their Grade Point Averages in their university studies (scale 1–4).



There appears to be a linear relationship between them. When we run a linear regression, the equation is:

$$\textit{University} = -1.51 + 0.151\ \textit{IB}$$

This means that on average, for every increase of 1 point in the total IB score, we expect an increase of 0.15 points in University Grade Point Average (GPA). If we want to predict the GPA of a student who scored 30 on an IB diploma, the model predicts, *on average*, a grade of:

$$\textit{University} = -1.51 + 0.151(30) = 3.02$$

The correlation coefficient of this relationship is $r = 0.758$, which is a relatively strong correlation. In addition, $r^2 = 57.5\%$. This means that changes in the IB score may help us explain 57.5% of the variation in the University GPA.

Does that mean high IB scores *cause* high university averages? The answer is no. They only help predict the future university averages.

Features of the regression line

- The **regression equation** can be used to predict the response variable according to values of the explanatory variable.
- The regression line must pass through the point $(\bar{x}, \bar{y})$.
- When the regression line is used for prediction and you substitute a specific value x_1 for the explanatory variable, the predicted value $\hat{y}_1$ of the response variable is an *average* value. For example, when we use the height–weight equation $Weight\,(kg) = 56.1 + 0.0966\,Height\,(cm)$ to predict the weight corresponding to a height of 182 cm, the value we get (73.68 kg) is an average weight of 19 year-old students of height 182 cm.

Exceptional cases of the regression line

If $r = 0$, the regression line is horizontal; its slope is zero.

If $r = 1$, all the points fall on a line with positive slope.

If $r = -1$, all the points fall on a line with negative slope.

Estimating the value of Y associated with a value of X that is larger than any of those observed, or smaller than any of those observed, is called **extrapolation**. Estimating the value of Y associated with a value of X that is within the range of the observed values of X but is not equal to any of the observed values of X is called **interpolation**.

Extrapolation is extremely suspect: without data in the range in which the estimate is wanted, there is no reason to believe that the relationship between X and Y is the same as it is in the region in which there are data.

Interpolation is sometimes reasonable when the scatter plot shows a strong relationship, especially if there are many data near the value of X or Y at which the estimate is sought.

Example 5

Here are the data for two variables. Draw the line of regression and indicate the distances, the sum of whose squares is minimized by the choice of the line of regression.

x	y
11	21
12	43
13	31
14	34
15	29
16	55
17	33

Solution

The scatter plot below shows the data and line of regression. The red distances are those required.

The line has an equation: $\hat{y} = 6.14 + 2.071\,x$

Look at the second table where we also introduced the value of each predicted y (Fit) and then calculated the distances (directed) whose squares were minimized.

x	y	Fit	Distance	Distance square
11	21	28.92857	−7.92857	62.8622449
12	43	31	12	144
13	31	33.07143	−2.07143	4.290816327
14	34	35.14286	−1.14286	1.306122449
15	29	37.21429	−8.21429	67.4744898
16	55	39.28571	15.71429	246.9387755
17	33	41.35714	−8.35714	69.84183673

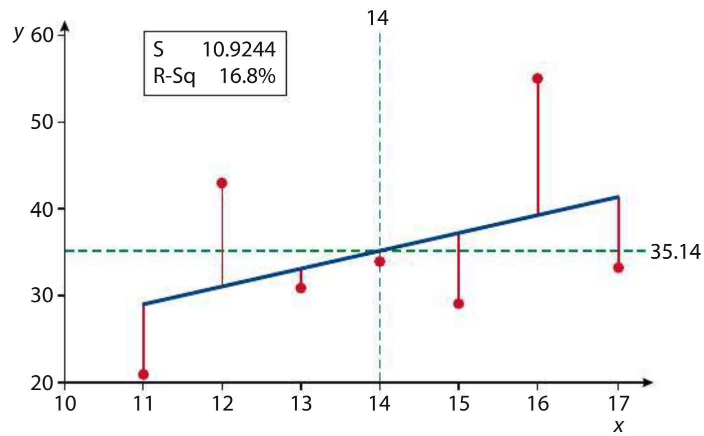
The minimum sum is 596.71. You can try to find any other line and you will notice that this is the minimum sum of the squares of distances.

Moreover, since $\overline{x} = 14$ and $\overline{y} = 35.14$, then:

$$35.14 = 6.14 + 2.071 \times 14$$

This indicates that the line contains the point $(\overline{x}, \overline{y})$.

Fitted line plot
$y = 6.14 + 2.071\,x$



If you regress x on y instead, the equation of regression is $\hat{x} = dy + c$.

The resulting formulae for the slope and intercept are $d = r\dfrac{S_x}{S_y}$, and $c = \overline{x} - d\overline{y}$.

A remarkable relationship appears here between the gradients of the regression line and r.

For example, $b = r\dfrac{S_y}{S_x}$ and $d = r\dfrac{S_x}{S_y}$, and hence, $bd = r\dfrac{S_y}{S_x} \cdot r\dfrac{S_x}{S_y} = r^2$.

Note: In cases where the explanatory variable is 'not controlled' we can regress x on y instead, the equation of regression is $\hat{x} = dy + c$.

Where the resulting formulas for the slope and intercept are

$$\begin{aligned} d &= \frac{\operatorname{cov}(X,Y)}{V(Y)} \\ &= \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2} \\ &= \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum y_i^2 - n\bar{y}^2} \\ &= r\frac{s_x}{s_y} \text{ and} \\ &= \bar{x} - d\bar{y}. \\ c &= \bar{x} - d\bar{y}. \end{aligned}$$

Example

The following data represent the volume in cubic mm and weight in grams of a certain fruit studied by a biologist.

Volume (x)	223	236	242	226	223	221	233	222	222	218	232	223
Weight (y)	165	171	173	170	168	172	168	167	162	166	164	164

Obtain the least-squares regression line of y on x as well as the regression line of x on y. Use the model to predict the weight of a 230-cubic mm fruit. Also, predict the volume of a 168 g fruit.

We will use software (you can use a GDC) for this calculation.

The least-squares regression of y on x is

$Y = 115 + 0.233x$

The predicted weight is $Y = 115 + 0.233(230) = 168.22$

The least squares regression of y on x is

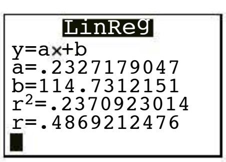
$X = 56.1 + 1.02y$

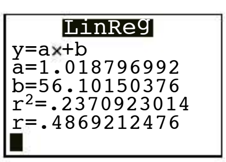
The predicted volume is

$X = 56.1 + 1.02(168) = 227.26$ cubic mm

You will also notice here that the product of the gradients (0.233) and (1.02) is 0.237 which is the same as the value of r^2 given by the software.

Using a TI-84, here are the results:





Notice how the values of r and r^2 are the same.

Hypothesis testing

When we claimed that there is some correlation between two variables we did that only by looking at the scatter plot. However, this is a matter of judgment sometimes. We can use our hypothesis testing procedures to check the validity of statements made about the correlation of two variables.

The hypothesis for claims of correlation are summarized below.

The null hypothesis is

H_0: $\rho = 0$ (that is there is no correlation)

The alternative hypothesis is one of the following:

H_1: $\rho > 0$ (That is, there is some positive correlations-upper tail test), or
H_1: $\rho < 0$ (That is, there is some negative correlations-lower tail test), or
H_1: $\rho \neq 0$ (That is, there is some correlations-two tail test).

To test the hypothesis, the test statistic is

$t = r\sqrt{\dfrac{n-2}{1-r^2}}$, which is obviously a t-distributed variable with $n - 2$ degrees of freedom.

Example 6

The data below represent the final exam scores in Mathematics and Physics for 10 students chosen at random at a large university. Test, at the 5% level of significance, whether there is some association between the scores in the two subjects.

Student	1	2	3	4	5	6	7	8	9	10
Mathematics	39	43	21	64	57	47	28	75	34	52
Physics	65	78	52	82	92	89	73	98	56	75

Solution

H_0: $\rho = 0$
H_1: $\rho \neq 0$
$R = 0.84$

The test statistic value is $t = 0.84\sqrt{\dfrac{8}{1-0.84^2}} = 4.375$

Rule: If $|t| > t_{\alpha/2}$, we reject the null hypothesis.

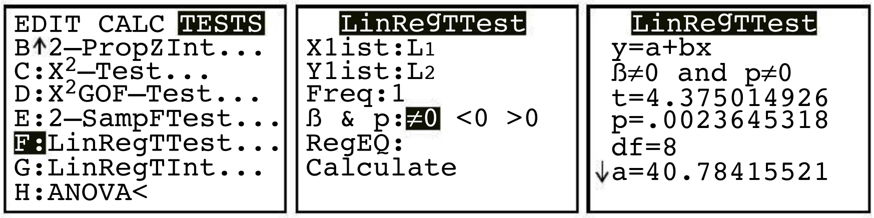
With 8 degrees of freedom, $t_{\alpha/2} = t_{0.025} = 2.306$

Thus, the test statistic lies in the rejection region.

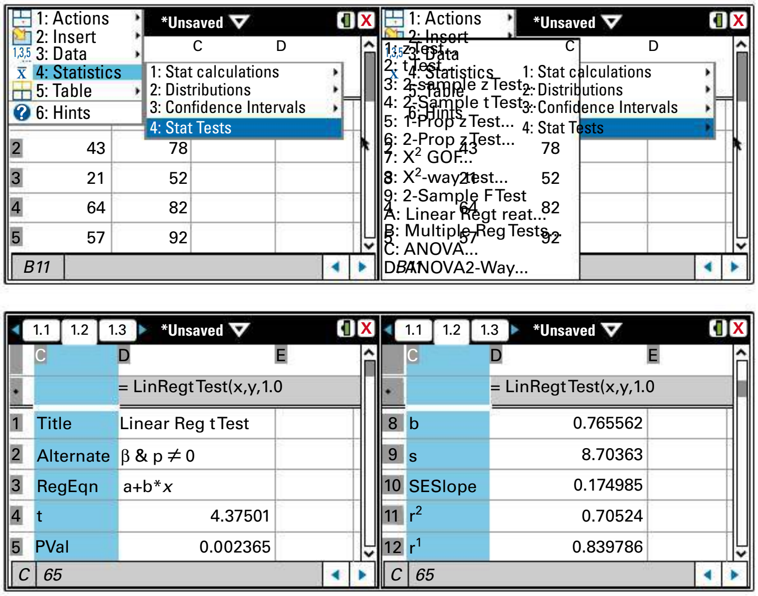
Conclusion: We have enough evidence to claim that there is some association between the scores in the two subjects.

A p-value approach can also be used. In this case the p-value is $0.00236 < 0.05$ and thus we also reject the null hypothesis.

Here is the output of a GDC:



And here are some screen shots of a TI-Nspire:



Note: You will notice in both screen shots that the test is both for the gradient of the regression line and for the correlation coefficient. This is so because it is the same test, i.e. when we test for correlation between two variables we will be testing also for the gradient of the line to be different from zero. Independence, as you recall will lead to zero correlation, and independence will also mean that the regression line is horizontal. Thus H_0: $\rho = 0$ is equivalent to H_1: $\rho = 0$

They also have the equivalent values of their test statistics because of the fact that $b = r\frac{s_y}{s_x}$ which means that b is a multiple of r. Thus, if $r = 0$, then so is b and if r is different from zero, so is b.

Exercise 7.2

1 Develop a regression model for each question in Exercise 7.1 and interpret the slope of each.

2 To test the benefit of using an online tutoring course for exam preparation, 20 students were given a test before they took part in the experiment and then afterwards. The tests were similar and the scores before and after the experiment were recorded. The intention was to find how improved the scores were due to participation in the experiment.

Analyze the data. For a student whose original score was 60, what do you expect, on average, the student's new score to be?

Student	Before	After
1	98	122
2	24	46
3	6	16
4	8	28
5	56	84
6	54	68
7	40	64
8	40	62
9	68	82
10	30	50
11	32	40
12	80	100
13	102	129
14	30	56
15	12	32
16	16	56
17	60	90
18	58	73
19	50	74
20	48	70

3 A large electronics company produces LCD monitors to be used in the computer industry. The monthly total cost of production over the period of one year, is given in the table below. (Number of units produced is in thousands and the cost is in 1000 euros.)

Number of units produced	Cost
16	1875
31	2586
57	3716
76	4712
13	1690
25	2191
49	3319
71	4362
20	2005
38	2775
63	4116
81	4860

a Draw a scatter plot of the data.

b Write down the equation of the regression line representing the association between units of production and sales. Draw the line on your scatter plot.

c Interpret the slope of the line and comment on the strength of this association.

d If the selling price of each unit during this year is 105 euros, what is the production level where the sales are equal to the cost?

4 The table shows the marks of 12 students sitting for IB Mathematics SL and IB Physics SL.

Mathematics	7	6	5	5	6	3	7	7	5	4	5	7
Physics	6	6	6	4	7	4	6	5	6	4	6	5

a Find the correlation coefficient and comment on your result.

b Find the regression equation that enables us to predict mathematics scores from the physics scores.

c What mark in mathematics would you expect for a candidate with a mark of 4 in physics?

5 Diamonds are usually priced according to weight. The carat is the usual measure and it is the weight of the diamond. 1 carat is equivalent to 200 milligrams. Some experts use points as the measure instead. 1 point is equivalent to 2 milligrams. Therefore, every carat is equivalent to 100 points. So, a 0.5 carat diamond is worth 50 points.

Here is the data for 20 diamonds and their prices.

Points	73	103	106	21	31	100	26	82	101	100	63	66
Price (€)	5909	15260	13640	1287	2177	12837	1911	6927	16143	10945	9117	6020

a Construct a scatter plot of the data. What type of trend do you observe?

b Write down the equation of a straight-line model relating the price to the number of points.

c Give a practical interpretation of the coefficients. If a practical interpretation is not possible, explain why.

d How well does the line fit the given data?

e Use the line you found to predict the price of a diamond with 63 points.

f Find the residual corresponding to your estimate in part e).

6 12 students in a graduating class take HL Mathematics and HL Physics. The marks they obtained on their mock exams in these subjects are given below

Student	1	2	3	4	5	6	7	8	9	10	11	12
Maths	53	48	83	70	39	51	73	47	24	61	43	54
Physics	56	45	80	63	42	38	72	45	32	46	48	50

a Find the product moment correlation coefficient for the scores and write down its p-value.

b Interpret the p-value in the context of the question.

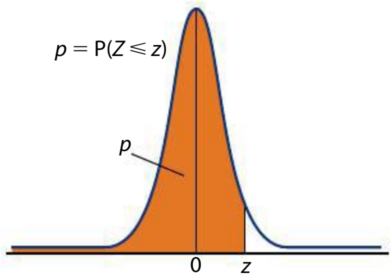
c Andrew obtained a grade of 64 in Mathematics. Predict his score in Physics according to the model.

d The same class sat mock exams in Economics SL and English SL and the correlation coefficient was 0.623. Using a 5% level of significance, determine whether the value indicates a positive association between the grades in Economics and English.

Tables

Normal distribution

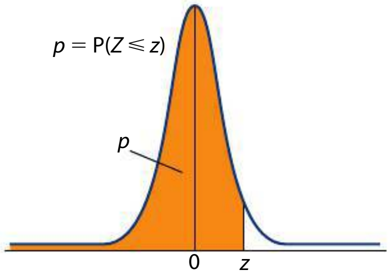
Area under the standard normal distribution



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Inverse normal distribution

Inverse normal numbers

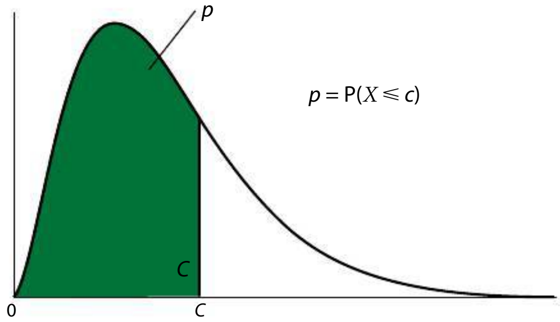


p	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.50	0.0000	0.0025	0.0050	0.0075	0.0100	0.0125	0.0150	0.0175	0.0201	0.0226
0.51	0.0251	0.0276	0.0301	0.0326	0.0351	0.0376	0.0401	0.0426	0.0451	0.0476
0.52	0.0502	0.0527	0.0552	0.0577	0.0602	0.0627	0.0652	0.0677	0.0702	0.0728
0.53	0.0753	0.0778	0.0803	0.0828	0.0853	0.0878	0.0904	0.0929	0.0954	0.0979
0.54	0.1004	0.1030	0.1055	0.1080	0.1105	0.1130	0.1156	0.1181	0.1206	0.1231
0.55	0.1257	0.1282	0.1307	0.1332	0.1358	0.1383	0.1408	0.1434	0.1459	0.1484
0.56	0.1510	0.1535	0.1560	0.1586	0.1611	0.1637	0.1662	0.1687	0.1713	0.1738
0.57	0.1764	0.1789	0.1815	0.1840	0.1866	0.1891	0.1917	0.1942	0.1968	0.1993
0.58	0.2019	0.2045	0.2070	0.2096	0.2121	0.2147	0.2173	0.2198	0.2224	0.2250
0.59	0.2275	0.2301	0.2327	0.2353	0.2378	0.2404	0.2430	0.2456	0.2482	0.2508
0.60	0.2533	0.2559	0.2585	0.2611	0.2637	0.2663	0.2689	0.2715	0.2741	0.2767
0.61	0.2793	0.2819	0.2845	0.2871	0.2898	0.2924	0.2950	0.2976	0.3002	0.3029
0.62	0.3055	0.3081	0.3107	0.3134	0.3160	0.3186	0.3213	0.3239	0.3266	0.3292
0.63	0.3319	0.3345	0.3372	0.3398	0.3425	0.3451	0.3478	0.3505	0.3531	0.3558
0.64	0.3585	0.3611	0.3638	0.3665	0.3692	0.3719	0.3745	0.3772	0.3799	0.3826
0.65	0.3853	0.3880	0.3907	0.3934	0.3961	0.3989	0.4016	0.4043	0.4070	0.4097
0.66	0.4125	0.4152	0.4179	0.4207	0.4234	0.4261	0.4289	0.4316	0.4344	0.4372
0.67	0.4399	0.4427	0.4454	0.4482	0.4510	0.4538	0.4565	0.4593	0.4621	0.4649
0.68	0.4677	0.4705	0.4733	0.4761	0.4789	0.4817	0.4845	0.4874	0.4902	0.4930
0.69	0.4959	0.4987	0.5015	0.5044	0.5072	0.5101	0.5129	0.5158	0.5187	0.5215
0.70	0.5244	0.5273	0.5302	0.5330	0.5359	0.5388	0.5417	0.5446	0.5476	0.5505
0.71	0.5534	0.5563	0.5592	0.5622	0.5651	0.5681	0.5710	0.5740	0.5769	0.5799
0.72	0.5828	0.5858	0.5888	0.5918	0.5948	0.5978	0.6008	0.6038	0.6068	0.6098
0.73	0.6128	0.6158	0.6189	0.6219	0.6250	0.6280	0.6311	0.6341	0.6372	0.6403
0.74	0.6433	0.6464	0.6495	0.6526	0.6557	0.6588	0.6620	0.6651	0.6682	0.6713
0.75	0.6745	0.6776	0.6808	0.6840	0.6871	0.6903	0.6935	0.6967	0.6999	0.7031
0.76	0.7063	0.7095	0.7128	0.7160	0.7192	0.7225	0.7257	0.7290	0.7323	0.7356
0.77	0.7388	0.7421	0.7454	0.7488	0.7521	0.7554	0.7588	0.7621	0.7655	0.7688
0.78	0.7722	0.7756	0.7790	0.7824	0.7858	0.7892	0.7926	0.7961	0.7995	0.8030
0.79	0.8064	0.8099	0.8134	0.8169	0.8204	0.8239	0.8274	0.8310	0.8345	0.8381
0.80	0.8416	0.8452	0.8488	0.8524	0.8560	0.8596	0.8633	0.8669	0.8705	0.8742
0.81	0.8779	0.8816	0.8853	0.8890	0.8927	0.8965	0.9002	0.9040	0.9078	0.9116
0.82	0.9154	0.9192	0.9230	0.9269	0.9307	0.9346	0.9385	0.9424	0.9463	0.9502
0.83	0.9542	0.9581	0.9621	0.9661	0.9701	0.9741	0.9782	0.9822	0.9863	0.9904
0.84	0.9945	0.9986	1.0027	1.0069	1.0110	1.0152	1.0194	1.0237	1.0279	1.0322
0.85	1.0364	1.0407	1.0450	1.0494	1.0537	1.0581	1.0625	1.0669	1.0714	1.0758

0.86	1.0803	1.0848	1.0893	1.0939	1.0985	1.1031	1.1077	1.1123	1.1170	1.1217
0.87	1.1264	1.1311	1.1359	1.1407	1.1455	1.1503	1.1552	1.1601	1.1650	1.1700
0.88	1.1750	1.1800	1.1850	1.1901	1.1952	1.2004	1.2055	1.2107	1.2160	1.2212
0.89	1.2265	1.2319	1.2372	1.2426	1.2481	1.2536	1.2591	1.2646	1.2702	1.2759
0.90	1.2816	1.2873	1.2930	1.2988	1.3047	1.3106	1.3165	1.3225	1.3285	1.3346
0.91	1.3408	1.3469	1.3532	1.3595	1.3658	1.3722	1.3787	1.3852	1.3917	1.3984
0.92	1.4051	1.4118	1.4187	1.4255	1.4325	1.4395	1.4466	1.4538	1.4611	1.4684
0.93	1.4758	1.4833	1.4909	1.4985	1.5063	1.5141	1.5220	1.5301	1.5382	1.5464
0.94	1.5548	1.5632	1.5718	1.5805	1.5893	1.5982	1.6072	1.6164	1.6258	1.6352
0.95	1.6449	1.6546	1.6646	1.6747	1.6849	1.6954	1.7060	1.7169	1.7279	1.7392
0.96	1.7507	1.7624	1.7744	1.7866	1.7991	1.8119	1.8250	1.8384	1.8522	1.8663
0.97	1.8808	1.8957	1.9110	1.9268	1.9431	1.9600	1.9774	1.9954	2.0141	2.0335
0.98	2.0537	2.0749	2.0969	2.1201	2.1444	2.1701	2.1973	2.2262	2.2571	2.2904
0.99	2.3263	2.3656	2.4089	2.4573	2.5121	2.5758	2.6521	2.7478	2.8782	3.0902

Chi-square table

Critical values of the χ^2 distribution



p	0.005	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990	0.995
$\nu = 1$	0.00004	0.0002	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

ν = number of degrees of freedom

Answers

Chapter 1

Exercise 1

1 a) $P(x \geqslant 2) = 0.5248$, $P(1 \leqslant x \leqslant 3) = 0.8448$
b) $E(X) = 1.6$, $Var(X) = 0.96$
c) $E(Y) = 5.8$, $Var(Y) = 3.84$

2 a) 0.193
b) $P(12 < x \leqslant 14) = 0.743$, $P(x \geqslant 14) = 0.263$
c) $E(X) = 13.452$, $Var(X) = 2.222$
d) $E(Y) = 26.904$, $Var(Y) = 8.888$
e) $E(Z) = 26.904$, $Var(Z) = 4.444$

3 a)

x	$p(x)$		y	$p(y)$
1	0.166 667		1	0.25
2	0.166 667		2	0.25
3	0.166 667		3	0.25
4	0.166 667		4	0.25
5	0.166 667			
6	0.166 667			

b) Mean (of x) = 3.5, variance = 2.917; mean (of y) = 2.5, variance = 1.25

c)

x	$p(x)$
2	0.041 667
3	0.083 333
4	0.125
5	0.166 667
6	0.166 667
7	0.166 667
8	0.125
9	0.083 333

d) Mean = 6, variance = 4.167

4 $E(V) = 3.5$, standard deviation = 0.285

5 a) 0.1 b) 3.2 c) 1.68 d) 16 e) 21.84

6 a) $E(X + Y) = 10$, $Var(X + Y) = 3$
b) $E(X - Y) = -4$, $Var(X - Y) = 3$
c) $E(2X + 3Y) = 27$, $Var(2X + 3Y) = 17$
d) $E(2X - 3Y) = -15$, $Var(2X - 3Y) = 17$

7 a) $E(X + Y) = \sqrt{7} + \sqrt{13}$, $Var(X + Y) = 5$
b) $E(X - Y) = \sqrt{7} - \sqrt{13}$, $Var(X - Y) = 5$
c) $E(2X + 3Y) = 2\sqrt{7} + 3\sqrt{13}$, $Var(2X + 3Y) = 35$
d) $E(2X - 3Y) = 2\sqrt{7} - 3\sqrt{13}$, $Var(2X - 3Y) = 35$

8 a) $E(2X + Y) = 2\sqrt{7} + 2$, $Var(2X + Y) = 22$
b) $E(X - 3Y) = \sqrt{7} - 6$, $Var(X - 3Y) = 23$
c) $E(2X + 3Y) = 2\sqrt{7} + 6$, $Var(2X + 3Y) = 38$
d) $E(2X - 3Y) = 2\sqrt{7} - 6$, $Var(2X - 3Y) = 38$

9 a) $E(l) = 1.01$, variance = 0.0024

b)

l	$P(l)$
2.1	0.36
2	0.48
1.9	0.16

$E(l) = 2.02$, variance = 0.0048

c)

l	$P(l)$
2.85	0.064
2.95	0.288
3.05	0.432
3.15	0.216

$E(l) = 3.03$, variance = 0.0072

10 a) 0.298 b) 0.227 c) 0.298

11 0.007

12 0.560

Practice questions 1

1 a) 0.841
b) (i) 0.0681 (ii) 0.0312 (iii) 0.932

2 0.164

3 a) $\lambda = 3$ b) 0.647 c) 0.265
d) (i) Mean = 7, variance = 11
(ii) Not Po

4 a) 10 b) 12 c) 7 d) 35

5 a) 0.944 b) Verify

6 a) (i) Mean = 0.5, variance = 0.13 (ii) 0.0828
b) 0.904

7 a) 0.0548 b) 0.993

8 a) (i) $\frac{3}{2}$ (ii) $\frac{11}{9}$ (iii) $\frac{125}{36}$
b) 0.432

Chapter 2

Exercise 2.1

1 a) 8 b) 16 c) $\frac{5}{7}$

2 a) $\frac{1}{5}$ b) Mean = 15, standard deviation = $2\sqrt{2}$
c) $\frac{3}{5}$

3 a) $\frac{1}{10}$ b) $E(V) = 4.5$, variance $= 8.25$

c) $\frac{1}{900}$ d) $\frac{2}{5}$

4 a) 6.5 b) 11.92 c) 0.076

5 a) 4.5 b) 5.25 c) 0.078

Exercise 2.2 and 2.3

1 a) 0.148 b) 0.538 c) 0.686 d) 3.125

2 a) 0.007 87 b) 0.0238 c) 0.984

3 $E(N) = 125$; no; standard deviation $= 124.5$

4 a) $\frac{3}{21}$ b) $\frac{6}{21}$ c) $\frac{6}{21}$

d) 4.33 e) 2.22 f) 6

5 a) 3.3 b) 1 c) 0.657

6 a) (i) 0.7 (ii) 0.6 (iii) 4.5 (iv) 8.25

b) (i) 0.059 (ii) 1 (iii) 2.64

7 a) 0.2 b) $E(X) = 5$, $\text{Var}(X) = 20$ c) 0.328

8 a) 0.141 b) 0.316

c) $E(N) = 4$, standard deviation $= 3.464$

9 a) 0.0527 b) 0.284

10 a) 8 b) 0.573 c) 0.0648 d) 0.714

11 a) 0.128 b) 0.107

Exercise 2.4

1 a) 0.1298 b) 0.1101

c) $E(N) = 13.33$, standard deviation $= 2.108$

2 a) 0.125 b) 0.125 c) 0.0938

3 a) 0.005 67 b) 0.052 92

4 a) 0.106 b) 0.0885 c) 0.1182

5 13.33

6 0.138

7 a) 0.080 b) $E(V) = 20$, standard deviation $= 46.7$

8 a) 0.09 b) 0.0437 c) 0.991

d) (i) Mean $= 1.11$, standard deviation $= 0.351$

(ii) Mean $= 3.33$, standard deviation $= 0.609$

9 a) (i) 0.6 (ii) 0.096

b) (i) 0.360 (ii) 0.092

c) 0.173

d) $E(N) = 1.67$, standard deviation $= 1.054$

e) $E(N) = 5$, standard deviation $= 1.826$

10 a) 0.081 b) 0.0098

c) $E(N) = 30$, standard deviation $= 16.43$

d) $E(N) = 1767$, standard deviation $= 739.35$

Exercise 2.5

1 a) 0.148 b) 0.439 c) 0.899

2 $E(N) = 2.60$, standard deviation $= 0.875$

3 a) 0.288 b) 0.216 c) 0.965 d) 0.251

4 1

5 a) 0.0951 b) 0.209 c) 0.890 d) 2

6 a) 0.491 b) 0.084 c) 0.088

7 a) (i) 0.0256 (ii) 0.154

(iii) 0.662 (iv) 0.462

b) 1.87

8 a) (i) 0.420 (ii) 0.028 (iii) 0.937

b) (i) 2.14 (ii) 14.29 hours

9 8

10 a) By chance, P(at most 1 non-native) = 0.187; no reason for doubt.

b) By chance, P(at most 2 females) = 0.314; no reason for doubt.

c) $E(N) = 2.4$, standard deviation $= 1.03$

d) $E(N) = 3$, standard deviation $= 1.05$

11 a) 0.491 b) 0.150

12 a)

x	0	1	2	3
$P(X = x)$	0.399	0.461	0.132	0.0088

b) $E(X) = 0.75$, $\text{Var}(X) = 0.503$

c) 0.601

Practice questions 2

1 a) Answers vary b) Mean $= 33.3$, variance $= 22.2$

c) 0.0768

2 a) 0.684 b) 0.0244 c) Answers vary

Chapter 3

Exercise 3

1 $1 - x + x^2 - x^3 + \cdots = \dfrac{1}{1+x}$

2 a) $1 + x^3 + x^6 + x^9 + \cdots = \dfrac{1}{1-x^3}$

b) This sequence is the same as in a) except for the two zeros at the start

$$x^2 + x^5 + x^8 + \cdots = x^2\left(1 + x^3 + x^6 + \cdots\right) = x^2\left(\frac{1}{1-x^3}\right).$$

3 Since X and Y are independent, then $E(XY) = E(X)E(Y) = (80 \times 0.25)(1.5) = 30$.

4 $G(t) = (q + pt)^n \Rightarrow G(0) = q^n = p(n = 0)$ as expected.

$G'(t) = np(q + pt)^{n-1} \Rightarrow G'(0) = npq^{n-1} = p(n = 1)$ as expected.

$G''(t) = n(n-1)p^2(q + pt)^{n-2} \Rightarrow G''(0)$

$= n(n-1)p^2q^{n-2} = 2p(n = 2)$ as expected.

5 $G(t) = e^{\lambda(t-1)} \Rightarrow G(0) = e^{\lambda(0-1)} = e^{-\lambda} = p(n = 0)$

$G'(t) = \lambda e^{\lambda(t-1)} \Rightarrow G'(0) = \lambda e^{-\lambda} = p(n = 1)$

$G''(t) = \lambda^2 e^{\lambda(t-1)} \Rightarrow G'(0) = \lambda^2 e^{-\lambda} = 2p(n = 2)$

6 a) $G(t)=\frac{1}{2}t+\left(\frac{1}{2}\right)^2 t^2+\left(\frac{1}{2}\right)^3 t^3+\ldots=\frac{\frac{t}{2}}{1-\frac{t}{2}}=\frac{t}{2-t}$

b) $G'(t)=\frac{2}{(2-t)^2}\Rightarrow G'(0)=\frac{2}{2^2}=\frac{1}{2}$ which is equal to $P(X=1)=\frac{1}{2}$.

$G''(t)=\frac{4}{(2-t)^3}\Rightarrow G''(0)=\frac{4}{2^3}=\frac{1}{2}$ which is equal to $2P(X=2)=2\cdot\frac{1}{2^2}=\frac{1}{2}$.

7 $G(t)=\frac{t}{6-5t}$

$G'(t)=\frac{6}{(5t-6)^2}\Rightarrow G'(1)=\frac{6}{(5-6)^2}=6=E(X)$

$G''(t)=\frac{60}{(6-5t)^3}\Rightarrow G''(1)=\frac{60}{(6-5)^3}=60$,

hence the variance is

$V(X)=G''(1)+G'(1)-(G'(1))^2$
$=60+6-36=30$

8 $G(t)=\frac{t}{9-8t}$

$G'(t)=\frac{9}{(8t-9)^2}\Rightarrow G'(1)=\frac{9}{(8-9)^2}=9=E(X)$

$G''(t)=\frac{144}{(9-8t)^3}\Rightarrow G''(1)=\frac{144}{(9-8)^3}=144$

$V(X)=G''(1)+G'(1)-(G'(1))^2$
$=144+9-81=72$

9 a) Since this is a pgf, then $G(1)=1$. Thus $G(1)=\frac{k}{(5-1)^2}\Rightarrow k=16$.

b) Since $G'(s)=\frac{32}{(5-s)^3}\Rightarrow G'(1)=E(Y)=\frac{32}{4^3}=\frac{1}{2}$

c) $G''(s)=\frac{96}{(s-5)^4}\Rightarrow G''(0)=\frac{96}{(-5)^4}=\frac{96}{625}=2P(Y=2)$

$\Rightarrow P(Y=2)=\frac{48}{625}$

10 a) For $G(s)$ to be a pgf, $G(1)=1$, i.e.,

$G(1)=\frac{m}{(7-4)^3}=1\Rightarrow m=27$.

b) $G'(s)=\frac{324}{(4s-7)^4}\Rightarrow G'(1)=\frac{324}{(4-7)^4}=4=E(X)$

$G''(s)=\frac{5184}{(7-4s)^5}\Rightarrow G''(1)=\frac{5184}{(7-4)^5}=\frac{64}{3}$

$V(X)=G''(1)+G'(1)-(G'(1))^2$

$=\frac{64}{3}+4-16=\frac{28}{3}$

c) $P(X=0)=G(0)=\frac{27}{(7)^3}=\frac{27}{343}$

$P(x=1)=G'(0)=\frac{324}{(-7)^4}=\frac{324}{2401}$

$P(X=2)=\frac{G''(0)}{2}=\frac{5184}{2(7)^5}=\frac{5184}{2(16\,807)}=\frac{2592}{16\,807}$

[This can also be done using the Binomial power series expansion in Option 3].

11 a) $X=1$ corresponds to Won solving correctly the first time; $P(X=1)=\frac{1}{3}$. $X=2$ corresponds to Kat solving correctly the first time, i.e. Won loses her first attempt and Kat wins; $P(X=2)=\frac{2}{3}\cdot\frac{1}{4}$, now with similar arguments, we have

$P(X=3)=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{3}$;

$P(X=4)=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{4}$;

$P(X=5)=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{3}$

Thus

$$G(t)=\frac{1}{3}t+\frac{2}{3}\cdot\frac{1}{4}t^2+\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{3}t^3+\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{4}t^4+\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{3}t^5$$

$$=\frac{1}{3}t\left(1+\frac{1}{2}t^2+\frac{1}{4}t^4+\ldots\right)+\frac{1}{6}t^2\left(1+\frac{1}{2}t^2+\ldots\right)$$

$$=\frac{1}{3}t\frac{1}{1-\frac{1}{2}t^2}+\frac{1}{6}t^2\frac{1}{1-\frac{1}{2}t^2}=\frac{2}{2-t^2}\left(\frac{1}{3}t+\frac{1}{6}t^2\right)=\frac{2t+t^2}{3(2-t^2)}$$

b) $G'(t)=\frac{2(t^2+2t+2)}{3(t^2-2)^2}\Rightarrow G'(1)=\frac{2(5)}{3(-1)^2}=\frac{10}{3}$

(Notice that $G'(0)=\frac{1}{3}=P(X=1)$!!

$G''(t)=\frac{4(t^3+3t^2+6t+2)}{3(2-t^2)^3}\Rightarrow G''(1)=16$

$V(X)=G''(1)+G'(1)-(G'(1))^2$

$=16+\frac{10}{3}-\frac{100}{9}=\frac{74}{9}$

(notice that $G''(0)=\frac{4(2)}{3(2)^3}\Rightarrow G''(0)=\frac{1}{3}$

$\Rightarrow P(X=2)=\frac{\frac{1}{3}}{2}=\frac{1}{6}$)

12 a) $G'(t)=\frac{2p^2q}{(1-qt)^3}\Rightarrow G'(1)=\frac{2p^2q}{(1-q)^3}=\frac{2q}{p}=E(X)$

$G''(t)=\frac{6p^2q^2}{(1-qt)^4}\Rightarrow G''(1)=\frac{6q^2}{p^2}$

$V(X)=G''(1)+G'(1)-(G'(1))^2$

$=\frac{6q^2}{p^2}+\frac{2q}{p}-\left(\frac{2q}{p}\right)^2=\frac{6q^2+2pq-4q^2}{p^2}$

$=\frac{2q(q+p)}{p^2}=\frac{2q}{p^2}$

b) Since $G'(t)=\frac{2p^2q}{(1-qt)^3}\Rightarrow G'(0)=2p^2q=P(X=1)$

Also, $G''(t) = \dfrac{6p^2q^2}{(1-qt)^4} \Rightarrow G''(0) = 6p^2q^2$

$\Rightarrow P(X = 2) = 3p^2q^2$

13 Let X represent the random variable. Here $x = 1, 2, 3, 4, 5, 6$ and $p(X = x) = \dfrac{1}{6}$.

In example 7 of the chapter, we found out that

$$G(t) = 0t^0 + \frac{1}{6}t^1 + \frac{1}{6}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{6}t^5 + \frac{1}{6}t^6$$
$$= \frac{1}{6}t\left(1 + t + \ldots + t^5\right)$$

The expression in brackets is a geometric series with first term 1 and a common ratio of t, thus:

$$G(t) = \frac{1}{6}t\left(1 + t + \ldots + t^5\right) = \frac{1}{6}t \cdot \frac{1-t^6}{1-t} = \frac{t\left(1-t^6\right)}{6(1-t)}.$$

14 The probability generating function is:

$$G(t) = \frac{1}{n}t^1 + \frac{1}{n}t^2 + \ldots + \frac{1}{n}t^n$$
$$= \frac{1}{n}t\left(1 + t + \ldots + t^{n-1}\right)$$

The expression in brackets is a geometric series with first term 1 and a common ratio of t, thus:

$$G(t) = \frac{1}{n}t\left(1 + t + \ldots + t^{n-1}\right) = \frac{t\left(1-t^n\right)}{n(1-t)}$$

Now attempting to use the closed form to find the expected value and variance is not appropriate here since the result is valid except when $t = 1$. Hence we need to consider the expanded form.

$$G(t) = \frac{1}{n}t + \frac{1}{n}t^2 + \ldots + \frac{1}{n}t^n = \frac{1}{n}\left(t + t^2 + \ldots + t^n\right)$$
$$G'(t) = \frac{1}{n}\left(1 + 2t + 3t^2 + \ldots + nt^{n-1}\right)$$
$$G'(1) = \frac{1}{n}(1 + 2 + 3 + \ldots + n) = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}$$

15 $\displaystyle\sum_{x=0}^{\infty} P(X = x) = 1 \Rightarrow \sum_{0}^{\infty} \frac{k}{e^x} = 1$

$$\Rightarrow k + \frac{k}{e} + \frac{k}{e^2} + \ldots = 1 \Rightarrow k\left(1 + \frac{1}{e} + \frac{1}{e^2} + \ldots\right) = 1$$
$$\Rightarrow k \cdot \frac{1}{1-\frac{1}{e}} = 1 \Rightarrow k = 1 - \frac{1}{e}$$

Now $P(X = x) = \dfrac{1-\frac{1}{e}}{e^x} = \dfrac{e-1}{e^{x+1}}$, thus:

$$G(t) = \sum_{\text{all } x} \frac{e-1}{e^{x+1}}t^x = \frac{e-1}{e} + \frac{e-1}{e^2}t + \frac{e-1}{e^3}t^2 + \ldots$$
$$= \frac{e-1}{e}\left(1 + \frac{t}{e} + \left(\frac{t}{e}\right)^2 + \ldots\right)$$

Now, with the condition that $t < e$, the expression in brackets is an infinite geometric series.

$$G(t) = \frac{e-1}{e}\left(1 + \frac{t}{e} + \left(\frac{t}{e}\right)^2 + \ldots\right) = \frac{e-1}{e} \frac{1}{1-\frac{t}{e}} = \frac{e-1}{e-t}.$$

$G'(t) = \dfrac{e-1}{(t-e)^2} \Rightarrow G'(1) = \dfrac{e-1}{(1-e)^2} = \dfrac{1}{e-1}$. Also

$G''(t) = \dfrac{2(1-e)}{(t-e)^3} \Rightarrow G''(1) = \dfrac{2(1-e)}{(1-e)^3} = \dfrac{2}{(e-1)^2}$

$V(X) = G''(1) + G'(1) - (G'(1))^2$

$$= \frac{2}{(e-1)^2} + \frac{1}{e-1} - \left(\frac{1}{e-1}\right)^2 = \frac{e}{(e-1)^2}$$

Chapter 4

Exercise 4

1 a) 0.338 b) 0.053

2 a) 0.0594 b) 0 c) 0.9973

3 a) 34.1% b) 36.65 weeks
c) Normal with $\mu = 38$, $\sigma = \dfrac{2}{\sqrt{120}}$
d) 0
e) a) and b) will change, while c) and d) will not

4 a) No
b) No, sample too small for the central limit theorem (CLT)
c) Yes, CLT applies, $p = 0.039$

5 a) 1 b) 4

6 5.06

7 a) 0.399 b) 0.154; the company's claim is fine.

8 a) 0.223 b) 0.460
c) Cannot find probability as the sample size is too small.

9 a) (i) 0.683 (ii) 0.904 (iii) 0.992

10 a) 0.001 87 b) [932.95, 987.05]

11 a) 0.837
b) No, as the sample size is too small for CLT to apply.

12 0.146

13 a) 0.009 52
b) This is so unlikely to happen. We can conclude that the claim may underestimate the true defective rate.

14 a) 0.004 40
b) This is so unlikely to happen. We can conclude that the claim may overestimate the true relief rate.

15 Approximately 0

16 22%

17 0.0548

18 a) 0.244 b) 0.271

19 0.003 35

20 a) 0.864 b) 0.941

21 1

22 a) 0.369 b) 0.004 91

23 $p \approx 7.37$, $\sigma \approx 1.72$

Chapter 5

Exercise 5

1 a) Mean = 79.333, standard deviation = 10.137
b) Mean = 0.276, standard deviation = 0.663

c) Mean = 73.067, standard deviation = 13.554
d) Mean = 47, standard deviation = 19.472
e) Mean = 66.692, standard deviation = 36.871

2 a) Mean = 499.54, standard deviation = 1.893
b) (498.50, 500.58)
c) Company's claim is acceptable
d) (498.07, 501.01)
e) 2.08, 2.94 f) (498.44, 500.64)

3 (−0.0412, 0.072 29)

4 92.5%

5 a) (995.88, 1000.1) b) 57.77%

6 a) (1002.48, 1072.52) b) 57

7 28 **8** (21.1, 21.6)

9 (0.743, 2.424) **10** (0.643, 0.734)

11 (13.84, 20.28) **12** 68

13 (0.1875, 0.2234) **14** 1068

15 16

16 a) 0.9996 b) 3

17 (0.106, 0.425)

18 a) $E(T) = kE\left(\hat{\theta}_1\right) + (1-k)\,E\left(\hat{\theta}_2\right) = k\theta + (1-k)\,\theta = \theta.$

b) $V(T) = k^2 V\left(\hat{\theta}_1\right) + (1-k)^2\,V\left(\hat{\theta}_2\right) = k^2\sigma_1^2 + (1-k)^2\,\sigma_2^2,$

To find the minimum, we find the first derivative of $V(T)$ with respect to k and equate it to zero:

$V'(T) = 2k\sigma_1^2 - 2(1-k)\,\sigma_2^2 = 0 \Rightarrow k = \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$

(We need to check for the minimum using first derivative test or second derivative test.)

19 a) Since X_1 is a value of the random variable itself, then $E\left(\hat{\theta}_1\right) = E(X_1) = \lambda \Rightarrow \hat{\theta}_1$ is unbiased, the other three are linear combinations of the values of the random variable, thus

$E\left(\hat{\theta}_2\right) = E\left(\frac{1}{2}(X_1 + X_2)\right) = \frac{1}{2}(\lambda + \lambda) = \lambda \Rightarrow \hat{\theta}_2$ is unbiased.

Similarly, $E\left(\hat{\theta}_3\right) = E\left(\frac{1}{3}(X_1 + 2X_2)\right) = \frac{1}{3}(\lambda + 2\lambda) = \lambda \Rightarrow \hat{\theta}_3$

is unbiased and so is $\hat{\theta}_4$

b) Again, since X_1 is a value of the random variable itself, then

$V\left(\hat{\theta}_1\right) = V(X_1) = \lambda^2$

$V\left(\hat{\theta}_2\right) = V\left(\frac{1}{2}(X_1 + X_2)\right) = \frac{1}{4}\left(\lambda^2 + \lambda^2\right) = \frac{\lambda^2}{2}$

$V\left(\hat{\theta}_3\right) = V\left(\frac{1}{3}X_1 + \frac{2}{3}X_2\right) = \frac{1}{9}\lambda^2 + \frac{4}{9}\lambda^2 = \frac{5\lambda^2}{9}$

and similarly

$V\left(\hat{\theta}_4\right) = \frac{\lambda^2}{3}$, and so obviously $\hat{\theta}_4$ is the most efficient.

20 a) We know that $E(X) = np$ and $V(X) = npq$ already. Thus

$E(v) = E(n\bar{p}\bar{q}) = E\left(n\frac{X}{n}\left(1 - \frac{X}{n}\right)\right) = E\left(\left(X - \frac{X^2}{n}\right)\right)$

$= E(X) - \frac{1}{n}E\left(X^2\right)$

Since $V(X) = E\left(X^2\right) - (E(X))^2 \Rightarrow E\left(X^2\right)$

$= V(X) + (E(X))^2 = npq + n^2p^2$

$E(v) = E(X) - \frac{1}{n}E\left(X^2\right) = np - \frac{1}{n}\left(npq + n^2p^2\right)$

$= (n-1)\,pq \neq V(X)$

b) The unbiased estimator should have expected value npq, so consider the estimator $\hat{v} = \frac{n}{n-1}(n\overline{pq})$.

This is so because $E(\hat{v}) = E\left(\frac{n}{n-1}(n\overline{pq})\right) = \frac{n}{n-1}E(n\overline{pq})$

$= \frac{n}{n-1}(n-1)\,pq = npq.$

Practice questions 5

1 984

2 (2.703, 2.707)

3 a) (i) 87.03 (ii) 215.58
b) (i) (86.22, 88.04) (ii) (86.37, 87.89)
c) Greater confidence leads to less precision

4 a) Mean = 33.18, variance = 3.22 b) (32.1, 34.2)

5 a) 96 b) 99.0%

6 a) (i) 0.45 (ii) 0.0144 (iii) (0.422, 0.478)
b) Random sampling

7 a) $(\bar{x} - 1.91, \bar{x} + 1.91)$ b) 99.0%

8 a) (11.8, 13.4) b) $\mu = 13.7$; inconsistent

9 a) (0.498, 0.557) b) 9576

10 a) 98.2% b) 10

Chapter 6

Exercise 6

1 There is evidence of change, p-value = 0.0339

2 There is no statistical evidence at the 1% level of significance, p-value = 1.51%

3 There is statistical evidence at the 2% level of significance, p-value = 0.274%

4 There is no statistical evidence at the 3% level of significance, p-value = 13.350%

5 There is no statistical evidence at the 5% level of significance to conclude that the wire is gold, p-value = 74.6%

6 a) There is no statistical evidence at the 5% level of significance (p-value = 38.8%) that the packs are underweight.
b) There is statistical evidence at the 5% level of significance (p-value = 2.64%) that the packs are underweight.

7 a) H_0: $p = 0.03$, H_1: $p > 0.03$. p-value = 42.7%; we do not have statistical evidence to conclude that the rate of cancer cases has increased.
b) Type II c) 73.1%

8 a) H_0: $p = 0.30$, H_1: $p > 0.30$. p-value = 0.02%; we have statistical evidence to conclude that the number of hospital stays has increased.
b) Type I. We conclude that hospital stays have increased when they actually did not.

c) 31.4%. We conclude that the number of hospital stays has not increased when it actually did.

9 a) H_0: $p = 0.54$, H_1: $p < 0.54$. p-value = 2.6%; we have statistical evidence at the 5% level of significance to conclude that consumer confidence is lower in 2009 than it was before.
b) 9.21%

10 a) H_0: $\mu = 3.2$, H_1: $\mu < 3.2$. Rejection region: $t < -1.761$, $t = -1.81$, p-value = 4.6%; we have statistical evidence to conclude that shop sales have decreased.
b) 79.7%. We conclude that the sales have not decreased when they actually did.

11 a) H_0: $\mu = 24.1$, H_1: $\mu > 24.1$. Rejection region: $t > 1.66$, $t = 1.71$, p-value = 4.5%; we have statistical evidence to conclude that the age of the consumer has increased.
b) 62.96%. We conclude that the average age has not increased when it actually did.

12 H_0: $\mu = 11.1$, H_1: $\mu > 11.1$. p-value = 0.2%; we have statistical evidence to conclude that the company's efforts are successful.

13 Matched pairs test. p-value = 2.4%; we have enough evidence that there is a difference in fuel consumption between the two car types.

14 Matched pairs test (absolute values!). p-value = 0; we conclude that the difference is more than 0.003 and hence they will not purchase the hydrostatic instruments. Type I error means that we will conclude that the difference is more than 0.003 and end up not purchasing the hydrostatic instruments; while Type II error means that we fail to see that the difference is more than 0.003 and end up purchasing the hydrostatic instruments.

15 a) Matched pairs test. p-value = 1.2%; we have statistical evidence to conclude that the passenger appears to have the worst seat.
b) 59%. We conclude that there is no difference in injury between the passenger and the driver when in fact there is a difference.

16 a) $\mathrm{P}\left(\bar{x} > 762.34 \mid \mu = 750, \sigma = \dfrac{30}{\sqrt{16}}\right) < 0.05$, and hence we reject H_0.
b) p-value = 2.28%, and hence we reject H_0.
c) 15.4%

17 $\bar{x} = \dfrac{896}{15} = 59.73$, $s_{n-1}^2 = \dfrac{15}{14}\left(\dfrac{54\,172}{15} - \left(\dfrac{896}{15}\right)^2\right) = 46.50$.
H_0: $\mu = 60$, H_1: $\mu < 60$. p-value = 44%; we do not have statistical evidence to reject the company's claim.

Practice questions 6

1 a) 0.369 b) 0.146 c) (i) 0.714 (ii) \$1716.60
d) No evidence of change of standards.
e) Cannot reject the hypothesis that the data is N(68, 9).

2 a) Differences (d): 1.5, 0.6, 0.3, −0.2, 2.0, 0.6, 1.5, 0.1, 0.5, −0.4.
b) (i) H_0: $\mu_d = 0$, H_1: $\mu_d < 0$
(ii) p-value = 0.0139 > 0.01; insufficient evidence to conclude that Puzzle 2 takes longer than Puzzle 1.

3 a) H_0: $\mu = 30$, H_1: $\mu \neq 30$
b) p-value = 0.114; do not reject H_0
c) t-test since population is normal and variance unknown.

4 a) H_0: $p = 0.5$, H_1: $p > 0.5$
b) (i) Critical region
(ii) Probability of finding a sample with $p \geqslant 0.733$ when the population has $p = 0.5$. The 'observed' significance level in this case is 0.0592.
c) P(Type II) = $\mathrm{P}(X \leqslant 10 \mid p = 0.6) = 0.783$
d) (i) Type II
(ii) Conclusion will be that the coin is fair when it is not.

5 a) H_0: $\mu_d = 5$, H_1: $\mu_d < 5$ (matched pairs)
b) (i) p-value = 0.0447; cannot reject at 1% level.
(ii) Reject at 10%
c) Randomness and normality

6 Matched pairs. H_0: $\mu_d = 0$, H_1: $\mu_d \neq 0$. p-value = 0.0320; claim cannot be justified.

7 Matched pairs. H_0: $\mu = 0$, H_1: $\mu > 0$. p-value = 0.004 09; there is enough evidence to support claim.

8 a) 0.0668 b) 9.53
c) H_0: $\mu = 75$, H_1: $\mu > 75$. p-value = 0.001 86; reject H_0.

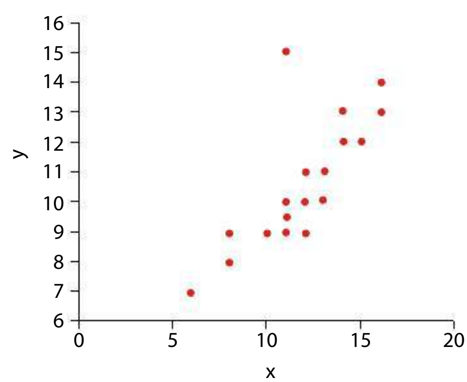
9 a) 65
b) In both cases, H_0: $p = 0.5$, H_1: $p \neq 0.5$.
(i) Amanda: $X \sim \mathrm{B}(3, 0.5)$;
P(Type I) = $\mathrm{P}(X = 0 \text{ or } 3) = 0.25$
Roger: $X \sim \mathrm{B}(8, 0.5)$;
P(Type I) = $\mathrm{P}(X \geqslant 6 \text{ or } X \leqslant 2) = 0.289$
Amanda has the smaller Type I probability.
(ii) P(Type II) = $\mathrm{P}(3 \leqslant X \leqslant 5 \mid p = 0.6) = 0.635$

10 a) Matched pairs. H_0: $\mu_d = 0$, H_1: $\mu_d > 0$.
b) p-value = 0.0295; we have enough evidence to conclude that practice sessions improve ability to memorize digits.

Chapter 7

Exercise 7.1

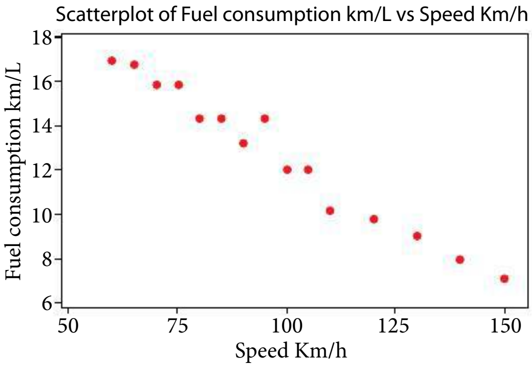
1



It appears that the data have a positive linear relationship. It is relatively strong except for an outlier apparently at (11, 15).

It appears that there is not much correlation in the data. This is confirmed by the low correlation coefficient of 0.260.

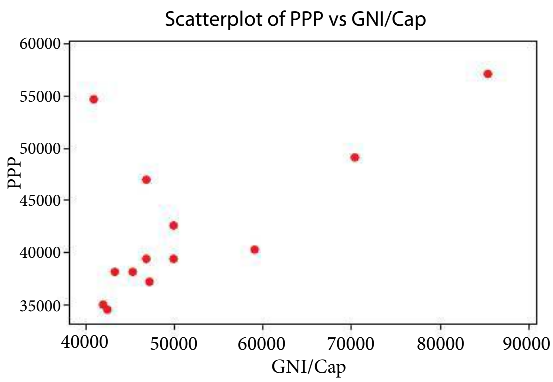
2 a)



b) We chose the speed as the explanatory variable because the car must first run to cause fuel consumption. Hence the speed helps explain the fuel consumption. The relationship appears to be negatively sloped because the consumption is measure by the distance travelled per litre of fuel.

c) The relationship appears to be a relatively strong negative one without any apparent outliers. The correlation coefficient is −0.986 which is very close to −1. A very strong relationship.

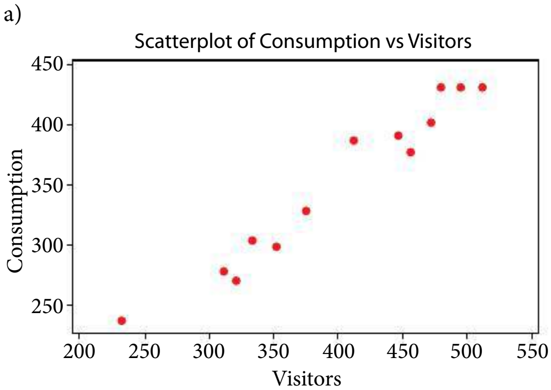
3 a)



b) The relationship appears to be a positive one except for an outlier which can be traced to be Singapore. We chose the explanatory variable to be the income, because the income level dictates how willing people are to pay for goods.

c) The relationship is relatively strong (weakened by Singapore's numbers). The correlation coefficient is 0.621. If we remove Singapore's data, then it becomes 0.886.

4 a)



b) There is obviously a positive relationship between the number of visitors and consumption. As the number of visitors increases the consumption will also increase.

c) The relationship seems to be strong and there is an absence of outliers. The correlation coefficient is 0.978 which is very close to 1.

Exercise 7.2

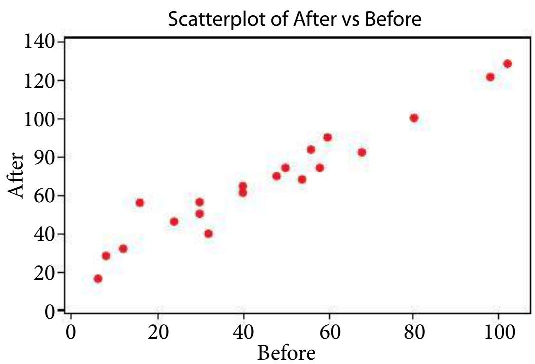
Ex 7.1, 1 The regression equation is: $y = 6.56 + 0.29\,x$. For every change of 1 unit in the x-values, the y-values will change, on average, by 0.29.

Ex 7.1, 2 The regression equation is: Fuel cons.km/L = 24.1 – 0.116 Speed km/h. For every increase of 1 km/h in speed, the average number of km per litre will decrease by 0.116 km/L, i.e. consumption will increase.

Ex 7.1, 3 The regression equation is: *PPP* = 24383 + 0.351 GNI/cap. For every increase of \$1 in GNI/cap, the *PPP* will increase, on average by \$0.351.

Ex 7.1, 4 The regression equation is: Consumption = 40.0 + 0.777 Visitors. For every increase of 1 visitor, we expect, on average, that consumption will increase by 0.777.

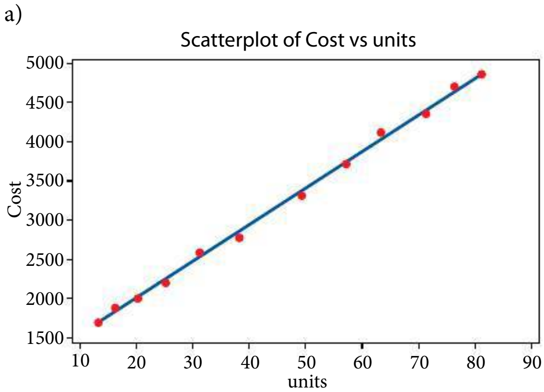
2



The scatter plot shows a strong positive relationship. That is the higher the 'Before' score the higher the 'After' score is. The regression equation is: After = 20.2 + 1.03 Before.

This means that, on average, for every change of 1 mark on the 'Before' test, the 'After' test is expected to change by 1.03. The correlation coefficient is 0.97 indicating a very strong linear relationship. For a student with 60 score on the 'Before' test, the model predicts, on average, a score of 81.90 on the 'After' test.

3 a)



b) The regression equation is: Cost = 1066 + 47.1 units.

c) For every increase of 1000 units in production, the cost, on average, will increase by 47100 Euros. The correlation coefficient is 0.999, which is almost perfect association. This is a strong linear relationship.

d) Let number of 1000 units be x, then: Cost = 1066 + 47.1 x

$\frac{Cost}{x} = \frac{1066}{x} + 47.1$ = cost per unit. If this cost is 105, then $105 = \frac{1066}{x} + 47.1 \Rightarrow x = 18.411$

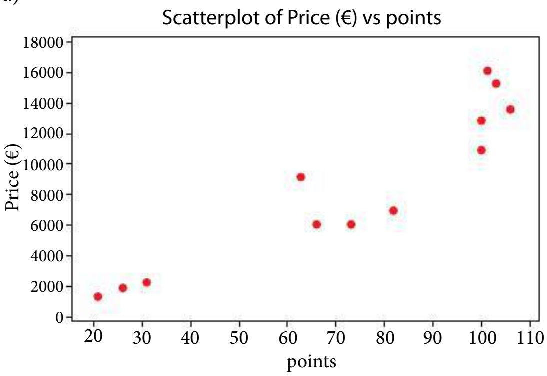
Thus the number of units will be 18 400 units.

4 a) R = 0.493. This is a relatively weak correlation between the two scores.

b) The regression equation is: Maths = 2.07 + 0.649 Physics

c) 4.7 (which can be rounded up to 5).

5 a)



Appears to be a positively sloped trend.

b) The regression equation is: Price (€) = −2689 + 154 points.

c) The intercept is meaningless as zero is not in the domain of the explanatory variable. On average, for every increase of 1 point, we expect the price to increase by 154 Euros.

d) $r = 0.93$ indicating a strong association between points and price.

e) The average price of a 63-point diamond is predicted to be 7024 Euros.

f) Residual = 2093.

6 a) Correlation coefficient = 0.905 and the p-value is approximately 0.

b) The p-value can tell us that on the assumption of a true null hypothesis, i.e. no correlation, the chance we get a sample with a coefficient as large as 0.905 is zero. Hence, we have strong evidence to reject the null hypothesis and conclude that there is a strong positive association between the scores of Maths and Physics.

c) The regression line of y on x is $y = 8.92 + 0.789x$, i.e. Physics grade = 8.92 + 0.789(Maths grade). Thus Andrew's Physics grade = 8.92 + 0.789(64) = 59.44 = 59.

d) This is a hypothesis test:

$H_0: \rho = 0$

$H_1: \rho > 0$

$$t = 0.623\sqrt{\frac{12-2}{1-0.623^2}} = 2.52$$

p-value = 0.0051, thus we reject H_0 and conclude that we have enough evidence that there is some positive association between the grades of Economics and English.

We can use a critical number approach too.

$T_{critical} = 1.81246$, and since our test statistic $t = 2.52 > 1.81246$, we reject the null hypothesis.

Sets, Relations and Groups

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Sets, Relations and Groups

Assessment statements

8.1 Finite and infinite sets.
Subsets.
Operations on sets; union; intersection; complement; set difference; symmetric difference.
De Morgan's laws; distributive, associative and commutative laws (for union and intersection).

8.2 Ordered pairs: the Cartesian product of two sets.
Relations; equivalence relations; equivalence classes.

8.3 Functions: injections; surjections; bijections.
Composition of functions and inverse functions.

8.4 Binary operations. Operation tables (Cayley tables).

8.5 Binary operations with associative, distributive and commutative properties.

8.6 The identity element e.
The inverse a^{-1} of an element a.
Proof that left-cancellation and right-cancellation by an element a hold, provided that a has an inverse.
Proofs of the uniqueness of the identity and inverse elements.

8.7 The definition of a group $\{G, *\}$.
The operation table of group is a Latin square but the converse is false. Abelian groups.

8.8 Examples of groups:
- $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ under addition
- integers under addition modulo n
- non-zero integers under muliplication, modulo p, where p is prime
- symmetries of plane figures including equilateral triangles and rectangles
- invertible functions under composition of functions.

8.9 The order of a group element and the order of a group.
Cyclic groups. Generators.
Proof that all cyclic groups are Abelian.

8.10 Permutations under composition of permutations.
Cycle notation for permutations.
Result that every permutation can be written as a composition of disjoint cycles.
The order of a combination of cycles.

8.11 Subgroups, proper subgroups.
Use and proof of subgroup tests.
Lagrange's theorem.
Use and proof of the result that the order of a finite group is divisible by the order of any element. (Corollary to Lagrange's theorem.)
Definition and examples of left and right cosets of a subgroup of a group.

8.12 Definition of a group homomorphism.
Definition of the kernel of a homomorphism.
Proof that the kernel and the range of a homomorphism are subgroups.
Proof of homomorphism properties for identities and inverses.
Isomorphism of groups.
The order of an element is unchanged by an isomorphism.

1 Sets

Review

We will start this option by reviewing and extending your knowledge of set theory. Many of the concepts you have already seen in the book. We will begin with a few definitions.

Definitions are essential in any subject matter because they help precision in discussion. However, if we try to define any term, we will be using other words which are defined using still other words that are not defined, and so on. That is why, in mathematics, like any other subject, new structures start with some terms that are 'understood' but are not defined.

A set is an **undefined** term in set theory. It is understood to be a '*well-defined*' collection of items or objects. Usually, the items in a set share some property. Any item that has the property is said to be a member (or an element) of the set and any item that does not have the property is not a member of the set.

Notation

We usually use capital letters to denote sets and the symbol $\in$ to denote membership in a set. Thus, $x \in A$ means that object x is an element or a member of set A, and $y \notin A$ means that item y is not a member or element of set A. Also, when we list the elements of a set, or when we describe it by a rule, we use braces to indicate the set, as you will see in the following example.

Let A be the set of numbers on the sides of a normal die. Then we can define the set A by either listing its elements:

$$A = \{1, 2, 3, 4, 5, 6\}$$

or by stating a rule:

$$A = \{x \mid x = \text{a number on a six-sided die}\}.$$

This is also called 'set-builder' notation.

(This is read as 'the set of x such that x is a number on a six-sided die' or any equivalent property.)

Notice that 5 is an element of A, and that is why we write

$$5 \in A$$

while 7 is not a member and we write

$$7 \notin A.$$

1.1 Basic set properties

What do we mean by a *well-defined* collection? When we define a set by a rule or by listing its elements, then **well defined** means that we should always be able to make a clear decision whether any object is, or is not, an element of the set.

For example, if we define set B as the set of the first 10 positive integers, i.e.

$B = \{x \mid x$ is one of the first 10 positive integers$\}$, or $B = \{1, 2, \ldots, 9, 10\}$

then, given any number, we can always say whether it is an element of B or not.

So, $2.999 \notin B$ while $3 \in B$.

If we define $C = \{y \mid y$ is one of 10 integers$\}$, can we say that $3 \in C$? The answer is no. 3 may or may not be an element of C. So, B is a well-defined collection and hence it is a set, and C is not well defined and hence it is not a set.

When we discuss objects we always have the set of all possible objects that we call the **universal set** and we denote it by U. A set that contains no element is called an **empty set** and it is denoted by $\varnothing$ or simply $\{\}$.

Note: Here is a list of sets that you already know but are mentioned here as a refresher.

$\mathbb{N}$ The set of natural numbers and zero, $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{Z}$ The set of integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
$\mathbb{Z}^+$ The set of positive integers, $\{1, 2, 3, \ldots\}$.
$\mathbb{Q}$ The set of rational numbers.
$\mathbb{Q}^+$ The set of positive rational numbers.
$\mathbb{R}$ The set of real numbers.
$\mathbb{R}^+$ The set of positive real numbers.
$\mathbb{C}$ The set of complex numbers.

Note: In many sources you may find a slight difference in the definition of these sets. Frequently we have

$\mathbb{N}$ The set of natural numbers, $\{1, 2, 3, \ldots\}$, while
$\mathbb{W}$ The set of natural numbers and zero, $\{0, 1, 2, 3, \ldots\}$.

Some sets can be defined using a rule:

$\mathbb{Q}$ (the set of rational numbers) can be defined as

$$\mathbb{Q} = \left\{ x \,\middle|\, x = \frac{a}{b}, a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

$\mathbb{Q}^+$ (the set of positive rational numbers) can be defined as

$$\mathbb{Q}^+ = \{ x \mid x \in \mathbb{Q}, x > 0 \}.$$

$\mathbb{C}$ (the set of complex numbers) can also be defined as

$$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R}, i^2 = -1 \}.$$

Some properties

1 No ordering is required for the elements of a set, thus $\{1, 2, 3, 4, 5, 6\}$ and $\{5, 1, 3, 2, 6, 4\}$ are the same set.

2 Each element of a set is listed only once; it is superfluous to list it again. Therefore, the set $\{1, 1, 2, 3, 4, 4, 5, 6\}$ is actually the set $\{1, 2, 3, 4, 5, 6\}$.

3 Two sets A and B are equal and we write $A = B$ if and only if they have the same elements.
For example, $\{1, 1, 2, 3\} = \{1, 2, 3\} = \{x | x \in \mathbb{Z}^+, x < 4\}$; or $A = B$, where $A = \{y | y = a + b, a, b \in \{1, 2, 3\}\}$ and $B = \{2, 3, 4, 5, 6\}$.

4 If there are exactly n distinct elements in a set A, where $n \in \mathbb{N}$, we say that A is a **finite** set and that n is the **cardinality** of A (the number of elements). Sometimes the number of elements is denoted by $|A|$ and sometimes as $n(A)$. If a set is not finite, then it is **infinite**.

For example, $A = \{1, 2, 3, 4, 5, 6\}$ is a finite set with $|A| = 6$, while $\mathbb{N}$ is an infinite set.

In many proofs, in this option or in other situations, when the statement is 'p if and only if q', denoted by p iff q, or $p \Leftrightarrow q$ then we need to prove that p implies q, **and** q implies p, i.e. $p \Rightarrow q$ and $q \Rightarrow p$. We will sometimes denote the situation by ($\Rightarrow$) and ($\Leftarrow$).

In proofs, we usually show that two sets are equal if elements from one set are also elements from the other set and vice versa. Thus, we write

$$(A = B) \Leftrightarrow ((\forall x \in A \Rightarrow x \in B) \text{ and } (\forall y \in B \Rightarrow y \in A)).$$

Here, we are borrowing a symbol that is used in logic to represent frequently used clauses such as 'for all elements from one set…', namely '$\forall$'. So, if we want to say, 'for every integer, x, $x^2 \geqslant 0$,' we write:

$$\forall x \in \mathbb{Z}, x^2 \geqslant 0.$$

Another quantifier that we may use in our discussion is the symbol for existence. So, if we want to say 'there is at least one element in A that is not in B,' then we write: $\exists x \in A$ such that $x \notin B$.

Example 1

List the elements of the following sets:

a) $A = \{x \in \mathbb{Z}^+ \mid -2 \leqslant x \leqslant 7\}$ b) $B = \{x \in \mathbb{Z} \mid x^2 < 16\}$

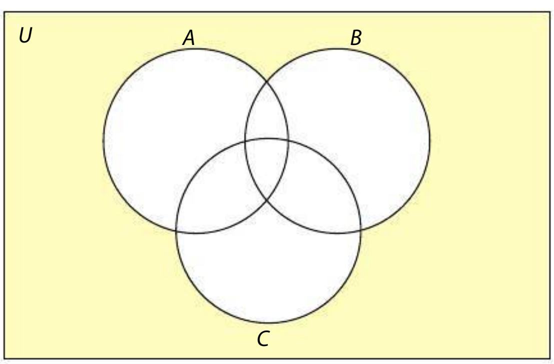
c) $C = \{x \in \mathbb{Q} \mid 3x^2 + 7x + 2 = 0\}$

Solution

a) $A = \{1, 2, 3, 4, 5, 6, 7\}$ b) $B = \{0, \pm 1, \pm 2, \pm 3\}$ c) $C = \left\{-2, -\frac{1}{3}\right\}$

1.2 Venn diagrams

Sets can also be represented graphically using Venn diagrams. In Venn diagrams the universal set U is usually represented by a rectangle. Inside this rectangle, circles (or other 'closed' curves) can be used to represent sets.



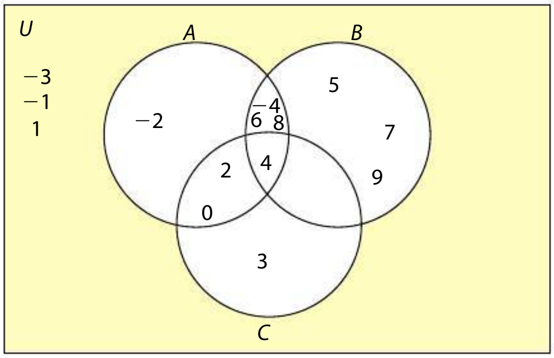
Venn diagrams are often used to indicate relationships between sets. We will show how a Venn diagram can be used in the following example.

Example 2

Given the universal set $U = \{x \in \mathbb{Z} \mid -4 \leqslant x \leqslant 9\}$, use a Venn diagram to show the following sets: $A = \{x \in U \mid x \text{ is even}\}$, $B = \{x \in U \mid |x| > 3\}$ and $C = \{x \in U \mid x^4 - 9x^3 + 26x^2 - 24x = 0\}$.

Solution

$A = \{-4, -2, 0, 2, 4, 6, 8\}$, $B = \{-4, 4, 5, 6, 7, 8, 9\}$, $C = \{0, 2, 3, 4\}$



Example 3

Write down the following sets in set-builder notation:

a) the set of all even integers

b) the set of all odd integers

c) the set of all integers divisible by 5

d) the set of all integers that have a remainder of 4 when divided by 7

e) the set of all integers that have a remainder of l when divided by a prime number p where $l < p$.

Solution

a) $A = \{2k \mid k \in \mathbb{Z}\}$

b) $B = \{2k - 1 \mid k \in \mathbb{Z}\}$

c) $C = \{5k \mid k \in \mathbb{Z}\}$

d) $D = \{7k + 4 \mid k \in \mathbb{Z}\}$

e) $E = \{pk + l \mid k \in \mathbb{Z}\}, 0 \leqslant l \leqslant p$

Example 4

Let M be the set $\{1, \{2, 3\}, 2, \varnothing\}$.

a) Find the number of elements of M.

b) Is $2 \in M$?

c) Is $3 \in M$?

d) Is $\{2, 3\} \in M$?

e) Is $\{\varnothing\} = \varnothing$?

Solution

a) 4

b) Yes.

c) No. $3 \in \{2, 3\}$ which is a member of M itself.

d) Yes.

e) No. $\{\varnothing\}$ is a set that contains the empty set as its only element, so it is not empty!

1.3 Subset

Definition 1

A set A is a **subset** of a set B, and we write $A \subseteq B$, if and only if every element of A is also an element of B. That means that the set A could be equal to the set B as well.

Formally, this means that for every x, if $x \in A$, then $x \in B$, or symbolically

$$A \subseteq B \Leftrightarrow \text{for every } x \in A \Rightarrow x \in B$$

In many cases, we can abbreviate 'for every …' by using the 'universal quantifier $\forall$' instead. So for the subset definition, we would restate it as:

$$A \subseteq B \Leftrightarrow \forall x \in A \Rightarrow x \in B$$

If $A \subseteq B$, but $A \neq B$, then A is called a **proper subset** of B and we write $A \subset B$.

When $A \subseteq B$, it is also common to say 'A is contained in B', or 'B is a **superset** of A', and we write $B \supseteq A$.

From the above definition, we can develop a method for showing that a set A is not a subset of a set B by observing that if $A \not\subset B$, then there is at least one $x \in A$ which is not in B. Notice here that if A is not a proper subset of B, it obviously cannot be a subset of B.

All the following statements are true.

✓ $\{x, y\} \subseteq \{x, y, z\}$

✓ $\{x, y\} \subset \{x, y, z\}$

✓ $\{x, y\} \subseteq \{x, \{x, y\}, y, z\}$

✓ $\{x, y\} \in \{x, \{x, y\}, y, z\}$

✓ $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Theorem 1

For any set A, $A \subseteq U$, $A \subseteq A$, and $\varnothing \subseteq A$.

Proof

- Since U is the universal set, it contains all elements, and hence it contains all elements that are in A.
- If $x \in A$ then $x \in A$, so $A \subseteq A$.
- The proof that $\varnothing \subseteq A$ can be done by contradiction. $\varnothing \subseteq A$ is a statement that is either true or false. Suppose it is false, that is, $\varnothing \not\subset A$, this means that not every $x \in \varnothing$ implies that $x \in A$, i.e. we can find some $x \in \varnothing$ such that $x \notin A$. This cannot be true because there is no $x \in \varnothing$ in the first place. So, our assumption that $\varnothing \not\subset A$ leads to a contradiction and hence cannot be true. Therefore, it has to be false, and $\varnothing \subseteq A$.

Equal sets revisited

With the definition of a subset, we can develop a new way of looking at equal sets.

By definition, A and B are equal if they have the same elements, i.e. every element of A is an element of B and every element of B is an element of A. Thus, we can now say

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, or equivalently in symbolic form

$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Please notice here that the statement above makes two claims:

$(\Rightarrow)$ If $A = B$, $\Rightarrow$ $A \subseteq B$ and $B \subseteq A$.

$(\Leftarrow)$ If $A \subseteq B$ and $B \subseteq A$, $\Rightarrow$ $A = B$.

Each of the following statements is true.

✓ $\{\varnothing\} \in \{\{\varnothing\}\}$

✓ $\varnothing \subseteq \{\{\varnothing\}\}$

✓ $\{\varnothing\} \not\subset \{\{\varnothing\}\}$

✓ $\{x\} \in \{\{x\}, y, z\}$

✓ $\{x\} \subset \{x, y, z\}$

✓ $\{x\} \not\subset \{\{x\}, y, z\}$

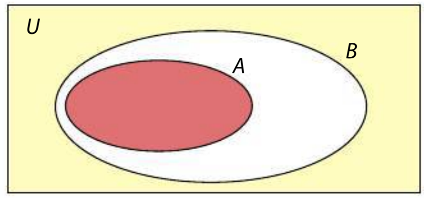
✓ $\varnothing \subseteq \{a, b, \varnothing\}$

✓ $\varnothing \in \{a, b, \varnothing\}$

✓ $\{\varnothing\} \notin \{a, b, \varnothing\}$

Venn diagrams for subsets

You can use Venn diagrams to show that one set is a subset of the other. Since, by definition, $A \subseteq B$ implies that every element of A is also an element of B, thus it is obvious that the Venn diagram for A is a part of the diagram for B.



Note: This diagram helps us understand the logic behind 'proof by using contra-positive' argument.

If A represents a proposition and B another one, then we can say that $A \Rightarrow B$; this is so because every element of A is automatically inside B. The contra-positive means that $\neg B \Rightarrow \neg A$. That is, if an element is not in B, it obviously cannot be in A.

$\neg$ is a negation symbol.
'$\neg$' is read as 'not'.

1.4 The power set

Definition 2

The **power set** of a set A, denoted as $\mathcal{P}(A)$, is the set of all subsets of A. Symbolically, this is written as

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}.$$

Example 5

Find the power set of $A = \{1, 2, 3\}$.

Solution

$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

Note: Notice here that $|A| = 3$ and $|\mathcal{P}(A)| = 8 = 2^3$. This is a surprising but true result.

Theorem 2

Let A be a set with n elements, $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof

In order to find $|\mathcal{P}(A)|$, we need to know how many subsets A has. Other than $\varnothing$ and A itself, the subsets of A have 1, 2, 3, …, or $n - 1$, elements each.

Recall from Chapter 4 of the textbook, that the number of subsets of size r that a set has, also known as combination of r elements out of n elements, is the binomial coefficient $\binom{n}{r}$. Thus,

$|\mathcal{P}(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$, where $\binom{n}{0}$ is the number of subsets with zero elements, i.e. $\varnothing$, and $\binom{n}{n}$ is the number of subsets with n elements, i.e. A.

However, applying the binomial theorem, we know that

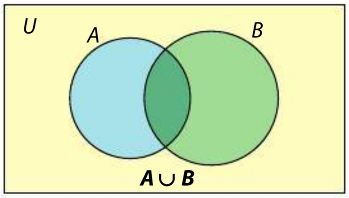
$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=(1+1)^n, \text{ and therefore } |\mathcal{P}(A)|=2^n.$$

1.5 Operations on sets

Union and intersection

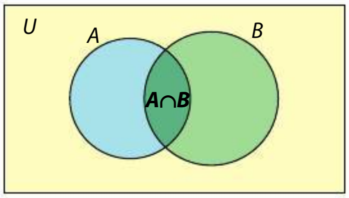
- If A and B are two sets of a universal set U, then **union** of A and B, written as $\boldsymbol{A \cup B}$, is the set of elements that belong to **$\boldsymbol{A}$, or $\boldsymbol{B}$, or *both*.** Symbolically, this is written as

  $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$.



- If A and B are two sets of a universal set U, then **intersection** of A and B, written as $\boldsymbol{A \cap B}$, is the set of elements that belong to ***both A and B.*** Symbolically, this is written as

  $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$.



For example, if $A = \{x, y, z\}$ and $B = \{m, x, n, y\}$, then

$A \cup B = \{m, n, x, y, z\}$ and $A \cap B = \{x, y\}$.

Also,

$A \cup \varnothing = A$

$A \cup U = U$

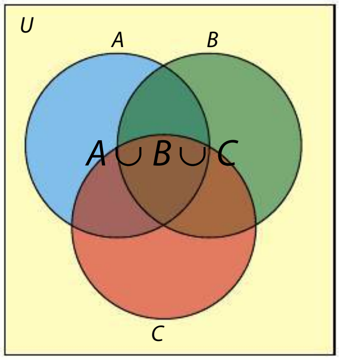
$A \cap U = A$

$A \cap \varnothing = \varnothing$.

The proof of each of the above is left as an exercise for you.

If $A \cap B = \varnothing$, then A and B are said to be **disjoint** sets.

Some properties of union and intersection



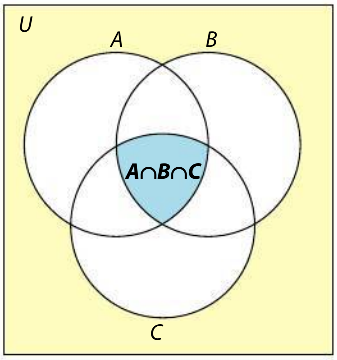
- Union of sets is associative.

  $A \cup (B \cup C) = (A \cup B) \cup C$

  Sometimes we write only $A \cup B \cup C$ as there is no need for parenthesis.

- Intersection of sets is associative.

  $A \cap (B \cap C) = (A \cap B) \cap C$

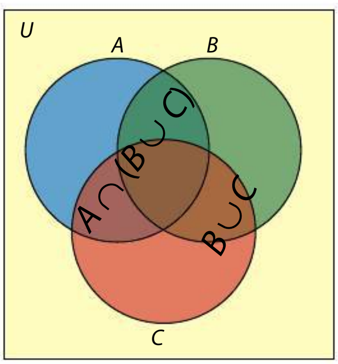


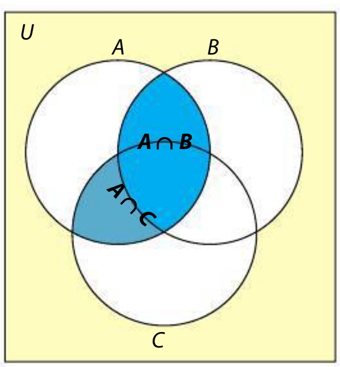
  Also here sometimes we write only $A \cap B \cap C$ as there is no need for parenthesis.

- Sometimes the union and intersection of sets can be utilized by several sets. It is helpful for you to get acquainted with two notations:
  - The union of n sets $A_1, A_2, A_3, \ldots, A_n$ can be written as $\bigcup_{i=1}^{n} A_i$.
  - The intersection of n sets $A_1, A_2, A_3, \ldots, A_n$ can be written as $\bigcap_{i=1}^{n} A_i$.

Distributive properties

- **Intersection is distributive over union.**





  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  Venn diagrams are helpful tools in understanding some set properties, but they are not proofs. For a property like this one, a formal proof is required and presented overleaf.

To show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we need to show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

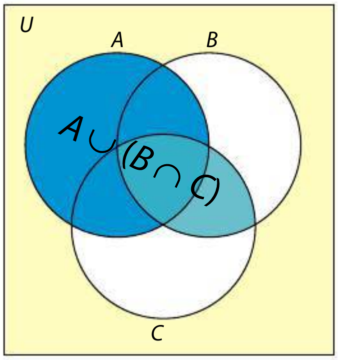
For all $x \in A \cap (B \cup C)$, $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, then $x \in B$ or $x \in C$. Now, if $x \in B$, then $x \in A \cap B$, or, if $x \in C$, then $x \in A \cap C$. Thus we have shown that $x \in A \cap B$ or $x \in A \cap C$. This by definition means that $x \in (A \cap B) \cup (A \cap C)$. This completes the first part of the proof.

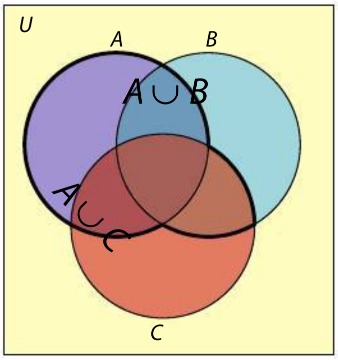
Now for every $x \in (A \cap B) \cup (A \cap C)$, $x \in (A \cap B)$ or $x \in (A \cap C)$. This means that $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. In both cases, x is an element of A and an element of either B or C, thus an element of $B \cup C$. Therefore, x belongs to both A and $B \cup C$, i.e. it belongs to $A \cap (B \cup C)$.

This completes the proof.

- Union is distributive over intersection.

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$





To show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, we need to show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

For all $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. Since $x \in B \cap C$, then $x \in B$ and $x \in C$. Now, if $x \in B$, then $x \in A \cup B$, and, if $x \in C$, then $x \in A \cup C$. Thus we have shown that $x \in A \cup B$ and $x \in A \cup C$. This by definition means that $x \in (A \cup B) \cap (A \cup C)$. This completes the first part of the proof.

Now for every $x \in (A \cup B) \cap (A \cup C)$, $x \in (A \cup B)$ and $x \in (A \cup C)$. This means that $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$. In both cases, if x is an element of A then it is an element of the union of A with any set, including $B \cap C$; and if x is not an element of A, then it must be an element of B and C, thus an element of $B \cap C$. Therefore, x belongs to A or $B \cap C$, i.e. it belongs to $A \cup (B \cap C)$.

This completes the proof.

- Union and intersection of sets are commutative operations.
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$

Example 6

Given that $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{3, 6, 9, 12\}$ and $C = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ find the following sets:

a) $A \cup B$ b) $C \cap (A \cup B)$ c) $C \cup (A \cap B)$

Solution

a) $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12\}$

b) $C \cap (A \cup B) = \{2, 3\}$

Notice here that
$C \cap A = \{2\}, C \cap B = \{3\} \Rightarrow (C \cap A) \cup (C \cap B) = \{2, 3\}$.

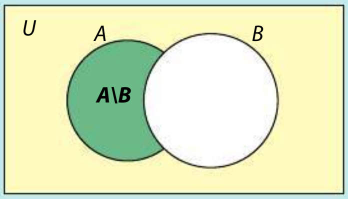
c) $C \cup (A \cap B) = \{2, 3, 5, 6, 7, 11, 12, 13, 17, 19, 23\}$ and
$C \cup B = \{2, 3, 5, 6, 7, 9, 11, 12, 13, 17, 19, 23\}$
$\Rightarrow (C \cup A) \cap (C \cup B) = \{2, 3, 5, 6, 7, 11, 12, 13, 17, 19, 23\}$.

1.6 Set differences

Definition 3

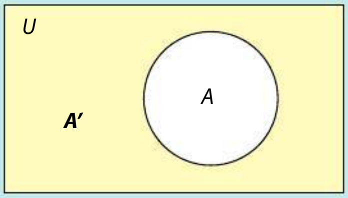
For any two sets *A* and *B*, the **difference** between set *A* and set *B*, denoted by ***A**B*** is the set of elements of *A* which are not in *B*. Symbolically,

$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}$.



For any set *A*, the **complement** of *A*, denoted by ***A′***, is the set of all elements in the universal set that are not in *A*.

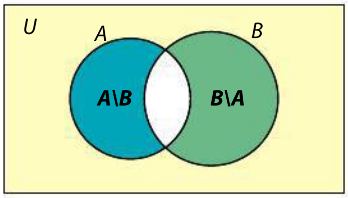
$A' = \{x \mid x \in U \text{ and } x \notin A\}$



From the definitions left, it becomes obvious that $A \cap A' = \phi$, or $A \cup A' = U$.

Note: If we start with the definition of difference, then the complement can be understood as $A' = U \backslash A$, and if we start with the definition of complement then the difference can be understood as $A \backslash B = A \cap B'$.

Symmetric difference



The **symmetric difference** of two sets A and B, denoted by $A \Delta B$, is the set of all elements **in A or in B but *not* in both.**

There are several ways of interpreting this difference:

$A \Delta B = \{x | x \in (A \cup B) \text{ and } x \notin (A \cap B)\}$

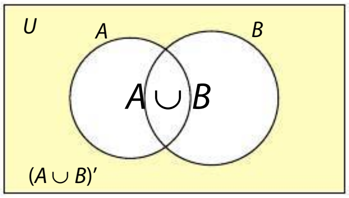
$A \Delta B = (A \cup B) \backslash (A \cap B)$

$A \Delta B = (A \backslash B) \cup (B \backslash A)$

De Morgan's laws

For any two sets A and B, the following two statements are true:

- $(A \cup B)' = A' \cap B'$, and
- $(A \cap B)' = A' \cup B'$.



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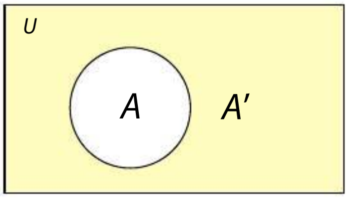
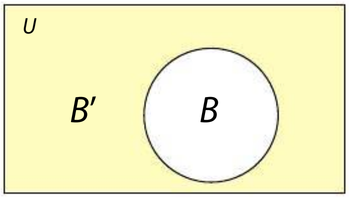
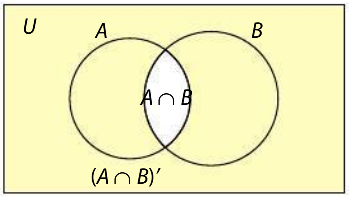


$\cap$



and



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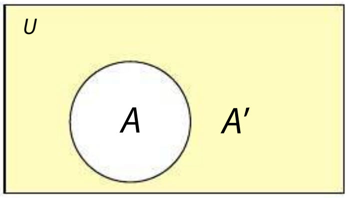
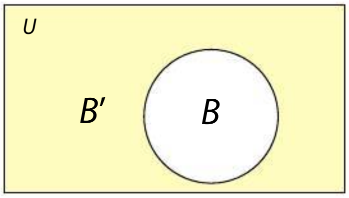


$\cup$



Proof (Optional – not required by IBO)

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$

(because if $x \in A$ then $x \in A \cup B$ which cannot be true here; similarly for B)

$\Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B'$, and thus

$(A \cup B)' \subseteq A' \cap B'$.

Also,

$x \in A' \cap B' \Rightarrow x \notin A$ and $x \notin B \Rightarrow x \notin (A \cup B)$

(because if $x \in A$ then $x \notin A'$, or if $x \in B$ then $x \notin B'$, which cannot be true here)

$\Rightarrow x \in (A \cup B)'$, and thus

$A' \cap B' \subseteq (A \cup B)'$.

This completes the proof.

The proof of the second part of De Morgan's rule is left as an exercise for you.

1.7 Summary of set properties

(Proofs of some of these properties may have been presented before, are obvious, or left as an exercise.)

1 Commutativity of union and intersection

$A \cup B = B \cup A;\ A \cap B = B \cap A$

2 Associativity of union and intersection

$(A \cup B) \cup C = A \cup (B \cup C);\ (A \cap B) \cap C = A \cap (B \cap C)$

3 Distributive properties

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4 Special cases

$A \cup \varnothing = A;\ A \cap \varnothing = \varnothing$

$A \cap U = A;\ A \cap U = U$

$A \cup A = A;\ A \cap A = A$

$A \cup (A \cap B) = A;\ A \cap (A \cup B) = A$

$(A')' = A;\ A \cap A' = \varnothing;\ A \cup A' = U$

$U' = \varnothing;\ \varnothing' = U$

Example 7

Simplify the following expressions:

a) $(A \cap B') \cap (A' \cap B)$

b) $(A \cup B') \cup (B \cup C') \cup (C \cup A')$

c) $A \cap (A' \cup B)$

d) $(A' \cup A)' \cup (A' \cup B)' \cap (A' \cup C)'$

Solution

a) $(A \cap B') \cap (A' \cap B) = A \cap B' \cap A' \cap B = A \cap A' \cap B' \cap B$
$= (A \cap A') \cap (B' \cap B) = \varnothing \cap \varnothing = \varnothing$

b) $(A \cup B') \cup (B \cup C') \cup (C \cup A') = A \cup B' \cup B \cup C' \cup C \cup A'$
$= A \cup A' \cup B \cup B' \cup C \cup C'$
$= (A \cup A') \cup (B \cup B') \cup (C \cup C')$
$= U \cup U \cup U = U$

c) $A \cap (A' \cup B) = (A \cap A') \cup (A \cap B) = \varnothing \cup (A \cap B) = A \cap B$

d) $(A' \cup A)' \cup (A' \cup B)' \cap (A' \cup C)' = U' \cup (A \cap B') \cap (A \cap C')$
$= \varnothing \cup (A \cap B' \cap A \cap C') = A \cap B' \cap C'$
$= A \cap (B \cup C)'$

Example 8

De Morgan's laws work for three or more sets. Show the following formulae to be true:

a) $(A \cup B \cup C)' = A' \cap B' \cap C'$

b) $(A \cap B \cap C)' = A' \cup B' \cup C'$

c) $\left(\bigcup_{i=1}^{n} A_i\right)' = \bigcap_{i=1}^{n} A_i',\ n \in \mathbb{Z}^+$

d) $\left(\bigcap_{i=1}^{n} A_i\right)' = \bigcup_{i=1}^{n} A_i',\ n \in \mathbb{Z}^+$

Solution

a) $(A \cup B \cup C)' = ((A \cup B) \cup C)' = (A \cup B)' \cap C' = A' \cap B' \cap C'$

b) $(A \cap B \cap C)' = ((A \cap B) \cap C)' = (A \cap B)' \cup C' = A' \cup B' \cup C'$

c) To prove this formula we need to use the method of mathematical induction.

(i) Basis step:
$n = 1 \Rightarrow A_1' = A_1'$

(ii) Inductive step:
We assume that the formula to be true for $n = k$, i.e.

$$\left(\bigcup_{i=1}^{k} A_i\right)' = \bigcap_{i=1}^{k} A_i'.$$

Now, we need show that the formula is true for $n = k + 1$.

$$\left(\bigcup_{i=1}^{k+1} A_i\right)' = \left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)' = \left(\bigcup_{i=1}^{k} A_i\right)' \cap A_{k+1}'$$

$$= \left(\bigcap_{i=1}^{k} A_i'\right) \cap A_{k+1}' = \bigcap_{i=1}^{k+1} A_i$$

(iii) Conclusion:
The formula is true for $n = 1$ and from the assumption that it is true for $n = k$ we have shown that it is true for $n = k + 1$. Therefore, we can deduce that the formula is for all $n \in \mathbb{Z}^+$.

d) In a similar manner to c), the proof is straightforward and is left for you to practise.

Example 9

Given the sets A, B and C show the following identities:

a) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$

b) $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$

c) $(A \backslash B) \backslash C = A \backslash (B \cup C)$

Solution

a) $A\backslash(B \cup C) = A \cap (B \cup C)' = A \cap B' \cap C'$

$= (A \cap B') \cap (A \cap C') = (A\backslash B) \cap (A\backslash C)$

b) $(A \cap B)\backslash C = (A \cap B) \cap C'$

$= (A \cap C') \cap (B \cap C') = (A\backslash C) \cap (B\backslash C)$

c) $(A\backslash B)\backslash C = (A \cap B') \cap C' = A \cap (B' \cap C')$

$= A \cap (B \cup C)' = A\backslash(B \cup C)$

Exercise 1

1 Determine which sets are equal.

a $A = \{3, 6, 7\}$, $B = \{6, 7, 3\}$ **b** $A = \{x \in \mathbb{Z} \mid x^2 = 8\}$, $B = \{y \in \mathbb{Z}^+ \mid y = 2\sqrt{2}\}$

c $A = \{2\}$, $B = \{x \in \mathbb{N} \mid x^2 = 4\}$ **d** $A = \{-2, \varnothing, 2\}$, $B = \{x \in \mathbb{Z} \mid x^2 = 4\}$

2 $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{1, 4, 5\}$. Find

a $A \cap (B \cup C)$ **b** $(A \cap B) \cup (A \cap C)$

c $(A \cup B)'$ **d** $A' \cup B'$

e $A' \cap B'$ **f** $A\backslash(B \cap C)$

g $A \Delta B$

3 Determine whether each of the following statements is true. Justify your response by stating a property/theorem that supports your argument.

a $\sqrt{2} \in \mathbb{Q}$ **b** $(\sqrt{-1})^2 \in \mathbb{Z}$

c $\{2\} \subseteq \{2\}$ **d** $\{a\} \subset \{a, b\}, a \neq b$

e $\mathbb{Z}^+ \subset \mathbb{Q}$ **f** $\{3, a, b, c\} = \{3, a, b, 3, c, b\}$

g $\{a, e\} \cup \{e, f\} \cup \{g, h\} = \{a, e, f, g, h\}$

h Let $a, b \in \mathbb{R}$, and $a < b$, then $[a, b] \cap \{a, b\} = \{a\} \cup \{b\}$.

i Let $a, b \in \mathbb{R}$, and $a < b$, $[a, b]\backslash]a, b[= \{a, b\}$.

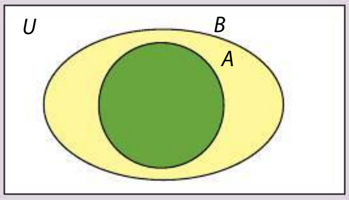
4 Let $A = \{a, \{2, a\}, \{4\}, \{\{2, 4\}\}, 4\}$. Determine which of the statements below are true and which are false.

a $a \in A$ **b** $\{a\} \notin A$ **c** $\{2, a\} \subseteq A$

d $\{\{4\}, 4\} \subseteq A$ **e** $\{2, 4\} \in A$ **f** $\{\{2, 4\}\} \subseteq A$

g $\{\{2, a\}\} \subseteq A$ **h** $\{2, a\} \notin A$ **i** $\varnothing \subseteq A$

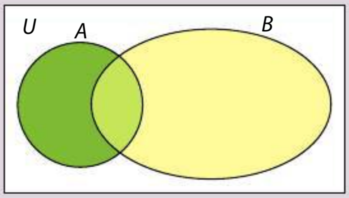
5 For each question part, copy the Venn diagram and shade the required region.



a $A \cap B$ **b** $A \cup B$ **c** $(A \cup B) \backslash (A \cap B)$

d $(A \cap B)'$ **e** $A \cap B'$ **f** $A' \cup B$

6 For each question part, copy the Venn diagram and shade the required region.

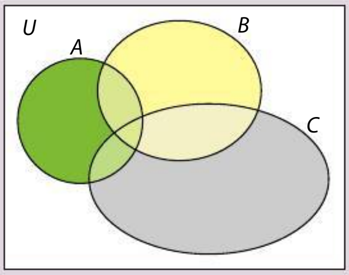


a $A \cap B$ **b** $A \cup B$ **c** $(A \cup B) \backslash (A \cap B)$

d $(A \cap B)'$ **e** $A \cap B'$ **f** $A' \cup B$

g $A \Delta B$

7 Three sets *A*, *B* and *C* are given. For each question part, copy the Venn diagram and shade the required region.

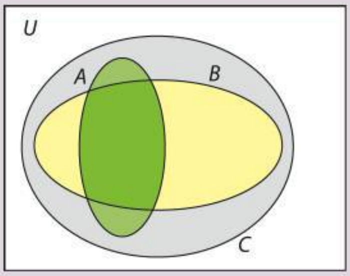


a $A \cap B'$ **b** $C' \cap B'$ **c** $B \cup (C \backslash A)$

d $(A \cup B)' \backslash C$ **e** $(A \cup B)' \backslash C$ **f** $(A \cap B)' \backslash C$

g $(A \cup B) \cap C'$

8 Three sets *A*, *B* and *C* are given. For each question part, copy the Venn diagram and shade the required region.



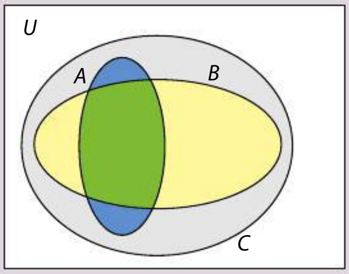
a $A \cap B'$ **b** $C' \cap B'$ **c** $B \cup (C \backslash A)$ **d** $(A \cup B') \backslash C$

e $(A \cup B)' \backslash C$ **f** $(A \cap B)' \backslash C$ **g** $(A \cup B) \cap C'$

9 Let $A = \{a \mid a \in \mathbb{Z} \text{ and } a^4 - a^2 = 0\}$ and $B = \{b \mid b \in \mathbb{Z}^+ \text{ and } b = a^2\}$. Find

a $A \backslash B$ **b** $B \backslash A$ **c** $A \cap B$ **d** $\mathcal{P}(A)$

10 Write an expression that describes the region shaded in blue.



11 In a class, 84 students are preparing for their IB exams. 56 study maths at HL, 60 study English at HL, and 10 do not study either of these two courses. How many students study both maths HL and English HL?

12 A and B are subsets of U.

$n(U) = 30$, $n(A \cup B) = 21$, $n(A \backslash B) = 10$, $n(B \backslash A) = 5$.

Find $n(B \cap A)'$.

13 We define $M_r \subseteq \mathbb{Z}^+$ for every $r \in \mathbb{N}$ by: $M_r = \{x \in \mathbb{Z}^+ \mid r \mid x\}$.

List the elements of each of the following sets.

a M_1 **b** M_2'

c $M_2 \cap M_3$ **d** $M_6 \backslash M_3$

14 What can you conclude if $A \cap B = A \cup B$? Justify your response.

15 Prove each of the following (all sets are subsets of a universal set U):

a $(P \cup Q) \backslash (P \cap R) = P \cap (Q \backslash R)$ **b** $(P \cup Q) \backslash (P \cap Q) = (P \backslash Q) \cup (Q \backslash P)$

c $M \times (N \cup P) = (M \times N) \cup (M \times P)$ **d** $(A' \cup B)' \cup (A \cap B) = A$

e $(A' \cup B) \cap (A \cup B) = B$ **f** $A \cup (B \cap A')' = (A' \cap B)'$

g $P \,\Delta\, Q = (P \cup Q) \cap (P \cap Q)'$ **h** $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$

i $[(A' \cup B) \cap (A \cup B')]' = (A \cap B)' \cap (A \cup B)$

j $(A' \cap B) \cup C' = (A \cap C)' \cap (B' \cap C)'$

k $[(A \cap B) \cup (A' \cap B')]' = (A \cup B) \cap (A' \cup B')$

l $(A \backslash B) \cap (B \backslash A) = (A \cup B) \backslash (A \cap B)$

16 A set A has n elements. A also has 21 subsets of size $(n - 2)$ each.

Find the number of subsets of A.

17 Prove each of the following (all sets are subsets of a universal set U):

a $A \cup B = A \Leftrightarrow B \subset A$ **b** $A \cap B = A \Leftrightarrow A \subset B$

c $A' \cup B = U \Leftrightarrow A \subset B$ **d** $A' \cap B = \varnothing \Leftrightarrow B \subset A$

e $A \subset B \Leftrightarrow B' \subset A'$

18 Let A and B be two non-empty subsets of a universal set U.

a Show that $A \subset B \Rightarrow \mathcal{P}(A) \subset \mathcal{P}(B)$.

b What is $(\varnothing)$? $\mathcal{P}(\mathcal{P}(\varnothing))$?

c What relation is there between $\mathcal{P}(A \cap B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$? Justify your response.

d What relation is there between $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A) \cup \mathcal{P}(B)$? Justify your response.

19 Find the following unions and intersections. Justify your work.

a $\bigcup\limits_{n\in\mathbb{N}} [n, n+1[$ **b** $\bigcap\limits_{n\in\mathbb{Z}^+} \left[-\frac{1}{n}, 0\right[$

c $\bigcup\limits_{n\in\mathbb{Z}^+} \left[\frac{1}{n}, 2+\frac{1}{n}\right[$ **d** $\bigcap\limits_{n\in\mathbb{Z}^+} \left[\frac{1}{n}, 2+\frac{1}{n}\right[$

20 If A and B are finite sets, determine whether $|A \cup B| = |A| + |B|$.

21 Prove each of the following, given that A, B and C are three non-empty sets of a universal set U.

a If $A \subseteq B$, then $A \cup C \subseteq B \cup C$. **b** If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

c $A \subseteq B$, iff $A \cap B = A$. **d** If $A \subseteq B$, then $B \backslash A \cup A = B$.

e $A \backslash B \subseteq A$ **f** $A \cup (B \backslash A) = A \cup B$

g $A \subseteq B' \Leftrightarrow A \cap B = \varnothing$ **h** $A \backslash B \subseteq B \Leftrightarrow A \subseteq B$

22 Let A and B be two sets. Consider the following conjectures and prove those that are true and give a counter example for each one that is not true.

a $(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ **b** $(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$

c $(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ **d** $(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

e $(A \cap B) \subseteq \mathcal{P}(A \cup B)$

Practice questions 1

1 $A - B$ is the set of all elements that belong to A but not to B.

a Use Venn diagrams to verify that $(A - B) \cup (B - A) = (A \cap B) - (A \cap B)$.

b Use De Morgan's laws to prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

2 Let A and B be two non-empty sets, and $A - B$ be the set of all elements of A which are not in B. Draw Venn diagrams for $A - B$ and $B - A$ and determine if $B \cap (A - B) = B \cap (B - A)$.

3 Let X be a set containing n elements (where n is a positive integer). Show that the set of all subsets of X contains 2^n elements.

4 a Use a Venn diagram to show that $(A \cup B)' = A' \cap B'$.

b Prove that $[(A' \cup B) \cap (A \cup B')]' = (A \cap B)' \cap (A \cup B)$.

5 The difference, $A - B$, of two sets A and B is defined as the set of all elements of A which do not belong to B.

a Show by means of a Venn diagram that $A - B = A \cap B'$.

b Using set algebra, prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

6 Use Venn diagrams to show that

a $A \cup (B \cap A')' = A \cup B'$

b $((A \cap B)' \cup B)' = \varnothing$.

7 Let A and B be subsets of the set U and let $C = A \cap B$, $D = A' \cup B$ and $E = A \cup B$.

a Draw separate Venn diagrams to represent the sets C, D and E.

b Using De Morgan's laws, show that $A = D' \cup C$.

c Prove that $B = D \cap E$.

8 Prove for sets A, B and C that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

9 For each $n \in \mathbb{Z}^+$, a subset of $\mathbb{Z}^+$ is defined by $S_n = \{x \in \mathbb{Z}^+ \mid n \text{ divides } x\}$.

a Express in simplest terms the membership of the following sets:

i S_1 **ii** S_2'

iii $S_2 \cap S_3$ **iv** $S_6 \backslash S_3$

b Prove that $(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B)$.

10 Prove that $(A \cup B) \backslash (A \cap C) = A \cap (B \backslash C)$ where A, B and C are three subsets of the universal set U.

Questions 1–10 © International Baccalaureate Organization

2 Relations and Functions

Please note:
The syllabus removed matrix examples from this option. Hence, they will not appear on exam papers. However, we will still use matrices in this book as examples to deepen your understanding of several concepts. Some questions (from old exam papers) may still contain matrices. These questions can be omitted if your teacher chooses to do so.

2.1 Relations

The Cartesian product

Definition 1

Let A and B be two subsets of U. The **Cartesian product** of A and B, denoted as $A \times B$, is defined by

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}.$$

From the definition above, we can interpret the Cartesian product as the set of all *ordered* pairs whose first component is a member of A and second component is a member of B.

Example 1

Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Find $A \times B$, $B \times A$, and $A \times A$.

Solution

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

Notice here that $A \times B \neq B \times A$.

$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$

Often, we are interested in the Cartesian product of a set with itself (as in the last question in Example 1) $A \times A$, which will be denoted by A^2. In general, we use A^n to include all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of members of set A.

Note: You may have seen by now that the Cartesian plane you use in graphing is called $\mathbb{R}^2$ since it is a Cartesian product of $\mathbb{R}$ with itself:

$$\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$$

The 3D space coordinate system is also known as $\mathbb{R}^3$:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

Example 2

A, B and C are subsets of U. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Solution

Since this is a Cartesian product, then elements of $A \times (B \cup C)$ are of the form (x, y).

Let $(x, y) \in A \times (B \cup C)$, then

$$x \in A \text{ and } y \in (B \cup C) \Rightarrow y \in B \text{ or } y \in C.$$

We know that $x \in A$ regardless of y, so, when $y \in B$, then we have $x \in A$ and $y \in B$, i.e. $(x, y) \in (A \times B)$; or when $y \in C$, then we have $x \in A$ and $y \in C$, i.e. $(x, y) \in (A \times C)$.

Thus, $(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$, and hence $(x, y) \in ((A \times B) \cup (A \times C))$. This proves that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Let $(x, y) \in ((A \times B) \cup (A \times C))$, then

$$(x, y) \in (A \times B) \text{ or } (x, y) \in (A \times C); \text{ hence,}$$

when $(x, y) \in (A \times B)$ then $x \in A$ and $y \in B$, or when $(x, y) \in (A \times C)$ then $x \in A$ and $y \in C$.

This in turn means that $x \in A$ and $y \in B$ or $y \in C$, and hence $y \in (B \cup C)$, thus

$$(x, y) \in A \times (B \cup C) \text{ and hence } (A \times B) \cup (A \times C) \subseteq A \times (B \cup C).$$

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Relations

If A and B are sets, as we defined earlier, the Cartesian product of A and B is the set

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

There are occasions when we are interested in only a part of $A \times B$. Take, for example, the set A to be the set of last year's HL maths students at your school,

$$A = \{\text{Marco, Roberto, Franz, George, Jin, Mara, } \ldots\},$$

and B the set of natural numbers $\mathbb{N}$. We may be interested in the scores that these students have on their IB exam, so we are interested in

$$\mathcal{R} = \{(x, y) \mid x \in A, y \in B, \text{ student } x \text{ has score } y\}.$$

For example, (Roberto, 7), (Franz, 3) and (Mara, 5) are elements of $\mathcal{R}$.

Generally, a relation is defined by a rule or description rather than by listing its ordered pairs.

Definition 2

Given two sets M and N, a relation $\mathcal{R}$ from M to N is a subset of $M \times N$.

In some sources, M is called the **domain** of the relation and N is the **range**.

Sometimes $\mathcal{R}$ is called a binary relation. Also, if we are given n sets $M_1, M_2, \ldots, M_n$, then an n-ary relation on $M_1 \times M_2 \times \ldots \times M_n$ is a subset of the Cartesian product $M_1 \times M_2 \times \ldots \times M_n$. If $M = N$ then $\mathcal{R}$ is a relation on set M and of course is a subset of $M \times M$.

Notation

There are several ways of writing a relation, two of which we state here.

- If $\mathcal{R}$ is a relation, then the following are equivalent descriptions: $(x, y) \in \mathcal{R} \leftrightarrow x\mathcal{R}y$.

  Let $A = \{3, 4, 5\}$ and $B = \{2, 4, 6\}$. Let $\mathcal{R}$, a relation from A to B, be defined by the rule:

  $x\mathcal{R}y \leftrightarrow x + y$ is a multiple of 3.

  We can write $3\mathcal{R}6$, or equivalently $(3, 6) \in \mathcal{R}$; $4\mathcal{R}2$, or equivalently $(4, 2) \in \mathcal{R}$, but we *cannot* write $(5, 2)$, $4\mathcal{R}6$, $(4\not\mathcal{R}6)$, etc.

- Let $\mathcal{T} = \left\{(x, y) \mid x, y \in \mathbb{Z}^+, \dfrac{x}{y} \in \mathbb{Z}^+\right\}$. This is a relation from $\mathbb{Z}^+$ to $\mathbb{Z}^+$. This can also be written as $x\mathcal{T}y$. So $15\mathcal{T}3$, but $3\not\mathcal{T}15$.

Equivalence relations

Definition 3

A relation $\mathcal{R}$ on a set M is **reflexive** if and only if $(x, x) \in \mathcal{R}$, or equivalently $x\mathcal{R}x$ for all $x \in M$.

Our major goal in this part is to discover particular properties of relations on a set. Thus, all the work in this part will involve subsets of $M \times M$ for some set M.

- $\mathcal{S} = \{(a, b) \in \mathbb{N}^2 \mid ab \geqslant 0\}$ is a reflexive relation on $\mathbb{N}$ since $aa = a^2 \geqslant 0$ for any number $a \in \mathbb{N}$.
- $\mathcal{W} = \left\{(x, y) \mid x, y \in (\mathbb{Z}\setminus\{0\}), \text{ and } \dfrac{x}{y} \in \mathbb{Z}\right\}$ is a reflexive relation since $\dfrac{x}{x} = 1 \in \mathbb{Z}$ for any non-zero integer x.
- $\mathcal{F} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x - y > 2\}$ is not reflexive since $x - x = 0 \not> 2$.

Definition 4

A relation $\mathcal{R}$ on a set M is **symmetric** if and only if for all $x, y \in M$,

$(x, y) \in \mathcal{R} \Longrightarrow (y, x) \in \mathcal{R}$,

or equivalently

$x\mathcal{R}y \Rightarrow y\mathcal{R}x$ for all $x, y \in M$.

- $\mathcal{S} = \{(a, b) \in \mathbb{N}^2 \mid ab \geqslant 0\}$ is symmetric since $ab \geqslant 0 \Leftrightarrow a \geqslant 0$ and $b \geqslant 0$, or $a \leqslant 0$ and $b \leqslant 0 \Leftrightarrow ba \geqslant 0$, i.e. $a\mathcal{S}b \Rightarrow b\mathcal{S}a$, or $(a, b) \in \mathcal{S} \Rightarrow (b, a) \in \mathcal{S}$.
- $\mathcal{P} = \{(a, b) \in \mathbb{R}^2 \mid a - b = 0\}$ is symmetric since $a - b = 0 \Rightarrow b - a = 0$, i.e. $(a, b) \in \mathcal{P} \Rightarrow (b, a) \in \mathcal{P}$ or $a\mathcal{P}b \Rightarrow b\mathcal{P}a$.

- $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$ is symmetric since addition over the set of real numbers is commutative, then $x^2 + y^2 = 4 \Rightarrow y^2 + x^2 = 4$ which implies that $x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
- $\rho = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 4\}$ is not symmetric since $x\rho y \Rightarrow x^2 - y^2 = 4 \Rightarrow y^2 - x^2 = -4 \not\Rightarrow y\rho x$; equivalently we may also write '$\Rightarrow y\not\rho x$'.

Example 3

A relation $\mathcal{P}$ on a set $M = \{0, 1, 2, 3, 4\}$ is given below. Determine whether it is reflexive or symmetric.

$$\mathcal{P} = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 1), (2, 2), (2, 3), (3, 3), (4, 3), (4, 4)\}$$

Solution

$\mathcal{P}$ is reflexive since for every element x in M, $x\mathcal{P}x$ – $(0, 0), (1, 1)$, etc.

$\mathcal{P}$ is not symmetric since there is at least one case where $x\mathcal{P}y$ but $y\not\mathcal{P}x$ – $(2, 3) \in \mathcal{P}$ but $(3, 2) \notin \mathcal{P}$.

Definition 5

A relation $\mathcal{R}$ on a set M is **antisymmetric** if and only if for all $x, y \in M$,

$(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R} \Rightarrow x = y$,

or equivalently,

for all $x, y \in M$, $x\mathcal{R}y$ and $y\mathcal{R}x \Rightarrow x = y$.

A relation $\rho = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant y\}$ is antisymmetric since $(x, y) \in \rho \Rightarrow x \geqslant y$ and $(y, x) \in \rho \Rightarrow y \geqslant x$, which can only be true if $x = y$.

Example 4

Is the relation $\mathcal{P}$ in Example 3 antisymmetric?

Solution

We have $(0, 1) \in \mathcal{P}$ and $(1, 0) \in \mathcal{P}$, but obviously $0 \neq 1$, so the relation is not antisymmetric.

Notice here that this relation is not symmetric and is not antisymmetric. This is to show that antisymmetric does not mean 'not symmetric'.

Definition 6

A relation $\mathcal{R}$ on a set M is **transitive** if and only if for all $x, y, z \in M$,

$(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$ or equivalently,

for all $x, y, z \in M$, $x\mathcal{R}y$ and $y\mathcal{R}z \Rightarrow x\mathcal{R}z$.

- A relation $\mathcal{T} = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant y\}$ is transitive since $(x, y) \in \mathcal{T} \Rightarrow x \geqslant y$ and $(y, z) \in \mathcal{T} \Rightarrow y \geqslant z$, which leads to the conclusion that $x \geqslant z$, i.e. $(x, z) \in \mathcal{T}$.

Example 5

M is the power set of a set A. Consider the following relation on this set:

$$\mathcal{S} = \{(X, Y) \mid X, Y \in M, \text{ and } X \subseteq Y\}.$$

Is $\mathcal{S}$ reflexive, symmetric, antisymmetric, or transitive?

Solution

Since $(X, X) \in \mathcal{S}$, i.e. $X \subseteq X$, then $\mathcal{S}$ is reflexive.

Since $(X, Y) \in \mathcal{S} \Rightarrow X \subseteq Y \not\Rightarrow Y \subseteq X$, then $\mathcal{S}$ is not symmetric.

Since $(X, Y) \in \mathcal{S}$ and $(Y, X) \in \mathcal{S} \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$, then $\mathcal{S}$ is antisymmetric.

Since $(X, Y) \in \mathcal{S}$ and $(Y, Z) \in \mathcal{S} \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z$, which means that $(X, Z) \in \mathcal{S}$, then $\mathcal{S}$ is transitive.

Example 6

Consider the relation $\mathcal{W} = \left\{(x, y) \mid x, y \in (\mathbb{Z} \setminus \{0\}), \text{and } \frac{x}{y} \in \mathbb{Z}\right\}$. Is this relation reflexive, symmetric, or transitive?

Solution

It has been shown on page 1236 that $\mathcal{W}$ is reflexive.

$(6, 3) \in \mathcal{W}$ because $\frac{6}{3} = 2 \in \mathbb{Z} \setminus \{0\}$, but $\frac{3}{6} = \frac{1}{2} \notin \mathbb{Z} \setminus \{0\} \Rightarrow (3, 6) \notin \mathcal{W}$, so $\mathcal{W}$ is not symmetric.

$(x, y) \in \mathcal{W}$ and $(y, z) \in \mathcal{W} \Rightarrow \frac{x}{y} = n$ and $\frac{y}{z} = m$, where m and n are non-negative integers, thus

$\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = nm$ is also a non-negative integer and hence $(x, z) \in \mathcal{W}$ and $\mathcal{W}$ is therefore transitive.

Example 7

Consider the relation $\mathcal{P}$ on a set $M = \{1, 2, 3, 4\}$ given below. Determine whether it is transitive.

$$\mathcal{P} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 3)\}$$

Solution

$\mathcal{P}$ is not transitive since $(2, 1)$ and $(1, 2)$ belong to $\mathcal{P}$ but $(2, 2)$ does not.

Definition 7

A relation $\mathcal{R}$ on a set M is called an **equivalence** relation if it is reflexive, symmetric and transitive.

Note: To prove a relation $\mathcal{R}$ is an equivalence relation, you will need to prove $\mathcal{R}$ to be

Reflexive: $x\mathcal{R}x$ for all $x \in M$.

Symmetric: for any $x, y \in M$, and if $x\mathcal{R}y$, then $y\mathcal{R}x$.

Transitive: for any $x, y, z \in M$, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

Consider the following relation over the set of integers, $\mathbb{Z}$:

$\mathcal{R} = \{(x, y) \in \mathbb{Z}^2 \mid x - y \text{ is a multiple of } 5\}$, i.e. $x\mathcal{R}y \Rightarrow x - y = 5k$ where $k \in \mathbb{Z}$.

$\mathcal{R}$ is reflexive since $x\mathcal{R}x \Rightarrow x - x = 0$, which is a multiple of 5.

$\mathcal{R}$ is symmetric since $x\mathcal{R}y \Rightarrow x - y = 5k \Rightarrow y - x = -5k$, which is also a multiple of 5.

$\mathcal{R}$ is transitive since $x\mathcal{R}y$ and $y\mathcal{R}z \Rightarrow x - y = 5k_1$ and $y - z = 5k_2$

$\Rightarrow x - z = (x - y) + (y - z) = 5(k_1 + k_2)$ is also a multiple of 5.

Therefore, $\mathcal{R}$ is an equivalence relation.

Example 8

Consider the relation $\mathcal{W} = \left\{(x, y) \mid x, y \in (\mathbb{Z}\setminus\{0\}), \text{and } \frac{x}{y} \in \mathbb{Z}\right\}$. Is $\mathcal{W}$ an equivalence relation?

Solution

We have shown above (Example 6) that $\mathcal{W}$ is reflexive and transitive, but not symmetric, and hence it is not an equivalence relation.

Example 9

Consider the set of triangles, $\mathcal{T}$, in a plane and define a relation, denoted by $\approx$, on this set by

$\approx = \left\{(X, Y) \in \mathcal{T}^2 \mid X \text{ is similar to } Y\right\}$. Is $\approx$ an equivalence relation?

Solution

To answer the question you need to recall the definition of similar triangles. One definition states that two triangles are similar if and only if their angles are congruent.

$X \approx X$ is obvious since the angles of a triangle are congruent to themselves.

If $X \approx Y$, then the angles of Y are naturally congruent to those of X, and hence $Y \approx X$.

If $X \approx Y$ and $Y \approx Z$, then the angles of X are also congruent to those of Z, and hence $X \approx Z$.

Therefore, $\approx$ is an equivalence relation.

$p|q$ means that q is a multiple of p.

There are other ways of defining congruence, and we will discuss them later in this publication.

Example 10 (Extremely important)

We define the relation called **congruence modulo 5**, denoted by $\equiv$, on the set of integers $\mathbb{Z}$ by

$a \equiv b \pmod 5$ if and only if 5 divides $(a - b)$, i.e. $5 | (a - b)$.

Is $\equiv$ an equivalence relation?

Solution

Reflexive: $a \equiv a \pmod 5$ since $5 | (a - a)$, i.e. since $a - a = 0$ is a multiple of 5.

Symmetric: If $a \equiv b \pmod 5$, then $(a - b)$ is a multiple of 5, i.e. $a - b = 5k$, where $k \in \mathbb{Z}$, thus $b - a = 5(-k)$. This in turn means that $b - a$ is a multiple of 5 since $-k \in \mathbb{Z}$, and hence $5 | (b - a)$ and $b \equiv a \pmod 5$.

Transitive: If $a \equiv b \pmod 5$ and $b \equiv c \pmod 5$, then $5 | (a - b)$ and $5 | (b - c)$, thus $a - b = 5k_1$ and $b - c = 5k_2$. Adding these two equations gives us $a - b + b - c = a - c = 5k_1 + 5k_2 = 5(k_1 + k_2)$, and hence $5 | (a - c)$, and therefore $a \equiv c \pmod 5$.

Therefore, we can conclude that congruence modulo 5 is an equivalence relation over the set of integers.

Equivalence classes

Example 10 is an instance of congruence modulo m, where m is any integer. A full discussion of congruence modulo m will appear later. Because of its significance, some important characteristics are worth studying. One question we can ask is: If we claim that $x \equiv a \pmod 5$ for a given integer a, is x a unique number or are there several such numbers?

Let us take $a = 0$, then the relation is $x \equiv 0 \pmod 5$. This implies that x can be 5, 10, …, $5k$ for an integer k.

This set of numbers $\{\ldots, -5, 0, 5, 10, \ldots\}$ is called the **congruence class of 0 modulo 5, and is denoted by** $[0]$. So,

$[0] = \{x \in \mathbb{Z} \mid x \equiv 0 \pmod 5\} = \{x \in \mathbb{Z} \mid 5 \mid (x - 0)\} = \{x \in \mathbb{Z} \mid 5 \mid x\}$

$= \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 5\} = \{\ldots, -5, 0, 5, 10, \ldots\}$.

Let us now take $a = 1$, then

$[1] = \{x \in \mathbb{Z} \mid x \equiv 1 \pmod 5\} = \{x \in \mathbb{Z} \mid 5 \mid (x - 1)\}$
$= \{x \in \mathbb{Z} \mid x \text{ is a multiple of 5 plus } 1\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$.

Similarly,

$[2] = \{x \in \mathbb{Z} \mid x \equiv 2 \pmod 5\} = \{x \in \mathbb{Z} \mid 5 \mid (x - 2)\}$
$= \{x \in \mathbb{Z} \mid x \text{ is a multiple of 5 plus } 2\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$

$[3] = \{x \in \mathbb{Z} \mid x \equiv 3 \pmod 5\} = \{x \in \mathbb{Z} \mid x \text{ is a multiple of 5 plus } 3\}$
$= \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

$[4] = \{x \in \mathbb{Z} \mid x \equiv 4 \pmod 5\} = \{x \in \mathbb{Z} \mid x \text{ is a multiple of 5 plus } 4\}$
$= \{\ldots, -6, -1, 4, 9, 14, \ldots\}$

$[5] = \{x \in \mathbb{Z} \mid x \equiv 5 \pmod 5\} = \{x \in \mathbb{Z} \mid x \text{ is a multiple of 5 plus } 5\}$
$= \{\ldots, -5, 0, 5, 10, 15, \ldots\}$.

We notice here that there is no need for [5] and we discover that [0] = [5]. Such classes like [0], [1], etc., are in general called **equivalence classes.**

Definition 8

If $\mathcal{R}$ is an equivalence relation on a set A for $a \in A$, the set $[a] = \{x \in A \mid x\mathcal{R}a\}$ of elements of A which are equivalent to a is called **the equivalence class of *a* with respect to $\mathcal{R}$**, or **the $\mathcal{R}$-equivalence class of *a***.

Example 11

Let $\mathcal{R}$ be the relation on set $\mathbb{Z}$ defined by

$\mathcal{R} = \{(a, b) \in \mathbb{Z}^2 \mid a - b \text{ is even}\}$.

Show that $\mathcal{R}$ is an equivalence relation and find the equivalence classes.

Solution

Reflexive: $a\mathcal{R}a$, since $a - a = 0$ is even.

Symmetric: $a\mathcal{R}b \Rightarrow a - b$ is even $\Rightarrow b - a$ is even $\Rightarrow b\mathcal{R}a$.

Transitive: $a\mathcal{R}b \Rightarrow a - b$ is even, $b\mathcal{R}c \Rightarrow b - c$ is even

$\Rightarrow a - b + b - c = a - c$ is even $\Rightarrow a\mathcal{R}c$.

The equivalence classes are

$[0] = \{\ldots, -2, 0, 2, \ldots\}$ and $[1] = \{\ldots, -3, -1, 1, 3, \ldots\}$.

Example 12

Let $\mathcal{D}$ be the relation on the set of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathcal{D} = \left\{(f, g) \in \mathbb{R}^2 \mid f'(x) = g'(x) \text{ for all } x \in \mathbb{R}\right\}.$$

Show that $\mathcal{D}$ is an equivalence relation and describe the equivalence classes.

Solution

Reflexive: $f\mathcal{D}f$ since $f'(x) = f'(x)$.

Symmetric: If $f\mathcal{D}g$, then $f'(x) = g'(x) \Rightarrow g'(x) = f'(x) \Rightarrow g\mathcal{D}f$.

Transitive: If $f\mathcal{D}g$ and $g\mathcal{D}h$, then

$f'(x) = g'(x)$ and $g'(x) = h'(x) \Rightarrow f'(x) = h'(x)$

$\Rightarrow f\mathcal{D}h$.

The equivalence class for a function f, $[f]$, is the set of all functions that differ from f by a constant, i.e. $[f] = \{g \in \mathbb{R} \mid g = f + C\}$, i.e. all anti-derivatives of $f'(x)$.

For example, $[x^3] = \{x^3 + C$, where C is an arbitrary real constant$\}$.

Theorem 1

If $\mathcal{R}$ is an equivalence relation on a set A, then any two equivalence classes $[a]$ and $[b]$ are either disjoint, or if they have any element in common then they must be equal. Stated differently, all three statements below are equivalent.

1 $a\mathcal{R}b$ **2** $[a] = [b]$ **3** $[a] \cap [b] \neq \varnothing$

Proof

1 If $a\mathcal{R}b$, now let $c \in [a] \Rightarrow c\mathcal{R}a$, but $a\mathcal{R}b$, and by transitive property, $c\mathcal{R}b \Rightarrow c \in [b]$, and hence $[a] \subseteq [b]$. Similarly, $[b] \subseteq [a]$, and therefore $[a] = [b]$. This means that (1) implies (2).

2 If $[a] = [b]$, then obviously $[a] \cap [b] \neq \varnothing$ as $[a]$ is non-empty because it is reflexive. This means that (2) implies (3).

3 If $[a] \cap [b] \neq \varnothing$, then there is at least an element $c \in [a] \cap [b]$. Now, $c \in [a] \Rightarrow c\mathcal{R}a$, and $c \in [b] \Rightarrow c\mathcal{R}b$, and hence by using symmetric and transitive properties we get $a\mathcal{R}b$. This means that (3) implies (1).

The theorem right leads us to the conclusion that $[a] \neq [b]$ if and only if $[a] \cap [b] = \varnothing$, i.e. $[a] \neq [b] \Rightarrow [a] \cap [b] = \varnothing$, and $[a] \cap [b] = \varnothing \Rightarrow [a] \neq [b]$.

Since (1) implies (2), (2) implies (3), and (3) implies (1), the statements must be equivalent.

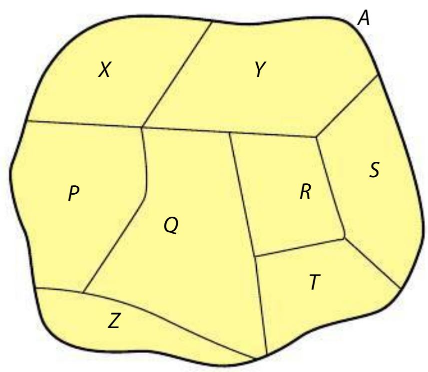
In the follow-up discussion to Example 10, we observed that $[5] = [0]$. One reason is that $0\mathcal{R}5$.

We are now in a position to investigate how an equivalence relation on a set A '*induces*' a **partition** of set A.

Definition 9

A **partition** of a set A is a collection of non-empty, disjoint subsets of A that are mutually exhaustive.

This means that the union of these subsets is the set A itself. A sample partition of a set A is shown below.



In general symbolic terms, a partition of a set A is a collection of n non-empty subsets of A such that

$$A_i \cap A_j = \varnothing, \text{for all } i \neq j, \text{ and } \bigcup_{i=1}^{n} A_i = A.$$

The last definition leads us to a very important theorem concerning equivalence relationships. We know that if a relation $\mathcal{R}$ is defined over a set A then the equivalence classes $[a_i]$ defined have the following properties:

$$[a_i] \neq \varnothing$$

$$\bigcup_{i=1}^{n} [a_i] = A$$

$$[a_i] \cap [a_j] = \varnothing, \text{for all } i \neq j.$$

This shows us that the equivalence relation created a partition of the set A whose subsets are the equivalence classes.

Theorem 2

If $\mathcal{R}$ is an equivalence relation on a set A, then the equivalence classes of $\mathcal{R}$ *induce* a partition of set A.

Proof

We need to prove two statements.

1 The equivalence classes form a partition of set A, and

2 A partition of set A forms an equivalence relation on set A.

1 This has been shown above depending on Theorem 1 and the definition of an equivalence class.

2 Suppose you have a partition containing n subsets of set A: $\{A_i | A_i \subseteq A \text{ for all } i \leq n\}$. Define a relation $\mathcal{R}$ on A such that $x\mathcal{R}y$ if x and y belong to the same subset of A.

$\mathcal{R}$ is reflexive since $x\mathcal{R}x$ for every $x \in A$, since x is in the same subset as itself!

$\mathcal{R}$ is symmetric since if $x\mathcal{R}y$ then x and y belong to the same subset of A. In that case obviously y and x belong to the same subset of A.

$\mathcal{R}$ is transitive since $x\mathcal{R}y$ and $y\mathcal{R}z$ imply that x and y belong to the same subset, say M, and y and z belong to the same subset N, and since y belongs to both subsets M and N, which are members of a partition and cannot have any element in common unless they are equal, then $M = N$ and therefore x and z are in the same subset.

Therefore, we have shown that the equivalence classes form a partition, and a partition generates an equivalence relation and hence we can say that equivalence classes of $\mathcal{R}$ *induce* a partition of set A.

Example 13

Consider the **congruence classes modulo 5** we generated in Example 10. Show that they form a partition of the set of integers.

Solution

Recall that the classes so created are: $[0]$, $[1]$, $[2]$, $[3]$ and $[4]$.

It is clear that $[a] \cap [b] = \varnothing$, unless $[a] = [b]$.

$[0] \cup [1] \cup [2] \cup [3] \cup [4] = \mathbb{Z}$

and hence the set of congruence classes mod 5 creates a partition of $\mathbb{Z}$.

Example 14

Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the following set $S = \{\{1, 4\}, \{6, 8, 2\}, \{3, 5\}, \{7\}\}$. Show that S is a partition of A.

Solution

Every element of S is non-empty.

All elements are mutually disjoint.

The union of all elements is A.

Therefore S is a partition of A.

Congruence (General)

So far you have seen some examples involving congruence for specific values. In this section we will discuss congruence in more general terms. This topic is important for this option as well as for the Discrete Mathematics option.

Definition 10

Let m be a positive integer. If a and b are integers, we say that a is congruent to b modulo m if $m|(a - b)$.

If a is congruent to b modulo m, then we write $a \equiv b \pmod{m}$. If a is not congruent to b modulo m, then we write $a \not\equiv b \pmod{m}$. The integer m is called the **modulus of congruence**.

- We have $24 \equiv 4 \pmod{5}$, since $5|(24 - 4)$. Similarly $5 \equiv -11 \pmod{8}$, since $8|(5 - (-11))$. On the other hand, $4 \not\equiv 17 \pmod{2}$ since $(4 - 17)$ is not divisible by 2.

Note: If $a, b \in \mathbb{Z}$, then $a \equiv b \pmod{m}$ for some positive integer m if and only if there exists an integer k such that $a = b + km$, since $m \mid (a - b)$ if and only if $a - b = km$ for some $k \in \mathbb{Z}$. So, we can summarize this result by stating:

Given a positive integer m and an integer b, integers which are congruent to b modulo m are obtained by adding an integer multiple of m to b.

As an illustration, let $m = 2$ and $b = 0$. Then the integers congruent to 0 modulo 2 are given by $a = 0 + 2k$, $k \in \mathbb{Z}$, i.e. $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$. If $b = 1$, then the collection of all integers congruent to 1 are $\{\ldots, -3, -1, 1, 3, \ldots\}$. We can observe that these two classes of integers are distinct and each one is associated to a remainder when we divide an arbitrary integer n by 2.

This discussion leads us to the following important theorem which explains the structure of congruence classes slightly more fully than we have done so far.

Theorem 3

If $a \equiv b \pmod{m}$ if and only if a and b leave the same remainder when we divide them by m.

Proof

$(\Rightarrow)$Let $a \equiv b \pmod{m}$. Then, by definition $m \mid (a - b)$.

Now, by the division algorithm, if we divide a by m, then we can find q_1 and r_1 such that

$$a = m \cdot q_1 + r_1, 0 \leqslant r_1 < m$$

and similarly, if we divide b by m, then we can find q_2 and r_2 such that

$$b = m \cdot q_2 + r_2, 0 \leqslant r_2 < m.$$

So, we now have

$$a - b = (m \cdot q_1 + r_1) - (m \cdot q_2 + r_2) = m(q_1 - q_2) + (r_1 - r_2).$$

However, $m \mid (a - b)$, and so m must divide the right-hand side, $m(q_1 - q_2) + (r_1 - r_2)$.

This leads to the fact that m must divide $(r_1 - r_2)$ too. But $0 \leqslant r_1 < m$ and $0 \leqslant r_2 < m$, and so $(r_1 - r_2)$ cannot divide m unless $r_1 - r_2 = 0$, i.e. $r_1 = r_2$.

Therefore, a and b leave the same remainder when we divide them by m.

$(\Leftarrow)$Let a and b leave the same remainder when we divide them by m.

Then we have

$a = m \cdot q_1 + r$ and $b = m \cdot q_2 + r$, and consequently $a - b = m(q_1 - q_2)$, which means that $m \mid (a - b)$ and therefore $a \equiv b \pmod{m}$.

The two previous theorems enable us to generalize the structure of congruence classes modulo m.

Since any two integers that leave the same remainder when divided by m, then the remainder itself will represent the equivalence class. This is so because if a leaves a remainder r when divided by m, then as we showed before:

$a = m \cdot q_1 + r \Rightarrow a - r = m \cdot q_1 \Rightarrow m \mid (a - r) \Rightarrow a \equiv r \pmod m$.

Also, since $r < m$, then it takes on all the values $\{0, 1, 2, 3, \ldots, m - 1\}$, and hence the congruence classes modulo m are $[0], [1], \ldots, [m - 1]$.

In some books, these classes are also called **residue classes mod *m***.

Theorem 4

Let $m \in \mathbb{Z}^+$. Then congruence modulo m is an equivalence relation.

Proof

1 **Reflexive property:** $a \equiv a \pmod m$ since $m \mid (a - a)$ for all $a \in \mathbb{Z}$.

2 **Symmetric property:** Suppose $a \equiv b \pmod m$. Then there is an integer k such that $a - b = km$. Hence, $b - a = (-k)m$ and $m \mid (b - a)$ $[-k$ is also an integer$]$. Thus $b \equiv a \pmod m$.

3 **Transitive property:** If $a \equiv b \pmod m$ and $b \equiv c \pmod m$, then $m \mid (a - b)$ and $m \mid (b - c)$. Hence, $m \mid \big((a-b)+(b-c)\big)$, i.e. $m \mid (a - c)$ and $a \equiv c \pmod m$.

Example 15

List the congruence classes mod 7.

Solution

Since the possible remainders when dividing by 7 are 0, 1, 2, …, 6, then the congruence classes are:

$[0] = \{\ldots, -7, 0, 7, 14, \ldots\}$

$[1] = \{\ldots, -6, 1, 8, 15, \ldots\}$

$\vdots$

$[6] = \{\ldots, -1, 6, 13, 20, \ldots\}$

2.2 Functions

The **function** concept has been discussed comprehensively in Chapter 2 of the HL book. We will present you here with a brief review of what you have seen there and a small number of bits and pieces that are not compulsory in the core part but essential for this option.

If f is a function from A to B, we also write $f: A \rightarrow B$; if $x \in A$, we also write $f: x \mapsto y$, where $y \in B$. (Notice the difference in symbols $\rightarrow$ between sets and $\mapsto$ between elements!)

Definition 11

If A and B are non-empty sets, a **function from *A* to *B*** is a relation f from A to B such that for all $x \in A$, there is a *unique* element $y \in B$ with $(x, y) \in f$.

In many instances, a function is also called a mapping (or simply map) from A to B. So, we say f is a mapping from A to B, or f maps x to $y = f(x)$.

The set A is the **domain** of the function f and the set B is the **codomain** of f. If $(x, y) \in f$, we write $y = f(x)$ and say that y is the **image** of x under f or the value of f at x, and we also say that x is **mapped** to $y = f(x)$ by the function f. Several other notations are used such as: x is called the **input**, or **preimage**, and y is the **output**.

Definition 12

If f is a function from A to B, then the **subset** of B defined by

$\{f(a) \mid a \in A\}$

is called the **image** of A and is denoted by $f(A)$. This is to say that the image of A is the subset of B that consists of the images of all elements of A.

So, the range is always a subset of the codomain: $f(A) \subseteq B$. That is, they are also equal in numerous cases. This is why several mathematicians only talk about range and do not mention codomain.

Additionally, if $f(A) = B$, then B is called the **range** of the function f. That is, if every element of B is an image of some element in A, then B is the range of f.

Example 16

Decide whether each of the following relations is a function. If the relation is a function, state its codomain and range.

a) $A = \{1, 2, 3\}$, $B = \{3, 4, 5, 6\}$, and $g = \{(1, 5), (2, 4), (3, 3)\}$

b) $A = \{1, 2, 3\}$, $B = \{3, 4, 5, 6\}$, and $h = \{(1, 5), (2, 4), (3, 3), (2, 6)\}$

c) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$

d) $g: \mathbb{R} \to [1, \infty[$ defined by $g(x) = x^2 + 1$

Solution

a) This is a function. Codomain is $\{3, 4, 5, 6\}$ and range is $\{3, 4, 5\}$.

b) This is not a function as 2 does not have a *unique* image.

c) This is a function. Codomain is $\mathbb{R}$ and range is $[1, \infty[$.

d) This is a function. Codomain = range = $[1, \infty[$.

Definition 13

A function $f: A \to B$ is a **surjection** if and only if for every $y \in B$, there is at least an $x \in A$ such that $f(x) = y$.

The function is also called **surjective** or **onto**.

The definition left is equivalent to saying that $f(A) = B$, i.e. the range is equal to the codomain!

Example 17

Consider each of the following and decide which of them is surjective.

a) $A = \{1, 2, 3\}$, $B = \{3, 4, 5, 6\}$, and $g = \{(1, 5), (2, 4), (3, 3)\}$

b) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$

c) $g: \mathbb{R} \to [1, \infty[$ defined by $g(x) = x^2 + 1$

Since every element of B must be an image for *at least* an element of A, then the number of elements of A, $n(A) = |A|$ must at least be the same as $n(B)$, i.e. if f is surjective, then $|A| \geqslant |B|$.

Solution

a) g is not onto since $6 \in B$ but there is no $x \in A$ such that $g(x) = 6$.

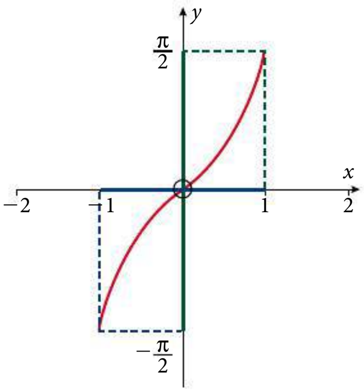
b) f is not surjective, since every $y < 1$ in B does not have an x in A such that $f(x) = y$.

c) g is a surjection, since the range and codomain are equal.

Example 18

Consider whether the function $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined by $f(x) = \arcsin x$ is a surjection.

Solution



Take any number $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. By definition, there is a sine value for each angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. there is an $x \in [-1, 1]$ such that $\sin y = x$, which implies that $y = \arcsin x$. Thus f is a surjection. You see that from the graph of $f(x) = \arcsin x$ (left) where it is clear that the codomain and range are the same.

Definition 14

A function $f: A \to B$ is an **injection** if and only if for any $x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. (Distinct inputs of f produce distinct outputs.)

The function is also called **injective**, **into**, or **1–1(one-to-one)**.

The above definition is equivalent to saying:

- For any $x_1, x_2 \in A$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. (Contra-positive of the basic definition and the one used frequently to prove functions are 1–1.)
- For every element y of the range $f(A)$ there is *exactly one* $x \in A$ such that $f(x) = y$. (For every output, there is *exactly* one input.)
- For every element y of the codomain there is *at most one* $x \in A$ such that $f(x) = y$. (For every output, there is *at most* one input.)

Note: Since *for every output, there is at most one input*, we can conclude that if f is injective, then every element in A must have an image in B, and hence $n(A) \leqslant n(B)$ or $|A| \leqslant |B|$.

Example 19

Consider each of the following and decide which of them is injective.

a) $A = \{1, 2, 3\}$, $B = \{3, 4, 5, 6\}$, and $g = \{(1, 5), (2, 4), (3, 3)\}$

b) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$

c) $g: [0, \infty[\to [1, \infty[$ defined by $g(x) = x^2 + 1$

Solution

a) g is an injection since $1, 2, 3 \in A$ all have different images in B.

b) f is not an injection since $f(-1) = 2 = f(1)$.

c) g is an injection, since the domain consists of non-negative real numbers only, then
$f(x_1) = f(x_2) \Rightarrow x_1^2 + 1 = x_2^2 + 1 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$.

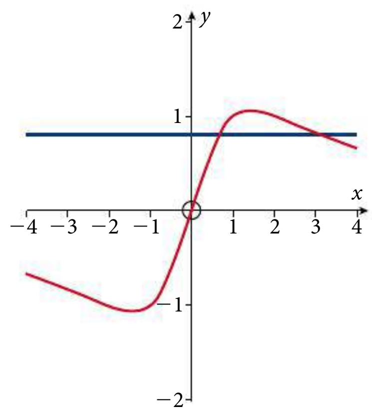
Example 20

Determine whether the function $g(x): \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \dfrac{3x}{x^2+2}$ is one-to-one.

Solution

$$g(x_1) = g(x_2) \Rightarrow \frac{3x_1}{x_1^2+2} = \frac{3x_2}{x_2^2+2} \Rightarrow 3x_1(x_2^2+2) = 3x_2(x_1^2+2)$$
$$\Rightarrow 3x_1x_2^2 + 6x_1 - 3x_2x_1^2 - 6x_2 = 0$$
$$\Rightarrow 3x_1(2 - x_1x_2) + 3x_2(x_1x_2 - 2) = 0$$
$$\Rightarrow (2 - x_1x_2)(3x_1 - 3x_2) = 0 \Rightarrow \text{either } x_2 = \frac{2}{x_1} \text{ or } x_1 = x_2$$

Since $g(x_1) = g(x_2) \not\Rightarrow x_1 = x_2$, the function is not an injection. Notice how the horizontal line intersects the graph of the function at two points, pointing to the fact that different input values do not necessarily have different output values.



Definition 15

A function $f: A \to B$ which is an **injection** as well as a **surjection** is a **bijection** from A to B.

The function is also called **1–1 correspondence** between A and B.

Since the bijection is a surjection, then $|A| \geq |B|$, and it is an injection, then $|A| \leq |B|$; therefore in this case we should have $|A| = |B|$.

Example 21

Consider whether the function $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined by $f(x) = \arcsin x$ is a bijection.

Solution

You have seen in Example 18 that this function is a surjection. We need to show that it is also an injection.

You may recall from your study of trigonometric functions that by restricting the range of this function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the following result is apparent.

$$f(x_1) = f(x_2) \Rightarrow \arcsin x_1 = \arcsin x_2 \Rightarrow x_1 = x_2$$

Therefore the function is a bijection. You can also observe that it is a bijection by noticing that on its graph (page 1248) the horizontal lines can intersect this function at one point, implying that for every y in the range there is exactly one x in the domain.

Example 22

Consider the function $h: \mathbb{Z} \to \mathbb{Z}$ defined by $h(n) = n^3 + n$. Is this function a bijection?

Solution

We need to show that the function is injective and surjective.

Injection: Consider $n_1, n_2 \in \mathbb{Z}$

$$h(n_1) = h(n_2) \Rightarrow n_1^3 + n_1 = n_2^3 + n_2 \Rightarrow n_1^3 - n_2^3 = n_2 - n_1$$
$$\Rightarrow (n_1 - n_2)(n_1^2 + n_1 n_2 + n_2^2) = n_2 - n_1$$

Now, if $n_2 \neq n_1$ then $n_1^2 + n_1 n_2 + n_2^2 = -1$. However, we have the following situations:

$n_1 n_2 > 0$, then $n_1^2 + n_1 n_2 + n_2^2 > 0$, or
$n_1 n_2 < 0$, then either $|n_1| > |n_2| \Rightarrow n_1^2 + n_1 n_2 > 0$ or
$|n_2| > |n_1| \Rightarrow n_2^2 + n_1 n_2 > 0$ and hence, in both cases,
$n_1^2 + n_1 n_2 + n_2^2 > 0$; therefore the only option is for $n_2 = n_1$.

Surjection: If h is surjective then given an element m in $\mathbb{Z}$, there should be n in $\mathbb{Z}$ such that $m = h(n) = n^3 + n$. However, $n^3 + n = n(n^2 + 1)$ is always even whatever the value of n is. Since if n is odd, then $n = 2k + 1$ for some integer k, and $n^3 + n = (2k + 1)(4k^2 + 4k + 2)$, which is the product of an odd number by an even number and is therefore even. Similarly, when n is even, this product is even. This means all the odd numbers in the codomain are not images of numbers in the domain. So, h is not surjective and hence it is not a bijection.

(Take $m = 3$, then it should be possible to write 3 as the sum of an integer and its cube. That is not possible.)

Example 23

Consider the function $h: \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^3 + x$. Is this function a bijection?

Solution

We need to show that the function is injective and surjective.

Injection: Similar to Example 22.

Surjection: If h is surjective then given an element y in $\mathbb{R}$, there should be x in $\mathbb{R}$ such that $y = h(x) = x^3 + x$. From your calculus chapters, you know that this function is increasing, and hence the horizontal line at y will intersect the graph at one point. Hence, there is always an x in the domain to correspond to every y in the codomain, and therefore it is surjective.

Thus h is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Example 24

Consider the function $i_A: A \to A$ defined by $i_A(x) = x$ for every $x \in A$. Show that function i_A is a bijection.

Solution

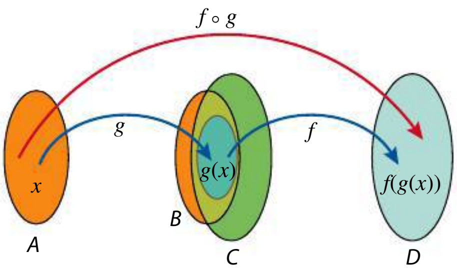
Since for every $x \in A$ there is an $x \in A$ such that $i_A(x) = x$, then i_A is a surjection.

Since $i_A(x_1) = i_A(x_2) \Rightarrow x_1 = x_2$, then i_A is an injection. Thus i_A is a bijection.

i_A is known as the **identity function** on A since it maps every element in A to itself.

Composition of functions

You may recall from the book that if the *outputs* of a function g are used as inputs of a function f, we are forming the **composition** of f with g. For this composition to be possible, the outputs of g must be elements of the domain of f, i.e. the range of g must be a subset of the domain of f.

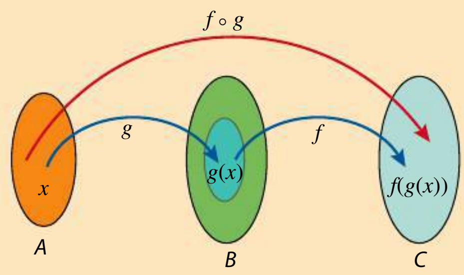


Definition 16

If $g: A \to B$ and $f: C \to D$ are functions from their respective domains, A and C, to their respective codomains, B and D respectively, and if $g(A) \subseteq C$, then the **composition** of f and g is the function

$$f \circ g: A \to D \text{ defined by } f \circ g(x) = f(g(x)).$$

In many cases, the codomain of the first function, B, does not have to be different from the domain of the second function C. Thus, you will have $g: A \to B$, $f: B \to C$ and $f \circ g: A \to C$ for example.



Note: Stated differently, suppose $g: A \to B$ and $f: C \to D$ are functions. Then for any $x \in A$, $g(x)$ is a member of $g(A)$ which is a subset of B. If $g(A)$ is also a subset of C, and we apply the function f to this value $g(x)$, the result is $f\left(g\left(x\right)\right)$, a member of D. Thus, taking an arbitrary element x of A, applying the function g, then applying the function f to $g(x)$ is the same as associating a unique element of D with x, i.e. we have created a function $A \to D$, called the **composition function** of f and g and denoted by $f \circ g$. Notice that with this notation, even though g is applied first, it appears second in the expression $f \circ g$.

Example 25

Let $g: [2, \infty[\to \mathbb{R}$ defined by $g(x) = x^2 - 2$, and $f: \left[1, \infty\right[\to \mathbb{R}$ defined by $f\left(x\right) = \sqrt{2x+2}$. If possible, find $f \circ g$. Also, if possible, find $g \circ f$.

Solution

Since the domain of f is $[1, \infty[$, the range of g must be a subset of this set. The range of g is $[2, \infty[$ too, and hence a subset of $[1, \infty[$, so we can find the composition.

$f \circ g: [2, \infty[\to \mathbb{R}$ defined by

$$f \circ g\left(x\right) = f\left(g\left(x\right)\right) = f\left(x^2 - 2\right) = \sqrt{2\left(x^2 - 2\right) + 2} = \sqrt{2x^2 - 2}.$$

The range of f is $[2, \infty[$ which is a subset of the domain of g, $[2, \infty[$, and thus

$g \circ f: [1, \infty[\to \infty[$ defined by

$$g \circ f\left(x\right) = g\left(f\left(x\right)\right) = g\left(\sqrt{2x+2}\right) = \left(\sqrt{2x+2}\right)^2 - 2 = 2x.$$

Note: In Example 25, you have seen that

$f \circ g\left(x\right) = \sqrt{2x^2 - 2} \neq g \circ f\left(x\right) = 2x$, i.e. composition of functions is **not commutative** (it is *not necessarily true* that $f \circ g = g \circ f$).

Example 26

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = |x - 3|$. Find

a) $f \circ g(2)$ b) $g \circ f(2)$

c) $f \circ g(1)$ d) $g \circ f(1)$

Solution

a) $f \circ g(2) = f(|2 - 3|) = f(1) = 2$ b) $g \circ f(2) = g(2^2 + 1) = g(5) = 2$

c) $f \circ g(1) = f(|1 - 3|) = f(2) = 5$ d) $g \circ f(1) = g(1^2 + 1) = g(2) = 1$

Notice here how in one case $f \circ g(x) = g \circ f(x)$ and in another $f \circ g(x) \neq g \circ f(x)$.

Example 27

Let $g: A \rightarrow B$ and $f: B \rightarrow C$ be two bijections. Show that $f \circ g$ is also a bijection.

Solution

To show that $f \circ g$ is a bijection, we need to show that it is surjective as well as injective.

Surjection: Recall that $f \circ g: A \rightarrow C$, so we must take a value $z \in C$ and show that it has a preimage $x \in A$ under $f \circ g$. Now, because f is surjective, then there is an element y in B such that $f(y) = z$. Also, because g is surjective, there is an element x in A such that $g(x) = y$. Thus,

$f \circ g(x) = f(g(x)) = f(y) = z$

and therefore $f \circ g$ is a surjection.

Injection: Assume that $f \circ g(x_1) = f \circ g(x_2) \Rightarrow f(g(x_1)) = f(g(x_2))$, but f is an injection, so

$g(x_1) = g(x_2)$. Now, g is also an injection, and hence

$g(x_1) = g(x_2) \Rightarrow x_1 = x_2$. Therefore,

$f \circ g(x_1) = f \circ g(x_2) \Rightarrow x_1 = x_2$, and $f \circ g$ is an injection.

The result follows.

Composition of functions is an associative operation.

That is, given $h: A \rightarrow B$, $g: B \rightarrow C$, and $f: C \rightarrow D$, then

$(f \circ g) \circ h = f \circ (g \circ h)$. To show that this is true, we can consider any element x in the domain of the composition, which is A, then

$(f \circ g) \circ h(x) = (f \circ g)(h(x)) = f(g(h(x)))$ by definition of composition.

Also,

$f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x)))$ by definition too.

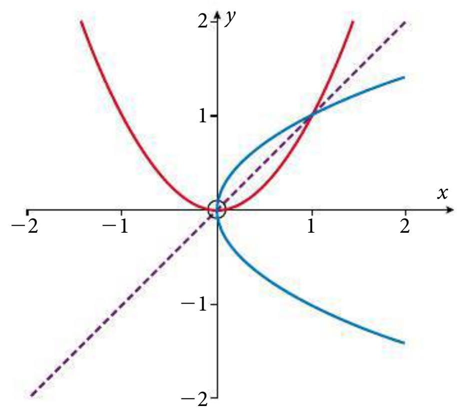
Therefore, $(f \circ g) \circ h = f \circ (g \circ h)$.

Inverse functions

Every relation $\mathcal{R}$ from set A to set B has an inverse relation $\mathcal{R}^{-1}$ from B to A formed by interchanging the order of the pairs in the relation $\mathcal{R}$:

$$\mathcal{R}^{-1} = \{(y, x) \in B \times A \mid (x, y) \in A \times B\} \Leftrightarrow y\mathcal{R}^{-1}x \text{ if and only if } x\mathcal{R}y.$$

Note: Recall that for relations over $\mathbb{R}$, interchanging the order of the pairs interchanges the horizontal and vertical coordinates of the points on the graphs of these relations. The result will be that graphs of relations and their inverses are reflections of each other with respect to the line $y = x$ (called the 'first bisector' or 'identity line').



Since functions are also relations, then each function has an inverse relation. The inverse relation of a function f may or may not be a function itself. If the inverse of a function is a function itself, then we call it the inverse function of f and denote it by f^{-1}.

Example 28

Consider the function f from $\{1, 2, 3, 4\}$ to $\{5, 6, 7\}$ defined by

$$f = \{(1, 5), (2, 5), (3, 6), (4, 7)\}.$$

a) Find the inverse f^{-1}.

b) Find the inverse of f^{-1}, that is find $(f^{-1})^{-1}$.

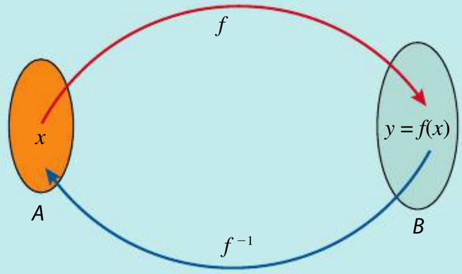
Solution

a) $f^{-1} = \{(5, 1), (5, 2), (6, 3), (7, 4)\}$. Notice here that the inverse f^{-1} is not a function itself.

b) $(f^{-1})^{-1} = \{(1, 5), (2, 5), (3, 6), (4, 7)\} = \{(1, 5), (2, 5), (3, 6), (4, 7)\} = f.$

Definition 17

Let $f: A \to B$ be a **bijection**. The inverse function of f is the function that assigns to an element $y \in B$ the unique element $x \in A$ such that $f(x) = y$.



The inverse function of f is denoted by f^{-1}. Thus,

$f^{-1}(y) = x$ when $f(x) = y$.

Note: Why does the function have to be a bijection in order to have an inverse function?

For f^{-1} to be a function, all elements in its domain, which is B, must have an image each. Hence, every $y \in B$ should be associated with some $x \in A$, and hence f is a surjection.

If f were not an injection, then there exists at least two elements x_1 and x_2 in A that have the same image $y \in B$. This means that for f^{-1}, there is an element $y \in B$ that is assigned two images x_1 and x_2 in A, implying that f^{-1} is not a function.

Theorem 5

If f is a function from A to B, the inverse relation f^{-1} is a function from B to A if and only if f is a bijection.

In general, when we are dealing with inverse functions it is customary to say 'the function has an inverse', or 'the function is invertible' to mean that the function has an inverse and that the inverse is a function.

The above discussion leads us to a very important property of inverse functions. Let us consider a function $f: A \to B$ and its inverse $f^{-1}: B \to A$. Then,

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y \Rightarrow f \circ f^{-1} = i_B.$$

Also,

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x \Rightarrow f^{-1} \circ f = i_A.$$

This observation provides us with a method to test whether two functions are inverses of each other.

Example 29

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4x^3$. Find the inverse of this function and check its correctness.

Solution

You have learned how to find the inverse of a function in Chapter 2 of the HL book. Recall that you switch the domain and range variables and solve the resulting equation for x.

$$f^{-1}(x) = \sqrt[3]{\frac{x}{4}}$$

To check our answer, we perform the composition as suggested in the note above.

$$f \circ f^{-1}(x) = f\left(\sqrt[3]{\frac{x}{4}}\right) = 4\left(\sqrt[3]{\frac{x}{4}}\right)^3 = 4 \cdot \frac{x}{4} = x, \text{ also}$$

$$f^{-1} \circ f(x) = f^{-1}(4x^3) = \sqrt[3]{\frac{(4x^3)}{4}} = \sqrt[3]{x^3} = x$$

Example 30

Show that the functions $f\colon \mathbb{R} \to]-2, \infty[$ and $h\colon]-2, \infty[\to \mathbb{R}$ defined by

$$f(x) = 5^{2x} - 2 \text{ and } h(x) = \frac{1}{2}\log_5(x+2)$$

are inverses of each other.

Solution

For any $x \in \mathbb{R}$,

$$h \circ f(x) = h(5^{2x} - 2) = \frac{1}{2}\log_5\left((5^{2x} - 2) + 2\right) = \frac{1}{2}\log_5(5^{2x}) = \frac{1}{2} \cdot 2x = x.$$

Also for any $x \in]-2, \infty[$,

$$f \circ h(x) = f\left(\frac{1}{2}\log_5(x+2)\right) = 5^{2\left(\frac{1}{2}\log_5(x+2)\right)} - 2 = 5^{\log_5(x+2)} - 2 = x + 2 - 2 = x.$$

Therefore f and h are inverses.

Example 31

If $f\colon A \to B$ and $g\colon B \to C$ are two invertible functions, show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution

To state the question differently, we can say that we need to show that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$.

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ i_B \circ g^{-1} = g \circ g^{-1} = i_C$$

Also,

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ i_B \circ f = f^{-1} \circ f = i_A.$$

Therefore $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, i.e. the inverse of the composition of two functions is the composition of their inverses in reverse order!

Exercise 2

1 Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. Find

a **i** $A \times (B \cup C)$

ii $A \times (B \cap C)$

iii $A \times (B \backslash C)$

iv $(A \times B) \cup (A \times C)$

v $(A \times B) \cap (A \times C)$

vi $(A \times B) \cap (A \times C')$

b Which of the above expressions are equal?

2 Which of the following relations are equivalence relations on the given set?

a $\mathbb{R}$, $x\mathcal{R}y \Leftrightarrow x = y$ or $x = -y$

b $\mathbb{Z}$, $x\mathcal{R}y \Leftrightarrow xy = 0$

c $\mathbb{R}$, $x\mathcal{R}y \Leftrightarrow x^2 + x = y^2 + y$

d $\mathbb{Z}^+$, $x\mathcal{R}y \Leftrightarrow xy$ is a square

e $\mathbb{R} \times \mathbb{R}$, $(x, y)\ \mathcal{R}\ (a, b) \Leftrightarrow x^2 + y^2 = a^2 + b^2$

3 In the previous problem, describe the equivalence classes for those relations that are equivalence relations.

4 Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $f: A \to A$ be a function defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x \neq 6 \\ 1, & \text{if } x = 6 \end{cases}$$

a Find $f(3)$, $f \circ f(3)$, and $f(f(2))$.

b Find a preimage of 4.

c Show that f is a bijection.

5 Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

a Define a relation $\mathcal{R}$ on S by

$A\mathcal{R}B \Leftrightarrow |A| = |B|$.

Determine whether $\mathcal{R}$ is an equivalence relation. If yes, describe the partition it induces on S. If not, justify why not.

b Define a relation $\mathcal{X}$ on S by

$A\mathcal{X}B \Leftrightarrow |A| \neq |B|$.

Determine whether $\mathcal{X}$ is an equivalence relation. If yes, describe the partition it induces on S. If not, justify why not.

6 Let $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $f(x)$ for all $x \in \mathbb{Z}^+$ in each of the cases below. Determine if f is an injection, a surjection, or both. Justify your answer.

a $f(x) = x + 1$

b $f(x) = 2x$

c $f(x) = x^2$

d $f(1) = 1$, $f(x) = x - 1$ for $x > 1$

7 Let $f: x \mapsto 3x + 4$.

a Is $f: \mathbb{R} \to \mathbb{R}$ a bijection? Justify.

b Is $f: \mathbb{N} \to \mathbb{N}$ a bijection? Justify.

c Is $f: \mathbb{Q} \to \mathbb{Q}$ a bijection? Justify.

8 Let E and F be two finite sets such that $|E| = m$ and $|F| = n$. In each of the following give some indication why you believe your conclusion to be correct.

a Determine the number of functions from E into F.

b If $m \leqslant n$, determine the number of injections of E into F.

c If $m = n$, determine the number of surjections of E into F.

9 Consider the two functions f and g from $\mathbb{Z}$ into $\mathbb{Z}$ defined by

$f(x) = 2x - 1$ and $g(x) = x^2 + 1$.

a Is f an injection? a surjection?

b Is g an injection? a surjection?

c If $A = [-4, 2]$ and $B = [0, 3]$, find

i $A \cup B$, $A \cap B$

ii $f(A \cup B)$, $f(A) \cup f(B)$, $f(A \cap B)$, $f(A) \cap f(B)$

iii $g(A \cup B)$, $g(A) \cup g(B)$, $g(A \cap B)$, $g(A) \cap g(B)$

10 Consider the function $f: \mathbb{N} \to \mathbb{N}$ defined by

$$x \mapsto \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ \dfrac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

Is f an injection? a surjection?

11 Let the two functions f and g be from A into A. Show that

a if $f \circ g$ is a surjection, then f is a surjection.

b if $f \circ g$ is an injection, then g is an injection.

12 Consider the set $A = \{a, b, c\}$ and define the function $f: A \to A$ such that

$f(a) = b$, $f(b) = c$, and $f(c) = a$.

a Show that f is a bijection from A into A.

b Calculate $f \circ f(a)$, $f \circ f(b)$, and $f \circ f(c)$.

c Determine $f \circ f \circ f$. What are the inverse functions of f and of $f \circ f$?

13 Let A and B be two subsets of a universal set U. Let $\mathcal{R}$ be an equivalence relation defined on the elements of B. You are also given a function $f: A \to B$.

Define a relation $\mathcal{S}$ in A such that $\forall\, x, y \in A$, $x\mathcal{S}y$ iff $f(x)\mathcal{R}f(y)$. Determine if $\mathcal{S}$ is an equivalence relation in A.

14 Define a relation $\mathcal{S}$ on $\mathbb{R}^2$ by: $(x_1, y_1)\mathcal{S}(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$.

a Show that $\mathcal{S}$ is an equivalence relation.

b Describe the partition that this relation induces on the Cartesian plane, and give the equivalence class for (1, 2).

15 The function h: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by h: $(a, b) \mapsto (2b - a, a + b)$.

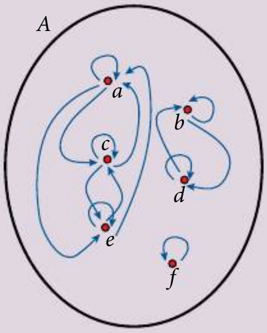
Determine whether h is injective, surjective, or both. If it has an inverse function find the inverse, and if does not have one, justify why not.

16 The relation $\mathcal{R}$ is defined over $\mathbb{Z} \times \mathbb{Z}^+$ by: $(x_1, y_1)\mathcal{R}(x_2, y_2) \Leftrightarrow x_1 y_2 = y_1 x_2$.

Show that $\mathcal{R}$ is an equivalence relation and describe the partition it induces.

17 A relation $\mathcal{S}$ on set A {a, b, c, d, e, f} is defined by the 'arrow diagram' below. (When there is an arrow from one element to the other then the elements are related, for example $a\mathcal{S}c$.)

Determine whether the relation is an equivalence relation, and if it is, describe the partition it induces on A.



18 Let $A = \{x | x \in \mathbb{N} \text{ and } 0 < x < 11\}$.

The relation $\mathcal{R}$ is defined on A by:

$x\mathcal{R}y \Leftrightarrow x^2 \equiv y^2 \pmod 5$.

Show that $\mathcal{R}$ is an equivalence relation on A, and write down all the equivalence classes.

19 Determine which of the following functions with domain and codomain $\mathbb{R}$ is a bijection. Justify your answer.

a $f(x) = 3x^2 + 1$

b $g(x) = 2x^3 + 1$

c $h(x) = \dfrac{3x^2 + 1}{x^2 + 2}$

20 If f: $A \rightarrow B$ is a bijection, and if h: $B \rightarrow C$ is a bijection, show that $h \circ f$ is also a bijection. Justify your response completely.

21 A relation φ is defined over the set of natural numbers by

$\varphi = \{(x, y) | x, y \in \mathbb{N} \text{ and } 3^x \equiv 3^y \pmod{10}\}$.

a Show that φ is an equivalence relation.

b Find the equivalence classes.

c Find the smallest possible value for 3^{101} mod(10).

22 Consider the function $h: \mathbb{Z} \to \mathbb{Z}$ defined by

$h(n) = 7n + 6.$

Determine whether h is

a injective

b surjective.

In both cases, justify your response.

23 Consider the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$f(x, y) = (x + 3y, 2x - 5y).$

Show that the function is bijective and find its inverse.

24 Let f and g be two mappings from a set A to A. Show that

a if $f \circ g$ is a surjection, then f is a surjection.

b if $f \circ g$ is an injection, then g is an injection.

25 Let $A = \{x \mid x \in \mathbb{Z}, x > 1\}$. A relation $\mathcal{R}$ is defined on A by

$x\mathcal{R}y \Leftrightarrow gcd(x, y) > 1.$

Show that the relation is reflexive, symmetric, but NOT transitive.

26 The function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$f(x) = e^{\cos x} + 1.$

a **i** Find the range, R, of f.

ii Show that the function is not an injection. Justify.

iii Determine, with reasons, whether the function is a surjection.

b We now restrict the function as follows:

$f: [0, k] \to R, k > 0.$

i Find the largest value of k for which the restricted function is a bijection.

ii Find an inverse for this restricted function.

27 Let $U = \{4, 8, 12, 16, 20, 24, 28, 32, 36\}$. A relation $\mathcal{S}$ is defined on U by

$x\mathcal{S}y \Leftrightarrow x^2 \equiv y^2 \pmod 7.$

a Show that $\mathcal{S}$ is an equivalence relation.

b Find the partition of U induced by $\mathcal{S}$ on U.

28 The relation $\mathcal{S}$ is represented by the table below. A '1' entry means that the element in the left column is related to the element in the top row; for example, $c\mathcal{S}d$. A zero entry implies that the two elements are not related, so $c\not{\mathcal{S}}e$.

Show that $\mathcal{S}$ is an equivalence relation and find all equivalence classes.

$\mathcal{S}$	a	b	c	d	e	f	g	h	i
a	1	0	0	0	0	1	0	1	0
b	0	1	0	0	1	0	0	0	1
c	0	0	1	1	0	0	0	0	0
d	0	0	1	1	0	0	0	0	0
e	0	1	0	0	1	0	0	0	1
f	1	0	0	0	0	1	0	1	0
g	0	0	0	0	0	0	1	0	0
h	1	0	0	0	0	1	0	1	0
i	0	1	0	0	1	0	0	0	1

29 The function h is defined by

$h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $h: (x, y) \mapsto (2x + 3y, y + 2x)$.

Show that h must have an inverse, and find that inverse, h^{-1}.

30 Determine whether the function g defined below is injective, surjective, or both. Justify your response.

$g: (\mathbb{R}^+)^2 \rightarrow (\mathbb{R}^+)^2$, where $g(x, y) = (2x + y, 2xy)$

31 A relation $\mathcal{R}$ is defined over $\mathbb{N}$ by: $x\mathcal{R}y \Leftrightarrow x^2 \equiv y^2 \pmod 5$.

a Show that $\mathcal{R}$ is an equivalence relation.

b Find the partition of $\mathbb{N}$ induced by $\mathcal{R}$ on $\mathbb{N}$.

32 **a** Show that the mapping $f: \mathbb{R}\backslash\{1\} \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{2x + 5}{x - 1}$$

is an injection.

b Find the value of a so that the function $f: \mathbb{R}\backslash\{1\} \rightarrow \mathbb{R}\backslash\{a\}$ becomes a bijection.

33 Consider a function $f: E \rightarrow F$. Let $A, B \subseteq E$ such that $A \cap B \neq \varnothing$. Show that

a $A \subset B \Rightarrow f(A) \subset f(B)$

b $f(A \cup B) = f(A) \cup f(B)$

c $f(A \cap B) \subset f(A) \cap f(B)$

d f is an injection $\Rightarrow f(A \cap B) = f(A) \cap f(B)$

34 If $\approxeq$ is an equivalence relation on a set A, prove each of the following.

a If $a, b \in A$ such that $a \not\approxeq b$, then $[a] \cap [b] = \varnothing$.

b If $a, b, c, d \in A$ such that $c \in [a]$, $d \in [b]$, and $[a] \neq [b]$, then $c \not\approxeq d$.

Practice questions 2

1 Let $S = \{(x, y) \mid x, y \in \mathbb{R}\}$, and let $(a, b), (c, d) \in S$. Define the relation Δ on S as follows:

$(a, b)\ \Delta\ (c, d) \Leftrightarrow a^2 + b^2 = c^2 + d^2$.

a Show that Δ is an equivalence relation.

b Find all ordered pairs (x, y) where $(x, y)\ \Delta\ (1, 2)$.

c Describe the partition created by this relation on the (x, y) plane.

2 Consider the set $\mathbb{Z} \times \mathbb{Z}^+$. Let R be the relation defined by the following:

For (a, b) and (c, d) in $\mathbb{Z} \times \mathbb{Z}^+$, $(a, b)\ R\ (c, d)$ if and only if $ad = bc$, where ab is the product of the two numbers a and b.

a Prove that R is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$.

b Show how R partitions $\mathbb{Z} \times \mathbb{Z}^+$, and describe the equivalence classes.

3 Let Y be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Define the relation R on Y by $aRb \Leftrightarrow a^2 - b^2 \equiv 0 \pmod 5$, where $a, b \in Y$.

a Show that R is an equivalence relation.

b **i** What is meant by 'the equivalence class containing a'?

ii Write down all the equivalence classes.

4 The relation R is defined on the non-negative integers a, b such that aRb if and only if $7^a \equiv 7^b$ (modulo 10).

a Show that R is an equivalence relation.

b By considering powers of 7, identify the equivalence classes.

c Find the value of 7^{503} (modulo 10).

5 Consider the functions f and g, defined by

$f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 5n + 4$, and

$g = \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ where $g(x, y) = (x + 2y, 3x - 5y)$.

a Explain whether the function f is

i injective

ii surjective.

b Explain whether the function g is

i injective

ii surjective.

c Find the inverse of g.

d Consider any functions $f\colon A \to B$ and $g\colon B \to C$. Given that $g \circ f\colon A \to C$ is surjective, show that g is surjective.

6 Let S = {integers greater than 1}. The relation R is defined on S by

$mRn \Leftrightarrow gcd(m, n) > 1$, for $m, n \in S$.

a Show that R is reflexive.

b Show that R is symmetric.

c Show using a counterexample that R is not transitive.

7 Let $a, b \in \mathbb{Z}^+$ and define $aRb \Leftrightarrow a^2 \equiv b^2$ (modulo 3).

a Show that R is an equivalence relation.

b Find all the equivalence classes.

8 We define the relation $(x, y)\ R\ (p, q)$ if and only if $x^2 - y^2 = p^2 - q^2$ where (x, y), $(p, q) \in \mathbb{R}^2$. Prove that R is an equivalence relation on $\mathbb{R}^2$. Describe geometrically the equivalence class of (1, 1).

9 Let $F(x) = x^2 - |x - 2|$.

a The function f is defined by

f: $]-\infty, 1] \to \mathbb{R}$, where $f(x) = F(x)$.

Find the range of f and determine whether it is an injection.

b The function g is defined by

g: $[1, \infty[\to [0, \infty[$, where $g(x) = F(x)$.

Show that g has an inverse and find this inverse.

10 The relation R is defined on ordered pairs by

$(a, b)R(c, d)$ if and only if $ad = bc$ where $a, b, c, d \in \mathbb{R}^+$.

a Show that R is an equivalence relation.

b Describe, geometrically, the equivalence classes.

3 Groups I

3.1 Binary operations

Operations on pairs of elements of sets arise in many contexts. In the set of integers, examples of such operations include the addition, subtraction, or multiplication of integers. In the set of 3×3 matrices, addition and multiplication of matrices are also operations. In such cases we speak of a **binary** operation. In general, a **binary operation** on a set A, denoted by any symbol of your choice, Δ for example, is a rule which assigns to each ordered pair of elements a and b from A a *uniquely* defined third element c and we write $a \Delta b = c$. Usually, we have a condition that c must also be an element of A; otherwise the operation is not called a binary operation.

Definition 1

A **binary operation** on a set A is a function from $A \times A$ into A. Thus a binary operation is a rule $*$ which assigns to every ordered pair $(a, b) \in A \times A$ exactly one element $c \in A$; this element is denoted by

$$a * b = c.$$

There are two, very important, points which *must* be checked to determine whether an operation is a binary operation on set A:

- The rule for the operation must be **well defined**: it must assign to every ordered pair (a, b) *exactly* one element c.
- The second condition is that the element c is an element of A. This is called the **closure** property. It is very important to know that there are a few sources (among which is the IB) that do not include closure as a condition for an operation to be a binary operation. So, in exams, you may be required to test the closure property separately. In the following examples, we will indicate whether you need to check closure.

Typical examples of binary operations are addition and multiplication over the set of real numbers, since when we add two real numbers we get another real number, the same for multiplication.

Example 1

Decide whether each operation is binary and whether each set is closed under the given operation.

a) The set of integers $\mathbb{Z}$ under subtraction.

b) The set of positive integers $\mathbb{Z}^+$ and division.

c) The set of 2×2 matrices with real coefficients $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$, and matrix addition.

Solution

a) Since the difference between two integers is a unique integer, the operation is a binary one and the set is closed under subtraction.

b) Since the quotient of any two positive integers is a unique real number, the operation is binary. However, the quotient $\frac{a}{b}$ is not always a positive integer and hence $\mathbb{Z}^+$ is not closed under division. (Please note here that in most books the operation is not considered a binary operation because the set is not closed under it.)

c) Take two arbitrary 2×2 matrices with real coefficients $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

Now since each entry in $\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ is real and unique for this sum the operation is binary. And since the resulting matrix is also an element of the set of 2×2 matrices with real coefficients, then it is closed under this operation. (Please note here too that the operation is considered binary because the result is a unique member of the set of real 2×2 matrices.)

Properties of binary operations

Definition 2

A binary operation $*$ on a set G is **associative** if and only if for all $a, b, c \in G$,

$a * (b * c) = (a * b) * c$.

A binary operation $*$ on a set G is **commutative** if and only if for all $a, b \in G$,

$a * b = b * a$.

A binary operation $*$ on a set G is **distributive** over another binary operation Δ if and only if for all $a, b, c \in G$,

$a * (b \,\Delta\, c) = (a * b) \,\Delta\, (a * c)$.

Example 2

Decide whether subtraction in the set of integers $\mathbb{Z}$ is associative or commutative.

Solution

Since $a - (b - c) = (a - b) + c \neq (a - b) - c$, the operation is not associative.

Also, $a - b \neq b - a$, except for $a = b = 0$, so the operation is not commutative.

Example 3

Decide whether the operation of intersection over the power set of a given set A is associative or commutative. Additionally, check if the operation of intersection is distributive over the union operation.

Solution

- Associativity: Let X, Y, and Z be subsets of A.

  For all $a \in X \cap (Y \cap Z) \Leftrightarrow a \in X$ and $a \in (Y \cap Z) \Leftrightarrow a \in X$ and $a \in Y$ and $a \in Z$

  $\Leftrightarrow (a \in X$ and $a \in Y)$ and $a \in Z \Leftrightarrow a \in (X \cap Y) \cap Z$. Therefore,

  $X \cap (Y \cap Z) = (X \cap Y) \cap Z$.
- Commutativity: If $a \in (X \cap Y) \Leftrightarrow a \in X$ and $a \in Y \Leftrightarrow a \in Y$ and $a \in X \Leftrightarrow a \in (Y \cap X)$. Therefore,

  $X \cap Y = Y \cap X$ and the operation is commutative.
- We proved in Chapter 1 that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Therefore, the operation 'intersection' is distributive over the operation 'union'.

Example 4

Decide whether matrix addition over the set of 2×2 matrices with real coefficients is associative and commutative.

Solution

- Associativity: Let $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ represent members of the set of

  2×2 matrices with real coefficients, i.e.

  $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \ldots$

$$M_1 + (M_2 + M_3) = M_1 + \begin{pmatrix} a_2 + a_3 & b_2 + b_3 \\ c_2 + c_3 & d_2 + d_3 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 + a_3 & b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 & d_1 + d_2 + d_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = (M_1 + M_2) + M_3$$

- Commutativity:

$$M_1 + M_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} = \begin{pmatrix} a_2 + a_1 & b_2 + b_1 \\ c_2 + c_1 & d_2 + d_1 \end{pmatrix} = M_2 + M_1$$

Sometimes, even if the operation itself is not commutative, you may still have some elements that are 'commutable'. For example, consider the following operation defined over $\mathbb{Z}^+$

$$a \circ b = a^b$$

In general $a \circ b \neq b \circ a$; for example
$2 \circ 5 = 2^5 = 32 \neq 5 \circ 2 = 5^2 = 25$,
however
$2 \circ 4 = 2^4 = 16 = 4 \circ 2 = 4^2 = 16$.

Operation (Cayley) tables

If S is a small finite set, it is often convenient to define the binary operation on S by means of a table, which is constructed as follows:

All the elements of the set S are written across the top row of the table and also vertically, in the same order down the leftmost column of the table, as shown. The element corresponding to $c * b$, for example, is at the intersection of the row containing c with the column containing b.

$*$	a	b	c	$\dots$
a				
b				
c		$c * b$		
$\vdots$				

Such operation tables are also called **Cayley tables**, after the British mathematician Arthur Cayley.

These tables have what is called the **Latin square property** (see page 1273).

Example 5

A binary operation Δ is defined over the set $S = \{m, n, r, s\}$ using the table below.

Show that the set is closed under this operation, decide whether it is commutative, and check on particular instances of associativity using n, r, and s.

Δ	m	n	r	s
m	m	n	r	s
n	n	r	s	m
r	r	s	m	n
s	s	m	n	r

When a set with a binary operation is given by a Cayley's table then the operation is commutative if and only if equal elements appear in all positions that are symmetrically placed relative to the main diagonal. That is, to check whether an operation defined by a Cayley's table is commutative, simply draw the main diagonal, and see if the table is symmetric about it. For example, the operation Δ defined by the table above is commutative.

Solution

- Since all elements in the table are elements of set S, S is closed under Δ.
- Since for all possible choices such as $n \Delta r = s = r \Delta n$, or $s \Delta r = n = r \Delta s$, etc. the operation is commutative.
- Consider $(n \Delta r) \Delta s = s \Delta s = r$, and $n \Delta (r \Delta s) = n \Delta n = r$; therefore, $(n \Delta r) \Delta s = n \Delta (r \Delta s)$.

  However, if we have to decide whether the operation is associative we have to consider all possible combinations, which is a very tedious task.

Example 6

Is the binary operation on $\mathbb{R}$ defined by $a * b = a + b - 1$ commutative? Is it associative?

Solution

- Since $a * b = a + b - 1$ and $b * a = b + a - 1 = a + b - 1 = a * b$, then the operation is commutative.
- $(a * b) * c = (a + b - 1) * c = (a + b - 1) + c - 1 = a + b + c - 2$, and

  $a * (b * c) = a * (b + c - 1) = a + (b + c - 1) - 1 = a + b + c - 2$; therefore $*$ is associative.

Example 7

Is the binary operation on $\mathbb{R}$ defined by $a * b = ab + 1$ commutative? Is it associative?

Solution

In some cases, you may find that associative behaviour holds for some elements of the set in question. However, we can only claim the associativity to hold if it does so for every element.

- Since $a * b = ab + 1$ and $b * a = ba + 1 = ab + 1 = a * b$, then the operation is commutative.
- $(a * b) * c = (ab + 1) * c = (ab + 1)c + 1 = abc + c + 1$, and

  $a * (b * c) = a * (bc + 1) = a(bc + 1) + 1 = abc + a + 1 \neq abc + c + 1$; therefore $*$ is not associative.

The identity element

In general, if we have a set S with a binary operation Δ on that set, then an element e of S is called a **left-identity** if $e \Delta a = a$ for every a in S. Similarly, it is called a **right-identity** if $a \Delta e = a$. e is called an **identity** if it is both a **left**- and a **right-identity**. This is given formally in the following definition.

Definition 3

An element e in a set S is an **identity element** (or **identity**) for an operation Δ defined over S if

$$e \Delta a = a \Delta e = a$$

for every element $a \in S$.

An element e is an identity if it leaves every element unchanged.

Theorem 1

If an operation $\circ$ admits a left-identity e_1 and a right-identity e_2, then these two identities are equal.

Theorem 1 means that there is a unique identity element i.e. there is one and only one identity element.

Proof

If we consider the left-identity e_1, then $e_1 \circ e_2 = e_2$. However, if we consider the right-identity e_2, then $e_1 \circ e_2 = e_1$. Thus $e_1 = e_2$ since they are both equal to $e_1 \circ e_2$.

- Addition over the integers has 0 as the identity element:

  For all $a \in \mathbb{Z}$, $a + 0 = 0 + a = a$.

- Multiplication over the set of non-zero integers has 1 as the identity element:

  For all $a \in \mathbb{Z}\backslash\{0\}$, $a \times 1 = a$ or $1 \times a = a$.

- The set A is the identity element for the operation of intersection over the power set of A:

  If $B \subseteq A$, then $A \cap B = B \cap A = B$.

- The empty set, $\varnothing$, is the identity for the operation of union over the power set of A:

  If $B \subseteq A$, then $\varnothing \cup B = B \cup \varnothing = B$.

- If we consider the set of real numbers and define the operation $*$ by $a * b = a^b$, then 1 is a right-identity only since $a * 1 = a^1 = a$, but $1 * a = 1^a \neq a$, so 1 is not a left-identity.

- The set of 2×2 matrices with real coefficients

  $M_2 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R}\right\}$ under matrix multiplication

  has $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as an identity element since

  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- The binary operation on $\mathbb{R}$ defined by $a * b = a + b - 13$ has 13 as an identity:

  $a * 13 = a + 13 - 13 = a$, and $13 * a = 13 + a - 13 = a$.

Notation

It is convenient when possible, to write ab when we mean $a \circ b$.

Theorem 2

If a binary operation $*$ on a set S admits an identity element e, then this element is unique.

Proof

Since e is an identity element, then for *any* $x \in S$:

$$x * e = e * x = x \tag{1}$$

Assume that there is at least another different identity element e', then for *any* $x \in S$:

$$x * e' = e' * x = x \tag{2}$$

Now, since (1) is true for *any* $x \in S$, it has to be true for $x = e'$, and thus:

$$x * e = e * x = x \Rightarrow e' * e = e * e' = e' \tag{3}$$

Also, since (2) is true for *any* $x \in S$, it has to be true for $x = e$, and thus:

$$x * e' = e' * x = x \Rightarrow e * e' = e' * e = e \tag{4}$$

By comparing (3) and (4) we notice that $e * e' = e'$ and $e * e' = e$, and hence $e = e'$.

Therefore, our assumption of the existence of an identity element other than e is false and we can conclude that the identity element e is unique.

- The binary operation on $\mathbb{Z}$ defined by $a * b = ab + 1$ has no identity. Assume e is an identity, then $a * e = ae + 1 = a \Rightarrow e = \dfrac{a-1}{a}$ which is not unique! Also, consider the case of $a = 1$, then $e = 0$, but $a * 0 = 0 + 1 \neq a$. So, this operation has no identity element.

The inverse element

In general, if we have a set S with a binary operation Δ on that set, then an element a of S has a **left-inverse** a' if $a' \,\Delta\, a = e$. Similarly, a has a **right-inverse** a'' if $a \,\Delta\, a'' = a$. An element that is both a **left-** and a **right-inverse** is called an **inverse** and we denote it by a^{-1}. This is formally given in the following definition.

Definition 4

An element a^{-1} in a set S is an **inverse element** (or **inverse**) for an operation Δ defined over S if

$$a^{-1} \,\Delta\, a = a \,\Delta\, a^{-1} = e$$

for any element $a \in S$.

Theorem 3

If, for an associative operation $\circ$, an element a admits a left-inverse a' and a right-inverse a'', then these two inverses are equal.

Proof

$a' \circ a \circ a'' = (a' \circ a) \circ a'' = e \circ a'' = a''$, also

$a' \circ a \circ a'' = a' \circ (a \circ a'') = a' \circ e = a'$, and therefore $a' = a''$.

- The set of integers $\mathbb{Z}$ under addition admits for each element an inverse; namely, for every $a \in \mathbb{Z}$, $-a$ is the inverse since $a + (-a) = (-a) + a = 0$.
- The set of non-negative real numbers under multiplication admits an inverse for each element; namely, for every $a \in \mathbb{R}\backslash\{0\}$, $\frac{1}{a}$ is the inverse since $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$.
- The set of invertible 2×2 matrices with real coefficients

  $$GL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$$

  under matrix multiplication admits an inverse

  $$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$ for each of its members since

$$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 4

If an operation $*$ defined on a set S has an identity element e, then every invertible element admits a *unique* inverse.

Proof

Let us take any invertible element $a \in S$. Assume that there is no unique inverse for a, then we can say that there are at least two inverses for a. Let the inverses of a be a_1 and a_2.

By definition:

$$a * a_1 = a_1 * a = e \qquad (1)$$

$$a * a_2 = a_2 * a = e \qquad (2)$$

By comparing (1) and (2), we can write

$a * a_1 = a_1 * a = e = a * a_2 = a_2 * a$, which implies that

$a * a_1 = e = a_2 * a$, and hence a_1 and a_2 are the right- and left-inverses of a which should be equal by Theorem 3.

Therefore, our assumption that there are at least two different inverses for a is false, and a admits a unique inverse, which we will denote here by a^{-1}.

Example 8

Consider the operation $*$ on the set of integers defined by $a * b = a + b - 13$. Does each element have an inverse?

Solution

Let a be an integer. Let b be a right-inverse of a. Recall that the identity for this operation is 13. Then $a * b = 13$. That is, $a + b - 13 = 13$. Solving for b we find $b = -a + 26$. This is also a left-inverse of a since $(-a + 26) * a = -a + 26 + a - 13 = 13$.

Cancellation laws

Theorem 5

Let $*$ be a binary operation that is defined on a non-empty set S with an identity element e and an inverse element a^{-1} for each element $a \in S$. The left and right cancellation laws hold, i.e.

if $a * b = a * c$, then $b = c$; and if $b * a = c * a$ then $b = c$.

Proof

Suppose $a * b = a * c$, and let a^{-1} be the inverse of a. Now operating with a^{-1} from the left we have

$a^{-1} * (a * b) = a^{-1} * (a * c) \Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$
$\Rightarrow e * b = e * c \Rightarrow b = c$; this is the left cancellation law.

Similarly, if $b * a = c * a$ we operate with a^{-1} from the right, and we have

$(b * a) * a^{-1} = (c * a) * a^{-1} \Rightarrow b * e = c * e \Rightarrow b = c$ (details are left for you as an exercise).

3.2 Groups

Definition 5

Let G be a non-empty set together with a binary operation $*$ that assigns to each ordered pair $(a, b) \in G^2$ an element denoted by $a * b$[1]. We say G is a **group** under this operation if the following four properties are satisfied. We usually write $(G, *)$ or $\{G, *\}$ to denote a group with an operation.

1. **Closure:** The set G is closed under this operation, i.e. $a * b \in G$.
2. **Associativity:** The operation is associative, i.e. $(a * b) * c = a * (b * c)$ for all a, b, c in G.
3. **Identity:** There is an element e in G, such that $a * e = e * a = a$ for all a in G. e is the identity element for the group under this operation.
4. **Inverses:** For each element a in G, there is an element b in G such that $a * b = b * a = e$. b is the inverse of a and every so often denoted by a^{-1}. (Notice that if b is the inverse of a, then a is the inverse of b. *Therefore, we can say that the inverse of the inverse is the original element itself* $\left(a^{-1}\right)^{-1} = a$.

[1] We usually consider that $a * b \in G$ by definition of a binary operation, but the IB syllabus does not define a binary operation to have this closure property. So, we will follow the syllabus in this publication and list the closure property separately.

If a group has the property that $a * b = b * a$, for every pair of elements a and b, we say the group is **Abelian** or **commutative**. A group is **non-Abelian** if there is at least one pair of elements a and b for which $a * b \neq b * a$.

A group G is said to be **finite** (or of **finite order**) if it has a finite (restricted) number of elements. In this case, the number of elements in G is called the **order** of G and is denoted by $|G|$. A group with infinitely many elements is said to have **infinite order**, or is **infinite**.

- $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all groups under ordinary addition. The identity is 0 and the inverse of a is $-a$. These are **infinite groups**.

Theorem 6 (Latin square property)

This property states that for all elements a and b in a group $(G, *)$, there exists a unique element c such that $a * c = b$.

The converse of Theorem 6 is not true, i.e. if for all elements a and b, there exists a unique element c such that $a * c = b$, it does not necessarily follow that the set under that operation is a group.

Proof

Existence: Let $c = a^{-1} * b$.

Since $a^{-1} \in G$ and $b \in G$, then by closure $a^{-1} * b \in G$, and

$a * c = a * \left(a^{-1} * b\right) = \left(a * a^{-1}\right) * b = e * b = b$ and so c exists and it satisfies $a * c = b$.

Uniqueness: Let d be another element such that $a * d = b$.

$$d = e * d = (a^{-1} * a) * d = a^{-1} * (a * d) = a^{-1} * b = c$$

We can prove, in a similar manner, that there exists a unique element g such that $g * a = b$.

The Latin square property gets its name from the fact that for a finite group $(G, *)$, it is possible to draw a Cayley table, which gives the element $a * b$ in the row corresponding to a and the column corresponding to b. This table will be a Latin square, a square display in which each possible value for a cell appears exactly once in each row, and exactly once in each column.

The set $\{1, -1, i, -i\}$ where $i^2 = -1$, is a group under complex multiplication.

Cayley's table is a good tool to use to check this group.

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Notice here that each element appears in the table, once every row and once every column, implying that the set is closed under multiplication and that the operation gives a unique element for every pair.

The row corresponding to 1 yields the same values as the top row, implying that 1 is the identity. This is confirmed by observing that the column corresponding to 1 is also the same.

1 appears in every row and column, implying that every element has an inverse. We will assume that multiplication of complex numbers is known to be associative. Finally, the table is symmetric around its main diagonal, and that is why it is an Abelian group. This group is **finite**.

In Cayley tables for **groups**, the following are true:

1 All entries must belong to the members of the group indicating closure.

2 Every entry appears exactly once in every column and every row. If a binary operation is well defined, then if $a * b = c$, then c is unique.

3 The identity element must appear in every row and column. Since every element has an inverse, then, for example, $a * a^{-1} = e$, implying that it is in the a-row and in the a^{-1}-row, and since the inverse is unique, then e appears only once in each.

Examples of groups

- $(\mathbb{Z}, \times)$ is not a group. It satisfies closure, identity, and associativity. However, not every element $a \in \mathbb{Z}$ has an inverse. For example, there is no integer b such that $3b = 1$.
- $(\mathbb{Q}^+, \times)$ is an Abelian group. The product of any two rational numbers is a rational number, so closure is satisfied; the identity is 1, which is a rational number, and every positive rational number a has an inverse $\frac{1}{a}$. Also, for every ordered pair $a \times b = b \times a$. The group is infinite.

- The set of 2×2 matrices with real coefficients $M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$ under matrix addition, $(M_2, +)$ is an Abelian group. It is closed since the sum of any two 2×2 matrices is a 2×2 matrix, the identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Addition of matrices is associative and therefore associativity is assumed. Also, as addition is commutative the group is Abelian. This group is infinite.
- The set of invertible 2×2 matrices with real coefficients $GL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$ under matrix multiplication, $(GL_2, \cdot)$.

  We have discussed this set in the discussion following Theorem 3, where we showed that it has an identity and every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse $\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$. Since the elements are matrices, we can assume that associativity of matrix multiplication holds here. We have not shown that the set is closed under multiplication yet. To show closure, we need to show that if we multiply two non-singular matrices, the answer should also be non-singular. Recall that for a matrix A to be non-singular, the determinant $(ad - bc)$ must be different from zero. Also, we need to recall that $\det(AB) = \det(A)\det(B)$, and if A and B are non-singular, their determinants are different from zero and hence $\det(AB) \neq 0$, which implies that AB is a member of GL_2, and closure is satisfied.

  Therefore, $(GL_2, \cdot)$ is a group. However, it is non-Abelian because multiplication of matrices is not commutative.

Please remember that for examinations starting 2014, questions containing matrices will not appear in official exam papers. Matrices are included here to explain certain concepts.

Theorem 7

If a and b are elements of a group $(G, *)$, then

1 $\left(a^{-1}\right)^{-1} = a$

2 $(a * b)^{-1} = b^{-1} * a^{-1}$

Proof

1 Since for every element a in G there is an inverse a^{-1}, such that $a * a^{-1} = a^{-1} * a = e$. Consider a^{-1} as an element in G, and hence $a^{-1} * a = a * a^{-1} = e$ implying that the inverse of a^{-1} is a, i.e. $\left(a^{-1}\right)^{-1} = a$.

2 We proved beforehand that the inverse of an element is unique.

$(a*b)(b^{-1}*a^{-1}) = a*(b*b^{-1})*a^{-1}$ using associativity, thus

$(a*b)(b^{-1}*a^{-1}) = a*e*a^{-1} = a*a^{-1} = e$; similarly

$(b^{-1}*a^{-1})(a*b) = b^{-1}*e*b = b^{-1}*b = e.$

Hence, $b^{-1}*a^{-1}$ is the unique inverse of $a*b$.

Example 9

Consider the set of invertible 2 × 2 matrices with real coefficients $SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$ under matrix multiplication, $(SL_2, \cdot)$.

a) Show that $(SL_2, \cdot)$ is a group.

b) If $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}$ are elements of this group, find $(A \cdot B)^{-1}$, $A^{-1} \cdot B^{-1}$, and $B^{-1} \cdot A^{-1}$.

Solution

a) The set is closed under matrix multiplication because for any two members A and B, AB (we will use AB to represent $A \cdot B$) will also be in the same set.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \Rightarrow AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$, and

since $\det(AB) = \det(A)\det(B)$, then $\det(AB) = 1 \times 1 = 1$ and AB is a member of this set. (You can also show that $\det(AB) = 1$ directly. With some algebra, you can write $\det(AB) = ad(eh - fg) + bc(fg - eh)$, but $eh - fg = 1$, and so $\det(AB) = ad - bc = 1$.)

The identity element $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a member of the set.

Moreover, every element has an inverse in the set.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ with $\det(A^{-1}) = da - cb = 1.$

And associativity is assumed.

b) $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 61 & 78 \\ 43 & 55 \end{pmatrix}$

$\Rightarrow (AB)^{-1} = \begin{pmatrix} 55 & -78 \\ -43 & 61 \end{pmatrix}$; also $A^{-1} = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}, B^{-1} = \begin{pmatrix} 9 & -5 \\ -7 & 4 \end{pmatrix}$

$\Rightarrow A^{-1}B^{-1} = \begin{pmatrix} 94 & -53 \\ -39 & 22 \end{pmatrix}$ and $B^{-1}A^{-1} = \begin{pmatrix} 55 & -78 \\ -43 & 61 \end{pmatrix}$

Notice here that this example demonstrates Theorem 7.2 above.

Notation

1 Since the binary operation in a group is an associative operation, the convention is to write $a * b * c$ instead of $(a * b) * c$ or $a * (b * c)$.

2 It is also the convention to write $\underbrace{a * a * \ldots * a}_{r\ times}$ as a^r, and we interpret this 'exponent' as the binary operation '$*$' applied r times. Hence, the laws of exponents such as a^{r+s} are also interpreted similarly, '$*$' applied r times and s times, i.e. $a^{r+s} = a^r * a^s$; and finally, $(a^r)^s = \underbrace{a^r * a^r * \ldots * a^r}_{s\ times} = a^{rs}$.

3 We also define $a^0 = e$, and $a^{-r} = \underbrace{a^{-1} * a^{-1} * \ldots * a^{-1}}_{r\ times}$.

Congruence revisited

In the previous chapter we defined congruence classes modulo m (*residue classes mod m*) and concluded that they partition the set of integers into m classes $[0], [1], \ldots, [m-1]$.

We define a congruence class as follows:

Definition 6

Let $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. The **congruence class of a modulo *n*** (denoted by $[a]$) is the set of all integers that are congruent to a modulo n, that is,

$$[a] = \{x \mid x \in \mathbb{Z} \text{ and } x \equiv a \pmod{n}\}.$$

Note: To say that $x \equiv a \pmod{n}$ means that $n \mid (x - a)$ or $x - a = kn$ for some integer k, or equivalently $x = a + kn$. Thus, a practical way of expressing a congruence class is

$[a] = \{x \mid x \equiv a \pmod{n}\} = \{x \mid x = a + kn, k \in \mathbb{Z}\}$, or in other words

$[a] = \{a + kn \mid k \in \mathbb{Z}\}$.

In congruence modulo 7, we have

$$[4] = \{4 + 7k \mid k \in \mathbb{Z}\} = \{4, 4 \pm 7, 4 \pm 14, 4 \pm 21, \ldots\}$$
$$= \{\ldots, -17, -10, -3, 4, 11, 18, 25, \ldots\}$$
$$[-3] = \{-3 + 7k \mid k \in \mathbb{Z}\} = \{-3, -3 \pm 7, -3 \pm 14, -3 \pm 21, \ldots\}$$
$$= \{\ldots, -24, -17, -10, -3, 4, 11, 18, \ldots\}$$

We observe that $[-3] = [4]$, which should not be surprising because we know that $-3 \equiv 4 \pmod 7$. This is an example of the following theorem.

Theorem 8

$a \equiv b \pmod{n}$ if and only if $[a] = [b]$.

Proof

($\Rightarrow$): Letting $a \equiv b \pmod{n}$, we show that $[a] \subseteq [b]$ first. Let $c \in [a]$, then $c \equiv a \pmod{n}$, but $a \equiv b \pmod{n}$; thus, by the transitive property, $c \equiv b \pmod{n}$ and $c \in [b]$ and therefore $[a] \subseteq [b]$.

Similarly we can show that $[b] \subseteq [a]$, and hence $[a] = [b]$.

($\Leftarrow$): Assume $[a] = [b]$.

Now $a \in [a]$ and hence $a \in [b]$ implying that $a \equiv b \pmod{n}$.

Note: We can use Theorem 8 to show that two congruence classes modulo n are either equal or disjoint.

If they are disjoint, there is nothing to prove. If they are not disjoint, then there is at least $x \in [a] \cap [b]$, which in turn means that $x \equiv a \pmod{n}$ and $x \equiv b \pmod{n}$. Thus, $a \equiv b \pmod{n}$ by transitive and symmetric properties, and $[a] = [b]$ by Theorem 6.

Theorem 9

There are precisely n different congruence classes modulo n, $[0], [1], [2], \ldots, [n-1]$.

Proof

(*Outline only*) Recall from the previous chapter that any integer $a \equiv r \pmod{n}$, where r is the remainder when dividing a by n. Hence, for all integers $[a] = [r]$. Since r must be non-negative and less than n, then the possible values are $0, 1, 2, \ldots, n-1$.

Definition 7

The set of all congruence classes modulo n is denoted by $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$. (It is read as 'Z mod n'.)

For example, $\mathbb{Z}_6 = \{[0], [1], [2], [3], [4], [5]\}$.

Theorem 10

Let $a, b, c, d \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. Then $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$ imply the following:

1 $a + c \equiv b + d \pmod{m}$

2 $a - c \equiv b - d \pmod{m}$

3 $ac \equiv bd \pmod{m}$

Proof

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $m \mid (a-b)$ and $m \mid (c-d)$. These imply that $m \mid ((a-b)+(c-d))$. But this is the same as $m \mid ((a+c)-(b+d))$. This proves (**1**). Proof of (**2**) is similar. To prove (**3**), note that $m \mid (a-b) \Rightarrow m \mid c(a-b)$ and $m \mid (c-d) \Rightarrow m \mid b(c-d)$. Thus $m \mid (c(a-b)+b(c-d))$, which is the same as $m \mid (ac-bd)$. This completes the proof.

Note: Theorem 10 can be applied to a simpler case too, which we state overleaf without proof.

If $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, such that $a \equiv b \pmod{m}$, then the following hold:

1 $a + c \equiv b + c \pmod{m}$

2 $a - c \equiv b - c \pmod{m}$

3 $ac \equiv bc \pmod{m}$

Example 10

Apply the previous theorems to $23 \equiv 7 \pmod 8$ using your own choice of numbers.

Solution

Let us consider adding 5 to both sides, i.e.

$$23 + 5 \equiv 7 + 5 \pmod 8 \Rightarrow 28 \equiv 12 \pmod 8$$

Subtract 9:

$$23 - 9 \equiv 7 - 9 \pmod 8 \Rightarrow 14 \equiv -2 \pmod 8$$

Multiply by 2:

$$23 \times 3 \equiv 7 \times 3 \pmod 8 \Rightarrow 69 \equiv 21 \pmod 8$$

Does the converse of the previous theorem work?

For (**1**) and (**2**), the answer is obviously yes:

$$a + c \equiv b + c \pmod{m} \Rightarrow a + c - c \equiv b + c - c \pmod{m} \Rightarrow a \equiv b \pmod{m}$$

and

$$a - c \equiv b - c \pmod{m} \Rightarrow a - c + c \equiv b - c + c \pmod{m} \Rightarrow a \equiv b \pmod{m}$$

For (**3**), let us take an example:

$33 \equiv 12 \pmod 7 \Leftrightarrow 3 \times 11 \equiv 3 \times 4 \pmod 7$. Cancel the 3 from both sides and you have

$11 \equiv 4 \pmod 7$, which is true!

However,

$$52 \equiv 12 \pmod 8 \Leftrightarrow 13 \times 4 \equiv 3 \times 4 \pmod 8 \text{ but } 13 \not\equiv 3 \pmod 8.$$

In fact if c and m are relatively prime, then $ac \equiv bc \pmod{m} \Rightarrow a \equiv b \pmod{m}$.

$$63 \equiv 15 \pmod 8 \Leftrightarrow 21 \times 3 \equiv 5 \times 3 \pmod 8 \text{ and } 21 \equiv 5 \pmod 8$$

Theorem 11

If $[a] = [b]$, and $[c] = [d]$ in $\mathbb{Z}_n$, then

$$[a + c] = [b + d], \text{ and } [ac] = [bd].$$

Proof

$[a] = [b] \Rightarrow a \equiv b \pmod{n}$, and $[c] = [d] \Rightarrow c \equiv d \pmod{n}$, and hence by Theorem 8 $a + c \equiv b + d \pmod{n}$, and $ac \equiv bd \pmod{n}$; and hence by Theorem 8 $[a + c] = [b + d]$, and $[ac] = [bd]$.

Now we can define two new operations on the set $\mathbb{Z}_n$.

Definition 8

Addition and multiplication in $\mathbb{Z}_n$ are defined by

$$[a] + [c] = [a + c] \text{ and } [a][c] = [ac].$$

Notation (1)

For convenience, and as long as it is clear from the context that we are in modulo n mode, we will use the symbol + for addition modulo n. For multiplication modulo n, we will place the numbers next to each other rather than use symbols, so ab will mean $a \times b$.

In many sources, you will find that authors choose to attach the mod to the operation symbol such as $+_n$ for addition modulo n and $\times_n$ for multiplication modulo n.

Example 11

In $\mathbb{Z}_7$, perform the following operations:

$[5] + [3]$, $[4][6]$

Solution

$[5] + [3] = [5 + 3] = [8] = [1]$ since $[8] = [1 + 7] = [1]$

$[4][6] = [4 \cdot 6] = [24] = [3]$ since $[24] = [3 + 21]$

Notation (2)

So far, we have been using $[a]$ to represent classes in $\mathbb{Z}_n$. However, whenever the context is clear that we are dealing with $\mathbb{Z}_n$, we will replace the class notation '$[a]$' with a. In $\mathbb{Z}_7$ for instance we write 5 to indicate $[5]$ and we might say $5 + 4 = 2$ since we mean the classes and not the numbers themselves.

For example, here are the Cayley tables for addition in $\mathbb{Z}_5$ and multiplication in $\mathbb{Z}_5$.

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Example 12

Determine whether $(\mathbb{Z}_6, +)$ is a group.

Solution

A Cayley table will be helpful in this exercise.

Closure has been discussed before. However, it is apparent from the table that all elements are members of $\mathbb{Z}_6$, so the set is closed under addition modulo 6.

The identity element is also clear – it is 0.

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Since 0 appears in every row and every column, then every element has its inverse. For example, the inverse of 2 is 4 and 3 is its own inverse.

Since we defined the addition of residue classes through addition of integers, the operation can be assumed to be associative.

Hence $(\mathbb{Z}_6, +)$ is a group.

Moreover, the operation is commutative and the group is an Abelian group.

Example 13

Determine whether the set $\{1, 3, 7, 9\}$ in $\mathbb{Z}_{10}$ with multiplication modulo 10 is a group.

Solution

Again a Cayley table is helpful.

$\times$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

The set is closed under multiplication modulo 10.

Associativity is assumed.

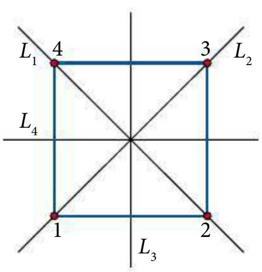
The identity element is 1 since $1 \times a = a$ for all a in this set. This is clear from the table as the first row and the first column demonstrate that multiplying by 1 left the elements untouched.

1 and 9 are their own inverses, 7 is the inverse of 3 and vice versa.

The group is also Abelian.

Extended examples of groups

Symmetries of a square

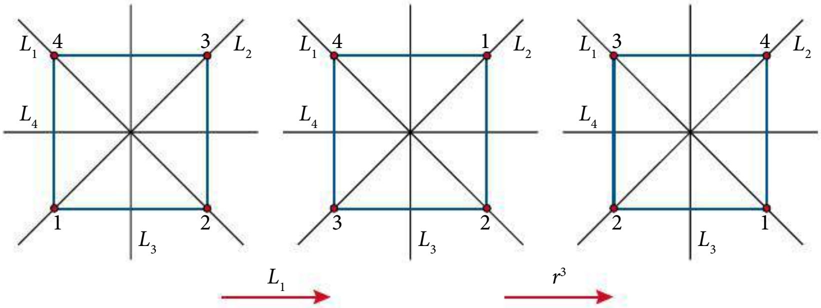


A square can be rotated counterclockwise through certain angles or reflected about certain lines, and it will end up with its original appearance. The corners, however, would have been moved. Rotation is centred at the centre of the square and the lines of reflection are the two lines through the diagonals, L_1 and L_2, and the two lines through the vertical axis of symmetry, L_3, and the horizontal axis, L_4. Rotation is through multiples of 90°: $e = R_0$, $r = R_{90}$, $r^2 = R_{180}$, or $r^3 = R_{270}$. Notice that $R_{360} = R_0$.

The table right gives the results of performing any of these 'symmetries'.

Symmetry	Before	$\rightarrow$	After
e	4 3 1 2	$\xrightarrow{R_0}$	4 3 1 2
r	4 3 1 2	$\xrightarrow{R_{90}}$	3 2 4 1
r^2	4 3 1 2	$\xrightarrow{R_{180}}$	2 1 3 4
r^3	4 3 1 2	$\xrightarrow{R_{270}}$	1 4 2 3
L_1	4 3 1 2	$\xrightarrow{L_1}$	4 1 3 2
L_2	4 3 1 2	$\xrightarrow{L_2}$	2 3 1 4
L_3	4 3 1 2	$\xrightarrow{L_3}$	3 4 2 1
L_4	4 3 1 2	$\xrightarrow{L_4}$	1 2 4 3

These rotations and reflections are known as the **symmetries of a square**. If a reflection or rotation is followed by another reflection or rotation, the result can be one of the eight symmetries listed. For example, if L_1 is followed by r^3, the result is equivalent to L_3, i.e. $r^3 \circ L_1 = L_3$. See figure below.



We call the set of symmetries $D_4 = \{e, r, r^2, r^3, L_1, L_2, L_3, L_4\}$. The operation we are using in this set is composition of transformations, $\circ$. Cayley's table for all possible compositions of these transformations is given below. Notice that all the entries in the table are members of D_4. This verifies the closure property for this set.

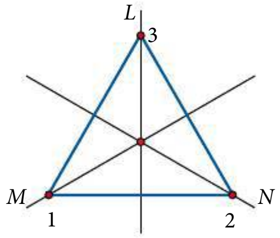
The composition of transformations is associative. Take, for example, $(rL_1)\,r^2$; this is r^2 followed by (rL_1), which in turn is L_1 followed by r, that is, the whole composition is r^2 followed by L_1 followed by r, which means rL_1r^2. We can argue similarly about $r(L_1r^2)$ and arrive at rL_1r^2. So, the operation is associative.

$\circ$	e	r	r^2	r^3	L_1	L_2	L_3	L_4
e	e	r	r^2	r^3	L_1	L_2	L_3	L_4
r	r	r^2	r^3	e	L_4	L_3	L_1	L_2
r^2	r^2	r^3	e	r	L_2	L_1	L_4	L_3
r^3	r^3	e	r	r^2	L_3	L_4	L_2	L_1
L_1	L_1	L_3	L_2	L_4	e	r^2	r	r^3
L_2	L_2	L_4	L_1	L_3	r^2	e	r^3	r
L_3	L_3	L_2	L_4	L_1	r^3	r	e	r^2
L_4	L_4	L_1	L_3	L_2	r	r^3	r^2	e

Clearly e, which in essence is doing nothing, is the identity and as is apparent from the table, every element has an inverse since e appears in every row and column. For example, the inverse of r is r^3 and vice versa, while each L_i is its own inverse.

Therefore $(D_4, \circ)$ is a group. Notice that $L_1 r = L_3$ while $rL_1 = L_4$ and so the group is not Abelian. Non-commutativity can also be seen by observing that the table is not symmetric about the main diagonal.

Symmetries of an equilateral triangle

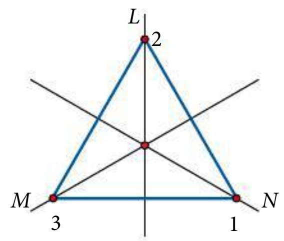


Another example of groups is the set of 'symmetries' in an equilateral triangle. There are three rotations, $I = R_0$, $R = R_{120}$, and $R^2 = R_{240}$, about the centroid, and there are three reflections around the lines through the three medians L, M, and N. We number the vertices as 1, 2, 3, so that you can discover what each transformation does.

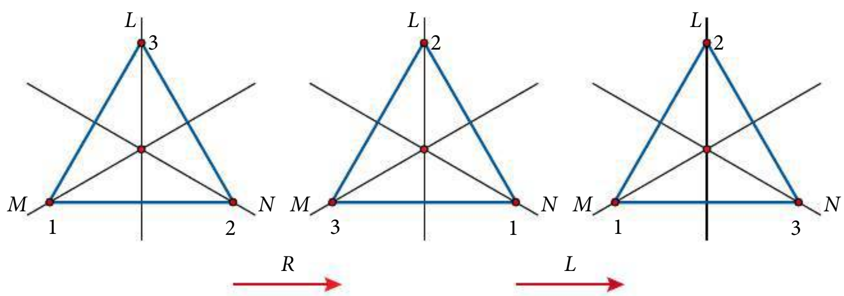
I does not change anything as expected.

R, for example, rotates the triangle around its centroid through an angle of 120° and so it takes 1 to the position taken by 2, 2 to the position of 3, and 3 to the position of 1 as shown in the diagram.

R^2 rotates the triangle through 240°. L reflects the triangle about its median L exchanging vertices 1 and 2 but keeping 3 untouched.



The composition of transformations can be looked at in a similar manner to the symmetries of the square and so the transformation LR is a rotation of 120° followed by a reflection in L, and so it is in essence a reflection in N and consequently we have $LR = N$. See figure below. (Remember that LR means that R is first, followed by L.)



Cayley's table below shows all possible compositions.

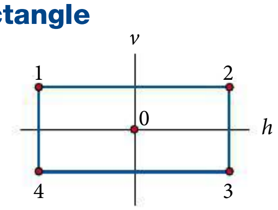
$\circ$	I	R	R^2	L	M	N
I	I	R	R^2	L	M	N
R	R	R^2	I	N	L	M
R^2	R^2	I	R	M	N	L
L	L	M	N	I	R	R^2
M	M	N	L	R^2	I	R
N	N	L	M	R	R^2	I

The set of six symmetries of the equilateral triangle with the operation of composition $\circ$, $(D, \circ)$ forms a group. Here is why.

The elements of the table are all members of the set and hence it is closed. Obviously, I is the identity element. The identity transformation I is included in every row and column and hence every element has an inverse. And associativity is assumed in the composition of transformations.

Notice, however, that $ML = R \neq LM = R^2$, and hence it is not Abelian. (Also, the table is not symmetric about the main diagonal.)

Symmetries of a rectangle



The last example of symmetries concerns the set of symmetries of a rectangle. Similar to what we have done with the square and triangle, we will label the vertices of the rectangle with integers and observe the outcome of each symmetry transformation.

There are two reflections in the rectangle, one about its horizontal axis of symmetry, h, and one about its vertical axis, v. There is one rotation of 180° counterclockwise around its centre, r. Obviously, there is the identity symmetry, e. In total therefore, we only have four symmetries for the rectangle, e, r, h, and v. The group of symmetries for the rectangle is then $(\{e, r, h, v\}, \circ)$.

Symmetry	Before	$\to$	After
e	$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$	$\xrightarrow{R_0}$	$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$
r	$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$	$\xrightarrow{R_{180}}$	$\begin{matrix} 3 & 4 \\ 2 & 1 \end{matrix}$
h	$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$	$\xrightarrow{h}$	$\begin{matrix} 4 & 3 \\ 1 & 2 \end{matrix}$
v	$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$	$\xrightarrow{v}$	$\begin{matrix} 2 & 1 \\ 3 & 4 \end{matrix}$

The table right gives the outcomes of these transformations.

Take rh for example; h results in $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, and when followed by r we get $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ which is nothing but the outcome of v. Cayley's table for this group is given below.

$\circ$	e	r	h	v
e	e	r	h	v
r	r	e	v	h
h	h	v	e	r
v	v	h	r	e

Notice that, similar to the other cases before, the set is closed under the composition operation, an identity element exists, the operation is associative, and each element has its inverse. As you see above, the identity appears in every row and column, and each element is its own inverse. You notice that in this case, the entries are symmetric about the main diagonal, and hence the operation is commutative. Therefore, this group is an Abelian group.

3.3 Permutations

Unfortunately, the convention used here is not universal. In some resources you will find that, in permutations, contrary to the traditional function composition, the operation is done 'left to right', i.e. $\alpha\beta$.

In this section, we study certain groups of functions, called *permutation* groups, from set S to itself. Although groups of permutations of any non-empty set S exist, we will focus on the case where S is finite, $|S| = n$.

Definition 9

If S is a set, then a **permutation** on S is a bijection $\alpha: S \to S$. The set of all permutations on a set S is denoted by S_n. If $\alpha, \beta \in S_n$, we simplify the notation by writing $\alpha\beta$ for $\alpha \circ \beta$, and $\alpha\beta$ is referred to as the *product* of α and β rather than α composed with β.

In Chapter 2, we learned that if two functions are bijective, then their composition is also bijective, so the product of permutations is a binary operation on S_n by definition 9, because if α and β are two such permutations, then $\alpha\beta$ will also be a permutation and hence we are assigning for the ordered pair (α, β) an element $\alpha\beta \in S_n$. Moreover, since $\alpha\beta \in S_n$ the set is closed under this operation. Also, since $\alpha: S \to S$ is a bijection, then $\alpha^{-1}: S \to S$ exists and is a bijection and hence $\alpha^{-1} \in S_n$. If we let e be the identity function on S, then the following hold:

1 If $\alpha, \beta \in S_n$, then $\alpha\beta \in S_n$.

2 If $\alpha, \beta, \gamma \in S_n$, then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

Associativity of composition of bijections.

3 The identity function e is in S_n.

4 If $\alpha \in S_n$ then $\alpha^{-1} \in S_n$.

This shows that S_n is a group under the binary operation of function composition. This is known as the **permutation group** on S. Also, since we are focusing on finite sets, and if S has n elements, then S_n is the **symmetric group** on n elements.

For example, consider the set $S = \{a_1, a_2, a_3, a_4, a_5\}$ and define the permutation $\alpha \in S_5$ by

$\alpha(a_1) = a_5,\ \alpha(a_2) = a_1,\ \alpha(a_3) = a_2,\ \alpha(a_4) = a_4,\ \alpha(a_5) = a_3$. That is, we have the following correspondence:

$$a_1 \mapsto a_5,\ a_2 \mapsto a_1,\ a_3 \mapsto a_2,\ a_4 \mapsto a_4,\ a_5 \mapsto a_3.$$

This can be simplified by using only the subscripts, i.e.

$\alpha(1) = 5,\ \alpha(2) = 1,\ \alpha(3) = 2,\ \alpha(4) = 4,\ \alpha(5) = 3$. Or

$$1 \mapsto 5,\ 2 \mapsto 1,\ 3 \mapsto 2,\ 4 \mapsto 4,\ 5 \mapsto 3.$$

So, nothing is lost by using this simplification, and since this process can be done for any permutation in S_n, then S can be replaced by $\{1, 2, 3, 4, 5\}$ or in general $S = \{a_1, a_2, a_3, a_4, \ldots, a_n\}$ can be replaced by $\{1, 2, 3, 4, \ldots, n\}$.

For example, when you have a list of items to sort, you are essentially faced with the problem of finding a permutation of the objects that will put them in order after the permutation.

If we consider permutations of n objects, there are $n!$ of them. To understand this, first think through where object number 1 ends up. There are n possibilities for that. After the outcome of object 1 is determined, there are only $n - 1$ possible outcomes for object number 2. Thus, there are $n(n - 1)(n - 2)... 3 \cdot 2 \cdot 1 = n!$ permutations of a set of n objects.

For example, if we consider all possible rearrangements of the set {1, 2, 3}, there are 3! = 6 of them. They are listed in the table below.

1	$1 \rightarrow 1$	$2 \rightarrow 2$	$3 \rightarrow 3$
2	$1 \rightarrow 2$	$2 \rightarrow 1$	$3 \rightarrow 3$
3	$1 \rightarrow 3$	$2 \rightarrow 2$	$3 \rightarrow 1$
4	$1 \rightarrow 1$	$2 \rightarrow 3$	$3 \rightarrow 2$
5	$1 \rightarrow 2$	$2 \rightarrow 3$	$3 \rightarrow 1$
6	$1 \rightarrow 3$	$2 \rightarrow 1$	$3 \rightarrow 2$

Here is one way to think about permutations (using permutations of three objects as an example). Imagine that there are three boxes labelled 1, 2, and 3. Initially, each contains a paper chip labelled with the same number: box 1 contains chip 1, and so on. A permutation is a rearrangement of the chips but in such a way that when you're done there is still only a single chip in each box.

In the table above, the notation $i \rightarrow j$ indicates that whatever was in box i moves to the box labelled j. So to apply permutation number 4 above means to take whatever chip is in box 2 and move it to box 3, to leave the contents of box 1 alone, and to take the chip from box 3 and put it into box 2. In other words, permutation number 4 above tells us to swap the contents of boxes 2 and 3.

The notation $i \rightarrow j$ is somewhat cumbersome to use, especially when the number of permutations is large. Below are the two possibilities for notation that we use in this book.

Notation

Two-row notation (array notation)

When we are investigating the permutation of objects in five boxes, we can write the permutation as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$$

This indicates that the contents of box 1 move to box 5, the chip in box 2 moves to box 1, the chip in box 3 moves to box 2, box 4 is unchanged, and the chip in box 5 moves to box 3.

The benefit of this notation is that it is very easy to discover where everything goes.

This notation indicates that each member of the first row is mapped onto the corresponding member in the second row (directly beneath it).

Product (composition) of permutations

This notation is used to find the product of any two permutations in the following manner:

In S_5, let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}$, then the product $\alpha\beta$ is the composition of α and β interpreted in the usual manner – β first, followed by α. So, for example,

$\alpha\beta(1) = \alpha(5) = 3$ and $\alpha\beta(3) = \alpha(2) = 1$, etc.

This process is done directly in the two-row notation.

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & \boxed{3} \end{pmatrix} \begin{pmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ \boxed{3} & 4 & 1 & 2 & 5 \end{pmatrix}$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$$

Note that $\alpha\beta \neq \beta\alpha$. S_5 is therefore not Abelian. This can be generalized for S_n.

Note: The identity element of S_n is written in **two-row** notation as

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

This notation helps you find the inverse of each permutation. To find the inverse of any permutation read from the bottom row to the top row rather than top to bottom – so if 3 appears below 2 in a permutation α then 2 must appear below 3 in α^{-1}. Thus if

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \text{ then}$$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

The shortcoming of the 2-row notation is that it requires writing down each number twice. Since the top row can always be put in order, however, there is no real need to write it, so simply listing the second row is sufficient (assuming there is an obvious way to put the boxes in order).

Cycle notation

We can write the example above as (1 5 3 2).

This indicates that the contents of box 1 move to box 5, the contents of box 5 to box 3, the contents of box 3 to box 2, and the contents of box 2 back moves back into box 1. The system is called **cycle notation** since the contents of the boxes in parentheses move in a cycle: 1 to 5, 5 to 3, 3 to 2, and 2 back to 1. Notice that 4 does not appear as the contents of box 4 were unchanged! However, you can also write the above permutation as (1 5 3 2)(4).

Another possible form of the cycle notation is (1, 5, 3, 2). This form may be helpful when we have 10 or more elements.

Permutations that do not move any items are often written as (1).

Some permutations have more than one cycle. For example, the cycle notation for the permutation corresponding to:

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

is

(1 3)(2 5)

There are two cycles: 1 to 3 and 3 moves back to 1, while the other cycle takes 2 to 5 and 5 back to 2.

In cycle notation, it is not convenient to have duplicate elements in the various cycles that make up the permutation, so something like (2 3)(2 5) is not usual. In such cases, the 'product' is simplified to give (2 5 3).

As another example for notation, consider the permutation (1 3 5)(2 7 6) of the numbers $\{1, 2, \ldots, 7\}$. Again, notice that 4 is not included here, as it stays fixed. However, if you want, you can clarify its position by writing (1 3 5)(2 7 6)(4).

Notice also that (1 3 5)(2 7 6) or (2 7 6)(1 3 5) are equivalent, i.e. the product of 'disjoint' permutations is commutative.

Note also that the ordering does not matter as long as each item to be permuted appears only once, and that you can list a cycle starting with any member of it. All of the following specify precisely the same permutation:

(1 4 6)(3 5 9 7 8); (1 4 6)(5 9 7 8 3); (4 6 1)(9 7 8 3 5)…

Product of permutations using cycle notation

Let us take α and β from the example above.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}; \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}$$

Written in cycle notation they are

α = (1 5 3 2) and β = (1 5)(2 4 3)

Now for $\alpha\beta$, as we know from composition of functions, β must be applied first. 1 goes to 5, and 5 in α goes to 3, so we have so far (1 3…)

Now 3 in β goes to 2, but 2 in α goes to 1, and so 3 in the composition must go to 1. This closes the first part of the new cycle. So it is (1 3). Next in β is 2, which goes to 4, followed by 4 in α, which is fixed. Thus, we have

(2 4). Hence, our final result will be (1 3)(2 4) which is the same result as above when written in two-row notation.

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

Similarly for $\beta\alpha$ we have: 1 goes to 5 in α and 5 goes to 1 in β and hence 1 goes to 1 in the composition, and so it is fixed. Next, 5 in α goes to 3 and 3 in β goes to 2, so we have so far (5 2…). However, 2 in α goes to 1 and 1 in β goes to 5 and so 2 goes to 5 in the composition, closing this part too, i.e. (2 5). Next in α we have 3, which goes to 2, but 2 in β goes to 4, and so 2 goes to 4 in the composition. Hence, we have (3 4…) in the composition. Knowing that 4 is fixed in α and 4 goes to 3 in β closes this part too and we have (3 4). Thus $\beta\alpha = (2\ 5)(3\ 4)$, which is the same result as above:

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$$

Example 14

Now try (1 3 4 2)(3 6 4 5)(1 6 2 3). Remember that we read from right to left!

Solution

$1 \to 6$, $6 \to 4$, $4 \to 2$, so $1 \to 2$

$2 \to 3$, $3 \to 6$, $6 \to 6$, so $2 \to 6$

$6 \to 2$, $2 \to 2$, $2 \to 1$, so $6 \to 1$, and this cycle closes, (1 2 6).

Next, we take the smallest number left in (1 6 2 3), 3.

$3 \to 1$, $1 \to 1$, $1 \to 3$, so $3 \to 3$, and 3 is fixed here.

$4 \to 4$, $4 \to 5$, $5 \to 5$, so $4 \to 5$

$5 \to 5$, $5 \to 3$, $3 \to 4$, and so $5 \to 4$, and this cycle closes too as (4 5).

Therefore, the product is:

(1 2 6)(4 5).

In 2-row notation, this could have been done in two stages:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 4 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 5 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 3 & 5 & 4 & 1 \end{pmatrix}$$

This is the same product as above.

Example 15

If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$, show that $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$ is the inverse of α.

Solution

If $\gamma = \alpha^{-1}$, then their product must be e.

$$\alpha\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e.$$

Inverse of a permutation

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 1 & 6 & 7 & 8 & 3 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 7 & 8 & 1 & 4 & 5 & 6 \end{pmatrix}$

Take the product $\alpha\beta$:

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = e$$

You can verify that this is true. It is also clear that

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = e$$

This is obviously an indication that α and β are inverses of each other. Comparing the two permutations, you can see clearly that in order to get the inverse of a permutation, you simply swap the two rows and rearrange the top row in numerical order!

In cycle notation, to find the inverse of a permutation, list the numbers in reverse order. For example α, written in cycle notation is $\alpha = (1\ 5\ 7\ 3)(4\ 6\ 8)$ and hence $\alpha^{-1} = (8\ 6\ 4)(3\ 7\ 5\ 1)$ which is β!

Inverse of a permutation

To find the inverse of a permutation α, we can use one of the two forms:

- If α is in the array form, then swap row 1 with row 2, then rearrange the new row 1 in numerical order.
- If α is in cycle form, write the representation of α down in reverse order. That is, reverse the order in which the numbers appear in each cycle as well as the order of the cycles themselves.

Example 16

Find the inverse of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$.

Solution

First swap rows.

$$\alpha^{-1} = \begin{pmatrix} 3 & 1 & 2 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

We now arrange the top row.

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

In cycle notation, $\alpha = (1\ 3\ 2)(4\ 5)$.

Hence, $\alpha^{-1} = (2\ 3\ 1)(5\ 4) = (1\ 2\ 3)(4\ 5)$, which is the same as above.

Inverse of a product of permutations

Since, as we have seen above, a permutation is a function, therefore it also obeys function rules.

Theorem 12

If α and β are two permutations defined on a set S, then $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$.

As a result of the property above, the **cancellation law** for permutation multiplication is valid. That is

$$\alpha\beta = \alpha\gamma \Leftrightarrow \beta = \gamma$$

The proof is straightforward: you multiply (from left) both sides of the equation by α^{-1}.

Proof

The proof follows from basic function rules.

$$(\alpha\beta)(\beta^{-1}\alpha^{-1}) = \alpha(\beta\beta^{-1})\alpha^{-1} \text{ associativity of composition}$$

$$= \alpha e \alpha^{-1} = \alpha\alpha^{-1} = e, \text{ also}$$

$$(\beta^{-1}\alpha^{-1})(\alpha\beta) = \beta^{-1}(\alpha^{-1}\alpha^{-1})\beta = e$$

Thus, $\beta^{-1}\alpha^{-1}$ is the inverse of $\alpha\beta$

Order of a permutation

Composing (multiplying) different permutations leads to the question of composing a permutation with itself. For a permutation α, taking its product with itself $\alpha\alpha$ can be written as α^2. In fact, the product of α with itself n-times is written as α^n.

Take for example the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}.$$

A few 'powers' of α are:

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

$$\alpha^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$\alpha^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$$

$$\alpha^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

$$\alpha^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e$$

Definition

For any permutation α, there exists a positive integer n such that $\alpha^n = e$. The smallest number n is called the order of the permutation.

In the previous example, the order of α is 6. We write $\text{ord}(\alpha) = 6$

Example

Consider the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$ as shown above earlier. Write it in cycle notation.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = (1\ 3\ 2)(4\ 5)$$

Notice here that we have a 2-cycle and a 3-cycle, while the order of the permutation is 6. This is a demonstration of the following theorem.

Theorem (Proof not included)

The order of a permutation written in disjoint cycle form is the *least common multiple of the lengths of the cycles.*

Example

Consider the permutation $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 4 & 3 & 8 & 1 & 2 \end{pmatrix}$ Write it in cycle notation, and verify that its order is 12.

The cycle notation for β is $(1\ 5\ 3\ 7)(2\ 6\ 8)$. 4 is fixed.

Since the length of the first cycle is 4 and the length of the second cycle is 3, then the order of β is 12.

$$\beta = (1\ 5\ 3\ 7)(2\ 6\ 8) \Rightarrow \beta^2 = (1\ 5\ 3\ 7)^2(2\ 6\ 8)^2 = (1\ 3)(5\ 7)(2\ 8\ 6)$$

$$\beta^3 = [(1\ 5\ 3\ 7)(2\ 6\ 8)]\,[(1\ 3)(5\ 7)(2\ 8\ 6)] = (1\ 7\ 3\ 5)$$

$$\beta^4 = [(1\ 5\ 3\ 7)(2\ 6\ 8)]\,[(1\ 7\ 3\ 5)] = (2\ 6\ 8)$$

$$\beta^8 = (2\ 6\ 8)\,(2\ 6\ 8) = (2\ 8\ 6)$$

Finally, $\beta^{12} = (2\ 8\ 6)\,(2\ 6\ 8) = e$.

Summary of properties of permutations

Here are some properties of permutations. Some have been discussed earlier and some are stated without formal proof.

1 Every permutation can be written as a product of disjoint cycles.

2 Disjoint cycles commute. That is, If $\alpha, \beta \in S_n$ and have no numbers in $\mathbb{Z}_n$ that are moved by both α and β then $\alpha\beta = \beta\alpha$. In other words, if the disjoint cycle form of α has no number in common with the disjoint cycle form of β, then α and β commute.

3 Since a permutation is a bijective mapping (injective and surjective function) and the product is a composition of function, then the product of permutations is associative. That is $\alpha(\beta\gamma) = (\alpha\beta)\gamma$, and thus we simply write $\alpha\beta\gamma$ for the product!

4 $|S_n| = n!$ That is, there are $n!$ different permutations for a set of size n.

5 The identity permutation is $e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$. Its cycle form is (1)

and when it is multiplied by any element of S_n the result is that element itself. Thus,

$e\alpha = \alpha e = \alpha$ for every $\alpha \in S_n$.

6 Every $\alpha \in S_n$ has an inverse α^{-1} such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = e$.

7 Permutation composition (multiplication) is not necessarily commutative.

8 The **cancellation law** for permutation multiplication is valid. That is $\alpha\beta = \alpha\gamma \Leftrightarrow \beta = \gamma$.

Example 17

Show that the number of elements in S_n is $n!$. (This is also the **order** of S_n.)

Solution

Any member of S_n is of the form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ \_ & \_ & \_ & \_ & \ldots & \_ \end{pmatrix}.$$

The number of elements in S_n is equal to the number of different ways we can place the numbers 1, 2, 3, …, n in the blanks of the second row. This is nothing but the number of permutations of n objects and hence it is $n!$. Permutation of objects without replacement has been covered in the core part of your course.

Example 18

Consider S_3, the symmetric group on 3 elements. Draw a Cayley table and verify that it is a group.

Solution

There are 3! elements for the set S_3. Let us use p_i to represent the different elements. For example,

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Here is a Cayley table for this group under function composition.

$\circ$	p_1	p_2	p_3	p_4	p_5	p_6
p_1	p_1	p_2	p_3	p_4	p_5	p_6
p_2	p_2	p_1	p_5	p_6	p_3	p_4
p_3	p_3	p_4	p_1	p_2	p_6	p_5
p_4	p_4	p_3	p_6	p_5	p_1	p_2
p_5	p_5	p_6	p_2	p_1	p_4	p_3
p_6	p_6	p_5	p_4	p_3	p_2	p_1

Notice that p_1 is the identity, since it leaves the other permutations 'untouched' when it is composed with each. Since p_1 appears in every row and column, then we can say that there is an inverse for each element. Associativity is assumed. Also, since the table is not symmetric about the main diagonal, we notice that the group is not Abelian.

Example 19

Let G be the set of functions $\{f, g, h, i, j, k\}$ defined below with the binary operation of function composition.

The functions are defined from $\mathbb{R}\backslash\{0, 1\}$ to $\mathbb{R}\backslash\{0, 1\}$.

$$f(x) = \frac{1}{1-x}, g(x) = \frac{x-1}{x}, h(x) = \frac{1}{x}, i(x) = x, j(x) = 1-x, k(x) = \frac{x}{x-1}.$$

Is $(G, \circ)$ a group?

Solution

$$f(g(x)) = \frac{1}{1-\frac{x-1}{x}} = \frac{x}{1} = x = i(x); f(h(x)) = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1} = k(x);$$

$$f(j(x)) = \frac{1}{1-(1-x)} = \frac{1}{x} = h(x); f(k(x)) = \frac{1}{1-\frac{x}{x-1}} = \frac{x-1}{-1} = 1-x = j(x)$$

Similarly, we can find the rest of the results. Here is the Cayley table for this group.

$\circ$	i	f	g	h	j	k
i	i	f	g	h	j	k
f	f	g	i	k	h	j
g	g	i	f	j	k	h
h	h	j	k	i	f	g
j	j	k	h	g	i	f
k	k	h	j	f	g	i

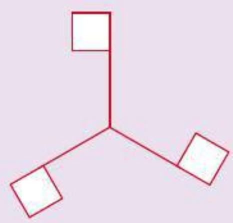
- The set is closed under composition.
- i is the identity element.
- Each element has an inverse as i appears in every row and column.
- Composition is associative.

The group is not Abelian as $g \circ h = j \neq h \circ g = k$.

Note: Try to see how this group is similar to S_3. One way is to set up some correspondence between the elements of this group and those of S_3. For example, $i \leftrightarrow p_1$, etc. We will leave that as an exercise for you.

Exercise 3

1 Suppose rotations of the figure (below) of $0, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$ are denoted by 0, 2, and 4 respectively.



a Show that the set {0, 2, 4} forms a group under the operation of transformation composition.

b Construct a Cayley table for the group.

c Is this an Abelian group?

2 Let the operation $\boxtimes$ be defined by $x \boxtimes y = xy^2$ over $\mathbb{Z}$.

a Find the value of

i $3 \boxtimes 5$ **ii** $5 \boxtimes 3$

iii $2 \quad 2$ **iv** $0 \boxtimes -4$

v $1 \boxtimes 3$ **vi** $3 \boxtimes 1$

vii $2 \boxtimes (3 \boxtimes 4)$ **viii** $(2 \boxtimes 3) \boxtimes 4$

b Is $x \quad y = y \quad x$ for all values? If not, for what values?

c Is $(x \quad y) \quad z = x \quad (y \quad z)$?

3 Show that addition modulo n is commutative and associative.

4 Find and set up a Cayley table for 'symmetries' in a rhombus.

5 Consider a set $A = \{a, b\}$. Let $M(A)$ be the set containing the following mappings on the elements of A:

$p(a) = a, p(b) = a; r(a) = a, r(b) = b; s(a) = b, s(b) = a; t(a) = b, t(b) = b.$

a Construct a Cayley table for composition '$\circ$' as an operation on $M(A) = \{p, r, s, t\}$.

b Which is the identity element? Why?

c Is $\circ$ commutative as an operation on $M(A)$?

d Which elements of $M(A)$ are invertible?

e Is $(M(A), \circ)$ a group?

6 Consider a set $A = \{a, b, c\}$. Let $M(A)$ be the set containing the following mappings on the elements of A:

$p(a) = a, p(b) = b, p(c) = c; r(a) = b, r(b) = a, r(c) = c; s(a) = a, s(b) = a, s(c) = a;$
$t(a) = b, t(b) = b, t(c) = b.$

a Construct a Cayley table for composition '$\circ$' as an operation on $M(A) = \{p, r, s, t\}$.

b Which is the identity element? Why?

c Is $\circ$ commutative as an operation on $M(A)$?

d Which elements of $M(A)$ are invertible?

e Is $(M(A), \circ)$ a group?

In questions 7–14, decide whether the given set forms a group under the given operation. If it does, describe the group, and if it does not, justify.

7 $\{-1, 1\}$ and multiplication.

8 $\{-1, 0, 1\}$ and addition.

9 $\{n \mid n = 10k \text{ where } k \in \mathbb{Z}\}$ and addition.

10 $\{x = 2^m \mid m \in \mathbb{Z}\}$ and multiplication.

11 $\{x = 2^m 3^n \mid m, n \in \mathbb{Z}\}$ and multiplication.

12 M, the set of all mappings from $\mathbb{R}$ to $\mathbb{R}$. Define the operation of addition $f + g$ for any mappings $f, g \in M$, by

$(f + g)(x) = f(x) + g(x)\ \forall\ x \in \mathbb{R}.$

13 $\mathbb{R} \backslash \{-1\}$, where the operation $*$ is defined by

$a * b = a + b + ab.$

14 $\left\{x \mid x = a + b\sqrt{2}\right\}$, where a and b are both rational numbers not both 0. The operation is ordinary multiplication.

15 Show that if a and b are in the same group $(G, *)$, then the equation $a * x = b$ has exactly one solution.

16 Let $(M, *)$ be a group with the rule that $\forall\ a, b \in M, a^2 * b^2 = (a * b)^2$. Show that $(M, *)$ is Abelian.

17 S_4 is the group of permutations of 4 elements under the operation of function composition.

a Find the order of the group and justify your answer.

b List all the elements of the group and construct a Cayley table for the operation.

c Show that the group is not Abelian.

18 Let M be the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $a, b, c \in \mathbb{R}$ and $a \neq 0$ and $c \neq 0$.

a Prove that M is a group under matrix multiplication.

b Show that this group is not Abelian.

c Consider the case where $a = c = 1$. Let N be the set of such matrices. Show that N under matrix multiplication is an Abelian group.

19 Consider the set $M = \{1, 3, 9, 11\}$ under multiplication modulo 16. Denote this multiplication simply by $\times$.

a Show that $3 \times (9 \times 11) = (3 \times 9) \times 11$.

b Show that $(M, \times)$ is a group.

c Is this a cyclic group? If yes, find all generators.

20 Complete the following table in a way that makes the operation commutative.

$\circ$	a	b	c	d
a	a	b		d
b		c		
c	c	d	a	b
d		a		c

Is the set $\{a, b, c, d\}$ an Abelian group under this operation?

21 Complete the following table in a way that makes the set $\{w, x, y, z\}$ an Abelian group under the operation given by the table.

$\propto$	w	x	y	z
w	y			x
x	z	w		
y				
z				w

22 Prove that the set of 2×2 matrices with real coefficients is an Abelian group under matrix addition.

23 Let $a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$.

Compute each of the following:

a $b \circ a$ **b** $a \circ b$ **c** a^{-1}

d b^{-1} **e** $b^{-1} \circ a^{-1}$ **f** $a^{-1} \circ b^{-1}$

g $(b \circ a)^{-1}$ **h** $(a \circ b)^{-1}$

24 Repeat question 23 using

$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$.

25 Let $E = \{x \mid x = 2k, k \in \mathbb{Z}\}$ and consider the binary operation $\odot$ defined by

$\forall\ (a, b), (c, d) \in \mathbb{Z} \times E, (a, b) \odot (c, d) = (a + c, b + d)$.

Prove that $(\mathbb{Z} \times E, \odot)$ is an Abelian group.

26 Let $(A, *)$ be an Abelian group with identity element e. Define on A a new binary operation $\odot$ defined by

$a \odot b = a * b * c, \forall\ a, b \in A$ and k is a specific element of A distinct from e.

Show that $(A, \odot)$ is an Abelian group.

27 Consider the group $(G, *)$ with identity element e. Define a relation, $\mathcal{R}$, on the elements of G:

If $a, b \in G$, then $b\mathcal{R}a$, if $\exists\ x \in G$ such that $b = x * a * x^{-1}$.

a Show that $\mathcal{R}$ is an equivalence relation.

b For a given element a, consider the function $f: G \rightarrow G$, such that

$f(x) = a^{-1} * x * a$.

Show that f is a bijection.

28 $(G, *)$ is a group such that $\forall\ x \in G, x * x = e$. Show that $(G, *)$ is Abelian.

29 $(G, *)$ is a group such that $\forall\ x, y \in G, (x * y)^2 = x^2 * y^2$. Show that $(G, *)$ is Abelian.

30 $(G, *)$ is a group such that $\forall\ x, y \in G, (x * y)^{-1} = x^{-1} * y^{-1}$. Show that $(G, *)$ is Abelian.

31 A teacher was typing a paper in which he wanted to include a list of 9 integers that form a group under multiplication modulo 91. Inadvertently he left out one of the integers and his list appeared with the following 8 numbers:

1, 9, 16, 22, 53, 74, 79, 81

Which integer was left out?

32 Find $\alpha\beta$ and $\beta\alpha$ when

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}.$$

33 If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$.

a Find α^2, α^4, and α^6.

b Write α in cycle notation.

c Find the inverse of α and verify your answer by multiplication of the two permutations.

34 Consider the following three permutations.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 7 & 1 & 5 & 8 & 2 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 7 & 1 & 3 & 2 & 8 \end{pmatrix}$$

Write each permutation in cycle form, and find each of the following.

a $\alpha\beta$ **b** $\alpha\beta\gamma$ **c** β^{-1} **d** $(\beta\gamma)^{-1}$

e $\gamma^{-1}\beta^{-1}$ **f** $\alpha^{-1}\gamma\alpha$ **g** $\text{ord}(\gamma)$ **h** $\text{ord}(\alpha^{-1}\gamma\alpha)$

35 Change to cycle notation.

a $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 7 & 4 & 5 & 6 & 3 & 1 & 2 & 10 & 9 \end{pmatrix}$

b $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 9 & 11 & 4 & 8 & 15 & 5 & 3 & 7 & 2 & 6 & 1 & 12 & 13 & 14 \end{pmatrix}$

36 Change the cycle notation of the S_9 members given below into two-row notation.

a (1 3 5 7 9)

b (1 5 2)(3 4)(7 8 9)

c (1 7 4 6)(3 5 9 8)

Practice questions 3

1 Show that the set $H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a = \pm 1, \text{ and } b \in Z \right\}$ forms a group under matrix multiplication.

(You may assume that matrix multiplication is associative.)

2 a Prove that the set of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$, is a group under matrix multiplication.

b Show that this group is Abelian if and only if there exists a real constant k such that $c = ka$.

3 The binary operation $a * b$ is defined by $a * b = \dfrac{ab}{a+b}$, where $a, b \in \mathbb{Z}^+$.

a Prove that $*$ is associative.

b Show that this binary operation does not have an identity element.

4 Let the matrix T be defined by $\begin{pmatrix} x & x+2 \\ x-5 & -x \end{pmatrix}$ such that $\det T = 1$.

a **i** Show that the equation for x is $2x^2 - 3x - 9 = 0$.

ii The solutions of this equation are a and b, where $a > b$.

Find a and b.

b Let A be the matrix where $x = 3$.

i Find A^2.

ii Assuming that matrix multiplication is associative, find the smallest group of 2×2 matrices which contains A, showing clearly that this is a group.

5 The set $S = \{a, b, c, d\}$ forms a group under each of two operations # and $*$, as shown in the following group tables.

#	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$*$	a	b	c	d
a	b			a
b		d		b
c				c
d	a	b		d

a Copy and complete the second table.

b Solve the following equations for x.

i $(b \# x) * c = d$

ii $(a * (x \# b)) * c = b$

6 Consider the group $(H, \bullet)$ with identity element e.

a For $x, y \in H$, show that $(x \bullet y)^{-1} = y^{-1} \bullet x^{-1}$.

b Given $x, y \in H$, the relation R is defined as follows:

$xRy \Leftrightarrow$ there exists $z \in H$ such that $x = z \bullet y \bullet z^{-1}$.

Determine whether or not R is an equivalence relation.

7 The permutations p_1 and p_2 of the integers {1, 2, 3, 4, 5} are given by

$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$.

a Find the order of p_1.

b i Find $p_2 p_1$, the composite permutation p_1 followed by p_2.

ii Determine whether or not p_1 and p_2 commute under composition of permutations.

c Find $(p_1^2 p_2)^{-1}$.

8 a and b are elements of the group G whose binary operation is multiplication.

a Use mathematical induction to prove that $(bab^{-1})^n = ba^nb^{-1}$, for all $n \in \mathbb{Z}^+$.

b Show that $(bab^{-1})^{-1} = ba^{-1}b^{-1}$.

c Use parts **a** and **b** to show that $(bab^{-1})^n = ba^nb^{-1}$ for all negative integers n.

9 The binary operation $*$ is defined for $a, b \in \mathbb{Z}^+$ by

$a * b = a + b - 2$.

a Determine whether or not $*$ is

i closed

ii commutative

iii associative.

b i Find the identity element.

ii Find the set of positive integers having an inverse under $*$.

10 a The relation aRb is defined on {1, 2, 3, 4, 5, 6, 7, 8, 9} if and only if ab is the square of a positive integer.

i Show that R is an equivalence relation.

ii Find the equivalence classes of R that contain more than one element.

b Given the group $(G, *)$, a subgroup $(H, *)$ and $a, b \in G$, we define $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation.

4 Groups II

4.1 Introduction

In this chapter, we will discuss further properties of groups along with subgroups and relations among groups.

Definition 1

An element a in a group $(G, *)$ is said to have a **finite order** if $a^m = e$ for some $m \in \mathbb{Z}^+$. In such cases, the **order of the element *a***, denoted by $|a|$, is the **smallest** positive integer n such that $a^n = e$. An element a is said to have **infinite order** if $a^m \neq e$ for **every** $m \in \mathbb{Z}^+$.

Example

- In the group $(\mathbb{R}\backslash\{0\}, \times)$, 3 has **infinite order** because $3^m \neq 1$ for **every** $m \in \mathbb{Z}^+$.
- In the group $\{1, -1, i, -i\}$ where $i^2 = -1$, under complex multiplication, the order of i is 4 because $i^4 = 1$, and the order of (-1) is 2 since $(-1)^2 = 1$. Obviously the order of 1 is 1.
- In the group $SL_2 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\right\}$,

  described in Chapter 3, the element $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

  has order 12 because 12 is the smallest positive integer where

  $A^{12} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}^{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$. So, we can write $|A| = 12$.
- In the additive group $(\mathbb{Z}_6, +)$ the element 2 has order 3 because $2 + 2 + 2 = 0$, while $|5| = 6$ since $5 + 5 + 5 + 5 + 5 + 5 = 0$.
- In the group $\{1, 3, 7, 9\}$ in $\mathbb{Z}_{10}$ with multiplication modulo 10, $|3| = 4$ since $3^4 = 1$ and $|9| = 2$.

Notice in the example above that the order of the identity is always 1. For example, $\left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 1$. Also, notice in the fifth instance of the example, $3^8 = 1, 3^{12} = 1$, etc. and $9^4 = 1, 9^6 = 1$, etc. These are manifestations of the following theorem.

Theorem 1

Let a be an element in a Group $(G, \cdot)$, then:

1 If a has a *finite order* $\boldsymbol{n}$, then $a^m = e$ if and only if $n \mid m$, i.e. m is a multiple of n.

2 $a^p = a^q$ if and only if $p \equiv q \pmod{n}$.

3 If a has infinite order, then all a^i (i is an integer) are different. (This means $a^i \neq a^j$ when $i \neq j$.)

Proof

1 If $n \mid m$, then we can write $m = kn$ for some integer k, and hence $a^m = a^{kn} = \left(a^n\right)^k = e^k = e$.

Conversely, if $a^m = e$, then by the division algorithm, $m = nq + r$ with $0 \leqslant r < n$, thus

$a^m = a^{nq+r} = a^{nq}a^r = \left(a^n\right)^q a^r = ea^r = a^r = e,$

but since n is the order of a, it is by definition the smallest integer with $a^n = e$. Hence, with $r < n$, $a^r = e$ is only possible if $r = 0$, and therefore $m = nq + 0$, i.e. $n \mid m$.

2 If $a^p = a^q$, then $a^p a^{-q} = a^q a^{-q} \Rightarrow a^{p-q} = a^0 = e$. By (**1**) $a^{p-q} = e$ is only possible if $n \mid (p - q)$, thus $p \equiv q \pmod{n}$ by definition of congruence modulo n.

3 We show this with indirect proof: suppose not all a^i are different, then there will be at least two values, x and y, with $x > y$ (you can also use $x < y$), such that

$a^x = a^y$ which implies that $a^{x-y} = e$ (using (**2**) above).

This in turn implies that $x \equiv y \pmod{n} \Rightarrow n$ is the order of the element, but the element has infinite order, which is a contradiction and therefore $a^x \neq a^y$.

Note: As a result of Theorem 1, we can conclude the following:

1 If $|a| = n$, and $n = kr$ with $r > 0$, then $|a^r| = k$.

2 If $a^x = a^y$ with $x \neq y$, then a must have a **finite order**.

Example

In the SL_2 group in the previous example, we used $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ and showed that $|A| = 12$.

Now, $B = A^4 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$. Now,

$$B^3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \text{ which verifies } (\mathbf{1}) \text{ above.}$$

The order of group elements has several uses in the following sections.

4.2 Subgroups

You may have noticed from examples or exercises that some groups are subsets of others with the same binary operation. The group SL_2 under matrix multiplication is a subset of the group GL_2 under matrix multiplication. The following definition describes this phenomenon.

Definition 2

If a non-empty subset H of a group G is itself a group under the binary operation of G, we call H a **subgroup** of G. If $H \subset G$ then H is a **proper subgroup** of G. If $H \subseteq G$, then H is a subgroup of G.

Every group has at least two **subgroups**, $(\{e\}, *)$ and $(G, *)$ itself. $(\{e\}, *)$ is usually called the **trivial subgroup**, and the rest of the subgroups are **non-trivial**. Aside from these two subgroups, all other subgroups are **proper**.

The **notation** for a subgroup can be the same as subsets, and thus the context of the discussion will determine whether $H \subseteq G$ refers to a subset or a group.

Example

Let G be the group $\{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition modulo 8. We will rearrange the elements in a Cayley table so that the subgroups will become apparent. Here is the table:

+	0	2	4	6	1	3	5	7
0	0	2	4	6	1	3	5	7
2	2	4	6	0	3	5	7	1
4	4	6	0	2	5	7	1	3
6	6	0	2	4	7	1	3	5
1	1	3	5	7	2	4	6	0
3	3	5	7	1	4	6	0	2
5	5	7	1	3	6	0	2	4
7	7	1	3	5	0	2	4	6

This group, as you notice, has two non-trivial subgroups: $A = \{0, 2, 4, 6\}$ and $B = \{0, 4\}$. B is a subgroup of A too.

Example

Consider the group of symmetries of the square $(D_4, \circ)$ which we developed in Chapter 3. Looking at the Cayley table, it is clear that rotations with the identity constitute a subgroup, while the reflections with the identity do not constitute a subgroup. Notice here that the subgroup of rotations consists of 'powers' of r. That is, the group is made up of $\{e = r^0, r, r^2, r^3\}$. Such a subgroup is called a **cyclic**[1] subgroup of D_4 generated by r.

$\circ$	e	r	r^2	r^3	L_1	L_2	L_3	L_4
e	e	r	r^2	r^3	L_1	L_2	L_3	L_4
r	r	r^2	r^3	e	L_4	L_3	L_1	L_2
r^2	r^2	r^3	e	r	L_2	L_1	L_4	L_3
r^3	r^3	e	r	r^2	L_3	L_4	L_2	L_1
L_1	L_1	L_3	L_2	L_4	e	r^2	r	r^3
L_2	L_2	L_4	L_1	L_3	r^2	e	r^3	r
L_3	L_3	L_2	L_4	L_1	r^3	r	e	r^2
L_4	L_4	L_1	L_3	L_2	r	r^3	r^2	e

Theorem 2

For any group $(G, *)$, if $x \in G$, then the subset of G, X defined by $X = \left\{x^k \mid k \in \mathbb{Z}\right\}$, is a subgroup of G and is known as the **cyclic subgroup** *generated* by x. x is also called the **generator** of this subgroup. This will be proved after the subgroup tests.

[1] Cyclic groups are discussed later in the chapter (page 1310).

Subgroup tests

When deciding whether a subset H of a group G is a subgroup of G, we do not need to apply the definition and verify the group axioms. There are a few theorems that will simplify the process.

Note: For the rest of this chapter, we will not be using any specific symbols to denote the operation. So for two elements a and b, we will write ab when we mean $a * b$.

Theorem 3

Let G be a group and H a non-empty subset of G. Then, H is a subgroup of G iff $ab^{-1} \in H$ whenever $a, b \in H$.

Proof

- If H is a subgroup of G: If $a, b \in H$, then b has an inverse $b^{-1} \in H$ by definition of a group, and since H is closed under the binary operation $ab^{-1} \in H$.
- Conversely, suppose that H is a non-empty subset of G where $ab^{-1} \in H$, whenever $a, b \in H$.
  - Let $a = b$, then whenever $a, b \in H$ and $ab^{-1} = aa^{-1} = e \in H$ and the identity axiom is verified.
  - Now, $e, a \in H$, and hence $ea^{-1} = a^{-1} \in H$, and the inverse axiom is verified.
  - Now, since H *includes* inverses, when $a, b \in H$, then $a, b^{-1} \in H$, and hence $a(b^{-1})^{-1} = ab \in H$. So the closure axiom is verified.
  - Associativity is inherited from G.

  Therefore, the set H is a subgroup of G.

Example 1

A group (M, Δ) has identity element i. N is a subset of M defined by

$$N = \{x \in M \mid x \,\Delta\, m = m \,\Delta\, x, \text{ for all } m \in M\}.$$

Show that N is a subgroup of M.

Solution

Let $a, b \in N$. We need to show that $a \,\Delta\, b^{-1} \in N$, i.e. we need to show that for all $m \in M$,

$$(a \,\Delta\, b^{-1}) \,\Delta\, m = m \,\Delta\, (a \,\Delta\, b^{-1}).$$

Now, let us first show that if $b \in N$ then $b^{-1} \in N$.

Since i is an element of M, then

$m \Delta i = i \Delta m \Rightarrow m \Delta (b \Delta b^{-1}) = (b \Delta b^{-1}) \Delta m$	Identity axiom
$\Rightarrow (m \Delta b) \Delta b^{-1} = b \Delta (b^{-1} \Delta m)$	Associativity
$\Rightarrow (b \Delta m) \Delta b^{-1} = b \Delta (b^{-1} \Delta m)$	Since $b \in N$
$\Rightarrow b \Delta (m \Delta b^{-1}) = b \Delta (b^{-1} \Delta m)$	Associativity
$\Rightarrow m \Delta b^{-1} = b^{-1} \Delta m$	Left cancellation
thus $b^{-1} \in N$	Definition of N

Now, since $a, b \in N \Rightarrow b^{-1} \in N$ and $m \Delta b^{-1} = b^{-1} \Delta m$, then

$$(a \Delta b^{-1}) \Delta m = a \Delta (b^{-1} \Delta m) = a \Delta (m \Delta b^{-1}) = (a \Delta m) \Delta b^{-1}$$
$$= (m \Delta a) \Delta b^{-1} = m \Delta (a \Delta b^{-1}),$$

which proves that whenever $a, b \in N$, then $a \Delta b^{-1} \in N$, and by Theorem 3, N is a subgroup of M.

Note: This proof will be done differently after the next theorem.

Theorem 4

Let G be a group and H a non-empty subset of G. Then, H is a subgroup of G if

1 $ab \in H$ whenever $a, b \in H$ (closure), and

2 $a^{-1} \in H$ whenever $a \in H$ (inverse).

Proof

- If H is a subgroup of G, it follows immediately, by definition, that the conditions are met.
- Conversely, if (1) and (2) hold, and $a, b \in H$, then by (2), $b^{-1} \in H$, and hence by (1) $ab^{-1} \in H$. Thus by Theorem 3, H is a subgroup of G.

Note: The importance of this theorem is that it reduces the number of characteristics we need to verify into two only.

Example 2

A group (M, Δ) has identity element i. N is a subset of M defined by

$$N = \{x \in M \mid x \Delta m = m \Delta x, \text{ for all } m \in M\}.$$

Show that N is a subgroup of M.

Solution

Let $a, b \in N$. We need to show that

a) $a \Delta b \in N$ whenever $a, b \in N$, and

b) $a^{-1} \in N$ whenever $a \in N$.

a) Since $a, b \in N$, then $a \Delta m = m \Delta a$, and $b \Delta m = m \Delta b$. We need to show that

$(a \Delta b) \Delta m = m \Delta (a \Delta b)$.

Now,

$$\begin{aligned}(a \Delta b) \Delta m &= a \Delta (b \Delta m) = a \Delta (m \Delta b) = (a \Delta m) \Delta b \\ &= (m \Delta a) \Delta b = m \Delta (a \Delta b).\end{aligned}$$

b) This has been proved in Example 2.

When dealing with finite groups, it is simpler to use the following theorem.

Theorem 5 (Finite subgroup test)

Let G be a group and H a ***finite*** non-empty subset of G. Then, H is a subgroup of G if H is closed under the operation of G.

Proof

This theorem is a special case of Theorem 4 applied to a finite subset of G. In essence it says that H is a subgroup of G if $ab \in H$ whenever $a, b \in H$.

Since the closure axiom has been proved by Theorem 4, we need only verify that under this condition $a^{-1} \in H$ whenever $a \in H$.

Now, if $a = e$, then $a^{-1} = a \in H$ and we are done. If $a \neq e$, and since H is finite, then a has an order n. Also, since H is closed, then all positive powers of a are in H. Not all these powers are different because n is finite and hence for any power $r > n$ there should be a power $s < n$ such that $a^{r-s} = e$, and since $a \neq e$, then $r - s > 1$. Thus $a^{r-s} = a \cdot a^{r-s-1} = e$, which implies that $a^{r-s-1} = a^{-1}$. But $r - s - 1 = m$, which is some positive integer implying that $a^{r-s-1} = a^{m}$ is a positive power of a and hence it has to be in H. So, we showed that whenever $a \in H$, then $a^{-1} \in H$, and that completes the proof.

Example 3

Show that the set $\{1, 3, 4, 5, 9\}$ under multiplication modulo 11 ($\times_{11}$) is a subgroup of $(\mathbb{Z}_{11} \backslash \{0\}, \times_{11})$.

Solution

Since the group is finite, it is enough to show the subset closed under this operation. There are 10 multiplications (rather than 25) to check:
$3 \times_{11} 4 = 1, 3 \times_{11} 5 = 4, 3 \times_{11} 9 = 5, 4 \times_{11} 5 = 9, 4 \times_{11} 9 = 3,$

$5 \times_{11} 9 = 1, 3^2 = 9, 4^2 = 5, 5^2 = 3, 9^2 = 4.$

Theorem 2 – proof

Recall that the claim is that subset X defined by $X = \{x^k | k \in \mathbb{Z}\}$ is a subgroup of G. (The **cyclic subgroup** *generated* by x. x is also called the **generator** of this subgroup.)

Since $x \in X$, then X is non-empty. Now, let $x^i, x^j \in X$. Then $i - j \in \mathbb{Z}$ and hence $x^{i-j} \in X$ by definition of X. This in turn means that $x^{i-j} = x^i x^{-j} = x^i \left(x^j\right)^{-1} \in X$; thus, letting $a = x^i$ and $b = x^j \Rightarrow ab^{-1} \in X$ whenever $a, b \in X$, and by Theorem 3, X is a subgroup of G.

The following example is a demonstration of the validity of this theorem.

Example

- Consider the group of symmetries in an equilateral triangle $(D, \circ)$ discussed in the previous chapter. Here is a reproduction of its Cayley table.

$\circ$	I	R	R^2	L	M	N
I	I	R	R^2	L	M	N
R	R	R^2	I	N	L	M
R^2	R^2	I	R	M	N	L
L	L	M	N	I	R	R^2
M	M	N	L	R^2	I	R
N	N	L	M	R	R^2	I

  Notice how R generates a cyclic subgroup of $(D, \circ)$.

- Consider the group $(\mathbb{Z}_5 \backslash \{0\}, \times)$. The group elements are $\{1, 2, 3, 4\}$. Take 2 for example.

  $2^2 = 4$, $2^3 = 3$, $2^4 = 1$, and hence 2 is a generator of a cyclic subgroup of $(\mathbb{Z}_5 \backslash \{0\}, \times)$. It is actually the group itself.

Example

Consider the group $(\mathbb{Z}_{11} \backslash \{0\}, \times)$. The group elements are $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Consider the element 3: $3^2 = 9$, $3^3 = 5$, $3^4 = 4$, $3^5 = 1$, and thus 3 generates a cyclic subgroup $\{1, 3, 4, 5, 9\}$ of the original group. The order of 3 in the group is 5, and so is the order of this subgroup. Notice that the order of this subgroup divides the order of the group itself, which is 10. 4, 5, or 9, will also generate this subgroup.

If you consider 2 or 6, you will see that they generate the whole group itself.

The previous example introduces us to the definition of **cyclic groups in Section 4.3.**

The centre of a group (Optional)

The **centre** of a group G is the subset $C(G)$ of all elements that commute with every element of G:

$$C(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$$

Theorem

For a group G the centre $C(G)$ is a subgroup of G.

Proof

Since e, the identity element commutes with all elements in G, it is an element of C according to the definition.

Also, if $a, b, \in C(G)$, then for any $g \in G$, $(ab)g = a(bg)$ by associativity.

Thus, $(ab)g = a(bg) = a(gb)$ since $b \in C(G)$.

Therefore, $(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab)$.

So $ab \in C(G)$.

Also, since $a \in C(G) \Rightarrow ag = ga \Rightarrow a^{-1}aga^{-1} = a^{-1}gaa^{-1} \Rightarrow ga^{-1} = a^{-1}g$.

Hence, $a^{-1} \in C(G)$.

Therefore, $C(G)$ is a subgroup of G by Theorem 4.

4.3 Cyclic groups

Definition 3

A group G is called **cyclic** if there is an element $a \in G$ such that $G = \{a^n \mid n \in \mathbb{Z}\}$. a is called a generator of G. Notice from the previous example that a generator is not unique. For instance, 2 and 6 are two of the generators of $(\mathbb{Z}_{11}\backslash\{0\}, \times)$.

Note: It is important to remember that in all cases, the identity element can be understood as $a^0 = e$, thus e is a member of every cyclic group too, but it cannot generate the groups except the trivial subgroup.

Theorem 6

All cyclic groups are Abelian.

Proof

If G is a cyclic group and x is a generator of order n, consider any two elements a and b in G. Since G is cyclic and generated by x, then there exists two integers r and s such that $a = x^r$ and $b = x^s$.

Now, $ab = x^r x^s = x^{r+s} = x^{s+r} = x^s x^r = ba$, and the group is Abelian.

Example

$(\mathbb{Z}, +)$ is cyclic. 1 is a generator. When the operation is addition, then 1^n is interpreted as $\underbrace{1 + 1 + \ldots + 1}_{n \text{ terms}}$.

- $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, $n \geqslant 1$ is a cyclic group under addition modulo n.

  1 is a generator. $-1 = n - 1$ is also a generator.
- $\mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\}$ is a specific example of such cyclic groups under addition modulo 8. 1, 3, 5, and 7 are generators.
  $3^8 = 3^0 = 0$, $3^3 = 3 + 3 + 3 = 1$, $3^6 = 2$, $3^1 = 3$, $3^4 = 3 + 3 + 3 + 3 = 4$,
  $3^7 = 5$, $3^2 = 3 + 3 = 6$, $3^5 = 7$.
- $A = \{1, 3, 7, 9\}$ under multiplication modulo 10 is cyclic with 3 and 7 as generators: $\{3^0, 3^1, 3^3, 3^2\}$; $\{7^0, 7^3, 7^1, 7^2\}$.
- Now consider the group $\{1, 3, 5, 7\}$ under multiplication modulo 8. We leave it for you to verify that this is a group. However, we will show you here that it is not cyclic. If it were cyclic, then we should be able to generate it with at least one of the elements, 1, 3, 5, or 7. However, 1, being the identity, does not generate it, and neither does 3 (since $3^2 = 1 \Rightarrow |3| = 2$), nor 5 ($|5| = 2$), nor 7 ($|7| = 2$).

Theorem 7 (Lagrange's theorem)

If H is a subgroup of a finite group G, then the order of H divides the order of G. That is, $|G|$ is a multiple of $|H|$.

Example

You have seen that the group $(\mathbb{Z}_{11}\backslash\{0\}, \times)$ with elements $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ has a subgroup $H = \{1, 3, 4, 5, 9\}$ whose order is 5. We also pointed out that the order 5 divides the order of the whole group, 10. This is a demonstration of Lagrange's theorem.

Example

In the group of symmetries of the square, $(D_4, \circ)$, we notice that the group $\{e, r, r^2, r^3\}$ is a subgroup. The order of the group is 8 and the order of the subgroup is 4.

Proof

To understand the proof, we need to introduce another concept, that of a **coset**.

Cosets

Consider H, a subgroup of a group G. Define a relation $\circ_H$ on G in the following manner:

$$a \circ_H b \Leftrightarrow a^{-1}b \in H$$

Stated differently, this relation means that $a \circ_H b$ iff $a^{-1}b = h$ for some $h \in H$. This can also be interpreted as saying $a \circ_H b$ iff $b = ah$ for some $h \in H$

The last interpretation of the relationship gives rise to the following theorem.

Theorem

If H is a subgroup of G, then the relation $a \circ_H b$ is an equivalence relation on G.

Since $\circ_H$ as defined is an equivalence relation, it gives rise to equivalence classes.

$a \circ_H b$ iff $b = ah$ for some $h \in H$ $\Rightarrow$ the equivalence class $[a]$ can be defined as

$$[a] = \{b{:}b = ah, h \in H\}.$$

Proof

To show that this relation is an equivalence relation, we need to show that it is reflexive, symmetric and transitive.

Reflexive: $a \circ_H a$ since $a^{-1}a = e \in H$ because H is a subgroup of G.

Symmetric: If $a \circ_H b \Leftrightarrow a^{-1}b \in H \Rightarrow (a^{-1}b)^{-1} \in H \Rightarrow b^{-1}a \in H \Rightarrow b \circ_H a$.

Transitive: If $a \circ_H b$ and $b \circ_H c \Leftrightarrow a^{-1}b \in H$ and $b^{-1}c \in H \Rightarrow (a^{-1}b)(b^{-1}c) = a^{-1}c \in H \Rightarrow a \circ_H c$.

This discussion gives rise to the following results.

Definition: Left coset

If H is a subgroup of G, and a any element in G then the left coset of H in G determined by a is the set $aH = \{ax \mid x \in H\}$. (We can define a right coset in a similar manner but we will only focus on left cosets for our purposes here.)

Example: Coset (1)

Let $G = (\mathbb{Z}_{11}\backslash\{0\}, \times)$ and $H = \{1, 3, 4, 5, 9\}$. The left cosets of H are:

$1H = H$, $3H = \{3, 9, 1, 4, 5\}$, this is also H, and so are $4H$, $5H$, and $9H$.

$2H = \{2, 6, 8, 10, 7\}$, $6H = \{6, 7, 2, 8, 10\}$, also equal to $2H$, and so are $8H$, $10H$, and $7H$.

So, we have 2 left cosets for this group. Notice that both cosets have the same order, namely 5, and that the order of the group is $10 = 5 \times 2$, and that once two cosets have an element in common, then they are equal, and finally, the union of the cosets is the group G itself.

Example: Coset (2)

Let G be the set of functions $\{f, g, h, i, j, k\}$ defined on page 1296 of Chapter 3. We reproduce its Cayley table here for reference.

$\circ$	i	f	g	h	j	k
i	i	f	g	h	j	k
f	f	g	i	k	h	j
g	g	i	f	j	k	h
h	h	j	k	i	f	g
j	j	k	h	g	i	f
k	k	h	j	f	g	i

Notice that it has a subgroup $\{i, h\}$, which we will consider as H.

The cosets are

$iH = H$, $fH = \{f, k\}$, $gH = \{g, j\}$, $hH = \{h, i\} = H$, $jH = \{j, g\} = gH$,
$kH = \{k, f\} = fH$.

Here we have 3 left cosets. Also notice that the cosets have the same order, namely 2, and that the order of the group is $6 = 2 \times 3$, and that once two cosets have an element in common, then they are equal, and the union of the cosets is the group G itself.

The two examples point to the following theorem.

Theorem: Lagrange

Let H be a subgroup of a group G.

1 H is a left coset of itself.

2 For every element a in G, $a \in aH$, i.e. a is a member of its own left coset.

3 $\bigcup_{a\in G} aH = G$. That is, G is the union of the left cosets of H.

4 Any two left cosets of H are either equal or disjoint ($aH = bH$, or $aH \cap bH = \varnothing$).

5 All left cosets have the same order, namely $|H|$.

Proof

1 $H = eH$

2 Since $e \in H$, $ae = a \in aH$.

3 Obviously $aH \subseteq G$ for all a because of the closure axiom.

And for every $a \in G$, we showed in (**2**) that $a \in aH$, which is a subset of $\bigcup_{a\in G} aH$, and thus

$G \subseteq \bigcup_{a\in G} aH$. Therefore $\bigcup_{a\in G} aH = G$.

4 Assume that $aH \cap bH \neq \varnothing$, thus we have at least an $x \in aH \cap bH$. Hence, because $x \in aH$ then

$x = ah_1$ for some $h_1 \in H$ by definition. Similarly, $x = bh_2$ for some $h_2 \in H$. This implies that

$x = ah_1 = bh_2$, which in turn implies that $a = bh_2(h_1)^{-1}$. Now for any $h \in H$, $ah \in aH$, but

$ah = bh_2(h_1)^{-1}h \in bH$ since $h_2(h_1)^{-1}h \in H$ by closure, and therefore $aH \subseteq bH$. A similar argument shows that $bH \subseteq aH$, and thus $aH = bH$.

5 Define a function $f: H \to aH$ by $f(h) = ah$. By definition of aH, any of its elements can be written as $f(h) = ah$, and hence f is surjective. Additionally, $f(h_1) = f(h_2) \Rightarrow ah_1 = ah_2 \Rightarrow h_1 = h_2$ (left cancellation), and the function is injective. Thus, f is bijective and its domain and range must have the same order.

One of the conclusions we can draw from the theorem above is that the different cosets corresponding to H form a partition of G.

Now, we can prove Lagrange's theorem:

Let $S_1, S_2, \ldots, S_k$ be the different cosets created by H. Since these cosets form a partition of G, then

$G = \bigcup_1^k S_i = S_1 \cup S_2 \cup \ldots \cup S_k$, and because these cosets are disjoint

$$|G| = \underbrace{|S_1| + |S_2| + \ldots + |S_k|}_{k\ times} = \underbrace{|H| + |H| + \ldots + |H|}_{k\ times} = k\,|H|.$$

Theorem 8

(Corollary to Lagrange's theorem)

Let G be a finite group, and x any element of G, then $|G|$ is a multiple of the order of x.

Proof

Recall from Theorem 2 that x generates a cyclic subgroup of G, which we denoted by $X = \{x^k | k \in \mathbb{Z}\}$, and using Lagrange's theorem, the order of G is a multiple of the order of X, which is the order of the element x itself.

Example 4

Show that if the order of a group G is a prime number, then the group is cyclic.

Solution

Let $|G| = n$ where n is a prime number.

Let x be any non-identity element in G, and by Theorem 2, it has an order k. But by Lagrange corollary, k must divide n which is not possible, and therefore $k = n$. Hence, G is a cyclic group generated by x.

Example 5

Consider $\mathbb{Z}_{12}$, the group of integers modulo 12 under addition and the subgroup $H = \{0, 3, 6, 9\}$. What are the left cosets?

Solution

The left cosets are

$0H = \{0, 3, 6, 9\}$. $3H$, $6H$, and $9H$ are all the same.

$1H = \{1, 4, 7, 10\} = 4H, 7H, 10H$ are all the same.

$2H = \{2, 5, 8, 11\} = 5H, 8H, 11H$ are all the same.

4.4 Homomorphism and isomorphism

The set of natural numbers as historically known is $\mathbb{N} = \{1, 2, 3, \ldots\}$. If we wanted to write it in different notation, Roman for example, then we have $\mathbb{N} = \{\text{I}, \text{II}, \text{III}, \ldots\}$. The two look different, but mathematically they are considered the same. The idea that eases the differences in names and notations is **isomorphism**. Isomorphism allows us to look at different groups as being equal regardless of the different appearances. For example, consider the subgroup A of S_3 represented by the table below and the group $\mathbb{Z}_3$ under addition modulo 3. A consists of the following 3 permutations:

$$i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Here are the tables.

$\circ$	i	α	β
i	i	α	β
α	α	β	i
β	β	i	α

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

As we said, the first group members are permutations and the operation is composition, while the second group's elements are congruence classes and the operation is addition modulo 3. However, close inspection shows us that they are alike. If we think of setting up a correspondence between the elements of the two groups as follows

$$i \leftrightarrow 0,\ \alpha \leftrightarrow 1,\ \beta \leftrightarrow 2,$$

then knowing one table of operations will enable us to fill the other one without performing the operation in question. That is, knowing the addition table and using this correspondence we can fill the first table without performing any composition of permutations.

Here is the definition of isomorphism that makes this possible.

Definition 4

Let G be a group with operation $*$, $(G, *)$, and let H be a group with operation Δ, (H, Δ).

1 A **homomorphism** of G into H is a mapping f: $G \to H$ such that

$$f(a * b) = f(a)\Delta f(b)$$

for every $a, b \in G$. G and H are said to be **homomorphic**.

2 An **isomorphism** of G into H is a bijective mapping f: $G \to H$ such that

$$f(a * b) = f(a)\Delta f(b)$$

for every $a, b \in G$.

G and H are said to be **isomorphic**. Notation differs among mathematicians. We will use $G \cong H$ to denote that the groups are isomorphic.

Notice here that an isomophosm is a homomorphism that is also bijective.

Example

Let k be an integer, and let $f: \mathbb{Z} \to \mathbb{Z}$ be a function defined by

$$f(n) = kn$$

f is a homomorphism from the group $(\mathbb{Z}, +)$ to itself, since

$$f(n_1 + n_2) = k(n_1 + n_2) = kn_1 + kn_2 = f(n_1) + f(n_2)$$

for all integers n_1 and n_2.

Example

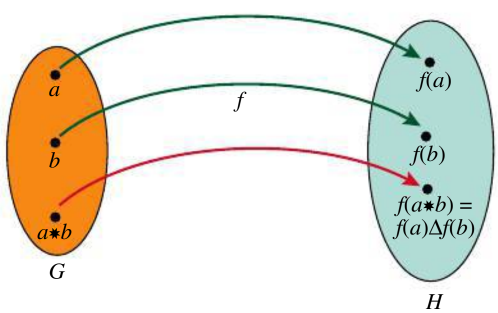
Let g: $\mathbb{Z} \to \mathbb{R}^+$ be defined by $g(x) = a^x$ where a is a positive real number, and consider the groups $(\mathbb{Z}, +)$ of integers under addition and $(\mathbb{R}^+, \times)$ of positive real numbers under multiplication.

g is a homomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{R}^+, \times)$.

For all integers x and y

$$g(x + y) = a^{x+y} = a^x \times a^y = g(x) \times g(y).$$

Note: Isomorphism is sometimes said to *preserve* the operation. It makes no difference whether we first operate in G and then apply f, or if we apply f first and then operate in H. See below.



For example, in the correspondence between Arabic notation and Roman notation, we get the same result if we add 2 + 3 = 5 and then translate that into Roman notation, $5 \to V$, or translate first, $2 \to II$ and $3 \to III$, and then add: $II + III = V$.

Since f is a bijection, then f^{-1} is also a bijection and it describes an isomorphism from H to G.

Example

Consider the example of the isomorphism described in the introduction to this section between the subgroup of S_3 and $\mathbb{Z}_3$.

The correspondence described, $i \leftrightarrow 0$, $\alpha \leftrightarrow 1$, $\beta \leftrightarrow 2$, defines the isomorphism between the two groups. Call the mapping g, then

$$g(i) = 0,\ g(\alpha) = 1,\ \text{and } g(\beta) = 2.$$

Then, for example,

$$g(\alpha \circ \beta) = g(i) = 0,$$

and

$$g(\alpha) + g(\beta) = 1 + 2 = 0;$$

thus

$$g(\alpha \circ \beta) = g(\alpha) + g(\beta).$$

You will need to check nine operations if you were to verify the definition for the whole operation. These are the entries in the Cayley table. In general, you need to check n^2 equations if G and H were finite of order n each.

Example

Consider the function $f: \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln x$ for each $x \in \mathbb{R}^+$.

$\mathbb{R}^+$ is a group with multiplication as the operation, $\mathbb{R}$ is a group with addition as the operation, and f is a bijection from $\mathbb{R}^+$ into $\mathbb{R}$ because it has an inverse $f^{-1}: \mathbb{R} \to \mathbb{R}^+$ defined by $f^{-1}(x) = e^x$. The mapping is an isomorphism because

$$f(xy) = \ln(xy) = \ln(x) + \ln(y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}^+.$$

Theorem 9

Let G be a group with operation $*$, $(G, *)$, and let H be a group with operation Δ, (H, Δ). If G and H are homomorphic with $f: G \to H$ as their homomorphism, then:

1 $f(e_G) = e_H$

2 $f(a^{-1}) = (f(a))^{-1}$, for each $a \in G$.

3 $f(a^n) = (f(a))^n$, for each $a \in G$ and each $n \in \mathbb{Z}$.

Proof

1 $e_G * e_G = e_G \Rightarrow f(e_G * e_G) = f(e_G) \Rightarrow f(e_G)\Delta f(e_G) = f(e_G)$,
but since $f(e_G) \in H$, then $f(e_G) = f(e_G)\Delta e_H$ as e_H is the identity in H;
thus $f(e_G)\Delta f(e_G) = f(e_G)\Delta e_H \Rightarrow f(e_G) = e_H$ by left cancellation.

2 $f(a * a^{-1}) = f(a)\Delta f(a^{-1})$, and $f(a * a^{-1}) = f(e_G) = e_H$
$\Rightarrow f(a)\Delta f(a^{-1}) = e_H$, but
$f(a), (f(a))^{-1} \in H \Rightarrow f(a)\Delta(f(a))^{-1} = e_H \Rightarrow$
$f(a)\Delta f(a^{-1}) = f(a)\Delta(f(a))^{-1} \Rightarrow f(a^{-1}) = (f(a))^{-1}$ by left cancellation.

3 We can use mathematical induction to prove this. We will prove it here for $n \geqslant 0$ and leave $n < 0$ as an exercise giving you a hint, $a^{-n} = (a^{-1})^n$.

The case $n = 0$ is obvious as $n = 0 \Rightarrow f(a^0) = (f(a))^0 \Rightarrow f(e_G) = e_H$ and also $n = 1$ is more obvious.

Now assume $f(a^k) = (f(a))^k$, then
$f(a^{k+1}) = f(a^k * a) = f(a^k)\Delta f(a) = (f(a))^k \Delta f(a) = (f(a))^{k+1}$.
Therefore, $f(a^n) = (f(a))^n$ is true for all integers by the principle of mathematical induction.

The following theorem will provide you with a few properties that are helpful in dealing with group relationships.

Two groups that are **isomorphic** are considered to be 'the same' in the sense that any group-theoretic claim about one is also true for the other. For example, if one is cyclic or Abelian, then the other is cyclic or Abelian.

Theorem 10

Let G be a group with operation $*$, $(G, *)$, and let H be a group with operation Δ, (H, Δ). If $G \cong H$ with $f: G \to H$ as their isomorphism and G is Abelian, then H is Abelian.

Proof

Consider any two elements $x, y \in H$, then since f is a bijection, there are two elements $a, b \in G$ such that $f(a) = x$ and $f(b) = y$.

Now,

$$x\Delta y = f(a)\Delta f(b) = f(a * b) = f(b * a) = f(b)\Delta f(a) = y\Delta x.$$

Thus H is Abelian.

Note: If two groups are isomorphic, then they must have the same order since their isomorphism is a bijection. This provides you with a convenient way of showing that two groups are not isomorphic. If $|G| \neq |H|$, then G and H cannot be isomorphic.

Here is a list of properties you can use in your proofs to quickly determine if two groups are not isomorphic:

G and H are groups, and $G \cong H$.

1 $|G| = |H|$

2 If G is Abelian, then H is Abelian.

3 If G is cyclic, then H is cyclic.

4 If G has a subgroup of order n, then H has a subgroup of order n ($n \in \mathbb{Z}^+$).

5 If G has an element of order n, then H has an element of order n.

(1) and (2) were discussed earlier. We will outline a proof for (3) here leaving the rest as exercises.

If G is cyclic, then there exists an element $a \in G$ which generates G, i.e. if the order of G is n, then it can be described as $\{a^0, a, a^2, \ldots, a^n\}$. Since $G \cong H$, Theorem 9(3) and the fact that f is a bijection enable us to say that there is $b \in H$, such that $b = f(a)$ and $f\left(a^k\right) = \left(f\left(a\right)\right)^k = b^k$ for all $k \leqslant n$. Hence, H can be described as $\{b^0, b, b^2, \ldots, b^n\}$, and therefore is cyclic with b as a generator.

Example

The previous example gave you an example of an isomorphism. Here is an extension to look at the properties too.

Recall that $\ln\left(ab\right) = \ln a + \ln b$. The logarithmic function is an example to show you that the operations in the two isomorphic groups can be quite different.

Additionally, you can really see how all the properties mentioned earlier are clearly demonstrated by the logarithmic function. For example, the identity for multiplication is 1 as you know, $f(1) = \ln(1) = 0$, which is the identity for addition. Also, if a is a positive real number, then $\frac{1}{a}$ is its inverse. If you find

$f\left(\frac{1}{a}\right) = \ln\frac{1}{a} = -\ln a = -f\left(a\right)$, so the image of the inverse is the inverse of the image!

Summary

Note: Isomorphism is a special case of what is called **group homomorphism**. Homomorphism is defined as:

Let G be a group with operation $*$, $(G, *)$, and let H be a group with operation Δ, (H, Δ). A **homomorphism** of G into H is a mapping $f: G \rightarrow H$ such that

$$f\left(a * b\right) = f\left(a\right) \Delta f\left(b\right)$$

for every $a, b \in G$.
G and H are said to be **homomorphic**.

The difference between homomorphism and isomorphism is that isomorphism requires the mapping to be a bijection while homomorphism does not.

Example 6

Consider the function $g(x)$: $(\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \times)$ defined by $g(x) = 2^x$. Show that this is an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \times)$.

Solution

- We need to show that the function is an injection:

  Suppose that $2^x = 2^y$, then $\log_2 2^x = \log_2 2^y \Rightarrow x = y$.

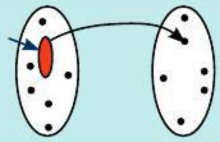
- To prove that it is a surjection, we need to show that for any positive real number y, we can find some real number x such that $g(x) = y$, i.e. $2^x = y$. Solving this equation for x gives us $x = \log_2 y$.
- To prove 'operation-preservation' we see that

  $g(x + y) = 2^{x+y} = 2^x \cdot 2^y = g(x)\ g(y)$.

Therefore, the function is an isomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \times)$.

Definition

If $f: G \to H$ is a group homomorphism, then the set $K = \{x \in G | f(x) = e_H\}$ is the **kernel** of f. The set K is often denoted by $\ker f$.



Theorem: Ker *f* is a subgroup

If $f: G \to H$ is a group homomorphism, then $\ker f = \{x \in G | f(x) = e_H\}$ with the group G operation is a subgroup of G.

> Remember that Theorem 3 states: Let G be a group and H a non-empty subset of G. Then, H is a subgroup of G iff $ab^{-1} \in H$ whenever $a, b \in H$.

Proof

Since $f(e_G) = e_H$, $e_G \in \ker f$ and thus $\ker f \neq \varnothing$. Also, if $x, y \in \ker f$, then $f(x) = f(y) = e_H$. Hence, by Theorem 9, $f(y^{-1}) = (f(y))^{-1} = e_H^{-1} = e_H$, and so $y^{-1} \in \ker f$.

Since f is a homomorphism, $f(xy^{-1}) = f(x)f(y^{-1}) = e_H e_H = e_H$.

So $xy^{-1} \in \ker f$.

Hence, by Theorem 3, $\ker f$ is a subgroup of G.

In the earlier discussion, we stated that a homomorphism between two groups does not need to be a bijection. Hence, if $f: G \to H$ is a group homomorphism, then f is not necessarily surjective. Thus the range of f is a subset of H and not necessarily equal to it. The following theorem helps characterize the range of a group homomorphism.

Theorem: Range of f is a subgroup

If $f: G \to H$ is a group homomorphism, then the range of f is a subgroup of H under group H operation.

Proof

Since $e_G \in G$, then $G \neq \varnothing$. If $x \in G$, then $f(x) \in f(G)$ and so $f(G) \neq \varnothing$. $f(G)$ is the range of f.

Let $f(x), f(y) \in f(G)$ where $x, y \in G$. Since $x, y \in G$, then $xy^{-1} \in G$ and so $f(xy^{-1}) \in f(G)$.

Since f is a homomorphism, $f(xy^{-1}) = f(x)\ f(y^{-1}) \in f(G)$ and also $f(y^{-1}) = (f(y))^{-1}$.

Since whenever $f(x), f(y) \in f(G)$, then $f(x)(f(y))^{-1} \in f(G)$.

Therefore, by Theorem 3, $f(G)$ is a subgroup of H.

Example 7

Let $(G, \times)$ be the multiplicative group of nonzero rational numbers and H the set of rational numbers different from 1. Define the binary operation $*$ on H by

$x * y = x + y - xy.$

a) Show that $(H, *)$ is a group.

b) Let $f: G \to H$ be defined by $f(x) = 1 - x$. Show that f is a group homomorphism.

Solution

a) If x, y are rational numbers different from 1, then $x + y - xy$ must also be a rational number different to 1. Otherwise, if $x + y - xy = 1$, then $x - xy = 1 - y \to x = \frac{1-y}{1-y} = 1$, which is a contradiction.

So, the set H is closed under $*$.

Let the identity element be e.

Hence,

$x * e = e * x = x + e - xe = x \Rightarrow e(1 - x) = 0$

Since $x \neq 1$, then $e = 0$.

So, the identity element is 0.

Now, if y is the inverse of x, then

$x * y = x + y - xy = 0 \Rightarrow y = \frac{x}{x-1}$

This is a rational number as it has a non-zero denominator and is different from 1. ($y = 1$ will lead to a contradiction; $0 = -1$)

So, every element has an inverse.

The associativity of the operation is left as an exercise:

$(x * y) * z = x * (y * z) = x + y + z - xy - xz - yz + xyz$

Therefore $(H, *)$ is a group.

b) Let $x, y \in G$, then

$f(xy) = 1 - xy; f(x) = 1 - x; f(y) = 1 - y;$

$f(x) * f(y) = 1 - x + 1 - y - (1 - x)(1 - y) = 1 - xy$

Hence, $f(xy) = f(x)*f(y)$ and the function f is a homomorphism.

Example 8

Consider the following two groups:

$\mathbb{R}$ under addition

$\mathbb{C}_1$ of complex numbers z with $|z| = 1$ under multiplication

Let $f: \mathbb{R} \to \mathbb{C}_1$ be the map defined by $f(x) = e^{2\pi ix}$.

Show that this is a homomorphism and find its kernel.

Solution

$f(x+y) = e^{2\pi i(x+y)} = e^{(2\pi ix + 2\pi iy)} = e^{2\pi ix} e^{2\pi iy} = f(x)f(y)$

Hence, f is a homomorphism.

To find ker f, we look for all $x \in \mathbb{R}$ such that $f(x) = e_{\mathbb{C}} = 1$ in this case,

$e^{2\pi ix} = 1 \Rightarrow 2\pi x$ must be a multiple of 2π. So, x must be an integer. Therefore ker $f = \mathbb{Z}$.

Example 9

Consider the group SL of 2×2 invertible matrices under matrix multiplication and the group of non-zero real Numbers $\mathbb{R}\backslash\{0\}$ under multiplication.

Define $f: SL \to \mathbb{R}\backslash\{0\}$ in the following manner.

If $A \in SL$, then $f(A) = \det A$.

Show that f is a homomorphism and find its kernel

Solution

$f(AB) = \det(AB) = \det A \cdot \det B = f(A)f(B)$

Hence, f is a homomorphism.

To find ker f, we look for all $A \in SL$, such that $f(A) = 1$,

So, A is any 2×2 matrix where $\det A = 1$. Thus ker f is SL_2 defined in the previous chapter.

Exercise 4

Note: In several questions, we will refer to the binary operation between two elements *a* and *b* by simply writing *ab*. This is done for convenience purposes and it does not mean that the operation is the usual multiplication of real numbers.

1 Show that $(\mathbb{Z}_5\backslash\{0\}, \times)$ is isomorphic to $(\mathbb{Z}_4, +)$.

2 Consider the set $M = \{[1], [3], [5], [9], [11], [13]\}$ under the operation $\times$, where $\times$ is multiplication modulo 14. (You may assume properties of multiplication modulo n in this problem.)

- **a** Show that $(5 \times 11) \times 3 = 5 \times (11 \times 3)$.
- **b** Show that $(M, \times)$ is a cyclic group and find all its generators.
- **c** Find all non-trivial proper subgroups of this group.

3 a $\left(\{e, x, x^2, x^3, x^4\}, \odot\right)$ is a cyclic group of order 5. Which elements generate the group?

b $\left(\{e, x, x^2, x^3, x^4, x^5\}, \odot\right)$ is a cyclic group of order 6. Which elements generate the group?

c Repeat part **b** for groups of order 7, 10, 15, and 20. How many generators does each have? Can you generalize?

4 Consider the group $S = \{I, R, R^2, L, M, N\}$ of symmetries of an equilateral triangle under transformation composition, $\circ$.

a Find the cyclic subgroup each of R, R^2, or L generates.

b Is $(S, \circ)$ cyclic? Justify your answer.

5 Let $U(n)$ be the set of integers less than n and relatively prime to n under multiplication modulo n.

For each group below, find the order of the group and the order of each of its elements. In each case explain how the order of the element is related to the order of the group.

a $(\mathbb{Z}_{12}, +_{12})$ **b** $(U(10), \cdot_{10})$ **c** $(U(12), \cdot_{12})$

d $(U(20), \cdot_{20})$ **e** D_4 (symmetries of the square)

6 Compute the orders of the following groups (all operations are modulo n):

a $U(3)$, $U(4)$, $U(12)$ **b** $U(5)$, $U(7)$, $U(35)$

c $U(4)$, $U(5)$, $U(20)$ **d** $U(3)$, $U(5)$, $U(15)$

Make a conjecture about the relationship among $|U(m)|$, $|U(n)|$, and $|U(mn)|$.

Now compute $|U(4)|$, $|U(10)|$, and $|U(40)|$. Do you need to adjust your conjecture?

7 Let $(G, *)$ be a group and $a \in G$. If $a^2 \neq e$ and $a^6 = e$, show that $a^4 \neq e$ and $a^5 \neq e$. What could be the order of a?

8 Let $(G, \cdot)$ be a group. Let $a \in G$ such that $|a| = 6$. Find $|a^2|$, $|a^3|$, $|a^4|$, and $|a^5|$. If $b \in G$ is such that $|b| = 9$, find $|b^i|$ for $i = 2, 3, \ldots, 8$.

9 Consider the group $(\mathbb{Z}_{11}\backslash\{0\}, \times_{11})$.

a Find the cyclic group each of 2, 3, 4, 6, or 10 generates.

b Is $(\mathbb{Z}_{11}\backslash\{0\}, \times_{11})$ cyclic? Justify your answer.

10 You are given the operation table for a set of 7 members.

	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	c	a	e	f	g	d
c	c	a	b	f	g	d	e
d	d	e	f	g	a	b	c
e	e	f	g	a	d	c	b
f	f	g	d	b	c	e	a
g	g	d	e	c	b	a	f

a Show that $\{a, b, c\}$ form a group.

b Show that the whole set cannot form a group.

11 Consider a group (M, Δ).

a If $x \in M$ has order 12, show that there is an element of M of order 3.

b If $|M| = 12$, show that (M, Δ) has a cyclic subgroup of order 2, 3, 4, or 6.

12 Show that a group with order p, where p is a prime number, must be cyclic.

13 A regular pentagon has 5 rotation symmetries I: R, which rotates the pentagon through an angle of 72°, R^2, an angle of 144°, R^3, an angle of 216°, and R^4, an angle of 288°.

Show that this group under composition of rotations is cyclic and that it is isomorphic to $(\mathbb{Z}_5, +)$.

14 Consider the set $N = \{1, 3, 5, 7, 9, 11, 13, 15\}$ under multiplication modulo 16. Denote this multiplication simply by $\times$.

a Show that $3 \times (9 \times 11) = (3 \times 9) \times 11$.

b Show that $(N, \times)$ is a group.

c Does N have any subgroups? What order should they be? Find all of them.

d Is this a cyclic group? If yes, find all generators.

15 Consider a group $(G, \circ)$ with an identity element i.

a $x \in G$ has order n. What should the order of x^{-1} be? Justify your answer.

b For $x, y, z \in (G, \circ)$, prove that $y = z^{-1}xz \Rightarrow y^n = z^{-1}x^n z$ for $n \in \mathbb{Z}^+$. (Hint: Use mathematical induction.)

16 Consider a group $(G, \bullet)$ with identity element e.

Consider also the set $H \subset G$ whose elements commute with all the elements of G, i.e.

$H = \{x \in G \mid \forall\, a \in G, ax = xa\}$.

Show that $(H, \bullet)$ is a subgroup of $(G, \bullet)$.

17 A group $(G, \cdot)$ is generated by two elements x and y subject only to the relations (every element of the group can be expressed as some product of x's and y's)

$x^3 = y^2 = (xy)^2 = 1$.

a List the different elements of the group.

b List all the subgroups of this group.

18 A group $(G, \cdot)$ is generated by two elements x and y subject only to the relations

$x^3 = y^2 = (xy)^3 = 1$.

a List 12 different elements of the group.

b List all the subgroups of this group.

19 Let Q be the group (under matrix multiplication) generated by the complex matrices

$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$.

Show that Q is a non-Abelian group of order 8.

20 Let T be the group (under matrix multiplication) generated by the real matrices
$u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
Show that Q is a non-Abelian group of order 8.

21 Let D be the group (under matrix multiplication) generated by the complex matrices
$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} e^{\frac{2\pi}{3}i} & 0 \\ 0 & e^{-\frac{2\pi}{3}i} \end{pmatrix}$, where $i^2 = -1$.
Show that D is a non-Abelian group of order 6.

22 If H and K are subgroups of a group $(G, *)$, then $H \cap K$ is also a subgroup of G.

Is the same true for $H \cup K$? Justify.

23 Let $(G, *)$ be a group, and $a, b, c \in G$. Show that the equation $a\,\boldsymbol{x}\,c = b$ has a *unique* solution in G.

24 Find all subgroups of $\{\mathbb{Z}_7 \backslash \{0\}, \times_7)$, of $(\{1, 3, 5, 7\}, \times_8)$, of $(\{1, 2, 4, 7, 8, 11, 13, 14\}, \times_{15})$.

25 Show that the group of matrices of the form
$\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$, $x \neq 0$
is a subgroup of the group $(GL_2, \cdot)$ of real 2×2 invertible matrices.

26 Determine the cyclic subgroups of the group $(GL_2, \cdot)$ of real 2×2 invertible matrices generated by
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

27 Prove that every subgroup of a cyclic group is cyclic. Show, by a counterexample, that the converse of this theorem is not true.

28 Let $(G, *)$ be a group, and $a \in G$ has infinite order. Show that $a^i = a^j$ if and only if $i = j$. That is, no two distinct powers of a are equal (integral exponents).

29 (Optional) Show that the determinant of a matrix defines a homomorphism from the group of 2×2 non-singular real matrices under matrix multiplication to the group of non-zero real numbers under normal multiplication.

30 Show that the group M of 2×2 matrices described below under matrix multiplication and the group of symmetries of the equilateral triangle are isomorphic.

$$M = \left\{ \begin{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{matrix} \right\}$$

31 Show that the group $(\{1, -1, i, -i\}, \times)$ is isomorphic to $(\mathbb{Z}_4, +)$.

32 Let G be a group with some operation and a is some *fixed* element of G. Show that the mapping h defined by

$h(x) = a\,x\,a^{-1}, \forall\, x \in G$

is an isomorphism from G into itself.

33 Consider the set $\{4, 8, 12, 16\}$. Show that this set is a cyclic group under multiplication modulo 20. Find its generators.

34 Consider the set $\{7, 35, 49, 77\}$. Show that this set is a group under multiplication modulo 84. Is this a cyclic group?

35 Let $G = \left\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\right\}$ and $H = \left\{\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{Q}\right\}$.

Show that $(G, +)$ and $(H, +)$ are isomorphic.

36 Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$.

Show that f is a homomorphism from the group of positive real numbers under multiplication to the group of real numbers under addition. Find its kernel.

37 Consider the absolute value function from the group of all non-zero real numbers (under multiplication) into the group of positive real numbers (under multiplication).

Show that it is a homomorphism and find its kernel.

38 Let $P[x]$ denote the group of all polynomials with real coefficients under addition. Define the mapping φ that assigns to every function its derivative, i.e.

for every $f \in P[x]$, $\varphi: P[x] \to P[x]$ such that $\varphi(f) = f'$.

Show that it is a homomorphism and find its kernel.

Practice questions 4

1 a Define an isomorphism between two groups $(G, \circ)$ and $(H, \bullet)$.

b Let e and e' be the identity elements of groups G and H respectively.

Let f be an isomorphism between these two groups. Prove that $f(e) = e'$.

c Prove that an isomorphism maps a finite cyclic group onto another finite cyclic group.

2 a Let f_1, f_2, f_3, f_4 be functions defined on $\mathbb{Q} - \{0\}$, the set of rational numbers excluding zero, such that $f_1(z) = z$, $f_2(z) = -z$, $f_3(z) = \frac{1}{z}$, and $f_4(z) = -\frac{1}{z}$, where $z \in \mathbb{Q} - \{0\}$.

Let $T = \{f_1, f_2, f_3, f_4\}$. Define $\circ$ as the composition of functions, *i.e.*

$(f_1 \circ f_2)(z) = f_1(f_2(z))$. Prove that $(T, \circ)$ is an Abelian group.

b Let $G = \{1, 3, 5, 7\}$ and $(G, \lozenge)$ be the multiplicative group under the binary operation $\lozenge$, multiplication modulo 8. Prove that the two groups $(T, \circ)$ and $(G, \lozenge)$ are isomorphic.

3 Let $S = \left\{x \mid x = a + b\sqrt{2};\, a, b \in \mathbb{Q},\, a^2 - 2b^2 \neq 0\right\}$.

a Prove that S is a group under multiplication, $\times$, of numbers.

b For $x = a + b\sqrt{2}$, define $f(x) = a - b\sqrt{2}$. Prove that f is an isomorphism from $(S, \times)$ onto $(S, \times)$.

4 a In any group, show that if the elements x, y, and xy have order 2, then $xy = yx$.

b Show that the inverse of each element in a group is unique.

c Let G be a group. Show that the correspondence $x \leftrightarrow x^{-1}$ is an isomorphism from G onto G if and only if G is **Abelian**.

5 Let $(S, \circ)$ be the group of all permutations of four elements a, b, c, d. The permutation that maps a onto c, b onto d, c onto a and d onto b is represented by $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$.

The identity element is represented by $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$.

Note that AB denotes the permutation obtained when permutation B is followed by permutation A.

a Find the inverse of the permutation $\begin{pmatrix} a & b & c & d \\ c & a & d & b \end{pmatrix}$.

b Find a subgroup of S of order 2.

c Find a subgroup of S of order 4, showing that it is a subgroup of S.

6 Let $S = \{f, g, h, j\}$ be the set of functions defined by

$f(x) = x$, $g(x) = -x$, $h(x) = \dfrac{1}{x}$, $j(x) = -\dfrac{1}{x}$, where $x \neq 0$.

a Construct the operation table for the group $\{S, \circ\}$, where $\circ$ is the composition of functions.

b The following are the operation tables for the groups $\{0, 1, 2, 3\}$ under addition modulo 4, and $\{1, 2, 3, 4\}$ under multiplication modulo 5.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

By comparing the elements in the two tables given plus the table constructed in part **a**, find which groups are isomorphic. Give reasons for your answers. State clearly the corresponding elements.

7 The group $(G, \times)$ has a subgroup $(H, \times)$. The relation R is defined on G $(xRy) \Leftrightarrow (x^{-1}y \in H)$, for $x, y \in G$.

a Show that R is an equivalence relation.

b Given that $G = \{e, p, p^2, q, pq, p^2q\}$, where e is the identity element, $p^3 = q^2 = e$, and $qp = p^2q$, prove that $qp^2 = pq$.

c Given also that $H = \{e, p^2q\}$, find the equivalence class with respect to R which contains pq.

8 a Find $\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b Let G be the set of matrices of the form

$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, for $a, b \in \mathbb{R}$.

Show that G is an Abelian group under matrix multiplication.

c Let F be the group of real ordered pairs under addition defined by

$(a, b) + (c, d) = (a + c, b + d)$.

Show that G is isomorphic to F.

9 a Show that the set S of numbers of the form $2^m \times 3^n$, where $m, n \in \mathbb{Z}$, forms a group $\{S, \times\}$ under multiplication.

b Show that $\{S, \times\}$ is isomorphic to the group of complex numbers $m + ni$ under addition, where $m, n \in \mathbb{Z}$.

10 a Draw the Cayley table for the set of integers $G = \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6, $+_6$.

b Show that $\{G, +_6\}$ is a group.

c Find the order of each element.

d Show that $\{G, +_6\}$ is cyclic and state its generators.

e Find a subgroup with three elements.

f Find the other proper subgroups of $\{G, +_6\}$.

Answers

Chapter 1

Exercise 1

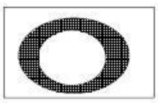
1 a) Equal b) Equal
c) Equal d) Not equal

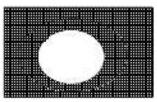
2 a) $\{1, 3, 4\}$ b) $\{1, 3, 4\}$ c) $\{6\}$
d) $\{1, 2, 5, 6\}$ e) $\{6\}$ f) $\{1, 2, 3\}$
g) $\{1, 2, 5\}$

3 a) False b) True c) True
d) True e) True f) True
g) True h) True i) True

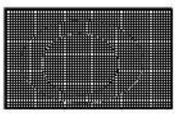
4 a) True b) True c) False
d) True e) False f) False
g) True h) False i) True

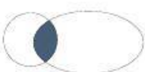
5 a) A b) B

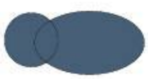
c) 

d) 

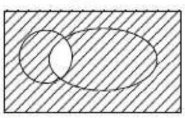
e) $\varnothing$

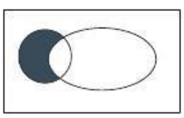
f) 

6 a) 

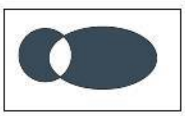
b) 

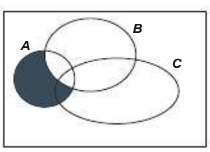
c) 

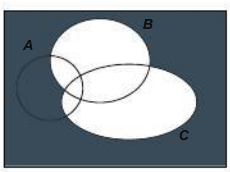
d) 

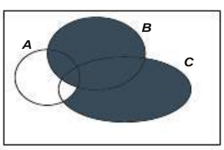
e) 

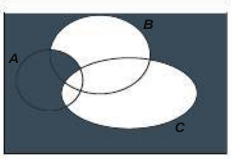
f) 

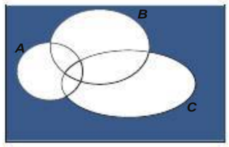
g) 

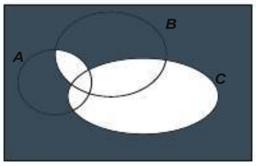
7 a) 

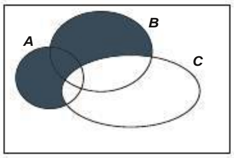
b) 

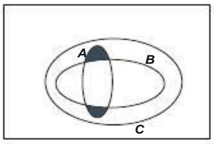
c) 

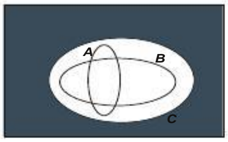
d) 

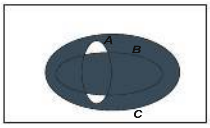
e) 

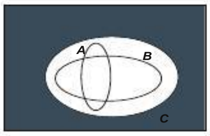
f) 

g) 

8 a) 

b) 

c) 

d-f) 

g) $\varnothing$

9 a) $\{-1\}$ b) $\varnothing$ c) $\{0, 1\}$
d) $\mathcal{P}(A) = \{\varnothing, \{0\}, \{-1\}, \{1\}, \{0, -1\}, \{0, 1\}, \{-1, 1\}, \{0, -1, 1\}\}$

10 $A \cap B'$ or $A \cap (C \setminus B)$

11 42

12 24

13 a) $\mathbb{Z}^+$ b) $\{1, 3, 5, ...\}$ c) M_6 d) $\varnothing$

14 $A = B$

15 a–l) Proof

16 128

17 a–e) Proof

18 a) Proof
b) $\{\varnothing\}$; $\{\varnothing, \{\varnothing\}\}$
c) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$
d) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

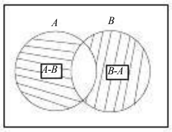
19 a) $[0, \infty[$ b) $\varnothing$ c) $[1, 3[$ d) $]0, 2]$

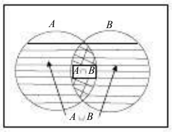
20 $|A \cup B| \neq |A| + |B|$

21 a–h) Proof

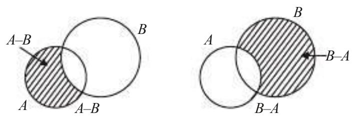
22 a–e) Proof

Practice questions 1

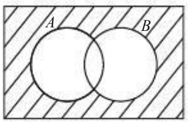
1 a) 



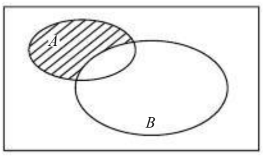
b) Proof

2


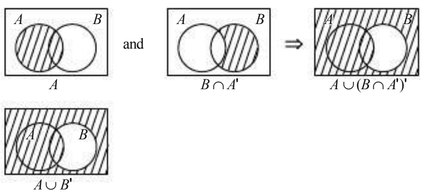
3 Proof

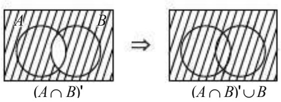
4 a)


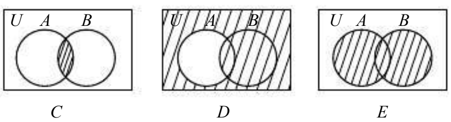
b) Proof

5 a)


b) Proof

6 a)


b)


7 a)


b–c) Proof

8 Proof

9 a) (i) $S_1 = x \in \mathbb{Z}^+ \,|1$ divides x

$= 1, 2, 3, \ldots = \mathbb{Z}^+$

(ii) $S_2 = x \in \mathbb{Z}^+ \,|2$ divides x

$= 2, 4, 6, \ldots$

Hence, $S_2' = 1, 3, 5, \ldots.$

(iii) $S_3 = x \in \mathbb{Z}^+ \,|3$ divides x

$= 3, 6, 9, \ldots$

Hence, $S_2 \cap S_3 = 6, 12, 18, \ldots.$

(iv) $S_6 = x \in \mathbb{Z}^+ \,|6$ divides x

$= 6, 12, 18, \ldots$

Hence, $S_6 \setminus S_3 = S_6 \cap S_2' = \varnothing.$

b) Proof

10 Proof

Chapter 2

Exercise 2

1 a) (i) $\{(1, a), (1, b), \ldots, (2, c), (1, x), \ldots, (3, z)\}$

(ii) $\varnothing$

(iii) $\varnothing$

(iv) $\{(1, a), (1, b), \ldots, (2, c), (1, x), \ldots, (3, z)\}$

(v) $\varnothing$

(vi) $A \times B$

b) i and iv; ii, iii, v, and vi

2 a, c, d, e

3 a) Points on the lines $y = x$ and $y = -x$ are symmetric with respect to the x- and y-axes. For example, $(2, 2)$, $(2, -2)$, $(-2, -2)$ and $(-2, 2)$.

c) Numbers of the form n and $-n - 1$.

d) Every complete square and its positive factors.

e) Concentric circles with O as centre.

4 a) 4, 5, 4

b) 3

c) Proof

5 a) $\mathcal{R}$ is an equivalence relation. Classes are: $\{1\}, \{2\}, \ldots, \{9\}$.

b) $\mathcal{X}$ is not an equivalence relation since it is not reflexive.

6 a) Injection b) Injection

c) Injection d) Surjection

7 a) Yes b) No c) Yes

8 a) nm b) $\dfrac{n!}{(n-m)!}$ c) $n!$

9 a) Yes; no

b) No; no

c) (i) $[-4, 3]$, $[0, 2]$

(ii) $[-9, 5]$, $[-9, 5]$, $[-1, 3]$, $[-1, 3]$

(iii) $[1, 17]$, $[1, 17]$, $[1, 5]$, $[1, 10]$

10 No; yes

11 a–b) Proof

12 a) $f(a) \neq f(b) \neq f(c)$

b) c, a, b

c) Identity; $f^{-1} = f \circ f$

13 $\mathcal{S}$ is an equivalence relation.

14 a) Proof

b) Concentric circles with centre at the origin. All points on the circle with radius $\sqrt{5}$.

15 Both. $h^{-1} : (a, b) \mapsto \left(\dfrac{2b - a}{3}, \dfrac{a + b}{3}\right)$.

16 Proof

17 $\mathcal{S}$ is an equivalence relation; $\{\{a, c, e\}, \{b, d\}, f\}$

18 $\{\{1, 4, 6, 9, 11\}, \{2, 3\}, \{5, 10\}, \{7, 8\}\}$

19 a) Not a bijection b) Bijection

c) Not a bijection

20 Proof

21 a) Proof b) $\{\{0, 4, 8\},\{1, 5, 9\},\{2, 6\},\{3, 7\}\}$

c) 3

22 a) Injective b) Not surjective

23 $f^{-1}(x, y) = \left(\dfrac{5x + 3y}{11}, \dfrac{2x - y}{11}\right)$

24 a–b) Proof

25 Proof

26 a) (i) $R = \left\{\dfrac{e+1}{e}, e + 1\right\}$

(ii) Proof (iii) Not a surjection

b) (i) $k = \pi$ (ii) $f^{-1}(x) = \arccos(\ln(x - 1))$

27 a) Proof

b) $\{\{4, 24, 32\}, \{8, 20, 36\}, \{12, 16\}, 28\}$

28 Proof

29 $h^{-1}(x, y) \mapsto \left(\dfrac{3y - x}{4}, \dfrac{x - y}{2}\right)$

30 Neither

31 a) Proof
b) $\{5k, \{1 + 5k, 4 + 5k\}, \{2 + 5k, 3 + 5k\}\}, k \in \mathbb{N}$

32 a) Proof b) $a = 2$

33 a–d) Proof

34 a–b) Proof

Practice questions 2

1 a) Proof
b) This is the set of ordered pairs (x, y) such that $x^2 + y^2 = 5$.
c) The partition is the set of all concentric circles in the plane with the origin as the centre.

2 a) Proof
b) The classes are those pairs (a, b) and (c, d) with $\dfrac{a}{b} = \dfrac{c}{d}$. The elements are on the same line going through the origin.

3 a) Proof
b) (i) Student explanation
(ii) $\{5, 10\}, \{1, 4, 6, 9\}, \{2, 3, 7, 8\}$

4 a) Proof
b) $\{0, 4, 8, \ldots\}, \{1, 5, 9, \ldots\}, \{2, 6, 10, \ldots\}, \{3, 7, 11, \ldots\}$
c) 3

5 a) (i–ii) f is injective but not surjective.
b) (i–ii) g is injective and surjective.
c) $g^{-1}(x, y) = \left(\dfrac{5x + 2y}{11}, \dfrac{3x - y}{11}\right)$
d) Proof

6 a–c) Proof

7 a) Proof
b) $3n - 2; 3n - 1; 3n; n \in \mathbb{Z}^+$

8 The equivalence class of $(1, 1)$ is a pair of straight lines through the origin with slopes ± 1.

9 a) Range is $\left[-\dfrac{9}{4}, \infty\right[$; not an injection
b) $g^{-1}(x) = +\sqrt{x + \dfrac{9}{4}} - \dfrac{1}{2}$ on $[0, 4]$

10 a) Proof
b) The equivalence classes are points lying, in the first quadrant, on straight lines through the origin.

Chapter 3

Exercise 3

1 a) Proof

b)

$\circ$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

c) Yes

2 a) (i) 75 (ii) 45 (iii) 8
(iv) 0 (v) 9 (vi) 3
(vii) 4608 (viii) 288
b) No; $x = 0, y = 0$, or $x = y$
c) No

3 Proof

4 e is the identity, s is the reflection with respect to the smaller diagonal, and l with respect to the larger diagonal, and r is a rotation of 180°.

$\circ$	e	r	s	l
e	e	r	s	l
r	r	e	l	s
s	s	l	e	r
l	l	s	r	e

5 a)

$\circ$	p	r	s	t
p	p	p	p	p
r	p	r	s	t
s	t	s	r	p
t	t	t	t	t

b) r is the identity.
c) No d) r, s e) No

6 a)

$\circ$	p	r	s	t
p	p	r	s	t
r	r	p	t	s
s	s	s	s	s
t	t	t	t	t

b) p is the identity.
c) No d) p, r e) No

7 A group with identity 1 and each element is self-inverse.

8 Not a group: $1 + 1 = 2 \notin \{-1, 0, 1\}$.

9 A group with identity 0 and inverse defined by $(10k)^{-1} = -10k$.

10 A group with identity 1 and inverse defined by $\left(2^m\right)^{-1} = 2^{-m}$.

11 A group with identity 1 and inverse defined by $\left(2^m 3^n\right)^{-1} = 2^{-m}3^{-n}$.

12 A group with identity $f(x) = 0$ and inverse defined by $f^{-1}(x) = -f(x)$.

13 A group with identity 0 and inverse defined by $a^{-1} = -\dfrac{a}{a + 1}$.

14 A group with identity 1 and inverse defined by $\left(a + b\sqrt{2}\right)^{-1} = \dfrac{a}{a^2 - 2b^2} - \dfrac{b}{a^2 - 2b^2}\sqrt{2}$.

15 Proof

16 Proof

17 a) 24
b) If we let $1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, $a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, ..., then the table will look like this:

$\circ$	1	a	b	c	$\cdots$
1	1	a	b	c	$\cdots$
a	a	1	c	b	$\cdots$
b	b	d	1	f	$\cdots$
c	c	f	a	d	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

c) For example: $a \circ b = c \neq b \circ a = d$

18 a–c) Proof

19 a–b) Proof

c) Yes; 3, 11

20

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

21

$\propto$	w	x	y	z
w	y	z	w	x
x	z	w	x	y
y	w	x	y	z
z	x	y	z	w

22 Proof

23 a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

24 a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

25 Proof

26 Proof

27 a–b) Proof

28 Proof

29 Proof

30 Proof

31 29

32 $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}, \beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$

33 a) $\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}, \alpha^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix},$

and $\alpha^6 = e$.

b) (13)(245)

c) $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$, and $\alpha\alpha^{-1} = \alpha^{-1}\alpha = e$.

34 a) $\alpha = (1365)(2478)$, $\beta = (18)(27)(36)(45)$,

$\gamma = (15)(247)(36)$

b) $\alpha\beta = (1283574)$

c) $\alpha\beta\gamma = (1783652)$

d) $\beta^{-1} = (18)(27)(36)(45) = \beta$

e) $(\beta\gamma)^{-1} = (18524)$

f) $\gamma^{-1}\beta^{-1} = (18524)$

g) $\alpha^{-1}\gamma\alpha = (13)(248)(56)$

h) $\text{ord}(\gamma) = 6$

i) $\text{ord}(\alpha^{-1}\gamma\alpha) = 6$

35 a) (1, 8, 2, 7)(3, 4, 5, 6)(9, 10)

b) (1, 10, 2, 9, 7, 5, 8, 3, 11, 6, 15, 14, 13, 12)

36 a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 1 & 4 & 3 & 2 & 6 & 8 & 9 & 7 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 5 & 6 & 9 & 1 & 4 & 3 & 8 \end{pmatrix}$

Practice questions 3

1 Proof

2 a–b) Proof

3 a–b) Proof

4 a (i) Proof

(ii) $a = 3,\ b = -\frac{3}{2}$

b (i) $A = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(ii) $\{A, A^2, A^3, I\}$

5 a)

$*$	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

b) (i) $x = d$

(ii) $x = a$

6 a) Proof

b) R is an equivalence relation.

7 a) 6

b) (i) $p_2 p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$

(ii) They do not commute.

c) $(p_1^2 p_2)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$

8 a–c) Proof

9 a) (i) Not closed (ii) Commutative

(iii) Not associative

b) (i) $e = 2$ (ii) $\{1, 2, 3\}$

10 a) (i) Proof

(ii) {2, 8}, {1, 4, 9}

b) Proof

Chapter 4

Exercise 4

1 Proof

2 a–b) Proof c) $\{1, 13\}, \{1, 9, 11\}$

3 a) $\{x, x^2, x^3, x^4\}$
b) $\{x, x^5\}$
c) 7 has 6 generators, 10 has 3, 15 has 8, and 20 has 8. The number of generators is the number of numbers less than or equal to the group order and is relatively prime to it.

4 a) $\{I, R, R^2\}, \{I, L\}$ b) No

5 a) 12, ([1], 12), ([2], 6), ([3], 4), ([4], 3), ([5], 12), ([6], 2), ([7], 12), ([8], 3), ([9], 3), ([10], 6), ([11], 12). Factors of 12.
b) 4, ([3], 4), ([7], 4), ([9], 2). Factors of 4.
c) 4, ([5], 2), ([7], 2), ([11], 2). Factors of 4.
d) 8, ([3], 4), ([7], 4), ([9], 2), ([11], 2), ([13], 4), ([17], 4), ([19], 2). Factors of 8.
e) 8, $(r, 4), (r^2, 2), (r^3, 4), (L_1, 2), (L_2, 2), (L_3, 2), (L_4, 2)$. Factors of 8.

6 a) $(U(3), 2), (U(4), 2), (U(12), 4)$
b) $(U(5), 4), (U(7), 6), (U(35), 24)$
c) $(U(4), 2), (U(5), 4), (U(20), 8)$
d) $(U(3), 2), (U(5), 4), (U(15), 8)$
$|U(mn)| = |U(m)|\cdot|U(n)|$; $(U(4), 2), (U(10), 4), (U(40), 16)$;
$|U(mn)| = |U(m)|\cdot|U(n)|$ iff m and n are relatively prime.

7 3 or 6

8 $|a^2| = 3, |a^3| = 2, |a^4| = 3, |a^5| = 6$.
$|b^2| = 9, |b^3| = 3, |b^4| = 9, |b^5| = 9, |b^6| = 3, |b^7| = 9, |b^8| = 9$.

9 a) 2 and 6 generate {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}; 3 and 4 generate {1, 3, 4, 5, 9}; 10 generates {1, 10}.
b) Yes

10 a–b) Proof

11 a–b) Proof

12 Proof

13 Proof

14 a–b) Proof
c) Yes; 2 or 4; {1, 7}, {1, 9}, {1, 11}, {1, 15}, {1, 3, 9, 11}, {1, 5, 9, 13}
d) No

15 a) n
b) Proof

16 Proof

17 a) $\{1, x, x^2, y, xy, x^2y\}$
b) $\{1, y\}, \{1, xy\}, \{1, x^2y\}, \{1, x, x^2\}$

18 a) $1, x, x^2, y, xy, yx^2, yx, x^2y, xyx, yxy, x^2yx, xyx^2$
b) $\{1\}, \{1, y\}, \{1, x^2yx\}, \{1, xyx^2\}, \{1, x, x^2\}, \{1, xy, yx^2\}, \{1, yx, x^2y\}, \{1, xyx, yxy\}$

19 Proof

20 Proof

21 Proof

22 No. Only if $H \subseteq K$ or $K \subseteq H$.

23 Proof

24 {1, 2, 4}, {1, 6}; {1, 3}, {1, 5}, {1, 7}; {1, 4}, {1, 11}, {1, 14}, {1, 2, 4, 8}, {1, 4, 7, 13}

25 Proof

26 $\left\{\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}, k \in \mathbb{N}\right\}, \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$

27 Proof

28 Proof

29 Proof

30 Proof

31 Proof

32 Proof

33 Generators: 8, 12

34 Not cyclic

35 Proof

36 If $x, y \in \mathbb{R}^+$ then $\ln(xy) = \ln x + \ln y$, thus f is a homomorphism. Since $f(x) = 0$ then $x = 1$, therefore $\ker f = \{1\}$.

37 If $x, y \in \mathbb{R}\backslash\{0\}$ then $|xy| = |x|\,|y|$, thus f is a homomorphism. Since $f(x) = 1$ then $x = \pm 1$, therefore $\ker f = \{-1, 1\}$.

38 If $f, g \in P[x]$, then $\varphi(f+g) = (f(x)+g(x))' = f'(x)+g'(x)$, thus φ is a homomorphism.
$\varphi(f) = 0 \Rightarrow f'(x) = 0 \Rightarrow f$ must be a constant. Hence $\ker\varphi$ is the set of all constant functions with real coefficients.

Practice questions 4

1 a–c) Proof

2 a–b) Proof

3 a–b) Proof

4 a–c) Proof

5 a) $\begin{pmatrix}a & b & c & d\\ b & d & a & c\end{pmatrix}$

b) For example: $\begin{pmatrix}a & b & c & d\\ a & b & c & d\end{pmatrix}; \begin{pmatrix}a & b & c & d\\ b & a & c & d\end{pmatrix}$

c) $\begin{pmatrix}a & b & c & d\\ a & b & c & d\end{pmatrix}$
$\begin{pmatrix}a & b & c & d\\ b & c & d & a\end{pmatrix}; \begin{pmatrix}a & b & c & d\\ c & d & a & b\end{pmatrix}; \begin{pmatrix}a & b & c & d\\ d & a & b & c\end{pmatrix}$

6 a)

$\circ$	f	g	h	j
f	f	g	h	j
g	g	f	j	h
h	h	j	f	g
j	j	h	g	f

b) $+_4$ is isomorphic with $\times_5$. Corresponding elements are: $0 \leftrightarrow 1, 1 \leftrightarrow 2, 2 \leftrightarrow 4, 3 \leftrightarrow 3$; or $0 \leftrightarrow 1, 1 \leftrightarrow 3, 2 \leftrightarrow 4, 3 \leftrightarrow 2$.

7 a–b) Proof c) $\{p^2, pq\}$

8 a) $\begin{pmatrix}1 & a & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -a & -b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

b) Proof c) Proof

9 a–b) Proof

10 a)

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

b) Proof

c)

Number	0	1	2	3	4	5
Order	1	6	3	2	3	6

d) Generators: 1 and 5

e) {0, 2, 4}

f) {0},{0, 3}

Calculus

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Calculus

Assessment statements

9.1 Infinite sequences of real numbers and their convergence or divergence.

9.2 Convergence of infinite series.

Tests for convergence: comparison test; limit comparison test; ratio test; integral test.

The p-series, $\sum \frac{1}{n^p}$.

Series that converge absolutely.

Series that converge conditionally.

Alternating series.

Power series: radius of convergence and interval of convergence. Determination of the radius of convergence by the ratio test.

9.3 Continuity and differentiability of a function at a point.

Continuous functions and differentiable functions.

9.4 The integral as a limit of a sum; lower and upper Riemann sums.

Fundamental theorem of calculus.

Improper integrals of the type $\int_a^\infty f(x)\,dx$.

9.5 First order differential equations.

Geometric interpretation using slope fields, including identification of isoclines.

Numerical solution of $\frac{dy}{dx} = f(x, y)$ using Euler's method.

Solving differential equations by method of separation of variables.

Homogenous differential equation $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ using the substitution $y = vx$.

Solution of $y' + P(x)y = Q(x)$, using the integrating factor.

9.6 Rolle's theorem. Mean value theorem.

Taylor polynomials; the Lagrange form of the error term.

Maclaurin series for e^x, $\sin x$, $\cos x$, $\ln(1 + x)$, $(1 + x)^p$, $p \in \mathbb{Q}$.

Use of substitution, products, integration and differentiation to obtain other series.

Taylor series developed from differential equations.

9.7 The evaluation of limits of the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ and $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$.

Use of L'Hôpital's rule or the Taylor series.

1 Sequences, Limits and Improper Integrals

Introduction

Important concepts regarding sequences, series and limits were covered in previous textbook chapters on the core syllabus. It would be helpful to go back and read through the first four sections of Chapter 4, especially the material on infinite geometric series in Section 4.4. The first section in Chapter 13 includes an informal approach to limits of functions and also covers properties of limits. Central to any discussion about sequences, series and limits is the concept of a function. Thus, it may also prove worthwhile to review some of the fundamental ideas, terminology and notation for functions covered in the first section of Chapter 2.

Arithmetic and geometric series, both finite and infinite, were discussed in Chapter 4. Much of the material in this chapter and the next two chapters is directly or indirectly involved with infinite series. As you will see, infinite series are mathematically interesting and have very useful applications. Our treatment of series in this option topic will require a more formal approach than taken in Chapter 4. In order to develop a more thorough treatment of infinite series, we must first consider **infinite sequences** of numbers.

Sequences and series are closely related, so you need to be careful to apply these words correctly. A sequence is an ordered list of numbers commonly written out with commas separating the numbers. A series is a sum of a sequence. The finite sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ is an ordered list whereas the closely related finite series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ is a sum that is precisely equal to the number $\frac{15}{8}$.

1.1 Infinite sequences

Sequences occur in many areas of mathematics. For example, the positive even numbers less than or equal to 10 form a sequence:

2, 4, 6, 8, 10.

This sequence is **finite** because the list of numbers ends with a specific number, 10 in this case. If a sequence does not end, it is **infinite**. We will be focusing on infinite sequences, so from now on if we use the word 'sequence' it is understood that we are referring to an infinite sequence.

Definition of a sequence

A sequence of numbers is a discrete function whose domain is the set of positive integers, $\mathbb{Z}^+$.

From the definition it is understood that an infinite sequence is a rule that associates a number to each positive integer. The number associated with the integer n is called the nth term of the sequence. Instead of using the familiar function notation $f(n)$ to represent the value (term) of a sequence f for a certain positive integer n, it is customary to use a subscripted letter, such as a_n or u_n. Hence, we will denote a sequence by $\{a_1, a_2, a_2, \ldots, a_n, \ldots\}$, or more simply with the notation $\{a_n\}, n \in \mathbb{Z}^+$. It follows that a_n is an **explicit formula** (sometimes called a closed formula) that is a function whose domain, n, is the set of positive integers and generates the value of the nth term of a sequence. The notation $\{a_n\}$ represents all the terms of a sequence, not just a single term. For example, for the sequence formed by the reciprocals of the positive integers, we can write $\{a_n\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and $a_n = \frac{1}{n}$.

Example 1 – Listing the terms of a sequence

a) The terms of the sequence $\{a_n\} = \left\{1 - \frac{1}{n}\right\}$ are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$.

b) The terms of the sequence $\{b_n\} = \left\{\frac{(-1)^{n+1}}{n}\right\}$ are $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$.

c) The terms of the sequence $\{c_n\} = \left\{\frac{2^{n-1}}{(n-1)!}\right\}$ are $1, \frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots$.

The first six terms of the sequence $\{c_n\}$ can be simplified to $1, 2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}, \ldots$. This highlights the fact that although it is often helpful to view some of the initial terms in an infinite sequence, knowing the explicit formula for the value of the nth term is even more useful. (**Note:** Evaluating the first term in the sequence $\{c_n\}$ required using the definition that $0! = 1$.)

Example 2 – A sequence defined by a recursive formula

It is not necessary for a sequence to be defined by an explicit formula, as in Example 1. The sequence $\{a_n\}$ defined by

$$a_1 = a_2 = 1, \ a_{n+2} = a_{n+1} + a_n \ \text{ for } n \geqslant 1$$

is a sequence that we saw in Chapter 4 of the book. The rule giving a_{n+2} in terms of a_{n+1} and a_n is an example of a recursion formula. It produces the famous Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, \ldots.$$

Although a bit complicated, an explicit formula exists for the nth term of the Fibonacci sequence. In general, the rules for sequences and series in this chapter will be explicit rather than recursive. See Chapter 4 of the book for discussion of explicit and recursive formulae for sequences.

Our foremost concern with a sequence $\{a_n\}$ is whether a_n has a limit L as n approaches infinity ($n \to \infty$). If it does, we say that $\{a_n\}$ **converges** to L; otherwise we say that $\{a_n\}$ **diverges.**

Since a sequence is a type of function, it seems appropriate that in our investigation of limits of sequences, we can apply the same ideas from our work with limits of functions in Chapter 13 of the book. A function f whose domain is the half-open interval $[1, \infty[$ can be converted into a sequence by restricting its domain to the integers in that interval, i.e. the positive integers $\mathbb{Z}^+$. Conversely, given a sequence $\{a_n\}$, it is often possible to define a function f on $[1, \infty[$ such that $f(n) = a_n$ for each integer $n > 0$. Thus, if it was established that $\lim\limits_{x \to \infty} f(x) = L$, it would necessarily follow that $\lim\limits_{n \to \infty} a_n = L$. Therefore, results obtained in Chapter 13 of the book for limits of functions are available for our work with limits of sequences. In our development of the derivative through a limit process, we stated an informal definition of a limit of a function and five properties of limits (Section 13.1).

Our earlier informal definition of a limit of a function said that if $f(x)$ becomes *arbitrarily close* to a unique finite number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L.

In Section 13.1 of the book, we used some algebraic techniques combined with some informal reasoning to find limits of rational functions. It seems reasonable to conjecture that for a sequence $\{a_n\}$ if the value of a_n matches a function f at every positive integer, and $f(x)$ approaches a limit L as $x \to \infty$, then the sequence will converge to the same limit L.

A sequence that has a limit **converges**, whereas a sequence that does not have a limit **diverges**.

Limit of a sequence theorem

Suppose that $f(x)$ is a function defined for all $x \geqslant k$, $k \in \mathbb{Z}^+$, and $\{a_n\}$ is a sequence such that $a_n = f(n)$ when $n \geqslant k$. If $\lim\limits_{x \to \infty} f(x) = L$, then $\lim\limits_{n \to \infty} a_n = L$.

Also, in Section 13.1 we presented a set of five properties for limits of functions. All of these can be translated into properties for limits of sequences. We list here the set of five corresponding properties of limits of sequences and an additional important property on the limit of a rational power of a sequence.

The converse of the limit of a sequence theorem is not true. That is, a convergent sequence does not imply that the associated real variable function must also converge.

Properties of limits of sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences such that $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} b_n = K$, and c is any real number, then:

1. Constant sequence: $\lim\limits_{n \to \infty} c = c$
2. Scalar multiple of a sequence: $\lim\limits_{n \to \infty} (c \cdot a_n) = cL$
3. Sum or difference of sequences: $\lim\limits_{n \to \infty} (a_n \pm b_n) = L \pm K$
4. Product of sequences: $\lim\limits_{n \to \infty} (a_n \cdot b_n) = LK$
5. Quotient of sequences: $\lim\limits_{n \to \infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{L}{K}, \quad K \neq 0$
6. Rational power of a sequence: $\lim\limits_{n \to \infty} (a_n)^p = L^p,\ p \in \mathbb{Q}$

These six properties of limits of sequences can be stated in words as follows:

1. The limit of a constant is equal to the constant.
2. The limit of a constant times a sequence is the constant times the limit of the sequence.

3. The limit of a sum/difference of sequences is the sum/difference of the limits of the sequences.
4. The limit of a product of sequences is the product of the limits of the sequences.
5. The limit of a quotient of sequences is the quotient of the limits of the sequences (given that the limit of the sequence in the denominator is not zero).
6. The limit of a rational power of a sequence is the rational power of the limit of the sequence.

In Chapter 13 of the book we reasoned informally that function values for functions in the form $f(x) = \frac{1}{x^k}$, where k is a rational number, approach zero as x goes to zero, i.e. $\lim\limits_{x \to \infty} \frac{1}{x^k} = 0,\ k \in \mathbb{Q}$. Thus, it makes sense that the result from Example 3, $\lim\limits_{n \to \infty} \left\{\frac{1}{n}\right\} = 0$, combined with property 6 for limits of sequences above, leads to the following intuitive rule for the limit of certain sequences.

If $r > 0,\ r \in \mathbb{Q}$, then $\lim\limits_{n\to\infty} \frac{1}{n^r} = 0$. **Note:** This rule is equivalent to $\lim\limits_{n\to\infty} n^r = 0$ if $r < 0$.

Example 3

Determine whether the sequence $\left\{\frac{3n^2 + 5n - 1}{2n^2 + 1}\right\}$ is convergent or divergent.

Solution

In Example 4, part d) of Section 13.1, we found $\lim\limits_{x\to\infty} \frac{3x^2 + 5x - 1}{2x^2 + 1}$ to be equal to $\frac{3}{2}$ as follows:

$$\lim_{x\to\infty} \frac{3x^2 + 5x - 1}{2x^2 + 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{5x}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}}$$

Dividing numerator and denominator by largest power of x, i.e. x^2.

$$= \lim_{x\to\infty} \frac{3 + \frac{5}{x} - \frac{1}{x^2}}{2 + \frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty} \frac{5}{x} - \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x^2}}$$

Applying $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ and $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$.

$$= \frac{3 + 0 - 0}{2 + 0}$$

Applying $\lim\limits_{x\to\infty} \frac{1}{x^k} = 0,\ k \in \mathbb{Q}$.

Hence, $\lim\limits_{x\to\infty}\dfrac{3x^2+5x-1}{2x^2+1}=\dfrac{3}{2}$.

Therefore, from the limit of a sequence theorem above, we can conclude that the sequence $\left\{\dfrac{3n^2+5n-1}{2n^2+1}\right\}$ is convergent and it converges to $\dfrac{3}{2}$.

In our discussion of the end behaviour of rational functions in Section 3.4 of the book, the following limit results were hinted at. We state them here because by means of the limit of a sequence theorem they can also be applied in finding limits of sequences with rules that are rational functions, such as the sequence in Example 3.

Limits of rational functions

Let R be the rational function given by

$$R(x)=\frac{f(x)}{g(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0}$$

1. If $n<m$, then $\lim\limits_{x\to\infty}R(x)=0$.
2. If $n=m$, then $\lim\limits_{x\to\infty}R(x)=\dfrac{a_n}{b_m}$.
3. If $n>m$, then $\lim\limits_{x\to\infty}R(x)=\infty$, i.e. does not exist.

Another useful limit theorem for functions that can be rewritten for sequences is the squeeze theorem from Section 13.2 where we used it to prove that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

The squeeze theorem for sequences

If $a_n \leqslant b_n \leqslant c_n$ for all n such that $n \geqslant N$, $N \in \mathbb{Z}^+$, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ then $\lim\limits_{n\to\infty} b_n = L$.
See Figure 1.2 below.

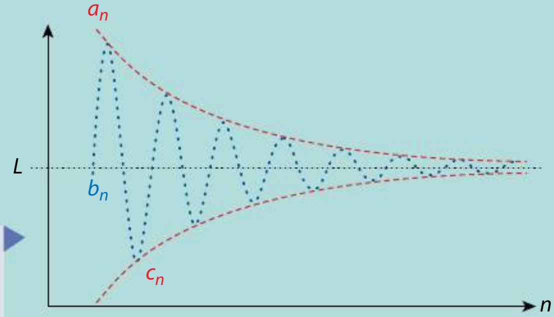


Figure 1.1 The sequences $\{a_n\}$ and $\{c_n\}$, both with limit of L, 'squeezing' the sequence $\{b_n\}$.

Note that the terms of sequence $\{b_n\}$ do not need to lie between $\{a_n\}$ and $\{c_n\}$ for all values of n. The requirement is that there must be some value of n for which all of the terms of $\{b_n\}$ beyond this value must lie between $\{a_n\}$ and $\{c_n\}$. This is illustrated in Example 5.

Example 4 – Applying the squeeze theorem

Show that each of the sequences converges, and find its limit.

a) $\left\{\dfrac{1}{2^n}\right\}$ b) $\left\{\dfrac{\cos n}{n}\right\}$

Solution

a) Because $2^n > 0$ and $2^n > n$ for all positive integers n, it follows that $0 \leqslant \frac{1}{2^n} \leqslant \frac{1}{n}$ for all integers $n \geqslant 1$. It is the case that $\lim\limits_{n\to\infty}\frac{1}{n} = 0$. Therefore, $\lim\limits_{n\to\infty}\frac{1}{2^n} = 0$ because $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty}\frac{1}{n} = 0$ and the sequence $\left\{\frac{1}{2^n}\right\}$ converges to zero.

b) Because $-1 \leqslant \cos x \leqslant 1$ for all real numbers x, it follows that $\frac{-1}{n} \leqslant \frac{\cos n}{n} \leqslant \frac{1}{n}$ for all integers $n \geqslant 1$. Therefore, $\lim\limits_{n\to\infty}\frac{\cos n}{n} = 0$ because $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = \lim\limits_{n\to\infty}\frac{1}{n} = 0$ and the sequence $\left\{\frac{\cos n}{n}\right\}$ converges to zero.

Example 5 – Applying the squeeze theorem for an alternating sequence

Consider the infinite sequence $\left\{\frac{(-1)^n}{n!}\right\}$.

a) Write out the first six terms of the sequence.

b) Use the squeeze theorem to show that the sequence converges to 0.

Solution

a) The first six terms of the sequence are $-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}, -\frac{1}{120}, \frac{1}{720}$. The sequence clearly alternates between positive and negative terms.

b) In order to apply the squeeze theorem, we need to find two convergent sequences that converge to 0 for which all terms for $n \geqslant N$ of the sequence $\left\{\frac{(-1)^n}{n!}\right\}$ will be between. Two sequences that will work in this case are $\left\{-\frac{1}{2^n}\right\}$ and $\left\{\frac{1}{2^n}\right\}$, both of which converge to 0.

The first six terms of these two sequences, respectively, are

$-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}, -\frac{1}{64}$ and $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$.

Observe that for $n = 1, 2$ and 3, the terms of $\left\{\frac{(-1)^n}{n!}\right\}$ are *not* between $\left\{-\frac{1}{2^n}\right\}$ and $\left\{\frac{1}{2^n}\right\}$; however they are for $n \geqslant 4$. That is,

$$-\frac{1}{2^n} \leqslant \frac{(-1)^n}{n!} \leqslant \frac{1}{2^n}, \ n \geqslant 4.$$

Therefore, by the squeeze theorem it follows that the sequence $\left\{\frac{(-1)^n}{n!}\right\}$ converges to zero.

The squeeze theorem can also be used to prove that the sequence of absolute values for the sequence in Example 5, $\left\{\left|\frac{(-1)^n}{n!}\right|\right\} = \left\{\frac{1}{n!}\right\}$, also converges to 0 since the inequality $0 \leqslant \frac{1}{n!} \leqslant \frac{1}{2^n}$ is true for all $n \geqslant 4$. In fact, there is a very useful theorem that states that if the absolute value sequence converges to 0, then the original sequence consisting of positive and/or negative terms also converges to 0. It is often more efficient to consider the sequence of absolute values and then apply the following theorem to the original sequence.

Absolute value theorem

For the sequence $\{a_n\}$, if $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Proof of the absolute value theorem is fairly straightforward. Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$; one with all positive terms and one with all negative terms. Because both of these sequences converge to 0 and $-|a_n| \leqslant a_n \leqslant |a_n|$ we can conclude by means of the squeeze theorem that $\{a_n\}$ must also converge to 0.

The converse of the absolute value theorem is not true. That is, if $\lim\limits_{n\to\infty} a_n = 0$ it does not necessarily follow that $\lim\limits_{n\to\infty} |a_n| = 0$.

The sequence $\left\{\frac{1}{2^n}\right\}$, equivalent to $\left\{\left(\frac{1}{2}\right)^n\right\}$, in Example 4 part a) is a geometric sequence with a common ratio, r, equal to $\frac{1}{2}$. It was shown to converge to zero. For what values of r, other than $\frac{1}{2}$, is the geometric sequence $\{r^n\}$ convergent? Figure 1.2 shows the graphs of geometric sequences, $\{r^n\}$, for different positive values of r.

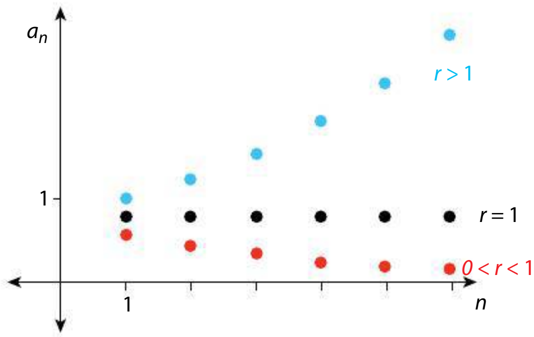


Figure 1.2 Graph of the sequence $\{r^n\}$ for different positive values of r.

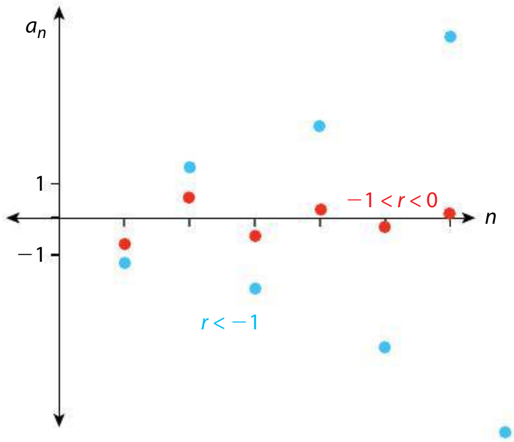
Note that the graph of a real-valued function, e.g. $f(x) = 2^x$, $x \in \mathbb{R}$, is a continuous smooth curve; however, the graph of a sequence, e.g. $\{a_n\} = \{2^n\}$, $n \in \mathbb{Z}^+$, is discrete points because the domain consists of only positive integers.

When $r > 1$ the sequence $\{r^n\}$ increases without bound, i.e. for $r > 1$, $\lim\limits_{n\to\infty} r^n = \infty$. Visually it appears that for $0 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$. In Example 4, part a), we used the squeeze theorem to prove that $\lim\limits_{n\to\infty} r^n = 0$ when $r = \frac{1}{2}$. We can use a similar argument to show that $\lim\limits_{n\to\infty} r^n = 0$ for any value of r in the interval $0 < r < 1$.

Thus, we have $\lim\limits_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$

What about negative values of r? Figure 1.3 (below) shows the graphs of geometric sequences, $\{r^n\}$, for different negative values of r. There is no graph of $\{r^n\}$ for $r = -1$. In this case, the terms would oscillate infinitely between 1 and -1, and clearly the sequence does not converge to any number.

Figure 1.3 Graph of the sequence $\{r^n\}$ for different negative values of r.



Clearly, when $r < -1$ the sequence $\{r^n\}$ alternates between positive and negative values that increase without bound. Thus, for $r < -1$, $\lim_{n\to\infty} r^n$ does not exist. Considering $-1 < r < 0$ we can also write the inequality as $0 < |r| < 1$. Additionally, $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n$. Using the result above that $\lim_{n\to\infty} r^n = 0$ for $0 < r < 1$, and since $0 < |r| < 1$, we can conclude that $\lim_{n\to\infty} |r|^n = 0$. Therefore, by the absolute value theorem and the obvious fact that $\lim_{n\to\infty} 0^n = 0$ it is true that $\lim_{n\to\infty} r^n = 0$ for the interval $-1 < r < 1$, which is equivalent to $|r| < 1$. It is also obvious that $\lim_{n\to\infty} 1^n = 1$. Thus the sequence $\{r^n\}$ is convergent for the interval $-1 < r \leqslant 1$ and divergent for other values of r. This result is summarized as follows.

Convergence of geometric sequences theorem

For $r \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ the geometric sequence $\{r^n\}$ is convergent for $-1 < r \leqslant 1$ such that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & r = 1 \end{cases}$$

Example 6 – The factorial function and exponential functions

Show that the sequence $\left\{\frac{x^n}{n!}\right\}$ converges to 0 for any real number x.

Solution

If $x < 0$, then the terms of the sequence will be alternately positive and negative. With the intention of applying the absolute value theorem, all that needs to be shown is that $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$. This takes a bit of work. We start by choosing some positive integer N such that $N > |x|$. It follows that the sequence $\left\{\left(\frac{|x|}{N}\right)^n\right\}$ is geometric. Because $N > |x|$ then $\frac{|x|}{N} < 1$

and it must follow that $\lim_{n\to\infty}\left(\frac{|x|}{N}\right)^n = 0$. We now focus our attention on all of the values of n such that $n > N$. For these values of n, we can write the following:

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1\times 2\times 3\ldots N\times\underbrace{(N+1)(N+2)\ldots n}_{(n-N)\text{ factors}}} \leqslant \frac{|x|^n}{N!N^{n-N}} = \frac{|x|^n N^N}{N!N^n} = \frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n$$

Hence, $0 \leqslant \frac{|x|^n}{n!} \leqslant \frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n$. The expression $\frac{N^N}{N!}$ is a constant and will not change as n changes. We know that $\lim_{n\to\infty}\left(\frac{|x|}{N}\right)^n = 0$, so applying the property $\lim_{n\to\infty}(c\cdot a_n) = c\lim_{n\to\infty} a_n$ we get $\lim_{n\to\infty}\frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n = \frac{N^N}{N!}\lim_{n\to\infty}\left(\frac{|x|}{N}\right)^n = \frac{N^N}{N!}(0) = 0$.

Thus, $0 \leqslant \frac{|x|^n}{n!} \leqslant \frac{N^N}{N!}\left(\frac{|x|}{N}\right)^n = 0$ and we can conclude that $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$.

Therefore, by the absolute value theorem $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, the sequence $\left\{\frac{x^n}{n!}\right\}$ converges to 0 for any real value of x.

Because we have shown that for any number x, $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, we can conclude that the factorial function increases faster than any exponential function.

1.2 L'Hôpital's rule

We have one more important theorem to consider that is an essential tool for helping to determine the limits of certain functions, and consequently the limits of certain sequences.

With limits of rational functions in Chapter 13 of the book, we were sometimes confronted with an expression of indeterminate form, commonly in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We handled these by performing some algebraic manipulations and applying limit theorems, as illustrated in Example 5 of Chapter 13. Not all limits can be managed in such a way. The following theorem specifically addresses limits of rational expressions that are of indeterminate form.

L'Hôpital's rule

Let f and g be functions whose derivative can be found at any value in an open interval $]a,b[$, except possibly at some value c where $a < c < b$. Assume that $g'(x) \neq 0$, except possibly at c. Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$; or $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$. (That is, the expression $\frac{f(x)}{g(x)}$ is in indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.)

Then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$ provided the limit on the right side exists (or is infinite).

L'Hôpital's rule first appeared in 1696 in a mathematical textbook entitled *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (Analysis of the Infinitely Small for the Understanding of Curves). The textbook was written by the French nobleman and mathematician Guillaume de L'Hôpital (1661–1704) and is considered the first textbook on differential calculus.

Although the method for evaluating limits of indeterminate forms presented here is attributed to L'Hôpital, it was actually first developed by the Swiss mathematician Johann Bernoulli (1667–1748). In fact, most of the mathematics in L'Hôpital's groundbreaking textbook is widely considered to be the work of Johann Bernoulli. L'Hôpital did acknowledge Bernoulli's contributions in the preface to the textbook. Nevertheless, the name of L'Hôpital is forever associated with the rule.

When you are applying l'Hôpital's rule make sure that you differentiate the numerator and denominator *separately*. **Do *not* use the quotient rule for differentiation.**

L'Hôpital's rule states simply that, given the right conditions, the limit of a quotient of functions is equal to the limit of the quotient of their derivatives. It is important to first verify the conditions regarding the limits of f and g before applying l'Hôpital's rule.

Example 7 – Applying l'Hôpital's rule

For each limit, use your GDC to conjecture a result, and then find the limit using l'Hôpital's rule.

a) $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}$

b) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sec x}{1+\tan x}$

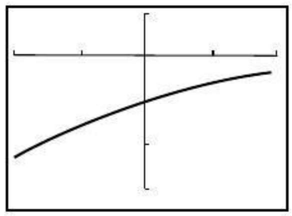
c) $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$

Solution

a) To visualize $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}$ we graph $f(x)=\dfrac{x}{1-e^x}$ as shown in the GDC images below.

```
Plot1  Plot2  Plot3
\Y1■X/(1-e^(X))
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
 Xmin=-2
 Xmax=2
 Xscl=1
 Ymin=-3
 Ymax=1
 Yscl=1
 Xres=1
```



Although $x=0$ is not in the domain of f, the graph appears to pass through the point $(0,-1)$ implying that $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}=-1$. Since $\lim\limits_{x\to 0}x=0$ and $\lim\limits_{x\to 0}\left(1-e^x\right)=0$, $\lim\limits_{x\to 0}\dfrac{x}{1-e^x}$ is in the indeterminate form $\dfrac{0}{0}$, and l'Hôpital's rule applies. Differentiating the numerator and denominator separately and evaluating the limit gives

$$\lim_{x\to 0}\frac{x}{1-e^x}=\lim_{x\to 0}\frac{1}{-e^x}=\frac{1}{-1}=-1.$$

b) Instead of viewing a graph of $f(x)=\dfrac{\sec x}{1+\tan x}$ to conjecture a value for $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sec x}{1+\tan x}$, let's use the GDC to construct a table of function values near $x=\dfrac{\pi}{2}\approx 1.5708$.

```
Plot1  Plot2  Plot3
\Y1■(1/cos(X))/(
 1+tan(X))
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
TABLE SETUP
 TblStart=1.5
 ΔTbl=.01
Indpnt: Auto Ask
Depend: Auto Ask
```

X	Y1	
1.54	.97057	
1.55	.97984	
1.56	.98938	
1.57	.9992	
1.58	1.0093	
1.59	1.0198	
1.6	1.0305	

X=1.6

The values in the table show that the function appears to be approaching 1 from either direction.

The values of $\sec x$ vanish to $+\infty$ when $x \to \frac{\pi}{2}$ from the left $\left(\text{i.e. } x \to \frac{\pi}{2}^{-}\right)$ and vanish to $-\infty$ when $x \to \frac{\pi}{2}^{+}$.

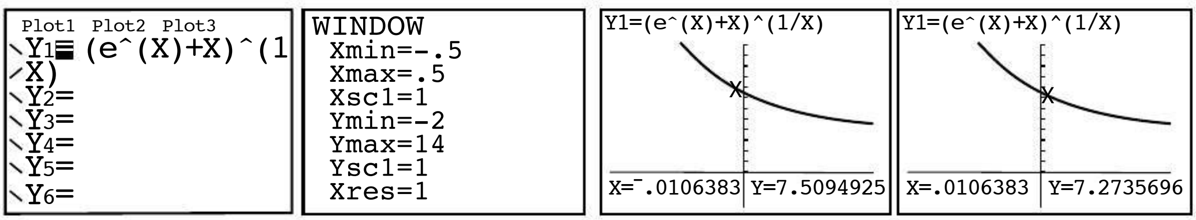
Similarly, $\lim\limits_{x \to \frac{\pi}{2}^{-}} (1 + \tan x) = +\infty$ and $\lim\limits_{x \to \frac{\pi}{2}^{+}} (1 + \tan x) = -\infty$. So when approaching $\frac{\pi}{2}$ from the left we have $\frac{+\infty}{+\infty}$, and $\frac{-\infty}{-\infty}$ when approaching from the right. L'Hôpital's rule also applies to one-sided limits. Applying the rule to the right-hand limit gives

$$\lim_{x \to \frac{\pi}{2}^{+}} \frac{\sec x}{1 + \tan x} = \lim_{x \to \frac{\pi}{2}^{+}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}^{+}} \sin x = 1.$$

$\left(\textbf{Note: } \frac{\sec x \tan x}{\sec^2 x} \text{ simplifies to } \sin x.\right)$

The left-hand limit is also 1; therefore the two-sided limit is equal to 1, i.e. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x} = 1$.

c) To visualize $\lim\limits_{x \to 0} \left(e^x + x\right)^{1/x}$ we graph $f(x) = \left(e^x + x\right)^{1/x}$ as shown in the GDC images below.



Tracing on the graph indicates that as $x \to 0$ the function approaches a value between 7.2735 and 7.5094. The exact value of the limit is not clear.

We observe that $\lim\limits_{x \to 0} \left(e^x + x\right)^{1/x}$ is in the indeterminate form 1^∞. However, by taking the logarithm of both sides of $f(x) = \left(e^x + x\right)^{1/x}$ and then taking the limit we can change the indeterminate form to $0/0$, to which we can apply l'Hôpital's rule.

$$\ln\left[f(x)\right] = \ln\left[\left(e^x + x\right)^{1/x}\right] = \frac{1}{x}\ln\left(e^x + x\right) = \frac{\ln\left(e^x + x\right)}{x}$$

Thus, $\ln\left[f(x)\right] = \frac{\ln\left(e^x + x\right)}{x}$, and taking the limit as $x \to 0$ of both sides produces $\lim\limits_{x \to 0} \ln\left[f(x)\right] = \lim\limits_{x \to 0} \frac{\ln\left(e^x + x\right)}{x}$

Right side in the form $0/0$; apply l'Hôpital's rule.

$$= \lim_{x \to 0} \frac{\frac{e^x + 1}{e^x + x}}{1} = \frac{e^0 + 1}{e^0 + 0} = 2.$$

Hence, $\lim\limits_{x \to 0} \ln\left[f(x)\right] = 2$.

Since $f(x) = (e^x + x)^{1/x}$, then $\lim_{x\to 0}(e^x + x)^{1/x} = \lim_{x\to 0} f(x)$

$= \lim_{x\to 0} e^{\ln f(x)}$ Applying the rule $e^{\ln a} = a$.

$= \lim_{x\to 0} e^2$. Using result $\lim_{x\to 0}\ln[f(x)] = 2$.

Therefore, $\lim_{x\to 0}(e^x + x)^{1/x} = e^2$.

$e^2 \approx 7.389$ (to 4 s.f.), so the limit is within the range estimated from the graph on the GDC.

L'Hôpital's rule should not be applied if the limit is not in indeterminate form. For example, consider the following limit: $\lim_{x\to 0}\frac{\sin x}{x+1}$. The limit is *not* indeterminate, because $\frac{\sin(0)}{0+1} = \frac{0}{1}$. Hence, the application of L'Hôpital's rule produces an incorrect result. L'Hôpital's rule gives the following result: $\lim_{x\to 0}\frac{\sin x}{x+1} = \lim_{x\to 0}\frac{\cos x}{1} = \frac{\cos(0)}{1} = \frac{1}{1} = 1$. The correct result can be obtained simply from direct substitution: $\lim_{x\to 0}\frac{\sin x}{x+1} = \frac{\sin(0)}{0+1} = \frac{0}{1} = 0$.

If, after applying L'Hôpital's rule, the quotient of the derivatives remains in indeterminate form, the rule can be applied more than once.

Example 8 – Repeated use of L'Hôpital's rule

Find $\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2}$.

Solution

Substituting $x = 1$ into the rational expression gives $\frac{1-1+\ln 1}{1-3\cdot 1+2} = \frac{0}{0}$. Thus the limit is in the indeterminate form $\frac{0}{0}$ and L'Hôpital's rule is applied:

$$\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2} = \lim_{x\to 1}\frac{-1+\frac{1}{x}}{3x^2-3}$$

Substituting $x = 1$ again gives the indeterminate form $\frac{0}{0}$, so L'Hôpital's rule is applied a second time, producing an expression that can be evaluated for $x = 1$:

$$\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2} = \lim_{x\to 1}\frac{-1+\frac{1}{x}}{3x^2-3} = \lim_{x\to 1}\frac{-\frac{1}{x^2}}{6x} = -\frac{1}{6}$$

Example 9 – Using l'Hôpital's rule to determine convergence of a sequence

Determine if the sequence $\{a_n\} = \left\{\frac{n^2+1}{3^n}\right\}$ converges. If it does, find its limit.

Solution

Consider the function $f(x) = \frac{x^2+1}{3^x}$, $x \in \mathbb{R}$, and its limit as $x \to \infty$.

Since $\lim_{x\to\infty}\frac{x^2+1}{3^x}$ is in indeterminate form of ∞/∞, we can apply l'Hôpital's rule.

$$\lim_{x\to\infty}\frac{x^2+1}{3^x}=\lim_{x\to\infty}\frac{2x}{(\ln 3)3^x}$$

But this limit is still in indeterminate form of ∞/∞, so we apply l'Hôpital's rule a second time.

$$\lim_{x\to\infty}\frac{x^2+1}{3^x}=\lim_{x\to\infty}\frac{2x}{(\ln 3)3^x}=\lim_{x\to\infty}\frac{2}{(\ln 3)^2 3^x}=0$$

Because the value of a_n matches the value of $f(x)$ for every positive integer, we can apply the limit of a sequence theorem and conclude that $\lim_{n\to\infty}\frac{n^2+1}{3^n}=0$.

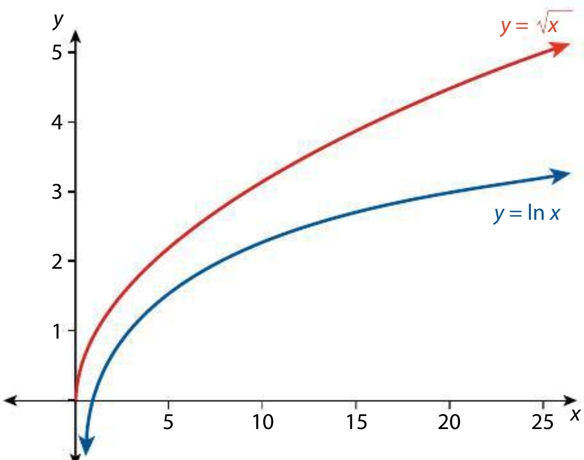
Therefore, the sequence $\left\{\frac{n^2+1}{3^n}\right\}$ converges to 0.

Example 10

Which sequence grows faster, $\{\ln n\}$ or $\{\sqrt{n}\}$?

Solution

We can gain some insight into this question by graphing the real-valued functions $y=\ln x$ and $y=\sqrt{x}$. The graph below implies that the sequence $\{\sqrt{n}\}$ grows faster than $\{\ln n\}$; that is, the infinite sequence $\left\{\frac{\ln n}{\sqrt{n}}\right\}$ converges to 0. Using l'Hôpital's rule to show that the limit of the function $f(x)=\frac{\ln x}{\sqrt{x}}$ is 0 as $x\to\infty$ will prove this result.



$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0 \text{ because } \lim_{x\to\infty}\frac{1}{x^k}=0,\ k\in\mathbb{Q}^+$$

Therefore, the sequence $\left\{\dfrac{\ln n}{\sqrt{n}}\right\}$ converges to 0, and we can conclude that $\left\{\sqrt{n}\right\}$ grows faster than $\{\ln n\}$.

1.3 Improper integrals

Previously we have defined the definite integral, $\int_a^b f(x)\,dx$, for a function f that is continuous (i.e. no 'gaps' in the domain) for the finite, bounded, interval $a \leqslant x \leqslant b$. In this section, we will look at ways of evaluating integrals where either one or both of the limits of integration (i.e. a and b) are infinite, or the function f has an infinite discontinuity in the interval $a \leqslant x \leqslant b$. An integral having either one of these characteristics is called an **improper integral**.

Let's look at an integral where one of the limits is infinite.

Example 11

Evaluate $\int_1^{\infty} \dfrac{1}{x^2}\,dx$ or show that it diverges.

Solution

We can replace the infinite limit of integration with a variable, say the variable b, and then take the limit of the integral as b approaches infinity.

Taking the limit as $b \to \infty$ gives $\int_1^{\infty} \dfrac{1}{x^2}\,dx = \lim\limits_{b\to\infty}\left(\int_1^b \dfrac{1}{x^2}\,dx\right) = \lim\limits_{b\to\infty}\left[-\dfrac{1}{x}\right]_1^b$

$= \lim\limits_{b\to\infty}\left(-\dfrac{1}{b}+1\right) = 0 + 1 = 1.$

Therefore, $\int_1^{\infty} \dfrac{1}{x^2}\,dx = 1.$

This result can be interpreted as indicating that the area under the curve $y = \dfrac{1}{x^2}$ from one to infinity is finite and is exactly equal to 1 (see Figure 1.4 below).

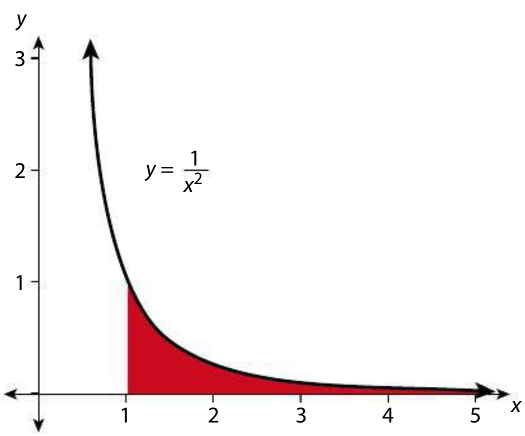


Figure 1.4 Area under the curve $y = \dfrac{1}{x^2}$ from 1 to ∞.

Certainly, not all improper integrals converge to a finite value.

Example 12

Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$ or show that it diverges.

Solution

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_1^{b} \frac{1}{x}\,dx\right) = \lim_{b\to\infty}\left[\ln x\right]_1^b = \lim_{b\to\infty}(\ln b - \ln 1) = \lim_{b\to\infty}(\ln b) = \infty$$

[or 'limit does not exist']

Therefore, the integral diverges. The area under the curve $y = \frac{1}{x}$ from 1 to infinity is infinite.

$\int \frac{1}{x}\,dx = \ln|x|$, but note that in Example 12 the absolute value is omitted because the integral is being evaluated from 1 to ∞, i.e. only positive numbers.

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists (as a finite number as in Example 11), and is called **divergent** if the limit does not exist (as in Example 12).

Example 13 – Using l'Hôpital's rule to evaluate an improper integral

Determine whether the integral $\int_1^{\infty} \frac{x}{e^x}\,dx$ converges or diverges; and if it converges, find its value.

Solution

We can rewrite the integral as a limit, $\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{x}{e^x}\,dx$;

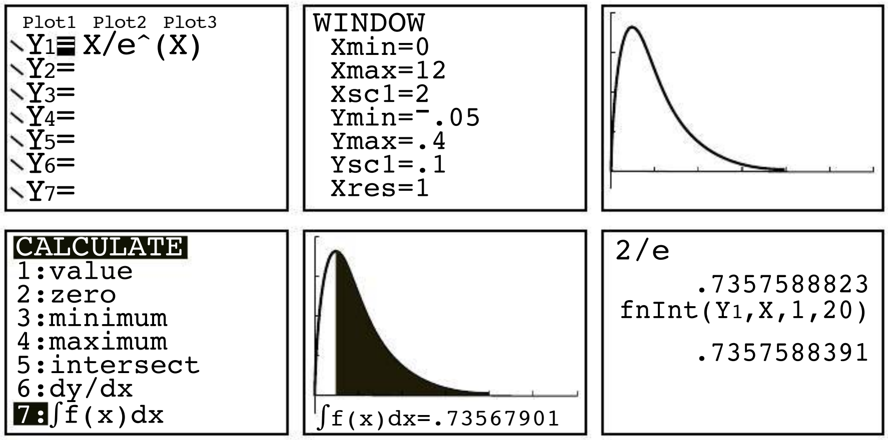
and now need to apply integration by parts to evaluate the integral.

Let $\begin{matrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{matrix}$; then $\lim_{b\to\infty}\int_1^{b} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\left[\left[-xe^{-x}\right]_1^b + \int_1^b e^{-x}\,dx\right]$

$$= \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b$$
$$= \lim_{b\to\infty}\left[-(x+1)e^{-x}\right]_1^b$$
$$= \lim_{b\to\infty}\left(\frac{-(b+1)}{e^b} + \frac{2}{e}\right)$$
$$= \lim_{b\to\infty}\left(\frac{-(b+1)}{e^b}\right) + \frac{2}{e}$$
$$= 0 + \frac{2}{e}$$

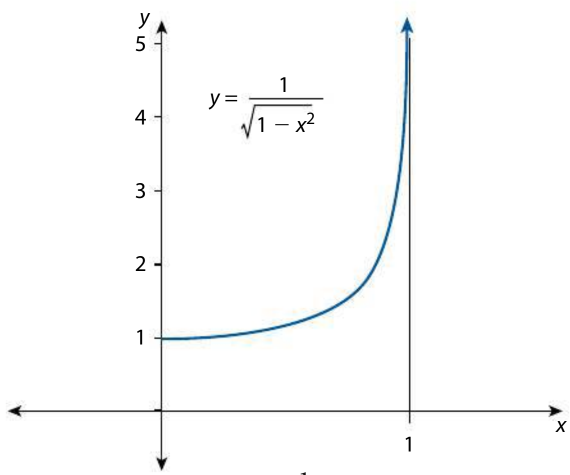
Therefore, $\int_1^{\infty} \frac{x}{e^x}\,dx = \frac{2}{e} \approx 0.735\,758\,8823$ (to ten significant figures).

The GDC images on the next page confirm our result. Note that even with an upper limit of just $x = 12$ the definite integral (computed on graph screen) agrees to three decimal places with the value of the 'improper' integral with an infinite upper limit; and when the upper limit is 20 (computed on home screen) the values agree to six decimal places. The integral converges at a fairly quick rate.



What is an infinite discontinuity? A function f has an infinite discontinuity at $x = c$ if either $\lim\limits_{x\to c} f(x) = \infty$ or $\lim\limits_{x\to c} f(x) = -\infty$ such that $x \to c$ from the right or left. For example, on the interval $0 \leqslant x \leqslant 1$, the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ has an infinite discontinuity at $x = 1$ because $\lim\limits_{x\to 1^-} \dfrac{1}{\sqrt{1-x^2}} = \infty$ (note: $x \to 1$ from the left) which can be observed in the graph in Figure 1.5.

Figure 1.5



The region under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $0 \leqslant x \leqslant 1$ is unbounded – and would, at first thought, have an infinite area. However, the unbounded region has a finite area and we can find the exact area as follows.

Example 14

Find the area, if possible (not possible if it's infinite), under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $0 \leqslant x \leqslant 1$.

Solution

We can replace the limit of integration where the infinite discontinuity occurs with a variable, say the variable b, and then take the limit of the integral as b approaches the value of x where the discontinuity occurs (approaching 1 from the left, in this case).

(Recall that the anti-derivative of $\dfrac{1}{\sqrt{1-x^2}}$ is $\arcsin x$.)

$$\text{Area} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\left(\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx\right) = \lim_{b\to 1^-}\left[\arcsin x\right]_0^b = \lim_{b\to 1^-}(\arcsin b - \arcsin 0)$$

$$= \lim_{b\to 1^-}(\arcsin b - 0) = \lim_{b\to 1^-}(\arcsin b) = \arcsin(1) = \frac{\pi}{2}$$

Therefore, the unbounded region under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $0 \leqslant x \leqslant 1$ has a finite area of exactly $\dfrac{\pi}{2}$.

Exercise 1

For questions 1–15, determine if the sequence converges or diverges. If it converges, find the limit of the sequence.

1 $\left\{\dfrac{7}{\sqrt[3]{n}}\right\}$ **2** $\left\{\dfrac{2n^2+n+1}{n^2+1}\right\}$ **3** $\left\{\dfrac{5n-13}{n^3+5n}\right\}$

4 $\{\cos n\pi\}$ **5** $\left\{\dfrac{(-1)^{n+1}}{2n-1}\right\}$ **6** $\left\{\left(-\dfrac{4}{5}\right)^n\right\}$

7 $\left\{\dfrac{e^n}{n^2}\right\}$ **8** $\left\{\dfrac{3\sqrt{n^2+1}}{4\sqrt[3]{n^2}-1}\right\}$ **9** $\left\{\sqrt{\dfrac{2n}{n+1}}\right\}$

10 $\left\{1+\dfrac{(-1)^n}{n}\right\}$ **11** $\left\{\dfrac{n}{1+\sqrt{n}}\right\}$ **12** $\left\{\left(1+\dfrac{2}{n}\right)^{\frac{1}{n}}\right\}$

13 $\left\{\dfrac{3^n}{n!}\right\}$ **14** $\left\{\dfrac{\ln 2n}{\ln n}\right\}$ **15** $\left\{\left(\dfrac{|n|+1}{n}\right)\right\}$

16 Use the squeeze theorem to show that $\displaystyle\lim_{n\to\infty}\frac{\sin 2n}{\sqrt{n}} = 0$.

17 Use the fact that $\displaystyle\lim_{n\to\infty}\frac{1}{n^2} = 0$ to prove that $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$.

For questions 18–20, use l'Hôpital's rule to find the value of each limit.

18 $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$ **19** $\displaystyle\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}$ **20** $\displaystyle\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

21 Determine whether the sequence $\left\{n\sin\dfrac{\pi}{n}\right\}$ is convergent or divergent. If convergent, find its limit. (Hint: Rewrite $n\sin\dfrac{\pi}{n}$ as $\dfrac{\sin\frac{\pi}{n}}{\frac{1}{n}}$.)

In questions 22–27, evaluate the limit.

22 $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x^2-4x-3}$ **23** $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1-x}-1}{x}$

24 $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}$ **25** $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x}-\cot x}{x}$

26 $\displaystyle\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}$ **27** $\displaystyle\lim_{x\to 0}\frac{a^x-b^x}{x}$, $a>0, b>0$

28 Given $f(x) = (1+x)^{1/x}$, find $\lim_{x\to\infty} f(x)$. (Hint: Start by taking the natural logarithm of both sides, converting the right side to the indeterminate form $\frac{0}{0}$. Then you can use l'Hôpital's rule.)

In questions 29–36, evaluate, or identify as divergent, the given integral.

29 $\int_0^1 \frac{1}{x^3}\,dx$

30 $\int_1^\infty \frac{1}{x^3}\,dx$

31 $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

32 $\int_0^\infty \frac{\sin x}{e^x}\,dx$

33 $\int_0^{\pi/2} \tan x\,dx$

34 $\int_0^\infty \frac{1}{1+e^x}\,dx$

35 $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

36 $\int_0^k \frac{x}{\sqrt{k^2-x^2}}\,dx$

37 Consider the *unbounded* region lying between the graph of $y = \frac{1}{x}$ and the x-axis for $x \geqslant 1$.

a Find the area of this region, if possible.

b Find the volume, if possible, of the solid generated by rotating this unbounded region about the x-axis.

c Comment on your results for **a** and **b**.

Practice questions 1

1 Show that $\lim_{x\to 1} \frac{x^2-1}{\sqrt{1+x^2}-\sqrt{2}} = 2\sqrt{2}$.

2 Use L'Hôpital's rule to find $\lim_{x\to 0} \frac{x\cos x - e^x + 1}{\cos^2 x}$.

3 Determine whether the integral $\int_{-1}^{0} \frac{e^x}{e^x-1}\,dx$ converges or diverges. If it converges, find its value.

4 Find the following.

a $\lim_{x\to 1} \frac{1-x^3}{2-\sqrt{x^2+3}}$

b $\lim_{x\to a} \frac{x-a}{x^3-a^3}$

5 Find the set of values of p for which the improper integral $\int_e^\infty \frac{\ln x}{x^p}\,dx$ converges.

6 Calculate each of the following limits.

a $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

b $\lim_{x\to 0}\left(\frac{\arctan x}{x}\right)$

7 Show that $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = 2\sqrt{3}$.

8 Calculate each of the following one-sided limits.

a $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

b $\lim_{x\to \frac{\pi}{2}^-} (\tan x - \sec x)$

9 a i Find $I_n = \int_{-n}^{\alpha n} \frac{x}{1+x^2}\,dx$ where α is a positive constant and n is a positive integer.

ii Determine $\lim_{n\to\infty} I_n$.

b Using L'Hôpital's rule to find $\lim_{x\to 0}\left(\frac{\tan \beta x - \beta \tan x}{\sin \beta x - \beta \sin x}\right)$ where β is a non-zero constant and $\beta \neq \pm 1$.

10 Show that $\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \frac{1}{2}$.

11 Giving a reason, state whether the following argument is correct or incorrect.

Using L'Hôpital's rule, $\lim_{x\to\pi^-}\left(\frac{\sin x}{1-\cos x}\right) = \lim_{x\to\pi^-}\left(\frac{\cos x}{\sin x}\right) = -\infty$.

12 For what values of k do the following converge?

a $\int_0^1 x^k\,dx$ **b** $\int_1^\infty x^k\,dx$

13 Find $\lim_{x\to 0}\left(\frac{\ln(a^2+x^2)}{\ln(a-x^3)}\right)$, where a is a positive constant, not equal to 1.

14 Show that $\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$.

15 Find the value of each limit.

a $\lim_{x\to 0}\left(\frac{2+x^2-2\cos x}{e^x+e^{-x}-2\cos x}\right)$

b $\lim_{x\to 0}\left(\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}\right)$

2 Series and Convergence

2.1 Infinite series

To start our study of infinite series in the option topic we consider using the terms of a sequence $\{a_n\}$ to form the sequence $\{s_n\}$ of **partial sums** of $\{a_n\}$ as follows:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

Here we have used the letter i as a subscript to indicate the ith term of a sequence; and have used the letter n as a subscript to indicate the nth partial sum. You need to be comfortable with using different letters for subscripts.

We can use sigma notation to write the general expression for s_n:

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

Definition of the sum of an infinite series

If the sequence of partial sums $\{s_n\} = \left\{\sum_{i=1}^{n} a_i\right\}$ converges, we say that its limit S is the sum of the infinite series $a_1 + a_2 + a_3 + \ldots$ and we write $S = \sum_{i=1}^{\infty} a_i$. If the sequence $\{s_n\}$ diverges then we say that the infinite series $\sum_{i=1}^{\infty} a_i$ also diverges.

As pointed out in Section 4.4 (of the textbook) in our discussion on infinite geometric series, the word 'sum' here is being used in a completely different way from how it is normally used. Ordinary addition of real numbers is a finite process; hence, it does not make sense to find the 'sum' of infinitely many terms. To be more precise, the 'sum' of an infinite series is a limit – that is, the limit of the partial sums for the series. We can write the sum as $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ but we must be careful not to assume that the '+' signs have the same properties to which we are accustomed. For example, as we will see, a rearrangement of the terms of a convergent series may change the value of its sum or even cause the series to diverge.

Example 1

a) Find the sum of the *finite* series $\sum_{n=1}^{6} (-1)^{n+1}$.

b) Consider the *infinite* series $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$.
Determine if the series converges to a sum or diverges.

Solution

a) Clearly, $\sum_{n=1}^{6}(-1)^{n+1} = 1-1+1-1+1-1 = 0.$

We can make the further observation that if the number of terms in this finite series was *any* even number, not just six, the sum is always 0; and, if the number of terms is odd the sum is always 1. In either case, we can 'pair up' consecutive terms to get zero. For example,

$$\sum_{n=1}^{6}(-1)^{n+1} = (1-1)+(1-1)+(1-1) = 0+0+0 = 0, \text{ or}$$

$$\sum_{n=1}^{7}(-1)^{n+1} = (1-1)+(1-1)+(1-1)+1 = 0+0+0+1 = 1.$$

b) It is very tempting to use the same strategy of 'pairing up' consecutive terms in this manner

$$\sum_{n=1}^{\infty}(-1)^{n+1} = (1-1)+(1-1)+(1-1)+\ldots$$

to argue that the sum of this infinite series is 0. However, this is erroneous. Consider that if we leave out the first term and start 'pairing up' from the second term we will obtain a different sum. The associative property of addition is what allowed us to 'pair up' the numbers for the finite sum in part a). Although the associative property works for finite sums it is clear that it does *not* work for infinite sums. The sum of an infinite series is defined to be the limit of the sequence of partial sums. For the sequence $\{s_n\} = \left\{\sum_{i=1}^{n}(-1)^{i+1}\right\}$, we have $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0,$ etc. Clearly this sequence is not converging to a limit. Therefore, the series has no sum and it diverges.

In studying infinite series, there are commonly two basic questions: Does a particular series converge or does it diverge? If it does converge, what is its sum?

Geometric series

There is one type of infinite series with which we are already familiar – and for which we know how to answer questions regarding convergence/divergence and computing sums; and this is **infinite geometric series** that we encountered in Chapter 4 of the textbook.

If a_1 represents the first term and r is the number that multiplies a term to obtain the next term in the series, then an infinite geometric series can be generalized as follows:

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1} + \ldots = \sum_{n=1}^{\infty}a_1r^{n-1},\ a_1 \neq 0$$

Let's consider three cases: $r = 1$, $r = -1$, and $r \neq \pm 1$.

If $r = 1$, then the nth partial sum is $s_n = a_1 + a_1 + a_1 + \ldots + a_1 = na_1$. Clearly the sequence of partial sums, $\{s_n\}$, will increase without bound and the geometric series diverges in this case.

If $r = -1$, then the nth partial sum is $s_n = a_1 - a_1 + a_1 - a_1 \ldots$. The sequence of partial sums, $\{s_n\}$, will behave in the same way as in Example 1 b) with $s_1 = a_1$, $s_2 = 0$, $s_3 = a_1$, $s_4 = 0$. The sequence of partial sums is not converging to a limit, so the geometric series also diverges for this case.

If $r \neq \pm 1$, then $s_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1}$.

Multiplying through by r gives

$$rs_n = a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + a_1 r^n.$$

Subtracting the second equation from the first produces

$$s_n - rs_n = a_1 - a_1 r^n.$$

Factorizing yields $s_n(1 - r) = a_1(1 - r^n)$.

Thus, the nth partial sum is

$$s_n = \frac{a_1(1 - r^n)}{1 - r}.$$

We know from the theorem for convergence of geometric sequences in the previous section that if $|r| < 1$ then r^n converges to 0 as $n \to \infty$. We can apply this fact and some properties of limits to give the following result:

$$\lim_{n\to\infty} \frac{a_1(1 - r^n)}{1 - r} = \lim_{n\to\infty} \frac{a_1 - a_1 r^n}{1 - r} = \lim_{n\to\infty} \frac{a_1}{1 - r} - \lim_{n\to\infty} \frac{a_1 r^n}{1 - r}$$
$$= \frac{a_1}{1 - r} - \left(\frac{a_1}{1 - r}\right) \lim_{n\to\infty} r^n = \frac{a_1}{1 - r} - 0 = \frac{a_1}{1 - r}$$

Therefore, if $|r| < 1$ then $\lim_{n\to\infty} s_n = \dfrac{a_1}{1 - r}$.

This rigorously confirms a result that appeared in Chapter 4, and we state

Convergence of geometric series

The geometric series with common ratio r

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

converges to the sum $\dfrac{a_1}{1 - r}$ if $|r| < 1$, and diverges if $|r| \geqslant 1$.

it again here. In this chapter, when we refer to a geometric series, it can be assumed that it is an infinite geometric series.

This result answers the two basic questions about geometric series. By identifying the value of the common ratio, r, we can determine which geometric series converge and which ones diverge; and for ones that converge we can easily compute the sum with the formula $S_\infty = \dfrac{a_1}{1 - r}$.

For any geometric series, the interval $|r| < 1$, which can also be written as $-1 < r < 1$, is known as its **interval of convergence**.

It is essential to understand that for any series $\sum a_n$ there are two important sequences for us to consider: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms.

Example 2

For each of the series, $\sum_{n=1}^{\infty} a_n$, below

(i) write the first four terms and find the limit (if it exists) of the sequence of its terms, $\lim_{n\to\infty} a_n$; and

(ii) write the first four terms of the sequence of its partial sums $\{s_n\}$ and find its limit (if it exists), i.e. the sum of the series.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ b) $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$

Solution

a) (i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \ldots$

The sequence of terms in the series is $\{a_n\} = \left\{\frac{(-1)^n}{3^n}\right\}$. This is a geometric sequence with $r = -\frac{1}{3}$ and because $-1 < -\frac{1}{3} < 1$ then it follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{3^n} = 0$.

(ii) The sequence of partial sums begins as follows:

$$s_1 = -\frac{1}{3}$$

$$s_2 = -\frac{1}{3} + \frac{1}{9} = -\frac{2}{9}$$

$$s_3 = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27} = -\frac{7}{27}$$

$$s_4 = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} = -\frac{20}{81}$$

Because the series is geometric such that $-1 < r < 1$, then the series converges to

$$\lim_{n\to\infty} s_n = \frac{a_1}{1-r} = \frac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)} = \frac{-\frac{1}{3}}{\frac{4}{3}} = -\frac{1}{4}.$$

Therefore, the sum of the series is $-\frac{1}{4}$.

b) (i) $\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = 4 + \frac{16}{5} + \frac{64}{25} + \frac{256}{125} + \ldots$

The series appears to be geometric with $r = \frac{4}{5}$. We can confirm this by simplifying the rule for the nth term:

$$2^{2n}5^{1-n} = \frac{(2^2)^n}{5^{n-1}} = \frac{4^1 4^{n-1}}{5^{n-1}} = 4\left(\frac{4}{5}\right)^{n-1}.$$

Hence, $\sum_{n=1}^{\infty} 2^{2n}5^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^{n-1}$ and its clear that the series is geometric with $a_1 = 4$ and $r = \frac{4}{5}$. Because $-1 < r < 1$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(4\left(\frac{4}{5}\right)^{n-1}\right) = 4\lim_{n\to\infty}\left(\frac{4}{5}\right)^{n-1} = 4\cdot 0 = 0.$$

(ii) The sequence of partial sums begins as follows:

$$s_1 = 4$$
$$s_2 = 4 + \frac{16}{5} = \frac{36}{5} = 7.2$$
$$s_3 = 4 + \frac{16}{5} + \frac{64}{25} = \frac{244}{25} = 9.76$$
$$s_4 = 4 + \frac{16}{5} + \frac{64}{25} + \frac{256}{125} = \frac{1476}{125} = 11.808$$

Because the series is geometric such that $-1 < r < 1$, then the series converges to

$$\lim_{n\to\infty} s_n = \frac{a_1}{1-r} = \frac{4}{1-\left(\frac{4}{5}\right)} = \frac{4}{\frac{1}{5}} = 20.$$

Therefore, the sum of the series is 20.

It is obvious that any series whose sequence of terms does not converge to zero, i.e. $\lim_{n\to\infty} a_n \neq 0$, will have a sequence of partial sums that diverges. In such a case, the magnitude (positive or negative) of terms will increase, causing the sequence of partial sums to increase without bound. We established that both series in Example 2 are convergent and also that $\lim_{n\to\infty} a_n = 0$ for both series. It seems reasonable to conjecture that a necessary *and* sufficient condition for an infinite series $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ to converge to a finite quantity is that the sequence, $\{a_n\}$, of individual terms a_n converges to zero. Is it possible for the sequence of terms of a series to converge to zero but the series itself does not converge, i.e. does not have a sum?

Example 3

Consider the series $\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. Determine whether the series converges or diverges.

Solution

Clearly, the sequence of terms converges to zero, i.e. $\lim_{n\to\infty}\frac{1}{n} = 0$. To answer the question about convergence of the series we need to look at the sequence of partial sums. Our analysis begins by bracketing the terms in the following way:

$$s_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) + \ldots$$

so that the final term in each bracketed group is the reciprocal of a power of two. Let's consider the sum of the first 2^n terms,

$$s_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} \ldots + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right)$$

$$\geqslant 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}$$

$$= 1 + n\left(\frac{1}{2}\right) = \frac{n+2}{2} \quad \Rightarrow \quad s_{2^n} \geqslant \frac{n+2}{2}$$

Clearly the sequence of these partial sums diverges, so s_{2^n} diverges.

Hence, the series $s_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is greater than a series that diverges, so it must also diverge.

Therefore, even though the sequence $\frac{1}{n} \to 0$ as $n \to \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This series is called the **harmonic series** – and we will encounter it often.

The clever method used in Example 3 is attributed to a French scholar, Nicole Oresme (1323–1382), who was the first to mathematically prove that the harmonic series diverges. Considering the state of mathematics in the 14th century, Oresme was well ahead of his time by inventing a type of coordinate geometry and using the idea of a fractional exponent – three centuries before Descartes developed coordinate geometry and Newton first invented our modern notation for fractional exponents.

With regard to his proof of the divergence of the harmonic series, Oresme's ingenious strategy involved replacing groups of fractions in the harmonic series with smaller fractions that have a sum of $\frac{1}{2}$. The following shows the heart of his strategy:

$$1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{3}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \frac{3}{2} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{4}{2}$$

$$1 + \frac{1}{2} + \ldots + \frac{1}{8} + \left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right) > \frac{4}{2} + \left(\frac{1}{16} + \ldots + \frac{1}{16}\right) = \frac{5}{2}$$

This process can be continued indefinitely, so that, in general, for any positive integer n we have

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} > \frac{n+1}{2}.$$

For example, if $n = 25$ then

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{33\,554\,432} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^{25}} > \frac{25+1}{2} = 13.$$

So Oresme's strategy shows that by taking enough terms of the harmonic series, we can guarantee that its sum will be greater than any finite number. Therefore, the series will diverge to infinity. It is interesting to note that although the harmonic series diverges, it does so very slowly. The sum of the harmonic series does not get above 10 until we have added 12 367 terms of the series!

The fact that the harmonic series diverges (Example 3) serves as a counterexample to our conjecture that $\lim_{n\to\infty} a_n = 0$ is both a necessary *and* sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge. It is true that convergence can only occur if $\lim_{n\to\infty} a_n = 0$ (i.e. a *necessary* condition), but $\lim_{n\to\infty} a_n = 0$ is NOT sufficient to guarantee convergence (i.e. not a *sufficient* condition). This leads to the following theorem.

nth term divergence test

If $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 4 – Using the nth term divergence test

Determine, if possible, whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1} = \frac{1}{2}+\frac{4}{5}+\frac{9}{10}+\frac{16}{17}+\ldots$ b) $\displaystyle\sum_{n=1}^{\infty} 3(-1)^{n+1} = 3-3+3-3+\ldots$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n+1} = \frac{1}{2}+\frac{1}{5}+\frac{1}{14}+\frac{1}{41}+\ldots$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3n!+1} = \frac{1}{4}+\frac{2}{7}+\frac{6}{19}+\frac{24}{73}+\ldots$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\frac{4}{17}+\ldots$

Solution

a) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{n^2/n^2}{n^2/n^2+1/n^2} = \lim_{n\to\infty}\frac{1}{1+1/n^2} = 1$

Therefore, by the nth term divergence test, the series is divergent.

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(3(-1)^{n+1}\right)$ does not exist because the terms alternate between $+3$ and -3.

Therefore, by the nth term divergence test, the series is divergent.

c) Certainly, $3^n+1 \to \infty$ as $n \to \infty$, so it follows that $\dfrac{1}{3^n+1} \to 0$ as $n \to \infty$. Hence, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2}{3^n+1} = 2\lim_{n\to\infty}\frac{1}{3^n+1} = 0$. Since the limit of the nth term is 0, the nth term divergence test does not apply and we are not able to make a conclusion about convergence or divergence. We can make an educated guess that it will probably converge because it is very similar to the convergent geometric series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3^n} = 2\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ with $r = \dfrac{1}{3}$. In the next section we will learn that it does in fact converge and recognizing that it is similar to a convergent geometric series is important.

d) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{3n!+1} = \lim_{n\to\infty} \frac{n!/n!}{3n!/n!+1/n!} = \frac{1}{3+0} = \frac{1}{3}$

Therefore, by the nth term divergence test, the series is divergent.

e) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n/n^2}{n^2/n^2+1/n^2} = \frac{0}{1+0} = 0$

We cannot apply the nth term divergence test since the limit of the nth term is 0. We will find in the next section that this series behaves like the harmonic series, that is, even though the sequence of its terms converges to 0 the series itself diverges.

Before moving onto the next section and investigating more thoroughly the convergence of infinite series, we state below some important properties of convergent series that are direct consequences of the properties of limits of sequences in Section 1.2 of the previous chapter.

Properties of convergent series

Given that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series, and c is a constant, then the following series are also convergent:

$\sum_{n=1}^{\infty} ca_n$, $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$.

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

2.2 Convergence tests

In this section, we develop some more sophisticated tests for convergence. These tests will allow us to efficiently determine convergence for a wide range of series. In Example 4 we were thwarted from determining whether the series in parts c) and e) were convergent or divergent. In general, it is not easy to find the exact sum of a series. We have been able to find exact sums for certain geometric series and telescoping series because we were able to obtain a formula for the sequence of partial sums, s_n. In this section, our purpose is to develop some tests that will let us determine whether a series is convergent or divergent without the need for a formula for the sequence of partial sums. Although in some cases the convergence test being employed will help us to find the sum of a series (or at least an approximation for the sum), in general, it is limited to finding out about convergence of a series without finding the sum. We will study four useful convergence tests that apply to series whose terms are **non-negative** and a fifth test that will apply to alternating series.

Integral test

From our discussion about improper integrals in the previous section, you may feel that there is a relationship between the convergence of an improper integral and the convergence of a series. We can take the formula for the nth term a_n of a series $\sum_{n=1}^{\infty} a_n$ and replace n by x to write a function $f(x)$. The relationship between $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ is explained in the following theorem.

The integral test for convergence

Let f be a function that is continuous, decreasing and positive for all $x \geqslant 1$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges. In other words:

1) If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

2) If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Before we can conduct a formal proof of the integral test we need to establish the definition of two words for which we have had a common-sense understanding up to now, and to state an important theorem.

Lower and upper bounds of a sequence

The number M is a **lower bound** of the sequence $\{a_n\}$ if $a_n \geqslant M$ for all positive integers n, and the number N is an **upper bound** of $\{a_n\}$ if $a_n \leqslant N$ for all positive integers n. A sequence $\{a_n\}$ is **bounded** if and only if it has a lower bound and an upper bound.

For the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, the sequence of its terms $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ any number greater than or equal to 1 is an upper bound, and any number that is less than or equal to zero is a lower bound. For the sequence $\left\{\frac{1}{n}\right\}$ we can call 1 the **least upper bound** and 0 the **greatest lower bound**. Another characteristic of the sequence $\left\{\frac{1}{n}\right\}$ is that the terms are always decreasing and it is not surprising that the sequence converges to its greatest lower bound. In our discussion of one-to-one functions in Chapter 2 of the book, we used the word **monotonic** to describe a function that is either always increasing or always decreasing. Also for the harmonic series, we established that the sequence of its partial sums, $\{s_n\}$, is divergent by essentially showing that $\{s_n\}$ does not have an upper bound, and hence is not bounded. It is sensible to conjecture that a bounded monotonic sequence will be convergent.

Bounded sequence theorem

A monotonic sequence converges if and only if it is bounded.

Before we conduct a formal proof of this theorem, we state an important property of the real numbers with the following postulate.

Completeness postulate

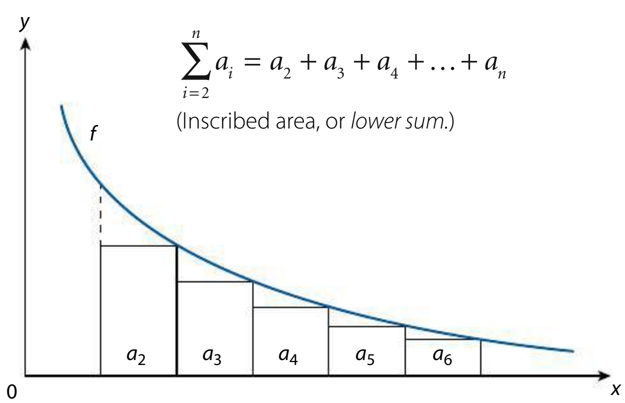
In the real numbers, every non-empty set that has an upper bound has a least upper bound.

Proof of the bounded sequence theorem

We prove the theorem for the case when the monotonic sequence, call it $\{a_n\}$, is increasing. If it converges to some limit L then it is bounded below by the first term of the sequence a_1 and above by L and is therefore bounded. Conversely, if $\{a_n\}$ is bounded, then the completeness postulate guarantees that $\{a_n\}$ has a least upper bound L. We now need to show that $\{a_n\}$ must converge to L. Firstly, since L is an upper bound for $\{a_n\}$ then it follows that $a_n \leqslant L$ for all n. Also, since L is the least upper bound then $L - \varepsilon$ is not an upper bound for any $\varepsilon > 0$. Hence, there exists an integer N such that $L - \varepsilon < a_N$. Because $\{a_n\}$ is always increasing then $a_N \leqslant a_n$ whenever $n > N$. Therefore, $L - \varepsilon < a_n \leqslant L$ and consequently $L - \varepsilon < a_n < L + \varepsilon$ which is equivalent to $-\varepsilon < a_n - L < \varepsilon$ and $|a_n - L| < \varepsilon$. This satisfies the $\varepsilon - N$ definition for the limit of a sequence and completes the proof for an increasing sequence $\{a_n\}$. A parallel argument can be written to prove the theorem for a decreasing sequence $\{a_n\}$.

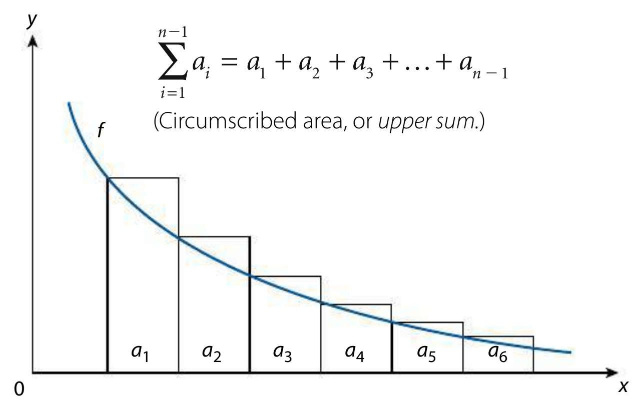
Proof of the integral test

The essential idea behind the proof is that the terms in a series $\sum_{n=1}^{\infty} a_n$ can be assigned to represent the area of ever decreasing rectangles of constant width and that the improper integral $\int_1^{\infty} f(x)\,dx$ is approximated by the sum of these rectangles. The total areas of the inscribed rectangles (Figure 2.1) and the circumscribed rectangles (Figure 2.2) are as follows:



$$\sum_{i=2}^{n} a_n \leqslant \int_1^n f(x)\,dx$$

Figure 2.1 Inscribed rectangles gives lower sum.



$$\int_1^n f(x)dx \leqslant \sum_{i=1}^{n-1} a_n$$

Figure 2.2 Circumscribed rectangles gives upper sum.

The exact area under the graph of f from $x = 1$ to n, i.e. the definite integral $\int_1^n f(x)\,dx$, lies between the inscribed and circumscribed areas. As Figures 2.1 and 2.2 illustrate,

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\,dx \leqslant \sum_{i=1}^{n-1} a_i.$$

Using the nth partial sum, $s_n = a_1 + a_2 + a_3 + \ldots + a_n$, we can write the inequality above as

$$s_n - a_1 \leqslant \int_1^n f(x)\,dx \leqslant s_{n-1}.$$

To prove part (1) we start by assuming $\int_1^n f(x)\,dx$ converges to L. Then it follows that for $n \geqslant 1$

$$s_n - a_1 \leqslant L$$

and consequently

$$s_n \leqslant L + a_1.$$

Hence, the sequence of partial sums $\{s_n\}$ is bounded and monotonic and it follows from the bounded sequence theorem that $\{s_n\}$ converges, and consequently the series $\sum_{n=1}^{\infty} a_n$ must also converge. For part (2) assume that the improper integral $\int_1^n f(x)\,dx$ diverges. Thus, $\int_1^n f(x)\,dx$ goes to infinity as $n \to \infty$, and given the inequality $s_{n-1} \geqslant \int_1^n f(x)\,dx$ it must follow that $\{s_n\}$ diverges which means that $\sum_{n=1}^{\infty} a_n$ also diverges.

Example 5 – Using the integral test

Determine the convergence or divergence of each series.

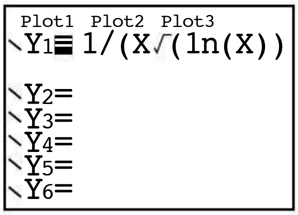
a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

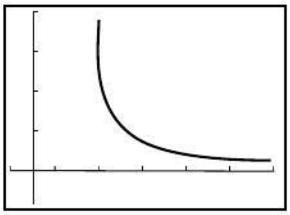
b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

c) $\displaystyle\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \ldots$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ [Example 4 part e)]

Solution





a) Graphing the function $f(x) = \dfrac{1}{x\sqrt{\ln x}}$ on our GDC provides us with a quick confirmation that f is continuous, decreasing and positive for all $x \geqslant 2$, thereby satisfying the conditions for applying the integral test. Recalling techniques for improper integrals from the first section of this chapter, we now need to evaluate $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ to see if it converges to a finite number or diverges to infinity. For this integral we will also need to apply the technique of u-substitution.

$$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int (\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\,dx\right) = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} \qquad \text{Let } u = \ln x, \text{ then } du = \frac{1}{x}dx.$$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{1}{x\sqrt{\ln x}}\,dx \qquad \text{Rewriting improper integral as a limit.}$$

$$= \lim_{b\to\infty}\left[2\sqrt{\ln x}\,\right]_2^b \qquad \text{Applying result from } u\text{-substitution.}$$

$$= \lim_{b\to\infty}\left[2\sqrt{\ln b} - 2\sqrt{\ln 2}\,\right]$$

$$= \infty$$

Therefore, the integral $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ diverges, and by the integral test the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ must also diverge.

b) For $f(x) = \frac{x}{e^x}$, it is clear that f is continuous, decreasing and positive for $x \geqslant 1$ because $e^x > 0$ and e^x grows faster than x; so the integral test applies. Using integration by parts:

$$\int \frac{x}{e^x}\,dx = \int xe^{-x}\,dx \qquad \text{Choose } u = x \Rightarrow du = dx \text{ and } dv = e^{-x} \Rightarrow v = -e^{-x}.$$

$$= -xe^{-x} - \int -e^{-x}\,dx \qquad \text{Substituting into formula } \int u\,dv = uv - \int v\,du.$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}$$

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \qquad \text{Rewriting improper integral as a limit.}$$

$$= \lim_{b\to\infty}\left[\left(-be^{-b} - e^{-b}\right) - \left(-e^{-1} - e^{-1}\right)\right]$$

$$= \lim_{b\to\infty}\left[-\frac{b+1}{e^b} + \frac{2}{e}\right] = -\lim_{b\to\infty}\left[\frac{b+1}{e^b}\right] + \lim_{b\to\infty}\left[\frac{2}{e}\right]$$

Applying l'Hôpital's rule to the first limit gives $\lim_{b\to\infty}\left[\frac{b+1}{e^b}\right] = \lim_{b\to\infty}\left[\frac{1}{e^b}\right] = 0$.

Therefore, $\int_1^{\infty} \frac{x}{e^x}\,dx = \frac{2}{e}$.

By the integral test, since the integral $\int_1^{\infty} \frac{x}{e^x}\,dx$ converges then the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ must also converge.

c) We need to find a rule for the nth term for the series that starts $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \ldots$.

Using some inductive reasoning we determine that the series expressed in summation notation is

As Example 5 a) illustrates, if the summation index for an infinite series starts at $n = k > 1$ rather than $n = 1$, the integral test can still be applied. The integral test can be modified as follows:

Let f be a function that is continuous, decreasing and positive for all $x \geqslant k$ such that $k > 1$ and $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges *if and only if* the improper integral $\int_k^{\infty} f(x)\,dx$ converges.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+\ldots+\frac{1}{n^2+1}+\ldots.$$

The function $f(x)=\frac{1}{x^2+1}$ satisfies the conditions of the integral test.

We need to recognize that the anti-derivative of $\frac{1}{x^2+1}$ is $\arctan x$ (a 'standard integral' in the IB formula booklet).

$$\int_1^{\infty}\frac{1}{x^2+1}dx=\lim_{b\to\infty}\int_1^b\frac{1}{x^2+1}dx$$ Rewriting improper integral as a limit.

$$=\lim_{b\to\infty}\left[\arctan x\right]_1^b$$
$$=\lim_{b\to\infty}\left[\arctan b-\arctan 1\right]$$
$$=\lim_{b\to\infty}\left[\arctan b\right]-\lim_{b\to\infty}\left[\frac{\pi}{4}\right]$$
$$=\frac{\pi}{2}-\frac{\pi}{4}$$

Therefore, $\int_1^{\infty}\frac{1}{x^2+1}dx=\frac{\pi}{4}$.

By the integral test, since the integral $\int_1^{\infty}\frac{1}{x^2+1}dx$ converges then the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ must also converge.

It is very important to know when using the integral test that the value of the improper integral is <u>not</u> equal to the sum of the series. The sum, expressed to ten significant figures, of the first 50 terms of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$, Example 5 d), is approximately 1.056 875 301; whereas $\int_1^{\infty}\frac{1}{x^2+1}dx=\frac{\pi}{4}\approx 0.7853981634$. Therefore, in general $\sum_{n=1}^{\infty}a_n\neq\int_1^{\infty}f(x)dx$.

d) $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ was the series in Example 4 e) for which the nth term divergence test was inconclusive because $\lim_{n\to\infty}\frac{n}{n^2+1}=0$. The function $f(x)=\frac{x}{x^2+1}$ satisfies the conditions of the integral test. The method of u-substitution will be useful to evaluate the integral $\int\frac{x}{x^2+1}dx$.

Let $u=x^2+1$ and it follows that $du=2xdx\Rightarrow\frac{1}{2}du=xdx$.

Substituting gives

$$\int\frac{x}{x^2+1}dx=\int\frac{1}{u}\cdot\frac{1}{2}du=\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}\ln u=\frac{1}{2}\ln\left(x^2+1\right).$$

Using this result we have:

$$\int_1^{\infty}\frac{1}{x^2+1}dx=\lim_{b\to\infty}\int_1^b\frac{1}{x^2+1}dx$$ Rewriting improper integral as a limit.

$$=\frac{1}{2}\lim_{b\to\infty}\left[\ln\left(x^2+1\right)\right]_1^b$$
$$=\frac{1}{2}\lim_{b\to\infty}\left[\ln\left(b^2+1\right)-\ln 2\right]$$
$$=\infty$$

By the integral test, since the integral $\int_1^{\infty}\frac{x}{x^2+1}dx$ diverges then the series $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ must also diverge.

p-series

Before we move onto the next convergence test, we can use the integral test to give us important results for any series that is in the form shown below, known as a ***p*-series**.

$$\sum_{1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots \text{ where } p \text{ is a constant.}$$

If $p = 1$, the p-series is the harmonic series which we know diverges. What about series for other values of p? The following example will lead to a simple test for the convergence of any p-series.

Example 6 – Convergence of *p*-series

For what values of p is the series $\sum_{1}^{\infty} \frac{1}{n^p}$ convergent?

Solution

Let's consider when $p < 0$, $p = 0$, and $p > 0$.

When $p < 0$, then $\frac{1}{n^p} \to \infty$ as $n \to \infty$. For example, if $p = -3$ then $\frac{1}{n^{-3}} = n^3$; and clearly n^3 increases without bound as $n \to \infty$.

When $p = 0$, then $\lim_{n \to 0} \frac{1}{n^p} = \frac{1}{n^0} = 1$.

In both of these cases, $\lim_{n \to 0} \frac{1}{n^p} \neq 0$ so the p-series diverges by the nth term divergence test.

When $p > 0$, the function $f(x) = \frac{1}{x^p}$ is continuous, decreasing and positive for $x \geqslant 1$ so we can use the integral test. We know from Example 3 in the previous section that the harmonic series $(p = 1)$ diverges, so let's assume that $p \neq 1$ and investigate the improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\,dx &= \lim_{b \to \infty} \int_1^b x^{-p}\,dx \\
&= \lim_{b \to \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b \\
&= \left(\frac{1}{1-p} \right) \lim_{b \to \infty} \left[x^{-p+1} \right]_1^b \\
&= \left(\frac{1}{1-p} \right) \lim_{b \to \infty} \left[b^{-p+1} - 1 \right]
\end{aligned}$$

If $p > 1$, then $-p + 1 < 0$ and consequently as $b \to \infty$, $b^{-p+1} \to 0$.

Hence, if $p > 1$ then $\int_1^{\infty} \frac{1}{x^p}\,dx = \left(\frac{1}{1-p} \right)(-1) = \frac{1}{p-1}$. Therefore the integral converges and the series $\sum_{1}^{\infty} \frac{1}{n^p}$ must also converge.

If $p < 1$, then $-p + 1 > 0$ and consequently as $b \to \infty$, $b^{-p+1} \to \infty$. Hence, if $p < 1$ then the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges and so does the series $\sum_{1}^{\infty} \frac{1}{n^p}$.

The results from Example 6 are summarized below.

Convergence of *p*-series

The *p*-series $\sum_{1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$

(i) **converges** if $p > 1$, and (ii) **diverges** if $p \leqslant 1$.

Note: When $p = 1$ this is the harmonic series.

Comparison test

The integral test compares a series consisting of all positive terms with an integral as a means of testing the convergence of the series. It is possible to use a second series in a similar way. If each term of a series of positive terms is less than or equal to the corresponding term of a known convergent series of positive terms, then the series is convergent. We will call this the comparison test and can state it as follows.

In the statement of the comparison test, $n \geqslant N$ means *from some term onward*. That is, eventually for some term and forever afterwards the terms of the series $\sum_{n=1}^{\infty} b_n$ are always greater than the corresponding terms of the series $\sum_{n=1}^{\infty} a_n$. This is often expressed by saying that $\sum_{n=1}^{\infty} b_n$ *dominates* $\sum_{n=1}^{\infty} a_n$.

Comparison test

Given $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$ for some integer N, it follows that

1 if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges;

2 if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Note: The comparison test can also be applied for the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ whenever there exists a positive constant c such that $0 \leqslant a_n \leqslant cb_n$ for all $n \geqslant N, N \in \mathbb{Q}^+$.

The comparison test significantly expands our ability to determine the convergence of a series with more complicated rules for the nth term. We achieve this by comparing a 'complicated' series to a 'simpler' series whose convergence or divergence is known.

Before proving both parts of the comparison test, we will find it helpful to state a corollary to the bounded sequence theorem that we recall says the following: A monotonic (always decreasing or always increasing) sequence converges if and only if it is bounded. If all the terms of an infinite series are positive, the sequence of partial sums is increasing. Therefore, the following theorem follows directly from the bounded sequence theorem.

Positive series convergence

A series of positive terms is convergent if and only if its sequence of partial sums has an upper bound.

Proof of comparison test

Proof of 1: Let $\{u_n\}$ and $\{v_n\}$ be sequences of the partial sums for the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively. Because $\sum_{n=1}^{\infty} b_n$ is a series of positive terms that is convergent, it follows from the positive series convergence theorem that the sequence $\{v_n\}$ has an upper bound – let's call it B. Since $a_n \leqslant b_n$ for all $n \geqslant 1$, we can conclude that $a_n \leqslant b_n \leqslant B$ for all $n \geqslant 1$. Thus, B is an upper bound of the sequence $\{u_n\}$. Because the terms of the series $\sum_{n=1}^{\infty} a_n$ are all positive then it follows from the positive series convergence theorem that $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof of 2: If $\sum_{n=1}^{\infty} a_n$ is divergent, then since $\{u_n\}$ is increasing $u_n \to \infty$. However, $b_n \geqslant a_n$, so $v_n \geqslant u_n$. It follows that $v_n \to \infty$ and, therefore, $\sum_{n=1}^{\infty} b_n$ must also diverge.

Example 7 – Using the comparison test

Determine the convergence or divergence of each series.

a) $\sum_{n=1}^{\infty} \frac{2}{3^n+1}$ [Example 4 c)]   b) $\sum_{n=1}^{\infty} \frac{1}{3+\sqrt{n}}$ c) $\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution

a) We can compare the given series

$$\frac{2}{4}+\frac{2}{10}+\frac{2}{28}+\frac{2}{82}+\ldots+\frac{2}{3^n+1}+\ldots$$

with the nth term of the geometric series

$$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\ldots+\frac{2}{3^n}+\ldots$$

which converges because its common ratio is between one and negative one; $r=\frac{1}{3}<1$.

It is clear that each term in the given series is less than its corresponding term in the geometric series. That is, $\frac{2}{3^n+1}<\frac{2}{3^n}$ for all $n \geqslant 1$.

Therefore, by the comparison test since the series $\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges the series $\sum_{n=1}^{\infty} \frac{2}{3^n+1}$ must also converge.

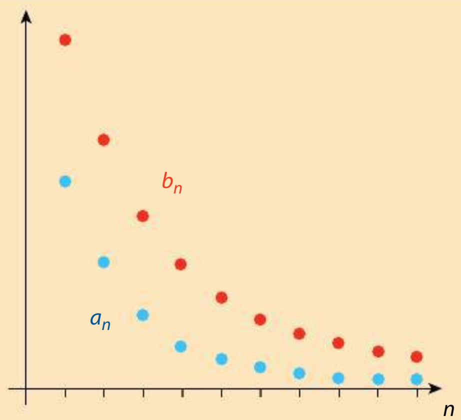


Figure 2.3

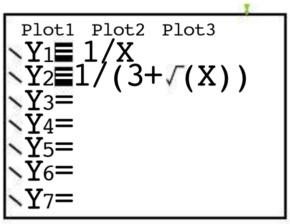
Note that part **1** and part **2** of the comparison test require that $0 \leqslant a_n \leqslant b_n$. You can think of $\sum a_n$ as the 'lower' series and $\sum b_n$ as the 'higher' series (see Figure 2.3). Thus, in a very informal sense the two parts of the comparison test say:

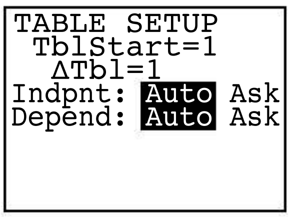
1. If the 'higher' series converges, then the 'lower' series must also converge.
2. If the 'lower' series diverges, then the 'higher' series must also diverge.

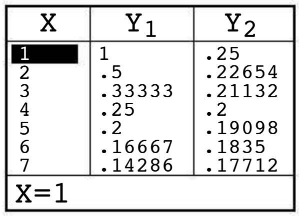
The 'higher' series *dominates* the 'lower' series.

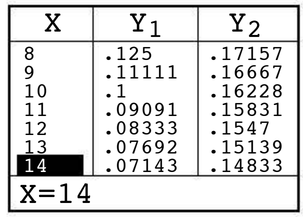
b) The series $\sum_{n=1}^{\infty}\frac{1}{3+\sqrt{n}}$ is similar to the p-series $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ which diverges because $p=\frac{1}{2}\leqslant 1$. If we compare the given series to this p-series we see that $\frac{1}{3+\sqrt{n}}<\frac{1}{\sqrt{n}}$ for all $n\geqslant 1$. However, the comparison test provides no conclusive result in this case where a series is *dominated* by a divergent series. Suspecting that the given series does in fact diverge we need to find a divergent series that the given series dominates. Let's compare it to the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. Remember, to satisfy the comparison test it is not necessary for $a_n\leqslant b_n$ to be true for all integers $n\geqslant 1$ but for all integers $n\geqslant N$ where N is some positive integer.

Our GDC is a handy tool to quickly compare the terms of the given series to the harmonic series. The screen images below show values for the first 14 terms of the two series in a table.









For the first five terms the terms in the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ are greater than those for $\sum_{n=1}^{\infty}\frac{1}{3+\sqrt{n}}$. However, it appears from the sixth term onwards that this reverses, that is,

$$\frac{1}{n}<\frac{1}{3+\sqrt{n}} \quad \text{for } n\geqslant 6.$$

Therefore, by the comparison test the series $\sum_{n=1}^{\infty}\frac{1}{3+\sqrt{n}}$ diverges.

How could we prove that the inequality $\frac{1}{n}<\frac{1}{3+\sqrt{n}}$ is true for $n\geqslant 6$? Try doing so by proving the inequality $3+\sqrt{n}<n$ for $n\geqslant 6$ by mathematical induction.

c) Consider the first few terms of the given series:

$$\sum_{n=0}^{\infty}\frac{1}{n!}=1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots$$

Now consider the first few terms of the convergent geometric series with $a_1=2$ and $r=\frac{1}{2}$.

$$\sum_{n=0}^{\infty}2\left(\frac{1}{2}\right)^n=2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

It appears that the terms of $\sum_{n=0}^{\infty}\frac{1}{n!}$ are less than or equal to the corresponding terms of the convergent geometric series for all $n\geqslant 1$. Recall that in Example 6 of the previous chapter we proved that $\lim_{n\to\infty}\frac{x^n}{n!}=0$ for any real number x. From that we concluded that the

In Example 7 c), we know that the sum of the infinite geometric series $\sum_{n=0}^{\infty}2\left(\frac{1}{2}\right)^n$ is $S_\infty=\frac{a_1}{1-r}=\frac{2}{1-\frac{1}{2}}=4$. Thus the sum $\sum_{n=0}^{\infty}\frac{1}{n!}$ must be less than 4. In fact, we will learn in the next section that this sum is exactly the number e. That is,

$$e=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\ldots+\frac{1}{n!}+\ldots$$

factorial function increases faster than any exponential function. Hence, $\frac{1}{n!} \leqslant 2\left(\frac{1}{2}\right)^n$ for $n \geqslant 1$. Therefore, by the comparison test the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Limit comparison test

In order for the comparison test to provide us with a conclusive result on the convergence or divergence of a series, the series being tested must be dominated by ('lower' than) a convergent series, or it must dominate ('higher' than) a divergent series. If these conditions are not met then the comparison test (sometimes called the direct comparison test) cannot be used. For example, consider the series $\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ that is nearly identical to the series $\sum_{n=1}^{\infty} \frac{2}{3^n + 1}$ that we proved is convergent in Example 7 a).

We strongly expect $\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ to also converge. However, the inequality $\frac{2}{3^n - 1} > \frac{2}{3^n}$ shows that the series dominates the convergent geometric series $\sum_{n=1}^{\infty} \frac{2}{3^n}$ so the comparison test does not apply. In a case like this another form of the comparison test, known as the **limit comparison test**, can be used. This test can be particularly useful in comparing a series to a p-series or a geometric series.

Limit comparison test

Given $a_n > 0$ and $b_n > 0$ for all $n \geqslant N$ for some integer N, it follows that:

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite and positive, then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ also diverges.

If applying the limit comparison test you get $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, this does ***not*** imply that the series $\sum_{n=1}^{\infty} a_n$ also diverges.

Proof

1. Let k and m be positive numbers such that $k < L < m$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ then there is a positive integer N, where $N > n$, such that

$$k < \frac{a_n}{b_n} < m.$$

It follows that

$$kb_n < a_n < mb_n.$$

If the series $\sum_{n=1}^{\infty} b_n$ converges then from the properties of series, the series

$\sum_{n=1}^{\infty} mb_n$ must also converge. Since $\sum_{n=1}^{\infty} mb_n$ dominates $\sum_{n=1}^{\infty} a_n$ then by the comparison test $\sum_{n=1}^{\infty} a_n$ must converge. Likewise, if the series $\sum_{n=1}^{\infty} b_n$ diverges then the series $\sum_{n=1}^{\infty} kb_n$ must also diverge, and since $\sum_{n=1}^{\infty} a_n$ dominates $\sum_{n=1}^{\infty} kb_n$ then by the comparison test $\sum_{n=1}^{\infty} a_n$ must diverge.

The proofs of parts 2 and 3 are left as exercises.

Example 8 – Using the limit comparison test

Determine the convergence or divergence of each series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{\sqrt{n}}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 7n}{3n^6 - n^3}$ d) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution

a) As mentioned above, this series resembles the convergent geometric series $\sum_{n=1}^{\infty} \frac{2}{3^n}$. Thus, we evaluate the following limit.

$$\lim_{n\to\infty} \frac{\frac{2}{3^n - 1}}{\frac{2}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n - 1}$$

$$= \lim_{n\to\infty} \frac{3^n/3^n}{3^n/3^n - 1/3^n}$$

$$= \lim_{n\to\infty} \frac{1}{1 - 1/3^n}$$

$$= 1$$

Since the limit is finite and positive and $\sum_{n=1}^{\infty} \frac{2}{3^n}$ converges then by the limit comparison test the series $\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$ must also converge.

b) The given series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+1}$ is similar to $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ which is a p-series best written as $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$. Since $p = \frac{2}{3} \leqslant 1$ we know that $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges. We then evaluate the following limit.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{n}}{n+1}}{\frac{1}{n^{2/3}}} = \lim_{n\to\infty} \frac{n^{1/3} \cdot n^{2/3}}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$
$$= 1$$

Since the limit is finite and positive and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges then by the limit comparison test the series $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+1}$ must also diverge.

c) As we saw in part b), it is possible to find a suitable p-series for comparison purposes by disregarding all but the highest powers of n in the numerator and denominator. Hence, for the given series $\sum_{n=1}^{\infty} \frac{n^2+7n}{3n^6-n^3}$ we can compare the series to $\sum_{n=1}^{\infty} \frac{n^2}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ which is a convergent p-series.

$$\lim_{n\to\infty} \frac{\frac{n^2+7n}{3n^6-n^3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^4\left(n^2+7n\right)}{3n^6-n^3}$$
$$= \lim_{n\to\infty} \frac{n^6+7n^5}{3n^6-n^3}$$
$$= \lim_{n\to\infty} \frac{n^6/n^6+7n^5/n^6}{3n^6/n^6-n^3/n^6}$$
$$= \frac{1+0}{3-0}$$
$$= \frac{1}{3}$$

Since the limit is finite and positive and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges then by the limit comparison test the series $\sum_{n=1}^{\infty} \frac{n^2+7n}{3n^6-n^3}$ must also converge.

d) Remember that in Section 13.2 of the book we proved $\lim_{n\to\infty} \frac{\sin x}{x} = 1$ by means of the squeeze theorem. So we can use the limit comparison theorem and compare the given series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Hence, $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$

Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges then $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges.

Ratio test

In a geometric series, the ratio of adjacent terms is constant. This can be expressed as

$$\frac{a_1 r^{n+1}}{a_1 r^n} = r.$$

We know that a geometric series converges if and only if this ratio is between −1 and 1. In other types of series, the ratio of adjacent terms does not remain constant but it can still give us helpful information about whether or not the series converges, as indicated in the following theorem.

Ratio test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms, and with

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then **1** the series *converges* if $L < 1$
2 the series *diverges* if $L > 1$
3 the test is *inconclusive* if $L = 1$.

Proof

1. For the case when $L < 1$, there must be a number r with $0 < r < 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leqslant r$$

for all n sufficiently large. Suppose that there exists some integer N such that $\frac{a_{n+1}}{a_n} \leqslant r$ if $n \geqslant N$.

Then
$$\left|\frac{a_{N+1}}{a_N}\right| \geqslant r \quad \Rightarrow \quad |a_{N+1}| \leqslant r|a_N|$$
$$\left|\frac{a_{N+2}}{a_{N+1}}\right| \geqslant r \quad \Rightarrow \quad |a_{N+2}| \leqslant r|a_{N+1}| \leqslant r^2|a_N|$$

and so on. Thus,

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \ldots \leqslant a_N|1 + r + r^2 + \ldots|$$

This shows that for $n \geqslant N$ the series $\sum_{n=1}^{\infty} a_n$ is dominated by the geometric series $a_N \sum_{n=1}^{\infty} r^{n-1}$. Because $0 < r < 1$ this geometric series converges and by the comparison test $\sum_{n=1}^{\infty} a_n$ must also converge.

2. For the case when $L > 1$, it must be true that $|a_{n+1}| > |a_n|$ for all n sufficiently large. Therefore, $\lim_{n\to\infty} a_n \neq 0$ and the series $\sum_{n=1}^{\infty} a_n$ must diverge by the nth term divergence test.

3. Applying the ratio test to the general p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ gives

$$\lim_{n\to\infty}\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p = 1.$$

We know that a p-series converges if $p > 1$ and diverges if $p \leqslant 1$. Hence, this shows that if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$ then it is possible to have a series that is either convergent or divergent. Therefore, the ratio test is inconclusive if $L = 1$.

The ratio test is particularly useful for testing series involving exponential expressions or expressions with factorials, as illustrated in the following example.

Example 9 – Using the ratio test

Determine the convergence or divergence of each series.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 3^{n+1}}{4^n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

a) All the terms of the given series are positive so we can do without the absolute value signs.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^3 3^{n+2}}{4^{n+1}}}{\dfrac{n^3 3^{n+1}}{4^n}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^3}{n^3} \cdot \frac{3^{n+2}}{3^{n+1}} \cdot \frac{4^n}{4^{n+1}} \right)$$

$$= \lim_{n\to\infty} \frac{3(n+1)^3}{4n^3}$$

$$= \frac{3}{4} < 1$$

Therefore, by the ratio test the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 3^{n+1}}{4^n}$ converges.

b) Again, the series has only positive terms so we can write the ratio test without absolute signs.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n$$

$$= e > 1$$

Therefore, by the ratio test the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

When applying the ratio test to series involving quotients of expressions with factorials, it is often necessary to perform simplification steps similar to those we did in Example 9:

$$\frac{n!}{(n+1)!}=\frac{\cancel{n!}}{(n+1)\cancel{n!}}=\frac{1}{n+1}$$

Although the ratio test worked in Example 9 part b) we could have used the *n*th term divergence test to prove that the series diverges by considering the following:

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}=1+\frac{2^2}{2}+\frac{3^3}{6}+\frac{4^4}{24}+\ldots \text{ and for the } n\text{th term } a_n=\frac{n\cdot n\cdot n\cdot\ldots\cdot n}{1\cdot 2\cdot 3\cdot\ldots\cdot n}\geqslant n$$

Thus as $n\to\infty$ the terms do not approach 0 and the series diverges by the *n*th term divergence test. It is often the case that we can determine whether or not a series converges by more than one test. The summary at the end of this section gives some tips on how to find the most efficient test to apply for a certain series.

As we will learn even further in the next section, the ratio test is useful in answering questions about convergence, as in the following example.

Example 10

For what values of x will the series $\sum_{n=1}^{\infty}\frac{2^n}{nx^n}$ converge?

Solution

Applying the ratio test gives the following inequality to solve.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{2^{n+1}}{(n+1)x^{n+1}}}{\dfrac{2^n}{nx^n}}\right|<1$$

$$\lim_{n\to\infty}\left|\frac{2^n 2}{(n+1)x^n x}\cdot\frac{nx^n}{2^n}\right|<1$$

$$\lim_{n\to\infty}\left|\frac{2n}{x(n+1)}\right|<1$$

$$\frac{1}{|x|}\cdot\lim_{n\to\infty}\left|\frac{2n}{n+1}\right|<1$$

$$\frac{2}{|x|}<1$$

$$|x|>2$$

Therefore, the series converges for any values of x such that $x<-2$ or $x>2$.

Note that when $x<-2$ the terms of $\sum_{n=1}^{\infty}\frac{2^n}{nx^n}$ will alternate between positive and negative. The ratio test is very useful in analyzing series with alternating terms that we will take a closer look at in the next section.

2.3 Alternating series and absolute convergence

Although we have encountered series whose terms alternate between positive and negative, the four convergence tests we have established – with the exception of the ratio test – apply only to series with positive terms.

We have encountered series with some negative terms, for example these alternating series:

$$-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}+\ldots+\frac{(-1)^n}{3^n}+\ldots \qquad \text{[Example 2 a)]}$$

$$3-3+3-3+3-3+\ldots+3(-1)^{n+1}+\ldots \qquad \text{[Example 4 b)]}$$

Series such as these, having terms that are alternately positive and negative, are called **alternating series.** The first series above is a geometric series with $r=-\frac{1}{3}$, so it converges to $\frac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)}=-\frac{1}{4}$, and the second series diverges by the nth term divergence test. But not all alternating series will be geometric nor satisfy the conditions of the nth term divergence test. The following test can be used to determine the convergence of a wider range of alternating series that satisfy certain conditions.

These examples show that an alternating series can be written in one of two forms: $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where the first term is positive, or $\sum_{n=1}^{\infty}(-1)^{n}a_n$ where the first term is negative such that a_n is a positive number. The alternating series test is stated using the first form. Since $(-1)\left[(-1)^n a_n\right]=(-1)^{n+1}a_n$ then the test also holds true for the second form by applying the property of convergent series: if $\sum_{n=1}^{\infty}a_n$ converges, and c is some real constant, then $\sum_{n=1}^{\infty}ca_n$ also converges.

Alternating series test

The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\ldots \qquad (a_n>0)$$

converges if both of the following conditions are satisfied.

1. $\lim_{n\to\infty}a_n=0$
2. $a_{n+1}\leqslant a_n$ for all $n\geqslant N$, for some positive integer N.

Proof

Consider the sequence of partial sums $\{s_n\}$ for an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_i$ such that $a_{n+1}\leqslant a_n$ and $\lim_{n\to\infty}a_n=0$. Figure 2.4 shows a graph of a few terms of $\{s_n\}$.

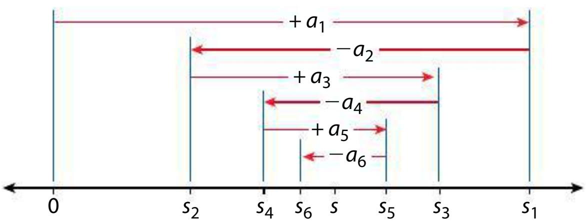


Figure 2.4 Convergence of the partial sums of an alternating series to their limit s.

Observe how the alternating signs of the terms of the series cause the partial sums to be alternately larger and smaller. As n increases the points corresponding to the nth partial sum 'jump' back and forth on either side of their limit s, gradually closing in as the value of the terms go to zero. Also observe that as these 'jumps' get smaller and smaller (because $a_{n+1} \leqslant a_n$), the odd-numbered terms in the sequence of partial sums, $\{s_{2n+1}\}$, form a decreasing sequence and the even-numbered terms, $\{s_{2n}\}$, form an increasing sequence. Furthermore, the decreasing sequence of odd-numbered terms has s_2 as a lower bound and thus, by the bounded sequence theorem, has a limit – call it L_1. Similarly, the sequence of even-numbered terms has s_1 as an upper bound and must also have a limit – call it L_2. If we can show that $L_1 = L_2$ then the series converges to a unique limit $s = L_1 = L_2$. With L_1 the limit of the sequence of odd-numbered terms in the sequence of partial sums and L_2 the limit of the even-numbered terms, it follows that

$$s_{2n} \leqslant L_2 \leqslant L_1 \leqslant s_{2n+1}$$

and

$$s_{2n+1} - s_{2n} = a_{2n+1}$$

Since

$$\lim_{n\to\infty} a_{2n+1} = 0$$

then it must follow that

$$L_1 = L_2$$

Therefore, the sequence of partial sums of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges.

When applying the alternating series test, it is best to verify condition (1) first. If condition (1) fails, that is, $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the nth term divergence test. If $\lim_{n\to\infty} a_n = 0$ but condition (2) fails then the alternating series test is inconclusive. There is another condition implied that must be met – that the series is truly (eventually) alternating. If this is not obvious by inspection then the easiest way to verify that a series written in the form $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ or $\sum_{n=1}^{\infty}(-1)^{n} a_n$ is alternating is to show that $a_n > 0$ for all $n \geqslant N$, for some positive integer N.

Example 11 – Using the alternating series test

Determine the convergence or divergence of each series.

a) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n^2+1}$ b) $\sum_{n=1}^{\infty}\dfrac{(-1)^n 2n}{3n-1}$ c) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\ln n}{n}$

Solution

a) The series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n^2+1}$ is alternating because $\dfrac{n}{n^2+1} > 0$ for $n \geqslant 1$.
Condition (1) is easily verified.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{n/n^2}{n^2/n^2+1/n^2} = \frac{0}{1+0} = 0$$

Now, let's attempt to satisfy condition (2) by proving the inequality $a_{n+1} \leqslant a_n$ for $a_n = \dfrac{n}{n^2+1}$.

$$\frac{n+1}{(n+1)^2+1} \leqslant \frac{n}{n^2+1}$$

$$(n+1)(n^2+1) \leqslant n\left[(n+1)^2+1\right]$$

Cross-multiplying; both denominators are positive.

$$n^3+n^2+n+1 \leqslant n^3+2n^2+2n$$

$$1 \leqslant n^2+n$$

$$n(n+1) \geqslant 1$$

Since $n \geqslant 1$, then the inequality $n(n+1) \geqslant 1$ is true. Hence, $a_{n+1} \leqslant a_n$ and condition (2) is satisfied. Therefore, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$ converges by the alternating series test.

b) The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{3n-1}$ is alternating since $\dfrac{2n}{3n-1} > 0$ for all $n \geqslant 1$, but

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{3n-1} = \frac{2}{3} \neq 0$ so condition (1) is not satisfied.

Applying the nth term divergence test, we need to find the limit of the nth term as $n \to \infty$.

$$\lim_{n\to\infty}\frac{(-1)^n 2n}{3n-1} = \lim_{n\to\infty}(-1)^n \cdot \lim_{n\to\infty}\frac{2n}{3n-1}$$

$\displaystyle\lim_{n\to\infty}\frac{2n}{3n-1} = \frac{2}{3}$ but $\displaystyle\lim_{n\to\infty}(-1)^n$ does not exist (Example 1 a)), so

$\displaystyle\lim_{n\to\infty}\frac{(-1)^n 2n}{3n-1}$ does not exist. Therefore, the series diverges by the nth term divergence test.

c) $a_n = \dfrac{\ln n}{n} > 0$ for all integers $n \geqslant 2$, so the series is alternating.

Checking condition (1) we can evaluate the following limit using l'Hôpital's rule because it has the indeterminate form $\dfrac{\infty}{\infty}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{d}{dx}(\ln n)}{\frac{d}{dx}(n)} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = 0$$

Hence, condition (1) is satisfied.

For condition (2) we must show that the sequence given by $a_n = \dfrac{\ln n}{n}$ is decreasing. It is not obvious whether this is true so we consider the derivative of the related function $f(x) = \dfrac{\ln x}{x}$.

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0 \text{ for all } x > e$$

Hence, f is decreasing for $x > e$ which means that $f(n+1) < f(n)$, so it follows that $a_{n+1} \leqslant a_n$ for $n \geqslant 3$.

Therefore, both conditions of the alternating series test have been satisfied and the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ is convergent.

Take another look at Figure 2.4 that was used in the proof of the alternating series test. Recalling that s is the limit of the partial sums, notice that $|s - s_3| < a_4$, $|s - s_4| < a_5$, $|s - s_5| < a_6$, etc. Furthermore, note that s is always between any two consecutive partial sums. This provides us with the means to estimate the error when we use the partial sum s_n to approximate the sum of an alternating series.

Alternating series estimation theorem

Suppose that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is a convergent alternating series that satisfies both conditions of the alternating series test and has an unknown sum of $\boldsymbol{s}$. When estimating $\boldsymbol{s}$ with the sum of the first n terms, the absolute value of the remainder R_n (i.e. the amount of error) is less than or equal to the first unused term. That is,

$$|R_n| = |s - s_n| \leqslant a_{n+1}.$$

In other words, the error generated in estimating the sum with the nth partial sum does not exceed the value of the $n + 1$ term.

Proof

As previously mentioned, the sum, s, of a convergent alternating series is always between any two consecutive partial sums. That is,

$s_n \leqslant s \leqslant s_{n+1}$, if n is even and $s_{n+1} \leqslant s \leqslant s_n$, if n is odd.

Whether n is even or odd, it follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n|.$$

Given that

$a_{n+1} = |s_{n+1} - s_n|$ Remember $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is an alternating series, so $a_{n+1} > 0$.

$|s - s_n| \leqslant a_{n+1}$

and therefore the proof is complete.

Note that the **alternating series estimation theorem** does **not** give a formula for the precise value of the error, but rather a *bound* for the error. Also note that this rule for the bound of the error when estimating s with s_n only applies to alternating series that satisfy the condition of the **alternating series test**.

Example 12

Show that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ converges, and find the sum of the series with error less than 0.0001.

Solution

Since $\lim_{n\to\infty}\frac{1}{n^4} = 0$ and $\frac{1}{(n+1)^4} \leqslant \frac{1}{n^4} \Rightarrow n^4 \leqslant (n+1)^4$ is true for all $n \geqslant 1$, the series satisfies both conditions of the alternating series test and therefore converges.

We know from the alternating series estimation theorem that the sum of the first nine terms will give an estimate for the sum with an error of at most

$$a_{9+1} = \frac{1}{10^4} = 0.0001.$$

Our GDC computes the ninth partial sum to be

$$s_9 = -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{8^4} - \frac{1}{9^4} \approx -0.947\,092\,5924.$$

```
-1+1/2^4-1/3^4+1
/4^4-1/5^4+1/6^4
-1/7^4+1/8^4-1/9
^4
                 -.947095924
```

This estimate of the sum of the series is accurate to three decimal places because an error of less than 0.0001 does not affect the third decimal place. Therefore, the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is $s \approx -0.947$, correct to three decimal places.

Example 13

Determine the convergence or divergence of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Solution

Applying the alternating series test we have

1 $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

and

2 $a_{n+1} \leqslant a_n \;\Rightarrow\; \frac{1}{n+1} \leqslant \frac{1}{n} \;\Rightarrow\; n \leqslant n+1$ which is true for all n.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

Absolute and conditional convergence

In the next section, we will learn that the alternating harmonic series converges to exactly $\ln 2$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

But more relevant to this section is that the result of Example 13 illustrates an important point to investigate further. We know that the harmonic series (a p-series with $p = 1$) diverges. However, if we take the harmonic series and change the sign of alternate terms to get the alternating harmonic series (Example 13), the positive and negative terms offset one another to produce a series that converges even though the series consisting of only positive terms diverges. The same situation is true of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You may recall that in Example 5 d) we used the integral test to prove that this series diverges. However, in Example 11 a) of this section we showed that the corresponding alternating series

Absolute and conditional convergence

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive and negative terms that is convergent.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent**.

If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is said to be **conditionally convergent**.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ converges. In contrast, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ (Example 12) converges and so does the corresponding series with positive terms $\sum_{n=1}^{\infty} \frac{1}{n^4}$ (a p-series with $p = 4 > 1$). The difference between these two situations requires us to define two types of convergence when considering the convergence of a series with positive and negative terms as occurs with any alternating series.

We have seen then that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (alternating harmonic series) and $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ are both **conditionally convergent** because for each the series composed of their terms all made positive diverges. Whereas $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is **absolutely convergent** because its corresponding series of positive terms also converges. You may wonder if it is possible for a series with positive terms, $\sum_{n=1}^{\infty} |a_n|$, to converge, but for a related series with some (or all) of the terms changed to negative, $\sum_{n=1}^{\infty} a_n$, to diverge. The answer is no, and we state the following theorem.

Absolute convergence theorem

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges, and therefore $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

The **absolute convergence theorem** essentially says that it is not possible to take a *convergent* series with *only positive* terms and change some of them to negative to create a new series that is divergent. However, as the alternating harmonic series demonstrates, it is possible to take a convergent series with *positive and negative* terms and change them all to positive to create a new series that is divergent.

Proof

It is true that $0 \leqslant a_n + |a_n| \leqslant 2|a_n|$ because by the definition of absolute value $|a_n|$ is either a_n or $-a_n$. A given condition for the theorem is that $\sum_{n=1}^{\infty} |a_n|$ converges, so $\sum_{n=1}^{\infty} 2|a_n|$ also converges. Therefore, by the comparison test $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. Since $a_n = (a_n + |a_n|) - |a_n|$, it follows from properties for convergent series that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$ where both series on the right converge. Therefore, $\sum_{n=1}^{\infty} a_n$ must converge. Q.E.D.

When trying to determine if an alternating series is absolutely convergent, conditionally convergent, or divergent, it is most effective to first check if the limit of the nth term is zero. If it is not then the series diverges, and you are finished. If the nth term divergence test is inconclusive then check whether the related series of positive terms converges (using any of the four tests for positive series). If it converges, then by the absolute convergence theorem, the series is absolutely convergent and you are finished. If it diverges, then test the alternating series using the alternating series test. It is inefficient to start by first applying the alternating series test.

Example 14

Classify each series as absolutely convergent, conditionally convergent, or divergent.

a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$

b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$

c) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

Solution

a) $\lim_{n\to\infty}\frac{(-1)^n}{n!}=0$, so result of nth term divergence test is inconclusive. We next consider the corresponding series with only positive terms $\sum_{n=1}^{\infty}\frac{1}{n!}$. Recall that in Example 7 c), we used the comparison test to show $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges. We now apply the alternating series test to $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$. Knowing $\lim_{n\to\infty}\frac{1}{n}=0$, and since $0<\frac{1}{n!}<\frac{1}{n}$ for all $n\geqslant 1$ then $\lim_{n\to\infty}\frac{1}{n!}=0$. Thus $\lim_{n\to\infty}a_n=0$. We now need to show that $\frac{1}{(n+1)!}\leqslant\frac{1}{n!}\Rightarrow n!\leqslant(n+1)!$. Rewriting $(n+1)!$ as $n!(n+1)$ gives $n!\leqslant n!(n+1)$ which is clearly true for all $n\geqslant 1$. Thus $a_{n+1}\leqslant a_n$, and we have satisfied both conditions of the alternating series test. Hence, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ converges and converges absolutely because $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n!}\right|=\sum_{n=1}^{\infty}\frac{1}{n!}$ also converges.

b) We can apply the nth term divergence test to show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$ diverges. Recall that the absolute value theorem stated that if $\lim_{n\to\infty}|a_n|=0$ then $\lim_{n\to\infty}a_n=0$. From this we can also say that if $\lim_{n\to\infty}|a_n|\neq 0$ then $\lim_{n\to\infty}a_n\neq 0$. We apply l'Hôpital's rule twice to prove that $\lim_{n\to\infty}\frac{2^n}{n^2}\neq 0$.

$$\lim_{n\to\infty}\frac{2^n}{n^2}=\lim_{n\to\infty}\frac{2^n\ln 2}{2n}=\lim_{n\to\infty}\frac{2^n(\ln 2)^2}{2}=\infty\text{ (does not exist)}$$

Therefore, by the nth term divergence test $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$ diverges.

c) In Example 8 d) we compared the series $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ to the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ and using the limit comparison test showed

Note: For Example 14 a), we could have been more efficient by applying the absolute convergence theorem since we have previously used the comparison test to show that $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n!}\right|=\sum_{n=1}^{\infty}\frac{1}{n!}$ converges.

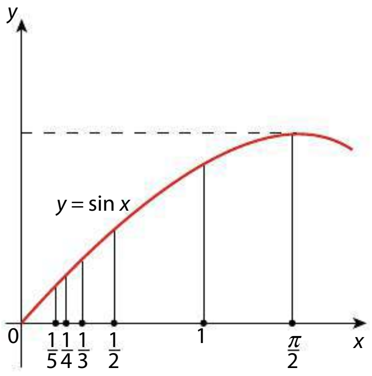


Figure 2.5

that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges. We now turn our attention to the given series and first need to confirm whether it is an alternating series. Since $\sin\left(\frac{1}{n}\right) > 0$ for all $n \geqslant 1$ then $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$ is an alternating series and we can apply the test for alternating series.

The graph shown in Figure 2.5 provides confirmation that not only $\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = 0$, but also that $\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$ for all $n \geqslant 1$. Thus the series satisfies the alternating series test and converges. Since the corresponding series of positive terms, $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$, diverges, $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$ converges conditionally.

Rearrangements of conditionally convergent series

The distinction between absolute and conditional convergence is important in many applications of infinite series. It seems perfectly logical that it is possible to rearrange a finite number of terms in an infinite series without affecting the sum. However, if we rearrange an infinite number of terms in an infinite series, the sum is unchanged only if the series is absolutely convergent. An extraordinary characteristic of series that are conditionally convergent is that their terms can be rearranged to form a divergent series, and even rearranged to form a series that converges to *any* predetermined sum. This is a direct consequence of the fact that the sum of an infinite series is defined to be the limit of the sequence of its partial sums. As mentioned previously, this means that operations (such as the associative property) that are valid for finite sums are not valid for infinite sums.

We can illustrate this paradoxical behaviour with the alternating harmonic series that is conditionally convergent. As stated earlier without explanation (next section), the sum of the alternating harmonic series is ln 2.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\ldots=\ln 2 \qquad (1)$$

Consider the following series:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots \qquad (2)$$

(2) consists of a rearrangement of the same terms as in (1). It is plausible to expect that the sum of the series in (2) is also ln 2.

Let's continue by dividing (1) by 2, giving:

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\ldots=\frac{1}{2}\ln 2 \qquad (3)$$

Now we add (3) and (1):

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\ldots=\ln 2 \qquad (1)$$

$$\frac{1}{2}\qquad-\frac{1}{4}\qquad+\frac{1}{6}\qquad-\frac{1}{8}\qquad+\frac{1}{10}\qquad-\ldots=\frac{1}{2}\ln 2 \qquad (3)$$

The result is

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots=\frac{3}{2}\ln 2 \qquad (4)$$

where the terms are arranged the same as in (2), but the sum is not what we expected. So which is correct, (1) or (4)? The answer is that they are both correct. Although both (1) and (4) are series containing the same terms, by rearranging the terms we have manipulated how the partial sums are formed which affects the limit of the partial sums and, consequently, affects the sum of the series.

Table 2.1 Tests for infinite series.

Test	Converges	Diverges	Notes
nth term divergence test $\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	Can only be used to show divergence
Geometric series $\sum_{n=0}^{\infty} a_1 r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geqslant 1$	$S_\infty = \frac{a_1}{1-r}$
p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leqslant 1$	Harmonic series when $p = 1$
Integral test $\sum_{n=1}^{\infty} a_n;\ a_n = f(n)$ f is continuous, positive and decreasing	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$s_n + \int_{n+1}^{\infty} f(x)\,dx$ and $s_n + \int_n^{\infty} f(x)\,dx$ are bounds for estimation of sum by s_n
Comparison test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ $0 \leqslant a_n \leqslant b_n$	$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges	$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges	Useful for series similar to p-series or geometric series
Limit comparison test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ $a_n > 0, b_n > 0$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$; if $0 < L < \infty \Rightarrow$ both behave the same $L = 0 \Rightarrow$ if b_n converges then a_n converges $L = \infty \Rightarrow$ if b_n diverges then a_n diverges		Useful if not able to show $0 \leqslant a_n \leqslant b_n$ for direct comparison
Ratio test $\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Inconclusive if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$
Alternating series test $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$\lim_{n\to\infty} a_n = 0$ and $0 < a_{n+1} < a_n$		s_n as estimate of sum remainder: $\lvert R_n\rvert < a_{n+1}$

Guidelines for testing series for convergence

Important questions to consider:

1. Is $\lim_{n\to\infty} a_n = 0$? If not, the series diverges by the nth term divergence test.
2. Is the series geometric, or similar to a geometric series? If similar, apply one of the comparison tests.
3. Is the series a p-series, or similar to a p-series? If similar, apply one of the comparison tests.
4. Consider $a_n = f(n)$. Is f a continuous, positive, decreasing function and is it possible to integrate f? If so, try integral test.
5. Does a_n involve n in a product or power, or has an expression with factorials? If so, try the ratio test.
6. Is the series an alternating series? If so, try the alternating series test. Remember that if $\sum |a_n|$ is convergent then $\sum a_n$ is absolutely convergent. Testing $\sum |a_n|$ makes more tests available.

Exercise 2

1 Using properties of convergent series and geometric series, find the sum of each of the series.

a $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{2^{3n}}$ **b** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{2}{3^n}\right)$ **c** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5^n + 3(2^{3n})}{9^n}\right)$

In questions 2–9, write the first four terms of the infinite series and determine whether the series is convergent or divergent. If the series is convergent, find its sum.

2 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$

4 $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$

5 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3}{2^n}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$

7 $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi)$

8 $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{5n+6}$

9 $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

10 a Find $\int xe^{-x}dx$ by using the method of integration by parts.

b Use the integral test to determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ is convergent or divergent.

11 Use the integral test to determine whether the series is convergent or divergent.

a $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ **b** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

12 Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+3}$ diverges by both of the following methods.

a Using the comparison test, compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$.

b Using the limit comparison test, compare the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

13 Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges by **a** the comparison test, and **b** the ratio test.

14 Give an example to show that the converse of the nth term divergence test is false. That is, find a series that diverges even though $\lim_{n\to\infty} a_n = 0$.

15 Use the ratio test to show that $\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$ converges.

In questions 16–29, determine the convergence or divergence of the series.

16 $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

17 $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

18 $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

19 $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

20 $\sum_{n=0}^{\infty} \frac{n!}{(n+2)!}$

21 $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

22 $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2n+1}$

23 $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$

24 $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

25 $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$

26 $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n^2+n}$

27 $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$

28 $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

29 $\sum_{n=0}^{\infty} \frac{1}{e^n}$

30 Use the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converges or diverges.

31 Find the sum of the following infinite series.

$$\frac{5}{1\times 2} + \frac{5}{2\times 3} + \frac{5}{3\times 5} + \ldots$$

32 For each series, use the sum of the first four terms to approximate the sum of the series. State an upper bound for the error of the approximation.

a $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)^2}$

b $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

33 **a** Express $n^2 + 2n + 2$ in the form $(n+a)^2 + b$ where a and b are integers.

b Use the integral test to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$ converges or diverges.

34 Determine whether $\sum_{n=1}^{\infty} \frac{\arctan n}{n}$ converges or diverges by comparing the series to $\sum_{n=1}^{\infty} \frac{1}{n}$ and applying the limit comparison test.

35 Use the alternating series estimation theorem to determine the minimum number of terms of the series $1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \ldots$ so that an approximation of the sum has an error less than 0.000 05.

36 Give an example of a series that is conditionally convergent. That is, a series that is convergent but not absolutely convergent.

In questions 37–42, determine whether each series converges absolutely, converges conditionally, or diverges.

37 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ **38** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

39 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(n+1)^2}$ **40** $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

41 $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$ **42** $\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}}$

43 Describe how the terms of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ can be rearranged so that its sum is 1.

44 What is the minimum number of terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ needed to approximate the sum of the series correct to three decimal places?

45 Prove parts 2 and 3 of the limit comparison test.

Practice questions 2

1 For each infinite series below, determine whether or not the infinite series converges or diverges. Clearly state/explain your reasoning.

a $1 + \frac{1}{1.1} + \frac{1}{1.21} + \frac{1}{1.331} + \frac{1}{1.4641} + \ldots + \frac{1}{(1.1)^{n-1}} + \ldots$

b $e + e^{\frac{1}{2}} + e^{\frac{1}{3}} + e^{\frac{1}{4}} + \ldots$

c $3 + \frac{3}{8} + \frac{3}{27} + \frac{3}{64} + \ldots + \frac{3}{n^3} + \ldots$

2 For each infinite series below, use the indicated convergence test to determine whether the infinite series converges or diverges.

a $\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots + \frac{n^3}{n!} + \ldots$ [Ratio test]

b $\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \ldots + \frac{n}{(2n-1)(2n+1)} + \ldots$ [Integral test]

3 By using the Limit Comparison Test, prove that the *general* harmonic series $\sum_{n=1}^{\infty} \frac{1}{an+b}$ diverges for any $a > 0$ and $b > 0$.

4 Test the convergence or divergence of the following infinite series, indicating the tests used to arrive at your conclusion.

a $\sum_{k=1}^{\infty} \frac{k+1}{3^k}$ **b** $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ **c** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$

5 Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1+\frac{1}{n}}} \right)$ converges.

6 **a** Describe how the integral test is used to show that a series is convergent. Clearly state all the necessary conditions.

b Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges.

7 Find the range of values of x for which the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

8 Determine whether the alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n+1}$ converges conditionally, converges absolutely or diverges.

9 Use the integral test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

10 Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

a Show that the series is convergent.

b i Express $\frac{1}{n(n+2)}$ in partial fractions.

ii Hence find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

11 Find the interval of convergence of the series $\sum_{n=1}^{\infty} \sin \frac{\pi}{n} x^n$.

12 Determine whether each of the following series converges or diverges.

a $\sum_{n=1}^{\infty} \frac{e}{\sqrt{n}}$ **b** $\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **c** $\sum_{n=1}^{\infty} \frac{2^{3n-1}}{3^{2n+4}}$

13 Show that the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ is convergent but not absolutely convergent.

14 Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{3n^2+1}$ is convergent.

15 Consider the infinite series $-1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{2}-\frac{1}{\sqrt{5}}+\cdots$

a Show that the series converges.

b Determine if the series converges absolutely or conditionally.

3 Power Series

3.1 Power series

Have you ever considered how your calculator computes values for certain functions? For functions such as $f(x) = 3x^2 - 5x + 8$, $g(x) = \dfrac{x^4 + 2x}{-4x^3 + x^2 - 6}$, and $h(x) = \sqrt{7x - 3}$ the method of evaluation is fairly straightforward because these are **algebraic functions**. As explained in Chapter 3 of the book, algebraic functions can be expressed as a finite number of sums, differences, multiples, quotients and radicals involving x^n. Polynomial functions, rational functions and functions involving radicals are examples of algebraic functions. But how does your calculator compute values for a function such as e^x? This is an example of a **transcendental function**. A transcendental function is non-algebraic, i.e. it *cannot* be expressed as a finite number of sums, differences, multiples, quotients and radicals involving x^n. Other familiar transcendental functions include the trigonometric and logarithmic functions.

Except for Example 10 in the previous chapter, all the series we have encountered thus far contained terms consisting of constants. A power series is essentially a polynomial function of infinite degree expressed in terms of a single variable (we will always use x).

Let's return to the primary question we wish to investigate. How does your calculator compute the values of transcendental functions, such as e^x? The manufacturers of the calculator had to decide on a computational algorithm. What computational method could be programmed into a calculator to evaluate e^x for a certain value of x?

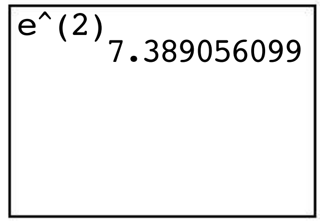
The answer lies in the fact that the calculator is summing up a type of series with variable terms, called a **power series**, that is representing e^x. In this section we will see that the power series for the function $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$. A calculator can only display a finite number of digits and thus it only sums enough terms to produce the necessary degree of accuracy. For example, suppose we wanted to use this series (we'll investigate its derivation later) to evaluate e^2 to three significant figures.

A power series is a very useful mathematical tool that can be used to represent a range of very important functions.

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \ldots$$

Shown below are successively more accurate approximations for the value of e^2 by summing the terms of the power series $\sum_{n=0}^{k} \frac{x^n}{n!}$ for $k = 2, 3, \ldots, 9$.

Once we get past the ninth term in the series we are no longer adding enough to change the first three digits. Thus, the first nine terms of the series are sufficient to give an approximation of e^2 accurate to three significant figures.



$$e^2 \approx 1 + 2 = 3$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} = 5$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} = 6\tfrac{1}{3} = 6.\overline{3}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} = 7$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} = 7\tfrac{4}{15} = 7.2\overline{6}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} = 7\tfrac{16}{45} = 7.3\overline{5}$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} = 7\tfrac{8}{21} = 7.3\overline{809523} \approx 7.38$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} = 7\tfrac{122}{315} = 7.38\overline{73015} \approx 7.39$$

$$e^2 \approx 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} + \frac{2^9}{9!} = 7\tfrac{1102}{2835} = 7.38871252205\ldots \approx 7.39$$

A calculator (see screen image above) computes to an accuracy of ten significant figures the value of e^2 to be 7.389 056 099. It certainly appears that the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ is converging to e^2. For any given value of x, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is an infinite series. This leads to an important question: For what values of x does the power series converge?

Before addressing this question, let's give a proper definition for a power series.

Performing such computations entirely by hand would be immensely tedious (and prone to error). However, this is not an impediment for an electronic computing device like a GDC. As we will see, the computation process is made more efficient by means of a formula that determines the number of terms required for a power series to produce a value to a given accuracy.

Definition of power series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$

is called a **power series centred at *c***, where c is a constant and a_n is the rule that determines each of the coefficients $a_0, a_1, a_2, \ldots$. Note that we have $(x-c)^\circ = 1$ even when $x = c$.

For any power series centred at $c = 0$, we have $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$.

Radius of convergence

At each value of x, a power series becomes a series of constants. In the previous section we gave a great deal of attention to such series, investigating whether they converge or diverge. The issue of convergence is very important for power series because for each value of x for which a power series converges, the series represents the number that is the sum of the series. Therefore, **a power series defines a function**. The function $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has as its domain all values of x for which the power series converges. It is evident that every power series is convergent for $x = c$. Some power series are only convergent at $x = c$ (see Example 3). Far more useful power series will converge for a finite interval with the same centre as the series (see Example 1), or converge for all x (see Example 2).

Example 1

For the general power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, if we let $a_n = 1$ for all n and 'centre' the series at $c = 0$, we get the geometric series

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$ having first terms $a_1 = 1$, and common ratio $r = x$.

The sum formula for geometric series assures us that this series converges to $\frac{1}{1-x}$ when $-1 < x < 1$, and consequently diverges when $|x| \geqslant 1$.
Therefore, we can write

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}, \qquad -1 < x < 1.$$

The expression on the right side of this equation defines a function whose domain is $x \in \mathbb{R}$, $x \neq 1$. The expression on the left side defines a function whose domain is the interval $-1 < x < 1$. The equation can only be true where both sides are defined, so its domain is $-1 < x < 1$, equivalent to $|x| < 1$. On this domain, the given power series is a **polynomial representation** of the function $f(x) = \frac{1}{1-x}$ (Figure 3.1, on next page). A power

series is best regarded as an attempt to describe a function *locally*, near where it is 'centred', i.e. near the value of c. To illustrate this point, partial sums of the series $\sum_{n=0}^{\infty} x^n$ with 3, 6 and 9 terms have been graphed in Figure 3.2. Figure 3.3 shows the same three partial sums along with $f(x) = \frac{1}{1-x}$ focused on the interval $-1 < x < 1$.

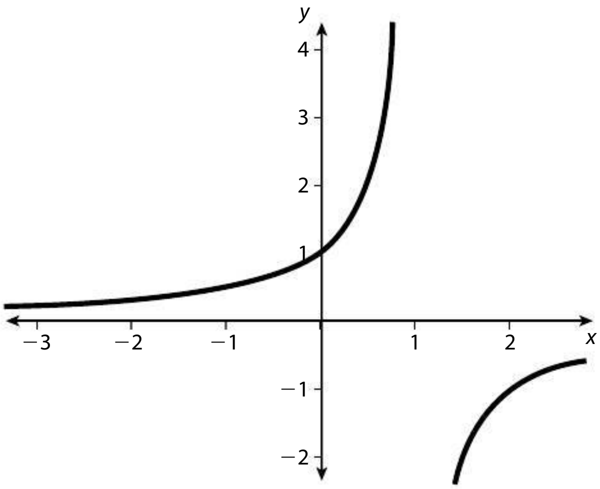


Figure 3.1 Graph of $y = \dfrac{1}{1-x}$.

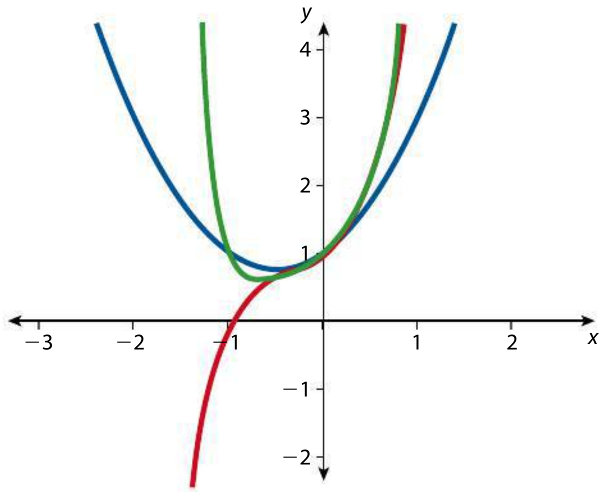


Figure 3.2 Graphs of the partial sums $1 + x + x^2$, $1 + x + x^2 + x^3 + x^4 + x^5$ and $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$.

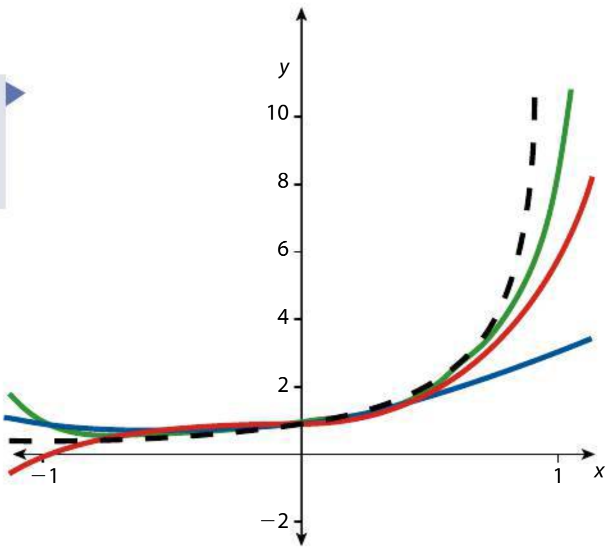


Figure 3.3 The partial sums $1 + x + x^2$, $1 + x + x^2 + x^3 + x^4 + x^5$, $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$ and $\dfrac{1}{1-x}$ (dashed).

Observe how in the interval $-1 < x < 1$ the graph of a partial sum of $\sum_{n=0}^{\infty} x^n$ gets closer to that of the graph of $f(x) = \dfrac{1}{1-x}$ as the number of terms increase, but are not close outside this interval.

Example 2

We've demonstrated that the power series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots + \dfrac{x^5}{n!} + \ldots$ represents the function $f(x) = e^x$. Find the values of x for which this power series converges.

Solution

Example 10 in the previous chapter illustrated that the ratio test is effective for answering this kind of question. Applying the ratio test gives the following inequality to solve.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{x^n}{n!}}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{x^n x}{(n+1)n!}\cdot\frac{n!}{x^n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{x}{n+1}\right| < 1$$

$$|x|\cdot\lim_{n\to\infty}\left|\frac{1}{n+1}\right| < 1$$

$$|x|\cdot 0 < 1$$

$$0 < 1$$

Therefore, the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is convergent for all real values of x.

This means that as we compute partial sums of more and more terms for $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ they will become ever more accurate representations of the function $f(x) = e^x$ over all real numbers, and not just constrained to a finite interval as occurred in the previous example. This is illustrated by comparing graphs of $f(x) = e^x$ (Figure 3.4) to graphs of partial sums of $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ with 4, 5 and 10 terms in Figure 3.5.

Figure 3.4 Graph of $y = e^x$.

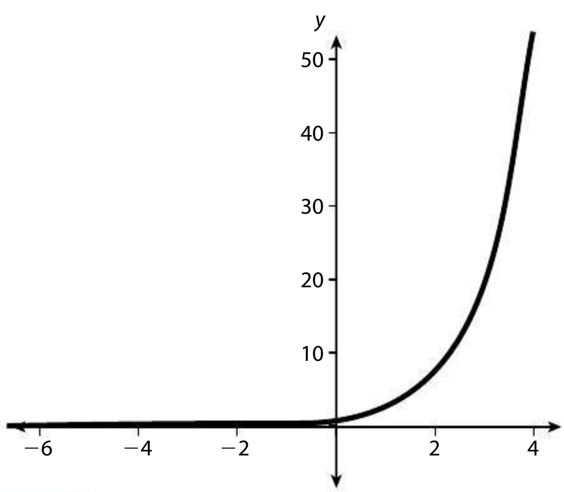
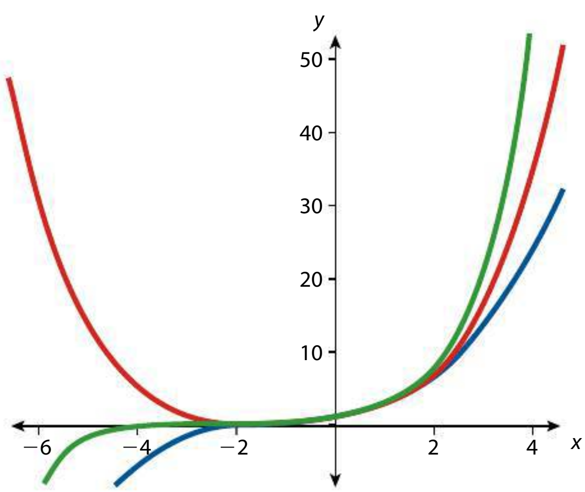


Figure 3.5 The partial sums $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}$.



Example 3

Find the values of x for which $\sum_{n=0}^{\infty} n!x^n$ is convergent.

Solution

Again, applying the ratio test gives:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}|x(n+1)| \\ &= |x|\lim_{n\to\infty}(n+1) \\ &= \infty, \quad x \neq 0\end{aligned}$$

Hence, the series diverges for all values of x, $x \neq 0$. We need to check the series when $x = 0$.

$$\sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \ldots = 1$$

Therefore, the power series $\sum_{n=0}^{\infty} n!x^n$ converges only at its centre, at $x = 0$.

As illustrated in the preceding three examples, the domain of a power series can be a single point, an interval of the real numbers centred at c, or all real numbers. The following theorem (which we present without proof) states that these are the only possibilities.

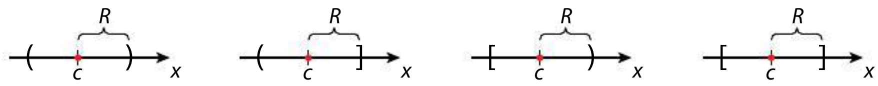
Convergence of a power series theorem

For a given power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ exactly one of the following is true:

1. The series converges only when $x = c$.
2. The series converges for all real values of x.
3. There is a positive number R such that the series converges for $|x - c| < R$, and diverges for $|x - c| > R$. The series may or may not converge at either of the endpoints $x = c - R$ and $x = c + R$ so we need to check for convergence at each of the endpoints.

The set of all values of x for which a given power series is convergent is called the **interval of convergence** of the power series. The number R of possibility **3** in the theorem is called the **radius of convergence** of the power series. If possibility **1** occurs, then $R = 0$; and if possibility **2** occurs, then $R = \infty$. For possibility **3**, there remains the question of what happens at the endpoints of the interval. Each endpoint must be tested separately to determine if the series converges or diverges for that value of x. Thus, if $0 < R < \infty$ the interval of convergence can be one of four different kinds, as illustrated in Figure 3.6.

Figure 3.6 Intervals of convergence.



$]c-R, c+R[$ $\quad]c-R, c+R]$ $\quad [c-R, c+R[$ $\quad [c-R, c+R]$

In Example 1 we showed that the interval of convergence for the power series $\sum_{n=0}^{\infty} x^n$ is $-1 < x < 1$, so $R = 1$. But we did not check the endpoints. At $x = -1$, $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$; and at $x = 1$, $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + \dots$. Both of these series diverge confirming that the interval of convergence is, in fact, $-1 < x < 1$. In Example 2, $R = \infty$; and, in Example 3, $R = 0$.

As we did in Examples 2 and 3, it is best to use the ratio test to determine the radius of convergence for a power series. The ratio test will fail when x is an endpoint of the interval of convergence, so you will need to check endpoints with one of the other tests from the previous chapter.

Why are we interested in power series? One reason is that power series share many of the desirable properties of polynomials. In particular, polynomial functions are generally much easier to differentiate and integrate. The power series representation of a particular function may enable us to perform some difficult operations on the function that would otherwise be quite difficult, for example differentiation and integration of the function.

Example 4

Find the radius of convergence and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n(3^n)}$.

Solution

We apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{2(n+1)(3^{n+1})} \cdot \frac{2n(3^n)}{(x-4)^n} \right| = \lim_{n\to\infty} \left| \frac{2n(x-4)}{3(2n+2)} \right|$$

$$= \lim_{n\to\infty} \left(\frac{2n}{6n+6} \right) |x-4|$$

$$= \frac{1}{3}|x-4|$$

Hence, the series converges for $\frac{1}{3}|x-4| < 1$ or $|x-4| < 3$.

The radius of convergence is $R = 3$. The series is centred at $c = 4$, so the series converges for

$$4 - 3 < x < 4 + 3 \text{ or } 1 < x < 7.$$

At $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-3)^n}{2n(3^n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$, which converges by the alternating series test.

At $x = 7$, the series is $\sum_{n=1}^{\infty} \frac{(3)^n}{2n(3^n)} = \sum_{n=1}^{\infty} \frac{1}{2n}$, which diverges by limit comparison with the harmonic series. Therefore, the interval of convergence is $1 \leqslant x < 7$, or $x \in [1, 7[$.

3.2 Maclaurin and Taylor series

In the preceding part of this chapter we were given the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

without any indication about its origin. In this section we will devise a general method for constructing a power series.

As we stated in our work for Example 1, 'a power series is best regarded as an attempt to describe a function *locally*, near where it is "centred", i.e. near the value of c', and that 'a power series is essentially a **polynomial function** of infinite degree'. With theses ideas in mind, we propose the following.

To find a polynomial function P that represents another function f, begin by choosing a number c in the domain of f at which f and P have the same value. This is the first requirement – that $P(c) = f(c)$. Thus, the graphs of f and P will pass through the same point, $(c, f(c))$. The approximating polynomial P is said to be *expanded about c* or *centred at c*. Of course, there will be many polynomial functions that will have the same value as f at $x = c$. We can make the graph of P further resemble that of f near the point they share by a second requirement: P has the same slope of f at $x = c$, that is, $P'(c) = f'(c)$. We can continue to improve how well P mimics the behaviour of f near c by additionally requiring that $P''(c) = f''(c)$, $P^{(3)}(c) = f^{(3)}(c)$, and so on.

Deriving the power series for $f(x) = e^x$ centred at 0

We start by finding a first degree polynomial $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of $f(x) = e^x$ at $x = 0$, that is, $P(0) = f(0)$ and $P'(0) = f'(0)$.

$$f(0) = e^0 = 1 \qquad P_1(0) = a_0 + a_1(0) = 1 \Rightarrow a_0 = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1 \qquad P_1'(x) = a_1 \Rightarrow P_1'(0) = a_1 = 1 \Rightarrow a_1 = 1$$

Therefore, $P_1(x) = 1 + x$.

Now to find the second degree polynomial approximation $P_2(x) = a_0 + a_1x + a_2x^2$ we require that $P''(0) = f''(0)$, knowing that $a_0 = 1$ and $a_1 = 1$.

$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1 \qquad P_2'(x) = a_1 + 2a_2x$$

$$P_2''(x) = 2a_2 \Rightarrow P_2''(0) = 2a_2 = 1 \Rightarrow a_2 = \frac{1}{2}$$

Therefore, $P_2(x) = 1 + x + \frac{1}{2}x^2$.

We continue and find the third degree polynomial $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and require that $P^{(3)}(0) = f^{(3)}(0)$ (third derivatives are equal).

$$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = e^0 = 1 \quad P_3'(x) = a_1 + 2a_2x + 3a_3x^2$$
$$P_3''(x) = 2a_2 + 6a_3x$$
$$P_3^{(3)}(x) = 6a_3 \Rightarrow P_3^{(3)}(0) = 6a_3 = 1 \Rightarrow a_3 = \frac{1}{6}$$

Therefore, $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

From the function values for $f(x) = e^x$ (six significant figures), $P_2(x)$ and $P_3(x)$ displayed in Table 3.1, we can make two observations: (**1**) the accuracy of the approximating polynomial improves as $x \to 0$; and we would expect, in general, as $x \to c$; and (**2**) the higher the degree of the approximating polynomial (more terms of the partial sum of the power series), the better the polynomial represents the function *f*.

x	−1.0	−0.2	−0.1	0	0.1	0.2	1.0
$f(x) = e^x$	0.367 879	0.818 731	0.904 837	1	1.105 17	1.221 40	2.718 28
$P_2(x)$	0.5	0.82	0.905	1	1.105	1.22	2.5
$P_3(x)$	$0.\overline{3}$	$0.818\overline{6}$	$0.904\,8\overline{3}$	1	$1.105\,1\overline{6}$	$1.221\overline{3}$	$2.\overline{6}$

Table 3.1 Comparing $P_2(x) = 1 + x + \frac{1}{2}x^2$, $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$, and $f(x) = e^x$.

Let's apply the procedure one more time to find $P_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

$$f^{(4)}(x) = e^x \Rightarrow f^{(4)}(0) = e^0 = 1 \quad P_4'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$
$$P_4''(x) = 2a_2 + 6a_3x + 12a_4x^2$$
$$P_4^{(3)}(x) = 6a_3 + 24a_4x$$
$$P_4^{(4)}(x) = 24a_4 \Rightarrow P_4^{(4)}(0) = 24a_4 = 1 \Rightarrow a_3 = \frac{1}{24}$$

Therefore, $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$.

By now we can see the pattern for $P_n(x)$.

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$$

We complete the derivation of the power series for $f(x) = e^x$ by determining its interval of convergence by means of the ratio test and checking interval endpoints, if they exist. In this way we derive the result that was presented to us at the start of this chapter. That is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

General result

We now wish to generalize the method for finding the power series representation for a given function *f*(*x*). For expansions about an arbitrary value of *c*, it is convenient to write the polynomial in the standard form from the definition.

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \ldots + a_n(x - c)^n$$

where $P_n(x)$ is the nth partial sum of the infinite series

$$\sum_{n=0}^{\infty} a_n(x - c)^n.$$

We will streamline the procedure that we used for deriving the power series for $f(x) = e^x$. Repeated differentiation of P_n produces

$$P_n'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots + na_n(x-c)^{n-1}$$
$$P_n''(x) = 2a_2 + 2(3a_3)(x-c) + \ldots + n(n-1)a_n(x-c)^{n-2}$$
$$P_n^{(3)}(x) = 2(3a_3) + \ldots + n(n-1)(n-2)a_n(x-c)^{n-3}$$
$$\vdots$$
$$P_n^{(n)}(x) = n(n-1)(n-2)\ldots(2)(1)a_n$$

Evaluating P_n and its first n derivatives at $x = c$, we obtain the following:

$$P_n(c) = a_0 \qquad P_n'(c) = a_1 \qquad P_n''(c) = 2a_2 \qquad \ldots \qquad P_n^{(n)}(c) = n!a_n$$

Because the value of f and its first n derivatives must agree with the value of P_n and its first n derivatives at $x = c$, it follows that:

$$a_0 = f(c) \qquad a_1 = f'(c) \qquad a_2 = \frac{f''(c)}{2!} \qquad \ldots \qquad a_n = \frac{f^{(n)}(c)}{n!}$$

Consequently, we assert that if f has a power series representation centred at $x = c$, that is, if

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n, \quad |x-c| < R$$

then its coefficients are given by the formula

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

Substituting this formula for a_n into $\sum_{n=0}^{\infty} a_n(x-c)^n$ we determine that if a function f has a power series representation centred at $x = c$, then it will take the form as defined below.

Definition of Taylor series and Maclaurin series

If a function f has derivatives of all orders at $x = c$, then the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

is called the **Taylor series of the function *f* centred at *c*.** As often occurs, if $c = 0$, then the series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

and is called the **Maclaurin series for *f*.**

For the simplicity of notation we agree to define the 'zeroth' derivative, $f^{(0)}$, to be the function f itself.

Although the English mathematician Brook Taylor (1685–1731) did not invent the process of using polynomial approximations for transcendental functions (others such as Leibniz, Johann Bernoulli and Abraham de Moivre had already used series that we would call Taylor series), Taylor was the first to provide a comprehensive study of their importance and applications in a textbook he wrote in 1715. Similarly, the Scottish mathematician Colin Maclaurin (1698–1746) did not invent the series named for him. He also wrote a textbook (in 1742) that greatly furthered applications of power series, so much so that Taylor series centred at $x = 0$ became known as Maclaurin series. Isaac Newton (1642–1727) was an early innovator with power series, especially his work in writing binomial expressions as series. A contemporary of Newton's was another brilliant Scottish mathematician, James Gregory (1638–1675). Gregory published power (Maclaurin) series for tan x, sec x, arctan x and arcsec x ten years before Maclaurin was born. Apparently Taylor was not aware of Gregory's mathematical work when writing his 1715 textbook.

Example 5

Find the Maclaurin series for $f(x) = \sin x$ and its interval of convergence.

Solution

To use the definition of a Maclaurin series we need to repeatedly differentiate $f(x) = \sin x$, and evaluate each derivative at $x = 0$, sufficiently to establish a pattern for the power series.

$$\begin{array}{ll}
f^{(0)}(x) = f(x) = \sin x & f(0) = 0 \\
f'(x) = \cos x & f'(0) = 1 \\
f''(x) = -\sin x & f''(0) = 0 \\
f^{(3)}(x) = -\cos x & f^{(3)}(0) = -1 \\
f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \\
f^{(5)}(x) = \cos x & f^{(5)}(0) = 1
\end{array}$$

The pattern that emerges repeats in blocks of four. Thus, the power series is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \\
&= 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots
\end{aligned}$$

Therefore, the Maclaurin series for $f(x) = \sin x$ is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Using the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| &= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| \\
&= \left| x^2 \right| \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} \\
&= \left| x^2 \right| \cdot 0 \\
&= 0 < 1
\end{aligned}$$

Hence, the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $\sin x$ converges for all real values of x.

Finding a Taylor series centred at c for a function $f(x)$

Step 1: Compute several derivatives of f. The 'zeroth' derivative, $f^{(0)}(x)$, is $f(x)$ itself.
Step 2: Evaluate the derivatives at $x = c$, and try to identify a pattern for the values of $f^{(n)}(c)$.
Step 3: Knowing that the rule for coefficients of a Taylor series is $a_n = \dfrac{f^{(n)}(c)}{n!}$, write the formula for the Taylor series, substituting into $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.
Step 4: Using the ratio test determine the radius of convergence for the series.
Step 5: If $0 < R < \infty$, use an appropriate convergence test to check for convergence/divergence at both of the interval endpoints $x = c - R$ and $x = c + R$. State the interval of convergence.

Finding a Taylor (or Maclaurin) series by this five-step process can prove to be difficult, primarily finding a pattern for the values of $f^{(n)}(c)$. We now establish some very useful properties of power series, giving us 'short cuts' for finding Taylor series from a known Taylor series.

3.3 Operations with power series

The power and versatility of representing functions with power series is due largely to the fact that they retain many of the properties of 'finite' polynomials. Two of the properties that make polynomials particularly useful in calculus are the ease with which they can be differentiated and integrated term by term. It is natural to ask whether the same holds true for power series. The answer is provided by the next theorem, which we state without proof.

Differentiation and integration of power series

If R is the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

then

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n (x-c)^n\right) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

for $c - R < x < c + R$

and

$$\int f(x)\,dx = \int\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)dx = \sum_{n=0}^{\infty}\left(\int a_n (x-c)^n\right)dx = \sum_{n=0}^{\infty} \frac{1}{n+1}(x-c)^{n+1}$$

for $c - R < x < c + R$.

The above theorem means that term-by-term differentiation or integration of a power series produces a power series that converges, respectively, to the derivative or integral of the function represented by the original series, given that we are in the interval of convergence of the original series. This

The radius of convergence of the series produced by differentiating or integrating a power series is equivalent to that of the original series. However, the interval of convergence may change due to convergence/divergence changing at one or both of the endpoints.

gives us another way to derive power series representations from a known power series, as the next example illustrates.

Example 6

Find the Maclaurin series for $f(x) = \cos x$.

Solution

We could apply the formula for a Maclaurin series as we did in Example 5 but it will be more efficient to simply differentiate the power series that we already established for $f(x) = \sin x$.

$$\begin{aligned}\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left((2n+1)\frac{(-1)^n x^{2n}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)\\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\end{aligned}$$

Note: $(2n+1)! = (2n+1)(2n)!$

The power series for $\sin x$ converges for all x, so the same is true for this power series for $\cos x$.

Therefore,

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots \text{ for } x \in \mathbb{R}.$$

For convenience, we state the operations with power series using series centred at 0, but they also apply for series centred at c, e.g. $\sum_{n=0}^{\infty} a_n(x-c)^n$.

A further beneficial property of power series is that algebraic manipulations valid for polynomials also work for power series. These are described below.

It is important to note that the operations for power series stated here can change the interval of convergence for the resulting series. In general, the interval of convergence of the resulting series for properties **1** and **2** will be the same as the original series; and for properties **3**, **4** and **5** it will be the intersection of the intervals of convergence of the two original series.

Properties of power series

Given the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and constant k, the following hold true:

1 $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2 $f(x^k) = \sum_{n=0}^{\infty} a_n (x^k)^n = \sum_{n=0}^{\infty} a_n x^{kn}$

3 $f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)x^n$

4 $f(x)\cdot g(x) = \sum_{n=0}^{\infty}(a_n b_n x)^n$

5 $\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty}\left(\frac{a_n}{b_n}\right)x^n, b_n \neq 0$

Example 7

a) Use the fact that the Maclaurin series for $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$ to find the Maclaurin series for $g(x) = \frac{1}{1+x}$.

b) Use the result from a) and integration of power series to find the Maclaurin series for $h(x) = \ln(1+x)$.

Solution

a) We can create the power series for $g(x) = \frac{1}{1+x}$ by simply substituting $-x$ for x into the power series for $f(x) = \frac{1}{1-x}$. Thus,

$$g(x) = f(-x) = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

and we can write this result as

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots \text{ for } -1 < x < 1.$$

Note: $\sum_{n=0}^{\infty}(-1)^n x^n$ diverges at both endpoints so the interval of convergence is the same as for the original series $\sum_{n=0}^{\infty} x^n$.

b) Knowing that $\int \frac{1}{1+x}\,dx = \ln(1+x)$, we can integrate the power series for $\frac{1}{1+x}$ to obtain the power series for $\ln(1+x)$.

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx \\ &= \sum_{n=0}^{\infty}\int\left((-1)^n x^n\right)dx \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}\right)x^{n+1} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\end{aligned}$$

Starting the index at $n = 1$ rather than $n = 0$.

Note: At $x = -1$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ becomes the divergent harmonic series and at $x = 1$ it becomes the convergent alternating harmonic series, so the interval of convergence changes to $-1 < x \leqslant 1$.

Therefore, we can write the result as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots \text{ for } -1 < x \leqslant 1.$$

Knowing that $\int \frac{1}{1+x^2}\,dx = \arctan x$, we can use the same approach used in Example 7 b) to show that the Maclaurin series for $\arctan x$ is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots \text{ for } -1 \leqslant x \leqslant 1.$$

We now have derived Maclaurin series for several important functions. We list these below with their corresponding intervals of convergence. These series can be used to construct other series.

Except for the Maclaurin series for $\frac{1}{1-x}$, the Maclaurin series listed here are given in the Mathematics HL Formula booklet.

Selected Maclaurin series

Function	Interval of convergence
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$-\infty < x < \infty$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$-1 < x \leqslant 1$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	$-\infty < x < \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	$-\infty < x < \infty$
$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$	$-1 \leqslant x \leqslant 1$

Example 8

a) Find the Maclaurin series for $f(x) = e^{x^2}$.

b) Hence, find a series for $\int e^{x^2}\,dx$.

c) Use the first four terms of the series in b) to approximate the value of $\int_0^1 e^{x^2}\,dx$.

Solution

a) We prefer not to derive a Taylor series (centred at 0, in this case) by the direct method used to find the series for $\sin x$ in Example 5, if there is an alternative approach. To find the Maclaurin series for e^{x^2} we can simply substitute x^2 in for x in the Maclaurin series for e^x.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \text{ for } -\infty < x < \infty$$

This series converges for all real values of x because the original series for e^x did so.

b) We can take the result from a) and integrate it term by term.

$$\int e^{x^2}\,dx = \int\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots + \frac{x^{2n}}{n!} + \ldots\right)dx$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} + \ldots + \frac{x^{2n+1}}{(2n+1)n!} + \ldots$$

That is,

$$\int e^{x^2}\,dx = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)n!}$$

and because the series for e^{x^2} converged for all real values of x, then this series does also.

c) The first four terms of the series for $\int e^{x^2}\,dx$ are

$$\sum_{n=0}^{3}\frac{x^{2n+1}}{(2n+1)n!} = x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} = x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}.$$

We use this to approximate the value of the definite integral $\int_0^1 e^{x^2}\,dx$.

$$\int_0^1 e^{x^2}\,dx \approx \left[x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}\right]_0^1$$

$$= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42}$$

$$= \frac{51}{35} = 1\frac{16}{35} \approx 1.457\,142\,857 \text{ (to ten significant figures)}$$

Also, to ten significant figures, a GDC computes the value of $\int_0^1 e^{x^2}\,dx$ to be 1.462651746.

The percentage difference between the value from the partial sum of the series for e^{x^2} and the calculator value is only about 0.377%.

We are unable to find a function that is an anti-derivative of e^{x^2} with the calculus techniques covered in this course. Helping us to integrate such functions is a very useful application of power series.

Continuing with the idea of using power series for approximation purposes, recall the discussion at the start of this chapter. As in the last example, when a power series is used (by your GDC, for example) to compute an approximate value for a function it does so with a suitable partial sum of the power series that, by definition, will be a polynomial. Given the definition of a Taylor series

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

it follows that $f(x)$ is the limit of the partial sums. In our derivation of the power series for $f(x) = e^x$ centred at 0 we found four partial sums that were polynomials of degree 1, 2, 3 and 4.

Taylor polynomials

In general the nth partial sum of a Taylor series is

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

which is a polynomial, typically called a **Taylor polynomial**, of degree n.

For instance, the result for Example 5 shows that the Taylor polynomials for $f(x) = \sin x$ centred at 0 for $n = 1$, 3 and 5 are:

$$P_1(x) = x \qquad P_3(x) = x - \frac{x^3}{3!} \qquad P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

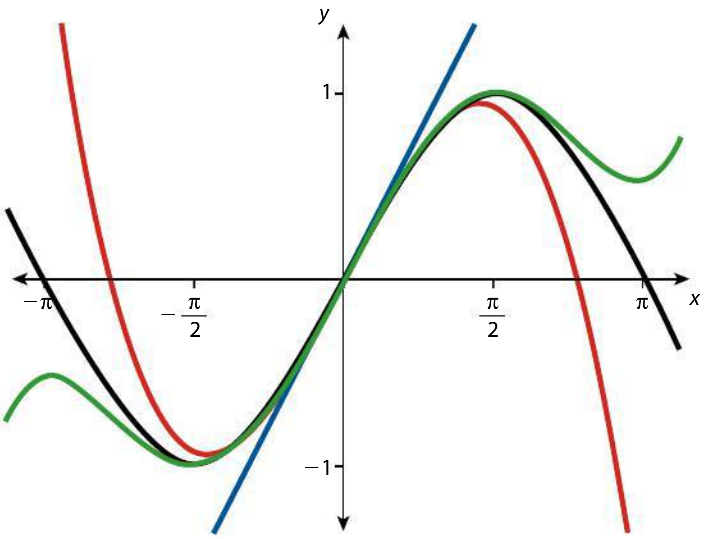


Figure 3.7 The sine function (in black) and the three Taylor polynomials $P_1(x) = x$, $P_3(x) = x - \frac{x^3}{3!}$ and $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

Since the error term for a Taylor polynomial results from 'cutting off' the series after a certain number of terms to a finite polynomial it is occasionally referred to as the **truncation error**.

In general, $f(x)$ is the limit of the sequence of its Taylor polynomials. That is,

$$f(x) = \lim_{n \to \infty} P_n(x).$$

Because any particular Taylor polynomial, $P_n(x)$, is an approximation to $f(x)$ we can write

$$f(x) = P_n(x) + R_n(x)$$

where $R_n(x)$ is the **remainder**, also called the **error term**.

We are now in a position to present the following important theorem that not only gives a formula for the Taylor polynomial for approximating a function f but also two formulae for computing the error term.

Although the Lagrange form cannot compute the error exactly, it does provide an efficient and accurate way to find the maximum error. By finding the value of b between c and x (inclusive) that maximizes $f^{(n+1)}(b)$, thereby maximizing $\frac{f^{(n+1)}(b)}{(n+1)!}(x-c)^{n+1}$, we can compute the maximum error from using an nth degree Taylor polynomial centred at c to approximate $f(x)$ for a specific value of x.

Taylor's theorem

If a function f has derivatives of all orders in an open interval I centred at c, then for each positive integer n and for each x in I,

$$f(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$ is the ***n*th degree Taylor polynomial** centred at $x = c$.

The **error term**, $R_n(x)$, can be computed with the following formula:

$$R_n(x) = \frac{f^{(n+1)}(b)}{(n+1)!}(x-c)^{n+1}$$ where b lies between c and x, inclusive (**Lagrange form**)

Example 9

a) Find the fifth degree Taylor polynomial centred at $x = 1$ for $f(x) = \ln x$.

b) Use this polynomial to approximate $\ln(1.3)$.

c) Use the Lagrange error term to determine an upper bound to the error in this approximation.

d) How many terms of the Taylor series centred at $c = 1$ are needed to approximate $\ln(1.3)$ so that the error is less than $0.000001 = 1.0 \times 10^{-6}$.

Solution

a) We can construct the Taylor series centred at $x = 1$ by substituting $x - 1$ for x in the Maclaurin series for $\ln(1 + x)$.

Since $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $-1 < x \leqslant 1$, then

$$\ln(1+(x-1)) = \ln x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots \text{ for } 0 < x \leqslant 2.$$

Hence, the fifth degree Taylor polynomial centred at $x = 1$ is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{4}.$$

b) $\ln(1.3) \approx (0.3) - \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} - \frac{(0.3)^4}{4} + \frac{(0.3)^5}{4} = 0.262\,461$

c) For the Lagrange form of the error term we have $x = 1.3$, $n = 5$, $c = 1$. By computing a few derivatives of $f(x) = \ln x$

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2}{x^3},\ f^{(4)}(x) = -\frac{6}{x^4}$$

we can see that the nth derivative is $f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}$. Therefore, the sixth derivative at $x = b$ is $f^{(6)}(b) = -\frac{5!}{b^6}$. We need to estimate its largest magnitude, that is, $\left|-\frac{5!}{b^6}\right| = \frac{120}{b^6}$ as b ranges from 1 to 1.3. Clearly it has a maximum when $b = 1$. Therefore,

$$R_5(1.3) \leqslant \frac{120}{(5+1)!}(1.3-1)^{5+1} = \frac{0.087\,48}{720} = 0.000\,1215.$$

The error is no larger than $0.000\,1215$. Given our approximation $\ln(1.3) \approx 0.262\,461$, we can say with absolute certainty that the exact value of $\ln(1.3)$ is somewhere between $0.262\,3395$ and $0.262\,5825$.

d) The magnitude of the $(n+1)$st derivative is given by $\left|f^{(n+1)}(b)\right| = \left|(-1)^n\frac{n!}{b^{n+1}}\right| = \frac{n!}{b^{n+1}}$. As in the previous part, we have $x = 1.3$

and $c = 1$, so $1 \leqslant b \leqslant 1.3$; and the $(n+1)$st derivative will be largest when $b = 1$. Therefore,

$$\max\left[R_n(1.3)\right] = \frac{n!}{(n+1)!}(1.3-1)^{n+1} = \frac{(0.3)^{n+1}}{n+1} < 1.0 \times 10^{-6}.$$

By trial and error (see GDC images below), we determine that the smallest value of n that satisfies this inequality is $n = 9$. So, the ninth degree Taylor polynomial would give us an error of less than 1.0×10^{-6} when approximating $\ln(1.3)$.

```
Plot1  Plot2  Plot3
\Y1■((.3)^(X+1))
/(X+1)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
Y1(7)
          8.20125E-6
Y1(8)
             2.187E-6
Y1(9)
            5.9049E-7
```

Example 10

a) Express the indefinite integral $\int \frac{\sin x}{x}\,dx$ as an infinite series.

b) Evaluate the definite integral $\int_0^1 \frac{\sin x}{x}\,dx$ accurate to three decimal places.

Solution

a)
$$\int \frac{\sin x}{x}\,dx = \int \frac{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \int \frac{x^{2n}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

$$= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \ldots$$

b)
$$\int_0^1 \frac{\sin x}{x}\,dx = \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}\right]_0^1$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)(2n+1)!}$$

$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{29400} + \ldots$$

We can use the alternating series estimation theorem that says that the error generated in using the nth partial sum of an alternating series to approximate its sum does not exceed the value of the $(n+1)$st term. Which is the first term to have a magnitude so that adding/subtracting it will not change the third decimal point of the sum?

$a_3 = \frac{1}{600} = 0.001\overline{6}$, $a_4 = \frac{1}{29400} \approx 0.000034$. Hence, the first three terms will give us an estimate with a maximum error of about 0.000 034, sufficient to guarantee an estimate accurate to three decimal places.

Therefore,

$$\int_0^1 \frac{\sin x}{x}\,dx \approx 1 - \frac{1}{18} + \frac{1}{600} \approx 0.946$$

A GDC confirms our result.

```
fnInt(sin(X)/X,X
,0,1)
        .9460830704
```

Exercise 3

In questions 1–14, determine the radius of convergence and interval of convergence of the power series.

1 $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1}$

2 $\displaystyle\sum_{n=1}^{\infty} n(x-2)^n$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$

5 $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

7 $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

8 $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n} x^n$

9 $\displaystyle\sum_{n=0}^{\infty} n!x^n$

10 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{n4^n}$

11 $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n!)} x^n$

12 $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$

13 $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$

14 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$

15 Given $k > 0$, express the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} (kx)^n$ in terms of k.

16 Find a power series for $\dfrac{1}{1+x}$ by replacing x with $-x$ in the geometric series for $\dfrac{1}{1-x}$ (Example 1). Also determine the interval of convergence for the series.

17 **a** Find the Maclaurin series for e^{-x^2} and determine its radius of convergence.

b Express $\int e^{-x^2}\,dx$ as a Maclaurin series and determine its radius of convergence.

c Use the first five terms of the series from **b** to estimate $\int_0^1 e^{-x^2}\,dx$. Show that the error for this estimate is less than 0.001.

18 Use multiplication or division of power series to find the first three non-zero terms in the Maclaurin series for each function.

a $f(x) = x\sin x$

b $g(x) = \tan x$ $\left[\text{Hint: } \tan x = \dfrac{\sin x}{\cos x}\right]$

c $f(x) = e^x \ln(1-x)$

19 Find a Maclaurin series for $\dfrac{1}{(1-x)^2}$ by differentiating the Maclaurin series for $\dfrac{1}{1-x}$. State the interval of convergence for the series.

20 **a** Find a power series for x^2e^{-x}.

b By differentiating term by term the power series in **a**, show that

$$\sum(-2)^{n+1}\frac{n+2}{n!} = 4.$$

21 a Write down the seventh degree Taylor polynomial for sin x.

b Use this polynomial to estimate $\sin\left(\frac{\pi}{12}\right)$.

c Use the Lagrange error term to determine an upper bound to the error in this approximation.

22 Determine all the values of x for which the series $\sum_{n=1}^{\infty} \frac{2^n x^n}{\ln(n+1)}$ converges.

23 Find the first four non-zero terms of the Taylor series centred at $x = 1$ for the function $f(x) = (x - 1)e^x$.

24 Find the Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$.

25 a Find the Maclaurin series for the function $f(x) = \frac{1}{1+x^2}$.

b Hence by integrating each term, show that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots .$$

c Show that this series converges for $-1 \leqslant x \leqslant 1$, and thus show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

d Use the first six terms of the series to estimate the value of ☒. Compute the maximum error for this estimate.

26 The function f is defined by $f(x) = \frac{e^x + e^{-x}}{2}$.

a Obtain an expression for $f^{(n)}(x)$, the nth derivative of $f(x)$ with respect to x.

b Hence, derive the Maclaurin series for $f(x)$ up to and including the term in x^4.

c Use the result to find a rational approximation to $f\left(\frac{1}{2}\right)$.

d Use the Lagrange error term to determine an upper bound to the error in this approximation.

27 Estimate the range of values of x for which the Maclaurin approximation

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is accurate to within 0.005.

28 Find a power series for xe^x and then integrate the resulting series term by term from 0 to 1 to show that $\sum_{n=1}^{\infty} \frac{1}{n!(n+2)} = \frac{1}{2}$.

29 Find a power series for $\sec^2 x$. (Hint: consider the derivative of tan x.)

30 Compute the Taylor series for each of the following functions $f(x)$ centred at the given point c.

a $f(x) = e^x,\ c = 2$

b $f(x) = \sin(x^3),\ c = 0$

c $f(x) = \frac{1}{(1-x)^3},\ c = 0$

d $f(x) = (x-1)^3 \ln x,\ c = 1$

31 Use series to evaluate each limit.

a $\lim_{x \to 0} \frac{1 - \cos x}{e^x - 1 - x}$

b $\lim_{x \to 0} \frac{x - \arctan x}{x^3}$

Practice questions 3

1 Find the fourth degree Taylor polynomial centred at $x = 0$ for the function $f(x) = \ln(\cos x),\ 0 \leqslant x < \frac{\pi}{2}$.

2 **a** Find the Maclaurin series for $\sin^2 x$ up to the term containing x^4.

b Hence, write down a series for $\cos^2 x$ up to the term containing x^4.

3 Find the first three non-zero terms in the Maclaurin expansion of $e^x \sin x$.

4 Find the first four terms of the Maclaurin series for the function $f(x) = e^{3x}$.

5 Find the Maclaurin series of $\sec x$ up to the term containing x^4.

6 **a** Write down the fourth degree Taylor polynomial centred at $x = 0$ for e^x.

b Determine the fourth degree Taylor polynomial centred at $x = 0$ for e^{x^2}.

c Hence, or otherwise, find the fourth degree Taylor polynomial centred at $x = 0$ for e^{x+x^2}.

7 Find the Maclaurin series for $\ln(2 + 3x)$ and find an expression for the error term $R_n(x)$.

8 **a** Find the fourth degree Taylor polynomial centred at $x = 0$ for the function $f(x) = \sqrt{4 + x}$.

b Find an expression for the error term $R_4(x)$, and find an error bound when $x = 0.1$.

9 Using the formula for the Lagrange form of the error term, determine how many terms of the Maclaurin series for $\cos x$ are needed to approximate $\cos(5°)$ correct to six decimal places. State this approximate value.

10 **a** Find the Maclaurin series for e^{-x^2}.

b Use the first three terms of this series to approximate $\int_0^1 e^{-x^2}\,dx$.

c Find an upper bound for the error in this approximation.

11 **a** Find the Maclaurin series for $\frac{1}{1+x^2}$. Express the result as an equation in the form $\frac{1}{1+x^2} = \sum_{n=1}^{\infty} a_n$ where a_n is in terms of x and n.

b By integrating both sides of the equation found in **a**, show that $\arctan x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$.

c Show that this series converges for $-1 \leqslant x \leqslant 1$.

d Hence, find the exact value to which the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ converges.

12 **a** Find the first four terms in the Maclaurin series for both of the following functions:

$f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1}{1-x}$

b Rewrite the function $h(x) = \frac{x+1}{x^2 - 5x + 6}$ as the sum of two fractions.

c Hence, find the first four terms in the Maclaurin series for the function $h(x) = \frac{x+1}{x^2 - 5x + 6}$.

13 **a** Write down the function represented by the power series $\sum_{n=1}^{\infty} x^{n-1}$ where the interval of convergence is $-1 < x < 1$.

b Using the result from **a** and the identity $\frac{1}{x} = \frac{-1}{1-(x+1)}$, find the Taylor series centred at $x = -1$ for the function $f(x) = \frac{1}{x}$. State the interval of convergence.

14 **a** Find the Maclaurin series of the function $g(x) = \sin(x^2)$ using the series expansion of $\sin x$, that is, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

b Using the Maclaurin series of $g(x) = \sin(x^2)$, evaluate the definite integral

$$\int_0^1 \sin(x^2)\,dx$$

correct to four decimal places.

15 **a** Find a Maclaurin series expansion for $f(x) = \ln(1 + x)$, for $0 \leqslant x < 1$.

b R_n is the error term in approximating $f(x)$ by taking the sum of the first $(n + 1)$ terms of its Maclaurin series. Prove

$$|R_n| \leqslant \frac{1}{n+1}, \ (0 \leqslant x < 1).$$

16 Find the range of values of x for which the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

17 Consider the function f defined by

$f(x) = \arcsin x$, for $|x| \leqslant 1$.

The derivatives of $f(x)$ satisfy the equation

$(1 - x^2)\, f^{(n+2)}(x) - (2n + 1)\, x\, f^{(n+1)}(x) - n^2 f^{(n)}(x) = 0$, for $n \geqslant 1$.

The coefficient of x^n in the Maclaurin series for $f(x)$ is denoted by a_n. You may assume that the series contains only odd powers of x.

a **i** Show that, for $n \geqslant 1$, $(n + 1)(n + 2)\, a_{n+2} = n^2 a_n$.

ii Given that $a_1 = 1$, find an expression for a_n in terms of n, valid for odd $n \geqslant 3$.

b Find the radius of convergence of this Maclaurin series.

c Find an approximate value for π by putting $x = \frac{1}{2}$ and summing the first three non-zero terms of this series. Give your answer to **four** significant figures.

18 Find the interval of convergence of the series $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right) x^n$.

19 **a** **i** State the domain and range of the function $f(x) = \arcsin(x)$.

ii Determine the first two non-zero terms in the Maclaurin series for $f(x)$.

b Use the small angle approximation

$$\cos(y) \approx 1 - \frac{y^2}{2} + \frac{y^4}{24}$$

to find a series for $\cos(\arcsin(x))$ up to and including the term in x^4.

c **i** Find the Maclaurin series for $(p + qx^2)^r$ up to and including the term in x^4 where $p, q, r \in \mathbb{R}$.

ii Find values of p, q and r such that your series in **c i** is identical to your answer to **b**. Comment on this result.

20 **a** Find the value of $\lim_{x \to 1}\left(\frac{\ln x}{\sin 2\pi x}\right)$.

b By using the series expansions for e^{x^2} and cos x evaluate $\lim_{x \to 1}\left(\frac{1-e^{x^2}}{1-\cos x}\right)$.

21 The function f is defined by $f(x) = \ln(1 + \sin x)$.

a Show that $f''(x) = \frac{-1}{1+\sin x}$.

b Determine the Maclaurin series for $f(x)$ as far as the term in x^4.

c Deduce the Maclaurin series for $\ln(1 - \sin x)$ as far as the term in x^4.

d By combining your two series, show that $\ln \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$.

e Hence, or otherwise, find $\lim_{x \to 0} \frac{\ln \sec x}{x\sqrt{x}}$.

22 **a** Find the first three terms of the Taylor series centred at $x = \frac{1}{2}$ for the function $f(x) = \sin(\pi x)$.

b Hence, find an approximate value to $\sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right)$, correct to three significant figures.

4 Calculus

Introduction

Many important ideas of differential and integral calculus have been presented and explained earlier in both the core syllabus (textbook) and in this option topic. Although we endeavoured to provide thorough explanations for the calculus methods developed and applied earlier in this course, this chapter will attempt to 'fill in the gaps' with regard to some important theorems that provide the theoretical foundation for much of the calculus ideas and methods previously encountered. We have made extensive use of derivatives and integrals to analyze functions, but this has mostly been done in an intuitive way while bypassing some of the fundamental theorems that make these analytical methods possible. In this chapter, we will look back at several fundamental ideas in calculus and present some important theorems. We will make use of material already covered in the textbook – in particular, some content from Chapter 16 (Integral Calculus). It will be very helpful to study this chapter in conjunction with the relevant parts of Chapter 16 that will be mentioned here.

4.1 Continuity and differentiability

The main difference between calculus and other branches of mathematics lies in the idea of a **limit** and the intimately related concept of **continuity**. We have made use of limits, continuity and the important concept of **differentiability** in the calculus topic in the core syllabus (Chapters 13, 15 and 16) and in this option topic. Our approach thus far has been informal and has relied on a visual interpretation of the graphs of functions. In Chapter 13 of the textbook (Section 13.3), a margin note stated the following:

Geometrically speaking a function is **continuous** if there is no break in its graph; and a function is **differentiable** (i.e. a derivative exists) at any point where its graph is 'smooth'.

In the first section of Chapter 3 Algebraic Functions, Equations and Inequalities, it was demonstrated that one of the properties of all polynomial functions is that they are continuous for all real numbers, i.e. the graph of a polynomial function never has a 'gap' or a 'hole' in it. Continuity is such a common feature of many familiar functions (such as polynomial functions) that to understand and recognize it we should look at some functions that lack this property, i.e. some **discontinuous** functions.

Consider the function $f(x) = \dfrac{x^2 + x - 6}{x^2 - x - 2}$. By factoring the numerator and denominator the function can also be expressed as $f(x) = \dfrac{(x-2)(x+3)}{(x-2)(x+1)}$.

The graph of f (Figure 4.1) clearly shows that there is a 'gap' at $x = -1$ and a 'hole' at $x = 2$. Thus at the points where $x = -1$ and $x = 2$ the function f is not continuous. It is **discontinuous**. It seems reasonable to say that the function is continuous everywhere else since the graph appears to have no other 'gaps' or 'holes'.

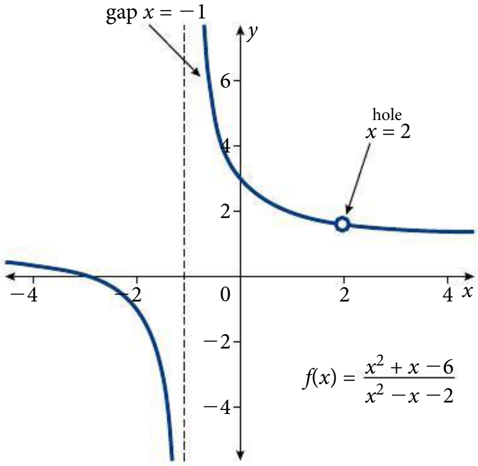
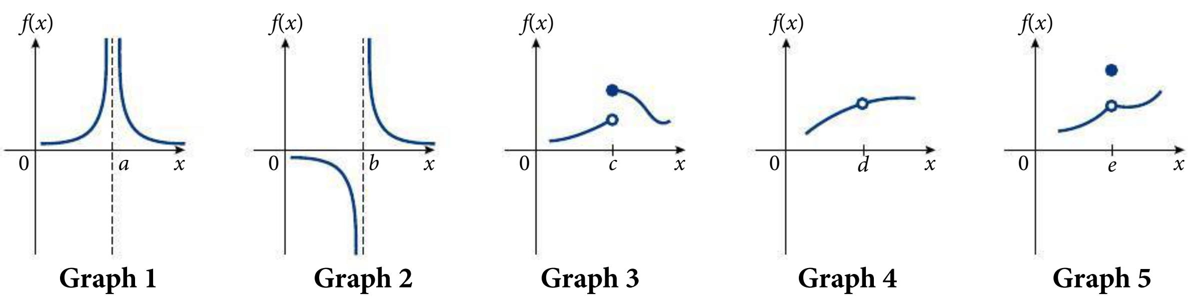


Figure 4.1

Figure 4.2 below shows examples of graphs of five functions that have different types of discontinuities. Respectively, the functions shown have points of discontinuity at $x = a, b, c, d$ and e.

Figure 4.2



The functions in Graphs 1 and 2 have vertical asymptotes at $x = a$ and $x = b$, so the functions are not defined for these values of x (as seen in the graph of the function f in Figure 4.1). This can be referred to as an **infinite discontinuity**.

The function in Graph 3 illustrates what can be described as a **step discontinuity**, where it is defined at $x = c$. However, the graph shows that a small change in x produces a 'jump' in the value of $f(x)$ so the function is not continuous at $x = c$.

The type of discontinuity seen in Graphs 4 and 5 is the same as the 'hole' that occurred at $x = 2$ in the graph of f in Figure 4.1. This type of discontinuity is often called a **removable discontinuity** because it can be *removed* by simply redefining the value of the function at the particular point where the 'hole' occurs.

We now need to develop a precise definition of continuity from the observations made in the preceding examples. From the examples, it is clear that the definition needs to incorporate the following two ideas:

1 Continuity is a *local* matter. In other words, a function can be continuous at some points and discontinuous at other points. Therefore, continuity cannot be defined for an *entire* function.

   We must define continuity *at a point*.

2 A function f is continuous at a point $x = c$ of its domain if $f(x)$ is near $f(c)$ when x is near c.

The second of these ideas is close to the definition we're looking for, but the idea of 'near' is not mathematically precise. In order to do so, we need to apply the formal concept of a limiting value. We also need to distinguish between a function being continuous at a point and a function being continuous at all points in a certain interval.

The functions in the Graphs 1, 2 and 4 in Figure 4.2 are discontinuous respectively at $x = a$, b and d because they do not satisfy the first condition for the definition of continuity. The function in Graph 4 is discontinuous at $x = c$ because it does not satisfy the second condition. In order for the limit of the function as x approaches c to exist, it must be true that the limit of the function as x approaches c from the left (**one-sided limit** from the left) equals the limit of the function as x approaches c from the right (**one-sided limit** from the right), i.e. $\lim\limits_{x\to c} f(x)$ exists if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)$. The function in Graph 5 is discontinuous at $x = e$ because it does not satisfy the third condition for the definition of continuity at a point.

Definition of continuity

1 **Continuity at a point:**
A function f is continuous at a point where $x = c$, if and only if the following three conditions are satisfied.

   i $f(c)$ exists **ii** $\lim\limits_{x\to c} f(x)$ exists **iii** $\lim\limits_{x\to c} f(x) = f(c)$

2 **Continuity on an interval:**
A function f is continuous on an interval of x-values, if and only if it is continuous at each value of x in that interval. At the endpoints of a closed interval (i.e. endpoints included in the interval), only the one-sided limits need to equal the function value.

Example 1

Consider the piece-wise function f, which is defined as follows.

$$f(x) = \begin{cases} |x| + 3 & \text{for } x < 1 \\ ax^2 + bx & \text{for } x \geqslant 1 \end{cases}$$

Find the values of a and b, such that f is continuous for all real numbers.

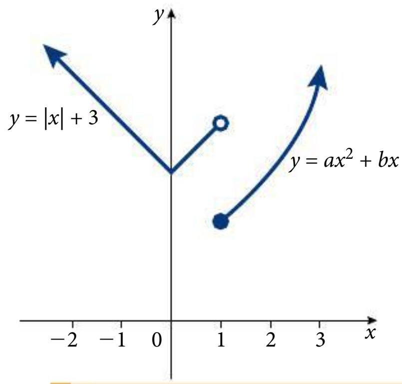
Solution

We know that:

for $x < 1$, the graph of f will be the typical 'v' shape of an absolute value function with a vertex at $(0, 3)$

for $x \geqslant 1$, the graph of f will be a parabola.

Although we do not know the values of a and b we can make a rough sketch of f (shown on the right). (Diagram not to scale)



We see that f satisfies all three conditions for continuity at all points except at $x = 1$. At this point, it satisfies the first condition, i.e. $f(1)$ exists, because $f(1) = a + b$. However, whether the second and third conditions are met depends on the values of a and b. The limit of $|x| + 3$ as x approaches 1 from the left is equal to $|1| + 3 = 4$. The limit of $ax^2 + bx$ as x approaches 1 from the right is equal to $a + b$. Thus, f will be continuous at all points if $a + b = 4$. Therefore, f will be continuous for all real numbers for any pair of values of a and b whose sum is 4.

An important property of functions that are continuous on an interval or intervals – and that makes them especially useful in various mathematical applications – is a property expressed in the following theorem.

For the purpose of consistency all intervals in this chapter are expressed using inequalities. For example, the closed interval $a \leqslant x \leqslant b$ could also be written as $x \in [a, b]$; and the open interval $a < x < b$ could also be written as $\in\ ]a, b[$. See Section 1.1 of the textbook for notation overview.

The intermediate value theorem

If a function f is continuous on the closed interval $a \leqslant x \leqslant b$ and N is a number between $f(a)$ and $f(b)$, then a number $x = c$ must exist such that $f(c) = N$.

It is important to mention that the intermediate value theorem guarantees the existence of at least one number c in the closed interval $a \leqslant x \leqslant b$. Of course, there may be more than one number c such that $f(c) = N$.

It is beyond the scope of this course to give a proof of the intermediate value theorem.

Example 2

Use the intermediate value theorem to show that the polynomial function $f(x) = 2x^3 - 4x^2 + 5x + 3$ has a zero in the closed interval $-1 \leqslant x \leqslant 0$.

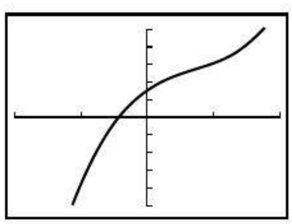
Solution

The function f is a polynomial function so it is continuous for $x \in \mathbb{R}$, and hence also continuous on the closed interval $-1 \leqslant x \leqslant 0$. With reference to the intermediate value theorem, we take $a = -1$, $b = 0$ and $N = 0$.

Since $f(-1) = 2(-1)^3 - 4(-1)^2 + 5(-1) + 3 = -2 - 4 - 5 + 3 = -8 < 0$ and $f(0) = 2(0)^3 - 4(0)^2 + 5(0) + 3 = 3 > 0$, it follows that $f(-1) < 0 < f(0)$. We can now apply the intermediate value theorem to conclude that there must be at least one number c in the interval $-1 \leqslant x \leqslant 0$ such that $f(c) = 0$ as shown in the GDC screen images below.

```
Plot1  Plot2  Plot3
\Y1= 2X³−4X²+5X+3
\Y2=
\Y3=
\Y4=
\Y5=
```

```
WINDOW
 Xmin=−2
 Xmax=2
 Xscl=1
 Ymin=⁻10
 Ymax=10
 Yscl=2
↓Xres=1
```



Of course, the intermediate value theorem is useful when access to a GDC is not allowed. The GDC images above are provided simply to confirm the result obtained from the intermediate value theorem.

The intermediate value theorem is an example of what is often referred to as an **existence theorem**. The theorem guarantees that a number *exists* with a certain property, but it does *not* provide a method for finding the value of the number. The following theorem is also an existence theorem where continuity of a function, or lack of it, plays an important role. It guarantees the existence, under certain conditions, of a solution to an extreme value (minimum/maximum) problem. Again, we will present this theorem without a formal proof.

The extreme value theorem

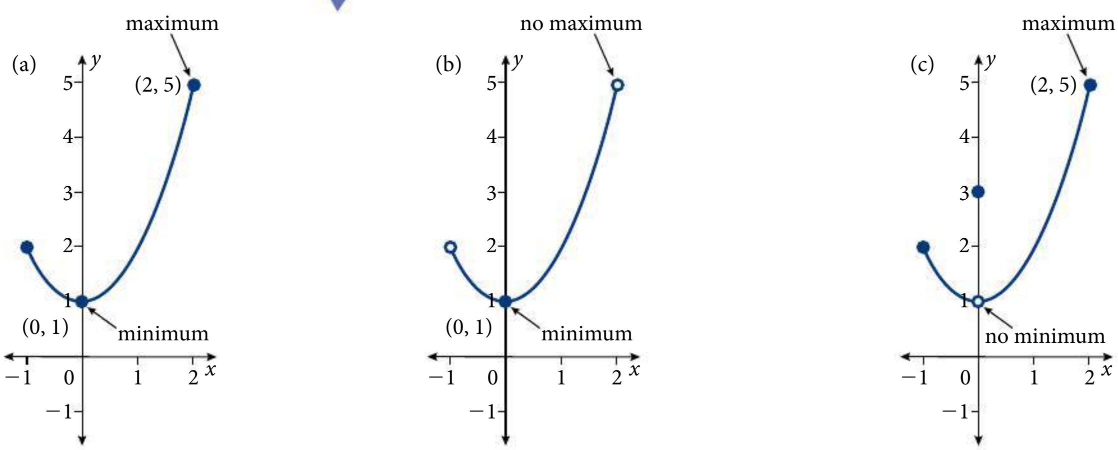
If a function f is continuous on a closed interval, then f has an absolute maximum and an absolute minimum on the closed interval.

The functions graphed in Figure 4.3 below illustrate some possibilities for a function having a maximum or a minimum on an interval. In (a), the function $y = x^2 + 1$ has both a maximum and a minimum on the **closed** interval $-1 \leqslant x \leqslant 2$. The maximum at the point $(2, 5)$ is an example of an extreme value (maximum in this case) that occurs at an endpoint. In (b), the function $y = x^2 + 1$ on the **open** interval $-1 < x < 2$ has a minimum but no maximum. In (c), the function is:

$$y = \begin{cases} x^2 + 1 & \text{for } x \neq 0 \\ 3 & \text{for } x = 0 \end{cases}$$

It is on the closed interval $-1 \leqslant x \leqslant 2$. It has a maximum but no minimum because of the **discontinuity** at $x = 0$.

Figure 4.3



$y = x^2 + 1, -1 \leqslant x \leqslant 2$

$y = x^2 + 1, -1 < x < 2$

$$y = \begin{cases} x^2 + 1 & \text{for } x > 0 \\ 3 & \text{for } x \neq 0 \end{cases}, -1 \leqslant x \leqslant 2$$

In (a) of Figure 4.3, since the function is continuous on a closed interval the extreme value theorem guarantees that an absolute minimum and an absolute maximum must exist.

As already mentioned, the fact that a function is differentiable at a point (i.e. a derivative exists for a function at a point) was described informally in Chapter 13 to be related to the 'smoothnesss' of the graph of the function. Recall the definition of the derivative of a function *f* from Section 13.2. The derivative at a point $x = c$, $f'(c)$, is given by $f'(c) = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

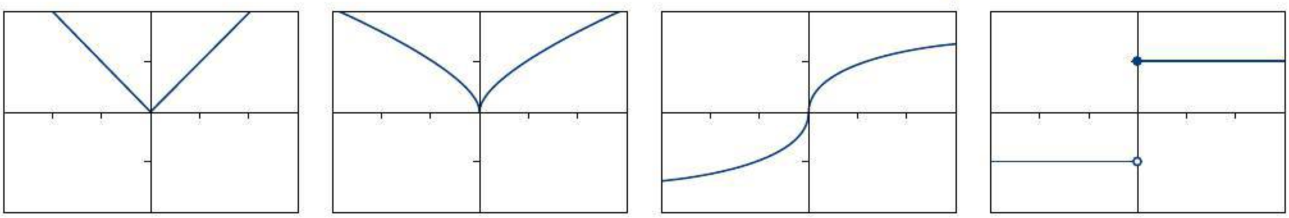
provided that this limit exists. The key phrase in this definition is 'provided that this limit exists'. The limit exists if the left-hand and right-hand limits are equal. Substituting $x - c$ for h in the limit definition for the derivative gives $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

This alternative limit form for the derivative is useful for determining whether or not a function is differentiable at a particular point where $x = c$. Thus, to show that a function *f* is *not* differentiable at $x = c$ we must show that the two one-sided limits (as x approaches c from either direction) for the definition of the derivative are *not* equal; that is, show

$\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \neq \lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$.

Graphically speaking, this means that a function *f* will not have a derivative at a point$(c, f(c))$ where the slopes of the secant lines fail to approach the same value as x approaches c from the right and from the left. This agrees with the previous informal description that a function is differentiable at a point where the graph of the function is 'smooth'. Also, a function will not be differentiable at a point of discontinuity because a discontinuity will cause one or both of the one-sided limits to be non-existent. The four graphs in Figure 4.4 illustrate four different types of situations where a function fails to be differentiable at a point.

Figure 4.4



at $x = 0$: (a) a sharp 'corner' (b) a sharp point or 'cusp' (c) a vertical tangent (d) a point of discontinuity

Each of the four functions shown in Figure 4.4 fail to have a derivative (i.e. not differentiable) at $x = 0$. A brief rationale is given for each.

- Function (a): The left-hand derivative and the right-hand derivative are not equal at $x = 0$. As x approaches 0 from the left the derivative approaches the value of -1, and as x approaches 0 from the right the derivative approaches the value of $+1$.
- Function (b): Both the left-hand derivative and the right-hand derivative do not exist at $x = 0$. As x approaches 0 from the left the derivative (slope of tangent) approaches $-\infty$, and as x approaches 0 from the right the derivative approaches $+\infty$.

- Function (c): Both the left-hand derivative and the right-hand derivative do not exist at $x = 0$. The derivative (slope of tangent) approaches $+\infty$ as x approaches 0 from both sides.
- Function (d): The function is discontinuous at $x = 0$ which will cause one or both of the one-sided derivatives to be non-existent. The function shown in (d) can be expressed in piecewise form as

  $$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x \geqslant 0 \end{cases}.$$

  Using the form of the limit definition of the derivative given earlier, $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$, we can show that the left-hand derivative does not exist at $x = 0$.

  $$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-1 - 1}{x} = \lim_{x \to 0^-} \frac{-2}{x} = \infty \text{ (increases without bound)}$$

Definition of differentiability

A function f is *differentiable* at a point where $x = c$ if the derivative, $f'(c)$, exists.

Example 3

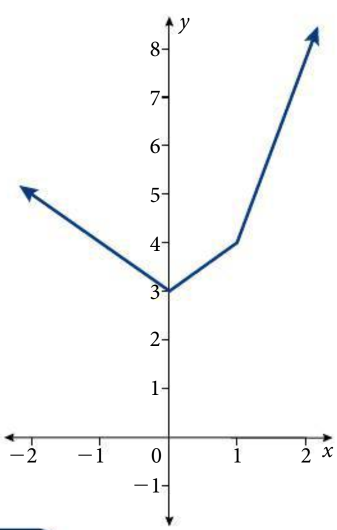
Consider the piece-wise function f from Example 1:

$$f(x) = \begin{cases} |x| + 3 & \text{for } x < 1 \\ ax^2 + bx & \text{for } x \geqslant 1 \end{cases}$$

a) Example 1 concluded that f is continuous for all real numbers if $a + b = 4$. Let $a = \frac{1}{2}$ and $b = \frac{7}{2}$. For these values of a and b, are there any values of x where f is not differentiable?

b) Find the values of a and b, such that f is differentiable for all x where $x \geqslant 0$.

Solution



a) From the graph below, the two points on the graph of f where $f'(c)$ may not exist (i.e. where the graph is not 'smooth') is at $x = 0$ and at $x = 1$. Let's consider both points separately.

For $x = 0$: The portion of f that is an absolute value function, $y = |x| + 3$, can be treated as a piecewise function – let's call it $g(x)$.

$$g(x) = \begin{cases} -x + 3 & \text{for } x \leqslant 0 \\ x + 3 & \text{for } x \geqslant 0 \end{cases}$$

We compute the derivatives of $y = -x + 3$ and $y = x + 3$.

$g(x) = -x + 3$ $\quad$ $g(x) = x + 3$

$g'(x) = -1$ $\quad$ $g'(x) = 1$

$g'(0) = -1$ $\quad$ $g'(0) = 1$

The left-hand derivative does not equal the right-hand derivative when $x = 0$. Thus, the function is **not differentiable** (does not have a derivative) at $x = 0$.

For $x = 1$: Left of $x = 1$ is the function $y = x + 3$ and right of $x = 1$ is the function $y = \frac{1}{2}x^2 + \frac{7}{2}x$.

We compute the left-hand and right-hand derivatives at $x = 1$.

left-hand derivative:	right-hand derivative:
$y = x + 3$	$y = \frac{1}{2}x^2 + \frac{7}{2}x$
$y' = 1$	$y' = x + \frac{7}{2}$
$y'(1) = 1$	$y'(1) = 1 + \frac{7}{2} = \frac{9}{2}$

The left-hand derivative does not equal the right-hand derivative when $x = 1$. Thus, the function is **not differentiable** at $x = 1$.

Therefore, the function $f(x) = \begin{cases} |x| + 3 & \text{for } x < 1 \\ \frac{1}{2}x^2 + \frac{7}{2}x & \text{for } x \geqslant 1 \end{cases}$

is not differentiable at $x = 0$ and at $x = 1$.

b) In order for f to be differentiable at $x = 1$ the left-hand and right hand derivatives must be equal at $x = 1$.

left-hand derivative:	right-hand derivative:
$y = x + 3$	$y = ax^2 + bx$
$y' = 1$	$y' = 2ax + b$
$y'(1) = 1$	$y'(1) = 2a + b = 1$

From Example 1, we know that $a + b = 4$ in order for $f(x)$ to be continuous at $x = 1$. Thus, solving simultaneous equations $a + b = 4$ and $2a + b = 1$ gives $a = -3$ and $b = 7$.

From the four functions graphed in Figure 4.4 and Example 3, we can conjecture that continuity of a function at a point does not imply that the function will also be differentiable at that point. However, differentiability does imply continuity, which is stated in the next theorem.

Differentiability implies continuity

If a function f is **differentiable** at a point $x = c$, then f is also **continuous** at $x = c$.

Proof

To prove that f is continuous at $x = c$ we must show that the three conditions of the definition of continuity are satisfied. That is, we must show that **i** $f(c)$ exists, **ii** $\lim\limits_{x \to c} f(x)$ exists, and **iii** $\lim\limits_{x \to c} f(x) = f(c)$.

i From the hypothesis of the *differentiability implies continuity* property, f is differentiable at $x = c$ so it must follow that $f(c)$ exists. From the definition of the derivative $f'(c) = \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c}$. It follows that $f(c)$ must exist otherwise this limit has no meaning.

ii and **iii** We can use the product rule for limits (Section 13.1 in textbook) which states that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, then

$\lim_{x\to a}\left[f(x)\cdot g(x)\right] = L\cdot K$ and knowing that $\lim_{x\to c}(x-c) = 0$ and that

$\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c} = f'(c)$ to perform the following:

$$\begin{aligned}\lim_{x\to c}\left[f(x)-f(c)\right] &= \lim_{x\to c}\left[(x-c)\cdot\frac{f(x)-f(c)}{x-c}\right]\\ &= \lim_{x\to c}(x-c)\cdot\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\ &= 0\cdot f'(x)\\ &= 0\end{aligned}$$

This result helps to produce the following:

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c}\left[f(x)-f(c)+f(c)\right]\\ &= \lim_{x\to c}\left[f(x)-f(c)\right]+\lim_{x\to c} f(c)\\ &= 0+\lim_{x\to c} f(c)\end{aligned}$$

Thus, $\lim_{x\to c} f(x)$ exists, and it is equal to $\lim_{x\to c} f(c)$. Therefore, all three conditions of the definition of continuity are satisfied and the theorem is proved.

One consequence of the property that differentiability implies continuity is proof that all polynomial functions are continuous for all real numbers.

The property that if a function is differentiable at a point then it must also be continuous at that point can be symbolized by writing: differentiable $\rightarrow$ continuous. It is worthwhile to point out that both the converse and the inverse of this property are false.

Converse: continuous $\rightarrow$ differentiable … *false*
Inverse: not differentiable $\rightarrow$ not continuous … *false*

Both of these false statements were illustrated in Example 3. However, the contrapositive of the property is true. That is …

Contrapositive: not continuous $\rightarrow$ not differentiable … *true*

In other words, if a function *f* is not continuous at a point then *f* is also not differentiable at that point. The property 'differentiable $\rightarrow$ continuous' and its contrapositive 'not continuous $\rightarrow$ not differentiable' provide an effective way to prove that a function is continuous or not differentiable at a particular point.

Example 4

Consider the function $g(x) = \dfrac{x^2-2x-3}{x-3}$.

a) Show that g is continuous at $x = 4$.

b) Show that g is not differentiable at $x = 3$.

Solution

a) In order to show that g is continuous at $x = 4$, we just need to show that a derivative exists for g at $x = 4$.

$$g'(x) = \frac{(x-3)(2x-2)-(x^2-2x-3)(1)}{(x-3)^2} = \frac{x^2-6x+9}{(x-3)^2}$$

$$= \frac{(x-3)^2}{(x-3)^2} = 1$$

for all values of x except $x = 3$.

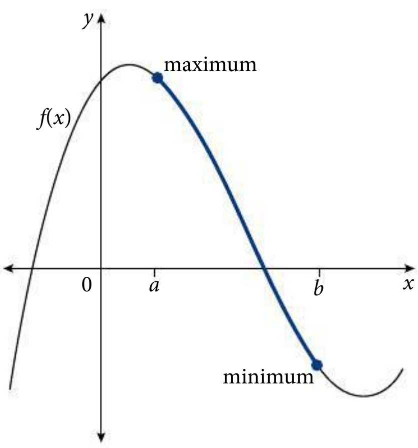
Thus, $g'(4) = 1$ and g is differentiable at $x = 4$. Since differentiability implies continuity then f is continuous at $x = 4$ … Q.E.D.

b) To prove that g is not differentiable we need to show that g is not continuous at $x = 3$.

The given function is equivalent to $g(x) = \frac{(x+1)(x-3)}{x-3}$. It's clear that g has a removable discontinuity at $x = 3$. Applying the contrapositive of the property that 'differentiability implies continuity' proves that since g is discontinuous at $x = 3$, then it is also not differentiable at $x = 3$.

4.2 Rolle's theorem and the mean value theorem

Figure 4.5
$f(x)$ on closed interval $a \leqslant x \leqslant b$



The extreme value theorem presented earlier in this chapter states that a function that is continuous on a closed interval must have both a minimum and a maximum on the interval. As mentioned, this is an example of an existence theorem. The theorem tells us that if a function satisfies a certain condition, then at least one minimum and at least one maximum must exist. The function does not tell us where these extreme values are located. Both of these extreme values could occur at the endpoints of the closed interval as illustrated in Figure 4.5. **Rolle's theorem**, named after the French mathematician Michel Rolle (1652–1719), is an existence theorem that states conditions that guarantee when a function must have at least one extreme value in the interior of a closed interval (i.e. an open interval).

Essentially what Rolle's theorem says is that between consecutive zeros of a function there must be at least one location where the derivative of the function is zero. Geometrically speaking, this means that between two zeros there must be at least one place where the graph of the function has a horizontal tangent.

Rolle's theorem

Let f be a function such that:

i it is continuous on the closed interval $a \leqslant x \leqslant b$;

ii it is differentiable on the open interval $a < x < b$;

iii $f(a) = 0$ and $f(b) = 0$.

Then there must exist a number c in the open interval $a < x < b$ such that $f'(c) = 0$.

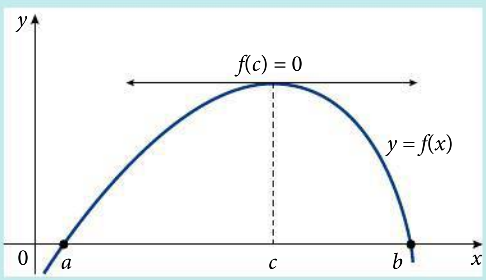
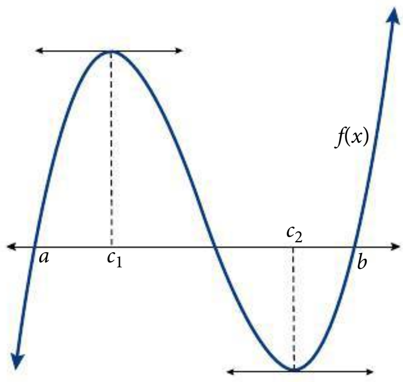


Figure 4.6



It is possible for a continuous function f to have more than location in the open interval $a < x < b$ where the derivative of f is zero. This is illustrated in Figure 4.6 where there is a horizontal tangent at $x = c_1$ and also at $x = c_2$. Thus, both $f'(c_1) = 0$ and $f'(c_2) = 0$.

Rolle's theorem is a special case of a more powerful existence theorem known as the **mean value theorem**. Recall the discussion in Section 2 of Chapter 13 (Differential Calculus I: Fundamentals) demonstrating that the derivative of a function (slope of tangent line) gives the **instantaneous rate of change** of the function at a point and that the slope of the secant line through two points gives the average rate of change between the two points. Over a particular interval in the domain of a function, the mean value theorem connects the average rate of change of the function with instantaneous rate of change of the function at a point within the interval. Although the mean value theorem can be used as an effective tool in solving certain problems, its importance lies in the fact that it has been used to prove several other important theorems in calculus. The theorem was briefly presented in the first section of Chapter 16 (Integral Calculus) where it was used to help establish the general rule for finding anti-derivatives (indefinite integrals) of functions. The mean value theorem plays an important role in the development of the fundamental theorem of calculus that is presented briefly at the end of this option topic chapter – and was thoroughly discussed in Section 16.4 (Area and definite integral) of the textbook.

The mean value theorem

Let f be a function such that:

i it is continuous on the closed interval $a \leqslant x \leqslant b$

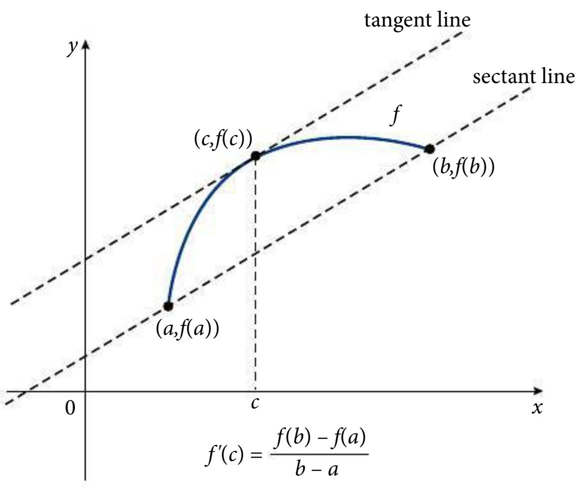
ii it is differentiable on the open interval $a < x < b$.

Then there must exist a number c in the open interval $a < x < b$ such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (See Figure 4.7)

Figure 4.7

The theorem presented on this page is sometimes referred to as the mean value theorem **for derivatives** to contrast it with another theorem involving the average (mean) value of a continuous function over an interval that is usually referred to as the mean value theorem **for integrals**. The word 'mean' in the theorem on this page refers to the average rate of change (slope of secant line) of function f in the interval $a \leqslant x \leqslant b$.



A geometric interpretation of the mean value theorem – as illustrated in Figure 4.7 – guarantees the existence of at least one tangent line to a function

f in the interval $a < x < b$ that is parallel to the secant line through the points $(a, f(a))$ and $(a, f(b))$. This is demonstrated in Example 5 below.

Example 5

Consider the function $f(x) = 6 - \frac{9}{x}$ over the open interval $1 < x < 9$. Find all values of c in this interval at which the conclusion of the mean value theorem is true. For any resulting value of c, verify the result by graphing f, the secant line through $(1, f(1))$ and $(9, f(9))$, and the tangent through $(c, f(c))$.

Solution

Firstly, $f(x)$satisfies the required conditions of the mean value theorem because the only point where f is not continuous and not differentiable is at $x = 0$ and f is being considered only over the interval $1 < x < 9$. Now need to find any value of c that satisfies $f'(c) = \frac{f(9) - f(1)}{9 - 1}$. Given that $f'(x) = \frac{9}{x^2}$, then $\frac{9}{c^2} = \frac{5-(-3)}{8} \Rightarrow \frac{9}{c^2} = 1 \Rightarrow c^2 = 9 \Rightarrow c = \pm 3$. Thus, $c = 3$.

Equation of secant line through $(1, -3)$ and $(9, 5)$:

$$\text{slope} = \frac{-3-5}{1-9} = \frac{-8}{-8} = 1$$

$$y - y_1 = m(x - x_1) \Rightarrow y - (-3) = 1(x - 1) \Rightarrow y + 3 = x - 1$$

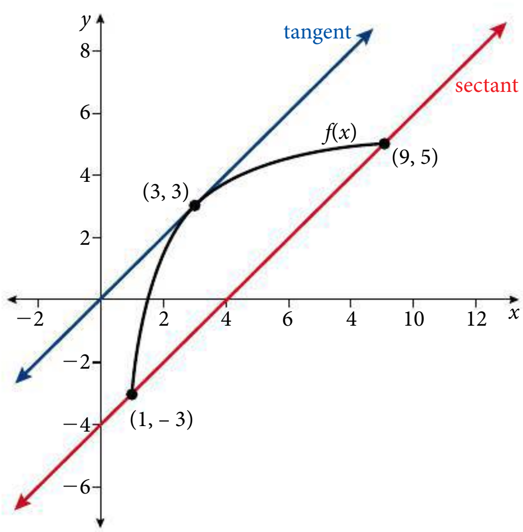
equation of secant line: $y = x - 4$

Equation of tangent line through $(3, f(3))$:

$$f(3) = 6 - \frac{9}{3} = 3; \text{ point of tangency is } (3, 3);\ f'(3) = \frac{9}{3^2} = 1$$

equation of tangent line: $y - 3 = 1(x - 3) \Rightarrow y = x$

Graph of f, secant line and tangent line:



The graph visually confirms the result in that the secant line and tangent line are parallel.

As mentioned, the mean value theorem can also be interpreted in terms of rates of change. The theorem guarantees the existence of at least one point in the open interval $a < x < b$ at which the instantaneous rate of change is equal to the average rate of change over the closed interval $a \leqslant x \leqslant b$. Example 6 illustrates the use of the mean value theorem in the context of rates of change.

Example 6

Two motion detectors that can measure the instantaneous rate of change of a toy car moving along a straight track are positioned 5 metres apart. As the toy car passes the first detector, its velocity is measured at 17 metres/minute. Fifteen seconds later the toy car passes the second detector and its velocity is measured at 19 metres/minute. Show that the velocity of the toy car must have been 20 metres/minute at some moment during the fifteen seconds that it traveled between the two detectors.

Solution

Since the instantaneous rates measured by the two detectors are measured in metres per minute – and that 15 seconds = $\frac{1}{4}$ minute – the motion of the toy car is being considered over the interval $0 < t < \frac{1}{4}$ with t in minutes. It makes sense to set the distance s in metres to be zero for $t = 0$, i.e. $s(0) = 0$; and then $s\left(\frac{1}{4}\right) = 5$ since the detectors are 5 metres apart. Thus, the average velocity for the toy car during the quarter minute that it took to travel 5 metres is given by

$$\text{average velocity} = \frac{s\left(\frac{1}{4}\right) - s(0)}{\frac{1}{4} - 0} = \frac{5-0}{\frac{1}{4}} = 20\,\text{metres/minute}$$

Assuming that the distance function $s(t)$ is differentiable over the interval, we can apply the mean value theorem to conclude that the toy car must have been traveling at a velocity of 20 metres/minute for at least one instant during the time it moved between the two detectors.

4.3 Riemann sums and the fundamental theorems of calculus

At the start of Section 16.4 (Area and definite integral) in the textbook we developed an informal, but logical, explanation for the area under a continuous function over a certain interval to be equal to the definite integral where the limits of integration are the endpoints of the interval. Critical to this explanation is the process of finding the sum of sets of rectangles of decreasing width to form better and better approximations of the area under the curve for a particular interval. Although the name is not used in Section 16,4, the sum of an infinite set of rectangles for the purpose of computing the area under a curve is called a **Riemann sum**. The discussion in Section 16.4 also presented two important theorems in calculus that are usually referred to as the **first fundamental theorem of calculus** and the **second fundamental theorem of calculus**. Before studying this section in the calculus option topic, it is very important that you go back and carefully read all of Section 16.4 in the textbook. What follows here is a review and brief description of material on the definite integral, Riemann sums and the fundamental theorems of calculus that are relevant to this HL option topic.

Riemann sums

In Section 16.4 we used the limits of sums of rectangles to define what we mean by the phrase *the area under a curve*. Figure 4.8 shows how we approximate this area with rectangles and also shows the notation we've chosen to use. The area being approximated is for the interval $a \leqslant x \leqslant b$. The interval is partitioned into n sub-intervals of equal width Δx. We then draw n rectangles each having a width of Δx and a height of $f(x_i^*)$ where x_i^* is an arbitrary point within the ith sub-interval.

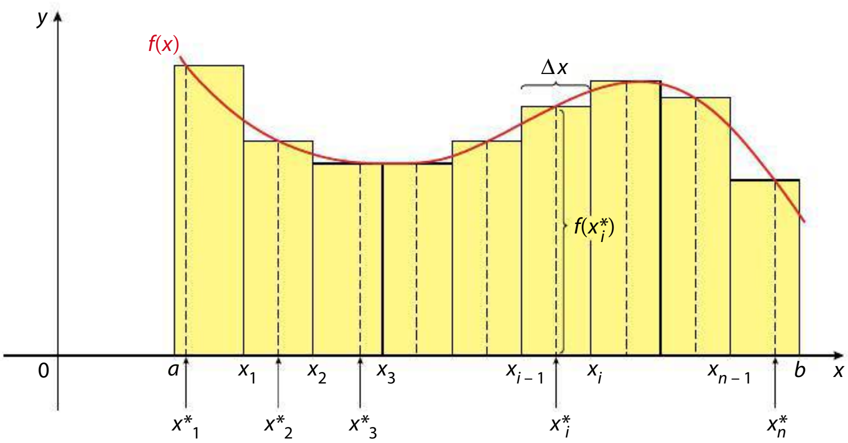


Figure 4.8

We learned that if we let the number of sub-intervals n (or rectangles) go to infinity – and simultaneously the width Δx go to zero – that the limit of the sum of the rectangles is equal to the area under the curve. This result is written as

$$\text{area} = \lim_{n\to\infty}\left[f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \ldots + f\left(x_n^*\right)\Delta x\right] = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

The sum $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$ is called a **Riemann sum** and is named after the German mathematician Bernhard Riemann (1826–1866). As we have encountered previously when computing areas with definite integrals in

Chapter 16, if the region whose area we are computing is below the x-axis then the 'heights' of the rectangles, i.e. $f(x_1^*)$, will be negative. Area is defined to be a positive value. Rather than changing the definition of area, mathematicians decided to call a Riemann sum a definite integral rather than an area.

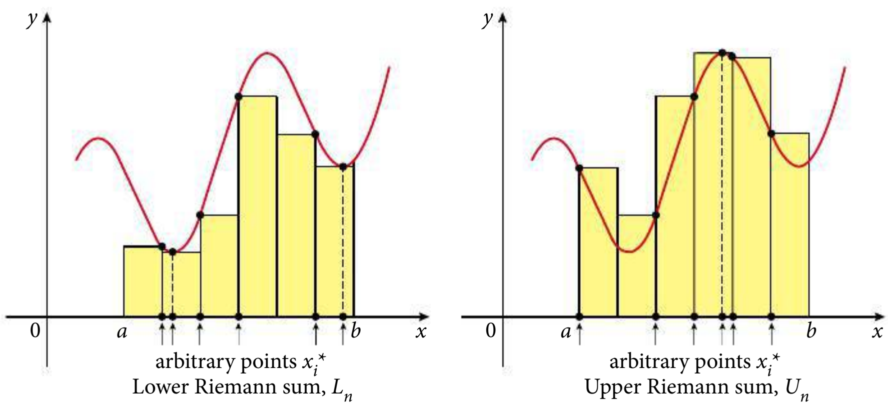
Therefore – as explained in Section 16.4 – the limit of a Riemann sum for a continuous function $f(x)$ on the interval $a \leqslant x \leqslant b$ is defined to be the definite integral of $f(x)$ from a to b; that is $\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \int_a^b f(x)\,dx$.

Although in forming a Riemann sum the widths of the rectangles does not need to be constant, most graphical illustrations of using rectangles to approximate the area of a region between a function and the x-axis (i.e. a Riemann sum) do use a constant width – as shown in Figure 4.8.

As you should understand at this point of your study of advanced mathematics, this is an enormously significant result in the development of calculus.

We will not prove it here, but it turns out that when forming a Riemann sum, it is not necessary for the rectangles to have a constant width. The width of the ith rectangle is denoted as Δx_i. As long as the function f is continuous and integrable over the given interval and the number of rectangles goes to infinity ($n \to \infty$) – thereby causing $\Delta x_i \to 0$ – then the limit of any Riemann sum will be equal to the definite integral $\int_a^b f(x)\,dx$.

It is possible to choose the location of each arbitrary point x_i^* located within the ith sub-interval so that height of the rectangle $f(x_i^*)$ is the lowest or highest in each sub-interval, as illustrated in Figure 4.9. The sum of the areas of the rectangles that are all the lowest possible is referred to as a **lower Riemann sum** (denoted L_n) and the sum of the area of the rectangle that are all the highest possible is referred to as an **upper Riemann sum** (denoted U_n).



The lower sum is a lower bound for the value of the definite integral and the upper sum is an upper bound, i.e. $L_n \leqslant \int_a^b f(x)\,dx \leqslant U_n$. The lower and upper sums will approach the same limit as $n \to \infty$ (and $\Delta x_i \to 0$) causing the value of the definite integral to be squeezed (recall the Squeeze theorem from the second section of Chapter 13) to this common limit, i.e. the definite integral.

Riemann sum and definition of a definite integral

If $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ is any Riemann sum, such that a closed interval $a \leqslant x \leqslant b$ is divided into n sub-intervals where the ith sub-interval has an arbitrary point x_i^* within it and has width Δx_i, and a function f is continuous and integrable on the same interval, then

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i = \int_a^b f(x)\,dx.$$

Continuity implies integrability

If a function f is continuous over the closed interval $a \leqslant x \leqslant b$, then f is also integrable over $a \leqslant x \leqslant b$.

One of the prerequisites for the definite integral of a function over a certain interval being defined as the limit of a Riemann sum is that the function be continuous and integrable (i.e. can be integrated) over the interval. In the first section of this chapter we thoroughly described and defined continuity of a function, but have not done so for integrability of a function. Fortunately, it can be proved that if a function is continuous over an interval then it must also be integrable over the interval. We will not present a proof because it is beyond the scope of this course.

Example 7

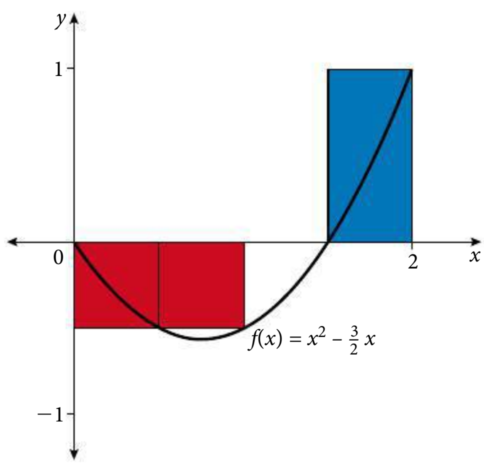
a) Evaluate the Riemann sum for $f(x) = x^2 - \frac{3}{2}x$ for the closed interval $0 \leqslant x \leqslant 2$ divided into 4 sub-intervals of equal width by evaluating the heights of the 4 rectangles at the right endpoint of each sub-interval. Comment on the result.

b) Using the same information from a), find the Riemann sum for f, but now dividing the interval into 6 sub-intervals. Comment on the result.

c) Using integration rules from earlier in the course, evaluate the exact value of the definite integral $\int_0^2\left(x^2 - \frac{3}{2}x\right)dx$. Comment on the result.

Solution

a)



Given that $n = 4$, then the width of each sub-interval is $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

The values of the endpoints of each of the 4 sub-intervals are $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$ and $x_4 = 2$. Thus the Riemann sum is:

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{4}\left(x_i^2 - \frac{3}{2}x_i\right)\cdot\frac{1}{2}$$

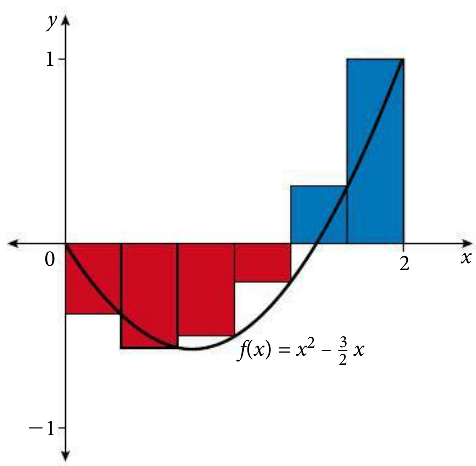
$$= \frac{1}{2}\cdot\left[\left(\frac{1}{2}\right)^2 - \frac{3}{2}\left(\frac{1}{2}\right) + (1)^2 - \frac{3}{2}(1) + \left(\frac{3}{2}\right)^2 - \frac{3}{2}\left(\frac{3}{2}\right) + (2)^2 - \frac{3}{2}(2)\right]$$

$$= \frac{1}{2}\cdot\left[-\frac{1}{2} - \frac{1}{2} + 0 + 1\right]$$

$$= 0$$

Clearly, the Riemann sum does not represent a sum of areas of rectangles. As shown in the figure above, the Riemann sum is the sum of the areas of the blue rectangles (above the x-axis) *minus* the sum of the red rectangles (below the x-axis). With the rectangles shown in the figure, it appears that the value of zero for the Riemann sum is an overestimate because the portion of the blue rectangle outside the region below the curve seems to be larger than the portion between the curve and the x-axis for the sub-interval $1 \leqslant x \leqslant \frac{3}{2}$.

b)



Given that $n = 6$, then the width of each sub-interval is $\Delta x = \frac{2-0}{6} = \frac{1}{3}$.

The values of the endpoints of each of the 6 sub-intervals are $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, $x_4 = \frac{4}{3}$, $x_5 = \frac{5}{3}$, and $x_6 = 2$. Thus, the Riemann sum is:

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{6}\left(x_i^2 - \frac{3}{2}x_i\right)\cdot\frac{1}{3}$$

$$= \frac{1}{3}\cdot\left[\begin{array}{l}\left(\frac{1}{3}\right)^2 - \frac{3}{2}\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 - \frac{3}{2}\left(\frac{2}{3}\right) + (1)^2 - \frac{3}{2}(1) \\ + \left(\frac{4}{3}\right)^2 - \frac{3}{2}\left(\frac{4}{3}\right) + \left(\frac{5}{3}\right)^2 - \frac{3}{2}\left(\frac{5}{3}\right) + (2)^2 - \frac{3}{2}(2)\end{array}\right]$$

$$= \frac{1}{3} \cdot \left[-\frac{7}{18} - \frac{5}{9} - \frac{1}{2} - \frac{2}{9} + \frac{5}{18} + 1 \right]$$

$$= \frac{1}{3} \cdot -\frac{7}{18}$$

$$= -\frac{7}{54}$$

From the figure above – showing the red rectangles that contribute negatively and the blue rectangles that contribute positively to the Riemann sum – it appears that the result of $-\frac{7}{54}$ is a much better approximation than the result in a) for the exact value of the Riemann sum. This should be expected because the number of rectangles has increased from 4 to 6.

c) $\int_0^2 \left(x^2 - \frac{3}{2}x \right) dx = \left[\frac{1}{3}x^3 - \frac{3}{4}x^2 \right]_0^2 = \left[\frac{1}{3}(2)^3 - \frac{3}{4}(2)^2 \right] - 0 = \frac{8}{3} - 3 = -\frac{1}{3}$

Therefore, the limit of the Riemann sum as $n \to \infty$ is exactly $-\frac{1}{3}$. The result of $-\frac{7}{54}$ in b) is a better estimate than the result in a) of 0 of the exact value of the definite integral. By computing the definite integral for the portion of the curve above the x-axis we can determine the exact area of the two regions bounded by the curve and the x-axis.

$$\int_{\frac{3}{2}}^2 \left(x^2 - \frac{3}{2}x \right) dx = \left[\frac{1}{3}x^3 - \frac{3}{4}x^2 \right]_{\frac{3}{2}}^2$$

$$= \left[\frac{1}{3}(2)^3 - \frac{3}{4}(2)^2 \right] - \left[\frac{1}{3}\left(\frac{3}{2}\right)^3 - \frac{3}{4}\left(\frac{3}{2}\right)^2 \right]$$

$$= -\frac{1}{3} - \left(-\frac{9}{16} \right) = \frac{11}{48}$$

Thus, the area of the bounded region above the x-axis is $\frac{11}{48}$ and consequently the area of the bounded region below the x-axis is $\frac{9}{16}$. Since the region below the x-axis has a negative value for the definite integral the exact result of $-\frac{1}{3}$ is confirmed by $\frac{11}{48} - \frac{9}{16} = \frac{11}{48} - \frac{27}{48} = -\frac{16}{48} = -\frac{1}{3}$.

Although the same notation is used for both, it is important to understand that a definite integral is **not** the same thing as an indefinite integral. A definite integral is a *number* while in contrast an indefinite integral is a *family of functions*.

Example 8

Express the following limit as a definite integral on the interval $0 \leqslant x \leqslant \pi$ where x_i is an arbitrary point in the ith sub-interval and Δx_i is the width of the ith sub-interval.

$$\lim_{n\to\infty} \sum_{i=1}^{n} (2x_i + \cos x_i)\Delta x_i$$

Solution

Comparing the limit $\lim_{n\to\infty}\sum_{i=1}^{n}(2x_i + \cos x_i)\Delta x_i$ to the limit in the definition of a definite integral, we can see that $f(x) = 2x + \cos x$. Since the endpoints of the closed interval are $a = 0$ and $b = \pi$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n}(2x_i + \cos x_i)\Delta x_i = \int_0^{\pi}(2x + \cos x)dx$$

Fundamental theorems of calculus

Look again at the computational work done in part c) of Example 7.

$$\int_0^2\left(x^2 - \frac{3}{2}x\right)dx = \left[\frac{1}{3}x^3 - \frac{3}{4}x^2\right]_0^2 = \left[\frac{1}{3}(2)^3 - \frac{3}{4}(2)^2\right] - 0 = \frac{8}{3} - 3 = -\frac{1}{3}$$

In Chapter 16, we learned methods of finding the anti-derivative (indefinite integral) of a function. In the work above, we had to know that the anti-derivative of x^2 is $\frac{1}{3}x^3$ and anti-derivative of $\frac{3}{2}x$ is $\frac{3}{4}x^2$. But how do we know the method for computing the numerical value of the definite integral? This method for computing a definite integral is given in the **second fundamental theorem of calculus** that was presented in the latter part of Section 16.4. This theorem follows from the **first fundamental theorem of calculus** that was also presented in Section 16.4 and is a consequence of the definition of the definite integral using Riemann sums. Collectively the two theorems are often referred to as **<u>the</u> fundamental theorem of calculus.** The development of these two theorems was thoroughly explained in Section 16.4 so there is no need to reproduce that discussion here. However, it is important that you go back and read that section of the textbook again. We consolidate the two theorems into one below.

The fundamental theorem of calculus

If a function f is continuous (and hence integrable) over the closed interval $a \leqslant x \leqslant b$, then both of the following statements are true.

1 If $g(x) = \int_a^x f(t)dt$, then
$g'(x) = f(x)$.

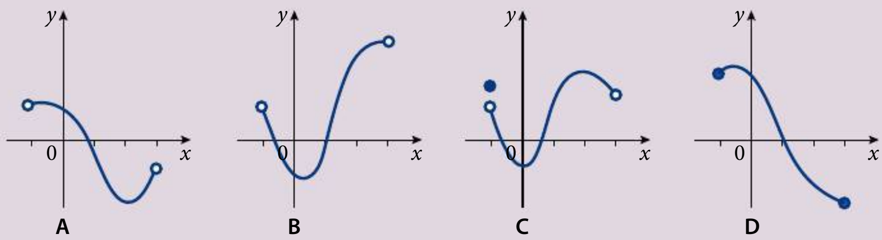
2 $\int_a^b f(x)dx = F(b) - F(a)$, where F is an anti-derivative of f, i.e.
$\frac{d}{dx}[F(x)] = f(x)$.

The first part of the theorem can also be written as $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$.

Thus, we can see that this part of the theorem very importantly establishes the fact that integration and differentiation are inverse processes. The second part of the theorem makes use of this fact resulting in the method for evaluating definite integrals. By showing that such dissimilar objects as the derivative and the integral are so closely intertwined, the fundamental theorem of calculus is certainly one of the major achievements in the development of mathematics and certainly the most important theorem in calculus.

Exercise 4

1 Given that a function g is continuous on the closed interval $-1 \leqslant x \leqslant 3$, which of the following could be a graph of g?



2 Consider the piece-wise function f defined as follows.

$$f(x) = \begin{cases} |x| + 2 & \text{for } x < 2 \\ ax^2 + bx & \text{for } x \geqslant 2 \end{cases}$$

Find the value(s) of b such that f is continuous for all real numbers.

3 State, in terms of a, the interval(s) on which the function g is continuous.

$$g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a} & \text{for } x \neq a \\ 2a & \text{for } x = a \end{cases}$$

4 Consider the function f defined below.

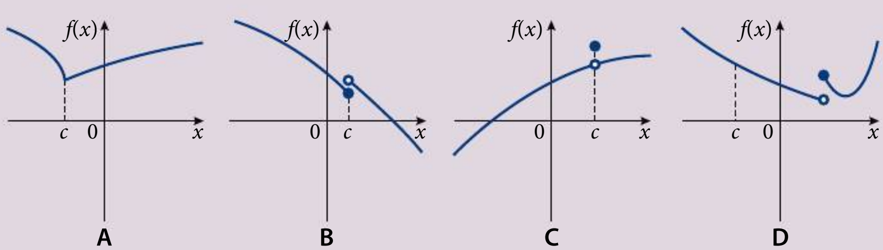
$$f(x) = \begin{cases} x^2 + x + 1 & \text{for } x \leqslant 1 \\ 2x + 1 & \text{for } x > 1 \end{cases}$$

At the point where $x = 1$, determine:

a whether f is continuous

b whether f is differentiable.

5 State whether each function graphed below is continuous or differentiable at $x = c$.



6 Find the value of a and the value of b, such that the function g is differentiable at $x = 2$.

$$g(x) = \begin{cases} ax^3 & \text{for } x \leqslant 2 \\ b(x-3)^2 + 10 & \text{for } x > 2 \end{cases}$$

7 Consider the function h defined below.

$$h(x)=\begin{cases}3x & \text{for } x \leqslant 1\\ ax^2+b & \text{for } x>1\end{cases}$$

a Find the relationship between a and b, such that h is continuous for all real numbers?

b Find the value of a and the value of b, such that h is both continuous and differentiable for all real numbers.

8 If $f(x) = x^3 - 3x^2 + x - 1$, find the point x_0 at which $f'(x)$ has its mean value in the interval $1 < x < 4$.

9 Consider the function $f(x) = x^2 + 1$ over the open interval $1 < x < 3$. Find the value of c in this interval at which the conclusion of the mean value theorem is true. For any resulting value of c, verify the result by graphing f, the secant line through $(1, f(1))$ and$(3, f(3))$, and the tangent through $(c, f(c))$.

10 If $g(x) = \cos x$, find the point x_0 where $g'(x)$ has its mean value in the interval $0 \leqslant x \leqslant \frac{\pi}{2}$.

11 Explain why the mean value theorem does not apply to the function $x^{\frac{2}{3}}$ on the interval $-1 \leqslant x \leqslant 8$.

12 The speed limit along a highway is 60 km per hour. Two police officers positioned 13 km from each other along the highway were monitoring the speed of cars. A car passed the first police officer and was recorded as travelling at 56 km per hour. 12 minutes later, the car passed the second officer who measured the car's velocity as 59 km per hour. Show work and give an explanation confirming whether or not the car broke the speed limit on the portion of highway between the two police officers.

13 Use the mean value theorem to show that $e^x \geqslant x + 1$ for $x > 0$.

14 Consider the portion of the function $f(x) = 2x - x^2$ that is above the x-axis, i.e. $y > 0$. Find the mean value of this function.

15 Use Rolle's theorem to show that the equation $x^3 + 2x + b = 0$, where b is a constant, cannot have more than one real zero.

For the functions in questions 16 and 17, find the value of c in the given interval at which the conclusion of the mean value theorem is true.

16 $f(x) = x^3 - 5x^2 - 3x,\ 0 < x < \frac{\pi}{2}$

17 $g(x) = \sqrt{1 - \sin x}\ , 0 < x < \frac{\pi}{2}$

18 Find the Riemann sum for the function $f(x) = 2x - x^2$ over the interval $0 \leqslant x \leqslant 2$. Use four sub-intervals. The arbitrary point for each sub-interval is the right endpoint of the sub-interval.

19 Find the lower and upper Riemann sums for the function $g(x) = x^2 + 3$ over the interval $0 \leqslant x \leqslant 2$, partitioning the interval into 4 sub-intervals.

In questions 20–22, express the limit as a definite integral on the given interval where x_i is an arbitrary point in the ith sub-interval and Δx_i is the width of the ith sub-interval.

20 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sqrt{x_i + 6}\ \Delta x_i, 0 \leqslant x \leqslant 4$

21 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{e^{x_i}}{x_i - 2} \Delta x_i, 3 \leqslant x \leqslant 5$

22 $\lim_{n\to\infty}\sum_{i=1}^{n}(3-\sin x_i)\,\Delta x_i,\ 0\leqslant x\leqslant 11$

23 Consider each of the integrals below.

a $\int_{2}^{6} x^3\,dx,\ n=4$

b $\int_{0}^{\pi}\sin x\,dx,\ n=3$

c $\int_{-2}^{2} 2^x\,dx,\ n=8$

i Estimate the definite integral (3 significant figures) by finding the value of the Riemann sum with n sub-intervals. Use the midpoint of each sub-interval as the arbitrary point for each sub-interval.

ii Find the exact value of the definite integral using the fundamental theorem of calculus (part 2).

iii State whether the estimate from **i** was an overestimate or underestimate and the percentage error for the estimate found in **i** compared to the exact value found in **ii**.

5 Differential Equations

Introduction

Equations involving an unknown function and its derivative(s) are called differential equations and frequently occur in mathematical models of real-life phenomena. Differential equations come in a great variety of forms, and many different procedures – analytic, graphical and numerical – exist for finding their solutions. The last section of Chapter 16 in the textbook (Section 16.9) is an optional section on differential equations. It provides an introduction to differential equations and also covers an analytic solution method for a certain class of differential equations (separable equations). In this chapter, we will explore differential equations further by considering two more classes of differential equations. Analytic methods are not always successful in solving a differential equation, so we will also investigate a graphical approach and a useful numerical method for approximating the solution to a differential equation.

There is a brief introduction to differential equations in Section 16.9 of the textbook. You are strongly encouraged to read through this section before working through this chapter.

A **differential equation** is an equation that relates an independent variable (commonly x or t), a dependent variable (usually y), and one or more derivatives of an unknown function $y = f(x)$ [or $y = f(t)$]. The general form of a differential equation (with independent variable x) can be written as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ky}{dx^k}\right) = 0$$

where the largest k for which $\frac{d^ky}{dx^k}$ occurs in the equation is called the **order** of the differential equation.

Here are some examples:

1 $x\dfrac{dy}{dx} + y\dfrac{dy}{dx} - y = 0$ — first order differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$

2 $\dfrac{dy}{dx} + \dfrac{y^2 - y}{x^2} = 0$ — first order differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$

3 $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 5y = 0$ — second order differential equation $F\left(x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}\right) = 0$

4 $\dfrac{dy}{dx} + y\sin x - e^{\cos x} = 0$ — first order differential equation $F\left(x, y, \dfrac{dy}{dx}\right) = 0$

5 $2\dfrac{d^3y}{dx^3} + (\ln x)\left(\dfrac{dy}{dx}\right)^2 + 4xy = 0$ — third order differential equation $F\left(x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}\right) = 0$

For this course, we only study **first order differential equations**, such as equations **1**, **2** and **4** above. In a first order differential equation, the first derivative, $\frac{dy}{dx}$, of the unknown function can be isolated on one side of the equation. Hence, a simpler general form for first order differential equations is

$$\frac{dy}{dx} = F(x, y)$$

where $\frac{dy}{dx}$ is expressed as a function in terms of x and y. Note that the first order differential equations **1**, **2** and **4** can all be re-written in this form. For example,

1. $x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x+y}$

The **solution** of a differential equation is the (initially unknown) function $y = f(x)$ whose derivative is $\frac{dy}{dx}$. Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x+1}, \quad x \neq -1.$$

Every solution of this equation is an anti-derivative of $\frac{1}{x+1}$.

$$y = \int \frac{1}{x+1}\,dx = \ln|x+1| + C, \quad x \neq -1$$

So the solution of the differential equation $\frac{dy}{dx} = \frac{1}{x+1}$ is the **explicitly defined function** $y = \ln|x+1| + C$ where C is an arbitrary constant. This is called a **general solution** because it is not a single function, but an infinite 'family' of functions dependent on the constant C. Figure 5.1 shows a few members of this family.

A differential equation may use symbols for the independent and dependent variables other than x and y. For the sake of simplicity, we will use x and y while we are developing theory and solution methods for differential equations. Also note that we are using F ('large F') to represent a two-variable function that when set equal to $\frac{dy}{dx}$ is the differential equation, and f ('small f') represents the unknown function whose slope at the point (x, y) is $\frac{dy}{dx}$.

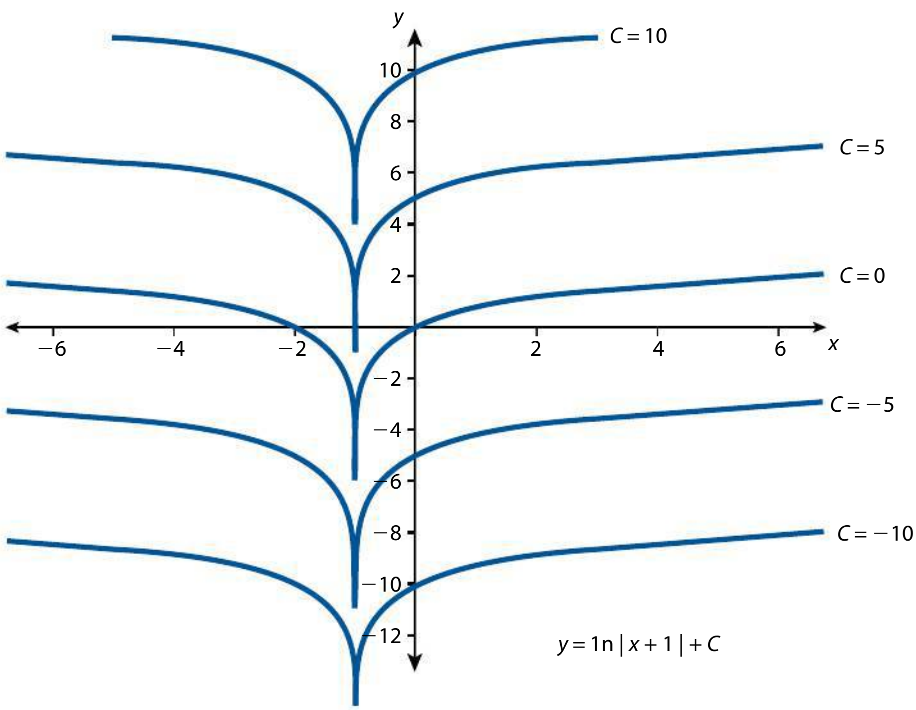


Figure 5.1

In general, we wish to find the **explicit solution** of a differential equation written in the form $y = f(x)$ where f is a known function. However, it is sometimes not possible to solve for y. In such a case we must settle for an **implicit solution** written in the form $g(y) = f(x)$ where g and f are known functions and $g(y) \neq y$.

In contrast, when we are given some **initial conditions** that allow us to evaluate a particular value for C we obtain a single function that we call a **particular solution** of the differential equation. For example, if we are given the initial conditions that $y = 5$ when $x = 0$ then we can solve for C, giving $C = 5$ and the particular solution of $y = \ln|x+1| + 5$.

Sometimes the solution of a differential equation will be expressed as an **implicitly defined function.** For example, the general solution to equation **1** is

$$\ln y = \frac{x}{y} + C.$$

It is an equation relating x and y and *implies* a function exists that defines y as a function of x.

To verify that this is a solution to **1**, we differentiate – applying implicit differentiation and the product rule:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{x}{y} + C\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(xy^{-1}\right) + \frac{d}{dx}(C)$$

$$\frac{1}{y}\frac{dy}{dx} = y^{-1} + x\left(-y^{-2}\frac{dy}{dx}\right) + 0$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}$$

$$y^2\left(\frac{1}{y}\frac{dy}{dx}\right) = y^2\left(\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right)$$

$$x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0$$

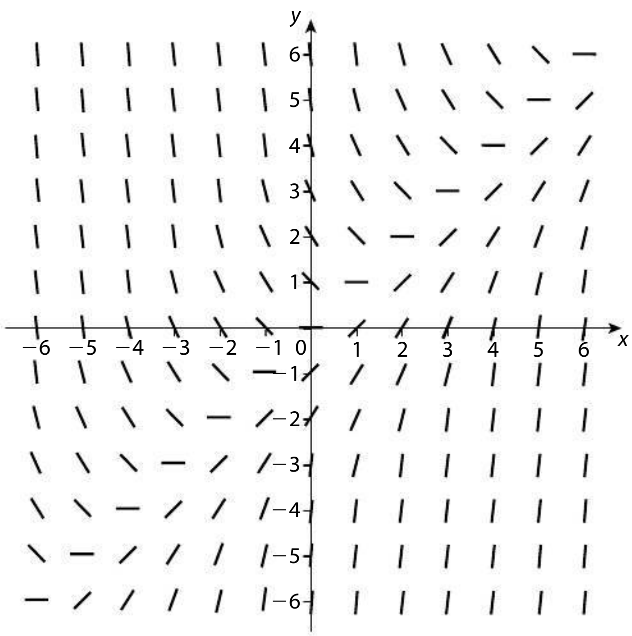
Therefore, for any real number C the function $\ln y = \frac{x}{y} + C$ is a solution, in implicit form, to the differential equation $x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0$. This means that the coordinates x and y of any point on the curve $\ln y = \frac{x}{y} + C$ combined with the value of the derivative $\frac{dy}{dx}$ at that point will solve the equation $x\frac{dy}{dx} + y\frac{dy}{dx} - y = 0$.

The only type of first order differential equation covered in Section 16.9 of the textbook is a class of differential equations referred to as **separable equations.** We solved these using a technique called **separation of variables.** One of our key goals in this chapter is to develop an analytic solution method for each of two further classes of first order differential equations. Before we delve into the details of these analytic methods, we examine a useful graphical method for helping us to sketch the function, or family of functions, that solves a differential equation.

5.1 Slope fields

Often the primary objective when solving a first order differential equation is to find an explicit solution. However, many differential equations used in mathematical models cannot be solved by means of an analytic method. For such equations, we must resort to graphical and/or numerical methods. Carried out by hand or by technology, a graphical method provides us with rough qualitative information about the graph of a solution to a differential equation.

A first order differential equation in the form $\frac{dy}{dx} = F(x, y)$ specifies the slope of the **solution curve** $y = f(x)$ at each point in the xy-plane where F is defined. We can use this fact to draw a short line segment whose slope is $F(x, y)$ at any point (x, y) in the plane. A plot of these line segments showing the slope (or direction) of the solution curve is a called a **slope field** (or direction field) for the first order differential equation. As a rule, the segments are drawn at representative points evenly spaced in both directions. Figure 5.2 shows a slope field for the equation $\frac{dy}{dx} = x - y$.

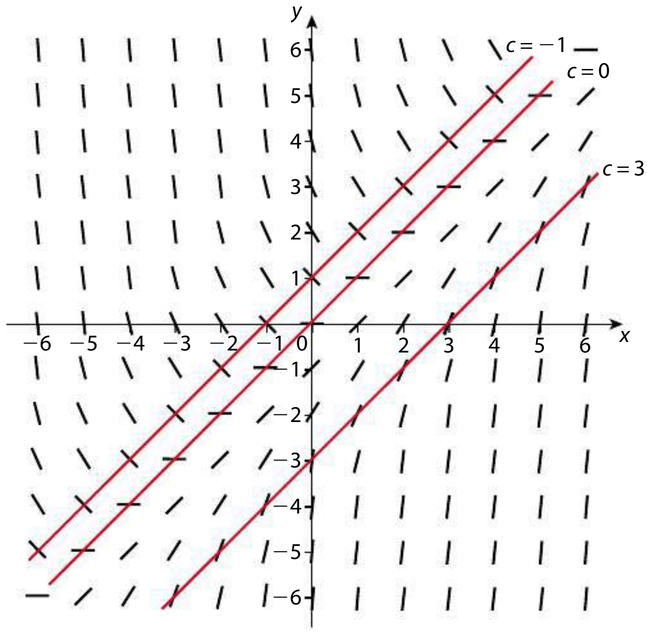


Although it looks fairly simple, the differential equation $\frac{dy}{dx} = x - y$ is not easy to solve. It can be solved analytically with one of the techniques that we develop later in the chapter. It is an example of a first order **linear** differential equation, and its general solution is $y = Ce^{-x} + x - 1$.

Figure 5.2 Slope field for $\frac{dy}{dx} = x - y$.

As you can imagine, it can be quite tedious to draw a slope field by hand. In practice, slope fields are easily generated by suitable graphing technology. However, there is a method that simplifies the process of doing it by hand. Rather than compute $\frac{dy}{dx}$ for a large number of x and y values, we look for points where $\frac{dy}{dx}$ has the same value. For some constant c, the graph of the equation $F(x, y) = c$ is a line, called an **isocline**, along which all the short line segments of a slope field have the same slope c. For the differential equation $\frac{dy}{dx} = x - y$, the isoclines are $x - y = c$. Figure 5.3, shows (in red) the isoclines for $c = -1$, 0 and 3. By first tracing in a few isoclines, we can create a slope field by easily drawing multiple line segments along it all having the same slope.

Figure 5.3 Slope field and three isoclines for $\frac{dy}{dx} = x - y$.



'Isocline' comes from 'iso-' meaning equal and '-cline' meaning slope. Be aware that isoclines themselves do not give any direct information about solution curves for the differential equation. They serve to ease the process of drawing a slope field. It is recommended that you draw isoclines lightly in pencil, and preferably dashed.

Solutions to a differential equation can be sketched by drawing in curves that are at each point tangent to the line segment at that point. Thus, a family of solution curves can be produced. To use a slope field to sketch a particular solution all we need to know is one point (an initial condition) that the solution curve passes through.

Isoclines are not always straight lines. Isoclines are analogous to contour lines on a map indicating land of equal elevation. Consider the differential equation $\frac{dy}{dx} = x^2 - y$ that has isoclines that are parabolas with equations of the form $y = x^2 - c$. When isocline curves are not lines, it is more difficult to use them to sketch a slope field.

Example 1

a) Draw a slope field for $\frac{dy}{dx} = -\frac{x}{y}$ on the xy-plane such that $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$. Sketch some sample solution curves. What shape are they?

b) Confirm that both $y = \sqrt{c^2 - x^2}$ and $y = -\sqrt{c^2 - x^2}$, where c is a constant, are each a general solution of the equation.

Solution

a) Rather than evaluating $\frac{dy}{dx} = -\frac{x}{y}$ for a large number of x and y values, we establish some isoclines by looking for points where $-\frac{x}{y}$ has a constant value.

If $\frac{dy}{dx} = -\frac{x}{y} = 0$ then $x = 0$. Hence, the y-axis is an isocline where all the line segments are horizontal.

If $y = 0$ (x-axis), then $\frac{dy}{dx}$ is undefined. Hence, the x-axis is an isocline where all the line segments are vertical (undefined slope).

If $\frac{dy}{dx} = -\frac{x}{y} = 1$ then $y = -x$ is an isocline where all the line segments have a slope of 1.

If $\frac{dy}{dx} = -\frac{x}{y} = -1$ then $y = x$ is an isocline where all the line segments have a slope of -1.

If necessary, we can continue in this manner and establish further isoclines, such as:

$y = 2x$ is an isocline where all the line segments have a slope of $-\frac{1}{2}$.

$y = -\frac{1}{2}x$ is an isocline where all the line segments have a slope of 2.

In fact, any line passing through the origin will be an isocline for the slope field for $\frac{dy}{dx} = -\frac{x}{y}$.

The resulting slope field – showing six lightly drawn isoclines – is shown below in Figure 5.4.

Drawing curves parallel to the line segments gives a family of solution curves that appear to be circles. Three members of the family are drawn in Figure 5.5.

Figure 5.4

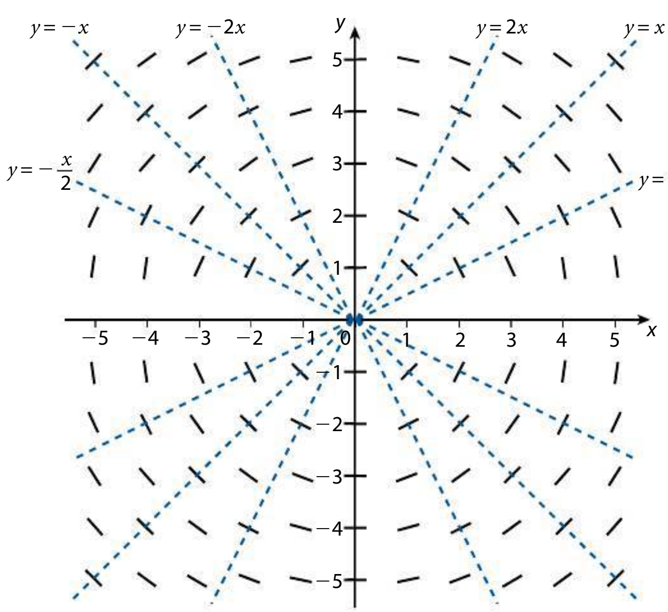
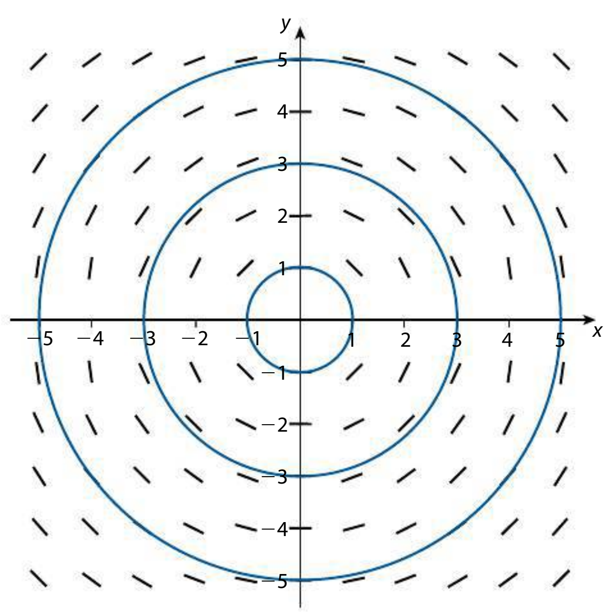


Figure 5.5



b) Checking that $y = \sqrt{c^2 - x^2}$ is a solution, we compute $\frac{dy}{dx}$ on the left side and substitute $\sqrt{c^2 - x^2}$ for y on the right side.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d}{dx}\left(\sqrt{c^2 - x^2}\right) = -\frac{x}{\sqrt{c^2 - x^2}}$$

$$\frac{1}{2}\left(c^2 - x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{c^2 - x^2}}$$

$$-\frac{x}{\sqrt{c^2 - x^2}} = -\frac{x}{\sqrt{c^2 - x^2}} \qquad \text{Q.E.D.}$$

Checking that $y = -\sqrt{c^2 - x^2}$ is a solution.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d}{dx}\left(-\sqrt{c^2 - x^2}\right) = -\left(\frac{x}{-\sqrt{c^2 - x^2}}\right)$$

$$-\frac{1}{2}\left(c^2 - x^2\right)^{-\frac{1}{2}}(-2x) = \frac{x}{\sqrt{c^2 - x^2}}$$

$$\frac{x}{\sqrt{c^2 - x^2}} = \frac{x}{\sqrt{c^2 - x^2}} \qquad \text{Q.E.D.}$$

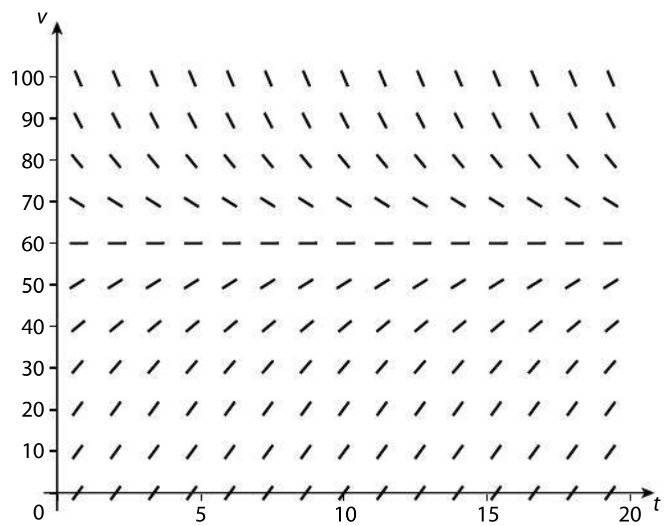
The solution $y = \sqrt{c^2 - x^2}$ is the family of curves consisting of the upper half of each circle, and the solution $y = -\sqrt{c^2 - x^2}$ is the family of curves consisting of the lower half of each circle.

Example 2

A model for the velocity v, in metres per second, at time t seconds of a 75 kg skydiver falling from an aeroplane is given by the equation

$$\frac{dv}{dt} = 10 - \frac{v^2}{360}.$$

Figure 5.6



a) From the direction field shown in Figure 5.6, sketch the solution curves with the following initial conditions:

(i) $v(0) = 0$, (ii) $v(0) = 35$, and (iii) $v(0) = 90$.

b) Explain why the value $v = 60$ is called the **terminal velocity** for this situation.

Solution

a) Solutions to $\dfrac{dv}{dt} = 10 - \dfrac{v^2}{360}$ satisfying $v(0) = 0$, $v(0) = 35$ and $v(0) = 90$ are sketched in Figure 5.7 below.

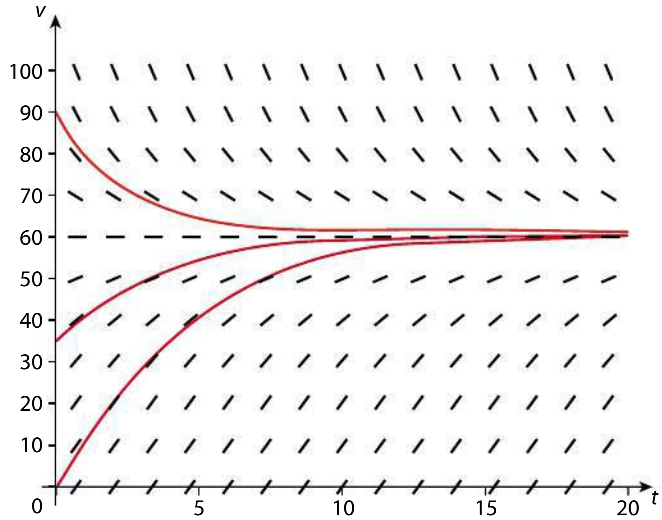
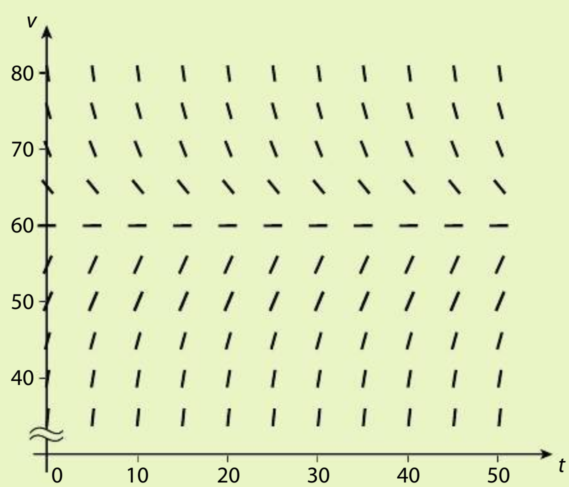


Figure 5.7

b) From the slope field it appears that all solutions have a limiting value of 60 as t goes to infinity. Due to increasing air resistance the skydiver reaches a maximum velocity, or terminal velocity, of 60 metres per second.

Note that the scales on the axes for the slope fields in Figures 5.2, 5.3, 5.4 and 5.5 are equal. Thus, the short line segments accurately depict the true slope for solution curves. The scales are not equal on the axes in Figures 5.6 and 5.7, so the line segments do not give a true indication of the slope. However, this is not an error. Sometimes, it is necessary to have unequal scales in order to show an appropriate interval of values for the independent and dependent variables. Figure 5.8 shows a portion of the same slope field given in Figures 5.6 and 5.7 but with equal scales on the axes.

Figure 5.8



5.2 Separable equations

A class of first order differential equations introduced in Section 16.9 of the textbook which can be solved analytically using integration is the class of **separable equations**. These are differential equations $\frac{dy}{dx} = F(x, y)$ that can be rewritten so that the variables x and y (along with their differentials dx and dy) are on opposite sides of the equation. For a first order differential equation where this separation of variables can be accomplished, the function $F(x, y)$ can be factored into a product of two functions – one involving only the independent variable (e.g. x) and the other involving only the dependent variable (e.g. y). That is,

$$\frac{dy}{dx} = F(x, y) = p(x)q(y).$$

Although there are two integrals in the equation $\int \frac{1}{q(y)} dy = \int p(x) dx + C$, only one constant of integration is needed. We could add a constant to both sides but they could then be combined into one constant.

Separable equation

A first order differential equation is considered separable if it can be written in the form

$$\frac{dy}{dx} = p(x)q(y).$$

The variables can then be separated by writing the equation in the form

$$\frac{1}{q(y)} dy = p(x) dx$$

and integrating both sides gives

$$\int \frac{1}{q(y)} dy = \int p(x) dx + C$$

which leads to a general solution.

It is not always obvious whether or not a differential equation is separable. Some algebraic manipulation is needed to confirm that the differential equation can, in fact, be written in the form $\frac{dy}{dx} = p(x)q(y)$. For example, $\frac{dy}{dx} = \frac{3}{xy} - \frac{x^2}{y}$ is separable because it can be written as $\frac{dy}{dx} = \frac{1}{y}\left(\frac{3}{x} - x^2\right)$; and $\frac{\tan x}{y}\frac{dy}{dx} = \frac{2}{\ln y}$ is also separable because it can be written as $\frac{dy}{dx} = \frac{2y}{\ln y}\cot x$. However, the equations $\frac{dy}{dx} = x^2 + y^2$ and $\frac{dy}{dx} = 1 + xy$ are *not* separable.

Example 3

Find the general solution of the differential equation

$$x^2 y\frac{dy}{dx} = x + 1, \quad x > 0, \; y > 0.$$

Solution

The equation is separable because algebraic rearrangements can be performed to write the equation as

$$\frac{dy}{dx} = \frac{1}{y}\left(\frac{x+1}{x^2}\right)$$

which is in the form $\frac{dy}{dx} = p(x)q(y)$ with $p(x) = \frac{x+1}{x^2}$ and $q(x) = \frac{1}{y}$.

We now separate the variables and integrate, giving:

$$y\,dy = \frac{x+1}{x^2}dx$$

$$\int y\,dy = \int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$\frac{1}{2}y^2 = \ln x - \frac{1}{x} + C$$

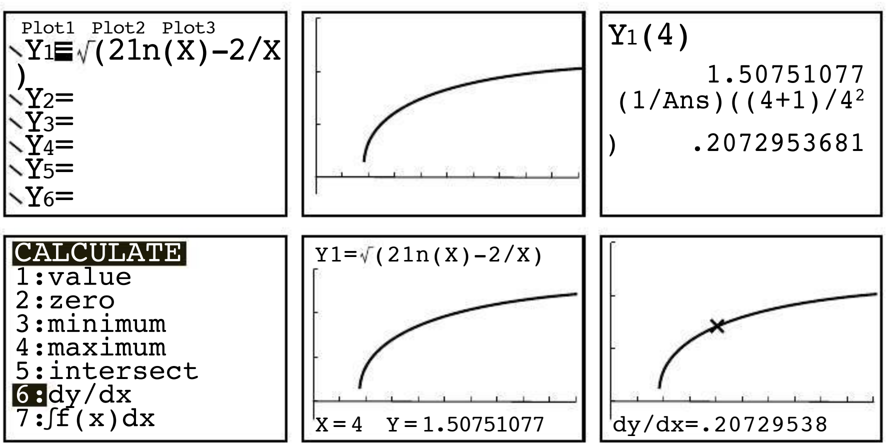
$$y = \sqrt{2\ln x - \frac{2}{x} + C}$$

This is the general solution of $x^2y\frac{dy}{dx} = x + 1$, $x > 0$, $y > 0$ in explicit form.

With some thinking we can use our GDC to help confirm this result.

$\frac{dy}{dx} = \frac{1}{y}\left(\frac{x+1}{x^2}\right)$ is the rule that gives us the slope of the graph of the function $y(x)$ at any point (x, y). In the GDC screen images below we enter the function $y = \sqrt{2\ln x - \frac{2}{x} + C}$; choose a value for x ($x = 4$, for example); use the GDC to find an approximate value for the corresponding y-coordinate; use the rule for $\frac{dy}{dx}$ to find the slope at that point; and then check to see if the same value for $\frac{dy}{dx}$ is given when evaluating it on the graph screen.

The GDC can also draw the tangent line at $x = 4$ and display its equation, confirming that the slope of the function at $x = 4$ is approximately 0.207 295 38, agreeing with the value computed by $\frac{dy}{dx} = \frac{1}{y}\left(\frac{x+1}{x^2}\right)$.



Here is an applied problem involving a separable differential equation.

Example 4

The rate of decay of a substance y at any time t is directly proportional to the amount of y and also directly proportional to the amount of another substance x. The constant of proportionality is $-\frac{1}{2}$ and the value of x at any time t is given by $x = \dfrac{4}{(1+t)^2}$.

a) Given the initial conditions that $y = 10$ when $t = 0$, find y as an explicit function of t.

b) Determine the amount of the substance remaining as t becomes very large.

Solution

a) The rate of decay of substance y is proportional to the product xy, and with the constant of proportionality having a value of $-\frac{1}{2}$ and $x = \dfrac{4}{(1+t)^2}$, this gives:

$$\frac{dy}{dt} = -\frac{1}{2}\left(\frac{4}{(1+t^2)}\right)y$$

$$\frac{1}{y}\,dy = \frac{-2}{(1+t)^2}\,dt$$ Separating variables.

$$\int \frac{1}{y}\,dy = -2\int \frac{1}{(1+t)^2}\,dt$$ Integrating both sides.

$$\ln y = \frac{2}{1+t} + C$$

$$y = e^{\frac{2}{1+t}+C}$$ Exponentiating; using e as the base.

$$y = e^C e^{\frac{2}{1+t}}$$

$$y = Ae^{\frac{2}{1+t}}$$ Let $A = e^c$, a convenient form for the arbitrary constant.

Solve for A knowing that initially $y = 10$ when $t = 0$:

$$10 = Ae^{\frac{2}{1+0}} \quad\Rightarrow\quad 10 = Ae^2 \quad\Rightarrow\quad A = 10e^{-2}$$

Substituting gives:

$$y = 10e^{-2}e^{\frac{2}{1+t}} \quad\Rightarrow\quad y = 10e^{\frac{2}{1+t}-2} \quad\Rightarrow\quad y = 10e^{\frac{-2t}{1+t}}$$

b) As $t \to \infty$, $\dfrac{-2t}{1+t} \to -2$; thus, as $t \to \infty$, $y \to 10e^{-2} \approx 1.36$

Example 5

Solve the differential equation $x\ dx + e^{x+y}\cos y\ dy = 0$.

Solution

As it is the equation cannot be written in the variables separable form $\frac{dy}{dx} = p(x)q(y)$. Since $e^{x+y} = e^x e^y$ we can make it so by multiplying both sides of the equation by e^{-x} and doing some rearrangement.

$$xe^{-x}dx + e^y \cos y\, dy = 0 \quad \Rightarrow \quad e^y \cos y\, dy = -xe^{-x}dx \;\Rightarrow\; \frac{dy}{dx} = -xe^{-x}\left(\frac{1}{e^y \cos y}\right)$$

Separating the variables and integrating both sides gives:

$$\int e^y \cos y\, dy = -\int xe^{-x}\, dx$$

$$\frac{e^y}{2}(\sin y + \cos y) = xe^{-x} + e^{-x} + C$$ Using integration by parts on both sides.

Therefore, the implicit function $\frac{e^y}{2}(\sin y + \cos y) = e^{-x}(x+1) + C$ is the general solution.

To finish this section we will find an explicit solution by the method of separation of variables for a relatively straightforward first order differential equation, but one whose solution will prove useful in developing another solution method.

Example 6

Find the general solution to the differential equation $\frac{dy}{dx} = -2xy$.

Solution

$$\frac{1}{y}dy = -2x\,dx$$ Separating variables; note loss of solution where $y = 0$.

$$\int \frac{1}{y}dy = -\int 2x\,dx$$ Integrating both sides.

$$\ln|y| = -x^2 + C_1$$

$$e^{\ln|y|} = e^{-x^2 + C_1}$$ Exponentiate both sides to solve for y.

$$|y| = e^{C_1}e^{-x^2}$$

$$|y| = C_2 e^{-x^2}$$ e^{C_1} is a positive constant; let $e^{C_1} = C_2$ and $C_2 > 0$.

If $y > 0$, then $|y| = y$ and the solution becomes

$$y = C_2 e^{-x^2}.$$

If $y < 0$, then $|y| = -y$ and the solution becomes

$$y = -C_2 e^{-x^2}.$$

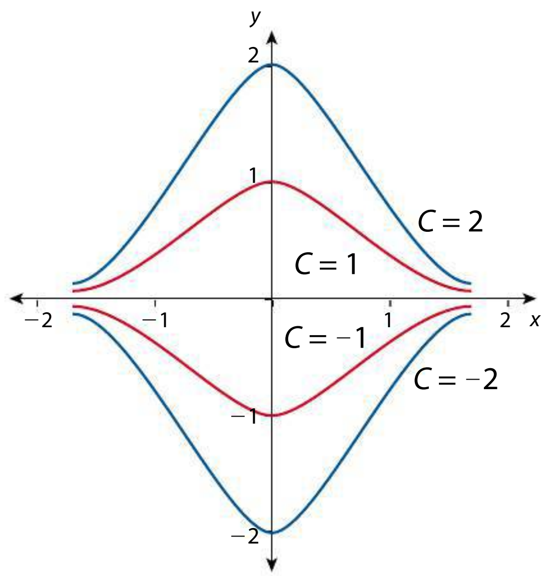
We can include both of these solutions, and also the 'lost' solution $y = 0$, by giving the general solution as

$$y = Ce^{-x^2}$$

with no restrictions on the constant C.

It is helpful for our review to recognize that the explicit solution $y = Ce^{-x^2}$ for Example 6 defines a 'family' of curves in the xy-plane. Some of these curves, with the corresponding value of C, have been graphed in Figure 5.9. In order to determine a specific curve from this 'family' we must impose an initial condition on the solution.

Figure 5.9



5.3 First order linear differential equations – use of integrating factor

As mentioned previously, a first order differential equation is called such because the first derivative $\frac{dy}{dx}$ appears in the equation. A differential equation is called *linear* when both $\frac{dy}{dx}$ and y appear only to the first power. The standard form for a *first order* ***linear*** *differential equation* is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We wish to develop a method to solve first order linear differential equations of this form (which could also be written as $y' + P(x)y = Q(x)$).

We start by considering a simple case when $Q(x) = 0$, so

$\frac{dy}{dx} + P(x)y = Q(x)$ becomes

$$\frac{dy}{dx} + P(x)y = 0.$$

This equation is variables separable, giving us

$$\frac{1}{y}\frac{dy}{dx} = -P(x).$$

This equation can be integrated in the same way as in Example 6 to give

$$\ln|y| = -\int P(x)\,dx + C_1$$

and following the same steps as in Example 6, we get

$$y = Ce^{-\int P(x)\,dx}$$

which is a general solution for the linear differential equation

$\frac{dy}{dx} + P(x)y = 0.$

However, we wish to find a general solution to the more general first order linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ where $Q(x)$ is not necessarily zero. By applying the product rule and implicit differentiation we observe that

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = \frac{dy}{dx}e^{\int P(x)\,dx} + yP(x)e^{\int P(x)\,dx}$$

$$= e^{\int P(x)\,dx}\left(\frac{dy}{dx} + P(x)y\right).$$

Thus, if we multiply both sides of $\frac{dy}{dx} + P(x)y = Q(x)$ by the factor $e^{\int P(x)\,dx}$ (called an **integrating factor**), we get

$$e^{\int P(x)\,dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)\,dx}Q(x).$$

From the working above, we can substitute $\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right)$ for $e^{\int P(x)\,dx}\left(\frac{dy}{dx} + P(x)y\right)$, yielding

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = e^{\int P(x)\,dx}Q(x).$$

Integrating both sides gives

$$ye^{\int P(x)\,dx} = \int e^{\int P(x)\,dx}Q(x)\,dx + C.$$

We can now solve for y, giving

$$y = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx}Q(x)\,dx + C\right].$$

Solution to first order linear differential equations

Given a first order linear differential equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

the general solution is

$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx}Q(x)\,dx + Ce^{-\int P(x)dx}$$

where C is an arbitrary constant.

Although the expression for the general solution given above looks quite complicated, the **basic steps for solving a first order linear differential equation** by means of an integrating factor are relatively simple.

Step 1: Make sure the equation is in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 2: Compute the integrating factor $e^{\int P(x)dx}$ by finding $\int P(x)\,dx$.

Step 3: Multiply both sides of the equation by the integrating factor.

Step 4: Integrate both sides of the equation. The left side will be $e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right)$ which is equivalent to $\frac{d}{dx}\left(ye^{\int P(x)dx}\right)$ and the integral of this expression is $ye^{\int P(x)dx}$.

Step 5: Obtain an explicit solution for y by dividing both sides by the integrating factor $e^{\int P(x)dx}$.

When computing the integrating factor $e^{\int P(x)dx}$, it is standard practice to omit the constant of integration from the indefinite integral of $P(x)$.

Let's illustrate the five basic solution steps with an example.

Example 7

Find the general solution of $x\frac{dy}{dx} - 2y = x^2$.

Solution

1. $\frac{x}{x}\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{x}$ Divide both sides by x to get equation into standard form.

$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$ Standard form $\frac{dy}{dx} + P(x)y = Q(x)$; $P(x) = -\frac{2}{x}$ and $Q(x) = x$.

2. Integrating factor: $e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$

3. $\frac{1}{x^2}\left[\frac{dy}{dx} - \left(\frac{2}{x}\right)y\right] = \frac{1}{x^2}(x)$ Multiply both sides by integrating factor.

$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x}$

4. $\int\left(\frac{1}{x^2}\frac{dy}{dx}-\frac{2y}{x^3}\right)dx=\int\frac{1}{x}dx$ Integrate both sides with respect to x.

$y\left(\frac{1}{x^2}\right)=\ln|x|+C$ $\frac{d}{dx}\left[y\left(\frac{1}{x^2}\right)\right]=\frac{1}{x^2}\frac{dy}{dx}-\frac{2y}{x^3}$, by product rule and implicit differentiation.

5. Therefore, $y=x^2\ln|x|+Cx^2$ is the general solution.

It is appropriate to call the differential equation $\frac{dy}{dx}+P(x)y=Q(x)$ **linear** because $\frac{dy}{dx}=-P(x)y+Q(x)$ is a **linear function** of y.

Example 8

Find the particular solution of

$$\left(x^2+1\right)\frac{dy}{dx}+xy=(1-2x)\sqrt{x^2+1}$$

given that $y=2$ when $x=1$.

Solution

1. $\frac{x^2+1}{x^2+1}\frac{dy}{dx}+\frac{xy}{x^2+1}=\frac{(1-2x)\sqrt{x^2+1}}{x^2+1}$

$\frac{dy}{dx}+\left(\frac{x}{x^2+1}\right)y=\frac{1-2x}{\sqrt{x^2+1}}$ Standard form with $P(x)=\frac{x}{x^2+1}$, $Q(x)=\frac{1-2x}{\sqrt{x^2+1}}$.

2. Integrating factor:

$$\int P(x)\,dx=\int\frac{x}{x^2+1}dx=\frac{1}{2}\ln\left(x^2+1\right)=\ln\sqrt{x^2+1}\ \Rightarrow\ e^{\ln\sqrt{x^2+1}}=\sqrt{x^2+1}$$

3. $\sqrt{x^2+1}\frac{dy}{dx}+\sqrt{x^2+1}\left(\frac{x}{x^2+1}\right)y=\frac{\sqrt{x^2+1}(1-2x)}{\sqrt{x^2+1}}$ Multiply both sides by integrating factor.

$$\sqrt{x^2+1}\frac{dy}{dx}+\left(\frac{x}{\sqrt{x^2+1}}\right)y=1-2x$$

4. $\int\left[\sqrt{x^2+1}\frac{dy}{dx}+\left(\frac{x}{\sqrt{x^2+1}}\right)y\right]dx=\int(1-2x)\,dx$ Integrate both sides.

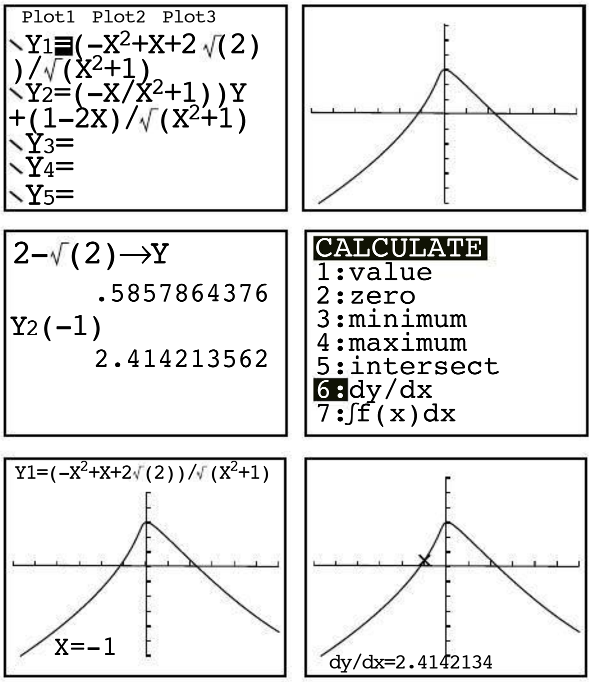
$$y\sqrt{x^2+1}=x-x^2+C$$

5. $y=\frac{-x^2+x+C}{\sqrt{x^2+1}}$ Divide both sides by integrating factor.

To solve for C, we substitute $y=2$ and $x=1$.

$$2=\frac{-1+1+C}{\sqrt{1+1}}\ \Rightarrow\ C=2\sqrt{2}$$

Therefore, the particular solution is $y=\frac{-x^2+x+2\sqrt{2}}{\sqrt{x^2+1}}$.



Once again, with a bit of effort, we can add some confidence to our result for Example 8 by using our GDC to graph the solution curve and then check to see if the original differential equation accurately describes its behaviour (shape).

Enter the solution curve for Y_1 and enter the differential equation in the form $\frac{dy}{dx} = -\left(\frac{x}{x^2+1}\right)y + \frac{1-2x}{\sqrt{x^2+1}}$ for Y_2. Turn Y_2 'off' (un-highlight) so that it is not graphed; only the solution curve is graphed. Choose a value for x that is in the graph window – say, $x = -1$; and evaluate the corresponding y-value for a point on the solution curve.

$y = \frac{-(-1)^2 - 1 + 2\sqrt{2}}{\sqrt{(-1)^2+1}} = \frac{-2+2\sqrt{2}}{\sqrt{2}} = 2-\sqrt{2}$; point $\left(-1, 2-\sqrt{2}\right)$ is on the solution curve. After setting y equal to $2-\sqrt{2}$, use Y_2 to find the value of $\frac{dy}{dx}$ at $\left(-1, 2-\sqrt{2}\right)$. Check that this value for the slope of the curve at $\left(-1, 2-\sqrt{2}\right)$, found to be approximately 2.414 213 562, agrees with the value found on the graph window. Both methods of finding $\frac{dy}{dx}$ at $\left(-1, 2-\sqrt{2}\right)$, from the differential equation and from the solution to the differential equation, give the same value, thus supporting our particular solution to the differential equation.

Example 9

In the earlier section on slope fields, we displayed a slope field (Figure 5.2) for the differential equation $\frac{dy}{dx} = x - y$. Find the general solution to this equation.

Solution

The equation first appears that it may be separable, but it cannot be expressed in the form $\frac{dy}{dx} = p(x)q(y)$. It is a first order linear differential equation because it can be rearranged to $\frac{dy}{dx} + y = x$ which puts it into the standard form $\frac{dy}{dx} + P(x)y = Q(x)$ such that $P(x) = 1$ and $Q(x) = x$. The integrating factor is $e^{\int dx} = e^x$, and multiplying through by this gives

$$e^x\frac{dy}{dx} + e^x y = e^x x$$

and continuing with the steps for solving a first order linear differential equation yields

$$\int\left(e^x \frac{dy}{dx} + e^x y\right) dx = \int e^x x \, dx$$

$$ye^x = e^x x - e^x + C \qquad \text{Using integration by parts on the right side.}$$

Thus, the general solution is $y = x - 1 + Ce^{-x}$. Figure 5.10 shows the same slope field displayed in Figure 5.2 for $\frac{dy}{dx} = x - y$ along with the graphs of three different solution curves generated from the general solution for $C = 1, \frac{1}{10}$ and -4.

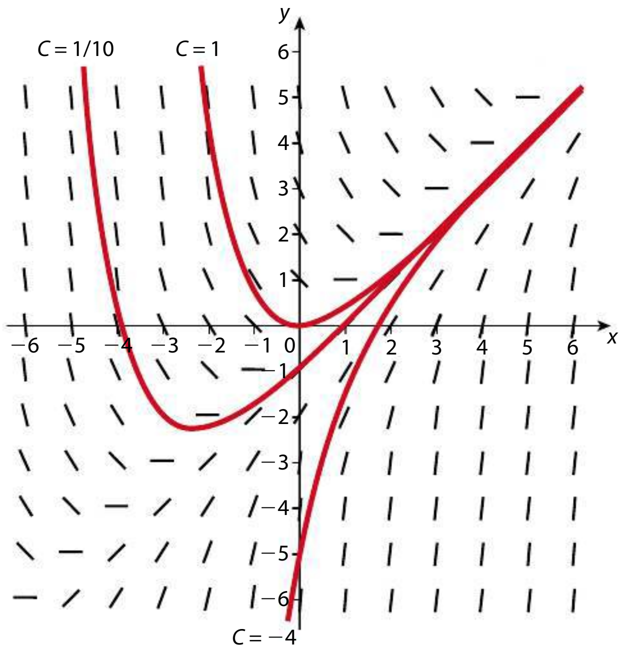


Figure 5.10

An analytic method for solving differential equations, such as those for separable equations and first order linear equations, demand fluency with a range of integration techniques, and differentiation – as the next example nicely illustrates.

Example 10

Find the particular solution to $(1 + \sin x)\frac{dy}{dx} - y\cos x = (1 + \sin x)^4$ given $y(0) = 1$.

Solution

Dividing through by $1 + \sin x$, the equation becomes

$$\frac{dy}{dx} - \left(\frac{\cos x}{1 + \sin x}\right) y = (1 + \sin x)^3.$$

The integrating factor is $e^{\int \frac{-\cos x}{1+\sin x} dx} = e^{-\ln(1+\sin x)} = e^{\ln\left[(1+\sin x)^{-1}\right]} = \dfrac{1}{1 + \sin x}$.

Multiplying both sides by the integrating factor gives

$$\frac{1}{1+\sin x}\frac{dy}{dx} - \left(\frac{\cos x}{(1+\sin x)^2}\right)y = (1+\sin x)^2.$$

Our experience of differentiating functions and familiarity with the solution pattern for first order differential equations, informs us that the left side is equal to the derivative of $\frac{y}{1+\sin x}$.

$$\frac{d}{dx}\left(\frac{y}{1+\sin x}\right) = \sin^2 x + 2\sin x + 1$$

We now integrate both sides. The integral of the left is simply $\frac{y}{1+\sin x}$ and integrating each term on the right is straightforward except for $\sin^2 x$. We need to take the double-angle identity $\cos 2x = 1 - 2\sin^2 x$ and rearrange it to give us $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

$$\int \frac{d}{dx}\left(\frac{y}{1+\sin x}\right)dx = \int\left(\frac{1}{2} - \frac{1}{2}\cos 2x + 2\sin x + 1\right)dx$$

$$\frac{y}{1+\sin x} = \frac{x}{2} - \frac{1}{4}\sin 2x - 2\cos x + x + C$$

$$y = (1+\sin x)\left(\frac{3x}{2} - \frac{1}{4}\sin 2x - 2\cos x + C\right)$$

Given $y(0) = 1$, it follows that $1 = (1+0)(0-0-2+C) \quad \Rightarrow \quad C = 3$

Therefore, the particular solution is

$$y = \frac{1}{4}(1+\sin x)(6x - \sin 2x - 8\cos x + 12).$$

5.4 Homogeneous differential equations

When a first order differential equation is not separable nor linear, it may still be possible to transform it by an appropriate substitution into an equation that we can solve analytically. One situation where this will always work is when the first order differential equation is **homogeneous**.

Homogeneous first order differential equations

The differential equation $\frac{dy}{dx} = F(x, y)$ is **homogeneous** if the right side can be expressed as a function of the ratio $\frac{y}{x}$ alone, that is, $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

The function F can be written as a function of $\frac{y}{x}$ if it can be expressed as a quotient of two **homogeneous functions of the same degree.** In general, a two-variable function is homogeneous of degree n if the sum of the powers of x and y in *each* term is n. For example: $g(x, y) = 2x^2 + xy - 5y^2$

is homogeneous of degree 2; and $h(x, y) = 3y^3 - xy^2$ is homogeneous of degree 3. The function $m(x, y) = 4x^2y^2 - x^3y^2$ is *not* homogeneous.

Thus, if we solve for $\frac{dy}{dx}$ and get it to be equal to a quotient in the form $\frac{M(x, y)}{N(x, y)}$, where M and N are homogeneous functions of the same degree, then the equation is a homogeneous differential equation. The equation $\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$ can be written as a function of $\frac{y}{x}$, i.e. $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, by dividing through both $M(x, y)$ and $N(x, y)$ by x^n, where n is the degree of M and N. Two examples are given below.

1. $\frac{dy}{dx} = \frac{6xy}{x^2 - y^2}$ is a homogeneous differential equation because both the numerator, $6xy$, and the denominator, $x^2 - y^2$, are homogeneous functions of degree 2. The right side can be expressed in terms of $\frac{y}{x}$ by dividing numerator and denominator by x^2.

$$\frac{dy}{dx} = \frac{\frac{6xy}{x^2}}{\frac{x^2 - y^2}{x^2}} = \frac{6\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)^2}$$

2. $\frac{dy}{dx} = \frac{3y^3 - xy^2}{x^3 + x^2y - xy^2}$ is a homogeneous differential equation because both the numerator, $3y^3 - xy^2$, and the denominator, $x^3 + x^2y - xy^2$, are homogeneous functions of degree 3. We divide numerator and denominator by x^3 to get

$$\frac{dy}{dx} = \frac{\frac{3y^3}{x^3} - \frac{xy^2}{x^3}}{\frac{4x^3}{x^3} + \frac{x^2y}{x^3} - \frac{2xy^2}{x^3}} = \frac{3\left(\frac{y}{x}\right)^3 - \left(\frac{y}{x}\right)^2}{4 + \frac{y}{x} - 2\left(\frac{y}{x}\right)^2}$$

Once a homogeneous differential equation is written in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ it can be solved analytically by making the substitution $y = vx$ $\left(\text{or } v = \frac{y}{x}\right)$ where v is a differentiable function of x. As we will see, this substitution transforms the differential equation into a separable equation for which we have a solution method.

Example 11

Find the particular solution for $xy^2\frac{dy}{dx} = x^3 + y^3$ given $y = 3$ when $x = 1$.

Solution

Dividing both sides by xy^2 reveals that the differential equation is homogeneous because both numerator and denominator on the right side are homogeneous functions of degree 3.

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

Dividing both numerator and denominator by x^3 expresses the derivative in terms of $\frac{y}{x}$.

$$\frac{dy}{dx} = \frac{\frac{x^3}{x^3} + \frac{y^3}{x^3}}{\frac{xy^2}{x^3}} = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

We now let $y = vx$ which means that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ by means of the product rule. Substituting v for $\frac{y}{x}$ and $v + x\frac{dv}{dx}$ for $\frac{dy}{dx}$ produces

$$v + x\frac{dv}{dx} = \frac{1 + v^3}{v^2}$$

which is a separable equation for the variables x and v because it can be written in the form $\frac{dy}{dx} = p(x)q(v)$, as shown below:

$$x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^3}{v^2} - \frac{v^3}{v^2} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{1}{v^2} \quad \Rightarrow \quad \frac{dv}{dx} = \left(\frac{1}{x}\right)\left(\frac{1}{v^2}\right)$$

Separating the variables and integrating:

$$v^2 dv = \frac{1}{x}dx$$

$$\int v^2\, dv = \int \frac{1}{x}dx$$

$$\frac{1}{3}v^3 = \ln|x| + C$$

If $y = 3$ when $x = 1$, then $v = \frac{y}{x} = \frac{3}{1} = 3$, and substituting gives

$9 = \ln 1 + C \quad \Rightarrow \quad C = 9$. Thus,

$$\frac{1}{3}v^3 = \ln|x| + 9 \quad \Rightarrow \quad v^3 = 3\ln|x| + 27$$

Substituting $\frac{y}{x}$ back in for v gives:

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + 27$$

$$y^3 = x^3\left(3\ln|x| + 27\right)$$

Therefore, the particular solution is $y = x\left(3\ln|x| + 27\right)^{\frac{1}{3}}$.

Using Example 11 as a guide we can outline the **basic steps for solving a first order homogeneous differential equation.**

Step 1: Confirm that, or rearrange it so that, the equation is in the form

$\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$, where M and N are homogeneous functions of the same degree.

Step 2: Divide both $M(x, y)$ and $N(x, y)$ by x^n, where n is the degree of M and N, so that the equation is in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

Step 3: Let $y = vx$ from which it follows that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and substitute v for $\frac{y}{x}$ and $v + x\frac{dv}{dx}$ for $\frac{dy}{dx}$ transforming the equation into a separable equation in terms of v and x.

Step 4: By applying the technique of separation of variables, find a solution in terms of v and x.

Step 5: Substitute $\frac{y}{x}$ back in for v and write the solution in terms of y and x.

Do not forget to perform *Step 5* – substituting $\frac{y}{x}$ back in for v – because you must express your final solution in terms of y and x.

Example 12

Consider the differential equation $\frac{dy}{dx} = \frac{x + y}{x - y}$ where $x > 0,\ y > 0$.

a) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = \frac{1 + v^2}{1 - v}$.

b) Hence, find the general solution of the differential equation, giving your answer in the form $C = f(x, y)$.

Solution

a) 1. The equation is already in the form $\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$ where, in this case, M and N are homogeneous of degree 1.

2. Divide numerator and denominator by x.

$$\frac{dy}{dx} = \frac{\frac{x}{x} + \frac{y}{x}}{\frac{x}{x} - \frac{y}{x}} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}}$$

3. Letting $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting v for $\frac{y}{x}$ and $v + x\frac{dv}{dx}$ for $\frac{dy}{dx}$, gives:

$$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

$$x\frac{dv}{dx} = \frac{1 + v}{1 - v} - \frac{v(1 - v)}{1 - v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{1 - v} \qquad \text{Q.E.D.}$$

b) 4. Separating the variables and integrating, yields

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx.$$

To integrate the left side we split up the fraction:

$$\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\arctan v - \frac{1}{2}\ln\left(1+v^2\right) = \ln x + C$$

5. Substituting $\frac{y}{x}$ back in for v gives

$$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln x + C.$$

Solving for C:

$$\arctan\left(\frac{y}{x}\right) - \left[\ln\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}} + \ln x\right] = C$$

$$\arctan\left(\frac{y}{x}\right) - \ln\left(x\sqrt{1+\frac{y^2}{x^2}}\right) = C$$

$$C = \arctan\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2}$$

This is the general solution such that y is an implicit function of x.

Example 13

a) Show that $\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{\sqrt{1+x^2}}$.

b) Show that the solution curve that satisfies the differential equation $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$ with initial conditions $y(0) = -1$ is the parabola $y = \frac{x^2}{4} - 1$. [Hint: Use the result from a) to integrate the separable equation that is in terms of v and x.]

Solution

a) $$\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{x+\sqrt{1+x^2}}\left[\frac{d}{dx}\left(x+\left(1+x^2\right)^{\frac{1}{2}}\right)\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{1+x^2}}$$ Q.E.D.

b) First, divide both sides by x to give $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$. The term $\sqrt{x^2+y^2}$ has a degree of 1, so both numerator and denominator are homogeneous functions of degree 1. Now divide numerator and denominator on right side by x in order to write the equation in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

$$\frac{dy}{dx} = \frac{\frac{y}{x}+\frac{\sqrt{x^2+y^2}}{\sqrt{x^2}}}{\frac{x}{x}} = \frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Letting $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting v for $\frac{y}{x}$ and $v + x\frac{dv}{dx}$ for $\frac{dy}{dx}$, gives:

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \quad \Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separating the variables and integrating:

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx$$

From part a) we know that $\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{\sqrt{1+x^2}}$. Therefore, $\int \frac{1}{\sqrt{1+v^2}}\,dv = \ln\left(v+\sqrt{1+v^2}\right)+C$. Using this result gives:

$$\ln\left(v+\sqrt{1+v^2}\right) = \ln|x| + \ln C$$ Setting arbitrary constant to ln C.

$$e^{\ln\left(v+\sqrt{1+v^2}\right)} = e^{\ln|x|+\ln C}$$ Exponentiating both sides using base of e.

$$v+\sqrt{1+v^2} = Cx$$

$$\left(\sqrt{1+v^2}\right)^2 = (Cx-v)^2$$

$$1+v^2 = C^2x^2 - 2Cxv + v^2$$

$$1 = C^2x^2 - 2Cxv$$

$$1 = C^2x^2 - 2Cx\left(\frac{y}{x}\right)$$ Substituting $\frac{y}{x}$ back in for v.

$$2Cy = C^2x^2 - 1$$

$$y = \frac{1}{2}Cx^2 - \frac{1}{2C}$$

Solve for C given the initial condition $y(0) = -1$:

$$-1 = 0 - \frac{1}{2C} \quad \Rightarrow \quad C = \frac{1}{2}$$

Hence,

$$y = \frac{1}{2}\left(\frac{1}{2}\right)x^2 - \frac{1}{2\left(\frac{1}{2}\right)}.$$

Therefore, the particular solution curve is the parabola $y = \frac{1}{4}x^2 - 1$.

5.5 Euler's method

We have established three analytic methods for solving different types of first order differential equations: separable equations, linear equations (integrating factor) and homogeneous equations (substitution $y = vx$). Also, earlier in this chapter we saw how a slope field is an effective graphical method that provides a rough idea about the behaviour of solutions to a differential equation, especially for an equation that we are not able to solve analytically. To roughly sketch a particular solution to a differential equation using a slope field, we need to know a point (initial condition) that the solution curve passes through in order to have a 'starting point' from which to sketch a curve that will be parallel to the short line segments drawn at representative points that indicate the slope of any solution. Several of the examples in this chapter have found particular solutions to what is referred to as an **initial-value problem** that is stated in the form

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0.$$

Consider the initial-value problem

$$\frac{dy}{dx} = x + y, \quad y(2) = 0.$$

Figure 5.11 shows the slope field for the differential equation $\frac{dy}{dx} = x + y$.

An approximation to the particular solution can be sketched by drawing a smooth curve through the point $(2, 0)$ that follows the slopes in the slope field, as shown in the figure.

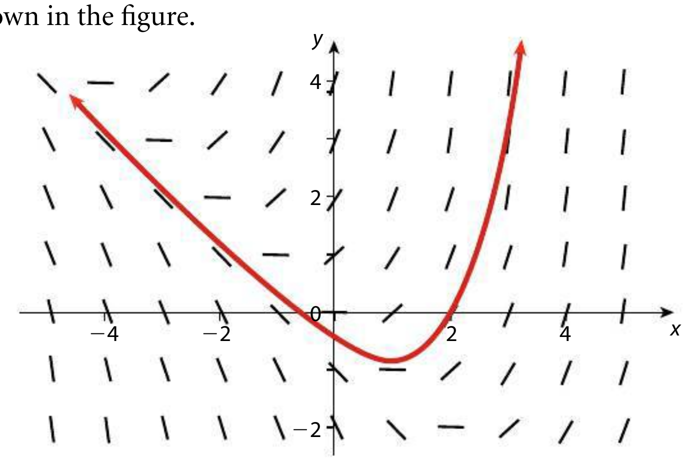


Figure 5.11 Slope field for $\frac{dy}{dx} = x + y$ and sketch of solution passing through (2, 0).

Let $y(x)$ represent the solution curve. To approximate a value of y for a specific value of x, for example y when $x = 3$, we could make an educated guess from the sketch of y made with the aid of the slope field. But if we want a more accurate approximation then we need to use a more refined method. The simplest numerical method is called **Euler's method**, after the prolific eighteenth-century mathematician who first devised this computational method to help him calculate the orbit of our Moon.

Euler's method uses the basic idea behind the construction of slope fields to find numerical approximations to solutions of differential equations. Let's illustrate the method with the initial-value problem that we have just been considering, namely:

$$\frac{dy}{dx} = x + y, \quad y(2) = 0$$

We know from the differential equation that the slope of the solution curve is 2 at the point $(2, 0)$ because $\frac{dy}{dx} = x + y = 2 + 0 = 2$. Hence, the line tangent to the solution curve at $(2, 0)$ has the equation: $y - 0 = 2(x - 2) \Rightarrow y = 2x - 4$. We can use this tangent line as a rough approximation to the solution curve (see Figure 5.12). This approximation clearly becomes less accurate as we move away from the point of tangency $(2, 0)$.

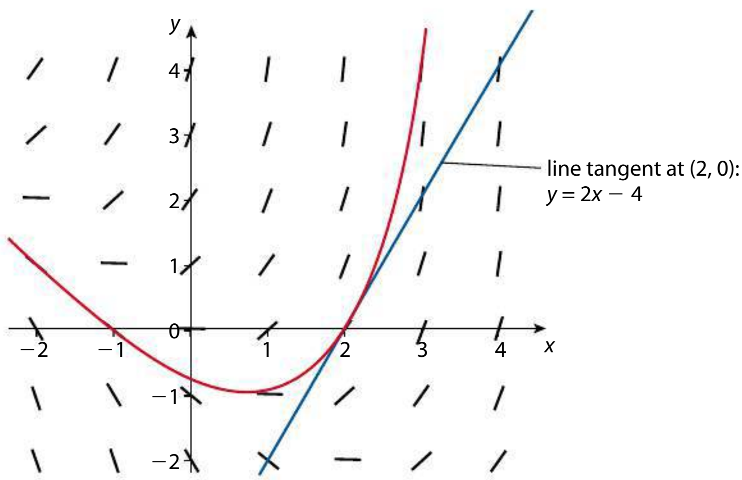


Figure 5.12

Euler's method improves this approximation by moving a short horizontal distance (the **step size** h) along this tangent line and then change direction according to the slope field. In this way we build an approximation to the curve by attaching little line segments together, each having the slope of the solution curve at its starting point.

In general, after being presented with an initial value problem: $\frac{dy}{dx} = F(x, y)$, $y(x_0) = y_0$ we choose a step size h. Starting at the point (x_0, y_0), for the interval $x_0 \leqslant x \leqslant x_0 + h$, we approximate the solution curve with the tangent line, i.e. the line with slope $F(x_0, y_0)$. This takes us as far as the point (x_1, y_1), whose coordinates are calculated as follows (see Figure 5.13):

$$x_1 = x_0 + h,\ y_1 = y_0 + hF(x_0, y_0)$$

Now we are at the starting point of the second line segment (x_1, y_1). We repeat the process, with the next line segment having slope $F(x_1, y_1)$. This takes us to the next point (x_2, y_2) on the Euler approximation where $x_2 = x_1 + h$ and $y_2 = y_1 + hF(x_1, y_1)$.

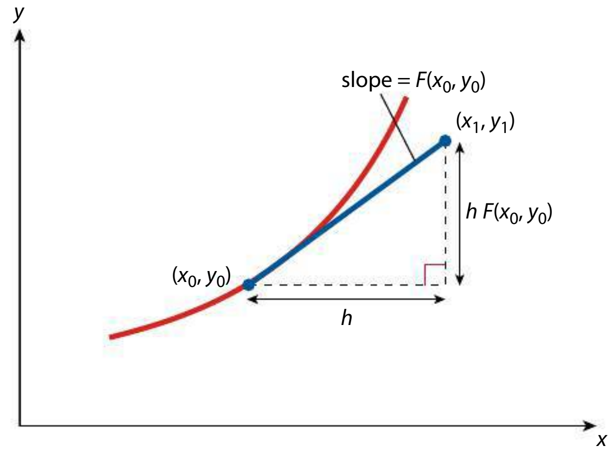
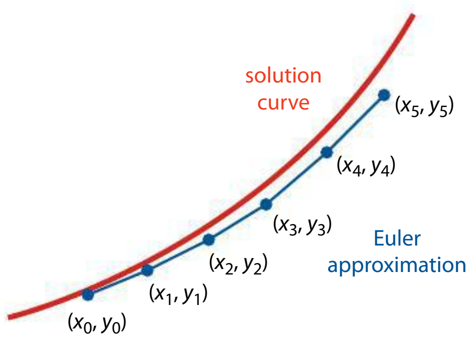


Figure 5.13 Euler's method starts at (x_0, y_0) on the solution curve and moves along a segment with slope $F(x_0, y_0)$ to define a new point (x_1, y_1) such that $x_1 = x_0 + h$ and $y_1 = y_0 + hF(x_0, y_0)$. The process is repeated with the new point.

Repeating this process we get an approximation to the solution curve consisting of line segments joining the points (x_0, y_0), (x_1, y_1), (x_2, y_2), etc. Each computed value y_n is an estimate of the corresponding 'true solution' y at $x = x_n$. The accuracy of the estimates depends on the choice of the step size h and the overall number of steps (iterations). Decreasing the step size while increasing the number of steps leads to increasingly more accurate estimates for solution values.

Figure 5.14 Further iterations of Euler's method build an approximation to the solution curve.



Euler's numerical method

For the differential equation $\frac{dy}{dx} = F(x, y)$ with the initial condition $y(x_0) = y_0$, the recursive formulae for generating the coordinates of the unknown $(n + 1)$st point (x_{n+1}, y_{n+1}) from the known nth point (x_n, y_n) on the approximate solution curve (Euler approximation) are:

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + hF(x_n, y_n) \qquad \text{for } n = 0, 1, 2, \ldots, N$$

where h, the step size, is a constant; and N is the total number of steps (iterations).

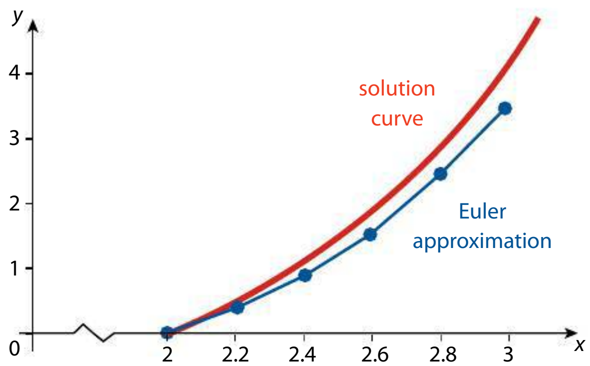
Let's now apply Euler's method to answer a question posed earlier for the initial-value problem presented at the start of this section.

Example 14

For the differential equation $\frac{dy}{dx} = x + y$ such that $y(2) = 0$, use Euler's method with a step value of 0.2 to find an approximate value of y when $x = 3$, giving your answer to two decimal places.

Solution

Figure 5.15



We use Euler's method to build an approximation to the 'true' solution curve starting at $x = 2$ and finishing at $x = 3$ by piecing together five short segments (Figure 5.15). We are given that $h = 0.2$, $x_0 = 2$, $y_0 = 0$ and $F(x, y) = x + y$. Using the appropriate formulae for x_n and y_n and iterating five times, we have:

$x_1 = x_0 + h = 2 + 0.2 = 2.2$

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2(2 + 0) = 0.4$

$x_2 = x_1 + h = 2.2 + 0.2 = 2.4$
$y_2 = y_1 + hF(x_1, y_1) = 0.4 + 0.2(2.2 + 0.4) = 0.92$

$x_3 = x_2 + h = 2.4 + 0.2 = 2.6$
$y_3 = y_2 + hF(x_2, y_2) = 0.92 + 0.2(2.4 + 0.92) = 1.584$

$x_4 = x_3 + h = 2.6 + 0.2 = 2.8$
$y_4 = y_3 + hF(x_3, y_3) = 1.584 + 0.2(2.6 + 1.584) = 2.4208$

$x_5 = x_4 + h = 2.8 + 0.2 = 3$
$y_5 = y_4 + hF(x_4, y_4) = 2.4208 + 0.2(2.8 + 2.4208) = 3.46496$

This process leads to an approximate (three decimal places) value of $y \approx 3.46$ when $x = 3$.

Because we will perform most of the calculations for each iteration on our GDC, it is often sufficient to simply display relevant results for each iteration in a table, as shown below.

n	x_n	y_n	$hF(x_n, y_n)$	x_{n+1}	y_{n+1}
0	2	0	0.4	2.2	0.4
1	2.2	0.4	0.52	2.4	0.92
2	2.4	0.92	0.664	2.6	1.584
3	2.6	1.584	0.8368	2.8	2.4208
4	2.8	2.4208	1.044 16	3.0	3.464 96

The first order differential equation in Example 14 is linear and hence can be solved by means of an integrating factor. Given $y(2) = 0$ the particular solution is $y = 3e^{x-2} - x - 1$. To three significant figures, the 'true' value of $y(3)$ is approximately 5.15. Thus, our approximation of 3.46 has an error of approximately 16.6%. Using a program on our GDC or a spreadsheet, we could easily decrease the step size (and increasing the number of steps) in order to improve the accuracy of the approximation. For example, if we used a step size of $h = 0.01$ (requiring 100 iterations) we would get an estimate of 5.11 (3 s.f.), reducing the error to less than 1%.

A numerical method like Euler's is especially useful when applied to a differential equation that cannot be solved by any known analytic methods, as we will do in the next example.

Example 15

Given that $\frac{dy}{dx} = \frac{x+1}{xy+2}$ and $y = 1$ when $x = 0$, use Euler's method with step size $h = 0.25$ to approximate the value of y when $x = 1$. Give the approximation to three significant figures.

Solution

We have that $x_0 = 0$, $y_0 = 1$, $h = 0.25$ and $F(x, y) = \frac{x+1}{xy+2}$. Thus the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x, y) = y_n + (0.25)\frac{x_n + 1}{x_n y_n + 2} \quad \Rightarrow \quad y_{n+1} = y_n + \frac{x_n + 1}{4x_n y_n + 8}$$

$n = 0$: $\quad x_1 = x_0 + h = 0 + 0.25 = 0.25$

$$y_1 = y_0 + \frac{x_0 + 1}{4x_0 y_0 + 8} = 1 + \frac{0 + 1}{4(0)(1) + 8} = \frac{9}{8} = 1.125$$

$n = 1$: $\quad x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$$y_2 = y_1 + \frac{x_1 + 1}{4x_1 y_1 + 8} = 1.125 + \frac{0.25 + 1}{4(0.25)(1.125) + 8} \approx 1.261986$$

$n = 2$: $\quad x_3 = x_2 + h = 0.5 + 0.25 = 0.75$

$$y_3 = y_2 + \frac{x_2 + 1}{4x_2y_2 + 8} = 1.261986 + \frac{0.5 + 1}{4(0.5)(1.261986) + 8} \approx 1.404518$$

$n = 3$: $\quad x_4 = x_3 + h = 0.75 + 0.25 = 1$

$$y_4 = y_3 + \frac{x_3 + 1}{4x_3y_3 + 8} = 1.404518 + \frac{0.75 + 1}{4(0.75)(1.404518) + 8} \approx 1.547801$$

Therefore, the approximate value of y when $x = 1$ is $y \approx 1.55$.

Exercise 5

1 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{\sqrt{1+x^2}}$. Given that $y = 1$ when $x = 0$, express y as an explicit function of x.

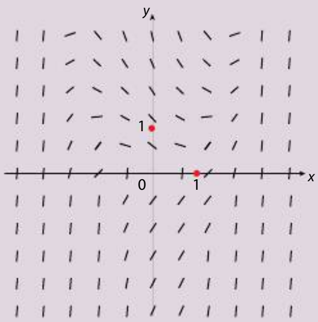
2 Find the particular solution to the differential equation $\dfrac{dy}{dx} = \sin x \cos^2 y$ given that $y = \dfrac{\pi}{4}$ when $x = \dfrac{\pi}{2}$.

3 The solution curve to the differential equation $x\dfrac{dy}{dx} = y(3 - y)$ passes through the point (2, 2). Find y as an explicit function of x.

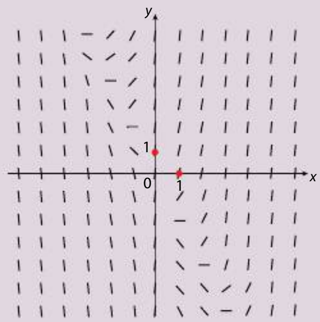
4 Show that the general solution to the differential equation $x\dfrac{dy}{dx} = y \ln x$ is $y = Cx^{\ln\sqrt{x}}$.

5 Match each slope field with its differential equation, listed below.

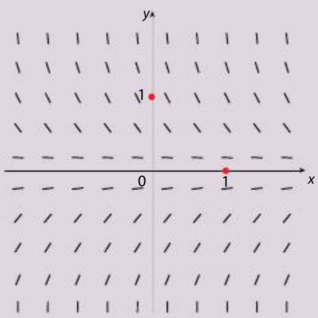
a



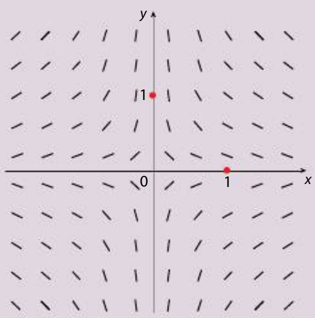
b



c



d



i $\dfrac{dy}{dx} = -2y$ $\qquad$ **ii** $\dfrac{dy}{dx} = x^2 - y$

iii $\dfrac{dy}{dx} = -\dfrac{y}{x}$ $\qquad$ **iv** $\dfrac{dy}{dx} = 2x + y$

6 All radioactive substances decay at a rate proportional to the amount of the substance that exists at any time. The half-life of radium is 1620 years. How much (accurate to 3 significant figures) of a 10-gram specimen of radioactive radium will remain after 25 years?

7 Solve the following separable differential equations.

a $\frac{dy}{dx} = \frac{2x}{y}$ **b** $\frac{dy}{dx} = \frac{y^2}{x^2}$ **c** $x^2\frac{dy}{dx} = y^2 - y$

d $x\frac{dy}{dx} = \tan y$ **e** $\frac{dy}{dx} = xy$ **f** $\sqrt{x^2+1}\frac{dy}{dx} = \frac{x}{y}$

g $\frac{dy}{dx} = \frac{y^2-1}{e^x}$ **h** $\ln y\frac{dy}{dx} = 1$

8 Using the method of separation of variables, show that an implicit solution for the differential equation $\frac{dy}{dx} = \frac{xy+y}{xy+x}$ is $ye^y = Axe^x$ where A is an arbitrary constant.

9 Find the general solution, in explicit form, to the differential equation $y\frac{dy}{dx} = \cos x$. Comment on the possible values of the constant C.

10 The equation for the rate of change of the population (in thousands), p, of a certain species is given by

$$\frac{dp}{dt} = 5p - 2p^2.$$

a Sketch the slope field.

b If the initial population is 4000 (that is, $p(0) = 4$), then what appears to be the limiting value of the population (that is, $\lim_{t\to\infty} p(t)$)?

c If $p(0) = 0.5$, what is $\lim_{t\to\infty} p(t)$?

d Comment on the long-term behaviour of the species' population growth.

11 Solve the initial-value problem:

$$\frac{dy}{dx} = \frac{2x + \sec^2 x}{2y}, \quad y(0) = -5$$

12 Consider the initial-value problem:

$$(1+x^2)\frac{dy}{dx} + 1 + y^2 = 0, \quad y(0) = -1$$

a Show that the implicit solution can be expressed as $\arctan y + \arctan x = \frac{\pi}{4}$.

b Use the formula for $\tan(A + B)$ to find the explicit solution.

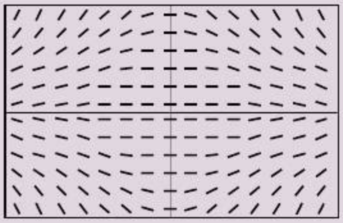
13 Solve the initial-value problem:

$$(1+x^2)\frac{dy}{dx} = 1 + y^2, \quad y(2) = 3$$

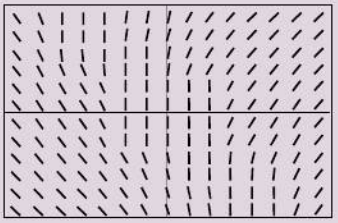
Write the solution in explicit form, expressing y in terms of x.

14 Match each slope field with its differential equation, listed below.

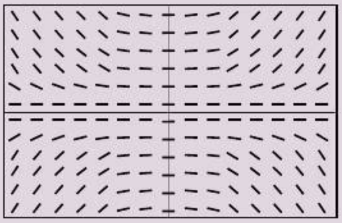
a



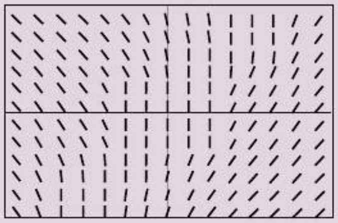
b



c



d



i $\dfrac{dy}{dx} = \dfrac{5}{x+y}$ **ii** $\dfrac{dy}{dx} = \dfrac{5}{x-y}$

iii $\dfrac{dy}{dx} = -\dfrac{xy}{10}$ **iv** $\dfrac{dy}{dx} = \dfrac{xy}{10}$

15 **a** Use the method of partial fractions to express $\dfrac{1}{x^2 - x - 2}$ as the sum of two fractions.

b Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x^2 - x - 2}$, $x > 2$ such that $y = 1$ when $x = 5$. Show that the solution is $2e^{\frac{3-3y}{y}} = \dfrac{x+1}{x-2}$.

16 Consider the differential equation $(1 - x^2)\dfrac{dy}{dx} + 2xy = 2x$.

a Find the general solution in the form $y = f(x)$ by the method of separation of variables.

b Write the differential equation in the standard form for a first order linear differential equation, $\dfrac{dy}{dx} + yP(x) = Q(x)$, and find the general solution by means of an integrating factor.

17 Solve each of the following first order linear differential equations.

a $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 6x^3$ **b** $\dfrac{dy}{dx} - xy = x$

c $\dfrac{dy}{dx} - \dfrac{y}{x} = x^3$ **d** $\dfrac{dy}{dx} + y\sin x = e^{\cos x}$

e $\dfrac{dy}{dx} - 3x^2 y = e^{x^3}$ **f** $x\dfrac{dy}{dx} = x + y$

18 Solve the first order linear differential equation

$\tan x\dfrac{dy}{dx} + y = \sec x$ giving your answer in the form $y = f(x)$.

19 Consider the initial-value problem:

$\dfrac{dy}{dx} - \dfrac{xy}{1-x^2} = 1$, $y(0) = 1$

a Show that the differential equation is a first order linear equation by writing it in the form $\dfrac{dy}{dx} + yP(x) = Q(x)$.

b Show that the integrating factor is $\sqrt{1-x^2}$.

c By using the substitution $x = \sin u$, show that

$$\int\sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C.$$

d Find the solution to the initial-value problem expressed in the form $y = f(x)$.

20 **a** Show that $\int \tan x\,dx = -\ln|\cos x|$.

b Show that $\frac{dy}{dx} = 1 + y\tan x$ is a first order linear differential equation.

c Find the general solution of $\frac{dy}{dx} = 1 + y\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

21 Find the particular solution to the differential equation $\frac{dy}{dx} = \frac{x^2\ln x - y}{x}$ given that $y = 1$ when $x = 1$.

22 Find the general solution, in explicit form, to the differential equation

$$x^2\frac{dy}{dx} - x^3 + xy = 0.$$

23 Find the general solution to the first order homogenous differential equation

$$\frac{dy}{dx} = \frac{3y-x}{3x-y}.$$

Write the answer in the form $C = f(x, y)$.

24 Solve each of the following first order homogeneous differential equations.

a $\frac{dy}{dx} = \frac{y}{x+1}$

b $\frac{dy}{dx} = \frac{x+2y}{x}$

c $x\frac{dy}{dx} = 2x + 3y$

d $\frac{dy}{dx} = -\frac{2x^2+y^2}{2xy+3y^2}$

e $xy\frac{dy}{dx} = x^2 - y^2$

f $x(y-x)\frac{dy}{dx} = y(x+y)$

25 Consider the differential equation $\frac{dy}{dx} = \frac{x+2y}{3y-2x}$, for $x > 0$.

a Use the substitution $y = vx$ to show that $v + x\frac{dv}{dx} = \frac{1+2v}{3v-2}$.

b Hence, find the solution of the differential equation, given that $y = 0$ when $x = 1$.

26 Use the substitution $y = vx$ to show that the general solution to the differential equation

$$y^2 - x^2 + xy\frac{dy}{dx} = 0 \text{ is } 2x^2y^2 - x^4 = C,$$ where C is a constant.

27 Consider the initial-value problem:

$$x\frac{dy}{dx} = y + \sqrt{x^2-y^2}, \quad y(1) = 1$$

a Use the substitution $y = vx$ to show that $v + x\frac{dv}{dx} = v + \sqrt{1-v^2}$.

b Hence, show that the solution is $\arcsin\left(\frac{y}{x}\right) = \ln|x| + \frac{\pi}{2}$.

28 Consider the differential equation $\frac{dy}{dx} = \frac{y^2+y}{x}$.

a Find the general solution.

b Given that $y = 1$ when $x = 1$, find a particular solution solved explicitly for y.

c Use Euler's method with step size $h = 0.2$ to approximate the solution at $x = 1.2, 1.4, 1.6$ and 1.8.

d Compute the percentage error for each of the approximate solutions found in **c** compared to the solution for the same value of x found using the explicit solution found in **b**.

29 Given that $\frac{dy}{dx} = xy^2$ and $y = 1$ at $x = 0$, use Euler's method with 5 steps to approximate the value of y at $x = 1$.

30 Use Euler's method with step size $h = 0.1$ to approximate the value of y when $x = 1$ for the differential equation $\frac{dy}{dx} = e^{xy}$ given that the solution curve passes through the point $(0, 1)$.

31 Use the substitution $y = vx$ to find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}.$$

32 Given that $\frac{dy}{dx} = x\sqrt{y}$ and $y = 4$ when $x = 1$, use Euler's method with step size $h = 0.1$ to approximate the solution at $x = 1.1, 1.2, 1.3, 1.4$ and 1.5.

33 Consider the initial-value problem:

$$\frac{dy}{dx} = x - y, \qquad y(0) = 0$$

a Show that the solution is $y = e^{-x} + x - 1$.

b Use Euler's method with 5 steps to find an approximate value of y when $x = 1$.

c Use Euler's method with 10 steps to find another approximation for $y(1)$.

d Compare the approximate values for $y(1)$ found in **b** and **c** to the actual value using the solution $y = e^{-x} + x - 1$. Comment.

Practice questions 5

1 Find the general solution to the differential equation $\frac{dy}{dx} = e^x\left(1 + y^2\right)$.

2 Show that the general solution to the differential equation $\frac{dy}{dx} = e^{x-y}$ is $y = \ln(Ce^x)$.

3 Find the general solution to the differential equation $\frac{dy}{dx} = -xy$.

4 The rate, in degrees Celsius per minute, at which the temperature of a cup of tea decreases is given by $-k(\alpha - 20)$ where α is the temperature in degrees Celsius and k is a constant. When $t = 0$ minutes $\alpha = 70°$, and when $t = 10$ minutes $\alpha = 50°$.

Find an equation for the temperature in terms of time t.

5 A curve that satisfies the differential equation $\frac{dy}{dx} = xy \sin x$ goes through the point $\left(\frac{\pi}{2}, 1\right)$. Show that the equation of the curve is $y = e^{\sin x - x\cos x - 1}$.

6 Consider the differential equation $x\dfrac{dy}{dx} - 3y = x^4$.

a Find the general solution.

b Given that $y = 2$ when $x = 1$, find the particular solution in explicit form.

7 Given that $y = 2$ when $x = 1$, solve the following differential equation explicitly for y.

$y\dfrac{dy}{dx} - 3x = x^4$

8 Find the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x^2}$, $x \neq 0$.

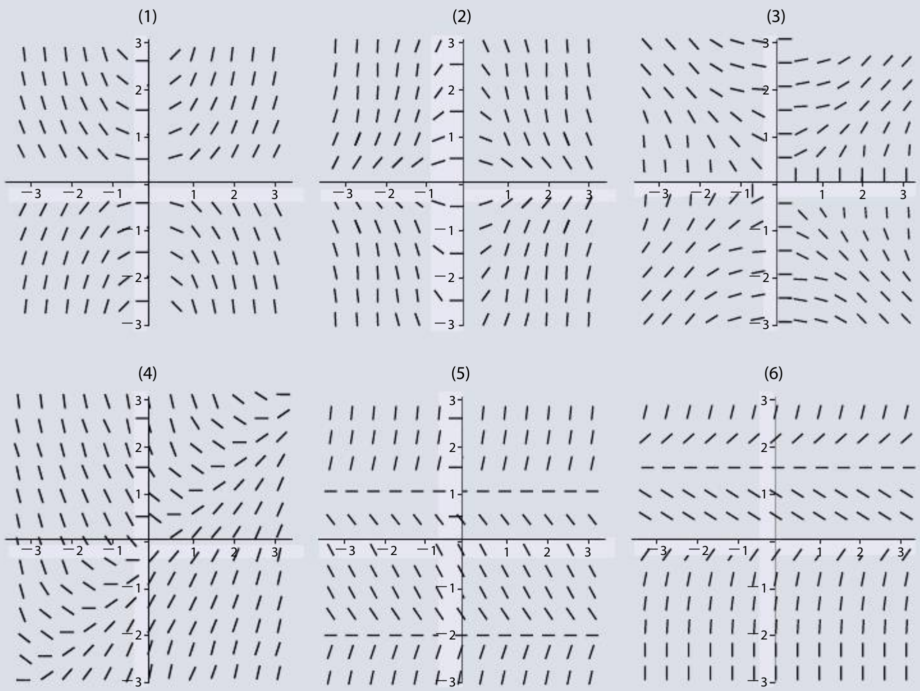
9 Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = \cos x$, $x \neq 0$, giving your answer in the form $y = f(x)$.

10 Consider the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$.

a Find the general solution.

b Given that $y = 20$ when $x = 4$, find an explicit solution for y in terms of x.

11 Match each of the differential equations with its direction field.



a $\dfrac{dy}{dx} = y(y - 1.5)$ **b** $\dfrac{dy}{dx} = xy$ **c** $\dfrac{dy}{dx} = -xy$

d $\dfrac{dy}{dx} = \dfrac{x}{y}$ **e** $\dfrac{dy}{dx} = x - y$ **f** $\dfrac{dy}{dx} = (y - 1)(y + 2)$

12 Find an equation for the curve that passes through the point $\left(\dfrac{\pi}{6}, 0\right)$ and for which the slope of the curve at any point (x, y) on the curve is $\dfrac{2y + 4}{\tan x}$.

13 For all positive values of x the slope of a curve at the point (x, y) is given by $\dfrac{y}{x^2 + x}$. The point $P(3, 6)$ lies on this curve. Find:

a the equation of the normal to the curve at P.

b the equation of the curve where y is expressed in terms of x.

14 Consider the differential equation $x\frac{dy}{dx}+2y=x^2-x+1$.

a Show that an integrating factor for solving the differential equation is x^2.

b Given that $y=\frac{1}{2}$ when $x=1$, solve the differential equation. Give the answer in the form $y=f(x)$.

15 Consider the differential equation $\frac{dy}{dx}=\frac{3y^2+x^2}{2xy}$, for $x>0$.

a Use the substitution $y=vx$ to show that $v+x\frac{dv}{dx}=\frac{3v^2+1}{2v}$.

b Hence, find the solution of the differential equation given that $y=2$ when $x=1$.

16 Consider the differential equation $x^2\frac{dy}{dx}=y^2+5xy+5x^2$ such that $y=-2$ when $x=1$. Using the substitution $y=vx$, show that the solution to the differential equation is $y=x\tan\left(\ln x+\frac{\pi}{4}\right)-x$.

17 Consider the differential equation $\frac{dy}{dx}=x^2+y^2$ where $y=2$ when $x=0$.

a Use Euler's method with step length 0.25 to find an approximate value of y when $x=1$.

b Write down, giving a reason, whether your approximate value for y is greater or less than the actual value of y.

18 Solve the differential equation $(x-y)\frac{dy}{dx}+x+y=0$ given that $y=0$ when $x=e$. Give the answer in the form $y=f(x)$.

19 Given that $\frac{dy}{dx}=\frac{y+2}{xy+1}$ and $y=1$ when $x=0$, use Euler's method with interval $h=0.5$ to find an approximate value of y when $x=1$.

20 a Show that the solution for the differential equation $\frac{dy}{dx}=\sec^2 x$ is $y=\tan x+c$.

b Consider the differential equation $(\cos x)\frac{dy}{dx}+(\sin x)y=2\cos^3 x\sin x-1$.

Write the differential equation in the form $\frac{dy}{dx}+P(x)y=Q(x)$, and find the integrating factor.

c Given $0\leqslant x<\frac{\pi}{2}$ and $y=3\sqrt{2}$ when $x=\frac{\pi}{4}$, show that the solution to the differential equation in (b) is $y=-\frac{1}{2}\cos x\cos 2x-\sin x+7\cos x$.

21 Consider the differential equation $xy\frac{dy}{dx}=3x^2+y^2$ such that $x>0$ and $y>0$.

Given that $y=2$ when $x=1$, show that the solution to the differential equation is $y=6x^2\ln x+4x^2$.

22 Consider the differential equation $\frac{dy}{dx}-2y=\sin x$ with boundary condition $y=1$ when $x=0$.

Use four steps of Euler's method starting at $x=0$, with interval $h=0.1$, to find an approximate value for y when $x=0.4$.

23 a Use integration by parts to show that

$$\int \sin x \cos x e^{-\sin x}\, dx = -e^{-\sin x}(1 + \sin x) + C.$$

Consider the differential equation $\frac{dy}{dx} - y\cos x = \sin x \cos x$.

b Find an integrating factor.

c Solve the differential equation given that $y = -2$ when $x = 0$. Give your answer in the form $y = f(x)$.

24 a Sketch on graph paper the slope field for the differential equation $\frac{dy}{dx} = x - y$ at the points (x, y) where $x \in \{0, 1, 2, 3, 4\}$ and $y \in \{0, 1, 2, 3, 4\}$. Use a scale of 2 cm for 1 unit on both axes.

b On the slope field sketch the curve that passes through the point (0, 3).

c Solve the differential equation to find the equation of this curve. Give your answer in the form $y = f(x)$.

25 Given that $\frac{dy}{dx} - 3e^x = y^2$ and $y = 2$ when $x = 0$, use Euler's method with a step length of 0.2 to find an approximation for the value of y when $x = 1$. Give all intermediate values with maximum possible accuracy.

26 Solve the differential equation $x\frac{dy}{dx} + 2y = \sqrt{1 + x^2}$ given that $y = 1$ when $x = \sqrt{3}$.

27 A curve that passes through the point (1, 2) is defined by the differential equation

$$\frac{dy}{dx} = 2x(1 + x^2 - y).$$

a i Use Euler's method to get an approximate value of y when $x = 1.3$, taking steps of 0.1. Show intermediate steps to four decimal places in a table.

ii How can a more accurate answer be obtained using Euler's method?

b Solve the differential equation, giving your answer in the form $y = f(x)$.

Answers

Chapter 1

Exercise 1

1 Converges to 0
2 Converges to 2
3 Converges to 0
4 Diverges
5 Converges to 0
6 Converges to 0
7 Diverges
8 Diverges
9 Converges to $\sqrt{2}$
10 Converges to 1
11 Diverges
12 Converges to 1
13 Converges to 0
14 Converges to 1
15 Converges to 1
16–17 Proof
18 $\frac{1}{2}$
19 2
20 $\frac{1}{2}$
21 Converges to ρ
22 -1
23 $-\frac{1}{3}$
24 $\frac{1}{6}$
25 $\frac{1}{3}$
26 $\ln 2$
27 $\ln\left(\frac{a}{b}\right)$
28 1
29 Divergent
30 $\frac{1}{2}$
31 ρ
32 $\frac{1}{2}$
33 Divergent
34 $\ln 2$
35 2
36 k
37 a) Area increases without bound, i.e. infinite
b) ρ units^3
c) The area of the region is infinite; however, the volume of the solid created by rotating the region about the x-axis is finite.

Practice questions 1

1 Proof
2 $\frac{1}{2}$
3 Diverges
4 a) 6 b) $\frac{1}{3a^2}$
5 $p > 1$
6 a) 0 b) 1
7 Proof
8 a) $\frac{1}{2}$ b) 0
9 a) (ii) $I_n = \frac{1}{2}\ln\left(\frac{1+\alpha^2n^2}{1+n^2}\right)$
(ii) $\lim\limits_{n\to\infty} I_n = \frac{1}{2}\ln\left(\alpha^2\right)$ or $\ln\alpha$
b) -2
10 Proof
11 Incorrect; $\frac{\sin x}{1-\cos x}$ is not of indeterminate form when $x=\pi$; $\lim\limits_{x\to\pi^-}\left(\frac{\sin x}{1-\cos x}\right)=0$.
12 a) $k > -1$ b) $k < -1$
13 2
14 Proof
15 a) 1 b) $\frac{1}{6}$

Chapter 2

Exercise 2

1 a) 8 b) -1 c) 25
2 $\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}+\frac{3}{\sqrt{10}}+\frac{4}{\sqrt{17}}+\ldots$; diverges by nth term divergence test
3 $3+\frac{3}{4}+\frac{3}{16}+\frac{3}{64}+\ldots$; converges to 4
4 $0+\ln\frac{1}{2}+\ln\frac{1}{3}+\ln\frac{1}{4}+\ldots$; diverges by nth term divergence test
5 $\frac{3}{2}-\frac{3}{4}+\frac{3}{8}-\frac{3}{16}+\ldots$; converges to 1
6 $\frac{1}{3}+\frac{2}{9}+\frac{2}{9}+\frac{8}{27}+\ldots$; diverges by nth term divergence test
7 $-1+1-1+1-\ldots$; diverges by nth term divergence test
8 $\frac{5}{11}+\frac{7}{16}+\frac{3}{7}+\frac{11}{26}+\ldots$; diverges by nth term divergence test
9 $\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}+\frac{1}{e^4}+\ldots$; converges to $\frac{1}{e-1}$
10 a) $\int xe^{-x}\,dx = -e^{-x}(x+1)+C$
b) $\int_1^\infty xe^{-x}\,dx = \frac{2}{e}$ and therefore the series is convergent.
11 a) Divergent b) Convergent
12–13 Proof
14 For $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, $\lim\limits_{n\to\infty} a_n = 0$ but it is a p-series with $p=\frac{1}{2}\leqslant 1$ so the series diverges.
15 Proof
16 Converges
17 Diverges
18 Converges
19 Converges
20 Converges
21 Diverges
22 Diverges
23 Diverges
24 Diverges
25 Diverges
26 Converges

27 Diverges

28 Converges

29 Converges

30 Diverges

31 5

32 a) $S_4 = \frac{10\,016}{11\,025} \approx 0.908\,48$; error $< \frac{1}{81}$

b) $S_4 = 0.095\,308\overline{3}$; error $< 0.000\,006$

33 a) $(n+1)^2 + 1$

b) $\int_1^\infty \frac{1}{(x+1)^2+1}dx = \lim_{b\to\infty}[\arctan(x+1)]_1^b = \frac{\pi}{2} - \arctan(2)$

$= \arctan\left(\frac{1}{2}\right)$; since $\int_1^\infty \frac{1}{(x+1)^2+1}dx$ converges to

$\arctan\left(\frac{1}{2}\right)$, then $\sum_{n=1}^{\infty}\frac{1}{n^2+2n+2}$ must also converge.

34 Diverges

35 11 terms

36 $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is conditionally convergent.

37 Converges absolutely

38 Converges conditionally

39 Diverges

40 Converges conditionally

41 Converges absolutely

42 Converges absolutely

43 $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{8} + \ldots$; the sum of this series is 1. The terms of the alternating harmonic series are rearranged such that consecutive positive terms are added until the sum is greater than 1, then consecutive negative terms are added until the sum is less than 1, and so on. Note that the difference between the partial sums and 1 is less than the last term used, so the series converges to 1.

44 7 terms

45 Proof

Practice questions 2

1 a) Converges; geometric series with $r = \frac{1}{1.1}$, so $|r| < 1$.

b) Diverges by nth term divergence test.

c) Converges; comparison test, compare to p-series with $p = 3$.

2 a) Converges

b) Diverges

3 Proof

4 a) Series converges by the ratio test.

b) Series converges by the integral test.

c) Series converges by the alternating series test.

5 Diverges by comparison with the harmonic series.

6 a) Integral test for $\sum a_n$: Let $a_n = f(n)$ where $f(x)$ is a continuous, positive and decreasing function for all $x \geqslant N$, where N is some positive integer. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^\infty f(x)\,dx$ both diverge or both converge. That is, if the integral is finite then $\sum a_n$ is finite, and if the integral is infinite then $\sum a_n$ is infinite.

b) Diverges by the integral test.

7 Ratio test gives interval of convergence as $-1 \leqslant x < 1$.

8 Converges conditionally.

9 Proof

10 a) Proof

b) (i) $\frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$

(ii) $\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

11 $-1 \leqslant x < 1$

12 a) Diverges

b) Converges

c) Converges

13 Proof

14 Proof

15 a) Proof

b) Converges conditionally.

Chapter 3

Exercise 3

1 $R = 1;\ -1 \leqslant x < 1$

2 $R = 1;\ 1 < x < 3$

3 $R = 2;\ 2 \leqslant x < 4$

4 $R = \infty;\ x \in \mathbb{R}$

5 $R = 1;\ -1 \leqslant x \leqslant 1$

6 $R = 1;\ 1 \leqslant x \leqslant 3$

7 $R = 1;\ 0 < x < 2$

8 $R = 1;\ -1 \leqslant x < 1$

9 $R = 0;\ x = 0$

10 $R = \frac{4}{3};\ -\frac{4}{3} \leqslant x < \frac{4}{3}$

11 $R = 4;\ -4 < x < 4$

12 $R = 3;\ -3 \leqslant x \leqslant 3$

13 $R = e;\ -e < x < e$

14 $R = 0;\ x = 4$

15 $-\frac{1}{k} < x < \frac{1}{k}$

16 $\sum_{n=0}^{\infty}(-1)^n x^n; -1 < x < 1$

17 a) $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n\frac{x^{2n}}{n!} + \ldots;\ R = \infty$

b) $\int e^{-x^2}dx = \int\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)\,n!} + \ldots\right)$

$= x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)n!} + \ldots;$

radius of convergence is also $R = \infty$.

c) $\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = \frac{5651}{7560} \approx 0.747;$

error $< a_6 = \frac{1}{11\cdot 5!} = 0.000\,\overline{75} < 0.001$

18 a) $x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$

b) $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

c) $x - \frac{1}{2}x^2 + \frac{7}{6}x^3$

19 $\sum_{n=0}^{\infty} nx^{n-1}$ for $-1 < x < 1$

20 a) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+2}}{n!}$

b) Proof

21 a) $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

b) $\sin\left(\frac{\pi}{12}\right) \approx 0.258\,819$

c) Error $< 1.4165 \times 10^{-10}$

22 $-\frac{1}{2} < x < \frac{1}{2}$

23 $(x-1)\,e + (x-1)^2\,e + \frac{(x-1)^3}{2}\,e + \frac{(x-1)^4}{6}\,e$

24 $\sum_{n=1}^{\infty} \frac{2}{(2n-1)} x^{2n-1} = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$

25 a) $\sum_{n=0}^{\infty} (-1)^n\, x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots$

b) Proof c) Proof d) $\pi \approx 2.976$; error $< 0.142\,86$

26 a) $f^{(n)}(x) = \frac{e^x + (-1)^n e^{-x}}{2}$

b) $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots$

c) $f\left(\frac{1}{2}\right) \approx \frac{433}{384} = 1.127\,604\,1\bar{6}$

d) Error $< 0.000\,136$

27 $-1.59 < x < 1.59$

28 $xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots$

29 $\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots$

30 a) $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

b) $\sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$

c) $-\frac{1}{2}\sum_{n=0}^{\infty} (n+1)\,nx^{n-1}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(x-1)^{n+4}$

31 a) $\lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots} = 1$ b) $\lim_{x\to 0} \frac{x - \arctan x}{x^3} = \frac{1}{3}$

Practice questions 3

1 $\ln(\cos x) \approx -\frac{x^2}{2} - \frac{x^4}{12}$

2 a) $\sin^2 x \approx x^2 - \frac{x^4}{3}$ b) $\cos^2 x \approx 1 - x^2 + \frac{x^4}{3}$

3 $e^x \sin x \approx x + x^2 + \frac{x^3}{3}$

4 $e^{3x} \approx 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2}$

5 $\sec x \approx 1 + \frac{x^2}{2} + \frac{5x^4}{24}$

6 a) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

b) $e^x \approx 1 + x^2 + \frac{x^4}{2!}$

c) $e^x \approx 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \frac{25x^4}{24}$

7 $\ln(2+3x) = \ln 2 + \frac{3}{2}x - \left(\frac{3}{2}\right)^2 \frac{x^2}{2} + \left(\frac{3}{2}\right)^3 \frac{x^3}{3} - \left(\frac{3}{2}\right)^4 \frac{x^4}{4} + \ldots;$

$R_n(x) = \frac{(-1)^n\, 3^{n+1}}{(n+1)\,(2+3c)^{n+1}}\, x^{n+1}$

8 a) $\sqrt{4+x} \approx 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} - \frac{5x^4}{16\,384}$

b) $R_4(x) = \frac{7}{256\,(4+x)^{9/2}}\, x^5$; since $2^9 < (4+0.1)^{9/2}$ then

$0 \leqslant R_4(x) \leqslant \frac{7}{256 \cdot 2^9}(0.1)^5 < 5.34 \times 10^{-10}$

9 2 terms needed; 0.996 195

10 a) $\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$

b) $\int_0^1 e^{-x^2}\,dx \approx \frac{23}{30}$

c) Error $< \frac{e}{42}$

11 a) $\frac{1}{1+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1}\, x^{2n-2}$

b) Proof c) Proof d) $\frac{\pi}{4}$

12 a) $\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \ldots$ and

$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \ldots$

b) $\frac{-3}{x-2} + \frac{4}{x-3}$

c) $\frac{x+1}{x^2-5x+6} \approx \frac{1}{6} + \frac{11x}{36} + \frac{49x^2}{216} + \frac{179x^3}{1296} + \ldots$

13 a) $\frac{1}{1-x}$

b) $\sum_{n=1}^{\infty}\left[-(x+1)^{n-1}\right] = -1 - (x+1) - (x+1)^2 - (x+1)^3 - \ldots,$

$-2 < x < 0$

14 a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ b) 0.3103

15 a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ b) Proof

16 Ratio test gives interval of convergence as $-1 \leqslant x < 1$.

17 a) (i) Proof

(ii) $a_n = \frac{1^2 \times 3^2 \times \ldots (n-2)^2}{n!}$, for odd $n \geqslant 3$

b) $R = 1$

c) $\pi \approx 3.139$

18 $-1 \leqslant x < 1$

19 a) (i) Domain $[-1, 1]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\arcsin x = x + \frac{x^3}{6} + \ldots$

b) $\cos(\arcsin x) = 1 - \frac{x^2}{2} - \frac{x^4}{8}$

c) (i) $p^r\left(1 + \frac{q}{p}x^2\right)^r = p^r\left(1 + r\frac{q}{p}x^2 + \frac{r(r-1)}{2}\frac{q^2}{p^2}x^4\right)$

(ii) $p = 1, q = -1,\ r = \frac{1}{2}$; hence, the series in b) and c) is $\left(1 - x^2\right)^{1/2}$ since $\cos(\arcsin x) = \cos\left(\arccos\sqrt{1 - x^2}\right) = \left(1 - x^2\right)^{1/2}$.

20 a) $\frac{1}{2\pi}$ b) -2

21 a) Proof

b) $\ln(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

c) $\ln(1 - \sin x) = -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \dots$

d) Proof

e) 0

22 a) $\sin(\pi x) \approx 1 - \frac{\pi^2\left(x - \frac{1}{2}\right)^2}{2!} + \frac{\pi^4\left(x - \frac{1}{2}\right)^4}{4!} - \dots$

b) 0.924

Chapter 4

Exercise 4

1 D

2 $b = 2 - 2a = 2(1 - a)$

3 $x < -a, x > -a$

4 a) Continuous at $x = 1$.

b) Not differentiable at $x = 1$.

5 a) Continuous, not differentiable.

b) Neither

c) Neither

d) Continuous and differentiable.

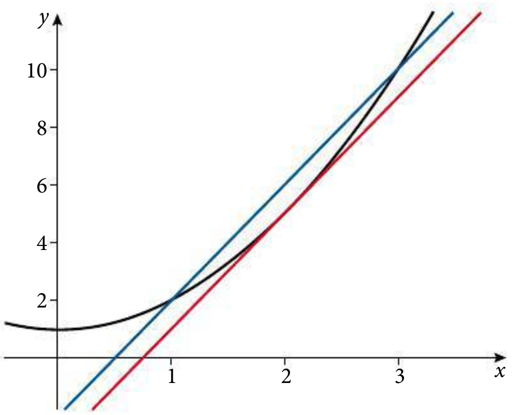
6 $a = \frac{5}{7}, b = -\frac{30}{7}$

7 a) $a + b = 3$

b) $a = \frac{3}{2}, b = \frac{3}{2}$

8 $x_0 = 1 + \sqrt{3}$

9 $c = 2$ (see graph)



10 $x_0 \approx 0.690$

11 $y = x^{2/3}$ is not differentiable at $x = 0$.

12 Along the 13 km portion of the highway the car's average speed was $\frac{13\text{ km}}{12\text{ min}} = \frac{13\text{ km}}{\frac{1}{5}\text{ hr}} = 65\frac{\text{km}}{\text{hr}}$. According to Mean Value Theorem, there was at least one instant in the 13 km portion when the car was travelling at $65\frac{\text{km}}{\text{hr}}$. This confirms that the car did break the speed limit.

13 Proof

14 $\frac{2}{3}$

15 Proof

16 $c = \frac{7}{3}$

17 $c \approx 0.670$

18 1.25

19 Lower sum $= \frac{31}{4}$; upper sum $= \frac{39}{4}$.

20 $\int_0^4 \sqrt{x + 6}\, dx$

21 $\int_3^5 \frac{e^x}{x - 2}\, dx$

22 $\int_0^{\pi} (3 - \sin x)\, dx$

23 a) (i) 316

(ii) 320

(iii) Underestimate; 1.25% error.

b) (i) $\frac{2\pi}{3} \approx 2.09$

(ii) 2

(iii) Overestimate; approx. 4.72% error.

c) (i) 5.38

(ii) $\frac{15}{4\ln 2}$

(iii) Underestimate; approx. 0.499% error.

Chapter 5

Exercise 5

1 $y = \frac{e^{\sqrt{1+x^2}}}{e}$

2 $y = \arctan(1 - \cos x)$

3 $y = \frac{3x^3}{x^3 + 4}$

4 Proof

5 (i) c (ii) a (iii) d (iv) b

6 24.7 grams

7 a) $2x^2 - y^2 = C$ b) $y = \frac{x}{1 - Cx}$

c) $\ln(y - 1) - \ln y + C_1 = -\frac{1}{x}$ or $\frac{y}{y - 1} = C_2 e^{1/x}$

d) $x = C_1 \sin y$ or $y = \arcsin(C_2 x)$

e) $y = Ce^{x^2/2}$ f) $y^2 = 2\sqrt{x^2 + 1} + C$

g) $\ln\sqrt{\frac{y - 1}{y + 1}} = e^x + C$ h) $x = y\ln y - y + C$

8 $\int \frac{y + 1}{y}\, dy = \int \frac{x + 1}{x}\, dx \Rightarrow \int\left(1 + \frac{1}{y}\right) dy = \int\left(1 + \frac{1}{x}\right) dx$

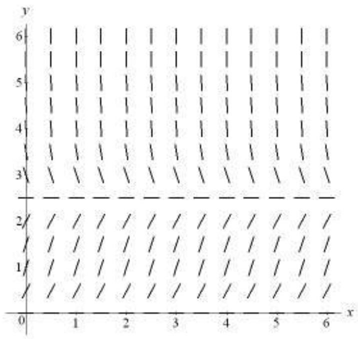
$\Rightarrow y + \ln|y| = x + \ln|x| + C$

$e^{y+\ln y} = e^{x+\ln x+C} \quad \Rightarrow \quad e^{\ln y}e^{y} = e^{\ln x}e^{x}e^{C} \quad \Rightarrow \quad ye^{y} = Axe^{x}$

9 $y = \pm\sqrt{2\sin x + C}$

The constant C cannot be completely arbitrary because $2\sin x + C \geqslant 0$. If $C < -1$, then $2\sin x + C$ will always be negative, regardless of the value of x. If $C > 1$, then $2\sin x + C$ will always be positive. If $-1 \leqslant C \leqslant 1$, then whether $2\sin x + C$ is positive or negative will depend on the value of x.

10 a)



b) $\frac{5}{2}$ c) $\frac{5}{2}$

d) Regardless of the initial value of the population, as time increases, the population stabilizes at 2 500.

11 $y = -\sqrt{x^2 + \tan x + 25}$

12 a) Proof

b) $y = \frac{x+1}{x-1}$

13 $y = \frac{7x+1}{7-x}$

14 (i) b (ii) d (iii) c (iv) a

15 a) $\frac{1}{3(x-2)} - \frac{1}{3(x+1)}$

b) proof

16 a) $y = C\left(x^2 - 1\right) + 1$

b) $\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right)y = \frac{2x}{1-x^2}$; integrating factor is $\left|\frac{1}{1-x^2}\right|$; leads to same solution as in part a)

17 a) $y = x^4 + \frac{C}{x^2}$ b) $y = Ce^{x^2/2} - 1$

c) $y = \frac{1}{3}x^4 + Cx$ d) $y = xe^{\cos x} + Ce^{\cos x}$

e) $y = xe^{x^3} + Ce^{x^3}$ f) $y = x\ln|x| + Cx$

18 $y = x\csc x + C\csc x$

19 a)–c) Proof

d) $y = \frac{x}{2} + \frac{\arcsin x}{2\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$

20 a)–b) Proof

c) $y = \tan x + C\sec x$

21 $y = \frac{1}{3}x^2 + \frac{C}{x}$

22 $y = \frac{1}{3}x^2\ln x - \frac{1}{9}x^2 + \frac{10}{9x}$

23 $C = \frac{y-x}{(y+x)^2}$

24 a) $y = Cx + C$ b) $y = Cx^2 - x$

c) $y = Cx^3 - x$ d) $2x^3 + 3xy^2 + 3y^3 = C$

e) $y^2 = \frac{x^2}{2} - \frac{C}{x^2}$ f) $y = x\ln(Cxy)$

25 a) Proof b) $x^2 + 4xy - 3y^2 - 1 = 0$

26 Proof

27 Proof

28 a) $\left|\frac{y}{y+1}\right| = C|x|$ b) $\left|\frac{y}{y+1}\right| = \frac{1}{2}|x|$

c)

x_n	y_n
1.2	1.400
1.4	1.960
1.6	2.789
1.8	4.110

d)

x_n	approx. y_n	exact y_n	% error
1.2	1.400	1.5	$6.\overline{6}$
1.4	1.960	$2.\overline{3}$	16
1.6	2.789	4	30.3
1.8	4.110	9	54.3

29 $y \approx 1.5405$ at $x = 1$

30 $y \approx 5.9584$ at $x = 1$

31 $y^2 = Cx^3 - x^2$

32

x_n	y_n
1.1	4.2
1.2	4.425 43
1.3	4.677 87
1.4	4.959 04
1.5	5.270 81

33 a) Proof

b) $y(1) \approx 0.327\,68$

c) $y(1) \approx 0.348\,678\,4401$

d) Actual value to 10 s.f. is $y(1) \approx 0.367\,879\,4412$; using more steps (and a smaller step size) gives a better approximation.

Practice questions 5

1 $y = \arctan\left(e^x + C\right)$

2 Proof

3 $y = Ce^{-\frac{1}{2}x^2}$

4 $\alpha = 20 + 50e^{-\frac{t}{10}\ln\frac{5}{3}}$

5 Proof

6 a) $y = (x + c)\,x^3$ b) $y = (x + 1)\,x^3$

7 $y = \sqrt{\dfrac{2x^5}{5} + \dfrac{6x^2}{5} + \dfrac{3}{5}}$

8 $y = Ce - \dfrac{1}{x}$

9 $y = \dfrac{C}{x} + \dfrac{\sin x}{x} - \cos x$

10 a) $y = \dfrac{C}{x} + \dfrac{x^3}{4}$ b) $y = \dfrac{16}{x} + \dfrac{x^3}{4}$

11 a) 6 b) 1 c) 2 d) 3 e) 4 f) 5

12 $y = 8\sin^2 x - 2$

13 a) $y = -2x + 12$ b) $y = \dfrac{8x}{x+1}$

14 a) Proof b) $y = \dfrac{1}{4}x^2 - \dfrac{1}{3}x + \dfrac{1}{2}x + \dfrac{1}{12x^2}$

15 a) Proof b) $5x = \dfrac{y^2}{x^2} + 1$ $\left(\text{or } y = x\sqrt{5x-1}\right)$

16 Proof

17 a) $y \approx 5.32$

b) Less than actual value; $\dfrac{dy}{dx} > 0$ so solution curve is curving upward; short segments from Euler's method to approximate solution curve will be below the actual solution curve.

18 $y = x - \sqrt{2x^2 - e^2}$

19 $y \approx 3.5$

20 a) Proof

b) $\dfrac{dy}{dx} + (\tan x)\,y = 2\cos^2 x \sin x - \sec x$; integrating factor is sec x.

c) Proof

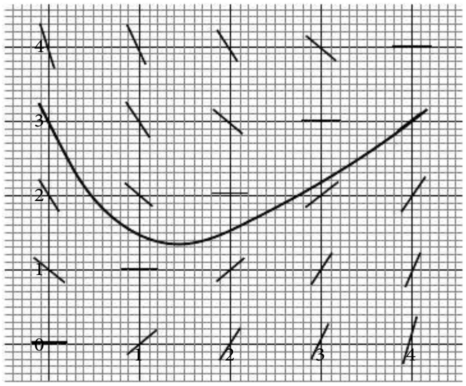
21 Proof

22 $y \approx 2.14$

23 a) Proof b) $e^{-\sin x}$

c) $y = -\sin x - 1 - e^{\sin x}$

24 a)–b)



c) $y = x - 1 + 4e^{-x}$

25

n	x_n	y_n
0	0	2
1	0.2	3.4
2	0.4	6.444841655
3	0.6	15.64713326
4	0.8	65.70696043
5	1	930.5232147

26 $yx^2 = \dfrac{1}{3}\left(1 + x^2\right)^{\frac{3}{2}} + \dfrac{1}{3}$

27 a) (i) $y(1.3) \approx 2.14$ (ii) Decrease the step size

b) $y = x^2 + e^{1-x^2}$

Discrete Mathematics

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Discrete Mathematics

Assessment statements

10.1	Strong induction.
	Pigeon-hole principle.
10.2	$a \mid b \Rightarrow b = na$ for some $n \in \mathbb{Z}$.
	The theorem $a \mid b$ and $a \mid c \Rightarrow a \mid (bx \pm cy)$ where $x, y \in \mathbb{Z}$.
	Division and Euclidean algorithms.
	The greatest common divisor, gcd(a, b), and the least common multiple, lcm(a, b), of integers a and b.
	Prime numbers; relatively prime numbers and the fundamental theorem of arithmetic.
10.3	Linear Diophantine equations $ax + by = c$.
10.4	Modular arithmetic.
	The solution of linear congruences.
	Solution of simultaneous linear congruences (Chinese remainder theorem).
10.5	Representation of integers in different bases.
10.6	Fermat's little theorem.
10.7	Graphs, vertices, edges, faces. Adjacent vertices, adjacent edges.
	Degree of a vertex, degree sequence.
	Handshaking lemma.
	Simple graphs; connected graphs; complete graphs; bipartite graphs; planar graphs; trees; weighted graphs, including tabular representation.
	Subgraphs; complements of graphs.
	Euler's relation: $v - e + f = 2$; theorems for planar graphs including $e \leqslant 3v - 6$, $e \leqslant 2v - 4$, leading to the results that K_5 and $K_{3,3}$ are not planar.
10.8	Walks, trails, paths, circuits, cycles.
	Eulerian trails and circuits.
	Hamiltonian paths and cycles.
10.9	Graph algorithms; Kruskal's; Dijkstra's.
10.10	Chinese postman problem.
	Not required: Graphs with more than four vertices of odd degree.
	Travelling salesman problem.
	Nearest-neighbour algorithm for determining an upper bound.
	Deleted vertex algorithm for determining a lower bound.
10.11	Recurrence relations. Initial conditions, recursive definition of a sequence.
	Solution of first- and second-degree linear homogeneous recurrence relations with constant coefficients.
	The first-degree linear recurrence relation $u_n = au_{n-1} + b$.
	Modelling with recurrence relations.

1 Number Theory I

1.1 Introduction

This option deals with two 'relatively' separate topics: number theory and graph theory. The name Discrete Mathematics is actually not a well-defined subject in the mathematics community. In some cases it includes number theory and in some it does not. However, your syllabus contains ideas from both, and that is what we will focus on. A common thread between the two parts is the requirement for relatively 'rigorous' proofs. We will start with **number theory**.

Number theory

Elementary number theory deals with the study of **integers** *in general* and the **positive integers** 1, 2, 3, … *in particular*. The set of positive integers is denoted by $\mathbb{Z}^+$, and that of integers is denoted by $\mathbb{Z}$, where

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (This is an IBO notation. In several mathematics sources, you will see that this set is called the set of natural numbers and is denoted by $\mathbb{N}$. Since you are preparing for an IB exam, we will follow this notation from this point onwards.)

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Of course, the integers are familiar to you from your primary school. You have worked with them hundreds of times and have formed an intuitive sense of many of their laws. This intuition carries some danger with it. It becomes hard to see the necessity to prove laws that we have become used to. However, we will assume some of the axioms we considered earlier as 'obvious' and will use them in the rest of the course.

Properties/axioms

On the set of integers, we can define the operations of addition and multiplication. As usual, we denote the sum and product of a and b by $a + b$ and $a \cdot b$, respectively. Following convention, we will also write ab for $a \cdot b$. Important properties of integers with respect to these two operations are mentioned below.

Closure property of addition: If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$.

Closure property of multiplication: If $a, b \in \mathbb{Z}$, then $ab \in \mathbb{Z}$.

Commutative property of addition: If $a, b \in \mathbb{Z}$, then $a + b = b + a$ for all $a, b \in \mathbb{Z}$.

Commutative property of multiplication: If $a, b \in \mathbb{Z}$, then $ab = ba$ for all $a, b \in \mathbb{Z}$.

Associative property of addition: If $a, b, c \in \mathbb{Z}$, then $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Z}$.

Associative property of multiplication: If $a, b, c \in \mathbb{Z}$, then $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{Z}$.

Distributive property of multiplication over addition: If $a, b, c \in \mathbb{Z}$, then $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{Z}$.

Additive identity property: For all $a \in \mathbb{Z}$, $a + 0 = a$.

Multiplicative identity property: For all $a \in \mathbb{Z}$, $a \cdot 1 = 1 \cdot a = a$.

Additive inverse property: For all $a \in \mathbb{Z}$, $a + (-a) = (-a) + a = 0$. Thus, $a + (-b)$ is written as $a - b$.

Cancellation property of multiplication: If $a, b, c \in \mathbb{Z}$, $a \neq 0$, then $ab = ac$ implies $b = c$.

These properties are also called axioms. An **axiom**, as you will recall, is a universally accepted principle, rule, or a proposition that is assumed without proof and serves as a starting point from which other statements are logically derived.

Here are some more properties, some of which can be proved by using the axioms mentioned before.

Cancellation property of addition: If $a, b, c \in \mathbb{Z}$, $a \neq 0$, then $a + b = a + c$ implies $b = c$.

This can be easily proved:

Given $a + b = a + c$, we add $-a$ to both sides. We get $(-a) + (a + b) = (-a) + (a + c)$.

By associative property of addition we get $((-a) + a) + b = ((-a) + a) + c$.

Now, by the additive inverse property, $0 + b = 0 + c$.

Using the additive identity property, we get $b = c$.

Ordering relation: On the system of integers $\mathbb{Z}$, there is an order relation 'less than', denoted by ' $<$ ', on the basis of which we have the following law:

Law of trichotomy: If $a \in \mathbb{Z}$ then exactly one of the following statements is true:

(i) $a < 0$ (ii) $a = 0$ (iii) $a > 0$.

Properties of inequality:

(i) If $a, b, c \in \mathbb{Z}$, and $a < b$, then $a + c < b + c$.

(ii) If $a, b, c \in \mathbb{Z}$, $a < b$, and $c > 0$, then $ac < bc$.

(iii) If $a, b, c \in \mathbb{Z}$, $a < b$, and $c < 0$, then $ac > bc$.

The following is an important property of positive numbers:

Well-ordering property: Every non-empty set of positive integers contains a least element.

The well-ordering property is a fundamental axiom of the system of positive integers. We can quickly verify that this property is quite an obvious one if we consider a finite set of positive integers like the ones mentioned below:

1 $S_1 = \{2, 5, 7, 9, 14, 21\}$

2 $S_2 = \{4, 29, 17, 3, 101\}$

In **1**, the least element is 2, because it is smaller than every other element in S_1.

In **2**, the least element is 3.

In this publication, we expect that you are familiar with these properties of integers from your earlier work with numbers. What we have mentioned here are a set of axioms which describe the properties of integers. We have neither tried to make these axioms independent of each other nor to mention a minimal number of axioms to develop the system of integers.

Next, we will demonstrate a few proofs for you to refresh your knowledge and to get started with proving statements yourself. Recall that a rational number is expressed as a ratio of two integers. Real numbers that are not rational are irrational. The sets $\mathbb{Q}$ and $\mathbb{R}$ denote the set of all rational numbers and real numbers, respectively.

Proofs

Most statements you will prove in this option are **implications**, i.e. assertions of the form 'if P, then Q', where P and Q are themselves statements. P is called the **hypothesis** and Q is called the **conclusion**. This is also written as $P \Rightarrow Q$. An example is:

S: If I have a free moment, then I will call you.

Here P is the statement 'I have a free moment' and Q is the statement 'I will call you'.

The implication 'if P, then Q' is considered to be true unless P is true and Q is false. Thus, my statement is truthful in each of the following cases:

- I have a free moment and I call you.
- I do not have a free moment and I do not call you.
- I do not have a free moment, but I call you anyway!

I would lie only if I have a free moment and I don't call you.

The meaning of 'if P, then Q' is summarized in the truth table right (where T is for true, and F for false).

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that 'if P, then Q' is not the same as 'if Q, then P'; and one of them could be true while the other false. The statement 'if Q, then P' is called the **converse** of 'if P, then Q'.

In the previous example, the converse would be 'If I call you, then I have a free moment'.

To disprove a statement we ordinarily use a counter example. For example, consider the statement:

If $a > b$, then $ac > bc$.

This could easily be disproved by letting $a = 1$, $b = -1$, and $c = -2$; obviously $a > b$, but $ac = -2 \not> bc = 2$.

In this option, you will be dealing mainly with two types of proofs:

- **Direct proof** is a proof in which logical arguments lead directly from the hypothesis to the conclusion. To prove $P \Rightarrow Q$ by direct proof, assume P holds, and show that Q must follow (see Example 1).
- **Indirect proof** is itself of two types: proof by **contradiction** and proof by **contrapostive.** In a proof by **contradiction**, we assume the statement is false and show that this leads to a contradiction, thereby showing that it is impossible for the statement to fail. A proof by **contrapositive** uses the fact that the implication $P \Rightarrow Q$ is logically equivalent to its contrapositive $\neg Q \Rightarrow \neg P$, thus proving the contrapositive will prove the statement itself (see Examples 2 and 3).

'$\neg$' is a negation symbol. '$\neg$' is read as 'not'.

Mathematical induction

In Section 4.7 of the textbook you worked extensively with one form of the principle of **mathematical induction** (MI). In this part of the option, we will prove the principle and introduce you to another form, which is called **strong mathematical induction**.

Proof of the mathematical induction principle (MI)

Statement 1: Let S be the set of positive integers such that

1 $1 \in S$

2 Whenever the integer $k \in S$, then $k + 1 \in S$.

Then S is $\mathbb{Z}^+$.

Proof

Let T be the set of integers not in S. Assume T to be non-empty. The well-ordering principle implies that T has a least element. Call the least element a. Since, by hypothesis, $1 \in S$, then $1 \notin T$ and hence $a > 1$.

Now, $a - 1 > 0$ and hence $a > a - 1 > 0$.

Since a is the smallest element in T, then $a - 1$ cannot be in T, and therefore $a - 1 \in S$. Now, if $a - 1 \in S$, by (**2**) above, $(a - 1) + 1 \in S$, i.e. $a \in S$. This contradicts the fact that $a \in T$. Therefore, we conclude that T must be empty and that S contains all positive integers.

Strong mathematical induction

A second version of the MI principle called '**second principle of MI**', or '**strong MI**', has the same structure except in the induction step:

Statement 2: Let S be the set of positive integers such that

1 $1 \in S$

2 Whenever the integers $1, 2, 3, \ldots, k \in S$, then $k + 1 \in S$.

Then S is $\mathbb{Z}^+$.

Proof

Let T be the set of integers not in S. Assume T to be non-empty. The well-ordering principle implies that T has a least element. Call the least element a. Since, by hypothesis, $1 \in S$, then $1 \notin T$ and hence $a > 1$.

Also, $1, 2, 3, \ldots, a - 1$ are all in S by hypothesis, and hence if $a - 1 \in S$, then by (**2**) above, $(a - 1) + 1 \in S$, i.e. $a \in S$. This contradicts the fact that $a \in T$. Therefore, we conclude that T must be empty and that S contains all positive integers.

Before we demonstrate how to use strong induction in specific examples, let us summarize the steps you need to follow:

To prove $S(n)$ true for all positive integers $n \geqslant n_0$, we complete the following two steps.

Basis Step: Verify that $S(n_0)$ is true. (In many cases $n_0 = 1$)

Inductive Step: Show that the implication $(S(n_0) \wedge S(n_0 + 1) \ldots \wedge S(k)) \rightarrow S(k+1)$ is true for all positive integers k.

Conclude: $S(n)$ is true for all positive integers larger than or equal to n_0.

Example 1

For any integer $n \geqslant 2$, n is divisible by a prime number.

Proof

Basis step:
$S(2)$ is true, since 2 is divisible by 2 and 2 is a prime number.

Inductive step:
Assume the statement is true for all $n = i$ with $2 \leqslant i \leqslant k$, i.e. $S(2) \wedge \ldots \wedge S(k)$ is true. (This is called the inductive hypothesis.)

Show that it is true for $n = k + 1$.

We must show that $n = k + 1$ is divisible by a prime number.

We consider two cases:

(i) $k + 1$ is prime, and in this case is divisible by itself, or,

(ii) $k + 1$ is composite, and hence

$k + 1$ can be written as a product of two integers x and y such that $2 \leqslant x \leqslant k$ as well as $2 \leqslant y \leqslant k$. However, with the assumption that all numbers between 2 and k are divisible by a prime, then x and y are divisible by a prime and hence by transitive property, $k + 1$ is also divisible by a prime.

Therefore $S(n)$ is true for all positive integers by the principle of strong induction.

Example 2

A sequence $\{a_n\}$ is defined by

$$\begin{cases} a_0 = 1, a_1 = 2, a_2 = 3 \\ a_n = a_{n-1} + a_{n-2} + a_{n-3} \ \forall n \in \mathbb{Z}, n \geqslant 3 \end{cases}$$

Show that $S(n)$: $a_n \leqslant 2^n$ for all non-negative integers.

Proof

Basis step:
$S(0)$ is true since $a_0 = 1 \leqslant 1 = 2^0$, $S(1)$ is true since $a_1 = 2 \leqslant 2 = 2^1$, and $S(2)$ is true since $a_2 = 3 \leqslant 4 = 2^2$.

Inductive step:
Assume the statement is true for all $n = i$ with $0 \leqslant i \leqslant k$, i.e. $S(0) \wedge \ldots \wedge S(k)$ is true, i.e., $a_0 \leqslant 2^0, \ldots, a_k \leqslant 2^k$.

Show that it is true for $n = k + 1$.

We must show that $a_{k+1} \leqslant 2^{k+1}$.

$a_{k+1} = a_k + a_{k-1} + a_{k-2} \leqslant 2^k + 2^{k-1} + 2^{k-2}$ which is based on the assumption above.

This leads to $a_{k+1} \leqslant 2^k + 2^{k-1} + 2^{k-2} \leqslant 2^k + 2^{k-1} + 2^{k-2} + 2^{k-3} + \ldots + 1$

But $2^k + 2^{k-1} + 2^{k-2} + 2^{k-3} + \ldots + 1 = \dfrac{1 - 2^{k+1}}{1 - 2} = 2^{k+1} - 1$ since it is a geometric series with $k + 1$ terms, first term equal to 1 and a common ratio of 2.

Hence, $a_{k+1} \leqslant 2^{k+1}$ and therefore $S(n)$ is true for all positive integers by the principle of strong induction.

Example 3

Fibonacci sequences are defined recursively by

$$\begin{cases} u_1 = 1, u_2 = 1 \\ u_n = u_{n-1} + u_{n-2}, n > 2. \end{cases}$$

Show that the closed form for the nth term of Fibonacci sequence is given by

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \text{ for } n > 2.$$

Proof

Basis step:

$S(1)$ is true, since $u_1 = \dfrac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2\sqrt{5}} = \dfrac{2\sqrt{5}}{2\sqrt{5}} = 1.$

$S(2)$ is true, since $u_2 = \dfrac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \dfrac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4\sqrt{5}}$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

Inductive step:

Assume the statement is true for all $n = i$ with $1 \leqslant i \leqslant k$, i.e. $S(1) \wedge \ldots \wedge S(k)$ is true.

Show that it is true for $n = k + 1$.

We must show that $u_{k+1} = \dfrac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$

We know that $u_{k+1} = u_k + u_{k-1}$ by definition of Fibonacci sequence.

By assumption, we know that $u_k = \dfrac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}$ and

$$u_{k-1} = \frac{(1+\sqrt{5})^{k-1} - (1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}.$$

Hence, $u_{k+1} = \dfrac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}} + \dfrac{(1+\sqrt{5})^{k-1} - (1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}$

$$= \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k + 2(1+\sqrt{5})^{k-1} - 2(1-\sqrt{5})^{k-1}}{2^k\sqrt{5}}$$

$$= \frac{\left((1+\sqrt{5})^k + 2(1+\sqrt{5})^{k-1}\right) - \left((1-\sqrt{5})^k + 2(1-\sqrt{5})^{k-1}\right)}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^k\left(1+\dfrac{2}{1+\sqrt{5}}\right) - (1-\sqrt{5})^k\left(1+\dfrac{2}{1-\sqrt{5}}\right)}{2^k\sqrt{5}}$$

By more algebraic manipulation we have:

$$u_{k+1} = \frac{(1+\sqrt{5})^k\left(1+\dfrac{2}{1+\sqrt{5}}\right)-(1-\sqrt{5})^k\left(1+\dfrac{2}{1-\sqrt{5}}\right)}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^k\left(\dfrac{1+\sqrt{5}}{2}\right)-(1-\sqrt{5})^k\left(\dfrac{1-\sqrt{5}}{2}\right)}{2^k\sqrt{5}} = \frac{(1+\sqrt{5})^{k+1}-(1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$$

Therefore, by the principle of strong induction, the closed form for the nth term of Fibonacci sequence is given by $u_n = \dfrac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}, n > 2.$

Example 4

Fibonacci sequences are defined recursively as in example 3.

Prove that $\sum_{i=1}^{n} u_i = u_{n+2} - 1$ for every $n \in \mathbb{Z}^+$.

Proof

Basis step:

$S(1)$ is true since for $n = 1$

$$\sum_{i=1}^{1} u_i = u_1 = 1 = u_3 - 1 = 2 - 1, \text{ which is true.}$$

As a check, we also know that $S(2)$ is true since

$$\sum_{i=1}^{2} u_i = u_1 + u_2 = 1 + 1 = u_4 - 1 = 3 - 1$$

Inductive step:

Assume the statement is true for $n = k$, show that it is true for $n = k + 1$.

We must show that $\sum_{i=1}^{k+1} u_i = u_{k+3} - 1$

$$\sum_{i=1}^{k+1} u_i = \sum_{i=1}^{k} u_i + u_{k+1} = u_{k+2} - 1 + u_{k+1}$$
$$= u_{k+3} - 1$$

Therefore, by the principle of mathematical induction, the statement is true for all positive integers.

Note: We could have used strong induction here in the following manner.

Assume the statement is true for all $n = i$ with $1 \leqslant i \leqslant k$, i.e. $S(1) \wedge \ldots \wedge S(k)$ is true, i.e.,

Show that it is true for $n = k + 1$

$$\sum_{i=1}^{k+1} u_i = \sum_{i=1}^{k-1} u_i + u_k + u_{k+1} = u_{k+1} - 1 + u_{k+2}$$
$$= u_{k+3} - 1$$

We suggest that you use the following format for proofs by strong (or mathematical) induction.

- Say what you are proving.
- Say that the proof is by strong mathematical induction, and make it clear what is playing the role of *n*.
- In the induction case, state the induction hypothesis (IH) and what you need to show (NTS).
- Divide the argument into cases, as needed.
- Indicate clearly where and how you use the inductive hypothesis.

Other methods of proofs – examples

Example 5

Prove that the product of two odd integers is an odd integer.

Proof

Given that a and b are odd integers, we need to prove that ab is an odd integer.

Let a and b be odd integers. Then you can find two integers m and n such that

$a = 2m + 1$ and $b = 2n + 1$.

The product ab is

$$ab = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$$
$$= 2(2mn + m + n) + 1 = 2k + 1$$

where $k = 2mn + m + n \in \mathbb{Z}$. Thus ab is odd.

Example 6

Prove that $\sqrt{2}$ is an irrational number.

Proof

Assume that $\sqrt{2}$ is rational. Then, by definition of rational numbers, $\sqrt{2}$ can be written as a **reduced** fraction $\frac{m}{n}$ where the two integers m and n, with $n \neq 0$, have no common divisor except 1.

$$\sqrt{2} = \frac{m}{n} \Rightarrow \left(\sqrt{2}\right)^2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2$$

This tells us that m^2 is even.

If m^2 is even, m must also be even (this is assumed true, but can be proved true easily), thus $m = 2k$ for some integer k.

This leads us to $m^2 = 4k^2$, and hence

$2n^2 = m^2 = 4k^2 \Rightarrow n^2 = 2k^2$, and thus n^2 is even, which in turn leads to n being even.

Thus both m and n are even, and hence they have another common factor, 2, which contradicts the definition of a rational number.

Therefore, assuming $\sqrt{2}$ to be rational leads us to a contradiction and so $\sqrt{2}$ cannot be rational.

Example 6 is a proof by contradiction. The next example will demonstrate the use of contrapostive in the proof.

Example 7

Let a be a positive real number. Prove that if a is an irrational number then $\sqrt{a}$ is also irrational.

Proof

Stated differently, we need to prove: $a \notin \mathbb{Q} \Rightarrow \sqrt{a} \notin \mathbb{Q}$.

We will use the contrapositive and attempt to prove $\sqrt{a} \in \mathbb{Q} \Rightarrow a \in \mathbb{Q}$.

Suppose $\sqrt{a} \in \mathbb{Q}$, then there are two integers m and n, with $n \neq 0$, such that $\sqrt{a} = \frac{m}{n}$ by definition of rational numbers. Thus $a = \left(\sqrt{a}\right)^2 = \frac{m^2}{n^2}$, and since m and n, with $n \neq 0$, are integers, then m^2 and n^2, with $n^2 \neq 0$, are also integers.

So, a can be written as the quotient of two integers, and hence it is a rational number, by definition.

By proving the contrapositive, the statement itself is true.

Note: There is a convention that is well known in mathematics and that is the use of the 'iff'. This word stands for 'if and only if,' which in turn means a logical equivalence. That is, if we say P iff Q, we mean P implies Q and Q implies P. Hence, in some proofs, we will have to prove both statements. In this publication, we will indicate the two-way process by using ($\Rightarrow$) for the first and ($\Leftarrow$) for the second.

Pigeonhole principle

As the name indicates, the idea stems from the following situation:

A flock of pigeons flies into a set of pigeonholes. If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with more than one pigeon (at least two pigeons).

Theorem: The pigeonhole principle

If $n + 1$ objects or more are placed into n positions, then there is at least one position that contains at least two of the objects.

Proof

Assume that no position has more than one object. Then there will be at most n objects. This is a contradiction since there are $n + 1$ or more objects.

Note: The pigeonhole principle is sometimes called the **Dirichlet drawer principle**, after the German mathematician Dirichlet.

Example 8

What is the minimum number of people in a room where at least two of them have the same birth month?

Solution

There should be at least 13 as there are only 12 possible months.

Example 9

True or false: In a HL IB class of 10 students, there will be at least two students with the same score.

Solution

True, since there are only seven grades possible in the mathematics examination.

Example 10

True or false: In a 5-digit number code situation given to a group larger than 10, there will be at least two codes that start with the same digit, end with the same digit, etc.

Solution

True, since there are only 10 digits possible!

1.2 Division algorithm

The sum, difference, and product of two integers is always an integer, but the quotient may not be. The concept of divisibility of one integer by another is central in number theory. We are not only interested to know the underlying reason for an integer to be divisible by another integer, but also interested to see how this concept is applied in different situations.

If a is a divisor of b so is $-a$, since $b = ac$ implies $b = (-a)(-c)$. So, the divisors of an integer at all times happen in pairs. To obtain all the divisors of a given integer, it is enough to get the positive divisors and then tag on to them the matching negative integers. In this book, we will usually limit our listing of divisors to the positive ones.

Definition 1

If a and b are integers with $a = 0$, then b is divisible by a if there exists an integer c such that $b = ac$.
In this case we say a divides b and denote this by $a|b$. a is called a divisor or factor of b and b is called a **dividend** or a **multiple** of a. If a does not divide b then we write $a \nmid b$.

Example

The following statements illustrate the concept of divisibility of integers:

$11 | 143$, $-4 | 28$, $19 | 133$, $5 | 0$, $3 \nmid 2$, and $15 \nmid 47$.

Example

The divisors of 8 are ±1, ±2, ±4, and ± 8. The divisors of 11 are ±1 and ± 11.

In subsequent sections, we will need some simple properties of divisibility which we now state and prove as theorems.

Theorem 1

If a, b, and c are integers with $a | b$ and $b | c$, then $a | c$.

Proof

Since $a | b$ and $b | c$, there exist integers m and n such that $b = am$ and $c = bn$. Hence, $c = (am)n = (mn)a$. Now, since mn is an integer, then, by definition, this shows that $a | c$.

Example

$3 | 6$ and $6 | 216$, then $3 | 216$; $5 | 15$ and $15 | 3375$, then $5 | 3375$; $11 | 44$ and $44 | 308$, then $11 | 308$.

Theorem 2

If $a | b$ and $a | c$, then $a | (b \pm c)$.

Proof

Since $a | b$ and $a | c$, then there exist integers m and n such that $b = ma$ and $c = na$.

Hence, $b \pm c = ma \pm na = (m \pm n)a$.

Now, since $m \pm n$ is an integer, $a | (b \pm c)$.

Corollary 1

If $a | b$ and $a | c$, then $a | (bx \pm cy)$, where a, b, x, and y are integers.

The corollary follows from Theorem 2 by recognizing that bx and cy are integers and can be substituted for b and c in the theorem.

This is to say that if a divides b and c then a divides any integer linear combination of b and c.

This property can be extended to sums of more than two integers. That is, if $a \mid b_j$ for $j = 1, 2, \ldots, n$, then

$a \mid (b_1x_1 + b_2x_2 + \ldots + b_nx_n)$ for all integers $x_1, x_2, \ldots, x_n$.

Example

$5 \mid 45$ and $5 \mid 60$, then $5 \mid (45 + 60)$, i.e. $5 \mid 105$; $5 \mid (7 \cdot 45 - 2 \cdot 60)$, i.e. $5 \mid 195$.

Theorem 3

If $a, b, c \in \mathbb{Z}$, then the following hold:

(i) $a \mid 0$, $1 \mid a$, and $a \mid a$.

(ii) $a \mid 1$ if and only if $a = \pm 1$.

(iii) If $a \mid b$, and $c \mid d$, then $ac \mid bd$.

(iv) $a \mid b$, and $b \mid a$, if and only if $a = \pm b$.

(v) If $a \mid b$, and $b \neq 0$, then $|a| \leqslant |b|$.

Proof

We will leave the proofs of parts (i)–(iv) as an exercise, and only prove (v) here.

If $a \mid b$, then there exists an integer c such that $b = ac$; moreover, $b \neq 0$ means that $c \neq 0$. Now, taking absolute values,

$$|b| = |ac| = |a|\,|c|.$$

Since $c \neq 0$, then $|c| \geqslant 1$, and therefore

$$|b| = |a|\,|c| \geqslant |a|.$$

Theorem 4: The division algorithm

If a and b are integers such that $b > 0$, then there exist unique integers q and r such that $a = bq + r$, with $0 \leqslant r < b$.

Note: We call q the **quotient** and r the **remainder**; we also call a the **dividend** and b the **divisor**.

Note that a is divisible by b if and only if the remainder in the division algorithm is zero.

Before we prove the division algorithm, let us consider some examples.

Example

If $a = 183$ and $b = 31$, then $q = 5$ and $r = 28$, since $183 = 31 \cdot 5 + 28$.

Also, $a = -183$ and $b = 31$, then $q = -6$ and $r = 3$, since $-183 = 31(-6) + 3$.

Note: It is natural for us to ask, given two numbers a and b, how can we find the quotient q and the remainder r mentioned in the division algorithm? As an illustration, let us take $a = 94$ and $b = 13$. In order to find the quotient q, multiply 13 successively by $\{1, 2, 3, \ldots\}$ until you reach a number larger than or equal to 91.

$13 \cdot 1 = 13, 13 \cdot 2 = 26, 13 \cdot 3 = 39, \ldots, 13 \cdot 7 = 91, 13 \cdot 8 = 104$

So, $q = 7$, and the remainder $r = 94 - 13 \cdot 7 = 3$.

This process is a result of the division algorithm itself:

$a = bq + r$, with $0 \leqslant r \leqslant b \Leftrightarrow \frac{a}{b} = q + \frac{r}{b}$, with $0 \leqslant \frac{r}{b} \leqslant 1$. This in turn can be interpreted as follows:

q is the integer part of the quotient of a by b, and $\frac{r}{b}$ is the decimal part, and hence q is nothing but the greatest integer function of $\frac{a}{b}$. So,

$q = \left[\frac{94}{13}\right] = [7.23] = 7$ and $r = 94 - 13(7) = 3$.

For instance, in the example above, we have

$q = \left[\frac{183}{31}\right] = [5.9] = 5$, and hence the remainder r is $183 - 31 \cdot 5 = 28$; also

$q = \left[\frac{-183}{31}\right] = [-5.9] = -6$, and $r = -183 - 31(-6) = 3$.

Example

$a = 121$ and $b = 9$, then $q = \left[\frac{121}{9}\right] = [13.4] = 13$, and $r = 121 - 13 \cdot 9 = 4$, and so $121 = 9 \cdot 13 + 4$. Also, if $a = -148$ and $b = 12$, then

$q = \left[\frac{-148}{12}\right] = [-12.3] = -13$, and $r = -148 - 12(-13) = 8$, and so

$-148 = 12(-13) + 8$.

We now present a proof of the division algorithm.

Proof of the division algorithm

This is an existence and uniqueness proof. First we have to prove that q and r exist, and then, if they exist, they are the only numbers that satisfy the division algorithm.

Existence:

Suppose the real number a/b is $q + k$, where q is an integer and $0 \leqslant k < 1$. Then

$$a = b(q + k) = bq + bk.$$

Now, since a is an integer and bq is an integer (product of two integers), it follows that bk is also an integer. Moreover, since $b > 0$, multiplying it with all sides of $0 \leqslant k < 1$ gives us $0 \leqslant bk < b$. With this in mind, we set $r = bk$, and thus we have

$$a = bq + r \text{ with } 0 \leqslant r < b.$$

Uniqueness:

Next we show that q and r are unique. Using an indirect proof, suppose they are not unique, then there exists at least another pair q_1 and r_1 that satisfy the division algorithm, and now we have

$$a = bq + r \text{ with } 0 \leqslant r < b, \text{ and}$$

$$a = bq_1 + r_1 \text{ with } 0 \leqslant r_1 < b.$$

Subtract the two equations and simplify:

$$r - r_1 = b(q - q_1) \text{ ....}(1)$$

Add the two inequalities $0 \leqslant r < b$ and $-b \leqslant -r_1 < 0$, and thus

$-b < r - r_1 < b.$

Divide all sides by b and we have

$-1 < \dfrac{r - r_1}{b} < 1.$

Since $\dfrac{r - r_1}{b} = q - q_1$ from equation (1), and since

$q - q_1$ is an integer, and the only integer between -1 and $+1$ is zero, then

$q - q_1 = 0$, which implies that $q = q_1$. Also, $\dfrac{r - r_1}{b} = 0 \Rightarrow r - r_1 = 0 \Rightarrow r = r_1$.

Therefore, q and r are unique.

Note: The result we established can also be applied when $b < 0$. For if $b < 0$, then $-b > 0$, and hence we can say that according to Theorem 4, there exist two integers q_1 and r such that

$a = (-b)q_1 + r$ with $0 \leqslant r < -b$, which can be rewritten as $a = b(-q_1) + r$ with $0 \leqslant r < -b$. Now take

$q = -q_1$, and we get $a = bq + r$ with $0 \leqslant r < -b$ and $q \in \mathbb{Z}$. This is the existence part of the theorem. Uniqueness follows the same approach as in the main theorem.

Combining this observation with the statement from Theorem 4, we obtain:

Corollary 2

If a and b are integers and $b \neq 0$, then there are unique integers q and r such that $a = qb + r$ with $0 \leqslant r < |b|$.

Example

a) Let $a = 51$ and $b = -9$, then $51 = (-9)(-5) + 6$. Here too, we can use the largest integer function in the following manner:

$$q = \left[\frac{a}{-b}\right] = \left[\frac{51}{9}\right] = [5.67] = 5, \text{ and } r = a - 9 \cdot 5 = 6.$$

b) Let $a = -51$ and $b = -9$, then $-51 = (-9)(6) + 3$. Here too, we can use the largest integer function:

$$q = \left[\frac{a}{-b}\right] = \left[\frac{-51}{9}\right] = [-5.67] = -6, \text{ and } r = a - 9 \cdot (-6) = 3.$$

Division algorithm with a GDC

The calculation we made above can also be performed with your GDC. Here are the solutions, i.e. q and r for the previous examples.

First, you go to the MATH menu, then to the 'NUM' submenu, then to the 'int(' function, which is the greatest integer function.

a)

```
MATH NUM CPX PRB
1:abs(
2:round(
3:iPart(
4:fPart(
5:int(
6:min(
7↓max(
```

```
int(51/9)
                  5
51-int(51/9)*9
                  6
```

b)

```
int(-51/9)
                 -6
-51-int(-51/9)*9
                  3
```

Example 11

Prove that if $a \in \mathbb{Z}$, then a^2 leaves a remainder of 0 or 1 when divided by 4.

Solution

By the division algorithm, $a = 4q + r$, where $0 \leqslant r < 4$. Thus,

$$a^2 = (4q + r)^2 = 16q^2 + 8qr + r^2.$$

Now the possible values of r are 0, 1, 2, or 3.

If $r = 0$, then $a^2 = 16q^2$, which is divisible by 4, so $r = 0$.

If $r = 1$, then $a^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q) + 1$, so $r = 1$.

If $r = 2$, then $a^2 = 16q^2 + 16q + 2^2 = 4(4q^2 + 4q + 1)$, which is divisible by 4, so $r = 0$.

If $r = 3$, then $a^2 = 16q^2 + 24q + 9 = 4(4q^2 + 6q + 2) + 1$, so $r = 1$.

Therefore, in all cases, $r = 0$ or 1.

Example 12

Show that the square of an odd integer is of the form $8k + 1$ for some integer k.

Solution

By the division algorithm, any integer is of the form $4q$, $4q + 1$, $4q + 2$, or $4q + 3$.

Hence, an odd integer can be of the form $4q + 1$ or $4q + 3$. When we square, we get

$$(4q+1)^2 = 16q^2 + 8q + 1 = 8(2q^2 + q) + 1 = 8k + 1, \text{ where } k = 2q^2 + q.$$

If the odd integer is of the form $4q + 3$, we have

$$\begin{aligned}(4q+3)^2 &= 16q^2 + 24q + 9\\ &= 8(2q^2 + 3q + 1) + 1 = 8k + 1, \text{ where } k = 2q^2 + 3q + 1.\end{aligned}$$

Example 13

Show that for all integers $a \geqslant 1$, $\dfrac{a(a^2+2)}{3}$ is an integer.

Solution

By the division algorithm, a is of the form $3q$, $3q + 1$ or $3q + 2$ for $q \in \mathbb{Z}$.

If $a = 3q$, then $\dfrac{a(a^2+2)}{3} = q(9q^2 + 2) \in \mathbb{Z}$.

If $a = 3q + 1$, then

$$\frac{a(a^2+2)}{3} = \frac{(3q+1)(9q^2+6q+3)}{3} = (3q+1)(3q^2+2q+1) \in \mathbb{Z}.$$

If $a = 3q + 2$, then

$$\frac{a(a^2+2)}{3} = \frac{(3q+2)(9q^2+12q+6)}{3} = (3q+2)(3q^2+4q+2) \in \mathbb{Z}.$$

Combining all three possibilities gives $\dfrac{a(a^2+2)}{3} \in \mathbb{Z}$ for $a \geqslant 1$.

Exercise 1.1–1.2

1 Find $a > 0$ where $a \mid 18$, $a \nmid 12$, and $\frac{36}{a} \nmid 10$.

2 Find $a > 0$ where $a \nmid 1000$, $5 \mid a$, $a \mid 60$, and $\frac{a}{2} \mid 75$.

3 Prove: If $m \neq 0$, then $a \mid b$ if and only if $ma \mid mb$.

4 Prove: $a \mid b$ and $b \mid a$ if and only if $a = \pm b$.

5 Prove: If $d \mid a$ and $a \neq 0$, then $|d| \leqslant |a|$.

6 Prove: If $c \mid a$ and $c \mid b$, then $c \mid (au + bv)$ for all $u, v \in \mathbb{Z}$.

7 Find the unique quotient and remainder when

a 1028 is divided by 34

b -380 is divided by 75

c 180 is divided by -31.

8 Show that the sum of an even integer and an odd integer is odd.

9 Show that the sum of two even integers or two odd integers is even.

10 Show that if a and b are odd integers and $b \nmid a$, then there exists k and l such that $a = bk + l$, where l is odd and $|l| < b$.

11 Show that if a, b, and c are integers with $b > 0$ and $c > 0$, such that when a is divided by b the quotient is q and the remainder is r, and when q is divided by c the quotient is u and the remainder is v, then when a is divided by bc, the quotient is u and the remainder is $bv + r$.

12 Show that if a and b are integers, then there are integers q, r, and $s = \pm 1$ such that $a = bq + sr$, where $\frac{-b}{2} < r \leqslant \frac{b}{2}$.

13 Prove that if u and v are integers with $v > 0$, then there exist unique integers s and t such that $u = sv + t$, where $2v \leqslant t < 3v$.

14 Use the division algorithm to prove that the cube of any integer has one of the following forms: $9k$, $9k + 1$, $9k + 8$ for some $k \in \mathbb{Z}$.

15 Use the division algorithm to prove that the fourth power of any integer is either of the form $5k$ or $5k + 1$ for $k \in \mathbb{Z}$.

16 Let a and b be non-zero integers.

a Prove that there exists unique integers q and r such that $a = bq + r$ with $\frac{-|b|}{2} < r \leqslant \frac{|b|}{2}$.

b Find the unique q and r given in **a** for $a = 49$ and $b = -6$.

17 For all odd integers m and n, if $mn = 4k + 1$, then m or n is of the form $4j - 1$.

18 Prove parts (i)–(iv) of Theorem 3.

19 Find positive integers x and y such that $x|y$ and $x2^x|y^2$, but $2^x > y$.

20 Find positive integers x and y such that $x|y$ and $2^x \leqslant y$, but $x2^x \nmid y^2$.

21 Find positive integers x and y such that $x2^x|y^2$ and $2^x \leqslant y$, but $x \nmid y$.

22 Prove that if $a|b$, and $b|c$, then $a|(ax + by + cz)$ for all $x, y, z \in \mathbb{Z}$.

In questions 23–29, prove each statement if it is true, or show that it is false either by reasoning or by finding a counter example.

23 For all integers a and b, $a + b$ is odd if and only if (iff) one of the numbers is odd and the other is even.

24 For all integers a and b, ab is even iff at least one of the numbers is even.

25 For all integers a and b, $a^3 - b^3$ is even iff $a - b$ is even.

26 For all integers n, $n^2 + n + 3$ is odd.

27 For all integers a, b, and c, $a|(b + c)$ iff $a|b$ and $a|c$.

28 For all integers a, b, and c, $a|(bc)$ iff $a|b$ and $a|c$.

29 For all integers a and b, $a^2|b^2$ iff $a|b$.

30 **a** If a group of eight students are chosen, what is the probability that two of them will be born on the same day of the week?

b Show that if any 11 numbers are chosen from the set of numbers {1, 2, 3, …, 20}, then one of them will be a multiple of another.

c Show that if any five points are chosen on or inside an equilateral triangle with side 1 cm, then two of them must be no more than 0.5 cm apart.

d Show that if any of seven points are chosen inside a hexagon with 1 cm sides, then two of them must be no more than 1 cm apart.

31 If Fibonacci numbers are denoted by F_n, and the golden ratio by $\varphi = \dfrac{1+\sqrt{5}}{2}$, prove that $\varphi^n = F_n\varphi + F_{n-1}$

32 Prove that $4 \mid 3^{2n-1} + 1$ for any integer $n \geq 1$.

33 Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geqslant \sqrt{n}, n \geqslant 1$.

34 Show that for all $n \in \mathbb{N}$, $n(n^2 + 5)$ is a multiple of 6.

1.3 Greatest common divisor/ Euclidean algorithm

If a, b, and c are integers and $c \neq 0$, then c is called a **common divisor** of a and b if $c \mid a$ and $c \mid b$. (In some cases, it is called a divisor of a and b.)

Let S be the set of all common divisors of a and b. S is a non-empty set, because ± 1 belong to the set.

If a and b are both non-zero, then the number of divisors of a and b is finite.

Hence, it makes sense to speak of the largest member of the set S.

Definition 2

If a and b are integers with at least one of them different from zero, then we define the **greatest common divisor** of a and b, denoted by $\gcd(a, b)$, as the largest positive integer which divides a and b.

Stated differently, the $\gcd(a, b)$ is a number d that satisfies the two conditions:

1 $d \mid a$ and $d \mid b$.

2 If c is a divisor of a and b, then $c \leqslant d$.

Example

- $\gcd(30, 80) = 10$. The positive divisors of 30 are: 1, 2, 3, 5, 6, 10, 15, 30. The divisors of 80 are: 1, 2, 4, 5, 8, 10, 20, 40, 80. Divisors of 30 and 80 are $\{1, 2, 5, 10\}$, and thus $\gcd(30, 80) = 10$. Notice that any other divisor must be less than 10.
- $\gcd(-30, 80) = 10$
- $\gcd(-30, 60) = 30$
- $\gcd(60, -75) = 15$
- $\gcd(25, 14) = 1$
- $\gcd(0, 23) = 23$

In defining the gcd, we can go as far as saying $d \mid |a|$ and $d \mid |b|$, i.e. in finding the gcd, we can ignore the sign!

The next theorem indicates that $\gcd(a, b)$ can be represented as a linear combination of a and b. That is, we can find two integers, x and y, such that $\gcd(a, b) = ax + by$.

For example, $\gcd(-24, 60) = 12$ implies that we can find two numbers x and y such that $12 = -24x + 60y$, and indeed $12 = -24 \cdot 2 + 60 \cdot 1$.

Theorem 5

If a and b are integers which are not both zero, then the greatest common divisor, $\gcd(a, b)$, of a and b is the smallest positive integer such that

$$\gcd(a, b) = ax + by$$

for $x, y \in \mathbb{Z}$.

Proof

Let S be the set of all positive integers of the form $ax + by$:
$S = \{ax + by \mid ax + by > 0;\ x, y \in \mathbb{Z}\}$.

S is non-empty, since $aa + bb = a^2 + b^2 > 0$. Hence, there is a smallest positive integer g such that

$$g = ax_1 + by_1$$

(by the well-ordering principle).

If either a or b is zero, the proof that $\gcd(a, b) = g$ is simple. For example, if $a = 0$, then $g = 0 + by_1 = b$ by taking $y_1 = 1$, and since $\gcd(0, b) = b$, thus $\gcd(a, b) = g$.

Assume that $a \neq 0$ and $b \neq 0$.

By the division algorithm,

$$a = gq + r \text{ with } 0 \leqslant r < g$$

and so

$$r = a - gq.$$

Hence,

$$r = a - (ax_1 + by_1)q = a(1 - x_1) + b(-qy_1).$$

Since $1 - x_1$ is an integer and $-qy_1$ is also an integer then r is of the form $ax + by$, which qualifies it to be a member of S. But r cannot be a member of S since $r < g$ and g is the smallest element in S, and therefore r must be zero.

This implies that $r = a - gq = 0$, and thus $a = gq$, or equivalently $g \mid a$. In a similar manner, we can show that $g \mid b$. Hence, g is a common divisor of a and b.

Let g_1 be any other common divisor of a and b, then Corollary 1 of Theorem 2 allows us to conclude

$$g_1 \mid (ax + by).$$

That is, $g_1 \mid g$, and by Theorem 3, part (v),

$$g_1 = |g_1| \leqslant |g| = g.$$

Thus, g is greater than any common divisor of a and b.

Finally, we can now claim that $g = \gcd(a, b)$.

The preceding theorem proved that the gcd exists and that it can be written as a linear combination of a and b. The theorem did not attempt to prove that g as found is unique. Below is a theorem that proves uniqueness.

Theorem 6

The greatest common divisor of two integers which are not both zero is unique.

Proof

Assume that g is not unique, then there is at least another integer g_1 that is also a gcd for a and b.

If g is the gcd, then any common divisor of a and b is a divisor of g, and hence $g_1 \mid g$, similarly $g \mid g_1$, and therefore $g_1 = g$.

Example

Let $a = 12$ and $b = 18$. Set S as described in the proof of Theorem 5 is

$$\begin{aligned} S &= \{ax + by \mid ax + by > 0;\ x, y \in \mathbb{Z}\} \\ &= \{12x + 18y\} \\ &= \{12(4) + 18(-2), 12(4) + 18(-1), 12(5) + 18(-3), \ldots\} \\ &= \{12, 30, 6, \ldots\}. \end{aligned}$$

The smallest element in this set is 6, which is the gcd of 12 and 18.

Now we know that $\gcd(a, b)$ is unique, and we know too that it is the smallest integer in the form $ax + by$. We have to decide how to efficiently calculate the $\gcd(a, b)$.

Theorem 7

If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.

Proof

Any common divisor of b and r also divides $bq + r = a$. Similarly, $r = a - bq$ implies that any common divisor of a and b also divides r. Thus, the two pairs of integers (a, b) and (b, r) have the same common divisors. So, they have the same greatest common divisor.

Example

- Let $a = 748$ and $b = 143$.

  We can write $748 = 143 \cdot 5 + 33$.

Now gcd(748, 143) = 11, and gcd(143, 33) = 11.

- Let $a = 954$ and $b = 216$.

  $954 = 216 \cdot 4 + 90$

  gcd(954, 216) = 18, and gcd(216, 90) = 18.

The Euclidean algorithm

Let a and b be two integers not both zero. Since $\gcd(|a|, |b|) = \gcd(a, b)$ there is no harm in assuming $a \geqslant b > 0$. By the division algorithm,

$$a = bq_1 + r_1, \text{ where } 0 \leqslant r_1 < b.$$

If $r_1 = 0$, then $b|a$ and $\gcd(a, b) = b$. If $r_1 \neq 0$, divide b by r_1 to produce integers q_2 and r_2 such that

$$b = r_1 q_2 + r_2, \text{ where } 0 \leqslant r_2 < r_1.$$

If $r_2 = 0$, then we stop and write $\gcd(a, b) = r_1$. If $r_2 \neq 0$, we continue the process. This results in the system of equations:

$$a = bq_1 + r_1,\ 0 < r_1 < b$$
$$b = r_1 q_2 + r_2,\ 0 < r_2 < r_1$$
$$r_1 = r_2 q_3 + r_3,\ 0 < r_3 < r_2$$
$$\vdots$$
$$\vdots$$
$$r_{n-1} = r_{n-1} q_n + r_n,\ 0 < r_n < r_{n-1}$$
$$r_{n-1} = r_n q_{n+1} + 0$$

Now, r_n, the last non-zero remainder, is the greatest common divisor of a and b by Theorem 7.

Example 14

Find the greatest common divisor of 306 and 657.

Solution

$657 = 306 \cdot 2 + 45$

$306 = 45 \cdot 6 + 36$

$45 = 36 \cdot 1 + 9$

$36 = 9 \cdot 4 + 0$

Thus, gcd(306, 657) = 9.

Example 15

Find the greatest common divisor of 7469 and −2387.

Solution

We know that gcd(−2387, 7469) = gcd(2387, 7469).

$$7469 = 3287 \cdot 3 + 308$$
$$2387 = 308 \cdot 7 + 321$$
$$308 = 231 \cdot 1 + 77$$
$$231 = 77 \cdot 3 + 0$$

Hence, gcd(−2387, 7469) = 77.

Application

Euclid's algorithm may be used to find integers x and y such that $\gcd(a, b) = ax + by$.

Example 16

Find $x, y \in \mathbb{Z}$ such that $\gcd(4147, 10\,672) = 4147x + 10\,672y$.

Solution

Using the Euclidean algorithm, we have

$$10\,672 = 4147 \cdot 2 + 2378 \ldots\ldots (0)$$
$$4147 = 2378 \cdot 1 + 1769 \ldots\ldots (1)$$
$$2378 = 1769 \cdot 1 + 609 \ldots\ldots (2)$$
$$1769 = 609 \cdot 2 + 551 \ldots\ldots (3)$$
$$609 = 551 \cdot 1 + 58 \ldots\ldots (4)$$
$$551 = 58 \cdot 9 + 29 \ldots\ldots (5)$$
$$58 = 29 \cdot 2 + 0$$

Thus, gcd(4147, 10 672) = 29. Now,

- From (5), $29 = 551 - 9(58)$.
- From (4), $29 = 551 - 9(609 - 551) = 10(551) - 9(609)$.
- From (3), $29 = 10(1769 - 2(609)) - 9(609) = 10(1769) - 29(609)$.
- From (2), $29 = 10(1769) - 29(2378 - 1769) = 39(1769) - 29(2378)$.
- From (1), $29 = 39(4147 - 2378) - 29(2378) = 39(4147) - 68(2378)$.
- From (0), $29 = 39(4174) - 68(10\,762 - 2(4147))$
  $= 175(4174) - 68(10\,762)$.

The last statement gives us the required expression, i.e. $29 = 175(4174) - 68(10\,672)$.

In this case, $x = 175$ and $y = -68$.

Example 17

Find $x, y \in \mathbb{Z}$ such that $\gcd(-180, 252) = -180x + 252y$.

Solution

Using the Euclidean algorithm, we have

$$252 = 180 \cdot 1 + 72$$
$$180 = 72 \cdot 2 + 36$$
$$72 = 36 \cdot 2 + 0$$

Hence, $\gcd(-180, 252) = \gcd(180, 252) = 36$. Now,

$$36 = 180 - 2(72) = 180 - 2(252 - 180) = 3(180) - 2(252).$$

So, $36 = -3(-180) - 2(252)$.

In this case, $x = -3$ and $y = -2$.

Example 18

Find $x, y \in \mathbb{Z}$ such that $\gcd(143, 252) = 143x + 252y$.

Solution

Using the Euclidean algorithm, we have

$$252 = 143 \cdot 1 + 109$$
$$143 = 109 \cdot 1 + 34$$
$$109 = 34 \cdot 3 + 7$$
$$34 = 7 \cdot 4 + 6$$
$$7 = 6 \cdot 1 + 1$$
$$6 = 1 \cdot 6 + 0$$

Hence, $\gcd(143, 252) = 1$ (143 and 252 are said to be relatively prime). Now,

$$\begin{aligned} 1 &= 7 - 6 = 7 - (34 - 7(4)) = 5(7) - 34 \\ &= 5(109 - 3(34)) - 34 = 5(109) - 16(34) \\ &= 5(109) - 16(143 - 109) = 21(109) - 16(143) \\ &= 21(252 - 143) - 16(143) = 21(252) - 37(143). \end{aligned}$$

So, $1 = 21(252) - 37(143)$ or $1 = -37(143) + 21(252)$.

Here, $x = -37$ and $y = 21$.

Example 18 triggers a new definition and a new theorem.

Definition 3

Two integers a and b, not both zero, are said to be relatively prime if $\gcd(a, b) = 1$.

So, 143 and 252 are relatively prime. 12 and 25 are relatively prime because $\gcd(12, 25) = 1$; however, 18 and 24 are not relatively prime because $\gcd(18, 24) = 6$.

Theorem 8

Let a and b be integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that $ax + by = 1$.

Proof

If a and b are relatively prime, so that $\gcd(a, b) = 1$, then Theorem 5 guarantees the existence of x and y satisfying $1 = ax + by$.

Now, suppose on the other hand, $1 = ax + by$ for some integers x and y. Let $g = \gcd(a, b)$. Since $g | a$ and $g | b$, then $g | (ax + by)$ by Corollary 1 of Theorem 2. This means that $g | 1$, which is only possible if $g = 1$, since g has to be positive.

Therefore, if a and b are relatively prime, then there exist two integers x and y such that $ax + by = 1$.

Example 19

Find $\gcd(14, 75)$ and write it in the form $14x + 75y$.

Solution

$$75 = 14 \cdot 5 + 5$$

$$14 = 5 \cdot 2 + 4$$

$$5 = 4 \cdot 1 + 1$$

So, $\gcd(14, 75) = 1$.

Now,

$$\begin{aligned} 1 &= 5 - 4 = 5 - (14 - 5(2)) = 3(5) - 14 = 3(75 - 14(5)) - 14 \\ &= 3(75) - 16(14) \\ &= -16(14) + 3(75). \end{aligned}$$

Example 20

Find $\gcd(49, 60)$ and write it in the form $49x + 60y$.

Solution

$$60 = 49 \cdot 1 + 11$$

$$49 = 11 \cdot 4 + 5$$

$$11 = 5 \cdot 2 + 1$$

So, $\gcd(49, 60) = 1$.

Now,

$$1 = 11 - 5 \cdot 2 = 11 - (49 - 11 \cdot 4) \cdot 2 = 9 \cdot 11 - 2 \cdot 49$$
$$= 9(60 - 49) - 2 \cdot 49 = 9(60) - 11(49) = -11(49) + 9(60).$$

Corollary 3

If $\gcd(a, b) = g$, then $\gcd\left(\frac{a}{g}, \frac{b}{g}\right) = 1$.

Proof

Since $\gcd(a, b) = g$, then by Theorem 5, it is possible to find integers x and y such that

$$g = ax + by.$$

Dividing both sides of the equation by g, we obtain

$$1 = \left(\frac{a}{g}\right)x + \left(\frac{b}{g}\right)y.$$

Now, using Theorem 8, we conclude that $\left(\frac{a}{g}\right)$ and $\left(\frac{b}{g}\right)$ are relatively prime, and hence

$$\gcd\left(\frac{a}{g}, \frac{b}{g}\right) = 1.$$

Note: Even though $\left(\frac{a}{g}\right)$ and $\left(\frac{b}{g}\right)$ appear as fractions, they are, in fact, integers because g is a divisor of both a and b.

Example

- $\gcd(180, 252) = 36 \Rightarrow \gcd\left(\frac{180}{36}, \frac{252}{36}\right) = \gcd(5, 7) = 1$
- $\gcd(4147, 10672) = 29 \Rightarrow \gcd\left(\frac{4147}{29}, \frac{10672}{29}\right) = \gcd(143, 368) = 1$
- $\gcd(-2387, 7469) = 77 \Rightarrow \gcd\left(\frac{-2387}{77}, \frac{7469}{77}\right) = \gcd(-31, 97) = 1$

Corollary 4

If $\gcd(a, b) = 1$, and if $a \mid c$ and $b \mid c$, then $ab \mid c$.

Proof

$a \mid c \Rightarrow c = ma$, and $b \mid c \Rightarrow c = nb$, and

$\gcd(a, b) = 1 \Rightarrow 1 = ax + by$ for some $x, y \in \mathbb{Z}$.

Multiplying the last equation by c renders

$c = cax + cby$, and with appropriate substitution of the values for c on the right-hand side, we have

$c = nbax + maby = ab(nx + my)$, which leads to the conclusion that $ab \mid c$.

Example

$\gcd(9, 14) = 1$, $9 \mid 756$ and $14 \mid 756$, then $9 \cdot 14 = 126 \mid 756$. In fact, $756 = 6 \cdot 126$.

Two other theorems of interest are detailed below.

Theorem 9

This is sometimes called Euclid's lemma.

If $a \mid bc$, and if $\gcd(a, b) = 1$, then $a \mid c$.

Proof

Since $1 = ax + by$, then $c = acx + bcy$. Obviously $a \mid ac$ and $a \mid bc$ which is given, and thus $a \mid (acx + bcy)$; therefore $a \mid c$.

Theorem 10

Let $a, b \in \mathbb{Z}$ not both zero. For a positive integer d, $d = \gcd(a, b)$ iff:

1 $d \mid a$ and $d \mid b$.

2 If $c \mid a$ and $c \mid b$, then $c \mid d$.

This is sometimes considered as an alternative to Theorem 5.

Proof

($\Rightarrow$) If $d = \gcd(a, b)$, then obviously $d \mid a$ and $d \mid b$. Also, $d = ax + by$, and if $c \mid a$ and $c \mid b$, then $c \mid (ax + by)$, i.e. $c \mid d$.

($\Leftarrow$) If $d \mid a$ and $d \mid b$, then d is a common divisor of a, and b. If $c \mid a$ and $c \mid b$, then $c \mid d$, then $d \geqslant c$, which means that d is greater than any divisor of a and b, and thus it is the greatest common divisor of a and b.

Note: The gcd can be extended to more than two integers. We can define it in a similar manner:

Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ with $a_1, a_2, \ldots, a_n$ not all zero. The greatest common divisor of $a_1, a_2, \ldots, a_n$, denoted $\gcd(a_1, a_2, \ldots, a_n)$, is the greatest integer d such that d divides $a_1, a_2, \ldots, a_n$.

For example, to find the greatest common divisor of $(18, 36, 63)$, we can perform the process by taking $\gcd(18, 36) = 18$, and then $\gcd(18, 63) = 9$. Or for $\gcd(30, 42, 70)$, we find $\gcd(30, 70) = 10$, and then $\gcd(10, 42) = 2$. Or for $\gcd(36, 48, 54, 126)$, we find $\gcd(36, 48) = 12$, and $\gcd(54, 126) = 18$, and so $\gcd(12, 18) = 6$.

Note:

- If $g = \gcd(a, b)$, and if k is an integer, then $\gcd(ka, kb) = kg$.
- If $g = \gcd(a, b)$, and if k is an integer, then $\gcd(a, b + ka) = g$.

The proofs are left for you as exercises.

Least common multiple

In this section we will discuss the smallest integer which is divisible by two given integers a and b.

We call such an integer the **least common multiple** of a and b. We will also investigate its relation with $\gcd(a, b)$.

Definition 4

Let $a, b, c \in \mathbb{Z}$ with $a, b > 0$. Then a **common multiple** of a and b is a number c such that $a|c$ and $b|c$.

Example

36 is a common multiple of 12 and 18 since $12|36$ and $18|36$.

Definition 5a

Let $a, b \in \mathbb{Z}$ and $a, b > 0$. Then the smallest positive integer l such that l is a multiple of a and b is called the **least common multiple** of a and b. l is denoted by $\operatorname{lcm}(a, b)$.

The existence of $l = \operatorname{lcm}(a, b)$ follows from the well-ordering principle. To see this, let S be the set of all positive multiples of a and b with $a, b > 0$. S is a non-empty set, since $a, b \in S$. By the well-ordering principle, S has a least element, say l. l is the $\operatorname{lcm}(a, b)$.

A slightly different definition of the lcm is given below. It may prove to be more appropriate for proofs later on.

Definition 5b

The **least common multiple** of two integers a and b, denoted by $\operatorname{lcm}(a, b)$, is the positive integer m satisfying the following:

1. $a|m$ and $b|m$.
2. If $a|c$ and $b|c$, with $c > 0$, then $m \leqslant c$.

Note: Given non-zero integers a and b, $\operatorname{lcm}(a, b)$ always exists and $\operatorname{lcm}(a, b) \leqslant |ab|$.

Theorem 11

For positive integers a and b,

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab.$$

Proof (Optional)

Let $e = \frac{a}{g}$ and $f = \frac{b}{g}$. Then $\frac{ab}{g^2} = ef \Leftrightarrow \frac{ab}{g} = gef$. Since a, b, and g are positive integers, gef is also a positive integer.

We show now that $gef = \text{lcm}(a, b)$.

Since $gef = (ge)f = af$ and $gef = egf = e(gf) = eb$, gef is a common multiple of a and b.

Now, let $l = \frac{ab}{g}$ and c be another common multiple of a and b.

Let $c = au$ and $c = bv$, where u and v are positive integers.

Also, by Theorem 5, there are integers x and y such that $g = ax + by$.

Hence,

$$\frac{c}{l} = \frac{cg}{ab} = \frac{c(ax+by)}{ab} = \left(\frac{c}{b}\right)x + \left(\frac{c}{a}\right)y = vx + uy.$$

Thus, $l \mid c$ and we conclude that $l \leqslant c$.

By the definition of $\text{lcm}(a, b)$, $l = \text{lcm}(a, b) = \frac{ab}{\gcd(a, b)}$.

Thus,

$$\gcd(a, b) \cdot \text{lcm}(a, b) = ab.$$

Example 21

Find

a) $\text{lcm}(36, 63)$

b) $\text{lcm}(396, 756)$

c) $\text{lcm}(2387, 7469)$.

Solution

a) Since $\gcd(36, 63) = 9$, then $\text{lcm}(36, 63) = \frac{36 \cdot 63}{9} = 252$.

b) Since $\gcd(396, 756) = 36$, then $\text{lcm}(396, 756) = \frac{396 \cdot 756}{36} = 8316$.

c) Since $\gcd(2387, 7469) = 77$, then $\text{lcm}(2387, 7469) = \frac{2387 \cdot 7469}{77} = 231\,539$.

Note: If $\text{lcm}(a, b) = l$, and if k is an integer, then $\text{lcm}(ka, kb) = kl$. The proof is left for you as an exercise.

Exercise 1.3

In questions 1–6 find the greatest common divisor by Euclidean algorithm.

1 $a = 172, b = 64$

2 $a = 167, b = 117$

3 $a = -323, b = 221$

4 $a = 1292, b = 884$

5 $a = 7469, b = -2387$

6 $a = 11\,143, b = 8749$

In questions 7–12 find integers x and y such that:

7 $2 = 32x + 78y$

8 $13 = 91x + 104y$

9 $6 = 3054x + 12\,378y$

10 $\gcd(-119, 272) = -119x + 272y$

11 $\gcd(1769, 2378) = 1769x + 2378y$

12 $\gcd(-2059, 2581) = -2059x + 2581y$

13 Do integers x and y exist such that $x + y = 100$ and $\gcd(x, y) = 8$?

14 Let a and b be relatively prime integers. Prove that $\gcd(a + b, a - b)$ is either 1 or 2.

15 Let $a, b \in \mathbb{Z}$ with a and b both non-zero. Prove that $\gcd(ca, cb) = |c|\gcd(a, b)$ for any non-zero integer c.

16 Let $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and $c \mid (a + b)$. Prove that $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$.

17 Find lcm(152, 236).

18 Find lcm(336, 746).

19 Find lcm(100, 105).

20 Find all pairs of positive integers whose greatest common divisor is 12 and the least common multiple is 360.

21 If two integers a and b have greatest common divisor 1, what can you say about $\text{lcm}(a, b)$? Give a reason for your answer.

22 You are given positive integers a, b, and c. If $\gcd(a, b, c) = g$, is it true that $\text{lcm}(a, b, c) = abc \div g$? If your answer is yes, find lcm(24, 42, 28).

23 Show that $\gcd(a, b) = \gcd(|a|, |b|)$.

24 Show that $\text{lcm}(a, b) = \text{lcm}(|a|, |b|)$.

25 Show why $\dfrac{ab}{\text{lcm}(a, b)}$ must be an integer when $a, b \neq 0$.

26 Prove that $\gcd(k, k + 2) = 2$ when k is even and $\gcd(k, k + 2) = 1$ when k is odd.

27 If $k \in \mathbb{Z}^+$, show that $\text{lcm}(k, k+2) = \dfrac{k(k+2)}{2}$ when k is even, and $\text{lcm}(k, k+2) = k(k+2)$ when k is odd.

28 If $k \in \mathbb{Z}^+$, show that $\gcd(a, a + k) = \gcd(a, k)$.

29 Let $a, b, c \in \mathbb{Z}\backslash\{0\}$. Show that if $a = bx + cy$, then $\gcd(b, c) \leqslant \gcd(a, b)$.

30 Let $a, b, c \in \mathbb{Z}\backslash\{0\}$. Show that if $a = bx + cy$, then $\gcd(b, c) \mid \gcd(a, b)$.

1.4 Fundamental theorem of arithmetic

Prime numbers

Consider the following numbers and their divisors:

Number	Divisors
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6
7	1, 7
8	1, 2, 4, 8
15	1, 3, 5, 15

You can clearly see that 2, 3, 5, and 7 each have two divisors, 1 and the number itself. The numbers 4, 6, 8, and 15 have additional divisors other 1 and the number itself. This leads to the following definition.

Definition 6

Every integer, p, greater than one which has only p and 1 for its divisors is called a **prime number**. If an integer $n > 1$ is not prime, then it is called a **composite number**.

For instance, integers 2, 3, 5, and 7 are prime numbers, while 4, 6, 8, and 15 are composite numbers.

Note:

- By definition, 1 is neither prime nor composite!
- 2 is the only even integer that is prime, all other even integers are composite. Every even integer can be written in the form $2n$, where n is an integer. As such, every integer has at least two divisors, 2 and n, different from 1 and itself.

  For instance, $6 = 2 \cdot 3$ has 2 and 3 as divisors in addition to 1 and 6; $18 = 2 \cdot 9$ has several divisors, but at least two are immediately apparent, 2 and 9. The other divisors of 18 are 3 and 6.

Example

Prime numbers between 2 and 100 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

Theorem 12

Every integer greater than 1 has a prime divisor.

Proof

We will present an indirect proof.

Suppose that not every integer greater than 1 has a prime divisor. Then there is an integer $n > 1$ which has no prime divisor. Let S be the collection of all integers greater than 1 with no prime divisors. Since, by assumption, $n > 1$ has no prime divisors, $n \in S$. S is a non-empty subset of natural numbers. By the well-ordering principle, S has a least element, say m. Since m has no prime divisors, m is not a prime. Hence, there exist $a, b \in \mathbb{Z}$ such that $m = ab$ with $1 < a < m$ and $1 < b < m$. Since $1 < a < m$, a has a prime divisor, say p. So $p | m$ which contradicts that m has no prime divisor. This proves that every integer greater than 1 has a prime divisor.

Example

Integers that are larger than 1 are even or odd. If a number m is even, then we can write it as $m = 2n$, and hence it has at least one prime divisor, 2. If the number is odd, then either it is a prime number, and that satisfies the theorem, or it has at least one of the following prime numbers as a divisor: 3, 5, 7, 11,…, and that satisfies the theorem too! Here are some numbers:

9 has 3 as a divisor, 11 is prime, 21 has 3 as a divisor, 143 has 11 as a divisor, 149 is prime.

Our next result shows that there are infinitely many primes. The proof of this result appears in Proposition 20 in Book IX of Euclid's *Elements*. This proof demonstrates a higher level of thinking and great mathematical ingenuity.

Theorem 13

There are infinitely many prime numbers.

Proof

Assume the result is not true. Then there are a finite number of primes. Let us label these primes $p_1, p_2, \ldots, p_n$. Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$.

Here N is an integer greater than 1. By Theorem 12, N has a prime divisor p.

Since $p_1, p_2, \ldots, p_n$ are all the primes, p has to be one of these, say p_i for some $i = 1, 2, \ldots, N$. Since $p_i | N$ and $p_i \mid p_1 \cdot p_2 \cdot \ldots \cdot p_n$, then $p_i \mid N - p_1 \cdot p_2 \cdot \ldots \cdot p_n$, i.e. $p_i | 1$, a contradiction, since $p_i > 1$. Hence, there are infinitely many primes.

Example

Mathematicians still compete to find the largest prime number. The following are some of the numbers discovered.

- $48\,047\,305\,725 \cdot 2^{172\,403} - 1$
- $34\,790! + 1$
- $2^{43\,112\,609} - 1$

Theorem 14

Let n be a composite number. Then n has a prime divisor p with $p \leqslant \sqrt{n}$.

Proof

Given that n is a composite number, there exists $a, b \in \mathbb{Z}$ such that $n = ab$, with $1 < a < n$ and $1 < b < n$.

Without loss of generality, let us assume $a \leqslant b \cdot n = ab$ implies that $a \leqslant \sqrt{n}$ because if $a > \sqrt{n}$, then $n = ab > \sqrt{n}\sqrt{n} = n$ which is impossible. By Theorem 12, a has a prime divisor. Let this divisor be p. Hence, $p | a$ and $n = ab$ implies $p | n$.

Furthermore, $p \leqslant a \leqslant \sqrt{n}$.

Theorem 14 provides a method of finding all prime numbers less than or equal to n. This was first given by the Greek mathematician Eratosthenes of Cyrene (276 BC–194 BC).

Example

Suppose that we wish to find all prime numbers less than or equal to 50. By Theorem 14, every composite number less than or equal to 50 has a prime divisor less than or equal to $\sqrt{50} = 7.07106\ldots$.

Such prime numbers are 2, 3, 5, and 7. Hence, from the list of integers from 2 to 50, we delete all multiples of 2, 3, 5, and 7, excluding 2, 3, 5, and 7. Applying this, we have

2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11
~~12~~	~~13~~	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~
~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~	31
~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~	41
~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~	

This method is called the **sieve of Eratosthenes**.

Any number which is in this list after removing the multiples of 2, 3, 5, and 7 cannot be composite by Theorem 14.

Note: Theorem 14 also provides an algorithm for testing whether a given positive integer $n > 1$ is prime or composite. To do this, determine all prime numbers less than or equal to $\sqrt{n}$, then test out if n is divisible by those primes. If n is divisible, then it is composite, otherwise it is a prime number.

Example 22

Test if 227 is a prime or composite number. Repeat with 456.

Solution

227: $\sqrt{227} = 15.066$. Hence, prime numbers less than 15 are 2, 3, 5, 7, 11, and 13. A simple divisibility test shows that 227 is not divisible by any of these numbers and thus it is prime.

456: $\sqrt{457} = 21.38$. Hence, prime numbers less than 21 are 2, 3, 5, 7, 11, 13, 17, and 19. A simple divisibility test shows that 457 is not divisible by any of these numbers and thus it is prime.

Note: If two prime numbers differ by two, then such pairs of prime numbers are called **twin primes**.

Examples of some twin primes are 3, 5; 5, 7; 11, 13; 17, 19; 29, 31; etc.

The twin prime conjecture

There are infinitely many prime numbers p such that $p + 2$ is also a prime number.

This is still an unsolved conjecture. At the time of writing, the largest known pair of twin primes are $65\,516\,468\,355 \cdot 2^{333\,333} \pm 1$.

Many problems in number theory deal with integers that are expressible in certain forms. For example, the even numbers 4, 6, 8, 10, 12, and 14 are expressed as the sum of two prime numbers, not necessarily distinct:

$$4 = 2 + 2, \quad 6 = 3 + 3, \quad 8 = 3 + 5, \quad 10 = 55, \quad 12 = 5 + 7, \quad 14 = 7 + 7.$$

This led Christian Goldbach to make the following conjecture in 1742.

The Goldbach conjecture

Every even integer greater than 2 can be expressed as the sum of two (not necessarily distinct) prime numbers.

Some extra problems

In this section we solve some additional problems to gain a better understanding of the methods previously outlined.

Example 23

Prove that if p is a prime and $p \mid a^k$ for some positive integer k, then $p \mid a$ and $p^k \mid a^k$. Is this valid if p is a composite number?

Solution

Since $a^k = a \cdot a \cdot \ldots \cdot a$ (k times), $p \mid a^k$ implies $p \mid a$. Hence, there is an integer q such that $a = pq$. Then $a^k = p^k q^k$ and consequently $p^k \mid a^k$.

This does not hold for all composite numbers. For example, take $p = 4$ and $a = 2$: $4 \mid p^k$ for $k = 2$, $4 \mid 2^2$, but $4 \nmid 2$.

Example 24

If $2^m + 1$ is prime, then prove that $m = 2^n$ for some integer $n \geqslant 0$.

Solution

We shall prove this by showing that if m is not a power of 2, then $2^m + 1$ is not a prime. If m is not a power of 2, then m has the form $2^n q$ for some odd integer $q > 1$.

$f(t) = t^q + 1$ is divisible by $t + 1$ (since $t^q + 1 = (t + 1)(t^{q-1} - t^{q-2} + \ldots + 1)$). Substituting $t = x^{2^n}$, we find that $2^{2^n} + 1$ divides $g(2) = 2^m + 1$. This implies that $2^m + 1$ cannot be a prime. This argument proves that when m is not a power of 2, $2^m + 1$ is not a prime. By using equivalence of statements, $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$, we complete the proof of the result.

The fundamental theorem of arithmetic

The fundamental theorem of arithmetic appeared in Proposition 14 in Book 1 of Euclid's *Elements*. This is the first big result in number theory and guarantees that any integer greater than 1 can be decomposed uniquely into a product of prime numbers.

Example

$12 = 2 \times 2 \times 3 = 2^2 \times 3, 56 = 2^3 \times 7, 124 = 2^2 \times 31, 11430 = 2 \times 3^2 \times 5 \times 127$

Theorem 15

Let $a, b, p \in \mathbb{Z}$, with p a prime number. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

Proof

Suppose $p \nmid a$. Then $\gcd(a, p) = 1$. Then there are integers m and n such that $ma + np = 1$. Also $p \mid ab$ means that there is an integer c such that $ab = pc$. Now multiplying both sides of $ma + np = 1$ by b, we get $mab + npb = b$. Using $ab = pc$, $mab + npb = pc$ reduces to $p(mc + nb) = b$. So $p \mid b$. This can be repeated for the case $p \nmid b$, and the conclusion would be $p \mid a$.

One may wonder if it is necessary that p be a prime in Theorem 15. In fact, the theorem fails to hold when p is a composite number. For example, take $p = 6$ and $a = 9$ and $b = 8$: $6 \mid (8 \cdot 9)$, but $6 \nmid 8$ and $6 \nmid 9$.

We can show that if $a_1, a_2, \ldots, a_n, p \in \mathbb{Z}$, with p a prime, and $p \mid a_1 \cdot a_2 \cdot \ldots \cdot a_n$, then $p \mid a_k$ for some $1 \leqslant k \leqslant n$.

We are now in a position to state the most important theorem of this section.

Theorem 16 (The fundamental theorem of arithmetic)

Every integer n greater than 1 can be expressed in the form $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_n^{a_n}$ with distinct prime numbers $p_1, p_2, \ldots, p_n$ and positive integers $a_1, a_2, \ldots, a_n$.

Proof (Outline – optional)

We must prove two things:

1 Every positive integer can be expressed as a product of primes.

2 The expression in **1** is unique.

First, we use strong induction to prove that every positive integer n is a product of primes. As a base case, $n = 1$ is the product of the empty set of primes. (A standard convention: the product of an empty set of numbers is defined to be 1, much as the sum of an empty set of numbers is defined to be 0. Without this convention the theorem would not be true for $n = 1$. In that case we can choose another value.) For the inductive step, suppose that every $k < n$ is a product of primes. We must show that n is also a product of primes.

We must show that n is also a product of primes. If n is itself prime, then this is true trivially. Otherwise, $n = ab$ for some $a, b < n$. By the induction assumption, a and b are both products of primes. Therefore, $a \cdot b = n$ is also a product of primes. Thus, the claim is proved by induction.

Second, we use the well-ordering principle to prove that every positive integer can be written as a product of primes in a unique way. The proof is by contradiction: assume, contrary to the claim, that there exist positive integers that can be written as products of primes in more than one way. By the well-ordering principle, there is a smallest integer with this property. Call this integer n, and let

$$n = p_1 \cdot p_2 \cdot \ldots \cdot p_j = q_1 \cdot q_2 \cdot \ldots \cdot q_k$$

be two of the (possibly many) ways to write n as a product of primes. Now, $p_1 \mid n$ and so

$$p_1 \mid q_1 \cdot q_2 \cdot \ldots \cdot q_k.$$

By the previous theorem, this implies that p_1 divides one of the primes q_i. But since q_i is a prime, it must be that $p_1 = q_i$. Deleting p_1 from the first product and q_i from the second, we find that n/ p_1 is a positive integer *smaller* that n that can also be written as a product of primes in two distinct ways. But this contradicts the definition of n as the smallest such positive integer. Thus, the assumption is false and we have one way of writing the product of primes.

Example

Prime factorization of $132 = 2^2 \cdot 3 \cdot 11$.

Prime factorization of $3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7$.

We can use the fundamental theorem to find the gcd and lcm of two or more integers.

Example 25

Find gcd(132, 3780) and lcm(132, 3780).

Solution

We have from the previous example:

$132 = 2^2 \cdot 3 \cdot 11$ and $3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7$

For gcd(132, 3780), we compare the exponents appearing on like prime numbers and choose the minimum exponent appearing in prime factorizations of 132 and 3780 (since gcd(132, 3780) is the largest common divisor of 132, 3780).

So, $\gcd(132, 3780) = 2^2 \cdot 3 = 12$.

Similarly for lcm(132, 3780), we compare the exponents appearing on like prime numbers and choose the maximum exponent appearing in their prime factorization.

Since $132 = 2^2 \cdot 3 \cdot 11 = 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 11^1$ and $3780 = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 11^0$, $\text{lcm}(132, 3720) = 2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 4180$.

We can now state what we have done in Example 18 as a theorem (proof not included here).

Theorem 17

Let $a, b \in \mathbb{Z}$ with $a, b > 1$. Let $a = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_n^{a}$ and $b = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_n^{b_n}$, where $p_1, p_2, \ldots, p_n$ are distinct prime numbers and $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are non-negative integers (some of these may be 0). Let m_i be the smaller and M_i be the larger of a_i and b_i for $i = 1, 2, \ldots, n$. Then,

$\gcd(a, b) = p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n}$, and

$\text{lcm}(a, b) = p_1^{M_1} p_2^{M_2} \ldots p_n^{M_n}$.

Example 26

Using the fundamental theorem of arithmetic, find gcd(1176, 936) and lcm (1176, 936).

Solution

$1176 = 2^3 \cdot 3 \cdot 7^2$; $936 = 2^3 \cdot 3^2 \cdot 13$, and hence:

$\gcd(936, 1176) = 2^3 \cdot 3 = 24$

$\text{lcm}(936, 1176) = 2^3 \cdot 3^2 \cdot 7^2 \cdot 13 = 45\,864$

This method of finding the gcd and lcm of two positive integers *a* and *b* is easily used to find the gcd and lcm of three or more positive integers. We consider the following as an illustration.

Example 27

Find gcd(132, 936, 1176) and lcm(132, 936, 1176).

Solution

$132 = 2^2 \cdot 3 \cdot 11, \quad 936 = 2^3 \cdot 3^2 \cdot 13, \quad 1176 = 2^3 \cdot 3 \cdot 7^2$

$\gcd(132, 936, 1176) = 2^2 \cdot 3 = 12$

$\operatorname{lcm}(132, 936, 1176) = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 13 = 504\,504$

Exercise 1.4

1 Prove that there are infinitely many primes of the form $4q + 3$, $q = 0, 1, \ldots$.

2 Prove that every prime $p \neq 3$ has the form $3q + 1$ or $3q + 2$ for some integer q.

3 Prove that there are infinitely many primes of the form $3q + 2$.

4 Prove that only for the prime number $p = 3$, $p^2 + 2$ is a prime.

5 If $2^p - 1$ is a prime number, then show that $2^{p-1}(2^r - 1)$ is equal to the sum of its proper divisors.

6 From $5 = 2^2 + 1$, $17 = 4^2 + 1$, $37 = 6^2 + 1$, $101 = 10^2 + 1$, and $197 = 14^2 + 1$, what kind of conjecture can you propose for primes of the form $n^2 + 1$?

7 Find the prime factorization of each integer given below.

a 87 **b** 361 **c** 945 **d** 1001 **e** 6992

8 Using the fundamental theorem of arithmetic, find the following:

a gcd(87, 361) and lcm(87, 361)

b gcd(361, 1001) and lcm(361, 1001)

c gcd(87, 361, 1001) and lcm(87, 361, 1001)

d gcd(87, 945, 6992) and lcm(87, 945, 6992)

9 Find five integers that are relatively prime (when taken together) such that no two of the integers are relatively prime when taken separately.

10 Let a and b be positive integers.

a Prove that $\gcd(a, b) \mid \operatorname{lcm}(a, b)$.

b Find and prove a necessary and sufficient condition for $\gcd(a, b) = \operatorname{lcm}(a, b)$.

c Prove that $\operatorname{lcm}(ca, cb) = c \operatorname{lcm}(a, b)$.

11 Let $\gcd(a, b) = g$. Show that if $a \mid bc$, then $a \mid gc$.

12 Show that if a and b are relatively prime, then a^2 and b^2 are also relatively prime.

In questions 13–16, use prime factors to decide whether $x|y$, to find gcd(x, y), and to find lcm(x, y).

13 $x = 585$, $y = 14\,157$

14 $x = 11\,500$, $y = 4232$

15 $x = 2277$, $y = 15\,939$

16 $x = 1870$, $y = 2275$

In questions 17–22, prove each statement if it is true, or show that it is false either by reasoning or by finding a counter example.

17 For all integers x, $x > 2$, $x^3 - 8$ is composite.

18 If $m^2 | n^2$ then $m | n$.

19 If $n | ab$ and $n \nmid a$, then $n | b$.

20 If $n | ab$ and $\gcd(n, a) = 1$, then $n | b$.

21 $\gcd(a, b) = \gcd(a, b + ka)$ for all $k \in \mathbb{Z}$.

22 $\gcd(a^n, b^n) = (\gcd(a, b))^n$.

23 What are the possible values of $\gcd(a, a + 3)$?

24 If a and b are relatively prime, then what are the possible values of $\gcd(a + b, a - b)$?

25 Under what conditions can we solve $ax + (a + 2)y = c$ for x and y?

2 Number Theory II

In Chapter 1 we dealt with all the theorems necessary to work on some applications of number theory. In this chapter we shall discuss a few of these applications.

2.1 Congruence

So far you have seen examples involving congruence for specific values. In this section we will discuss congruence in more general terms. This topic is important for this option, as well as for the abstract algebra option.

Definition 1

Let m be a positive integer. If a and b are integers, we say that a is congruent to b modulo m if $m \mid (a - b)$.

If a is congruent to b modulo m, then we write $a \equiv b \pmod{m}$. If a is not congruent to b modulo m, then we write $a \not\equiv b \pmod{m}$. The integer m is called the **modulus of congruence**.

Example

We have $24 \equiv 4 \pmod{5}$, since $5 \mid (24 - 4)$. Similarly, $5 \equiv -11 \pmod{8}$, since $8 \mid (5 - (-11))$. On the other hand, $4 \not\equiv 17 \pmod{2}$, since $(4 - 17)$ is not divisible by 2.

Theorem 1

If $a, b \in \mathbb{Z}$, then $a \equiv b \pmod{m}$ for some positive integer m if and only if there exists an integer k such that $a = b + km$.

Proof

($\Rightarrow$) Since $m \mid (a - b)$ if and only if $a - b = km$ for some $k \in \mathbb{Z}$, then $a = b + km$.

($\Leftarrow$) If for some $k \in \mathbb{Z}$, $a = b + km$, $km = a - b$. Hence, $m \mid (a - b)$, and consequently $a \equiv b \pmod{m}$.

So, we can summarize this result by stating: Given a positive integer m and an integer b, integers which are congruent to b modulo m are obtained by adding integer multiples of m to b.

As an illustration, let $m = 2$ and $b = 0$. Then the integers congruent to 0 modulo 2 are given by $a = 0 + 2k$, $k \in \mathbb{Z}$, i.e. $\{\dots, -4, -2, 0, 2, 4, \dots\}$.

If $b = 1$, then the collection of all integers congruent to 1 are $\{\ldots, -3, -1, 1, 3, \ldots\}$. We can observe that these two classes of integers are distinct and each one is associated to a remainder when we divide an arbitrary integer n by 2.

This discussion leads us to the following important theorem which explains how congruence partitions the set of integers into different sets like the ones above. These are called **congruence classes modulo *m*.**

Theorem 2

$a \equiv b \pmod{m}$ if and only if a and b leave the same remainder when we divide them by m.

Proof

($\Rightarrow$) Let $a \equiv b \pmod{m}$. Then, by definition, $m \mid (a - b)$.

Now, by the division algorithm, if we divide a by m, we can find q_1 and r_1 such that

$$a = m \cdot q_1 + r_1, 0 \leqslant r_1 < m$$

and similarly, if we divide b by m, then we can find q_2 and r_2 such that

$$b = m \cdot q_2 + r_2, 0 \leqslant r_2 < m.$$

So, we now have

$$a - b = (m \cdot q_1 + r_1) - (m \cdot q_2 + r_2) = m(q_1 - q_2) + (r_1 - r_2).$$

However, $m \mid (a - b)$, and so m must divide the right-hand side, $m(q_1 - q_2) + (r_1 - r_2)$.

This leads to the fact that m must divide $(r_1 - r_2)$ too. But $0 \leqslant r_1 < m$ and $0 \leqslant r_2 < m$, and so $(r_1 - r_2)$ cannot divide m unless $r_1 - r_2 = 0$, i.e. $r_1 = r_2$.

Therefore, a and b leave the same remainder when we divide them by m.

($\Leftarrow$) Let a and b leave the same remainder when we divide them by m.

Then we have

$a = m \cdot q_1 + r$ and $b = m \cdot q_2 + r$, and consequently

$a - b = m(q_1 - q_2)$, which means that $m \mid (a - b)$ and therefore $a \equiv b \pmod{m}$.

Theorem 3

Let $m \in \mathbb{Z}^+$. Then congruence modulo m is an equivalence relation. (See Option 2 Chapter 2 for review.)

Proof

1 **Reflexive property:** $a \equiv a \pmod{m}$ since $m \mid (a - a)$ for all $a \in \mathbb{Z}$.

2 **Symmetric property:** Suppose $a \equiv b \pmod{m}$. Then there is an integer k such that $a - b = km$. Hence, $b - a = (-k)m$ and $m \mid (b - a)$ [$-k$ is also an integer]. Thus $b \equiv a \pmod{m}$.

3 **Transitive property:** If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $m \mid (a - b)$ and $m \mid (b - c)$. Hence, $m \mid \big((a-b)-(b-c)\big)$, i.e. $m \mid (a-c)$ and $a \equiv c \pmod{m}$.

Note: The two previous theorems enable us to generalize the structure of congruence classes modulo m. Since any two integers that leave the same remainder when divided by m, the remainder itself will represent the equivalence class. This is so because if a leaves a remainder r when divided by m, then as we showed before:

$$a = m \cdot q_1 + r \Rightarrow a - r = m \cdot q_1 \Rightarrow m \mid (a - r) \Rightarrow a \equiv r \pmod{m}.$$

Also, since $r < m$, then it takes on all the values $\{0, 1, 2, 3, \ldots, m - 1\}$, and hence the congruence classes modulo m are

$$[0], [1], \ldots, [m - 1].$$

These classes are also called residue classes mod m. Also each *value of r* is called a least residue modulo m.

Example 1

List the congruence classes mod 7.

Solution

Since the possible remainders when dividing by 7 are 0, 1, 2, …, 6, then the congruence classes are:

$[0] = \{\ldots, -7, 0, 7, 14, \ldots\}$

$[1] = \{\ldots, -6, 1, 8, 15, \ldots\}$

$\vdots$

$[6] = \{\ldots, -1, 6, 13, 20, \ldots\}$

Given a positive integer m, the set of integers $\mathbb{Z}$ is partitioned into $m - 1$ congruence classes. If we pick two members of a congruence class then they are congruent modulo m. Further, $[a] = [b]$ if and only if $a \equiv b \pmod{m}$.

For a given $m \geqslant 1$, we denote the congruence classes by $\mathbb{Z}_m$, called the set of **residue classes** modulo m (also called the set of **integers modulo *m*** or the set of **least residues**). So, $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$. For convenience purposes, once we make it clear that we are working with residue classes, we use the digits $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ to represent the classes.

Next, we show how to do arithmetic with these congruence classes, so that $\mathbb{Z}_m = \{k \mid k = 0, 1, \ldots, m-1\}$ behaves like a system of numbers.

For this purpose, we define arithmetic in the congruence classes as modular arithmetic.

First we know that an addition, subtraction or multiplication of both sides of a congruence preserves the congruence.

Theorem 4

If $a, b, c, m \in \mathbb{Z}$ and $m > 0$, such that $a \equiv b \pmod{m}$, then the following holds:

(i) $a + c \equiv b + c \pmod{m}$

(ii) $a - c \equiv b - c \pmod{m}$

(iii) $ac \equiv bc \pmod{m}$

Proof

$a \equiv b \pmod{m}$ implies that $m \mid (a - b)$.

Since $(a - b) = (a + c) - (b + c)$, $m \mid (a + c) - (b + c)$. Hence (i) holds.

In the same manner, (ii) follows from $(a - c) - (b - c)$.

To prove (iii), we use $ac - bc = c(a - b)$ and the fact that $m \mid (a - b)$ implies $m \mid (a - b)c$, i.e. $m \mid (ac - bc)$.

Example

Since $23 \equiv 7 \pmod{8}$, from Theorem 3,
$28 \equiv 23 + 5 \equiv 7 + 5 \pmod{8} \equiv 12 \pmod{8}$.

Also, $14 \equiv 23 - 9 \equiv (7 - 9) \bmod 8 \equiv -2 \pmod{8}$, and
$69 \equiv 23(3) \equiv 7(3) \bmod 8 \equiv 21 \pmod{8}$.

It is natural to ask if division upholds such a property – we see that it is not the case.

Example

$12 = 6 \cdot 2 \equiv 3 \cdot 2 \pmod{6}$. But $6 \not\equiv 3 \pmod{6}$. So we cannot cancel 2.

Similarly, $14 = 7 \cdot 2 \equiv 4 \cdot 2 \pmod{6}$. But $7 \not\equiv 4 \pmod{6}$.

Our next result is similar to Theorem 3. However, it generalizes the theorem.

Theorem 5

Let $a, b, c, d, m \in \mathbb{Z}$ and $m > 0$. Then $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ imply the following:

(i) $a + c \equiv b + d \pmod{m}$

(ii) $a - c \equiv b - d \pmod{m}$

(iii) $ac \equiv bd \pmod{m}$

Proof

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $m \mid a - b$ and $m \mid c - d$.

These imply that $m \mid [(a - b) + (c - d)]$. But this is the same as $m \mid [(a + c) - (b + d)]$. This proves (i).

Proof of (ii) is similar.

To prove (iii), note that $m \mid (a - b)$ implies $m \mid c(a - b)$ and $m \mid (c - d)$ implies $m \mid b(c - d)$.

Thus, $m \mid [c(a - b) + b(c - d)]$, which is the same as $m \mid (ac - bd)$. This completes the proof.

Example

Since $31 \equiv 9 \pmod{11}$ and $15 \equiv 4 \pmod{11}$, by Theorem 4, we have

$31 + 15 \equiv 9 + 4 \pmod{11} \Rightarrow 46 \equiv 13 \pmod{11}$, and

$31 \times 15 \equiv 9 \times 4 \pmod{11} \Rightarrow 465 \equiv 36 \pmod{11}$.

Theorem 6

Let $a, b, c, m \in \mathbb{Z}$ with $m > 0$, and $d = \gcd(c, m)$, then

$ac \equiv bc \pmod{m} \Rightarrow a \equiv b \left(\bmod m/d\right)$.

Proof

If $ac \equiv bc \pmod{m}$, then we know $m \mid (ac - bc)$ or $m \mid c(a - b)$. Hence, there is an integer k such that $c(a - b) = km$. Divide both sides by d:

$$\frac{c}{d}(a - b) = k\frac{m}{d} \;\ldots\ldots\ldots\ldots(1)$$

Since, from Chapter 1 (Corollary 3), we know $\gcd\left(\frac{c}{d}, \frac{m}{d}\right) = 1$, then we know that $\frac{m}{d}$ divides the right-hand side of equation (1), so it has to divide the left-hand side, and since it is relatively prime to $\frac{c}{d}$, it should divide $(a - b)$ by Theorem 9 of Chapter 1. Therefore, $a \equiv b \left(\bmod m/d\right)$.

Example

$70 \equiv 40 \pmod{15}$, and $\gcd(10, 15) = 5$, then $7 \equiv 4 \pmod{3}$.

The following corollary is also helpful in solving congruence problems.

Corollary 1

Let $a, b, c, m \in \mathbb{Z}$ with $m > 0$, and $\gcd(c, m) = 1$, then

$ac \equiv bc \pmod{m} \Rightarrow a \equiv b \pmod{m}$.

The proof is a simple application of Theorem 6 when $d = 1$.

Example

$54 \equiv 24 \pmod 5$ implies that $\frac{54}{3} \equiv \frac{24}{3} \pmod 5$, i.e. $18 \equiv 8 \pmod 5$, since $\gcd(3, 5) = 1$.

Theorem 7

Let $a, b, c, m \in \mathbb{Z}$ with $c, m > 0$, then

$a \equiv b \pmod m \Rightarrow a^c \equiv b^c \pmod m$.

Proof

$a \equiv b \pmod m \Rightarrow m \mid (a-b)$. Also,

$a^c - b^c = (a-b)\left(a^{c-1} + a^{c-2}b + \ldots + ab^{c-2} + b^{c-1}\right)$, then

$m \mid (a-b), (a-b) \mid (a^c - b^c) \Rightarrow m \mid (a^c - b^c)$.

Hence, $a^c \equiv b^c \pmod m$.

Example

$8 \equiv 3 \pmod 5$ implies $64 \equiv 9 \pmod 5$, or $512 \equiv 27 \pmod 5$, etc.

Theorem 8

If $a \equiv b \pmod{m_1}$, $a \equiv b \pmod{m_2}$, …, $a \equiv b \pmod{m_k}$, where $a, b, m_1, \ldots, m_k \in \mathbb{Z}$ and $m_1, \ldots, m_k > 0$, then $a \equiv b \pmod l$, where $l = \operatorname{lcm}(m_1, \ldots, m_k)$.

Proof

$a \equiv b \pmod{m_1}$, $a \equiv b \pmod{m_2}$, …, $a \equiv b \pmod{m_k}$ imply that $m_1 \mid (a-b)$, $m_2 \mid (a-b)$, …, $m_k \mid (a-b)$. This in turn implies that $\operatorname{lcm}(m_1, \ldots, m_k) \mid (a-b)$. (Proof is left as an exercise.)

Consequently,

$a \equiv b \pmod l$.

Note: A consequence of Theorem 8 is the situation where $m_1, \ldots, m_k$ are pairwise relatively prime. In such a case we will have

$a \equiv b \pmod{m_1 \cdot m_2 \cdot \ldots \cdot m_k}$.

Example

$342 \equiv 12 \pmod 5$, $342 \equiv 12 \pmod{10}$, $342 \equiv 12 \pmod{15}$, and $342 \equiv 12 \pmod 6$.

Since $\operatorname{lcm}(5, 10, 15, 6) = 30$, then we can conclude that $342 \equiv 12 \pmod{30}$, which is indeed true, as $342 - 12 = 330 = 30 \cdot 11$.

$342 \equiv 12 \pmod 5$, $342 \equiv 12 \pmod 2$, $342 \equiv 12 \pmod 3$, and $342 \equiv 12 \pmod{11}$. Since the moduli are pairwise relatively prime, then $342 \equiv 12 \pmod{5\cdot 2\cdot 3\cdot 11}$, i.e. $342 \equiv 12 \pmod{330}$.

Exercise 2.1

1 Say whether each statement is true or false.

a $16 \equiv 49 \pmod{11}$ **b** $72 \equiv 24 \pmod 9$

c $87 \equiv 303 \pmod{16}$ **d** $-25 \equiv 215 \pmod{12}$

2 Find the least residue (mod 31) of $33 \cdot 26^2$.

3 Show that if $a \equiv b \pmod m$ and $d \mid m$, then $a \equiv b \pmod d$.

In questions 4–16, find the least residue of a modulo m.

4 $a = 114$, $m = 7$

5 $a = 85$, $m = 8$

6 $a = 67$, $m = 50$

7 $a = 60$, $m = 51$

8 $a = -62$, $m = 50$

9 $a = -81$, $m = 51$

10 $a = -114$, $m = 7$

11 $a = 72 \cdot 73 \cdot 74$, $m = 71$

12 $a = 80 \cdot 81 \cdot 85$, $m = 82$

13 $a = 100^6$, $m = 49$

14 $a = 49^4$, $m = 23$

15 $a = 50^{99}$, $m = 7$

16 $a = 50^{99}$, $m = 17$

17 If $x \equiv 2 \pmod{17}$, $y \equiv 4 \pmod{17}$, and $z \equiv 5 \pmod{17}$, find the least residue of $x + yz \pmod{17}$.

18 If $x \equiv 2 \pmod{17}$, $y \equiv 4 \pmod{17}$, and $z \equiv 5 \pmod{17}$, find the least residue of $x^2 + y^2 + z^2 \pmod{17}$.

19 Prove that $7^n \equiv 6n + 1 \pmod{36}$ for all $n \in \mathbb{Z}^+$.

20 Prove that $2 \cdot 7^n \equiv 2^n(5n + 2) \pmod{25}$ for all $n \in \mathbb{Z}^+$.

21 Prove that $2^n + 3^n \equiv 5^n \pmod 6$ for all $n \in \mathbb{Z}^+$.

22 Prove that $16^n \equiv 1 - 10n \pmod{25}$ for all $n \in \mathbb{Z}^+$.

23 Prove that $3 \mid (4^n - 1)$ for all $n \in \mathbb{Z}^+$.

24 Let f_n be the nth term of a Fibonacci sequence. Prove that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \text{ for all } n \in \mathbb{Z}^+.$$

25 Prove that $2^{2^n} + 1 \equiv 5 \pmod{12}$ for all $n \in \mathbb{Z}^+$.

26 Prove that $(-4)^n \equiv 1 - 5n \pmod{25}$ for all $n \in \mathbb{Z}^+$.

27 Prove that $5^n \equiv 1 + 4n \pmod{16}$ for all $n \in \mathbb{Z}^+$.

28 Prove that $8^n \mid (4n)!$ for all $n \in \mathbb{Z}^+$.

29 Show that $31 \mid 2^{5n} - 1$ for $n \in \mathbb{Z}^+$.

In questions 30–33, prove each statement if it is true, or show that it is false either by reasoning or by finding a counter example.

30 If a, b, and c are three consecutive integers, then $a + b + c \equiv 0 \pmod 3$.

31 If a and b are two even integers, then $ab \equiv 0 \pmod 4$.

32 If $n \in \mathbb{Z}$, $n > 1$, then $n^2 \not\equiv 3 \pmod 4$.

33 If $n \in \mathbb{Z}$, n is odd, then $n^4 - 1 \equiv 0 \pmod{16}$.

34 Find all values of a such that $a \equiv 307 \pmod{17}$, $0 \leqslant a \leqslant 33$.

35 Find all values of a such that $a \equiv 971 \pmod{23}$, $-20 \leqslant a \leqslant 50$.

36 Find all values of n such that $342 \equiv 573 \pmod n$.

37 Show that any integer is congruent modulo 17 to any multiple of 7.

38 Show that if $x^2 \equiv y^2 \pmod p$, where p is a prime, then $|a| \equiv |b| \pmod p$.

39 Show that $a \equiv b \pmod n$ implies that $\gcd(a, n) = \gcd(b, n)$.

40 The multiplicative inverse of a number a mod n is the number b such that $ab \equiv 1 \pmod n$. Find the multiplicative inverse, if any, of $7 \pmod{19}$, $39 \pmod{95}$ and $91 \pmod{191}$.

41 With p a prime number different from 2, show that $(p + 1)/2$ is an integer and that it is the multiplicative inverse of 2 modulo p.

42 With p a prime number different from 2, show that $(p + 1)^2/4$ is an integer and that it is the multiplicative inverse of 4 modulo p.

2.2 The Diophantine equation $ax + by = c$

The Greek mathematician Diophantus who lived in Alexandria around 250 AD considered linear equations with integer solutions. In honour of him, any equation with one or more unknowns, which is to be solved over the set of integers, is called a Diophantine equation. The simplest sort of Diophantine equation that we will consider is the linear Diophantine equation in two unknowns, $ax + by = c$.

Definition 2

A simple linear Diophantine equation in two unknowns is of the form $ax + by = c$, where a, b, and c are integers and a and b are not both zero.

A solution of the linear equation is a pair of integers x_0 and y_0 such that $ax_0 + by_0 = c$.

Before we consider the general method of solving such equations, let us consider the simple equation $2x + 4y = 16$. One solution is $x = 6$ and $y = 1$. Another solution is $x = 12$ and $y = -2$. In view of this, we expect that a linear Diophantine equation may have more than one pair of solutions. A fundamental question to ask is: Does every linear Diophantine equation have a solution? The equation $2x + 4y = 5$ does not have a solution for any integers x and y. This follows from Theorem 9 below.

Theorem 9

A linear Diophantine equation $ax + by = c$, where a, b, and c are integers and a and b are not both zero, has a solution if and only if $\gcd(a, b) \mid c$.

Proof

($\Rightarrow$) Suppose $g = \gcd(a, b)$. Then there are integers r and s such that $a = gr$ and $b = gs$.

If $ax + by = c$ has a solution (x_0, y_0), then $ax_0 + by_0 = c$. Thus, $c = ax_0 + by_0 = grx_0 + gsy_0 = g(rx_0 + sy_0)$.

This implies that $g \mid c$.

($\Leftarrow$) Conversely, assume that $g \mid c$, i.e. there exists an integer t such that $c = gt$.

By Theorem 5 of Chapter 1, there are integers u and v such that $au + bv = g$.

Hence, $atu + btv = tg = c$. Therefore, $x = tu$ and $y = tv$ form a particular solution of the equation $ax + by = c$. This completes the proof.

Our next result shows how to get all solutions of $ax + by = c$ when we know a particular solution (x_0, y_0).

Theorem 10

If $x = x_0$ and $y = y_0$ is a particular solution of the linear Diophantine equation $ax + by = c$, then other solutions are given by $x = x_0 + \left(\frac{b}{g}\right)t$ and $y = y_0 - \left(\frac{a}{g}\right)t$, where $g = \gcd(a, b)$ and t is an arbitrary integer.

Proof (Optional)

Suppose we have found a solution (x_0, y_0) of the equation $ax + by = c$. If (x_0', y_0') is any other solution of $ax + by = c$, then $ax_0 + by_0 = c = ax_0' + by_0'$, which is equivalent to $a(x_0' - x_0) = b(y_0 - y_0')$. We know that there are relatively prime integers r and s such that $a = gr$ and $b = gs$. Using these, we obtain

$$gr(x_0' - x_0) = gs(y_0 - y_0')$$

or

$$r(x_0' - x_0) = s(y_0 - y_0') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From (1), we see that $r \mid s(y_0 - y_0')$ with $\gcd(r, s) = 1$, and we have, by Euclid's lemma,

$r \mid (y_0 - y_0')$, and thus $(y_0 - y_0') = rl$ for some integer l.

Now substituting this in (1), we get

$x_0' - x_0 = sl$.

Thus, $x_0' = x_0 + sl = x_0 + \left(\frac{b}{g}\right)l$ and $y_0' = y_0 - rl = y_0 - \left(\frac{a}{g}\right)l$.

$$\begin{aligned} ax_0' + by_0' &= a\left(x_0 + \left(\frac{b}{g}\right)l\right) + b\left(y_0 - \left(\frac{a}{g}\right)l\right) \\ &= ax_0 + by_0 + \left(\frac{ab}{g} - \frac{ab}{g}\right)l = ax_0 + by_0 = c \end{aligned}$$

since (x_0, y_0) is a solution of $ax + by = c$.

Thus, if a linear Diophantine equation has a solution, it has an infinite number of solutions.

The following is a direct result of Theorem 10.

Corollary 2

If a and b are relatively prime, then $ax + by = c$ has solutions given by

$x = x_0 + bt$ and $y = y_0 - at$,

where (x_0, y_0) is a particular solution of $ax + by = c$ and t is any integer.

Theorem 9 and 10 combined

Let $a, b, c \in \mathbb{Z}$. Consider the Diophantine equation

$ax + by = c$.

If $\gcd(a, b) \nmid c$, there are no solutions to the equation.

If $\gcd(a, b) \mid c$, there are infinitely many solutions of the form

$x = x_0 + \frac{b}{g}t$ and $y = y_0 - \frac{a}{g}t$,

where $g = \gcd(a, b)$, (x_0, y_0) is a particular solution, and t is any integer.

Theorems 9 and 10 are usually combined into one theorem which may be more meaningful. We used two separate theorems for the sake of easing up the proof!

Example 2

Solve $6x + 9y = 21$.

Solution

Since $\gcd(6, 9) = 3$, and $3 \mid 21$, there are an infinite number of solutions. To find them, we first attempt to find one by trial and error.
$x_0 = -4$ and $y_0 = 5$ is a particular solution.

Hence, the general solution is

$$x = -4 + \frac{9}{3}t = -4 + 3t \text{ and } y = 5 - \frac{6}{3}t = 5 - 2t.$$

How do we find a particular solution?

There is no unique answer to this question. There are a few approaches that work relatively well.

1 Trial and error, as in Example 2.

2 Using linear congruence (which you will study later in more detail).

The equation $ax + by = c$ can be rewritten as $ax - c = -by$, which implies that $ax \equiv c \pmod{b}$, which is simpler to solve.

For example:
$6x + 9y = 21 \Rightarrow 6x \equiv 21 \pmod{9}$
$\Rightarrow 2x \equiv 7 \pmod{3}$ [Theorem 6]
$\Rightarrow 2x \equiv (6 + 1) \pmod{3} \Rightarrow 2x \equiv 1 \pmod{3}$

Here we can find $x_0 = 2$ (or any number in its residual class!). Hence,

$y_0 = 1$, and our general solution is

$x = 2 + 3t$ and $y = 1 - 2t$.

When $t = -2$, we get $x = -4$ and $y = 5$, which is the solution found in Example 2.

3 Using 'reverse' Euclidean algorithm.

We know that $\text{gdc}(6, 9) = 3$, but to find a linear combination of 3 in terms of 6 and 9, we have to perform the algorithm first so that we can reverse it afterwards (as we did in the previous chapter). Otherwise, finding the linear combination will again be guesswork.

$9 = 1 \cdot 6 + 3$ and $6 = 2 \cdot 3 + 0$, so

$3 = 1 \cdot 9 - 6$, and now we multiply both sides by 7 to get $21 = 7 \cdot 9 - 7 \cdot 6$; so we choose $x_0 = -7$ and $y_0 = 7$ to be a particular solution.

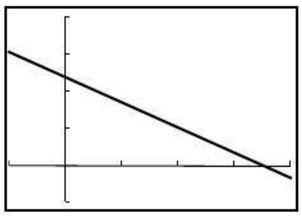
Hence, the general solution is

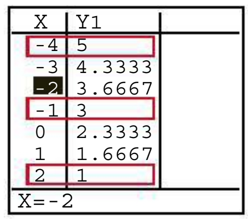
$x = -7 + 3t$ and $y = 7 - 2t$.

(Notice that if we substitute $t = 1$, we get the solution in **1** (Example 2) and if we substitute $t = 3$, we get the solution in **2**.)

Notice that the three solutions can be consolidated, and eventually they yield the same set of numbers.

Note: Since the solution for the equation, *if it exists*, is always an integer, and since this type deals with two variables, but gives only one equation, it is natural to expect an infinite number of solutions. One way to look at the solutions is to get an idea of the solution through a graph of the equation. As you know, $ax + by = c$ is the equation of a straight line. The line consists of all ordered pairs (x, y) that satisfy the equation. Not all of them are integers of course. By graphing and producing a table, you may be able to find a particular solution, after which the general solution is very simple.





Notice how you can find three particular solutions: $(-1, 3)$, $(-4, 5)$, and $(2, 1)$.

Example 3

Solve $12x + 25y = 331$.

Solution

We will use two methods to demonstrate their application and leave the trial and error for you to investigate. You might find the task easier if you set up a spreadsheet.

a) Euclidean algorithm:

We notice that 12 and 25 are relatively prime.

$25 = 2 \cdot 12 + 1$, and so $1 = 1 \cdot 25 - 2 \cdot 12$

$331 = 331 \cdot 25 - 662 \cdot 12$ Multiply both sides by 331.

$x_0 = -662$ and $y_0 = 331$ A particular solution.

$x = -662 + 25t$ and $y = 331 - 12t$ The general solution to this equation.

b) Linear congruence:

$12x + 25y = 331 \Rightarrow 12x \equiv 331 \pmod{25}$

$\Rightarrow 12x \equiv (325 + 6) \pmod{25} \Rightarrow 12x \equiv 6 \pmod{25}$

$\Rightarrow 2x \equiv 1 \pmod{25}$ [Corollary 1]

Here we find $x_0 = 13$ and therefore $y_0 = 7$ to be a particular solution.

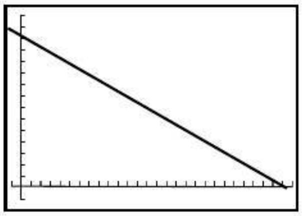
The general solution would be

$x = 13 + 25t$ and $y = 7 - 12t$.

Notice that if we substitute $t = -27$, we get

$x = -662$ and $y = 331$.

Using a GDC here too helps you recognize (13, 7) as a solution.



X	Y1
9	8.92
10	8.44
11	7.96
12	7.48
13	7
14	6.52
15	6.04

Note: Sometimes a constraint is added to the request of finding a solution. For instance, in Example 3, a condition is imposed that our solution must be positive. Luckily enough b) gave us a positive answer, but a) did not. However, to guarantee that it happens, we solve a system of two inequations.

$-662 + 25t > 0$ and $331 - 12t > 0$

$$\left.\begin{array}{l} -662 + 25t > \Rightarrow t > \dfrac{662}{25} = 26\dfrac{12}{25} \\ 331 - 12t > 0 \Rightarrow t < \dfrac{331}{12} = 27\dfrac{7}{12} \end{array}\right\} \Rightarrow 26\frac{12}{25} < t < 27\frac{7}{12}$$

$t = 27$ is the only possibility, and hence $x = 13$ and $y = 7$.

Example 4

Solve the equation $6x + 51y = 22$.

Solution

Since $\gcd(6, 51) = 3 \nmid 22$, there is no solution.

Summary of the process of solving $ax + by = c$

Step 1: Calculate $g = \gcd(a, b)$.

Step 2: Check if $g | c$. If it is not true, then there are no solutions, so stop here. If $g | c$, then write $c = gk$.

Step 3: If $g | c$, then find integers u and v such that $au + bv = g$. Then $x_0 = uk$ and $y_0 = vk$ is a particular solution of $ax + by = c$. Use one of the three methods we discussed.

Step 4: Write the general solution $x = x_0 + \left(\frac{b}{g}\right)t$ and $y = y_0 - \left(\frac{a}{g}\right)t$ for all $t \in \mathbb{Z}$.

Example 5

Find the number of \$20 bills and the number of \$50 bills which will together make \$510.

Solution

The problem is equivalent to the Diophantine equation $20x + 50y = 510$, where x is the required number of \$20 bills and y is the required number of \$50 bills.

$\gcd(20, 50) = 10$, and $10 \mid 510$. So, $510 = 10 \cdot 51$.

$10 = 20 \cdot (-2) + 50 \cdot 1$ Using any of three methods discussed.

This implies that $10 \cdot 51 = 20 \cdot (-2 \cdot 51) + 50 \cdot 51$,
i.e. $510 = 20\,(-102) + 50 \cdot 51$

Thus, $x_0 = -102$ and $y_0 = 51$ is a particular solution.

The general solution of the Diophantine equation is

$x = -102 + \left(\frac{50}{10}\right)t = -102 + 5t$ and $y_0 = 51 - \left(\frac{20}{10}\right)t = 51 - 2t$.

We want to choose values of t so that x and y are positive.

Hence, we need $-102 + 5t \geqslant 0$ and $51 - 2t \geqslant 0$, which implies that

$\frac{102}{5} = 20\frac{2}{5} \leqslant t \leqslant \frac{51}{2} = 25\frac{1}{2}$.

Hence, only $t = 21, 22, 23, 24$, and 25 can be used.

Substituting these values of t into the expressions for x and y, we get the number of \$20 and \$50 bills which will make \$510 to be:
$(x, y) = (3, 9), (8, 7), (13, 5), (18, 3)$ and $(23, 1)$.

Example 6

a) Find the general solution of the linear Diophantine equation $172x + 20y = 1000$.

b) Find the positive integer solutions of this equation.

Solution

a) $\gcd(172, 20) = 4$ Use any method of your choice.

$172x \equiv 1000 \pmod{20} \Rightarrow 43x \equiv 250 \pmod{5} \Rightarrow (40 + 3)x \equiv 250 \pmod{5}$

$\Rightarrow 3x \equiv 0 \pmod{5}$; thus $x = 0$ (or any of its residue class mod 5)

A particular solution is $x_0 = 0$ and $y_0 = 50$.   Substitute $x = 0$ into the equation.

A general solution is $x = 0 + 5t$ and $y = 50 - 43t$. $20 \div 4$ and $172 \div 4$.

If you choose to use the Euclidean algorithm (presented here for comparison purposes), then

$$\left.\begin{aligned} 172 &= 8(20) + 12 \\ 20 &= 1(12) + 8 \\ 12 &= 1(8) + 4 \\ 8 &= 2(4) + 0 \end{aligned}\right\} \gcd(172, 20) = 4$$

Now, we express $4 = 172u + 20v$.
From the calculations for finding gcd(172, 20), we have

$$4 = 12 - 8 = 12 - (20 - 12) = 2(12) - 20 = 2(172 - 8(20)) - 20 = 2(172) + (-17)20.$$

Hence, $u = 2$ and $v = -17$.

Since $\frac{1000}{4} = 250$, the particular solution (x_0, y_0) is given by

$x_0 = 2(250) = 5000$ and $y_0 = (-17)(250) = -4250$.

Hence, the general solution is given by

$$x = 500 + \left(\frac{20}{4}\right)t = 500 + 5t \text{ and } y = -4250 - \left(\frac{172}{4}\right)t = -4250 - 43t,\ t \in \mathbb{Z}.$$

Notice how there is an apparent difference in the solutions between the two methods. However, we leave it as an exercise for you to consolidate the two answers by the appropriate choice of values of t.

b) To find the positive integer solutions, t must be chosen so that $5t > 0$ and $50 - 43t > 0$.

This implies $0 < t < \frac{50}{43}$.

Thus, $t = 1$ is the only possible value, and so we have $x = 5$ and $y = 7$.

In the Euclidean method case:
$5t + 500 > 0$ and $-4250 - 43t > 0$.

This implies that $-100 < t < -98\frac{36}{43}$.

Hence, we take $t = -99$.

Thus, $x - 500 + 5(-99) = 5$ and $y = -4250 - 43(-99) = 7$, which is the same result as before.

Example 7

Solve the Diophantine equation $1492x + 1066y = -4$.

Solution

$\gcd(1492, 1066) = 2$. Since $2 \mid -4$, the Diophantine equation has infinitely many solutions.

Now, $2 = (-5)1492 + 7(1066)$. Since $-4 = 2 \cdot (-2)$, the particular solution (x_0, y_0) is given by $x_0 = (-5)(-2) = -10$ and $y_0 = 7(-2) = -14$.
Using (x_0, y_0), the general solution has the form

$$x = 10 + \left(\frac{1066}{2}\right)t = 10 + 533t \text{ and } y = -14 - \left(\frac{1492}{2}\right)t = 14 - 746t,\ t \in \mathbb{Z}.$$

Example 8

Find the smallest positive integer n such that the Diophantine equation $533x + 299y = 10\,000 + n$ has a solution, and for this value of n find the positive integer solutions.

Solution

$\gcd(533, 299) = 13$. In order for the linear Diophantine equation $533x + 299 = 10\,000 + n$ to have a solution, $10\,000 + n$ must be divisible by 13. Thus, $1000 + n \equiv 0 \pmod{13} \Rightarrow 9997 + 3 + n \equiv 0 \pmod{13}$ $\Rightarrow 3 + n \equiv 0 \pmod{13} \Rightarrow n = 10$.

Hence, the equation to be solved is

$533x + 299y = 10\,010 \Rightarrow 533x \equiv 10\,010 \pmod{299} \Rightarrow 41x \equiv 770 \pmod{23}$(Why?)

$\Rightarrow 18x \equiv 11 \pmod{23} \Rightarrow x = 7$, since $18 \cdot 7 = 126 - 11 = 115 = 5 \cdot 23$.

By back substitution into the equation, we have $y = 21$.

Notice the difference if we were to use the Euclidean algorithm method.

Knowing $\gcd(533, 299) = 13$, we need to find u and v such that $13 = 533u + 277v$.

We can find that $u = 9$ and $v = 16$.

A particular solution is given by $x_0 = \left(\frac{10\,010}{13}\right)9 = 6930$ and

$y_0 = \left(\frac{10\,010}{13}\right)(-16) = -12\,320$.

Hence, the general solution is given by $x = 6930 + 23t$ and $y = -12\,320 - 41t$.

For positive integer solutions both x and y are positive, so

$6930 + 23t > 0$ or $t > \frac{-6930}{23}$ and $-12\,320 - 41t > 0$ which implies

$-12\,320 > 41t$ or $41t < -12\,320$ or $t < \frac{-12\,320}{41}$.

Hence, $-301.304 < t < -300.975$.

On taking $t = -301$, $x = 6930 + 32(-301) = 7$ and
$y = -12\,320 - 41(-301) = 21$.

Exercise 2.2

1 Determine which of the following Diophantine equations have a solution.

a $51x + 6y = 22$

b $14x + 33y = 115$

c $35x + 14y = 93$

2 Determine the general solution of the following Diophantine equations.

a $13x - 7y = 21$

b $221x + 35y = 11$

c $1485x + 1745y = 15$

3 Determine the positive integer solutions of the linear Diophantine equations.

a $5x - 11y = 29$

b $32x + 55y = 71$

c $62x + 11y = 788$

4 A grocer orders apples and oranges for \$16.78. If apples cost him 25 cents each and oranges cost him 18 cents each, how many of each type of fruit did he order?

5 Kate spent €100.64 on posters. Some of the posters cost €4.98 each and some €5.98. How many did she buy?

6 A person has \$4.55 in change composed of dimes and quarters. Set up the linear Diophantine equation and find the maximum and the minimum number of coins that the person can have.

7 David collected \$75 at the market by selling chickens and geese. He got \$4 for each chicken and \$7 for each goose. How many of each did he sell?

8 A farmer purchased one hundred head of livestock for a total cost of \$4000. Calves, lambs, and piglets cost \$120, \$50, and \$25 each, respectively. If the farmer bought at least one animal of each type, how many of each type did he buy?

9 Roberto bought three dozen oranges and two dozen apples. He paid €8.04 in total. Each orange costs more than 10 cents, while an apple costs more than 15 cents. How much did he pay for the oranges?

10 Marco has a small grocery shop. He buys tomatoes from farmer Antonio in large boxes and then repackages them in smaller boxes. Marco bought 11 large boxes and sold 39 small boxes. A small box contains less than 12 tomatoes. At the end of the day, Marco was left with 19 tomatoes. How many tomatoes does each large box contain?

11 Farmer Josip owes farmer Tim €10. Neither of the two has any cash, but Josip has 14 sheep valued at €185 each. He suggests paying Tim in sheep with Tim paying the change in pigs, which are valued at €110 each. Is this possible? If yes, how; if not, why not?

In questions 12–34, either find all integral (integer) solutions to the given equation or show that it has none.

12 $3x + 2y = 1$

13 $3x - 2y = 1$

14 $17x + 14y = 4$

15 $33x - 12y = 9$

16 $91x + 221y = 15$

17 $361x + 2109y = 1000$

18 $401x + 503y = 20$

19 $26x + 14y = 2$

20 $27x + 15y = 3$

21 $217x + 341y = 62$

22 $117x + 247y = 39$

23 $2x + 3y = 50;\ x, y > 0$

24 $3x + 4y = 60;\ x, y > 0$

25 $4x + 6y = 60;\ x, y > 0$

26 $6x + 9y = 91;\ x, y > 0$

27 $4x + 6y = 25$

28 $3x + 5y = 50\,001$

29 $6x + 9y = 60\,001$

30 $21x - 14y = 10\,000$

31 $42y - 12x = 366$

32 $66x + 51y = 300$

33 $55x + 200y = -100$

34 $121x + 561y = 13\,200;\ x, y > 0$

35 $a, b \in \mathbb{Z}^+$, show that there exist $x, y \in \mathbb{Z}$ such that $\frac{1}{\text{lcm}(a, b)} = \frac{x}{a} + \frac{y}{b}$.

36 Show that if a and b are relatively prime, and $c \neq 0$, then $\gcd(ac, bc) \mid c$.

2.3 Linear congruences

A congruence of the form

$ax \equiv b \pmod{m}$, where x is an unknown integer,

is called a linear congruence in one variable. As you have seen in the previous section, the study of such congruences is similar to the work with linear Diophantine equations in two variables. In fact, we used linear congruences to solve some of these equations.

Example 9

Find a solution to linear congruence $3x \equiv 4 \pmod{7}$.

Solution

For now, let us try and find the solution by trial and error and some knowledge of congruence.

One way to approach this is to resort to the definition of congruence:

$3x \equiv 4 \pmod{7}$ implies that $7 \mid (3x - 4)$.

In other words, $3x - 4 = 7k$ for some integer k.

This means that $3x - 4$ should be equal to one of the multiples of 7 $\{0, \pm 7, \pm 14, \pm 21, \pm 28, \ldots\}$.

When $x = 6$, $3x - 4 = 14$ and we have a solution. If we let $x = -1$, $3x - 4 = -7$ and we have another solution. However, you know that $6 \equiv -1 \pmod{7}$. So, it appears that all members of the residue class of 6 will be solutions too.

If you recall some of the rules we learned earlier, you can solve the problem without guessing!

Multiply the equation by 5. This gives you $15x \equiv 20 \pmod{7}$.

This, in turn, means $(14 + 1)x \equiv (14 + 6) \pmod{7}$, which simplifies to $x \equiv 6 \pmod{7}$.

From the previous discussion, you notice that if we have $x = x_0$ as a solution to the congruence $ax \equiv b \pmod{m}$, and if $x_1 \equiv x_0 \pmod{m}$, then $ax_1 \equiv ax_0 \equiv b \pmod{m}$, and hence x_1 is also a solution. Thus, if one member of a residue class modulo m is a solution, then the entire class is made up of solutions. The question remains: How many different '*incongruent*' solutions does the congruence have?

The following theorem tells you when to expect a solution and how many incongruent solutions modulo m the congruence has.

Theorem 11

Let $a, b, m \in \mathbb{Z}$, with $m > 0$ and $\gcd(a, m) = g$. If $g \nmid b$, then $ax \equiv b \pmod{m}$ has no solutions. If $g \mid b$, then $ax \equiv b \pmod{m}$ has exactly g '*incongruent*' solutions modulo m.

Proof

$ax \equiv b \pmod{m}$ can be written as $ax - b = my$, where y is an integer. (Definition of congruence.)

The last equation can be rewritten as $ax - my = b$. This is a Diophantine equation!

The Diophantine equation, by Theorem 9 and 10 combined, has no solution if $g \nmid b$, while it has infinitely many solutions if $g \mid b$. These solutions are given by

$$x = x_0 + \frac{m}{g}t \text{ and } y = y_0 - \frac{-a}{g}t = y_0 + \frac{a}{g}t,$$

where $x = x_0$ and $y = y_0$ is a particular solution of the equation. The values of x given above,

$$x = x_0 + \frac{m}{g}t,$$

are the solutions to the linear congruence. There are infinitely many of them, but they are congruent, as you notice from the equation.

To find out how many incongruent solutions there are, let us first look at the conditions under which two solutions like $x_1 = x_0 + \frac{m}{g}t$ and $x_2 = x_0 + \frac{m}{g}s$ are congruent modulo m.

Now, $x_1 \equiv x_2 \pmod{m} \Rightarrow x_0 + \frac{m}{g}t \equiv x_0 + \frac{m}{g}s \pmod{m} \Rightarrow \frac{m}{g}t \equiv \frac{m}{g}s \pmod{m}$. (1)

Now, $\gcd\left(m, \frac{m}{g}\right) = \frac{m}{g}$ since $\left(\frac{m}{g}\right) \mid m$, so by Theorem 6 we now have

$t \equiv s \pmod{g}$ [we divide (1) by (m/g)].

Therefore, to have a complete set of incongruent solutions $x = x_0 + \frac{m}{g}t$, we need to consider all residue classes modulo g. This proves the theorem.

Note: When $\gcd(a, m) = 1$, there is exactly one unique solution modulo m.

Example 10

Solve each of the following linear congruences.

a) $14x \equiv 13 \pmod{21}$

b) $9x \equiv 15 \pmod{21}$

c) $8x \equiv 7 \pmod{13}$

d) $9x \equiv 12 \pmod{15}$

e) $7x \equiv 1 \pmod{31}$

f) $7x \equiv 22 \pmod{31}$

g) $18x \equiv 30 \pmod{42}$

Solution

a) $\gcd(14, 21) = 7$, and $7 \nmid 13$, so the equation has no solution.

b) $\gcd(9, 21) = 3$, and $3 \mid 15$, so we have three incongruent solutions modulo 21.

Theorem 6 helps us rewrite the equation as

$3x \equiv 5 \pmod{7} \Rightarrow 3x \equiv (5 + 7) \pmod{7} \Rightarrow 3x \equiv 12 \pmod{7} \Rightarrow x \equiv 4 \pmod{7}$

This implies that the solutions to the equation are of the form

$x = x_0 + \frac{m}{g}t = 4 + 7t$, with $t = 0, 1,$ and 2.

Thus, the solutions are: $x \equiv 4, 11, 18 \pmod{21}$.

c) $\gcd(8, 13) = 1$, so we have one solution modulo 13.

$8x \equiv 7 \pmod{13} \Rightarrow 8x \equiv (7 + 13) \pmod{13} \Rightarrow 8x \equiv 20 \pmod{13}$
$\Rightarrow 2x \equiv 5 \pmod{13}$, and again

$2x \equiv 5 \pmod{13} \Rightarrow 2x \equiv 18 \pmod{13} \Rightarrow x \equiv 9 \pmod{13}$, which is the solution.

d) $\gcd(9, 15) = 3$, and $3 \mid 12$, so we have exactly three incongruent solutions modulo 15.

Rewrite the equation: $3x \equiv 4 \pmod{5}$ Divided by 3.

$3x \equiv (4 + 20) \pmod 5 \Rightarrow 3x \equiv 24 \pmod 5 \Rightarrow x \equiv 8 \pmod 5$

$x = 8 + 5t$, with $t = 0, 1,$ and 2.

Therefore, the solutions are given by

$x \equiv 8 \pmod{15}$, $x \equiv 13 \pmod{15}$, and $x \equiv 18 \equiv 3 \pmod{15}$.

e) $\gcd(7, 31) = 1$, so there is exactly one solution modulo 31.

$7x \equiv 1 \pmod{31}$ Multiply by 9.

$63x \equiv 9 \pmod{31} \Rightarrow (62x + x) \equiv 9 \pmod{31} \Rightarrow x \equiv 9 \pmod{31}$

Hint: $x \equiv 9 \pmod{31}$ is called an inverse of 7 modulo 31.

f) $\gcd(7, 31) = 1$, so there is exactly one solution.

$7x \equiv 22 \pmod{31}$ Multiply by 9.

$63x \equiv 198 \pmod{31} \Rightarrow x \equiv 12 \pmod{31}$ (Why?)

g) $\gcd(18, 42) = 6$, so we have six incongruent solutions modulo 42.

$18x \equiv 30 \pmod{42} \Rightarrow 3x \equiv 5 \pmod 7$

$\Rightarrow 3x \equiv 12 \pmod 7 \Rightarrow x \equiv 4 \pmod 7$

$x = 4 + 7t$, with $t = 0, 1, 2, 3, 4,$ and 5.

Therefore, the solutions are given by

$x \equiv 4, 11, 18, 25, 32,$ and $39 \pmod{42}$.

The Chinese remainder theorem

An old Chinese puzzle poses a question as follows:

Find a number that leaves a remainder of 1 when divided by 3, a remainder of 2 when divided by 5, and a remainder of 3 when divided by 7. Interpreting this puzzle using congruences, we get the following system:

$x \equiv 1 \pmod 3$

$x \equiv 2 \pmod 5$

$x \equiv 3 \pmod 7$

Even though systems with more than one variable can be solved, this section focuses on systems of simultaneous congruences with one variable but different moduli, like the one above.

The following theorem will provide us with a method for finding all solutions of simultaneous congruences similar to the given example.

Theorem 12: The Chinese remainder theorem

Let $m_1, m_2, \ldots, m_r$, be positive integers which are pairwise relatively prime, i.e. $\gcd(m_i, m_j) = 1, \forall i \neq j,\ i, j = 1, 2 \ldots, r$.

The system of congruences

$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_2}$$
$$\vdots$$
$$x \equiv a_r \pmod{m_r}$$

has a unique solution modulo $M = m_1 m_2 \ldots m_r$.

Proof (Optional)

Let $M_k = \dfrac{M}{m_k} = m_1 m_2 \ldots m_{k-1} m_{k+1} \ldots m_r$.

In words, M_k is the product of all the moduli m_i, with the modulus m_k omitted.

By hypothesis, all the m_i are relatively prime in pairs, so the $\gcd(M_k, m_k) = 1$. According to the previous section's theorems, it is possible to solve the congruence $M_k x \equiv 1 \pmod{m_k}$. Call that *unique* solution x_k. That is, $M_k x_k \equiv 1 \pmod{m_k}$.

Our aim now is to prove that the integer

$$x = a_1 M_1 x_1 + a_2 M_2 x_2 + \ldots + a_r M_r x_r$$

is a simultaneous solution of the given system.

To show this, we need to show that $x \equiv a_k \pmod{m_k}$ for $k = 1, 2, \ldots, r$.

Since $m_k \mid M_j$ whenever $j \neq k$, $M_j \equiv 0 \pmod{m_k}$. Thus, in the sum for x, all terms except the kth term are congruent to $0 \pmod{m_k}$.

Hence, $x \equiv a_k M_k x_k \pmod{m_k}$, with $M_k x_k \equiv 1 \pmod{m_k}$ implying that $x \equiv a_k \pmod{m_k}$.

This proves the existence of the solution.

Now, let y be another solution to the system.

Then for each k, $y \equiv x \equiv a_k \pmod{m_k}$, which means that $m_k \mid (x - y)$.

Then using Theorem 8, we see that $M = m_1 m_2 \ldots m_r \mid (x - y)$.

Therefore, $y \equiv x \pmod{M}$.

Example 11

Solve the system:

$$x \equiv 1 \pmod{3}$$
$$x \equiv 2 \pmod{5}$$
$$x \equiv 3 \pmod{7}$$

Solution

$M = 3 \cdot 5 \cdot 7 = 105$

$M_1 = \frac{105}{3} = 35$; $M_2 = \frac{105}{5} = 21$; $M_3 = \frac{105}{7} = 15$

Now, to determine x_1, we solve $35x_1 \equiv 1 \pmod 3$, which simplifies to $x_1 \equiv 2 \pmod 3$.

For x_2, $21x_2 \equiv 1 \pmod 5$, we have $x_2 \equiv 1 \pmod 5$, and finally

$15x_3 \equiv 1 \pmod 7$, which gives $x_3 \equiv 1 \pmod 7$.

Therefore, our solution x is

$$x \equiv 1 \cdot 35 \cdot 2 + 2 \cdot 21 \cdot 1 + 3 \cdot 15 \cdot 1 \equiv 157 \equiv 52 \pmod{105}.$$

Checking back in the original system, you see that this solution satisfies the system:

$52 \equiv 1 \pmod 3$, since $51 = 3 \cdot 17$; $52 \equiv 2 \pmod 5$, since $50 = 10 \cdot 5$; and $52 \equiv 3 \pmod 7$, since $49 = 7 \cdot 7$.

Example 12

Solve the system:

$$x \equiv 2 \pmod 3$$

$$x \equiv 5 \pmod 4$$

$$x \equiv -3 \pmod 7$$

Solution

3, 4, and 7 are pairwise relatively prime.

$M = 3 \cdot 4 \cdot 7 = 84$

$M_1 = \frac{84}{3} = 28$; $M_2 = \frac{84}{4} = 21$; $M_3 = \frac{84}{7} = 12$

Now, to determine x_1, we solve $28x_1 \equiv 1 \pmod 3$, which simplifies to $x_1 \equiv 1 \pmod 3$.

For x_2, $21x_2 \equiv 1 \pmod 4$, we have $x_2 \equiv 1 \pmod 4$, and

$12x_3 \equiv 1 \pmod 7$, which gives $x_3 \equiv 3 \pmod 7$.

Therefore, our solution x is

$$x \equiv 1 \cdot 28 \cdot 2 + 1 \cdot 21 \cdot 5 + 3 \cdot 12 \cdot (-3) \equiv 53 \pmod{84}.$$

Again, checking back in the original system, you see that this solution satisfies the system:

$53 \equiv 2 \pmod 3$, since $51 = 17 \cdot 3$; $53 \equiv 5 \pmod 4$, since $48 = 12 \cdot 4$; and $53 \equiv -3 \pmod 7$, since $56 = 8 \cdot 7$.

The following example offers a slight variation on the same theme.

Example 13

Solve the linear congruence

$$3x \equiv 11 \pmod{2275}.$$

Solution

Since $\gcd(3, 2275) = 1$, the linear congruence has a unique solution modulo 2275.

We will approach the problem differently because of the size of the modulus.

Since $2275 = 5^2 \cdot 7 \cdot 13$, the original congruence may be replaced by the system:

$$3x \equiv 11 \pmod{25}$$

$$3x \equiv 11 \pmod{7}$$

$$3x \equiv 11 \pmod{13}$$

$M = 25 \cdot 7 \cdot 13 = 2275$

$$M_1 = \frac{2275}{25} = 91;\ M_2 = \frac{2275}{7} = 325;\ M_3 = \frac{2275}{13} = 175$$

Now, to determine x_1, we solve $91x_1 \equiv 16x_1 \equiv 1 \pmod{25}$, which simplifies to $x_1 \equiv 11 \pmod{25}$.

Verify.

For x_2, $325x_2 \equiv 3x_2 \equiv 1 \pmod{7}$, we have $x_2 \equiv 5 \pmod{7}$, and

$175x_3 \equiv 6x_3 \equiv 1 \pmod{13}$, which gives $x_3 \equiv 11 \pmod{13}$.

We still need to determine the particular solutions, a_is, since the linear congruences are not in the standard $x \equiv a_i \pmod{m_i}$ form.

$3x \equiv 11 \pmod{25}$ will give $a_1 = 12$.

$3x \equiv 11 \pmod{7}$ will give $a_2 = 6$.

$3x \equiv 11 \pmod{13}$ will give $a_3 = 8$.

Thus, the solution to the original congruence is now given by

$$x \equiv 12 \cdot 91 \cdot 11 + 6 \cdot 325 \cdot 5 + 8 \cdot 175 \cdot 11 \equiv 37\,162 \equiv 762 \pmod{2275}.$$

What we observe here is that, even though we had to solve six congruences, the moduli of these congruences are relatively small as compared to 2275 and could mostly be solved by mere inspection. This method offers a way to perform computer arithmetic with large integers.

Alternative method of solution

There is also a method similar to solving systems of equations by substitution that you are familiar with from early years.

This is an iterative method where we find a general solution for the variable in one congruence and substitute that value into another congruence, until we finish. We will demonstrate this method with an example.

Example 14

Solve the system:

$x \equiv 1 \pmod 5$ ....(1)

$x \equiv 2 \pmod 6$ ....(2)

$x \equiv 3 \pmod 7$ ....(3)

Solution

Rewrite (1) using the definition of congruence, i.e. $x - 1 = 5t$ with $t \in \mathbb{Z}$, which leads to $x = 5t + 1$. Now, for this solution to serve as a solution to the system, it must satisfy the second congruence:

$5t + 1 \equiv 2 \pmod 6$, i.e. $5t \equiv 1 \pmod 6$.

This can be solved to give $t \equiv 5 \pmod 6$.

So, $t = 5 + 6k$, where $k \in \mathbb{Z}$, and hence $x = 5t + 1 = 5(5 + 6k) + 1 = 30k + 26$.

This x in turn must satisfy the third congruence, and hence

$30k + 26 \equiv 3 \pmod 7$, i.e. $2k + 5 \equiv 3 \pmod 7 \Rightarrow 2k \equiv -2 \pmod 7$
$\Rightarrow k \equiv -1 \pmod 7$, and thus $k \equiv 6 \pmod 7$.

Hence, $k = 6 + 7u$, where $u \in \mathbb{Z}$. Finally,

$x = 30k + 26 = 30(6 + 7u) + 26 = 210u + 206$, which is equivalent to saying

$x \equiv 206 \pmod{210}$, which is the simultaneous solution.

This method demonstrates that a system of simultaneous congruences can be solved by successively solving linear congruences. This can be done even if the moduli are not pairwise relatively prime.

Example 15

Solve the linear congruence

$17x \equiv 9 \pmod{276}$.

Solution

Observe that $276 = 3 \cdot 4 \cdot 23$, and hence the congruence is equivalent to the following system:

$17x \equiv 9 \pmod 3 \Rightarrow x \equiv 0 \pmod 3$ ....(1)

$17x \equiv 9 \pmod 4 \Rightarrow x \equiv 1 \pmod 4$ ....(2)

$17x \equiv 9 \pmod{23} \Rightarrow 17x \equiv 9 \pmod{23}$ ....(3)

We will approach this problem using the iterative method.

From (1) we have $x = 3k$, where $k \in \mathbb{Z}$. Now, we substitute this into (2):

$3k \equiv 1 \pmod 4 \Rightarrow 9k \equiv 3 \pmod 4 \Rightarrow k \equiv 3 \pmod 4$

Thus, $k = 3 + 4i$, with $i \in \mathbb{Z}$, and hence $x = 3k = 3(3 + 4i) = 9 + 12i$.

From (3), we have
$17x \equiv 9 \pmod{23} \Rightarrow 17(9 + 12i) \equiv 9 \pmod{23} \Rightarrow 153 + 204i \equiv 9 \pmod{23}$
$\Rightarrow 204i \equiv -144 \pmod{23} \Rightarrow 3i \equiv 6 \pmod{23} \Rightarrow i \equiv 2 \pmod{23}$, and so
$i = 2 + 23t$.

Therefore, $x = 9 + 12i = 9 + 12(2 + 23t) = 33 + 276t$, and finally

$x \equiv 33 \pmod{276}$ is the solution to the system of congruences, and hence a solution to $17x \equiv 9 \pmod{276}$.

Systems of linear congruences

We will consider systems of two congruences involving two unknowns. The modulus will also be the same in both congruences. Of course, more congruences and more unknowns are possible, but they go beyond the scope of this publication.

The process we follow in trying to solve such systems is equivalent to what we do in solving systems of simultaneous equations in algebra. We will explain the method through the use of an example.

Example 16

Find the solution to:

$$3x + 4y \equiv 5 \pmod{13}$$
$$2x + 5y \equiv 7 \pmod{13}$$

Solution

Multiply the first congruence by 5 and the second by 4 to obtain

$$15x + 20y \equiv 25 \pmod{13}$$
$$8x + 20y \equiv 28 \pmod{13}$$

By subtraction, we have

$7x \equiv -3 \pmod{13}$, which will give us a solution for x.

$x \equiv 7 \pmod{13}$ We leave the verification as an exercise.

If we multiply the first congruence by 2 and the second by 3, we have

$$6x + 8y \equiv 10 \pmod{13}$$
$$6x + 15y \equiv 21 \pmod{13}$$

By subtraction, we have

$7y \equiv 11 \pmod{13}$, which in turn will yield

$y \equiv 9 \pmod{13}$.

The solution to the system is therefore

$(x \equiv 7 \pmod{13}, y \equiv 9 \pmod{13})$.

Theorem 13 (Optional)

Let $a, b, c, d, e, f, m \in \mathbb{Z}$ with $m > 0$. The system of congruences

$$ax + by \equiv e \pmod{m}$$

$$cx + dy \equiv f \pmod{m}$$

will have a unique solution if $\gcd(ad - bc, m) = 1$.

Exercise 2.3

In questions 1–13, find all solutions of each of the linear congruences.

1 $5x \equiv 2 \pmod{7}$

2 $6x \equiv 3 \pmod{9}$

3 $17x \equiv 30 \pmod{40}$

4 $5x \equiv 9 \pmod{49}$

5 $107x \equiv 333 \pmod{888}$

6 $490x \equiv 750 \pmod{800}$

7 $2x \equiv 3 \pmod{7}$

8 $12x \equiv 6 \pmod{18}$

9 $19x \equiv 16 \pmod{24}$

10 $15x \equiv 9 \pmod{25}$

11 $128x \equiv 833 \pmod{1001}$

12 $14x \equiv 5 \pmod{45}$

13 $3x \equiv 2 \pmod{78}$

14 For what integer values of k, where $k \in [0, 36[$, does the congruence $16x \equiv k \pmod{36}$ have solutions? When it has solutions, how many incongruent solutions are there?

In questions 15–19, attempt to use both methods, the Chinese remainder and the iterative methods, in solving each system.

15 Solve: $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{4}$

16 Solve: $x \equiv 7 \pmod{9}$, $x \equiv 13 \pmod{23}$, $x \equiv 1 \pmod{2}$

17 Solve: $2x \equiv 3 \pmod{5}$, $4x \equiv 3 \pmod{7}$

18 Solve: $6x \equiv 8 \pmod{10}$, $15x \equiv 30 \pmod{55}$

19 Solve: $x \equiv 0 \pmod{2}$, $x \equiv 0 \pmod{3}$, $x \equiv 1 \pmod{5}$, $x \equiv 6 \pmod{7}$

20 Find an integer that leaves a remainder of 9 when divided by 10 or 11, but is divisible by 13.

21 Find the solution of

$x + 2y \equiv 1 \pmod 5$

$2x + y \equiv 1 \pmod 5$

22 Find the solution of

$x + 3y \equiv 1 \pmod 5$

$3x + 4y \equiv 2 \pmod 5$

23 Find the solution of

$4x + y \equiv 2 \pmod 5$

$2x + 3y \equiv 1 \pmod 5$

24 Find the solution of

$2x + 3y \equiv 5 \pmod 7$

$x + 5y \equiv 6 \pmod 7$

2.4 Integer representations and operations

We usually use the decimal notation to represent integers. It is a positional numeral system with base 10.

In this section, we shall show that any positive integer can be uniquely represented in a base b, where b is a positive integer. When $b = 2$, the representation is called a **binary representation**; when $b = 16$, the representation is called the **hexadecimal expansion**. We will describe a method of finding the base b representation of an integer, and describe a procedure to carry out integer arithmetic.

Use of bases other than ten is known from the history of mathematics (see Howard Eves, *An Introduction to the History of Mathematics*, 6th edition (Thomson Brooks/Cole, 1990) pages 19–27). Between 2000 to 500 BCE, the Babylonians evolved a sexagesimal system (base 60). The Mayan numerical system used base 20, but a positional system of its own. Some African tribes used base 5, and base 2 appears in Chinese mathematics. Some of the Egyptian calculations used base 7.

Before we discuss representation of an integer in an arbitrary base, we examine our familiar decimal system and build the rest of our work on that.

Decimal representation of integers

1765 in base 10 is written as

$$1765 = 1000 + 700 + 60 + 5 = 1 \cdot 10^3 + 7 \cdot 10^2 + 6 \cdot 10 + 5 \cdot 10^0.$$

In general, if n is a natural number whose decimal representation is

$a_r a_{r-1} \ldots a_1 a_0$, where $0 \leqslant a_k \leqslant 9$, $k = 0, 1, \ldots, r$, then

$$n = a_r \cdot 10^r + a_{r-1} \cdot 10^{r-1} + \ldots + a_1 \cdot 10^1 + a_0 \cdot 10^0 = \sum_{k=0}^{r} a_k 10^k.$$

Each a_k is called a decimal digit of n.

For another example, when we write 54 273, we mean

$$5 \cdot 10^4 + 4 \cdot 10^3 + 2 \cdot 10^2 + 7 \cdot 10 + 3.$$

Theorem 14

Let b be a positive integer with $b > 1$. Then every positive integer n can be written uniquely in the form

$$n = a_r \cdot b^r + a_{r-1} \cdot b^{r-1} + \ldots + a_1 \cdot b^1 + a_0 \cdot b^0 = \sum_{k=0}^{r} a_k b^k$$

where r and a_r are non-negative integers, with $a_r \leqslant b - 1$ for $k = 0, 1, 2, \ldots, r$, and the initial coefficient $a_r \neq 0$.

Proof (Optional)

We obtain an expression of the desired type by applying the division algorithm in sequence in the following manner:

Divide n by b to get $n = bq_0 + a_0$, $0 \leqslant a_0 < b$. If $q_0 \neq 0$, continue dividing by b to get:

$$q_0 = bq_1 + a_1, 0 \leqslant a_1 < b$$

We continue this process to obtain:

$$q_1 = bq_2 + a_2, 0 \leqslant a_2 < b$$

$$q_2 = bq_3 + a_3, 0 \leqslant a_3 < b$$

$$\vdots$$

$$q_{r-2} = bq_{r-1} + a_{r-1}, 0 \leqslant a_{r-1} < b$$

$$q_{r-1} = b \cdot 0 + a_r, 0 \leqslant a_r < b$$

The last step of the process is achieved when a quotient of 0 is obtained.

Now, as you recall from the division algorithm,

$$n > q_0 > q_1 > \ldots \geqslant 0.$$

Since this sequence is a decreasing sequence of non-negative integers which continues as long as its terms are positive, the last term is 0.

Now, combining what we obtained above, we get

$$n = bq_0 + a_0 = b(bq_1 + a_1) + a_0 = b(b(bq_2 + a_2) + a_1) + a_0$$
$$= b(b(b(bq_3 + a_3) + a_2) + a_1) + a_0, \text{ and so on.}$$

$$n = a_r \cdot b^r + a_{r-1} \cdot b^{r-1} + \ldots + a_1 \cdot b^1 + a_0 \cdot b^0$$

The uniqueness can also be proved, but will not be included here.

Note: When a number is expressed in a base different from decimal, it is a convention to write it as

$$(a_r a_{r-1} \ldots a_1 a_0)_b.$$

b is usually called the **base** or **radix** of the system or expansion. Recall that our system, with base 10, is called the decimal system. Base 2 is the binary system, base 8 is the octal system, and base 16 is the hexadecimal system (or *hex* for short).

Example 17

Follow the outlined process in Theorem 13 to find an expression for 1948 in base 2 and in base 5.

Solution

Base 2: $1948 = 2 \cdot 974 + 0$

$974 = 2 \cdot 487 + 0$

$487 = 2 \cdot 243 + 1$

$243 = 2 \cdot 121 + 1$

$121 = 2 \cdot 60 + 1$

$60 = 2 \cdot 30 + 0$

$30 = 2 \cdot 15 + 0$

$15 = 2 \cdot 7 + 1$

$7 = 2 \cdot 3 + 1$

$3 = 2 \cdot 1 + 1$

$1 = 2 \cdot 0 + 1$

Therefore, the number in base 2 is $(11110011100)_2$.

Base 5: $1948 = 5 \cdot 389 + 3$

$389 = 5 \cdot 77 + 4$

$77 = 5 \cdot 15 + 2$

$15 = 5 \cdot 3 + 0$

$3 = 5 \cdot 0 + 3$

Therefore, the number in base 5 is $(30243)_5$.

To verify, we can change these numbers back into decimal by writing their base expansion:

$(11110011100)_2 = 1 \cdot 2^{10} + 1 \cdot 2^9 + 1 \cdot 2^8 + 1 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 = 1948$

$(30243)_5 = 3 \cdot 5^4 + 0 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 3 = 1948$

If systems use more digits than the decimal system, then they need more digits. No-one so far has invented new digits. Number theorists have been using letters to represent the extensions. For example, in base 16, the digits used are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F. The letters A, B, C, D, E, and F are used to represent the digits that correspond to 10, 11, 12, 13, 14, and 15 (written in decimal notation). Next is an example to demonstrate the conversion between the two systems.

Example 18

a) Convert $(A35B0F)_{16}$ to decimal notation.

b) Convert 38609905 to hex.

Solution

a) $(A35B0F)_{16} = A \cdot 16^5 + 3 \cdot 16^4 + 5 \cdot 16^3 + B \cdot 16^2 + 0 \cdot 16^1 + F$

$= 10 \cdot 16^5 + 3 \cdot 16^4 + 5 \cdot 16^3 + 11 \cdot 16^2 + 0 \cdot 16^1 + 15$

$= 10705679_{10}$

b) $38609905 = 16 \cdot 2413119 + 1$

$2413119 = 16 \cdot 150819 + 15\ (F)$

$150819 = 16 \cdot 9426 + 3$

$9426 = 16 \cdot 589 + 2$

$589 = 16 \cdot 36 + 13\ (D)$

$36 = 16 \cdot 2 + 4$

$2 = 16 \cdot 0 + 2$

Therefore, $38609905_{10} = (24D23F1)_{16}$.

Note: A simple conversion is possible between binary and hexadecimal notations. Each hex digit is written as a block of four binary digits according to the following table.

Hex digit	Binary	Hex digit	Binary	Hex digit	Binary
0	0000	6	0110	C	1100
1	0001	7	0111	D	1101
2	0010	8	1000	E	1110
3	0011	9	1001	F	1111
4	0100	A	1010		
5	0101	B	1011		

Example 19

a) Convert from hex to binary: $(3FCB9)_{16}$

b) Convert from binary to hex: $(11011111011010100111 00)_2$

Solution

a) We simply replace each digit with its binary equivalent. However, for the first digit to the left, if it starts with zeros, then they should be omitted (similar to decimal representation when we are talking about 0213, we mean 213).

$(3FCB9)_{16} = (\cancel{00}111111110010111001)_2 = (111111110010111001)_2$

b) We break the number into blocks of four, starting from the right. If the last block is missing digits, we add the initial zeros.

$(1101111011010100111 00)_2$

$= (000110111110110101001 1100)_2$

$= (1BDA9C)_{16}$

Operations in different systems

The operations of addition, subtraction, and multiplication can be performed using similar methods to those you learned in the decimal system. We will explain a few operations using examples.

Example 20: Addition in base 4

Add: $(32032)_4 + (10203)_4$

Solution

Before you perform any operation, it is advisable that you set up a table for that operation. So, for addition in base 4, here is the addition table.

	1	2	3
1	2	3	10
2	3	10	11
3	10	11	12

1			1	1	
	3	2	0	3	2
	1	0	2	0	3
1	0	2	3	0	1

Starting at the right: $2 + 3 = 11$ Write 1, and retain 1.

$1 + 3 + 0 = 10$ Write 0, and retain 1; and so on.

Therefore, $(32032)_4 + (10203)_4 = (102301)_4$.

Example 21: Multiplication in base 6

Find the product $(352)_6 \times (524)_6$.

Solution

We set up a multiplication table to make our task simple.

	1	2	3	4	5
1	1	2	3	4	5
2	2	4	10	12	14
3	3	10	13	20	23
4	4	12	20	24	32
5	5	14	23	32	41

We arrange the numbers in a similar manner to decimal multiplication.

$$\begin{array}{cccccc} & & & 5 & 2 & 4 \\ & & & 3 & 5 & 2 \\ \hline & & 1 & 4 & 5 & 2 \\ & 4 & 3 & 1 & 2 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ \hline 3 & 3 & 1 & 0 & 1 & 2 \end{array}$$

Start at the right.

$2 \times 4 = 12$ Write 2, and retain 1 to the next step.

$2 \times 2 = 4, 4 + 1 = 5$ Write 5.

$2 \times 5 = 14$ Write 14 as it is the last product on this line.

Next, you shift left one digit and do the multiplication by 5.

Finally, you add, in base 6, all the products you found.

Therefore, $(352)_6 \times (524)_6 = (331012)_6$.

Some divisibility rules

Rule 1: divisibility by 10^n

Consider an integer a written in decimal notation.

$$a = a_n a_{n-1} a_{n-2} \ldots a_1 a_0$$

This number, as discussed earlier, is a notation for the following decimal expansion:

$$a = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + \ldots + a_1 \cdot 10 + a_0$$

We can split this number into two parts as follows:

$$\begin{aligned} a &= a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + \ldots + a_1 \cdot 10 + a_0 \\ &= a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \ldots a_1 a_0 \\ &= 10^{n-1}(10a_n + a_{n-1}) + a_{n-2} \ldots a_1 a_0 \\ &= 10^{n-1} \cdot k + \underbrace{a_{n-2} \ldots a_1 a_0}_{(n-1)\text{ digits}} \end{aligned}$$

$k = 10a_n + a_{n-1}$ is an integer because it is the sum of two integers.

Now, if we let $m = n - 1$, a can now be written as

$$a = 10^m \cdot k + \underbrace{a_{m-1} \ldots a_1 a_0}_{m\text{ digits}} = 10^m \cdot k + p.$$

Therefore, a can be written as the sum of a multiple of mth power of 10 and a number p represented by the last m digits of a.

Now, we know that

$10^m \cdot k \equiv 0 \pmod{10^m}$, and hence

$10^m \cdot k + p \equiv p \pmod{10^m}$, and thus

$a \equiv p \pmod{10^m}$,

and this means that a and p have the same remainder when divided by 10^m.

We can conclude that the remainder when dividing any integer by 10^m is the number formed by its last m digits from the right.

For instance, the remainder of dividing 34 527 by 1000 is 527.

As a direct consequence, a number is divisible by 10^m if its last m digits are zeros.

Rule 2: divisibility by 2 and 5

As a consequence of the previous result, we can claim that every integer a can be written as

$a = 10 \cdot k + p$, and hence p represents the last digit!

Now,

$10 \equiv 0$ (mod 2 or 5)

$\Rightarrow 10 \cdot k + p \equiv p$ (mod 2 or 5), and so

$a \equiv p$ (mod 2 or 5).

Therefore, any integer has the same remainder when divided by 2 or 5 as its last digit. Consequently, a number is divisible by 2 or 5 if the last digit is divisible by 2 or 5.

The remainder of dividing 23 456 789 by 2 is 1 since the remainder of dividing 9 by 2 is 1.

The number 123 455 is divisible by 5 because the last digit is divisible by 5.

Rule 3: divisibility by 4 and 25

$a = 100k + p$, where p represents the last two digits.

Similarly to previous discussions,

$a \equiv p$ (mod 4 or 25), which leads to the rule:

The remainder of dividing any integer by 4 or 25 is the same as the remainder of the number representing the last two digits. Similarly the case with divisibility.

8 and 125 have similar rules, but with the last three digits!

The number 123 432 is divisible by 4 since 32 is divisible by 4.

The number 123 432 leaves a remainder of 7 when divided by 25 because 32 does!

Rule 4: divisibility by 3 and 9

Since a can be written as

$a = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + \ldots + a_1 \cdot 10 + a_0$, and since

$10 \equiv 1$ (mod 3 or 9), which also implies that $10^k \equiv 1^k$ (mod 3 or 9), then

$$\begin{aligned} a_n \cdot 10^n &\equiv a_n \pmod{3 \text{ or } 9} \\ a_{n-1} \cdot 10^{n-1} &\equiv a_{n-1} \pmod{3 \text{ or } 9} \\ a_{n-2} \cdot 10^{n-2} &\equiv a_{n-2} \pmod{3 \text{ or } 9} \\ &\vdots \\ a_1 \cdot 10 &\equiv a_1 \pmod{3 \text{ or } 9} \\ a_0 &\equiv a_0 \pmod{3 \text{ or } 9} \end{aligned}$$

Hence, $a \equiv a_n + a_{n-1} + a_{n-2} + \ldots + a_1 + a_0$ (mod 3 or 9).

Therefore, the remainder of dividing a number by 3 or 9 is the same as the remainder of dividing the sum of its digits by 3 or 9.

Similarly, we can say that a number is divisible by 3 or 9 iff the sum of its digits is divisible by 3 or 9.

Rule 5: divisibility by 11

Since $10 \equiv -1$ (mod 11),

$10^2 \equiv 1$ (mod 11), and hence

$10^{2k} \equiv 1$ (mod 11), and $10^{2k+1} \equiv -1$ (mod 11), and thus

$a \equiv (a_0 + a_2 + \ldots + a_{2k} + \ldots) - (a_1 + a_3 + \ldots + a_{2k+1} + \ldots)$ (mod 11).

This means that the remainder of dividing a number by 11 is equal to the remainder when the difference between the sum of its digits with even position and the sum of its digits with odd position is divided by 11. Similarly, the number is divisible by 11 if the difference between these sums is divisible by 11.

For example, 6 570 289 is divisible by 11 because
$(9 + 2 + 7 + 6) - (8 + 0 + 5) = 11$.

Exercise 2.4

1 Convert $(2009)_{10}$ to base 7 notation.

2 Convert $(3060)_7$ to decimal notation.

3 Convert $(452091)_{10}$ to base 8 notation.

4 Convert $(713060)_8$ to decimal notation.

5 Convert $(1001110011010)_2$ to base 10 notation.

6 Convert $(2010)_{10}$ to binary notation.

7 Convert $(2012452091)_{10}$ to hex notation.

8 Convert $(7B1CE3060)_{16}$ to decimal notation.

9 Convert $(10001111001)_2$ to hex notation.

10 Convert $(11101001110)_2$ to hex notation.

11 Convert $(FECDB)_{16}$ to binary notation.

12 Convert $(7DEFACED89)_{16}$ to binary notation.

13 A number N in base 10 consists of the same digit a repeated n times. For example, 4444444.

a When does $11 | N$? **b** When does $3 | N$? **c** When does $2 | N$?

2.5 Fermat's little theorem

When working with congruences relating to exponents, the next theorem is of great value.

Theorem 15: Fermat's little theorem

If p is prime and a is a positive integer with $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

For example, $6^{7-1} \equiv 1 \pmod{7}$, i.e. $6^6 - 1$ is a multiple of 7.

Proof (Optional)

We begin by considering the first $p - 1$ positive multiples of a:

$a, 2a, 3a, 4a, \ldots, (p-1)a$

None of these numbers is congruent to any other modulo p, nor is any congruent to zero. Since if that were the case, then with $1 \leqslant r \leqslant s \leqslant p - 1$, $ra \equiv sa \pmod{p}$. Then using the cancellation law as $\gcd(a, p) = 1$, we will have

$r \equiv s \pmod{p}$, which cannot happen as both s and r are smaller than p.

Hence, the set of integers $a, 2a, 3a, 4a, \ldots, (p-1)a$ would each leave a remainder when divided by p, and the set of these remainders constitute the $p - 1$ residue classes $1, 2, 3, \ldots, p - 1$. Thus,

$$a \cdot 2a \cdot 3a \cdot 4a \cdot \ldots \cdot (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-1) \pmod{p}$$

$$a \cdot a \cdot a \cdot a \cdot \ldots \cdot a\,(1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (p-1)) \equiv 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-1) \pmod{p},$$

$$\Rightarrow a^{p-1} \cdot (p-1)! \equiv (p-1)! \pmod{p}$$

and since $\gcd(p, (p-1)!) = 1$, we can cancel $(p-1)!$, and therefore

$$a^{p-1} \equiv 1 \pmod{p}.$$

Another version of this theorem is also used:

If p is prime and a is a positive integer with $p \nmid a$, then $a^p \equiv a \pmod{p}$.

Example

This example demonstrates the proof of Fermat's little theorem.

Let $p = 7$ and $a = 5$.

We will consider the first six multiples of 5:

$$1 \cdot 5 \equiv 5 \pmod 7, \quad 2 \cdot 5 \equiv 3 \pmod 7, \quad 3 \cdot 5 \equiv 1 \pmod 7,$$
$$4 \cdot 5 \equiv 6 \pmod 7, \quad 5 \cdot 5 \equiv 4 \pmod 7, \quad 6 \cdot 5 \equiv 2 \pmod 7$$

Hence,

$$(1 \cdot 5)(2 \cdot 5)(3 \cdot 5)(4 \cdot 5)(5 \cdot 5)(6 \cdot 5) \equiv 5 \cdot 3 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \pmod 7$$
$$\Rightarrow (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot 5^6 \equiv 6! \pmod 7$$
$$\Rightarrow (6!) \cdot 5^6 \equiv 6! \pmod 7$$

Since $\gcd(6!, 7) = 1$, cancel $6!$, and therefore

$$5^6 \equiv 1 \pmod 7.$$

Example 22

Show that $5^{38} \equiv 4 \pmod{11}$.

Solution

We know that $5^{10} \equiv 1 \pmod{11}$, and so $5^{30} \equiv 1 \pmod{11}$.

Also $5^2 \equiv 3 \pmod{11}$, giving us $5^8 \equiv 3^4 \equiv 4 \pmod{11}$.

Therefore, $5^{38} \equiv 1 \cdot 4 \equiv 4 \pmod{11}$.

Example 23

Find the least positive residue of 3^{201} mod 11.

Solution

We know that $3^{10} \equiv 1 \pmod{11}$, and hence $3^{201} = (3^{10})^{20} \cdot 3 \equiv 3 \pmod{11}$.

Example 24

Solve $7x \equiv 3 \pmod{13}$ for x.

Solution

Since $7^{12} \equiv 1 \pmod{13}$, then $7^{11} \cdot 7x \equiv 7^{11} \cdot 3 \pmod{13}$, and hence $x \equiv 7^{11} \cdot 3 \pmod{13}$. Therefore, $x \equiv 2 \cdot 3 \pmod{13} \equiv 6 \pmod{13}$.

Note: Example 24 can be generalized to solve linear congruencies of the form $ax \equiv b \pmod{p}$ when p is prime in the following manner:

If $ax \equiv b \pmod{p}$, then $a^{p-2} \cdot ax \equiv a^{p-2} \cdot b \pmod{p}$, which implies that $a^{p-1}x \equiv a^{p-2} \cdot b \pmod{p}$, and knowing that $a^{p-1} \equiv 1 \pmod{p}$, we will have $x \equiv a^{p-2} \cdot b \pmod{p}$.

Exercise 2.5

1 Find x such that $3^{12} \equiv x \pmod{11}$.

2 Find x such that $3^{21} \equiv x \pmod{11}$.

3 Find the value of $5^{173} \pmod{13}$.

4 Find the value of $6^{47} \pmod{17}$.

5 Find the value of $10^{321} \pmod{11}$.

6 Solve $8x \equiv 7 \pmod{17}$.

7 Solve $3x \equiv 10 \pmod{17}$.

8 Solve $7x \equiv 12 \pmod{17}$.

9 Solve $3x \equiv 4 \pmod{11}$.

10 Solve $3^{14} \equiv x \pmod{13}$.

11 Solve $3^{45} \equiv x \pmod{13}$.

12 **a** Use Fermat's little theorem to calculate: $7^{2009} \pmod{11}$, $7^{2009} \pmod{13}$, and $7^{2009} \pmod{17}$.

b Hence, calculate: $7^{2009} \pmod{2431}$.

13 Find the remainder upon dividing 512^{372} by 13.

14 Find the remainder upon dividing 3444^{3233} by 17.

15 Find the remainder upon dividing 314^{159} by 31.

16 **a** Show that if p is a prime number then $(a + 1)^p \equiv a^p + 1 \pmod{p}$, where a is an integer.

b Hence, derive Fermat's little theorem.

17 Show that $11^{104} + 1$ is a multiple of 17.

18 Let x and 35 be relatively prime numbers. Show that $x^{12} \equiv 1 \pmod{35}$.

19 Let x and 42 be relatively prime numbers. Show that $x^6 \equiv 1 \pmod{168}$.

20 Show that each of the following is true:

a $b^{21} \equiv b \pmod{15}$, for all integers b.

b $b^7 \equiv b \pmod{42}$, for all integers b.

c $b^9 \equiv b \pmod{30}$, for all integers b.

2.6 Recurrence relations

Sometimes it is difficult to define a function, a sequence, or a set explicitly. However, it may be easier to define it in terms of itself! This process is called recursion.

For instance, you recall from Chapter 4 in your book that we can use recursion to define sequences. For example, the arithmetic sequence is defined, recursively by stating the first term a_1 and by writing down the rule for finding any term of the sequence from the previous one. In the case of the arithmetic sequence, this rule is:

$a_n = a_{n-1} + d$, where d is the common difference.

Similarly, you know that a geometric sequence is defined by stating the first term g_1 and the rule for finding each term from previous ones, namely:

$g_n = g_1 r^{n-1}$, where r is the common ratio.

The arithmetic and geometric sequences have their explicit forms of course. Moving between an explicit form and a recursive form is a necessity in many cases. Specifically, the explicit form of these types is easier to work with in cases where the value of a large term is required. Imagine that you need to find the value of the 100th term in an arithmetic sequence. Using the recursive definition means that you have to know the 99th term in order to get to your target, while knowing the explicit form enables you to find the requested term by simply substituting 100 for n in the explicit formula

$a_n = a_1 + (n - 1)d$.

Consider the following situation. You are given a sequence with $a_0 = 1$ and $a_n = 3a_{n-1}$ for $n > 0$. By looking at a few terms you can easily recognize this sequence as that of the powers of 3 i.e. $a_n = 3^n$ for $n \geqslant 0$. Of course, it is simpler to work with the latter form.

Example 25

Find an explicit formula for the following sequence.

$a_0 = 1$
$a_n = na_{n-1}$ for $n > 0$

Solution

The first few terms will give you an idea of what the explicit form of the definition is

$a_0 = 1,\ a_1 = 1 \times 1,\ a_2 = 2 \times 1 = 2,\ a_3 = 3 \times 2 = 6,\ a_4 = 4 \times 6 = 24,\ \ldots$

This in fact is nothing but $n!$

To prove this, we can use mathematical induction.

Basis step

$a_1 = 1 \times 1!$

Inductive step

Assume that the statement is true for $n = k$, i.e. $a_k = k!$. We prove that it is true for $n = k + 1$.

By definition, $a_{k+1} = (k+1)a_k = (k+1) \cdot k! = (k+1)!$

Recurrence relations

As we discussed above, you notice that a recursive definition of a sequence identifies one or more early terms and a law for defining later terms from those preceding them. Such rules are called **recurrence relations**. So, when the problem is to discover an explicit formula for a recursively defined sequence, the recursive formula is called a **recurrence relation**. Remember that to define a sequence well, a recursive formula must be supplemented by information about some earlier terms of the sequence. This information is called the **initial condition(s)** for the sequence.

Definition 1

A **recurrence relation** for a sequence $\{a_n\}$ is a formula that expresses a_n in terms of one or more of the previous terms of the sequence: a_{n-1}, a_{n-2}, etc.…

Definition 2

A sequence is called a **solution** of a recurrence relation, if its terms satisfy the recurrence relation.

Definition 3

Initial conditions are explicitly given values for a certain number of the terms of the sequence.

Examples

i The recurrence relation $a_n = 2a_{n-1} + 3$ for $n > 1$ with $a_1 = 5$ defines the sequence 13, 29, 61, 125, ….

ii The recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n > 2$ with **initial conditions** $F_1 = 1$ and $F_2 = 1$ describes the well-known Fibonacci sequence 1, 1, 2, 3, 5, 8, …

iii In Example 25 above, the recurrence relation is $a_n = na_{n-1}$ for $n > 0$ and the initial condition $a_0 = 1$ describes the sequence $a_n = n!$

Example 26

Consider the recurrence relation $u_{n+1} = 2u_n - u_{n-1}$ for $n \geqslant 1$. Which of the following is a solution of this relation?

a) $u_n = 3n$ b) $u_n = 2^n$ c) $u_n = 5$

Solution

a) For $u_n = 3n$ with $n \geqslant 1$, we see that according to the recurrence relation, if u_n is a solution, then

$$u_n = 2u_{n-1} - u_{n-2} = 2(3(n-1)) - 3(n-2) = 6n - 6 - 3n + 6 = 3n$$

$\therefore\ u_n = 3n$ is a solution.

b) For $u_n = 2^n$ with $n \geqslant 1$, we see that according to the recurrence relation, if u_n is a solution, then

$$u_n = 2u_{n-1} - u_{n-2} = 2(2^{(n-1)}) - 2^{(n-2)} = 2^n - 2^{n-2} \neq 2^n$$

$\therefore\ u_n = 2^n$ is not a solution.

c) For $u_n = 5$ with $n \geqslant 1$, we see that according to the recurrence relation, if u_n is a solution, then

$$u_n = 2u_{n-1} - u_{n-2} = 2(5) - 5 = 5$$

$\therefore\ u_n = 5$ is a solution.

Consider this sequence: $u_n = 3n + 5$.

If u_n is a solution, then

$$\begin{aligned} u_n &= 2u_{n-1} - u_{n-2} \\ &= 2(3(n-1) + 5) - 3(n-2) - 5 \\ &= 6n - 6 + 10 - 3n + 6 - 5 \\ &= 3n + 5 \end{aligned}$$

$\therefore\ u_n = 3n + 5$ is a solution.

This demonstrates a theorem which we will prove in Section 2.8 that if u and v are solutions of a linear recurrence relation, then $au + bv$ where a and b are arbitrary constants, is also a solution.

Example 27

Consider the recurrence relation $a_{n+1} = 2a_n + 1$ with the initial condition $a_1 = 7$.

a) Find a_2, a_3, and a_4.

b) Show that $a_n = 2^{n+2} - 1$ is a solution to this recurrence relation.

Solution

a) $a_2 = a_{1+1} = 2a_1 + 1 = 15$

$a_3 = a_{2+1} = 2a_2 + 1 = 31$

$a_4 = a_{3+1} = 2a_3 + 1 = 63$

b) Notice that $a_2 = 15 = 2^{2+2} - 1$, $a_3 = 31 = 2^{3+2} - 1$, and $a_4 = 63 = 2^{4+2} - 1$

Now, substituting $a_n = 2^{n+2} - 1$ into a_{n+1} will give us:

$$a_{n+1} = 2^{n+1+2} - 1 = 2^{n+3} - 1 = 2\,(2^{n+2}) - 1 = 2(2^{n+2} - 1) + 1 = 2a_n + 1$$

2.7 Modelling with recurrence relations

We can use recurrence relations to model a diverse range of situations. Such situations include counting bit strings with specific properties, compound interest, counting growth of populations under specific constraints, and some counting related to recreational mathematics! Here are some examples.

Compound interest

Consider that a person makes a one-off deposit of an amount of P_0 in a savings account that pays r in annual interest. (r is in decimal notation. For example for 5% , $r = 0.05$)

How much money will be in the account after n years?

Solution

Let P_n denote the amount in the account after n years. Then P_n is equal to the amount that has accumulated over the last $n - 1$ years, P_{n-1}, plus the interest earned during the nth year, $r\,P_{n-1}$.

Therefore, $P_n = P_{n-1} + r\,P_{n-1} = (1 + r)\,P_{n-1}$.

To find an explicit formula for the amount of money, we can use an iterative approach for that purpose. (It is also called **backtracking**.)

$$P_1 = (1 + r)\,P_0$$

$$P_2 = (1 + r)\,P_1 = (1 + r)(1 + r)\,P_0 = (1 + r)^2\,P_0$$

$$P_3 = (1 + r)^3\,P_0$$

$$\vdots$$

$$P_n = (1 + r)^n\,P_0$$

You have seen this formula in Chapter 4 of the textbook too. We can use mathematical induction to establish its validity.

Basis step
For $n = 0$, $P_0 = (1 + r)^0 P_0 = P_0$

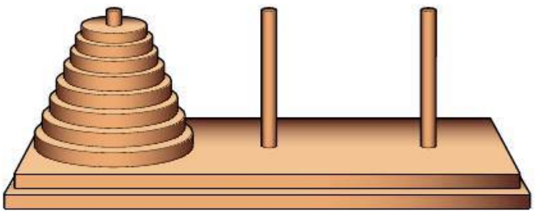
Inductive step
Assume this to be true for $n = k$, i.e. $P_k = (1 + r)^k\,P_0$

For $n = k + 1$

$$P_{k+1} = (1 + r)\,P_k = (1 + r)\,(1 + r)^k\,P_0 = (1 + r)^{k+1}\,P_0$$

$$\therefore\ P_n = (1 + r)^n\,P_0 \text{ for all possible values of } n.$$

Tower of Hanoi



The Tower of Hanoi puzzle involves moving a pile of different-sized disks from one peg to another, using an intermediate peg. Only one disk at a time can be moved, a disk can only be moved if it is the top disk on a pile, and a larger disk can never be placed on a smaller one. Our task is to find the number of moves needed to move all the n disks from peg 1 to peg 3 for example.

Solution

Let d_n represent the number of moves required to move the disks from peg 1 to 3, using peg 2 as an auxiliary 'stop'.

We can move the top $n - 1$ disks, following the rules of the game, from peg 1 to peg 2, leaving the largest disk at peg 1. This can be done in d_{n-1} ways.

Now we move the largest disk, in one move from peg 1 to peg 3. The next step is then to move the $n - 1$ disks from peg 2 to peg 3, which can be done in d_{n-1} ways again. Hence, the total number of moves is now

$$d_n = d_{n-1} + d_{n-1} + 1 = 2d_{n-1} + 1$$

This is the recurrence relation leading us to the solution. The initial condition here is $d_1 = 1$, because one disk requires one move only to be transferred from peg 1 to peg 3.

We can use an iterative method (backtracking) to solve this recurrence relation

$$\begin{aligned}
d_n &= 2d_{n-1} + 1 \\
&= 2(2d_{n-2} + 1) + 1 = 2^2d_{n-2} + 2 + 1 \\
&= 2^2(2d_{n-3} + 1) + 2 + 1 = 2^3d_{n-3} + 2^2 + 2 + 1 \\
&= 2^3(2d_{n-4} + 1) + 2^2 + 2 + 1 = 2^4d_{n-4} + 2^3 + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1}d_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1
\end{aligned}$$

However, $d_1 = 1$, and so

$$d_n = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1$$

The right-hand side of this equation is a geometric series, with first term 1 and common ratio 2, and hence

$$d_n = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 = 1 \cdot \frac{1 - 2^n}{1 - 2} = 2^n - 1$$

The formula can be proved using mathematical induction:

Basis step
For $n = 1$, $d_1 = 2^1 - 1 = 1$, which is true.

Inductive step
Assume this to be true for $n = k$, i.e. $d_k = 2^k - 1$.

For $n = k + 1$,

$d_{k+1} = 2d_k + 1$. according to the recurrence relation, and thus
$d_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$ as required.

Fibonacci's Rabbits

The imaginative problem that Fibonacci probed (in the year 1202) was about how fast rabbits could breed in *ideal* settings.

Presume that a newborn pair of rabbits, one male and one female, are put in a field. Rabbits are able to mate at the age of 1 month so that at the end of its second month a female can produce another pair of rabbits. Assume that our rabbits **never die** and that the female always produces **one new pair (one male, one female) every month** from the second month on. The puzzle that Fibonacci posed was this:

How many pairs will there be in 1 year?

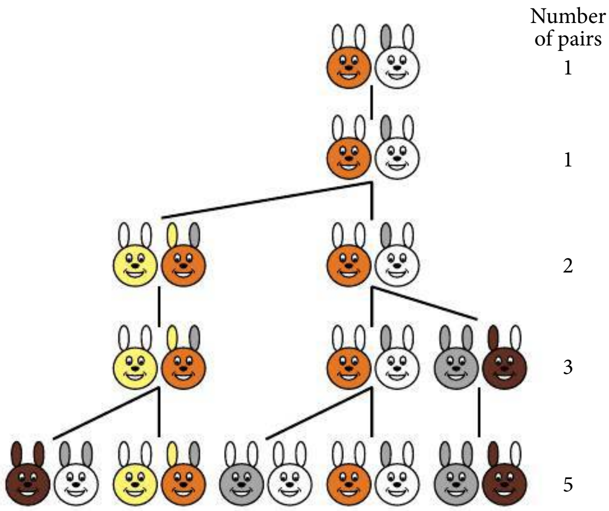
Our task here is to find a recurrence relation for the number of pairs of rabbits after n months.

Solution

Consider the situation according to the way it is set up.

- By the end of the first month, there is only one pair, the original.
- At the end of the second month, they mate, but there is still one only 1 pair.
- At the end of the third month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the fourth month, the original female produces a second pair, making 3 pairs in all in the field, the newborn pair mate but no new children yet.
- At the end of the fifth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

Now let r_n be the number of pairs of rabbits at the end of n months. So, at the end of the first month there is only one pair, i.e., $r_1 = 1$. At the end of the second month, still one pair, i.e., $r_2 = 1$. At the end of the third month there are two pairs, i.e., $r_3 = 2$ and so on.



To find the number after n months, we add the number in the field in the previous month, r_{n-1}, and the number of the newborn pairs, which will be

r_{n-2}, since each newborn pair comes from a pair at least 2 months old.

Consequently:

$$r_n = r_{n-1} + r_{n-2}$$

This, along with the initial conditions $r_1 = 1$and $r_2 = 1$ describes the Fibonacci sequence which you know already.

2.8 Solving linear recurrence relations

As you have seen earlier, some of the recurrence relations can be solved using iteration (backtracking), others can be solved by some other improvised techniques, and a specific type known as **linear homogeneous recurrence relations with constant coefficients** can be solved explicitly in a systematic manner.

Definition 1

A **linear homogeneous recurrence relations of degree *k* with constant coefficients** is a recurrence relation of the form

$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$,

where $c_1, c_2, \dots, c_k$ are real numbers with $c_k \neq 0$.

This relation is linear because the right-hand side is a linear combination of the earlier terms of the sequence, and homogeneous because *all* terms are multiples of the a_is. The coefficients of the terms are all constants rather than functions of n. The degree of the relation is k because a_n is expressed in terms of the previous k terms of the sequence.

Note: In this book, we will limit our discussion to linear recurrence relations of at most second degree.

Example 28

Which of the following recurrence relations are linear homogeneous?

a) $s_n = 3s_{n-1}$

b) $f_{n+1} = f_n + f_{n-1}$

c) $b_n = 2b_{n-1}\,b_{n-2}$

d) $a_n = 2a_{n-1} + 5n$

e) $A_n = 1.09A_{n-1}$

f) $c_n = 2c_{n-1} - c^2{}_{n-2}$

Solution

a) This is linear homogeneous since the nth term is a constant multiple of the previous term.

b) This is linear homogeneous since the nth term is the sum of the previous two terms.

c) This is not linear homogeneous since the nth term is the product of the previous two terms and not a constant multiple of one of them.

d) This is not linear homogeneous since the right-hand side contains a function of n rather than a constant.

e) This is linear homogeneous since the nth term is a constant multiple of the previous term.

f) This is not linear homogeneous since the right-hand side contains a power of one term that is higher than 1.

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = x^n$, where x is a constant.

Obviously $a_n = x^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

if and only if

$$x^n = c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_k x^{n-k}$$

Multiplying both sides by x^{k-n} and simplifying will yield the equation

$$x^k = c_1 x^{k-1} + c_2 x^{k-2} + \dots + c_k$$

Note

$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_k$ is known as the **characteristic polynomial**.

This is called the **characteristic equation** of the recurrence relation. Obviously too, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution to the recurrence relation if and only if r is a solution of the characteristic equation.

We will demonstrate the general method of solving linear homogeneous relations with constant coefficients by finding an explicit solution to a second order relation first.

Example 29

Solve the recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$ with initial conditions $a_0 = 4$, $a_1 = 10$.

Solution

The associated characteristic equation is:

$$x^2 - 2x - 8 = 0$$

Solving this equation, we have two solutions.

$$r_1 = 4 \text{ or } r_2 = -2.$$

At this point, we have two solutions of the recursive relation.

$$s_n = 4^n \text{ or } t_n = (-2)^n$$

In Example 27 of Section 2.6, we verified a theorem that if u and v are solutions, then a linear combination of u and v will also be a solution. Thus

$$a_n = b(4^n) + d(-2)^n$$

is a solution to the relation.

To satisfy the initial conditions, we must have

$$a_0 = 4 \Rightarrow b(4^0) + d(-2)^0 = 4 \Rightarrow b + d = 4$$

$$a_1 = 10 \Rightarrow b(4^1) + d(-2)^1 = 10 \Rightarrow 4b - 2d = 10$$

Solving this system we find that $b = 3$ and $d = 1$, and thus

$$a_n = 3(4^n) + (-2)^n$$

is the solution to the recurrence relation.

Notice that

$$\begin{aligned} 2a_{n-1} + 8a_{n-2} &= 2(3(4^{n-1}) + (-2)^{n-1}) + 8(3(4^{n-2}) + (-2)^{n-2}) \\ &= 6(4^{n-1}) + 2\,(-2)^{n-1} + 24\,(4^{n-2}) + 8\,(-2)^{n-2} \\ &= 6(4^{n-1}) + 2\,(-2)^{n-1} + 6\,(4^{n-1}) - 4\,(-2)^{n-1} \\ &= 12(4^{n-1}) - 2\,(-2)^{n-1} \\ &= 3(4^n) + (-2)^n \\ &= \mathrm{a}_n \end{aligned}$$

This verifies that a_n is a solution to the recurrence relation.

Theorem 1

If u_n and v_n are solutions to the second order linear homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$, then $t_n = bu_n + dv_n$ is also a solution.

Proof

Since u_n and v_n are solutions then

$$u_n = c_1u_{n-1} + c_2u_{n-2} \text{ then } v_n = c_1v_{n-1} + c_2v_{n-2}$$

Thus

$$\begin{aligned} t_n &= bu_n + dv_n = b(c_1u_{n-1} + c_2u_{n-2}) + d(c_1v_{n-1} + c_2v_{n-2}) \\ &= c_1(bu_{n-1} + dv_{n-1}) + c_2(bu_{n-2} + dv_{n-2}) \\ &= c_1t_{n-1} + c_2t_{n-2} \end{aligned}$$

Therefore t_n is a solution to $a_n = c_1a_{n-1} + c_2a_{n-2}$.

Theorem 2

1 If the characteristic polynomial $x^2 - c_1x - c_2$ of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ has two distinct zeros r_1 and r_2, then $a_n = br_1{}^n + dr_2{}^n$ where b and d depend on the initial conditions, is the explicit formula for the solution sequence.

2 If the characteristic polynomial $x^2 - c_1x - c_2$ of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ has a single zero r, then $a_n = br^n + dnr^n$ where b and d depend on the initial conditions, is the explicit formula for the solution sequence.

3 If the characteristic polynomial $x^2 - c_1x - c_2$ of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ has two conjugate complex zeros z_1 and z_2, then we express these zeros in polar form where $z_1 = (r, \theta)$ and $z_2 = (r, -\theta)$ and the solution will be of the form $a_n = r^n(b\cos(n\theta) + d\sin(n\theta))$ where b and d depend on the initial conditions.

Proof

1 Suppose that r_1 and r_2 are zeros of $x^2 - c_1x - c_2$, so $r_1^2 - c_1r_1 - c_2 = 0, r_2^2 - c_1r_2 - c_2 = 0$, and $a_n = br_1^n + dr_2^n$, for $n \geqslant 1$. We show that this definition of a_n defines the same sequence as $a_n = c_1a_{n-1} + c_2a_{n-2}$.

First we note that b and d are chosen so that the initial conditions are satisfied. That is

$a_1 = br_1 + dr_2$ and $a_2 = br_1^2 + dr_2^2$.

Thus

$$\begin{aligned} a_n &= br_1^n + dr_2^n \\ &= br_1^{n-2}r_1^2 + dr_2^{n-2}r_2^2 \end{aligned}$$

Now, using the fact that r_1 and r_2 are zeros of $x^2 - c_1x - c_2$, we have

$$r_1^2 - c_1r_1 - c_2 = 0 \Rightarrow r_1^2 = c_1r_1 + c_2$$

and $$r_2^2 - c_1r_2 - c_2 = 0 \Rightarrow r_2^2 = c_1r_2 + c_2$$

Thus

$$\begin{aligned} a_n &= br_1^{n-2}r_1^2 + dr_2^{n-2}r_2^2 \\ &= br_1^{n-2}(c_1r_1 + c_2) + dr_2^{n-2}(c_1r_2 + c_2) \\ &= c_1(br_1^{n-1} + dr_2^{n-1}) + c_2(br_1^{n-2} + dr_2^{n-2}) \\ &= c_1a_{n-1} + c_2a_{n-2} \end{aligned}$$

2 This part may be proved in a similar manner and is left as an exercise.

3 The proof of this part is beyond the scope of this book.

Example 30

Find the solution to the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, where $a_1 = 5$, $a_2 = 3$.

Solution

The characteristic equation associated with this relation is

$$x^2 - 3x + 2 = 0$$

The characteristic roots are 1 and 2.

Thus, the solution to the relation is of the form

$$a_n = br_1^n + dr_2^n = b(1^n) + d(2^n),$$

With the initial conditions, we have

$$\left.\begin{aligned} b + 2d &= 5 \\ b + 4d &= 3 \end{aligned}\right\} \Rightarrow b = 7, d = -1$$

Therefore the solution is

$$a_n = 7 - 2^n$$

Note: Notice here that using $a_n = 7 - 2^n$, we find that the first 5 terms are: 5, 3, –1, –9, and –25 and using the recurrence relation, we have 5, 3, –1, –9, and –25.

Example 31

Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$, where $u_0 = 1$, $u_1 = 1$.

Solution

The associated characteristic equation is

$$x^2 - 4x + 4 = 0$$

This has one solution, $x = 2$.

According to theorem 2, the solution to this equation has the form

$$u_n = bx^n + dnx^n$$

Thus, the solution for this relation is

$$u_n = b2^n + dn2^n$$

The initial conditions yield

$$\left.\begin{aligned} 1 &= b \\ 1 &= 2b + 2d \end{aligned}\right\} \Rightarrow b = 1, d = -\frac{1}{2}$$

Therefore, the solution is

$$u_n = 2^n - \frac{1}{2}n2^n = 2^n - n2^{n-1}$$

Example 32

Solve the recurrence relation $v_n = 2v_{n-1} - 2v_{n-2}$, where $v_0 = 1$, $v_1 = 3$.

Solution

The characteristic equation for the recurrence relation is

$$t^2 - 2t + 2 = 0$$

The characteristic roots are then

$$z_1 = 1 + i, z_2 = 1 - i$$

When written in polar form, the roots are

$$z_1 = \sqrt{2}cis\frac{\pi}{4},\ z_2 = \sqrt{2}cis\frac{-\pi}{4}$$

Thus any solution of the relation is of the form

$$v_n = \left(\sqrt{2}\right)^n\left(b\cos\left(n\frac{\pi}{4}\right) + d\sin\left(n\frac{\pi}{4}\right)\right)$$

With the initial conditions we have

$$\left.\begin{array}{l} v_0 = 1 = \left(\sqrt{2}\right)^0\left(b\cos\left(0\cdot\frac{\pi}{4}\right) + d\sin\left(0\cdot\frac{\pi}{4}\right)\right) = b \\ v_1 = 3 = \left(\sqrt{2}\right)^1\left(b\cos\left(\frac{\pi}{4}\right) + d\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}\left(b\frac{1}{\sqrt{2}} + d\frac{1}{\sqrt{2}}\right)\end{array}\right\} \Rightarrow \left.\begin{array}{l} b = 1 \\ b + d = 3\end{array}\right\} \Rightarrow b = 1;\ d = 2$$

The solution of the recurrence equation is then

$$v_n = \left(\sqrt{2}\right)^n\left(\cos\left(n\frac{\pi}{4}\right) + 2\sin\left(n\frac{\pi}{4}\right)\right)$$

Example 33

Consider the Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ for $n > 2$ with initial conditions $F_1 = 1$ and $F_2 = 1$.

Find an explicit expression for F_n.

Solution

The characteristic equation associated with this is

$$x^2 - x - 1 = 0$$

The characteristic roots are then

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}.$$

Thus, any solution to Fibonacci's sequence is of the form

$$F_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Now using the initial conditions we have

$$\left.\begin{array}{l} F_1 = b\left(\frac{1+\sqrt{5}}{2}\right)^1 + d\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \\ F_2 = b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1\end{array}\right\} \Rightarrow b = \frac{1}{\sqrt{5}}, d = \frac{-1}{\sqrt{5}}$$

Hence, Fibonacci's nth term is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Linear non-homogeneous recurrence relations with constant coefficients

We have seen how to solve linear homogeneous recurrence relations by using characteristic polynomials and some other relations by using iteration. This section explores techniques that can be used to solve non-homogeneous relations.

For example, $a_n = 2a_{n-1} + 3n$ is a recurrence relation but not homogeneous.

Definition

A recurrence relation of the form

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + f(n)$

where c_i for $i = 1, 2, \ldots, k$ are real numbers and $f(n)$ is a function of n not identically zero is a **linear non-homogeneous recurrence relation with constant coefficients**.

The recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ is called the associated homogeneous recurrence relation. It plays a very important role in the solution of the non-homogeneous recurrence relation.

Examples

Each of the following recurrence relations are linear non-homogeneous.

i $a_n = 2a_{n-1} + 3n$

ii $b_n = b_{n-1} - 3b_{n-2} + n^2 + 2n$

iii $u_n = 2u_{n-1} + u_{n-2} + 2n5^n$

Each of the following is the associated linear homogeneous recurrence relation.

i $a_n = 2a_{n-1}$

ii $b_n = b_{n-1} - 3b_{n-2}$

iii $u_n = 2u_{n-1} + u_{n-2}$

The importance of the associated homogeneous relations in the solution of the non-homogeneous relations is shown by the following theorem.

Theorem 3 (Optional)

If p_n is a particular solution of the linear non-homogeneous recurrence relation with constant coefficients $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + f(n)$ and if h_n is a solution of the associated homogeneous relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$, then every solution of the non-homogeneous relation is of the form $p_n + h_n$.

Proof

If p_n is a solution of the non-homogeneous relation, then

$$p_n = c_1 p_{n-1} + c_2 p_{n-2} + \ldots + c_k p_{n-k} + f(n).$$

Suppose that q_n is another solution of the non-homogeneous equation, then

$$q_n = c_1 q_{n-1} + c_2 q_{n-2} + \ldots + c_k q_{n-k} + f(n)$$

Subtracting the two equations gives

$$q_n - p_n = c_1 (q_{n-1} - p_{n-1}) + c_2 (q_{n-2} - p_{n-2}) + \ldots + c_k (q_{n-k} - p_{n-k})$$

This shows that $q_n - p_n$ is a solution of the associated homogeneous relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Call this solution h_n and so: $h_n = a_n = q_n - p_n \Rightarrow q_n = h_n + p_n$

Example 34 (Optional)

Find a solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.

Solution

The associated homogeneous relation is

$$a_n = 2a_{n-1}$$

This is easily spotted to be a geometric sequence and hence has a solution

$$h_n = b2^n$$

To find a particular solution, we can attempt $p_n = dn \cdot 2^n$

To find d we substitute p_n back into the original non-homogeneous relation

$$dn \cdot 2^n = 2d\left((n-1)2^{n-1}\right) + 3 \cdot 2^n$$

Simplify the equation by dividing through by 2^{n-1} to get

$$2dn = 2d(n-1) + 3 \cdot 2 \Rightarrow d = 3$$

Thus, the particular solution we seek is $p_n = 3n \cdot 2^n$ and hence the general solution of the non-homogeneous relation is the sum of the solution to the homogeneous relation and this one: $h_n = b2^n + 3n2^n = (b + 3n)2^n$.

Exercise 2.6–2.8

In questions 1–4, give the first five terms and identify the recurrence relation as linear homogeneous or not. If the relation is linear homogeneous, then what is its degree?

1 $b_n = \frac{5}{2} b_{n-1};\ b_1 = 6$

2 $a_n = -3a_{n-1} - 2a_{n-2};\ a_1 = -2,\ a_2 = 4$

3 $a_n = 2^{n-1} a_{n-1};\ a_1 = 5$

4 $b_n = 5b_{n-1} + 3;\ b_1 = 1$

In questions 5–10, solve each of the recurrence relations.

5 $b_n = \frac{5}{2}b_{n-1}; b_1 = 4$

6 $a_n = 5a_{n-1} + 3; a_1 = 3$

7 $a_n = a_{n-1} + n; a_1 = 4$

8 $b_n = -\frac{11}{10}b_{n-1}; b_1 = 10$

9 $a_n = a_{n-1} - 2; a_1 = 0$

10 $b_n = nb_{n-1}; b_1 = 8$

In questions 11–13, solve each of the recurrence relations.

11 $b_n = 4b_{n-1} + 5b_{n-2}; b_1 = 6,\ b_2 = 6$

12 $a_n = -3a_{n-1} - 2a_{n-2}; a_1 = -2,\ a_2 = 4$

13 $a_n = 2a_{n-1} - 2a_{n-2}; a_1 = 1,\ a_2 = 4$

14 Develop a general explicit formula for a recurrence relation of the form $u_n = au_{n-1} + b$ where a and b are real numbers.

Apply the result to the situations above that fit that model.

Practice questions 2

- Practice questions 1–10 cover work from Chapters 1–2 inclusive.

1 For any positive integers a and b, let gcd(a, b) and lcm (a, b) denote the greatest common divisor and the least common multiple of a and b, respectively. Prove that

$a \times b = (\gcd(a, b)) \times (\text{lcm}(a, b))$.

2 **a** Using Euclid's algorithm, find integers x and y such that $17x + 31y = 1$.

b Given that $17p + 31q = 1$, where $p, q \in \mathbb{Z}$, show that

$|p| \geqslant 11$ and $|q| \geqslant 6$.

3 Find the remainder when 67^{101} is divided by 65.

4 **a** Convert the number 95 from base 10 to base 6.

b Working in base 6, square your answer to part **a**.

c Convert your answer to part **b** to a base 10 number.

5 The function $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by $f(x) = \gcd(x, 6)$.

a Find the range of the function f.

b Show that the function f is periodic and find its period.

c Find the set of positive integers satisfying $f(x) = 2$.

6 **a** Use the Euclidean algorithm to find the greatest common divisor of 43 and 73.

Consider the equation $43x + 73y = 7$, where $x, y \in \mathbb{Z}$.

b **i** Find the general solution of this equation.

ii Find the solution which minimizes $|x| + |y|$.

7 **a** Use the Euclidean algorithm to show that 275 and 378 are relatively prime.

b Find the general solution to the Diophantine equation $275x + 378y = 1$.

8 **a** Define what is meant by the statement $x \equiv y \pmod{n}$, where $x, y, n \in \mathbb{Z}^+$.

b Hence, prove that if $x \equiv y \pmod{n}$ then $x^2 \equiv y^2 \pmod{n}$.

c Determine whether or not $x^2 \equiv y^2 \pmod{n}$ implies that $x \equiv y \pmod{n}$.

9 **a** **i** Given that $a \equiv d \pmod{n}$ and $b \equiv c \pmod{n}$, prove that

$(a + b) \equiv (c + d) \pmod{n}$.

ii Hence, solve the system:

$$\begin{cases} 2x + 5y \equiv 1 \pmod{6} \\ x + y \equiv 5 \pmod{6} \end{cases}$$

b Show that $x^{97} - x + 1 \equiv 0 \pmod{97}$ has no solution.

10 **a** Given that $ax \equiv b \pmod{p}$, where $a, b, p, x \in \mathbb{Z}^+$, p is prime and a is not a multiple of p, use Fermat's little theorem to show that

$x \equiv a^{p-2}b \pmod{p}$.

b Hence, solve the simultaneous linear congruences

$3x \equiv 4 \pmod{5}$

$5x \equiv 6 \pmod{7}$

giving your answer in the form $x \equiv c \pmod{d}$.

3 Graphs

Terminology

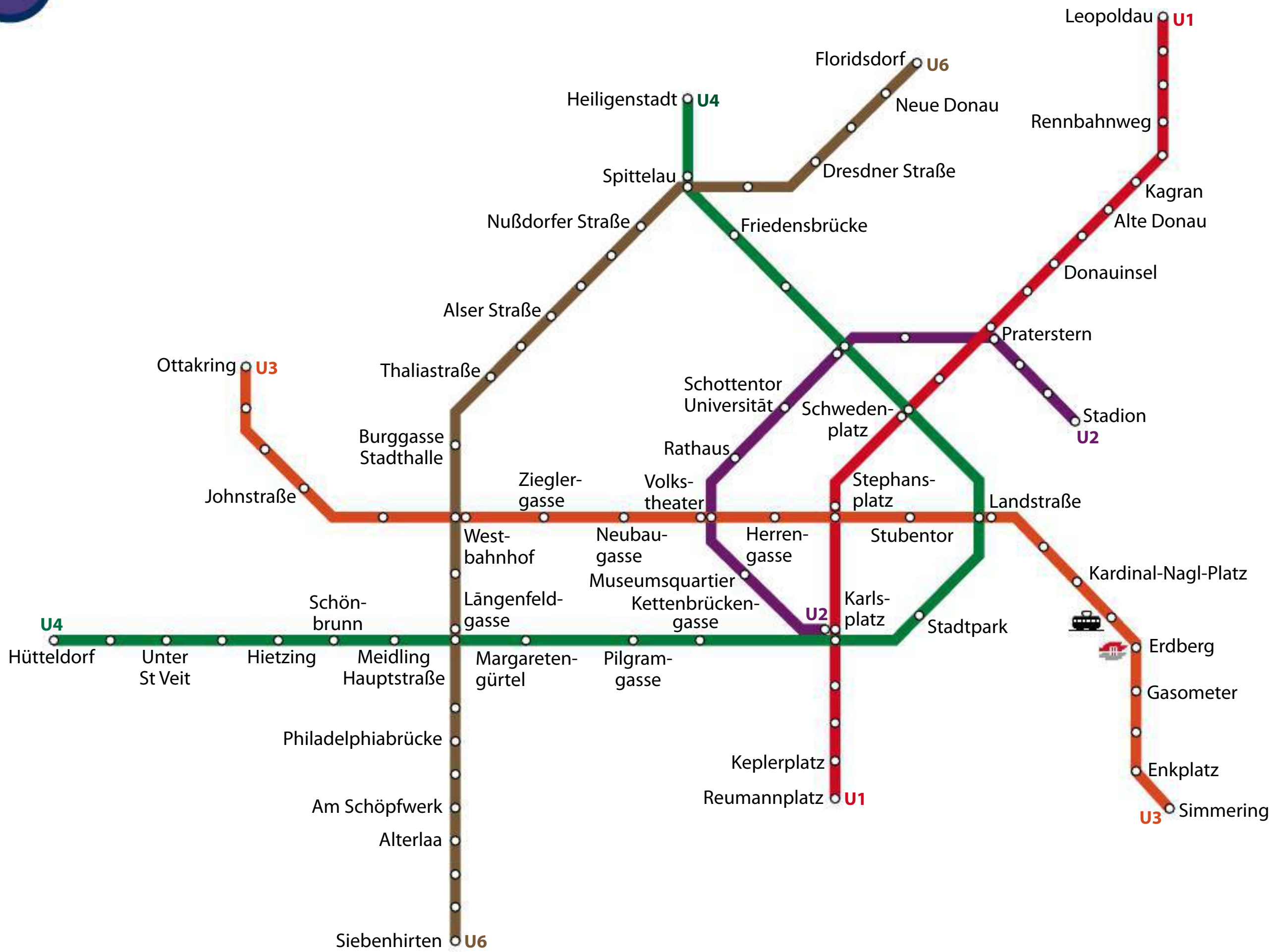
You should be aware that many different terminologies exist in graph theory and that different textbooks may employ different combinations of these.

In IB examination questions, the terminology used will be as it appears in the syllabus. A summary of the terminology is provided below.

Graph	Consists of a set of vertices and a set of edges; an edge joins its endpoints (vertices).
Subgraph	A graph within a graph.
Weighted graph	A graph in which each edge is allocated a number or weight.
Loop	An edge whose endpoints are joined to the same vertex.
Multiple edges	Multiple edges occur if more than one edge joins the same pair of vertices.
Walk	A sequence of linked edges.
Trail	A walk in which no edge appears more than once.
Path	A walk with no repeated vertices.
Circuit	A walk that begins and ends at the same vertex, and has no repeated edges.
Cycle	A walk that begins and ends at the same vertex, and has no other repeated vertices.
Hamiltonian path	A path that contains all the vertices of the graph.
Hamiltonian cycle	A cycle that contains all the vertices of the graph.
Eulerian trail	A trail that contains every edge of a graph.
Eulerian circuit	A circuit that contains every edge of a graph.
Degree of a vertex	The number of edges joined to the vertex; a loop contributes two, one for each of its endpoints.
Simple graph	A graph without loops or multiple edges.
Complete graph	A simple graph where every vertex is joined to every other vertex.

Connected graph	A graph that has a path joining every pair of vertices.
Disconnected graph	A graph that has at least one pair of vertices not joined by a path.
Tree	A connected graph that contains no cycles.
Weighted tree	A tree in which each edge is allocated a number or weight.
Spanning tree of a graph	A subgraph containing every vertex of the graph, which is also a tree.
Minimum spanning tree	A spanning tree of a weighted graph that has the minimum total weight.
Complement of a graph G	A graph with the same vertices as G but which has an edge between any two vertices if and only if G does not.
Graph isomorphism between two simple graphs G and H	A one-to-one correspondence between vertices of G and H such that a pair of vertices in G is adjacent if and only if the corresponding pair in H is adjacent.
Planar graph	A graph that can be drawn in the plane without any edge crossing another.
Bipartite graph	A graph whose vertices can be divided into two sets and in which edges always join a vertex from one set to a vertex from the other set.
Complete bipartite graph	A bipartite graph in which every vertex in one set is joined to every vertex in the other set.
Adjacency matrix of G, denoted by A_G	The adjacency matrix, A_G, of a graph G with n vertices, is the $n \times n$ matrix in which the entry in row i and column j is the number of edges joining the vertices i and j. Hence, the adjacency matrix will be symmetric about the diagonal.
Cost adjacency matrix of G, denoted by C_G	The cost adjacency matrix, C_G, of a graph G with n vertices is the $n \times n$ matrix in which the entry in row i and column j is the weight of the edges joining the vertices i and j.

3.1 Introduction



The diagram above is a map of Vienna's underground. Maps like this one do not generally correspond to the real geographic sites in the city but rather the way in which the different stations are organized. This way, a passenger using the underground can plan a route from one station to another. The map as presented is simply a diagrammatic means of representing how the stations are interconnected.

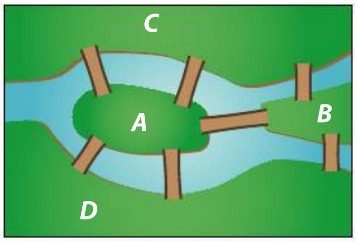
The above situation is one simple application of graph theory. The theory has many applications, including chemical molecules, floor plans, electrical and computer networks, and many others. We will begin with some basic definitions.

3.2 Graphs: definitions

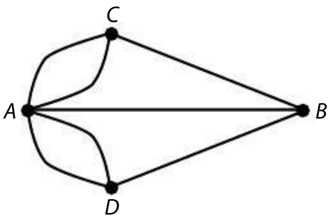
When we are using a map, we are more concerned with seeing how to get from one point to another using the routes available. Consequently, we are dealing with two sets of objects: locations and routes. Such situations involving two sets give rise to relations between the elements of the sets.

If V denotes the set of vertices (also called **nodes** or **points**) and E denotes the set of edges (routes, lines), graph G is the non-empty set consisting of vertices and edges, as shown below.

Figure 3.1



a)



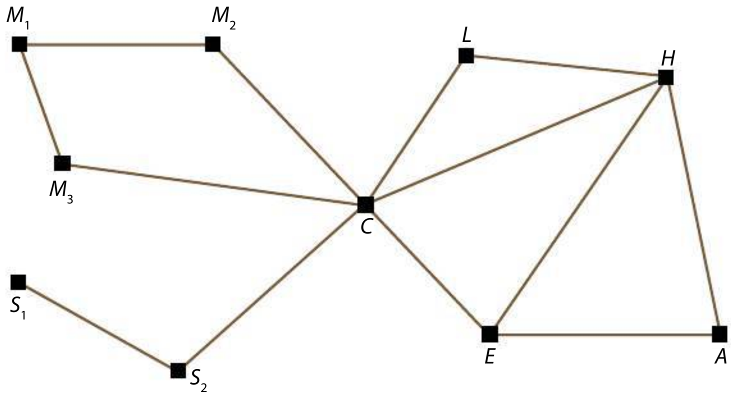
b) Graph G

Related to the discussion above is the Königsberg bridge problem (Figure 3.1a). The Pregel river passes through the Prussian city of Königsberg and divides it into two banks and two islands in the middle. Seven bridges connect the four land areas of the city. Residents of the city had a problem – namely to determine whether it was possible to walk through the city using each of the bridges exactly once.

The Königsberg problem inspired Euler to find a solution which appeared in his paper *Solutio problematis ad geometriam situs pertinentis*, published in 1736. Euler realized that the physical layout of land, water and bridges could be modelled by the graph shown in Figure 3.1b). The land parts are represented by points A, B, C, and D, and the bridges by lines (edges) which could be curved. By means of such a graph, the real problem is transformed into a mathematical one: given the graph in Figure 3.1b), is it possible to choose a vertex, traverse the edges one after the other, and return to the starting vertex using every edge only once? Euler showed that it was impossible. This is a problem we will visit later in the chapter.

Consider Figure 3.2 below, representing a school network. Each computer is connected to the network by one cable. In this network, there is at most one cable between any two computers and there is no cable that connects a computer to itself. This network can be modelled by a **simple graph**, which consists of vertices that represent the computers and undirected edges that represent the cables. Each edge connects two different vertices and no two edges connect the same pair of vertices.

Figure 3.2



Definition 1

A **graph $G = (V, E)$** consists of two sets: V, a non-empty set of **vertices** (**nodes** or **points**), and E, a set of unordered pairs of different elements of V called **edges** (**arcs** or **sides**).

1 In this publication, all graphs are assumed to be **finite graphs**, which means that they consist of a finite number of vertices and edges.

2 Edges in a graph are allowed to cross each other without intersecting at a vertex. See Figure 3.3 right.

3 A graph with no direction assigned to its edges is **undirected.**

4 **Notation:** Vertices are denoted by single letters or by numbers, so we can say vertex A or a, or 1, and edges connecting two vertices u and v by either (u, v), u-v, uv, or by a single variable such as e_1. See Figure 3.3.

5 A graph where all pairs of adjacent vertices are connected by only one edge are **simple graphs.** The graph in Figure 3.3 is simple.

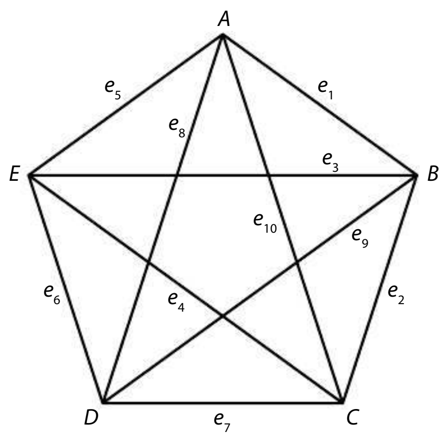
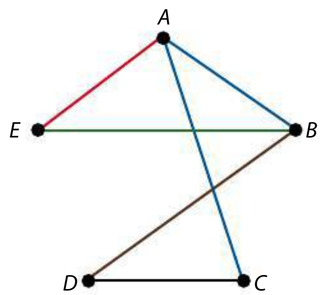
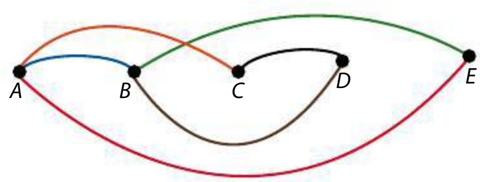


Figure 3.3

Note: In graph theory we do not concern ourselves with the shape of edges or position of the vertices. What is important is which vertices are connected by which edges. The same graph in Figure 3.4 (below) can be represented in different ways, two of which are shown. We consider those two graphs as equivalent.

Figure 3.4





Definition 2

1 Two vertices A and B in an undirected graph G are called **adjacent** (or neighbours) if $u = \{A, B\}$ is an edge in G. The edge u is said to be **incident** with vertices A and B. The edge u is also said to **connect** A and B. A and B are also called the **endpoints** of $\{A, B\}$. Two edges are said to be **adjacent** if they have a vertex in common.

2 If an edge has only one endpoint, then the edge joins the vertex to itself and is called a **loop**.

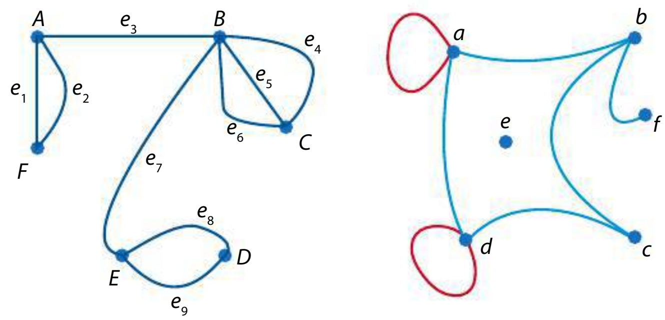
3 If two edges have the same endpoints, they are called **multiple edges** or **parallel edges**.

4 The degree of a vertex in an undirected graph is the number of edges incident with it. The loop, however, contributes two degrees to the vertex it is incident with. The degree of a vertex a is denoted by deg(a). A vertex with degree 0, is said to be **isolated** and a vertex with degree 1 is **pendant**. Vertices with odd degrees are called **odd vertices** and those with even degrees are **even vertices**.

Example 1

Identify the elements of the two graphs below.

a) b)



Solution

a) A is adjacent to B and F, while F and B are not adjacent. B is adjacent to C and E but not to D.

e_1 is incident with F and A, and so is e_2. e_1 and e_2 are multiple (parallel) edges. Also, e_4, e_5, and e_6 are multiple (parallel) edges, as are e_8 and e_9. There are no loops. Deg$(A) = 3$, deg$(B) = 5$, and deg$(E) = 3$. A, B, C, and E are odd, while F and D are even. e_1 and e_3 are adjacent since they have A as a common vertex. e_6 and e_7 are also adjacent.

b) a and d have loops incident with them. Deg$(a) = 4$, with 2 degrees from the loop! Edges cd and cb are adjacent since they have vertex c in common. Vertex e with deg$(e) = 0$ is isolated while vertex f with deg$(f) = 1$ is pendant.

Now we give a formal definition of a simple graph.

Definition 3

A **simple graph** $G = (V, E)$ is a graph that contains no loops or parallel edges. If there is more than one edge adjacent to two vertices, the graph is called a **multiple graph** or a **multigraph**.

For instance, the graphs in Example 1 above are multigraphs while the graphs in Figures 3.2, 3.3, or 3.4 are simple.

Theorem 1 (The handshaking theorem)

Let $G = (V, E)$ be a graph with e edges, i.e. $|E| = e$. Then the sum of all degrees of the vertices in V is twice the number of edges. That is,

$$\sum_{v \in V} \deg(v) = 2e.$$

Note: This applies even if the graph is a multigraph.

Proof

Every edge contributes 2 to the sum of the degrees of the vertices, since every edge is incident with exactly two vertices (they may be equal!). So by adding all the vertex degrees we count each edge twice.

For instance, in Example 1, graph a) has 9 edges and $3 + 5 + 3 + 2 + 3 + 2 = 18$ degrees. Graph b) has 7 edges and $4 + 3 + 2 + 4 + 0 + 1 = 14$ degrees.

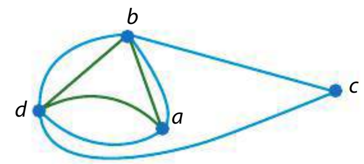
This is called the **Handshaking theorem**, because of the resemblance between an edge having two endpoints and a handshake involving two hands!

Example 2

In a graph with four vertices a, b, c, and d, the degrees are as follows: $\deg(a) = 4$, $\deg(b) = \deg(d) = 5$, and $\deg(c) = 2$. Is this graph possible? If yes, draw a representation, and if not, justify why not.

Solution

Since the sum of the degrees is 16, there is a possible graph with $16/2 = 8$ edges. On the right is a demonstration of such a graph.



Theorem 1 gives rise to another important theorem.

Theorem 2

An undirected graph $G = (V, E)$ can only have an even number of odd vertices.

Proof

The degree of a vertex is either odd or even. Let V_O consist of all odd vertices in V, and V_E consist of all even vertices in V.

Since $V = V_O \cup V_E$ and $V_O \cap V_E = \varnothing$, then

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_O} \deg(v) + \sum_{v \in V_E} \deg(v).$$

Since $2e$ is even, the right-hand side of the equation must be even. Also, the even vertices will have an even sum! Thus, the odd vertices can only have an even sum since the sum of odd numbers cannot be even, and since all the terms in this sum are odd, there must be an even number of them. Thus, there is an even number of odd vertices.

Example

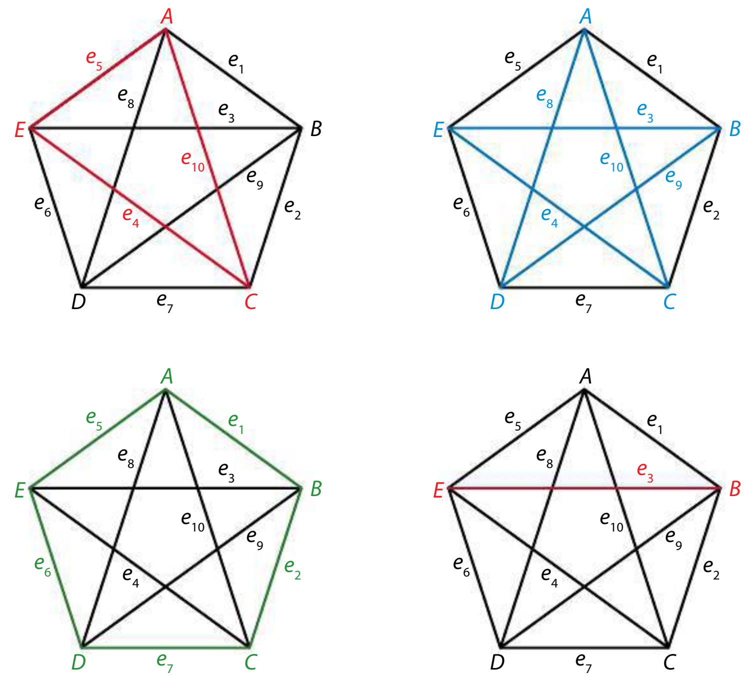
In Figure 3.2, the graph has two odd vertices, S_1 and E; in Figure 3.4, A and B are odd vertices; in Example 1, A, B, C, and E are the odd vertices in graph a), while b and f are odd in graph b); and finally, in Example 2, b and d are the odd vertices.

Definition 4: Subgraphs

Given that $G = (V, E)$ is a **graph**, then, $G_1 = (V_1, E_1)$ is called a **subgraph** of G if $V_1 \subseteq V$, $E_1 \subseteq E$, and $V_1 \neq \varnothing$.

Example

The following are subgraphs of the graph in Figure 3.3. The subgraphs are coloured to distinguish them from the parent one.

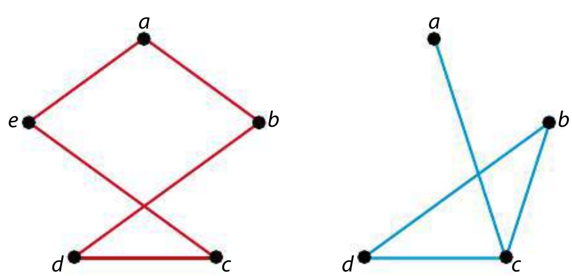


Definition 5: Union (optional)

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example 3

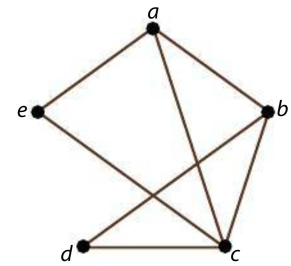
Find the union of the graphs G_1 and G_2 shown below.



G_1 G_2

Solution

The vertex set of the union $G_1 \cup G_2$ is the union of the two vertex sets. So, $E = E_1 \cup E_2 = \{a, b, c, d, e\}$. The edge set is the union of the two edge sets, i.e. $V = V_1 \cup V_2 = \{ae, ab, ac, bc, bd, cd, ce\}$. The union is displayed on the right.



$G_1 \cup G_2$

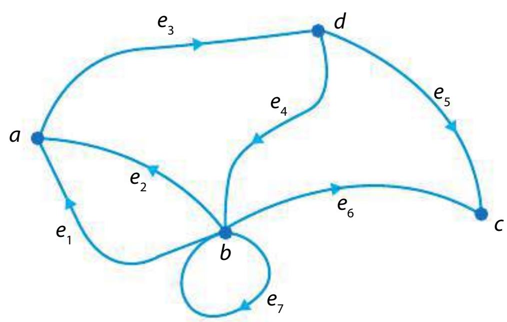
Some special graphs

So far we have only considered **undirected** graphs. Adding direction to edges gives us a new look at a slightly different graph, the **directed graph** or simply **digraph**. The difference from the previous discussion is that edges in a directed graph have directions. That is, for example, the edge *ab* is not the same as the edge *ba*.

Definition 6: Digraphs

A **directed graph** or **digraph** $G = (V, E)$ consists of two sets: V, a non-empty set of **vertices** (**nodes** or **points**) and E, a set of **ordered** pairs of different elements of V called **edges** (arcs or sides).

Here is a representation of a digraph. Notice that the difference from a graph is that each edge e_i is represented by an arrow rather than simply an arc.



G consists of four vertices a, b, c, and d; and seven arcs: $e_1 = (b, a)$, $e_2 = (b, a)$, $e_3 = (a, d)$, $e_4 = (d, b)$, $e_5 = (d, c)$, $e_6 = (c, b)$, and $e_7 = (b, b)$. Each directed arc has an **initial vertex** and a **terminal vertex**. So, e_3 has a as its initial point and d as its terminal point. e_7 is a loop with the same initial and terminal vertex b. e_1 and e_2 are called **parallel** edges since they have the same initial vertex b and terminal vertex a.

Definition 7: Degrees in digraphs

In a digraph, the **in-degree** of a vertex v, $\deg^-(v)$, is the number of edges with v as their terminal vertex. The **out-degree** of v, $\deg^+(v)$, is the number of edges with v as their initial vertex.

Note: According to the definition, a loop contributes one in-degree and one out-degree for the vertex.

In the graph for a digraph on the previous page, for example, $\deg^-(a) = 2$ and $\deg^+(a) = 1$. Also, $\deg^-(b) = 2$ [one degree from e_4 and one from e_7], while $\deg^+(b) = 4$. Moreover, $\deg^-(c) = 2$ and $\deg^+(c) = 0$.

Theorem 3

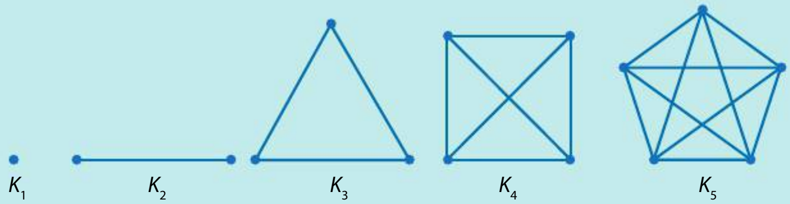
In a digraph $G = (V, E)$, $|E| = \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v)$.

Proof

Since each edge has an initial vertex and a terminal vertex, the sum of the in-degrees is the same as the number of edges. The same is true for the out-degrees.

Definition 8: Complete graphs

A **simple graph** $G = (V, E)$ is called a **complete graph** if for all $a, b \in V$ there is an edge $\{a, b\}$. A complete graph with n vertices is denoted by K_n.
Here are the graphs of K_n, where $n = 1, 2, \ldots, 5$.



Theorem 4

The number of edges in a **complete graph** K_n is given by $|K_n| = \frac{n(n-1)}{2}$.

Proof

The number of vertices is n and each edge connects two vertices; therefore, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

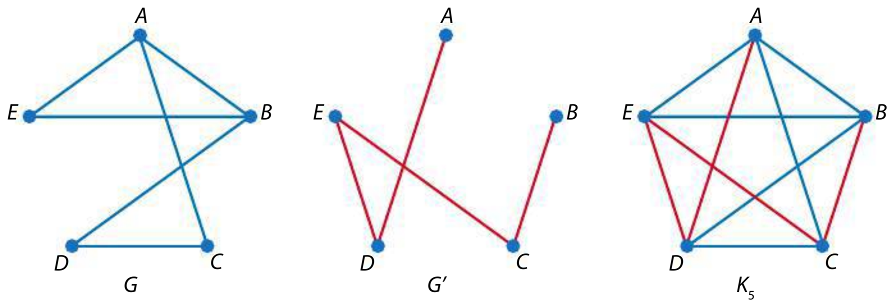
Definition 9: Complement

Let $G = (V, E)$ be a **simple graph**. Then the **complement** of G, denoted by G', is a graph that contains the same set of vertices as the graph G and contains all the edges that are not in G.

When dealing with sets, the complement of a set A is the set containing the elements of the universal set U that are not in the given set itself. The complete graphs here play a similar role to the universal set. The complement of G which has n vertices is the subgraph of K_n consisting of the n vertices in G and all the edges that *are not* in G. So, two vertices are adjacent in G' if and only if they are not adjacent in G.

Example

The graph G presented in the figure below is coloured in blue, while G' is coloured in red.



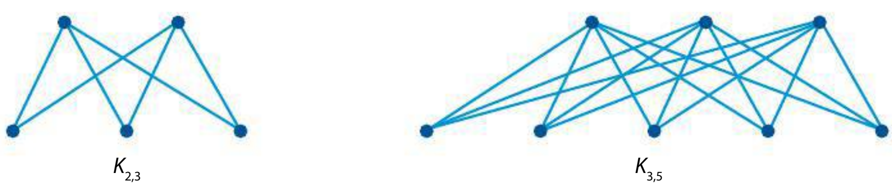
We notice that the graphs G and G' together form a K_5. In some books it is said that those two graphs complement each other to a complete graph.

Another similarity with the complement of a set can be seen here when we look for the complement of K_n. K_n's complement consists of all the vertices and no edges and it is called a **null graph**. This is similar to the case when we look for the complement of U. It is the empty set.

Definition 10: Bipartite graphs

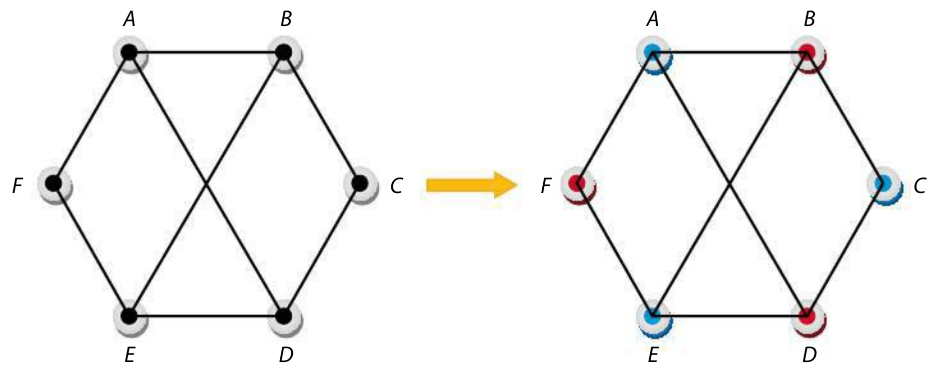
A simple graph $G = (V, E)$ is said to be a **bipartite graph** if the vertex set V can be separated into two subsets V_1 and V_2 such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \varnothing$, often called a **partition**, and all the edges for the set E are of the form $\{X, Y\}$ such that $X \in V_1$ and $Y \in V_2$ (no edge in G connects either two vertices in V_1 nor two vertices in V_2).
A bipartite graph is said to be a **complete bipartite graph** if every vertex from V_1 is adjacent to every vertex from V_2. The most common notation of a complete bipartite graph is $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$.

Here are some examples of complete bipartite graphs.

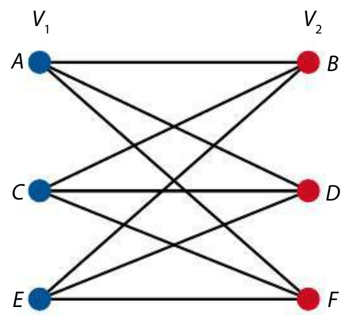


Example

The graph on the following figure (page 1588) is bipartite. As we carefully investigate it we notice that the vertices can be split into two disjoint sets and no edge connects two vertices from the same set. If we simply colour vertices with different colours (red and blue), we observe that no blue vertex is adjacent to a red vertex; therefore, two possible partitions are $V_1 = \{A, C, E\}$ and $V_2 = \{B, D, F\}$.

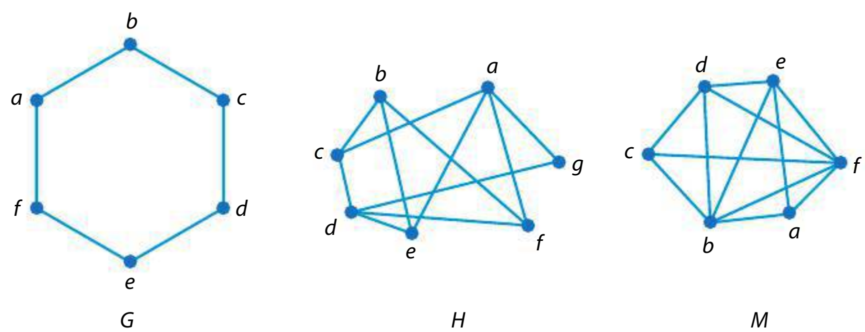


This can be made clearer by rearranging the graph without changing the way the vertices are connected. With this, it becomes obvious that we have a bipartite graph.



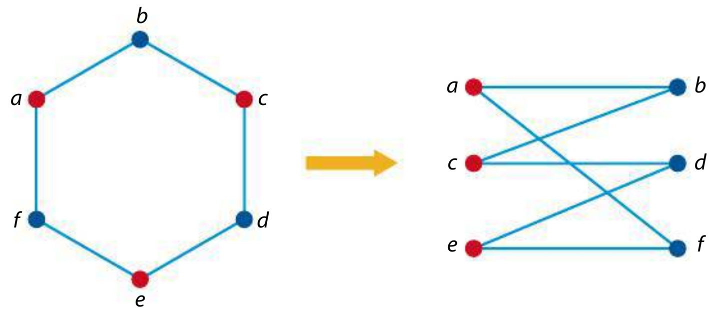
Example 4

Which of the following graphs are bipartite?

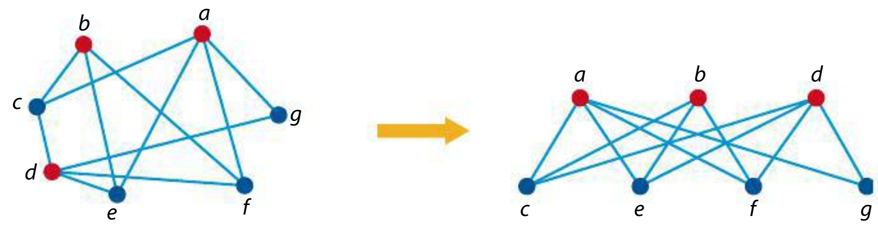


Solution

G: If we colour the vertices with two different colours, we notice that we can do that without any two adjacent vertices sharing a colour. By rearranging the vertices, you can clearly see that we are able to separate them into two sets. So, *G* is bipartite.



H: Doing the same thing here will also yield a bipartite graph.



M: *M* cannot be bipartite. If you consider vertex *b* and vertex *f*, they cannot be in the same subset as they are adjacent. So, they should be in different subsets. Now, *a* can either be in the subset containing *b*, but it cannot since the two are adjacent; or *a* could be in the subset containing *f*, but that cannot happen either.

Exercise 3.1 and 3.2

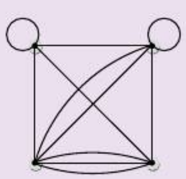
1 For each graph write down:

i the number of vertices

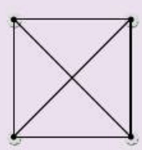
ii the number of edges

iii the degree of each vertex.

a



b



c

2 Consider a group of 5 people at a party. Is it possible for each of them to chat with:

a 3 other people from the group

b 4 other people from the group?

If possible, represent the solution in the form of a graph.

3 What is the minimum number of edges a simple connected graph with n vertices can have?

4 A graph has n vertices. What is the number of edges if the graph is complete?

5 Find the number of vertices and edges for the following graphs:

a $K_{3,4}$ **b** $K_{13,17}$ **c** $K_{m,n}$

6 A complete bipartite graph $K_{m,n}$ has altogether 24 vertices and 128 edges. Find the number of vertices in each partition.

7 A graph is called ***r*-regular** if all the vertices have the same degree r.

a How many vertices does a 3-regular graph have if it has 12 edges?

b Is it possible to have a regular simple graph with 14 edges? Explain your solution.

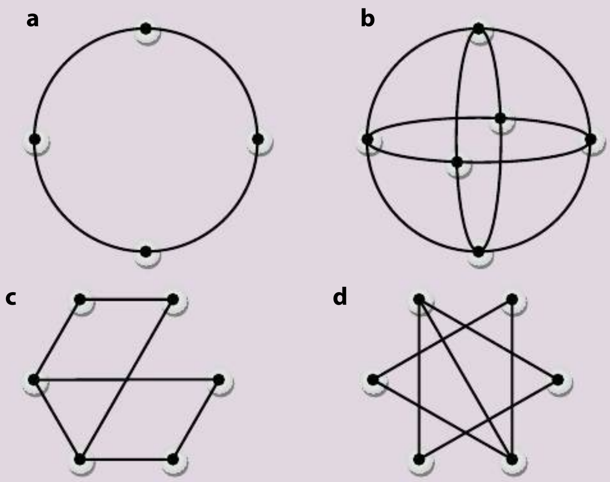
c How many regular simple graphs are there with p edges, where p is a prime number?

d If the number of edges in a graph is e and vertices v, show that, if the graph is simple and connected, then $v-1 \leqslant e \leqslant \dfrac{v(v-1)}{2}$.

8 Show that in a simple connected graph there are at least two vertices of the same degree.

9 Prove that any subgraph of a bipartite graph must be bipartite.

10 Explain which of the following graphs are bipartite:



11 A graph with $v = 7$ has the following vertex degrees: 2, 3, 3, 3, 4, 4, 5. What is the number of edges of this graph?

12 In each of the following, determine whether it is possible to have a simple graph. If yes, draw it. If not, explain why not.

a Number of vertices $v = 5$, vertex degrees: 1, 3, 3, 4, 4

b Number of vertices $v = 6$, vertex degrees: 1, 3, 3, 4, 4, 5

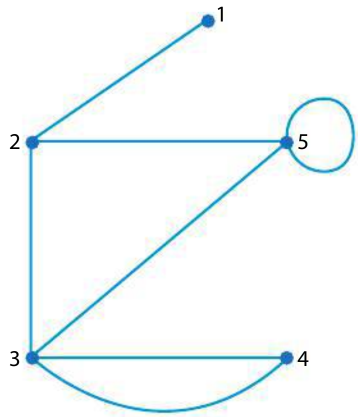
c Number of vertices $v = 6$, vertex degrees: 1, 2, 2, 3, 3, 3

3.3 Graph representation

Diagrams are very helpful and useful in representing graphs and sometimes they are the best way to understand them. However, there are other methods used to represent graphs and a few of these may at times be more convenient. In this section we will see how we can represent graphs in different ways.

Adjacency matrices

For any graph, we can store information about the number of edges connecting each pair of vertices in matrix form. Consider the graph given below with the matrix at the right.



$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} 1 \rightarrow \\ 2 \rightarrow \\ 3 \rightarrow \\ 4 \rightarrow \\ 5 \rightarrow \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

The syllabus does not mention matrices. The term used is 'adjacency tables', which is an equivalent but is not universally used. In this publication, we will continue to use the adjacency matrix notation. In some cases, the table (matrix) may use 'T' for 1 and 'F' for 0.

Every row corresponds to a vertex and every column corresponds to a vertex too. The entries in each row correspond to the number of edges connecting that vertex to the vertices represented by the columns. For example, row 1 has only 1 in the second entry. This is because there is one edge connecting vertex 1 to vertex 2. Row 3, for another example, has 0 in the first entry because vertex 3 has no edges with vertex 1 (i.e. they are not adjacent), has 1 in entry 2 because there is 1 edge connecting vertex 3 to vertex 2, and has 2 in entry 4 because there are 2 edges connecting vertices 3 and 4. Notice that row 5 has an entry corresponding to column 5, because there is a loop at vertex 5.

The following definition formalizes the idea and introduces some notation.

Definition 11

The **adjacency matrix A_G** of a **simple graph** $G = (V, E)$ with n vertices is an $n \times n$ matrix containing 1 or 0 in such a way that any entry of the matrix

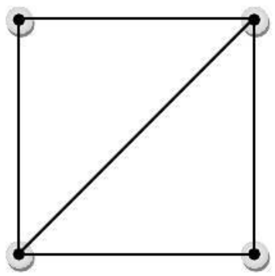
$$a_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge from } E \\ 0 & \text{otherwise} \end{cases}$$

Note: For a multigraph, the definition can be adjusted to reflect the fact that there could be more than one edge between two vertices. So, for a multigraph, we can say that the adjacency matrix has the property

$$a_{i,j} = \begin{cases} k(i, j) & k = \text{number of edges between } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}$$

Example 5

a) Use an adjacency matrix to represent the given graph.



b) Draw a graph represented by the given adjacency matrix.

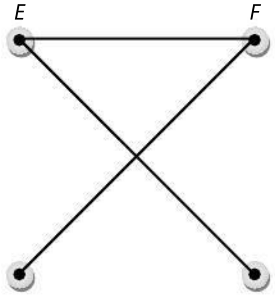
$$B_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Solution

a)

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

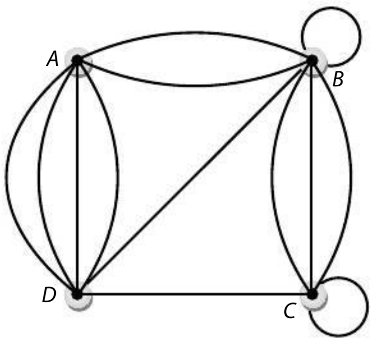
b)



We notice that in a simple graph all the entries on the main diagonal of its adjacency matrix are 0. This is the case since there are no loops in a simple graph. The matrix is also symmetric about its main diagonal since the simple graph is not a digraph, and thus when there is an edge between v_i and v_j this contributes 1 to the (i, j) entry. Similarly, when the edge is between v_j and v_i, this contributes 1 to the (j, i) entry. In the case of a multigraph that contains loops and multiple edges, the entries on the leading diagonal will be 1 if there is a loop at that vertex, whilst multiple edges will contribute correspondingly to a non-diagonal, and hence the matrix may not be symmetric.

Example 6

Use an adjacency matrix to represent the following multigraph.



Solution

$$A_G = \begin{pmatrix} 0 & 2 & 0 & 4 \\ 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 1 \\ 4 & 1 & 1 & 0 \end{pmatrix}$$

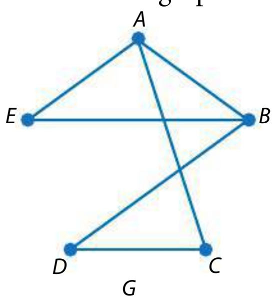
Note: We notice that adjacency matrices of complete graphs have all entries equal to 1 except on the main diagonal where they are all 0. For example,

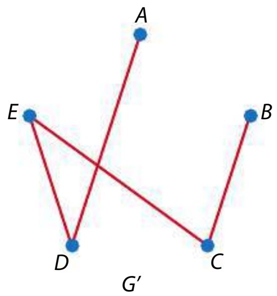
the adjacency matrix of K_3 is $A_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Note: The adjacency matrices of complementary graphs each have the main diagonal as 0, but all the other entries are complementary 1 and 0. That means whenever there is a 1 in one matrix it is 0 in the other matrix and vice versa, apart from the main diagonal, of course. When we add them we obtain an adjacency matrix of a complete graph.

Example 7

Consider the graphs G and G' below and write their adjacency matrices.





Solution

$$G \Rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \text{and } G' \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Incidence matrices (Optional)

Another way that can be helpful in comparing different graphs to check if they have similar structures is the **incidence matrix**. The incidence matrix consists of n rows corresponding to the vertices that a graph has, and k columns corresponding to the edges that this graph has. The matrix will have a 1 in the entry (i, j) if the edge e_j is incident with the vertex v_i.

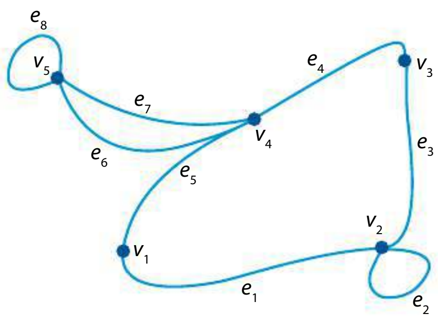
Definition 12

The **incidence matrix** $\boldsymbol{I}_G$ of a **simple** graph $G = (V, E)$ with n vertices and k edges is an $n \times k$ matrix containing 1 or 0 in such a way that any entry of the matrix

$$a_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example 8

Represent the graph shown below with an incidence matrix.



Solution

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left(\begin{array}{ccccc|c|c|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \boxed{1} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} \end{array}\right) \end{array}$$

Notice how multiple edges are represented by columns with identical entries while loops are the only columns with exactly one entry equal to 1.

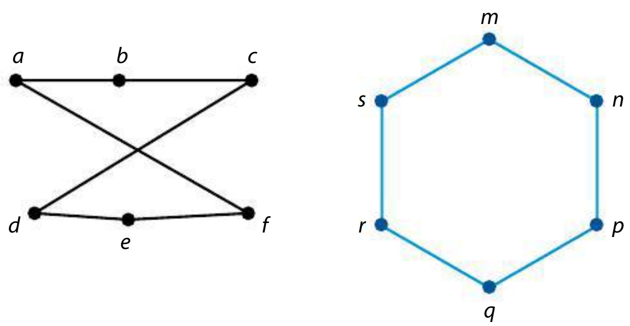
In the case of simple graphs, the row totals give the degree of each vertex of the graph. In multigraphs, however, the entries corresponding to loops should be multiplied by 2 to give the degree of the vertex involved.

Isomorphic graphs

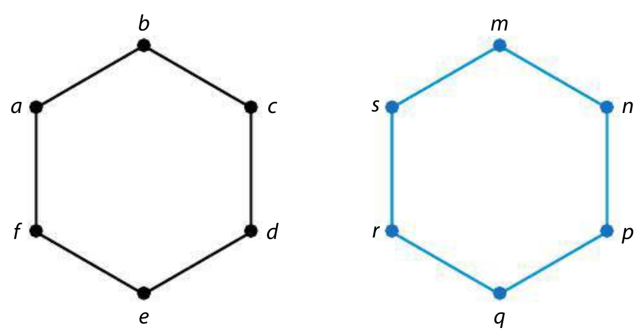
Isomorphic comes from the Greek words ***iso*** (the same as) and ***morphe*** (form).

Try the following experiment with two of your classmates:
Give one of them the following instructions: 'Draw and label the six vertices *a*, *b*, *c*, *d*, *e*, and *f* of a graph *G*. Now connect *a* to *b*, *c* to *b*, *c* to *d*, *d* to *e*, *f* to *e*, and *a* to *f*.' Now give the other the following instructions: 'Draw and label the six vertices *m*, *n*, *p*, *q*, *r*, and *s* of a graph *H*. Now connect *m* to *n*, *n* to *p*, *p* to *q*, *q* to *r*, *r* to *s*, and *s* to *m*.'

An experiment that was performed in one class produced the following two graphs.



You may have noticed already that these two graphs define the same situation. However, they appear to be different. If we rearrange the way we graphed them, you will see that they are equivalent. Here is a rearrangement.



Such graphs are said to be **isomorphic**. You can set up a one-to-one correspondence between the vertices of the two graphs, keeping the adjacent vertices in one graph and the images of the adjacent vertices in the other. For example, here we can match *a* with *m*, *b* with *n*, and so on. This way, any two vertices that are adjacent in one graph have their images adjacent in the same way. We say that the two graphs have the same structure.

Although the syllabus does not include isomorphic graphs, we will still use them here because they help to make some operations more efficient. Obviously this will not jeopardize your chances of earning marks. All sound mathematical methods are acceptable in exams. Moreover, 'isomorphism' is still on the list of terms in the syllabus.

One-to-one correspondence means that the function f is **surjective** and **injective**, i.e. it is a **bijection**.

Definition 13

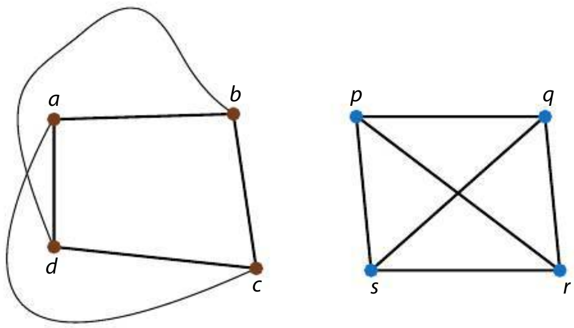
Let $G = (V, E)$ and $G' = (V', E')$ be two **simple graphs**. If there is a one-to-one correspondence $f: V \to V'$, such that for every pair of vertices v_i and v_j that are adjacent in the graph G vertices $f(v_i)$ and $f(v_j)$ are adjacent in G', then the graphs G and G' are said to be **isomorphic**. The function f is called a **graph isomorphism**.

Stated differently, when two graphs are isomorphic, there is a bijection between the vertices of the two graphs that *preserves* the *adjacency* association. In the previous example, the bijection could be defined by

$$g(a) = m, g(b) = n, g(c) = p, g(d) = q, g(e) = r, \text{ and } g(f) = s.$$

Example 9

Consider the graphs G and H given below. Examine whether the two graphs are isomorphic.



Solution

We set up the following function: $f(a) = p, f(b) = q, f(c) = r, f(d) = s$.

This function preserves adjacency as is easily verified, and hence it is an isomorphism. Take the adjacent vertices a and b, for example, $f(a) = p$ is adjacent to $f(b) = q$. The rest can clearly be seen.

Hence, the two graphs can be considered the same, as far as graph structure is concerned.

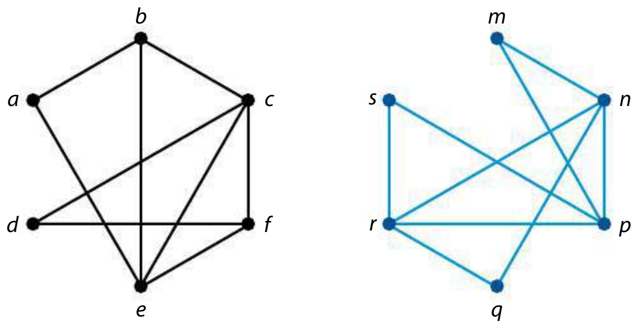
Note: If we set up the adjacency matrices for the two graphs above, we get:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \Leftrightarrow \begin{array}{c} \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

It is important to note that when you arrange the matrices of two isomorphic graphs in such a way that the corresponding vertices occupy the same rows and columns, the adjacency matrices of both are identical, as you see above.

Example 10

Consider the following two graphs and examine whether they are isomorphic.



Solution

If we consider the adjacency matrices for both, we get:

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a\\ b\\ c\\ d\\ e\\ f \end{array} & \begin{pmatrix} 0&1&0&0&1&0\\ 1&0&1&0&1&0\\ 0&1&0&1&1&1\\ 0&0&1&0&0&1\\ 1&1&1&0&0&1\\ 0&0&1&1&1&0 \end{pmatrix} \end{array} \text{ and } \begin{array}{cc} & \begin{array}{cccccc} m & n & p & q & r & s \end{array} \\ \begin{array}{c} m\\ n\\ p\\ q\\ r\\ s \end{array} & \begin{pmatrix} 0&1&1&0&0&0\\ 1&0&1&1&1&0\\ 1&1&0&0&1&1\\ 0&1&0&0&1&0\\ 0&1&1&1&0&1\\ 0&0&1&0&1&0 \end{pmatrix} \end{array}$$

Since these graphs are simple, then the column/row totals are the degrees of each vertex. We can clearly see that the degree sequence of the first graph is 2, 2, 3, 3, 4, 4, while the second graph is 2, 2, 2, 4, 4, 4. This means that we cannot set up a correspondence to preserve adjacency, and hence the two graphs are not isomorphic.

The **degree sequence** of a graph is the list of degrees of the vertices of the graph, listed from smallest (largest) degree to largest (smallest).

Example 10 leads us to the following theorem.

Theorem 5

Let $G = (V, E)$ and $G' = (V', E')$ be two **isomorphic graphs** and $f: V \to V'$ a **graph isomorphism**. If a is any vertex from set V, then $\deg(a) = \deg(f(a))$.

Stated differently, corresponding vertices in an isomorphism must have the same degree.

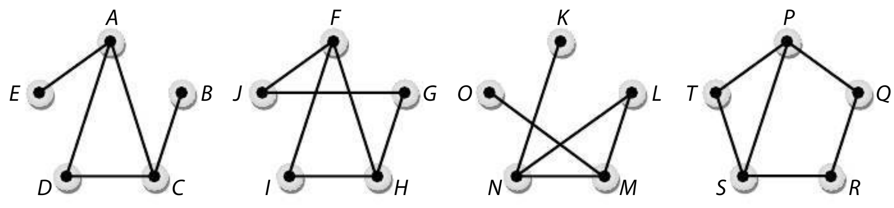
Proof

Assume that $\deg(a) \neq \deg(f(a))$, then we have two cases to consider.

The first case is $\deg(a) > \deg(f(a))$, and if this is true, then there is a vertex b such that b is adjacent to a in G, but $f(b)$ is not adjacent to $f(a)$, which is a contradiction to the definition of **graph isomorphism** f. A similar argument is true for the case when $\deg(a) < \deg(f(a))$. Therefore, $\deg(a) = \deg(f(a))$.

Example 11

Determine which pairs of graphs are isomorphic.



Solution

Looking at a table showing the degrees of the corresponding vertices of the graphs, we can try to construct a graph isomorphism.

Graph	1					2					3					4				
Vertex	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
Degree	3	1	3	2	1	3	2	3	2	2	1	2	3	3	1	3	2	2	3	2

Obviously graphs **1** and **3** have the same degree sequence: 1, 1, 2, 3, 3. Therefore, we would proceed in trying to find an isomorphism between them.

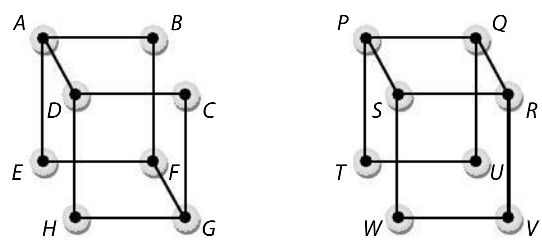
One possible isomorphism between **1** and **3** is $f(A) = M$, $f(B) = K$, $f(C) = N$, $f(D) = L$, $f(E) = O$.

Note that we have to be careful with respect to the vertices with degree one because if we assign $f(A)$ to M then we must assign $f(E)$ to O, since A and E are adjacent in **1**. Another alternative is to assign $f(E)$ to K which would give us a contradiction to Theorem 5, since M and K are not adjacent in graph **3**.

Similarly, **2** and **4** have the same degree sequence: 2, 2, 2, 3, 3. An isomorphism between graphs **2** and **4** could be $g(F) = P$, $g(G) = Q$, $g(H) = S$, $g(I) = T$, $g(J) = R$. Again, here we need to be careful not to assign two adjacent vertices of degree 2 in graph **4** to vertices I and J in **2** which are not adjacent. If we do, we will be violating Theorem 5's conclusion.

Example 12

Determine whether the following pair of graphs are isomorphic.



Graph G Graph H

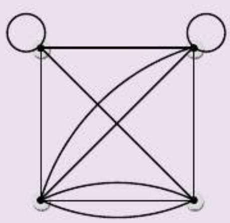
Solution

These two graphs are not isomorphic even though they have an equal number of vertices of degree 2, as well as degree 3. The problem arises with the fact that in graph G all the vertices of degree 2 are adjacent only to vertices of degree 3, while in graph H all the vertices of degree 2 are connected to one vertex of degree 3 and one of degree 2. Let's take one such pair, for example, B and U. Both have a degree of 2. B is adjacent to vertices A and F both of which are of degree 3, while U in graph H is adjacent to Q with degree 3 and T with degree 2. A function that matches vertex B, for example, to vertex U will have to match A and F to T and Q. Since A and F have degree 3, one of them will be matched with T which is of degree 2. This will contradict Theorem 4. Any attempt to set up a correspondence will meet the same obstacle, and therefore there is no isomorphism between graphs G and H.

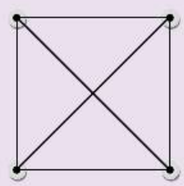
Exercise 3.3

1 For each graph, write down its adjacency matrix.

a



b



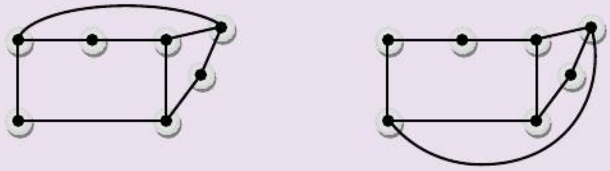
c

2 Draw the graph for each adjacency matrix and determine pairs of isomorphic graphs.

a $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

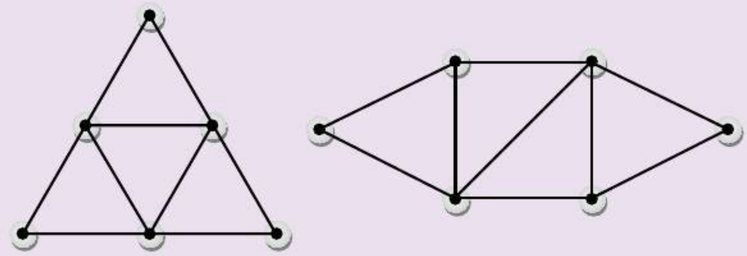
d $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ **f** $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

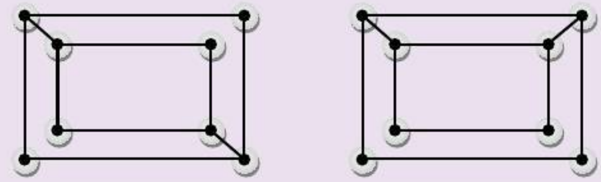
3 Determine whether the following graphs are isomorphic. Explain your answer.

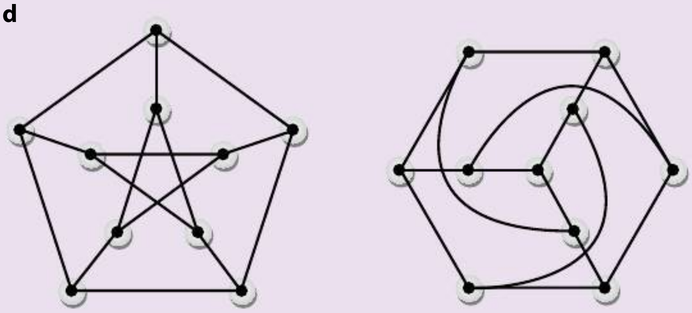


4 Determine whether the following pairs of graphs are isomorphic.

a


b


c


d


5 Draw two non-isomorphic graphs with three vertices and two edges. How many such non-isomorphic graphs are possible?

6 Draw two non-isomorphic graphs with four vertices and three edges. How many such non-isomorphic graphs are possible?

7 Draw all possible non-isomorphic simple regular graphs with four vertices.

3.4 Paths, walks and trails

Many of the applications of graph theory have to do with paths formed by travelling along the edges of graphs. The example of the Königsberg bridges (page 1580) is one of the oldest. Some current applications include network links, how messages travel between different nodes, postal routes, refuse collection, etc.

We will start this section by stating a few additional necessary definitions.

Definition 14: Walks

Let $G = (V, E)$ be a **graph**. A **walk** is a sequence of alternating vertices and edges that starts and ends with a vertex and where each edge is adjacent to its neighbouring vertices. Stated slightly differently, a $v_0 - v_n$ walk in G is a finite alternating sequence

$$v_0, e_1, v_1, e_2, v_2, \ldots, e_{n-1}, v_{n-1}, e_n, v_n$$

of vertices and edges starting at vertex v_0 and ending at vertex v_n and involving the n edges

$$e_i = \{v_{i-1}, v_i\}, \text{ where } 1 \leqslant i \leqslant n.$$

v_0 and v_n do not have to be different.

The **length of a walk**, n, is the number of edges used in the sequence.

Note: A walk may repeat both edges and vertices.

Note: Like several things in graph theory, unfortunately there is still no unique way of labelling walks. For example, if a graph G has the set of vertices $V = \{a, b, c, \ldots\}$, then a walk can be described as

$$a, \{a, b\}, b, \{b, c\}, \ldots$$

or simply as

$$\{a, b\}, \{b, c\}, \ldots$$

or as

$$a, b, c, \ldots$$

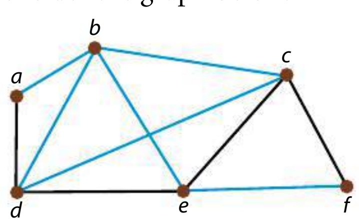
or as

$$abc\ldots$$

We will use the following example to introduce slight variations to the above definition.

Example

Consider the graph below.



The blue coloured walk is the **walk** *abdcbef*. Notice here that vertex b has been visited twice. The length of this walk is 6. No edge has been visited more than once.

The walk *abdcedb* has a length of 6 and uses the edge bd twice and the vertices b and d are used twice.

A walk like the first one is known as a **trail**.

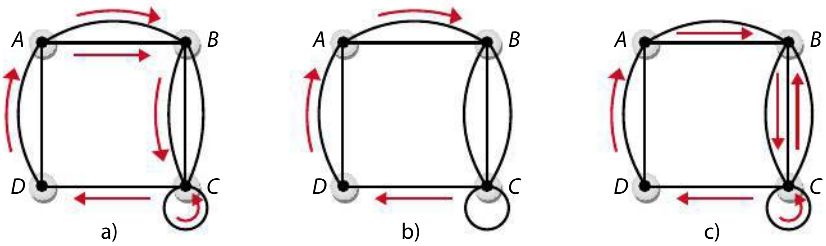
Definition 15

1 A **trail** is a walk in which no edge appears more than once. A trail (like *abcebda*) which begins and ends at the same vertex is called a **circuit**.

2 A walk (like *abef*) where no vertex is visited more than once is called a **path**. A path (like *abceda*) which begins and ends at the same vertex is called a **cycle**.

Example 13

Determine whether each sequence shown is a walk, a path or a trail.



a) $A, \{A, B\}_{\text{lower}}, B, \{B, C\}, C, \{C, C\}_{\text{loop}}, C, \{C, D\}, D, \{D, A\}, A, \{A, B\}_{\text{upper}}, B$

b) $C, \{C, D\}, D, \{D, A\}, A, \{A, B\}_{\text{upper}}, B$

c) $C, \{C, C\}_{\text{loop}}, C, \{C, D\}, D, \{D, A\}, A, \{A, B\}_{\text{upper}}, B, \{B, C\}_{\text{middle}}, C, \{C, B\}_{\text{middle}}, B$

Solution

a) The sequence is a trail since no edge has been repeated. Starting at vertex A to vertex B we used the lower edge, while at the end of the sequence again from vertex A to vertex B we used the upper edge. This sequence cannot be a path since vertices C, A, and B have been repeated.

b) The sequence is a path since no vertex has been repeated.

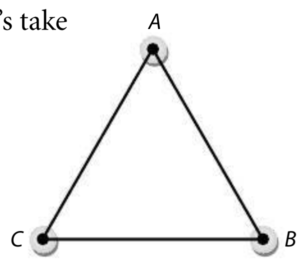
c) The sequence is a walk, since it cannot be a trail as the middle edge from B to C has been repeated twice.

Note: Every path is a trail, while a trail can be a path only in a simple graph.

Adjacency matrices and walks

Adjacency matrices can be very useful in determining the number of possible walks in a graph. Let's take a K_3 and its adjacency matrix for example.

$$A_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$



The adjacency matrix also represents walks of length 1.

How many different walks of length 2 can we have in K_3?

We observe that since this graph is regular, all the vertices will be treated equally. Start walking from A and note where we can arrive after travelling through two edges:

> A **regular graph** is a graph where all vertices have the same degree.

$A, \{A, B\}, B, \{B, C\}, C$ $\quad$ $A, \{A, B\}, B, \{B, A\}, A$

$A, \{A, C\}, C, \{C, B\}, B$ $\quad$ $A, \{A, C\}, C, \{C, A\}, A$

We notice that two walks of length 2 will end up back at A, while only one walk of length 2 will end up at B or C.

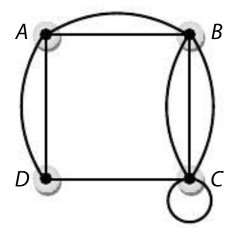
Now, look at the square of the adjacency matrix:

$$A_{K_3}^2 = A_{K_3} \cdot A_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

We notice that the entries are the number of walks of length 2 in K_3. Two walks from each vertex back to the same vertex and one walk from each vertex to each of the other two.

Example

Consider the multigraph given right.



Its adjacency matrix is $A_G = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$ and the square of the matrix is $A_G^2 = \begin{pmatrix} 8 & 0 & 8 & 0 \\ 0 & 13 & 3 & 7 \\ 8 & 3 & 11 & 1 \\ 0 & 7 & 1 & 5 \end{pmatrix}$.

Here, for example, the matrix suggests that there are eight walks of length 2 from A to C. We will not list them, we will just explain how to find them. There are 2 edges from A to B and then 3 edges to get from B to C. Therefore, by the counting principle, there are $3 \times 2 = 6$ walks from A to C through B. On the other hand, there are 2 edges from A to D and only 1 edge from D to C. Therefore, there are 2 ways from A to C through D. Now, the total number of walks from A to C is then $6 + 2 = 8$, which is suggested by the matrix. On the other hand, it looks like there are so many walks of length 2 from C back to itself. There are 3 edges to B and 3 edges back, and therefore nine walks through B altogether. There is only one walk to D and back. At the end there is a loop at C; therefore, if we go through it twice that is the last possible walk, which sums up to 11.

To summarize both generalizations we will state the following theorem.

Theorem 6

Let G be a graph containing v vertices and A_G be its adjacency matrix. The number of walks of length n from vertex v_i to v_j is given by the (i, j)th entry of A_G^n, $n \in \mathbb{Z}^+$.

Proof

We will conduct the proof by using mathematical induction on n.

Basis step: Every entry in the adjacency matrix is the number of edges from A_i to A_j; therefore, walks of length 1. The statement is true for $n = 1$.

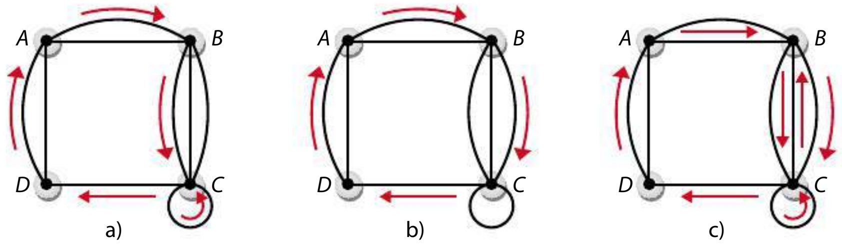
Inductive step: We will assume that every entry of matrix A_G^k is the number of walks of length k between two vertices. Since $A_G^{k+1} = A_G^k \cdot A_G$ then the (i, j)th entry of the matrix A_G^{k+1} is calculated in the following way:

$c_{ij} = b_{i1} \times a_{1j} + b_{i2} \times a_{2j} + \ldots + b_{in} \times a_{nj}$, where b_{ik} is the number of walks of length k from vertex v_i to v_k, and a_{ki} is the number of walks of length 1 from vertex v_k to v_j, giving the total number of walks of length $k + 1$ from vertex v_i to v_j through the vertex v_k. When we add up all the walks from vertex v_i to v_j through different vertices v_k, we get the total sum of all possible walks of length $k + 1$ from vertex v_i to v_j.

Conclusion: Since the statement is true for $n = 1$ and $S(k) \Rightarrow S(k + 1)$, by the principle of mathematical induction, we can conclude that the statement is true for all $n \in \mathbb{Z}^+$.

Example 14

Determine whether each sequence shown below is a closed walk, a cycle or a circuit.



a) $C, \{C, C\}_{\text{loop}}, C, \{C, D\}, D, \{D, A\}, A, \{A, B\}_{\text{upper}}, B, \{B, C\}_{\text{left}}, C$

b) $D, \{D, A\}, A, \{A, B\}_{\text{upper}}, B, \{B, C\}_{\text{right}}, C, \{C, D\}, D$

c) $A, \{A, B\}_{\text{lower}}, B, \{B, C\}_{\text{middle}}, C, \{C, B\}_{\text{right}}, B, \{B, C\}_{\text{middle}},$
$C, \{C, C\}_{\text{loop}}, C, \{C, D\}, D, \{D, A\}, A$

Solution

a) The sequence is a circuit since it is closed and no edge has been repeated. This sequence cannot be a cycle because of the loop at C.

b) The sequence is a cycle since it is closed and no vertex has been repeated.

c) The sequence is a closed walk, since it cannot be a circuit as the middle edge from B to C has been repeated twice.

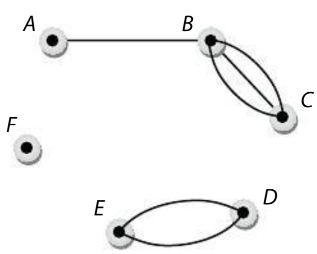
Every cycle is always a circuit, while a circuit can be a cycle only in a simple graph.

Definition 16

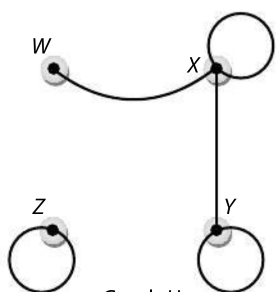
Let V be a non-empty set of **vertices** and E be a non empty set of **edges**. The graph $G = (V, E)$ is called a **connected graph** if there is a **path** between any two vertices from the set V.

Example

The graphs presented by all the figures so far are connected. The following graphs G and H are not connected since they contain vertices or even subgraphs that are isolated. Note that in the case of the vertex Z, even though it is isolated, the degree is not equal to zero unlike the vertex F.



Graph G



Graph H

The graphs G and H have the following adjacency matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We notice that the adjacency matrix of a graph that is not connected contains only zeroes in a row and a column of the isolated vertex, or contains only one 1 at the diagonal position in that row or column. On the other hand, disconnected subgraphs can be shown as diagonal matrices where all the other entries are zeroes.

Properties of connected graphs

We will state some properties of connected graphs that will be helpful in later discussions. However, they are not required for examination purposes and their proofs are not supplied in this publication.

Property 1

Let $G = (V, E)$ be a **simple connected graph**, and let a and b be two vertices in G that are not adjacent. If a graph G_1 is formed by *adding* the edge ab to G, then G_1 has a cycle that contains the edge ab.

Property 2

When an edge is removed from a cycle in a connected graph, the result is a graph that is still connected.

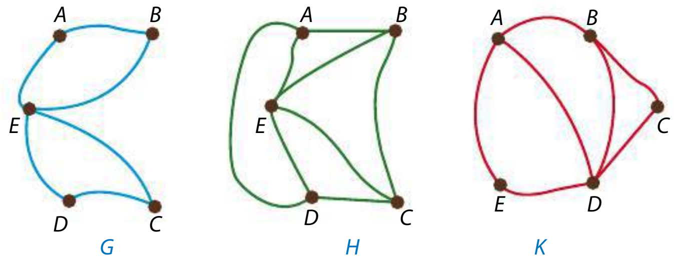
Eulerian graphs

Definition 17

Let $G = (V, E)$ be a **connected graph**. A **trail** where every edge of G appears only once is called an **Eulerian trail**. A **circuit** where every edge of G appears only once is called an **Eulerian circuit**. A connected graph with an Eulerian circuit is called an **Eulerian graph**.

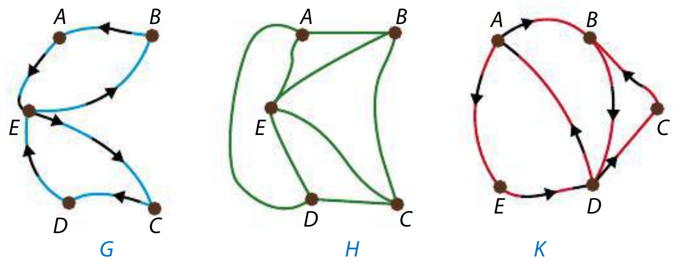
Example 15

Which of the undirected graphs below have an Eulerian circuit? Which have an Eulerian trail only?



Solution

Graph G has an Eulerian circuit. Look at *AECDEBA*, for example.



You can verify that H has neither an Eulerian circuit nor trail. You will be able to confirm this later in the chapter.

Graph K does not have an Eulerian circuit, but it has an Eulerian trail, *AEDCBDAB*.

Theorem 7

Let $G = (V, E)$ be a **connected graph**. G has an **Eulerian circuit** if and only if every vertex has an *even* degree.

Proof

($\Rightarrow$) Suppose G has an Eulerian circuit. This means the circuit starts at a vertex v_0 (say) and continues with an edge v_0v_1 incident to it, and carries on with the rest of the vertices until it gets back to v_0, i.e. $v_0, v_1, v_2, v_3, \ldots, v_{n-1}, v_0$. Now, v_0v_1 contributes one degree to v_0 and one degree to v_1, but v_1v_2 contributes another degree to v_1, which implies that the circuit contributes two degrees to every vertex it visits. Also, $v_{n-1}v_0$ contributes another degree to v_0, making the total for v_0 at least 2 degrees. Thus, the degree of every vertex, including v_0, is an even integer.

($\Leftarrow$) Conversely (a short argument that can be expanded), if we assume that each vertex has an even degree, then the circuit can visit each vertex through one edge and leave it using another unused edge. Thus, we can form an Eulerian circuit since the graph is connected.

Example

Refer to Example 15. Graph G has $\deg(A) = \deg(B) = \deg(C) = \deg(D) = 2$, and $\deg(E) = 4$. That is why G is Eulerian.

Graph H has $\deg(A) = \deg(B) = \deg(C) = \deg(D) = 3$, and $\deg(E) = 4$. Only one of the vertices is even while the rest are all odd; thus it cannot be Eulerian.

Graph K has $\deg(C) = \deg(E) = 2$, and $\deg(D) = 4$, while $\deg(A) = \deg(B) = 3$. This is why it does not have an Eulerian circuit. We know however that it has an Eulerian trail. This can be confirmed using the following theorem.

Theorem 8

Let $G = (V, E)$ be a **connected graph**. G has an **Eulerian trail** but not an Eulerian circuit, if and only if it has *exactly two* vertices of *odd* degree.

Proof

($\Rightarrow$) Suppose G has an Eulerian trail. This means the trail starts at a vertex v_0 (say) and continues with an edge v_0v_1 incident to it, and carries on with the rest of the vertices until it gets to v_n, i.e. $v_0, v_1, v_2, v_3, \ldots, v_n$. Now, v_0v_1 contributes one degree to v_0 and one degree to v_1, but v_1v_2 contributes another degree to v_1, which implies that the trail contributes two degrees to every (internal) vertex it visits. However, since it stops at v_n, then it only contributes one degree to v_n. Thus, the degree of exactly two vertices is odd.

($\Leftarrow$) Conversely, suppose G contains exactly two vertices of odd degree, say v_0v_n. Now, add a new (auxiliary) edge v_0v_n to the graph and the result will be a new graph G_1 with all even degrees. Hence, G_1 has an Eulerian circuit. Removing the auxiliary edge from the circuit leaves you with a trail.

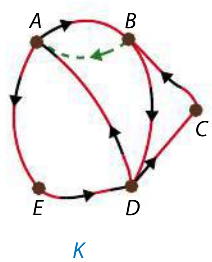
Note: An Eulerian trail must begin and end with a vertex of odd degree!

This is an informal approach to Theorems 7 and 8.

Consider an Eulerian walk W as a sequence of edges $e_1e_2e_3, \ldots, e_n$. Consider a vertex v. Each edge incident with v is used exactly once in the walk. Say v is not the first or last vertex of the walk. Let's walk along W. Each time we arrive at v, say along edge e_i, we must leave along edge e_{i+1}. Thus, each time we visit v we use two edges. Say the number of times we visit v is k. Then v has degree $2k$, an even number. What if v is the first or last vertex? Then the same reasoning applies except for the first or last edge in the walk. If the walk is closed (circuit), then the first and last edge both visit v and we still have an even number. If the walk is open (trail), then either the first or last edge visits v, but not both and we see that v has an odd degree. Thus, the first and last vertices of W have odd degree and we have two vertices of odd degree.

Example

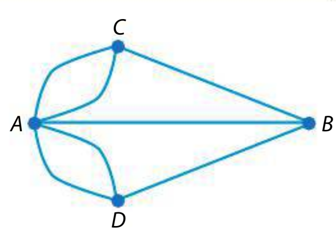
Consider the graph K in Example 15.



By adding an edge BA, we are able to have the circuit $AEDCBDABA$. By removing the edge BA, we get the trail $AEDCBDAB$.

Example 16

Consider the Königsberg bridge problem again (page 1580).
Can we solve it?



Solution

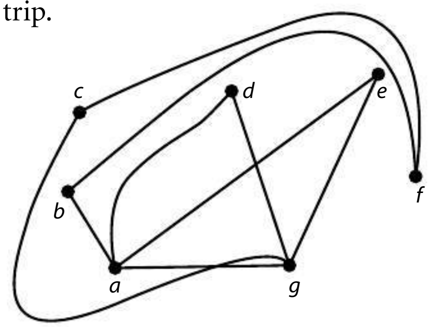
Notice here that $\deg(B) = \deg(C) = \deg(D) = 3$, and $\deg(A) = 5$.

Thus, by Theorems 7 and 8, no Eulerian circuit is possible in such a graph, nor an Eulerian trail.

The next example will offer a way in which an Eulerian circuit can be constructed in an Eulerian graph.

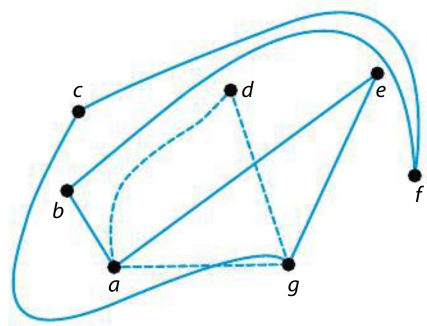
Example 17

The vertices in the following graph are the roads connecting several cities that you want to visit on a short holiday. You don't want to use the same road twice and you want to return home to city *a*. Find a route for your trip.



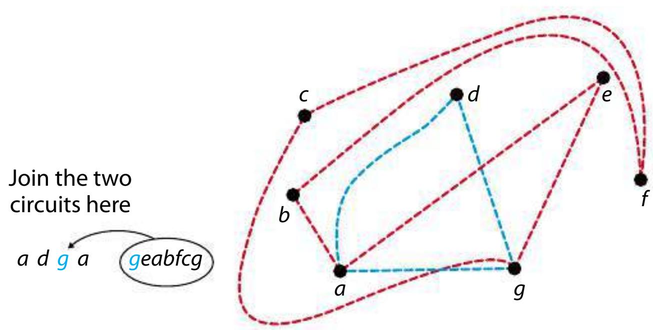
Solution

This is asking you to find an Eulerian circuit for the given graph.



This is an Eulerian graph since all vertex degrees are even.

First construct a circuit C beginning with a (say); *adga* is such a circuit. Since it does not include all edges, it is not Eulerian. Next, look for a vertex in C that is adjacent to a non-used edge; a and g are such vertices. Beginning with g, for example, construct a circuit using unused edges; *geabfcg* is such a circuit. Use a broken line as before.



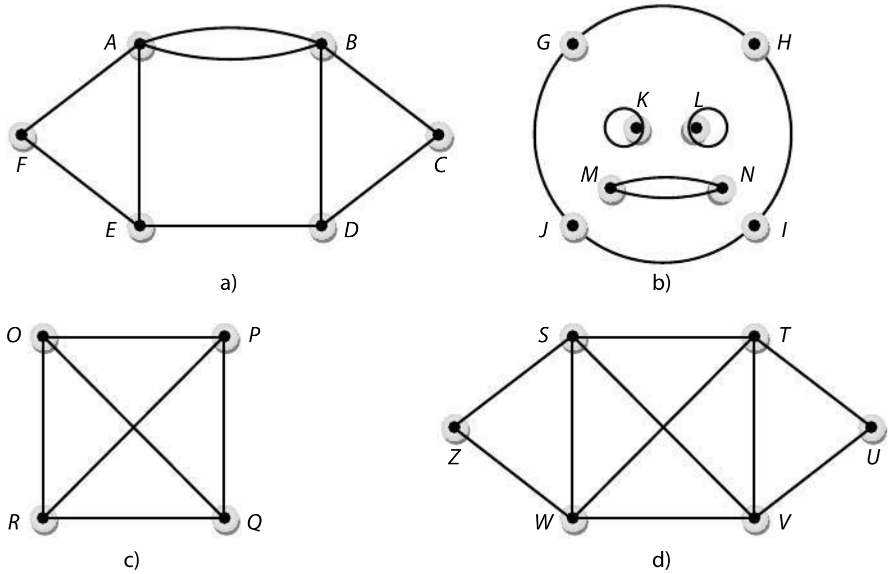
Since no more solid edges remain, the procedure stops here. To combine the two circuits, join them at vertex g where the second circuit started.

Thus, the Eulerian circuit for the graph is

adgeabfcga.

Example 18

In which of the following graphs is it possible to find an Eulerian trail or an Eulerian circuit? When possible, find an example of the trail or circuit. When not possible, explain the reasons for the absence of an Eulerian trail or circuit.



Solution

a) Looking at vertex degrees, we have:

$\deg(A) = \deg(B) = 4$, $\deg(C) = \deg(F) = 2$ and $\deg(D) = \deg(E) = 3$

Since two vertices have odd degrees, it is possible to find a trail.

We need to start from a vertex of an odd degree, so one possible Eulerian trail would be:

$D, \{D, C\}, C, \{C, B\}, B, \{B, D\}, D, \{D, E\}, E, \{E, F\}, F, \{F, A\}, A, \{A, B\}_{\text{upper}}, B, \{B, A\}_{\text{lower}}, A, \{A, E\}, E.$

b) Even though all vertices are of an even degree (2) the graph is not connected; therefore, it is not possible to find either an Eulerian trail or an Eulerian circuit.

c) All the vertices are of the same degree (3), so it is not possible to find either an Eulerian trail or an Eulerian circuit.

d) Looking at vertex degrees we have:

$\deg(S) = \deg(T) = \deg(V) = \deg(W) = 4$ and $\deg(U) = \deg(Z) = 2$

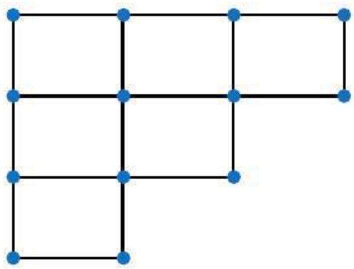
Thus, it is possible to find a circuit. We can start from any vertex, so one of the possible Eulerian circuits would be:

STVWSVUTWZS.

If we apply the algorithm presented in Example 17 above, we can start with a circuit *SZWS*, for example. Then *WTVW*, and lastly *VUTSV*. Now we join the first two at *W*, getting a new circuit *SZWTVWS*. Lastly, we join this circuit with the third one at *V*, thus getting *SZWTVUTSVWS* as our Eulerian circuit.

Hamiltonian graphs

Below is a graph where the vertices represent locations of postal boxes where mail has to be picked up every day. Postal services must find a route so that mail can be picked up from each of these boxes. Would an Eulerian circuit suffice for this job?



The answer is No! An Eulerian circuit would not provide a good solution since the primary goal is simply visiting each vertex rather than travelling each edge. In this problem, it would be very inefficient to require each edge to be travelled since this would force multiple visits to the same vertex.

In general, Eulerian circuits/paths are not the appropriate tool for analyzing problems where it is only important to visit each vertex. For problems of this type, whether an edge is travelled is not important.

Remember, Eulerian circuits/paths deal with situations where it is important to travel every edge.

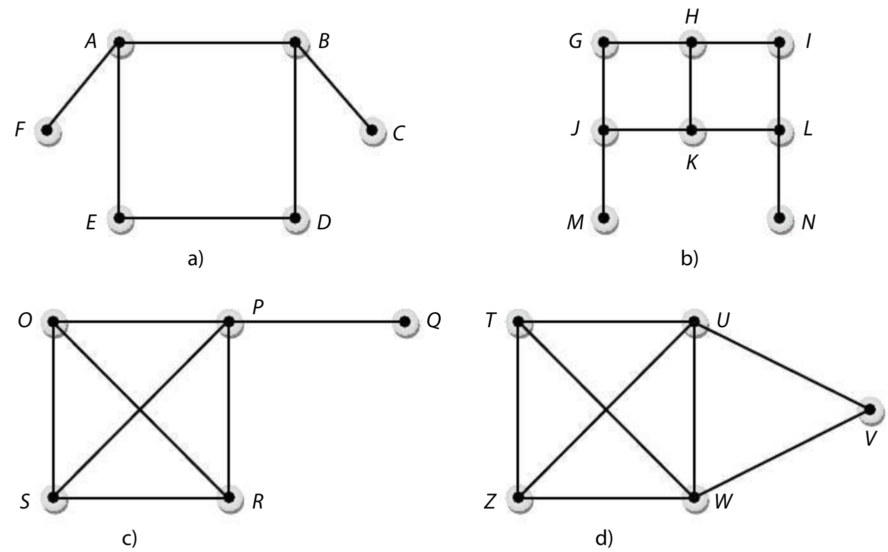
We have found some conditions for the existence of trails and circuits containing all the edges of a graph only once. Can we do a similar task with vertices? Is it possible to find a path or a cycle that contains all the vertices in a given graph?

Definition 18

Let $G = (V, E)$ be a **connected graph**. A **path** that contains all vertices of G is called a **Hamiltonian path**. A **cycle** that contains all vertices of G is called a **Hamiltonian cycle**. A connected graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.

Example 19

In which of the following graphs is it possible to find a **Hamiltonian path** or a **Hamiltonian cycle**? When possible, find an example of the path or cycle; and when not, explain the reasons for the absence of a Hamiltonian path or cycle.



Solution

a) Two vertices have a degree of 1, so if we leave either of these two vertices we cannot come back to them; therefore, it is not possible to find a cycle. A possible Hamiltonian path would be:

$F, \{F, A\}, A, \{A, E\}, E, \{E, D\}, D, \{D, B\}, B, \{B, C\}, C.$

b) It is not possible to find a Hamiltonian cycle because there are two vertices of degree 1. Neither is it possible to find a Hamiltonian path since at the end there are two non-adjacent vertices that we need to connect.

c) There is only one vertex of a degree of 1; therefore, it is not possible to find a cycle. A possible Hamiltonian path would be *QPOSR*.

d) It is possible to find a Hamiltonian cycle. We can start from any vertex, so one such possible cycle would be *VUZTWV*.

Unlike the situation with Eulerian trails and circuits, there is no well-known test, or listing of requisites, that can be employed to establish whether a graph contains a Hamiltonian path or cycle. In its place, there are some negative tests, which can explain that a certain graph cannot contain such a cycle or path. There are some theorems that establish either necessary conditions or sufficient conditions for a graph to have a Hamiltonian path or cycle. We will examine some of these in the following pages. When faced with certain graphs, however, we will time and again resort to trial and error.

Theorem 9 (Optional but extremely helpful)

The proof is beyond the scope of this publication.

Let $G = (V, E)$ be a **simple connected graph**. If $|V| = n$, $n \geqslant 3$ and, for each vertex $A \in V$, $\deg(A) \geqslant \dfrac{n}{2}$, then the graph G has a Hamiltonian cycle. This fact is known as **Dirac's theorem**.

Note: We can easily see that this is not a necessary condition. The dodecahedron graph corresponding to Hamilton's original game has $n = 20$ and $\deg(v) = 3$ for every vertex v, yet the graph is Hamiltonian.

Theorem 10 (Optional)

Let $G = (V, E)$ be a **simple connected graph**. If $|V| = n$, $n \geqslant 3$ and, for each pair of **non-adjacent** vertices $A, B \in V$, $\deg(A) + \deg(B) \geqslant n$, then the graph G has a Hamiltonian cycle. This fact is known as **Ore's theorem**. This is a generalization of Dirac's theorem.

Proof

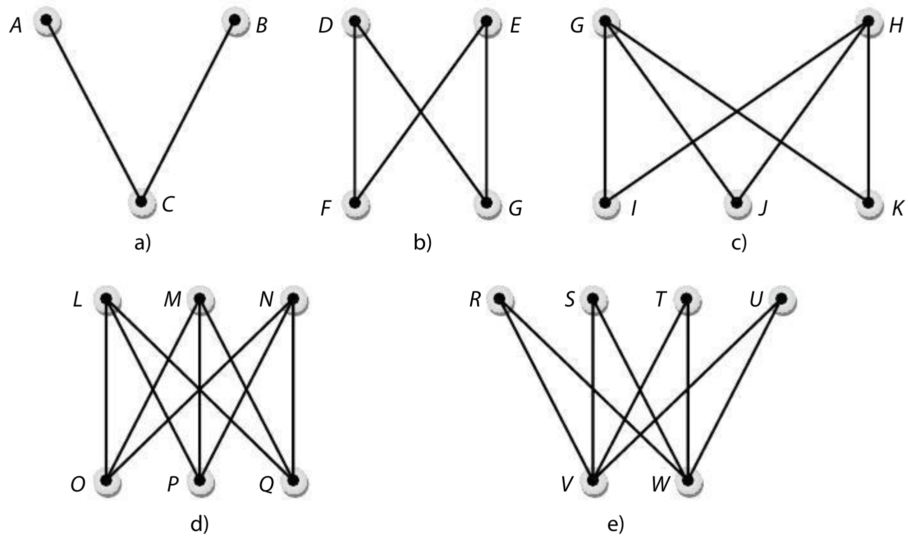
It can be proved by Dirac's theorem. Since for any two vertices A and B on graph G

$\deg(A) \geqslant \frac{n}{2}$, $\deg(B) \geqslant \frac{n}{2} \Rightarrow \deg(A) + \deg(B) \geqslant \frac{n}{2} + \frac{n}{2} = n$, so this must be true for two non-adjacent vertices too.

Unfortunately these two theorems give us only **sufficient** conditions, not **necessary** conditions for the statement. Also, once we know of the existence of a Hamiltonian cycle, there is no guidance for finding that cycle or how to find a Hamiltonian path.

Example 20

In which of the following bipartite graphs is it possible to find a Hamiltonian path or a Hamiltonian cycle? If possible, find an example of it and if not possible, give a reason why not.



Solution

a) There is a Hamiltonian path A, $\{A, C\}$, C, $\{C, B\}$, B, but no cycle. We can see that the vertices don't satisfy the conditions of Theorems 9 or 10.

b) There is a Hamiltonian cycle. One such possible cycle would be:

$D, \{D, F\}, F, \{F, E\}, E, \{E, G\}, G, \{G, D\}, D.$

We can observe that all four vertices have a degree of 2 and they satisfy the conditions of Theorems 9 and 10.

c) There is a Hamiltonian path but no cycle. To find one such path we need to start from a vertex of a degree 2 and not repeat a vertex before we travel through all of them. One possible path is $J, \{J, G\}, G, \{G, I\}, I, \{I, H\}, H, \{H, K\}, K$. We notice that the vertices don't satisfy the conditions of the theorems since vertices I, J, and K have a degree of 2, which is less than 2.5. Also, taken two at a time, the sum of their degrees is 4, which is less than 5.

d) There is a Hamiltonian cycle. One such possible cycle would be:

$L, \{L, O\}, O, \{O, M\}, M, \{M, P\}, P, \{P, N\}, N, \{N, Q\}, Q, \{Q, L\}, L.$

We notice that all four vertices have a degree of 3 and they satisfy the conditions of Theorems 9 and 10.

e) There is no Hamiltonian path nor cycle. The problem is that every time we visit a 2-degree vertex, we need to leave it, revisiting a 4-degree vertex. And hence there is no Hamiltonian cycle.

The above example points to two possible negative tests.

Bipartite graphs – negative tests

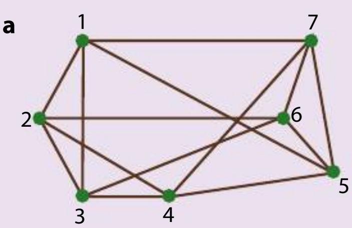
G is a bipartite graph with V_1 and V_2 subsets of vertices. Let subset 1 have m vertices and subset 2, n *vertices.*

- If $m \neq n$, G cannot have a Hamiltonian cycle. The case with Example 20 a), c), and e).
- If m and n differ by 2 or more, there is no Hamiltonian path. The case with Example 20 e).

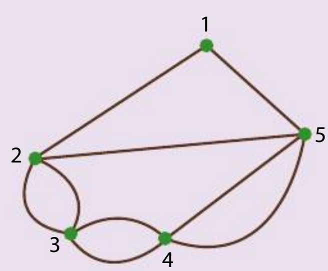
Exercise 3.4

1 Explain why each of the following graphs is Eulerian and find an Eulerian circuit for each.

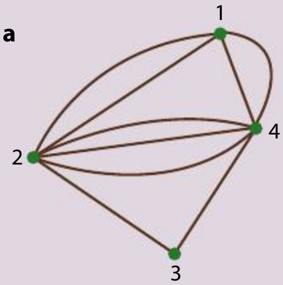
a

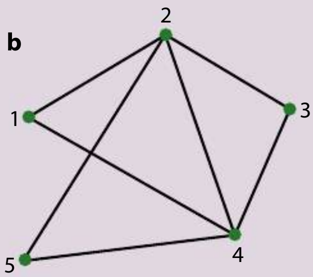


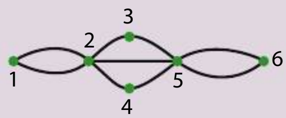
b



2 In each of the graphs below, find an Eulerian circuit or explain why no Eulerian circuit exists.

a 

b 

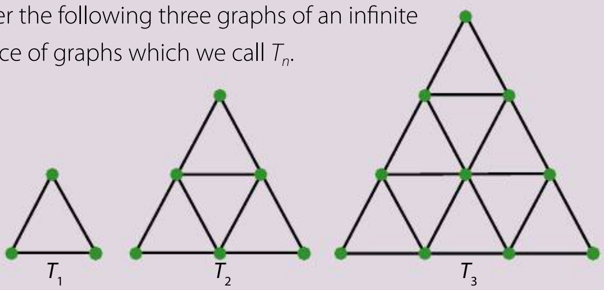
c 

3 Under what conditions would each of the following be Eulerian? Justify your answer.

a K_n **b** $K_{m,n}$

4 Are the graphs in questions 1 and 2 Hamiltonian? If one is not Hamiltonian but has a Hamiltonian path, find it.

5 Consider the following three graphs of an infinite sequence of graphs which we call T_n.

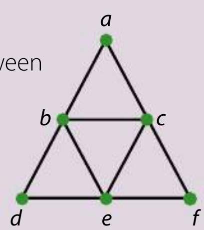


a Find an Eulerian circuit when possible, or justify why not when one does not exist.

b Find a Hamiltonian cycle when possible, or justify why not when one does not exist.

c When is T_n Eulerian? Hamiltonian?

6 How many walks of length 1, 2, 3, or 4 are there between a and e in the simple graph right?



7 Find the number of walks of length x between the vertices in K_5 when x is

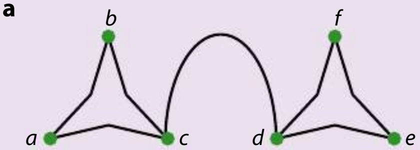
a 4 **b** 5 **c** 6

8 Consider the graph $K_{3,4}$. Let a and b be two vertices in the subset of three non-adjacent vertices. Find the number of walks of length x between these vertices when x is

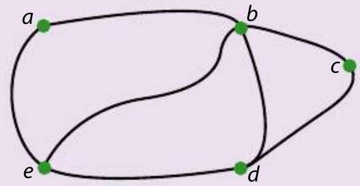
a 4 **b** 5 **c** 6 **d** 7

9 In each of the following, determine whether the given graph has a Hamiltonian cycle. If it does, find one such cycle. If it does not, justify why not. For those graphs that do not have a cycle, do any of them have a Hamiltonian path? If yes, find it and if not, justify why not.

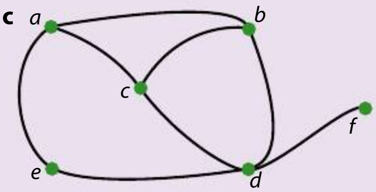
a



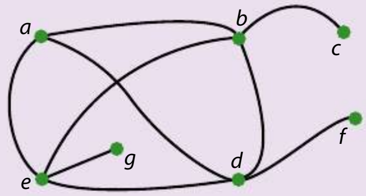
b



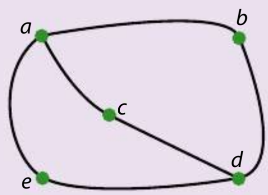
c



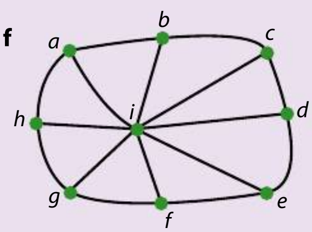
d



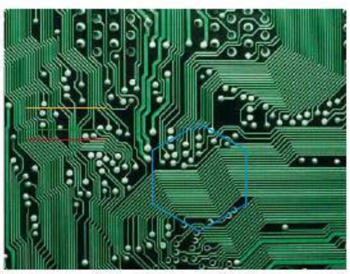
e



f



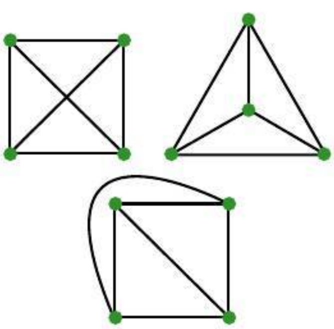
3.5 Planar graphs



One of the applications of graph theory is in the design of electronic components. In cases of computer chips, electronic components are assembled using printed circuits, where the conducting strips are printed onto boards of insulating material. The conducting strips may not cross, since that would lead to a malfunction of the component because of short circuits. Complex circuits where crossing strips are unavoidable have to be printed on several boards which are then packed together. Naturally, manufacturers want to print circuits onto the minimum number of boards, for obvious reasons. This is an application where graphs that represent components of circuits have to be **planar**.

Definition 19

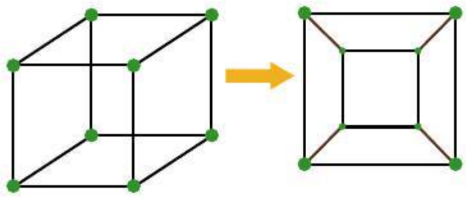
A **planar graph** is a graph that can be represented by a diagram in which no edges cross. Such a diagram is called a **plane diagram** (also known as **planar representation** or **embedding**). For example, K_4 is a planar graph.



For instance, two diagrams of K_4 are shown left. The first is not a plane diagram, while the second and third are.

Example 21

Is the graph known as the 3-cube, Q_3 shown below, planar?

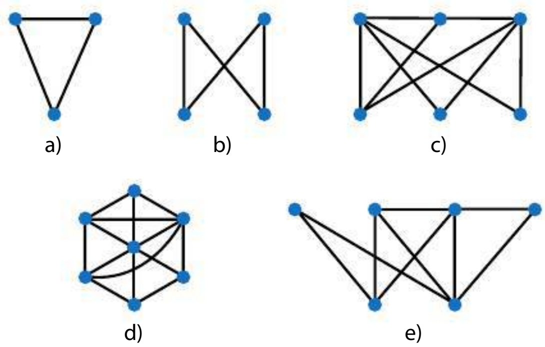


Solution

Q_3 is planar because it can be drawn without any edges crossing, as you can see in the accompanying plane diagram.

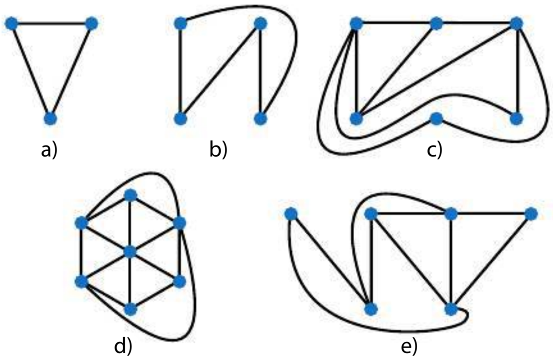
Example 22

Below are the plane graphs of a few graphs. Show that they are planar.



Solution

Here are the plane graphs redrawn to show that no two edges in any of the graphs cross. Hence, they are planar.

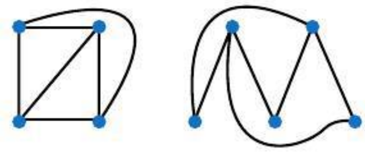


Example 23 (Important)

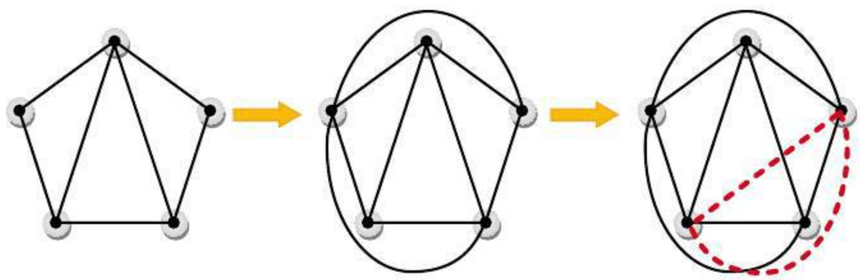
Investigate which of the complete graphs K_n and complete bipartite graphs $K_{m,n}$ are planar.

Solution

It is obvious that the following complete graphs are planar: K_1, K_2, K_3, $K_{2,1}$, and $K_{2,2}$ (as shown in Example 22). It is not very difficult to find the planar embedding for K_4 and $K_{3,2}$, as shown in the following figure.

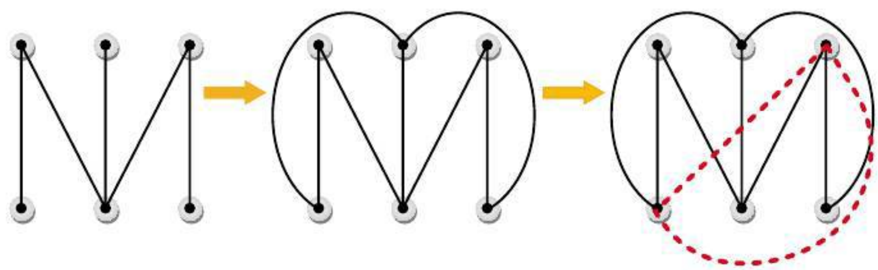


Whether K_5 and $K_{3,3}$ are planar needs to be further investigated. Start with K_5. After drawing the pentagon and all the diagonals from one vertex, proceed with drawing one edge at a time.



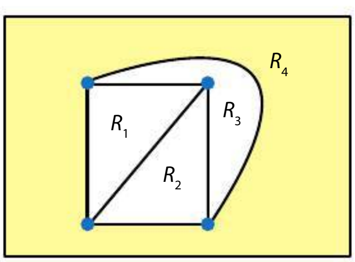
It becomes clear that in order to draw the last edge we must cross one of the previously drawn edges; therefore, it is not possible to find a planar representation of K_5.

Apply a similar approach to find a plane diagram of $K_{3,3}$.



You can see that before reaching the last edge, there is no way to draw any edge left without crossing some other edge. Thus, $K_{3,3}$ is not planar.

Euler's formula



A planar representation of a graph partitions the plane into separate regions. For example, the graph diagram K_4 is given left, and, as you notice, it splits the plane into four **regions** (known as **faces** in IB documents). Euler showed that all graph diagrams of the same graph partition the plane into the same number of regions. He accomplished this by finding a relationship between the number of regions, the number of edges and the number of vertices of a planar graph.

Theorem 11 (Euler's formula)

Let $G = (V, E)$ be a **connected planar simple graph** (**multigraph**) where $|V| = v, |E| = e$, and f is the number of faces or regions this graph's **planar embedding** establishes in the plane, then

$$v - e + f = 2.$$

Proof (By induction)

$P(e)$: For every embedding of a connected planar graph with e edges, v vertices, and f faces, $v - e + f = 2$.

Basis step: $P(0)$: The formula is true for a graph with zero edges. This means the graph is made of one vertex only. $v = 1, f = 1$ (since the vertex does not partition the plane!) and $e = 0$. Since $1 - 0 + 1 = 2$, so $P(0)$ is true. We can also consider $P(1)$. That means one edge, thus $v = 2$ and $f = 1$. Thus $2 - 1 + 1 = 2$, which indicates that $P(1)$ is true. (If the edge is a loop, it is a similar argument with $f = 2$, $v = 1$, and $e = 1$.)

Inductive step: Let $k > 1$ be given such that $P(k)$ is true. That is, we have a connected planar graph with k edges, v vertices, and f faces where the formula is true, $v - k + f = 2$. Now, consider a graph G with $k + 1$ edges, v vertices, and f faces. G either has a cycle or does not have one.

Case 1: G has no cycle. Since there are no cycles, the graph is not closed and there is only one unbounded face. (See Figure 3.1.) $v = k + 2$. In an open graph, every edge has two vertices, but since it is connected, every two edges share one vertex, and hence each edge contributes one to the number of vertices available, except either the first or last edge, and hence

$$v - e + f = k + 2 - (k+1) + 1 = 2.$$

Case 2: G has a cycle. Let a be an edge in this cycle. Now, create a graph G_1 by deleting the edge a from G. (Deleting an edge merges two regions R_1 and R_2, for example, together.) This subgraph contains k edges and $f - 1$ faces. Using the fact that $P(k)$ is true and can be applied to G_1, then

$$v - k + f - 1 = 2 \Rightarrow v - (k+1) + f = v - e + f = 2.$$

Thus, by the principle of mathematical induction, $P(0)$ is true, and assuming $P(k)$ to be true, we showed that $P(k + 1)$ is true, and thus the relation is true for all $e \in \mathbb{N}$.

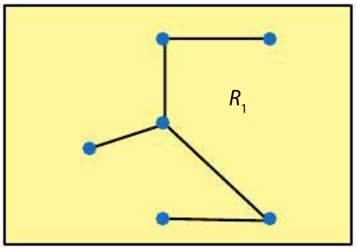
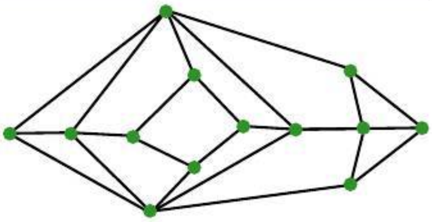


Figure 3.1

There will be more about this in the next chapter.

Example 24

Verify Euler's formula for the connected planar graph given right.



Solution

The graph has 13 vertices, 23 edges, and 12 regions.

So, $13 - 23 + 12 = 25 - 23 = 2$.

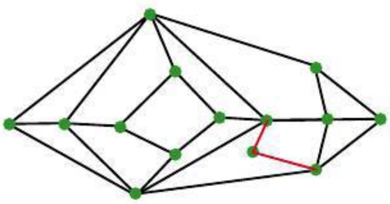
Example 25

A connected planar graph has 24 edges, dividing the plane into 12 regions. How many vertices does this graph have? Create such a graph.

Solution

$$v - 25 + 13 = 2 \Rightarrow v = 14.$$

We took the liberty of using the previous graph and added one vertex!



Theorem 12

If G is a connected simple planar graph with e edges and $v > 2$ vertices, then $e \leqslant 3v - 6$.

Proof

Given that we need at least three edges to form two regions or faces[1] in a simple graph then $2e \geqslant 3f$. Then, by using Euler's formula, we obtain the following:

$$\left.\begin{array}{l} 2+e-v=f \\ 2e \geqslant 3f \end{array}\right\} \Rightarrow 2e \geqslant 3(2+e-v) \Rightarrow 2e \geqslant 6+3e-3v \Rightarrow e \leqslant 3v-6$$

Example 26

Show that K_5 is not planar.

Solution

K_5 is a simple connected graph with $e = 10$ and $v = 5$. If it were planar, then

$$e = 10 \leqslant 3v - 6 = 15 - 6 = 9,$$

which is not true. Thus, K_5 is not planar.

[1] There are some other considerations we chose not to include here. For more information, see Ralph Grimaldi, *Discrete and Combinatorial Mathematics*, 5th edition (Addison-Wesley, 2003).

Theorem 13

If G is a connected simple planar graph with e edges and $v > 2$ vertices, and no circuits of length 3, then $e \leqslant 2v - 4$.

Proof

The proof is similar to that of Theorem 12. Since there are no circuits of degree 3, then we need at least four edges to form two regions. Hence, $2e \geqslant 4f$. Thus,

$$\left.\begin{array}{l} 2+e-v=f \\ 2e \geqslant 4f \end{array}\right\} \Rightarrow 2e \geqslant 4(2+e-v) \Rightarrow 2e \geqslant 8+4e-4v \Rightarrow 2e \leqslant 4v-8 \Rightarrow e \leqslant 2v-4$$

Example 27

Show that $K_{3,3}$ is not planar.

Solution

$K_{3,3}$ is a simple connected graph with no circuit of length 3. $v = 6$ and $e = 9$. If it were planar, then

$$e = 9 \leqslant 2v - 4 = 12 - 4 = 8,$$

which is not true. Thus, $K_{3,3}$ is not planar.

Important

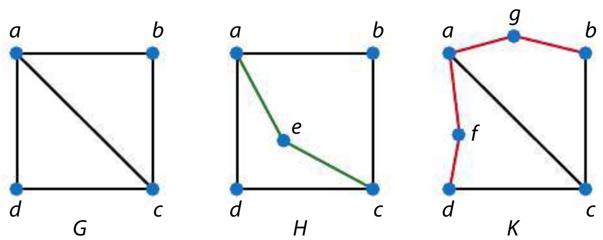
Since $K_{3,3}$ is a simple connected graph, if we were to apply Theorem 12, then we have $e = 9 \leqslant 3v - 6 = 18 - 6 = 12$, which is true! It would be **an error to conclude that $K_{3,3}$ is planar**. This is using the converse of the theorem without proving it. Unfortunately, the theorem we proved is necessary but not sufficient. That is, if the graph is planar, then the relation is true.

Note: Since K_5 and $K_{3,3}$ are not planar, it is obvious that all the graphs containing K_5 or $K_{3,3}$ as subgraphs are also not planar. Moreover, all the graphs that contain a subgraph that can be obtained from K_5 or $K_{3,3}$ using certain permitted operations are not planar.

Homeomorphic graphs

If we remove an edge, let's call it $\{A, B\}$, from a graph and we add another vertex C together with the edges $\{A, C\}$ and $\{B, C\}$, such an operation is called an **elementary subdivision**. Graphs are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

To understand the idea consider the graphs in the following figure.



Graph H is obtained from G by one elementary subdivision: remove edge ac from G, then add the edges ae and ec to the graph. Graph K is obtained from G by two elementary subdivisions: remove ab and add ag and gb, and remove ad and add af and fd. Thus, H and K are homeomorphic.

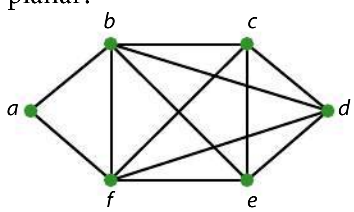
The following theorem is a useful result of the previous discussion.

Theorem 14 (Kuratowski's theorem)

A graph $G = (V, E)$ is not a planar graph if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

Example 28

Is the following graph planar?



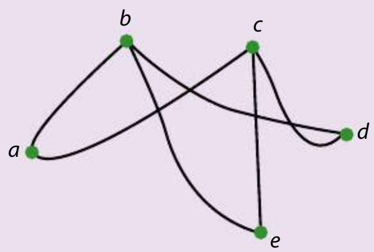
Solution

The graph is not planar since K_5 is a subgraph. ($bcdef$) is K_5.

Exercise 3.5

For each graph in questions 1–4, decide whether the graph is planar. If it is, give a reason for your decision and draw a planar representation. If it is not, justify why not.

1



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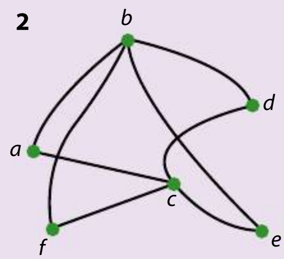

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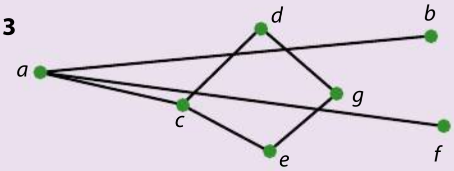
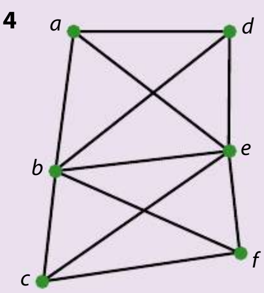


4



5 A connected planar graph contains 10 vertices and partitions the plane into seven regions. What is the number of edges in the graph?

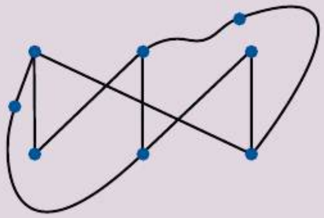
6 What is the maximum number of edges in a simple connected planar graph with 7 vertices? 8 vertices?

7 Find the minimum number of vertices in a simple connected planar graph with 14 edges? 21 edges?

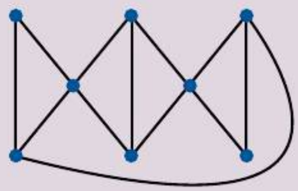
8 A connected planar graph has 8 vertices with 3 degrees each. How many regions are created by a planar embedding of this graph?

In questions 9–10, determine whether the graphs are planar.

9



10



Practice questions 3

1 Explain whether or not it is possible to have a cycle of odd length in a bipartite graph.

2 a A complete graph K_n contains subgraphs isomorphic to K_m, where $m < n$. How many isomorphic subgraphs does K_n contain if:

i $m = 2$ ii $m = 3$ iii $m, m = 1, \ldots, n$

b For what value(s) of m would the number of isomorphic graphs be the largest?

3 Given a complete graph K_5, find the number of trails no longer than 3 between two vertices.

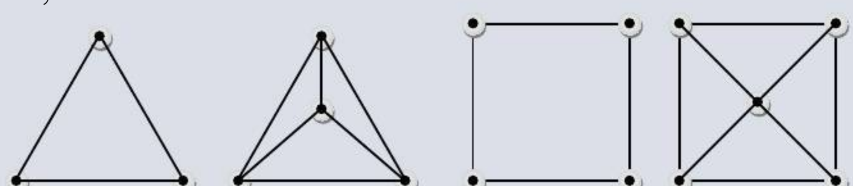
4 Given the complete graph K_4, and a walk of length l between any two vertices in the graph, find the number of different walks when

a $l = 2$ b $l = 3$.

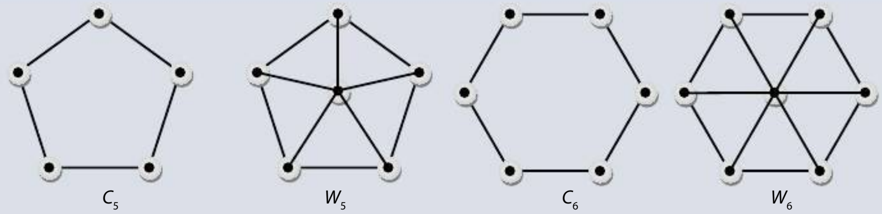
5 Given the complete bipartite graph $K_{3,3}$ and a walk of length l between any two non-adjacent vertices in the graph, find the number of different walks when

a $l = 3$ b $l = 4$.

6 **Cycle** C_n, $n \geqslant 3$, is a graph in which every vertex has an order of 2.
Wheel W_n, $n \geqslant 3$, is a graph that consists of a cycle C_n and an additional point that is connected to all the vertices in the cycle. Below are some examples of cycles and wheels:



C_3 W_3 C_4 W_4



a Show that the number of edges in a wheel W_n is twice the number of edges in a cycle C_n .

b Are any of these graphs, C_n or W_n, isomorphic to a complete graph K_n?

c Show that in C_4 there are 2^{n-1} paths of length n between

i adjacent vertices when n is odd

ii non-adjacent vertices when n is even.

7 Show that a cycle graph C_n, $n \geqslant 3$, is bipartite if and only if n is even.

8 Explain why no wheel graph W_n, $n \geqslant 3$, can be bipartite.

9 Draw the complementary graph of C_5. Is the complementary graph isomorphic to the original graph? If yes, construct an isomorphism between those two graphs.

10 A graph is called **self-complementary** if it is isomorphic to its complementary graph. Is it possible to find a self-complementary graph with

a 4 vertices **b** 6 vertices?

If possible, draw the graph and its complementary graph.

11 A parent-teacher organization (PTO) at an international school has six people working for it. They are Adam, Bernard, Cecile, Donatella, Eva, and Flor. They can communicate in at least one language according to the following table.

Name	English	Spanish	French	German
Adam	✓	✓	✓	
Bernard	✓		✓	✓
Cecile			✓	
Donatella		✓	✓	
Eva				✓
Flor		✓		

a Draw a graph indicating which people can communicate with each other.

b Cecile ordinarily communicates with Flor with the help of Donatella. Unfortunately, Donatella has gone to visit her mother. Can Cecile still communicate with Flor? Write down how it can be done.

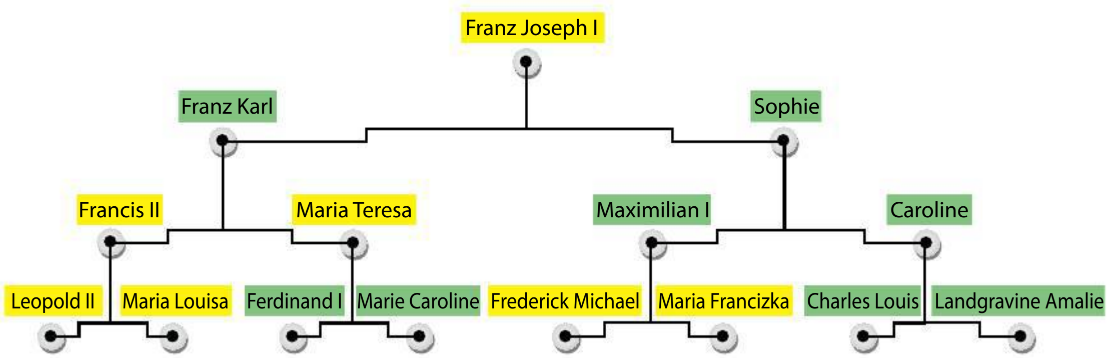
c Who is the most important person without whom it is not possible to communicate with all the members of the PTO? Give your reasons.

4 Trees and Algorithms

4.1 Introduction

Trees are among the most, if not the most, important class of graphs and they make fine modelling tools. In 1847, Gustav Kirchhof, a German scientist, used them to solve systems of equations for electrical networks. In 1857, the English mathematician Arthur Cayley used them to count the different isomers of the saturated hydrocarbons. Today, trees are widely used in mathematics, computer science, and many other fields including social sciences.

For example, a common representation of the genealogical charts of a family is called a family tree. In the form of a graph, vertices represent the family members, whilst edges represent the parent-child relationship. Here is a tree that represents the ancestors of the Austrian Emperor Franz Joseph I.



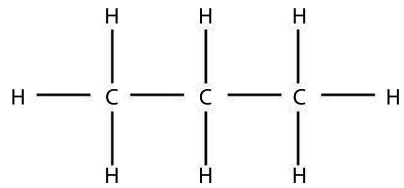
4.2 Trees

You are familiar with trees in graph theory. In Chapter 3, we discussed several instances of connected graphs that do not contain cycles. These are trees. As in graph theory, tree terminology is unfortunately not standard. We will use the IBO terminology in this publication.

Definition 1

Let $T = (V, E)$ be a **connected simple graph**. If T contains no **cycles**, it is a **tree**. A **subtree** is a **subgraph** of a **tree** that is a **tree** itself.

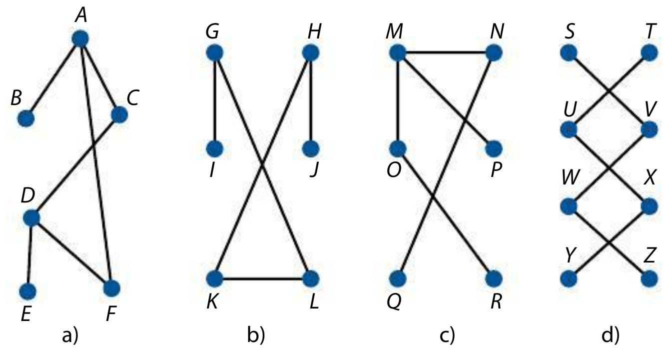
For example, the compound propane (C_3H_8) has this structure:



The structure has no cycle, so it is a tree.

Example 1

Which of the graphs are trees? Give your reasons.



Solution

Graphs b) and c) are trees. Graph a) contains a cycle, *ACDFA*, while graph d) is not connected.

Theorem 1

A graph $T = (V, E)$ is a **tree** if and only if there is **a unique simple path** between any pair of vertices.

Proof

($\Rightarrow$) If graph T is a tree, then it is connected with no cycles; thus, for any two vertices, there is a simple path between those two vertices. The uniqueness of the path can be proven by contradiction. Assume that there are two different paths between two vertices, but then those two paths together would form a cycle which is a contradiction, since T is a tree.

($\Leftarrow$) Now, assume that there is a unique simple path between any two vertices of the graph T. Given that there is a path then graph T is connected. Now, if graph T contains a cycle, then between two vertices in that cycle we can find two different paths, which contradicts the uniqueness of the path.

In many applications of trees, such as the family tree we discussed earlier, organizational trees, computer file systems, networks, etc., a vertex is designated as the **root.** Since there is a unique path from 'the root' to each vertex of the tree by Theorem 1, we direct each edge away from the root in a manner described by Figure 4.1. A tree with its root produces a graph called a **rooted tree**.

Definition 2

Let $T = (V, E)$ be a **tree**. Let v_i be a vertex such that every edge is directed away from it. T is called a **rooted tree**.

As you notice from the definition above, we can change any tree into a rooted tree by the choice of the **root**.

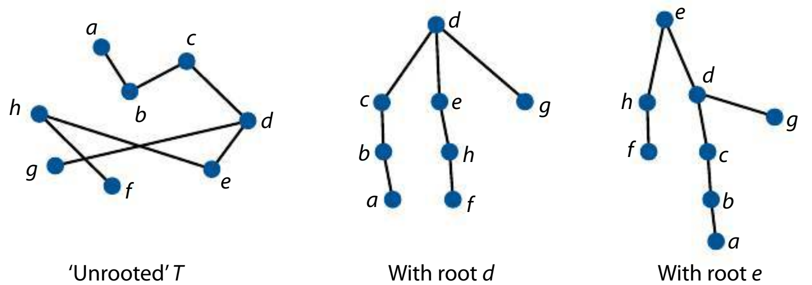


Figure 4.1

In a rooted tree, the starting vertex is the **root** while the other vertices are called **parent**, **child**, **siblings**, **ancestors**, and **descendants**. A vertex of a tree with no children is called a **leaf**. Vertices that have children are called **internal vertices**.

In Figure 4.1 above, for the tree with root d, b is a parent of a and a is a child of b. Vertices c, e, and g are siblings, since they have the same parent d. Ancestors of f are d, e, and h, whereas a and f have no descendants – therefore each of them is a leaf. We can say that all the vertices in the tree are descendants of the root. An internal vertex in a rooted tree is said to be at a level i when the path connecting it to the root is i. For example, in the tree with root d, c, e, and g are at level 1, while a and f are at level 3. In the tree with root e, h and d are at level 1, while a is at level 4.

Note: All vertices in a rooted tree have each a degree at least 2, except for the leaves. Each leaf has a degree of 1.

Theorem 2

A **tree** $T = (V, E)$ with n vertices has $n - 1$ edges.

Proof

We will conduct the proof by mathematical induction.

Statement: $S(n)$: a tree with n vertices has $n - 1$ edges.

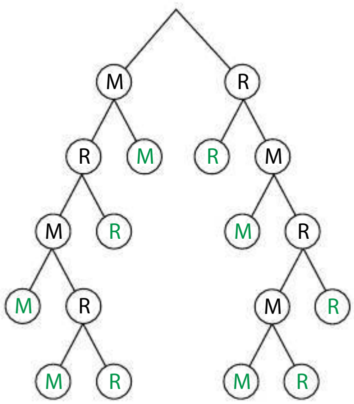
Basis step: When a tree has only one vertex, it has no edges. The statement is thus true for $n = 1$.

Inductive step: Assume that every tree with k vertices has $k - 1$ edges.

Now, consider a tree that has $k + 1$ vertices. Let vertex a be a leaf of T and let vertex b be the parent of a. Removing vertex a from the tree removes the edge $\{a, b\}$ too and leaves us with a **subtree** that has k vertices. By assumption, this subtree has $k - 1$ edges. However, T has one more edge than its subtree, and therefore has k edges. Thus, tree T that has $k + 1$ vertices has $(k + 1) - 1$ edges.

Conclusion: Since the statement is true for $n = 1$ and $S(k) \Rightarrow S(k + 1)$, by the principle of mathematical induction, the statement is true for all $n \in \mathbb{Z}^+$.

Example 2



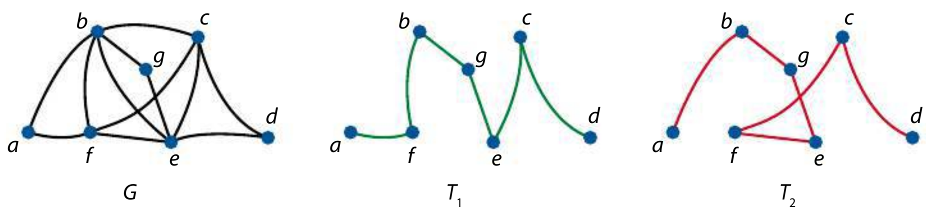
Marco and Roberto play a tennis game. They agree that whoever wins a total of three games first or two games in a row will be declared the winner. How many outcomes are possible, and what is the maximum number of games they will play?

Solution

The situation can be represented by a tree. There could be 10 possible outcomes corresponding to the vertices with degree 1 in the tree. The number of possible games corresponds to the layers of the tree we have, that is, five games.

4.3 Spanning trees

All connected graphs have trees that span them. Consider the following situation: In a small mountainous area, winter is harsh and snow sometimes makes it difficult to keep all the towns connected to the rest of the world. Because of the cost involved and the amount of equipment needed for the task, the authorities try to make sure that a minimum number of roads between the towns are accessible by ploughing as few roads as possible. Graph G below shows the road network on the left and two possible networks of ploughed roads to the right (T_1 and T_2). These subgraphs of G are called **spanning trees** of G.



Definition 3

Let $G = (V, E)$ be a **connected graph**. A subgraph H of G is a **spanning tree** of G if H is a tree which contains every vertex of G.

Theorem 3

Every connected graph has a spanning tree.

Proof

Let G be a connected graph. If G has no cycles, then it is a tree and we are done.

If G is not a tree, it must contain at least one cycle. Remove an edge from the cycle. The graph is still connected. If the new graph is acyclic (with no cycles), then it is a tree, and hence a spanning tree since it visits all vertices. Otherwise, it must have another cycle. Repeat the process with another edge from a cycle, until a subgraph T is acyclic. Since T is acyclic, connected, and contains every vertex, then it is a spanning tree.

How to find a spanning tree

Spanning trees can be constructed in two ways, either by removing edges (vertices are not removed) which form cycles or by building a tree one edge at a time. The two methods are described below.

Method 1: Edge removal

Assume that $G = (V, E)$ is a **connected graph**. Edges are removed one at a time in such a way that the resulting graph always remains connected. If this is done until no further edges can be removed, then the resulting graph is a **spanning tree**.

Method 2: Edge addition

Assume that $G = (V, E)$ is a **connected graph**. Start with the subgraph containing all the vertices from the set V. Adjoin the edges, one edge at a time, in such a way that the resulting graph has no cycle. If this is done until no further edge can be added, then the resulting graph is a **spanning tree**.

We will present here three algorithms for constructing spanning trees. They all proceed by successively adding edges that have not already been used. We will consider non-programming sets of instructions for these algorithms. One of these is Kruskal's algorithm which makes use of Theorem 2 of Section 4.2.

Kruskal's algorithm

Given that a graph $G = (V, E)$ is a simple connected graph, and $|V| = n$, find a spanning tree T for G.

Algorithm

Set the counter $i = 0$. (i is the number of edges of the sought tree. Every time we add an edge, we increase this number by 1.)

Step 1: Select an edge, e_1. If e_1 does not create a cycle, add it to the tree, set $i = 1$, and add e_1 to the tree T.

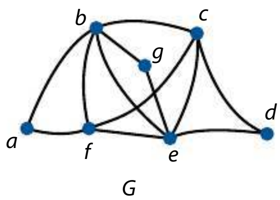
Step 2: For $1 \leqslant i \leqslant n - 2$, if edges $e_1, e_2, \ldots, e_i$ have been selected, then select edge e_{i+1} from the remaining edges so that the subgraph determined by $e_1, e_2, \ldots, e_{i+1}$ contains no cycles.

Step 3: Replace i by $i + 1$.

If $i = n - 1$, the subgraph T determined by $e_1, e_2, \ldots, e_{i+1}$ is connected with $n - 1$ edges and n vertices, and hence is a spanning tree.

If $i < n - 1$, return to step 2.

Example 3

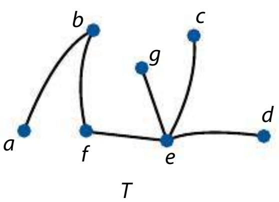


Apply Kruskal's algorithm to find a spanning tree for graph G given left.

Solution

We will construct a spanning tree using the steps in Kruskal's algorithm and summarize the steps in the table below. Observe that the number of vertices is seven.

Edge in G	Cycle formed?	Edges in tree	Number of edges in tree	Notes
ab	no	*ab*	1	
bf	no	*ab*, *bf*	2	
fa	yes	*ab*, *bf*	2	no edges added
fe	no	*ab*, *bf*, *fe*	3	
eg	no	*ab*, *bf*, *fe*, *eg*	4	
gb	yes	*ab*, *bf*, *fe*, *eg*	4	no edges added
ec	no	*ab*, *bf*, *fe*, *eg*, *ec*	5	
ed	no	*ab*, *bf*, *fe*, *eg*, *ec*, *ed*	6	stop, $i = 7 - 1$



The figure left gives the spanning tree so constructed. Notice though that this is not a unique tree and we could have created a different one if we made different choices at f, for example.

> **The depth-first search algorithm (DFS)**
>
> Here is an outline of the steps in this algorithm.
>
> 1. Start at a vertex v_i, and mark it as visited.
> 2. Pick a vertex v_{i+1}, adjacent to v_i and not yet visited.
> 3. Add edge $v_i v_{i+1}$ to the tree, and replace i by $i + 1$.
> 4. Repeat steps 2 and 3, until you reach a vertex that has no adjacent vertices.
> 5. Backtrack to a vertex that has adjacent vertices that have not been visited, and repeat step 4.
> 6. Stop when all vertices have been visited.

Example 4

Refer to the same graph G given in Example 3. Find a spanning tree using DFS.

Solution

1. Start at a, $i = 1$.
2. Go to f.
3. $T = \{af\}$, $i = 2$.
4. Go to c: $T = \{af, fc\}$, $i = 3$.

   Go to d: $T = \{af, fc, cd\}$, $i = 4$ (path marked in green).
5. Backtrack to c and

   go to e: $T = \{af, fc, cd, ce\}$, $i = 5$ (new edge in blue).

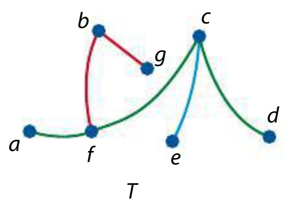
   Backtrack to f and

   go to b: $T = \{af, fc, cd, ce, fb\}$, $i = 6$.

   go to g: $T = \{af, fc, cd, ce, fb, bg\}$, $i = 7$ (in red).

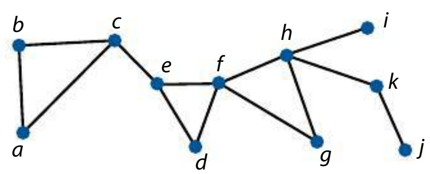
   Stop, all vertices added.

The figure is shown right.



Example 5

Find a spanning tree using DFS for the graph below.

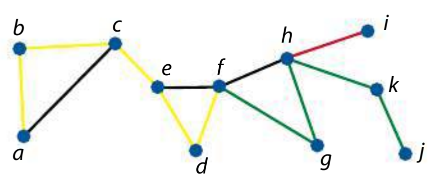


Solution

1. Start at f, $i = 1$.
2. Go to g.
3. $T = \{fg\}$, $i = 2$.
4. Continue to h, k, and j; now $i = 5$ (in green).
5. Backtrack to h, then go to i (in red).

   Now, backtrack to f, then go to d, e, c, b, and a (in yellow).

On the left is the resulting spanning tree.



BFS as given here is in outline only. If you are interested in a detailed algorithm, check the algorithm given at the end of this section.

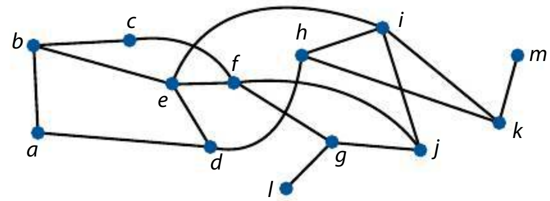
The breadth-first search algorithm (BFS)

Here is an outline of the steps in this algorithm. In this algorithm, we visit the vertices, level by level, until all vertices are visited.

1. Start at a vertex v_i, and mark it as visited.
2. Pick a vertex v_{i+1}, adjacent to v_i and not yet visited.
3. Add edge $v_i v_{i+1}$ to the tree, and replace i by $i + 1$.
4. Visit all unvisited vertices adjacent to v_i.
5. Repeat step 4 until all vertices are visited.
6. When $i = n$, stop. All vertices are added.

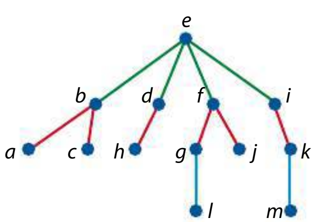
Example 6

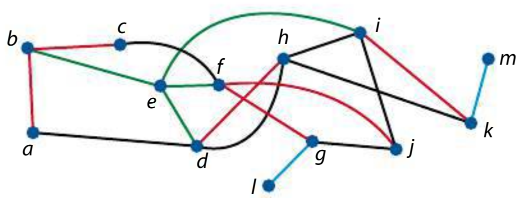
Find a spanning tree using BFS for the graph below.



Solution

1. Start at *e*.
2. Add *b*, *d*, *f*, and *i*. There are no more vertices adjacent to *e*. These are at level 1.
3. Go to *b*, add *a* and *c*. There are no more vertices adjacent to *b*.
4. Go to *d*, add *h*. No more vertices adjacent to *d*.
5. Go to *f*, add *g*, and *j*.
6. Go to *i*, add *k*. Now level 1 vertices are exhausted. Go to level 2 vertices.
7. At *a*, *c*, *h*, and *j* we cannot add any new vertices. At *g* add *l* and at *k* add *l*, and stop.





On the left is a plan of the algorithm, with the corresponding spanning tree.

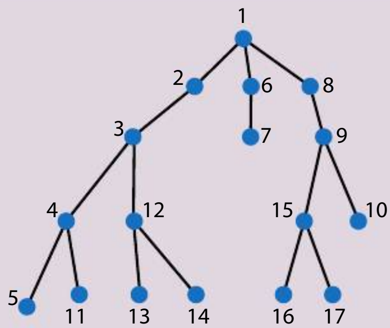
BFS algorithm

procedure BFS(G: Connected graph with vertices $v_1, v_2, \ldots, v_n$)[1]
T := tree consisting only of vertex v_1
L := empty list
Put v_1 in L (list of unprocessed vertices)
while L is not empty
    remove the first vertex, v, from L
    for each neighbour w of v
        if w is not in L and not in T **then**
            add w to the end of list L
            add w and edge (v, w) to T

[1] Kenneth Rosen, *Discrete Mathematics and its Applications*, 7th edition (McGraw-Hill Higher Education, 2012) p. 759

Exercise 4.1–4.3

1 Consider the tree on the right.



a List the leaves of this tree.

b List the parents of 4, 8, and 15.

c List the descendants of 3, 7, and 15.

d List the siblings of 4, 7, and 9.

2 Let $T(u, e)$ and $S(v, f)$ be two trees, where u and v are the set of vertices and e and f are the sets of edges for the two trees. If $|e| = 17$ and $|v| = 2|u|$, find $|u|$, $|v|$, and $|f|$.

3 $G = (V, E)$ is a connected undirected graph with $|E| = 30$. What is the maximum number of vertices?

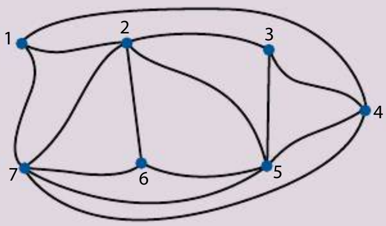
4 $T = (V, E)$ is a tree with n vertices, where $n \geqslant 2$. How many different paths are there in T?

5 a Find two non-isomorphic spanning trees for $K_{2,3}$. How many such trees are there?

b How many non-isomorphic spanning trees are there for $K_{2,n}$, $n \in \mathbb{Z}^+$?

In questions 6–8, find a spanning tree for the graph shown. In each question use an edge removal process.

6



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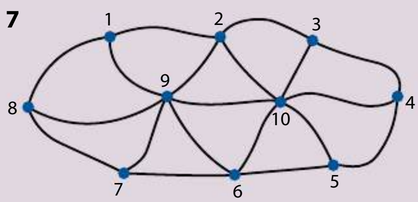

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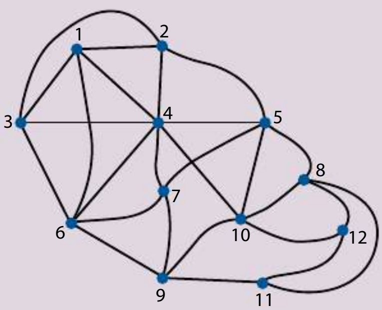


In questions 9–11, use Kruskal's algorithm to produce a spanning tree for each graph.

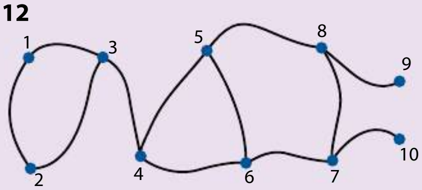
9 Find a spanning tree for the graph in questions 6.

10 Find a spanning tree for the graph in question 7.

11 Find a spanning tree for the graph in question 8.

In questions 12–14, use DFS to produce a spanning tree for each graph. Consider 1 to be the root.

12



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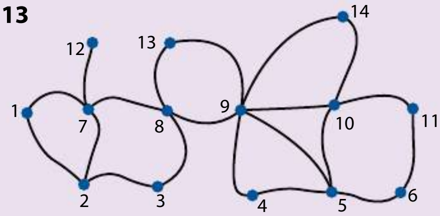

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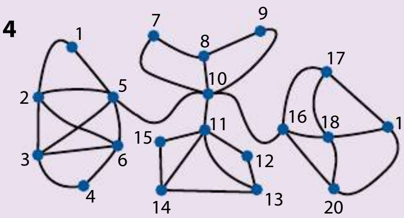


In questions 15–17, use **a** BFS and **b** Kruskal's algorithm to produce a spanning tree for each graph. Consider 1 to be the root.

15 Find a spanning tree for the graph in question 12.

16 Find a spanning tree for the graph in question 13.

17 Find a spanning tree for the graph in question 14.

18 **Cycle** C_n, $n \geqslant 3$, is a graph in which every vertex has an order of 2.
Wheel W_n, $n \geqslant 3$, is a graph that consists of a cycle C_n and an additional point that is connected to all the vertices in the cycle.

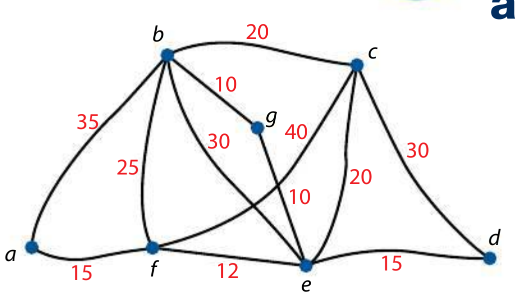
Use **a** DFS and **b** BFS to find a spanning tree for each of the following:

i W_6 starting at the centre vertex

ii K_5

iii $K_{3,4}$ starting at a vertex with degree 3

4.4 Weighted graphs and greedy algorithms



Several real situations can be modelled using graphs with weights assigned to their edges.

Consider the roads in the mountainous area discussed in Example 3. However, now we have the distances between the towns (see left). To minimize cost, we will have to minimize the total distance travelled. Airlines use such graphs to represent distances and times between different airports; networks utilize such graphs to represent the response time between different nodes; and there are many other applications. These graphs are called **weighted graphs**.

Definition 4

Let $G = (V, E)$ be a **graph**. If a numerical value or a weight is assigned to every edge of G, then we say that G is a **weighted graph**.
The **weight of a path** would be the sum of all the weights of all the edges in that path.

Representation

A convenient way of representing the weights that are assigned to the different edges is to use a special type of adjacency matrix called the **cost adjacency matrix** C_G. The entry (i, j) corresponds to the weight of the path from vertex i to vertex j. So, for example, the entry corresponding to (a, b) in the cost adjacency matrix for the graph above is 35. Below is the cost adjacency matrix for that graph.

$$C_G = \begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline a & - & 35 & - & - & - & 15 & - \\ b & 35 & - & 20 & - & 30 & 25 & 10 \\ c & - & 20 & - & 30 & 20 & 40 & - \\ d & - & - & 30 & - & 15 & - & - \\ e & - & 30 & 20 & 15 & - & 12 & 10 \\ f & 15 & 25 & 40 & - & 12 & - & - \\ g & - & 10 & - & - & 10 & - & - \end{array}$$

We use the convention that where there is no connection, we put a dash (–). (In some books, 0 is used instead.)

The cost adjacency matrix is a good tool for storing data and retrieving weights of edges when needed, without getting lost in looking at the numbers next to each edge.

Weighted graphs are associated with spanning trees that have a minimum weight. In the examples in this section, we are interested in finding a spanning tree with minimum weight. Such trees are called **minimal** (or **minimum**) **spanning trees**. There are a few algorithms that help us find such trees. These are called **greedy algorithms**. Two of these will be discussed in this section: **Kruskal's algorithm** and **Prim's algorithm**.

Kruskal's algorithm

Kruskal's algorithm for minimal spanning trees is an extension of his algorithm for spanning trees, introduced on page 1628. In this algorithm, we keep track of the weight of the edge. Here is an outline:

Given that a graph $G = (V, E)$ is a simple, weighted, connected graph, and $|V| = n$, find a spanning tree T for G.

Algorithm

Set the counter $i = 0$. (i is the number of edges of the sought tree. Every time we add an edge, we increase this number by 1.)

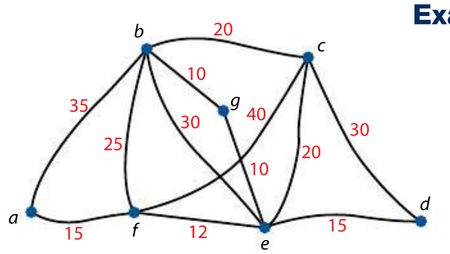
Step 1: Select an edge, e_1, where e_1 does not create a cycle and has the smallest possible weight, add it to the tree, set $i = 1$, and add e_1 to the tree T.

Step 2: For $1 \leqslant i \leqslant n - 2$, if edges $e_1, e_2, \ldots, e_i$ have been selected, then select edge e_{i+1} from the remaining edges so that the subgraph determined by $e_1, e_2, \ldots, e_{i+1}$ contains no cycles and the weight of e_{i+1} is the smallest possible.

Step 3: Replace i by $i + 1$.

If $i = n - 1$, the subgraph T determined by $e_1, e_2, \ldots, e_{i+1}$ is connected with $n - 1$ edges and n vertices, and hence is a spanning tree.

If $i < n - 1$, return to step 2.

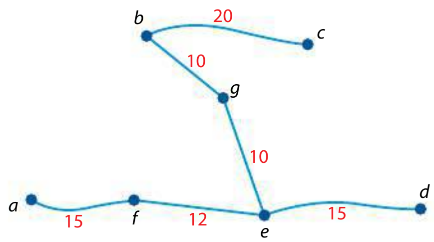


Example 7

Apply Kruskal's algorithm to find a minimal spanning tree for graph G given left.

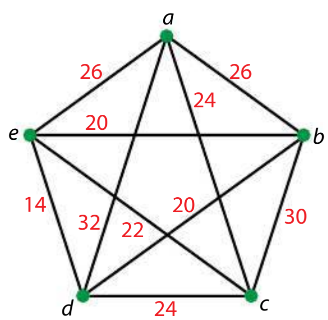
Solution

Here too we can use a table to summarize our steps. However, we will not use a table as we want you to experience applying the algorithm in as many different ways as possible.



1. Select edge bg as it has the lowest weight (ge too); $i = 1$, weight is 10.
2. Now select ge with smallest possible weight of 10, no cycle formed, add it to the tree; weight is 20, $i = 2$.
3. Now select fe with weight 12, no cycle, add it to T; weight is 32, $i = 3$.
4. Select af, then ed, add to T; weight is $32 + 15 + 15 = 62$, $i = 3 + 1 + 1 = 5$.
5. Select bc (or ed), add to T; weight is $62 + 20 = 82$, $i = 6$. Stop.

The tree is $T = \{bg, ge, fe, af, ed, bc\}$ with minimal weight of 82. The minimal spanning tree is shown in the figure left.



This procedure of applying Kruskal's algorithm is very helpful especially in graphs with a relatively small number of edges.

Example 8

Find a minimal spanning tree for the network left.

Solution

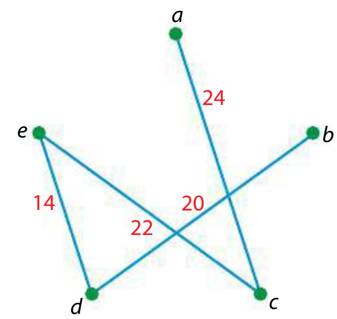
We will arrange the weights in non-decreasing order to make it easier to choose the edges to be added.

Weight	14	20	20	22	24	24	26	26	30	32
Edge	de	db	eb	ec	dc	ac	ea	ab	bc	ad

1. Select de, weight 14, add to T, $i = 1$.
2. Select db, no cycle formed, weight 20, add to T, $i = 2$.
3. Select eb, cycle formed, reject.

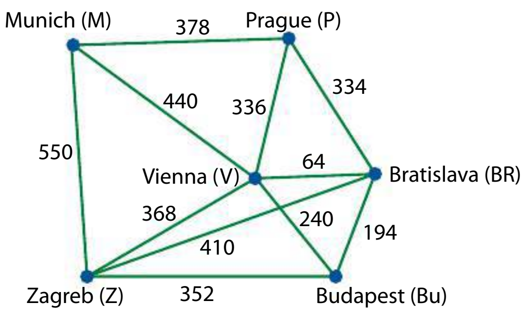
4. Select ec, no cycle formed, weight 22, add to T, $i = 3$.
5. Select dc, cycle formed, reject.
6. ac, no cycle formed, weight 24, add to T, $i = 4$. Stop.
7. Tree is formed and has a weight of $14 + 20 + 22 + 24 = 80$.

The diagram to the right shows the resulting minimal spanning tree.



Example 9

Use Kruskal's algorithm to find a minimum spanning tree for the graph below.



Solution

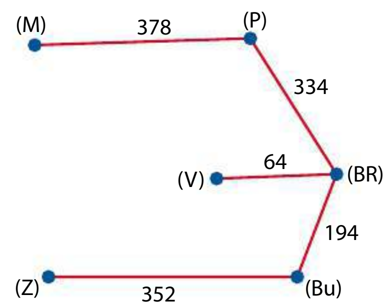
We will list all the edges in a table and then sort them in non-descending order. Then we decide whether or not we are going to include them in the minimum spanning tree.

Edge	Weight
V-Br	64
V-Bu	240
V-P	336
V-Z	368
V-M	440
Br-Bu	194
Bu-Z	352
Z-M	550
M-P	378
P-Br	334
Br-Z	410

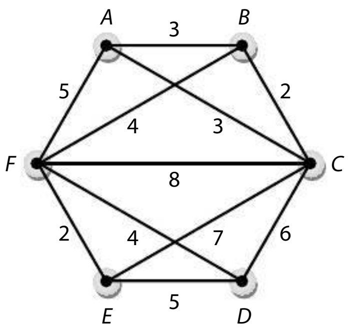
Edge	Weight	Decision
V-Br	64	yes, $i = 1$
Br-Bu	194	yes, $i = 2$
V-Bu	240	no, the cycle V-Br-Bu-V
P-Br	334	yes, $i = 3$
V-P	336	no, the cycle V-Br-P-V
Bu-Z	352	yes, $i = 4$
V-Z	368	no, the cycle V-Bu-P-V
M-P	378	yes, $i = 5$, STOP
Br-Z	410	
V-M	440	
Z-M	550	

The three edges left form a cycle with the edges already included in the spanning tree, and hence are not included. Also, after we included the fifth edge, we stop since a tree with six vertices contains five edges. We know that any additional edge to the tree will form a cycle with some of the existing edges.

By Kruskal's algorithm, the minimum spanning tree appears right.



So, the **minimum spanning tree** has a **weight** of
$64 + 194 + 334 + 352 + 378 = 1322$.



Example 10

Use Kruskal's algorithm to find the weight of a minimum spanning tree in the graph left.

Solution

Edge	Weight
$\{A, B\}$	3
$\{A, C\}$	3
$\{A, F\}$	5
$\{B, C\}$	2
$\{B, F\}$	4
$\{C, D\}$	6
$\{C, E\}$	7
$\{C, F\}$	8
$\{D, F\}$	4
$\{D, E\}$	5
$\{E, F\}$	2

Edge	Weight	Decision
$\{B, C\}$	2	yes, $i = 1$
$\{E, F\}$	2	yes, $i = 2$
$\{A, B\}$	3	yes, $i = 3$
$\{A, C\}$	3	no, creates cycle $BCAB$
$\{B, F\}$	4	yes, $i = 4$
$\{D, F\}$	4	yes, $i = 5$, STOP
$\{A, F\}$	5	
$\{D, E\}$	5	
$\{C, D\}$	6	
$\{C, E\}$	7	
$\{C, F\}$	8	

So, the **minimum spanning tree** has a **weight** of $2 + 2 + 3 + 4 + 4 = 15$.

It is also possible that instead of the edge $\{A, B\}$ we include the edge $\{A, C\}$.

Notice here that edge $\{E, F\}$ was added, even though it was not adjacent to any existing edge in the tree. The algorithm will guarantee that the tree will eventually be formed by focusing on $n - 1$ edges with no cycles.

Prim's algorithm (Optional)

Prim's algorithm is similar to Kruskal's with the exception that it requires the added edges to be adjacent to existing edges of the tree.

This topic is no longer in the IB Syllabus for 2014.

Algorithm

Set the counter $i = 0$. (i is the number of edges of the sought tree. Every time we add an edge, we increase this number by 1.)

Step 1: Select an edge, e_1, where e_1 does not create a cycle and has the smallest possible weight, add it to the tree, set $i = 1$, and add e_1 to the tree T.

Step 2: For $1 \leq i \leq n - 2$, if edges $e_1, e_2, \ldots, e_i$ have been selected, then select edge e_{i+1} from the remaining edges which is adjacent to one of the edges in the tree and so that the subgraph determined by $e_1, e_2, \ldots, e_{i+1}$ contains no cycles and the weight of e_{i+1} is the smallest possible.

Step 3: Replace i by $i + 1$.

If $i = n - 1$, the subgraph T determined by $e_1, e_2, \ldots, e_{i+1}$ is connected with $n - 1$ edges and n vertices, and hence is a spanning tree.

If $i < n - 1$, return to step 2.

Example 11

Use Prim's algorithm to find a minimum spanning tree in the graph in Example 9. The data from the figure can be stored into the following cost adjacency matrix.

$$\begin{array}{c|c} & \begin{array}{cccccc} \text{V} & \text{Br} & \text{Bu} & \text{Z} & \text{M} & \text{P} \end{array} \\ \begin{array}{c} \text{V} \\ \text{Br} \\ \text{Bu} \\ \text{Z} \\ \text{M} \\ \text{P} \end{array} & \begin{pmatrix} - & 64 & 240 & 368 & 440 & 336 \\ 64 & - & 194 & 410 & - & 334 \\ 240 & 194 & - & 352 & - & - \\ 368 & 410 & 352 & - & 550 & - \\ 440 & - & - & 550 & - & 378 \\ 336 & 334 & - & - & 378 & - \end{pmatrix} \end{array}$$

Solution

Again we are going to start with the Vienna–Bratislava edge that has a length of 64 and then we will add one edge at a time. Once we reach five edges in the set we will stop. (*wt* corresponds to weight.)

Step 1: $T = \{\{\text{V}, \text{Br}\}\}, \quad wt(\{\text{V}, \text{Br}\}) = 64$

Step 2: $T = \{\{\text{V}, \text{Br}\}, \{\text{Br}, \text{Bu}\}\}, \quad wt(\{\text{Br}, \text{Bu}\}) = 194$

Step 3: $T = \{\{\text{V}, \text{Br}\}, \{\text{Br}, \text{Bu}\}, \{\text{Br}, \text{P}\}\}, \quad wt(\{\text{Br}, \text{P}\}) = 334$

Step 4: $T = \{\{\text{V}, \text{Br}\}, \{\text{Br}, \text{Bu}\}, \{\text{Br}, \text{P}\}, \{\text{Bu}, \text{Z}\}\}, \quad wt(\{\text{Bu}, \text{Z}\}) = 352$

Step 5: $T = \{\{\text{V}, \text{Br}\}, \{\text{Br}, \text{Bu}\}, \{\text{Br}, \text{P}\}, \{\text{Bu}, \text{Z}\}, \{\text{P}, \text{M}\}\}, \quad wt(\{\text{P}, \text{M}\}) = 378$
STOP

So, we have the same **minimum spanning tree** with a **weight** of 1322.

Notice how in Example 11 step 2, we added {Br, Bu} because it is adjacent to {V, Br} and in step 4 {Bu, Z} because it is adjacent to {Br, Bu}. This is not a requirement of Kruskal's algorithm. In this specific example, both algorithms happened to add the edges in the same order. This is not always the case. Notice how in Example 10 step 2, we added $\{E, F\}$ even though it is not adjacent to $\{B, C\}$, which is in the tree already. To show the difference between the two algorithms, the next example will apply Prim's algorithm to the same graph.

Example 12

Apply Prim's algorithm to the graph given in Example 10. For demonstration purposes, the cost adjacency matrix is produced here.

$$C_G = \begin{array}{c|c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} - & 3 & 3 & - & - & 5 \\ 3 & - & 2 & - & - & 4 \\ 3 & 2 & - & 6 & 7 & 8 \\ - & - & 6 & - & 5 & 4 \\ - & - & 7 & 5 & - & 2 \\ 5 & 4 & 8 & 4 & 2 & - \end{pmatrix} \end{array}$$

Solution

> **Note:** $\{A, C\}$ is added as it is adjacent to $\{B, C\}$. Notice that at this stage in Kruskal's algorithm, we add $\{E, F\}$ instead because it is the next 'lightest' edge.

Since there are two edges with the same weight of 2, we can start with either of them. We will start with the edge $\{B, C\}$.

Step 1: $T = \{\{B, C\}\}, \quad wt(\{B, C\}) = 2$

Step 2: $T = \{\{B, C\}, \{A, C\}\}, \quad wt(\{A, C\}) = 3$

Step 3: $T = \{\{B, C\}, \{A, C\}, \{B, F\}\}, \quad wt(\{B, F\}) = 4$

Step 4: $T = \{\{B, C\}, \{A, C\}, \{B, F\}, \{F, E\}\}, \quad wt(\{F, E\}) = 2$

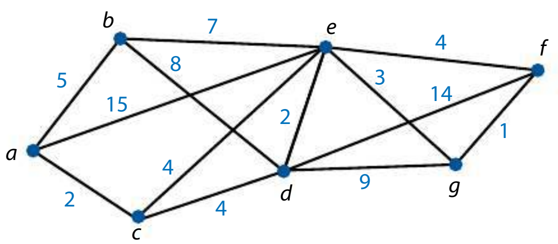
Step 5: $T = \{\{B, C\}, \{A, C\}, \{B, F\}, \{F, E\}, \{F, D\}\}, \quad wt(\{F, D\}) = 4$

STOP

So, the **minimum spanning tree** has the same **weight** of 15, but the process of adding edges to the tree had a different order.

Note: Kruskal's algorithm appears to be the easier of the two. However, this is only true for small graphs. As the graph size increases, spotting a cycle in Kruskal's algorithm is more difficult than in Prim's algorithm.

Example 13



Apply Kruskal's and Prim's algorithms to find a minimum spanning tree for the graph left.

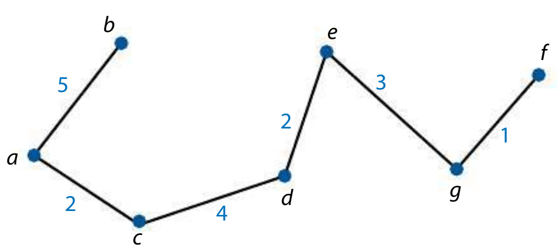
Solution

In both cases, since we have seven vertices, we will stop after finding six edges. We will set up a table of weights that will help us in finding the spanning trees we need.

Weight	1	2	2	3	4	4	4	5	7	8	9	14	15
Edge	*fg*	*de*	*ac*	*eg*	*ef*	*cd*	*ce*	*ab*	*be*	*bd*	*dg*	*df*	*ae*

Kruskal's algorithm

Weight	Edge	Cycle	Tree	Total weight	*i*
1	*fg*	no	*fg*	1	1
2	*de*	no	*fg, de*	3	2
2	*ac*	no	*fg, de, ac*	5	3
3	*eg*	no	*fg, de, ac, eg*	8	4
4	*ef*	yes, reject	*fg, de, ac, eg*	8	4
4	*cd*	no	*fg, de, ac, eg, cd*	12	5
4	*ce*	yes, reject	*fg, de, ac, eg, cd*	12	5
5	*ab*	no	*fg, de, ac, eg, cd, ab*	17	6
		Stop	Tree found	17	



On the left is the minimum spanning tree.

Prim's algorithm

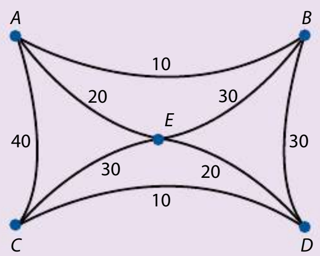
Weight	Edge	Adjacent	Cycle	Tree	Total weight	*i*
1	*fg*		no	*fg*	1	1
2	*de*, *ac*	no		*fg*	1	1
3	*eg*	yes	no	*fg*, *eg*	4	2
2	*de*	yes	no	*fg*, *eg*, *de*	6	3
2	*ac*	no		*fg*, *eg*, *de*	6	3
4	*ef*	yes	yes, reject	*fg*, *eg*, *de*	6	3
4	*cd*	yes	no	*fg*, *eg*, *de*, *cd*	10	4
2	*ac*	yes	no	*fg*, *eg*, *de*, *cd*, *ac*	12	5
4	*ce*	yes	yes, reject	*fg*, *eg*, *de*, *cd*, *ac*	12	5
5	*ab*	yes	no	*fg*, *eg*, *de*, *cd*, *ac*, *ab*	17	6
			Stop	Tree found	17	

Notice that we found a minimum spanning tree with the same weight as Kruskal's. In this specific example, it turned out to be the same tree. However, this must not be the case. The only common result should be the weight of the tree. Also worth noting here is that in Kruskal's algorithm, once you finish investigating a minimum weight you move to the next level, while in Prim's algorithm, if the adjacency test fails, then you need to revisit the level at a later stage, as happened to edges *ac* and *de* (weight of 2) and *ce* (weight 4).

Exercise 4.4

For questions 1–5, use Kruskal's algorithm to find a minimum spanning tree (mst) for each given weighted graph.

1



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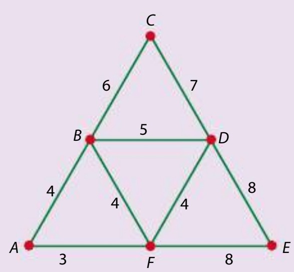

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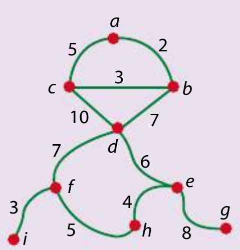

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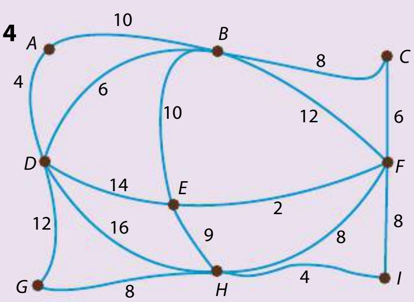

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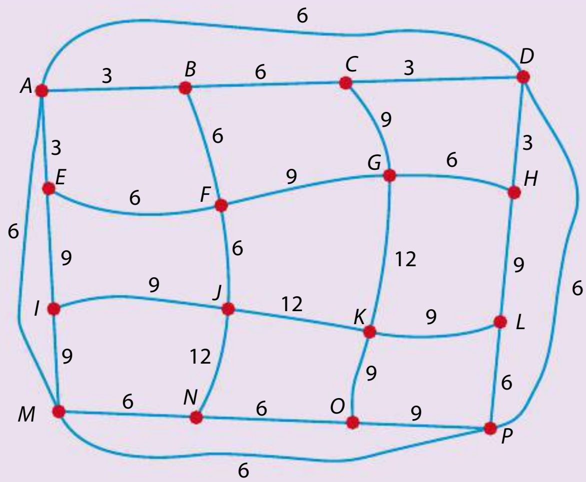


For questions 6–10 (optional), use Prim's algorithm to find a minimum spanning tree (mst) for each given weighted graph.

6 Find a mst for the graph in question 1.

7 Find a mst for the graph in question 2.

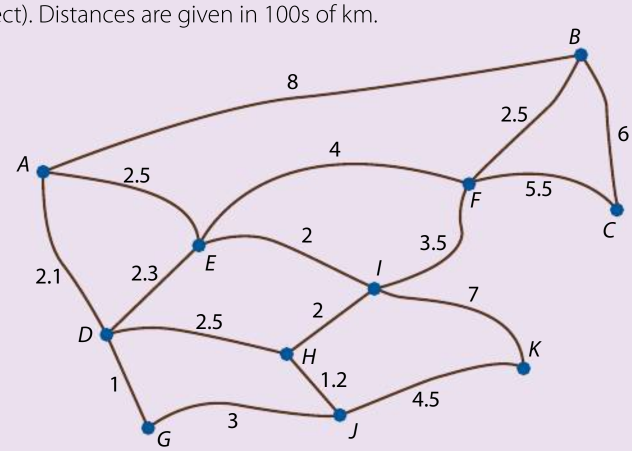
8 Find a mst for the graph in question 3.

9 Find a mst for the graph in question 4.

10 Find a mst for the graph in question 5.

11 (Optional) Describe the differences between the results of questions 1 and 6, 2 and 7, 3 and 8, 4 and 9, as well as 5 and 10.

12 The following is the network for a large bus company. To minimize cost, some routes must be discontinued. Find out which routes should be kept to ensure that transport between all the cities is still possible (though not necessarily direct). Distances are given in 100s of km.

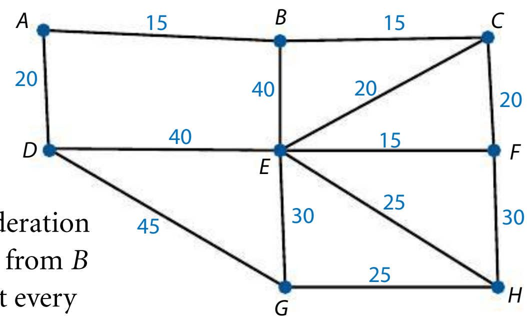


4.5 Shortest path, route inspection and the travelling salesman problem

A **shortest path** is a path from one vertex to another in a weighted graph, using the smallest possible weight. As a path, no edges or vertices are visited more than once. The shortest path, especially in complex networks, is not always evident. That is why Edsger Dijkstra, a Dutch mathematician, in his shortest-path algorithm created a way for finding the shortest path. In this section, we will discuss the algorithm and apply it to a few situations. However, you need to keep in mind that in textbook examples, the solution may be readily obvious by inspection or trial and error. However, by learning the algorithms, like many other aspects of graph theory, you are developing the skills which can later be used in more complex situations. So, even if you can immediately spot the solution to a problem, we strongly recommend that you follow the algorithm's steps in order to understand how to apply it later.

Example 14

In the weighted graph right we are required to find a path between vertices *A* and *H* which has the smallest total weight.

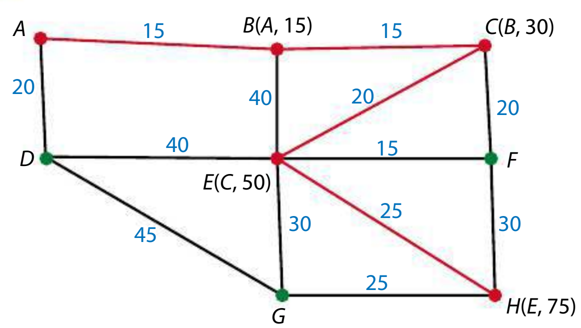


Solution

We can proceed from *A* to the 'nearest' vertices, taking into consideration the least weight possible. So, from *A* we can go to *B* or to *D*. Then from *B* we can go to *C* or *E*, while from *D* we can go to *E* or *G*. Arriving at every new vertex, we look at the total weight of the path. If there is a new path to arrive at the old vertex, we consider the total weight; if it is smaller than the one we already have, we cross out the old path and adopt the new one instead. The whole process is given in the table below. (Several ways of arranging your work are available and will be demonstrated.)

Step 1	Step 2	Step 3	Step 4	Step 5
A	*B*(*A*, 15)	*C*(*B*, 30)	*F*(*C*, 50)	
			E(*C*, 50)	
	D(*A*, 20)	~~*E*(*B*, 55)~~		*H*(*E*, 75)
		G(*D*, 65)	~~*H*(*G*, 90)~~	

Note that for every vertex we visit, we label it with a temporary label, which includes the previously visited vertex and the total weight, so far. In the third step, we labelled *E*(*B*, 55) because, so far, this is the smallest weight (coming through *B*), but then in the fourth step, once we reached *E* with a smaller weight of 50, we cross out *E*(*D*, 55). The same happens to the paths of the vertex *H* in the fifth step.



So, the path with the smallest weight is *ABCEH*.

Example 14 demonstrates the general rule used in Dijkstra's algorithm. It proceeds by finding the shortest path from *A* to its adjacent vertices, then the shortest path to a second 'level' set of vertices, and so on until the length of the shortest path to *H* is found.

The algorithm performs a sequence of iterations. A key set of vertices is assembled by adding one vertex at each iteration. A labelling process is executed at each iteration. In this labelling process, a vertex w is labelled with the length of a shortest path from A to w that contains only vertices from the key set. The vertex added to the set is one with the minimal label among those vertices not already members of the set. In the next few paragraphs, we give a formal statement of the algorithm followed by a description of the algorithm.

Dijkstra's algorithm[1]

procedure *Dijkstra* (G: weighted connected simple graph, with all weights positive)

{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$, where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G.}

for $i := 1$ **to** n

$L(v_i) := \infty$

$L(a) := 0$

$S := \varnothing$

{The labels are now initialized so that the label of a is zero and all other labels are ∞, and S is the empty set.}

while $z \notin S$

begin

$u :=$ a vertex not in S with $L(u)$ minimal

$S := S \cup \{u\}$

for all vertices v not in S

if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$

{This adds a vertex to S with minimal label and updates the labels of vertices not in S.}

end {$L(z)$ = length of shortest path from a to z}

Interpretation of Dijkstra's algorithm

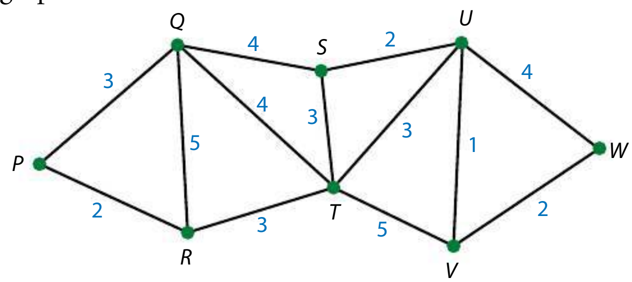
We need to find the shortest path from a to z. The algorithm begins by labelling a with 0 and the other vertices with ∞. We use the notation $L(v)$ to represent the shortest path from the source, a, to the present vertex v. S is the key set containing all vertices with minimum path length discovered so far. We begin with $S = \varnothing$. Every iteration will update the set S by adding a new vertex u with the smallest label. Once this is done, we update the labels of all vertices not in S, say v, such that $L(v)$ is the length of the shortest path to v through vertices already in S. This process is iterated successively adding vertices to the key set until z is added. In the following example, we will demonstrate the use of this algorithm. There are several interpretations of how to keep track of the successive steps; we will use the following convention:

[1] Kenneth Rosen, *Discrete Mathematics and its Applications*, 5th edition (McGraw-Hill, 2003) p. 597.

each vertex, v, is labelled with an ordered pair (x, l), where x represents the vertex just preceding v and l is the shortest length of the path from a. All labels are temporary, until the algorithm identifies their path as shortest and they are changed into permanent labels, which we will denote by circling the vertex. Any temporary label that does not become permanent will be crossed out. We will also use tables to demonstrate the steps.

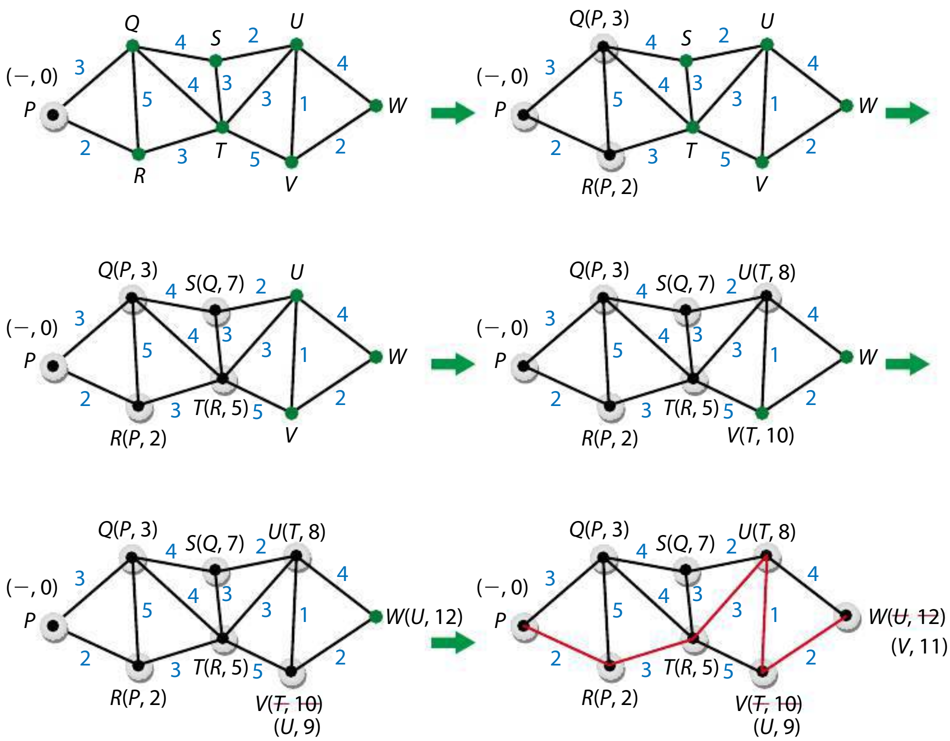
Example 15

Use Dijkstra's algorithm to find the shortest path between P and W in the following graph.



Solution

Note that only in this example will we draw the graph at different stages. You would not have to do that if you were performing the algorithm. In the diagrams below, we use the convention that if a vertex is not labelled, then it has the label $(-, \infty)$.



Below is the table with the steps. Each cell contains the length of the path and the preceding vertices. The highlighted cells are the ones describing the shortest path. Each cell also lists the path lengths that are calculated at this stage.

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7
$L(P) = 0$						
$L(R) = \infty$	$L(R) = 2, \{P\}$					
$L(Q) = \infty$	$L(Q) = \infty$	$L(Q) = 3, \{P\}$				
$L(T) = \infty$	$L(T) = \infty$	$L(T) = 5, \{P, R\}$				
$L(S) = \infty$	$L(S) = \infty$	$L(S) = \infty$	$L(S) = 7, \{P, Q\}$			
$L(U) = \infty$	$L(U) = \infty$	$L(U) = \infty$	$L(U) = \infty$	$L(U) = 8, \{P, R, T\}$		
$L(V) = \infty$	$L(V) = \infty$	$L(V) = \infty$	$L(V) = \infty$	$L(V) = 10, \{P, R, T\}$	$L(V) = 9, \{P, R, T, U\}$	
$L(W) = \infty$	$L(W) = \infty$	$L(W) = \infty$	$L(W) = \infty$	$L(W) = \infty$	$L(W) = 12, \{P, R, T, U\}$	$L(W) = 11, \{P, R, T, U, V\}$

Step 1: We start by labelling $P\,(-, 0)$ since there is no vertex to precede it. Make it permanent.

Step 2: From A there are two unlabelled vertices, Q and R. Since $L(P) = 0$, vertex R gives the smallest $L(P) + w(P, R) = 0 + 2$, then we label $R(P, 2)$ and we add it to the path S. Make it permanent.

Step 3: Now S has two vertices, P and R. They have two unlabelled adjacent vertices, Q and T. Vertex Q has the smallest $L(P) + w(P, Q) = 0 + 3 = 3$ $(L(R) + w(R, T) = 2 + 3 = 5$, and $(L(R) + w(R, Q) = 2 + 5 = 7)$. We make $Q(P, 3)$ permanent.

Step 4: Now S has three vertices, P, R, and Q. They have two unlabelled adjacent vertices, S and T. Similar to the previous process, we make $T(R, 5)$ permanent.

Step 5: Now S has four vertices, P, R, Q, and T. They have three unlabelled adjacent vertices, S, U, and V. Similar to the previous process, we make $S(Q, 7)$ permanent.

Step 6: Now S has five vertices, P, R, Q, T, and S. They have one unlabelled adjacent vertex, W. Similar to the previous process, we make $U(T, 8)$ permanent and *update* $L(V)$.

Step 7: Similar to above, we make $V(U, 9)$ permanent, and *update* $L(W)$ and make it permanent.

So, the shortest path is *PRTUVW* and it has a length of 11.

A practical interpretation of Dijkstra's algorithm

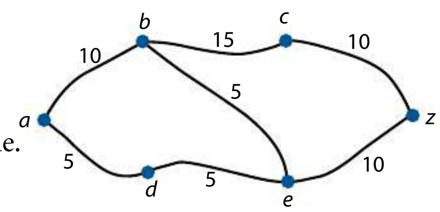
To find a shortest path from vertex a to vertex z in a weighted graph, proceed as follows:

1. Set $v_1 = a$ and assign to this vertex the label $(-, 0)$. Assign every other vertex a temporary label of ∞, where ∞ is reckoned to be larger than any real number!

2. Until z has been assigned a *permanent* label, do the following:
   (i) Take the vertex v_i that most recently acquired a permanent label, say d. For each vertex that is adjacent to v_i which has not yet received a permanent label, if $d + w(v_i v) < L(v)$, the current temporary label of v, update $L(v)$ to $d + w(v_i v)$.
   (ii) Take a vertex v that has a temporary label smallest among all temporary labels in the graph and make its temporary label permanent. If there are several vertices v that tie for the smallest temporary label, make any choice.

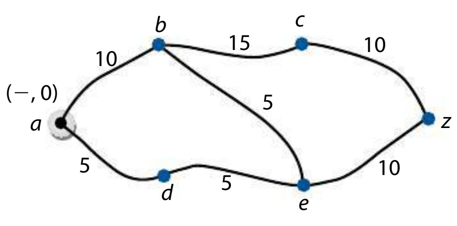
Example 16

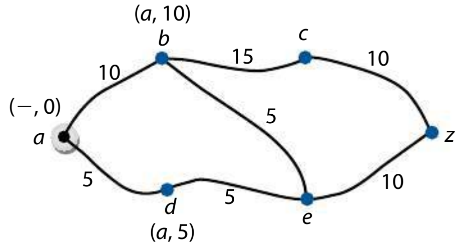
Find a shortest path from a to z in the graph on the right.



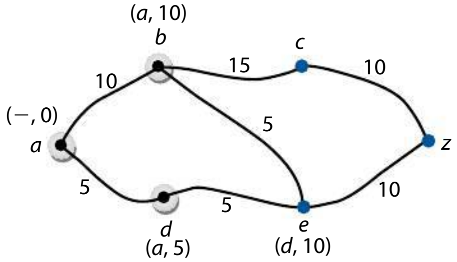
Solution

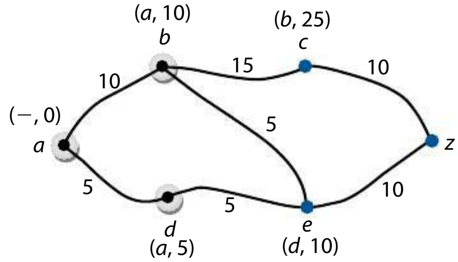
We will follow the algorithm by labelling the graph without a table this time.



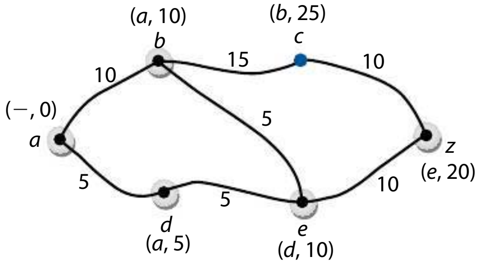


First we label and make a permanent. Next, we label vertex d with $(a, 5)$ to indicate the length of the path and that it is visited through a. Similarly, we label b $(a, 10)$.





Next we make d permanent and update vertex e. Then we make b permanent and update vertices c and e (no change in e).



Next we make e permanent and update z. At this point, we can make the label at z permanent; a shortest path has been found.

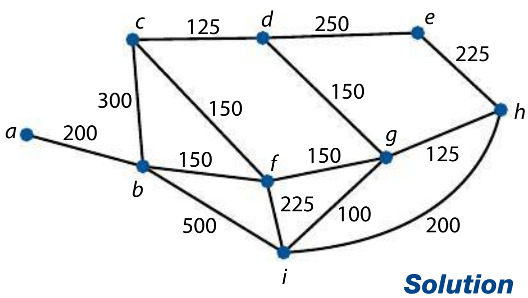
Notice in the above example that it is not necessary to change the label of a vertex v if $d + w(v_i v) \not> L(v)$, and that it is also unnecessary to make all vertices in the graph permanent as long as they don't contribute towards a shortest path.

The Chinese postman problem

This is also known as the **route inspection problem**. Contrary to its name, this has little to do with a 'real' Chinese postman. The reference is to the Chinese mathematician Kwan Mei-Ko who, in 1962, posed an inspection problem in terms of a postman covering each road of a network exactly once and coming back to his starting point.

We will start this subsection with an example.

Example 17



A cable network has to be inspected for possible faulty wires. The diagram left represents a sketch of the wires along with the length of each section (in metres) and the junction names. We would like to inspect every cable at least once and come back to the starting junction, a.

Solution

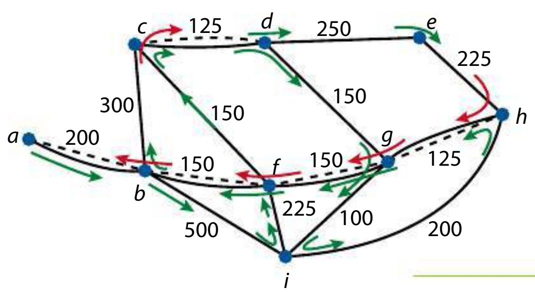
The problem is similar to finding an Eulerian circuit. However, this is not possible since we have four vertices with odd degree: a, c, d, and h.

Since we are starting at a, ab has to be retraced. This makes b also with odd degree. Knowing that we have to get back to b to reach a, leaves us now with four vertices with odd degree. To be able to inspect the cables, we need to retrace some of the paths between these junctions. We will consider all possible pairings that result in shortest lengths.

bc and dh: $300 + 275 = 575$

bd and ch: $425 + 400 = 825$

bh and cd: $425 + 125 = 550$



So, bh and cd is the shortest, and hence we will retrace these paths.

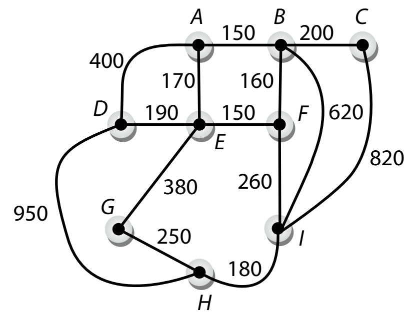
The original network has 2850 metres, and we will retrace $ab = 200$ and $bh + cd = 550$, giving a total length of 3600 metres. Such a route is: *abifcdgihgfbcdehgfba*. The route is given left.

As you may have noticed, when the number of edges to be inspected is high, the process will be tedious to follow. The algorithm proposed by Kwan Mei-Ko makes the process more systematic.

Chinese postman algorithm

1. Find all vertices of odd degree.
2. For each pair of odd vertices, find the path of shortest length.
3. Pair up all odd vertices from step 2, so that the sum of the lengths is minimum.
4. In the original graph, duplicate the shortest-length paths found in step 3.
5. Find an Eulerian circuit containing all edges of the 'new graph'.

Example 18



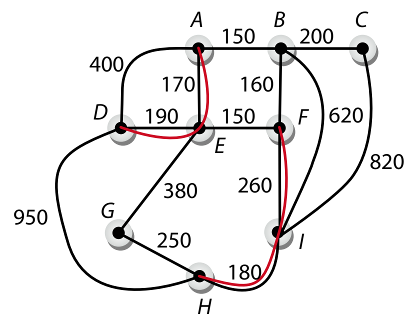
A guard patrols a campus of a large school as given by the graph right. The weights of the edges are distances given in metres. If the guard must pass through each street at least once during his shift, find the minimum distance he will cover.

Solution

The sum of all the distances in the graph is 4880 metres.

There are four odd vertices: *A*, *D*, *F*, and *H*. We need to investigate all the possible pairings and then choose the shortest paths between pairs of vertices.

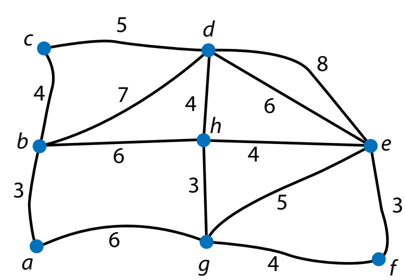
Pairing	Shortest path	Distance (m)
A, *D*	*AED*	360
A, *F*	*ABF*	310
A, *H*	*ABFIH*	750
D, *F*	*DEF*	360
D, *H*	*DEGH*	820
F, *H*	*FIH*	440



Now, we need to look at the pairings that will include all four vertices and give us the minimum sum of the distances. The pairings are *A*, *D* and *F*, *H*, and the paths that we will repeat are *AED* and *FIH* with their distances of 360 and 440 metres.

So, the minimum distance the guard will cover in one shift is $4880 + 360 + 440 = 5680$ metres. We leave tracing a path with length of 5680 for you as an exercise.

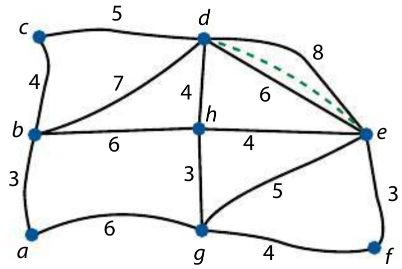
Example 19

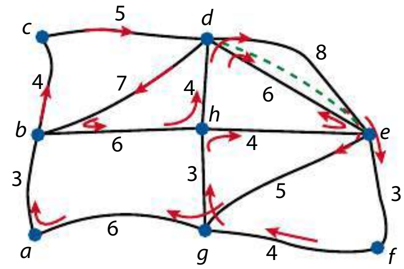


A truck has to visit a neighbourhood with a street network as shown. What is a possible route that minimizes the distance it has to travel? Distances are in kilometres.

Solution

Vertices *d* and *e* are odd. So, we first duplicate the shortest path between them which is 6, and then try to find the minimum distance to be travelled.





Since all vertices are even by now, the graph is Eulerian. We can use the algorithm developed in Example 6 of Chapter 3, or any other method, to find the circuit. If we start at *b*, we can create a cycle *bcdb*, which can be joined at *b* with *hdefgh*, which can be joined at *h* with *edegab*. Our route is then *bcdbhdefghedegab* with length of 68 + 6 (retracing *de*) = 74 km. This is not unique. You can find other circuits with the same minimum length of 74 km.

The travelling salesman problem

Given a set of cities and the cost of travel between each pair of them, the travelling salesman problem, or TSP for short, is to find the cheapest way of visiting all of the cities and returning to your starting point.

The simplicity of the statement of the problem is misleading. The TSP is one of the most considered problems in computational mathematics and yet no successful solution method is known for the general case.

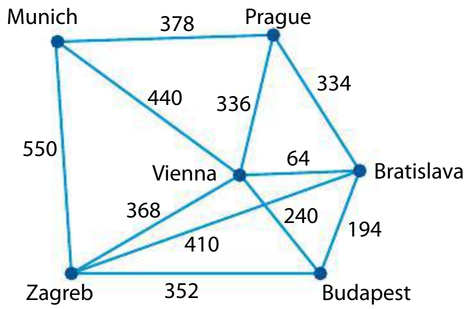
The TSP naturally arises in many transportation and logistics applications; for example, practical uses for the TSP include routing trucks for package pickups and material handling in warehouses. Other applications involve the scheduling of service calls at communications businesses.

Although transportation applications are the most natural setting for the TSP, the simplicity of the model has led to many interesting applications in other areas. A classic example is the scheduling of a machine to drill holes in a circuit board. In this case the holes to be drilled are the cities, and the cost of travel is the time it takes to move the drill head from one hole to the next. The technology for drilling varies from one industry to another, but whenever the travel time of the drilling device is a significant portion of the overall manufacturing process, then the TSP can play a role in reducing costs.

Basically, the travelling salesman problem is related to the search for Hamiltonian cycles in a graph. We will start with a simple example.

Example 20

A travelling salesman lives in Vienna. He needs to go on a business trip by car, visiting the following cities: Prague, Munich, Zagreb, Budapest, and Bratislava. On the figure right the distances between the cities are given in kilometres. (Not all routes have been included in the diagram.)



Solution

There are several possible Hamiltonian cycles and for each we calculate the total distance travelled.

Cycle	Distance (km)
V Br P M Z Bu V	1918
V P M Z Bu Br V	1874
V M Z Bu Br P V	2206
V Z Bu Br P M V	2066
V Bu Br P M Z V	2064

The shortest cycle is the second one from Vienna to Prague, Munich, Zagreb, Budapest, Bratislava, and back to Vienna, which has a total length of 1874 km. Since every cycle can have two directions, it is possible to visit all the cities in reverse order.

The solution presented in the example is a trial and error approach. Are there other approaches?

Remembering that a Hamiltonian cycle is a cycle that visits every vertex in a connected graph exactly once, we see that the classical TSP is a Hamiltonian cycle with minimum length. However, similar to the Chinese postman problem, we allow vertices to be visited more than once.

As you observed in the previous example, if you can inspect all possible routes involved in the TSP, you will be able to find the minimum total weight. However, as the number of vertices increases, checking all possibilities becomes impractical, if we don't say impossible. There is an assumption that the graph under consideration is *complete*, and as such, theoretically, the number of possible routes to inspect for a graph with n vertices will be $\frac{(n-1)!}{2}$ (considering routes in reverse order). For example,

if you have five cities, then the number of routes to be inspected will be 12 and if you have 10 cities the number will jump to 181 440. If there are 20 cities, the number will be $6.0 \cdot 10^{17}$. Even if you have a fast computer that can calculate 1 000 000 routes per second, it will take such a computer approximately 19 years to finish the task! So far, there is no known solution to the general TSP problem. Mathematicians resorted to finding near-minimum-weight solutions. Many algorithms have been developed. The nearest neighbour algorithm, nearest insertion algorithm, cutting-plane methods, and branch-and-cut methods are a few such algorithms. The IB syllabus does not require you to use such algorithms and thus we will not discuss these concepts in detail in this publication. We will just demonstrate the use of two of the algorithms without requiring you to do them.

In discussing the TSP in this publication, we will consider complete graphs with at least three vertices. Such graphs will have a Hamiltonian cycle. Moreover, since the number of vertices is finite, then the number of Hamiltonian cycles will also be finite. Thus, there must be at least one with minimum weight.

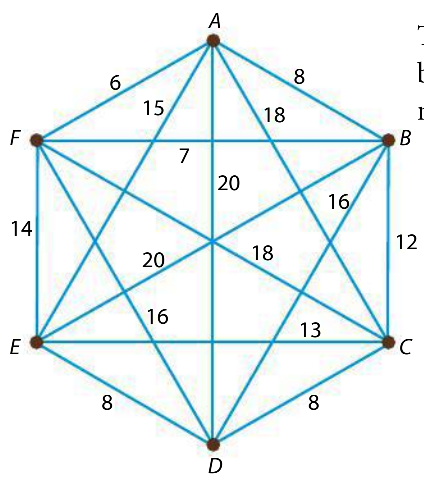
Also, since the weights of the edges in the complete graphs represent the shortest distances between the nodes of the original route network, the complete graph must satisfy the **triangle inequality**. As you recall from geometry, the sum of two sides of a triangle must be larger than or equal to the third side. Thus, for every choice of three vertices, v_i, v_j, and v_k, the following must be true:

$$w(v_i, v_j) + w(v_i, v_k) \geqslant w(v_j, v_k)$$

The nearest neighbour algorithm

1. Choose a starting vertex.
2. Consider the edge of smallest weight incident to this vertex. If the other end of this edge is not visited yet, add it to the tour.
3. Repeat step 2 until all vertices have been visited.
4. Add the edge connecting the last visited vertex to the starting vertex.

where $w(v_i, v_j)$ is the weight of the corresponding edge.

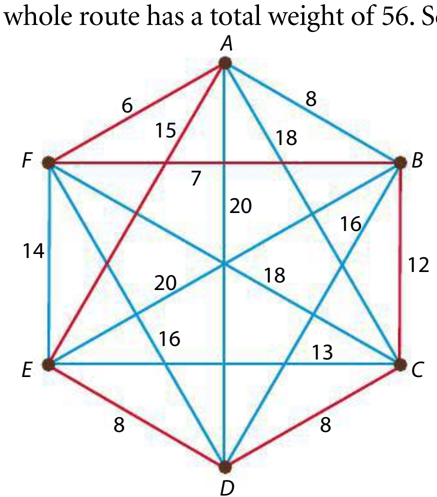


This algorithm will *sometimes* produce a minimal Hamiltonian cycle, but, in general, it may produce cycles with a considerably greater than minimum weight.

We will use the complete graph on the left to demonstrate the algorithm. The salesman is to start and end at A.

Starting at A, the first edge is AF since 6 is the minimum among 6, 8, 15, 18, and 20.

With the same reasoning, FB is chosen, with weight 7. BC is next with a weight of 12, followed by CD with 8, DE with 8 and finally, we get back to A with 15. The whole route has a total weight of 56. See the figure below.



The nearest insertion algorithm

This algorithm creates a cycle in the graph and then enlarges it to include a vertex which is closest to the given cycle.

1. Choose a starting vertex, u_1.
2. Consider the edge of smallest weight incident to this vertex. Add it to the cycle C. The vertex at the other end of this edge is added to C; call it u_2.
3. Select an edge with minimum weight that joins a vertex in C to one not in C; call the new vertex v.
4. Next, we enlarge the cycle to include the new vertex v. Now we consider the following expression:

   $x = w(u_i, v) + w(u_j, v) - w(u_i, u_j)$

   We choose the pair of vertices u_i and u_j for which x is minimum. We then include the edges (u_i, v) and (u_j, v), and we remove (u_i, u_j). (x represents the increase in the weight of the cycle when we add v.)
5. Repeat steps 3 and 4 until we include all vertices in the cycle.

Let us apply the algorithm to the previous graph.

We start with AF as it is the smallest, then we add B. Now we have a cycle AFB as shown in the first diagram overleaf.

Now consider all possible cycle expansions by comparing the x values for adding any of the remaining vertices. Here are the values:

Consider vertex C:
$AC + CF - AF = 18 + 18 - 6 = 30$, $AC + CB - AB = 18 + 12 - 8 = 22$,
$BC + CF - FB = 18 + 12 - 7 = 23$

Consider vertex D:
$20 + 16 - 8 = 28$, $16 + 20 - 6 = 30$, $16 + 16 - 7 = 25$

Consider vertex E:
$14 + 15 - 6 = 23$, $15 + 20 - 8 = 27$, $14 + 20 - 7 = 27$

So, 22 is the minimum, and since it corresponds to the connection of C and A and B, we add AC and BC and remove AB. Now the cycle is $AFBCA$ as shown in the second diagram.

Repeat the same steps for the new cycle:

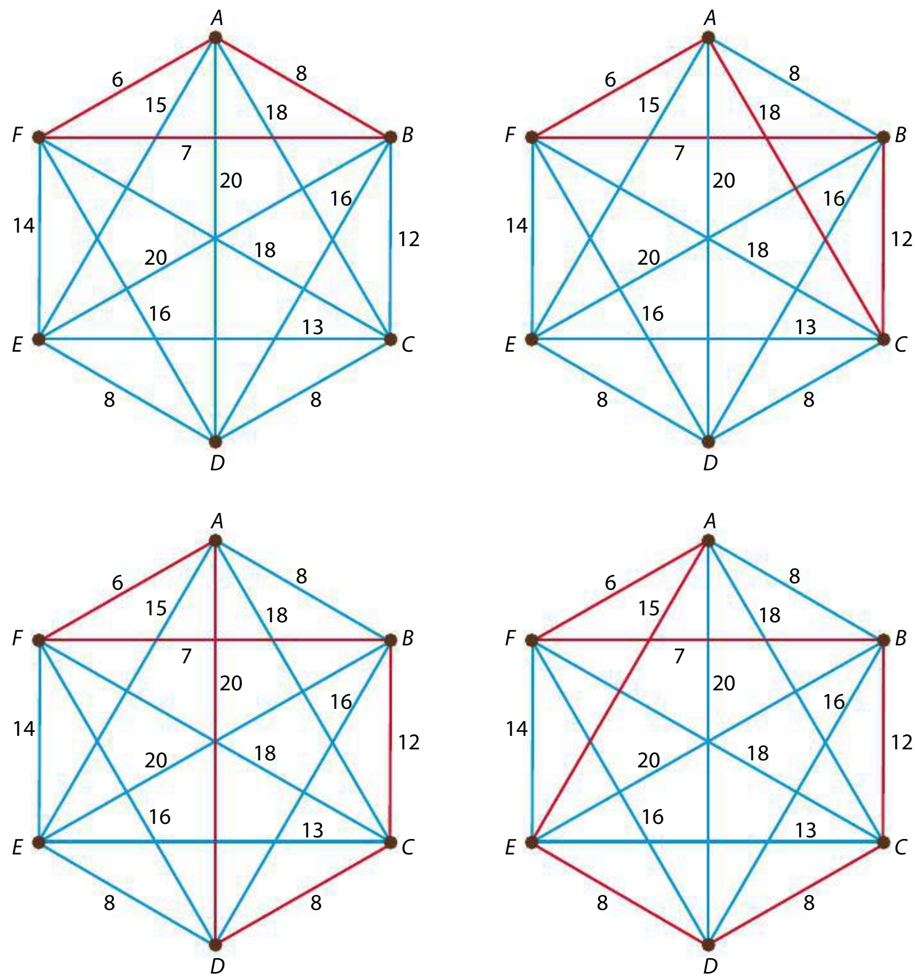
Consider vertex D:
$8 + 16 - 12 = 12$, $8 + 20 - 18 = 10$, $16 + 20 - 6 = 30$, $16 + 16 - 7 = 25$

Consider vertex E:
$14 + 15 - 6 = 23$, $14 + 20 - 7 = 27$, $15 + 13 - 18 = 10$, $20 + 13 - 12 = 21$

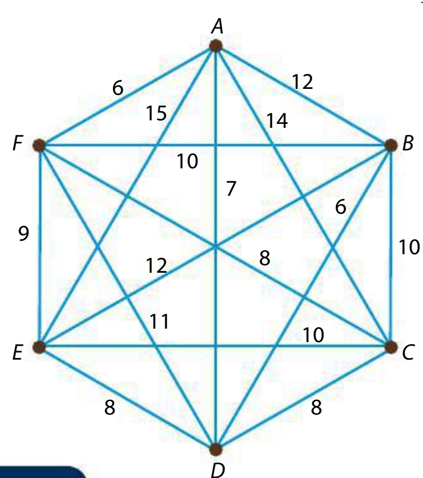
Thus, DC and DA are added and AC removed. (Notice that we could have added E at this stage instead of D. See the third diagram.)

Lastly, consider E:
$15 + 8 - 20 = 3$, $14 + 20 - 7 = 27$, $14 + 15 - 6 = 23$, $20 + 13 - 12 = 21$, $13 + 8 - 8 = 13$

Thus, we add ED and EA and remove DA. The route now has a weight of 56 as before.



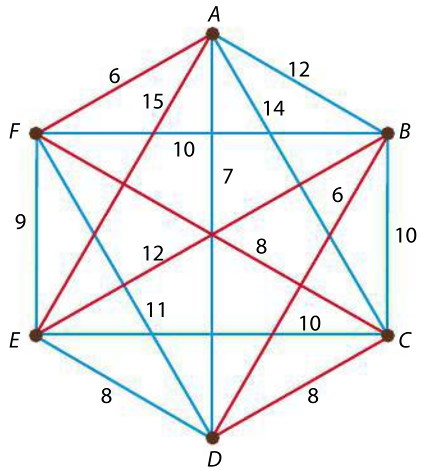
Caution: The equality between the routes created by these two algorithms are not always equal. And neither of them will definitely produce a minimum weight Hamiltonian cycle.



Example 21

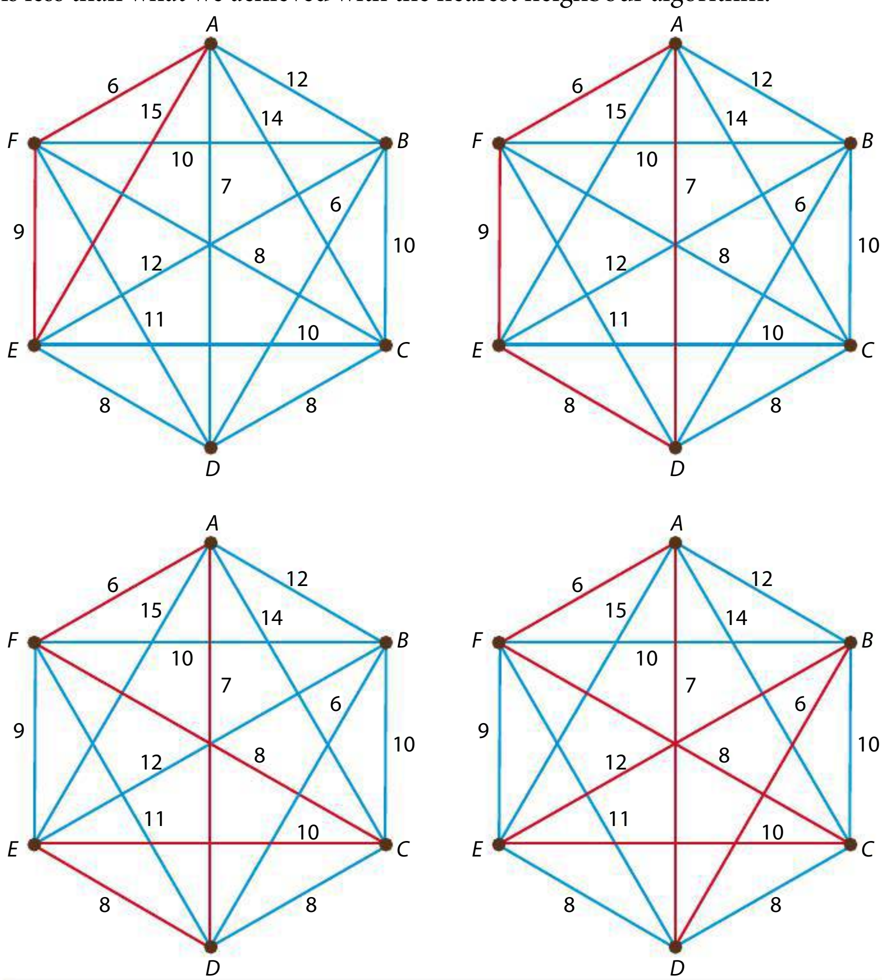
Consider the graph in the figure left and use the nearest neighbour and nearest insertion algorithms to find a minimum TSP tour.

Nearest neighbour algorithm
Starting at A, the next vertex must be F. From F, the edge with smallest weight leads to C, then similarly from C to D, then to B, to E, and finally back to A. The total weight is 55.



Nearest insertion algorithm

First cycle could be AFE with weight of 30. Considering x values for possible expansion, we find that can be achieved by adding vertex D with $x = 0$. We add AD and DE and remove AE. The weight so far is 30. Applying the algorithm again, we can add C to the cycle by adding FC and CE and removing FE. The cycle $AFCEDA$ has a weight of 39 so far. Lastly, we expand the cycle to include B by adding BD and BE, they have an x value of 10, and removing ED. So, the cycle now is $AFCEBDA$ with a total weight of 49, which is less than what we achieved with the nearest neighbour algorithm.



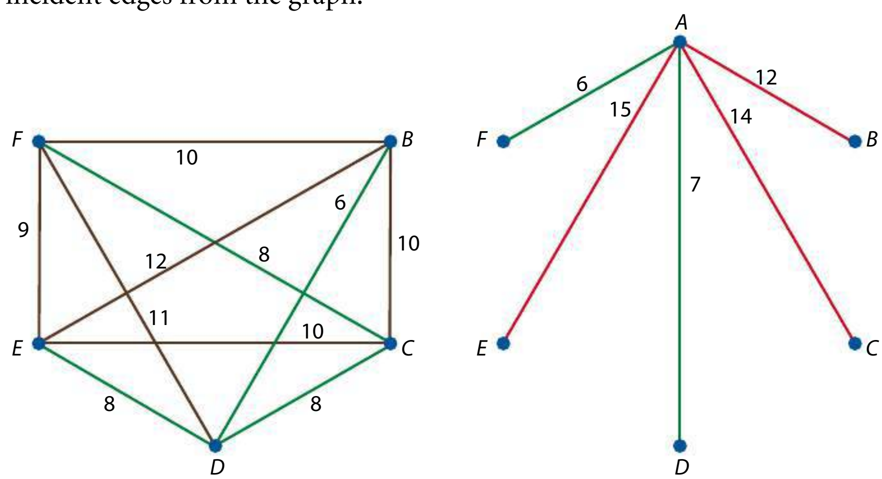
Lower and upper bounds

As Example 21 shows, the nearest neighbour algorithm, for example, did not lead us to a Hamiltonian cycle with minimum possible weight. As you observed, we were able to have an improved cycle. How far can we go?

A lower bound can be found by using algorithms that help us find minimal spanning trees. The argument is as follows: If we have a minimum weight Hamiltonian cycle in a complete graph, then we can remove one vertex v, for example, and all edges incident to it. Then we have a minimal spanning tree passing through the rest of the vertices. The weight of the Hamiltonian cycle is the weight of this minimal spanning tree plus the total weight of the edges we just removed. This argument leads us to the following **lower bound algorithm**.

1. Choose a vertex v in the complete graph and find the total of the two smallest edge weights incident to v.
2. Find the total weight of a minimum spanning tree going through all the remaining vertices.
3. The sum of the row totals is a lower bound.

Let us take the graph in Example 21 for instance. Remove A and its incident edges from the graph.



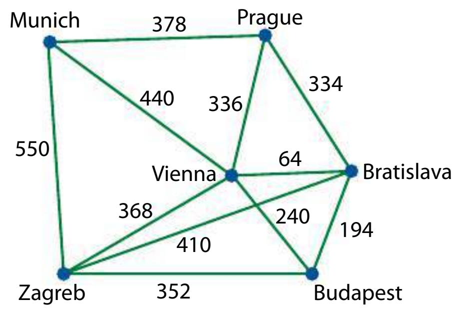
A minimum spanning tree for the remaining vertices is marked in green and has a weight of 30. The two edges with minimum total weight are AF and AD with a weight of 13. Hence, a lower bound for the cycle is 43, which is less than the smallest we found, 49.

So, now we can say that the minimum weight for a Hamiltonian cycle lies between 43 and 49.

As you notice from above, we used the weight of the Hamiltonian cycle we found earlier as an upper bound. There are a few ways of looking at an upper bound. One is to say the upper bound is the length of any cycle you manage to find, or, in general, is twice the length of a minimal spanning tree. The reason for this is a worst-case scenario. That is, the travelling salesman would visit every city and return that way, tracing each edge of the spanning tree twice.

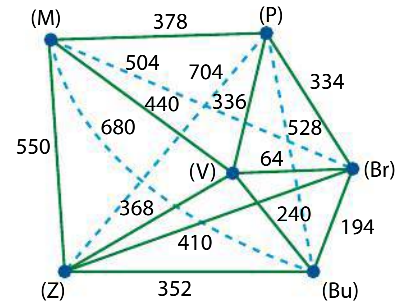
Example 22

We will try to find a lower bound, an upper bound, and a possible shortest route for the Vienna salesman in Example 20.



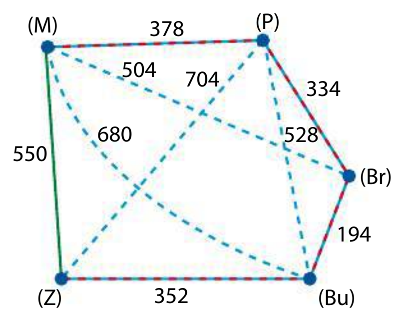
Solution

Note: As you may notice, the graph is not complete. However, in TSP we are allowed to add new edges which represent the minimum weight between two vertices that are not adjacent in the original graph. For example, Budapest and Prague are not directly connected; however, a path of minimum length of 334 + 194 = 528 through Bratislava can be added. Similarly, Budapest–Munich can have an extra edge of length 440 + 240 = 680 added, as well as Prague–Zagreb with 704 and Munich–Bratislava with 504. The new complete graph is given right.



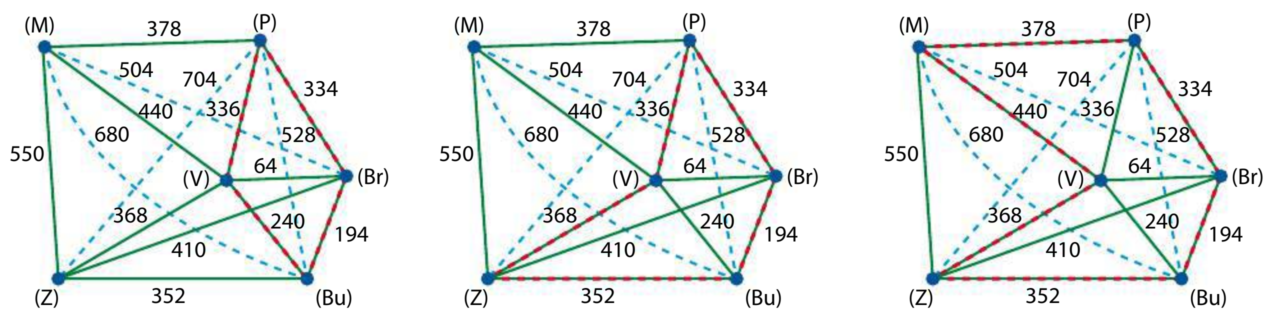
To find a lower bound, remove Vienna, for example, and all edges incident to it, and then find a minimum spanning tree for the rest. The tree weight is 1258.

The minimum total weight of two of the edges from Vienna is 64 + 240 = 304 and together with the minimum spanning tree this gives us a lower bound of 1562. Notice that if we remove another city, we may receive a different lower bound! An upper bound may be the route weight of 1874 that we found earlier.



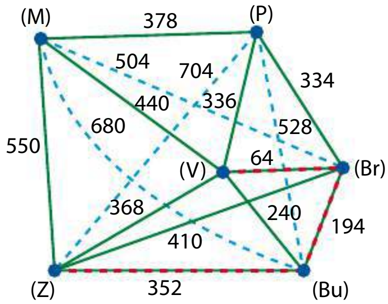
Thus, we are confident that our minimal Hamiltonian cycle would be between 1562 and 1874.

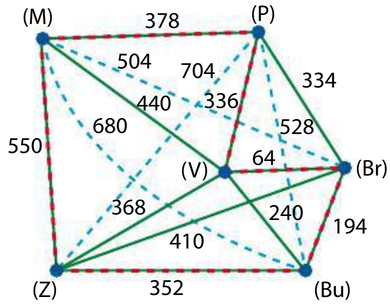
Apply the nearest insertion algorithm. You will expand the cycles; starting with V, Br, Bu, you will get the following sequence:



Unfortunately, the algorithm here did not yield the best results. The length of the route is 2066, which is greater than the upper bound.

Applying the nearest neighbour algorithm yields the following:

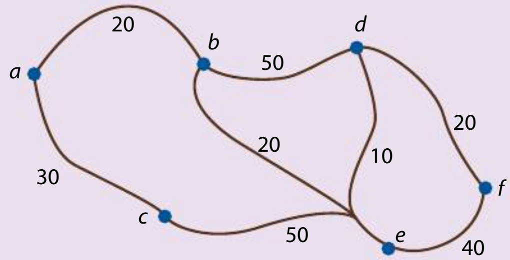




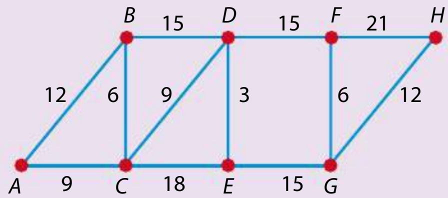
The total weight of this route is 1874, the same as that obtained by the 'brute force' method we used at the outset of this section and which we used as a lower bound. Notice here that the nearest neighbour algorithm gave better results than the nearest insertion algorithm. This again points to the fact that we do not have a unique solution to the TSP.

Exercise 4.5

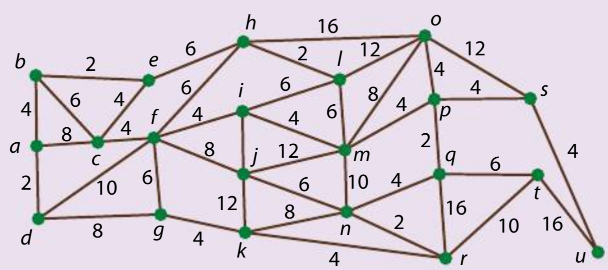
1 Find the length of the shortest path between *a* and *f* in the following weighted graph. Write down the path you suggest.



2 Find the length of the shortest path between *A* and *H* in the following weighted graph. Write down the path you find.



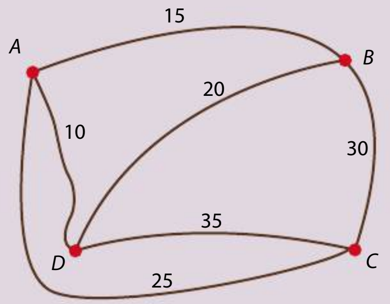
3 A circuit board has the following sub-network with the time, in millionths of a second, it takes a DC signal to flow through. Find the minimum time it takes a signal to go from *a* to *u*. Write down the path that gives this time.



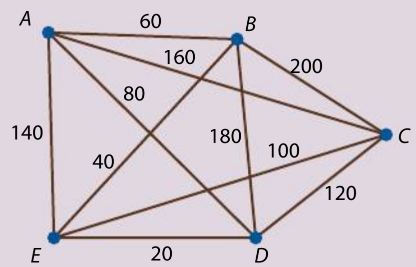
4 In question 1, find the shortest route between *a* and *d*.

5 In question 2, find the shortest route between *A* and *F*, and between *B* and *H*.

6 Solve the TSP for the following graph.

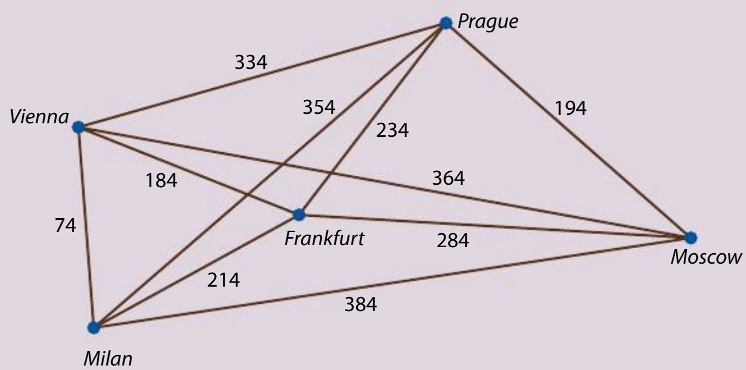


7 Solve the TSP for the following graph.



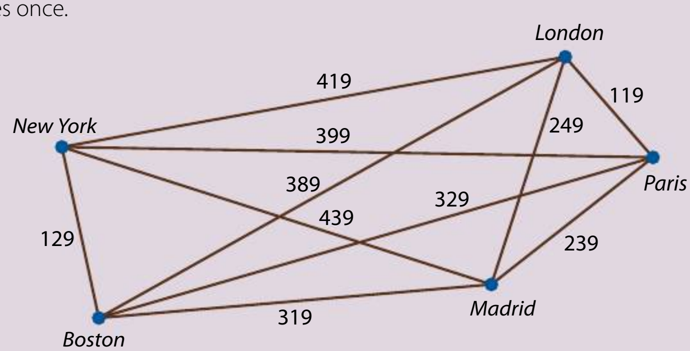
8 The flight paths between cities is given by the graph below. The weight on each edge is the cheapest possible two-way flight between the two cities. The prices are in Euros.

Find the route with the minimum total cost for a tourist who wants to visit each of the cities once.



9 The flight paths between cities is given by the graph below. The weight on each edge is the cheapest possible two-way flight between the two cities. The prices are in Euros.

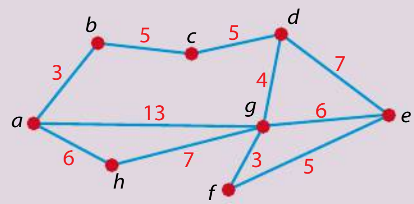
Find the route with the minumum total cost for a tourist who wants to visit each of the cities once.



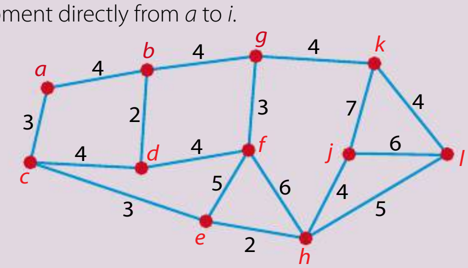
10 The nodes A, B, C, D, and E in a network have to be connected with the minimum length of cable. The distances between the nodes are given below. Find the most efficient connection route.

	A	**B**	**C**	**D**	**E**
A		100	90	80	110
B	100		130		120
C	90	130		120	
D	80		120		130
E	110	120		130	

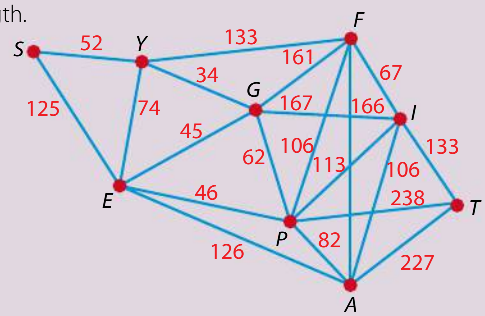
11 Use a shortest path algorithm to find the shortest route from a to e.



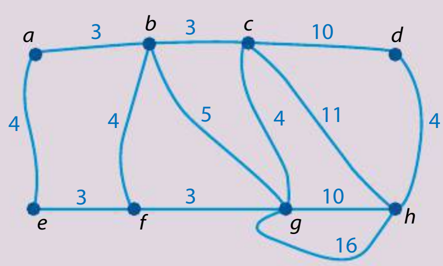
12 The graph below is the network of a transport company where the weights of the edges are distances in 10 km units. A shipment has to be transported from a to i. However, a part of the shipment has to be delivered to f first. Find the most efficient route for this delivery. Compare your result to the distance travelled when delivering the whole shipment directly from a to i.



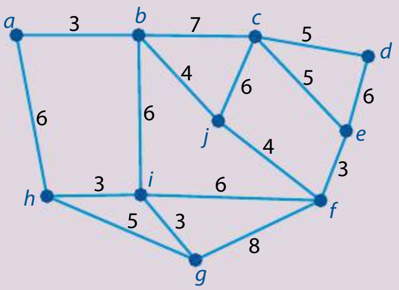
13 You are in charge of organizing the campaign tour for a politician. The following is a map showing the distances between the different cities that he must visit. He is based in E and needs to return there at the end of the tour. Find a suitable tour of minimum length.



14 A road sweeping truck has to sweep all the streets in a block of the city whose map is supplied. Distances are in 100s of metres. Find a route of minimum total length.



15 A local telephone network has to be inspected for possible defects. Find the shortest possible inspection tour to ensure that all cables have been checked. The sketch gives the length of each cable in 100s of metres.



Review questions 4

1 Show that if we delete an edge from a tree the remaining graph is not connected. The two unconnected components are subtrees to the original tree.

2 Show that if we add an edge between two non-adjacent vertices in a tree then the new graph contains only one cycle.

3 Show that a graph G contains a subgraph that is a tree if and only if it is connected and contains at least two vertices.

4 Let $T = (V, E)$ be a tree. Given that $|E| = 43$, find $|V|$.

5 **a** Let T be a tree with seven vertices. Find the number of all possible paths between the vertices in the tree (or subtrees).

b Find the formula for the number of all possible subtrees in a tree with n vertices.

6 Given that T is a tree, show that T contains at least two vertices of a degree 1.

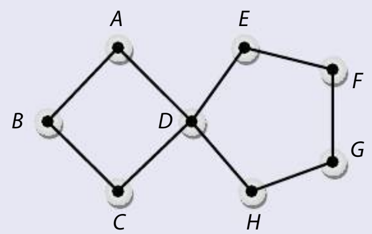
7 Given a complete graph with four vertices K_4, is it possible to find a spanning tree whose complement is also a spanning tree? Is it possible to find such a spanning tree in K_5?

8 Show that a complete bipartite graph $K_{m,n}$ contains a spanning tree with $m + n - 1$ edges.

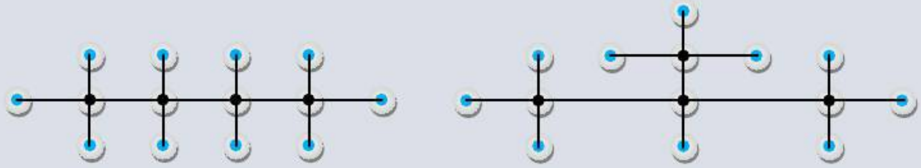
9 Given a complete bipartite graph $K_{2,2}$, is it possible to find a spanning tree whose complement is also a spanning tree? Is it possible to find such a spanning tree in $K_{2,3}$?

10 Draw all possible non-isomorphic trees with five vertices.

11 Find how many different spanning trees (some might be isomorphic) there are in the following graph.



12 The following graphs represent two molecules of chemical isomers of the saturated hydrocarbon C_4H_{10} (butane and isobutane). Each vertex that has a degree of 4 represents a carbon atom, C, whilst each vertex that has a degree of 1 represents a hydrogen atom, H. Explain why these two graphs are non-isomorphic.

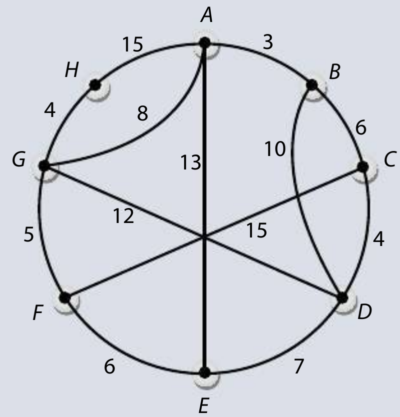


13 Given that a molecule (a tree) of a saturated hydrocarbon contains n carbon atoms (vertices of a degree 4), find how many hydrogen atoms (vertices of degree 1) there are.

14 Molecules of chemical isomers of the saturated hydrocarbon C_5H_{12} are called pentane, isopentane, and neopentane. Draw the trees representing those tree molecules, and give reasons why those three trees are mutually non-isomorphic.

15 Show that a complete binary tree with n internal vertices has $n + 1$ leaves.

16 Use Dijkstra's algorithm to find the shortest path between the vertices B and F in the following weighted graph.



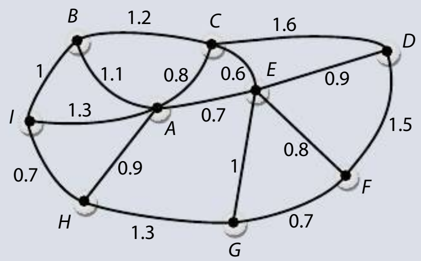
17 The bus routes connecting various cities and the cost of the tickets in dollars are given in the table below.

Cities	A	B	C	D	E	F
A	–	25	42	–	55	28
B		–	15	63	–	17
C			–	12	20	–
D				–	22	40
E					–	10
F						–

a Draw the weighted graph that represents all the routes between the cities.

b Jerry would like to travel from A to D. Determine the cheapest route and find how much will Jerry pay for his travel.

18 Ravi and his band have an upcoming concert in a club. He needs to display the concert posters in his neighbourhood. The following graph represents the plan of the posts where the posters can be displayed. Ravi's home is denoted by the vertex A. The distances between the posts are given in kilometres.



Find the shortest distance Ravi will need to travel in order to put the posters on all the posts before returning home.

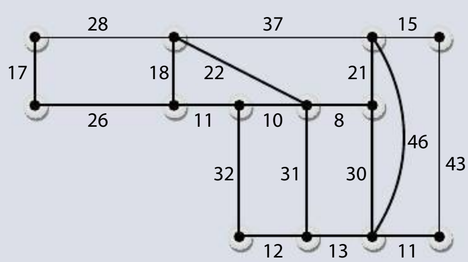
19 Jenny collects air miles and has earned 230 000 free miles through her air company. The cost of the plane tickets in free miles between the cities she visits is given in the matrix below. Each entry represents thousands of miles.

$$C_G = \begin{bmatrix} 0 & 0 & 10 & 0 & 20 & 25 & 10 \\ 0 & 0 & 10 & 18 & 54 & 0 & 0 \\ 10 & 10 & 0 & 8 & 0 & 50 & 0 \\ 0 & 18 & 8 & 0 & 0 & 0 & 45 \\ 20 & 54 & 0 & 0 & 0 & 28 & 32 \\ 25 & 0 & 50 & 0 & 28 & 0 & 16 \\ 10 & 0 & 0 & 45 & 32 & 16 & 0 \end{bmatrix}$$

a Draw a weighted graph representing the possible flights between the cities with the corresponding cost in free miles.

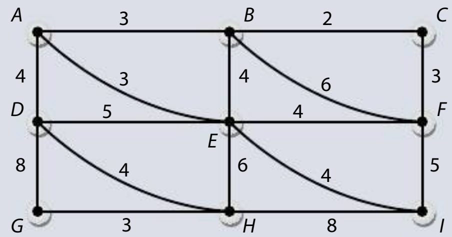
b Jenny would like to make a round trip and visit all the cities. What is the cheapest route and will she have enough miles for such a trip or she will need to buy some additional miles to pay for the trip?

20 Jack is a security guard. During the night shift he must patrol every single corridor of a warehouse. The plan of the corridors is given below. The time needed to patrol each corridor is given in minutes.



Is it possible for Jack to patrol the whole warehouse during his night shift from 10 p.m. till 6 a.m.? If yes, how many minutes will he have for a break? If not, how much longer would he need to stay in order to fulfil his duty?

21 Apply Kruskal's and Prim's algorithms to find the minimum spanning tree for the following graph.



Show all the decision steps in both algorithms. Draw the minimum spanning tree and state its weight.

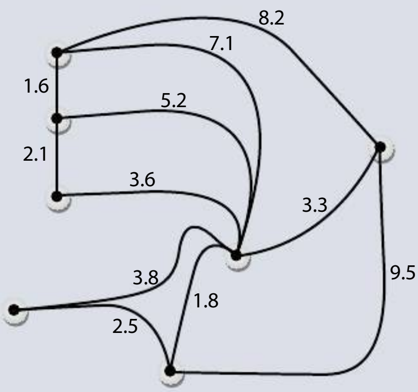
22 Adapt Kruskal's and Prim's algorithms to devise an algorithm to determine the maximum spanning tree in question 21.

23 Information on the distances between the cities in a country are provided in the table below. Each distance is given in kilometres.

Cities	P	Q	R	S	T	U
P	–	–	–	–	–	–
Q	200	–	–	–	–	–
R	292	487	–	–	–	–
S	333	465	222	–	–	–
T	86	282	203	257	–	–
U	333	509	133	97	235	–

The government would like to construct a system of highways to connect all the cities. Determine which highways should be built so that the cost of the construction is minimal. Assume that the cost of a kilometre of highway is constant.

24 Peter needs to install sockets that will be connected by an optical cable in his apartment so that he can watch TV and use the phone and internet in the rooms. The positions of the sockets are shown on the following graph. The distances between the sockets are given in metres.



Given that the cost of optical cable is 70 cents per metre, find the minimum price Peter will pay for buying the cable.

25 Christian plays a computer game in which he must enter rooms in order to collect some points. The points in the first level of the game are given in the following matrix.

$$C_G = \begin{bmatrix} 0 & 2 & 3 & 4 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 2 & 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 4 & 0 & 0 & 3 \\ 4 & 2 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 & 5 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 4 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 6 & 0 & 5 \\ 0 & 0 & 3 & 0 & 0 & 2 & 0 & 5 & 0 \end{bmatrix}$$

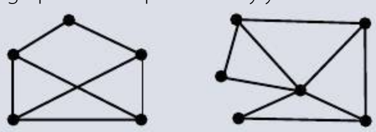
In order to advance to the higher level of the game, he must visit all the rooms in the shortest possible time. Find the maximum possible points Christian can collect at the first level.

Practice questions 4

Practice questions 1–10 cover work from Chapters 3–4 inclusive.

1 a Prove that if two graphs are isomorphic they have the same degree sequence.

b Are the following graphs isomorphic? Justify your answer.



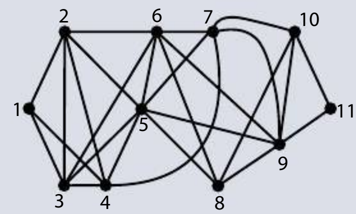
2 In an offshore drilling site for a large oil company, the distances between the planned wells are given below in metres.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	30									
3	40	60								
4	90	190	130							
5	80	200	10	160						
6	70	40	20	40	130					
7	60	120	50	90	30	60				
8	50	140	90	70	140	70	40			
9	40	170	140	60	50	90	50	70		
10	200	80	150	110	90	30	190	90	100	
11	150	30	200	120	190	120	60	190	150	200

a It is intended to construct a network of paths to connect the different wells in a way that minimizes the sum of the distances between them.

Use Prim's algorithm to find a network of paths of minimum total length that can span the whole site.

b Pipes are laid under water. Well 1 has the largest amount of oil to be pumped per day, and Well 11 is designed to be the main transportation hub. The only possible connections to be made between wells are shown in the diagram below.

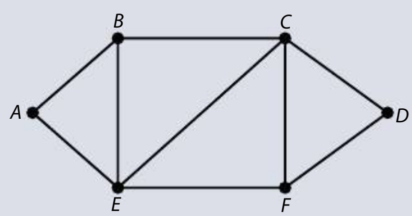


The associated cost for each pipe, in 100-thousand dollars, is given in the table below. Use Dijkstra's algorithm to find the path with minimum cost that can transport oil from Well 1 to Well 11.

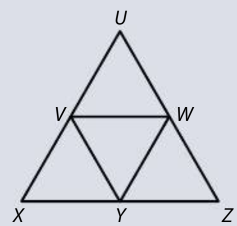
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	6									
3	3									
4	8	7	2							
5		14	12	6						
6		16	19		7					
7				24	20	29				
8					23	15				
9					56	30	41	50		
10							42	25	40	
11									32	22

3 a Define the isomorphism of two graphs *G* and *H*.

b Determine whether the two graphs below are isomorphic. Give a reason for your answer.



Graph *G*



Graph *H*

c Find an Eulerian trail for graph *G* starting with vertex *B*.

d State a result which shows that graph *H* has an Eulerian circuit.

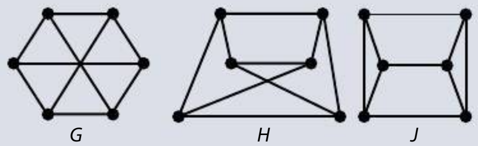
4 a Define the following terms.

i A bipartite graph.

ii An isomorphism between two graphs, *M* and *N*.

b Prove that an isomorphism between two graphs maps a cycle of one graph into a cycle of the other graph.

c The graphs G, H, and J are drawn below.

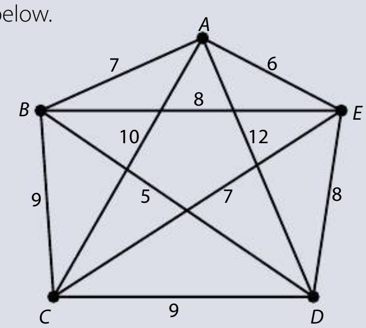


i Giving a reason, determine whether or not G is a bipartite graph.

ii Giving a reason, determine whether or not there exists an isomorphism between graphs G and H.

iii Using the result in part **b**, or otherwise, determine whether or not graph H is isomorphic to graph J.

5 Let G be the graph below.



a Find the total number of Hamiltonian cycles in G starting at vertex A. Explain your answer.

b i Find a minimum spanning tree for the subgraph obtained by deleting A from G.

ii Hence, find a lower bound for the travelling salesman problem for G.

c Give an upper bound for the travelling salesman problem for the graph above.

d Show that the lower bound you have obtained is not the best possible for the solution to the travelling salesman problem for G.

6 a Show that the sum of the degrees of all the vertices of a graph is even.

b There are nine men at a party. By considering an appropriate graph, show that it is impossible for each man to shake hands with exactly five other men.

c For a connected planar graph, prove Euler's relation, $v - e + f = 2$.

7 Consider the following adjacency matrices for the graphs G_1 and G_2.

$$\begin{array}{c} \\ p\\ q\\ r\\ s\\ t \end{array}\begin{array}{c} \begin{array}{ccccc} p & q & r & s & t \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ p\\ q\\ r\\ s\\ t \end{array}\begin{array}{c} \begin{array}{ccccc} p & q & r & s & t \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

G_1 $\qquad$ G_2

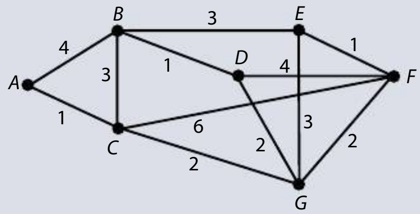
a Draw the graphs of G_1 and G_2.

b For each graph, giving a reason, determine whether or not it

i	is simple	**ii**	is connected
iii	is bipartite	**iv**	is a tree

v has an Eulerian trail, giving an example of a trail if one exists.

8 Let H be the weighted graph drawn below.



a **i** Name the two vertices of odd degree.

ii State the shortest path between these two vertices.

iii Using the route inspection algorithm, or otherwise, find a walk, starting and ending at A, of minimum total weight which includes every edge at least once.

iv Calculate the weight of this walk.

b Write down a Hamiltonian cycle in H.

9 A graph G has e edges and n vertices.

a Show that the sum of the degrees of the vertices is twice the number of edges.

b Deduce that G has an even number of vertices of odd degree.

c **i** Graph G is connected, planar and divides the plane into exactly four regions. If $(n - 1)$ vertices have degree 3 and exactly one vertex has degree d, determine the possible values of (n, d).

ii For each possible (n, d), draw a graph which satisfies the conditions described in **i**.

10 **a** **i** Let M be the adjacency matrix of a bipartite graph. Show that the leading diagonal entries in M^{37} are all zero.

ii What does the (i, j)th element of $M + M^2 + M^3$ represent?

b Prove that a graph containing a triangle cannot be bipartite.

c Prove that the number of edges in a bipartite graph with n vertices is less than or equal to $\frac{n^2}{4}$.

Answers

Chapter 1

Exercise 1.1–1.2

1 9 **2** 30 **3–6** Proof

7 a) Q = 30, R = 8
b) Q = −6, R = 70
c) Q = −5, R = 25

8–15 Proof

16 a) Proof b) $q = -8, r = 1$

17–18 Proof

19 $x = 4, y = 8$ **20** $x = 3, y = 9$

21 $\varnothing$ **22** Proof

23 True **24** True

25 True **26** True

27 False **28** False

29 True **30** a) 1 b)–d) proof

31 Proof **32** Proof

33 Proof **34** Proof

Exercise 1.3

1 4 **2** 1 **3** 17 **4** 68
5 77 **6** 1

7 $x = -17, y = 7$ **8** $x = -1, y = 1$

9 $x = -535, y = 132$ **10** $x = 9, y = 4$

11 $x = -1769, y = -29$ **12** $x = 5, y = 4$

13 No **14–16** Proof

17 8968 **18** 125 328

19 2100

20 (12, 360), (24, 180), (36, 120), (60, 72)

21 $\operatorname{lcm}(a, b) = ab$ **22** No **23–30** Proof

Exercise 1.4

1–5 Proof

6 For example, they end with 1 or 7.

7 a) $3 \cdot 29$ b) 19^2 c) $3^3 \cdot 5 \cdot 7$
d) $7 \cdot 11 \cdot 13$ e) $2^4 \cdot 19 \cdot 23$

8 a) $\gcd = 1, \operatorname{lcm} = 3 \cdot 29 \cdot 19^2$
b) $\gcd = 1, \operatorname{lcm} = 19^2 \cdot 7 \cdot 11 \cdot 13$
c) $\gcd = 1, \operatorname{lcm} = 3 \cdot 29 \cdot 19^2 \cdot 7 \cdot 11 \cdot 13$
d) $\gcd = 1, \operatorname{lcm} = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 29$

9 6, 10, 15, 42, 70

10–12 Proof

13 $x \nmid y, \gcd = 3^2 \cdot 13, \operatorname{lcm} = 3^2 \cdot 5 \cdot 11^2 \cdot 13$

14 $x \nmid y, \gcd = 2^2 \cdot 23, \operatorname{lcm} = 2^3 \cdot 5^3 \cdot 23^2$

15 $x | y, \gcd = 3^2 \cdot 11 \cdot 23, \operatorname{lcm} = 3^2 \cdot 7 \cdot 11 \cdot 23$

16 $x \nmid y, \gcd = 5, \operatorname{lcm} = 2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

17–22 Proof

23 1, 3

24 1, 2

25 When a is odd, always; when a is even, only when c is even.

Chapter 2

Exercise 2.1

1 a) True b) False c) False d) True

2 19 **3** Proof **4** 2

5 5 **6** 17 **7** 9

8 38 **9** 19 **10** 5

11 6 **12** 6 **13** 15

14 12 **15** 1 **16** 16

17 5 **18** 11 **19–33** Proof

34 1, 18 **35** −18, 5, 28

36 3, 7, 11, 21, 33, 77, 231

37–39 Proof **40** 11, 39, 21 **41–42** Proof

Exercise 2.2

1 a) No solution b) Solution c) No solution

2 a) $x = 7 - 7t, y = 10 - 13t$
b) $x = 1 + 35t, y = -6 - 221t$
c) $x = -141 + 349t, y = 120 - 297t$

3 a) $x = 8 - 11t, y = 1 - 5t$, with $t \in \{..., -2, -1, 0\}$
b) No positive solutions
c) (1, 66), (12, 4)

4

Apples	16	34	52
Oranges	71	46	21

5 7 of the €4.98 posters and 11 of the €5.98 posters.

6 $10d + 25q = 455$; minimum = 20, maximum = 44

7

Chicken	3	10	17
Geese	9	5	1

8 (Calves, lambs, piglets): (5, 41, 54), or (10, 22, 68), or (15, 3, 82)

9 €3.96

10 23

11 Minimum number of sheep required = 16. Transaction is not possible.

12 $(1 + 2t, -1 - 3t)$ **13** $(1 - 2t, 1 - 3t)$

14 $(6 + 14t, -7 - 17t)$

15 $(1 - 4t, 2 - 11t)$ or $(1 + 4t, 2 + 11t)$

16 None
17 None
18 $(345 + 503t, -275 - 401t)$
19 $(6 + 7t, -11 - 13t)$
20 $(4 + 5t, -7 - 9t)$
21 $(5 + 11t, -3 - 7t)$
22 $(13 + 19t, -6 - 9t)$
23 $(1 + 3t, 16 - 2t), 0 \leqslant t < 8$
24 $(4 + 4t, 12 - 3t), 0 \leqslant t < 4$
25 $(3 + 3t, 8 - 2t), 0 \leqslant t < 4$
26 None
27 None
28 $(2 + 5t, 9999 - 3t)$
29 None
30 None
31 $(1 + 7t, 9 + 2t)$
32 $(3 + 17t, 2 - 22t)$
33 $(20 + 40t, -6 - 11t)$
34 (21, 19) or (72, 8)
35–36 Proof

Exercise 2.3

1 $2 + 7k$
2 $2 + 3k$
3 $33 + 40k$
4 $41 + 49k$
5 $111 + 888k$
6 $75 + 80k$
7 $5 + 7k$
8 $2 + 3k$
9 $16 + 24k$
10 No solution
11 $812 + 1001k$
12 $10 + 45k$
13 No solution
14 $k \in (0, 4, 8, 12, \ldots, 32\}$; 4
15 11 (mod 12)
16 151 (mod 414)
17 34 (mod 35)
18 13 (mod 55)
19 6 (mod 210)
20 559 (mod 1430)
21 (2 (mod 5), 2 (mod 5))
22 No solution
23 (k (mod 5), $2 + k$ (mod 5))
24 (k (mod 7), $4 + 4k$ (mod 7))

Exercise 2.4

1 $(5600)_7$
2 $(1071)_{10}$
3 $(1562773)_8$
4 $(235056)_{10}$
5 $(5018)_{10}$
6 $(11111011010)_2$
7 $(77F394FB)_{16}$
8 $(33047851104)_{10}$
9 $(479)_{16}$
10 $(74E)_{16}$
11 $(1111111011001101110)_2$
12 $(1111101111011111010110011101101100010 01)_2$
13 a) When n is even.
b) When either a is a multiple of 3 or n is a multiple of 3.
c) When a is even.

Exercise 2.5

1 9
2 3
3 5
4 10
5 10
6 3 (mod 17)
7 9 (mod 17)
8 9 (mod 17)
9 5 (mod 11)
10 9 (mod 13)
11 1
12 a) 8 (mod 11), 11 (mod 13), 10 (mod 17)
b) 1064 (mod 2431)
13 1
14 10
15 8
16–20 Proof

Exercise 2.6–2.8

1 $b_1 = 6, b_2 = 15, b_3 = \frac{75}{2}, b_4 = \frac{375}{4}, b_5 = \frac{1875}{8}$; linear homogeneous of degree 1.

2 $a_1 = -2, a_2 = 4, a_3 = -8, a_4 = 16, a_5 = -32$; linear homogeneous of degree 2.

3 $a_1 = 5, a_2 = 10, a_3 = 40, a_4 = 320, a_5 = 5120$; not linear.

4 $b_1 = 1, b_2 = 8, b_3 = 43, b_4 = 218, b_5 = 1093$, not homogeneous.

5 $b_n = \frac{5}{2} b_{n-1}; b_1 = 4 \Rightarrow b_n = 4\left(\frac{5}{2}\right)^{n-1}$

6 $a_n = 5a_{n-1} + 3; a_1 = 3 \Rightarrow a_n = 3\left(5^{n-1}\right) + \frac{3}{4}\left(5^{n-1} - 1\right) = \frac{3}{4}\left(5^n - 1\right)$

7 $a_n = a_{n-1} + n; a_1 = 4 \Rightarrow a_n = 3 + \frac{n(n+1)}{2}$

8 $b_n = -\frac{11}{10} b_{n-1}; b_1 = 10 \Rightarrow b_n = 10\left(-\frac{11}{10}\right)^{n-1}$

9 $a_n = a_{n-1} - 2; a_1 = 0 \Rightarrow a_n = 2 - 2n$

10 $b_n = nb_{n-1}; b_1 = 8 \Rightarrow b_n = 8n!$

11 $\begin{cases} b_n = 4b_{n-1} + 5b_{n-2}; b_1 = 6, b_2 = 6 \Rightarrow r^2 - 4r - 5 = 0 \Rightarrow r \in \{-1, 5\} \\ \Rightarrow b_n = \frac{2}{5} \cdot 5^n - 4(-1)^n \end{cases}$

12 $\begin{cases} a_n = -3a_{n-1} - 2a_{n-2}; a_1 = -2, a_2 = 4 \Rightarrow r^2 + 3r + 2 = 0 \\ \Rightarrow r \in \{-1, -2\} \Rightarrow a_n = (-2)^n \end{cases}$

13 $\begin{cases} a_n = 2a_{n-1} - 2a_{n-2}; a_1 = 1, a_2 = 4 \Rightarrow r^2 - 2r + 2 = 0 \\ \Rightarrow r \in \{1 - i, 1 + i\} \Rightarrow a_n = \left(\sqrt{2}\right)^n \left(-\cos\left(n\frac{\pi}{4}\right) + 2\sin\left(n\frac{\pi}{4}\right)\right) \\ \Rightarrow \text{or } a_n = \frac{-1 - 2i}{2}(1 + i)^n + \frac{-1 + 2i}{2}(1 - i)^n \end{cases}$

14 $u_n = au_{n-1} + b \Rightarrow u_n = a^{n-1} \cdot u_1 + b \cdot \frac{a^{n-1} - 1}{a - 1}, a \neq 1.$

Practice questions 2

1 Proof
2 a) $x = 11, y = -6$ b) Proof
3 32
4 a) 235 b) 105 441 c) 9025
5 a) $\{1, 2, 3, 6\}$ b) 6
c) $6k - 4$ or $6k - 2, k \in \mathbb{Z}^+$
6 a) 1
b) (i) $x = 119 - 73k, y = -70 + 43k$ (ii) $(-27, 16)$
7 a) Proof
b) $x = 11 + 378n, y = -8 - 275n$, where $n \in \mathbb{Z}$
8 Definition and proof
9 a) (i) Proof (ii) (0, 5), (2, 3), (4, 1) (mod 6)
b) Proof
10 a) Proof b) $x \equiv 18 \pmod{35}$

Chapter 3

Exercise 3.1 and 3.2

1 a) (i) 4 (ii) 9 (iii) {5, 6, 5, 6}
b) (i) 4 (ii) 6 (iii) {3, 3, 3, 3}
c) (i) 5 (ii) 5 (iii) {2, 1, 3, 2, 2}

2 a) No b) Yes, K_5

3 $n - 1$

4 $\dfrac{n(n-1)}{2}$

5 a) $v = 7, e = 12$ b) $v = 30, e = 221$
c) $v = m + n, e = mn$

6 8, 16

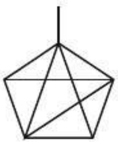
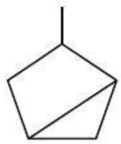
7 a) 8 b) Yes; $r = 2, |v| = 14$, or $r = 4, |v| = 7$
c) $\left\lfloor \dfrac{p}{2} \right\rfloor$ d) Proof

8–9 Proof

10 a, c

11 12

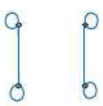
12 a) No, $|E|$ is not even.

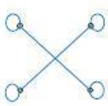
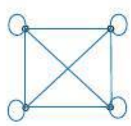
b) Yes,  c) Yes, 

Exercise 3.3

1 a) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

2 a)  b)  c) 

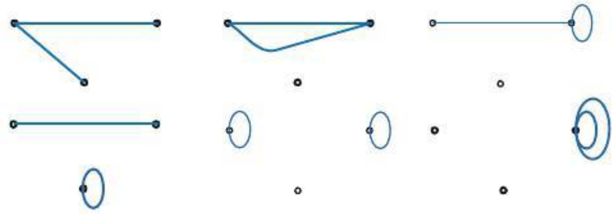
d)  e)  f) 

Graphs a) and c), and b) and e), are isomorphic.

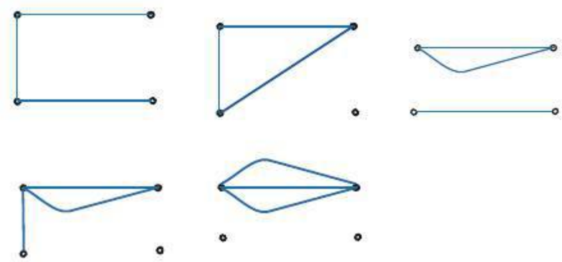
3 Isomorphic. Label the nodes, in both graphs, clockwise *a*, *b*, *c*, *d*, *e*, *f*, *g*. The correspondence $a \leftrightarrow g$, $b \leftrightarrow f$, $c \leftrightarrow e$, $d \leftrightarrow d$, $e \leftrightarrow c$, $f \leftrightarrow b$, $g \leftrightarrow a$ is a homomorphism because when you rearrange the vertices in the second graph, you will have the same adjacency matrix as the first one.

4 a) No b) No c) No d) Yes

5 2 without loops, 6 with loops



6 5 without loops, 15 with loops



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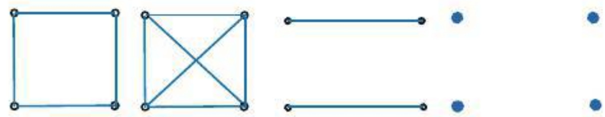


Exercise 3.4

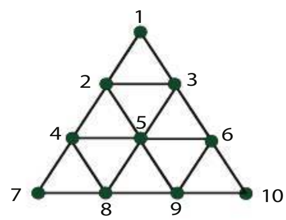
1 Vertices have even degrees.
a) 123174263456751 b) 1234543251

2 a) 1234214241 b) 12345241
c) Vertices 2 and 5 have degree 5 each.

3 a) When n is odd. b) When m and n are both even.

4 Graph 1(a) Hamiltonian: 12345671; graph 1(b) Hamiltonian: 123451.
Graph 2(a) Hamiltonian: 12341; graph 2(b) Hamiltonian path: 12345; graph 2(c) neither.

5 a) (10, 9, 6, 5, 9, 8, 5, 4, 8, 7, 4, 2, 5, 3, 2, 1, 3, 6, 10)
b) (10, 9, 8, 7, 4, 5, 2, 1, 3, 6, 10)
c) An Eulerian circuit is always possible ($n \geqslant 3$), because the degree of every vertex is even. A Hamiltonian cycle is also possible using the same plan as above: visit all vertices except one side, and then go back along that side.



6 Length 1 = 0; length 2 = 2; length 3 = 3, and length 4 = 10.

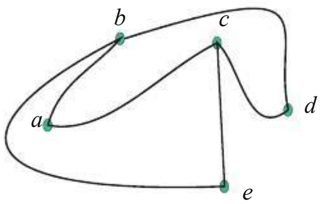
7 a) 51 between vertices not on the main diagonal, 52 for vertices on the diagonal
b) 205 between vertices not on the main diagonal, 204 for vertices on the diagonal
c) 819 between vertices not on the main diagonal, 820 for vertices on the diagonal

8 a) 48 among vertices of the 3-part, and 36 among the 4-part
b) 144 from vertices of 3-part to vertices of 4-part
c) 576 among vertices of the 3-part, and 432 among the 4-part
d) 1728 from vertices of 3-part to vertices of 4-part

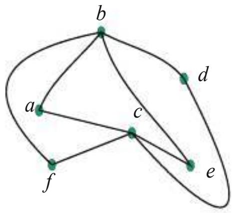
9 a) No cycle. If you start at the left, you will need to visit *c* and *d* twice. Path: *abcdef*.
b) Cycle: *abcdea*.
c) No cycle since *f* has degree 1. Path: *eabcdf*.
d) Neither cycle nor path as three vertices have degree 1.
e) No cycle, because in any of them *a* or *d* would have to be visited twice. Path: *eacdb*.
f) Cycle: *ahgfedcbia*.

Exercise 3.5

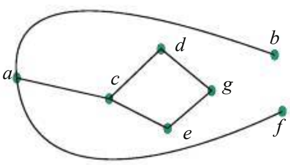
1 Planar. Redraw:



2 Planar. Redraw:



3 Planar. Redraw:



4 Not planar. *bf* and *ce* must cross, so must *ae* and *bd*.

5 15 **6** 15, 18

7 7, 9 **8** 6

9 Not planar **10** Planar

Practice questions 3

1 No, because there will be an edge connecting two vertices in the same component.

2 a) (i) $\binom{n}{2}$ (ii) $\binom{n}{3}$ (iii) $\binom{n}{m}$

b) $\frac{n+2}{2}$ or $\frac{n+1}{2}$

3 10

4 a) 2 b) 7

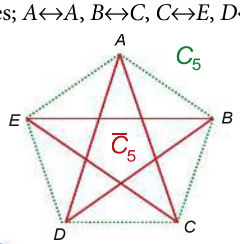
5 a) 0 b) 27

6 a) Proof b) Only C_3 is isomorphic to K_3 and W_3 to K_4.
c) Proof

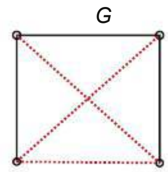
7 Proof

8 They contain odd cycles (size 3).

9 Yes; $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow E$, $D \leftrightarrow B$, $E \leftrightarrow D$.

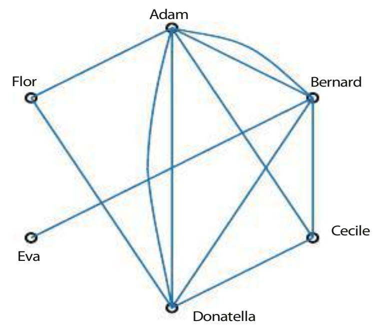


10 a) Yes:



b) No $\bar{G}$

11 a)



b) Yes, through Adam.
c) Bernard, as without him Eva is isolated.

Chapter 4

Exercise 4.1–4.3

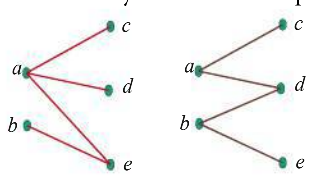
1 a) 5, 7, 10, 11, 13, 14, 16, 17
b) 3, 1, 9
c) 3: 12, 13, 14; 7: no descendants; 15: 16, 17
d) 4: 12; 7: no siblings; 9: no siblings

2 $|u| = 18$, $|v| = 36$, $|f| = 35$

3 31

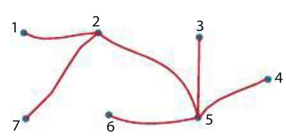
4 $\binom{n}{2}$

5 a) These are the only two non-isomorphic trees.



b) $\left\lfloor \frac{n+1}{2} \right\rfloor$

6



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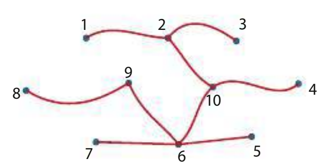

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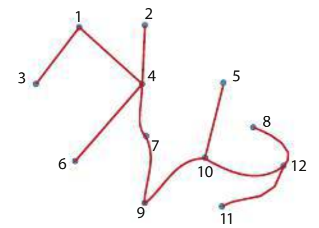


9 12, 23, 34, 45, 56, 67

10 12, 23, 34, 45, 56, 67, 78, 89, 9(10)

11 12, 24, 45, 58, 8(12), (12)(11), (11)9, 9(10), 47, 76, 63

12 13, 34, 45, 58, 89, 46, 67, 7(10), 12

13 17, 78, 89, 9(10), (10)(11), (11)6, 65, 54, (10)(14), 9(13), 83, 32

14 12, 23, 34, 46, 65, 5(10), (10)9, 98, 87, (10)(11), (11)(12), (12)(13), (13)(14), (14)(15), (10)(16), (16)(17), (17)(19), (19)(20), (20)(18)

15 a) 13, 12, 34, 45, 46, 67, 78, 7(10), 89

b) 12, 23, 34, 45, 56, 67, 78, 89, 7(10)

16 a) 12, 17, 7(12), 78, 83, 8(13), 89, 94, 95, 9(10), 9(14), (10)(11), (11)6

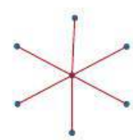
b) 12, 23, 38, 89, 94, 45, 56, 6(11), (11)(10), (10)(14), 9(13), 87, 7(12)

17 a) 12, 15, 23, 26, 34, 5(10), (10)7, (10)8, (10)9, (10)(11), (10)(16), (11)(12), (11)(13), (11)(14), (11)(15), (16)(17), (16)(18), (16)(20), (20)(19)

b) 12, 23, 34, 46, 65, 5(10), (10)7, 78, 89, (10)(11), (11)(12), (12)(13), (13)(14), (14)(15), (10)(16), (16)(17), (17)(18), (18)(20), (20)(19)

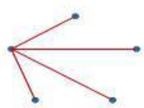
18 a) (i)

b) (i)



a) (ii)

b) (ii)

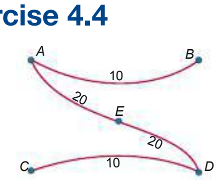


a) (iii)

b) (iii)

Exercise 4.4

1



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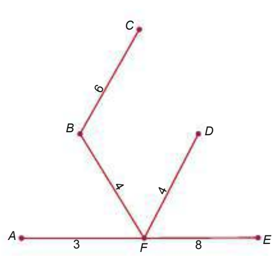

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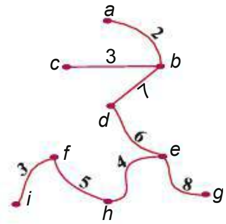
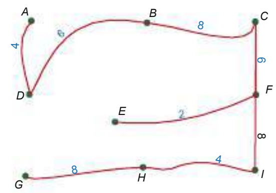


4



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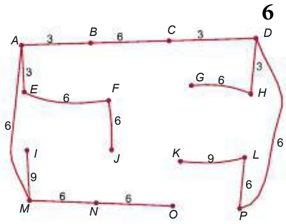

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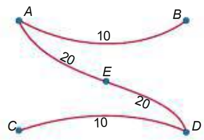

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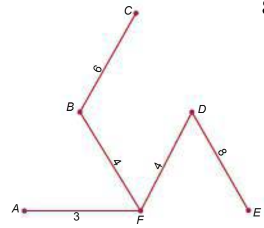

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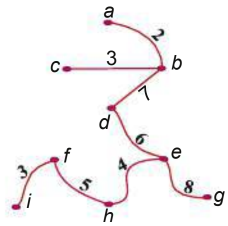

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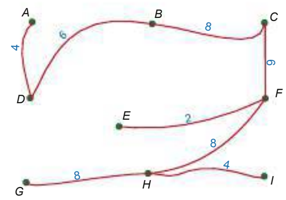
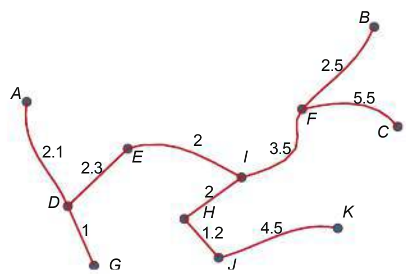


10 A few shapes are possible, one of which is similar to the answer to question 5.

11 1 and 6 have the same final tree. However, when building the tree using Kruskal's algorithm, *AB* and *CD* were added first. When using Prim's algorithm, *AB* was followed by *AE*, *ED*, and then *CD*.
With 2 and 7, there is no apparent difference. The different shapes are due to random choices.
3 and 8 have the same final tree too. Using Kruskal's algorithm, the order of addition to the tree is: *ab*, *bc*, *fi*, *he*, *fh*, *ed*, *bd*, and *eg*. Using Prim's algorithm, the order is: *ab*, *bc*, *bd*, *ed*, *he*, *fh*, *fi* and *eg*.
4 and 9 may have the same tree too. However, using Kruskal's algorithm, the order of edge addition is: *ef*, *ad*, *hi*, *cf*, *db*, *bc*, *fi*, and *gh*. Using Prim's algorithm, the order is: *ef*, *fc*, *fh*, *ih*, *cb*, *bd*, *da*, and *gh*.

5 and 10 may have the same tree too. However, using Kruskal's algorithm, the order of edge addition is: *AB*, *AE*, *CD*, *DH*, *BC*, …. Using Prim's algorithm, the order is: *AB*, *AE*, *BC*, *CD*, *DH*, ….

12



Exercise 4.5

1 70, *abedf*

2 48, *ACDEGH*

3 32, *acfimpsu*

4 *abed*

5 *A–F*: *ACDF*; *B–H*: *BCDEGH*

6 *ADBCA*, 85

7 *EDCABE* or *DEBACD*, 400

8 Vienna–Frankfurt–Prague–Moscow–Milan–Vienna: €1070.

9 New York–Paris–London–Madrid–Boston–New York: €1215.

10 *DACBED*, 550

11 *age*, 19

12 *abdfhi*, 21; *acehi*, 13

13 Without visiting any city twice: *ESYFITAPGE*, 926. Visiting *Y* twice: *EGYSYFITAPE*, 871.

14 *abcdhghcgbfgfea*, 8300

15 *abcdecjfefibjfgihgha*, 9200

Review questions 4

1–3 Proof

4 44

5 a) 21 b) $\binom{n}{2}$

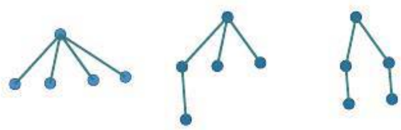
6 Proof

7 Yes; no

8 Proof

9 No; no

10

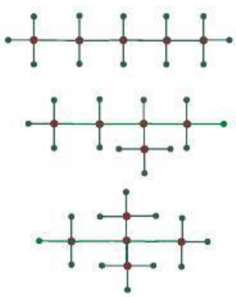


11 20

12 On the left there are 2 carbon atoms adjacent to 3 hydrogen atoms each, while on the right 3 carbon atoms have this property.

13 $2n + 2$

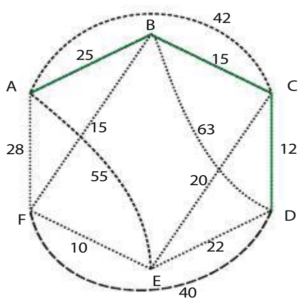
14



15 Proof

16 *BAGF*, 16

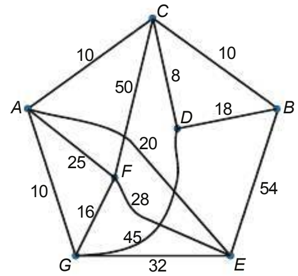
17 a)



b) ABCD, $52

18 *ACEDFGHIBA*, 8.6 km

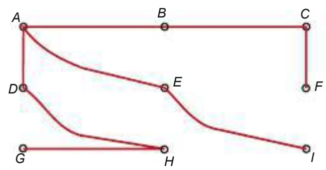
19 a)



b) Sample: *ACBDCAEFGA* with 130 000 free miles, which she can afford.

20 Yes; he will have a 20-minute break.

21 Sample for Kruskal's algorithm: *BC*, *AB*, *AE*, *CF*, *GH*, *AD*, *DH*, *EI*. Sample for Prim's algorithm: *BC*, *AB*, *AE*, *CF*, *AD*, *DH*, *GH*, *EI*. Weight = 26.



22 Sample for Kruskal's algorithm: *DG*, *HI*, *BF*, *EH*, *DE*, *FI*, *AD*, *FC*. Sample for Prim's algorithm: *DG*, *DE*, *EH*, *HI*, *IF*, *BF*, *AD*, *CF*. Weight = 45.

23 PT, SU, RU, PQ, TR, total distance of 719 km

24 1043 cents (10.43 dollars)

25 35

Practice questions 4

1 a) Proof

b) Not isomorphic; one has a vertex of degree 4, the other does not.

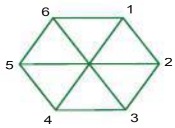
2 a)

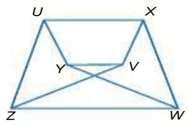
Vertices added to the tree	Edge added	Weight
3	Ø	0
5	3, 5	10
6	3, 6	20
7	5, 7	30
10	6, 10	30
1	3, 1	40
2	1, 2	30
11	2, 11	30
9	1, 9	40
4	6, 4	40
8	7, 8	40
		310

b) Any of two paths: 1–3–4–5–6–8–10–11 or 1–3–4–5–6–9–11, with weight 80.

3 a) Student definition
b) Not isomorphic; G has a vertex of degree 3, while H has not.
c) $BAEBCEFCDF$
d) All vertices have even degree.

4 a) Student definition b) Proof
c) (i) G is bipartite since if we label the vertices clockwise as 1, 2, 3, …, the two components will be {1, 3, 5} and {2, 4, 6}.



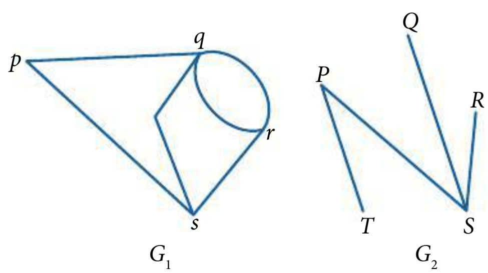


(ii) G and H are isomorphic: $1\leftrightarrow U$, $2\leftrightarrow X$, $3\leftrightarrow V$, $4\leftrightarrow Y$, $5\leftrightarrow W$, $6\leftrightarrow Z$.
(iii) No; H is bipartite, J is not.

5 a) 24
b) (i) $BDEC$
(ii) 33
c) $DBAEC$ is a minimum spanning tree of weight 26. Upper bound = 26 × 2 = 52.
d) A minimum tour is 34; 33 cannot be achieved.

6 a) Every edge creates 2 degrees, with n edges there are $2n$ degrees.
b) Each vertex will have a degree of 5, 45 in total, which is not even. Hence, it is not possible.
c) See Chapter 3, page 121.

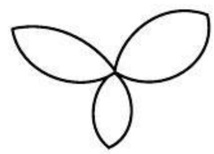
7 a)



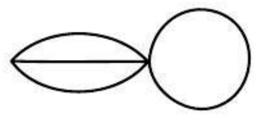
b) (i) G_1 is not simple, G_2 is simple.
(ii) Both are connected.
(iii) Both are bipartite. G_1: components are {p, r, t} and {q, s}. G_2: components are {P, R, Q} and {T, S}.
(iv) G_1 is not a tree, as it has a cycle. G_2 is a tree.
(v) G_1 contains an Eulerian trail: $rqpsrqts$. G_2 does not have an Eulerian trail since four vertices have odd degrees.

8 a) (i) D, E
(ii) EBD
(iii) Example: $ABEFGCBDBEGDFCA$
(iv) 36
b) Example: $ABEFDGCA$

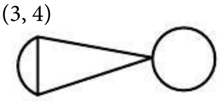
9 a) Every edge creates 2 degrees, with e edges there are $2e$ degrees.
b) Student deduction
c) (i) (n, d) = (1, 6), (2, 5), (3, 4), (5, 2) or (6, 1)
(ii) (1, 6)



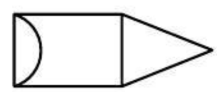
(2, 5)



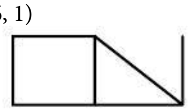
(3, 4)



(5, 2)



(6, 1)



10 a) (i) Proof
(ii) Number of paths from v_i to v_j with a maximum length of 3.
b)–c) Proof