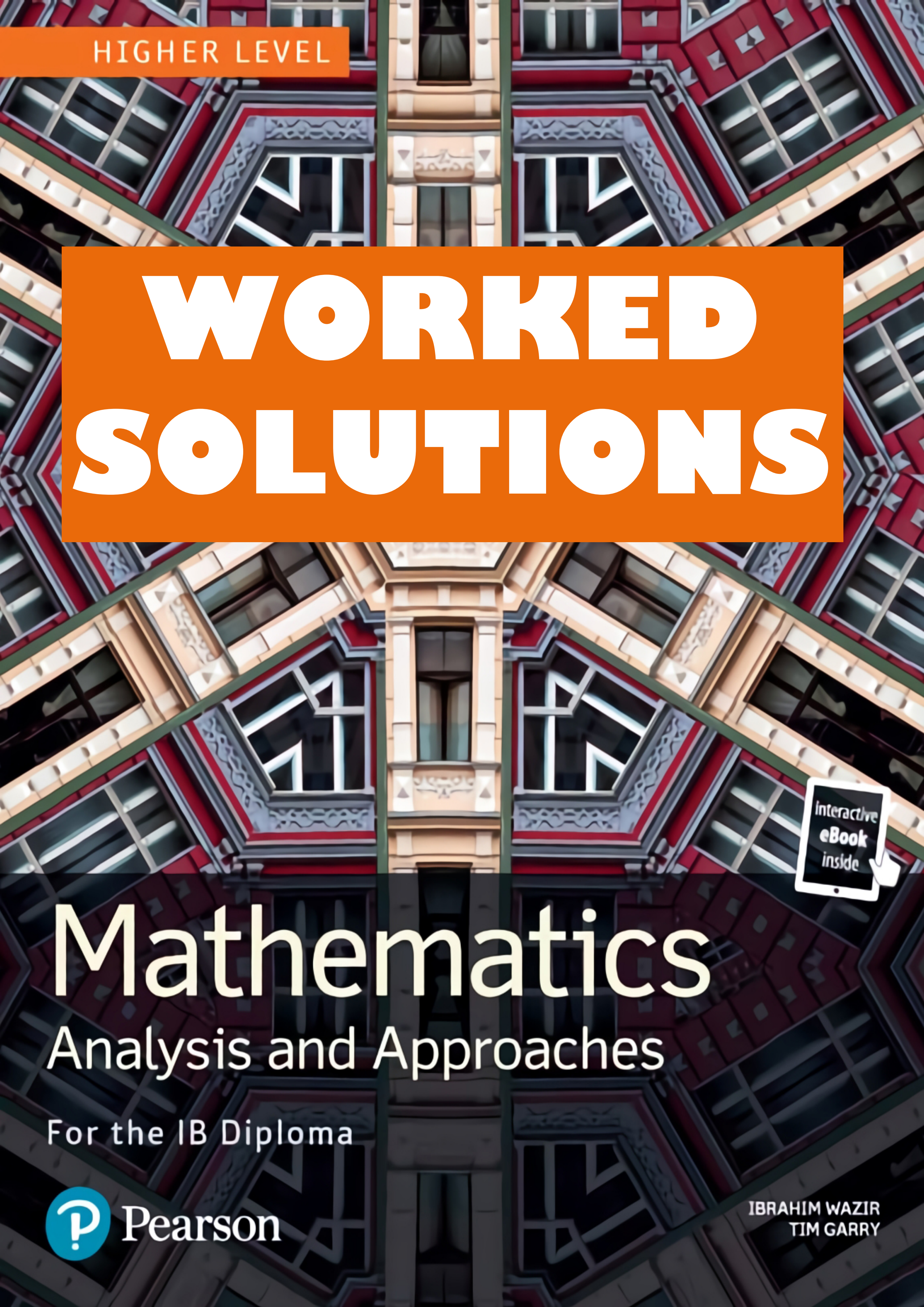

HIGHER LEVEL

WORKED SOLUTIONS

interactive eBook inside

Mathematics

Analysis and Approaches

For the IB Diploma

Pearson

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Exercise 1.1

1. In each of the following, make the indicated letter the subject of the left side of the equality:

(a) $m(h-x)=n \Rightarrow h-x=\frac{n}{m} \Rightarrow -x=\frac{n}{m}-h \Rightarrow x=h-\frac{n}{m}$

(b) $v=\sqrt{ab-t} \Rightarrow v^2=ab-t \Rightarrow v^2+t=ab \Rightarrow a=\frac{v^2+t}{b}$

(c) $A=\frac{h}{2}(b_1+b_2) \Rightarrow \frac{2A}{h}=b_1+b_2 \Rightarrow b_1=\frac{2A}{h}-b_2$

(d) $A=\frac{1}{2}r^2\theta \Rightarrow \frac{2A}{\theta}=r^2 \Rightarrow r=\pm\sqrt{\frac{2A}{\theta}}$; if r is a length, then $r=\sqrt{\frac{2A}{\theta}}$

(e) $\frac{f}{g}=\frac{h}{k} \Rightarrow fk=gh \Rightarrow k=\frac{gh}{f}$

(f) $at=x-bt \Rightarrow at+bt=x \Rightarrow t(a+b)=x-h \Rightarrow t=\frac{x}{a+b}$

(g) $V=\frac{1}{3}\pi r^3h \Rightarrow r^3=\frac{3V}{\pi h} \Rightarrow r=\sqrt[3]{\frac{3V}{\pi h}}$

(h) Factor out k at the denominator of the fraction:

$$F=\frac{g}{k(m_1+m_2)} \Rightarrow Fk(m_1+m_2)=g \Rightarrow k=\frac{g}{F(m_1+m_2)}$$

2. In parts **(a)** to **(d)**, find m, the slope of the line, using $m=\frac{y_2-y_1}{x_2-x_1}$, then find the y-intercept c, by choosing one of the given points and substituting its coordinates and the value of the slope into the slope-intercept form of the equation of a line, $y=mx+c$.

(a) $m=\frac{-7-1}{3-(-9)}=\frac{-8}{12}=-\frac{2}{3}$

$$(-9,1) \Rightarrow 1=-\frac{2}{3}(-9)+c \Rightarrow 1=6+c \Rightarrow c=-5$$

$$\Rightarrow y=-\frac{2}{3}x-5$$

(b) EITHER: $m=\frac{-4-(-4)}{10-3}=\frac{0}{7}=0$

$(10,-4) \Rightarrow -4=0(-4)+c \Rightarrow c=-4 \Rightarrow y=-4.$

OR: The given points have the same y-coordinate, so the line is horizontal, the equation of the line is $y=-4$.

(c) $m = \frac{11-(-9)}{4-(-12)} = \frac{20}{16} = \frac{5}{4}$

$$(4{,}11) \Rightarrow 11 = \frac{5}{4}(4) + c \Rightarrow 11 = 5 + c \Rightarrow c = 6$$

$$\Rightarrow y = \frac{5}{4}x + 6$$

(d) By inspection, the given points have the same x-coordinate, so the line is vertical, and its equation is $x = \frac{7}{3}$

(e) $4x + y - 3 = 0 \Rightarrow y = -4x + 3$

Parallel lines have the same slope, consequently the slope of the required line is $m = -4$.

$$(7, -17) \Rightarrow -17 = (-4)(7) + c \Rightarrow c = 11$$

$$\Rightarrow y = -4x + 11$$

(f) $2x - 5y - 35 = 0 \Rightarrow -5y = -2x + 35 \Rightarrow y = \frac{2}{5}x - 7$

The slope of a line perpendicular to another line with slope $\frac{2}{5}$ is $-\frac{5}{2}$

$$(-5, \frac{11}{2}) \Rightarrow \frac{11}{2} = (-\frac{5}{2})(-5) + c \Rightarrow c = \frac{11}{2} = \frac{25}{2} + c \Rightarrow c = -7$$

$$\Rightarrow y = -\frac{5}{2}x - 7$$

3. In each of these, substitute the coordinates of the given points in the relevant formulae:

(a) $d = \sqrt{(-4-4)^2 + (10-(-5))^2} = \sqrt{64 + 225} = \sqrt{289} = 17$

$M(\frac{-4+4}{2}, \frac{10+(-5)}{2}) = M(0, \frac{5}{2})$

(b) $d = \sqrt{(-1-5)^2 + (2-4)^2} = \sqrt{36 + 4} = \sqrt{40}$

$M(\frac{-1+5}{2}, \frac{2+4}{2}) = M(2{,}3)$

(c) $d = \sqrt{(\frac{1}{2} - (-\frac{5}{2}))^2 + (1 - \frac{4}{3})^2} = \sqrt{9 + \frac{1}{9}} = \sqrt{\frac{82}{9}} = \frac{\sqrt{82}}{3}$

$M\left(\frac{\frac{1}{2}+(-\frac{5}{2})}{2}, \frac{1+\frac{4}{3}}{2}\right) = M(-1, \frac{7}{6})$

(d) $d = \sqrt{(12-(-10))^2 + (2-9)^2} = \sqrt{484 + 49} = \sqrt{533}$

$M(\frac{12+(-10)}{2}, \frac{2+9}{2}) = M(1, \frac{11}{2})$

4. Substitute the coordinates of the given points in the distance formula and set it equal to 5:

(a) $5 = \sqrt{(5-k)^2 + (-1-2)^2} \Rightarrow 25 = (5-k)^2 + 9$

$$\Rightarrow 16 = (5-k)^2 \Rightarrow \pm 4 = 5 - k$$

$$\Rightarrow k = 5 \pm 4 \Rightarrow k = 1, k = 9$$

(b) $5 = \sqrt{(-2-1)^2 + (-7-k)^2} \Rightarrow 25 = 9 + (-7-k)^2$

$$\Rightarrow 16 = (-7-k)^2 \Rightarrow \pm 4 = -7 - k$$

$$\Rightarrow k = -7 \pm 4 \Rightarrow k = -11, k = -3$$

5. Use the distance formula to show that properties regarding the length of sides for the indicated shapes hold true.

(a) Let $A(4,0)$, $B(2,1)$ and $C(-1,-5)$ be the vertices of triangle ABC.
The lengths of the sides are:

$$AB = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$
$$AC = \sqrt{(4-(-1))^2 + (0-(-5))^2} = \sqrt{25+25} = \sqrt{50}$$

$$BC = \sqrt{(2-(-1))^2 + (1-(-5))^2} = \sqrt{9+36} = \sqrt{45}$$

If the sides of a triangle verify Pythagoras' theorem ($AC^2 = AB^2 + BC^2$), then $\Delta ABC$is a right-angled triangle.

$$AC^2 = 50$$

$AB^2 + BC^2 = 5 + 45 = 50 \Rightarrow \Delta ABC$is a right-angled triangle.

(b) Let $A(1,-3)$, $B(3,2)$ and $C(-2,4)$ be the vertices of triangleABC.

$$AB = \sqrt{(1-3)^2 + (-3-2)^2} = \sqrt{4+25} = \sqrt{29}$$

$$AC = \sqrt{(1-(-2))^2 + (-3-4)^2} = \sqrt{9+49} = \sqrt{58}$$

$$BC = \sqrt{(3-(-2))^2 + (2-4)^2} = \sqrt{25+4} = \sqrt{29}$$

$\Rightarrow AB = AC \Rightarrow \Delta ABC$is isosceles, because two of its sides have the same length.

(c) Let $A(0,1)$, $B(3,7)$, $C(4,4)$ and $D(1,-2)$ be the vertices of quadrilateral $ABCD$.

If $AB = DC$ and $AD = BC$, then $ABCD$ is a parallelogram,

$$AB = \sqrt{(0-3)^2 + (1-7)^2} = \sqrt{9+36} = \sqrt{45}$$

$$DC = \sqrt{(1-4)^2 + (-2-4)^2} = \sqrt{9+36} = \sqrt{45} \Rightarrow AB = DC$$

$$AD = \sqrt{(0-1)^2 + (1-(-2))^2} = \sqrt{1+9} = \sqrt{10}$$

$$BC = \sqrt{(3-4)^2 + (7-4)^2} = \sqrt{1+9} = \sqrt{10} \Rightarrow AD = BC$$

$AB = DC, AD = BC \Rightarrow ABCD$is a parallelogram.

6. **(a)** Subtract the two equations to eliminate *x*:

$x + 3y = 8$

$\underline{x - 2y = 3}$

$5y = 5 \Rightarrow y = 1$

Substitute $y = 1$ into either of the original equations to find *x*:

$x + 3(1) = 8 \Rightarrow x = 5$

The solution is $(5, 1)$.

(b) Multiply the second equation by 3 and add it to the first equation to eliminate *y*:

$3x + 2y = 13 \rightarrow 9x + 6y = 39$

$\underline{x - 6y = 1}$

$10x = 40 \Rightarrow x = 4$

Substitute $x = 4$ into either of the original equations to find *x*:

$4 - 6y = 1 \Rightarrow -6y = -3 \Rightarrow y = \frac{1}{2}$

The solution is $(4, \frac{1}{2})$.

(c) Multiply the first equation by 4 and the second equation by 3, then subtract the two equations, to eliminate *y*:

$6x + 3y = 6 \rightarrow \quad 24x + 12y = 24$

$5x + 4y = -1 \rightarrow \underline{15x + 12y = -3}$

$9x = 27 \Rightarrow x = 3$

Substitute $x = 3$ into either of the original equations to find *x*:

$5(3) + 4y = -1 \Rightarrow 4y = -16 \Rightarrow y = -4$

The solution is $(3, -4)$.

(d) Multiply the second equation by 4 and add it to the first equation to eliminate *x*:

$-2x + 3y = 2 \rightarrow -8x + 12y = 8$

$\underline{8x - 12y = 4}$

$0 = 12$

This is not true, so the system of equations has no solution.

(e) Multiply the first equation by 5 and the second equation by 7, then add the two equations, to eliminate *y*:

$$\begin{array}{rl} 5x+7y=9 \rightarrow & 25x+35y=45 \\ -11x-5y=1 \rightarrow & \underline{-77x-35y=7} \\ & \quad -52x=52 \Rightarrow x=-1 \end{array}$$

Substitute $x=-1$ into either of the original equations to find *y*:

$$5(-1)+7y=9 \Rightarrow 7y=14 \Rightarrow y=2$$

The solution is $(-1,2)$

7. **(a)** Make *y* the subject in the first equation:

$$2x+y=1 \Rightarrow y=1-2x$$

Substitute the expression of *y* in the second equation and solve for *x*:

$$3x+2(1-2x)=3 \Rightarrow 3x+2-4x=3 \Rightarrow -x=1 \Rightarrow x=-1$$

Substitute $x=-1$ into either of the original equations to find *y*:

$$2(-1)+y=1 \Rightarrow -2+y=1 \Rightarrow y=3$$

The solution is $(-1,3)$

(b) Make *y* the subject in the second equation:

$$5x-y=-7 \Rightarrow y=5x+7$$

Substitute the expression of *y* in the first equation and solve for *x*:

$$3x-2(5x+7)=7 \Rightarrow 3x-10x-14=7 \Rightarrow -7x=21 \Rightarrow x=-3$$

Substitute $x=-3$ into the expression of *y*:

$$y=5(-3)+7 \Rightarrow y=-8$$

The solution is $(-3,-8)$

(c) Divide the first equation by 2:

$$2x+8y=-6 \Rightarrow x+4y=-3$$

Make *x* the subject in this equation:

$$x+4y=-3 \Rightarrow x=-3-4y$$

Substitute the expression of *x* in the second equation and solve for *y*:

$-5(-3-4y)-20y=15 \Rightarrow 15+20y-20y=15 \Rightarrow 15=15$, true for all $y \in \mathbb{R}$.

This means that there are an infinite number of solutions, due to the fact that the two equations are multiples of each other (the lines representing the two equations are coincident). It follows that the solution of this system is the set of all points on the line with equation $2x+8y=-6$ (or $-5x-20y=25$, or $y=-\frac{1}{4}x-\frac{3}{4}$)

(d) Make *y* the subject in the second equation:

$x + y = 20 \Rightarrow y = 20 - x$

Substitute the expression of *y* in the first equation and solve for *x*:

$\frac{x}{5} + \frac{20-x}{2} = 8 \Rightarrow \frac{2x+100-5x}{10} = 8 \Rightarrow -3x + 100 = 80 \Rightarrow -3x = -80 \Rightarrow x = \frac{20}{3}$

Substitute $x = \frac{20}{3}$ into the expression of *y*:

$y = 20 - \frac{20}{3} \Rightarrow y = \frac{40}{3}$

The solution is $(\frac{20}{3}, \frac{40}{3})$

(e) Make *y* the subject in the second equation:

$4x + y = 5 \Rightarrow y = 5 - 4x$

Substitute the expression of *y* in the first equation and solve for *x*:

$2x - (5 - 4x) = -2 \Rightarrow 2x - 5 + 4x = -2 \Rightarrow 6x = 3 \Rightarrow x = \frac{1}{2}$

Substitute $x = \frac{1}{2}$ into the expression of *y*:

$y = 5 - 4(\frac{1}{2}) \Rightarrow y = 3$

The solution is $(\frac{1}{2}, 3)$

(f) Multiply the second equation by 10:

$0.25x + 0.1y = -0.25 \Rightarrow 2.5x + y = -2.5$

Make *y* the subject in this equation:

$2.5x + y = -2.5 \Rightarrow y = -2.5 - 2.5x$

Substitute the expression of *y* in the first equation and solve for *x*:

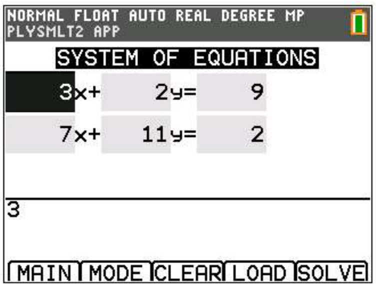
$$0.4x + 0.3(-2.5 - 2.5x) = 1 \Rightarrow 0.4x - 0.75 - 0.75x = 1$$
$$\Rightarrow -0.35x = 1.75 \Rightarrow x = -5$$

Substitute $x = -5$ into the expression of *y*:

$y = -2.5 - 2.5(-5) \Rightarrow y = 10$

The solution is $(-5, 10)$

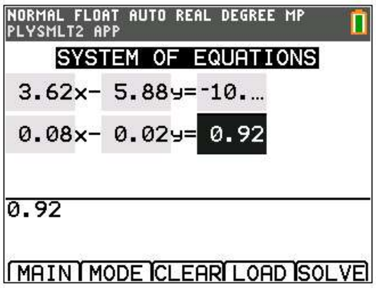
8. **(a)** The solution is $(5, -3)$



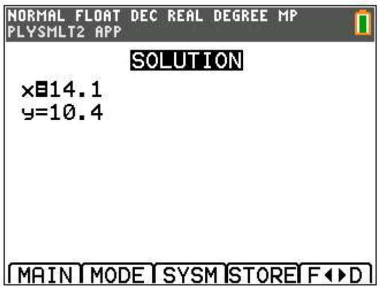
SYSTEM OF EQUATIONS
3x+ 2y= 9
7x+ 11y= 2
3

SOLUTION
x=5
y=-3

(b) The solution is $(14.1, 10.4)$

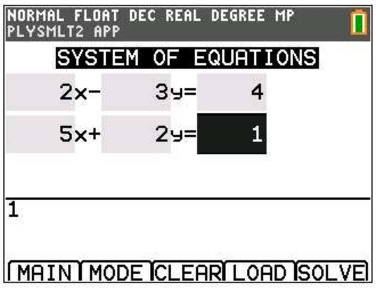


SYSTEM OF EQUATIONS
3.62x− 5.88y=-10.…
0.08x− 0.02y= 0.92
0.92



SOLUTION
x=14.1
y=10.4

(c) The solution is $(\frac{11}{19}, -\frac{18}{19})$



SYSTEM OF EQUATIONS
2x− 3y= 4
5x+ 2y= 1
1

SOLUTION
x=11/19
y=-18/19

9. **(a)**

$$\begin{pmatrix} 4 & -1 & 1 & -5 \\ 2 & 2 & 3 & 10 \\ 5 & -2 & 6 & 1 \end{pmatrix} \xrightarrow{2R_2-R_1} \begin{pmatrix} 4 & -1 & 1 & -5 \\ 0 & 5 & 5 & 25 \\ 5 & -2 & 6 & 1 \end{pmatrix} \xrightarrow{R_2\div 5} \begin{pmatrix} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 5 & -2 & 6 & 1 \end{pmatrix}$$

$$\xrightarrow{5R_1-4R_3} \begin{pmatrix} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 3 & -19 & -29 \end{pmatrix} \xrightarrow{R_3-3R_2} \begin{pmatrix} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -22 & -44 \end{pmatrix}$$

$$\Rightarrow 4x - y + z = -5$$

$$y + z = 5$$

$$-22z = -44 \Rightarrow z = 2 \Rightarrow y + 2 = 5 \Rightarrow y = 3$$

$$\Rightarrow 4x - 3 + 2 = -5 \Rightarrow 4x = 4 \Rightarrow x = 1$$

The solution is $(-2, 4, 3)$

(b)

$$\begin{pmatrix}4 & -2 & 3 & -2\\ 2 & 2 & 5 & 16\\ 8 & -5 & -2 & 4\end{pmatrix}\xrightarrow{2R_2-R_1}\begin{pmatrix}4 & -2 & 3 & -2\\ 0 & 6 & 7 & 34\\ 8 & -5 & -2 & 4\end{pmatrix}\xrightarrow{2R_2-R_1}\begin{pmatrix}4 & -2 & 3 & -2\\ 0 & 6 & 7 & 34\\ 8 & -5 & -2 & 4\end{pmatrix}$$

$$\xrightarrow{R_3-2R_1}\begin{pmatrix}4 & -2 & 3 & -2\\ 0 & 6 & 7 & 34\\ 0 & -1 & -8 & 8\end{pmatrix}\xrightarrow{6R_3+R_2}\begin{pmatrix}4 & -2 & 3 & -2\\ 0 & 6 & 7 & 34\\ 0 & 0 & -41 & 82\end{pmatrix}$$

$\Rightarrow 4x - 2y + 3z = -2$

$6y + 7z = 34$

$-41z = 82 \Rightarrow z = -2 \Rightarrow 6y + 7(-2) = 34 \Rightarrow 6y = 48 \Rightarrow y = 8$

$\Rightarrow 4x - 2(8) + 3(-2) = -2 \Rightarrow 4x = -20 \Rightarrow x = -5$

The solution is $(-5, 8, -2)$

(c)

$$\begin{pmatrix}5 & -3 & 2 & 2\\ 2 & 2 & -3 & 3\\ 1 & -7 & 8 & -4\end{pmatrix}\xrightarrow{5R_2-2R_1}\begin{pmatrix}5 & -3 & 2 & 2\\ 0 & 16 & -19 & 11\\ 1 & -7 & 8 & -4\end{pmatrix}\xrightarrow{5R_3-R_1}\begin{pmatrix}5 & -3 & 2 & 2\\ 0 & 16 & -19 & 11\\ 1 & -32 & 38 & -22\end{pmatrix}$$

$$\xrightarrow{2R_2+R_3}\begin{pmatrix}5 & -3 & 2 & 2\\ 0 & 16 & -19 & 11\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$\Rightarrow 0x + 0y + 0z = 0 \Rightarrow$ infinite solutions

(d)

$$\begin{pmatrix}3 & -2 & 1 & -29\\ -4 & 1 & -3 & 37\\ 1 & -5 & 1 & -24\end{pmatrix}\xrightarrow{3R_2+4R_1}\begin{pmatrix}3 & -2 & 1 & -29\\ 0 & -5 & -5 & -5\\ 1 & -5 & 1 & -24\end{pmatrix}\xrightarrow{3R_3-R_1}\begin{pmatrix}3 & -2 & 1 & -29\\ 0 & -5 & -5 & -5\\ 0 & -13 & 2 & -43\end{pmatrix}$$

$$\xrightarrow{R_2\div(-5)}\begin{pmatrix}3 & -2 & 1 & -29\\ 0 & 1 & 1 & 1\\ 0 & -13 & 2 & -43\end{pmatrix}\xrightarrow{R_3+13R_2}\begin{pmatrix}3 & -2 & 1 & -29\\ 0 & 1 & 1 & 1\\ 0 & 0 & 15 & -30\end{pmatrix}$$

$\Rightarrow 3x - 2y + z = -29$

$y + z = 1$

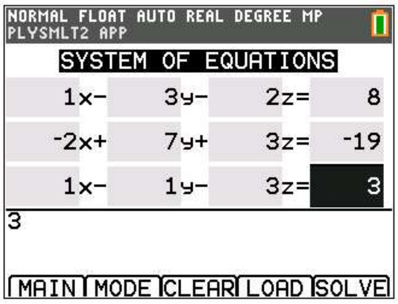
$15z = -30 \Rightarrow z = -2 \Rightarrow y + (-2) = 1 \Rightarrow y = 3$

$\Rightarrow 3x - 2(3) + (-2) = -29 \Rightarrow 3x = -21$

$\Rightarrow x = -7$

The solution is $(-7, 3, -2)$

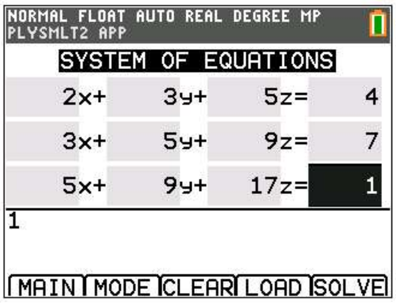
10. **(a)** The solution is $(4, -2, 1)$



```
SYSTEM OF EQUATIONS
  1x-   3y-   2z=    8
 -2x+   7y+   3z=  -19
  1x-   1y-   3z=    3
3
```

```
SOLUTION
x=4
y=-2
z=1
```

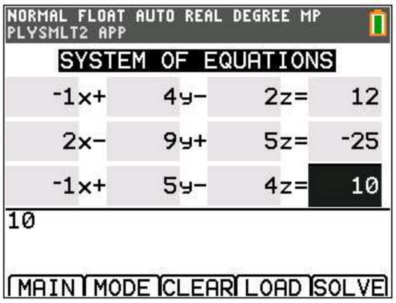
(b) The system of equations has no solution.



```
SYSTEM OF EQUATIONS
  2x+   3y+   5z=    4
  3x+   5y+   9z=    7
  5x+   9y+  17z=    1
1
```

```
NO SOLUTION FOUND
```

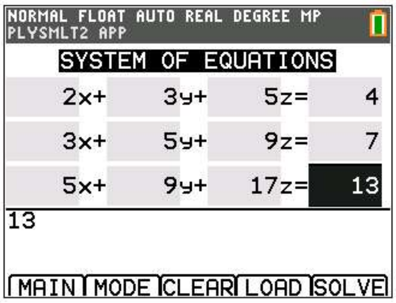
(c) The solution is $(-2, 4, 3)$



```
SYSTEM OF EQUATIONS
 -1x+   4y-   2z=   12
  2x-   9y+   5z=  -25
 -1x+   5y-   4z=   10
10
```

```
SOLUTION
x=-2
y=4
z=3
```

(d) Infinite solutions.



```
SYSTEM OF EQUATIONS
  2x+   3y+   5z=    4
  3x+   5y+   9z=    7
  5x+   9y+  17z=   13
13
```

```
SOLUTION SET
x=-1+2z
y=2-3z
z=z
```

11. $$\begin{pmatrix} 1 & 1 & 0 & 3-k \\ k & 0 & -1 & -3 \\ 6 & 2 & -3 & 1 \end{pmatrix} \xrightarrow{R_2-kR_1} \begin{pmatrix} 1 & 1 & 0 & 3-k \\ 0 & -k & -1 & -3-k(3-k) \\ 6 & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3-6R_1} \begin{pmatrix} 1 & 1 & 0 & 3-k \\ 0 & -k & -1 & k^2-3k-3 \\ 0 & -4 & -3 & 6k-17 \end{pmatrix}$$

$$\xrightarrow{kR_3-4R_2} \begin{pmatrix} 1 & 1 & 0 & 3-k \\ 0 & -k & -1 & k^2-3k-3 \\ 0 & 0 & -3k+4 & k(6k-17)-4(k^2-3k-3) \end{pmatrix}$$

The last equation is:

$(-3k+4)z = k(6k-17)-4(k^2-3k-3) \Rightarrow (-3+4k)z = 2k^2-5k+12$

In order for the system to have no solutions, the coefficient of z must be 0, and the right side of the equation should not be equal to 0.

$-3k+4=0 \Rightarrow k=\frac{4}{3}$

When $k=\frac{4}{3}$, the right-hand side of the equation is:

$2(\frac{4}{3})^2-5(\frac{4}{3})+12=\frac{80}{9}\neq 0 \Rightarrow$the required value is $k=\frac{4}{3}$

Exercise 1.2

1. **(a)** **(i)** Graph G **(ii)** $y=2x$ represents a straight line with a positive slope ($m=2$), passing through the origin ($c=0$)

(b) **(i)** Graph L **(ii)** $y=-3$ represents a horizontal line

(c) **(i)** Graph H **(ii)** $x-y=2 \Rightarrow y=x-2$ represents a straight line with positive slope ($m=1$), and with a negative y-intercept ($c=-2$)

(d) **(i)** Graph K **(ii)** $x^2+y^2=4 \Rightarrow y=\pm\sqrt{4-x^2}$ represents a circle centred at the origin and with radius 2 (this is not a function)

(e) **(i)** Graph J **(ii)** $y=2-x$ represents a straight line with a negative slope ($m=-1$), and with a positive y-intercept ($c=2$)

(f) **(i)** Graph C **(ii)** $y=x^2+2$ represents a parabola opening upwards, with vertex at $(0,2)$

(g) **(i)** Graph A **(ii)** $y^3=x \Rightarrow y=\sqrt[3]{x}$

(h) **(i)** Graph I **(ii)** $y=\frac{2}{x}$ represents a rational function with asymptotes $x=0$ and $y=0$

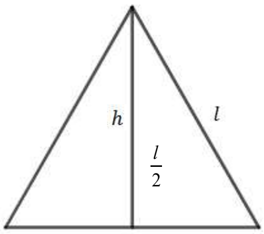
(i) **(i)** Graph F **(ii)** $x^2 + y = 2 \Rightarrow y = 2 - x^2$ represents a parabola opening downwards, with vertex at $(0, 2)$

2. $C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$

$A = \pi r^2 \Rightarrow A = \pi(\frac{C}{2\pi})^2 \Rightarrow A = \pi\frac{C^2}{4\pi^2} \Rightarrow A = \frac{C^2}{4\pi}$

3. The area of a triangle is $A = \frac{1}{2}bh$, where b and h are the base and the height of the triangle, respectively. Let l be the side of the equilateral triangle.

When drawing one of the heights, a right-angled triangle is formed with sides h, l and $\frac{l}{2}$. Use Pythagoras' theorem to find h:



$$h^2 + (\frac{l}{2})^2 = l^2 \Rightarrow h = \sqrt{l^2 - \frac{l^2}{4}} \Rightarrow h = \sqrt{\frac{3l^2}{4}} \Rightarrow h = \frac{l\sqrt{3}}{2}$$

$$A = \frac{1}{2} \cdot l \cdot \frac{l\sqrt{3}}{2} \Rightarrow A = \frac{l^2\sqrt{3}}{4}$$

4. The area of the pavement can be calculated as the difference between the areas of two rectangles:

$A = (18 + 2x)(12 + 2x) - 12 \times 18 \Rightarrow A = 216 + 60x + 4x^2 - 216 \Rightarrow A = 4x^2 + 60x$

5. Using Pythagoras' theorem:

$h^2 = x^2 + x^2 \Rightarrow h^2 = 2x^2 \Rightarrow h = x\sqrt{2}$

6. **(a)** Substitute the given values to find k: $23.5 = k\frac{375}{150} \Rightarrow k = \frac{150 \times 23.5}{375} = 9.4$

(b) $P = 9.4\frac{T}{V} \Rightarrow PV = 9.4T \Rightarrow PV = 9.4(375) \Rightarrow PV = 3525 \Rightarrow V = \frac{3525}{P}$

7. **(a)** $F = kx$

(b) $25 = k(16 - 12) \Rightarrow 25 = 4k \Rightarrow k = 6.25$

(c) $F = 6.25(18 - 12) \Rightarrow F = 37.5$

8. **(a)** The domain is the set of all x-values: $\{-6.2, -1.5, 0.7, 3.2, 3.8\}$

(b) The radius is a length, so $r > 0$

(c) f is a linear function, so $x \in \mathbb{R}$

(d) h is a quadratic function, so $x \in \mathbb{R}$

(e) The radicand must be greater or equal to zero, so $3 - t \geq 0 \Rightarrow t \leq 3$

(f) All real numbers can be cube rooted, so $t \in \mathbb{R}$

(g) f is a rational function; its denominator cannot be equal to 0.
$x^2 - 9 \neq 0 \Rightarrow x^2 \neq 9 \Rightarrow x \neq \pm 3$

(h) There are two conditions: x cannot be 0, as it is at the denominator of a fraction, and $\frac{1}{x^2} - 1 \geq 0 \Rightarrow 1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$, consequently the domain is $-1 \leq x \leq 1, x \neq 0$.

9. No, because a line with equation $x = c$, where c is a constant, fails the requirement of a function to map one x-value to only one y-value.

10. **(a)** Substitute the given x-value into the expression of the function:

(i) $h(21) = \sqrt{21-4} = \sqrt{17}$

(ii) $h(53) = \sqrt{53-4} = \sqrt{49} = 7$

(iii) $h(4) = \sqrt{4-4} = 0$

(b) The radicand must be negative for the function to be undefined,
so $x - 4 < 0 \Rightarrow x < 4$

(c) The domain of h is given by $x - 4 \geq 0 \Rightarrow x \geq 4$, the range is $y \geq 0$
(the square root of any positive number is positive, the square root of 0 is 0).

11. **(a)** **(i)** $f(x) = \frac{1}{x-5}$ is a rational function, its denominator cannot be equal to 0, so $x - 5 \neq 0 \Rightarrow$ the domain is $\{x: x \in \mathbb{R}, x \neq 5\}$. The range of f is $\{y: y \in \mathbb{R}, y \neq 0\}$, as the function can take all real values, except 0, as $\lim\limits_{x\to\infty} \frac{1}{x-5} = 0$.

(ii) To find the x-intercept, set y (or $f(x)$) equal to 0, and solve for x:

$0 = \frac{1}{x-5} \Rightarrow$ no solution, so no x-intercept.

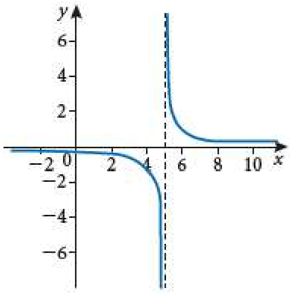
To find the y-intercept, set x equal to 0, and calculate y:

$y = \frac{1}{0-5} \Rightarrow y = -\frac{1}{5} \Rightarrow$ the y-intercept is $(0, -\frac{1}{5})$.

$x - 5 = 0 \Rightarrow$ the equation of the vertical asymptote is $x = 5$.

$\lim\limits_{x\to\infty} \frac{1}{x-5} = 0 \Rightarrow$ the equation of the horizontal asymptote is $y = 0$.

The graph of the function is:



(b) **(i)** $g(x) = \frac{1}{\sqrt{x^2-9}}$ is a rational function with a square root at the denominator, so the radicand has to be strictly positive, so $x^2 - 9 > 0 \Rightarrow x < -3, x > 3$, meaning the domain is $\{x: x < -3, x > 3\}$. The values given by function *g* will be strictly positive, because the square root at the denominator will always result in a positive number, so the range of *g* is $\{y: y > 0\}$.

(ii) To find the *x*-intercept, set *y* (or $g(x)$) equal to 0, and solve for *x*:

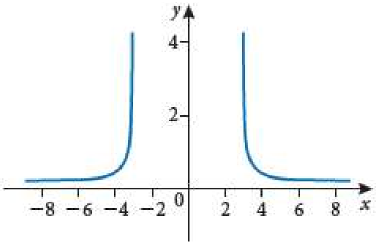
$0 = \frac{1}{\sqrt{x^2-9}} \Rightarrow$ no solution, so no *x*-intercept.

The *y*-intercept is found when $x = 0$, but this value for *x* is not in the domain of the function, so there is no *y*-intercept.

$x^2 - 9 = 0 \Rightarrow x = \pm 3$ there are two vertical asymptotes: $x = 3, x = -3$.

$\lim\limits_{x\to\infty} \frac{1}{\sqrt{x^2-9}} = 0 \Rightarrow$the equation of the horizontal asymptote is $y = 0$.

The graph of the function is:



(c) **(i)** $h(x) = \frac{2x-1}{x+2}$ is a rational function, its denominator cannot be equal to 0, so $x + 2 \neq 0 \Rightarrow$ the domain is $\{x: x \in \mathbb{R}, x \neq -2\}$.

$\lim\limits_{x\to\infty} \frac{2x-1}{x+2} = 2$, this means that the range of *h* is $\{y: y \in \mathbb{R}, y \neq 2\}$.

(ii) To find the *x*-intercept, set *y* (or $h(x)$) equal to 0, and solve for *x*:

$0 = \frac{2x-1}{x+2} \Rightarrow 2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$, so the *x*-intercept is $(\frac{1}{2}, 0)$.

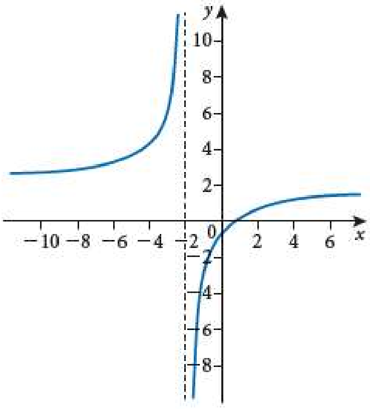
To find the *y*-intercept, set *x* equal to 0, and calculate *y*:

$y = \frac{2(0)-1}{0+2} \Rightarrow y = -\frac{1}{2} \Rightarrow$ the *y*-intercept is $(0, -\frac{1}{2})$.

$x + 2 = 0 \Rightarrow$ the equation of the vertical asymptote is $x = -2$.

$\lim\limits_{x\to\infty} \frac{2x-1}{x+2} = 2 \Rightarrow$the equation of the horizontal asymptote is $y = 2$.

The graph of the function is:



(d) **(i)** $p(x) = \sqrt{5 - 2x^2}$ is a square root function, so the radicand must be positive or 0:

$$5 - 2x^2 \geq 0 \Rightarrow 5 \geq 2x^2 \Rightarrow x^2 \leq \frac{5}{2}$$

$$\Rightarrow -\sqrt{\frac{5}{2}} \leq x \leq \sqrt{\frac{5}{2}} \Rightarrow -\frac{\sqrt{10}}{2} \leq x \leq \frac{\sqrt{10}}{2} \Rightarrow \text{the domain is } \left\{x: -\frac{\sqrt{10}}{2} \leq x \leq \frac{\sqrt{10}}{2}\right\}.$$

The values given by function g will be positive, because the square root at the denominator will always result in 0 or a positive number, but, because the radicand is a quadratic expression, the range of $y = 5 - 2x^2$ must also be considered. The x-coordinate of the vertex of the downwards parabola representing $y = 5 - 2x^2$ is $x = 0$, consequently the y-coordinate is $y = 5$, meaning the y-values given by g are always less than 5, $y \leq 5$. This means that the range of p is $\{y: 0 \leq y \leq \sqrt{5}\}$.

(ii) To find the x-intercept, set y (or $p(x)$) equal to 0, and solve for x:

$$5 - 2x^2 = 0 \Rightarrow 5 = 2x^2 \Rightarrow x^2 = \frac{5}{2} \Rightarrow x = \pm\sqrt{\frac{5}{2}} \Rightarrow x = \pm\frac{\sqrt{10}}{2} \Rightarrow$$

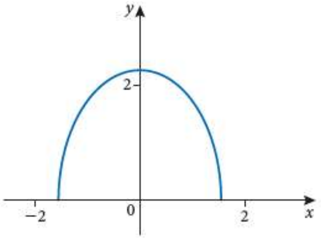
the x-intercepts are $(-\frac{\sqrt{10}}{2}, 0)$ and $(\frac{\sqrt{10}}{2}, 0)$.

To find the y-intercept, set x equal to 0, and calculate y:

$y = \sqrt{5 - 2(0)^2} \Rightarrow y = \sqrt{5} \Rightarrow$ the y-intercept is $(0, \sqrt{5})$.

This function does not have any asymptotes.

The graph of the function is:



(e) **(i)** $f(x) = \frac{1}{x} - 4$ is a rational function, its denominator cannot be equal to 0, so $x \neq 0 \Rightarrow$ the domain is $\{x: x \in \mathbb{R}, x \neq 0\}$.

$\lim\limits_{x \to \infty}(\frac{1}{x} - 4) = -4$, this means that the range of *f* is $\{y: y \in \mathbb{R}, y \neq -4\}$.

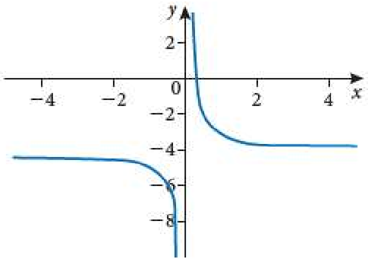
(ii) To find the *x*-intercept, set *y* (or $f(x)$) equal to 0, and solve for *x*:

$0 = \frac{1}{x} - 4 \Rightarrow \frac{1}{x} = 4 \Rightarrow x = \frac{1}{4} \Rightarrow$the *x*-intercept is $(\frac{1}{4}, 0)$.

The *y*-intercept is found when $x = 0$, but this value for *x* is not in the domain of the function, so there is no *y*-intercept.

The equation of the vertical asymptote is $x = 0$, the equation of the horizontal asymptote is $y = -4$ $(\lim\limits_{x \to \infty}(\frac{1}{x} - 4) = -4)$.

The graph of the function is:



Exercise 1.3

1. **(a)** $(f \circ g)(5) = f(g(5)) = f(\frac{1}{5-3}) = f(\frac{1}{2}) = 2 \cdot \frac{1}{2} = 1$

(b)$(g \circ f)(5) = g(f(5)) = g(2 \cdot 5) = g(10) = \frac{1}{10-3} = \frac{1}{7}$

(c) $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x-3}) = 2 \cdot \frac{1}{x-3} = \frac{2}{x-3}$

(d) $(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2x-3}$

2. **(a)** $(f \circ g)(0) = f(g(0)) = f(2 - 0^2) = f(2) = 2 \cdot 2 - 3 = 1$

(b)$(g \circ f)(0) = g(f(0)) = g(2 \cdot 0 - 3) = g(-3) = 2 - (-3)^2 = 2 - 9 = -7$

(c) $(f \circ f)(4) = f(f(4)) = f(2 \cdot 4 - 3) = f(5) = 2 \cdot 5 - 3 = 7$

(d) $(g \circ g)(-3) = g(g(-3)) = g(2 - (-3)^2) = g(-7) = 2 - (-3)^2 = 2 - 49 = -47$

(e) $(f \circ g)(-1) = f(g(-1)) = f(2 - (-1)^2) = f(1) = 2 \cdot 1 - 3 = -1$

(f) $(g \circ f)(-3) = g(f(-3)) = g(2 \cdot (-3) - 3) = g(-9) = 2 - (-9)^2 = 2 - 81 = -79$

(g) $(f \circ g)(x) = f(g(x)) = f(2 - x^2) = 2 \cdot (2 - x^2) - 3 = 4 - 2x^2 - 3 = 1 - 2x^2$

(h) $(g \circ f)(x) = g(f(x)) = g(2x - 3) = 2 - (2x - 3)^2$

$$= 2 - (4x^2 - 12x + 9) = 2 - 4x^2 + 12x - 9$$

$$\Rightarrow (g \circ f)(x) = -4x^2 + 12x - 7$$

(i) $(f \circ f)(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 6 - 3 = 4x - 9$

(j) $(g \circ g)(x) = g(g(x)) = g(2 - x^2) = g(2 - (2 - x^2)^2)$

$$= 2 - (4 - 4x^2 + x^4) = 2 - 4 + 4x^2 - x^4$$

$$\Rightarrow (g \circ g)(x) = -x^4 + 4x^2 - 2$$

3. In each question, when finding the domain of $f \circ g$, check the following two conditions:

- the input x is in the domain of g, because the first rule to be applied to x is g, the inside function of $f(g(x))$.
- the output $g(x)$ is in the domain of f (the range of g must be either equal to or a subset of the domain of f).

The same applies when finding the domain of $g \circ f$: x must be in the domain of f, because the first rule to be applied to x is f, the inside function of $g(f(x))$, and $f(x)$ is in the domain of g.

(a)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(2 + 3x) = 4(2 + 3x) - 1 = 8 + 12x - 1 = 7 + 12x$

$x \in \mathbb{R} \Rightarrow 2 + 3x \in \mathbb{R}$, this set of values is the same as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(4x - 1) = 2 + 3(4x - 1) = 2 + 12x - 3 = 12x - 1$

$x \in \mathbb{R} \Rightarrow 4x - 1 \in \mathbb{R}$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(b)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \geq 1$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(-2x) = (-2x)^2 + 1 = 4x^2 + 1$

$x \in \mathbb{R} \Rightarrow -2x \in \mathbb{R}$, this set of values is the same set of values as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(x^2+1) = -2(x^2+1) = -2x^2-2$

$x \in \mathbb{R} \Rightarrow 1 + x^2 \geq 1$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(c)

	f	g
Domain	$x \in \mathbb{R}$, $x \geq -1$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \geq 0$	$y \in \mathbb{R}$, $y \geq 1$

$(f \circ g)(x) = f(g(x)) = f(1+x^2) = \sqrt{1+x^2+1} = \sqrt{x^2+2}$

$x \in \mathbb{R} \Rightarrow 1 + x^2 \geq 1$, this set of values is a subset of the domain of f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x+1}) = 1 + (\sqrt{x+1})^2 = 1 + x + 1 = x + 2$

$x \geq -1 \Rightarrow \sqrt{x+1} \geq 0$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also $x \geq -1$.

(d)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq -4$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}, y \neq 0$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(x-1) = \frac{2}{x-1+4} = \frac{2}{x+3}$

$x \in \mathbb{R} \Rightarrow x - 1 \in \mathbb{R}$, but this set of values is not a subset of the domain of f. To be able to compose the two functions, the x-value, which is the input of g resulting in an output of $y = -4$, must be excluded from the domain of g.

$x - 1 = -4 \Rightarrow x = -3$, so the domain of $(f \circ g)(x)$ is $x \in \mathbb{R}, x \neq -3$.

$$(g \circ f)(x) = g(f(x)) = g(\frac{2}{x+4}) = \frac{2}{x+4} - 1 = \frac{2-x-4}{x+4} = \frac{-x-2}{x+4} = -\frac{x+2}{x+4}$$

$x \in \mathbb{R}, x \neq -4 \Rightarrow \frac{2}{x+4} \neq 0$, this set of values is a subset of the domain of g, so the domain of $(g \circ f)(x)$ is also $x \in \mathbb{R}, x \neq -4$.

(e)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

$(f \circ g)(x) = f(g(x)) = f(\frac{x-5}{3}) = 3 \cdot \frac{x-5}{3} + 5 = x - 5 + 5 = x$

$x \in \mathbb{R} \Rightarrow \frac{x-5}{3} \in \mathbb{R}$, this set of values is the same as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(3x + 5) = \frac{3x+5-5}{3} = \frac{3x}{3} = 3$

$x \in \mathbb{R} \Rightarrow 3x - 5 \in \mathbb{R}$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(f)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \leq 1,$ $y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{1 - x^2}) = 2 - (\sqrt[3]{1 - x^2})^3$$
$$= 2 - (1 - x^2) = 2 - 1 + x^2 = x^2 + 1$$

$x \in \mathbb{R} \Rightarrow \sqrt[3]{1 - x^2} \leq 1$, this set of values is a subset of the domain of f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

$$(g \circ f)(x) = g(f(x)) = g(2 - x^3) = \sqrt[3]{1 - (2 - x^3)^2} = \sqrt[3]{1 - (4 - 4x^3 + x^6)}$$
$$= \sqrt[3]{-x^6 + 4x^3 - 3}$$

$x \in \mathbb{R} \Rightarrow 2 - x^3 \in \mathbb{R}$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}$.

(g)

	f	g
Domain	$x \in \mathbb{R},$ $x \neq 4$	$x \in \mathbb{R},$ $x \neq 0$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R},$ $y \neq 0$

$$(f \circ g)(x) = f(g(x)) = f(\frac{1}{x^2}) = \frac{2\frac{1}{x^2}}{4-\frac{1}{x^2}} = \frac{\frac{2}{x^2}}{\frac{4x^2-1}{x^2}} = \frac{2}{4x^2-1}$$

$x \in \mathbb{R}, x \neq 0 \Rightarrow \frac{1}{x^2} > 0$, but this set of values is not a subset of the domain of f. To be able to compose the two functions, the x-value, which is the input of g resulting in an output of $y = 4$, must be excluded from the domain of g.

$\frac{1}{x^2} = 4 \Rightarrow \frac{1}{4} = x^2 \Rightarrow x = \pm\frac{1}{2}$, so the domain of $f \circ g$ is: $x \in \mathbb{R}, x \neq 0, x \neq \pm\frac{1}{2}$.

$$(g \circ f)(x) = g(f(x)) = g(\frac{2x}{4-x}) = \frac{1}{\left(\frac{2x}{4-x}\right)^2} = \frac{(4-x)^2}{4x^2}$$

$x \in \mathbb{R}, x \neq 4 \Rightarrow \frac{2x}{4-x} \neq 0$, but this is not a subset of the domain of g. To be able to compose the two functions, the x-value, which is the input of f resulting in an output of $y = 4$, must be excluded from the domain of f.

$\frac{2x}{4-x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$, so the domain of $g \circ f$ is: $x \in \mathbb{R}, x \neq 4, x \neq 0$.

(h)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq -3$	$x \in \mathbb{R}$, $x \neq -3$
Range	$y \in \mathbb{R}$, $y \neq -3$	$y \in \mathbb{R}$, $y \neq -3$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{2}{x+3} - 3\right) = \frac{2}{\frac{2}{x+3}-3+3} - 3 = 2 \cdot \frac{x+3}{2} = xx \in \mathbb{R}$,

$x \neq -3 \Rightarrow \frac{2}{x+3} - 3 \neq -3$, this set of values is the same as the domain for f, so the domain of $f \circ g$ is also $x \in \mathbb{R}, x \neq -3$.

$(g \circ f)(x) = g(f(x)) = g(\frac{2}{x+3} - 3) = x$

$x \in \mathbb{R}, x \neq -3 \Rightarrow \frac{2}{x+3} - 3 \neq -3$, this set of values is the same as the domain for g, so the domain of $g \circ f$ is also $x \in \mathbb{R}, x \neq -3$.

(i)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 1$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 1$	$y \in \mathbb{R}$, $y \geq -1$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \frac{x^2 - 1}{x^2 - 1 - 1} = \frac{x^2 - 1}{x^2 - 2}$$

$x \in \mathbb{R} \Rightarrow x^2 - 1 \geq -1$, but this set of values is not a subset of the domain of f. To be able to compose the two functions, the x-value, which is the input of g resulting in an output of $y = 1$, must be excluded from the domain of g.

$x^2 - 1 = 1 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$, so the domain of $f \circ g$ is $x \in \mathbb{R}, x \neq \pm\sqrt{2}$.

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 - 1 = \frac{x^2}{(x-1)^2} - 1$$

$$= \frac{x^2 - (x^2 - 2x + 1)}{(x-1)^2} = \frac{2x-1}{(x-1)^2}$$

$x \in \mathbb{R}, x \neq 1 \Rightarrow \frac{x}{x-1} \neq 1$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also$x \in \mathbb{R}, x \neq 1$.

4. **(a)**

	g	h
Domain	$x \in \mathbb{R}$, $x \geq 1$	$x \in \mathbb{R}$
Range	$y \geq 0$	$y \in \mathbb{R}$, $y \leq 10$

$(g \circ h)(x) = g(h(x)) = g(10 - x^2) = \sqrt{10 - x^2 - 1} = \sqrt{9 - x^2}$

$x \in \mathbb{R} \Rightarrow 10 - x^2 \leq 10$, this is not a subset of the domain of *f*. To be able to compose the two functions, only the *x*-values which are common to both the domain of *g* (real numbers greater than or equal to 1) and the range of *h* (real numbers less than or equal to 10) are acceptable, so only real numbers between 1 and 10, including 1 and 10, can be inputs for *g*. This means that the domain of $g \circ h$ will not contain the *x*-values that will lead to outputs which are less than 1.

$10 - x^2 < 1 \Rightarrow 9 < x^2 \Rightarrow x < -3, x > 3$, so the domain of $g \circ h$ is: $-3 \leq x \leq 3$.

The outputs of $g \circ h$, corresponding to inputs taking values from the set $\{x: -3 \leq x \leq 3\}$, will be elements of the set $\{y: 0 \leq y \leq 3\}$, so the range of $g \circ h$ is $0 \leq y \leq 3$.

(b) $(h \circ g)(x) = h(g(x)) = h(\sqrt{x - 1}) = 10 - (\sqrt{x - 1})^2 = 10 - x + 1 = -x + 11$

$x \geq 1 \Rightarrow \sqrt{x - 1} \geq 0$, this is a subset of the domain of *h*, so the domain of $h \circ g$

is also $x \geq 1$.

The range of $h \circ g$ is $y \leq 10$, as the restriction on the domain of *h* does not impact the range of the quadratic $y = 10 - x^2$.

5. **(a)**

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 0$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 0$	$y \in \mathbb{R}$, $y \leq 10$

$(f \circ g)(x) = f(g(x)) = f(10 - x^2) = \frac{1}{10 - x^2}$

$x \in \mathbb{R} \Rightarrow 10 - x^2 \leq 10$, this set of values is not a subset of the domain of *f*. To be able to compose the two functions, the *x*-value, which is the input of *g* resulting in an output of $y = 0$, must be excluded from the domain of *g*.

$10 - x^2 = 0 \Rightarrow 10 = x^2 \Rightarrow x = \pm\sqrt{10}$, so the domain of $f \circ g$ is: $x \in \mathbb{R}, x \neq \pm\sqrt{10}$.

The outputs of $f \circ g$ will be elements of the set $\{y: y \in \mathbb{R}, y \neq 10\}$, as the input of *g* cannot be 0, so the range of $f \circ g$ is: $y \in \mathbb{R}, y \neq 0$.

(b) $(g \circ f)(x) = g(f(x)) = g(\frac{1}{x}) = 10 - \frac{1}{x^2}$

$x \in \mathbb{R}, x \neq 0 \Rightarrow \frac{1}{x} \neq 0$, this set of values is a subset of the domain of g, so the domain of $g \circ f$ is also $x \in \mathbb{R}, x \neq 0$.

The outputs of $g \circ f$, corresponding to inputs taking values from the set $\{x: x \in \mathbb{R}, x \neq 0\}$, will be elements of the set $\{y: y \in \mathbb{R}, y < 10\}$, ($y = 10$ is obtained for $x = 0$, which is not in the domain), so the range of $g \circ f$ is: $y \in \mathbb{R}, y < 10$.

6. In each of the following, try to identify the rules transforming the input x into the expression given by $f(x)$, and take into consideration the order in which the two functions are combined.

(a) $f(x) = (x+3)^2 \Rightarrow$ 3 is added to x, and the result is squared
$\Rightarrow h(x) = x + 3, g(x) = x^2$

(b) $f(x) = \sqrt{x-5} \Rightarrow$ 5 is subtracted from x, and the result is square rooted
$\Rightarrow h(x) = x - 5, g(x) = \sqrt{x}$

(c) $f(x) = 7 - \sqrt{x} \Rightarrow x$ is square rooted, and the result is subtracted from 7
$\Rightarrow h(x) = \sqrt{x}, g(x) = 7 - x$

(d) $f(x) = \frac{1}{x+3} \Rightarrow$ 3 is added to x, and the reciprocal of the result is computed
$\Rightarrow h(x) = x + 3, g(x) = \frac{1}{x}$

(e) $f(x) = 10^{x+1} \Rightarrow$1 is added to x, this result is then the power to which 10 is raised
$\Rightarrow h(x) = x + 1, g(x) = 10^x$

(f) $f(x) = \sqrt[3]{x-9} \Rightarrow$ 9 is subtracted from x, and the result is cube rooted
$\Rightarrow h(x) = x - 9, g(x) = \sqrt[3]{x}$

(g) $f(x) = |x^2 - 9| \Rightarrow$ 9 is subtracted from the square of x, and the absolute value of result is taken $\Rightarrow h(x) = x^2 - 9, g(x) = |x|$

(h) $f(x) = \frac{1}{\sqrt{x-5}} \Rightarrow$ the square root of the difference between x and 5, and the reciprocal of the result is computed $\Rightarrow h(x) = \sqrt{x-5}, g(x) = \frac{1}{x}$

7. **(a)**

	f	g
Domain	$x \in \mathbb{R}$, $x \geq 0$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \geq 0$	$y \in \mathbb{R}$, $y \geq 1$

$$(f \circ g)(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1+1} = \sqrt{x^2+2}$$

$x \in \mathbb{R} \Rightarrow x^2 + 1 \geq 1$, this is a subset of the domain of f, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

(b)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq 0$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 0$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(x+3) = \frac{1}{x+3}$$

$x \in \mathbb{R} \Rightarrow x+3 \in \mathbb{R}$, but this is not a subset of the domain of *f*. To be able to compose the two functions, the *x*-value which is the input of *g* resulting in an output of $y = 0$ must be excluded from the domain of *g*.

$x+3=0 \Rightarrow x=-3$, so the domain of $f \circ g$ is $x \in \mathbb{R}, x \neq -3$

(c)

	f	g
Domain	$x \in \mathbb{R}$, $x \neq \pm 1$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$, $y \neq 0$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(x+1) = \frac{3}{(x+1)^2-1} = \frac{3}{x^2+2x+1-1} = \frac{3}{x^2+2x}$$

$x \in \mathbb{R} \Rightarrow x+1 \in \mathbb{R}$, but this set of values is not a subset of the domain of *f*. To be able to compose the two functions, the *x*-values, which are the inputs of *g* resulting in the outputs $y = \pm 1$, must be excluded from the domain of *f*.

$x+1 = \pm 1 \Rightarrow x = -1 \pm 1 \Rightarrow x = 0, x = -2$,
so the domain of $f \circ g$ is $x \in \mathbb{R}, x \neq -2, x \neq 0$

(d)

	f	g
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(\tfrac{x}{2}) = 2 \cdot \tfrac{x}{2} + 3 = x+3$$

$x \in \mathbb{R} \Rightarrow \frac{x}{2} \in \mathbb{R}$, this set of values is the same set of values as the domain for *f*, so the domain of $f \circ g$ is also $x \in \mathbb{R}$.

Exercise 1.4

1. **(a)** For $f: x = 2 \Rightarrow y = -5$, so for $f^{-1}: x = -5 \Rightarrow y = 2$, this means $f^{-1}(-5) = 2$

(b) For $f: x = 6 \Rightarrow y = 10$, so for $f^{-1}: x = 10 \Rightarrow y = 6$, this means $f^{-1}(10) = 6$

2. **(a)** For $f: x = -1 \Rightarrow y = 13$, so for $f^{-1}: x = 13 \Rightarrow y = -1$, this means $f^{-1}(13) = -1$

(b) For $f: x = b \Rightarrow y = a$, so for $f^{-1}: x = a \Rightarrow y = b$, this means $f^{-1}(b) = a$

3. If $y = f^{-1}(x)$, then $x = f^{-1}(y)$, so $y = 5$ is an output for the original function *f*.

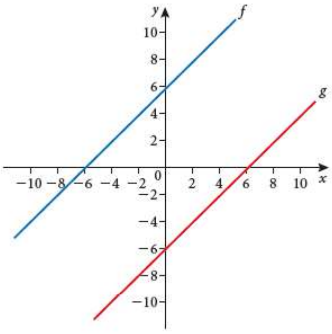
$\Rightarrow 5 = 3x - 7 \Rightarrow 3x = 12 \Rightarrow x = 4 \Rightarrow g^{-1}(5) = 4$

4. $-12 = x^2 - 8x \Rightarrow x^2 - 8x + 12 = 0 \Rightarrow (x-6)(x-2) = 0 \Rightarrow x = 2$ or $x = 6$, but $x \geq 4$, so $x = 6 \Rightarrow h^{-1}(-12) = 6$

5. **(a)** **(i)** $(f \circ g)(x) = f(g(x)) = f(x-6) = x - 6 + 6 = x$

$(g \circ f)(x) = g(f(x)) = g(x+6) = x + 6 - 6 = x$

(ii)

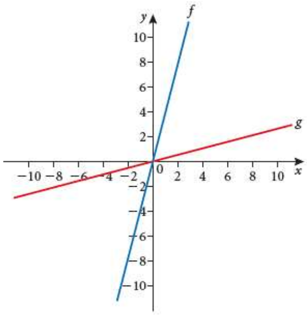


The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(b) **(i)** $(f \circ g)(x) = f(g(x)) = f(\frac{x}{4}) = 4 \cdot \frac{x}{4} = x$

$(g \circ f)(x) = g(f(x)) = g(4x) = \frac{4x}{4} = x$

(ii)

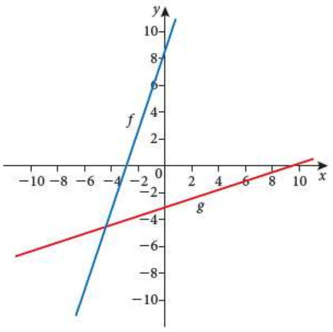


The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(c) **(i)** $(f \circ g)(x) = f(g(x)) = f(\frac{1}{3}x - 3) = 3 \cdot (\frac{1}{3}x - 3) + 9 = x - 9 + 9 = x$

$(g \circ f)(x) = g(f(x)) = g(3x - 9) = \frac{1}{3}(3x + 9) - 3 = x + 3 - 3 = x$

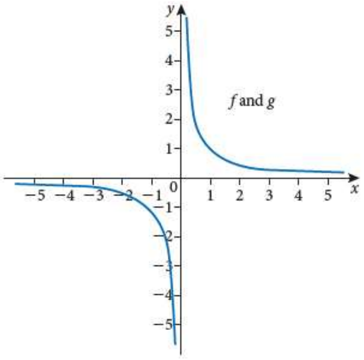
(ii)



The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(d) **(i)** $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x = (g \circ f)(x)$

(ii)

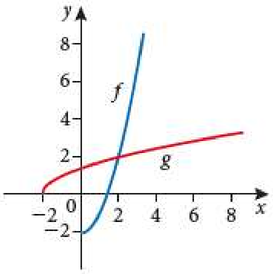


The graph of $y = \frac{1}{x}$ is symmetrical about the line $y = x$, so the reflected graph overlaps with the original graph.

(e) **(i)** $(f \circ g)(x) = f(g(x)) = f(\sqrt{x + 2}) = (\sqrt{x + 2})^2 - 2 = x + 2 - 2 = x$

$(g \circ f)(x) = g(f(x)) = g(x^2 - 2) = \sqrt{x^2 - 2 + 2} = \sqrt{x^2} = |x| = x, x \geq 0.$

(ii)

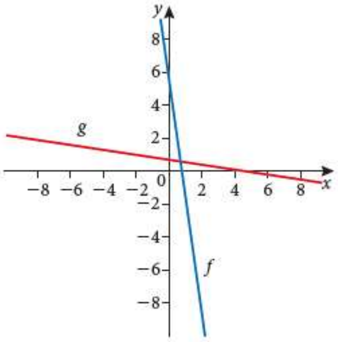


The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(f) **(i)** $(f \circ g)(x) = f(g(x)) = f(\frac{5-x}{7}) = 5 - 7 \cdot \frac{5-x}{7} = 5 - (5 - x) = x$

$(g \circ f)(x) = g(f(x)) = g(5 - 7x) = \frac{5-(5-7x)}{7} = \frac{7x}{7}x = x$

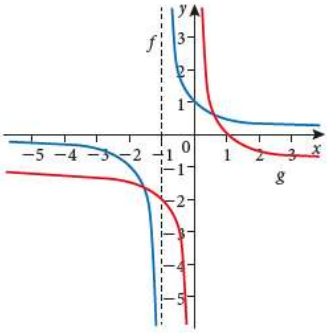
(ii)



The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(g) **(i)** $(f \circ g)(x) = f(g(x)) = f(\frac{1-x}{x}) = \frac{1}{1+\frac{1-x}{x}} = \frac{1}{\frac{x+1-x}{x}} = \frac{1}{\frac{1}{x}} = x$

$(g \circ f)(x) = g(f(x)) = g(\frac{1}{1+x}) = \frac{1-\frac{1}{1+x}}{\frac{1}{1+x}} = \frac{\frac{1+x-1}{1+x}}{\frac{1}{1+x}} = \frac{\frac{x}{1+x}}{\frac{1}{1+x}} = \frac{x}{1+x} \cdot \frac{1+x}{1} = x$

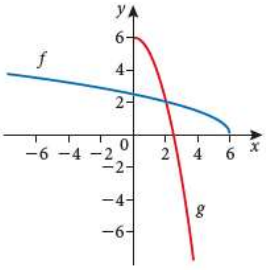


The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(h) **(i)** $(f \circ g)(x) = f(g(x)) = f(6 - x^2) = (6 - (6 - x^2))^{\frac{1}{2}} = (x^2)^{\frac{1}{2}} = |x| = x, x \geq 0.$

$(g \circ f)(x) = g(f(x)) = g((6 - x)^{\frac{1}{2}}) = 6 - ((6 - x)^{\frac{1}{2}})^2 = 6 - (6 - x) = x$

(ii)



The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

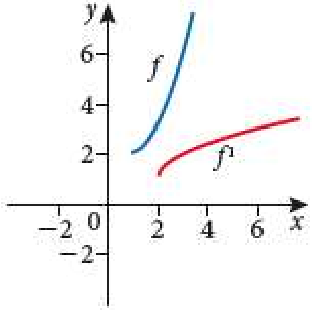
(i) **(i)** $(f \circ g)(x) = f(g(x)) = f(1 + \sqrt{x-2}) = (1 + \sqrt{x-2})^2 - 2(1 + \sqrt{x-2}) + 3$

$= 1 + 2\sqrt{x-2} + x - 2 - 2 - 2\sqrt{x-2} + 3 = x$

$(g \circ f)(x) = g\big(f(x)\big) = g(x^2 - 2x + 3) = 1 + \sqrt{x^2 - 2x + 3 - 2}$

$= 1 + \sqrt{x^2 - 2x + 1} = 1 + \sqrt{(x-1)^2} = 1 + |x - 1| = 1 + x - 1 = x, x \geq 1.$

(ii)

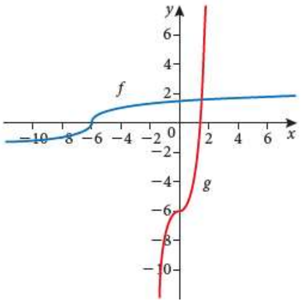


The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

(j) **(i)** $(f \circ g)(x) = f(g(x)) = f(2x^3 - 6) = \sqrt[3]{\frac{2x^3-6+6}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$

$(g \circ f)(x) = g(f(x)) = g(\sqrt[3]{\frac{x+6}{2}}) = 2(\sqrt[3]{\frac{x+6}{2}})^3 - 6 = 2(\frac{x+6}{2}) - 6 = x + 6 - 6 = x$

(ii)



The two graphs are reflections of each other in the line $y = x$, so *f* and *g* are inverse functions.

6. The fact that the domain of f^{-1} is the same as the range of f is used when answering the following questions.

(a) $f(x) = 2x - 3$

$y = 2x - 3$

$x = 2y - 3$

$x + 3 = 2y$

$y = \frac{x+3}{2} \Rightarrow f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$

The domain of f^{-1} is $x \in \mathbb{R}$.

(b) $f(x) = \frac{x+7}{4}$

$y = \frac{x+7}{4}$

$x = \frac{y+7}{4}$

$4x = y + 7$

$y = 4x - 7 \Rightarrow f^{-1}(x) = 4x - 7$

The domain of f^{-1} is $x \in \mathbb{R}$.

(c) $f(x) = \sqrt{x}$

$y = \sqrt{x}$

$x = \sqrt{y}$

$x^2 = y \Rightarrow f^{-1}(x) = x^2$

The domain of f^{-1} is $x \in \mathbb{R}, x \geq 0$.

(d) $f(x) = \frac{1}{x+2}$

$y = \frac{1}{x+2}$

$x = \frac{1}{y+2}$

$y + 2 = \frac{1}{x}$

$y = \frac{1}{x} - 2 \Rightarrow f^{-1}(x) = \frac{1}{x} - 2$

The domain of f^{-1} is $x \in \mathbb{R}, x \neq 0$.

(e) $f(x) = 4 - x^2$

$y = 4 - x^2$

$x = 4 - y^2$

$y^2 = 4 - x$

$y = \pm\sqrt{4 - x}$

$y \geq 0 \Rightarrow f^{-1}(x) = \sqrt{4 - x}$

The domain of f^{-1} is $x \leq 4, x \in \mathbb{R}$.

(f) $f(x) = \sqrt{x-5}$

$y = \sqrt{x-5}$

$x = \sqrt{y-5}$

$x^2 = y - 5$

$y = x^2 + 5 \Rightarrow f^{-1}(x) = x^2 + 5$

The domain of f^{-1} is $x \geq 0, x \in \mathbb{R}$.

(g) $f(x) = ax + b$

$y = ax + b$

$x = ay + b$

$x - b = ay$

$y = \frac{x-b}{a} \Rightarrow f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}$

The domain of f^{-1} is $x \in \mathbb{R}$.

(h) $f(x) = x^2 + 2x$

$y = x^2 + 2x$

$x = y^2 + 2y$

$y^2 + 2y - x = 0$

$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(-x)}}{2(1)}$

$y = \frac{-2 \pm \sqrt{4+4x}}{2}$

$y \geq -1 \Rightarrow y = \frac{-2 + \sqrt{4+4x}}{2}$

$y = \frac{-2 + \sqrt{4(1+x)}}{2}$

$y = \frac{-2 + 2\sqrt{1+x}}{2}$

$y = -1 + \sqrt{1+x} \Rightarrow f^{-1}(x) = -1 + \sqrt{1+x}$

The domain of f^{-1} is $x \geq -1, x \in \mathbb{R}$.

(i) $f(x) = \frac{x^2-1}{x^2+1}$

$y = \frac{x^2-1}{x^2+1}$

$x = \frac{y^2-1}{y^2+1}$

$x(y^2+1) = y^2 - 1$

$xy^2 + x = y^2 - 1$

$x + 1 = y^2 - xy^2$

$x + 1 = y^2(1-x)$

$y^2 = \frac{x+1}{1-x}$

$y = \pm\sqrt{\frac{x+1}{1-x}}$

$y \leq 0 \Rightarrow y = -\sqrt{\frac{x+1}{1-x}}$

$f^{-1}(x) = -\sqrt{\frac{x+1}{1-x}}$

The domain of f^{-1} is $-1 \leq x \leq 1, x \in \mathbb{R}$.

(j) $f(x) = x^3 + 1$

$y = x^3 + 1$

$x = y^3 + 1$

$y^3 = x - 1$

$y = \sqrt[3]{x-1} \Rightarrow f^{-1}(x) = \sqrt[3]{x-1}$

The domain of f^{-1} is $x \in \mathbb{R}$.

7. **(a)** The graph of $f(x) = \frac{2x+3}{x-1}, x \neq 1, x \in \mathbb{R}$, shows that *f* is a decreasing function on its domain (no need for restricting the domain). This means that *f* is a one-to-one function, consequently f^{-1}exists.

Next, find the expression of f^{-1}:

$y = \frac{2x+3}{x-1}$

$x = \frac{2y+3}{y-1}$

$x(y-1) = 2y + 3$

$xy - x = 2y + 3$

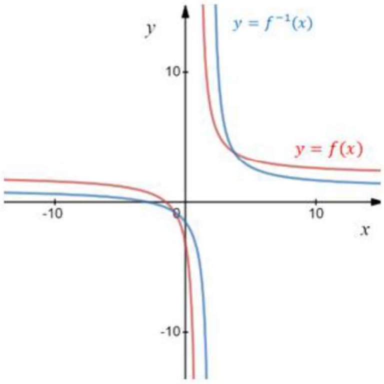
$xy - 2y = x + 3$

$y(x-2) = x+3$

$y = \frac{x+3}{x-2} \Rightarrow f^{-1}(x) = \frac{x+3}{x-2}$

The domain of f^{-1} is the range of f, namely $x \neq 2, x \in \mathbb{R}$.

The two graphs are shown below:



(b) The graph of $f(x) = (x-2)^2, x \in \mathbb{R}$, shows that f is not a one-to-one function on its domain. In order for f^{-1} to exist, the domain of f must be restricted to either $x \leq 2$ or $x \geq 2$.

Next, find the expression of f^{-1} for $x \geq 2$:

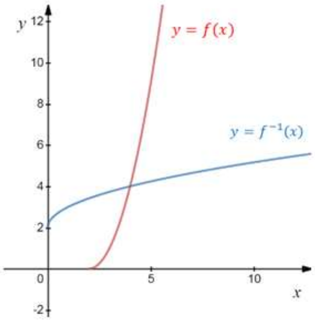
$y = (x-2)^2$

$x = (y-2)^2$

$\sqrt{x} = y-2$

$y = \sqrt{x} + 2 \Rightarrow f^{-1}(x) = \sqrt{x} + 2$

The domain of f^{-1} is the range of f, namely $x \geq 0, x \in \mathbb{R}$. The graphs are shown below for a domain for f restricted to $x \geq 2$:



(c) The graph of $f(x) = \frac{1}{x^2}, x \neq 0, x \in \mathbb{R}$, shows that f is not a one-to-one function on its domain. In order for f^{-1} to exist, the domain of f must be restricted to either $x < 0$ or $x > 0$.

Next, find the expression of f^{-1} for $x > 0$:

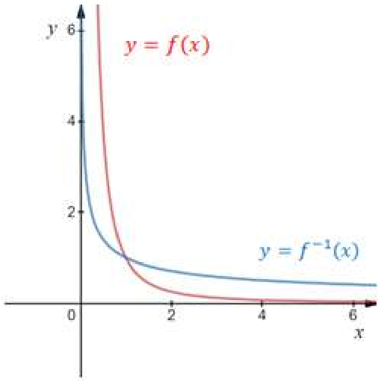
$y = \frac{1}{x^2}$

$x = \frac{1}{y^2}$

$y^2 = \frac{1}{x}$

$y = \frac{1}{\sqrt{x}} \Rightarrow f^{-1}(x) = \frac{1}{\sqrt{x}}$

The domain of f^{-1} is the range of f, namely $x > 0, x \in \mathbb{R}$. The graphs are shown below for a domain for f restricted to $x > 0$:



(d) The graph of $f(x) = 2 - x^4, x \in \mathbb{R}$, shows that f is not a one-to-one function on its domain. In order for $f^{-1}$to exist, the domain of f must be restricted to either $x \leq 0$ or $x \geq 0$.

Next, find the expression of f^{-1} for $x > 0$:

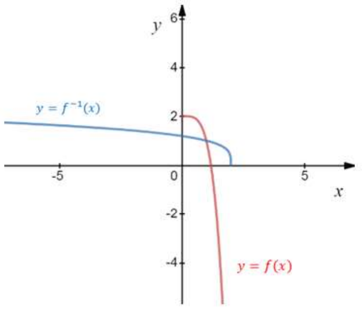
$y = 2 - x^4$

$x = 2 - y^4$

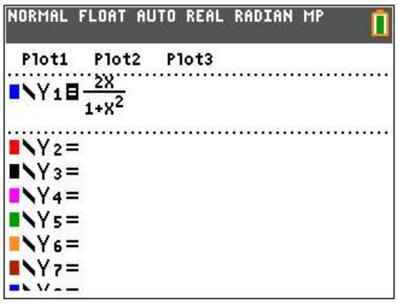
$y^4 = 2 - x$

$y = \sqrt[4]{2 - x} \Rightarrow f^{-1}(x) = \sqrt[4]{2 - x}$

The domain of f^{-1} is the range of f, namely $x \leq 2, x \in \mathbb{R}$. The graphs are shown below for a domain for f restricted to $x \geq 0$:



8.



NORMAL FLOAT AUTO REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = \frac{2X}{1+X^2}$

$Y_2=$

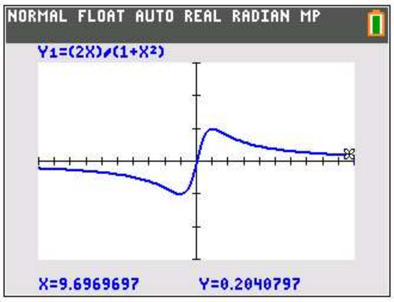
$Y_3=$

$Y_4=$

$Y_5=$

$Y_6=$

$Y_7=$



NORMAL FLOAT AUTO REAL RADIAN MP

Y1=(2X)/(1+X²)

X=9.6969697 Y=0.2040797

The graph of $f(x) = \frac{2x}{1+x^2}$ has a minimum point at $x = -1$ and a maximum point at

$x = 1$. Only a one-to-one function has an inverse, so the three intervals on which *f* satisfies this requirement are: $x < -1$ (*f* is decreasing), $-1 \leq x \leq 1$ (*f* is increasing), and $x > 1$ (*f* is decreasing).

9. First, the inverse functions of *g* and *h* need to be found.

$g(x) = x + 3$

$y = x + 3$

$x = y + 3$

$y = x - 3 \Rightarrow g^{-1}(x) = x - 3$

$h(x) = 2x - 4$

$y = 2x - 4$

$x = 2y - 4$

$x + 4 = 2y$

$y = \frac{x+4}{2} \Rightarrow h^{-1}(x) = \frac{1}{2}x + 2$

(a) $(g^{-1} \circ h^{-1})(5) = g^{-1}(h^{-1}(5)) = g^{-1}(\frac{1}{2}(5) + 2) = g^{-1}(\frac{9}{2}) = \frac{9}{2} - 3 = \frac{3}{2}$

(b) $(h^{-1} \circ g^{-1})(9) = h^{-1}(g^{-1}(5)) = h^{-1}(9 - 3) = h^{-1}(6) = \frac{1}{2}(6) + 2 = 5$

(c) $(g^{-1} \circ g^{-1})(2) = g^{-1}(g^{-1}(2)) = g^{-1}(2 - 3) = g^{-1}(-1) = -1 - 3 = -4$

(d) $(h^{-1} \circ h^{-1})(2) = h^{-1}(h^{-1}(2)) = h^{-1}(\frac{1}{2}(2) + 2) = h^{-1}(3) = \frac{1}{2}(3) + 2 = \frac{7}{2}$

(e) $(g^{-1} \circ h^{-1})(x) = g^{-1}(h^{-1}(x)) = g^{-1}(\frac{1}{2}x + 2) = \frac{1}{2}x + 2 - 3 = \frac{1}{2}x - 1$

(f) $(h^{-1} \circ g^{-1})(x) = h^{-1}(g^{-1}(x)) = h^{-1}(x - 3) = \frac{1}{2}(x - 3) + 2 = \frac{1}{2}x + \frac{1}{2}$

(g) $(g \circ h)(x) = g(h(x)) = g(2x - 4) = 2x - 4 + 3 = 2x - 1$

$y = 2x - 1$

$x = 2y - 1$

$2y = x + 1$

$y = \frac{x+1}{2} \Rightarrow (g \circ h)^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$

(h) $(h \circ g)(x) = h(g(x)) = h(x - 3) = 2(x + 3) - 4 = 2x + 6 - 4 = 2x + 2$

$y = 2x + 2$

$x = 2y + 2$

$2y = x - 2$

$y = \frac{x-2}{2} \Rightarrow (h \circ g)^{-1}(x) = \frac{1}{2}x - 1$

10. $f(x) = \frac{a}{x+b} - b, a \neq 0$

$y = \frac{a}{x+b} - b$

$x = \frac{a}{y+b} - b$

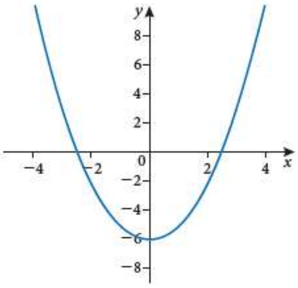
$x + b = \frac{a}{y+b}$

$y + b = \frac{a}{x+b}$

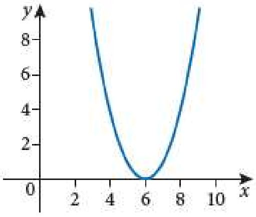
$y = \frac{a}{x+b} - b \Rightarrow f^{-1}(x) = \frac{a}{x+b} - b \Rightarrow f^{-1}(x) = f(x)$

Exercise 1.5

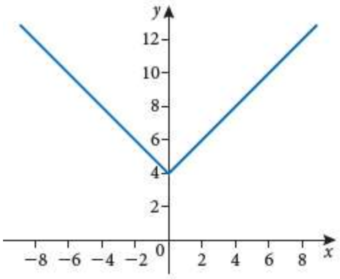
1. **(a)** The graph of $f(x) = x^2 - 6$ is a vertical translation of the parabola representing the basic function $y = x^2$. The transformed graph is obtained by shifting all points on the original parabola 6 units down, so the new vertex will be at $(0, -6)$, and the new x-intercepts are at $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.



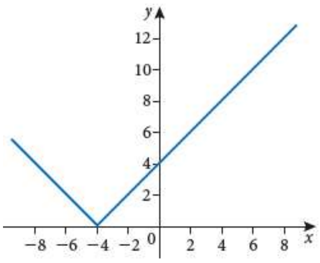
(b) The graph of $f(x) = (x-6)^2$ is a horizontal translation of the basic function $y = x^2$. All points on the original parabola will shift 6 units right, including the original vertex, $(0, 0)$, which will now be located at $(6, 0)$.



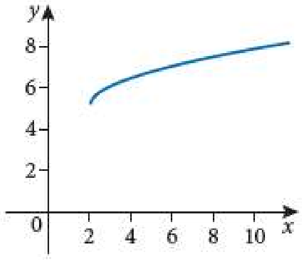
(c) The graph of $f(x) = |x| + 4$ is a vertical translation of the graph representing the basic function $y = |x|$. The transformed graph is obtained by shifting all points on the original graph 4 units up, including the original x-intercept, $(0, 0)$, now at $(0, 4)$.



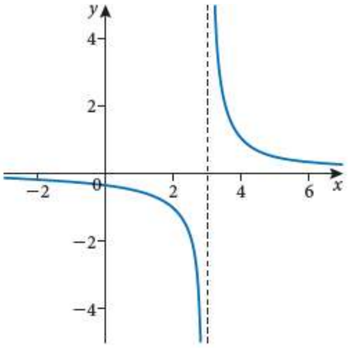
(d) The graph of $f(x) = |x + 4|$ is a horizontal translation of the graph representing the basic function $y = |x|$. The transformed graph is obtained by shifting all points on the original graph 4 units left, including the original x-intercept, $(0, 0)$, now at $(-4, 0)$.



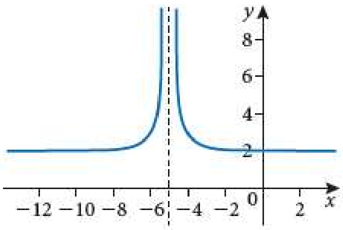
(e) The graph of $f(x) = 5 + \sqrt{x - 2}$ is obtained by translating all points on the graph of the basic function $y = \sqrt{x}$ by 2 units right and 5 units up. The original *x*-intercept, $(0, 0)$, will now be located at $(2, 5)$.



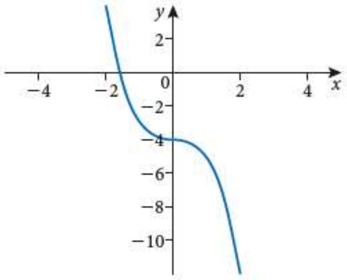
(f) The graph of $f(x) = \frac{1}{x-3}$ is a horizontal translation of the basic function $y = \frac{1}{x}$. All points on the original graph will shift 3 units right, including the original vertical asymptote, $x = 0$, which will now have equation $x = 3$. The horizontal asymptote is the same, $y = 0$.



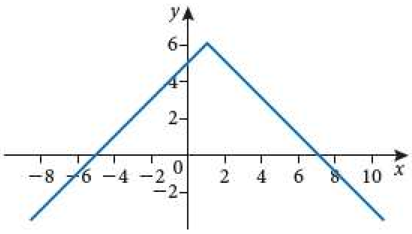
(g) The graph of $f(x) = \frac{1}{(x+5)^2} + 2$ is a translation of the graph of the basic function $y = \frac{1}{x^2}$. All points on the original graph will shift 5 units left and 2 units up, including the original asymptotes: the equations of the new vertical and horizontal asymptotes are $x = -5$ and $y = 2$, respectively.



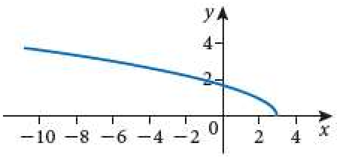
(h) The graph of $f(x) = -x^3 - 4$ is obtained by transforming the graph of the basic function $y = x^3$. First, the original graph is reflected in the *x*-axis, then all points shift 4 units down, including the original *x*-intercept, which is now at $(0, -4)$.



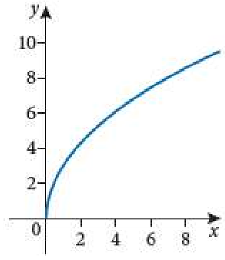
(i) The graph of $f(x) = -|x - 1| + 6$ is obtained by transforming the graph representing the basic function $y = |x|$. First, shift all points on the original graph 1 unit right, then reflect the graph in the *x*-axis, and shift it 6 units up. The original *x*-intercept, $(0, 0)$, will first shift to $(1, 0)$, then to $(1, 6)$. The reflection in the *x*-axis has no effect on point $(1, 0)$, as it is on the line of reflection.



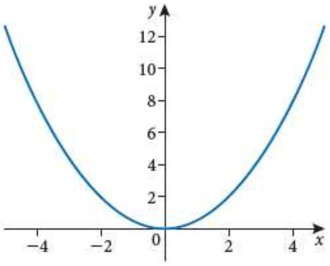
(j) The graph of $f(x) = \sqrt{-x + 3}$ is obtained by transforming the graph of the basic function $y = \sqrt{x}$ as follows: first, shift all points on the graph 3 units left, then reflect it in the *x*-axis. The transformation can also be performed in a different order if we consider the equivalent form of the equation of f, $f(x) = \sqrt{-(x - 3)}$: first, reflect the graph in the *x*-axis and then shift it right 3 units, you will obtain the same graph as before. In both cases, the original *x*-intercept,$(0, 0)$, will end up at $(0, 3)$.



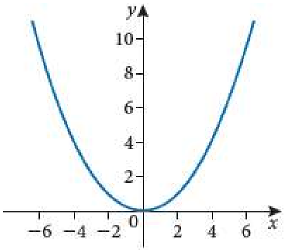
(k) The graph of $f(x) = 3\sqrt{x}$ is obtained by transforming the graph of the basic function $y = \sqrt{x}$ by stretching it vertically using a scale factor of 3 (the *y*-coordinates of all points on the original graph are multiplied by 3). The *x*-intercept, $(0, 0)$, is not impacted by this transformation, it stays at $(0, 0)$.



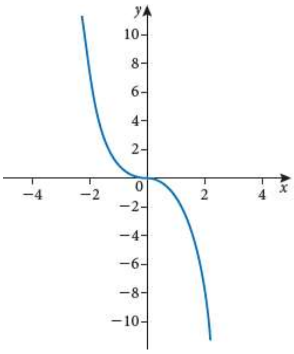
(l) The graph of $f(x) = \frac{1}{2}x^2$ is obtained by transforming the graph of the basic quadratic function $y = x^2$ by stretching it vertically using a scale factor of $\frac{1}{2}$ (the *y*-coordinates of all points on the original graph are multiplied by $\frac{1}{2}$). The *x*-intercept, $(0, 0)$, is not impacted by this transformation, it stays at $(0, 0)$.



(m) The graph of $f(x) = (\frac{1}{2}x)^2$ is obtained by transforming the graph of the basic quadratic function $y = x^2$ by stretching it horizontally using a scale factor of 2 (the *x*-coordinates of all points on the original graph are multiplied by 2).
The transformation can also be performed in a different way if we consider the equivalent form of the equation of f, $f(x) = \frac{1}{4}x^2$, this can be done by vertically stretching the graph of $y = x^2$ with a scale factor of $\frac{1}{4}$ (the *y*-coordinates of all points on the original graph are multiplied by $\frac{1}{4}$). In both cases, the original *x*-intercept,$(0, 0)$, will not be impacted by the transformation, it stays at $(0, 0)$.



(n) The graph of $f(x) = (-x)^3$is obtained by transforming the graph of the basic function$y = x^3$ by reflecting the original graph in the *y*-axis. Alternatively, if we consider the equivalent form of the equation of *f*, $f(x) = -x^3$, then the graph of $y = x^3$ should be reflected in the *x*-axis. In both cases, the original *x*-intercept,$(0, 0)$, will not be impacted by the transformation; it stays at$(0, 0)$.



2. **(a)** The given graph represents a parabola, so the transformations are applied to the graph of the basic quadratic function $y = x^2$. Based on the given graph, the transformations are: reflection in the *x*-axis, followed by a vertical shift of 5 units up.

$y = x^2 \to y = -x^2 \to y = -x^2 + 5 \Rightarrow$ the equation of the graph is $y = -x^2 + 5$

(b) The shape of the given graph suggests that the basic function to work with is $y = \sqrt{x}$. The graph is a reflection in the *y*-axis, as checked when considering the point $(-4, 2)$. This is the image of point $(4, 2)$, so the required equation is $y = \sqrt{-x}$.

(c) The basic graph in this case is $y = |x|$, the applied transformations are: reflection in the *x*-axis, followed by a horizontal shift of 1 unit left.

$y = |x| \to y = -|x| \to y = -|x + 1| \Rightarrow$ the equation of the graph is $y = -|x + 1|$

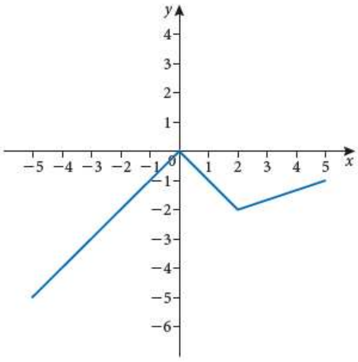
(d) The graph to be transformed is $y = \frac{1}{x}$, the applied transformations are: a horizontal translation of 2 units right, followed by a vertical shift of 3 units down (this is easily deduced when looking at the original vertical and horizontal asymptotes, the lines $x = 0$ and $y = 0$ have changed to $x = 2$ and $y = -3$, respectively).

$y = \frac{1}{x} \to y = \frac{1}{x-2} \to y = \frac{1}{x-2} - 3 \Rightarrow$ the equation of the graph is $y = \frac{1}{x-2} - 3$

3. In the following questions, first recognise the transformations to be applied by analysing the given equation, then transform the important points of the original graph: the two ends, $(-5, -2)$ and $(5, 2)$, the maximum point, $(0, 3)$, and the minimum point, $(2, 1)$, then plot and join these images to obtain the final graph.

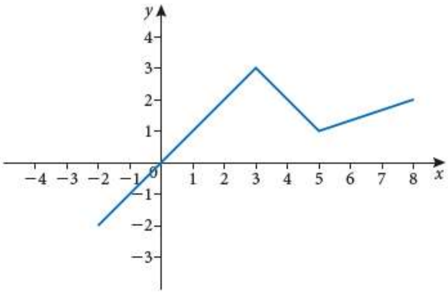
(a) The transformation to be applied is a vertical shift of 3 units down.

$$(-5, -2) \to (-5, -2 - 3) = (-5, -5), (5, 2) \to (5, 2 - 3) = (5, -1),$$
$$(0, 3) \to (0, 3 - 3) = (0, 0), (2, 1) \to (2, 1 - 3) = (2, -2)$$



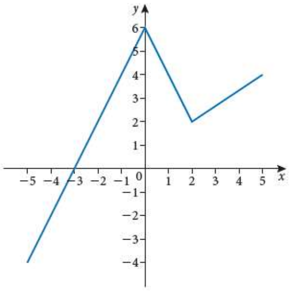
(b) The transformation to be applied is a horizontal shift of 3 units to the right.

$$(-5,-2) \to (-5+3,-2) = (-2,-2),\ (5,2) \to (5+3,2) = (8,2),$$
$$(0,3) \to (0+3,3) = (3,3),\ (2,1) \to (2+3,1) = (5,1)$$



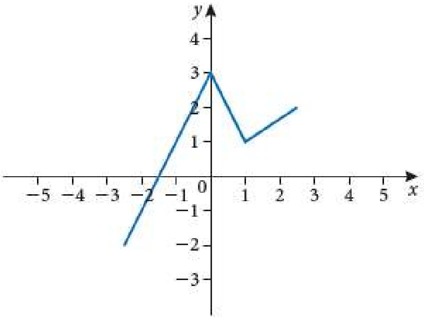
(c) The transformation to be applied is a vertical stretch of scale factor 2.

$$(-5,-2) \to (-5,-2\cdot 2) = (-5,-4),\ (5,2) \to (5,2\cdot 2) = (5,4),$$
$$(0,3) \to (0,3\cdot 2) = (0,6),\ (2,1) \to (2,1\cdot 2) = (2,2)$$



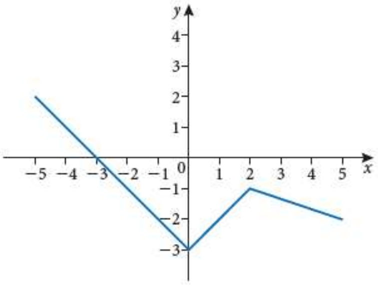
(d) The transformation to be applied is a horizontal stretch of scale factor $\frac{1}{2}$.

$$(-5,-2) \to (-5\cdot\frac{1}{2},-2) = (-\frac{5}{2},-2),\ (5,2) \to (5\cdot\frac{1}{2},2) = (\frac{5}{2},2),$$
$$(0,3) \to (0\cdot\frac{1}{2},3) = (0,3),\ (2,1) \to (2\cdot\frac{1}{2},1) = (1,1)$$



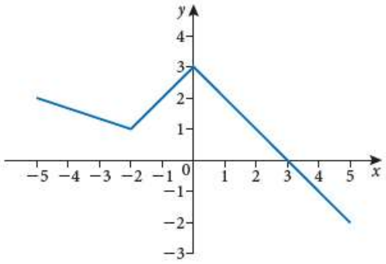
(e) The transformation to be applied is a reflection in the *x*-axis.

$(-5,-2) \rightarrow (-5,2)$, $(5,2) \rightarrow (5,-2)$, $(0,3) \rightarrow (0,-3)$, $(2,1) \rightarrow (2,-1)$



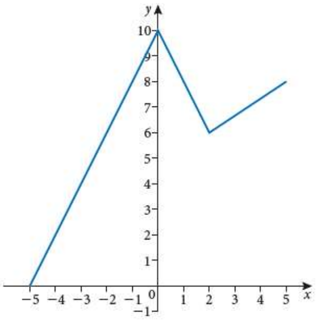
(f) The transformation to be applied is a reflection in the *y*-axis.

$(-5,-2) \rightarrow (5,-2)$, $(5,2) \rightarrow (-5,2)$, $(0,3) \rightarrow (0,3)$, $(2,1) \rightarrow (-2,1)$



(g) The transformations to be applied are: a vertical stretch of scale factor 2, followed by a vertical translation of 4 units up.

$(-5,-2) \rightarrow (-5,-2 \cdot 2+4) = (-5,0)$,$(5,2) \rightarrow (5,2 \cdot 2+4) = (5,8)$,
$(0,3) \rightarrow (0,3 \cdot 2+4) = (0,10)$, $(2,1) \rightarrow (2,1 \cdot 2+4) = (2,6)$



4. **(a)** $y = x^2 \to y = (x-3)^2 \to y = (x-3)^2 + 5 \Rightarrow$ the transformations to be applied are a horizontal shift of 3 units right, followed by a vertical translation of 5 units up (the order of the translations does not matter).

(b) $y = x^2 \to y = -x^2 \to y = -x^2 + 2 \Rightarrow$ the transformations to be applied are a reflection in the x-axis,, followed by a vertical translation of 2 units up (the order of the transformations does not matter).

(c) $y = x^2 \to y = (x+4)^2 \to y = \frac{1}{2}(x+4)^2 \Rightarrow$ the transformations to be applied are a horizontal translation of 4 units left, followed by a vertical stretch with scale factor $\frac{1}{2}$ (the order of the transformations does not matter).

(d) $y = x^2 \to y = (3x)^2 \to y = (3(x-1))^2 \to y = (3(x-1))^2 - 6 \Rightarrow$ the transformations to be applied are: a horizontal stretch with scale factor $\frac{1}{3}$, followed by a horizontal translation of 1 unit right and a vertical translation of 6 units down.

5. **(a)** **(i)** The function $\frac{1}{f(x)}$ is not defined where $f(x) = 0$:

$0 = \frac{1}{2}x - 4 \Rightarrow x = 8$

This means that the domain of $\frac{1}{f(x)}$ will be $\{x : x \in \mathbb{R}, x \neq 8\}$ and its graph will have a vertical asymptote, $x = 8$. The behaviour of the graph about $x = 8$ should be investigated:

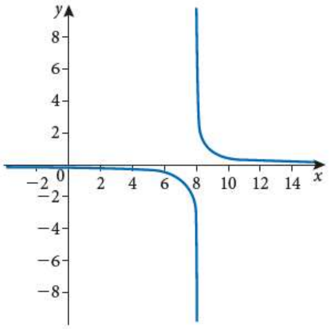
$x \to 8^- \Rightarrow \frac{1}{f(x)} \to -\infty$ (because $f(x) < 0$ for $x < 8$)

$x \to 8^+ \Rightarrow \frac{1}{f(x)} \to \infty$ (because $f(x) > 0$ for $x > 8$)

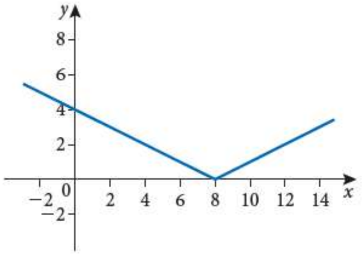
As x becomes infinitely large, either positively or negatively, the values of $\frac{1}{f(x)}$ will decrease and will approach 0, this means that the graph of $\frac{1}{f(x)}$ will have a horizontal asymptote, $y = 0$.

The y-intercept can easily be calculated, $y = \frac{1}{\frac{1}{2}(0)-4} \Rightarrow y = -\frac{1}{4}$

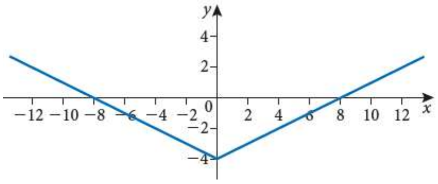
Consequently, the graph of $\frac{1}{f(x)}$ is:



(ii) To obtain the graph of $|f(x)|$, the part of the graph of $f(x)$ which is below the x-axis will be reflected above the x-axis. The resulting graph is:



(iii) To obtain the graph of $f(|x|)$, the part of the graph of $f(x)$ to the left of the y-axis is eliminated and the remaining part is reflected in the y-axis. The resulting graph is will be symmetrical about the y-axis:



(b) **(i)** The function $\frac{1}{f(x)}$ is not defined where $f(x) = 0$:

$0 = (x - 4)(x + 2) \Rightarrow x = 4, x = -2$

This means that the domain of $\frac{1}{f(x)}$ will be $\{x: x \in \mathbb{R}, x \neq -2, x \neq 4\}$ and its graph will have two vertical asymptotes, $x = -2$ and $x = 4$. The behaviour of the graph of $\frac{1}{f(x)}$ about $x = -2$ and $x = 4$ should be investigated:

$x \to -2^- \Rightarrow \frac{1}{f(x)} \to \infty$ (because $f(x) > 0$ for $x < -2$)

$x \to -2^+ \Rightarrow \frac{1}{f(x)} \to -\infty$ (because $f(x) < 0$ for $-2 < x < 4$)

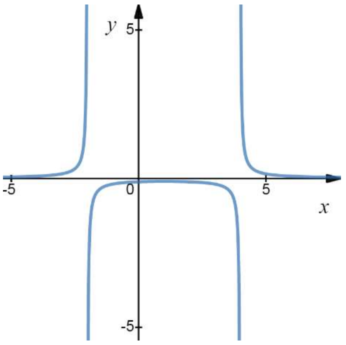
$x \to 4^- \Rightarrow \frac{1}{f(x)} \to -\infty$, (because $f(x) < 0$ for $-2 < x < 4$)

$x \to 4^+ \Rightarrow \frac{1}{f(x)} \to \infty$ (because $f(x) > 0$ for $x > 4$)

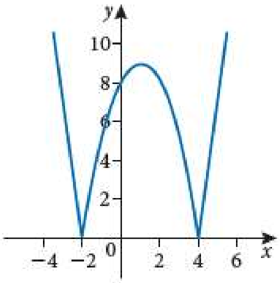
As x becomes infinitely large, either positively or negatively, the values of $\frac{1}{f(x)}$ will decrease and will approach 0, this means that the graph of $\frac{1}{f(x)}$ will have a horizontal asymptote, $y = 0$.

The y-intercept can easily be calculated, $y = \frac{1}{(0-4)(0+2)} \Rightarrow y = -\frac{1}{8}$

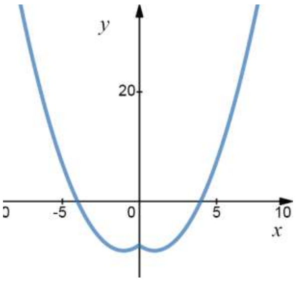
Consequently, the graph of $\frac{1}{f(x)}$ is:



(ii) To obtain the graph of $|f(x)|$, the part of the graph of $f(x)$ which is below the *x*-axis will be reflected above the *x*-axis. The resulting graph is:



(iii) To obtain the graph of $f(|x|)$, the part of the graph of $f(x)$ to the left of the *y*-axis is eliminated and the remaining part is reflected in the *y*-axis. The resulting graph is will be symmetrical about the *y*-axis:



(c) **(i)** The function $\frac{1}{f(x)}$ is not defined where $f(x) = 0$:

$0 = x^3 \Rightarrow x = 0$

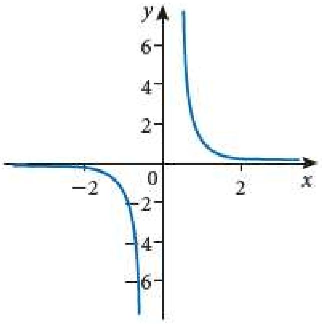
This means that the domain of $\frac{1}{f(x)}$ will be $\{x: x \in \mathbb{R}, x \neq 0\}$ and its graph will have a vertical asymptote, $x = 0$. The behaviour of the graph about $x = 0$ should be investigated:

$x \to 0^- \Rightarrow \frac{1}{f(x)} \to -\infty$ (because $f(x) < 0$ for $x < 0$)

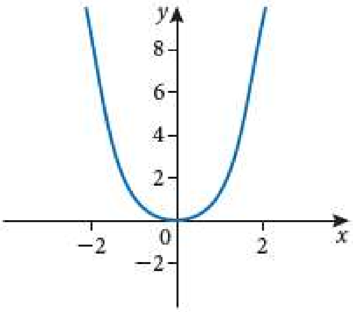
$x \to 0^+ \Rightarrow \frac{1}{f(x)} \to \infty$ (because $f(x) > 0$ for $x > 0$)

As x becomes infinitely large, either positively or negatively, the values of $\frac{1}{f(x)}$ will decrease and will approach 0, this means that the graph of $\frac{1}{f(x)}$ will have a horizontal asymptote, $y = 0$.

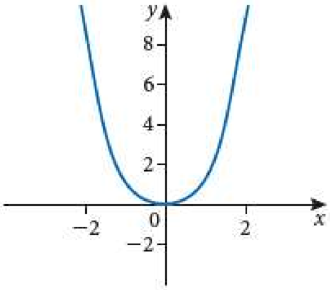
Consequently, the graph of $\frac{1}{f(x)}$ is:



(ii) To obtain the graph of $|f(x)|$, the part of the graph of $f(x)$ which is below the x-axis will be reflected above the x-axis. The resulting graph is:



(iii) To obtain the graph of $f(|x|)$, the part of the graph of $f(x)$ to the left of the y-axis is eliminated and the remaining part is reflected in the y-axis. The resulting graph is will be symmetrical about the y-axis:



Chapter 1 practice questions

1. **(a)**

	f	g
Domain	$x \in \mathbb{R}$, $x \geq 3$	$x \in \mathbb{R}$
Range	$y \geq 0$	$y \in \mathbb{R}$, $y \geq -1$

$(f \circ g)(x) = f(g(x)) = f(x^2 + 2x) = \sqrt{x^2 + 2x - 3}$

$x \in \mathbb{R} \Rightarrow x^2 + 2x \geq -1$, this is not a subset of the domain of *f*. To be able to compose the two functions, only the *x*-values which are common to both the domain of *f* (real numbers greater than or equal to 3) and the range of *g* (real numbers greater than or equal to -1) are acceptable. This means only real numbers greater than or equal to 3 can be inputs for *f*. Consequently, the domain of $f \circ g$ will not contain the *x*-values that will lead to outputs for *g* which are less than 3.

$x^2 + 2x < 3 \Rightarrow x^2 + 2x - 3 < 0 \Rightarrow (x - 1)(x + 3) < 0$

$\Rightarrow -3 < x < 1 \Rightarrow x \in]-3,1[$

It follows that the domain of $f \circ g$ is $x \leq -3, x \geq 1$, so $a = -3$ and $b = 1$.

(b) The range of $f \circ g$ is $y \geq 0$, as the outputs are square roots of positive numbers or 0.

2. **(a)** $g^{-1}(3) = x \Leftrightarrow 3 = g(x)$

$2x - 7 = 3 \Rightarrow 2x = 10 \Rightarrow x = 5 \Rightarrow g^{=1}(3) = 5$

(b) $(h \circ g)(6) = h(g(6)) = h(2(6) - 7) = h(5) = 3(2 - 5) = -9$

3. **(a)** $g(x) = \frac{4-x}{3}$

$y = \frac{4-x}{3}$

$x = \frac{4-y}{3}$

$3x = 4 - y$

$y = -3x + 4 \Rightarrow g^{-1}(x) = -3x + 4$

(b) $(f \circ g^{-1})(x) = 8 \Rightarrow f(g^{-1}(x)) = 8 \Rightarrow f(-3x + 4) = 8 \Rightarrow 5(-3x + 4) - 2 = 8$

$\Rightarrow -15x = -10 \Rightarrow x = \frac{2}{3}$

4. **(a)** $(g \circ h)(x) = g(h(x)) = g(2x) = 2x - 3 \Rightarrow (g \circ h)(x) = 2x - 3$

(b) First, find the expressions of the required inverse functions, substitute the given *x*-values, then calculate the value of the left side of the given conjecture, to see if it equals 24.

$g(x) = x - 3$

$y = x - 3$

$x = y - 3$

$x + 3 = y$

$y = x + 3 \Rightarrow g^{-1}(x) = x + 3$

$g(x) = 2x$

$y = 2x$

$x = 2y$

$y = \frac{x}{2} \Rightarrow h^{-1}(x) = \frac{x}{2}$

$y = \frac{x}{2} \Rightarrow h^{-1}(x) = \frac{x}{2}$

$\text{LHS} = g^{-1}(14) + h^{-1}(14) = (14 + 3) + \frac{14}{2} = 17 + 7 = 24 = \text{RHS}$

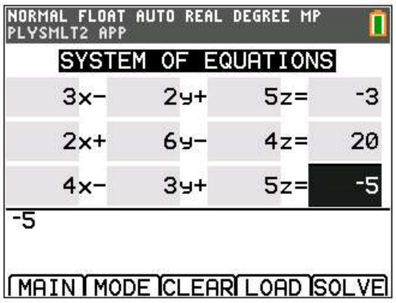
5.

$$\begin{pmatrix} 2 & -1 & 3 & 2 \\ 3 & 1 & 2 & -2 \\ -1 & 2 & -3 & -4 \end{pmatrix} \xrightarrow{2R_2-3R_1} \begin{pmatrix} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & -10 \\ -1 & 2 & -3 & -4 \end{pmatrix} \xrightarrow{2R_3+R_1} \begin{pmatrix} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & -10 \\ 0 & 3 & -3 & -6 \end{pmatrix}$$

$$\xrightarrow{3R_2-5R_3} \begin{pmatrix} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow 0x + 0y + 0z = 0 \Rightarrow$ infinite solutions, as the equation holds true for any $x, y, z \in \mathbb{R}$.

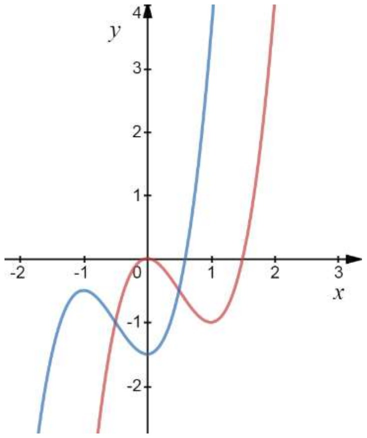
6. The solution is $(1, 3, 0)$.



NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
SYSTEM OF EQUATIONS
3x− 2y+ 5z= -3
2x+ 6y− 4z= 20
4x− 3y+ 5z= -5
-5

NORMAL FLOAT AUTO REAL DEGREE MP
PLYSMLT2 APP
SOLUTION
x=1
y=3
z=0

7. **(a)** The transformation $f(x) \to f(x+1) - \frac{1}{2}$ represents a horizontal translation of 1 unit left followed by a vertical shift of $\frac{1}{2}$ unit down.



(b) Minimum point: $(1,-1) \to (1-1,-1-\frac{1}{2}) = (0,-\frac{3}{2})$

Maximum point: $(0,0) \to (0-1,0-\frac{1}{2}) = (-1,-\frac{1}{2})$

8. Reflection in the line $y = 0$: $y = x^2 \to y = -x^2$

Vertical stretch with scale factor k: $y = -x^2 \to y = -kx^2$

Horizontal shift of p units left, $p > 0$: $y = -k(x+p)^2$

Vertical shift of q units up, $q > 0$: $y = -k(x+p)^2 \to y = -k(x+p)^2 + q$

When comparing $y = -k(x+p)^2 + q$ to $y = -\frac{1}{2}(x+5)^2 + 3$, it follows that:

(a) $k = \frac{1}{2}$

(b) $p = 5$

(c) $q = 3$

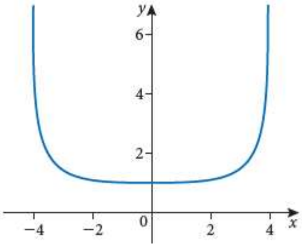
9. **(a)** $f(x) = \frac{4}{\sqrt{16-x^2}}$ is a rational function with a square root at the denominator, it follows that the radicand has to be strictly positive (it cannot be equal to 0), so $16 - x^2 > 0 \Rightarrow -4 < x < 4$. This set of values coincides with the given domain.

To find the x-intercept, set y (or $f(x)$) equal to 0, and solve for x:

$0 = \frac{4}{\sqrt{16-x^2}} \Rightarrow$ no solution, so no x-intercept.

The y-intercept is found when $x = 0$: $y = \frac{4}{\sqrt{16-0^2}} \Rightarrow y = 1$

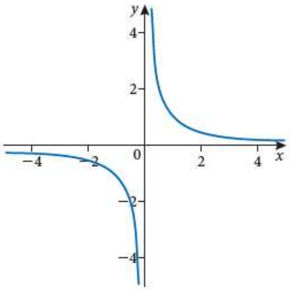
$16 - x^2 = 0 \Rightarrow x = \pm 4 \Rightarrow$ there are two vertical asymptotes: $x = 4, x = -4$.



(b) $x = 4, x = -4$

(c) The values given by function *f* will be strictly positive and greater or equal to 1, as seen in the graph, so the range of *g* is $\{y : y \geq 1\}$.

10. **(a)** This is the basic rational function, it has asymptotes $x = 0, y = 0$ and no axes intercepts.



(b) Translation of 4 units left: $y = \frac{1}{x} \rightarrow y = \frac{1}{x+4}$

Translation of 2 units down: $y = \frac{1}{x+4} \rightarrow y = \frac{1}{x+4} - 2$

The expression of function *h* is $h(x) = \frac{1}{x+4} - 2$

(c) **(i)** To find the *x*-intercept, set *y* (or $h(x)$) equal to 0, and solve for *x*:

$$0 = \frac{1}{x+4} - 2 \Rightarrow 2 = \frac{1}{x+4} \Rightarrow x + 4 = \frac{1}{2} \Rightarrow x = \frac{1}{2} - 4 \Rightarrow x = -\frac{7}{2}$$

The *x*-intercept is $(-\frac{7}{2}, 0)$.

The *y*-intercept is found by substituting $x = 0$ into the expression of the function:
$\Rightarrow y = \frac{1}{0+4} - 2 \Rightarrow y = \frac{1}{4} - 2 \Rightarrow y = -\frac{7}{4}$

The *y*-intercept is $(0, -\frac{7}{4})$.

(ii) The equation of the vertical asymptote is found by setting the denominator equal to 0 and solving for *x*:

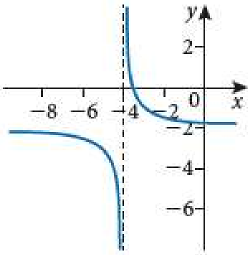
$x + 4 = 0 \Rightarrow x = -4 \Rightarrow$ the equation of the vertical asymptote is $x = -4$.

The equation of the horizontal asymptote is found after evaluating $\lim_{x\to\infty} h(x)$.

$$\lim_{x\to\infty}\left(\frac{1}{x+4} - 2\right) = 0 - 2 = -2$$

This means that as $x \to \infty$, the values of $h(x)$ approach -2, so there is a horizontal asymptote and its equation is $y = -2$.

(iii) Plot the axes intercepts and draw the asymptotes, then sketch the graph of h.



11. **(a)** **(i)** $f(8) = \sqrt{8+3} = \sqrt{11}$

(ii) $f(8) = \sqrt{46+3} = \sqrt{49} = 7$

(iii) $f(8) = \sqrt{-3+3} = \sqrt{0} = 0$

(b) f is undefined when the radicand is negative: $x + 3 < 0 \Rightarrow x < -3$, so f is undefined for $x < -3$.

(c) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x+3}) = (\sqrt{x+3})^2 - 5 = x + 3 - 5 = x - 2$

12. **(a)** First, find the expression of g^{-1}, the inverse function of g:

$g(x) = \frac{x-8}{2}$

$y = \frac{x-8}{2}$

$x = \frac{y-8}{2}$

$2x = y - 8$

$y = 2x + 8 \Rightarrow g^{-1}(x) = 2x + 8$

$\Rightarrow g^{-1}(-2) = 2(-2) + 8 = 4$

(b) $(g^{-1} \circ h)(x) = g^{-1}\big(h(x)\big) = g^{-1}(x^2 - 1)$

$= 2(x^2 - 1) + 8 = 2x^2 - 2 + 8 = 2x^2 + 6$

(c) $2x^2 + 6 = 22 \Rightarrow 2x^2 = 16 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8} \Rightarrow x = \pm 2\sqrt{2}$

13. **(a)** $f(x) = 3x - 1$

$y = 3x - 1$

$x = 3y - 1$

$x + 1 = 3y$

$y = \frac{x+1}{3} \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$

(b) $(f \circ g)(x) = f(g(x)) = f(\frac{4}{x}) = 3 \cdot \frac{4}{x} - 1 = \frac{12}{x} - 1$

(c) $(f \circ g)(x) = \frac{12}{x} - 1$

$y = \frac{12}{x} - 1$

$x = \frac{12}{y} - 1$

$x + 1 = \frac{12}{y}$

$y = \frac{12}{x+1} \Rightarrow (f \circ g)^{-1}(x) = \frac{12}{x+1}$

(d) $(g \circ g)(x) = g(g(x)) = g(\frac{4}{x}) = \frac{4}{\frac{4}{x}} = 4 \cdot \frac{x}{4} = x$

14. **(a)** The equation of the vertical line MN is $x = -3$ (from graph). This means that the denominator of the function takes value 0 when $x = -3$.

$0 = -3 - b \Rightarrow b = -3$

If $A(-4, -8)$ is on the graph it means that its coordinates satisfy the equation of the curve:

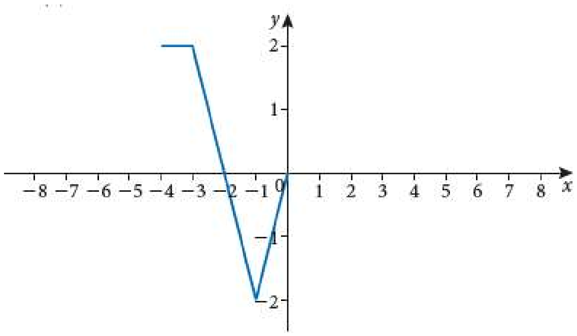
$-8 = \frac{a}{-4-(-3)} \Rightarrow -8 = \frac{a}{-1} \Rightarrow a = 8$

(i) $a = 8$

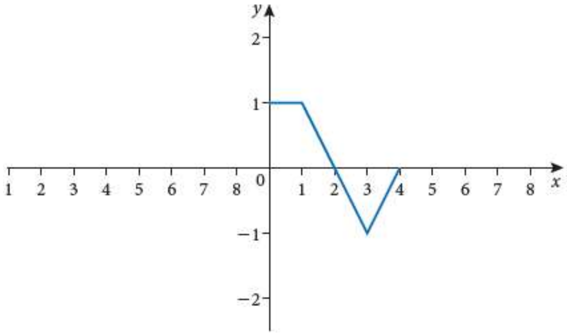
(ii) $b = -3$

(b) $A(-4, -8) \rightarrow A'(-4, 8)$ represents a reflection in the *x*-axis, as the only change in the coordinates is that the original *y*-value changes to its opposite.

15. **(a)** The first transformation represents a vertical stretch with scale factor 2 (the *y*-coordinates of all points on the graph are multiplied by 2).



The second transformation represents a horizontal translation of 4 units right (all *x*-coordinates are increased by 5).



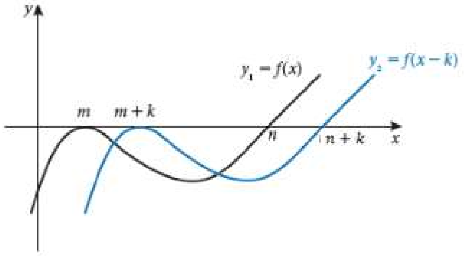
(b) $y = f(x) \rightarrow y = -f(x)$ represents a reflection in the x-axis.

$y = -f(x) \rightarrow y = -f(x) - 1$ represents a vertical translation of 1 unit down.

$(-4, -8) \rightarrow (-4, 8) \rightarrow (-4, 8-1) = (-4, 7)$

The coordinates of A' are $A'(-4, 7)$.

16. The transformation $y_1 = f(x) \rightarrow y_2 = f(x-k)$ represents a horizontal translation of k units to the right. As $0 < k < n - m$, the first x-intercept of y_2 will not be greater than the second x-intercept of y_1.



The first x-intercept of y_2 is the image of the first x-intercept of y_1: $(m, 0) \rightarrow (m+k, 0)$

The second x-intercept of y_2 is the image of the second x-intercept of y_1: $(n, 0) \rightarrow (n+k, 0)$

17. $(f \circ g)(x) = f(g(x)) = f(x^3) = x^3 + 1$

$(f \circ g)(x) = x^3 + 1$

$y = x^3 + 1$

$x = y^3 + 1$

$x - 1 = y^3$

$y = \sqrt[3]{x-1} \Rightarrow (f \circ g)^{-1}(x) = \sqrt[3]{x-1}$

18. **(a)** $g(x) = (f \circ f)(x) = f(f(x)) = f(\frac{x}{x+1}) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1}$

(b) $(g \circ g)(2) = g(g(2)) = g(\frac{2}{2(2)+1}) = g(\frac{2}{5}) = \frac{\frac{2}{5}}{\frac{2}{5}+1} = \frac{\frac{2}{5}}{\frac{7}{5}} = \frac{2}{7}$

19. **(a)** In order for f to have real values the radicand has to be positive or 0:

$$\frac{1}{x^2} - 2 \geq 0 \Rightarrow \frac{1}{x^2} \geq 2 \Rightarrow \frac{1}{2} \geq x^2 \Rightarrow -\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}} \Rightarrow -\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}$$

In order for f to take finite values the denominator of the function should not take the value 0:

$$x^2 \neq 0 \Rightarrow x \neq 0$$

The required set of values is: $-\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}, x \neq 0$

(b) f is a square root function, so its outputs will be either positive or 0, so the range of f is: $y \geq 0$.

20. $f(x) = \frac{2x+1}{x-1}$

$y = \frac{2x+1}{x-1}$

$x = \frac{2y+1}{y-1}$

$x(y-1) = 2y + 1$

$xy - x = 2y + 1$

$xy - 2y = x + 1$

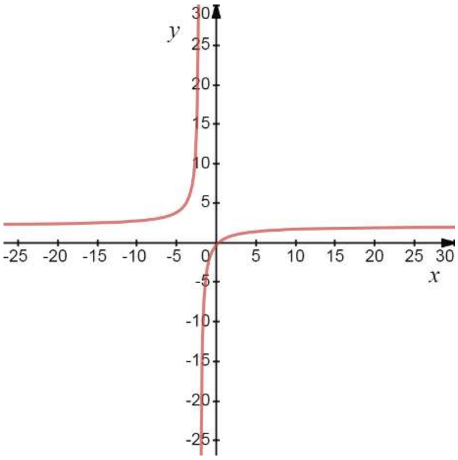
$y(x-2) = x + 1$

$y = \frac{x+1}{x-2} \Rightarrow f^{-1}(x) = \frac{x+1}{x-2}$

The domain of f^{-1} is the same as the range of f, so $y \in \mathbb{R}, y \neq 2$ (f has a horizontal asymptote, $y = 2$, so 2 will not be an output for f).

21. **(a)** $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+1}{x-1} = 2 \Rightarrow f$ has a horizontal asymptote, $y = 2$, so the range of f will not include the value $y = 2$.

The graph of the function is shown below:



Based on the graph, if $x > 0$, the values of *f* will not exceed 2, but they will be greater than $f(0)$.

$f(0) = \frac{2(0)-1}{0+2} = -\frac{1}{2} \Rightarrow$ the range of *f* is: $A = \left\{y: -\frac{1}{2} < y < 2, y \in \mathbb{R}\right\}$

(b) $f(x) = \frac{2x-1}{x+2}$

$y = \frac{2x-1}{x+2}$

$x = \frac{2y-1}{y+2}$

$x(y+2) = 2y - 1$

$xy + 2x = 2y - 1$

$xy - 2y = -2x - 1$

$y(x-2) = -2x - 1$

$y = \frac{-2x-1}{x-2} \Rightarrow f^{-1}(x) = \frac{-2x-1}{x-2}$

22. **(a)** $(f \circ g)(x) = f(g(x)) = (g(x))^3$

$\Rightarrow x + 1 = (g(x))^3 \Rightarrow g(x) = \sqrt[3]{x+1}$

(b) $(g \circ f)(x) = g(f(x)) = g(x^3)$

$\Rightarrow g(x^3) = x + 1$

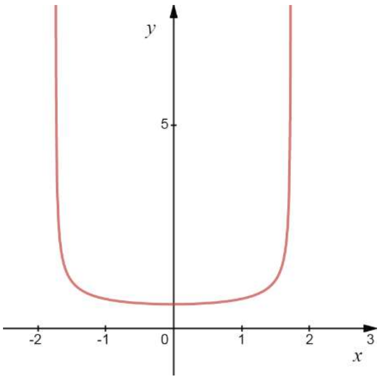
g must have a rule that will transform x^3 into $x + 1$. This means it first has to cube root x^3 to obtain x, then 1 should be added to x to obtain $x + 1$.

$\Rightarrow g(x) = \sqrt[3]{x} + 1$

23. **(a)** In order for *f* to have real values the radicand has to be positive, but not 0, as the square root is at the denominator of the function:

$3 - x^2 > 0 \Rightarrow 3 \geq x^2 \Rightarrow -\sqrt{3} < x < \sqrt{3} \Rightarrow S = \{x: -\sqrt{3} < x < \sqrt{3}, x \in \mathbb{R}\}$.

(b) The graph of the function is shown below:



Based on the shape of the graph, the y-values will be greater than or equal to $f(0)$.

$f(0) = \frac{1}{\sqrt{3-0^2}} = \frac{1}{\sqrt{3}} \Rightarrow$ the range of f is: $\{y: y \geq \sqrt{3}, y \in \mathbb{R}\}$

24. $(f \circ g)(x) = f(g(x)) = f(2x-1)$

$f(2x-1) = \frac{x+1}{2}$

f must have a rule that will transform $2x-1$ into $\frac{x+1}{2}$. This means it first has to add 1, then divide by 2 to obtain x, then 1 to be added to x, and the result divided by 2 to obtain $\frac{x+1}{2}$.

$\Rightarrow f(x) = (\frac{x+1}{2} + 1) \div 2 \Rightarrow f(x) = (\frac{x+3}{2}) \cdot \frac{1}{2} = \frac{x+3}{4}$

$\Rightarrow f(x-3) == \frac{x-3+3}{4} \Rightarrow f(x-3) = \frac{x}{4}$

25. Points A, B, C and D are (from graph): $A(-3, 25), B(0,0), C(3,-35), D(6,0)$

(a) The transformation to be applied to the graph of $y = f(x)$ is a horizontal shift of 4 units right.

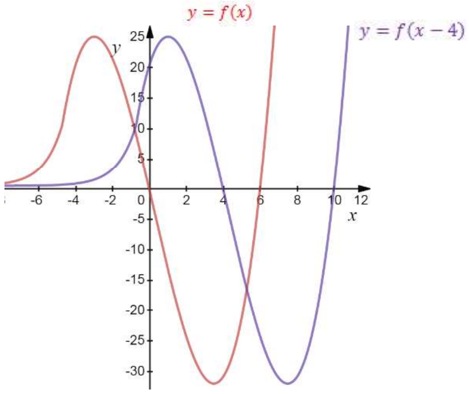
$A(-3, 25) \to A'(-3+4, 25) = A'(1, 25)$

$B(0,0) \to B'(0+4, 0) = B'(4,0)$

$C(3,-35) \to C'(3+4, -35) = C'(7, -35)$

$D(6,0) \to D'(6+4, 0) = D(10,0)$

The graphs of both the original and the transformed function are shown below:



(b) The transformations to be applied to the graph of $y = f(x)$ are, in any order, a reflection in the x-axis and a horizontal stretch with scale factor $\frac{1}{3}$.

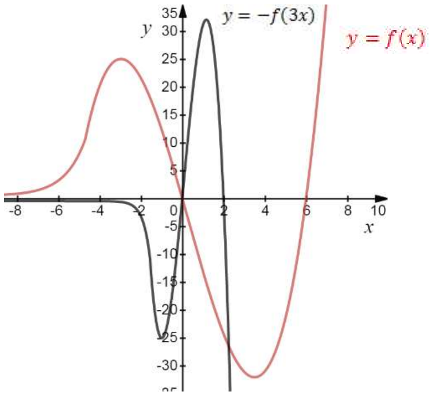
$A(-3, 25) \to (-3, -25) \to (3 \cdot \frac{1}{3}, -25) = (-1, -25) \Rightarrow A''(-1, -25)$

$B(0,0) \to B''(0,0)$

$C(3, -35) \to (3, 35) \to (3 \cdot \frac{1}{3}, 35) = (1,35) \Rightarrow C''(1, 35)$

$D(6,0) \to (6 \cdot \frac{1}{3}, 0) = (2,0) \Rightarrow D''(2,0)$

The graphs of both the original and the transformed function are shown below:



OR:

The transformations to be applied to the graph of $y = f(x)$ are, in any order, a reflection in the *y*-axis and a horizontal stretch with scale factor $\frac{1}{3}$.

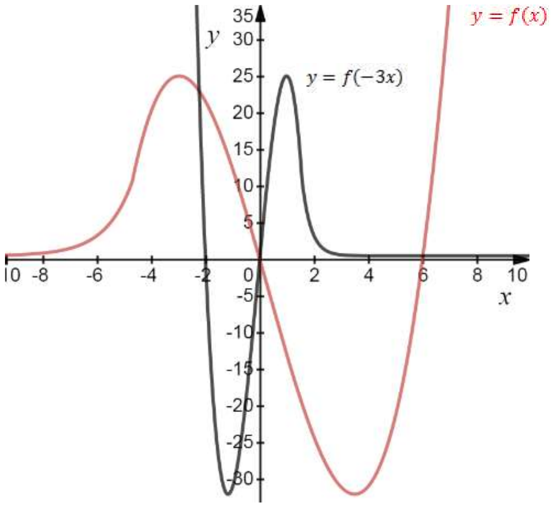
$A(-3,25) \to (3,25) \to (3 \cdot \frac{1}{3}, 25) = (1,25) \Rightarrow A''(1,25)$

$B(0,0) \to B'(0+4,0) = B'(4,0)$

$C(3,-35) \to C'(3+4,-35) = C'(7,-35)$

$D(6,0) \to D'(6+4,0) = D(10,0)$

The graphs of both the original and the transformed function are shown below:



Exercise 2.2

1. **(a)** $f : x \mapsto x^2 - 10x + 32$

Converting to vertex form: $h = -\frac{b}{2a} = -\frac{-10}{2} = 5$,

$k = c - \frac{b^2}{4a} = 32 - \frac{(-10)^2}{4} = 32 - 25 = 7$

$\therefore f(x) = (x-5)^2 + 7$

(i) axis of symmetry: $x = 5$, coordinates of the vertex: $(5, 7)$

(ii) horizontal translation 5 units right, vertical translation 7 units up

(iii) since the coefficient of the leading term is positive, the function *f* has minimum value 7 for $x = 5$.

(b) $f : x \mapsto x^2 + 6x + 8$

Converting to vertex form: $h = -\frac{b}{2a} = -\frac{6}{2} = -3$, $k = c - \frac{b^2}{4a} = 8 - \frac{6^2}{4} = 8 - 9 = -1$

$\therefore f(x) = (x+3)^2 - 1$

(i) axis of symmetry: $x = -3$, coordinates of the vertex: $(-3, -1)$

(ii) horizontal translation 3 units left, vertical translation 1 unit down

(iii) since the coefficient of the leading term is positive, the function *f* has minimum value -1 for $x = -3$.

(c) $f : x \mapsto -2x^2 - 4x + 10$

Converting to vertex form: $h = -\frac{b}{2a} = -\frac{-4}{-4} = -1$, $k = c - \frac{b^2}{4a} = 10 - \frac{(-4)^2}{-8} = 10 + 2 = 12$

$\therefore f(x) = -2(x+1)^2 + 12$

(i) axis of symmetry: $x = -1$, coordinates of the vertex: $(-1, 12)$

(ii) horizontal translation 1 unit left, reflection in the *x*-axis, vertical stretch by factor 2, vertical translation 12 units up

(iii) since the coefficient of the leading term is negative, the function *f* has maximum value 12 for $x = -1$.

(d) $f: x \mapsto 4x^2 - 4x + 9$

Converting to vertex form: $h = -\frac{b}{2a} = -\frac{-4}{8} = \frac{1}{2}$, $k = c - \frac{b^2}{4a} = 9 - \frac{(-4)^2}{16} = 9 - 1 = 8$

$\therefore f(x) = 4\left(x - \frac{1}{2}\right)^2 + 8$

(i) axis of symmetry: $x = \frac{1}{2}$, coordinates of the vertex: $\left(\frac{1}{2}, 8\right)$

(ii) horizontal translation $\frac{1}{2}$ unit right, vertical stretch by factor 4, vertical translation 8 units up

(iii) since the coefficient of the leading term is positive, the function *f* has minimum value 8 for $x = \frac{1}{2}$.

(e) $f: x \mapsto \frac{1}{2}x^2 + 7x + 26$

Converting to vertex form: $h = -\frac{b}{2a} = -\frac{7}{1} = -7$, $k = c - \frac{b^2}{4a} = 26 - \frac{7^2}{2} = 26 - \frac{49}{2} = \frac{3}{2}$

$\therefore f(x) = \frac{1}{2}(x+7)^2 + \frac{3}{2}$

(i) axis of symmetry: $x = -7$, coordinates of the vertex: $\left(-7, \frac{3}{2}\right)$

(ii) horizontal translation 7 units left, vertical shrink by factor $\frac{1}{2}$, vertical translation $\frac{3}{2}$ units up

(iii) since the coefficient of the leading term is positive, the function *f* has minimum value $\frac{3}{2}$ for $x = -7$.

2. **(a)** $x^2+2x-8=0 \Leftrightarrow (x+4)(x-2)=0 \Leftrightarrow x+4=0$ or $x-2=0 \Leftrightarrow x=-4$ or $x=2$

(b) $x^2=3x+10 \Leftrightarrow x^2-3x-10=0 \Leftrightarrow (x-5)(x+2)=0$

$x-5=0$ or $x+2=0 \Leftrightarrow x=5$ or $x=-2$

(c) $6x^2-9x=0 \Leftrightarrow 3x(2x-3)=0 \Leftrightarrow x=0$ or $2x-3=0 \Leftrightarrow x=0$ or $x=\frac{3}{2}$

(d) $3x^2+11x-4=0 \Leftrightarrow (3x-1)(x+4)=0 \Leftrightarrow 3x-1=0$ or $x+4=0 \Leftrightarrow x=\frac{1}{3}$ or $x=-4$

(e) $3x^2+18=15x \Leftrightarrow 3x^2-15x+18=0 \Leftrightarrow 3(x^2-5x+6)=0$

$\Leftrightarrow 3(x-3)(x-2)=0 \Leftrightarrow x-3=0$ or $x-2=0 \Leftrightarrow x=3$ or $x=2$

(f) $9x-2=4x^2 \Leftrightarrow 4x^2-9x+2=0 \Leftrightarrow (4x-1)(x-2)=0$

$4x-1=0$ or $x-2=0 \Leftrightarrow x=\frac{1}{4}$ or $x=2$

3. **(a)** $x^2+4x-3=0$

$(x+2)^2-7=0$

$(x+2+\sqrt{7})(x+2-\sqrt{7})=0$

$x+2+\sqrt{7}=0$ or $x+2-\sqrt{7}=0$

$x=-2-\sqrt{7}$ or $x=-2+\sqrt{7}$

(b) $x^2-4x-5=0$

$x^2-4x+4-9=0$

$(x-2)^2-9=0$

$(x-2-3)(x-2+3)=0$

$x-5=0$ or $x+1=0$

$x=5$ or $x=-1$

(c) $x^2-2x+3=0$

$x^2-2x+1+2=0$

$(x-1)^2+3=0$

No real solution

(d) $2x^2+16x+6=0$

Divide both sides by 2:

$x^2+8x+3=0$

$x^2+8x+16-13=0$

$(x+4)^2-13=0$

$(x+4+\sqrt{13})(x+4-\sqrt{13})=0$

$x+4+\sqrt{13}=0$ or $x+4-\sqrt{13}=0$

$x=-4-\sqrt{13}$ or $x=-4+\sqrt{13}$

(e) $x^2+2x-8=0$

$x^2+2x+1-9=0$

$(x+1)^2-9=0$

$(x+1+3)(x+1-3)=0$

$x+4=0$ or $x-2=0$

$x=-4$ or $x=2$

(f) $-2x^2+4x+9=0$

Divide both sides by -1:

$2x^2-4x-9=0$

$2(x^2-2x)-9=0$

$2(x^2-2x+1-1)-9=0$

$2\left[(x-1)^2-1\right]-9=0$

$2(x-1)^2-11=0 \Leftrightarrow (x-1)^2-\frac{11}{2}=0$

$$\left(x-1+\frac{\sqrt{11}}{\sqrt{2}}\right)\left(x-1-\frac{\sqrt{11}}{\sqrt{2}}\right)=0$$

$$\left(x-1+\frac{\sqrt{22}}{2}\right)\left(x-1-\frac{\sqrt{22}}{2}\right)=0$$

$$\left(x-\frac{2-\sqrt{22}}{2}\right)\left(x-\frac{2+\sqrt{22}}{2}\right)=0$$

$x-\frac{2-\sqrt{22}}{2}=0$ or $x-\frac{2+\sqrt{22}}{2}=0$

$x=\frac{2-\sqrt{22}}{2}$ or $x=\frac{2+\sqrt{22}}{2}$

4. $f(x)=x^2-4x-1$

(a) $\Delta=b^2-4ac=(-4)^2-4(1)(-1)=20,\ \sqrt{\Delta}=\sqrt{20}=2\sqrt{5}$

Quadratic formula: $x=\frac{-b\pm\sqrt{\Delta}}{2a}$. Therefore the zeros are:

$x_1=\frac{4-2\sqrt{5}}{2}=2-\sqrt{5},\quad x_2=\frac{4+2\sqrt{5}}{2}=2+\sqrt{5}$

(b) Axis of symmetry passes through the vertex. The x-coordinate of the vertex:

$x=\frac{x_1+x_2}{2}=\frac{2-\sqrt{5}+2+\sqrt{5}}{2}=2$. Therefore, equation of the axis of symmetry: $x=2$

(c) Since the coefficient of the leading term is positive, the function f has minimum value $f(2)=2^2-4(2)-1=-5$

5. **(a)** $x^2+3x+2=0$

$\Delta=b^2-4ac=3^2-4(1)(2)=9-8=1$. $\Delta>0$, therefore two real and distinct solutions.

(b) $2x^2-3x+2=0$

$\Delta=b^2-4ac=(-3)^2-4(2)(2)=9-16=-7$. $\Delta<0$, therefore no real solutions.

(c) $x^2-1=0$

$\Delta=b^2-4ac=0^2-4(1)(-1)=4$. $\Delta>0$, therefore two distinct solutions.

(d) $2x^2-\frac{9}{4}x+1=0$

$\Delta=b^2-4ac=\left(-\frac{9}{4}\right)^2-4(2)(1)=\frac{81}{16}-8=-\frac{47}{16}$. $\Delta<0$, therefore no real solutions.

6. The equation $2x^2+px+1=0$ has one real solution if its discriminant is equal to 0.

$\Delta=b^2-4ac=p^2-4(2)(1)=p^2-8$

$\Delta=0\Leftrightarrow p^2-8=0\Leftrightarrow\left(p+2\sqrt{2}\right)\left(p-2\sqrt{2}\right)=0\Leftrightarrow p+2\sqrt{2}=0$ or $p-2\sqrt{2}=0$

$p=-2\sqrt{2}$ or $p=2\sqrt{2}$

7. The equation $x^2+4x+k=0$ has two distinct real solutions if its discriminant is greater than 0.

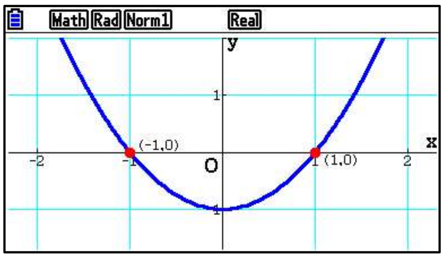
$\Delta=b^2-4ac=4^2-4(1)k=16-4k$

$\Delta>0\Leftrightarrow16-4k>0\Leftrightarrow k<4$

8. If the equation $x^2-4kx+4=0$ has two distinct real solutions, then its discriminant is greater than 0.

$\Delta=b^2-4ac=(4k)^2-4(1)4=16k^2-16$

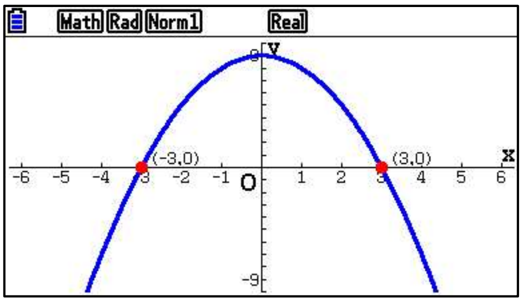
$\Delta>0\Leftrightarrow16k^2-16>0\Leftrightarrow k^2-1>0\Leftrightarrow k\in\left]-\infty,-1\right[\cup\left]1,+\infty\right[$ as seen on the graph:



9. The graph of the function $g: x \mapsto mx^2 + 6x + m$ will have no common point with the *x*-axis if $m \neq 0 \wedge \Delta < 0$.

$\Delta = b^2 - 4ac = 6^2 - 4m^2 = 36 - 4m^2$

$\Delta < 0 \Leftrightarrow 36 - 4m^2 < 0 \Leftrightarrow 9 - m^2 < 0 \Leftrightarrow m \in \left]-\infty, -3\right[\cup \left]3, +\infty\right[$ as seen on the graph:



10. The inequality $3x^2 - 12x + k > 0$ is true for all real values of *x* if the quadratic function $f(x) = 3x^2 - 12x + k$ has no real zeros. Since the coefficient of the leading term is positive, the discriminant of the function must be negative.

$\Delta = b^2 - 4ac = (-12)^2 - 4(3)(k) = 144 - 12k$

$\Delta < 0 \Leftrightarrow 144 - 12k < 0 \Leftrightarrow k > 12$

11. The expression can be rewritten in the form $f(x) = -x^2 + x - 2$. The discriminant of the quadratic function $\Delta = b^2 - 4ac = 1^2 - 4(-1)(-2) = 1 - 8 = -9 < 0$. The coefficient of the leading term is negative. Consequently, the graph of the function opens downwards and the function has no real zeros. Therefore $f(x) < 0$ for all real values of *x*.

12. (a) Since two zeros of the function are given, then $y = a(x+1)(x-4)$. The graph of the function passes through the point $(0, 8)$, so

$8 = a(0+1)(0-4) \Leftrightarrow -4a = 8 \Leftrightarrow a = -2$

Now, $y = -2(x+1)(x-4) = -2(x^2 - 3x - 4)$, or $y = -2x^2 + 6x + 8$

(b) Since two zeros of the function are given, then $y = a\left(x - \frac{1}{2}\right)(x-3)$. The graph of the function passes through the point $(-1, 4)$, so

$4 = a\left(-1 - \frac{1}{2}\right)(-1-3) \Leftrightarrow 6a = 4 \Leftrightarrow a = \frac{2}{3}$

Now, $y = \frac{2}{3}\left(x - \frac{1}{2}\right)(x-3) = \frac{2}{3}\left(x^2 - \frac{7}{2}x + \frac{2}{3}\right)$, or $y = \frac{2}{3}x^2 - \frac{7}{3}x + 1$

13. The equation $2x^2+(3-k)x+k+3=0$ has two imaginary solutions $\Delta<0$.

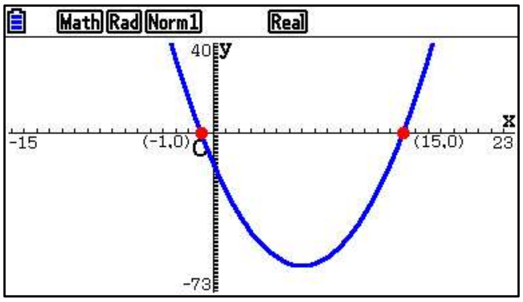
$\Delta=b^2-4ac=(3-k)^2-4(2)(k+3)=9-6k+k^2-8k-24=k^2-14k-15$

$\Delta<0 \Leftrightarrow k^2-14k-15<0$

$k^2-14k-15=(k+1)(k-15)=0$

$k_1=-1,\ k_2=15$

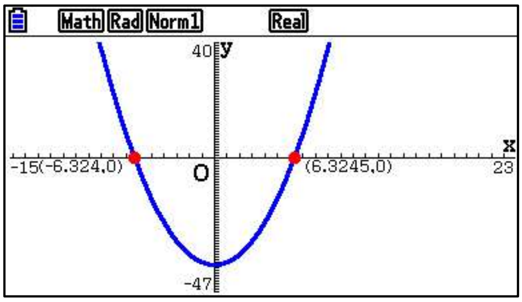
Therefore $k\in\left]-1,\ 15\right[$



14. The function $f(x)=5x^2-kx+2$ has two distinct real zeros if $\Delta>0$.

$\Delta=b^2-4ac=(-k)^2-4(5)(2)=k^2-40$

$\Delta>0 \Leftrightarrow k^2-40>0 \Leftrightarrow \left(k+2\sqrt{10}\right)\left(k-2\sqrt{10}\right)>0$



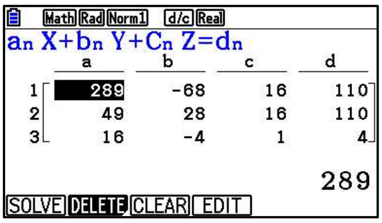
As can be seen on the graph,
$k\in\left]-\infty,\ -2\sqrt{10}\right[\cup\left]2\sqrt{10},\ +\infty\right[$

15. If the graph of a quadratic function $f(x)=ax^2+bx+c,\ a\neq 0$ passes through the points $\left(-\frac{17}{4},\ \frac{55}{8}\right),\ (-4,\ 4),\ \left(\frac{7}{4},\ \frac{55}{8}\right)$, their coordinates have to satisfy the formula of the function. Therefore:

$$\begin{cases}\left(-\frac{17}{4}\right)^2 a-\frac{17}{4}b+c=\frac{55}{8}\\ \left(\frac{7}{4}\right)^2 a+\frac{7}{4}b+c=\frac{55}{8}\\ (-4)^2 a-4b+c=4\end{cases}$$

Simplifying: $\begin{cases}\frac{289}{16}a-\frac{17}{4}b+c=\frac{55}{8}\\ \frac{49}{16}a+\frac{7}{4}b+c=\frac{55}{8}\\ 16a-4b+c=4\end{cases}$, or $\begin{cases}289a-68b+16c=110\\ 49a+28b+16c=110\\ 16a-4b+c=4\end{cases}$

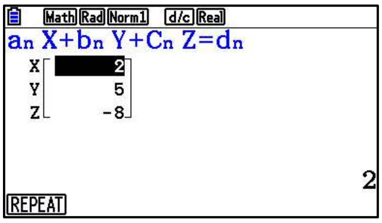
Using a GDC we get $a=2,\ b=5,\ c=-8$ as can be seen on following pictures:



$a_n X+b_n Y+C_n Z=d_n$

$$\begin{array}{c|cccc} & a & b & c & d\\ \hline 1 & 289 & -68 & 16 & 110\\ 2 & 49 & 28 & 16 & 110\\ 3 & 16 & -4 & 1 & 4\end{array}$$

289



$a_n X+b_n Y+C_n Z=d_n$

$$\begin{array}{cc} X & 2\\ Y & 5\\ Z & -8\end{array}$$

2

Therefore $f(x)=2x^2+5x-8$

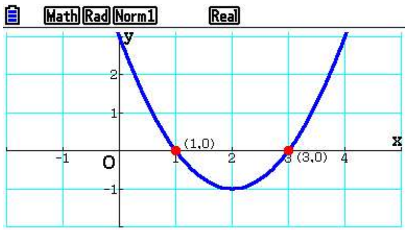
16. If $f(3)=f(-1)=2$, then the x-coordinate of the vertex $x=\frac{3-1}{2}=1$ (by symmetry of the graph). The function can be now expressed in the form $f(x)=a(x-1)^2+10$. Since $f(3)=2$, then $f(3)=a(3-1)^2+10=2\ \Rightarrow\ 4a=-8\Leftrightarrow a=-2$ and $f(x)=-2(x-1)^2+10$. Now $f(2)=-2(2-1)^2+10=8$.

17. $4x+1<x^2+4$

$x^2-4x+3>0$

$(x-3)(x-1)>0$

$x=3$ or $x=1$



Therefore $x\in\left]-\infty,\,1\right[\cup\left]3,\,+\infty\right[$

18. The discriminant of the quadratic function $f(x)=2x^2+(2-t)x+t^2+3$ is:
$\Delta=b^2-4ac=(2-t)^2-4(2)(t^2+3)=4-4t+t^2-8t^2-24=-7t^2-4t-20$.
Since the discriminant is a quadratic itself, then:
$\Delta_1=b^2-4ac=(-4)^2-4(-7)(-20)=16-560=-544$

Since $\Delta_1<0$ and the coefficient of the leading term in the expression for Δ is negative, the expression is negative for all values of *t*. It follows that there is no value of *t* for which the equation $2x^2+(2-t)x+t^2+3=0$ has real roots.

19. The equation $ax^2+bx+a=0$ must have two distinct roots.

In a quadratic equation $ax^2+bx+c=0$ the product of the roots is $\frac{c}{a}$, which in this case is $\frac{a}{a}=1$. Alternatively, the quadratic formula: $x_1=\frac{-b-\sqrt{b^2-4a^2}}{2a}=-\frac{b+\sqrt{b^2-4a^2}}{2a}$ and $x_2=\frac{-b+\sqrt{b^2-4a^2}}{2a}=-\frac{b-\sqrt{b^2-4a^2}}{2a}$. Let's calculate the product of the roots:

$$x_1x_2=\left(-\frac{b+\sqrt{b^2-4a^2}}{2a}\right)\left(-\frac{b-\sqrt{b^2-4a^2}}{2a}\right)=\frac{\left(b+\sqrt{b^2-4a^2}\right)\left(b-\sqrt{b^2-4a^2}\right)}{4a^2}$$

$$x_1x_2=\frac{b^2-\left(b^2-4a^2\right)}{4a^2}=\frac{b^2-b^2+4a^2}{4a^2}=\frac{4a^2}{4a^2}=1$$

(or, $x_1x_2=\frac{a}{a}=1$)

Therefore the two roots of the equation are reciprocals of each other.

20. (a) $2x^2+6x-5=0$

$$x_1+x_2=-\frac{b}{a}=-\frac{6}{2}=-3$$

$$x_1x_2=\frac{c}{a}=\frac{-5}{2}=-\frac{5}{2}$$

(b) $x^2=1-3x$

$$x^2+3x-1=0$$

$$x_1+x_2=-\frac{b}{a}=-\frac{3}{1}=-3$$

$$x_1x_2=\frac{c}{a}=\frac{-1}{1}=-1$$

(c) $4x^2-6=0$

$$x_1+x_2=-\frac{b}{a}=-\frac{0}{4}=0$$

$$x_1x_2=\frac{c}{a}=\frac{-6}{4}=-\frac{3}{2}$$

(d) $x^2+ax-2a=0$

$$x_1+x_2=-\frac{b}{a}=-\frac{a}{1}=-a$$

$$x_1x_2=\frac{c}{a}=\frac{-2a}{1}=-2a$$

(e) $m(m-2)=4(m+1)$

$$m^2-6m-4=0$$

$$x_1+x_2=-\frac{b}{a}=-\frac{-6}{1}=6$$

$$x_1x_2=\frac{c}{a}=\frac{-4}{1}=-4$$

(f) $3x-\frac{2}{x}=1,\ \ x\neq 0$

$$3x^2-x-2=0$$

$$x_1+x_2=-\frac{b}{a}=-\frac{-1}{3}=\frac{1}{3}$$

$$x_1x_2=\frac{c}{a}=\frac{-2}{3}=-\frac{2}{3}$$

21. Let the new equation be $ax^2+bx+c=0,\ a\neq 0$. If $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ are its roots, then:

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)=1$$

Since α and β are roots of the equation $2x^2-3x+6=0$, then $\alpha+\beta=\dfrac{3}{2}$ and $\alpha\beta=\dfrac{6}{2}=3$.

Now $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\left(\frac{3}{2}\right)^2-2(3)}{3}=\dfrac{\frac{9}{4}-6}{3}=-\dfrac{5}{4}\Rightarrow-\dfrac{b}{a}=-\dfrac{5}{4}\Rightarrow b=\dfrac{5}{4}a$ and

$\left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\beta}{\alpha}\right)=\dfrac{c}{a}=1\Rightarrow c=a$. Substituting in the new equation we get $ax^2+\dfrac{5}{4}ax+a=0$ and finally $4x^2+5x+4=0$.

22. If α and β are roots of the equation $3x^2+5x+4=0$, then:

(a) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\dfrac{b^2}{a^2}-\dfrac{2c}{a}=\dfrac{b^2-2ac}{a^2}=\dfrac{5^2-2(3)(4)}{3^2}=\dfrac{1}{9}$

(b) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\alpha^2+\beta^2}{\alpha\beta}=\dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\dfrac{\frac{1}{9}}{\frac{4}{3}}=\dfrac{1}{12}$

(c) $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)=(\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right]$

$$=\left(-\frac{b}{a}\right)\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)=-\frac{b}{a}\left(\frac{b^2-3ac}{a^2}\right)=-\frac{5}{3}\left(\frac{5^2-3(3)(4)}{3^2}\right)=\frac{55}{27}$$

23. (a) Given the equation $x^2+8x+k=0$ and that $x_1=3x_2$ we can write $x_1+x_2=-8$ and $x_1x_2=k$.

Now: $3x_2+x_2=-8$

$4x_2=-8$

$x_2=-2,\ x_1=-6$

(b) $k=x_1x_2=(-2)(-6)=12$

24. If α and β are roots of the equation $x^2+x+4=0$ then:

(a) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-\frac{b}{a}}{\frac{c}{a}}=-\frac{b}{c}=-\frac{1}{4}$

(b) If the roots of a quadratic equation $ax^2+bx+c=0,\ a\neq 0$ are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ then

$$-\frac{b}{a}=\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{1}{4}\Rightarrow b=\frac{1}{4}a \text{ and } \frac{c}{a}=\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)=\frac{1}{\alpha\beta}=\frac{1}{4}\Rightarrow c=\frac{1}{4}a$$

Substituting in the above equation we get $ax^2+\frac{1}{4}ax+\frac{1}{4}a=0$ and finally $4x^2+x+1=0$

25. If α and β are roots of the equation $5x^2-3x-1=0$ then $\alpha+\beta=\frac{3}{5}$ and $\alpha\beta=-\frac{1}{5}$.

Let the new equation be $ax^2+bx+c=0,\ a\neq 0$.

(a) $-\frac{b}{a}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{\left(\frac{3}{5}\right)^2-2\left(-\frac{1}{5}\right)}{\left(-\frac{1}{5}\right)^2}=19\Rightarrow b=-19a$

$$\frac{c}{a}=\left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right)=\frac{1}{(\alpha\beta)^2}==\frac{1}{\left(-\frac{1}{5}\right)^2}=25\Rightarrow c=25a$$

Therefore $ax^2-19ax+25a=0$ and finally $x^2-19x+25=0$

(b) $-\frac{b}{a}=\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right]}{\alpha\beta}=\frac{\left(\frac{3}{5}\right)\left[\left(\frac{3}{5}\right)^2-3\left(-\frac{1}{5}\right)\right]}{-\frac{1}{5}}=-\frac{72}{25}\Rightarrow b=\frac{72}{25}a$

$$\frac{c}{a}=\left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right)=\frac{(\alpha\beta)^2}{\alpha\beta}=\alpha\beta=-\frac{1}{5}\Rightarrow c=-\frac{1}{5}a$$

Therefore $ax^2+\frac{72}{25}ax-\frac{1}{5}a=0$ and finally $25x^2+72x-5=0$

Exercise 2.3

1. **(a)**

$$\begin{array}{r} 3x-4 \\ x+3\overline{)\,3x^2+5x-5} \\ \underline{-3x^2-9x} \\ -4x-5 \\ \underline{4x+12} \\ 7 \end{array}$$

Then $3x^2+5x-5=(x+3)(3x-4)+7$

(b)

$$\begin{array}{r} 3x^3-2x^2-4x+1 \\ x-2\overline{)\,3x^4-8x^3+9x+5} \\ \underline{-3x^4+6x^3} \\ -2x^3+9x \\ \underline{2x^3-4x^2} \\ -4x^2+9x \\ \underline{4x^2-8x} \\ x+5 \\ \underline{-x+2} \\ 7 \end{array}$$

Then $3x^4-8x^3+9x+5=(x-2)(3x^3-2x^2-4x+1)+7$

(c)

$$\begin{array}{r} x^2-x-1 \\ x-4\overline{)\,x^3-5x^2+3x-7} \\ \underline{-x^3+4x^2} \\ -x^2+3x \\ \underline{x^2-4x} \\ -x-7 \\ \underline{x-4} \\ -11 \end{array}$$

Then $x^3-5x^2+3x-7=(x-4)(x^2-x-1)-11$

(d)

$$\begin{array}{r} 3x^2+5x \\ 3x-1\overline{)\,9x^3+12x^2-5x+1} \\ \underline{-9x^3+3x^2} \\ 15x^2-5x \\ \underline{-15x^2+5x} \\ +1 \end{array}$$

Then $9x^3+12x^2-5x+1=(3x-1)(3x^2+5x)+1$

2. The binomial $x-1$ is a factor of $f(x)=2x^3-17x^2+22x-7$. Divide $f(x)$ by $x-1$:

$$
\begin{array}{rr}
 & 2x^2-15x+7 \\ \cline{2-2}
x-1 \big) & 2x^3-17x^2+22x-7 \;\square \\
 & \underline{-2x^3+2x^2} \qquad\qquad\quad \\
 & -15x^2+22x \qquad\;\; \\
 & \underline{15x^2-15x} \qquad\;\; \\
 & 7x-7 \\
 & \underline{-7x+7} \\
 & 0
\end{array}
$$

The expression $2x^2-15x+7$ can be further factored out to give $2x^2-15x+7=(2x-1)(x-7)$. Therefore $f(x)=2x^3-17x^2+22x-7=(x-7)(x-1)(2x-1)$

3. The binomial $2x+1$ is a factor of $f(x)=6x^3-5x^2-12x-4$. Divide $f(x)$ by $2x+1$:

$$
\begin{array}{rr}
 & 3x^2-4x-4 \\ \cline{2-2}
2x+1 \big) & 6x^3-5x^2-12x-4 \;\square \\
 & \underline{-6x^3-3x^2} \qquad\qquad\quad \\
 & -8x^2-12x \qquad\;\; \\
 & \underline{8x^2+4x} \qquad\;\; \\
 & -8x-4 \\
 & \underline{8x+4} \\
 & 0
\end{array}
$$

The expression $3x^2-4x-4$ can be further factored out to give $3x^2-4x-4=(3x+2)(x-2)$. Therefore $f(x)=6x^3-5x^2-12x-4=(x-2)(2x+1)(3x+2)$

4. The binomial $x+\frac{2}{3}$ is a factor of $f(x)=3x^4+2x^3-36x^2+24x+32$. Divide by $x+\frac{2}{3}$:

$$\begin{array}{r} 3x^3-36x+48 \\ x+\frac{2}{3}\overline{\big)3x^4+2x^3-36x^2+24x+32}\;\square \\ \underline{-3x^4-2x^3} \\ -36x^2+24x \\ \underline{36x^2+24x} \\ 48x+32 \\ \underline{-48x-32} \\ 0 \end{array}$$

Now,

$$\begin{aligned} 3x^3-36x+48 &= 3(x^3-12x+16)=3(x^3-4x-8x+16)=3\left[x(x^2-4)-8(x-2)\right] \\ &= 3\left[x(x-2)(x+2)-8(x-2)\right]=3(x-2)(x^2+2x-8) \\ &= (x-2)(x-2)(x+4)=(x-2)^2(x+4) \end{aligned}$$

Therefore

$$f(x)=3x^4+2x^3-36x^2+24x+32=3(x-2)^2(x+4)\left(x+\frac{2}{3}\right)=(x-2)^2(x+4)(3x+2)$$

5. **(a)**

$$\begin{array}{r} x-2 \\ x-2\overline{\big)\;x^2-5x+4} \\ \underline{-x^2+3x} \\ -2x+4 \\ \underline{2x-6} \\ -2 \end{array}$$

Quotient = $x-2$
Remainder = –2

(b)

$$\begin{array}{r} x^2+2 \\ x+2\overline{\big)\;x^3+2x^2+2x+1} \\ \underline{-x^3-2x^2} \\ 2x+1 \\ \underline{-2x-4} \\ -3 \end{array}$$

Quotient = x^2+2
Remainder = –3

(c)

$$\begin{array}{r} 3 \\ 3x^2-7x\overline{)\,9x^2-x+5} \\ \underline{-9x^2+21x} \\ 20x+5 \end{array}$$

Quotient = 3
Remainder = $20x+5$

(d)

$$\begin{array}{r} x^4+x^3+4x^2+4x+4 \\ x-1\overline{)\,x^5+3x^3-6\qquad\qquad\quad} \\ \underline{-x^5+x^4}\qquad\qquad\qquad\quad \\ x^4+3x^3\qquad\qquad\quad \\ \underline{-x^4+x^3}\qquad\qquad\quad \\ 4x^3-6\qquad\quad \\ \underline{-4x^3+4x^2}\qquad\quad \\ 4x^2-6\quad \\ \underline{-4x^2+4x}\quad \\ 4x-6 \\ \underline{-4x+4} \\ -2 \end{array}$$

Quotient = $x^4+x^3+4x^2+4x+4$
Remainder = -2

6. **(a)** By the remainder theorem, the required remainder is equal $P(2)$

$$P(2)=2(2)^3-3(2)^2+4(2)-7=5$$

(b) By the remainder theorem, the required remainder is equal $P(-1)$

$$P(-1)=(-1)^5-2(-1)^4+3(-1)^2-20(-1)+3=23$$

(c) By the remainder theorem, the required remainder is equal $P(-7)$

$$P(-7)=5(-7)^4+30(-7)^3-40(-7)^2+36(-7)+14=-483$$

(d) By the remainder theorem, the required remainder is equal $P\left(\frac{1}{4}\right)$

$$P\left(\frac{1}{4}\right)=\left(\frac{1}{4}\right)^3-\frac{1}{4}+1=\frac{49}{64}$$

7. If $x = 2$ is a double root of the polynomial $P(x) = x^4 - 5x^3 + 7x^2 - 4$ then the polynomial $P(x)$ is exactly divisible by $(x-2)^2 = x^2 - 4x + 4$.

$$\begin{array}{r}
x^2 - x - 1 \\
x^2 - 4x + 4\overline{\smash{\big)}\,x^4 - 5x^3 + 7x^2 - 4} \\
\underline{-x^4 + 4x^3 - 4x^2 \phantom{{}-4}} \\
-x^3 + 3x^2 - 4 \\
\underline{x^3 - 4x^2 + 4x \phantom{{}-4}} \\
-x^2 + 4x - 4 \\
\underline{x^2 - 4x + 4} \\
0
\end{array}$$

The remaining roots are the roots of $f(x) = x^2 - x - 1$

$$f(x) = 0 \Leftrightarrow x^2 - x - 1 = 0$$

$$\Delta = b^2 - 4ac = (-1)^2 - 4(1)(-1) = 5 \Rightarrow x = \frac{1+\sqrt{5}}{2} \text{ or } x = \frac{1-\sqrt{5}}{2}$$

8. If $x = 3$ is a zero of the polynomial $f(x) = x^3 + x^2 + kx + 1$, then $f(3) = 0$

Therefore $3^3 + 3^2 + 3k + 1 = 0 \Leftrightarrow 3k = -37 \Leftrightarrow k = -\frac{37}{3}$

9. Since 1 and 4 are zeros of the polynomial $f(x) = 2x^4 - 5x^3 - 14x^2 + ax + b$, the following conditions must hold: $f(1) = 0$ and $f(4) = 0$

$$f(1) = 2(1)^4 - 5(1)^3 - 14(1)^2 + a(1) + b = a + b - 17 = 0$$

$$f(4) = 2(4)^4 - 5(4)^3 - 14(4)^2 + a(4) + b = 4a + b - 32 = 0$$

We need to solve the system of equations:

$a + b = 17$

$4a + b = 32$

Subtracting the first equation from the second one we get $3a = 15$ or $a = 5$

Then $b = 17 - a = 17 - 5 = 12$

10. (a) The polynomial can be written in the product form as $f(x) = a(x+2)(x-1)(x-4)$. We can assume $a = 1$. Then

$$f(x) = (x^2 + x - 2)(x-4) = x^3 - 4x^2 + x^2 - 4x - 2x + 8 = x^3 - 3x^2 - 6x + 8$$

(b) The polynomial can be written in the product form as $f(x)=a(x+1)(x-3)^2(x+2)$

We can assume $a=1$. Then $f(x)=(x^2+3x+2)(x^2-6x+9)=x^4-3x^3-7x^2+15x+18$

(c) The polynomial can be written in the product form as $f(x)=a(x-2)^3$

We can assume $a=1$. Then $f(x)=(x-2)^3=x^3-6x^2+12x-8$

11. (a) The polynomial $P(x)=6x^3+7x^2+ax+b$ satisfies the following conditions: $P(2)=72$ and $P(-1)=0$ (by the remainder theorem and the factor theorem respectively). Then:

$6(2)^3+7(2)^2+2a+b=2a+b+76=72$ and $6(-1)^3+7(-1)^2-a+b=-a+b+1=0$

We need to solve the system of equations:

$2a+b=-4$

$-a+b=-1$

Subtracting second equation from the first one we get $3a=-3 \Leftrightarrow a=-1$. Then $b=-1+a=-2$

(b) The polynomial can be written as $P(x)=6x^3+7x^2-x-2$. If $(2x-1)$ is a factor of $P(x)$ then $P\left(\frac{1}{2}\right)$ must be equal to 0.

$P\left(\frac{1}{2}\right)=6\left(\frac{1}{2}\right)^3+7\left(\frac{1}{2}\right)^2-\frac{1}{2}-2=\frac{6}{8}+\frac{7}{4}-\frac{1}{2}-2=0$. Therefore $(2x-1)$ is a factor.

To find the third factor we can divide $P(x)$ by $(2x-1)(x+1)=2x^2+x-1$

$$\begin{array}{r|l}
 & 3x+2 \\ \hline
2x^2+x-1 & 6x^3+7x^2-x-2 \\
 & \underline{-6x^3-3x^2+3x} \\
 & \quad 4x^2+2x-2 \\
 & \quad \underline{-4x^2-2x+2} \\
 & \qquad\qquad 0
\end{array}$$

Since the remainder is equal to 0, then the remaining factor of $P(x)$ is $3x+2$

12. By the remainder theorem:

$p(-1)=[a(-1)+b]^3=(-a+b)^3=-1$ and $p(2)=[a(2)+b]^3=(2a+b)^3=27$

Now we need to solve the following system of equations:

$(-a+b)^3=-1$ and $(2a+b)^3=27$

Taking the third root of both sides we get

$-a+b=-1$ and $2a+b=3$

Subtracting the second equation from the first one gives $-3a=-4$, so $a=\frac{4}{3}$ and $b=a-1=\frac{4}{3}-1=\frac{1}{3}$

13. Let $P(x)=x^3+ax^2+bx+c$. By the factor theorem and by the remainder theorem the following conditions must be satisfied simultaneously:

$P(2)=0$

$P(-2)=0$

$P(3)=10$

Therefore:

$$P(2)=2^3+a(2)^2+b(2)+c=4a+2b+c+8=0$$

$$P(-2)=(-2)^3+a(-2)^2+b(-2)+c=4a-2b+c-8=0$$

$$P(3)=3^3+a(3)^2+b(3)+c=9a+3b+c+27=10$$

To determine values of a, b and c we need to solve the following system of equations:

$$4a+2b+c=-8$$

$$4a-2b+c=8$$

$$9a+3b+c=-17$$

Subtracting the second equation from the first one we get $4b=-16 \Leftrightarrow b=-4$ and $c=-4a$. Substituting these values into the third equation we have $9a-12-4a=-17 \Leftrightarrow 5a=-5 \Leftrightarrow a=-1$. Therefore $a=-1,\ b=-4,\ c=4$.

14. Let $P(x)=5x^3-3x^2+ax+7$ and $Q(x)=4x^3+ax^2+7x-4$. By the remainder theorem:

$$P(-2)=5(-2)^3-3(-2)^2+a(-2)+7=-2a-45=R$$

$$Q(-2)=4(-2)^3+a(-2)^2+7(-2)-4=4a-50=2R$$

Substituting $R=-2a-45$ into the second equation we have $4a-50=2(-2a-45) \Leftrightarrow 4a-50=-4a-90 \Leftrightarrow 8a=-40 \Leftrightarrow a=-5$

15. Let $f(x)=x^3-19x^2+bx-216$. If the roots of the equation $f(x)=0$ are consecutive terms of a geometric sequence, we can write

$$a+ar+ar^2=19$$

$$a(ar)(ar^2)=216$$

where a is the first term of a geometric sequence and r is the common ratio.

Simplifying the system of equations, we get

$$\begin{cases} a+ar+ar^2=19 \\ a^3r^3=216 \end{cases} \Leftrightarrow \begin{cases} a+ar+ar^2=19 \\ ar=6 \end{cases} \Leftrightarrow \begin{cases} a+6+ar^2=19 \\ a=\dfrac{6}{r} \end{cases} \Leftrightarrow \begin{cases} a+ar^2=13 \\ a=\dfrac{6}{r} \end{cases}$$

Substituting $a=\frac{6}{r}$ into the first equation we have

$\frac{6}{r}+\frac{6}{r}(r^2)=13 \Leftrightarrow \frac{6}{r}+6r=13 \Leftrightarrow 6r^2-13r+6=0$. Factoring out gives $6r^2-13r+6=0 \Leftrightarrow (3r-2)(2r-3)=0 \Leftrightarrow r=\frac{2}{3}$ or $r=\frac{3}{2}$. Then $a=9$ or $a=4$ respectively. Therefore, the roots of the given equation are $(9, 6, 4)$ or $(4, 6, 9)$. Since these are the same roots, we can write

$$x^3-19x^2+bx-216=(x-4)(x-6)(x-9)$$

$$x^3-19x^2+bx-216=(x^2-10x+24)(x-9)$$

$$x^3-19x^2+bx-216=x^3-19x^2+114x-216$$

from which it follows that $b=114$

16. (a) Since a polynomial $P(x)$ is divided by first degree binomial $(ax+b)$, the remainder can only be a constant. We can write

$$P(x)=(ax-b)Q(x)+R$$

Let $x=\frac{b}{a}$, $a\neq 0$. Then $P\left(\frac{b}{a}\right)=\left[a\left(\frac{b}{a}\right)-b\right]Q\left(\frac{b}{a}\right)+R=(b-b)Q\left(\frac{b}{a}\right)+R=0+R=R$

Therefore, when a polynomial $P(x)$ is divided by $(ax-b)$, the remainder is $R=P\left(\frac{b}{a}\right)$.

(b) The remainder when $P(x)=9x^3-x+5$ is divided by $(3x+2)$ is equal $P\left(-\frac{2}{3}\right)$.

$$P\left(-\frac{2}{3}\right)=9\left(-\frac{2}{3}\right)^3-\left(-\frac{2}{3}\right)+5=3$$

17. (a) By the formula, the sum of the roots of the equation $x^4-\frac{2}{3}x^3+3x^2-2x+5=0$ is $-\frac{a_{n-1}}{a_n}=-\frac{-\frac{2}{3}}{1}=\frac{2}{3}$ and the product of the roots is $(-1)^n\frac{a_0}{a_n}=(-1)^4\left(\frac{5}{1}\right)=5$

(b) Expanding the equation $(x-2)^3=x^4-1$ we have

$$x^3-6x^2+12x-8=x^4-1$$
$$x^4-x^3+6x^2-12x+7=0$$

and by the formula: the sum of the roots is $-\frac{a_{n-1}}{a_n}=-\frac{-1}{1}=1$ and the product of the roots is $(-1)^n\frac{a_0}{a_n}=(-1)^4\left(\frac{7}{1}\right)=7$

(c) Convert the equation $\frac{3}{x^2+2}=\frac{2x^2-x}{2x^5+1},\ x\neq-\frac{1}{\sqrt[5]{2}}$ into an algebraic sum.

$$3\left(2x^5+1\right)=\left(x^2+2\right)\left(2x^2-x\right)$$

$$6x^5+3=2x^4-x^3+4x^2-2x$$

$$6x^5-2x^4+x^3-4x^2+2x+3=0$$

and by the formula: the sum of the roots is $-\frac{a_{n-1}}{a_n}=-\frac{-2}{6}=\frac{1}{3}$, the product of the roots is $(-1)^n\frac{a_0}{a_n}=(-1)^5\left(\frac{3}{6}\right)=-\frac{1}{2}$

18. The given equation can be written as $ax^3+bx^2+cx+d=a(x-\alpha)(x-\beta)(x-\gamma)$.
Expand the right-hand side of the equation:

$$a(x-\alpha)(x-\beta)(x-\gamma)=a\left[x^2-(\alpha+\beta)x+\alpha\beta\right](x-\gamma)$$

$$=a\left[x^3-(\alpha+\beta+\gamma)x^2+(\alpha+\beta)\gamma x+\alpha\beta x-\alpha\beta\gamma\right]$$
$$=a\left[x^3-(\alpha+\beta+\gamma)x^2+(\alpha\gamma+\beta\gamma+\alpha\beta)x-\alpha\beta\gamma\right]$$

$$=ax^3-a(\alpha+\beta+\gamma)x^2+a(\alpha\gamma+\beta\gamma+\alpha\beta)x-a\alpha\beta\gamma$$

Comparing coefficients of x in both forms of the given function we have:

$$a(\alpha\gamma+\beta\gamma+\alpha\beta)=b\Leftrightarrow\ \alpha\gamma+\beta\gamma+\alpha\beta=\frac{b}{a}$$

19. Let $\alpha,\ 2\alpha,\ \beta$ be the three roots of the equation $x^3-63x+162=0$. Then:

$$\begin{cases}\alpha+2\alpha+\beta=0\\ \alpha(2\alpha)(\beta)=-162\end{cases} \Leftrightarrow \begin{cases}3\alpha+\beta=0\\ 2\alpha^2\beta=-162\end{cases} \Leftrightarrow \begin{cases}\beta=-3\alpha\\ 2\alpha^2\beta=-162\end{cases}$$

Now, $2\alpha^2(-3\alpha)=-162 \Leftrightarrow -6\alpha^3=-216 \Leftrightarrow \alpha^3=27 \Leftrightarrow \alpha=3$

Then $\beta=-3\alpha=-3(3)=-9$ and the zeros are 3, 6, and –9

20. Let the zeros of the equation $x^3-6x^2-24x+64=0$ be represented by $\frac{\alpha}{r},\ \alpha,\ \alpha r$ for some constant *r*. Then:

$$\begin{cases}\frac{\alpha}{r}+\alpha+\alpha r=6\\ \frac{\alpha}{r}(\alpha)(\alpha r)=-64\end{cases} \Leftrightarrow \begin{cases}\frac{\alpha}{r}+\alpha+\alpha r=6\\ \alpha^3=-64\end{cases} \Leftrightarrow \begin{cases}\frac{\alpha}{r}+\alpha+\alpha r=6\\ \alpha=-4\end{cases}$$

and $\frac{\alpha}{r}+\alpha+\alpha r=6 \Leftrightarrow \frac{-4}{r}-4-4r=6 \Leftrightarrow \frac{-4}{r}-4r-10=0 \Leftrightarrow 2r^2+5r+2=0$

Factorising the last equation, we have

$2r^2+5r+2=0 \Leftrightarrow (2r+1)(r+2)=0 \Leftrightarrow r=-\frac{1}{2}$ or $r=-2$ and

$\frac{\alpha}{r}=\frac{-4}{-\frac{1}{2}}=8,\ \alpha=-4,\ \alpha r=-4\left(-\frac{1}{2}\right)=2$ or $\frac{\alpha}{r}=\frac{-4}{-2}=2,\ \alpha=-4,\ \alpha r=-4(-2)=8$

Therefore, the three zeros of the equation $x^3-6x^2-24x+64=0$ are 2, –4, 8.

21. Zeros of the equation $x^3-6x^2+kx+10=0$ can be represented as $\alpha,\ \alpha+d,\ \alpha+2d$ for some constant *d*. Then

$$\begin{cases}\alpha+\alpha+d+\alpha+2d=6\\ \alpha(\alpha+d)(\alpha+2d)=-10\end{cases} \Leftrightarrow \begin{cases}3\alpha+3d=6\\ \alpha(\alpha+d)(\alpha+2d)=-10\end{cases} \Leftrightarrow \begin{cases}\alpha+d=2\\ \alpha(\alpha+d)(\alpha+2d)=-10\end{cases}$$

and $\begin{cases}\alpha=2-d\\ \alpha(2)(\alpha+2d)=-10\end{cases} \Leftrightarrow \begin{cases}\alpha=2-d\\ \alpha(\alpha+2d)=-5\end{cases}$.

Now, $(2-d)(2-d+2d)=-5 \Leftrightarrow (2-d)(2+d)=-5 \Leftrightarrow 4-d^2=-5 \Leftrightarrow d^2=9$

Therefore $d=-3$ or $d=3$ and $\alpha=2-(-3)=5$ or $\alpha=2-3=-1$ respectively. Then

$\alpha=5,\ \alpha+d=5-3=2,\ \alpha+2d=5-6=-1$ or

$\alpha=-1,\ \alpha+d=-1+3=2,\ \alpha+2d=-1+6=5$

and the zeros of the given equation are 5, 2, and −1.

The given polynomial can be converted to product form:

$$x^3-6x^2+kx+10=(x-5)(x-2)(x+1)$$

Then:

$$x^3-6x^2+kx+10=(x^2-7x+10)(x+1)$$

$$x^3-6x^2+kx+10=x^3-6x^2+3x+10$$

It follows that $k=3$

Note: If we had ignored the given hint and replaced the roots by $\alpha-d,\ \alpha,\ \alpha+d$, the solution would have been more straightforward!

22. Let the zeros of the equation $x^3+x^2+2x+k=0$ be represented by $\frac{\alpha}{r},\ \alpha,\ \alpha r$ for some constant r. Then, using result of question **18** and key facts from this section:

$$\begin{cases}\frac{\alpha}{r}+\alpha+\alpha r=-1\\ \frac{\alpha}{r}(\alpha)+\frac{\alpha}{r}(\alpha r)+\alpha(\alpha r)=2\\ \frac{\alpha}{r}(\alpha)(\alpha r)=-k\end{cases}\Leftrightarrow\begin{cases}\frac{\alpha}{r}+\alpha+\alpha r=-1\\ \frac{\alpha^2}{r}+\alpha^2+\alpha^2 r=2\\ \alpha^3=-k\end{cases}\Leftrightarrow\begin{cases}\alpha\left(\frac{1}{r}+1+r\right)=-1\\ \alpha^2\left(\frac{1}{r}+1+r\right)=2\\ k=-\alpha^3\end{cases}$$

Dividing the second equation by the first one side by side we get

$$\frac{\alpha^2\left(\frac{1}{r}+1+r\right)}{\alpha\left(\frac{1}{r}+1+r\right)}=\frac{2}{-1}\Leftrightarrow \alpha=-2\text{. Therefore, } k=-\alpha^3=-(-2)^3=8$$

Note: This question is an investigation that goes beyond the syllabus coverage. Consider it a challenge! The roots are not all real numbers.

Exercise 2.4

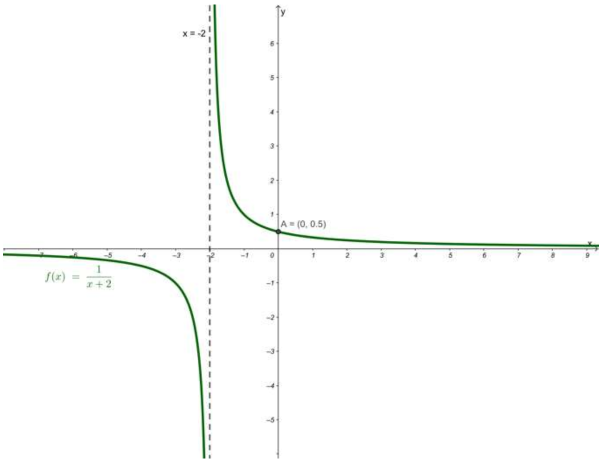
1. **(a)** $f(x)=\dfrac{1}{x+2}$

Domain: $\mathbb{R}-\{-2\}$, so there is a vertical asymptote $x=-2$. As $x\to\pm\infty$, $f(x)\to 0$, so there is a horizontal asymptote $y=0$.

$f(x)\neq 0$ on its domain, so the graph of $f(x)$ has no *x*-intercept.

As $x=0,\ f(0)=\dfrac{1}{0+2}=\dfrac{1}{2}$

The *y*-intercept has coordinates $\left(0,\ \dfrac{1}{2}\right)$



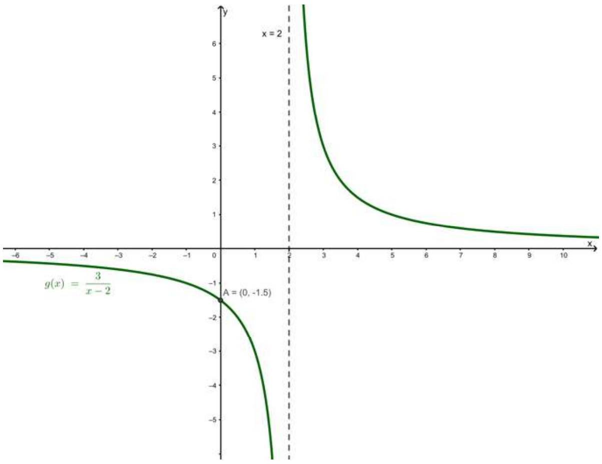
(b) $g(x)=\dfrac{3}{x-2}$

Domain: $\mathbb{R}-\{2\}$, so there is a vertical asymptote $x=2$. As $x\to\pm\infty$, $g(x)\to 0$, so there is a horizontal asymptote $y=0$.

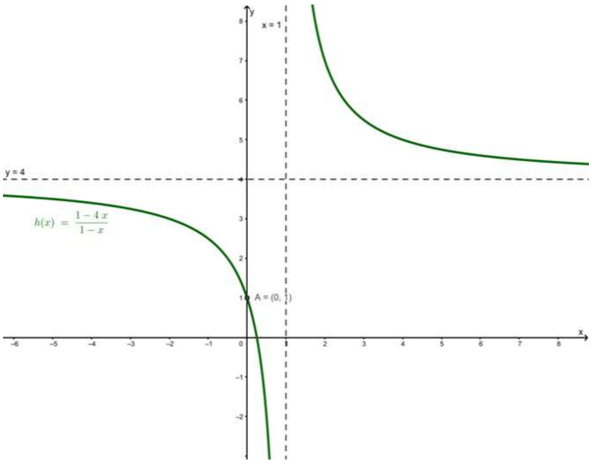
$g(x)\neq 0$ on its domain, so the graph of $g(x)$ has no *x*-intercept.

As $x=0,\ g(0)=\dfrac{3}{0-2}=-\dfrac{3}{2}$

The *y*-intercept has coordinates $\left(0,\ -\dfrac{3}{2}\right)$



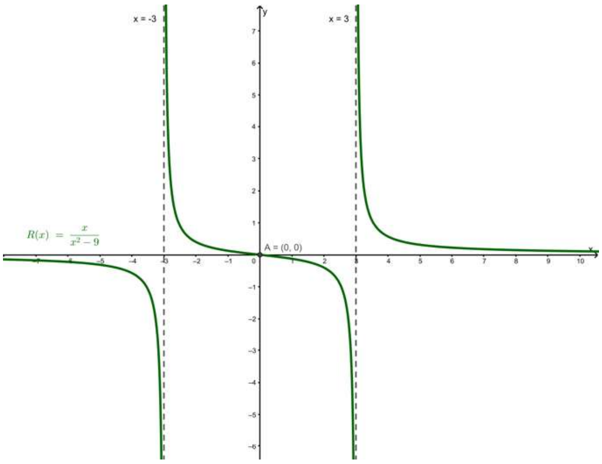
(c) $h(x)=\frac{1-4x}{1-x}$



Domain: $\mathbb{R}-\{1\}$, so there is a vertical asymptote $x=1$. The function $h(x)$ can be expressed in the form $h(x)=\frac{\frac{1}{x}-4}{\frac{1}{x}-1}$. As $x\to\pm\infty$, $\frac{1}{x}\to 0$, so $h(x)\to 4$, so there is a horizontal asymptote $y=4$.

$h(x)=0\Leftrightarrow 1-4x=0\Leftrightarrow x=\frac{1}{4}$, so the graph of $h(x)$ has x-intercept $\left(\frac{1}{4},\ 0\right)$. As $x=0,\ h(0)=\frac{1-4(0)}{1-0}=1$. The y-intercept has coordinates $(0,\ 1)$.

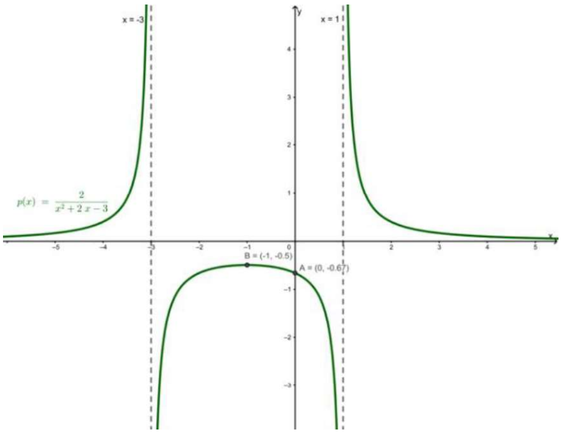
(d) $R(x)=\frac{x}{x^2-9}$



Domain: $\mathbb{R}-\{-3,\ 3\}$, so vertical asymptotes are $x=-3$ and $x=3$.The function $R(x)$ can be written in the form $R(x)=\frac{\frac{1}{x}}{1-\frac{9}{x^2}}$. For $x\to\pm\infty$, $\frac{1}{x}\to 0$ and $\frac{9}{x^2}\to 0$, so the graph of $R(x)$ has a horizontal asymptote $y=0$. Also, $R(x)=0\Leftrightarrow x=0$, so the graph of $R(x)$ has x-intercept $(0,\ 0)$. For $x=0,\ R(0)=0$, so the y-intercept has coordinates $(0,\ 0)$.

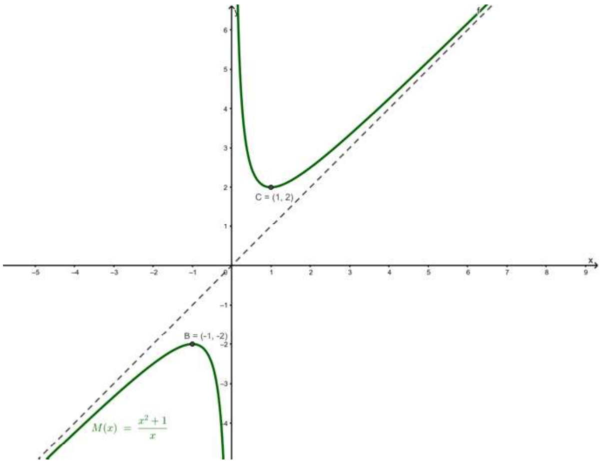
(e) $p(x) = \dfrac{2}{x^2+2x-3} = \dfrac{2}{(x+3)(x-1)}$

Domain: $\mathbb{R} - \{-3, 1\}$, so vertical asymptotes are $x=-3$ and $x=1$. As $x \to -3^-$, $p(x) \to +\infty$, as $x \to -3^+$, $p(x) \to -\infty$. As $x \to 1^-$, $p(x) \to +\infty$, as $x \to 1^+$, $p(x) \to -\infty$. Since the function $y = x^2+2x-3$ for $x=-1$ assumes the minimum value $y_{\min} = (-1)^2 + 2(-1) - 3 = -4$, then $p(x)$ for $x=-1$ assumes its maximum value $p_{\max} = p(-1) = \dfrac{2}{-4} = -\dfrac{1}{2}$. For $x \to \pm\infty$, $\dfrac{2}{x^2+2x-3} \to 0$, so the graph of $p(x)$ has horizontal asymptote $y=0$. The graph of $p(x)$ has no x-intercept. For $x=0$, $p(0) = -\dfrac{2}{3}$, so the y-intercept has coordinates $\left(0, -\dfrac{2}{3}\right)$.



(f) $M(x) = \dfrac{x^2+1}{x}$

Domain: $\mathbb{R} - \{0\}$, so vertical asymptote has equation $x=0$. As $x \to 0^-$, $M(x) \to -\infty$, as $x \to 0^+$, $p(x) \to +\infty$. Since the degree of the numerator is one more than the degree of the denominator, the graph of $M(x)$ has oblique asymptote: $\dfrac{x^2+1}{x} = x + \dfrac{1}{x} \Rightarrow y = x$. The graph of $M(x)$ has no x-intercept.

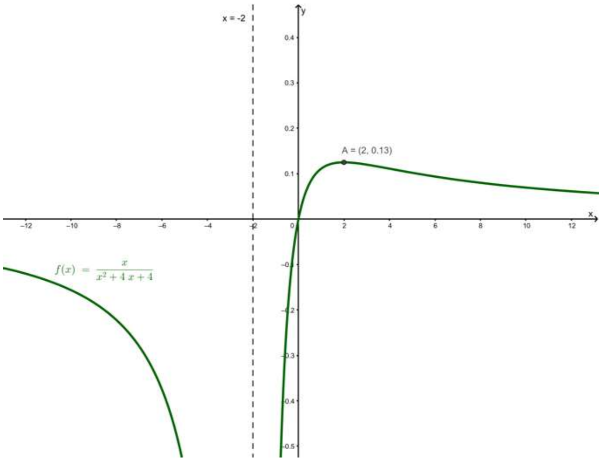


For $x=0$, $M(0)$ is indeterminate so there is no y-intercept.

(g) $f(x)=\dfrac{x}{x^2+4x+4}=\dfrac{x}{(x+2)^2}$

Domain: $\mathbb{R}-\{-2\}$, so vertical asymptote has equation $x=-2$. As $x\to-2^-$, $f(x)\to-\infty$, as $x\to-2^+$, $f(x)\to-\infty$. The degree of the numerator is one less than the degree of the denominator, so the horizontal asymptote has equation $y=0$. If $x=0$, $f(x)=0$, so the x-intercept of the graph of $f(x)$ is $(0,\ 0)$.

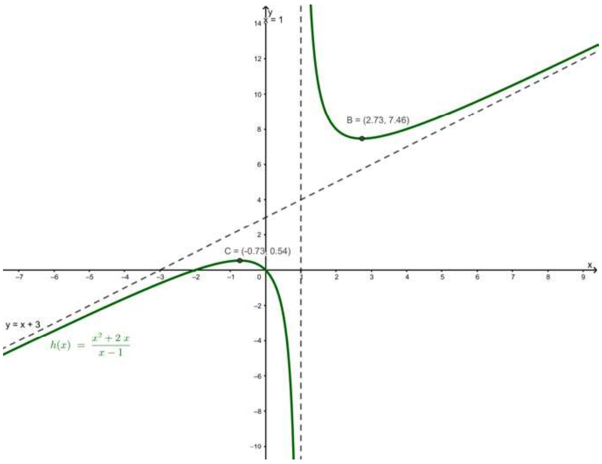
For $x=0$, $f(0)=0$ so the y-intercept of the graph of $f(x)$ is $(0,\ 0)$.



(h) $h(x)=\dfrac{x^2+2x}{x-1}$

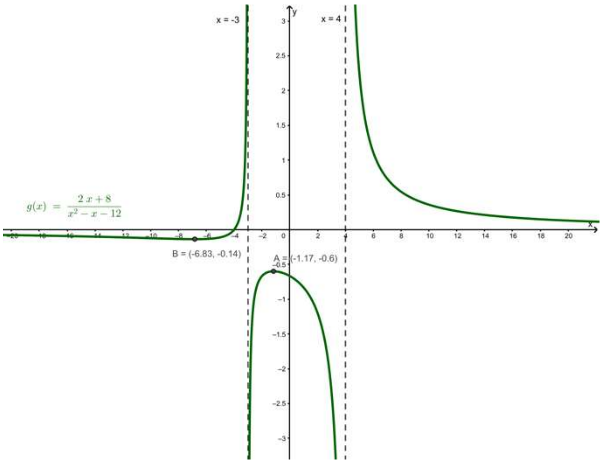
Domain: $\mathbb{R}-\{1\}$, so vertical asymptote has equation $x=1$. As $x\to1^-$, $h(x)\to-\infty$, as $x\to1^+$, $h(x)\to+\infty$. The degree of the numerator is one more than the degree of the denominator, so the graph of $h(x)$ has an oblique asymptote $y=ax+b$. Dividing $x-1$ into x^2+2x we get:

$$\begin{array}{r|l} & \quad\ \ x+3 \\ \hline x-1 & x^2+2x \\ & \underline{-x^2+x} \\ & \qquad 3x \\ & \quad \underline{-3x+3} \\ & \qquad\quad 3 \end{array}$$



Therefore, the oblique asymptote has equation $y=x+3$. $h(x)=0$ if $x=-2$ or $x=0$, so the x-intercepts of the graph of $h(x)$ are $(-2,\ 0)$ and $(0,\ 0)$. For $x=0$, $h(0)=0$ so the y-intercept of the graph of $h(x)$ is $(0,\ 0)$.

(i) $g(x)=\dfrac{2x+8}{x^2-x-12}=\dfrac{2x+8}{(x-4)(x+3)}$

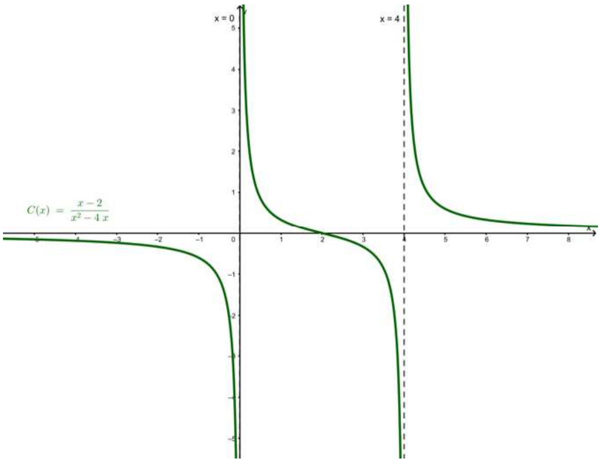


Domain: $\mathbb{R}-\{-3,\ 4\}$, so vertical asymptotes have equations $x=-3$ and $x=4$. As $x\to -3^-$, $g(x)\to +\infty$, as $x\to -3^+$, $g(x)\to -\infty$. As $x\to 4^-$, $g(x)\to -\infty$, as $x\to 4^+$, $g(x)\to +\infty$. The degree of the numerator is one less than the degree of the denominator, so the graph of $g(x)$ has horizontal asymptote $y=0$.

If $x=0,\ g(0)=-\dfrac{2}{3}$, so the y-intercept has coordinates $\left(0,\ -\dfrac{2}{3}\right)$.

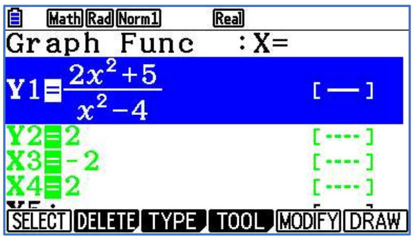
$g(x)=0 \Leftrightarrow 2x+8=0 \Leftrightarrow x=-4$. The x-intercept is $(-4,\ 0)$.

(j) $C(x)=\dfrac{x-2}{x^2-4x}=\dfrac{x-2}{x(x-4)}$



Domain: $\mathbb{R}-\{0,\ 4\}$, so vertical asymptotes have equations $x=0$ and $x=4$. As $x\to 0^-$, $C(x)\to -\infty$, as $x\to 0^+$, $C(x)\to +\infty$. As $x\to 4^-$, $C(x)\to -\infty$, as $x\to 4^+$, $C(x)\to +\infty$. The degree of the numerator is one less than the degree of the denominator, so the graph of $C(x)$ has horizontal asymptote $y=0$. The graph of $C(x)$ has no y-intercept. $C(x)=0 \Leftrightarrow x-2=0 \Leftrightarrow x=2$. The x-intercept is $(2,\ 0)$.

2. **(a)**



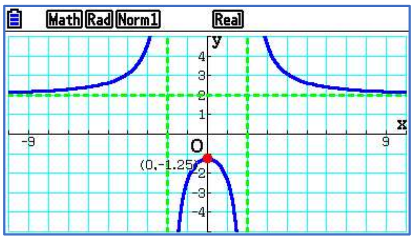
Graph Func :X=

$Y1 = \dfrac{2x^2+5}{x^2-4}$ [—]

$Y2 = 2$ [----]

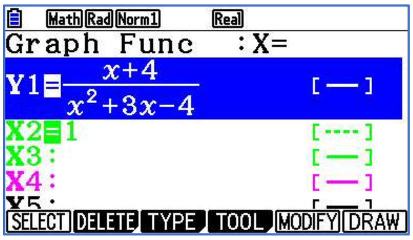
$X3 = -2$ [----]

$X4 = 2$ [----]



Domain: $\mathbb{R}-\{-2,\ 2\}$, Range: $\left]-\infty,\ -\dfrac{5}{4}\right]\cup\left]2,\ +\infty\right[$

(b)



Graph Func :X=

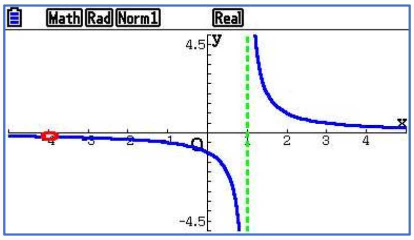
$Y1 = \frac{x+4}{x^2+3x-4}$ [—]

X2 = 1 [----]

X3 : [—]

X4 : [—]

Y5 :

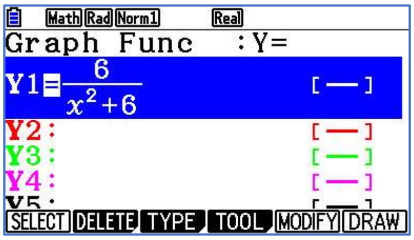


The function $f(x) = \frac{x+4}{x^2+3x-4}$ can be written in the form $f(x) = \frac{x+4}{(x+4)(x-1)}$.

If $x \neq -4$, then $f(x) = \frac{1}{x-1}$ and it would assume the value of $-\frac{1}{5}$ if x could be equal to –4. Therefore:

Domain: $\mathbb{R} - \{-4, 1\}$, Range: $\left]-\infty, -\frac{1}{5}\right[\cup \left]-\frac{1}{5}, 0\right[\cup \left]0, +\infty\right[$

(c)



Graph Func :Y=

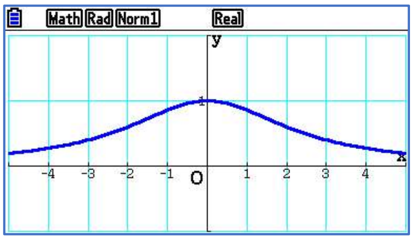
$Y1 = \frac{6}{x^2+6}$ [—]

Y2 : [—]

Y3 : [—]

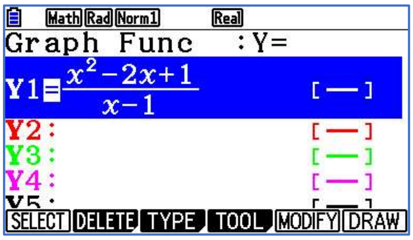
Y4 : [—]

Y5 :



Domain: $\mathbb{R}$, Range: $]0, 1]$

(d)



Graph Func :Y=

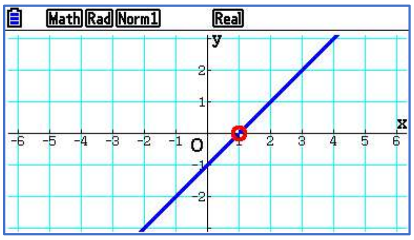
$Y1 = \frac{x^2-2x+1}{x-1}$ [—]

Y2 : [—]

Y3 : [—]

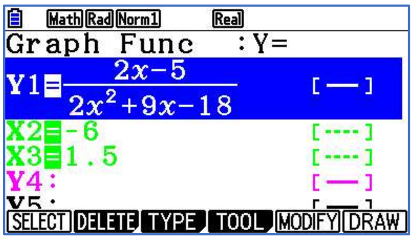
Y4 : [—]

Y5 :



Domain: $\mathbb{R} - \{1\}$, Range: $\mathbb{R} - \{0\}$

3. (a)



Graph Func :Y=

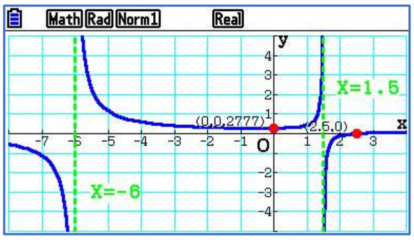
$Y1 = \frac{2x-5}{2x^2+9x-18}$ [—]

X2 = -6 [----]

X3 = 1.5 [----]

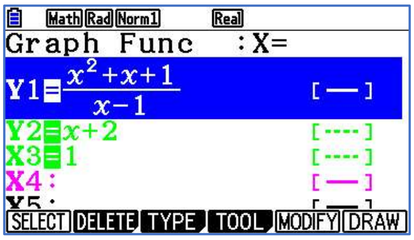
Y4 : [—]

Y5 :



The x-intercept: $\left(\frac{5}{2},\ 0\right)$, the y-intercept: $\left(0,\ \frac{5}{18}\right)$, vertical asymptotes: $x=-6$ and $x=\frac{3}{2}$, horizontal asymptote: $y=0$.

(b)



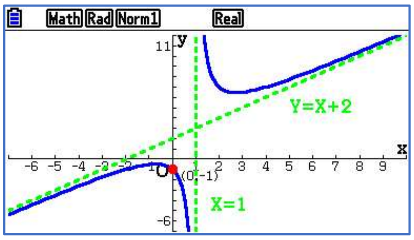
Graph Func :X=

$Y1 = \dfrac{x^2+x+1}{x-1}$ [—]

$Y2 = x+2$ [----]

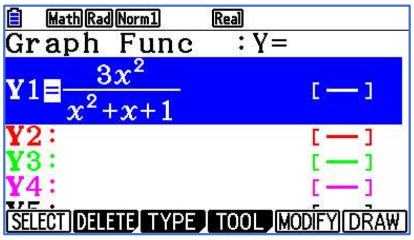
$X3 = 1$ [----]

X4: [—]



The x-intercept: none, the y-intercept: $(0,\ -1)$, vertical asymptote: $x=1$, oblique asymptote: $y=x+2$

(c)



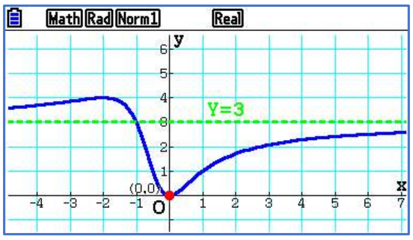
Graph Func :Y=

$Y1 = \dfrac{3x^2}{x^2+x+1}$ [—]

Y2: [—]

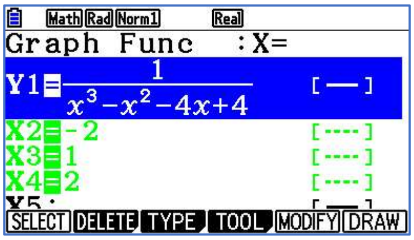
Y3: [—]

Y4: [—]



The x-intercept and the y-intercept: $(0,\ 0)$, horizontal asymptote: $y=3$

(d)



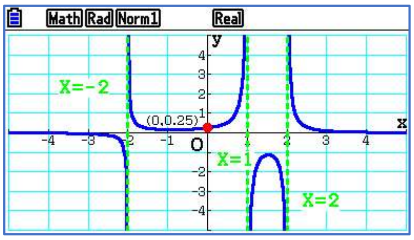
Graph Func :X=

$Y1 = \dfrac{1}{x^3-x^2-4x+4}$ [—]

$X2 = -2$ [----]

$X3 = 1$ [----]

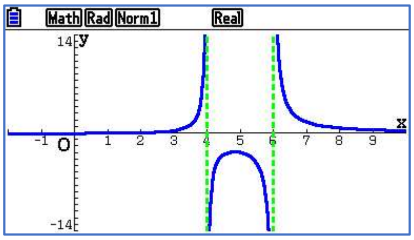
$X4 = 2$ [----]



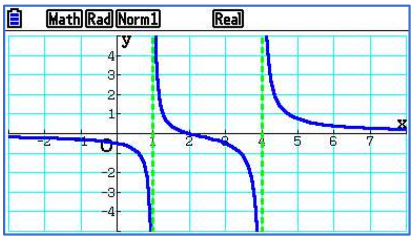
The x-intercept: none, the y-intercept: $\left(0,\ \frac{1}{4}\right)$, vertical asymptotes: $x=-2, x=1$ and $x=2$, horizontal asymptote: $y=0$

4. The function $y=\dfrac{x-a}{(x-b)(x-c)}$:

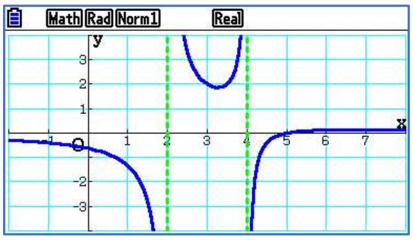
(a) for $a<b<c$ (say, $a=1,\ b=4,\ c=6$)



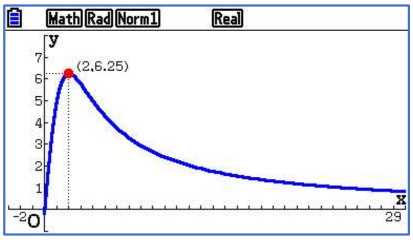
(b) for $b<a<c$ (say, $a=2,\ b=1,\ c=4$)



(c) for $b<c<a$ (say, $a=5,\ b=2,\ c=4$)



5. **(a)** **(b)**



The highest concentration occurs at $t=2$ minutes and is equal to 6.25 mg/L.

(c) Concentration of the drug in patient's bloodstream eventually decreases to zero.

(d) $C(t)<0.5 \Leftrightarrow \dfrac{25t}{t^2+4}<0.5$

$$25t<0.5\left(t^2+4\right)$$

$0.5t^2-25t+2>0$ and by the quadratic formula:

$$t=\frac{25-\sqrt{25^2-4(0.5)(2)}}{2(0.5)}=0.080 \text{ or } t=\frac{25+\sqrt{25^2-4(0.5)(2)}}{2(0.5)}=49.9$$

Therefore, it takes 50 minutes for the concentration of the drug in patient's bloodstream to drop below 0.5 mg/L.

Exercise 2.5

1. (a) $\sqrt{x+6}+2x=9$

$\sqrt{x+6}=9-2x$

Squaring both sides of the equation we get

$x+6=81-36x+4x^2 \Leftrightarrow 4x^2-37x+75=0$

$x=\dfrac{37\pm\sqrt{(-37)^2-4(4)(75)}}{2(4)} \Rightarrow x=3 \text{ or } x=6.25$

When $x=3,\ \sqrt{3+6}+2(3)=3+6=9$

When $x=6.25,\ \sqrt{6.25+6}+2(6.25)=3.5+12.5=16\neq 9$

The only solution is $x=3$

(b) $\sqrt{x+7}=x-5$

Squaring both sides of the equation we get

$x+7=x^2-10x+25 \Leftrightarrow x^2-11x+18=0$

Factorising: $(x-2)(x-9)=0 \Leftrightarrow x=2 \text{ or } x=9$

When $x=2,\ \sqrt{2+7}+5=3+5=8\neq 2$

When $x=9,\ \sqrt{9+7}+5=9$

The only solution is $x=9$

(c) $\sqrt{7x+14}=x+2$

Squaring both sides of the equation we get

$7x+14=x^2+4x+4 \Leftrightarrow x^2-3x-10=0$

Factorising: $(x-5)(x+2)=0 \Leftrightarrow x=5 \text{ or } x=-2$

When $x=-2,\ \sqrt{7(-2)+14}-2=-2$

When $x=5,\ \sqrt{7(5)+14}-2=7-2=5$

The solutions are $x=-2 \text{ or } x=5$

(d) $\sqrt{2x+3}-\sqrt{x-2}=2$

$\sqrt{2x+3}=2+\sqrt{x-2}$

Squaring both sides of the equation we get

$2x+3=4+4\sqrt{x-2}+x-2 \Rightarrow 4\sqrt{x-2}=x+1$

Squaring again we get $16x-32=x^2+2x+1 \Leftrightarrow x^2-14x+33=0$

Factorising: $(x-11)(x-3)=0 \Leftrightarrow x=11$ or $x=3$

When $x=11$, $\sqrt{2(11)+3}-\sqrt{11-2}=5-3=2$

When $x=3$, $\sqrt{2(3)+3}-\sqrt{3-2}=3-1=2$

The solutions are $x=3$ or $x=11$

(e) $\frac{5}{x+4}-\frac{4}{x}=\frac{21}{5x+20} \Leftrightarrow \frac{5}{x+4}-\frac{4}{x}=\frac{21}{5(x+4)}$

Multiplying by the least common denominator $5x(x+4)$ we get

$25x-20(x+4)=21x \Leftrightarrow 25x-20x-80=21x \Leftrightarrow -16x=80$

Therefore, $x=-5$

Check: $\frac{5}{-5+4}-\frac{4}{-5}=-\frac{5}{1}+\frac{4}{5}=-\frac{21}{5}$ and $\frac{21}{5(-5)+20}=\frac{21}{-5}=-\frac{21}{5}$

The solution is $x=-5$

(f) $\frac{x+1}{2x+3}=\frac{5x-1}{7x+3}$

Multiplying by the least common denominator $(2x+3)(7x+3)$ we get

$(x+1)(7x+3)=(5x-1)(2x+3) \Leftrightarrow 7x^2+10x+3=10x^2+13x-3 \Leftrightarrow 3x^2+3x-6=0$

Factorising we get $3(x^2+x-2)=0 \Leftrightarrow (x+2)(x-1)=0 \Leftrightarrow x=-2$ or $x=1$

When $x=-2$, $\frac{-2+1}{2(-2)+3}=\frac{-1}{-1}=1$ and $\frac{5(-2)-1}{7(-2)+3}=\frac{-11}{-11}=1$

When $x=1$, $\frac{1+1}{2(1)+3}=\frac{2}{5}$ and $\frac{5(1)-1}{7(1)+3}=\frac{4}{10}=\frac{2}{5}$

The solutions are $x=-2$ or $x=1$

(g) $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{x+4}$

Multiplying by the least common denominator $x(x+1)(x+4)$ we get

$(x+1)(x+4)-x(x+4)=x(x+1)\Leftrightarrow x^2+5x+4-x^2-4x=x^2+x\Leftrightarrow x^2-4=0$

Therefore, $(x-2)(x+2)=0\Leftrightarrow x=-2$ or $x=2$

When $x=-2$, $\frac{1}{-2}-\frac{1}{-2+1}=-\frac{1}{2}\ +1=\frac{1}{2}$ and $\frac{1}{-2+4}=\frac{1}{2}$

When $x=2$, $\frac{1}{2}-\frac{1}{2+1}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$ and $\frac{1}{2+4}=\frac{1}{6}$

The solutions are $x=-2$ or $x=2$

(h) $\frac{2x}{1-x^2}+\frac{1}{x+1}=2$

Multiplying by the least common denominator $1-x^2$ we get

$2x+1-x=2-2x^2\Leftrightarrow 2x^2+x-1=0$

Factorising, $(2x-1)(x+1)=0\Leftrightarrow x=\frac{1}{2}$ or $x=-1$

When $x=\frac{1}{2}$, $\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}+\frac{1}{\left(\frac{1}{2}\right)+1}=\frac{1}{\frac{3}{4}}+\frac{1}{\frac{3}{2}}=\frac{4}{3}+\frac{2}{3}=2$

When $x=-1$ both denominators are equal to 0 so $x=-1$ is not a solution.

The only solution is $x=\frac{1}{2}$

(i) $x^4-2x^2-15=0\Leftrightarrow\left(x^2\right)^2-2x^2-15=0$

Let $t=x^2$, $t\geq 0$. Then $2t^2-2t-15=0$

Factorising, $(t+3)(t-5)=0\Leftrightarrow t=-3<0$ or $t=5$. Now, $x^2=5\Leftrightarrow x=\pm\sqrt{5}$

The solutions are $x=-\sqrt{5}$ or $x=\sqrt{5}$

(j) $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 15 = 0 \Leftrightarrow 2\left(x^{\frac{1}{3}}\right)^2 - x^{\frac{1}{3}} - 15 = 0$

Let $t = x^{\frac{1}{3}}$. Then $2t^2 - t - 15 = 0$

Factorising, $(2t+5)(t-3) = 0 \Leftrightarrow t = -\frac{5}{2}$ or $t = 3$

Now, $x^{\frac{1}{3}} = -\frac{5}{2} \Leftrightarrow x = -\frac{125}{8}$ or $x^{\frac{1}{3}} = 3 \Leftrightarrow x = 27$

The solutions are $x = -\frac{125}{8}$ or $x = 27$

(k) $x^6 - 35x^3 + 256 = 0 \Leftrightarrow (x^3)^2 - 35x^3 + 256 = 0$

Let $t = x^3$. Then $t^2 - 35t + 216 = 0$

Factorising, $(t-8)(t-27) = 0 \Leftrightarrow t = 8$ or $t = 27$
Now, $x^3 = 8 \Leftrightarrow x = 2$ or $x^3 = 27 \Leftrightarrow x = 3$

When $x = 2$, $x^3 = 8$ and $8^2 - 35(8) + 216 = 64 - 280 + 216 = 0$

When $x = 3$, $x^3 = 27$ and $27^2 - 35(27) + 216 = 729 - 945 + 216 = 0$

The solutions are $x = 2$ or $x = 3$

(l) $5x^{-2} - x^{-1} - 2 = 0 \Leftrightarrow 5(x^{-1})^2 - x^{-1} - 2 = 0$

Let $t = x^{-1}, t \neq 0$. Then $5t^2 - t - 2 = 0$

The discriminant, $\Delta = (-1)^2 - 4(5)(-2) = 1 + 40 = 41$.
Now, $t = \frac{1-\sqrt{41}}{10}$ or $t = \frac{1+\sqrt{41}}{10}$.

When

$$x = t^{-1} = \frac{10}{1-\sqrt{41}} = \frac{10(1+\sqrt{41})}{(1-\sqrt{41})(1+\sqrt{41})} = \frac{10(1+\sqrt{41})}{1-41} = -\frac{10(1+\sqrt{41})}{40} = -\frac{1+\sqrt{41}}{4} \text{ or}$$

$$x = t^{-1} = \frac{10}{1+\sqrt{41}} = \frac{10(1-\sqrt{41})}{(1+\sqrt{41})(1-\sqrt{41})} = \frac{10(1-\sqrt{41})}{1-41} = -\frac{10(1-\sqrt{41})}{40} = -\frac{1-\sqrt{41}}{4}$$

Since x can be any real number except 0, the solutions are
$x = -\frac{1+\sqrt{41}}{4}$ or $x = -\frac{1-\sqrt{41}}{4}$

(m) $|3x+4|=8 \Leftrightarrow 3x+4=-8 \text{ or } 3x+4=8$

Now:

$3x=-12 \text{ or} 3x=4$

$x=-4 \text{ or } x=\dfrac{4}{3}$

(n) $|x+6|=|3x-24|$

Since both sides of the equation are non-negative:

$(x+6)^2=(3x-24)^2$

$x^2+12x+36=9x^2-144x+576 \Leftrightarrow 8x^2-156x+540=0 \Leftrightarrow 2x^2-39x+135=0$

The discriminant, $\Delta=(-39)^2-4(2)(135)=1521-1080=441$

Now, $x=\dfrac{39-\sqrt{441}}{2(2)}=\dfrac{39-21}{4}=\dfrac{18}{4}=\dfrac{9}{2}$ or $x=\dfrac{39+\sqrt{441}}{2(2)}=\dfrac{39+21}{4}=15$

The solutions are: $x=\dfrac{9}{2}$ or $x=15$

(o) $|5x+1|=2x$

When $5x+1\geq 0 \Leftrightarrow x\geq -\dfrac{1}{5}$, $5x+1=2x \Leftrightarrow 3x=-1 \Leftrightarrow x=-\dfrac{1}{3}<-\dfrac{1}{5}$

When $5x+1<0 \Leftrightarrow x<-\dfrac{1}{5}$, $-(5x+1)=2x \Leftrightarrow -5x-1=2x \Leftrightarrow -7x=1 \Leftrightarrow x=-\dfrac{1}{7}>-\dfrac{1}{5}$

Therefore, there is no solution.

(p) $|x-1|+|x|=3 \Leftrightarrow |x-1|=3-|x|$

Squaring both sides:

$(x-1)^2=9-6|x|+x^2 \Leftrightarrow 6|x|=2x+8 \Leftrightarrow 3|x|=x+4$

When $x\geq 0$: $3x=x+4 \Leftrightarrow 2x=4 \Leftrightarrow x=2$

When $x<0$: $-3x=x+4 \Leftrightarrow -4x=4 \Leftrightarrow x=-1$

The solutions are $x=-1$ or $x=2$

(q) $\left|\dfrac{x+1}{x-1}\right| = 3 \Leftrightarrow \dfrac{x+1}{x-1} = -3$ or $\dfrac{x+1}{x-1} = 3$ and $x \neq 0$

$x+1 = -3(x-1)$ or $x+1 = 3(x-1)$

$x+1 = -3x+3$ or $x+1 = 3x-3$

$x = \dfrac{1}{2}$ or $x = 2$

When $x = \dfrac{1}{2}$, $\left|\dfrac{\frac{1}{2}+1}{\frac{1}{2}-1}\right| = \left|\dfrac{\frac{3}{2}}{-\frac{1}{2}}\right| = |-3| = 3$

When $x = 2$, $\left|\dfrac{2+1}{2-1}\right| = \left|\dfrac{3}{1}\right| = 3$

The solutions are: $x = \dfrac{1}{2}$ or $x = 2$

(r) In the equation $\sqrt{x} - \dfrac{6}{\sqrt{x}} = 1$ clearly $x > 0$

Multiplying both sides of the equation by $\sqrt{x}$ we get

$x - 6 = \sqrt{x} \Leftrightarrow x - \sqrt{x} - 6 = 0$

Let $t = \sqrt{x},\ t > 0$. Then $t^2 - t - 6 = 0$.

Factorising, $(t+2)(t-3) = 0 \Leftrightarrow t = -2 < 0$ or $t = 3$. Now, $\sqrt{x} = 3 \Leftrightarrow x = 9$.

The solution is $x = 9$.

(s) $\sqrt{4-x} - \sqrt{6+x} = \sqrt{14+2x}$

$\sqrt{4-x} = \sqrt{14+2x} + \sqrt{6+x}$

Squaring both sides of the equation we get

$4 - x = 14 + 2x + 2\sqrt{(14+2x)(6+x)} + 6 + x$

$2\sqrt{(14+2x)(6+x)} = -16 - 4x \Leftrightarrow \sqrt{(14+2x)(6+x)} = -8 - 2x$

Squaring again we get
$84 + 26x + 2x^2 = 64 + 32x + 4x^2 \Leftrightarrow 2x^2 + 6x - 20 = 0 \Leftrightarrow x^2 + 3x - 10 = 0$

Factorising: $(x+5)(x-2) = 0 \Leftrightarrow x = -5$ or $x = 2$

When $x=-5$, $\sqrt{4-(-5)}-\sqrt{6+(-5)}=\sqrt{9}-\sqrt{1}=2$ and $\sqrt{14+2(-5)}=2$

When $x=2$, $\sqrt{4-2}-\sqrt{6+2}=\sqrt{2}-2\sqrt{2}=-\sqrt{2}$ and $\sqrt{14+2(2)}=\sqrt{18}=3\sqrt{2}$

The solution is $x=-5$

(t) In the equation $\dfrac{6}{x^2+1}=\dfrac{1}{x^2}+\dfrac{10}{x^2+4}$, $x\neq 0$

Multiplying by the least common denominator $x^2\left(x^2+1\right)\left(x^2+4\right)$ we get

$6x^2\left(x^2+4\right)=\left(x^2+1\right)\left(x+4\right)+10x^2\left(x^2+1\right)$

Therefore, $6x^4+24x^2=x^4+5x^2+4+10x^4+10x^2 \Leftrightarrow 5x^4-9x^2+4=0$

Let $t=x^2$, $t\geq 0$. Then $5t^2-9t-4=0$.

Factorising, $(5t-4)(t-1)=0 \Leftrightarrow t=\dfrac{4}{5}$ or $t=1$

Now, $x^2=\dfrac{4}{5} \Leftrightarrow x=\pm\dfrac{2}{\sqrt{5}}=\pm\dfrac{2\sqrt{5}}{5}$ or $x^2=1 \Leftrightarrow x=\pm 1$

The solutions are $x=\pm\dfrac{2\sqrt{5}}{5}$ or $x=\pm 1$

(u) $x-\sqrt{x+10}=0 \Leftrightarrow x=\sqrt{x+10}$

Squaring both sides of the equation we get

$x^2=x+10 \Leftrightarrow x^2-x-10=0$

The discriminant $\Delta=(-1)^2-4(1)(-10)=1+40=41$

Now, $x=\dfrac{1-\sqrt{41}}{2}<0$ or $x=\dfrac{1+\sqrt{41}}{2}$

Since $x=\dfrac{1-\sqrt{41}}{2}<0$ and $\sqrt{x+10}=\sqrt{\dfrac{1-\sqrt{41}}{2}+10}>0$,

the only solution is $x=\dfrac{1+\sqrt{41}}{2}$

(v) In the equation $6x-37\sqrt{x}+56=0$ clearly $x\geq 0$

Let $t=\sqrt{x},\ t\geq 0$. Then $6t^2-37t+56=0$

The discriminant, $\Delta=(-37)^2-4(6)(56)=1369-1344=25$

Now, $t=\dfrac{37-\sqrt{25}}{2(6)}=\dfrac{32}{12}=\dfrac{8}{3}$ or $t=\dfrac{37+\sqrt{25}}{2(6)}=\dfrac{42}{12}=\dfrac{7}{2}$

Therefore, $x=t^2=\left(\dfrac{8}{3}\right)^2=\dfrac{64}{9}$ or $x=t^2=\left(\dfrac{7}{2}\right)^2=\dfrac{49}{4}$

The solutions are $x=\dfrac{64}{9}$ or $x=\dfrac{49}{4}$

2. **(a)** $3x^2-4<4x$

$3x^2-4x-4<0$

$(3x+2)(x-2)<0$

		$-\frac{2}{3}$		2	
$3x+2$	–	0	+	+	+
$x-2$	–	–	–	0	+
$(3x+2)(x-2)$	+	0	–	0	+

Answer: $x\in\left]-\dfrac{2}{3},\ 2\right[$

(b) $\dfrac{2x-1}{x+2}\geq 1$

$\dfrac{2x-1}{x+2}-1\geq 0 \Leftrightarrow \dfrac{2x-1-x-2}{x+2}\geq 0 \Leftrightarrow \dfrac{x-3}{x+2}\geq 0$

		-2		3	
$x+2$	–	0	+	+	+
$x-3$	–	–	–	0	+
$\dfrac{x-3}{x+2}$	+	X	–	0	+

Answer: $x\in\left]-\infty,\ -2\right[\cup\left[3,\ +\infty\right[$

(c) $2x^2+8x\le 120 \Leftrightarrow x^2+4x-60\le 0 \Leftrightarrow (x+10)(x-6)\le 0$

		-10		6	
$x+10$	$-$	0	$+$	$+$	$+$
$x-6$	$-$	$-$	$-$	0	$+$
$(x+10)(x-6)$	$+$	0	$-$	0	$+$

Answer: $x\in[-10,\ 6]$

(d) $|1-4x|>7$

We can write $1-4x<-7$ or $1-4x>7$. Then $-4x<-8$ or $-4x>6$ and $x>2$ or $x<-\frac{3}{2}$. The answer is $x\in\left]-\infty,\ -\frac{3}{2}\right[\cup\left]2,\ +\infty\right[$

(e) $|x-3|>|x-14|$

Since the expressions on both sides of the inequality are non-negative,

$(x-3)^2>(x-14)^2$

$(x-3)^2-(x-14)^2>0$

$(x-3-x+14)(x-3+x-14)>0$

$11(2x-17)>0\Leftrightarrow x>\frac{17}{2}$

The answer is $x\in\left]\frac{17}{2},\ +\infty\right[$

(f) $\left|\frac{x^2-4}{x}\right|\le 3\Leftrightarrow -3\le\frac{x^2-4}{x}\le 3$ and $x\ne 0$

$\frac{x^2-4}{x}\ge -3$ and $\frac{x^2-4}{x}\le 3$

$\frac{x^2-4}{x}+3\ge 0$ and $\frac{x^2-4}{x}-3\le 0$

$\frac{x^2+3x-4}{x}\ge 0$ and $\frac{x^2-3x-4}{x}\le 0$

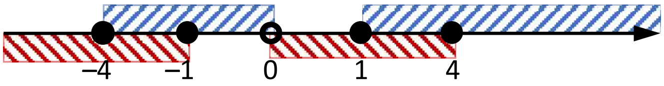
$\frac{(x+4)(x-1)}{x}\ge 0$ and $\frac{(x-4)(x+1)}{x}\le 0$

		–4		0		1	
$x+4$	–	0	+	+	+	+	+
x	–	–	–	X	+	+	+
$x-1$	–	–	–	–	–	0	+
$\dfrac{(x+4)(x-1)}{x}$	–	0	+	X	–	0	+

$x\in[-4,\ 0[\cup[1,\ +\infty[$

		–1		0		4	
$x+1$	–	0	+	+	+	+	+
x	–	–	–	X	+	+	+
$x-4$	–	–	–	–	–	0	+
$\dfrac{(x-4)(x+1)}{x}$	–	0	+	X	–	0	+

$x\in]-\infty,\ -1]\cup]0,\ 4]$



$x\in[-4,\ -1]\cup[1,\ 4]$

(g) $\dfrac{x}{x-2}>\dfrac{1}{x+1}\Leftrightarrow\dfrac{x}{x-2}-\dfrac{1}{x+1}>0$

$$\frac{x(x+1)-(x-2)}{(x-2)(x+1)}>0$$

$$\frac{x^2+x-x+2}{(x-2)(x+1)}>0\Leftrightarrow\frac{x^2+2}{(x-2)(x+1)}>0$$

Since $x^2+2>0$ for all real values of x, then the inequality holds when $(x-2)(x+1)>0$

		–1		2	
$x+1$	–	X	+	+	+
$x-2$	–	–	–	X	+
$(x+1)(x-2)$	+	X	–	X	+

$x\in]-\infty,\ -1[\cup]2,\ +\infty[$

(h) $\dfrac{4x-1}{x^2-2x-3}<3\Leftrightarrow\dfrac{4x-1}{x^2-2x-3}-3<0$

$$\frac{4x-1-3x^2+6x+9}{x^2-2x-3}<0\Leftrightarrow\frac{-3x^2+10x+8}{(x+1)(x-3)}<0\Leftrightarrow\frac{3x^2-10x-8}{(x+1)(x-3)}>0$$

Factorising the numerator, we get $\dfrac{(3x+2)(x-4)}{(x+1)(x-3)}>0$

		-1		$-\frac{2}{3}$		3		4	
$x+1$	$-$	X	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$3x+2$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	X	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$\frac{(3x+2)(x-4)}{(x+1)(x-3)}$	$+$	X	$-$	0	$+$	X	$-$	0	$+$

$$x \in]-\infty, -1[\cup \left]-\frac{2}{3}, 3\right[\cup]4, +\infty[$$

3. The function $f(x) = 3kx^2 - (k+3)x + k - 2$ has no real zeros if $k \neq 0$ and the discriminant $\Delta < 0$ (note that for $k = 0$ the function has one real zero).

$\Delta = [-(k+3)]^2 - 4(3k)(k-2) = k^2 + 6k + 9 - 12k^2 + 24k = -11k^2 + 30k + 9$

Now $-11k^2 + 30k + 9 < 0 \Leftrightarrow 11k^2 - 30k - 9 > 0 \Leftrightarrow (11k+3)(k-1) > 0$

		$-\frac{3}{11}$		3	
$11k+3$	$-$	0	$+$	$+$	$+$
$k-3$	$-$	$-$	$-$	0	$+$
$(11k+3)(k-3)$	$+$	0	$-$	0	$+$

$$k \in \left]-\infty, -\frac{3}{11}\right[\cup]3, +\infty[$$

4. **(a)** The equation $px^2 - 3x + 1 = 0$ has one real solution when $p = 0$ or $p \neq 0$ and the discriminant $\Delta = 0$.

$\Delta = (-3)^2 - 4p(1) = 9 - 4p$

Now $9 - 4p = 0 \Leftrightarrow p = \frac{9}{4}$. Therefore, one real solution when $p = 0$ or $p = \frac{9}{4}$

(b) The equation has two real solutions when $p \neq 0$ and the discriminant $\Delta > 0$.

$9 - 4p > 0 \Leftrightarrow p < \frac{9}{4}$ but also $p \neq 0$. Therefore, $p \in]-\infty, 0[\cup \left]0, \frac{9}{4}\right[$

(c) The equation has no solution when $p \neq 0$ and the discriminant $\Delta < 0$.

$9 - 4p < 0 \Leftrightarrow p > \frac{9}{4}$. Therefore, $p \in \left]\frac{9}{4}, +\infty\right[$

5. The function $f(x)=x^2+x(k-1)+k^2>0$ for all real values of x when $\Delta<0$.

$\Delta=(k-1)^2-4k^2=k^2-2k+1-4k^2=-3k^2-2k+1$

Now $-3k^2-2k+1<0 \Leftrightarrow 3k^2+2k-1>0 \Leftrightarrow (3k-1)(k+1)>0$

		-1		$\frac{1}{3}$	
$k+1$	$-$	0	$+$	$+$	$+$
$3k-1$	$-$	$-$	$-$	0	$+$
$(k+1)(3k-1)$	$+$	0	$-$	0	$+$

$$k\in\left]-\infty,-1\right[\cup\left]\frac{1}{3},+\infty\right[$$

6. **(a)** Assume $m>n>0$, prove $m+\frac{1}{n}\geq 2$.

The inequality is equivalent to $mn-2n+1\geq 0$.

For $m\geq n$ we have $mn\geq n^2$ and $mn-2n+1\geq n^2-2n+1=(n-1)^2$ which is true for all real values of n.

(b) Notice that $(m+n)\left(\frac{1}{m}+\frac{1}{n}\right)=(m+n)\frac{m+n}{mn}=\frac{(m+n)^2}{mn}$. We need to show that $\frac{(m+n)^2}{mn}\geq 4$ for all real numbers m and n such that $m>0$ and $n>0$.

Since $m>0$ and $n>0$:

$(m+n)^2\geq 4mn \Leftrightarrow m^2+2mn+n^2\geq 4mn \Leftrightarrow m^2-2mn+n^2\geq 0 \Leftrightarrow (m-n)^2\geq 0$ which is true for all real positive values of m and n.

Alternative solution:

$$(m-n)^2\geq 0 \Leftrightarrow m^2-2mn+n^2\geq 0 \Leftrightarrow m^2-2mn+n^2+4mn\geq 4mn$$

$$\Leftrightarrow (m+n)^2\geq 4mn \Leftrightarrow \frac{(m+n)^2}{mn}\geq 4$$

Now, $\frac{(m+n)(m+n)}{mn}\geq 4 \Leftrightarrow (m+n)\frac{m+n}{mn}\geq 4 \Leftrightarrow (m+n)\left(\frac{1}{m}+\frac{1}{n}\right)\geq 4$

7. $(x^2+x)^2 = 5x^2+5x-6$

$(x^2+x)^2 = 5(x^2+x)-6$

$(x^2+x)^2 - 5(x^2+x)+6=0$

Let $x^2+x=t$. We have:

$t^2-5t+6=0 \Leftrightarrow (t-3)(t-2)=0 \Leftrightarrow t=3$ or $t=2$

Now, $x^2+x=3$ or $x^2+x=2$

$x^2+x-3=0$ or $x^2+x-2=0$

$x=\dfrac{-1\pm\sqrt{1-4(1)(-3)}}{2(1)}$ or $(x+2)(x-1)=0$

$x=\dfrac{-1\pm\sqrt{13}}{2}$ or $x=-2$ or $x=1$

8. Assumption: $a, b, c>0$ and $a\neq b\neq c$

Show $(a+b+c)^2<3(a^2+b^2+c^2)$

If $a, b, c>0$ and $a\neq b\neq c$, then $(a-b)^2+(a-c)^2+(b-c)^2>0$. We have:

$a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2>0$, so $2ab+2bc+2ac<2a^2+2b^2+2c^2$

Adding $a^2+b^2+c^2$ to both sides of the last inequality we get

$a^2+b^2+c^2+2ab+2bc+2ac<3a^2+3b^2+3c^2$

which is equivalent to $(a+b+c)^2<3(a^2+b^2+c^2)$

9. **(a)** $\left|\dfrac{2x-3}{x}\right|<1,\ x\neq 0$. This inequality is equivalent to $|2x-3|<|x|$.

Now, squaring both sides of the inequality we get:

$(2x-3)^2<x^2$

$4x^2-12x+9<x^2$

$3x^2-12x+9<0 \Leftrightarrow x^2-4x+3<0 \Leftrightarrow (x-3)(x-1)<0$

		1		3	
$x-1$	–	0	+	+	+
$x-3$	–	–	–	0	+
$(x-3)(x-1)$	+	0	–	0	+

$x \in]1, 3[$

(b) $\dfrac{3}{x-1} - \dfrac{2}{x+1} < 1$, where $x \neq -1$ and $x \neq 1$

$$\frac{3x+3-2x+2}{(x-1)(x+1)} < 1 \Leftrightarrow \frac{x+5}{(x-1)(x+1)} - 1 < 0 \Leftrightarrow \frac{x+5}{(x-1)(x+1)} - 1 < 0$$

$$\frac{x+5-x^2+1}{(x-1)(x+1)} < 0 \Leftrightarrow \frac{-x^2+x+6}{(x-1)(x+1)} < 0 \Leftrightarrow \frac{x^2-x-6}{(x-1)(x+1)} > 0$$

$$\frac{(x-3)(x-2)}{(x-1)(x+1)} > 0$$

		–2		–1		1		3	
$x+2$	–	0	+	+	+	+	+	+	+
$x+1$	–	–	–	X	+	+	+	+	+
$x-1$	–	–	–	–	–	X	+	+	+
$x-3$	–	–	–	–	–	–	–	0	+
$\dfrac{(x+2)(x-3)}{(x+1)(x-1)}$	+	0	–	X	+	X	–	0	+

$x \in]-\infty, -2[\cup]-1, 1[\cup]3, +\infty[$

10. Since both sides of the inequality $|a+b| \leq |a|+|b|$ are nonnegative, we can square both sides to get $a^2+2ab+b^2 \leq a^2+2|ab|+b^2 \Leftrightarrow 2ab \leq 2|ab| \Leftrightarrow |ab| \geq ab$.

If a and b have the same sign, then $|ab| = ab$ and the inequality above is satisfied.

If a and b have opposite signs, then $|ab| = -ab > 0$ and $ab < 0 \Rightarrow |ab| > ab$. Therefore, $|ab| \geq ab$ for any real values of a and b which means that $|a+b| \leq |a|+|b|$ for any real values of a and b.

Exercise 2.6

1. **(a)** $\dfrac{5x+1}{x^2+x-2} \equiv \dfrac{5x+1}{(x-1)(x+2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+2}$

Multiply both sides of the identity by $x-1$. We get $\dfrac{5x+1}{x+2} \equiv A + \dfrac{B(x-1)}{x+2}$

Now, let $x=1$. Then $\dfrac{5(1)+1}{1+2} = A + \dfrac{B(1-1)}{1+2} \;\; \Rightarrow \;\; A = \dfrac{6}{3} = 2$

Multiply both sides of the identity by $x+2$. We get $\dfrac{5x+1}{x-1} \equiv \dfrac{A(x+2)}{x-1} + B$

Now, let $x=-2$. Then $\dfrac{5(-2)+1}{-2-1} = \dfrac{A(-2+2)}{-2-1} + B \Rightarrow B = \dfrac{-9}{-3} = 3$

Therefore $\dfrac{5x+1}{x^2+x-2} = \dfrac{2}{x-1} + \dfrac{3}{x+2}$

(b) $\dfrac{x+4}{x^2-2x} \equiv \dfrac{x+4}{x(x-2)} \equiv \dfrac{A}{x} + \dfrac{B}{x-2}$

Multiply both sides of the identity by x. We get $\dfrac{x+4}{x-2} \equiv A + \dfrac{Bx}{x-2}$. Now, let $x=0$.

Then $\dfrac{0+4}{0-2} = A + \dfrac{B(0)}{0-2} \;\; \Rightarrow A = \dfrac{4}{-2} = -2$

Multiply both sides of the identity by $x-2$. We get $\dfrac{x+4}{x} \ldots \dfrac{A(x-2)}{x} + B$.

Now, let $x=2$. Then $\dfrac{2+4}{2} = \dfrac{A(2-2)}{2} + B \Rightarrow B = \dfrac{6}{2} = 3$

Therefore $\dfrac{x+4}{x^2-2x} = \dfrac{3}{x-2} - \dfrac{2}{x}$

(c) $\dfrac{x+2}{x^2+4x+3} \equiv \dfrac{x+2}{(x+1)(x+3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+3}$

Multiply both sides of the identity by $x+1$. We get $\dfrac{x+2}{x+3} \equiv A + \dfrac{B(x+1)}{x+3}$.

Now, let $x=-1$. Then $\dfrac{-1+2}{-1+3} \equiv A + \dfrac{B(-1+1)}{-1+3} \;\; \Rightarrow A = \dfrac{1}{2}$

Multiply both sides of the identity by $x+3$. We get $\dfrac{x+2}{x+1} \equiv \dfrac{A(x+3)}{x+1} + B$.

Now, let $x=-3$. Then $\frac{-3+2}{-3+1}\equiv\frac{A(-3+3)}{-3+1}+B \Rightarrow B=\frac{-1}{-2}=\frac{1}{2}$

Therefore $\frac{x+2}{x^2+4x+3}=\frac{1}{2x+2}+\frac{1}{2x+6}=\frac{1}{2(x+1)}+\frac{1}{2(x+3)}$

(d) $\frac{5x^2+20x+6}{x^3+2x^2+x}\equiv\frac{5x^2+20x+6}{x(x^2+2x+1)}\equiv\frac{5x^2+20x+6}{x(x+1)^2}\equiv\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

Multiply both sides of the identity by $x(x+1)^2$:
$5x^2+20x+6\equiv A(x+1)^2+Bx(x+1)+Cx$

Now, let $x=-1$. Then $5x^2+20x+6\equiv A(x+1)^2+Bx(x+1)+Cx \Rightarrow -C=-9 \Rightarrow C=9$

Let $x=0$. Then $5(0)^2+20(0)+6\equiv A(0+1)^2+B(0)(0+1)+C(0) \Rightarrow A=6$

Let $x=1$. Then $5(1)^2+20(1)+6=6(1+1)^2+B(1)(1+1)+9(1)$

$\Rightarrow 24+2B+9=31 \Rightarrow 2B=-2 \Rightarrow B=-1$

Therefore $\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}$

(e) $\frac{2x^2+x-12}{x^3+5x^2+6x}\equiv\frac{2x^2+x-12}{x(x^2+5x+6)}\equiv\frac{2x^2+x-12}{x(x+2)(x+3)}\equiv\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x+3}$

Multiply both sides of the identity by $x(x+2)(x+3)$:
$2x^2+x-12\equiv A(x+2)(x+3)+Bx(x+3)+Cx(x+2)$

Now, let $x=0$. Then $2(0)^2+0-12=A(0+2)(0+3)+B(0)(0+3)+C(0)(0+2)$
$\Rightarrow 6A=-12 \Rightarrow A=-2$

Let $x=-2$. Then
$2(-2)^2-2-12=A(-2+2)(-2+3)+B(-2)(-2+3)+C(-2)(-2+2)$
$\Rightarrow -2B=-6 \Rightarrow B=3$

Let $x=-3$. Then
$2(-3)^2-3-12=A(-3+2)(-3+3)+B(-3)(-3+3)+C(-3)(-3+2)$

$\Rightarrow 3C=3 \Rightarrow C=1$

Therefore, $\frac{2x^2+x-12}{x^3+5x^2+6x}=\frac{1}{x+3}+\frac{3}{x+2}-\frac{2}{x}$

(f) $\frac{4x^2+2x-1}{x^3+x^2} \equiv \frac{4x^2+2x-1}{x^2(x+1)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$

Multiply both sides of the identity by $x^2(x+1)$:
$4x^2+2x-1 \equiv Ax(x+1)+B(x+1)+Cx^2$

Now, let $x=0$. Then $4(0)^2+2(0)-1 = A(0)(0+1)+B(0+1)+C(0)^2 \Rightarrow B=-1$

Let $x=-1$. Then $4(-1)^2+2(-1)-1 = A(-1)(-1+1)+B(-1+1)+C(-1)^2 \Rightarrow C=1$

Let $x=1$. Then $4(1)^2+2(1)-1 \equiv A(1)(1+1)+(-1)(1+1)+1(1)^2$
$\Rightarrow 2A-1=5 \Rightarrow 2A=6 \Rightarrow A=3$

Therefore, $\frac{4x^2+2x-1}{x^3+x^2} = \frac{3}{x}-\frac{1}{x^2}+\frac{1}{x+1}$

(g) $\frac{3}{x^2+x-2} \equiv \frac{3}{(x+2)(x-1)} \equiv \frac{A}{x+2}+\frac{B}{x-1}$

Multiply both sides of the identity by $(x+2)(x-1)$:
$3 \equiv A(x-1)+B(x+2)$. Now, let $x=-2$
Then $3 \equiv A(-2-1)+B(-2+2) \Leftrightarrow 3=-3A \Rightarrow A=-1$

Now, let $x=1$. Then $3 = A(1-1)+B(1+2) \Leftrightarrow 3=3B \Rightarrow B=1$

Therefore, $\frac{3}{x^2+x-2} = \frac{1}{x-1}-\frac{1}{x+2}$

(h) $\frac{5-x}{2x^2+x-1} \equiv \frac{5-x}{(2x-1)(x+1)} \equiv \frac{A}{x+1}+\frac{B}{2x-1}$

Multiply both sides of the identity by $(2x-1)(x+1)$:
$5-x \equiv A(2x-1)+B(x+1)$. Now, let $x=\frac{1}{2}$. Then
$5-\frac{1}{2} = A\left[2\left(\frac{1}{2}\right)-1\right]+B\left(\frac{1}{2}+1\right) \Leftrightarrow \frac{9}{2}=\frac{3}{2}B \Rightarrow B=3$

Now, let $x=-1$. Then $5-(-1) = A[2(-1)-1]+B(-1+1) \Leftrightarrow 6=-3A \Rightarrow A=-2$

Therefore, $\frac{5-x}{2x^2+x-1} = \frac{3}{2x-1}-\frac{2}{x+1}$

(i) $\dfrac{3x+4}{(x+2)^2} \equiv \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$

Multiply both sides of the identity by $(x+2)^2$:

$3x+4 \equiv A(x+2)+B$. Now, let $x=-2$. Then $3(-2)+4 \equiv A(-2+2)+B \Rightarrow B=-2$

Now, let $x=0$. Then $3(0)+4 = A(0+2)-2 \Leftrightarrow 2A=6 \Rightarrow A=3$

Therefore, $\dfrac{3x+4}{(x+2)^2} \equiv \dfrac{3}{x+2} - \dfrac{2}{(x+2)^2}$

(j) $\dfrac{12}{x^4-x^3-2x^2} \equiv \dfrac{12}{x^2(x^2-x-2)} \equiv \dfrac{12}{x^2(x-2)(x+1)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} + \dfrac{D}{x+1}$

Multiply both sides of the identity by $x^2(x-2)(x+1)$:
$12 \equiv Ax(x-2)(x+1) + B(x-2)(x+1) + Cx^2(x+1) + Dx^2(x-2)$

Now, let $x=0$. Then
$12 = A(0)(0-2)(0+1) + B(0-2)(0+1) + C(0)^2(0+1) + D(0)^2(0-2) \Leftrightarrow -2B=12$
$\Rightarrow B=-6$

Let $x=-1$. Then
$12 = A(-1)(-1-2)(-1+1) + B(-1-2)(-1+1) + C(-1)^2(-1+1) + D(-1)^2(-1-2) \Leftrightarrow -3D=12$
$\Rightarrow D=-4$

Let $x=2$. Then
$12 = A(2)(2-2)(2+1) + B(2-2)(2+1) + C(2)^2(2+1) + D(2)^2(2-2) \Leftrightarrow 12C=12$
$\Rightarrow C=1$

Let $x=1$. Then $12 = A(1)(1-2)(1+1) - 6(1-2)(1+1) + (1)(1)^2(1+1) - 4(1)^2(1-2)$
$\Leftrightarrow 12 = -2A+12+2+4 = 12 \Leftrightarrow -2A=-6 \Rightarrow A=3$

Therefore, $\dfrac{12}{x^4-x^3-2x^2} = \dfrac{1}{x-2} - \dfrac{4}{x+1} + \dfrac{3}{x} - \dfrac{6}{x^2}$

(k) $\dfrac{2}{x^3+x} \equiv \dfrac{2}{x(x^2+1)} \equiv \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

Let's multiply both sides of the identity by $x(x^2+1)$:
$2 \equiv A(x^2+1) + (Bx+C)x \equiv (A+B)x^2 + Cx + A$

Now, $A=2$ and $C=0$ and $A+B=0 \Rightarrow B=-2$

Therefore, $\dfrac{2}{x^3+x}=\dfrac{2}{x}-\dfrac{2x}{x^2+1}$

(l) $\dfrac{x+2}{x^3+3x}\equiv\dfrac{x+2}{x\left(x^2+3\right)}\equiv\dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$

Multiply both sides of the identity by $x\left(x^2+3\right)$:

$x+2\equiv A\left(x^2+3\right)+\left(Bx+C\right)x\equiv\left(A+B\right)x^2+Cx+3A$

Now, $A=\dfrac{2}{3}$ and $C=1$ and $A+B=0\ \Rightarrow B=-\dfrac{2}{3}$

Therefore, $\dfrac{x+2}{x^3+3x}=\dfrac{2}{3x}+\dfrac{3-2x}{3\left(x^2+3\right)}$

(m) $\dfrac{3x+2}{x^3+6x}\equiv\dfrac{3x+2}{x\left(x^2+6\right)}\equiv\dfrac{A}{x}+\dfrac{Bx+C}{x^2+6}$

Multiply both sides of the identity by $x\left(x^2+6\right)$:

$3x+2\equiv A\left(x^2+6\right)+\left(Bx+C\right)x\equiv\left(A+B\right)x^2+Cx+6A$

Now, $A=\dfrac{1}{3}$ and $C=3$ and $A+B=0\ \Rightarrow B=-\dfrac{1}{3}$

Therefore, $\dfrac{3x+2}{x^3+6x}=\dfrac{1}{3x}+\dfrac{9-x}{3\left(x^2+6\right)}$

(n) $\dfrac{2x+3}{x^3+8x}\equiv\dfrac{2x+3}{x\left(x^2+8\right)}\equiv\dfrac{A}{x}+\dfrac{Bx+C}{x^2+8}$

Multiply both sides of the identity by $x\left(x^2+8\right)$:

$2x+3\equiv A\left(x^2+8\right)+\left(Bx+C\right)x\equiv\left(A+B\right)x^2+Cx+8A$

Now, $A=\dfrac{3}{8}$ and $C=2$ and $A+B=0\ \Rightarrow B=-\dfrac{3}{8}$

Therefore, $\dfrac{2x+3}{x^3+8x}=\dfrac{3}{8x}+\dfrac{16-3x}{8\left(x^2+8\right)}$

(o) $\frac{x+5}{x^3-5x^2+4x} \equiv \frac{x+5}{x(x^2-5x+4)} \equiv \frac{x+5}{x(x-1)(x-4)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-4}$

Let's multiply both sides of the identity by $x(x-1)(x-4)$:
$x+5 \equiv A(x-1)(x-4) + Bx(x-4) + Cx(x-1)$

Now, let $x=0$.

Then $0+5 \equiv A(0-1)(0-4) + B(0)(0-4) + C(0)(0-1) \Leftrightarrow 5 \equiv 4A \Rightarrow A = \frac{5}{4}$

Let $x=1$.
Then $1+5 \equiv A(1-1)(1-4) + B(1)(1-4) + C(1)(1-1) \Leftrightarrow -3B = 6 \Rightarrow B = -2$

Let $x=4$.

Then $4+5 \equiv A(4-1)(4-4) + B(4)(4-4) + C(4)(4-1) \Leftrightarrow 12C = 9 \Rightarrow C = \frac{3}{4}$

Therefore, $\frac{x+5}{x^3-5x^2+4x} = \frac{5}{4x} - \frac{2}{x-1} + \frac{3}{4(x-4)}$

Chapter 2 practice questions

1. $x^2-(a+3b)x+3ab=0$
This appears to be a factorable equation.

$(x-3b)(x-a)=0$

Therefore, the solutions are $x=3b$ or $x=a$

2. $\frac{3x-2}{5}+3 \geq \frac{4x-1}{3}$. Multiplying both sides of the inequality by 15 we get:
$3(3x-2)+3(15) \geq 5(4x-1) \Leftrightarrow 9x-6+45 \geq 20x-5 \Leftrightarrow -11x \geq -44 \Leftrightarrow x \leq 4$

3. If the vertex of the parabola $y=3x^2-8x+c$ is $\left(\frac{4}{3},\ -\frac{1}{3}\right)$, then $-\frac{1}{3}=3\left(\frac{4}{3}\right)^2-8\left(\frac{4}{3}\right)+c$.

Therefore, $-\frac{1}{3}=3\left(\frac{16}{9}\right)-\frac{32}{3}+c \Leftrightarrow -\frac{1}{3}=\frac{16}{3}-\frac{32}{3}+c \Rightarrow c=5$.

4. The quadratic function $f(x)=ax^2+bx+c$ can be expressed in the turning point form $f(x)=a(x-p)^2+q$, where p and q are coordinates of the vertex.
So, $f(x)=a(x-4)^2+6$.

Since the graph of the function passes through the point $(2,\ 4)$ then $4=a(2-4)^2+6$.

It follows that $4=4a+6$ and $a=-\frac{1}{2}$. Therefore,

$$f(x)=-\frac{1}{2}(x-4)^2+6=-\frac{1}{2}(x^2-8x+16)+6=-\frac{1}{2}x^2+4x-2.$$

We need to verify that $x=4+2\sqrt{3}$ is a zero of the quadratic function:

$$f(4+2\sqrt{3})=-\frac{1}{2}(4+2\sqrt{3}-4)^2+6=-\frac{1}{2}(12)+6=0,$$

so $x=4+2\sqrt{3}$ is the zero of the function. Therefore, $a=-\frac{1}{2},\ b=4,\ c=-2$

5. If the roots of the polynomial $f(x)=x^3+5x^2+px+q$ are $\omega,\ 2\omega$ and $\omega+3$ then:

$$\omega+2\omega+\omega+3=4\omega+3=-\frac{b}{a}=-5\Rightarrow\omega=-2$$

If $\omega=-2$ and $\omega+3=1$ are roots of the equation, then

$$f(-2)=(-2)^3+5(-2)^2-2p+q=12-2p+q=0$$
$$f(1)=1+5+p+q=6+p+q=0$$

If we subtract the two equations, we get $p = 2$, and consequently $q=-8$.

6. **(a)** The equation $mx^2-2(m+2)x+m+2=0$ has two real roots if and only if $m\neq 0$ and $\Delta\geq 0$ (there can be a double root, which satisfies requirements of the problem).

$$\Delta=4(m+2)^2-4m(m+2)=4m^2+16m+16-4m^2-8m=8m+16$$

$\Delta\geq 0\Leftrightarrow 8m+18\geq 0\Leftrightarrow m\geq -2$ but also $m\neq 0$. Therefore, $m\in[-2,\ 0[\cup]0,\ +\infty[$

(b) The equation $mx^2-2(m+2)x+m+2=0$ has two real roots with opposite signs if $m\neq 0$ and $\Delta\geq 0$, additionally, $x_1x_2=\frac{m+2}{m}<0$.

		-2		0	
$m+2$	$-$	0	$+$	$+$	$+$
m	$-$	$-$	$-$	0	$+$
$\frac{m+2}{m}$	$+$	0	$-$	X	$+$

$m\in]-2,\ 0[$

Now, the three conditions, $m\neq 0$, $m\in[-2,\ 0[\cup]0,\ +\infty[$ and $m\in]-2,\ 0[$ must be satisfied simultaneously. Therefore, $m\in]-2,\ 0[$

7. Since $x-1$ and $x+1$ are factors of the polynomial $f(x)=x^3+ax^2+bx+c=0$ then by the factor theorem: $f(-1)=0$ and $f(1)=0$. Additionally, by the remainder theorem $f(2)=12$. We have:

$$\begin{cases} 1^3+a(1)^2+b(1)+c=0 \\ (-1)^3+a(-1)^2+b(-1)+c=0 \\ 2^3+a(2)^2+b(2)+c=12 \end{cases} \Leftrightarrow \begin{cases} a+b+c=-1 \\ a-b+c=1 \\ 4a+2b+c=4 \end{cases} \Leftrightarrow \begin{cases} 2b=-2 \\ a-b+c=1 \\ 4a+2b+c=4 \end{cases}$$

It follows, that $b=-1$ and $\begin{cases} a+c=0 \\ 4a+c=6 \end{cases}$

Solving the system of equations, we get $\begin{cases} c=-a \\ 4a-a=6 \end{cases} \Leftrightarrow \begin{cases} c=-a \\ 3a=6 \end{cases} \Leftrightarrow \begin{cases} c=-2 \\ a=2 \end{cases}$

Therefore $a=2,\ b=-1,\ c=-2$

8. Squaring both sides of the inequality $|x|<5|x-6|$ we get

$$x^2<25(x^2-12x+36)$$

$$x^2<25x^2-300x+900$$

$$24x^2-300x+900>0$$

$$2x^2-25x+75>0 \Leftrightarrow (2x-15)(x-5)>0$$

		5		$\frac{15}{2}$	
$x-5$	$-$	0	$+$	$+$	$+$
$2x-15$	$-$	$-$	$-$	0	$+$
$(2x-15)(x-5)$	$+$	0	$-$	0	$+$

$$x\in\left]-\infty,\ 5\right[\cup\left]\frac{15}{2},\ +\infty\right[$$

9. The equation $2x^2+(3-k)x+k+3=0$ has two imaginary roots when $\Delta<0$.

$$\Delta=(3-k)^2-4(2)(k+3)=9-6k+k^2-8k-24=k^2-14k-15$$

$$\Delta<0 \Leftrightarrow k^2-14k-15=(k+1)(k-15)<0$$

		-1		15	
$k+1$	$-$	0	$+$	$+$	$+$
$k-15$	$-$	$-$	$-$	0	$+$
$(k+1)(k-15)$	$+$	0	$-$	0	$+$

$$k\in\left]-1,\ 15\right[$$

10. (a) $f(x)=\dfrac{2x^2+8x+7}{x^2+4x+5}=\dfrac{2x^2+8x+10-3}{x^2+4x+5}=\dfrac{2(x^2+4x+5)}{x^2+4x+5}-\dfrac{3}{x^2+4x+5}$

$f(x)=2-\dfrac{3}{(x+2)^2+1}$

(b) (i) $\lim\limits_{x\to+\infty} f(x)=2$, **(ii)** $\lim\limits_{x\to-\infty} f(x)=2$ since in both cases as $x\to\infty$, $(x+2)^2+1\to\infty$ and $\dfrac{3}{(x+2)^2+1}\to 0$

(c) $f(x)$ has a minimum when the expression $\dfrac{3}{(x+2)^2+1}$ assumes the maximum value possible. This happens when $y=(x+2)^2+1$ has the minimum value. The minimum value of the parabola $y=(x+2)^2+1$ occurs at the vertex, whose coordinates are $(-2, 1)$.

11. The equation $(k-2)x^2+4x-2k+1=0$ has two distinct real roots if $k-2\neq 0\Leftrightarrow k\neq 2$ and the discriminant $\Delta>0$.

$\Delta=4^2-4(k-2)(-2k+1)=16+8k^2-20k+8=8k^2-20k+24$

$\Delta>0\Leftrightarrow 8k^2-20k+24>0\Leftrightarrow 2k^2-5k+6>0$

The discriminant of $2k^2-5k+6$, $\Delta_1=(-5)^2-4(2)(6)=25-48=-23<0$ and the coefficient of k^2 is positive, so $\Delta>0$ for all real values of *k*. Considering that $k\neq 2$, the equation $(k-2)x^2+4x-2k+1=0$ has two distinct real roots when $k\in\mathbb{R}-\{2\}$

12. By the remainder theorem $f(-1)=-20$. Then:

$6(-1)^4+11(-1)^3-22(-1)^2+a(-1)+6=-20$

$6-11-22-a+6=-20\Leftrightarrow a=-1$

13. By the remainder theorem, the polynomial $p(x)=(ax+b)^3$ must satisfy the following conditions: $p(-1)=-1$ and $p(2)=27$. Then: $(a(-1)+b)^3=-1$ and $(a(2)+b)^3=27$ which is equivalent to $-a+b=-1$ and $2a+b=3$. Subtracting the first equation from the second we get $3a=4\Leftrightarrow a=\dfrac{4}{3}$ and $b=a-1=\dfrac{4}{3}-1=\dfrac{1}{3}$. Therefore $a=\dfrac{4}{3}$, $b=\dfrac{1}{3}$.

14. By the remainder theorem $f(2)=f(-1)$. Then:

$2^3+3(2)^2+a(2)+b=(-1)^3+3(-1)^2+a(-1)+b$

$8+12+2a=-1+3-a\Leftrightarrow 3a=-18\Leftrightarrow a=-6$

15. By the remainder theorem, the polynomial $f(x)=x^4+ax+3$ must satisfy $f(1)=8$. Then: $1^4+a(1)+3=8\Leftrightarrow a=4$.

16. Let $f(x)=x^3+ax^2-3x+b$. By the factor theorem, $f(2)=0$ and by the remainder theorem, $f(-1)=6$. We have: $2^3+a(2^2)-3(2)+b=0$ and $(-1)^3+a(-1)^2-3(-1)+b=6$. Simplifying, we get $4a+b=-2$ and $a+b=4$. Subtracting the second equation from the first we get $3a=-6\Rightarrow a=-2$ and $b=4-a=4-(-2)=6$.

17. If x^2-4x+3 is a factor, then $x-1$ is also a factor, thus

$f(1)=1+(a-4)+(3-4a)+3=0\Rightarrow a=1$

Alternatively, $x-3$ is a factor leading to the same answer.

Another approach, but much longer is:

The polynomial $f(x)=x^3+(a-4)x^2+(3-4a)x+3$ can be written as $x^3+(a-4)x^2+(3-4a)x+3=(x^2-4x+3)(x-m)$, where $x-m$ is the result of division of $f(x)$ by x^2-4x+3. Now, after expanding and simplifying of the right-hand side: $x^3+(a-4)x^2+(3-4a)x+3=x^3-(m+4)x^2+(4m+3)x-3m$.

The following conditions must be satisfied: $\begin{cases}a-4=-m-4\\3-4a=4m+3\\3=-3m\end{cases}\Leftrightarrow\begin{cases}a=-m\\-a=m\\-1=m\end{cases}$.

Therefore, $a=1$.

18. By the factor theorem, $f(-2)=0$. Then:

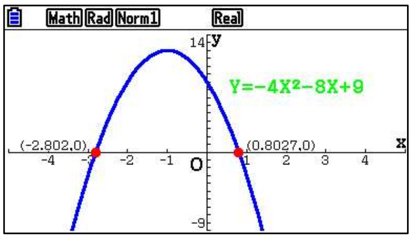
$(-2)^3-2(-2)^2-5(-2)+k=0\Leftrightarrow -8-8+10+k=0\Leftrightarrow k=6$

19. The equation $kx^2-3x+(k+2)=0$ has two distinct real roots if $k\neq 0$ and the discriminant $\Delta>0$. Then:

$\Delta=(-3)^2-4k(k+2)=9-4k^2-8k$

$\Delta>0\Leftrightarrow -4k^2-8k+9>0$

This is a parabola, concave downwards, and it is above the *x*-axis between its roots, as seen in the GDC screenshot:



$$k \in \left]\frac{-2-\sqrt{13}}{2}, \frac{-2+\sqrt{13}}{2}\right[\approx \left]-2.80, 0.803\right[\text{ (to 3 s. f.)}$$

20. The equation $(1+2k)x^2 - 10x + k - 2 = 0,\ k \in \mathbb{R}$ has real roots if $1+2k = 0 \Leftrightarrow k = -\frac{1}{2}$ or $k \neq -\frac{1}{2}$ and the discriminant $\Delta \geq 0$

$$\Delta = (-10)^2 - 4(1+2k)(k-2) = 100 - 4\left(2k^2 - 3k - 2\right) = -8k^2 + 12k + 108$$

$$\Delta \geq 0 \Leftrightarrow -8k^2 + 12k + 108 \geq 0 \Leftrightarrow 8k^2 - 12k - 108 \leq 0 \Leftrightarrow 2k^2 - 3k - 27 \leq 0$$

Factorising, we get $(2k-9)(k+3) \leq 0$

		-3		$\frac{9}{2}$	
$k+3$	–	0	+	+	+
$2k-9$	–	–	–	0	+
$(k+3)(2k-9)$	+	0	–	0	+

$$k \in \left[-3, \frac{9}{2}\right]$$

21. The inequality $m(x+1) \leq x^2$ can be written as $x^2 - mx - m \geq 0$. Since the coefficient of x^2 is positive, the inequality is true for all $x \in \mathbb{R}$, when the discriminant $\Delta \leq 0$.

$$\Delta = (-m)^2 - 4(1)(-m) = m^2 + 4m = m(m+4)$$

$$\Delta \leq 0 \Leftrightarrow m(m+4) \leq 0$$

		-4		0	
$m+4$	–	0	+	+	+
m	–	–	–	0	+
$m(m+4)$	+	0	–	0	+

$$m \in [-4, 0]$$

22. Since both sides of the inequality $|5-3x| \leq |x+1|$ are non-negative, we can write $(5-3x)^2 \leq (x+1)^2$. Now we have:

$$25 - 30x + 9x^2 \leq x^2 + 2x + 1$$

$$8x^2 - 32x + 24 \leq 0$$

$$x^2 - 4x + 3 \leq 0$$

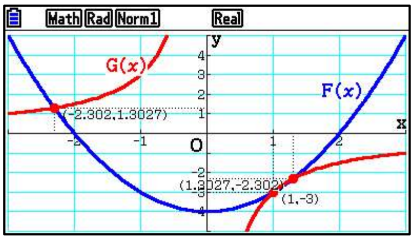
$$(x-3)(x-1) \leq 0$$

		1		3	
$x-1$	$-$	0	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	0	$+$
$(x-1)(x-3)$	$+$	0	$-$	0	$+$

$x \in [1,\ 3]$

23. The inequality $x^2-4+\frac{3}{x}<0$, where $x \neq 0$ can be written $x^2-4 \leq -\frac{3}{x}$

Let $f(x)=x^2-4$ and $g(x)=-\frac{3}{x}$. By GDC:



$$x^2-4 \leq -\frac{3}{x} \Leftrightarrow x \in \left]-2.30,\ 0\right[\cup \left]1,\ 1.30\right[\quad \text{(to 3 s. f.)}$$

A more algebraic approach is also possible:

$$x^2-4+\frac{3}{x}<0 \Rightarrow \frac{x^3-4x+3}{x}<0 \Rightarrow \frac{(x-1)\left(x^2+x-3\right)}{x}<0$$

This can be solved by setting up a table as in previous problems; the exact solution will be

$$x \in \left]-\frac{1+\sqrt{13}}{2},\ 0\right[\cup \left]1,\ \frac{-1+\sqrt{13}}{2}\right[$$

24. Since both sides of the inequality $|x-2| \geq |2x+1|$ are non-negative, we can write $(x-2)^2 \geq (2x+1)^2$

Now we have: $x^2-4x+4 \geq 4x^2+4x+1$

$$3x^2+8x-3 \leq 0 \Leftrightarrow (3x-1)(x+3) \leq 0$$

		-3		$\frac{1}{3}$	
$x+3$	$-$	0	$+$	$+$	$+$
$3x-1$	$-$	$-$	$-$	0	$+$
$(3x-1)(x+3)$	$+$	0	$-$	0	$+$

$x \in \left[-3,\ \frac{1}{3}\right]$

25. $f(x) \le g(x) \Leftrightarrow \frac{x+4}{x+1} \le \frac{x-2}{x-4}$ or $\frac{x+4}{x+1} - \frac{x-2}{x-4} \le 0$. Now we have:

$$\frac{(x+4)(x-4)-(x-2)(x+1)}{(x+1)(x-4)} \le 0 \Leftrightarrow \frac{x^2-16-x^2+x+2}{(x+1)(x-4)} \le 0$$

$$\frac{x-14}{(x+1)(x-4)} \le 0$$

		-1		4		14	
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-14$	$-$	$-$	$-$	$-$	$-$	0	$+$
$\frac{x-14}{(x+1)(x-4)}$	$-$	X	$+$	X	$-$	0	$+$

$$x \in]-\infty, -1[\cup]4, 14]$$

26. The inequality $\left|\frac{x+9}{x-9}\right| \le 2$, where $x \ne 9$, can be written as $|x+9| \le 2|x-9|$.

Since both sides of the inequality are nonnegative:

$$(x+9)^2 \le 4(x-9)^2$$

$$x^2+18x+81 \le 4x^2-72x+324$$

$$3x^2-90x+243 \ge 0 \Leftrightarrow x^2-30x+81 \ge 0 \Leftrightarrow (x-3)(x-27) \ge 0$$

		3		27	
$x-3$	$-$	0	$+$	$+$	$+$
$x-27$	$-$	$-$	$-$	0	$+$
$(x-3)(x-27)$	$+$	0	$-$	0	$+$

$$x \in]-\infty, 3] \cup [27, +\infty[$$

27. The inequality $\frac{2x}{|x-1|} < 1$, where $x \ne 1$,

For $x < 1$: $\frac{2x}{-x+1} - 1 < 0 \Rightarrow \frac{3x-1}{1-x} < 0 \Rightarrow x < \frac{1}{3}$.

For $x > 1$: $\frac{2x}{x-1} - 1 < 0 \Rightarrow \frac{x+1}{x-1} < 0 \Rightarrow -1 < x < 1$, $\Rightarrow$ no solution exists when $x > 1$.

Therefore, the inequality is satisfied for $x < \frac{1}{3}$

28. $\dfrac{2x-5}{x^2+x-2}=\dfrac{2x-5}{(x+2)(x-1)}=\dfrac{A}{x+2}+\dfrac{B}{x-1}$

Multiplying both sides by $x+2$ we get: $\dfrac{2x-5}{x-1}=A+\dfrac{B(x+2)}{x-1}$

Now, let $x=-2$. Then $A=\dfrac{2x-5}{x-1}=\dfrac{2(-2)-5}{-2-1}=\dfrac{-9}{-3}=3$

Multiplying both sides by $x-1$ we get: $\dfrac{2x-5}{x+2}=\dfrac{A(x-1)}{x+2}+B$

Now, let $x=1$. Then $B=\dfrac{2(1)-5}{1+2}=\dfrac{-3}{3}=1$ Therefore, $\dfrac{2x-5}{x^2+x-2}=\dfrac{3}{x+2}-\dfrac{1}{x-1}$

29. $\dfrac{2x-48}{x^2-9}=\dfrac{2x-5}{(x+3)(x-3)}=\dfrac{A}{x+3}+\dfrac{B}{x-3}$

Multiplying both sides by $x+3$ we get: $\dfrac{2x-48}{x-3}=A+\dfrac{B(x+3)}{x-3}$

Now, let $x=-3$. Then $A=\dfrac{2x-48}{x-3}=\dfrac{2(-3)-48}{-3-3}=\dfrac{-54}{-6}=9$

Multiplying both sides by $x-3$ we get: $\dfrac{2x-48}{x+3}=\dfrac{A(x-3)}{x+3}+B$

Now, let $x=3$. Then $B=\dfrac{2(3)-48}{3+3}=\dfrac{-42}{6}=-7$

Therefore, $A=9,\ B=-7$

30. $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{A}{x-a}+\dfrac{B}{x-b},\ x\neq a$ and $x\neq b$

Multiplying both sides by $x-a$ we get: $A+\dfrac{B(x-a)}{x-b}=\dfrac{a-b}{x-b}$

Now, let $x=a$. Then $A=\dfrac{a-b}{a-b}-\dfrac{B(a-a)}{a-b}=1$

Multiplying both sides by $x-b$ we get: $\dfrac{a-b}{x-a}=\dfrac{A(x-b)}{x-a}+B$

Now, let $x=b$. Then $B=\dfrac{a-b}{b-a}-\dfrac{A(b-b)}{b-a}=\dfrac{a-b}{-(a-b)}=-1$

Therefore, $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{1}{x-a}-\dfrac{1}{x-b},\ x\neq a$ and $x\neq b$

Exercise 3.1

1. In parts **(a) – (d)**, substitute n (or k) = 1, 2, … , 5 into the given formula.

(a) $s(n)=2n-3 \;\Rightarrow s(1)=-1,\ s(2)=1,\ s(3)=3,\ s(4)=5,\ s(5)=7$

(b) $g(k)=2^k-3 \;\Rightarrow g(1)=-1,\ g(2)=1,\ g(3)=5,\ g(4)=13,\ g(5)=29$

(c) $f(n)=3\times 2^{-n} \;\Rightarrow f(1)=\frac{3}{2},\ f(2)=\frac{3}{4},\ f(3)=\frac{3}{8},\ f(4)=\frac{3}{16},\ f(5)=\frac{3}{32}$

(d) $a_n=(-1)^n(2^n)+3 \;\Rightarrow a_1=1,\ a_2=7,\ a_3=-5,\ a_4=19,\ a_5=-29$

(e) $\begin{cases} a_1=5 \\ a_n=a_{n-1}+3 \end{cases} \Rightarrow a_1=5,\ a_2=8,\ a_3=11,\ a_4=14,\ a_5=17$

(f) $\begin{cases} b_1=3 \\ b_n=b_{n-1}+2n \end{cases} \Rightarrow b_1=3,\ b_2=7,\ b_3=13,\ b_4=21,\ b_5=31$

2. In parts **(a) – (d)**, simply substitute n = 1, 2, …, 5 and n = 50 into the formula.

(a) $-1,1,3,5,7 \qquad a_{50}=97$

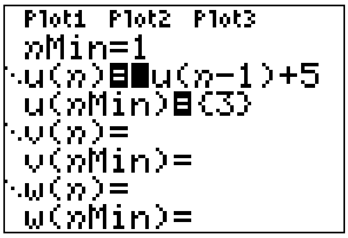
(b) $2,6,18,54,162 \qquad b_{50}=2\cdot 3^{49}=4.786\times 10^{23}$

(c) $\frac{2}{3},-\frac{2}{3},\frac{6}{11},-\frac{4}{9},\frac{10}{27} \qquad u_{50}=-\frac{100}{2502}=-\frac{50}{1251}$

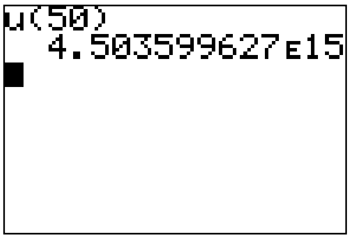
(d) $1,2,9,64,625 \qquad a_{50}=50^{49}=1.776\times 10^{83}$

In parts **(e) – (h)**, start with the first term and substitute it in the given formula to find the second term, and so on. To find the 50th term, we will use a GDC in Sequential mode. Be aware that some GDCs start with u_{n+1} rather than u_n, as shown in the second set of screen shots. In this case, you start with n = 0 and end with n = 49.

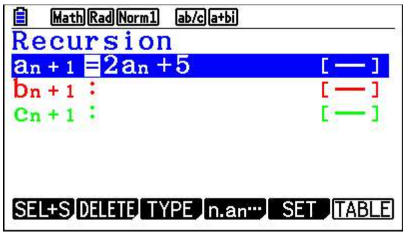
(e) $3,11,27,59,123 \qquad a_{50}=4.50\times 10^{15}$



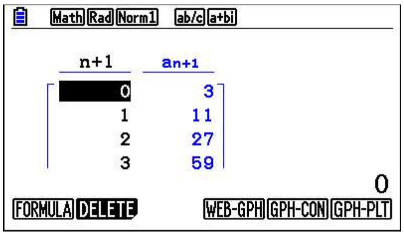
```
nMin=1
·u(n)=u(n-1)+5
 u(nMin)={3}
·v(n)=
 v(nMin)=
·w(n)=
 w(nMin)=
```



```
u(50)
  4.503599627E15
```

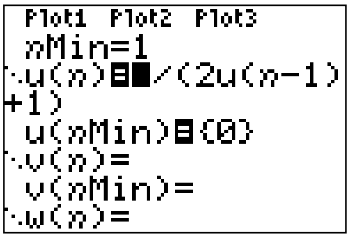


```
Recursion
an+1 =2an +5        [—]
bn+1 :              [—]
cn+1 :              [—]
```

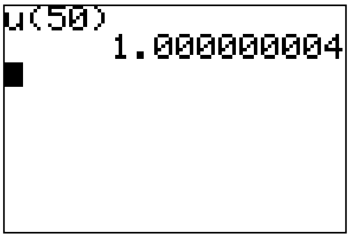


```
 n+1   an+1
   0      3
   1     11
   2     27
   3     59
              0
```

(f) $0,\ 3,\ \frac{3}{7},\ \frac{3}{2\cdot\frac{3}{7}+1}=\frac{21}{13},\ \frac{39}{55}$ $u_{50}\approx 1.00$

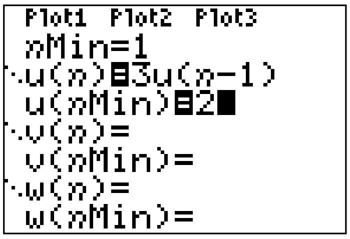


```
Plot1 Plot2 Plot3
nMin=1
·u(n)=■/(2u(n-1)
+1)
 u(nMin)={0}
·v(n)=
 v(nMin)=
·w(n)=
```

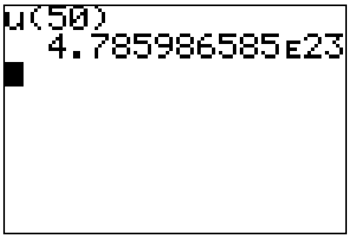


```
u(50)
      1.000000004
■
```

(g) 2, 6, 18, 54, 162 $b_{50}\approx 4.786\times 10^{23}$

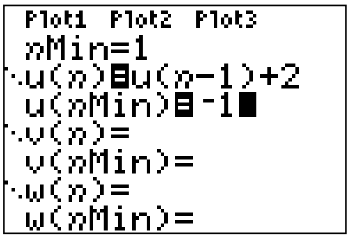


```
Plot1 Plot2 Plot3
nMin=1
·u(n)=3u(n-1)
 u(nMin)=2■
·v(n)=
 v(nMin)=
·w(n)=
 w(nMin)=
```

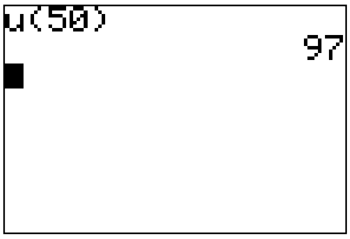


```
u(50)
   4.785986585E23
■
```

(h) $-1, 1, 3, 5, 7$ $a_{50}=97$



```
Plot1 Plot2 Plot3
nMin=1
·u(n)=u(n-1)+2
 u(nMin)=-1■
·v(n)=
 v(nMin)=
·w(n)=
 w(nMin)=
```



```
u(50)
                97
■
```

3. In this question, you need to observe and spot the pattern, perhaps using trial and error.

(a) $u_n=\frac{1}{4}u_{n-1},\quad u_1=\frac{1}{3}$

(b) $u_n=\frac{4a^2}{3}u_{n-1},\quad u_1=\frac{1}{2}a$

(c) $u_n=u_{n-1}+a+k,\quad u_1=a-5k$

4. In this question, you need to observe and spot the pattern, perhaps using trial and error.

(a) $u_n=n^2+3$

(b) $u_n=3n-1$

(c) $u_n=\frac{2n-1}{n^2}$

(d) $u_n=\frac{2n-1}{n+3}$

5. For the Fibonacci sequence $1,1,2,3,5,8,13,21,34,55,89,...$ we have:

$$a_1=\frac{F_2}{F_1}=\frac{1}{1}=1,\ a_2=\frac{F_3}{F_2}=\frac{2}{1}=2,\ a_3=\frac{F_4}{F_3}=\frac{3}{2},\ a_4=\frac{F_5}{F_4}=\frac{5}{3},$$

(a) $$a_5=\frac{F_6}{F_5}=\frac{8}{5},\ a_6=\frac{F_7}{F_6}=\frac{13}{8},\ a_7=\frac{F_8}{F_7}=\frac{21}{13},\ a_8=\frac{F_9}{F_8}=\frac{34}{21},$$

$$a_9=\frac{F_{10}}{F_9}=\frac{55}{34},\ a_{10}=\frac{F_{11}}{F_{10}}=\frac{89}{55}$$

(b) $$a_n=\frac{F_{n+1}}{F_n}=\frac{F_{n-1}+F_n}{F_n}=\frac{F_n}{F_n}+\frac{F_{n-1}}{F_n}=1+\frac{1}{\frac{F_n}{F_{n-1}}}=1+\frac{1}{a_{n-1}}$$

6. For $F_n=\frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n}{2^n}\right)$ we have:

(a) Substitute n = 1, 2, …, 10 into the equation.
You can use computer software for this as well.

$$F_1=\frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)-\left(1-\sqrt{5}\right)}{2}\right)=1$$

$$F_2=\frac{1}{\sqrt{5}}\left(\frac{\left(1+2\sqrt{5}+5\right)-\left(1-2\sqrt{5}+5\right)}{4}\right)=1$$

$$F_3=\frac{1}{\sqrt{5}}\left(\frac{\left(1+3\sqrt{5}+15+5\sqrt{5}\right)-\left(1-3\sqrt{5}+15-5\sqrt{5}\right)}{8}\right)=2$$

$$F_4=\frac{1}{\sqrt{5}}\left(\frac{\left(1+4\sqrt{5}+30+20\sqrt{5}+25\right)-\left(1-4\sqrt{5}+30-20\sqrt{5}+25\right)}{16}\right)=3$$

Similarly, $F_5=5, F_6=8, F_7=13, F_8=21, F_9=34,$ and $F_{10}=55$

This sequence is equal to the Fibonacci sequence.

(b) By symbolic manipulation or by taking each term and multiplying it by a fraction whose numerator and denominators are the conjugates of the given number, or simplifying the right-hand side, you get:

$$3\pm\sqrt{5}=\frac{6\pm2\sqrt{5}}{2}=\frac{\left(1\pm2\sqrt{5}+5\right)}{2}=\frac{\left(1\pm\sqrt{5}\right)^2}{2}$$

(c) $$F_{n-1}+F_n=\frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}}\right)+\frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n}{2^n}\right)$$

Factor out $\frac{1}{\sqrt{5}}$ and express both fractions with the same denominator, then collect like-terms:

$$F_{n-1}+F_n=\frac{1}{\sqrt{5}}\left[\frac{2\left(1+\sqrt{5}\right)^{n-1}+\left(1+\sqrt{5}\right)^n}{2^n}-\frac{2\left(1-\sqrt{5}\right)^{n-1}+\left(1-\sqrt{5}\right)^n}{2^n}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n-1}\left(2+1+\sqrt{5}\right)}{2^n}-\frac{\left(1-\sqrt{5}\right)^{n-1}\left(2+1-\sqrt{5}\right)}{2^n}\right]$$

Simplify and use result of **(b)**

$$F_{n-1}+F_n=\frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n-1}\left(3+\sqrt{5}\right)}{2^n}-\frac{\left(1-\sqrt{5}\right)^{n-1}\left(3-\sqrt{5}\right)}{2^n}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n-1}}{2^n}\cdot\frac{\left(1+\sqrt{5}\right)^2}{2}-\frac{\left(1-\sqrt{5}\right)^{n-1}}{2^n}\cdot\frac{\left(1-\sqrt{5}\right)^2}{2}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n+1}}{2^{n+1}}-\frac{\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}}\right]=F_{n+1}$$

Exercise 3.2

1. If there are four means between 3 and 7, there are six terms in this sequence such that

$a_1=3, a_6=7$

Apply the nth term formula to find d and then simplify:

$$a_6=a_1+5d \Rightarrow 7=3+5d \Rightarrow d=\frac{4}{5}=0.8$$

The sequence is $3,\frac{19}{5},\frac{23}{5},\frac{27}{5},\frac{31}{5},7$

2. **(a)** Arithmetic: $a_{n+1}-a_n=\left[2(n+1)-3\right]-(2n-3)=2 \Rightarrow d=2$

$\Rightarrow a_{50}=a_1+49d=-1+49\cdot 2=97$

(b) Arithmetic: $b_{n+1}-b_n=(n+1+2)-(n+2)=1 \Rightarrow d=1$

$\Rightarrow b_{50}=b_1+49d=3+49\cdot 1=52$

(c) Arithmetic: $c_1=-1, c_2=1, c_3=3 \Rightarrow d=2 \Rightarrow c_{50}=c_1+49d=-1+49\cdot 2=97$

(d) There is no constant common difference, so the sequence is not arithmetic.

$u_n - u_{n-1} = 3u_{n-1} + 2 - 3u_{n-2} - 2 = 3\left(u_{n-1} - u_{n-2}\right)$

Also, u_1 is not defined, so the sequence cannot be determined.

(e) $e_2 - e_1 = 5 - 2 = 3,\ e_3 - e_2 = 7 - 5 = 2$

There is no constant common difference, so the sequence is not arithmetic.

(f) Arithmetic: $f_2 - f_1 = f_3 - f_2 = f_4 - f_3 = -7 \Rightarrow d = -7$

$\Rightarrow f_{50} = f_1 + 49d = 2 + 49\cdot(-7) = -341$

3. **(a)** **(i)** $a_1 = -2,\ d = 4$: $a_8 = a_1 + (8-1)d = -2 + 7\cdot 4 = 26$

(ii) $a_n = -2 + (n-1)\cdot 4 = 4n - 6$

(iii) $a_n = a_{n-1} + 4,\ a_1 = -2$

(b) **(i)** $a_1 = 29,\ d = -4$: $a_8 = a_1 + 7d = 29 + 7\cdot(-4) = 1$

(ii) $a_n = 29 + (n-1)\cdot(-4) = -4n + 33$

(iii) $a_n = a_{n-1} - 4,\ a_1 = 29$

(c) **(i)** $a_1 = -6,\ d = 9 : a_8 = a_1 + 7d = -6 + 7\cdot 9 = 57$

(ii) $a_n = -6 + (n-1)\cdot 9 = 9n - 15$

(iii) $a_n = a_{n-1} + 9,\ a_1 = -6$

(d) **(i)** $a_1 = 10.07,\ d = -0.12 : a_8 = a_1 + 7d = 10.07 + 7\cdot(-0.12) = 9.23$

(ii) $a_n = 10.07 + (n-1)\cdot(-0.12) = -0.12n + 10.19$

(iii) $a_n = a_{n-1} - 0.12,\ a_1 = 10.07$

(e) **(i)** $a_1 = 100,\ d = -3 : a_8 = a_1 + 7d = 100 + 7\cdot(-3) = 79$

(ii) $a_n = 100 + (n-1)\cdot(-3) = -3n + 103$

(iii) $a_n = a_{n-1} - 3,\ a_1 = 100$

(f) **(i)** $a_1 = 2,\ d = -\dfrac{5}{4} : a_8 = a_1 + 7d = 2 + 7\cdot(-\dfrac{5}{4}) = -\dfrac{27}{4}$

(ii) $a_n = 2 + (n-1)\cdot(-\dfrac{5}{4}) = -\dfrac{5}{4}n + \dfrac{13}{4}$

(iii) $a_n = a_{n-1} - \dfrac{5}{4},\ a_1 = 2$

4. There are five means between 13 and -23, so there are seven terms in this sequence such that

$a_1 = 13, a_7 = -23$

Apply the *n*th term formula to find *d* and then simplify:

$a_7 = a_1 + 6d \Rightarrow -23 = 13 + 6d \Rightarrow d = -6$

Thus, the sequence is $13, 7, 1, -5, -11, -17, -23$

5. Similar to question **4**:

$$a_1 = 299, a_5 = 300 \Rightarrow a_5 = a_1 + 4d \Rightarrow 300 = 299 + 4d \Rightarrow d = \frac{1}{4} = 0.25$$

The sequence is 299, 299.25, 299.5, 299.75, 300

6. We need to find the first term and the common difference.

$a_5 = 16, a_{14} = 42 \Rightarrow a_1 + 4d = 16$, and

$a_{14} = 42 \Rightarrow a_1 + 13d = 42$, and solving the system

$$d = \frac{26}{9},\ a_1 = -\frac{40}{9} \Rightarrow a_n = -\frac{40}{9} + (n-1)\cdot\frac{26}{9} = \frac{1}{9}(26n - 66)$$

7. Similar to question **6**:

$$\left.\begin{aligned} a_3 &= -40 \\ a_9 &= -18 \end{aligned}\right\} \Rightarrow \begin{cases} a_1 + 2d = -40 \\ a_1 + 8d = -18 \end{cases}$$

$$\Rightarrow d = \frac{11}{3},\ a_1 = -\frac{142}{3} \Rightarrow a_n = -\frac{142}{3} + (n-1)\cdot\frac{11}{3} = \frac{11}{3}n - 51$$

8. Use the *n*th term formula for each part.

(a)
$a_1 = 3, d = 6, a_n = 525$
$a_n = a_1 + (n-1)d \Rightarrow 525 = 3 + (n-1)\cdot 6 \Rightarrow n = 88$

(b)
$a_1 = 9, d = -6, a_n = -201$
$a_n = a_1 + (n-1)d \Rightarrow -201 = 9 + (n-1)\cdot(-6) \Rightarrow n = 36$

(c)
$a_1 = 3\frac{1}{8}, d = 4\frac{1}{4} - 3\frac{1}{8} = \frac{9}{8}, a_n = 14\frac{3}{8}$
$a_n = a_1 + (n-1)d \Rightarrow \frac{115}{8} = \frac{25}{8} + (n-1)\cdot\frac{9}{8} \Rightarrow n = 11$

(d)
$a_1 = \frac{1}{3}, d = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, a_n = 2\frac{5}{6}$
$a_n = a_1 + (n-1)d \Rightarrow \frac{17}{6} = \frac{1}{3} + (n-1)\cdot\frac{1}{6} \Rightarrow n = 16$

(e)
$$a_1 = 1-k, d = (1+k)-(1-k) = 2k, a_n = 1+19k$$
$$a_n = a_1 + (n-1)d \quad \Rightarrow 1+19k = 1-k+(n-1)\cdot 2k$$
$$\Rightarrow 20k = (n-1)\cdot 2k \Rightarrow n = 11$$

9.
$$a_1 = 15, a_7 = -21$$
$$a_7 = a_1 + 6d \quad \Rightarrow -21 = 15 + 6d \quad \Rightarrow d = -\frac{36}{6} = -6$$

The sequence is $15, 9, 3, -3, -9, -15, -21$

10.
$$a_1 = 99, a_5 = 100$$
$$a_5 = a_1 + 4d \quad \Rightarrow 100 = 99 + 4d \quad \Rightarrow d = \frac{1}{4}$$

The sequence is $99, 99.25, 99.5, 99.75, 100$

11.
$$\begin{cases} a_3 = 11 \\ a_{12} = 47 \end{cases} \Rightarrow \begin{cases} a_1 + 2d = 11 \\ a_1 + 11d = 47 \end{cases} \Rightarrow d = 4,\ a_1 = 3$$

The sequence is defined by: $a_1 = 3,\ a_n = 3 + (n-1)\cdot 4 = 4n-1$ for $n > 1$

12.
$$\begin{cases} a_7 = -48 \\ a_{13} = -10 \end{cases} \Rightarrow \begin{cases} a_1 + 6d = -48 \\ a_1 + 12d = -10 \end{cases} \Rightarrow d = \frac{19}{3},\ a_1 = -86$$

The sequence is defined by: $a_1 = -86,\ a_n = -86 + (n-1)\cdot \frac{19}{3} = \frac{19n-277}{3}$ for $n > 1$

13.
$$a_{30} = 147, d = 4$$
$$a_{30} = a_1 + 29d \quad \Rightarrow 147 = a_1 + 29\cdot 4 \quad \Rightarrow a_1 = 31$$
$$a_n = a_1 + (n-1)d = 31 + (n-1)\cdot 4 = 4n + 27$$

14.
$$a_1 = -7, d = 3, a_n = 9803$$
$$a_n = a_1 + (n-1)d \quad \Rightarrow 9803 = -7 + (n-1)\cdot 3 \quad \Rightarrow n = 3271$$

Yes, 9803 is the 3271th term of the sequence.

15.

$a_1 = 9689, a_{100} = 8996$

$a_n = a_1 + (n-1)d \quad \Rightarrow 8996 = 9689 + 99d \quad \Rightarrow d = -7$

$a_{110} = a_1 + 109d = 9689 + 109 \cdot (-7) = 8926$

$a_n = 1 \quad \Rightarrow 9689 + (n-1) \cdot (-7) = 1 \quad \Rightarrow n = 1385$

Yes, 1 is the 1385th term of the sequence.

16.

$a_1 = 2, a_{30} = 147$

$a_n = a_1 + (n-1)d \ \Rightarrow 147 = 2 + 29d \ \Rightarrow d = 5$

$a_n = 995 \ \Rightarrow 2 + (n-1) \cdot 5 = 995 \ \Rightarrow n = \frac{998}{5}$

As a fractional result is not possible for *n,* we conclude that 995 is not a term of this sequence.

Exercise 3.3

1. **(a)** **(i)** $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots$ The sequence is geometric.

(ii) $r = \frac{u_n}{u_{n-1}} = \frac{3^{a+1}}{3} = \ldots = \frac{3^{3a+1}}{3^{2a+1}} = 3^a$

(iii) $u_{10} = u_1 r^9 = 3 \cdot \left(3^a\right)^9 = 3^{9a+1}$

(b) **(i)** $0, 3, 6, 9 \ldots$ The sequence is arithmetic.

(ii) $d = a_n - a_{n-1} = (3n-3) - \left[3(n-1) - 3\right] = 3$

(iii) $a_{10} = a_1 + 9d = 0 + 9 \cdot 3 = 27$

(c) **(i)** $8, 16, 32, 64, \ldots$ The sequence is geometric.

(ii) $r = \frac{b_n}{b_{n-1}} = \frac{16}{8} = \ldots = 2$

(iii) $b_{10} = b_1 r^9 = 8 \cdot 2^9 = 4096$

(d) **(i)** $-1, -4, -10, -22, \ldots$ Neither arithmetic nor geometric.

(e) **(i)** $4, 12, 36, 108, \ldots$ The sequence is geometric.

(ii) $r = \frac{u_n}{u_{n-1}} = \frac{12}{4} = \ldots = 3$

(iii) $u_{10} = u_1 r^9 = 4 \cdot 3^9 = 78\,732$

(f) **(i)** $2, 5, 12.5, 31.25, 78.125...$ The sequence is geometric.

(ii) $r = \frac{u_n}{u_{n-1}} = \frac{5}{2} = \frac{12.5}{5} = ... = 2.5$

(iii) $u_{10} = u_1 r^9 = 2 \cdot 2.5^9 \approx 7629.39$

(g) **(i)** $2, -5, 12.5, -31.25, 78.125...$ The sequence is geometric.

(ii) $r = \frac{-5}{2} = \frac{12.5}{-5} = ... = -2$

(iii) $u_{10} = u_1 r^9 = 2 \cdot (-2.5)^9 \approx -7629.39$

(h) **(i)** $2, 2.75, 3.5, 4.25, 5...$ The sequence is arithmetic.

(ii) $d = 2.75 - 2 = 3.5 - 2.75 = 4.25 - 3.5 = 5 - 4.25 = 0.75$

(iii) $u_{10} = u_1 + 9d = 2 + 9 \cdot 0.75 = 8.75$

(i) **(i)** $18, -12, 8, -\frac{16}{3}, \frac{32}{9}...$ The sequence is geometric.

(ii) $r = \frac{-12}{18} = \frac{8}{-12} = ... = -\frac{2}{3}$

(iii) $u_{10} = u_1 r^9 = 18 \cdot \left(-\frac{2}{3}\right)^9 = -\frac{1024}{2187} \approx -0.468$

(j) **(i)** $52, 55, 58, 61, ...$ The sequence is arithmetic.

(ii) $d = 55 - 52 = ... = 61 - 58 = 3$

(iii) $u_{10} = u_1 + 9d = 52 + 9 \cdot 3 = 79$

(k) **(i)** $-1, 3, -9, 27, -81, ...$ The sequence is geometric.

(ii) $r = \frac{3}{-1} = \frac{-9}{3} = ... = -3$

(iii) $u_{10} = u_1 r^9 = (-1) \cdot (-3)^9 = 19683$

(l) **(i)** $0.1, 0.2, 0.4, 0.8, 1.6, 3.2, ...$ The sequence is geometric.

(ii) $r = \frac{0.2}{0.1} = ... = \frac{3.2}{1.6} = 2$

(iii) $u_{10} = u_1 r^9 = 0.1 \cdot 2^9 = 51.2$

(m) **(i)** $3, 6, 12, 18, 21, 27, ...$ Neither arithmetic nor geometric.

(n) **(i)** $6,14,20,28,34,...$ Neither arithmetic nor geometric.

(o) **(i)** $2.4,3.7,5,6.3,7.6,...$ The sequence is arithmetic.

(ii) $d=3.7-2.4=...=7.6-6.3=1.3$

(iii) $u_{10}=u_1+9d=2.4+9\cdot1.3=14.1$

2. **(a)** **(i)** Arithmetic: $d=2-(-3)=...=5\Rightarrow a_8=a_1+7d=-3+7\cdot5=32$

(ii) $a_n=-3+(n-1)\cdot5=5n-8$

(iii) $a_1=-3,\ a_n=a_{n-1}+5$ for $n>1$

(b) **(i)** Arithmetic: $d=15-19=...=-4\Rightarrow a_8=a_1+7d=19+7\cdot(-4)=-9$

(ii) $a_n=19+(n-1)\cdot(-4)=23-4n$

(iii) $a_1=19,\ a_n=a_{n-1}-4$ for $n>1$

(c) **(i)** Arithmetic: $d=3-(-8)=...=11\Rightarrow a_8=a_1+7d=-8+7\cdot11=69$

(ii) $a_n=-8+(n-1)\cdot11=11n-19$

(iii) $a_1=-8,\ a_n=a_{n-1}+11$ for $n>1$

(d) **(i)** Arithmetic:

$d=9.95-10.05=...=-0.1\Rightarrow a_8=a_1+7d=10.05+7\cdot(-0.1)=9.35$

(ii) $a_n=10.05+(n-1)\cdot(-0.1)=10.15-0.1n$

(iii) $a_1=10.05,\ a_n=a_{n-1}-0.1$ for $n>1$

(e) **(i)** Arithmetic: $d=99-100=...=-1\Rightarrow a_8=a_1+7d=100+7\cdot(-1)=93$

(ii) $a_n=100+(n-1)\cdot(-1)=101-n$

(iii) $a_1=100,\ a_n=a_{n-1}-1$ for $n>1$

(f) **(i)** Arithmetic: $d=\frac{1}{2}-2=...=-\frac{3}{2}\Rightarrow a_8=a_1+7d=2+7\cdot\left(-\frac{3}{2}\right)=-\frac{17}{2}$

(ii) $a_n=2+(n-1)\cdot\left(-\frac{3}{2}\right)=\frac{7-3n}{2}$

(iii) $a_1=2,\ a_n=a_{n-1}-\frac{3}{2}$ for $n>1$

(g) **(i)** Geometric: $r = \frac{6}{3} = \ldots = 2 \Rightarrow a_8 = a_1 \cdot r^7 = 3 \cdot 2^7 = 384$

(ii) $a_n = 3 \cdot 2^{n-1}$

(iii) $a_1 = 3,\ a_n = 2a_{n-1}$ for $n > 1$

(h) **(i)** Geometric: $r = \frac{12}{4} = \ldots = 3 \Rightarrow a_8 = a_1 \cdot r^7 = 4 \cdot 3^7 = 8748$

(ii) $a_n = 4 \cdot 3^{n-1}$

(iii) $a_1 = 4,\ a_n = 3a_{n-1}$ for $n > 1$

(i) **(i)** Geometric: $r = \frac{-5}{5} = \frac{5}{-5} = -1 \Rightarrow a_8 = a_1 \cdot r^7 = 5 \cdot (-1)^7 = -5$

(ii) $a_n = 5 \cdot (-1)^{n-1}$

(iii) $a_1 = 5,\ a_n = -a_{n-1}$ for $n > 1$

(j) **(i)** Geometric: $r = \frac{-6}{3} = \frac{12}{-6} = \ldots = -2 \Rightarrow a_8 = a_1 \cdot r^7 = 3 \cdot (-2)^7 = -384$

(ii) $a_n = 3 \cdot (-2)^{n-1}$

(iii) $a_1 = 3,\ a_n = -2a_{n-1}$ for $n > 1$

(k) The sequence is neither arithmetic nor geometric.

(l) **(i)** Geometric: $r = \frac{3}{-2} = \ldots = -\frac{3}{2} \Rightarrow a_8 = a_1 \cdot r^7 = (-2) \cdot \left(-\frac{3}{2}\right)^7 = \frac{2187}{64}$

(ii) $a_n = -2 \cdot \left(-\frac{3}{2}\right)^{n-1} = \frac{3^{n-1}}{(-2)^{n-2}}$

(iii) $a_1 = -2,\ a_n = \left(-\frac{3}{2}\right)a_{n-1}$ for $n > 1$

(m) **(i)** Geometric: $r = \frac{25}{35} = \ldots = \frac{5}{7} \Rightarrow a_8 = a_1 \cdot r^7 = 35 \cdot \left(\frac{5}{7}\right)^7 \approx 3.32$

(ii) $a_n = 35 \cdot \left(\frac{5}{7}\right)^{n-1} = \frac{5^n}{7^{n-2}}$

(iii) $a_1 = 35,\ a_n = \frac{5}{7}a_{n-1}$ for $n > 1$

(n) **(i)** Geometric: $r = \frac{-3}{-6} = \ldots = \frac{1}{2} \Rightarrow a_8 = a_1 \cdot r^7 = (-6) \cdot \left(\frac{1}{2}\right)^7 = -\frac{3}{64}$

(ii) $a_n = -6 \cdot \left(\frac{1}{2}\right)^{n-1} = -\frac{3}{2^{n-2}}$

(iii) $a_1 = -6,\ a_n = \frac{1}{2} a_{n-1}$ for $n > 1$

(o) **(i)** Geometric: $r = \frac{19}{9.5} = \ldots = 2 \Rightarrow a_8 = a_1 \cdot r^7 = 9.5 \cdot 2^7 = 1216$

(ii) $a_n = 9.5 \cdot (2)^{n-1}$

(iii) $a_1 = 9.5,\ a_n = 2a_{n-1}$ for $n > 1$

(p) **(i)** Geometric: $r = \frac{95}{100} = \ldots = 0.95 \Rightarrow a_8 = a_1 \cdot r^7 = 100 \cdot 0.95^7 \approx 69.83$

(ii) $a_n = 100 \cdot (0.95)^{n-1}$

(iii) $a_1 = 100,\ a_n = 0.95a_{n-1}$ for $n > 1$

(q) **(i)** Geometric: $r = \frac{\frac{3}{4}}{2} = \ldots = \frac{3}{8} \Rightarrow a_8 = a_1 \cdot r^7 = 2 \cdot \left(\frac{3}{8}\right)^7$

(ii) $a_n = 2 \cdot \left(\frac{3}{8}\right)^{n-1}$

(iii) $a_1 = 2,\ a_n = \frac{3}{8} a_{n-1}$ for $n > 1$

3. The sequence will have six terms including 3 and 96:

$a_1 = 3, a_6 = a_1 r^5 = 96$
$\Rightarrow 3r^5 = 96 \Rightarrow r^5 = 32 \Rightarrow r = 2$

Thus, the sequence will be $3, 6, 12, 24, 48, 96$

4. Similar to question **3**:

$a_1 = 7, a_5 = a_1 r^4 = 4375$
$\Rightarrow 7r^4 = 4375 \Rightarrow r^4 = 625 \Rightarrow r^4 = 5^4 \Rightarrow r = \pm 5$

Thus, the sequence will be $7, 35, 175, 875, 4375$ or $7, -35, 175, -875, 4375$

5. Similar to question **4**:

$$a_1 = 16, a_3 = a_1 r^2 = 81 \Rightarrow 16r^2 = 81 \Rightarrow r^2 = \frac{81}{16} \Rightarrow r = \pm\frac{9}{4}$$

Thus, the sequence will be 16, 36, 81 or 16, – 36, 81

6. Similar to question **5**:

$$a_1 = 7, a_6 = a_1 r^5 = 1701 \Rightarrow 7r^5 = 1701 \Rightarrow r^5 = 243 \Rightarrow r = 3$$

Thus, the sequence will be $7, 21, 63, 189, 567, 1701$

7. Similar to question **6**:

$$a_1 = 9, a_3 = a_1 r^2 = 64 \Rightarrow 9r^2 = 64 \Rightarrow r^2 = \frac{64}{9} \Rightarrow r = \pm\frac{8}{3}$$

Thus, the sequence will be 9, 24, 64 or 9, – 24, 64

8. There are 4 terms, thus:

$$a_1 = 24, a_4 = a_1 r^3 = 3 \Rightarrow 24r^3 = 3 \Rightarrow r^3 = \frac{3}{24} = \frac{1}{8} \Rightarrow r = \frac{1}{2}$$

$$\Rightarrow a_5 = a_1 r^4 = 24 \cdot \left(\frac{1}{2}\right)^4 = \frac{24}{16} = \frac{3}{2}$$

The *n*th term: $a_n = a_1 r^{n-1} = 24\left(\frac{1}{2}\right)^{n-1} = \frac{3}{2^{n-4}}$

9. $a_1 = 24, a_3 = a_1 r^2 = 6 \Rightarrow 24r^2 = 6 \Rightarrow r^2 = \frac{6}{24} = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2}$

There are two solutions:

$$\text{For } r = \frac{1}{2}: \; a_4 = a_1 r^3 = 24 \cdot \left(\frac{1}{2}\right)^3 = \frac{24}{8} = 3; a_n = 24\left(\frac{1}{2}\right)^{n-1} = \frac{3}{2^{n-4}}$$

$$\text{For } r = -\frac{1}{2}: \; a_4 = a_1 r^3 = 24 \cdot \left(-\frac{1}{2}\right)^3 = -\frac{24}{8} = -3; a_n = 24\left(-\frac{1}{2}\right)^{n-1}$$

10. We use the *n*th term formula for the 4th term.

$$r = \frac{2}{7}, a_4 = a_1 r^3 = \frac{14}{3} \Rightarrow a_1 = \frac{a_4}{r^3} = \frac{14}{3} / \left(\frac{2}{7}\right)^3 = \frac{2401}{12}$$

$$\Rightarrow a_3 = a_1 r^2 = \frac{2401}{12} \cdot \left(\frac{2}{7}\right)^2 = \frac{49}{3}$$

11. Let 118 098 be the nth term.

$$6,18,54,\ldots \Rightarrow a_1 = 6,\ r = 3$$
$$a_n = a_1 r^{n-1} \Rightarrow 118098 = 6 \cdot 3^{n-1} \Rightarrow 3^{n-1} = 19683$$
$$\Rightarrow 3^{n-1} = 3^9 \Rightarrow n-1 = 9 \Rightarrow n = 10$$

118 098 is the 10th term.

12. Using the nth term formula for the 4th and 7th terms:

$$a_4 = 18 \Rightarrow a_1 r^3 = 18 \quad \cdots\cdots(1)$$
$$a_7 = \frac{729}{8} \Rightarrow a_1 r^6 = \frac{729}{8} \quad \cdots\cdots(2)$$

We can divide (2) by (1) to obtain:

$$r^3 = \frac{81}{16} \Rightarrow r = \sqrt[3]{\frac{81}{16}} = \frac{3}{2}\sqrt[3]{\frac{3}{2}}, \text{ and so } a_1 = \frac{18}{r^3} = \frac{18}{\frac{81}{16}} = \frac{32}{9}$$

Now we can proceed as in the previous question:

$$\frac{59049}{128} = a_n = a_1 r^{n-1} \Rightarrow \frac{59049}{128} = \frac{32}{9} \cdot \left(\left(\frac{3}{2}\right)^{\frac{4}{3}}\right)^{n-1} \Rightarrow \left(\frac{3}{2}\right)^{\frac{4n-4}{3}} = \frac{531441}{4096}$$
$$\Rightarrow \left(\frac{3}{2}\right)^{\frac{4n-4}{3}} = \left(\frac{3}{2}\right)^{12} \Rightarrow \frac{4n-4}{3} = 12 \Rightarrow n = 10$$

Since n is a natural number, we can conclude that $\frac{59049}{128}$ is the 10th term of the sequence.

13. Using the nth term formula for the 3rd and 6th terms:

$$a_3 = 18 \Rightarrow a_1 r^2 = 18 \quad \cdots\cdots(1)$$
$$a_6 = \frac{243}{4} \Rightarrow a_1 r^5 = \frac{243}{4} \quad \cdots\cdots(2)$$

We can divide (2) by (1) to obtain:

$$r^3 = \frac{27}{8} \Rightarrow r = \frac{3}{2} \Rightarrow a_1 = \frac{18}{r^2} = \frac{18}{\frac{9}{4}} = 8$$

Now we can proceed as in the previous question:

$$\frac{19683}{64} = a_n = a_1 r^{n-1} \Rightarrow \frac{19683}{64} = 8 \cdot \left(\frac{3}{2}\right)^{n-1} \Rightarrow \left(\frac{3}{2}\right)^{n-1} = \frac{19683}{512}$$
$$\Rightarrow \left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^{9} \Rightarrow n-1 = 9 \quad \Rightarrow n = 10$$

Since n is a natural number, we can conclude that $\frac{19683}{64}$ is the 10th term of the sequence.

14. The interest is paid $n=2$ times per year. For $t = 10$ years, the annual interest rate $r=0.04$, and the principal $P=1500$, and so we have:

$$A=P\left(1+\frac{r}{n}\right)^{nt}=1500\left(1+\frac{0.04}{2}\right)^{2\times 10}=1500\cdot 1.02^{20}\approx 2228.92$$

Vitoria will have €2228.92 in her account.

15.

$$P=500,\ r=0.04,\ \ n=4,\ \ t=16$$

$$A=P\left(1+\frac{r}{n}\right)^{nt}=500\left(1+\frac{0.04}{4}\right)^{4\times 16}\approx 945.23$$

Jane will have £945.23 on her 16th birthday.

16.

$$A=4000,\ \ t=6,\ \ n=4,\ r=0.05$$

$$A=P\left(1+\frac{r}{n}\right)^{nt}\Rightarrow 4000=P\left(1+\frac{0.05}{4}\right)^{24}$$

$$\Rightarrow P=\frac{4000}{\left(1+\frac{0.05}{4}\right)^{24}}\approx 2968.79$$

You should invest €2968.79 now.

17. This situation can be modelled by a geometric sequence whose first term is 7554 and whose common ratio is 1.005. Since we count the population of 2017 among the terms, the number of terms is 6.

$$a_6=7554\cdot 1.005^5\approx 7744.748$$

The population in 2022 would be 7745.

18.

$$r=\frac{3}{7},a_4=\frac{14}{3}\Rightarrow a_2=\frac{a_4}{r^2}=\frac{\frac{14}{3}}{\left(\frac{3}{7}\right)^2}=\frac{686}{27}$$

19.

$$a_1=7,r=3\Rightarrow a_n=137781=7\cdot 3^{n-1}$$

$$\Rightarrow 3^{n-1}=19683\ \ \Rightarrow\ \ 3^{n-1}=3^9\ \ \Rightarrow\ \ n=10$$

137 781 is the 10th term.

20.

$$P = 1000,\ n = 4,\ t = 18,\ r = 0.06$$

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 1000\left(1+\frac{0.06}{4}\right)^{4\cdot 18} \approx 2921.16$$

Erik will have £2921.16 on his 18th birthday.

Exercise 3.4

1. First, we need to determine the number of terms.

$$a_1 = 11,\ d = 6, a_n = 365$$
$$a_n = a_1 + (n-1)d \Rightarrow 365 = 11 + (n-1)\cdot 6 \Rightarrow n = 60$$

The sum of the sequence is $S_{60} = \frac{60}{2}\left(11+365\right) = 11280$

2. The series is geometric. First, we need to determine the number of terms.

$$a_1 = 2,\ r = -\frac{3}{2}, a_n = -\frac{177147}{1024}$$
$$a_n = a_1 r^{n-1} \Rightarrow -\frac{177147}{1024} = 2\cdot\left(-\frac{3}{2}\right)^{n-1}$$
$$\Rightarrow \left(-\frac{3}{2}\right)^{n-1} = -\frac{177147}{2048} \Rightarrow \left(-\frac{3}{2}\right)^{n-1} = \left(-\frac{3}{2}\right)^{11} \Rightarrow n = 12$$

The sum of the series is $S_{12} = \dfrac{2\left(\left(-\frac{3}{2}\right)^{12} - 1\right)}{-\frac{3}{2} - 1} = -\dfrac{105469}{1024} \approx -103$

3. $\displaystyle\sum_{k=0}^{13}(2-0.3k) = 2 + 1.7 + 1.4 + ... + (-1.9)$

The series is arithmetic with 14 terms, $a_1 = 2$, and $d = -0.3$

The sum of the sequence is $S_{14} = \frac{14}{2}\left(2+(-1.9)\right) = 0.7 = \frac{7}{10}$

4. $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + ...$ is an infinite geometric series with $a_1 = 2$ and $r = -\frac{2}{5}$

The sum is $S_\infty = \dfrac{2}{1-\left(-\frac{2}{5}\right)} = \dfrac{10}{7}$

5. $\frac{1}{3}+\frac{\sqrt{3}}{12}+\frac{1}{16}+\frac{\sqrt{3}}{64}+\frac{3}{256}+\ldots$ is an infinite geometric series with $a_1=\frac{1}{3}$ and $r=\frac{\sqrt{3}}{4}$

The sum is $S_\infty=\dfrac{\frac{1}{3}}{1-\frac{\sqrt{3}}{4}}=\dfrac{4}{3(4-\sqrt{3})}\cdot\dfrac{4+\sqrt{3}}{4+\sqrt{3}}=\dfrac{16+4\sqrt{3}}{39}$

6. In each case we have an infinite geometric series:

(a)

$$0.\overline{52}=0.52\,52\,52\,52\ldots=0.52+0.0052+0.000052+\ldots$$
$$=52\cdot10^{-2}+52\cdot10^{-4}+52\cdot10^{-6}+\ldots$$
$$a_1=\frac{52}{100},\ r=\frac{1}{100}\Rightarrow 0.\overline{52}=S_\infty=\frac{\frac{52}{100}}{1-\frac{1}{100}}=\frac{52}{99}$$

(b)

$$0.4\overline{53}=0.4\,53\,53\,53\,53\ldots=0.4+0.053+0.00053+0.0000053+\cdots$$
$$=0.4+53\cdot10^{-3}+53\cdot10^{-5}+53\cdot10^{-7}+\ldots=0.4+S_\infty$$
$$a_1=\frac{53}{1000},\ r=\frac{1}{100}$$
$$0.4\overline{53}=0.4+S_\infty=0.4+\frac{\frac{53}{1000}}{1-\frac{1}{100}}=\frac{4}{10}+\frac{53}{990}=\frac{449}{990}$$

(c)

$$3.01\overline{37}=3.01\,37\,37\,37\ldots=3.01+0.0037+0.00\,0037+0.00000037+\ldots$$
$$=3.01+37\cdot10^{-4}+37\cdot10^{-6}+37\cdot10^{-8}+\ldots=3.01+S_\infty$$
$$a_1=\frac{37}{10<t>000},\ r=\frac{1}{100}\Rightarrow$$
$$3.01\overline{37}=3.01+S_\infty=3.01+\frac{\frac{37}{10\,000}}{1-\frac{1}{100}}=\frac{301}{100}+\frac{37}{9900}=\frac{29836}{9900}=\frac{7459}{2475}$$

7. Maggie invests \$150 (R) at the beginning of every month for six years, so we are calculating future value (FV) for annuity due. For an annual rate of $r=0.06$ and $m=6\times12=72$ periods:

$$i=\frac{0.06}{12}=0.005$$
$$FV=R\left(\frac{(1+i)^{m+1}-1}{i}-1\right)=150\left(\frac{(1+0.005)^{73}-1}{0.005}-1\right)=13026.135$$

There will be \$13,026.14 in her account after six years.

8. **(a)** The series $9+13+17+...+85$ is arithmetic.
We need to determine the number of terms in the series.
$a_1=9,\ d=4,\ a_n=85$
$a_n=a_1+(n-1)d\ \Rightarrow 85=9+(n-1)\cdot 4\ \Rightarrow n=20$
The partial sum of the series is $S_{20}=\frac{20}{2}(9+85)=940$

(b) The series $8+14+20+...+278$ is arithmetic.
We need to determine the number of terms in the series.
$a_1=8,\ d=6,\ a_n=278$
$a_n=a_1+(n-1)d\ \Rightarrow 278=8+(n-1)\cdot 6\ \Rightarrow n=46$
The sum of the series is $S_{46}=\frac{46}{2}(8+278)=6578$

(c) The series $155+158+161+...+527$ is arithmetic.
We need to determine the number of terms in the series.
$a_1=155,\ d=3,\ a_n=527$
$a_n=a_1+(n-1)d\ \Rightarrow 527=155+(n-1)\cdot 3\ \Rightarrow n=125$
The sum of the series is $S_{125}=\frac{125}{2}(155+527)=42625$

9.
$a_k=2+3k\ \Rightarrow a_1=2+3\cdot 1=5,\ a_n=2+3n$
$S_n=\frac{n}{2}[5+2+3n]=\frac{n(3n+7)}{2}$

10. For the arithmetic series $17+20+23...$ we have:
$a_1=17,\ d=3\Rightarrow S_n=\frac{n}{2}[2a_1+(n-1)d]=\frac{n}{2}[34+(n-1)\cdot 3]=\frac{n(3n+31)}{2}$
$S_n>678\ \Rightarrow \frac{n(3n+31)}{2}>678\ \Rightarrow 3n^2+31n-1356>0$
The solutions of the quadratic equation $3n^2+31n-1356=0$ are 16.71 and -27.05, so the solutions of the inequality are $n>16.71$ or $n<-27.05$. Since $n\in\mathbb{N}$, we conclude that we need to add 17 terms to exceed 678.

11. For the arithmetic series $-18-11-4...$ we have:
$a_1=-18,\ d=7\Rightarrow S_n=\frac{n}{2}[2a_1+(n-1)d]=\frac{n}{2}[-36+(n-1)\cdot 7]=\frac{n(7n-43)}{2}$
$S_n>2335\ \Rightarrow \frac{n(7n-43)}{2}>2335\ \Rightarrow 7n^2-43n-4670>0$
The solutions of the quadratic equation $7n^2-43n-4670=0$ are 29.08 and -22.94, so the solutions of the inequality are $n>29.08$ or $n<-22.94$. Since $n\in\mathbb{N}$, we conclude that we need to add 30 terms to exceed 2335.

12. First sequence:

$$a, a+2d, a+4d, \ldots, a+98d$$

$$S_{50}=\frac{50}{2}(a+a+98d)=25(2a+98d)=T$$

Second sequence:

$$a+d, a+3d, a+5d, \ldots, a+99d$$

Combined sequence:

$$a, a+d, a+2d, a+3d, \ldots, a+99d$$

$$S_{100}=\frac{100}{2}(a+a+99d)=50(2a+99d)=S$$

Then:

$$2T+200=S \quad \Rightarrow 2\cdot 25(2a+98d)+200=50(2a+99d)$$

$$\Rightarrow 100a+4900d+200=100a+4950d \quad \Rightarrow 50d=200 \Rightarrow d=4$$

13. **(a)** For the arithmetic sequence $3, 7, 11, \ldots, 999$ we have:

$$a_1=3,\ d=4,\ a_n=999$$

$$a_n=a_1+(n-1)d \quad \Rightarrow 999=3+(n-1)\cdot 4 \quad \Rightarrow n=250$$

$$S_{250}=\frac{250}{2}(3+999)=125250$$

(b) The removed terms, $11, 23, 35, \ldots, 995$ form an arithmetic sequence with 83 terms and $b_1=11$ and $d=12 \Rightarrow S_{83}=\frac{83}{2}[2\cdot 11+82\cdot 12]=41749$

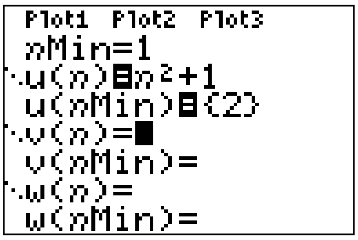
The sum of the remaining terms is then $125250-41749=83501$

14. We have the following system of simultaneous equations that can be solved by any method of your choice:

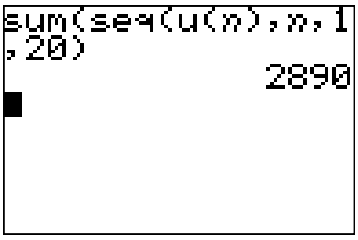
$$\begin{cases} a+(a+d)+(a+2d)+\ldots+(a+9d)=235 \\ (a+10d)+(a+11d)+\ldots+(a+19d)=735 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{10}{2}[a+(a+9d)]=235 \\ \frac{10}{2}[(a+10d)+(a+19d)]=735 \end{cases} \Rightarrow \begin{cases} 2a+9d=47 \\ 2a+29d=147 \end{cases} \Rightarrow d=5,\ a=1$$

15. **(a)** For $\sum_{k=1}^{20}(k^2+1)$, using a GDC in Sequential mode:

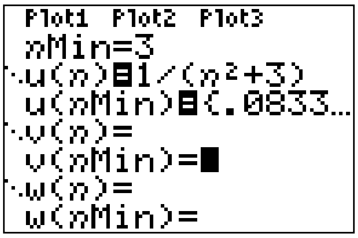


```
Plot1 Plot2 Plot3
nMin=1
·u(n)■n²+1
 u(nMin)■{2}
·v(n)=■
 v(nMin)=
·w(n)=
 w(nMin)=
```

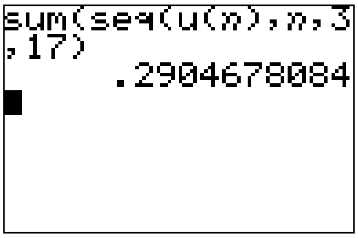


```
sum(seq(u(n),n,1
,20)
              2890
■
```

(b) For $\sum_{i=3}^{17}\left(\frac{1}{i^2+3}\right)$:

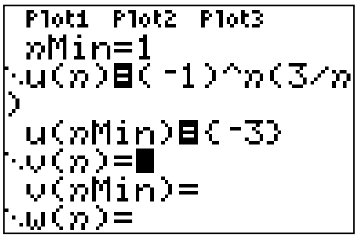


```
Plot1 Plot2 Plot3
nMin=3
·u(n)=1/(n²+3)
 u(nMin)={.0833…
·v(n)=
 v(nMin)=█
·w(n)=
 w(nMin)=
```

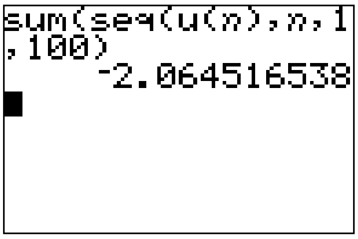


```
sum(seq(u(n),n,3
,17)
       .2904678084
█
```

(c) For $\sum_{n=1}^{100}(-1)^n\frac{3}{n}$:



```
Plot1 Plot2 Plot3
nMin=1
·u(n)=(-1)^n(3/n
)
 u(nMin)={-3}
·v(n)=█
 v(nMin)=
·w(n)=
```



```
sum(seq(u(n),n,1
,100)
      -2.064516538
█
```

16. For the arithmetic series $13+19+...+367$ we have:

$a_1=13,\ d=6, a_n=367$

$a_n=a_1+(n-1)d\ \Rightarrow 367=13+(n-1)\cdot 6\ \Rightarrow n=60$

$$S_{60}=\frac{60}{2}(13+367)=11400$$

17. For the geometric series $2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+.....-\frac{4096}{177147}$ we have:

$a_1=2,\ r=-\frac{2}{3}, a_n=-\frac{4096}{177147}$

$a_n=a_1r^{n-1}\ \Rightarrow 2\cdot\left(-\frac{2}{3}\right)^{n-1}=-\frac{4096}{177147}\ \Rightarrow \left(-\frac{2}{3}\right)^{n-1}=\left(-\frac{2}{3}\right)^{11}\ \Rightarrow n=12$

$$S_{12}=a_1\frac{1-r^{12}}{1-r}=2\frac{1-\left(-\frac{2}{3}\right)^{12}}{1-\left(-\frac{2}{3}\right)}\approx 1.191$$

18. $\sum_{k=0}^{11}(3+0.2k)=3+3.2+3.4+...+5.2$ is an arithmetic series with:

$a_1=3,\ d=0.2,\ n=12.$ So: $\sum_{k=0}^{11}(3+0.2k)=S_{12}=\frac{12}{2}(3+5.2)=49.2$

19. For the infinite geometric series $2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+\cdots$ we have:

$$a_1=2,\ r=-\frac{2}{3}\Rightarrow S_\infty=\frac{a_1}{1-r}=\frac{2}{1+\frac{2}{3}}=\frac{6}{5}$$

20. For the infinite geometric series $\frac{1}{2}+\frac{\sqrt{2}}{2\sqrt{3}}+\frac{1}{3}+\frac{\sqrt{2}}{3\sqrt{3}}+\frac{2}{9}\ldots$ we have:

$$a_1=\frac{1}{2},\ r=\frac{\sqrt{2}}{\sqrt{3}}$$

$$S_\infty=\frac{a_1}{1-r}=\frac{\frac{1}{2}}{1-\frac{\sqrt{2}}{\sqrt{3}}}=\frac{\sqrt{3}}{2\left(\sqrt{3}-\sqrt{2}\right)}=\frac{\sqrt{3}}{2\left(\sqrt{3}-\sqrt{2}\right)}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{3+\sqrt{6}}{2}$$

21. **(a)** For $u_n=\frac{3}{5^n}$ we have a geometric series:

$$S_1=\frac{3}{5};\ S_2=\frac{3}{5}+\frac{3}{25}=\frac{18}{25};\ S_3=\frac{3}{5}+\frac{3}{25}+\frac{3}{125}=\frac{93}{125}$$
$$S_4=\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}=\frac{468}{625}$$
$$S_n=3\frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}}=\frac{15}{4}\left(1-\frac{1}{5^n}\right)$$

(b) For this series, we need to spot the pattern (or use partial fractions) and try to relate it to the term number, n. Thus, $v_n=\frac{1}{n^2+3n+2}=\frac{1}{n+1}-\frac{1}{n+2}$

$$S_1=\frac{1}{6}=\frac{1}{2\cdot 3};\ S_2=\frac{1}{6}+\frac{1}{12}=\frac{3}{12}=\frac{1}{4}=\frac{2}{8}$$
$$S_3=\frac{1}{6}+\frac{1}{12}+\frac{1}{20}=\frac{6}{20}=\frac{3}{10};\ S_4=\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}=\frac{10}{30}=\frac{1}{3}=\frac{4}{12}$$

The pattern should be apparent by now: $S_n=\frac{n}{2n+4}$. Alternatively, you can list the terms and simplify as shown on the next page:

$$\frac{1}{6}=\frac{1}{2}-\cancel{\frac{1}{3}}$$

$$\frac{1}{12}=\cancel{\frac{1}{3}}-\frac{1}{4}$$

$$\vdots=\vdots-\cancel{\vdots}$$

$$\frac{1}{n^2+3n+2}=\cancel{\frac{1}{n+1}}-\frac{1}{n+2}$$

$$\sum\frac{1}{n^2+3n+2}=\frac{1}{2}-\frac{1}{n+2}$$

(c) For $u_n=\sqrt{n+1}-\sqrt{n}$ we have:

$$S_1=\sqrt{2}-\sqrt{1}=\sqrt{2}-1;\ S_2=\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)=\sqrt{3}-1$$

$$S_3=\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)=\sqrt{4}-1=1$$

$$S_4=\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)+\left(\sqrt{5}-\sqrt{4}\right)=\sqrt{5}-1$$

$$S_n=\sqrt{n+1}-1$$

22. The heights that the ball reaches after each bounce form an infinite geometric sequence: $16\cdot 0.81, 16\cdot 0.81^2,\ldots$

(a) After the10th bounce: $16\cdot 0.81^{10}\approx 1.945\,\text{m}$

(b) $16+2\cdot\left(16\cdot 0.81+16\cdot 0.81^2+16\cdot 0.81^3+\ldots.\right)=16+2\cdot\dfrac{12.96}{1-0.81}\approx 152.42\,\text{m}$

23. The shaded area in the first square is made up of two right-angled triangles that measure 8 cm on each side. When the two triangles are joined at their hypotenuse, they make a square with side length 8. Thus, the shaded area is $8^2=64\text{ cm}^2$.

In each successive square, each of the shaded triangles is half the size of the previous ones, thus the new shaded area is one half of the shaded area in the previous square.

The shaded area in the second square is $\frac{1}{2}\cdot 64=32\text{ cm}^2$, in the third square it is 16 cm^2, etc.

Total shaded area forms a geometric series with $a_1=64,\ r=\frac{1}{2}$

(a) $S_{10}=64\cdot\dfrac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}}=\dfrac{1023}{8}=127.875$

(b) $S_\infty=\dfrac{64}{1-\frac{1}{2}}=128$

24. **(a)** The first shaded area is $4\cdot 2-2\cdot 1=6$

The second shaded area is $1\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}=\frac{3}{8}$

The third shaded area is $\frac{1}{4}\cdot\frac{1}{8}-\frac{1}{8}\cdot\frac{1}{16}=\frac{3}{128}$

Total shaded area is $6+\frac{3}{8}+\frac{3}{128}=\frac{819}{128}$

(b) If the process were repeated indefinitely, the total shaded area forms an infinite geometric sequence with $a_1=6,\ r=\frac{1}{16}$:

$$S_\infty=\frac{6}{1-\frac{1}{16}}=\frac{32}{5}$$

25. **(a)** Arithmetic series:

$a_1=7,\ d=5,\ a_n=342$

$a_n=a_1+(n-1)d\ \Rightarrow 342=7+(n-1)\cdot 5\ \Rightarrow n=68$

$$S_{68}=\frac{68}{2}\left(7+342\right)=11866$$

(b) Arithmetic series:

$a_1=9486,\ d=-7,\ a_n=8912$

$a_n=a_1+(n-1)d\ \Rightarrow 8912=9486+(n-1)\cdot(-7)\ \Rightarrow n=83$

$$S_{83}=\frac{83}{2}\left(9486+8912\right)=763517$$

(c) Geometric series:

$a_1=2,\ r=3, a_n=9565938$

$a_n=a_1r^{n-1}\ \Rightarrow 2\cdot 3^{n-1}=9565938\ \Rightarrow 3^{n-1}=4782969\ \Rightarrow 3^{n-1}=3^{14}\ \Rightarrow n=15$

$$S_{15}=a_1\frac{r^{15}-1}{r-1}=2\cdot\frac{3^{15}}{3-1}=14348907$$

(d) Geometric series:

$$a_1 = 120,\ r = \frac{1}{5}, a_n = \frac{24}{78125}$$

$$a_n = a_1 r^{n-1} \Rightarrow 120\cdot\left(\frac{1}{5}\right)^{n-1} = \frac{24}{78125} \Rightarrow \left(\frac{1}{5}\right)^{n-1} = \frac{1}{390\,625} \Rightarrow \left(\frac{1}{5}\right)^{n-1} = \left(\frac{1}{5}\right)^{8} \Rightarrow n = 9$$

$$S_9 = 120\cdot\frac{1-\left(\frac{1}{5}\right)^{9}}{1-\frac{1}{5}} \approx 150$$

Exercise 3.5

Note: In all calculations, we will use the following equivalent notations interchangeably:

$${}_nC_x = {}^nC_x = \binom{n}{x} = \frac{n!}{x!(n-x)!}$$

1. **(a)** Using the 5th row of Pascal's triangle:

$$\begin{aligned}(x+2y)^5 &= 1x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + 1(2y)^5\\ &= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5\end{aligned}$$

(b) Using the 4th row:

$$\begin{aligned}(a-b)^4 &= 1a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + 1(-b)^4\\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$

(c) Using the 6th row:

$$\begin{aligned}(x-3)^6 &= 1x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + 1(-3)^6\\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729\end{aligned}$$

(d) Using the 4th row:

$$\begin{aligned}(2-x^3)^4 &= 1\cdot 2^4 + 4\cdot 2^3(-x^3) + 6\cdot 2^2(-x^3)^2 + 4\cdot 2(-x^3)^3 + 1\cdot(-x^3)^4\\ &= 16 - 32x^3 + 24x^6 - 8x^9 + x^{12}\end{aligned}$$

(e) Using the 7th row:

$$\begin{aligned}(x-3b)^7 &= 1\cdot x^7 + 7x^6(-3b) + 21x^5(-3b)^2 + 35x^4(-3b)^3\\ &\quad + 35x^3(-3b)^4 + 21x^2(-3b)^5 + 7x(-3b)^6 + (-3b)^7\\ &= x^7 - 21x^6b + 189x^5b^2 - 945x^4b^3 + 2835x^3b^4\\ &\quad - 5103x^2b^5 + 5103xb^6 - 2187b^7\end{aligned}$$

(f) Using the 6th row:

$$\begin{aligned}\left(2n+\frac{1}{n^2}\right)^6 &= 1\cdot(2n)^6+6(2n)^5\frac{1}{n^2}+15(2n)^4\left(\frac{1}{n^2}\right)^2+20(2n)^3\left(\frac{1}{n^2}\right)^3\\ &\quad +15(2n)^2\left(\frac{1}{n^2}\right)^4+6(2n)\left(\frac{1}{n^2}\right)^5+1\cdot\left(\frac{1}{n^2}\right)^6\\ &= 64n^6+192n^3+240+\frac{160}{n^3}+\frac{60}{n^6}+\frac{12}{n^9}+\frac{1}{n^{12}}\end{aligned}$$

(g) Using the 4th row:

$$\begin{aligned}\left(\frac{3}{x}-2\sqrt{x}\right)^4 &= 1\cdot\left(\frac{3}{x}\right)^4+4\left(\frac{3}{x}\right)^3(-2\sqrt{x})+6\left(\frac{3}{x}\right)^2(-2\sqrt{x})^2\\ &\quad +4\left(\frac{3}{x}\right)^1(-2\sqrt{x})^3+1\cdot(-2\sqrt{x})^4\\ &= \frac{81}{x^4}-\frac{216\sqrt{x}}{x^3}+\frac{216}{x}-96\sqrt{x}+16x^2\end{aligned}$$

2. **(a)** $$\binom{8}{3}=\frac{8!}{3!5!}=\frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot\cancel{6}\cdot 7\cdot 8}{(1\cdot\cancel{2}\cdot\cancel{3})(\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5})}=56$$

(b) $$\binom{18}{5}-\binom{18}{13}=\frac{18!}{5!13!}-\frac{18!}{13!5!}=0$$

(c) $$\binom{7}{4}\binom{7}{3}=\frac{7!}{4!3!}\cdot\frac{7!}{3!4!}=\left(\frac{7!}{3!4!}\right)^2=\left(\frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot 5\cdot\cancel{6}\cdot 7}{(1\cdot\cancel{2}\cdot\cancel{3})(\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4})}\right)^2=35^2=1225$$

(d) $$\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}=(1+1)^5=32$$

(e) $$\binom{6}{0}-\binom{6}{1}+\binom{6}{2}-\binom{6}{3}+\binom{6}{4}-\binom{6}{5}+\binom{6}{6}=(1-1)^6=0$$

3. **(a)**

$$
\begin{aligned}
(x-2y)^7 &= \sum_{i=0}^{7}\binom{7}{i}x^{n-i}(-2y)^i \\
&= x^7 + \binom{7}{1}x^6(-2y)^1 + \binom{7}{2}x^5(-2y)^2 + \binom{7}{3}x^4(-2y)^3 + \binom{7}{4}x^3(-2y)^4 \\
&\quad + \binom{7}{5}x^2(-2y)^5 + \binom{7}{6}x^1(-2y)^6 + (-2y)^7 \\
&= x^7 - 14x^6y + 84x^5y^2 - 280x^4y^3 + 560x^3y^4 \\
&\quad - 672x^2y^5 + 448xy^6 - 128y^7
\end{aligned}
$$

(b)

$$
\begin{aligned}
(2a-b)^6 &= \sum_{i=0}^{6}\binom{6}{i}(2a)^{6-i}(-b)^i \\
&= \binom{6}{0}(2a)^6 + \binom{6}{1}(2a)^{6-1}(-b)^1 + \binom{6}{2}(2a)^{6-2}(-b)^2 + \binom{6}{3}(2a)^{6-3}(-b)^3 \\
&\quad + \binom{6}{4}(2a)^{6-4}(-b)^4 + \binom{6}{5}(2a)^{6-5}(-b)^5 + \binom{6}{6}(-b)^6 \\
&= 64a^6 - 192a^5b + 240a^4b^2 - 160a^3b^3 + 60a^2b^4 - 12ab^5 + b^6
\end{aligned}
$$

(c)

$$
\begin{aligned}
(x-4)^5 &= \sum_{i=0}^{5}\binom{5}{i}x^{5-i}(-4)^i \\
&= \binom{5}{0}x^{5-i}(-4)^i + \binom{5}{1}x^{5-1}(-4)^1 + \binom{5}{2}x^{5-2}(-4)^2 + \binom{5}{3}x^{5-3}(-4)^3 \\
&\quad + \binom{5}{4}x^{5-4}(-4)^4 + \binom{5}{5}x^{5-5}(-4)^5 \\
&= x^5 - 20x^4 + 160x^3 - 640x^2 + 1280x - 1024
\end{aligned}
$$

(d)

$$
\begin{aligned}
(2+x^3)^6 &= \sum_{i=0}^{6}\binom{6}{i}2^{6-i}(x^3)^i \\
&= \binom{6}{0}2^6 + \binom{6}{1}2^{6-1}(x^3)^1 + \binom{6}{2}2^{6-2}(x^3)^2 + \binom{6}{3}2^{6-3}(x^3)^3 \\
&\quad + \binom{6}{4}2^{6-4}(x^3)^4 + \binom{6}{5}2^{6-5}(x^3)^5 + \binom{6}{6}(x^3)^6 \\
&= 64 + 192x^3 + 240x^6 + 160x^9 + 60x^{12} + 12x^{15} + x^{18}
\end{aligned}
$$

(e)

$$\begin{aligned}(3x-b)^7 &= \sum_{i=0}^{7}\binom{7}{i}(3x)^{7-i}(-b)^i\\ &= \binom{7}{0}(3x)^{7-0}(-b)^0+\binom{7}{1}(3x)^{7-1}(-b)^1+\binom{7}{2}(3x)^{7-2}(-b)^2+\binom{7}{3}(3x)^{7-3}(-b)^3\\ &\quad+\binom{7}{4}(3x)^{7-4}(-b)^4+\binom{7}{5}(3x)^{7-5}(-b)^5+\binom{7}{6}(3x)^{7-6}(-b)^6+\binom{7}{7}(3x)^{7-7}(-b)^7\\ &= 2187x^7-5103x^6b+5103x^5b^2-2835x^4b^3+945x^3b^4-189x^2b^5+21xb^6-b^7\end{aligned}$$

(f)

$$\begin{aligned}\left(2n-\frac{1}{n^2}\right)^6 &= \sum_{i=0}^{6}\binom{6}{i}(2n)^{6-i}\left(-\frac{1}{n^2}\right)^i\\ &= \binom{6}{0}(2n)^6+\binom{6}{1}(2n)^{6-1}\left(-\frac{1}{n^2}\right)^1+\binom{6}{2}(2n)^{6-2}\left(-\frac{1}{n^2}\right)^2+\binom{6}{3}(2n)^{6-3}\left(-\frac{1}{n^2}\right)^3\\ &\quad+\binom{6}{4}(2n)^{6-4}\left(-\frac{1}{n^2}\right)^4+\binom{6}{5}(2n)^{6-5}\left(-\frac{1}{n^2}\right)^5+\binom{6}{6}\left(-\frac{1}{n^2}\right)^6\\ &= 64n^6-192n^3+240-\frac{160}{n^3}+\frac{60}{n^6}-\frac{12}{n^9}+\frac{1}{n^{12}}\end{aligned}$$

(g)

$$\begin{aligned}\left(\frac{2}{x}-3\sqrt{x}\right)^4 &= \sum_{i=0}^{4}\binom{4}{i}\left(\frac{2}{x}\right)^{4-i}(-3\sqrt{x})^i\\ &= \binom{4}{0}\left(\frac{2}{x}\right)^{4-0}+\binom{4}{1}\left(\frac{2}{x}\right)^{4-1}(-3\sqrt{x})^1+\binom{4}{2}\left(\frac{2}{x}\right)^{4-2}(-3\sqrt{x})^2\\ &\quad+\binom{4}{3}\left(\frac{2}{x}\right)^{4-3}(-3\sqrt{x})^3+\binom{4}{4}(-3\sqrt{x})^4\\ &= \frac{16}{x^4}-\frac{96}{x^{\frac{5}{2}}}+\frac{216}{x}-216\sqrt{x}+81x^2\end{aligned}$$

(h)

$$
\begin{aligned}
\left(1+\sqrt{5}\right)^4+\left(1-\sqrt{5}\right)^4 &= \sum_{i=0}^{4}\binom{4}{i}1^{4-i}(\sqrt{5})^i+\sum_{i=0}^{4}\binom{4}{i}1^{4-i}(-\sqrt{5})^i \\
&= \binom{4}{0}(\sqrt{5})^0+\binom{4}{1}(\sqrt{5})^1+\binom{4}{2}(\sqrt{5})^2+\binom{4}{3}(\sqrt{5})^3+\binom{4}{4}(\sqrt{5})^4 \\
&\quad -\left[\binom{4}{0}(\sqrt{5})^0-\binom{4}{1}(\sqrt{5})^1+\binom{4}{2}(\sqrt{5})^2-\binom{4}{3}(\sqrt{5})^3+\binom{4}{4}(\sqrt{5})^4\right] \\
&= 2\left(\binom{4}{0}(\sqrt{5})^0+\binom{4}{2}(\sqrt{5})^2+\binom{4}{4}(\sqrt{5})^4\right)=2(1+30+25)=112
\end{aligned}
$$

(i)

$$
\begin{aligned}
\left(\sqrt{3}+1\right)^8-\left(\sqrt{3}-1\right)^8 &= \sum_{i=0}^{8}\binom{8}{i}(\sqrt{3})^{8-i}1^i-\sum_{i=0}^{8}\binom{8}{i}(\sqrt{3})^{8-i}(-1)^i \\
&= (\sqrt{3})^8+\binom{8}{1}(\sqrt{3})^7+\binom{8}{2}(\sqrt{3})^6+\binom{8}{3}(\sqrt{3})^5+\binom{8}{4}(\sqrt{3})^4 \\
&\quad +\binom{8}{5}(\sqrt{3})^3+\binom{8}{6}(\sqrt{3})^2+\binom{8}{7}(\sqrt{3})^1+(\sqrt{3})^0 \\
&\quad -\left[\begin{array}{l}(\sqrt{3})^8-\binom{8}{1}(\sqrt{3})^7+\binom{8}{2}(\sqrt{3})^6-\binom{8}{3}(\sqrt{3})^5+\binom{8}{4}(\sqrt{3})^4 \\ \quad -\binom{8}{5}(\sqrt{3})^3+\binom{8}{6}(\sqrt{3})^2-\binom{8}{7}(\sqrt{3})^1+(\sqrt{3})^0\end{array}\right] \\
&= 2\left[\binom{8}{1}(\sqrt{3})^7+\binom{8}{3}(\sqrt{3})^5+\binom{8}{5}(\sqrt{3})^3+\binom{8}{7}(\sqrt{3})^1\right] \\
&= 2\left[216\sqrt{3}+504\sqrt{3}+168\sqrt{3}+8\sqrt{3}\right]=1792\sqrt{3}
\end{aligned}
$$

4. **(a)** and **(c)**

$$
\begin{aligned}
\left(x-\frac{2}{x}\right)^{45} &= \sum_{i=0}^{45}\binom{45}{i}x^{45-i}\left(-\frac{2}{x}\right)^i \\
&= \binom{45}{0}x^{45}+\binom{45}{1}x^{44}\left(-\frac{2}{x}\right)^1+\binom{45}{2}x^{43}\left(-\frac{2}{x}\right)^2+ \\
&\quad \ldots+\binom{45}{43}x^2\left(-\frac{2}{x}\right)^{43}+\binom{45}{44}x^1\left(-\frac{2}{x}\right)^{44}+\binom{45}{45}x^0\left(-\frac{2}{x}\right)^{45} \\
&= x^{45}-90x^{43}+3960x^{41}+\ldots-\frac{990\cdot 2^{43}}{x^{41}}+\frac{45\cdot 2^{44}}{x^{43}}-\frac{2^{45}}{x^{45}}
\end{aligned}
$$

(b) For the constant term, i should be:

$$x^{45-i}=\frac{1}{x^i}\Rightarrow x^{45-i}=x^i\Rightarrow 45-i=i\Rightarrow i=\frac{45}{2}$$

Since i is a natural number this is not possible,
i.e. there is no constant term in the expansion.

(d) For the term containing x^3, i should be:

$$x^{45-i}\frac{1}{x^i}=x^3\Rightarrow x^{45-2i}=x^3\Rightarrow 45-2i=3\Rightarrow i=21$$

For $i=21$, the 22nd term is: $\binom{45}{21}x^{45-21}\left(-\frac{2}{x}\right)^{21}=-\binom{45}{21}2^{21}x^3$

5. $$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n!}{(n-k)!(n-(n-k))!}=\binom{n}{n-k}$$

6.
$$2^n=(1+1)^n=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}\cdot 1+\ldots+\binom{n}{n}1^n$$
$$=1+\binom{n}{1}+\ldots+\binom{n}{n}$$
$$\Rightarrow\binom{n}{1}+\ldots+\binom{n}{n}=2^n-1$$

7. **(a)** $$k!=k(k-1)\cdot(k-2)\cdots 3\cdot 2\cdot 1=k\big((k-1)\cdot(k-2)\cdots 3\cdot 2\cdot 1\big)=k(k-1)!$$

(b)
$$(n-k+1)!=(n-k+1)\cdot(n-k)\cdots 3\cdot 2\cdot 1$$
$$=(n-k+1)\big((n-k)\cdots 3\cdot 2\cdot 1\big)=(n-k+1)(n-k)!$$

(c) $${}_nC_{r-1}+{}_nC_r=\frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!}$$

By multiplying the first fraction by $\frac{r}{r}$ and the second by $\frac{n-r+1}{n-r+1}$,

both denominators will be equal:

$${}_nC_{r-1}+{}_nC_r=\frac{n!\cdot r}{r\cdot(r-1)!(n-r+1)!}+\frac{n!\cdot(n-r+1)}{r!(n-r)!\cdot(n-r+1)}$$
$$=\frac{n!\cdot r}{r!(n-r+1)!}+\frac{n!\cdot(n-r+1)}{r!(n-r+1)!}=\frac{n!\cdot r+n!\cdot(n-r+1)}{r!(n-r+1)!}$$

Now factor out $n!$ in the numerator:

$$_nC_{r-1} + {}_nC_r = \frac{n!(r+n-r+1)}{r!(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}_{n+1}C_r$$

8. This appears to be the binomial expansion of $\left(\frac{1}{3}+\frac{2}{3}\right)^6$, i.e.,

$$\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \ldots + \binom{6}{6}\left(\frac{2}{3}\right)^6 = \left(\frac{1}{3}+\frac{2}{3}\right)^6 = 1$$

9. $$\binom{8}{0}\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right) + \binom{8}{2}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^2 + \ldots + \binom{8}{8}\left(\frac{3}{5}\right)^8 = \left(\frac{2}{5}+\frac{3}{5}\right)^8 = 1$$

10. $$\binom{n}{0}\left(\frac{1}{7}\right)^n + \binom{n}{1}\left(\frac{1}{7}\right)^{n-1}\left(\frac{6}{7}\right) + \binom{n}{2}\left(\frac{1}{7}\right)^{n-2}\left(\frac{6}{7}\right)^2 + \ldots + \binom{n}{n}\left(\frac{6}{7}\right)^n = \left(\frac{1}{7}+\frac{6}{7}\right)^n = 1^n = 1$$

11. $$\left(x^2-\frac{1}{x}\right)^6 = \sum_{i=0}^{6}\binom{6}{i}\left(x^2\right)^{6-i}\left(-\frac{1}{x}\right)^i$$

To determine i, we have to solve the exponential equation:

$$\left(x^2\right)^{6-i}\left(\frac{1}{x}\right)^i = 1 \quad \Rightarrow x^{12-2i-i} = x^0 \quad \Rightarrow 12-3i=0 \quad \Rightarrow i=4$$

So, for $i = 4$, we have the term:

$$\binom{6}{4}\left(x^2\right)^{6-4}\left(-\frac{1}{x}\right)^4 = 15\left(x^4\right)\left(\frac{1}{x^4}\right) = 15$$

12. $$\left(3x-\frac{2}{x}\right)^8 = \sum_{i=0}^{8}\binom{8}{i}(3x)^{8-i}\left(-\frac{2}{x}\right)^i$$

To determine i, we have to solve the exponential equation:

$$(x)^{8-i}\left(\frac{1}{x}\right)^i = 1 \quad \Rightarrow x^{8-i-i} = x^0 \quad \Rightarrow 8-2i=0 \quad \Rightarrow i=4$$

So, for $i = 4$, we have the term:

$$\binom{8}{4}(3x)^{8-4}\left(-\frac{2}{x}\right)^4 = 70\left(81x^4\right)\left(\frac{16}{x^4}\right) = 90\,720$$

13. $$\left(2x-\frac{3}{x^3}\right)^8 = \sum_{i=0}^{8}\binom{8}{i}(2x)^{8-i}\left(-\frac{3}{x^3}\right)^i$$

To determine i, we have to solve the exponential equation:

$$(x)^{8-i}\left(\frac{1}{x^3}\right)^i = 1 \quad \Rightarrow x^{8-i-3i} = x^0 \quad \Rightarrow 8-4i = 0 \quad \Rightarrow i = 2$$

So, for $i = 2$, we have the term:

$$\binom{8}{2}(2x)^{8-2}\left(-\frac{3}{x^3}\right)^2 = 28\left(64x^6\right)\left(\frac{9}{x^6}\right) = 16128$$

14. $$(1+x)^{10} = 1+\binom{10}{1}x+\binom{10}{2}x^2+\cdots = 1+10x+45x^2+\cdots$$

(a) $1.01^{10} = (1+0.01)^{10} \approx 1+10\cdot 0.01+45\cdot 0.01^2 = 1.1045$

(b) $0.99^{10} = (1-0.01)^{10} \approx 1-10\cdot 0.01+45\cdot 0.01^2 = 0.9045$

15.

$$\begin{aligned}\binom{n}{r-1}+2\binom{n}{r}+\binom{n}{r+1} &= \frac{n!}{(n-r+1)!(r-1)!}+\frac{2n!}{(n-r)!r!}+\frac{n!}{(n-r-1)!(r+1)!}\\ &= \frac{n!}{(n-r-1)!(r-1)!}\left(\frac{1}{(n-r+1)(n-r)}+\frac{2}{(n-r)r}+\frac{1}{r(r+1)}\right)\\ &= \frac{n!}{(n-r-1)!(r-1)!}\left(\frac{r(r+1)+2(n-r+1)(r+1)+(n-r+1)(n-r)}{(n-r+1)(n-r)r(r+1)}\right)\\ &= \frac{n!(n^2+3n+2)}{(n-r+1)!(r+1)!} = \frac{n!(n+1)(n+2)}{(n-r+1)!(r+1)!}\\ &= \frac{(n+2)!}{[(n+2)-(r+1)]!(r+1)!} = \binom{n+2}{r+1}\end{aligned}$$

The sum of the entry in row n column $r-1$ plus twice the entry in nth row and rth column, and the entry in the nth row and $(r+1)$th column is equal to the entry two rows directly below the last entry.

16. In each case we have an infinite geometric series.

(a)

$$0.\overline{7} = 0.7777... = 0.7 + 0.07 + 0.007 + \cdots$$

$$\Rightarrow a_1 = \frac{7}{10},\ r = \frac{1}{10}$$

$$\Rightarrow 0.\overline{7} = S_\infty = \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{7}{9}$$

(b)

$$0.3\overline{45} = 0.3 + 0.0454545... = 0.3 + 0.045 + 0.00045 + \cdots$$

$$\Rightarrow a_1 = \frac{45}{1000},\ r = \frac{1}{100}$$

$$\Rightarrow 0.3\overline{45} = 0.3 + S_\infty = 0.3 + \frac{\frac{45}{1000}}{1 - \frac{1}{100}} = \frac{3}{10} + \frac{45}{990} = \frac{19}{55}$$

(c)

$$3.21\overline{29} = 3.21 + 0.00292929... = 3.21 + 0.0029 + 0.000029 + \cdots$$

$$\Rightarrow a_1 = \frac{29}{10000},\ r = \frac{1}{100}$$

$$\Rightarrow 3.21\overline{29} = 3.21 + S_\infty = 3.21 + \frac{\frac{29}{10000}}{1 - \frac{1}{100}} = \frac{321}{100} + \frac{29}{9900} = \frac{7952}{2475}$$

17.

$$(2x-3)^9 = \sum_{i=0}^{9} \binom{9}{i} (2x)^{9-i} (-3)^i$$

$$i = 3 \Rightarrow \binom{9}{3} (2x)^{9-3} (-3)^3 = 84 \cdot 64 \cdot x^6 (-27) = -145152x^6$$

The coefficient of x^6 is -145152

18.

$$(ax+b)^7 = \sum_{i=0}^{7} \binom{7}{i} (ax)^{7-i} (-3)^i$$

$$\Rightarrow i = 4 \Rightarrow \binom{7}{4} (ax)^{7-4} b^4 = 35a^3x^3b^4$$

The coefficient of x^3b^4 is $35a^3$

19.

$$\left(\frac{2}{z^2}-z\right)^{15}=\sum_{i=0}^{15}\binom{15}{i}\left(\frac{2}{z^2}\right)^{15-i}(-z)^i$$

$$\Rightarrow i=10 \Rightarrow \binom{15}{10}\left(\frac{2}{z^2}\right)^{15-10}(-z)^{10}=3003\cdot\frac{32}{z^{10}}z^{10}=96096$$

20.

$$(3n-2m)^5=\binom{5}{0}(3n)^5(-2m)^0+\binom{5}{1}(3n)^4(-2m)^1+\binom{5}{2}(3n)^3(-2m)^2$$

$$+\binom{5}{3}(3n)^2(-2m)^3+\binom{5}{4}(3n)^1(-2m)^4+\binom{5}{5}(3n)^0(-2m)^5$$

$$=243n^5-810n^4m+1080n^3m^2-720n^2m^3+240nm^4-32m^5$$

21.

$$\left(4+3r^2\right)^9=\sum_{i=0}^{9}\binom{9}{i}4^{9-i}\left(3r^2\right)^i$$

$$\Rightarrow i=5 \Rightarrow \binom{9}{5}4^4\left(3r^2\right)^5=126\cdot256\cdot243r^{10}=7838208r^{10}$$

The coefficient of r^{10} is 7 838 208

22. In the expansion of $(2-kx)^5$, the coefficient of x^3 is ${}_5C_3 2^2(-k)^3=-40k^3$

Thus, $40k^3=1080 \Rightarrow k^3=27 \Rightarrow k=3$

Exercise 3.6

1. **(a)** $${}_5P_5=\frac{5!}{(5-5)!}=\frac{1\cdot2\cdot3\cdot4\cdot5}{1}=120$$

(b) $$5!=1\cdot2\cdot3\cdot4\cdot5=120$$

(c) $${}_{20}P_1=\frac{20!}{(20-1)!}=\frac{1\cdot2\cdot3\cdot\ldots19\cdot20}{1\cdot2\cdot3\cdot\ldots19}=20$$

(d) $${}_8P_3=\frac{8!}{(8-3)!}=\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4\cdot5}=6\cdot7\cdot8=336$$

2. **(a)** $${}_5C_5=\binom{5}{5}=\frac{\cancel{5!}}{(5-5)!\cancel{5!}}=1$$

(b) $${}_5C_0=\binom{5}{0}=\frac{\cancel{5!}}{\cancel{(5-0)!}0!}=1$$

(c) $${}_{10}C_3=\binom{10}{3}=\frac{10!}{(10-3)!3!}=\frac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10}{\cancel{(1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7)}1\cdot2\cdot3}=120$$

(d) $${}_{10}C_7=\binom{10}{7}=\frac{10!}{(10-7)!7!}=\frac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10}{(1\cdot2\cdot3)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}}=120$$

3. **(a)** $$\binom{7}{3}+\binom{7}{4}=\frac{7!}{(7-3)!3!}+\frac{7!}{(7-4)!4!}=2\cdot\frac{7!}{4!3!}=2\cdot\frac{\cancel{1\cdot2\cdot3\cdot4}\cdot5\cdot6\cdot7}{\cancel{(1\cdot2\cdot3\cdot4)}(1\cdot2\cdot3)}=70$$

(b) $$\binom{8}{4}=\frac{8!}{(8-4)!4!}=\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{(1\cdot2\cdot3\cdot4)^2}=70$$

(c) $$\binom{10}{6}+\binom{10}{7}=\frac{10!}{(10-6)!6!}+\frac{10!}{(10-7)!7!}=$$

$$\frac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6}\cdot7\cdot8\cdot9\cdot10}{(1\cdot2\cdot3\cdot4)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6}}+\frac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10}{(1\cdot2\cdot3)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}}=210+120=330$$

(d) $$\binom{11}{7}=\frac{11!}{(11-7)!7!}=\frac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10\cdot11}{(1\cdot2\cdot3\cdot4)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}}=330$$

4. **(a)** $$\binom{8}{5}-\binom{8}{3}=\frac{8!}{(8-5)!5!}-\frac{8!}{(8-3)!3!}=\frac{8!}{3!5!}-\frac{8!}{5!3!}=0$$

(b) $$11\cdot10!=11\cdot10\cdot9\ldots2\cdot1=11!=39916800$$

(c) $$\binom{10}{3}-\binom{10}{7}=\frac{10!}{(10-3)!3!}-\frac{10!}{(10-7)!7!}=\frac{10!}{7!3!}-\frac{10!}{3!7!}=0$$

(d) $$\binom{10}{1}=\frac{10!}{(10-1)!1!}=\frac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}\cdot10}{(\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9})\cdot1}=10$$

5. **(a)** $\frac{10!}{5!}=\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4\cdot5}=30240;\ 2!=2\Rightarrow\frac{10!}{5!}\neq2!$

(b) $(5!)^2=(1\cdot2\cdot3\cdot4\cdot5)^2=14400$

$25!=1\cdot2\cdot3\cdot\cdots\cdot24\cdot25=1.551121\times10^{25}\Rightarrow\ (5!)^2\neq25!$

(c) $\binom{101}{8}=\frac{101!}{(101-8)!8!}=\frac{101!}{93!8!};\binom{101}{93}=\frac{101!}{(101-93)!93!}=\frac{101!}{8!93!}\ \Rightarrow\binom{101}{8}=\binom{101}{93}$

6. Using the fundamental principle of counting, there are $3\cdot2\cdot4=24$ different systems to choose from.

7. Using the fundamental principle of counting, there are $3\cdot4\cdot2\cdot3=72$ different choices.

8. Using the fundamental principle of counting, there are $8\cdot3\cdot13=312$ different combinations of choices.

9. Using the fundamental principle of counting, there are $\Rightarrow\underbrace{4\cdot4\cdots4}_{12\,\text{times}}=4^{12}=16777216$ ways to answer all the questions.

10. Using the fundamental principle of counting, there are $\underbrace{2\cdot2\cdots2}_{6\,\text{times}}\cdot\underbrace{4\cdot4\cdots4}_{6\,\text{times}}=2^6 4^6=262144$ ways to answer all the questions.

11. Using the fundamental principle of counting, each letter can be chosen in 26 ways and each digit in 10 ways, so there are $26^3\cdot10^5=1757600000$ different passwords.

12. The first and last digit can be chosen in 9 ways (cannot be 0) and the three middle digits can be chosen in 10 ways, so there are $9\cdot10^3\cdot9=81000$ such numbers.

13. **(a)** Eight people can be seated in a row in $8!=40320$ different ways (permutations).

(b) If every member of each couple likes to sit together, four couples can be seated in a row in $4!$ different ways, and for each member of a couple there are 2 possible places, so altogether there are $4!\cdot2^4=384$ different ways that they can be seated.

14. **(a)** Eight children can be arranged in single file in $8!=40320$ different ways (permutations).

(b) If the girls must go first, the five girls can be arranged in $5!$ ways and the three boys in $3!$ ways, so altogether there are $5!3!=720$ different orders.

15. In alphabetical order:

AEJN, AENJ, AJEN, AJNE, ANEJ, ANJE, EAJN, EANJ, EJAN, EJNA, ENAJ, ENJA, JAEN, JANE, JEAN, JENA, JNAE, JNEA, NAEJ, NAJE, NEAJ, NEJA, NJAE, NJEA.

16.

ACG	AGC	CAG	CGA	GAC	GCA
ACI	AIC	CAI	CIA	IAC	ICA
ACM	AMC	CAM	CMA	MAC	MCA
AGI	AIG	GAI	GIA	IAG	IGA
AGM	AMG	GAM	GMA	MAG	MGA
AIM	AMI	IAM	IMA	MAI	MIA
CGI	CIG	GCI	GIC	ICG	IGC
CGM	CMG	GCM	GMC	MCG	MGC
CIM	CMI	ICM	IMC	MCI	MIC
GIM	GMI	IGM	IMG	MGI	MIG

17. **(a)** Three letters can be chosen in 26^3 ways and four digits in 10^4 ways. Altogether there are $26^3 \cdot 10^4 = 175760000$ possible codes.

(b) The letters can be chosen in $26^3 - 97$ ways and the four digits in 10^4 ways.

Altogether there are $(26^3 - 97) \cdot 10^4 = 174790000$ possible codes.

18. **(a)** The president can be chosen in 17 ways, the deputy in 16 ways, and the treasurer in 15 ways. Altogether there are $17 \cdot 16 \cdot 15 = 4080$ ways.

(b) If the president is male, he can be chosen in 7 ways, thus the deputy can be

chosen in 16 ways and the treasurer in 15 ways; so, altogether, there are $7 \cdot 16 \cdot 15 = 1680$ ways.

(c) The deputy (male) can be chosen in 7 ways, the treasurer (female) in 10 ways, and the president in 15 ways; so, altogether, there are $7 \cdot 10 \cdot 15 = 1050$ ways.

(d) If the president and deputy are both male, then they can be chosen in $7 \cdot 6$ ways; if they are both female in $10 \cdot 9$ ways. The treasurer can be chosen in 15 ways. Altogether there are $(7 \cdot 6 + 10 \cdot 9) \cdot 15 = 1980$ ways.

19. Since the order is not important, we have combinations.

(a) Three officers of the same specialisation can be chosen as 3 mathematicians out of 8

(in ${}_8C_3$ ways), or 3 computer scientists out of 12 (in ${}_{12}C_3$ ways), or 3 engineers out of

6 (in ${}_6C_3$ ways). Altogether there are ${}_8C_3 + {}_{12}C_3 + {}_6C_3 = 56 + 220 + 20 = 296$ ways.

(b) There are a total of ${}_{26}C_3$ ways to choose three officers from 26 people. Three officers that are not engineers can be chosen as 3 out of 20 mathematicians and computer scientists in ${}_{20}C_3$ ways. So, there are ${}_{26}C_3 - {}_{20}C_3 = 2600 - 1140 = 1460$ combinations with at least one engineer.

(c) Two mathematicians can be chosen in ${}_{8}C_2$ ways, and the third member in 18 ways out of 12 computer scientists and 6 engineers; so, altogether, there are ${}_{8}C_2 \cdot 18 = 28 \cdot 18 = 504$ ways.

20. Three numbers, each in the range 1 to 50.

(a) There are $50^3 = 125000$ different combinations.

(b) There are ${}_{50}P_3 = 50 \cdot 49 \cdot 48 = 117\,600$ combinations without duplicates.

(c) If the first and second number are matching, then the first number can be chosen in 50 ways, the second in 1 way (must be the same as the first), and the third in 49 ways; so, altogether there are $50 \cdot 1 \cdot 49 = 2450$ combinations.

(d) If two out of three numbers are matching, it may be the first and the second, or the first and the third, or the second and the third; so, altogether, there are $3 \cdot 2450 = 7350$ combinations.

21. Five couples can be permutated in $5!$ ways, but, as they are sitting around a circle, all circular permutations that come in groups of five (ABCDE, BCDEA, CDEAB,…) are equivalent, so there are actually $\frac{5!}{5}$ arrangements. For each couple, there are 2 different ways of sitting (male right or left of female), so, altogether, there are $\frac{5!}{5} \cdot 2^5 = 768$ different seating arrangements.

22. **(a)** Two elements out of nine can be chosen in ${}_{9}C_2 = 36$ ways, so there are 36 two-element subsets.

(b) There are ${}_{9}C_1$ one-element subsets, ${}_{9}C_3$ three-element subsets, ${}_{9}C_5$ five-element subsets, ${}_{9}C_7$ seven-element subsets, and ${}_{9}C_9$ nine-element subsets. Altogether, we have:

${}_{9}C_1 + {}_{9}C_3 + {}_{9}C_5 + {}_{9}C_7 + {}_{9}C_9 = 256$ subsets with an odd number of elements.

23. **(a)** Four out of $9 + 12 = 21$ members can be chosen in $\binom{21}{4} = 5985$ ways.

(b) Two women out of 12 can be chosen in $\binom{12}{2}$ ways, and two men out of 9 can be chosen in $\binom{9}{2}$ ways; so, altogether, there are $\binom{12}{2} \cdot \binom{9}{2} = 66 \cdot 36 = 2376$ teams.

(c) More women than men can be realised if there are 3 or 4 women, so there are

$\binom{12}{3}\cdot\binom{9}{1}+\binom{12}{4}=220\cdot 9+495=2475$ such teams.

(d) **(a)** Teams with only Tim or Gwen: $2\cdot {}_{20}C_3=2\cdot 1140=2280$

(b) Two women (one is Gwen) can be chosen in $1\cdot\binom{11}{1}$ ways, and two men (not Tim) can be chosen in $\binom{8}{2}$ ways; two women (not Gwen) can be chosen in $\binom{11}{2}$ ways, and two men (one is Tim) can be chosen in $1\cdot\binom{8}{1}$ ways.

In total, there are $1\cdot\binom{11}{1}\cdot\binom{8}{2}+\binom{11}{2}\cdot 1\cdot\binom{8}{1}=1\cdot 11\cdot 28+55\cdot 1\cdot 8=748$ teams.

(c) More women than men can be realised in the following cases:

3 women (one is Gwen), 1 man (not Tim) $\Rightarrow 1\cdot\binom{11}{2}\cdot\binom{8}{1}$

3 women (not Gwen), 1 man (Tim) $\Rightarrow\binom{11}{3}\cdot 1$

4 women (one is Gwen) $\Rightarrow 1\cdot\binom{11}{3}$

Altogether: $1\cdot\binom{11}{2}\cdot\binom{8}{1}+\binom{11}{3}\cdot 1+1\cdot\binom{11}{3}=55\cdot 8+165+165=770$ teams.

24. **(a)** A sample of 6 disks out of 100 can be chosen in ${}_{100}C_6=1\,192\,052\,400$ ways.

(b) All four defective disks could be in ${}_{96}C_2\cdot{}_4C_4=4560$ samples, which gives us $\frac{4560}{1\,192\,052\,400}=0.00000383,$ i.e. 0.000383% of the total.

(c) At least one defective disk could be in ${}_{96}C_5\cdot{}_4C_1+{}_{96}C_4\cdot{}_4C_2+{}_{96}C_3\cdot{}_4C_3+{}_{96}C_2\cdot{}_4C_4=265\,004\,096$ samples,

which gives us $\frac{265\,004\,096}{1\,192\,052\,400}=0.2223,$ i.e. 22.23% of the total.

25. **(a)** 6 people out of $10+8+4=22$ can be chosen in ${}_{22}C_6=74613$ ways.

(b) Two members out of each party can be chosen in ${}_{10}C_2\cdot{}_8C_2\cdot{}_4C_2=7560$ ways.

26. If we place the boys in a row with spaces in between, we have the following setup:

$$\_\,B\,\_\,B\,\_\,B\,\_\,B\,\_\,B\,\_\,B\,\_\,B\,\_\,B\,\_\,B\,\_$$

We have 10 spaces where a girl can stand. Thus, we have ${}_{10}P_6$ different ways of arranging the girls. In each case, the boys can be arranged in 9! different ways, and therefore there will be ${}_{10}P_6\cdot 9!=54867456000$ ways of arranging the group with no two girls next to each other.

Chapter 3 practice questions

1. $$a_1=4, a_4=19, a_n=99 \Rightarrow \left.\begin{matrix}4+3d=19\\ 4+(n-1)d=99\end{matrix}\right\}\Rightarrow d=5 \text{ and} n=20$$

2. $$A=3000,\ r=0.06,\ \ n=4,\ \ t=6; A=P\left(1+\frac{r}{n}\right)^{nt}$$

$$\Rightarrow 3000=P\left(1+\frac{0.06}{4}\right)^{4\cdot 6}\Rightarrow P=\frac{3000}{\left(1+\frac{0.06}{4}\right)^{4\cdot 6}}=2098.63$$

You should invest \$2,098.63 now.

3. Nick's studying hours form an arithmetic sequence with first term and common difference $d=2$. Maxine's studying hours form a geometric sequence with first term $b_1=12$ and common ratio $r=1.1$

(a) $$a_5=a_1+4d=12+4\cdot 2=20$$
$$b_5=b_1r^4=12\cdot 1.1^4\approx 17.57$$

In week 5, Nick studied for 20 hours and Maxine studied for 17.57 hours.

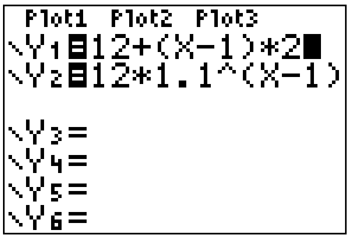
(b) $$S_{arithmetic15}=\frac{15}{2}\left[2\cdot 12+(15-1)2\right]=390$$
$$S_{geometric15}=12\frac{1.1^{15}-1}{1.1-1}\approx 381.27$$

For the 15 weeks, Nick studied for a total of 390 hours and Maxine studied for a total of 381.27 hours.

(c) $b_n > 40 \Rightarrow 12\cdot 1.1^{n-1} > 40 \Rightarrow n > \dfrac{\log\dfrac{40}{12}}{\log 1.1}+1 \Rightarrow n > 13.6$

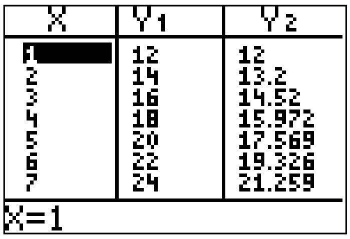
Maxine will exceed 40 hours of study per week in the 14th week.

(d) We need to determine *n* so that $b_n \geq a_n$. The easiest way of doing this is by entering *n* (as X), a_n (as Y1), and b_n (as Y2) into TABLE in a GDC.

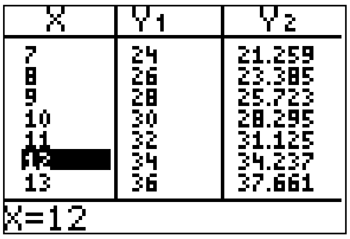


```
Plot1 Plot2 Plot3
\Y1=12+(X-1)*2
\Y2=12*1.1^(X-1)

\Y3=
\Y4=
\Y5=
\Y6=
```



```
  X     Y1      Y2
1       12      12
2       14      13.2
3       16      14.52
4       18      15.972
5       20      17.569
6       22      19.326
7       24      21.259
X=1
```



```
  X     Y1      Y2
7       24      21.259
8       26      23.385
9       28      25.723
10      30      28.295
11      32      31.125
12      34      34.237
13      36      37.661
X=12
```

We see that in week 12 Nick will study for 34 hours while Maxine studies for 34.24 hours. So, Maxine will catch up with Nick in week 12.

4. Plan A forms an arithmetic sequence with first term $a_1 = 1000$ and common difference $d = 80$

Plan B forms a geometric sequence with first term $b_1 = 1000$ and common ratio $r = 1.06$

(a)

$b_2 = 1000\cdot 1.06 = 1060$ g

$b_3 = 1000\cdot 1.06^2 = 1123.6$ g

(b)

$a_{12} = a_1 + 11d = 1000 + 11\cdot 80 = 1880$ g

$b_{12} = b_1 \cdot 1.06^{11} \approx 1898.3$ g

(c) **(i)** $S_{A12} = \dfrac{12}{2}\left(2a_1 + 11d\right) = 6\left(2\cdot 1000 + 11\cdot 80\right) = 17\,280$ g

(ii) $S_{B12} = b_1 \dfrac{r^n - 1}{r - 1} = 1000\dfrac{1.06^{12} - 1}{1.06 - 1} \approx 16869.9$ g

5. **(a)** The initial amount forms a geometric sequence with $a_1 = 500$ and common ratio

$r = 1.06$ (fixed rate 6% per annum).

After 10 years, it will be worth $a_{11} = a_1 r^{10} = 500\cdot 1.06^{10} =$ €895.42 = €895 to the nearest euro.

(b) The future value is a partial sum of a geometric sequence:

$$FV = a_1\left(\frac{r^{11}-1}{r-1}-1\right) = 500\left(\frac{1.06^{11}-1}{1.06-1}-1\right) = €\,6985.82 = €6{,}986 \text{ to the nearest euro.}$$

6. 6, 9.5, 13, … is an arithmetic sequence with $a_1 = 6$ and $d = 3.5$

(a) $a_{40} = a_1 + 39d = 6 + 39\cdot 3.5 = 142.5$

(b) $S_{103} = \frac{103}{2}\left[2a_1 + (103-1)d\right] = \frac{103}{2}\left(2\cdot 6 + 102\cdot 3.5\right) = 19003.5$

7. For $a_n = \sqrt[3]{8 - a_{n-1}^3}$:

(a) $a_1 = 1,\ a_2 = \sqrt[3]{8-1^3} = \sqrt[3]{7},\ a_3 = \sqrt[3]{8-(\sqrt[3]{7})^3} = 1,\ a_4 = \sqrt[3]{8-1^3} = \sqrt[3]{7}$

$\{a_n\}$ alternates between values of 1 and $\sqrt[3]{7}$

(b) $a_1 = 2,\ a_2 = \sqrt[3]{8-2^3} = 0,\ a_3 = \sqrt[3]{8-0^3} = 2,\ a_4 = \sqrt[3]{8-2^3} = 0$

$\{a_n\}$ alternates between values of 2 and 0

8. The training program forms an arithmetic sequence with $a_1 = 2$ and $d = 0.5$

(a) $a_n = 20 \Rightarrow 2 + (n-1)\cdot 0.5 = 20 \Rightarrow n = 37$

She first runs a distance of 20 km on the 37th day of her training.

(b) $S_{37} = \frac{37}{2}\left(2\cdot 2 + 36\cdot 0.5\right) = 407$

The total distance run during 37 days of training would be 407 km.

9. **(a)** $r = \frac{2400}{1600} = \frac{3600}{2400} = \frac{3}{2}$

(b) **(i)** The number of new participants in 2022 is the 13th term in the sequence.

$$a_{13} = a_1 r^{12} = 1600\cdot\left(\frac{3}{2}\right)^{12} = 207594$$

(ii)

$$a_n > 50000 \Rightarrow 1600\left(\frac{3}{2}\right)^{n-1} > 50000 \Rightarrow \left(\frac{3}{2}\right)^{n-1} > 31.25$$

$$\Rightarrow n > \frac{\log 31.25}{\log 1.5} + 1 \Rightarrow n > 9.489$$

The 10th term of the sequence will be greater than 50 000; therefore, the number of new participants will first exceed 50 000 in 2019.

(c) $S_{13} = a_1 \frac{r^{13}-1}{r-1} = 1600 \times \frac{1.5^{13}-1}{1.5-1} = 619582$

(d) This trend in growth would not continue due to market saturation.

10.
$a_1 = 25,\ a_4 = 13 \Rightarrow a_1 + 3d = 13 \Rightarrow 25 + 3d = 13 \Rightarrow d = -4$
$a_n = -11995 \Rightarrow a_1 + (n-1)d = -11995 \Rightarrow 25 + (n-1)(-4) = -11995 \Rightarrow n = 3006$

11. **(a)** $MN = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$

(b) $\text{Area}_{\square MNPQ} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

(c) **(i)** $RS = \sqrt{\left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{8}+\frac{1}{8}} = \frac{1}{2} \Rightarrow \text{Area}_{\square RSTU} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \ldots \Rightarrow r = \frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

(d) **(i)** $\text{Area}_{10} = 1\left(\frac{1}{2}\right)^9 = \frac{1}{512}$

(ii) $S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

12. Aristede's swimming program forms an arithmetic sequence with $a_1 = 200$ and $d = 20$

(a) $a_{52} = a_1 + 51d = 200 + 51\cdot 20 = 1220$

Aristede will swim 1220 metres in the final week.

(b) $S_{52} = \frac{52}{2}(2a_1 + 51d) = \frac{52}{2}(2\cdot 200 + 51\cdot 20) = 36920$

Altogether, Aristede swims 36 920 metres.

13. **(a)** $\text{Area}_{\square A} = \left(\frac{3}{3}\right)^2 = 1;\ \text{Area}_{\square B} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

(b) $\text{Area}_{\square C} = \left(\frac{1}{9}\right)^2 = \frac{1}{81}$

(c)

$$\text{Shaded Area}_2 = 1 + 8 \cdot \frac{1}{9} = 1 + \frac{8}{9}$$

$$\text{Shaded Area}_3 = \text{Shaded Area}_2 + 8 \cdot \frac{8}{81} = 1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2$$

(d)

$$\text{Shaded Area}_\infty = 1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2 + \ldots = \frac{1}{1 - \frac{8}{9}} = 9$$

$$\text{Unshaded Area}_\infty = 9 - \text{Shaded Area}_\infty = 0$$

14. **(a)** **(i)** The series 2 + 22 + 222 + 2222 + … is neither arithmetic nor geometric.

(ii) The series $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$ is geometric with $r = \frac{2}{3} < 1$; thus converging.

(iii) The series $0.8 + 0.78 + 0.74 + 0.74 + \ldots$ is arithmetic with $d = -0.02$.

(iv) The series $2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \ldots$ is geometric with $r = \frac{4}{3} > 1$; thus diverging.

(b) For series **(ii)** we have $S_\infty = \frac{2}{1 - \frac{2}{3}} = 6$

15. The Kell scheme forms an arithmetic sequence with $a_1 = 18000$ and $d = 400$

The YBO scheme forms a geometric sequence with $b_1 = 17000$ and $r = 1.07$

(a) All answers are in euros.

(i)

Kell: $a_2 = 18000 + 400 = 18400,\ a_3 = a_2 + 400 = 18800$

YBO: $b_2 = 17000 \cdot 1.07 = 18190,\ \ b_3 = b_2 \cdot 1.07 = 19463.3$

(ii)

Kell: $S_{10} = \frac{10}{2}(2 \cdot 18000 + 9 \cdot 400) = 198000$

YBO: $S_{10} = 17000 \frac{1.07^{10} - 1}{1.07 - 1} = 234879.62$

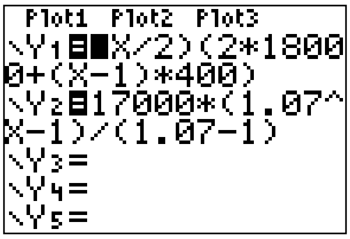
(iii)

Kell: $a_{10} = 18000 + 9 \cdot 400 = 21600$

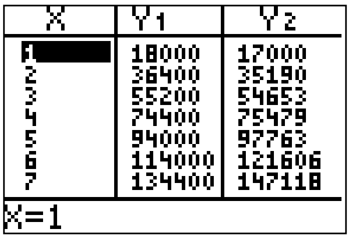
YBO: $b_{10} = 17000 \cdot 1.07^9 = 31253.81$

(b) **(i)** From **(a)** **(ii)** we can see that $b_3 > a_3$, so Merijayne will start earning more than Tim in the third year.

(ii) We can compare their total earnings with the help of a GDC. If X denotes the year, Y1 represents total earnings for Tim, and Y2 for Merijayne we have:



```
 Plot1  Plot2  Plot3
\Y1=■X/2)(2*1800
0+(X-1)*400)
\Y2=17000*(1.07^
X-1)/(1.07-1)
\Y3=
\Y4=
\Y5=
```



```
   X   |  Y1    |  Y2
 1     | 18000  | 17000
 2     | 36400  | 35190
 3     | 55200  | 54653
 4     | 74400  | 75479
 5     | 94000  | 97763
 6     | 114000 | 121606
 7     | 134400 | 147118
X=1
```

In the fourth year, Merijayne's total earnings will exceed those of Tim.

16. The number of seats in each row forms an arithmetic sequence with $a_1 = 16$ and $d = 2$

(a) $a_{24} = a_1 + 23d = 16 + 23 \cdot 2 = 62$

(b) $S_{24} = 16 + 18 + ... + 62 = \frac{24}{2}(16 + 62) = 936$

17. The values of the investment after each year form a geometric sequence with $a_1 = 7000$, $r = 1.0525$, and a_{n+1} represents the value after n years.

(a) Value of investment $= 7000 \cdot 1.0525^t$

(b) $7000 \cdot 1.0525^t = 10000 \Rightarrow 1.0525^t = \frac{10}{7} \Rightarrow t = \frac{\log\left(\frac{10}{7}\right)}{\log(1.0525)} = 6.97$

The minimum number of years is 7.

(c) If the rate of 5% is compounded quarterly, the value of the investment over 7 years would be $7000 \cdot \left(1 + \frac{5}{4 \cdot 100}\right)^{7 \cdot 4} = 9911.95$

For 5.25% compounded annually, the value of the investment would be $7000 \cdot 1.0525^7 = 10015.04$

Therefore, the investment at 5.25% annually is better.

18. **(a)** $S_1 = 9 \Rightarrow a_1 = 9$

$S_2 = 20 \Rightarrow a_1 + a_2 = 20 \Rightarrow 9 + a_2 = 20 \Rightarrow a_2 = 11$

(b) $d = a_2 - a_1 = 11 - 9 = 2$

(c) $a_4 = a_1 + 3d = 9 + 3 \cdot 2 = 15$

19. $$\begin{cases} a_2 = a + d = 7 \\ S_4 = \frac{4}{2}[2a + 3d] = 12 \end{cases} \Rightarrow \begin{cases} a + d = 7 \\ 2a + 3d = 6 \end{cases} \Rightarrow a = 15,\ d = -8$$

20. We can expand the left-hand side to the second powers

$$(1+x)^5(1+ax)^6 \equiv (1+5x+10x^2+\cdots)(1+6ax+15a^2x^2+\cdots)$$

Multiply, simplify and compare it to the right-hand side:

$$(1+x)^5(1+ax)^6 \equiv 1+5x+6ax+10x^2+15a^2x^2+30ax^2+\cdots$$
$$=1+(5+6a)x+(10+15a^2+30a)x^2+\cdots \equiv 1+bx+10x^2+.....+a^6x^{11}$$

By comparing the coefficients of the same powers of *x* we get:

$$\begin{cases} 5+6a=b \\ 10+15a^2+30a=10 \end{cases}$$
$$a(a+2)=0 \quad \Rightarrow a=-2,\, b=-7$$

21. $$\begin{cases} \dfrac{a_5}{a_{12}}=\dfrac{6}{13} \\ a_1\cdot a_3=32 \end{cases} \Rightarrow \frac{a+4d}{a+11d}=\frac{6}{13} \quad \Rightarrow a=2d$$

$$a_1\cdot a_3=32 \Rightarrow a(a+2d)=32 \Rightarrow 2d\cdot 4d=32 \quad \Rightarrow d=2 \text{(all terms positive)},\ a=4$$

$$S_{100}=\frac{100}{2}(2\cdot 4+99\cdot 2)=10<t>300$$

22. $a_1=5; a_2=a_1+d=13 \Rightarrow d=8$

(a) $a_n=a_1+(n-1)d=5+(n-1)\cdot 8=8n-3$

(b) $a_n<400 \Rightarrow 8n-3<400 \Rightarrow n<50.375$

There are 50 terms that are less than 400.

23. $$(2+3x)^{10}=\sum_{i=0}^{10}\binom{10}{i}2^{10-i}(3x)^i$$
$$i=7 \Rightarrow \binom{10}{7}2^{10-7}(3x)^7=120\cdot 8\cdot 2187x^7=2099520x^7$$

The coefficient of x^7 is 2099520.

24. $$S_n=3n^2-2n,\ S_{n-1}=3(n-1)^2-2(n-1)=6n-5,$$
$$\Rightarrow u_n=S_n-S_{n-1}=6n-5$$

25. Six people can be ordered in 6! ways, but, as they are seated around a circular table, all circular permutations that come in groups of six (ABCDEF, BCDEFA, CDEFAB…) are equivalent, so there are actually $\frac{6!}{6}$ ways. As Mr Black and Mrs White should not sit together, we must subtract all circular permutations in which this pair is regarded as one person, but multiplied by 2, because a male can be on the left or right of the female.

Altogether, there are $\frac{6!}{6}-\frac{5!}{5}\cdot 2=120-24\cdot 2=72$ ways.

26. Firstly, we must determine which is the last positive term in the sequence.

$$a_1=85,\ d=-7$$
$$a_n=85+(n-1)(-7)>0\ \Rightarrow 92-7n>0\Rightarrow n<13.14$$

For $n=13\ \Rightarrow S_{13}=\frac{13}{2}\left[2\cdot 85+12\cdot(-7)\right]=559$

27. $\left(x+\frac{1}{kx^2}\right)^7=\sum_{i=0}^{7}\binom{7}{i}x^{7-i}\left(\frac{1}{kx^2}\right)^i$

The term in x corresponds to $i = 2$:

$$i=2\ \Rightarrow\binom{7}{2}x^{7-2}\left(\frac{1}{kx^2}\right)^2=21x^5\cdot\frac{1}{k^2x^4}=\frac{21x}{k^2}\Rightarrow\frac{21}{k^2}=\frac{7}{3}\ \Rightarrow k^2=9\Rightarrow k=3,\cancel{k=-3}$$

28. $S_\infty=\frac{a}{1-r}=\frac{27}{2}\quad\Rightarrow a=\frac{27(1-r)}{2}$

$$S_3=a\frac{1-r^3}{1-r}=13\Rightarrow\frac{27\cancel{(1-r)}}{2}\cdot\frac{1-r^3}{\cancel{1-r}}=13\ \Rightarrow r^3=\frac{1}{27}\ \Rightarrow r=\frac{1}{3},\ a=9$$

29. Student A can get 1 book in ${}_6C_1$ ways. The rest of the books go to student B.

Student A can get 2 books in ${}_6C_2$ ways, 3 books in ${}_6C_3$ ways, 4 books in ${}_6C_4$ ways, and 5 books in ${}_6C_5$ ways.

Altogether, there are ${}_6C_1+{}_6C_2+{}_6C_3+{}_6C_4+{}_6C_5=62$ ways.

30. This is an infinite geometric series with: $a_1=-12, r=-\frac{2}{3}$

$$S_\infty=\frac{a_1}{1-r}=\frac{-12}{1+\frac{2}{3}}=\frac{-36}{5}$$

31. For $u_n=3(4)^{n+1}, n\in\mathbb{Z}^+$:

(a) $u_1=48, r=4$ **(b)** $S_n=48\frac{4^n-1}{4-1}=16(4^n-1)$

32. **(a)** For this infinite geometric series $a_1 = 1, r = \frac{2x}{3}$

The series converges for $|r| < 1 \Rightarrow \left|\frac{2x}{3}\right| < 1 \Rightarrow -\frac{3}{2} < x < \frac{3}{2}$

(b) $x = 1.2 \Rightarrow r = \frac{2 \cdot 1.2}{3} = \frac{4}{5} \Rightarrow S_\infty = \frac{1}{1-\frac{4}{5}} = 5$

33. A four-digit number cannot have a zero in the 10 000 position.
There are $9 \times 10^3 = 9000$ four-digit numbers.

Without digit 3, there are $8 \times 9^3 = 5832$ numbers.

So, with at least one digit 3, there are $9000 - 5832 = 3168$ numbers.

34. **(a)** For this series we: $a_1 = 2, d = 3 \Rightarrow S_n = \frac{n}{2}[2 \cdot 2 + (n-1) \cdot 3] = \frac{n(3n+1)}{2}$

(b) $S_n = 1365 \Rightarrow \frac{n(3n+1)}{2} = 1365 \Rightarrow 3n^2 + n - 2730 = 0 \Rightarrow n = 30, \cancel{-\frac{91}{3}}$

35. $(1-\frac{1}{2}x)^8 = \sum_{i=0}^{8}\binom{8}{i}\left(-\frac{1}{2}x\right)^i; i = 3 \Rightarrow \binom{8}{3}\left(-\frac{1}{2}x\right)^3 = 56\left(-\frac{1}{8}x^3\right) = -7x^3$

The coefficient of x^3 is -7.

36. $\sum_{r=1}^{50}\ln(2^r) = \sum_{r=1}^{50} r\ln(2) = \ln 2\sum_{r=1}^{50} r = (\ln 2)\left[\frac{50}{2}(2+49)\right] = 1275\ln 2$

37. **(a)** $a_2 = 3a_1 - 2a_0 = 4; a_3 = 3a_2 - 2a_1 = 8; a_4 = 3a_3 - 2a_2 = 16$

(b) **(i)** $a_n = 2^n$

(ii) $3a_n - 2a_{n-1} = 3 \cdot 2^n - 2 \cdot 2^{n-1} = 3 \cdot 2^n - 2^n = 2 \cdot 2^n = 2^{n+1} = a_{n+1}$

38. **(a)** $\begin{cases} S_2 = a + ar = 15 \\ S_\infty = \frac{a}{1-r} = 27 \end{cases} \Rightarrow \begin{cases} a(1+r) = 15 \\ a = 27(1-r) \end{cases} \Rightarrow 27(1-r)(1+r) = 15$

$\Rightarrow 27(1-r^2) = 15 \Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \frac{2}{3}$ (all terms positive)

(b) $a = 27\left(1 - \frac{2}{3}\right) = 9$

39. For the arithmetic sequence $2, a-b, 2a+b+7, a-3b$ the difference must be constant.

So, we have the following system of equations:

$$\begin{cases}(a-b)-2=(2a+b+7)-(a-b)\\(a-3b)-(2a+b+7)=(2a+b+7)-(a-b)\end{cases}\Rightarrow\begin{cases}a-b-2=a+2b+7\\-a-4b-7=a+2b+7\end{cases}$$

$$\Rightarrow\begin{cases}3b=-9\\a+3b=-7\end{cases}\Rightarrow a=2, b=-3$$

40. There are ${}_8C_4$ ways of selecting teams of 4 members. Also, there are ${}_6C_2$ ways of selecting teams that include both oldest members. Thus, the required number is ${}_8C_4-{}_6C_2=55$ ways.

41. As $a, 1, b$ form an arithmetic progression, the difference is the same: $1-a=b-1$

As $1, a, b$ form a geometric progression, the ratio is the same: $\frac{a}{1}=\frac{b}{a}$

We solve the following system of equations:

$$\begin{cases}1-a=b-1\\\frac{a}{1}=\frac{b}{a}\end{cases}\Rightarrow\begin{cases}a+b=2\\a^2=b\end{cases}\Rightarrow a^2+a-2=0\Rightarrow a=1 \text{ or } a=-2$$

As $a=1$ gives $b=1$, the solution is $a=-2, b=4$

42. We can see that:

$$OB=OA=1$$
$$OB_1=OA\cos\theta=\cos\theta=OA_1$$
$$OB_2=OA_2=OA_1\cos\theta=\cos\theta\cdot\cos\theta=\cos^2\theta$$
$$OB_3=OA_3=OA_2\cos\theta=\cos^3\theta$$

As the length of the arc is equal to $\theta\cdot$radius, the sum of the arc lengths is:

$$AB+A_1B_1+A_2B_2+A_3B_3+.....=\theta+\theta\cos\theta+\theta\cos^2\theta+\theta\cos^3\theta+\cdots$$

This is a geometric series with first term θ common ratio $\cos\theta$. Thus, the required sum is the sum to infinity of this series: $\frac{u_1}{1-r}=\frac{\theta}{1-\cos\theta}$

43. For $S_n=2n^2-n, n\in\mathbb{Z}^+$ we have:

(a) $u_1=S_1=2\cdot1^2-1=1; u_2=S_2-u_1=2\cdot2^2-2-1=5$

$u_3=S_3-S_2=2\cdot3^2-3-6=9$

(b) $u_n=S_n-S_{n-1}=2n^2-n-\left[2(n-1)^2-(n-1)\right]=4n-3$

44. **(a)**

$$
\begin{aligned}
(2+x)^5 &= \sum_{i=0}^{5}\binom{5}{i}2^{5-i}x^i \\
&= \binom{5}{0}2^5+\binom{5}{1}2^4x+\binom{5}{2}2^3x^2+\binom{5}{3}2^2x^3+\binom{5}{4}2^1x^4+\binom{5}{5}2x^5 \\
&= 32+5\cdot 16x+10\cdot 8x^2+10\cdot 4x^3+5\cdot 2x^4+x^5 = 32+80x+80x^2+40x^3+10x^4+x^5
\end{aligned}
$$

(b)

$$
\begin{aligned}
2.01^5 &= (2+0.01)^5 = 32+80\cdot 0.01+80\cdot 0.01^2+40\cdot 0.01^3+10\cdot 0.01^4+0.01^5 \\
&= 32+0.8+0.008+0.00004+0.0000001+0.0000000001 = 32.8080401001
\end{aligned}
$$

45. Interest is paid yearly. For *t* years, $r = 0.063$ rate, and principal $P = 5000$ after *t* full years:

(a) $A_t = P(1+r)^t = 5000(1.063)^t$

(b) $A_5 = 5000(1.063)^5 = 6786.35$

(c) $5000(1.063)^n > 10\,000 \Rightarrow (1.063)^n > 2 \Rightarrow n > \dfrac{\log 2}{\log 1.063} = 11.35$

The value will exceed $10,000 after 12 full years.

46. For an arithmetic sequence with $S_n = 4n^2 - 2n$ we have:

$$
\begin{aligned}
u_2 &= S_2 - S_1 = (4\cdot 2^2 - 2\cdot 2) - (4\cdot 1^2 - 2\cdot 1) = 12 - 2 = 10 \\
u_m &= S_m - S_{m-1} = \left(4m^2 - 2m\right) - \left[4(m-1)^2 - 2(m-1)\right] = 8m - 6 \\
u_{32} &= S_{32} - S_{31} = (4\cdot 3 = 2^2 - 2\cdot 32) - (4\cdot 31^2 - 2\cdot 31) = 250
\end{aligned}
$$

As they are consecutive terms in a geometric sequence, the ratio is the same.

$$\frac{8m-6}{10} = \frac{250}{8m-6} \Rightarrow (8m-6)^2 = 2500 \Rightarrow 8m-6 = 50 \Rightarrow m = 7$$

47. $u_9 = 0 = u_1 + 8d$ and $S_{16} = 12 = 8\left(2u_1 + 15d\right)$

This leads to a system of equations:

$$\begin{cases} 8\left(2u_1 + 15d\right) = 12 \\ 0 = u_1 + 8d \end{cases} \Rightarrow \begin{cases} 2u_1 + 15d = \dfrac{3}{2} \\ 2u_1 + 16d = 0 \end{cases} \Rightarrow d = -\frac{3}{2}, u_1 = 12$$

48. **(a)**

$$\begin{aligned}(1-x)^n &= 1 - {}_nC_1x^{n-1} + {}_nC_2x^{n-2} - {}_nC_3x^{n-3} + \cdots \\ &= 1 - nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} - \frac{n(n-1)(n-2)}{6}x^{n-3} + \cdots\end{aligned}$$

(b) **(i)** In an arithmetic sequence, the common difference is constant:

$$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} \Rightarrow n^3 - 9n^2 + 14n = 0$$

(ii) $n^3 - 9n^2 + 14n = 0 \Rightarrow n = 0, 2, 7$. Since there are 4 terms, $n = 7$.

49. **(a)** We are selecting four students out of 11: ${}_{11}C_4 = 330$

(b) ${}_5C_2 \times {}_6C_2 = 150$

(c) 'At least one junior' is the complement of 'No junior'.

${}_{11}C_4 - {}_5C_4 = 325$

50. **(a)** $(3+x)^4 = 81 + 108x + 54x^2 + 12x^3 + x^4$

(b) $3.1^4 = (3+0.1)^4 = 81 + 108\times 0.1 + 54\times 0.1^2 + 12\times 0.1^3 + 0.1^4 = 92.3521$

51. The number of ways the youngest will receives 3 books is ${}_7C_3 = 35$. Every time, the other 2 students will share the 4 books left in ${}_4C_2 = 6$ ways. Thus, the 7 books may be shared by all three in $35\times 6 = 210$ ways.

52. **(a)** After 10, the first number divisible by 7 is 14 and the last number is 294.

This is an arithmetic sequence whose first term is 14 and last term is 294.

$294 = 14 + 7(n-1) \Rightarrow n = 41$

(b) $S_{41} = \frac{41}{2}(14 + 294) = 6314$

(c) We can use the sum of the arithmetic sequence formula:

$$\begin{aligned}&S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right) \\ &\Rightarrow \frac{n}{2}\left(2\times 1000 - 7(n-1)\right) < 0 \\ &\Rightarrow n > 286.71 \Rightarrow n = 287\end{aligned}$$

53. **(a)** If the largest is a 5, there will be 3 cards to be chosen from the 4 numbers below it: ${}_4C_3 = 4$. If the largest is a 6, there will be 3 cards to be chosen from the 5 numbers below it: ${}_5C_3 = 10$. If the largest is a 7, there will be 3 cards to be chosen from the 6 numbers below it: ${}_6C_3 = 20$. Thus, there will be 34 selections possible.

(b) 'At least two even numbers' is the complement of 'at most one even number'.

We have ${}_9C_4 = 126$ selections altogether.

There are ${}_5C_4 = 5$ selections with no even and ${}_5C_3 \times 4 = 40$ with only one even.

Therefore, there are $126 - 5 - 40 = 81$ selections with at least one even number.

54. **(a)** Consider the wives to be one unit. Then there will be 4 units to be arranged in 4! different ways. However, within the unit of wives, there will be 3! ways of arrangement. Thus, the number of ways this can happen is $4! \times 3! = 144$

(b) There are 4 spaces around the husbands where the wives can sit, they can arrange themselves in ${}_4P_3 = 24$ different ways. However, for every arrangement the husbands can be arranged in 3! ways. Thus, the number of ways this can happen is $4! \times 3! = 144$

55. **(a)** **(i)** $\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}} = 2^{u_{n+1} - u_n} = 2^d$

(ii) $v_1 = 2^{u_1} = 2^a$

(iii) $v_n = 2^{u_n} = 2^{a+(n-1)d}$

(b) v_n is a geometric sequence with first term $v_1 = 2^a$ and common ratio $\frac{v_{n+1}}{v_n} = 2^d$

(i) $S_n = v_1 \frac{1 - r^n}{1 - r} = 2^a \frac{1 - 2^{nd}}{1 - 2^d}$

(ii) The series converges if the common ratio is less than 1, i.e.,

$2^d < 1 \Rightarrow d < 0$

(iii) $S_\infty = \frac{v_1}{1 - r} = \frac{2^a}{1 - 2^d}$

(iv) $S_\infty = \frac{2^a}{1 - 2^d} = 2^{a+1} \Rightarrow \frac{1}{1 - 2^d} = 2 \Rightarrow d = -1$

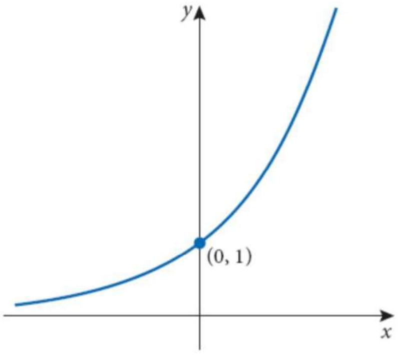
Exercise 4.1 & 4.2

1.

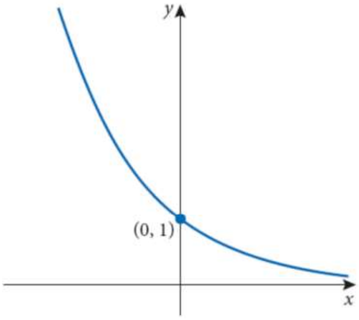
(a) $f(x) = b^x; b > 0$, an $b \neq 1$
(if $b = 1$, the function is equivalent to the horizontal line $y = 1$)

(b) Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}^+$

(c) **(i)** When $b > 1$, the function is increasing for all values of *x*.
y-intercept can be found at $x = 0$
$\therefore f(0) = b^0 = 1$
$\therefore$ *y*-intercept at $(0, 1)$

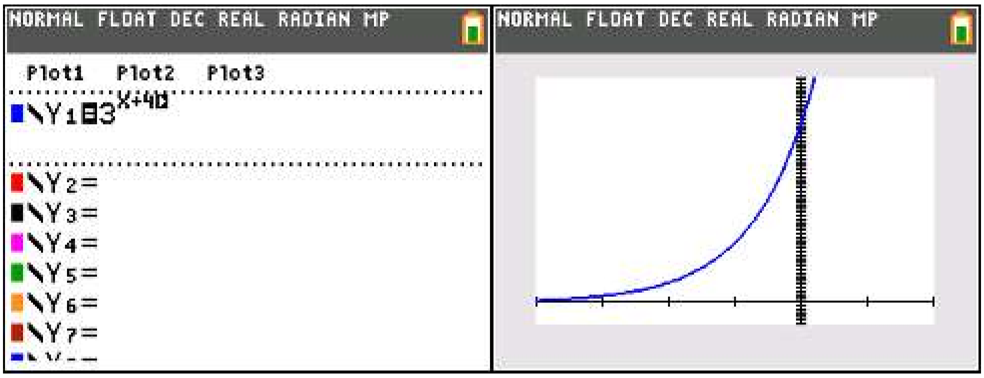


(ii) When $0 < b < 1$, function is decreasing for all values of *x*.
y-intercept can be found at $x = 0$
$\therefore f(0) = b^0 = 1$
$\therefore$ *y*-intercept at $(0, 1)$



2.

(a) Sketch:



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = 3^{X+4}$

$Y_2=$

$Y_3=$

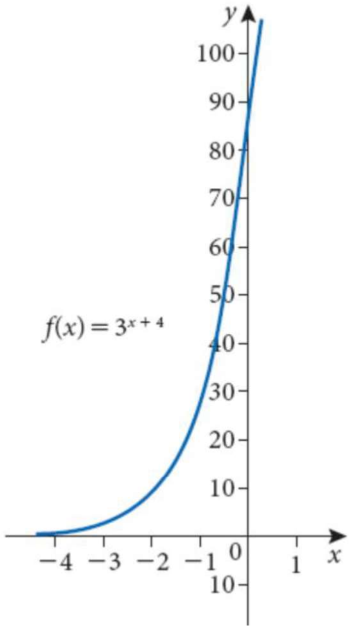
$Y_4=$

$Y_5=$

$Y_6=$

$Y_7=$

NORMAL FLOAT DEC REAL RADIAN MP

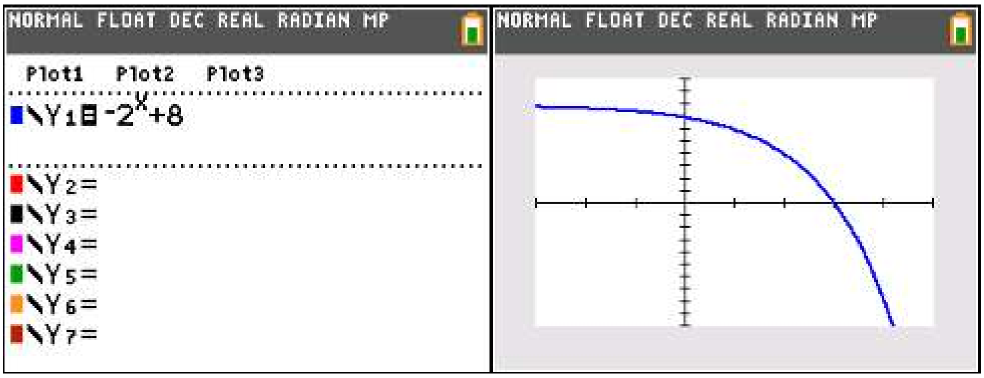


i. y-intercept can be found at $x = 0$
$f(0) = 3^{0+4} = 3^4 = 81$
$\therefore$ y-intercept at $(0, 81)$
x-intercept can be found at $y = 0$
$0 = 3^{x+4}$ impossible
$\therefore$ no x-intercept
Alternatively, asymptote of $f(x)$ at $y = 0$, $\therefore$ no x-intercept possible

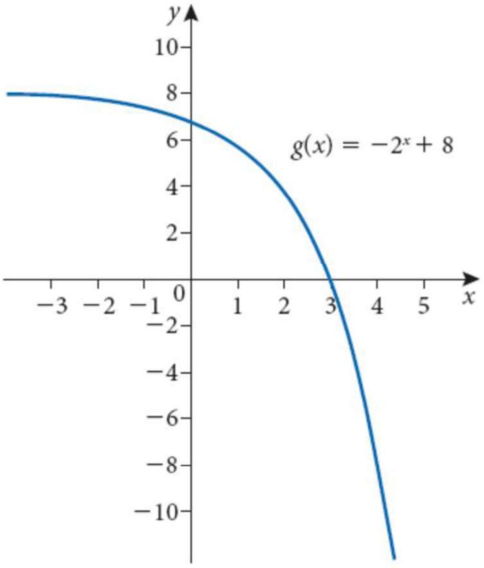
ii. Horizontal asymptote of $f(x)$ at $y = 0$, as no upward translation

iii. Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}^+$

(b) Sketch:



Y1 = -2^X+8
Y2 =
Y3 =
Y4 =
Y5 =
Y6 =
Y7 =

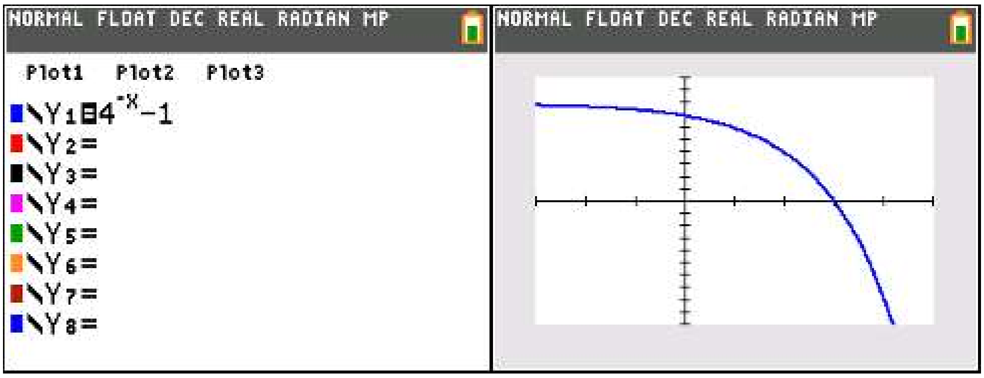


i. y-intercept can be found at $x = 0$
$g(0) = -2^0 + 8 = -1 + 8 = 7$
$\therefore$ y-intercept at $(0, 7)$
x-intercept can be found at $y = 0$
$0 = -2^x + 8$
$-8 = -2^x \Rightarrow 8 = 2^x$
$\therefore x = 3$ and x-intercept is $(3, 0)$

ii. Upward translation of 8 units $\therefore$ horizontal asymptote of $g(x)$ at $y = 8$

iii. Domain: $x \in \mathbb{R}$
Range: $y < 8$

(c) Sketch:

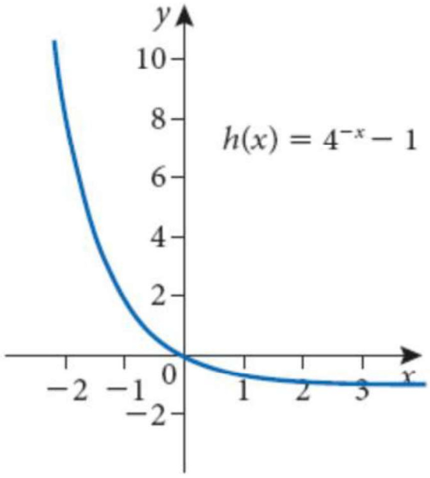


NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=4^-X-1
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=

NORMAL FLOAT DEC REAL RADIAN MP

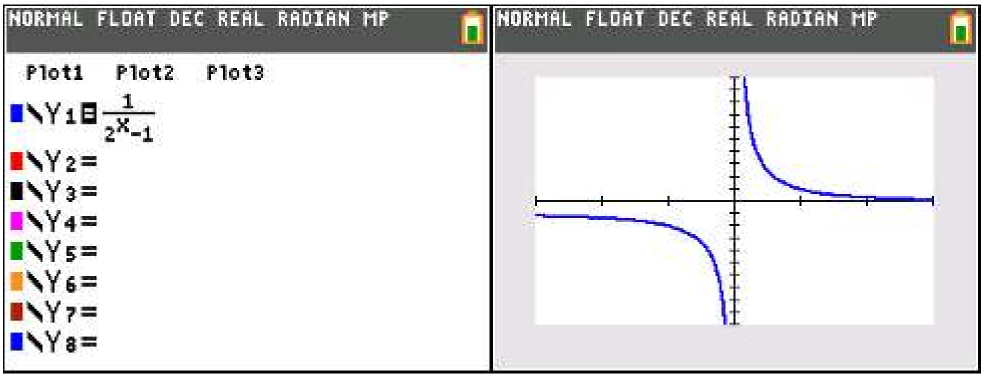


i. y-intercept can be found at $x = 0$
$h(0) = 4^{-(0)} - 1 = 1 - 1 = 0$
∴ y-intercept at $(0, 0)$
∴ x-intercept also at $(0, 0)$

ii. Downward translation of 1 unit ∴ horizontal asymptote of $h(x)$ at $y = -1$

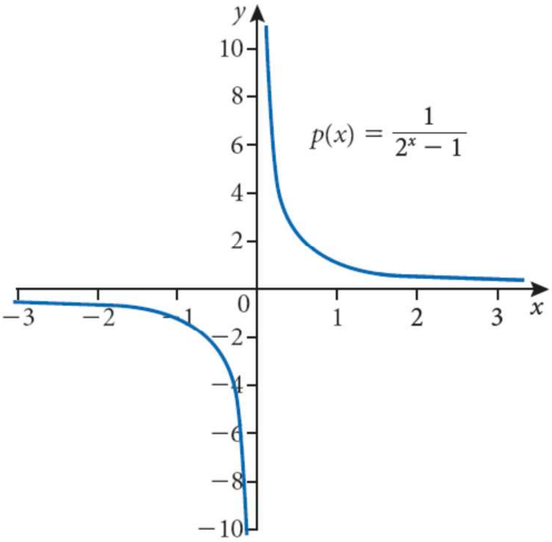
iii. Domain: $x \in \mathbb{R}$
Range: $y > -1$

(d) Sketch:



NORMAL FLOAT DEC REAL RADIAN MP
Plot1 Plot2 Plot3
■\Y1 = $\frac{1}{2^X-1}$
■\Y2=
■\Y3=
■\Y4=
■\Y5=
■\Y6=
■\Y7=
■\Y8=

NORMAL FLOAT DEC REAL RADIAN MP



i. y-intercept can be found at $x = 0$

$g(0) = \frac{1}{2^0 - 1} = \frac{1}{1-1} =$ undefined (this implies a vertical asymptote at $x = 0$)

$\therefore$ no y-intercept

x-intercept can be found at $y = 0$

$0 = \frac{1}{2^x - 1}$ impossible as numerator can never equal zero

(this implies a horizontal asymptote at $y = 0$)

$\therefore$ no x-intercept

ii. $\lim\limits_{x \to +\infty} \frac{1}{2^x - 1} = \frac{1}{\infty} \therefore$ horizontal asymptote at $y = 0$

$\lim\limits_{x \to -\infty} \frac{1}{2^x - 1} = \frac{1}{0-1} = -1 \therefore$ second horizontal asymptote at $y = -1$

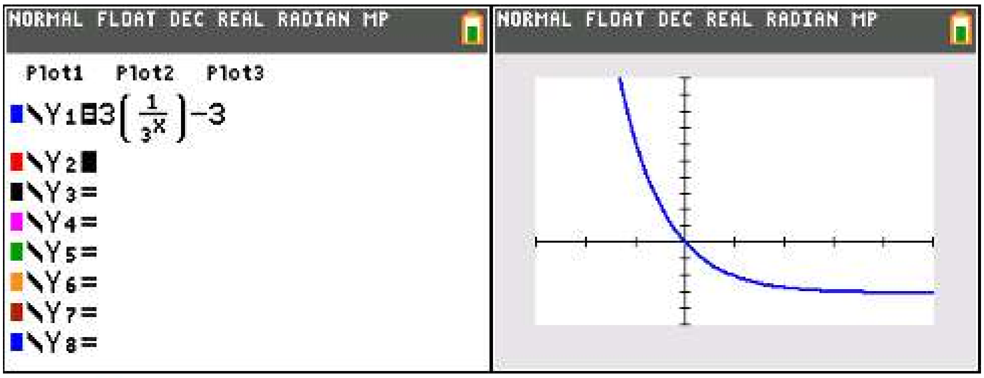
vertical asymptote at $x = 0$

(see part **i** for further explanation on vertical asymptote)

iii. Domain: $x \in \mathbb{R}, x \neq 0$

Range: $y > 0, y < -1$

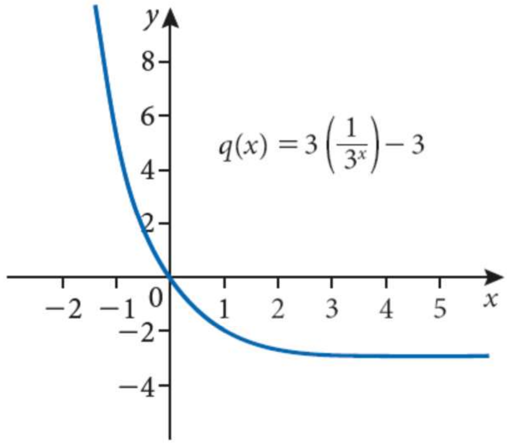
(e) Sketch:


NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=3(1/3^X)-3
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=

NORMAL FLOAT DEC REAL RADIAN MP



i. y-intercept can be found at $x = 0$

$q(0) = 3\left(\frac{1}{3^0}\right) - 3 = 3 - 3 = 0$

$\therefore$ y-intercept at $(0, 0)$

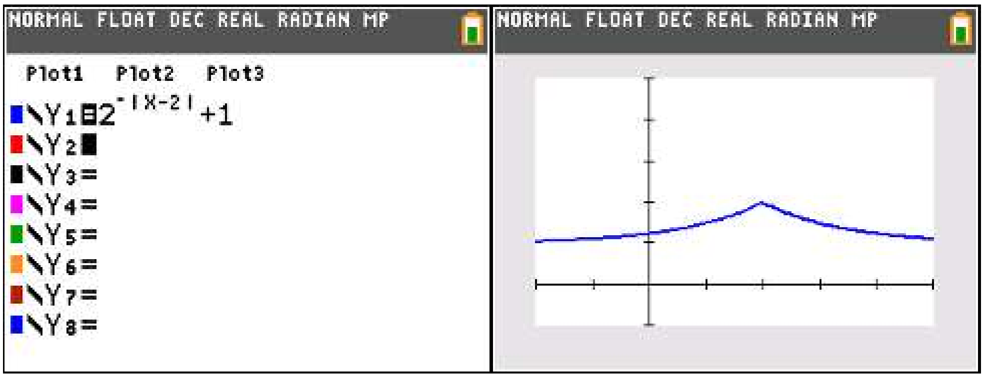
$\therefore$ x-intercept also at $(0, 0)$

ii. Downward translation of 3 units $\therefore$ horizontal asymptote of $q(x)$ at $y = -3$

iii. Domain: $x \in \mathbb{R}$

Range: $y > -3$

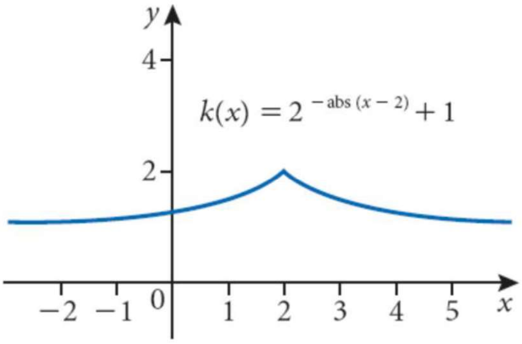
(f) Sketch:



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=$2^{-|X-2|}+1$
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=



i. y-intercept can be found at $x = 0$

$k(0) = 2^{-|0-2|} + 1 = 2^{-|-2|} + 1 = 2^{-2} + 1 = \frac{1}{4} + 1 = \frac{5}{4}$

$\therefore$ y-intercept at $(0, \frac{5}{4})$

x-intercept can be found at $y = 0$

$0 = 2^{-|x-2|} + 1$

$-1 = 2^{-|x-2|}$ impossible

$\therefore$ no x-intercept

ii. Upward translation of 8 units $\therefore$ horizontal asymptote of $k(x)$ at $y = 1$

iii. Domain: $x \in \mathbb{R}$

Function maximum occurs at $(2, 2)$

$\therefore$ Range: $1 < y < 2$

3. $f(x) = a(b)^{x-c} + d$, has domain $x \in \mathbb{R}$

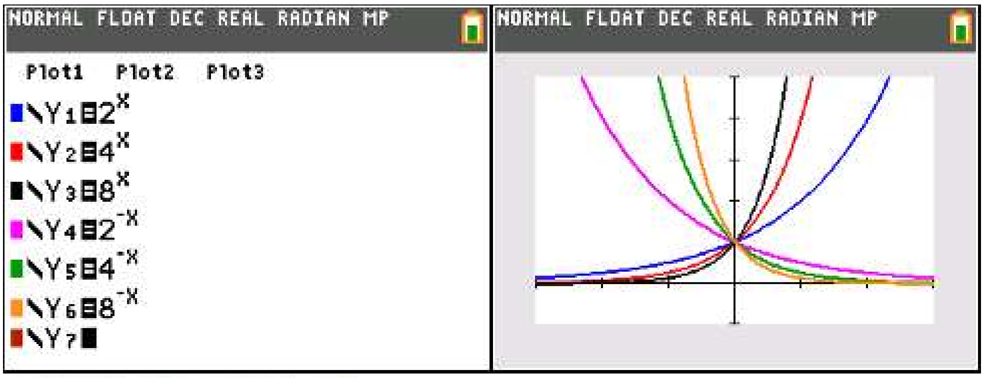
An upward translation of d units, will shift the asymptote to $y = d$

If $a > 0$, this will create a range of $y > d$.

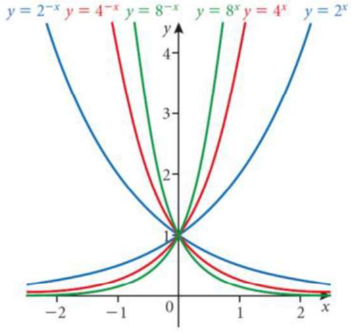
Alternatively, if $a < 0$, the function will reflect about the x-axis, resulting in the range $y < d$.

y-intercept occurs at $x = 0 \Rightarrow f(0) = a(b)^{0-c} + d$ $\therefore$ y-intercept at $(0, a \cdot b^{-c} + d)$

4.



```
NORMAL FLOAT DEC REAL RADIAN MP
  Plot1  Plot2  Plot3
■\Y1■2^X
■\Y2■4^X
■\Y3■8^X
■\Y4■2^-X
■\Y5■4^-X
■\Y6■8^-X
■\Y7■
```



5.

(a) $y = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

(b) $y = 4^{-x} = (4^{-1})^x = \left(\frac{1}{4}\right)^x$

(c) $y = 8^{-x} = (8^{-1})^x = \left(\frac{1}{8}\right)^x$

6. Given that $1 < a < b$ for $y = a^x$ and $y = b^x$, both functions will be continually increasing, however $y = b^x$ will be increasing at a faster rate. Therefore, b^x will be steeper than a^x.

7. Let growth factor per year be written as x. Given that population triples after 25 years, $x^{25} = 3$, giving an annual growth factor of $x = \sqrt[25]{3} = 3^{\frac{1}{25}} \approx 1.04$

$\therefore P(t) = 100\,000 \cdot 3^{\frac{t}{25}}$, where t represents the number of years.

(a) $P(50) = 100\,000 \cdot 3^{\frac{50}{25}} = 100\,000 \cdot 3^2 = 900\,000$

(b) $P(70) = 100\,000 \cdot 3^{\frac{70}{25}} = 100\,000 \cdot 3^{2.8} \approx 2\,167\,402$

(c) $P(50) = 100\,000 \cdot 3^{\frac{100}{25}} = 100\,000 \cdot 3^4 = 8\,100\,000$

8. Let growth factor per minute be written as x. Given that population doubles after 3 minutes, $\therefore\ x^3 = 2$, giving an annual growth factor of $x = \sqrt[3]{2} = 2^{\frac{1}{3}} \approx 1.26$

$\therefore N(t) = 10^4 \cdot 2^{\frac{t}{3}}$, gives the number of bacteria after t minutes.

(a) $N(3) = 10^4 \cdot 2^{\frac{3}{3}} = 10^4 \cdot 2^1 = 2 \cdot 10^4 = 20\,000$

(b) $N(9) = 10^4 \cdot 2^{\frac{9}{3}} = 10^4 \cdot 2^3 = 8 \cdot 10^4 = 80\,000$

(c) $N(27) = 10^4 \cdot 2^{\frac{27}{3}} = 10^4 \cdot 2^9 = 512 \cdot 10^4 = 5.12 \cdot 10^6 = 5\,120\,000$

(d) $N(60) = 10^4 \cdot 2^{\frac{60}{3}} = 10^4 \cdot 2^{20} = 1048576 \cdot 10^4 \approx 1.05 \cdot 10^{10}$

9.

(a) Equation $A(t) = A_0(r)^t$ where A_0 is the initial investment amount, r is the annual rate and t is years. Doubling in 10 years implies that $r^{10} = 2 \;\therefore r = \sqrt[10]{2} = 2^{\frac{1}{10}} \approx 1.07$

$\therefore A(t) = A_0(2)^{\frac{t}{10}} \approx A_0(1.07)^t$

(b) In equation $r = 1 + r_s$, where r_s would be the simple interest rate

$\therefore r_s = 1 - \mathrm{r} = 1 - \sqrt[10]{2} = 1 - 2^{\frac{1}{10}} \approx 0.0718$ or 7.18%

10. For investment problems, one can use the following expression: $FV = PV\left(1 + \frac{r}{n}\right)^{nt}$, where PV represents the present (original) value of the investment, r represents the investment rate, n represents the annual amount of compounding periods, t represents the time in years, and FV represents the final value of the investment after t years. Using this information we can calculate the sub-sections of this question:

(a) As quarterly implies 4 compounding periods per year, the equations take the following form: $FV = 10\ 000\left(1 + \frac{0.11}{4}\right)^{4(5)} \approx \$17{,}204.28$ (rounded to 2 d.p.)

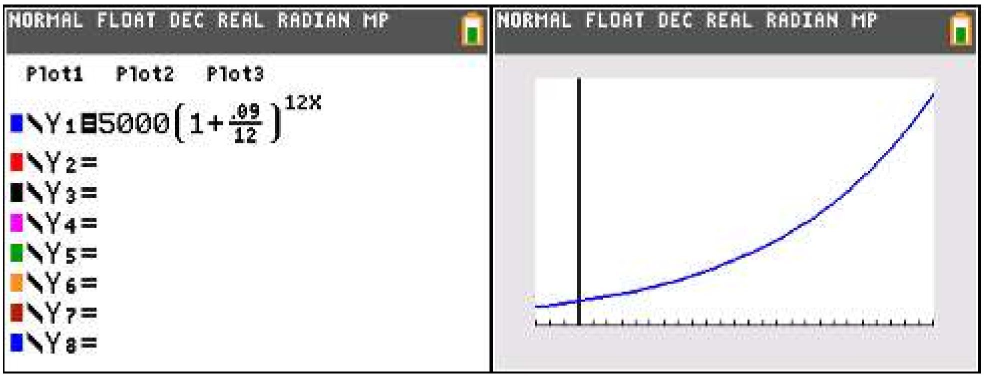
(b) $FV = 10\ 000\left(1 + \frac{0.11}{4}\right)^{4(10)} \approx \$29{,}598.74$ (rounded to 2 d.p.)

(c) $FV = 10\ 000\left(1 + \frac{0.11}{4}\right)^{4(15)} \approx \$50{,}922.51$ (rounded to 2 d.p.)

11. See question **10** for explaination of investment formula for n compounding periods per year.

(a) Monthly compounding periods implies 12 compounding periods per year, therefore: $A(t) = 5000\left(1 + \frac{0.09}{12}\right)^{12t}$, for t years.

(b)



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = 5000\left(1+\frac{.09}{12}\right)^{12X}$

$Y_2 =$

$Y_3 =$

$Y_4 =$

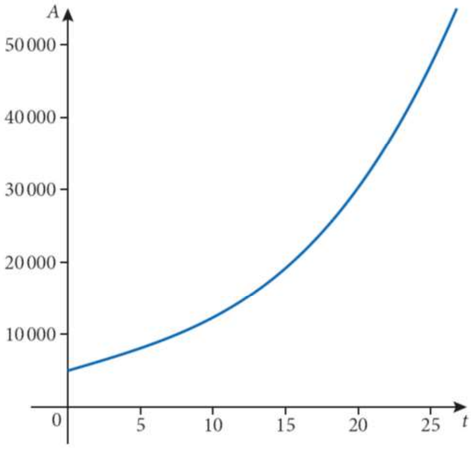
$Y_5 =$

$Y_6 =$

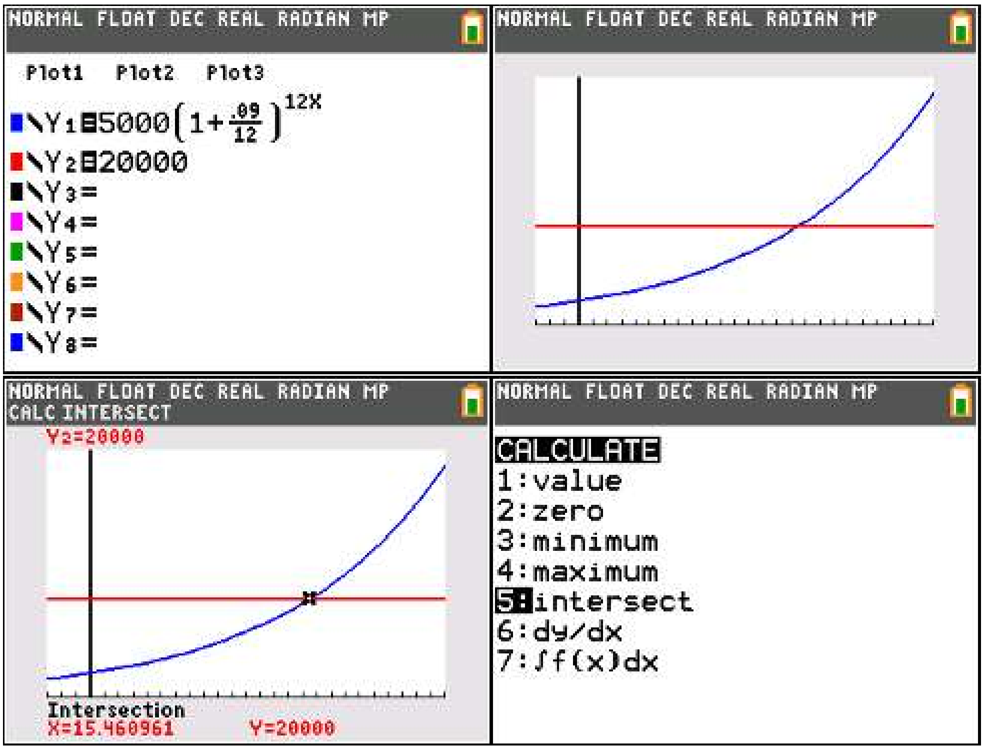
$Y_7 =$

$Y_8 =$

NORMAL FLOAT DEC REAL RADIAN MP



(c)



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = 5000\left(1+\frac{.09}{12}\right)^{12X}$

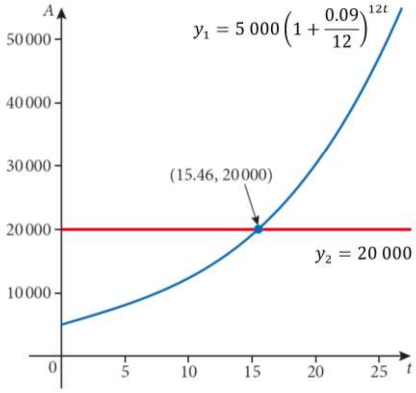
$Y_2 = 20000$

$Y_3=$
$Y_4=$
$Y_5=$
$Y_6=$
$Y_7=$
$Y_8=$

CALC INTERSECT

$Y_2=20000$

Intersection
X=15.460961 Y=20000



Intersection occurs at $(15.5,\ 20\,000)$

∴ After 16 years, the investment will have a value greater than \$20,000.

12. See question **10** for explanation of investment formula for n compounding periods per year.

(a) Annual compounding implies $n = 1$

$$\therefore\ FV = 10\,000\left(1+\frac{0.11}{1}\right)^{1(5)} \approx \$16{,}850.58 \text{ (rounded to 2 d.p.)}$$

(b) Monthly compounding implies $n = 12$

$$\therefore\ FV = 10\,000\left(1+\frac{0.11}{12}\right)^{12(5)} \approx \$17{,}289.16 \text{ (rounded to 2 d.p.)}$$

(c) Daily compounding implies $n = 365$

$$\therefore\ FV = 10\,000\left(1+\frac{0.11}{365}\right)^{365(5)} \approx \$17{,}331.09 \text{ (rounded to 2 d.p.)}$$

(d) Hourly compounding implies $n = 365 \cdot 24 = 8750$

$$\therefore\ FV = 10\,000\left(1+\frac{0.11}{8750}\right)^{8750(5)} \approx \$17{,}332.47 \text{ (rounded to 2 d.p.)}$$

13. See question **10** for explanation of investment formula for n compounding periods per year. In this question interest rate of 100% implies $r = 1$.

(a) Annual compounding implies $n = 1$

$\therefore\ FV = 1\left(1 + \frac{1}{1}\right)^{1(1)} = \$2$

(b) Monthly compounding implies $n = 12$

$\therefore\ FV = 1\left(1 + \frac{1}{12}\right)^{12(1)} \approx \$2.61$

(c) Daily compounding implies $n = 365$

$\therefore\ FV = 1\left(1 + \frac{1}{365}\right)^{365(1)} \approx \$2.71$

(d) Hourly compounding implies $n = 365 \cdot 24 = 8750$

$\therefore\ FV = 1\left(1 + \frac{1}{8750}\right)^{8750(1)} \approx \$2.72$

(e) Compounding every minute implies $n = 8750 \cdot 60 = 525000$

$\therefore\ FV = 1\left(1 + \frac{1}{525\,000}\right)^{525\,000(1)} \approx \$2.72$

14. Increase of 3.2% implies growth rate of 1.032, meaning $P(t) = P_0 \cdot 1.032^t$, where P_0 is the initial deer population and $P(t)$ is the population after t years.

(a) $248\,000 = P_1 \cdot 1.032^1$, where P_1 is the population one year ago.

$\therefore P_1 = \frac{248\,000}{1.032} \approx 240\,310$

(b) $248\,000 = P_8 \cdot 1.032^8$, where P_8 is the population eight years ago.

$\therefore P_8 = \frac{248\,000}{1.032^8} \approx 192\,758$ (rounded to the nearest unit)

15. The general formula for half-life decay is $A_f = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where A_0 is the initial amount, h is the half-life of the material, and A_f is the final amount after t years.

Given the half-life of carbon is 5730, to find percentage remaining after 20 000 years, we compute $\left(\frac{1}{2}\right)^{\frac{20\,000}{5730}} \approx 0.088978\ldots$

$\therefore$ 8.90% of the original carbon will be left.

16. See question **15** for explanation of half-life formula.

$A_f = A_0\left(\frac{1}{2}\right)^{\frac{t}{1.5}}$ where t is in days

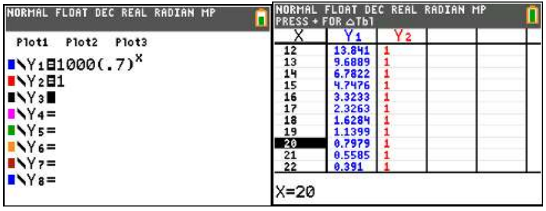
$\therefore \left(\frac{1}{2}\right)^{\frac{5}{1.5}} \approx 0.0992 = 9.92\%$ remaining in the blood stream

17.

(a) Loss of 30% per week, $\therefore 100 - 30 = 70\% = 0.7$

$\therefore A(w) = 1000(0.7)^w$

(b) $1000(0.7)^w < 1$



```
NORMAL FLOAT DEC REAL RADIAN MP
 Plot1  Plot2  Plot3
■\Y1⊟1000(.7)^X
■\Y2⊟1
■\Y3■
■\Y4=
■\Y5=
■\Y6=
■\Y7=
■\Y8=
```

```
NORMAL FLOAT DEC REAL RADIAN MP
PRESS + FOR △Tbl
  X     Y1      Y2
  12    13.841  1
  13    9.6889  1
  14    6.7822  1
  15    4.7476  1
  16    3.3233  1
  17    2.3263  1
  18    1.6284  1
  19    1.1399  1
  20    0.7979  1
  21    0.5585  1
  22    0.391   1
X=20
```

$\therefore$ 20 weeks

18. Exponential growth is defined for functions $f(x) = b^x; b > 1$ as a value of $b = 1$ would result in a constant function equivalent to $y = 1$.
Alternatively, exponential decay can be defined for functions $f(x) = b^x; 0 < b < 1$.
Exponential functions cannot take the form $b < 0$ as this would result in a discontinuous function.

19. **Case I** resembles an arithmetic sequence with $a_0 = 1$ and common difference $d = 1$.
Therefore, the sum of an arithmetic sequence can be used:
$S_{30} = \frac{30}{2}(2(1) + 1(30 - 1)) = \$465$
Case II resembles a geometric sequence with $g_0 = 0.01$ common ratio, *r*= 2.
Therefore, the sum of a geometric sequence can be used:
$S_{30} = \frac{0.01(1-2^{30})}{(1-2)} = \frac{0.01(1-1\,073\,741\,824)}{(1-2)} = \frac{-10\,737\,418.24}{-1} = \$10{,}737{,}418.24$

20.

(a) From the graph, $f(1) = 6 \Rightarrow 6 = k(a)^1 = ka$
Also from the graph, $f(3) = 24 \Rightarrow 24 = k(a)^3 = ka \cdot a^2$
By substitution: $24 = 6 \cdot a^2 \Rightarrow a^2 = 4 \therefore a = 2$
(Note: a must be positive due to the nature of the function.)
By substitution: $6 = k(2) \therefore k = 3$
Final function: $f(x) = 3(2)^x$

(b) From the graph, $f(0) = 2 \Rightarrow 2 = k(a)^0 = k$
$\therefore k = 2$
Also from the graph, $f(2) = \frac{2}{9} \Rightarrow \frac{2}{9} = k(a)^2 = 2 \cdot a^2 \Rightarrow \frac{1}{9} = a^2$
$\therefore a = \frac{1}{3}$
(Note: a must be positive due to the nature of the function.)
Final function: $f(x) = 2\left(\frac{1}{3}\right)^x$

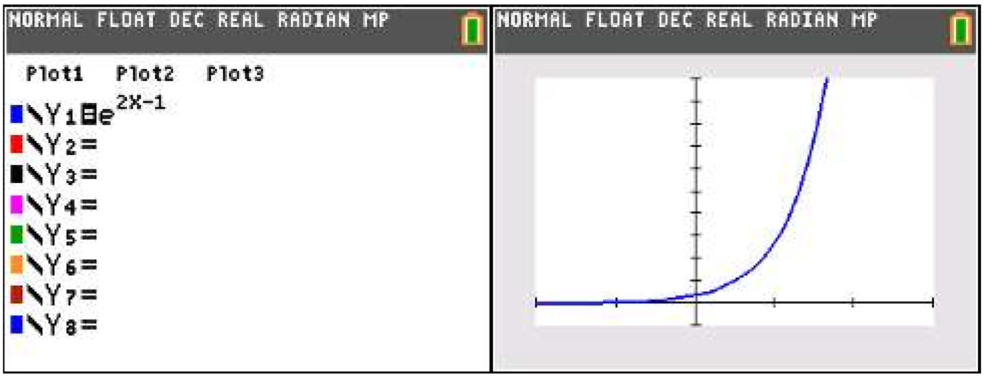
(c) From the graph, $f(-1) = -\frac{4}{3} \Rightarrow -\frac{4}{3} = k(a)^{-1} = \frac{k}{a}$
$\therefore k = -\frac{4a}{3}$
Also from the graph, $f(1) = -12 \Rightarrow -12 = k(a)^1 = ka$
By substitution: $-12 = \left(-\frac{4a}{3}\right) \cdot a \Rightarrow a^2 = \frac{12 \cdot 3}{4} = 9$
$\therefore a = 3$
(Note: a must be positive due to the nature of the function.)
By substitution: $k = -\frac{4(3)}{3}$
$\therefore k = -4$
Final function: $f(x) = -4(3)^x$

(d) From the graph, $f(1) = 15 \Rightarrow 15 = k(a)^1 = ka$
Also from the graph, $f(2) = 150 \Rightarrow 150 = k(a)^2 = ka \cdot a^2$
By substitution: $150 = 15 \cdot a$
$\therefore a = 10$
By substitution: $k = \frac{15}{10}$
$\therefore k = 1.5$
Final function: $f(x) = 1.5(10)^x$

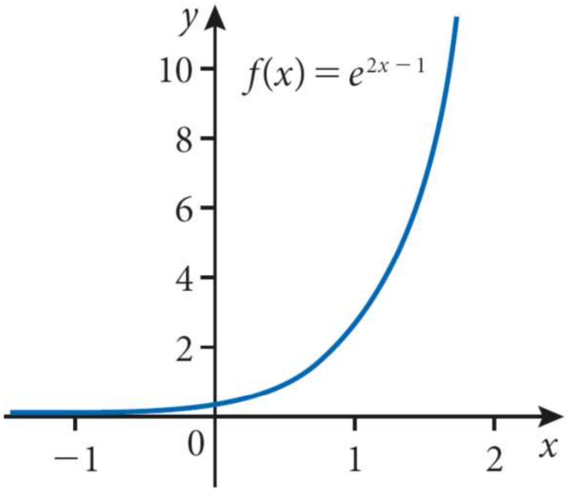
Exercise 4.3

1.

(a) Sketch:



```
NORMAL FLOAT DEC REAL RADIAN MP
Plot1  Plot2  Plot3
\Y1=e^(2X-1)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
```



i. y-intercept can be found at $x = 0$

$f(0) = \mathrm{e}^{2(0)-1} = \mathrm{e}^{-1} = \frac{1}{\mathrm{e}}$

$\therefore$ y-intercept at $(0, \frac{1}{\mathrm{e}})$

x-intercept can be found at $y = 0$

$0 = \mathrm{e}^{2x-1}$ impossible

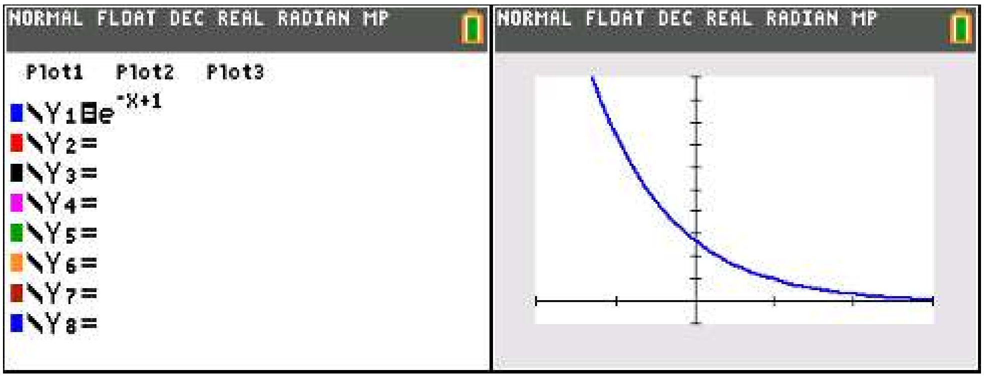
$\therefore$ no x-intercept

ii. Horizontal asymptote of $f(x)$ at $y = 0$ as there is no x-intercept OR as there has been no upward translation

iii. Domain: $x \in \mathbb{R}$

Range: $y > 0$

(b) Sketch:

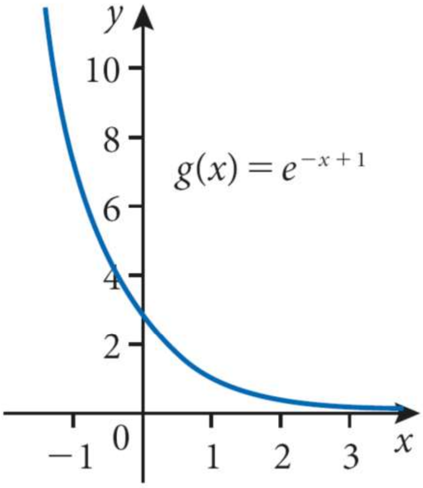


NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=e^-X+1
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=

NORMAL FLOAT DEC REAL RADIAN MP

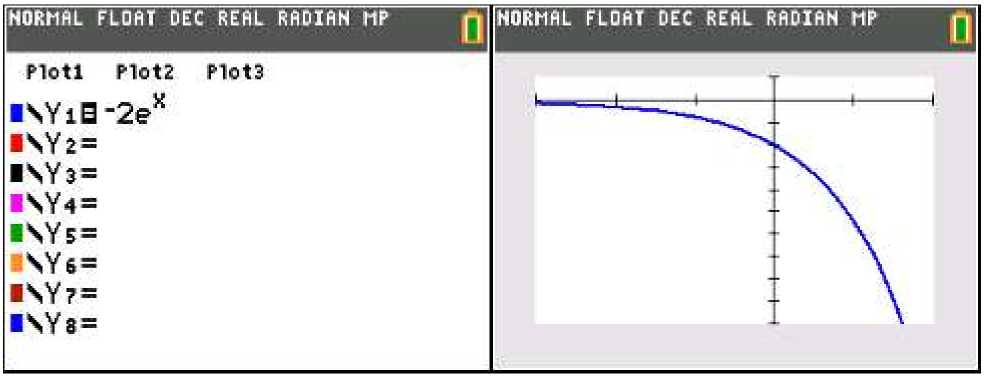


i. y-intercept can be found at $x = 0$
$g(0) = \mathrm{e}^{-(0)+1} = \mathrm{e}^1 = \mathrm{e}$
$\therefore$ y-intercept at $(0,\ \mathrm{e})$
x-intercept can be found at $y = 0$
$0 = \mathrm{e}^{-x+1}$ is impossible
$\therefore$ no x-intercept

ii. Horizontal asymptote of $h(x)$ at $y = 0$ as there is no x-intercept OR as there has been no upward translation

iii. Domain: $x \in \mathbb{R}$
Range: $y > 0$

(c) Sketch:

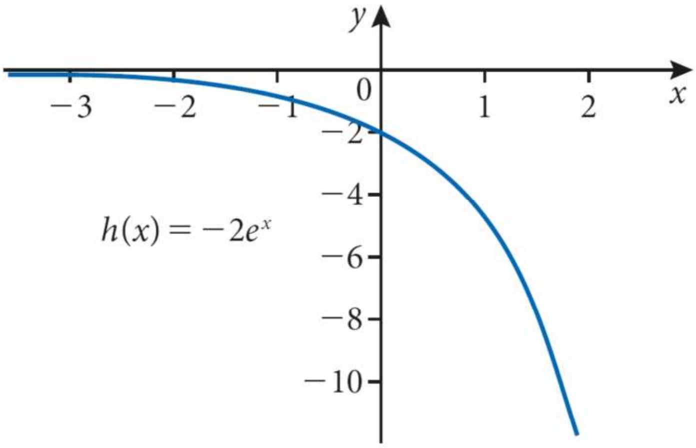


NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=-2e^X
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=

NORMAL FLOAT DEC REAL RADIAN MP

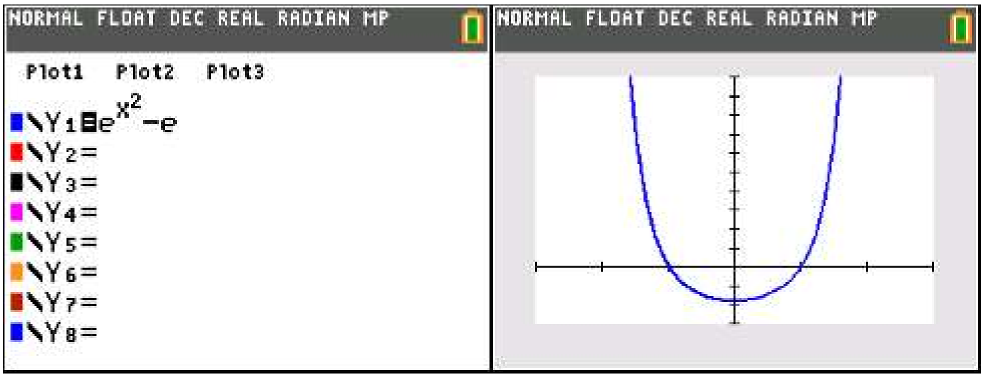


i. y-intercept can be found at $x = 0$
$h(0) = -2e^0 = -2 \cdot 1 = -2$
$\therefore$ y-intercept at $(0,\ -2)$
x-intercept can be found at $y = 0$
$0 = -2e^x$ is impossible
$\therefore$ no x-intercept

ii. Horizontal asymptote of $h(x)$ at $y = 0$ as there is no x-intercept OR as there has been no upward translation

iii. Domain: $x \in \mathbb{R}$
Range: $y < 0$ (as there has been a reflection about the x-axis)

(d) Sketch:



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = e^{x^2} - e$

$Y_2 =$

$Y_3 =$

$Y_4 =$

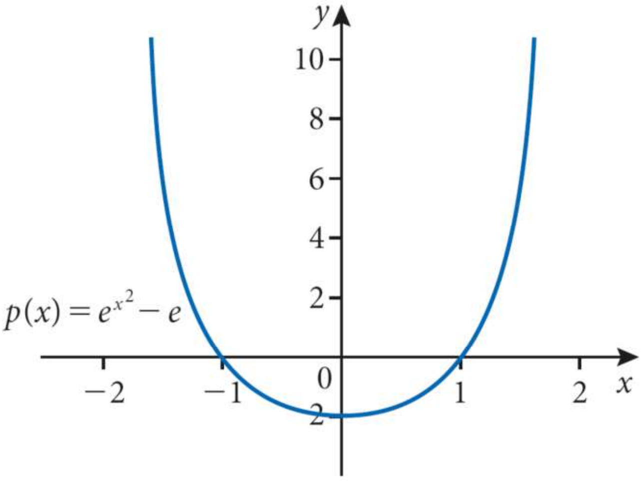
$Y_5 =$

$Y_6 =$

$Y_7 =$

$Y_8 =$

NORMAL FLOAT DEC REAL RADIAN MP



i. y-intercept can be found at $x = 0$

$p(0) = \mathrm{e}^{(0)^2} - \mathrm{e} = \mathrm{e}^0 - \mathrm{e} = 1 - \mathrm{e}$

$\therefore$ y-intercept at $(0, 1 - \mathrm{e})$

x-intercept can be found at $y = 0$

$0 = \mathrm{e}^{x^2} - \mathrm{e}$

$\mathrm{e} = \mathrm{e}^{x^2} \Rightarrow x^2 = 1$

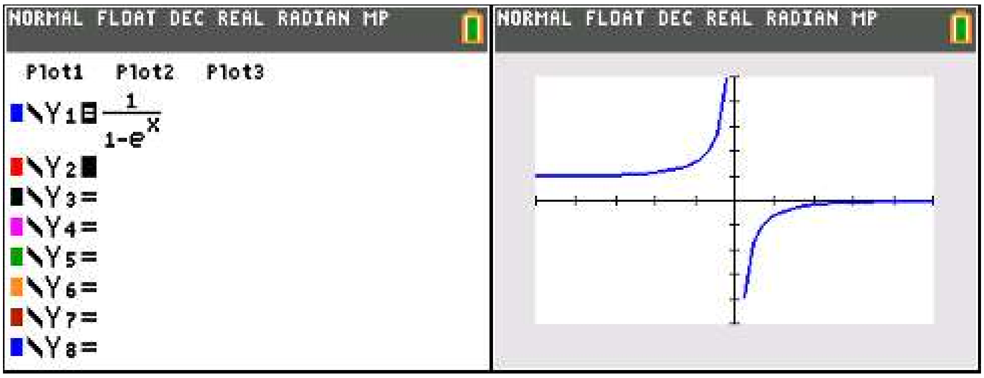
$\therefore x = \pm 1$ and x-intercept are at $(-1, 0)$ and $(1, 0)$

ii. No asymptotes (just a minimum point)

iii. Domain: $x \in \mathbb{R}$

Range: $y > (1 - \mathrm{e})$

(e) Sketch:



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = \frac{1}{1-e^X}$

$Y_2 =$

$Y_3 =$

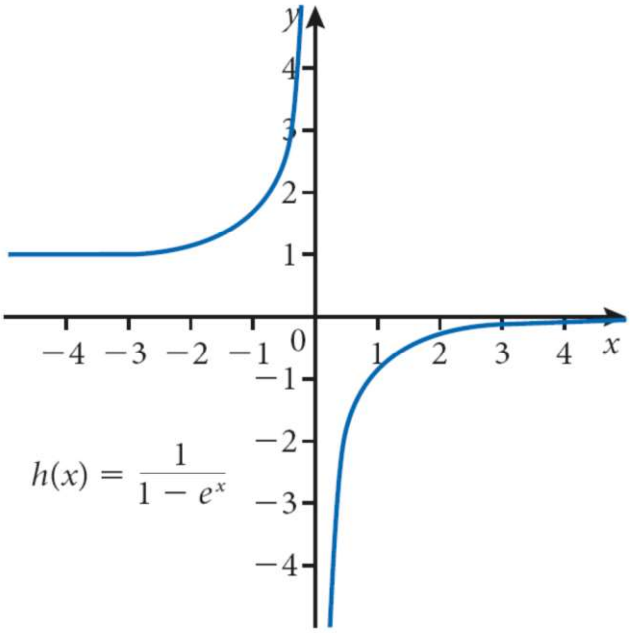
$Y_4 =$

$Y_5 =$

$Y_6 =$

$Y_7 =$

$Y_8 =$



i. y-intercept can be found at $x = 0$

$h(0) = \frac{1}{1-e^0} = \frac{1}{1-1} = \frac{1}{0} =$ undefined

$\therefore$ no y-intercept (implies an asymptote at $x = 0$)

x-intercept can be found at $y = 0$

$0 = \frac{1}{1-e^x}$ impossible as numerator can never be zero

$\therefore$ no x-intercept (implies an asymptote at $y = 0$)

ii. $\lim\limits_{x\to+\infty} \frac{1}{1-e^x} = \frac{1}{-\infty}$

$\therefore$ horizontal asymptote at $y = 0$

$\lim\limits_{x\to-\infty} \frac{1}{1-e^x} = \frac{1}{1-0} = 1$

$\therefore$ second horizontal asymptote at $y = 1$

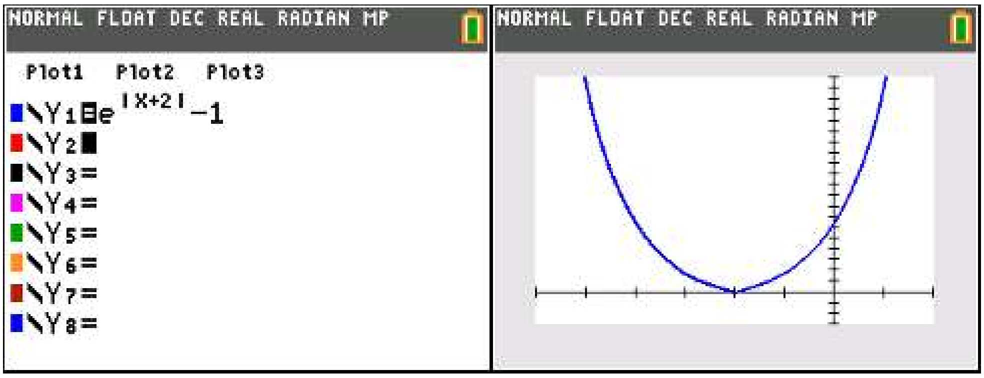
vertical asymptote at $x = 0$

(see part **i** for further explanation on vertical asymptote)

iii. Domain: $x \in \mathbb{R}, x \neq 0$

Range: $y < 0, y > 1$

(f) Sketch:



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=e^|X+2|−1

Y2=

Y3=

Y4=

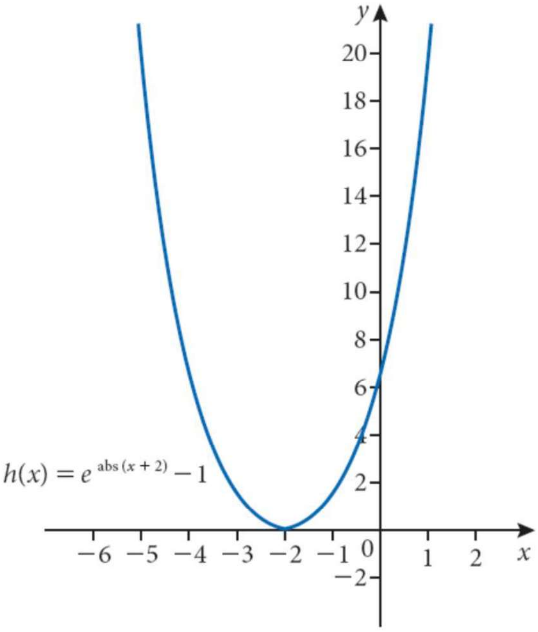
Y5=

Y6=

Y7=

Y8=

NORMAL FLOAT DEC REAL RADIAN MP



i. y-intercept can be found at $x = 0$

$h(0) = \mathrm{e}^{|0+2|} - 1 = \mathrm{e}^2 - 1$

$\therefore$ y-intercept at $(0, \mathrm{e}^2 - 1)$

x-intercept can be found at $y = 0$

$0 = \mathrm{e}^{|x+2|} - 1$

$1 = \mathrm{e}^{|x+2|} \Rightarrow |x+2| = 0$

$\therefore x = -2$ and x-intercept is $(-2, 0)$

ii. No asymptote (only a minimum)

iii. Domain: $x \in \mathbb{R}$

Range: $y > 0$

2.

(a) $\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

(b) $\left(1 - \frac{1}{100}\right)^{100} = (0.99)^{100} = 0.366032341273$

$\left(1 - \frac{1}{10\,000}\right)^{10\,000} = (0.9999)^{10000} = 0.367861046433$

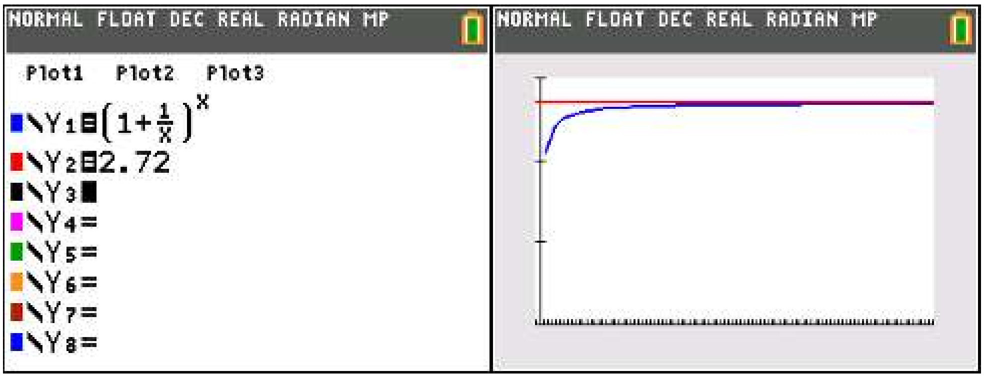
$\left(1 - \frac{1}{1\,000\,000}\right)^{1\,000\,000} = (0.999999)^{1\,000\,000} = 0.367879257232$

(c) $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n \approx 0.36788$

(d) $\frac{1}{0.36788} \approx \mathrm{e}$

$\therefore$ it is the reciprocal of e

3.



Plot1 Plot2 Plot3
$Y_1 = (1+\frac{1}{X})^X$
$Y_2 = 2.72$
$Y_3 =$
$Y_4 =$
$Y_5 =$
$Y_6 =$
$Y_7 =$
$Y_8 =$

$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e} \approx 2.71828\ldots$

As a result, the graph of $y=\left(1+\frac{1}{x}\right)^x$ has an asymptote at $y=\mathrm{e}$.

As a result, it will never reach $y = 2.72$

$\therefore$ there will never be an intersection between the two graphs.

4.

(a) Bank A: $FV = 500\cdot\left(1+\frac{0.0685}{12}\right)^{3\cdot 12} = 613.71$

$\therefore$ interest $= 613.71 - 500 = €113.71$

(b) Bank A: $FV = 500\cdot \mathrm{e}^{0.0685\cdot 3} = 614.07$

$\therefore$ interest $= 614.07 - 500 = €114.07$

5.

(a) Blue Star: $FV = 1000\cdot\left(1+\frac{0.0613}{52}\right)^{5\cdot 52} \approx \$1{,}358.42$

Red Star: $FV = 1000\cdot \mathrm{e}^{0.0595\cdot 5} \approx \$1{,}346.49$

$\therefore$ Blue Star account will result in the greatest total after 5 years

(b) $\approx \$1{,}358.42$ (see part **a** for explanation)

(c) $1{,}358.42 - 1{,}346.49 = \$11.93$

6.

(a) $A(1) = A_0\cdot \mathrm{e}^{-0.0239(1)} = A_0\cdot 0.97638334\ldots$

$\therefore$ percentage remaining $\approx 97.6\ \%$

(b) $A(10) = A_0\cdot \mathrm{e}^{-0.0239(10)} = A_0\cdot 0.78741488\ldots$

$\therefore$ percentage remaining $\approx 78.7\ \%$

(c) $A(100) = A_0\cdot \mathrm{e}^{-0.0239(100)} = A_0\cdot 0.09162968\ldots$

$\therefore$ percentage remaining $\approx 9.16\ \%$

(d) $A(250) = A_0\cdot \mathrm{e}^{-0.0239(250)} = A_0\cdot 0.002541502\ldots$

$\therefore$ percentage remaining $\approx 0.254\ \%$

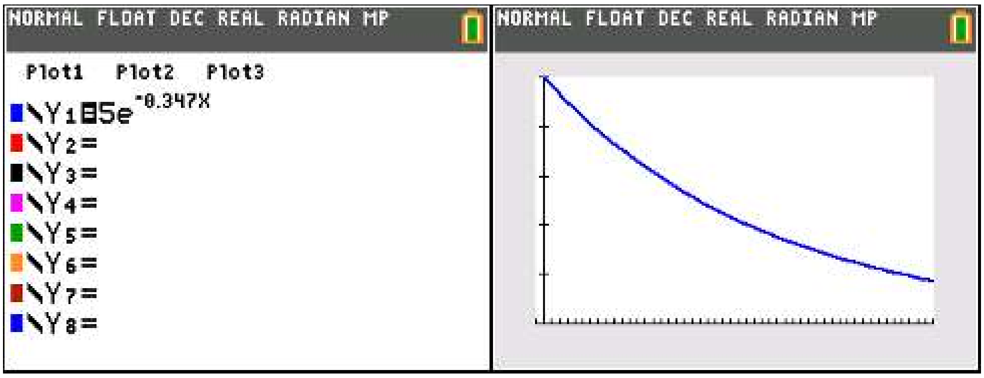
7.

(a) $A(0) = 5 \cdot \mathrm{e}^{-0.0347(0)} = 5 \cdot 1 = 5$ kg

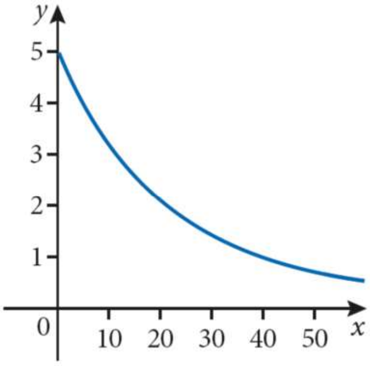
(b) $A(10) = A_0 \cdot \mathrm{e}^{-0.0347(10)} = A_0 \cdot 0.7068053\ldots$

$\therefore$ percentage remaining $\approx 70.7\ \%$

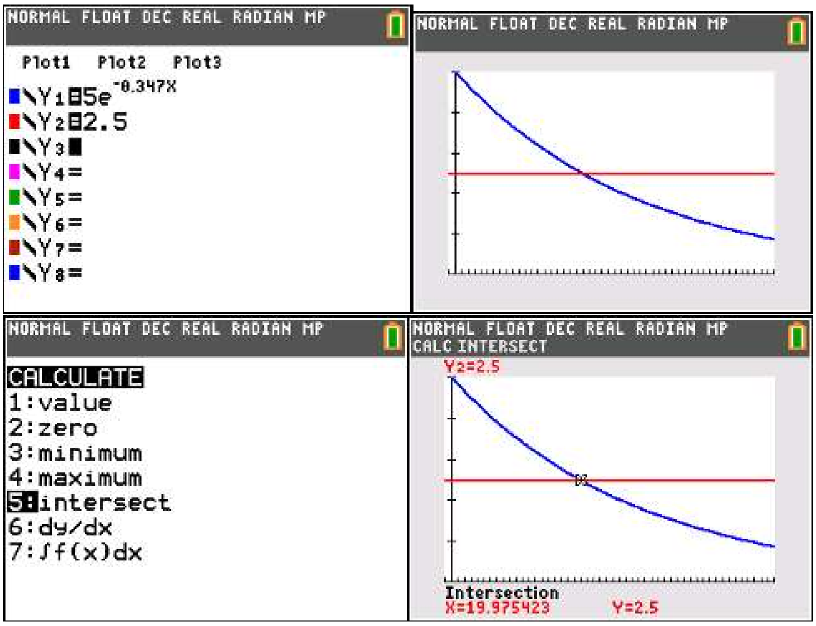
(c)



```
NORMAL FLOAT DEC REAL RADIAN MP
 Plot1  Plot2  Plot3
 Y1=5e^(-0.347X)
 Y2=
 Y3=
 Y4=
 Y5=
 Y6=
 Y7=
 Y8=
```



(d) Half-life occurs at half initial amount $\therefore$ 2.5 kg



```
NORMAL FLOAT DEC REAL RADIAN MP
 Plot1  Plot2  Plot3
 Y1=5e^(-0.347X)
 Y2=2.5
 Y3=
 Y4=
 Y5=
 Y6=
 Y7=
 Y8=
```

```
NORMAL FLOAT DEC REAL RADIAN MP
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

```
NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT
Y2=2.5
Intersection
X=19.975423    Y=2.5
```

From observation, approximately 20 days

8. Option A: $FV = PV \cdot \left(1+\frac{0.085}{2}\right)^2 = PV \cdot 1.08680625 \Longrightarrow$ increase of $\approx 8.68\%$

Option B: $FV = PV \cdot \left(1+\frac{0.0825}{4}\right)^4 = PV \cdot 1.08508761943\ldots \Longrightarrow$ increase of $\approx 8.51\%$

Option C: $FV = PV \cdot \mathrm{e}^{0.08} = PV \cdot 1.08328706768\ldots \Longrightarrow$ increase of $\approx 8.33\%$

$\therefore$ Option A provides the best increase on the initial investment

9.

(a) Given that $A(t) = B(t) \Longrightarrow 20 \cdot \mathrm{e}^{0.068t} = 20 \cdot r^t \Longrightarrow (\mathrm{e}^{0.068})^t = r^t$

$\therefore r = \mathrm{e}^{0.068} = 1.07026530848 \ldots \approx 1.07$

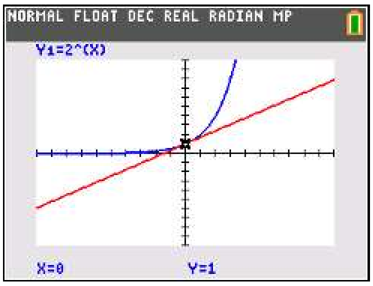
(b) r is the growth rate per minute

$1.07026530848 \ldots - 1 \approx 0.07026530848\ldots$

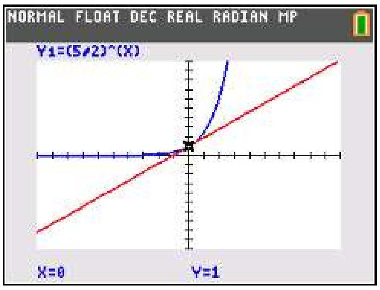
$\therefore 7.03\%$

10.

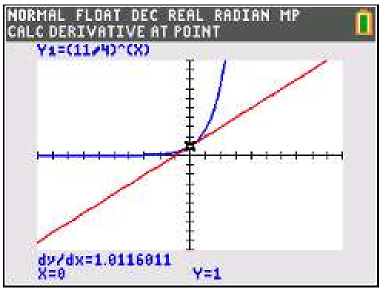
(a) Gradient < 1


NORMAL FLOAT DEC REAL RADIAN MP
Y1=2^(X)
X=0 Y=1

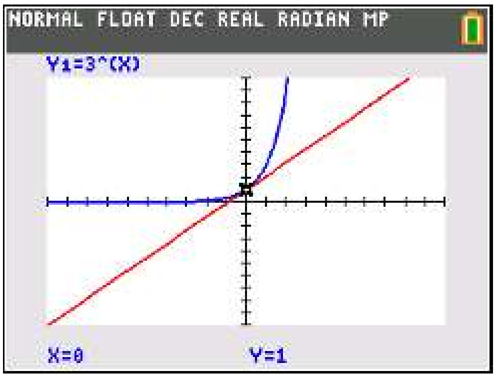
(b) Gradient < 1


NORMAL FLOAT DEC REAL RADIAN MP
Y1=(5/2)^(X)
X=0 Y=1

(c) Gradient > 1


NORMAL FLOAT DEC REAL RADIAN MP
CALC DERIVATIVE AT POINT
Y1=(11/4)^(X)
dy/dx=1.0116011
X=0 Y=1

(d) Gradient > 1



11.

(a) After 10 years: $FV = 1000 \cdot \mathrm{e}^{0.045 \cdot 10} \approx £1{,}568.31$

(b) $2000 = 1000 \cdot \mathrm{e}^{0.045 \cdot t} \Longrightarrow 2 = \mathrm{e}^{0.045 \cdot t} \Longrightarrow 0.045 \cdot t = \ln 2$

$\therefore t = \frac{\ln 2}{0.045} \approx 15.4$ years

(c) $10\,000 = 5000 \cdot \mathrm{e}^{0.045 \cdot t} \Longrightarrow 2 = \mathrm{e}^{0.045 \cdot t} \Longrightarrow 0.045 \cdot t = \ln 2$

$\therefore t = \frac{\ln 2}{0.045} \approx 15.4$ years

(d) The answers are the same, as the time to double depends only on the interest rate and the amount of compounding periods, and these are constant between **(b)** and **(c)**.

Exercise 4.4

1.

(a) $\log_2 16 = 4 \Longrightarrow$
$2^4 = 16$

(b) $\ln 1 = 0 \Longrightarrow$
$\mathrm{e}^0 = 1$

(c) $\log 100 = 2 \Longrightarrow$
$10^2 = 100$

(d) $\log 0.01 = -2 \Longrightarrow$
$10^{-2} = 0.01$

(e) $\log_7 343 = 3 \Longrightarrow$
$7^3 = 343$

(f) $\ln\left(\frac{1}{\mathrm{e}}\right) = -1 \Longrightarrow$
$\mathrm{e}^{-1} = \frac{1}{\mathrm{e}}$

(g) $\log 50 = y \Longrightarrow$
$10^y = 50$

(h) $\ln x = 12 \Longrightarrow$
$\mathrm{e}^{12} = x$

(i) $\ln(x + 2) = 3 \Longrightarrow$
$\mathrm{e}^3 = x + 2$

2.

(a) $2^{10} = 1024 \Longrightarrow$
$\log_2 1024 = 10$

(b) $10^{-4} = 0.0001 \Longrightarrow$
$\log 0.0001 = -4$

(c) $4^{-\frac{1}{2}} = \frac{1}{2} \Longrightarrow$
$\log_4 \left(\frac{1}{2}\right) = -\frac{1}{2}$

(d) $3^4 = 81 \Longrightarrow$
$\log_3 81 = 4$

(e) $10^0 = 1 \Longrightarrow$
$\log 1 = 0$

(f) $\mathrm{e}^x = 5 \Longrightarrow$
$\ln 5 = x$

(g) $2^{-3} = 0.125 \Longrightarrow$
$\log_2 0.125 = -3$

(h) $\mathrm{e}^4 = y \Longrightarrow$
$\ln y = 4$

(i) $10^{x+1} = y \Longrightarrow$
$\log y = x + 1$

3.

(a) $\log_2 64 \Longrightarrow$
$2^x = 64$
$x = 6$
$\therefore \log_2 64 = 6$

(b) $\log_4 64 \Longrightarrow$
$4^x = 64$
$x = 3$
$\therefore \log_4 64 = 3$

(c) $\log_2 \left(\frac{1}{8}\right) \Longrightarrow$
$2^x = \frac{1}{8}$
$x = -3$
$\therefore \log_2 64 = -3$

(d) $\log_3(3^5) \Longrightarrow$
$3^x = 3^5$
$x = 5$
$\therefore \log_3(3^5) = 5$

(e) $\log_{16} 8 \Longrightarrow$
$16^x = 8$
$(2^4)^x = 2^3$
$4x = 3$
$\therefore \log_{16} 8 = \frac{3}{4}$

(f) $\log_{27} 3 \Longrightarrow$
$27^x = 3$
$(3^3)^x = 3$

$x = \frac{1}{3}$

$\therefore \log_{27} 3 = \frac{1}{3}$

(g) $\log_{10} 0.001 \Longrightarrow$

$10^x = 0.001$

$x = -3$

$\therefore \log_{10} 0.001 = -3$

(h) $\ln \mathrm{e}^{13} \Longrightarrow$

$\mathrm{e}^x = \mathrm{e}^{13}$

$x = 13$

$\therefore \ln \mathrm{e}^{13} = 13$

(i) $\log_8 1 \Longrightarrow$

$8^x = 1$

$x = 0$

$\therefore \log_8 1 = 0$

(j) $10^{\log 6} = 10^{\log_{10} 6} = 6$

(k) $\log_3 \left(\frac{1}{27}\right) \Longrightarrow$

$3^x = \dfrac{1}{27}$

$3^x = 3^{-3}$

$x = -3$

$\therefore \log_3 \left(\frac{1}{27}\right) = -3$

(l) $\mathrm{e}^{\ln\sqrt{2}} = \mathrm{e}^{\log_e \sqrt{2}} = \sqrt{2}$

(m) $\log 1000 = \log_{10} 1000 \Longrightarrow$

$10^x = 1000$

$x = 3$

$\therefore \log 1000 = 3$

(n) $\ln \sqrt{\mathrm{e}} \Longrightarrow$

$\mathrm{e}^x = \sqrt{\mathrm{e}}$

$\mathrm{e}^x = \mathrm{e}^{\frac{1}{2}}$

$x = \frac{1}{2}$

$\therefore \ln \sqrt{\mathrm{e}} = \frac{1}{2}$

(o) $\ln \left(\frac{1}{\mathrm{e}^2}\right) \Longrightarrow$

$\mathrm{e}^x = \dfrac{1}{\mathrm{e}^2}$

$\mathrm{e}^x = \mathrm{e}^{-2}$

$x = -2$

$\therefore \ln \left(\frac{1}{\mathrm{e}^2}\right) = -2$

(p) $\log_3(81^{22}) \Longrightarrow$

$3^x = 81^{22}$

$3^x = (3^4)^{22} \Longrightarrow$

$x = 4 \cdot 22 = 88$

$\therefore \log_3(81^{22}) = 88$

(q) $\log_4 2 \Longrightarrow$
$4^x = 2 \ \Longrightarrow$
$(2^2)^x = 2^1$
$2x = 1$
$x = \frac{1}{2}$
$\therefore \log_4 2 = \frac{1}{2}$

(r) $3^{\log_3 18} = 18$

(s) $\log_5\left(\sqrt[3]{5}\right) \Longrightarrow$
$5^x = \sqrt[3]{5}$
$5^x = 5^{\frac{1}{3}}$
$x = \dfrac{1}{3}$
$\therefore \log_5\left(\sqrt[3]{5}\right) = \frac{1}{3}$

(t) $10^{\log \pi} = 10^{\log_{10} \pi} = \pi$

4.

(a) $\log\sqrt{3} = 0.23856 \ldots \approx 0.239$
(b) $\ln\sqrt{3} = 0.549306 \ldots \approx 0.549$
(c) $\log 25 = 1.39794 \ldots \approx 1.40$
(d) $\log\left(\frac{1+\sqrt{5}}{2}\right) = 0.2089876 \ldots \approx 0.209$
(e) $\ln(100^3) = 13.81551 \ldots \approx 13.8$

5. For the function $f(x) = \log x$, domain: $x > 0$

(a) $y = \log(x - 2)$ is a translation of $f(x)$ by two units to the right OR
$x - 2 > 0$
$\therefore$ domain: $x > 2$

(b) $y = \log(x^2)$
$x^2 > 0$ which is true for all real numbers not equal to zero
$\therefore$ domain: $x \in \mathbb{R}, x \neq 0$

(c) $y = \log(x) - 2$ is a translation of $f(x)$ by two units downward
(which has no effect on the domain)
$\therefore$ domain: $x > 0$

(d) $y = \log_7(8 - 5x)$
$8 - 5x > 0$
$8 > 5x$
$\therefore$ domain: $x < \frac{8}{5}$

(e) $y = \sqrt{x + 2} - \log_3(9 - 3x)$
Due to square root: $x + 2 \geq 0 \Rightarrow$
$x \geq -2$
Due to logarithm: $9 - 3x > 0 \Rightarrow$
$9 > 3x \Longrightarrow$
$x < 3$
$\therefore$ domain: $-2 \leq x < 3$

(f) $y = \sqrt{\ln(1-x)}$
Due to square root: $\ln(1-x) \geq 0 \Rightarrow$
$\mathrm{e}^0 \leq 1 - x \Longrightarrow$
$x \leq 0$
Due to logarithm: $1 - x > 0 \Rightarrow$
$x < 1$
$\therefore$ domain: $x < 1$

6.

(a) $y = \frac{1}{\ln x}$
Due to denominator: $\ln x \neq 0 \Rightarrow$
$x \neq \mathrm{e}^0 \Longrightarrow$
$x \neq 1$
Due to logarithm: $x > 0 \Rightarrow$
$\therefore$ domain: $x > 0, x \neq 1$
Function can never take the value of zero due to the numerator
$\therefore$ range: $y \in \mathbb{R}, y \neq 0$

(b) $y = |\ln(x-1)|$
$x - 1 > 0 \Rightarrow$
$\therefore$ domain: $x > 1$
Function can never take a negative value due to absolute value
$\therefore$ range: $y \geq 0$

(c) $y = \frac{x}{\log x}$
Due to denominator: $\log x \neq 0 \Rightarrow$
$x \neq 10^0 \Longrightarrow$
$x \neq 1$
Due to logarithm: $x > 0 \Rightarrow$
$\therefore$ domain: $x > 0, x \neq 1$
Function can never take the value of zero as a zero numerator is outside the domain
$\therefore$ range: $y \in \mathbb{R}, y \neq 0$

7. All functions in this question take the form $f(x) = \log_b x$

(a) From the graph: $f(4) = 1 \Longrightarrow$
$\log_b 4 = 1 \Longrightarrow$
$b^1 = 4$
$\therefore f(x) = \log_4 x$

(b) From the graph: $f\left(\frac{1}{2}\right) = -1 \Longrightarrow$
$\log_b \frac{1}{2} = -1 \Longrightarrow$
$b^{-1} = \frac{1}{2} \Longrightarrow$
$b = 2$
$\therefore f(x) = \log_2 x$

(c) From the graph: $f(10) = 1 \Longrightarrow$
$\log_b 10 = 1 \Longrightarrow$
$b^1 = 10$
$\therefore f(x) = \log_{10} x$

(d) From the graph: $f(9) = 2 \Longrightarrow$
$\log_b 9 = 2 \Longrightarrow$
$b^2 = 9 \Longrightarrow$
$b = 3$
$\therefore f(x) = \log_3 x$

8.

(a) $\log_2(2m) = \log_2 2 + \log_2 m = 1 + \log_2 m$

(b) $\log\left(\frac{9}{x}\right) = \log 9 - \log x$

(c) $\ln \sqrt[5]{x} = \ln(x^{\frac{1}{5}}) = \frac{1}{5}\ln x$

(d) $\log_3(ab^3) = \log_3 a + \log_3(b^3) = \log_3 a + 3\log_3 b$

(e) $\log[10x(1+r)^t] = \log 10 + \log x + \log[(1+r)^t] = 1 + \log x + t\log(1+r)$

(f) $\ln\left(\frac{m^3}{n}\right) = \ln(m^3) - \ln n = 3\ln m - \ln n$

9.

(a) $\log_b(pqr) = \log_b p + \log_b q + \log_b r$

(b) $\log_b\left(\frac{p^2q^3}{r}\right) = \log_b(p^2) + \log_b(q^3) - \log_b r = 2\log_b p + 3\log_b q - \log_b r$

(c) $\log_b(\sqrt[4]{pq}) = \log_b\left[(pq)^{\frac{1}{4}}\right] = \frac{1}{4}\log_b(pq) = \frac{1}{4}\log_b p + \frac{1}{4}\log_b q$

(d) $\log_b\left(\sqrt{\frac{qr}{p}}\right) = \log_b\left[\left(\frac{qr}{p}\right)^{\frac{1}{2}}\right] = \frac{1}{2}\log_b\left(\frac{qr}{p}\right) = \frac{1}{2}\log_b q + \frac{1}{2}\log_b r - \frac{1}{2}\log_b p$

(e) $\log_b\left(\frac{p\sqrt{q}}{r}\right) = \log_b p + \log_b \sqrt{q} - \log_b r = \log_b p + \frac{1}{2}\log_b q - \log_b r$

(f) $\log_b\left(\frac{(pq)^3}{\sqrt{r}}\right) = \log_b[(pq)^3] - \log_b \sqrt{r} = 3\log_b(pq) - \log_b\left(r^{\frac{1}{2}}\right)$
$= 3\log_b p + 3\log_b q - \frac{1}{2}\log_b r$

10.

(a) $\log(x^2) + \log\left(\frac{1}{x}\right) = \log\left(\frac{x^2}{x}\right) = \log x$

(b) $\log_3 9 + 3\log_3 2 = \log_3(9 \cdot 2^3) = \log_3 72$

(c) $4\ln y - \ln 4 = \ln(y^4) - \ln 4 = \ln\left(\frac{y^4}{4}\right)$

(d) $\log_b 12 - \frac{1}{2}\log_b 9 = \log_b 12 - \log_b\left(9^{\frac{1}{2}}\right) = \log_b\left(\frac{12}{\sqrt{9}}\right) = \log_b\left(\frac{12}{3}\right) = \log_b 4$

(e) $\log x - \log y - \log z = \log\left(\frac{x}{yz}\right)$

(f) $2\ln 6 - 1 = \ln(6^2) - \ln \mathrm{e} = \ln\left(\frac{36}{\mathrm{e}}\right)$

11.

(a) $\log_2 1000 = \frac{\ln 1000}{\ln 2} = 9.965784\ldots \approx 9.97$

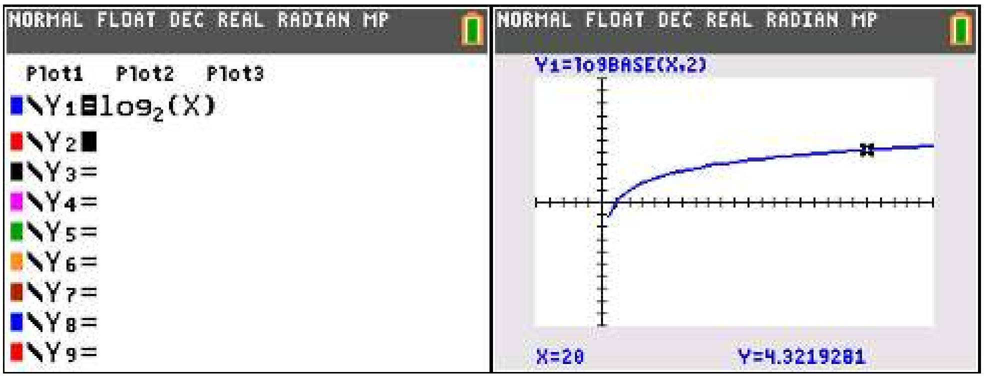
(b) $\log_{0.5} 40 = \frac{\ln 40}{\ln 0.5} = -5.321928\ldots \approx -5.32$

(c) $\log_6 40 = \frac{\ln 40}{\ln 6} = 2.0588028\ldots \approx 2.06$

(d) $\log_5 0.75 = \frac{\ln 0.75}{\ln 5} = -0.1787469\ldots \approx -0.179$

12.

(a) $f(x) = \log_2 x = \frac{\ln x}{\ln 2}$



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3
Y1=log₂(X)
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
Y9=

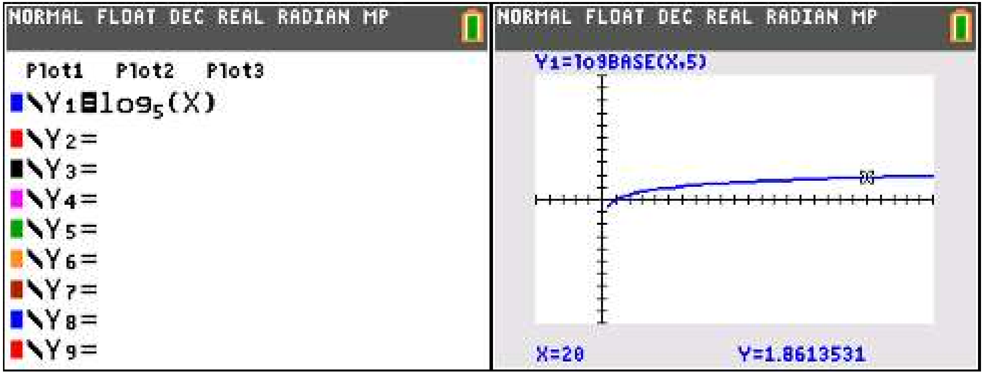
NORMAL FLOAT DEC REAL RADIAN MP

Y1=logBASE(X,2)

X=20 Y=4.3219281

$f(20) \approx 4.32$

(b) $f(x) = \log_5 x = \frac{\ln x}{\ln 5}$



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3
Y1=log₅(X)
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
Y9=

NORMAL FLOAT DEC REAL RADIAN MP

Y1=logBASE(X,5)

X=20 Y=1.8613531

$f(20) \approx 1.86$

13. $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$

14. $\log \mathrm{e} = \log_{10} \mathrm{e} = \frac{\ln \mathrm{e}}{\ln 10} = \frac{1}{\ln 10}$

15. $dB = 10\log\left(\frac{I}{10^{-16}}\right) = 10(\log I - \log(10^{-16})) = 10(\log I - (-16)) = 10\log I + 160$

For $I = 10^{-4} \Longrightarrow$

$dB = 10\log(10^{-4}) + 160 = 10(-4) + 160 = 120$ decibels

16.

(a) $\log f(x) = \log[5(2^x)] = \log 5 + \log 2^x = \log 5 + x\log 2$

$\therefore y = \log 5 + x\log 2$

(b) $\log f(x) = \log[a(b^x)] = \log a + \log b^x = \log a + x\log b$

In the form $y = mx + c = \log b\, x + \log a$

$\therefore m = \log b\,, c = \log a$

Exercise 4.5

1.

(a) $10^x = 5 \Longrightarrow$
$x = \log 5 = 0.698970\ldots \approx 0.699$

(b) $4^x = 32 \Longrightarrow$
$x = \log_4 32 = 2.5$

(c) $8^{x-6} = 60 \Longrightarrow$
$x - 6 = \log_8 60 \Longrightarrow$
$x = 6 + \log_8 60 = 7.96896\ldots \approx 7.97$

(d) $2^{x+3} = 100 \Longrightarrow$
$x + 3 = \log_2 100 \Longrightarrow$
$x = -3 + \log_2 100 = 3.643856\ldots \approx 3.64$

(e) $\left(\frac{1}{5}\right)^x = 22 \Longrightarrow$
$x = \log_{\frac{1}{5}} 22 = -1.92057\ldots \approx -1.92$

(f) $\mathrm{e}^x = 15 \Longrightarrow$
$x = \ln 15 = 2.70805\ldots \approx 2.71$

(g) $10^x = \mathrm{e} \Longrightarrow$
$x = \log \mathrm{e} = 0.43429\ldots \approx 0.434$

(h) $3^{2x-1} = 35 \Longrightarrow$
$2x - 1 = \log_3 35 \Longrightarrow$
$x = \frac{1+\log_3 35}{2} = 2.1181\ldots \approx 2.12$

(i) $2^{x+1} = 3^{x-1} \Longrightarrow$
$\ln 2^{x+1} = \ln 3^{x-1} \Longrightarrow$
$(x+1)\ln 2 = (x-1)\ln 3 \Longrightarrow$
$x\ln 2 + \ln 2 = x\ln 3 - \ln 3 \Longrightarrow$
$\ln 3 + \ln 2 = x\ln 3 - x\ln 2 \Longrightarrow$
$\ln 3 + \ln 2 = x(\ln 3 - \ln 2) \Longrightarrow$
$x = \frac{\ln 3+\ln 2}{\ln 3-\ln 2} = 4.41902\ldots \approx 4.42$

(j) $2\mathrm{e}^{10x} = 19 \Longrightarrow$
$\mathrm{e}^{10x} = \frac{19}{2} \Longrightarrow$
$10x = \ln\left(\frac{19}{2}\right) \Longrightarrow$
$x = \frac{1}{10}\ln\left(\frac{19}{2}\right) = 0.225129\ldots \approx 0.225$

(k) $6^{\frac{x}{2}} = 5^{1-x} \Longrightarrow$
$\ln 6^{\frac{x}{2}} = \ln 5^{1-x} \Longrightarrow$
$\left(\frac{x}{2}\right)\ln 6 = (1-x)\ln 5 \Longrightarrow$
$x\left(\frac{1}{2}\ln 6\right) = \ln 5 - x\ln 5 \Longrightarrow$
$x\left(\ln 6^{\frac{1}{2}}\right) + x\ln 5 = \ln 5 \Longrightarrow$
$x(\ln\sqrt{6} + \ln 5) = \ln 5 \Longrightarrow$
$x = \frac{\ln 5}{\ln\sqrt{6}+\ln 5} = 0.6424087\ldots \approx 0.642$

(l) $\left(1+\frac{0.05}{12}\right)^{12x} = 3 \Longrightarrow$
$\left(\frac{1205}{1200}\right)^{12x} = 3 \Longrightarrow$
$12x = \log_{\left(\frac{1205}{1200}\right)}(3) \Longrightarrow$
$x = \frac{1}{12}\log_{\left(\frac{1205}{1200}\right)}(3) = 22.017989\ldots \approx 22.0$

2.

(a) $4^x - 2^{x+1} = 48 \Longrightarrow$
$(2^2)^x - 2\cdot 2^x = 48 \Longrightarrow$
$\therefore 2^{2x} - 2\cdot 2^x - 48 = 0 \Longrightarrow$
If $a = 2^x \Longrightarrow$
$a^2 - 2\cdot a - 48 = 0 \Longrightarrow$
$(a+6)(a-8) = 0 \Longrightarrow$
$a = -6, 8$
Resubstituting $a = 2^x$ gives
Either $2^x \neq -6$ (impossible)
OR $2^x = 8$
$\therefore x = 3$

(b) $2^{2x+1} - 2^{x+1} + 1 = 2^x$
$2\cdot 2^{2x} - 2\cdot 2^x - 2^x + 1 = 0$
$\therefore 2\cdot 2^{2x} - 3\cdot 2^x + 1 = 0$
If $a = 2^x \Longrightarrow$
$2a^2 - 3a + 1 = 0 \Longrightarrow$
$(2a-1)(a-1) = 0 \Longrightarrow$
$a = 1, \frac{1}{2}$
Resubstituting $a = 2^x$ gives
Either $2^x = 1 \Longrightarrow$
$x = 0$
OR $2^x = \frac{1}{2} \Longrightarrow$
$x = -1$
$\therefore x = 0, -1$

(c) $6^{2x+1} - 17(6^x) + 12 = 0$
$\therefore 6\cdot 6^{2x} - 17\cdot 6^x + 12 = 0$
If $a = 6^x \Longrightarrow$
$6a^2 - 17a + 12 = 0 \Longrightarrow$
$(2a-3)(3a-4) = 0 \Longrightarrow$
$a = \frac{3}{2}, \frac{4}{3}$
Resubstituting $a = 6^x$ gives
Either $6^x = \frac{3}{2} \Longrightarrow$
$x = \log_6\left(\frac{3}{2}\right) = \frac{\ln\left(\frac{3}{2}\right)}{\ln 6}$
OR $6^x = \frac{4}{3} \Longrightarrow$
$x = \log_6\left(\frac{4}{3}\right) = \frac{\ln\left(\frac{4}{3}\right)}{\ln 6}$
$\therefore x = \log_6\left(\frac{3}{2}\right), \log_6\left(\frac{4}{3}\right)$

(d) $3^{2x+1} + 3 = 10(3^x)$
$\therefore 3 \cdot 3^{2x} - 10 \cdot 3^x + 3 = 0$
If $a = 3^x \Longrightarrow$
$3a^2 - 10a + 3 = 0 \Longrightarrow$
$(3a - 1)(a - 3) = 0 \Longrightarrow$
$a = \frac{1}{3}, 3$
Resubstituting $a = 3^x$ gives
Either $3^x = 3 \Longrightarrow$
$x = 1$
OR $3^x = \frac{1}{3} \Longrightarrow$
$x = -1$
$\therefore x = \pm 1$

3.

(a) $FV = 5000 \cdot \left(1 + \frac{0.075}{4}\right)^{4 \cdot 3} \approx \$6{,}248.58$

(b) $10\,000 = 5000 \cdot \left(1 + \frac{0.075}{4}\right)^{4x}$
$2 = (1.01875)^{4x}$
$4x = \log_{1.01875} 2$
$x = \frac{1}{4}\log_{1.01875} 2 \approx 9.33$ years
It will take 9 and $\frac{1}{4}$ years to double the investment.

4. $1500 = 500 \cdot \mathrm{e}^{0.085x}$
$3 = \mathrm{e}^{0.085x}$
$0.085x = \ln 3$
$x = \frac{\ln 3}{0.085} \approx 12.9$ years

5. Doubling every hour implies a population $P(t) = 1 \cdot 2^t$ after t hours
$1\,000\,000 = 1 \cdot 2^t$
$t = \log_2 1\,000\,000 \approx 19.9$
$\therefore$ 20 hours for the population to exceed 1 million bacteria

6.

(a) $2 = (1.03)^x$
$x = \log_{1.03} 2 \approx 23.4$ years or 24 years

(b) $2 = (1.06)^x$
$x = \log_{1.06} 2 \approx 11.9$ years or 12 years

(c) $2 = (1.09)^x$
$x = \log_{1.03} 2 \approx 8.04$ years or 9 years

7. $100\% - 11\% = 89\%$
$\frac{1}{2} = (0.89)^x$
$x = \log_{0.89} \frac{1}{2} \approx 5.95$ years
Thus, in 6 years

8. $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{2.46\cdot 10^5}}$

(a) $A(1000) = 1\cdot\left(\frac{1}{2}\right)^{\frac{1000}{2.46\cdot 10^5}} \approx 0.997$

(b) $0.7 = \left(\frac{1}{2}\right)^{\frac{t}{2.46\cdot 10^5}}$

$\frac{t}{2.46\cdot 10^5} = \log_{\frac{1}{2}} 0.7$

$t = 2.46\cdot 10^5 \cdot \log_{\frac{1}{2}} 0.7 = 126\,585$ years

9.

(a) $16(1.18)^5 \approx 36.6$

37 dogs

(b) $70 < 16(1.18)^t$

$4.375 < 1.18^t$

$t > \log_{1.18} 4.375$

$t > 8.92$ years

$\therefore$ 9 years

10.

(a) $V(10) = 1000(0.925)^{10} \approx 459$ litres

(b) $500 = 1000(0.925)^t$

$0.5 = (0.925)^t$

$t = \log_{0.925} 0.5 \approx 8.89$ minutes

8 minutes and 53 seconds

(c) $0.05 = (0.925)^t$

$t = \log_{0.925} 0.05 \approx 38.4$ minutes

$\therefore$ 39 minutes

11.

(a) $m_0 = 5\mathrm{e}^{-0.13(0)} = 5\mathrm{e}^0 = 5$ kilograms

(b) $0.5 = 5\mathrm{e}^{-0.13t}$

$0.1 = \mathrm{e}^{-0.13t}$

$-0.13t = \ln 0.1$

$t = \frac{\ln 0.1}{-0.13} = 17.7$ years

12.

(a) $\log_2(3x-4) = 4$

$3x - 4 = 2^4$

$3x = 16 + 4$

$x = \frac{20}{3}$

(b) $\log(x-4) = 2$

$x - 4 = 10^2$

$x = 104$

(c) $\ln x = -3$

$x = \mathrm{e}^{-3} = \frac{1}{\mathrm{e}^3}$

(d) $\log_{16} x = \frac{1}{2}$

$x = 16^{\frac{1}{2}} = \sqrt{16} = \pm 4$; $x > 0$ due to nature of the function

$\therefore x = 4$

(e) $\log\sqrt{x+2} = 1$
$\sqrt{x+2} = 10$
$x + 2 = 10^2$
$x = 98$

(f) $\ln(x^2) = 16$
$x^2 = \mathrm{e}^{16}$
$x = \mathrm{e}^{\frac{16}{2}} = \pm\mathrm{e}^8$

(g) $\log_2(x^2+8) = \log_2 x + \log_2 6$
$\log_2(x^2+8) = \log_2 6x$
$x^2 + 8 = 6x$
$x^2 - 6x + 8 = 0$
$(x-4)(x-2) = 0$
$\therefore x = 4, 2$

(h) $\log_3(x-8) + \log_3 x = 2$
$\log_3\big(x(x-8)\big) = 2$
$3^2 = x(x-8)$
$0 = x^2 - 8x - 9$
$0 = (x-9)(x+1)$
$x = -1, 9$
$x > 0$ due to nature of the function
$\therefore x = 9$

(i) $\log 7 - \log(4x+5) + \log(2x-3) = 0$
$\log\big(7(2x-3)\big) = \log(4x+5)$
$7(2x-3) = 4x+5$
$14x - 21 = 4x + 5$
$10x = 26$
$x = \frac{26}{10} = \frac{13}{5} = 2.6$

(j) $\log_3 x + \log_3(x-2) = 1$
$\log_3\big(x(x-2)\big) = 1$
$3^1 = x(x-2)$
$0 = x^2 - 2x - 3$
$0 = (x-3)(x+1)$
$x = 3, -1$
$x > 0$ due to nature of the function
$\therefore x = 3$

(k) $\log(x^8) = (\log x)^4$
By observation $x = 1$, OR
$8\log x = (\log x)^4$
$$8 = \frac{(\log x)^4}{\log x}$$
$8 = (\log x)^3$
$\sqrt[3]{8} = \log x$
$2 = \log x$
$x = 10^2 = 100$
$\therefore x = 1,\ 100$

13.

(a) $5\log_4 x + 2 > 0$

$\log_4 x > -\dfrac{2}{5}$

$x > 4^{-\frac{2}{5}}$ OR $x > \frac{1}{\sqrt[5]{16}}$

(b) $2\log(x^2) - 3\log x < \log(8x) - \log(4x)$

$\log(x^4) - \log(x^3) < \log(8x) - \log(4x)$

$\log\left(\dfrac{x^4}{x^3}\right) < \log\left(\dfrac{8x}{4x}\right)$

$\log(x) < \log(2)$

$x < 2$

(c) $(\mathrm{e}^x - 2)(\mathrm{e}^x - 3) < 2\mathrm{e}^x$

$\mathrm{e}^{2x} - 5\mathrm{e}^x + 6 < 2\mathrm{e}^x$

$\mathrm{e}^{2x} - 7\mathrm{e}^x + 6 < 0$

$(\mathrm{e}^x - 6)(\mathrm{e}^x - 1) < 0$

$\mathrm{e}^x - 6 < 0$

$\mathrm{e}^x < 6$

$x < \ln 6$

OR

$\mathrm{e}^x - 1 < 0$

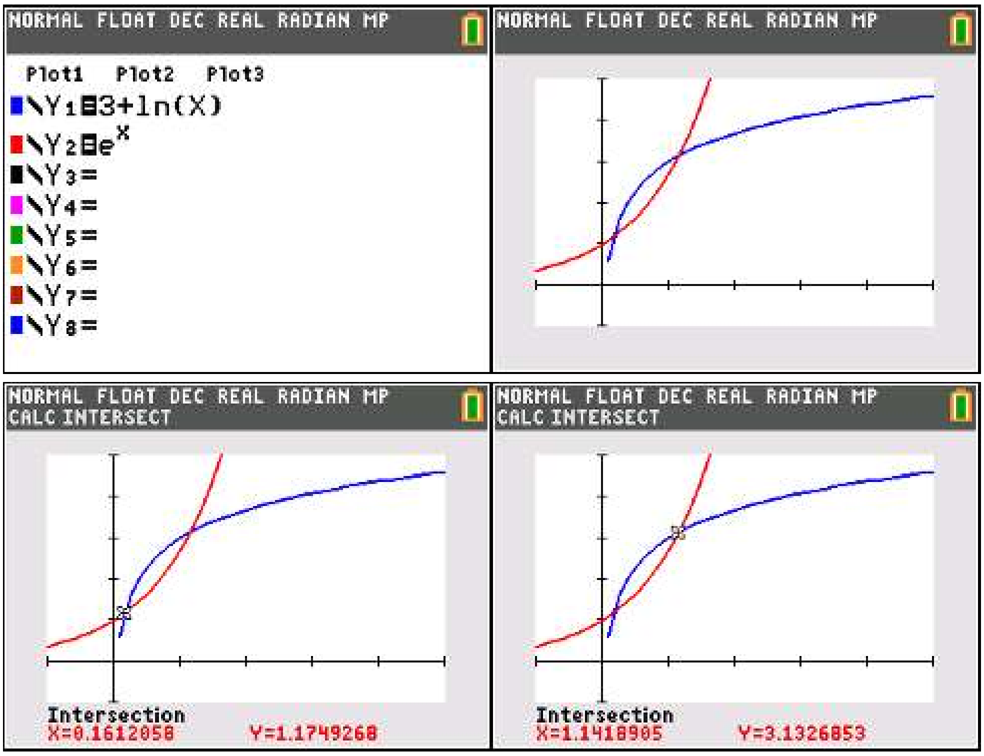
$\mathrm{e}^x < 1$

$x < 0$

x-value	$x < 0$	$0 < x < \ln 6$	$x > \ln 6$
$(\mathrm{e}^x - 6)(\mathrm{e}^x - 1) < 0$	$(-)(-) > 0$	$(-)(+) < 0$	$(+)(+) > 0$

$\therefore\ 0 < x < \ln 6$

(d) $3 + \ln x > \mathrm{e}^x$



Y1=3+ln(X)
Y2=e^X
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=

Intersection X=0.1612058 Y=1.1749268

Intersection X=1.1418905 Y=3.1326853

$\therefore\ 0.161 < x < 1.14$ (to 3 s.f.)

Chapter 4 practice questions

1.

(a) Let point P be $(P_x, 0)$ as it is the x-intercept.
x-intercept can be found at $y = 0$
$0 = 2 - \log_3(P_x + 1)$
$\log_3(P_x + 1) = 2$
$P_x + 1 = 3^2$
$\therefore P_x = 8$

(b) Let point Q be $\left(0, Q_y\right)$ as it is the y-intercept.
y-intercept can be found at $x = 0$
$Q_y = 2 - \log_3(0 + 1) = 2 - \log_3(1) = 2 - 0$
$\therefore Q_y = 2$

(c) Let point R be $(R_x, 3)$ as it is the intersection with the line $y = 3$.
$3 = 2 - \log_3(R_x + 1)$
$\log_3(R_x + 1) = -1$
$R_x + 1 = 3^{-1}$
$R_x + 1 = \frac{1}{3}$
$R_x = -\frac{2}{3}$
$\therefore$ point R be $\left(-\frac{2}{3}, 3\right)$

2.

(a) $5 = A_0 e^{-0.0045(800)}$
$A_0 = \frac{5}{e^{-0.0045(800)}} \approx 183$ grams

(b) $\frac{1}{2} = e^{-0.0045t}$
$-0.0045t = \ln\left(\frac{1}{2}\right)$
$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.0045} \approx 154$ years

3. $\log_2(5x^2 - x - 2) = 2 + 2\log_2 x$
$\log_2(5x^2 - x - 2) = \log_2 4 + \log_2 x^2$
$\log_2(5x^2 - x - 2) = \log_2(4x^2)$
$5x^2 - x - 2 = 4x^2$
$x^2 - x - 2 = 0$
$(x - 2)(x + 1) = 0$
$x = 2, -1$
$x > 0$ due to the nature of the function
$\therefore x = 2$

4. Solving $\log_2 4\sqrt{2} = x$, gives:
$x = \log_2 4 + \log_2 \sqrt{2} = \log_2 4 + \log_2\left(2^{\frac{1}{2}}\right)$
$x = 2 + \frac{1}{2}$
$\therefore x = \frac{5}{2}$
Substituting into $y = 4x^2 - 2x - 6 + z$, gives:

$y = 4\left(\frac{5}{2}\right)^2 - 2\left(\frac{5}{2}\right) - 6 + z = 25 - 5 - 6 + z$

$y = 14 + z$

Rearranging $\log_z y = 4$, gives:

$z^4 = y$

Re-substituting into $y = 14 + z$, gives:

$z^4 = 14 + z$

$\therefore z = 2$

$y = 14 + 2$

$\therefore y = 16$

5. $2\mathrm{e}^{3t} - 7\mathrm{e}^{2t} + 7\mathrm{e}^{t} - 2 = 0$

Let $a = \mathrm{e}^t$

$2a^3 - 7a^2 + 7a - 2 = 0$

$$\begin{array}{r|rrrr} 1 & 2 & -7 & 7 & -2 \\ \hline & \downarrow & 2 & -5 & 2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

$(a-1)(2a^2 - 5a + 2) = 0$

$(a-1)(2a-1)(a-2) = 0$

$a = 1, \frac{1}{2}, 2$

$\mathrm{e}^t = 1 \Longrightarrow$

$t = \ln 1 = 0$

$\mathrm{e}^t = \frac{1}{2} \Longrightarrow$

$t = \ln \frac{1}{2}$

$\mathrm{e}^t = 2 \Longrightarrow$

$t = \ln 2$

$\therefore t = 0, \ln\frac{1}{2}, \ln 2$

6. $8\mathrm{e}^2 - 2\mathrm{e}\ln x = (\ln x)^2$

$8\mathrm{e}^2 - 2\mathrm{e}\ln x - (\ln x)^2 = 0$

$(4\mathrm{e} + \ln x)(2\mathrm{e} - \ln x) = 0$

Either: $4\mathrm{e} + \ln x = 0 \Longrightarrow$

$-4\mathrm{e} = \ln x \Longrightarrow$

$x = \mathrm{e}^{-4\mathrm{e}}$

OR: $2\mathrm{e} - \ln x = 0 \Longrightarrow$

$2\mathrm{e} = \ln x \Longrightarrow$

$x = \mathrm{e}^{2\mathrm{e}}$

$\therefore x = \mathrm{e}^{-4\mathrm{e}}, \mathrm{e}^{2\mathrm{e}}$

7.

(a) $\log_3 x - 4\log_x 3 + 3 = 0$

$\log_3 x + 3 - 4\dfrac{\log_3 3}{\log_3 x} = 0$

$\log_3 x\left(\log_3 x + 3 - 4\dfrac{1}{\log_3 x}\right) = 0(\log_3 x)$

$(\log_3 x)^2 + 3\log_3 x - 4 = 0$

Let $\log_3 x = a$:
$a^2 + 3a - 4 = 0$
$(a+4)(a-1) = 0$
Resubstituting $a = \log_3 x$
$(\log_3 x + 4)(\log_3 x - 1) = 0$
Either $\log_3 x + 4 = 0 \Rightarrow$
$\log_3 x = -4 \Rightarrow$
$x = 3^{-4} = \frac{1}{81}$
OR $\log_3 x - 1 = 0 \Rightarrow$
$\log_3 x = 1 \Rightarrow$
$x = 3^1 = 3$
$\therefore x = \frac{1}{81}, 3$

(b) $\log_2(x-5) + \log_2(x+2) = 3$
$\log_2[(x-5)(x+2)] = 3$
$(x-5)(x+2) = 2^3$
$x^2 - 3x - 10 = 8$
$x^2 - 3x - 18 = 0$
$(x-6)(x+3) = 0 \Rightarrow$
$x = 6, -3$
$x + 2 > 0$ due to the domain of log functions
$\therefore x = 6$

8.

(a) $2\log a + 3\log b - \log c$
$= \log a^2 + \log b^3 - \log c$
$= \log\left(\frac{a^2b^3}{c}\right)$

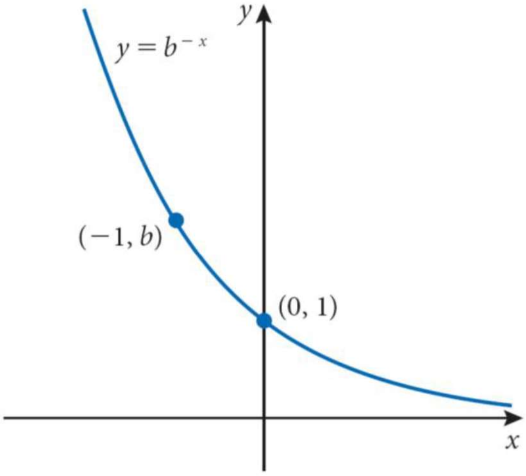
(b) $3\ln x - \frac{1}{2}\ln y + 1$
$= \ln x^3 - \ln\sqrt{y} + \ln \mathrm{e}$
$= \ln\frac{\mathrm{e}x^3}{\sqrt{y}}$

9. $0.79 = \mathrm{e}^{-0.000124\mathrm{t}} \Rightarrow$
$-0.000124t = \ln 0.79$
$t = \frac{\ln 0.79}{-0.000124} \approx 1900$ years

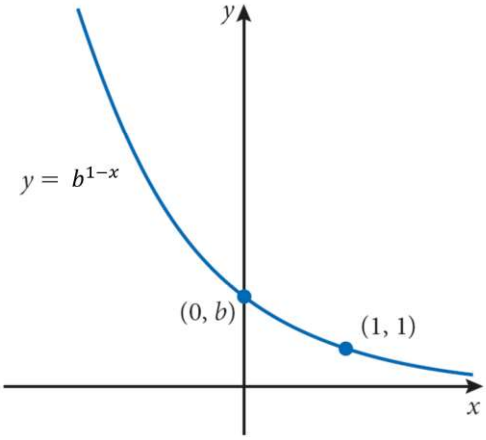
10. $0 = \log_3(2c-3) - 4$
$4 = \log_3(2c-3)$
$2c - 3 = 3^4$
$2c = 81 + 3$
$\therefore c = 42$

11.

(a) $y = b^{-x}$ is a reflection of $y = b^{x}$ about the *y*-axis.



(b) $y = b^{1-x}$ is a translation of $y = b^{-x}$ by one unit to the right.



12.

(a) $A(1600) = \frac{1}{2}A_0 \Longrightarrow$

$\frac{1}{2}A_0 = A_0\mathrm{e}^{-kt}$

$\frac{1}{2} = \mathrm{e}^{-k(1600)}$

$-1600k = \ln\frac{1}{2}$

$k = \frac{\ln\frac{1}{2}}{-1600} \approx 4.33 \times 10^{-4}$

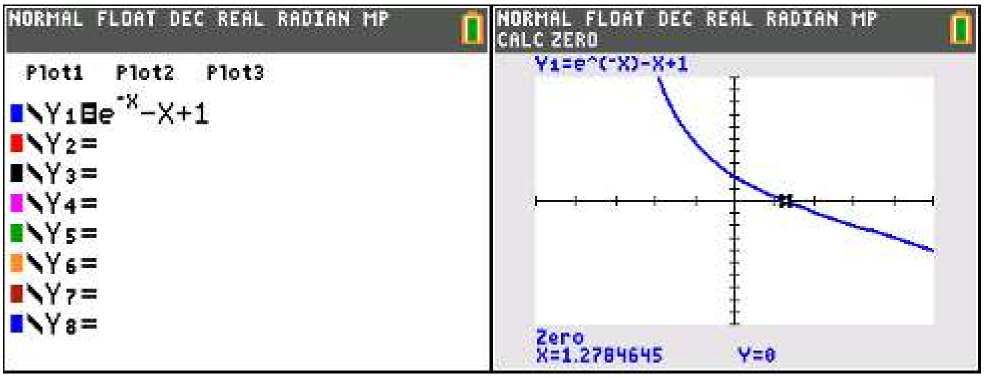
$\therefore k = 0.000433$

(b) $A(4000) = A_0\mathrm{e}^{-0.000433(4000)} \Longrightarrow$

$= A_0 \times 0.176930 \ldots$

$\therefore \approx 17.7\%$ is remaining

13.



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = e^{-X} - X + 1$

$Y_2 =$

$Y_3 =$

$Y_4 =$

$Y_5 =$

$Y_6 =$

$Y_7 =$

$Y_8 =$

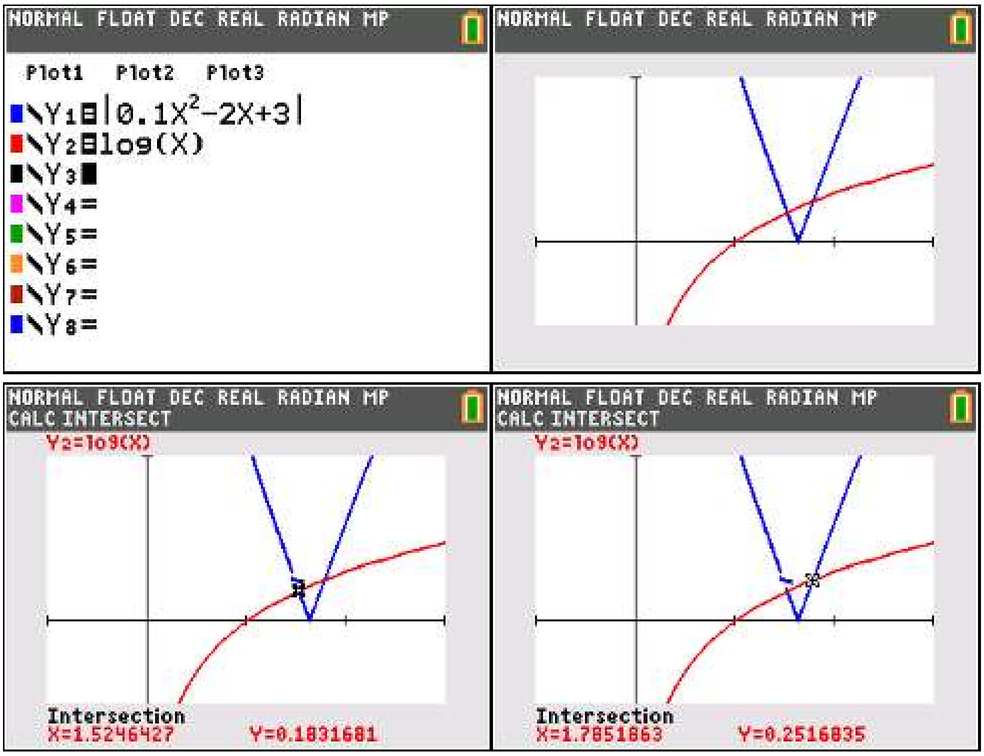
NORMAL FLOAT DEC REAL RADIAN MP
CALC ZERO

Y1=e^(-X)-X+1

Zero
X=1.2784645 Y=0

$\therefore x \approx 1.28$

14. $|0.1x^2 - 2x + 3| < \log x$



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = |0.1X^2 - 2X + 3|$

$Y_2 = \log(X)$

$Y_3 =$

$Y_4 =$

$Y_5 =$

$Y_6 =$

$Y_7 =$

$Y_8 =$

NORMAL FLOAT DEC REAL RADIAN MP

NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT

Y2=log(X)

Intersection
X=1.5246427 Y=0.1831681

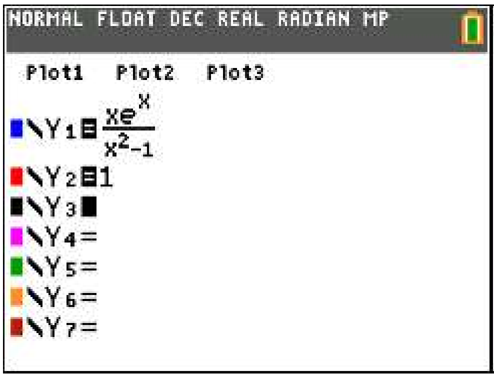
NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT

Y2=log(X)

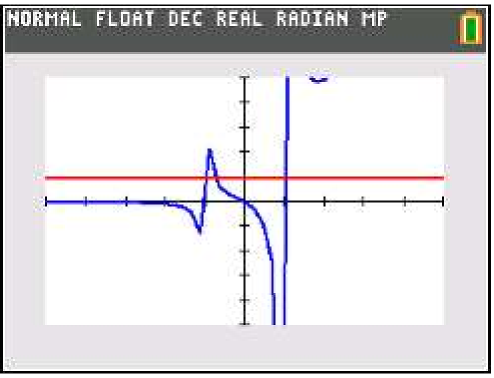
Intersection
X=1.7851863 Y=0.2516835

$\therefore 1.52 < x < 1.79$ (to 3 s.f.)

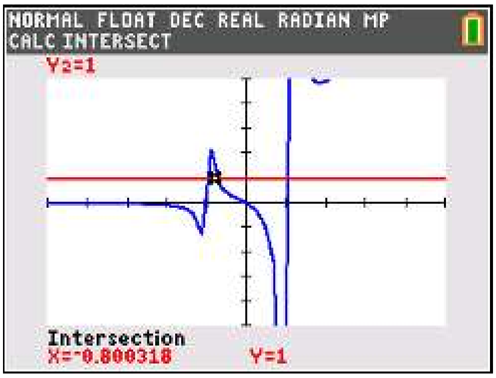
15. $\frac{xe^x}{x^2-1} \geq 1$



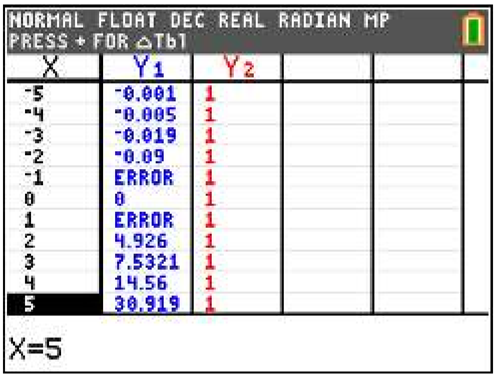
```
NORMAL FLOAT DEC REAL RADIAN MP
 Plot1  Plot2  Plot3
■\Y1■xe^X/(X²-1)
■\Y2■1
■\Y3■
■\Y4=
■\Y5=
■\Y6=
■\Y7=
```



```
NORMAL FLOAT DEC REAL RADIAN MP
```



```
NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT
Y2=1
Intersection
X=-0.800318        Y=1
```



```
NORMAL FLOAT DEC REAL RADIAN MP
PRESS + FOR △Tbl
  X  |  Y1     | Y2
 ⁻5  | ⁻0.001  | 1
 ⁻4  | ⁻0.005  | 1
 ⁻3  | ⁻0.019  | 1
 ⁻2  | ⁻0.09   | 1
 ⁻1  | ERROR   | 1
 0   | 0       | 1
 1   | ERROR   | 1
 2   | 4.926   | 1
 3   | 7.5321  | 1
 4   | 14.56   | 1
 5   | 30.919  | 1
X=5
```

The GDC table shows two asymptotes of $y = \frac{xe^x}{x^2-1}$ at $x = \pm 1$

$\therefore -1 < x < -0.800$ OR $x > 1$ (to 3 s.f.)

16.

(a) $4^x[2(4^x) + 4^{-x}] = 4^x[3]$

$2(4^{2x}) + 1 = 3(4^x)$

$2(4^{2x}) - 3(4^x) + 1 = 0$

Let $4^x = a$

$2a^2 - 3a + 1 = 0$

$(2a - 1)(a - 1) = 0$

Either $2a - 1 = 0 \Longrightarrow$

$a = \frac{1}{2}$

OR $a - 1 = 0 \Longrightarrow$

$a = 1$

Resubstituting $a = 4^x$

Either $4^x = \frac{1}{2} \Longrightarrow$

$x = -\frac{1}{2}$

OR $4^x = 1 \Longrightarrow$

$x = 0$

$\therefore x = -\frac{1}{2}, 0$

(b)

i. $a^x = \mathrm{e}^{2x+1}$

$2x + 1 = \ln a^x$

$2x + 1 = x \ln a$

$\dfrac{2x+1}{x} = \dfrac{x \ln a}{x}$

$2 + \dfrac{1}{x} = \ln a$

$\dfrac{1}{x} = -2 + \ln a$

$\therefore x = \dfrac{1}{-2 + \ln a}$

ii. $-2 + \ln a \neq 0$

$\ln a \neq 2$

$a \neq \mathrm{e}^2$

$\therefore$ when $a = \mathrm{e}^2$ this equation has no solutions.

17. $2^{2x+3} = 2^{x+1} + 3$

$2^3 \cdot 2^{2x} = 2 \cdot 2^x + 3$

$8 \cdot 2^{2x} - 2 \cdot 2^x - 3 = 0$

Let $a = 2^x$

$8a^2 - 2a - 3 = 0$

$(4a - 3)(2a + 1) = 0$

Either $4a - 3 = 0 \Rightarrow$

$a = \frac{3}{4}$

OR $2a + 1 = 0 \Rightarrow$

$a = -\frac{1}{2}$

Resubstituting $a = 2^x$ gives:

Either $2^x = \frac{3}{4} \Rightarrow$

$x = \log_2 \frac{3}{4}$

OR $2^x = -\frac{1}{2}$ impossible

$\therefore\; x = \log_2 \dfrac{3}{4}$

This must be given in the form $a + \log_2 b$

$= \log_2 3 - \log_2 4 = \log_2 3 - 2$

$\therefore a = -2$ and $b = 3$

18. $2(\ln x)^2 = 3 \ln x - 1$

$2(\ln x)^2 - 3 \ln x + 1 = 0$

$(2 \ln x - 1)(\ln x - 1) = 0$

Either: $2 \ln x - 1 = 0 \Rightarrow$

$\ln x = \frac{1}{2} \Rightarrow x = \mathrm{e}^{\frac{1}{2}} = \sqrt{\mathrm{e}}$

OR: $\ln x - 1 = 0 \Rightarrow$

$\ln x = 1 \;\Rightarrow$

$x = \mathrm{e}^1 = \mathrm{e}$

$\therefore x = \sqrt{\mathrm{e}}, \mathrm{e}$

19.

(a) $V = 100\,(1 + 0.05)^{20} \approx \$265.33$

(b) $265.33 < 100\left(1 + \frac{0.05}{12}\right)^{12t}$

$2.6533 < \left(\frac{1205}{1200}\right)^{12t} \Rightarrow$

$12t > \log_{\frac{1205}{1200}} 2.6533 \Rightarrow$

$t > \frac{1}{12}\log_{\frac{1205}{1200}} 2.6533$

$t > 19.556713$ ... years

≈ 235 months

20. $9\log_5 x = 25\log_x 5$

$9\log_5 x = 25\frac{\log_5 5}{\log_5 x}$

$9\log_5 x = \frac{25}{\log_5 x}$

$(\log_5 x)^2 = \frac{25}{9}$

$\log_5 x = \sqrt{\frac{25}{9}}$

$\log_5 x = \pm\frac{5}{3}$

$\therefore x = 5^{\pm\frac{5}{3}}$

21. $|\ln(x + 3)| = 1$

Either: $\ln(x + 3) = 1 \Rightarrow$

$x + 3 = \mathrm{e}^1 \Rightarrow$

$x = \mathrm{e} - 3$

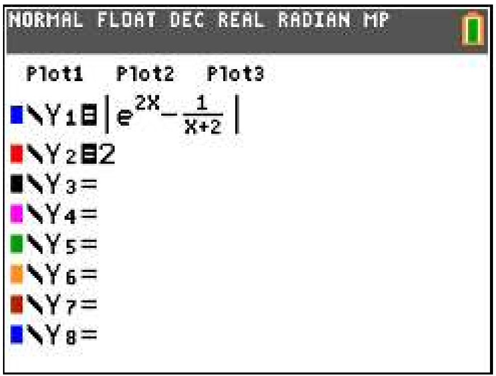
OR: $\ln(x + 3) = -1 \Rightarrow$

$x + 3 = \mathrm{e}^{-1} \Rightarrow$

$x = \frac{1}{\mathrm{e}} - 3$

$\therefore x = \mathrm{e} - 3$ OR $x = \frac{1}{\mathrm{e}} - 3$

22. $\left|\mathrm{e}^{2x}-\frac{1}{x+2}\right|=2$



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

$Y_1 = \left|e^{2X}-\frac{1}{X+2}\right|$

$Y_2 = 2$

$Y_3 =$

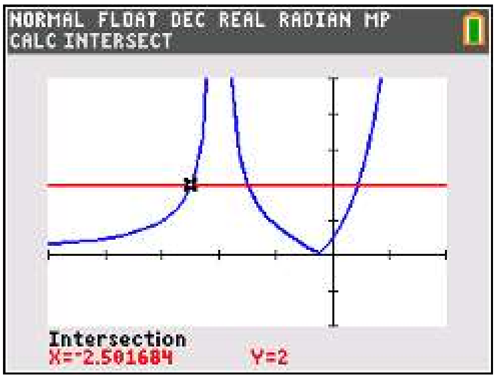
$Y_4 =$

$Y_5 =$

$Y_6 =$

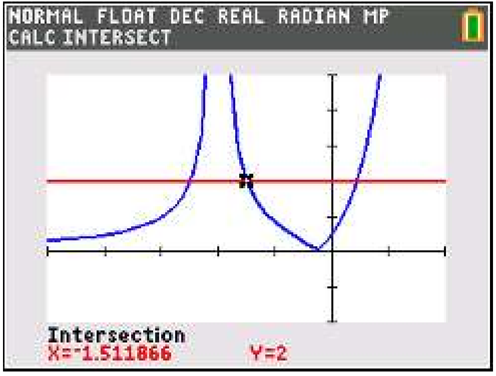
$Y_7 =$

$Y_8 =$



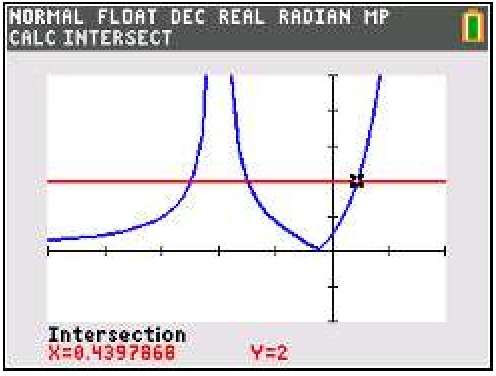
NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT

Intersection
X=-2.501684 Y=2



NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT

Intersection
X=-1.511866 Y=2



NORMAL FLOAT DEC REAL RADIAN MP
CALC INTERSECT

Intersection
X=0.4397868 Y=2

$\therefore x=-2.50,-1.51,0.440$ (to 3 s.f.)

23. e^{kt} is the growth factor

When $t=20$ the growth is double.

$2=\mathrm{e}^{k(20)}$

$20k=\ln 2$

$\therefore k=\dfrac{\ln 2}{20}$

24.

(a) $f(x)=\ln x+\ln(x-2)-\ln(x^2-4)=\ln\left(\frac{x(x-2)}{x^2-4}\right)=\ln\left(\frac{x(x-2)}{(x-2)(x+2)}\right)$

$\therefore f(x)=\ln\left(\frac{x}{x+2}\right)$

(b) $x=\ln\left(\frac{y}{y+2}\right)\Rightarrow$

$\frac{y}{y+2}=\mathrm{e}^x$

$y=(y+2)\,\mathrm{e}^x$

$y=y\mathrm{e}^x+2\mathrm{e}^x$

$y-y\mathrm{e}^x=2\mathrm{e}^x$

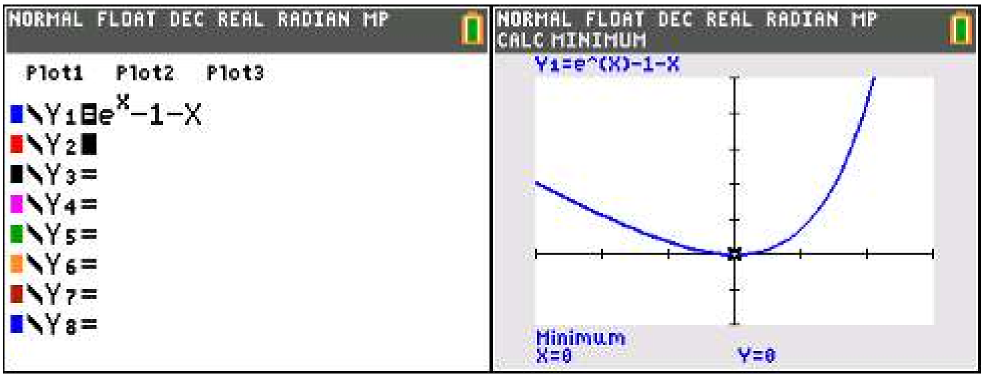
$y(1-\mathrm{e}^x)=2\mathrm{e}^x$

$y=\frac{2\mathrm{e}^x}{1-\mathrm{e}^x}$

$\therefore f^{-1}(x)=\dfrac{2\mathrm{e}^x}{1-\mathrm{e}^x}$

25.

(a) $f: x \to \mathrm{e}^{x} - 1 - x$



NORMAL FLOAT DEC REAL RADIAN MP

Plot1 Plot2 Plot3

Y1=e^X−1−X
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=

NORMAL FLOAT DEC REAL RADIAN MP
CALC MINIMUM
Y1=e^(X)-1-X

Minimum
X=0 Y=0

∴ minimum of $f = 0$

(b) As the range of $f: x \to \mathrm{e}^{x} - 1 - x$ is $f: x \geq 0 \Rightarrow$
$\mathrm{e}^{x} - 1 - x \geq 0$
$\therefore\ \mathrm{e}^{x} \geq 1 + x$

Exercise 5.1

1. **(a)** Yes, since it is a sentence that asserts a true fact.
(b) No, since it may be true or false depending on circumstances.
(c) No, since it does not assert any fact.
(d) Yes, since it is a sentence that asserts a true fact.
(e) Yes, since it is a sentence that asserts a true fact.

2. **(a)** $a = 7$, since $3 < 4$ is true, and both parts of this conjunction must be true.
(b) No value makes this conjunction true since one part is false.
(c) $a = 7$, since one part in this disjunction is false, the second one must be true.
(d) Any value. This is a conditional of the form $P \Rightarrow Q$, and Q is true, therefore any value of P will make it true.
(e) $a \neq 7$, since this is a conditional with a false consequent and only $F \Rightarrow F$ can be true.
(f) $a \neq 7$, since this is a conditional with a false consequent and only $F \Rightarrow F$ can be true.

3. Convention: a = antecedent; c = consequent.

(a) a, c **(b)** a, c **(c)** a, c **(d)** a, c

(e) c, a **(f)** c, a **(g)** c, a

4. **(a)** Converse: If triangles have four sides, then quadrilaterals have three sides.
Contrapositive: If triangles do not have four sides, then quadrilaterals do not have three sides.
(b) Converse: If $\sqrt{3}$ is a rational number, then the moon is made of butter.
Contrapositive: If $\sqrt{3}$ is an irrational number, then the moon is not made of butter.
(c) Converse: b divides 30 only if b divides 5.
Contrapositive: b does not divide 30 only if b does not divide 5.
(d) Converse: f to be continuous is sufficient for the differentiability of f.
Contrapositive: f to be discontinuous is sufficient for the non-differentiability of f.
(e) Converse: A sequence a is convergent whenever a is bounded.
Contrapositive: A sequence a is not convergent whenever a is not bounded.
(f) Converse: A function f is integrable if f is bounded.
Contrapositive: A function f is not integrable if f is not bounded.
(g) Converse: $3 + 3 = 6$ is necessary for $3 + 2 = 5$.
Contrapositive: $3 + 3 \neq 6$ is necessary for $3 + 2 \neq 5$.

5. We use the truth table for $P \Rightarrow Q$.

(a) Q can be true or false, since $F \Rightarrow T$ and $T \Rightarrow T$ are true.

(b) Q must be true, since, in this case, only $T \Rightarrow T$ will give true.

(c) Q must be false.

6. We use the truth table for $P \Rightarrow Q$.

(a) F, since $T \Rightarrow F$ is false

(b) T, since $F \Rightarrow F$ is true

(c) T, since $F \Rightarrow T$ is true

(d) T, since $F \Rightarrow T$ or $F \Rightarrow F$ are true

(e) F, since $T \Rightarrow F$ is false

Exercise 5.2 & 5.3

1. **(a)** This is an existence statement. It is enough to find such an integer, e.g. $20 = 4 + 6 + 10$

(b) Direct proof: Let an even integer be $2m$ where m is any integer. m can be written as the sum of any 3 integers, p, q and r. Thus $2m = 2(p+q+r) = 2p + 2q + 2r$, which are even integers.

(c) Not true. Let $(2n + 1)$ and $(2k + 1)$ be any two odd integers. Then, $(2n + 1) + (2k + 1) = 2(n + k + 1)$ which is even. Thus, the sum cannot be odd.

(d) Let $m = 2n + 1$ be any odd integer.
$2n$ is even, thus it can be the sum of three even integers, so,
$m = 2p + 2q + 2r + 1 = (2p+1) + (2q-1) + (2r+1)$, which are three odd integers.

(e) Let $n, n+1,$ and $n+2$ be three consecutive integers, then,
$n + (n+1) + (n+2) = 3n + 3 = 3(n+1)$, which is a multiple of 3.

(f) Not true. A counter example is enough: $n + (n+1) + (n+2) + (n+3) = 4n + 6$.
If $n = 1$, this sum is 10, which is not divisible by 4.

(g) True. Every three consecutive numbers should have at least one even number and every three consecutive numbers should have one multiple of 3 (multiples of 3 are periodic with period 3). Thus, the product should be a multiple of 6.

(h) Not true since $a = -b$ is another possibility.

2. **(a)** We need to find the perfect square less than or equal to 1871. This is 1849 or 43^2. That is, he turned 43 in year 1849, so he was born in 1849 – 43 = 1806.

(b) Again, we need to find the perfect square less than or equal to 2018. The closest perfect square less than 2018 is 1936 = 44^2. This means that your friend turned 44 in year 1936, which means he/she will be 126 in year 2018. This is not likely given our life expectancy! (The greatest fully authenticated age to which any human has ever lived is 122 years 164 days by Frenchwoman Jeanne Louise Calment.)

3. If $n \geq 2$, then $n!$ contains at least one even factor. Thus, for example we factor 2 out of both terms $n(n-1)\times(n-2)\times\cdots\times 2\times 1\ +\ 2\ =\ 2\big(n(n-1)\times(n-2)\times\cdots\times 1+\ 1\big)=2Q$, which is even.

4. **(a)** Needs to be proven in both directions.
If $5n + 3$ even, then if you add any odd number to it, the resulting number will be odd. So, adding the odd integer $2n – 5$ to $5n + 3$ will result in an odd integer. That is, $5n + 3 + 2n – 5 = 7n – 2$ is odd. Similarly, if $7n – 2$ is odd, adding the odd integer $5 – 2n$ will result in an even number. That is, $7n – 2 + 5 – 2n = 5n + 3$ is even.

(b) $7n – 2$ will be even. If $5n + 3$ is odd, then $5n$ must be even ⇒ n is even ⇒$7n – 2$ is even.

5. $m^2 + n^2$ even ⇒ m^2 and n^2 are both even or both odd ⇒ m and n have the same parity because if, for example, m^2 is even, m must be even as the product of two odd numbers cannot be even. Similarly, if m^2 is odd, then m must be odd.

6. By contradiction:
Assume that $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$. Square both sides and simplify, $0 = xy$. Thus, one of x or y must be zero, which contradicts the fact that both are positive.

7. If $x = 0$, or $y = 0$, then $xy = 0$ is obvious.
In the opposite direction: If $xy = 0$. Assume that x and y are both different from zero.
But there are no nonzero real numbers that can have a product of zero, thus a contradiction.

8. Let O be the set of odd numbers.

(a) $\exists\, x \in O$ such that $x = k^2$ where, $k \in \mathbb{Z}$

(b) $\forall\, x \in O, x \neq k^2$ where, $k \in \mathbb{Z}$

(c) True. Since it is an existence statement, $x = 81 = 9^2$

9. **(a)** Statement: $\forall\, x \in \mathbb{Z}^+$, $13 \mid x$. Negation: $\exists\, x \in \mathbb{Z}^+$, $13 \nmid x$

(b) False, a counter example: $x = 10$, $13 \nmid 10$

10. By contradiction: Assume $a^2(b^2 - 2b)$ is odd, but at least a or b is even.
Say a is even, then, $a = 2k$ and $a^2(b^2 - 2b) = 4k^2(b^2 - 2b)$ is even. Contradiction.

11. Contrapositive: $5 \mid m$ and $5 \mid n \Rightarrow m = 5r$ and $n = 5s \Rightarrow mn = 25rs \Rightarrow 25 \mid mn$

12. Prove by cases: $m + n$ is even $\Rightarrow$ m and n must both be odd or both be even.

1. If both are odd: m^2 is odd and n^2 is odd $\Rightarrow m^2 + n^2$ is even.
2. If both are even: m^2 is even and n^2 is even $\Rightarrow m^2 + n^2$ is even.

Or, direct: $m + n$ is even $\Rightarrow m + n = 2k \Rightarrow m^2 + n^2 = (m + n)^2 - 2mn = 4k^2 - 2mn = 2N$

13. If n is even, then $n = 2k \Rightarrow n^2 + 2n + 9 = 4(k^2 + k) + 9$, which is odd.

If $n^2 + 2n + 9$ is odd, then $n^2 + 2n$ must be even. But $2n$ is even, so n^2 is even, and so, n must be even. (Or, by using the contrapositive method.)

14. Let $a = 2k + 1$ and simplify. $a^2 + 3a + 5 = 2(2k^2 + 5k + 4) + 1 = 2N + 1$, which is odd.

Exercise 5.4

In the solutions to the following exercises, for convenience we have excluded some relatively obvious calculations. Conventions for this section: MI = mathematical induction.

1. You can either recall arithmetic series, or by inspection the sum: $S_n = n(n+1)$

Basis step: $S_1 = 2 = 1(1+1)$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,
$S_k = 2 + 4 + \cdots + 2k = k(k+1)$, and we prove the formula true for $n = k + 1$

$$S_{k+1} = [2 + 4 + \cdots + 2k] + (2k + 2) = S_k + 2k + 2$$
$$= k(k+1) + 2(k+1) = (k+1)(k+2)$$

This shows that S_{k+1} is true whenever S_k is true, which completes the inductive step of the proof.

2. We will provide a proof using MI.

Let $P(n)$ be the statement that $a_n = 3^{n-1}$

Basis step: $a_1 = 1 = 3^{1-1}$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$a_k = 3^{k-1}$, and we prove this to be true for $n = k + 1$.

By definition of the sequence,

$a_{k+1} = 3a_k$ and by assumption $a_k = 3^{k-1}$

Therefore $a_{k+1} = 3 \cdot 3^{k-1} = 3^k$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for all positive integers.

3. We will provide a proof using MI.

Let $P(n)$ be the statement that $a_n = 4n - 3$

Basis step: $a_2 = 1 + 4 = 5 = 4 \times 2 - 3$. So, the formula is true for $n = 2$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$a_k = 4k - 3$, and we prove this to be true for $n = k + 1$.

By definition of the sequence,

$a_{k+1} = a_k + 4$, and by assumption $a_k = 4k - 3$

Therefore $a_{k+1} = a_k + 4 = 4k - 3 + 4 = 4(k+1) - 3$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 2$.

4. We will provide a proof using MI.

Let $P(n)$ be the statement that $a_n = 2^n - 1$

Basis step: $a_2 = 2 \times 1 + 1 = 3 = 2^2 - 1$. So, the formula is true for $n = 2$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$a_k = 2^k - 1$, and we prove this to be true for $n = k + 1$.

By definition of the sequence,

$a_{k+1} = 2a_k + 1$, and by assumption $a_k = 2^k - 1$

Therefore $a_{k+1} = 2 \cdot (2^k - 1) + 1 = 2^{k+1} - 1$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 2$.

5. We will provide a proof using MI.

Let $P(n)$ be the statement that $a_n = \frac{n}{n+1}$

Basis step: $a_1 = \frac{1}{2} = \frac{1}{1+1}$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$a_k = \frac{k}{k+1}$ and we prove this to be true for $n = k + 1$.

By definition of the sequence,

$a_{k+1} = a_k + \frac{1}{(k+1)(k+2)}$ and by assumption $a_k = \frac{k}{k+1}$

Therefore $a_{k+1} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

6. Either by inspection or realising that this is a geometric series: $S(n) = 1 - \left(\frac{1}{2}\right)^n$

We will provide a proof using MI.

Basis step: $S(1) = \frac{1}{2} = 1 - \frac{1}{2}$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$S(k) = 1 - \left(\frac{1}{2}\right)^k$ and we prove this to be true for $n = k + 1$.

By definition of the sequence, the kth term is $\frac{1}{2^k}$,

$S(k+1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$ and by assumption $S(k) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = 1 - \left(\frac{1}{2}\right)^k$

Therefore

$$S(k+1) = S(k) + \frac{1}{2^{k+1}} = 1 - \left(\frac{1}{2}\right)^k + \frac{1}{2^{k+1}}$$
$$= 1 - \left(\frac{1}{2}\right)^k + \frac{1}{2}\left(\frac{1}{2}\right)^k = 1 - \left(\frac{1}{2}\right)^{k+1}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

7. We will provide a proof using MI.

Let $P(n)$ be the statement that $S(n) = 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

Basis step: $S(0) = 1 = 2^{0+1} - 1$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$S(k) = 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$ and we prove this to be true for $n = k + 1$.

By definition of the sequence,

$a_{k+1} = 2^{k+1}$ and by assumption $S(k) = 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$,

therefore

$$S(k+1) = 1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = S(k) + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2 \times 2^{k+1} - 1 = 2^{k+2} - 1$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true, thus, by the principle of MI, $S(n)$ is true for all non-negative integers.

8. We will provide a proof using MI.

Let $P(n)$ be the statement that $a_n = a_1 r^{n-1}$

Basis step: $a_1 = a_1 r^{1-1} = a_1,$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$a_k = a_1 r^{k-1}$ and we prove this to be true for $n = k + 1$.

By definition of the geometric sequence,

$a_{k+1} = r a_k$ and by assumption $a_k = a_1 r^{k-1}$,

therefore $a_{k+1} = r \cdot a_k = r \cdot a_1 r^{k-1} = a_1 r^k$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

9. We will provide a proof using MI.

Let $P(n)$ be the statement that $S_n = \dfrac{a - ar^n}{1 - r}$

Basis step: $S_1 = a = \dfrac{a - ar}{1 - r}$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$S_k = \dfrac{a - ar^k}{1 - r}$ and we prove this to be true for $n = k + 1$.

By definition of the geometric sequence,

$a_{k+1} = r a_k$, and by assumption $S_k = \dfrac{a - ar^k}{1 - r}$

Therefore $S_{k+1} = S_k + ra_k = \frac{a - ar^k}{1-r} + r \cdot ar^{k-1} = \frac{a - ar^k + ar^k\left(1-r\right)}{1-r} = \frac{a - ar^{k+1}}{1-r}$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

10. We will provide a proof using MI.

Let $P(n)$ be the statement that $2^n < n!$

Basis step: $n = 4, 2^4 = 16 < 4! = 24,$. So, the formula is true for $n = 4$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$P(k): 2^k < k!$ and we prove this to be true for $n = k + 1$.

We need to show that $P(k+1): 2^{k+1} < (k+1)!$ is true and by assumption $2^k < k!$

$2^k < k! \Rightarrow 2 \cdot 2^k < 2 \cdot k! < \left(k+1\right) \cdot k!$ since we know that $k > 3 \Rightarrow 2 < k+1$

Thus, $2^{k+1} < \left(k+1\right)!$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 4$.

11. We will provide a proof using MI.

Let $P(n)$ be the statement that $2^n > n^2$

Basis step: $n = 5,\ 2^5 = 32 > 5^2 = 25$. So, the formula is true for $n = 5$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$P(k): 2^k > k^2$ and we prove this to be true for $n = k + 1$.

We need to show that $P(k+1): 2^{k+1} > (k+1)^2$ is true and by assumption $2^k > k^2$

$2^k > k^2 \Rightarrow 2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2$, but $k > 4 \Rightarrow k^2 > 4k = 2k + 2k > 2k + 1$

Thus, $2^{k+1} > 2k^2 = k^2 + k^2 > k^2 + 2k + 1 = \left(k+1\right)^2$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 5$.

12. We will provide a proof using MI.

Let $P(n)$ be the statement that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = \left(n+1\right)! - 1$

Basis step: $n = 1,\ 1 \cdot 1! = \left(1+1\right)! - 1$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$P(k): 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = \left(k+1\right)! - 1$ and we prove this to be true for $n = k + 1$.

We need to show that $P(k+1): 1\cdot 1! \cdots + k\cdot k! + (k+1)\cdot(k+1)! = (k+2)! - 1$

with the assumption that $P(k)$ is true.

$$P(k+1): 1\cdot 1! \cdots + k\cdot k! + (k+1)\cdot(k+1)!$$
$$= (k+1)! - 1 + (k+1)\cdot(k+1)!$$
$$= (k+1)![1+k+1] - 1 = (k+2)! - 1$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

13. We will provide a proof using MI.

Let $P(n)$ be the statement $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Basis step: $n = 1$, $\frac{1}{1\cdot 2} = \frac{1}{1+1}$. So, the formula is true for $n = 1$.

Inductive step: assume the formula is true for $n = k$, i.e.,

$P(k): \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k\cdot(k+1)} = \frac{k}{k+1}$ and we prove this to be true for $n = k + 1$.

We need to show that $P(k+1): \frac{1}{1\cdot 2} + \cdots + \frac{1}{k\cdot(k+1)} + \frac{1}{(k+1)\cdot(k+2)} = \frac{k+1}{k+2}$

with the assumption that $P(k)$ is true.

$$P(k+1): \frac{1}{1\cdot 2} + \cdots + \frac{1}{k\cdot(k+1)} + \frac{1}{(k+1)\cdot(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)\cdot(k+2)} = \frac{k(k+2)+1}{(k+1)\cdot(k+2)} = \frac{(k+1)^2}{(k+1)\cdot(k+2)} = \frac{k+1}{k+2}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

14. We will provide a proof using MI.

Let $P(n)$ be the statement that $n^3 - n = 3q$, i.e., $n^3 - n$ is divisible by 3.

Basis step: $n = 1$, $1^3 - 1 = 0$, which is divisible by 3. So, the statement is true for $n = 1$.

Inductive step: assume the statement is true for $n = k$, i.e.,

$P(k): k^3 - k = 3M$ and we prove this to be true for $n = k + 1$.

We need to show that $P(k+1): (k+1)^3 - (k+1)$ is divisible by 3 with the assumption $P(k)$ is true.

$$P(k+1): (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$
$$= k^3 - k + 3(k^2 + k) = 3M + 3N$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

15. We will provide a proof using MI.
Let $P(n)$ be the statement that $n^5 - n = 5q$, i.e., $n^5 - n$ is divisible by 5.
Basis step: $n = 1,\ 1^5 - 1 = 0$, which is divisible by 5. So, the statement is true for $n = 1$.
Inductive step: assume the statement is true for $n = k$, i.e.,
$P(k): k^5 - k = 5M$ and we prove this to be true for $n = k + 1$.
We need to show that $P(k+1): (k+1)^5 - (k+1)$ is divisible by 5 with the assumption $P(k)$ is true.

$$P(k+1): (k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$
$$= \left(k^5 - k\right) + 5\left(k^4 + 2k^3 + 2k^2 + k\right) = 5M + 5N$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

16. We can prove this by several ways. We choose proof by cases:
$n^3 - n = (n-1)n(n+1)$

- If n is even, then it is divisible by 2, and either $n - 1$ or $n + 1$ must be divisible by 3.
- If n is odd, then $n - 1$ and $n + 1$ are divisible by 2. Either n is divisible by 3 or it leaves a remainder of 2 when divided by 3. Thus $n + 1$ would be divisible by 3.

17. We can prove this by several ways. We choose proof by cases:
$n^2 + n = n(n + 1)$.

- If n is even, then it is divisible by 2.
- If n is odd, then $n + 1$ is divisible by 2.

18. Using the binomial expansion:

$$5^n = (4+1)^n = \sum_0^n \binom{n}{i} 4^i = 1 + \sum_1^n \binom{n}{i} 4^i$$
$$\Rightarrow 5^n - 1 = \sum_1^n \binom{n}{i} 4 \cdot 4^{i-1} = 4\sum_1^n \binom{n}{i} 4^{i-1}$$

19. We will provide a proof using MI.

Let $P(n)$ be the statement that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

Basis step: $n=1$, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^1 = \begin{pmatrix} a^1 & 0 \\ 0 & b^1 \end{pmatrix}$. So, the statement is true for $n = 1$.

Inductive step: assume the statement is true for $n = k$, i.e.,

$P(k): \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}$ and we prove this to be true for $n = k + 1$.

We need to show that $P(k+1): \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{k+1} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$ with the assumption $P(k)$ is true.

$$P(k+1): \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{k+1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^k \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$$

This shows that $P(k+1)$ is true whenever $P(k)$ is true, thus, by the principle of MI, $P(n)$ is true for $n \geq 1$.

20. In all three cases we will provide brief MI proofs.

(a) $n=1: \sum_{i=1}^{1}(2i+4) = 6 = 1+5$

$n=k: \sum_{i=1}^{k}(2i+4) = k^2+5k$

$\Rightarrow n=k+1: \sum_{i=1}^{k+1}(2i+4) = \sum_{i=1}^{k}(2i+4) + 2(k+1)+4 = k^2+5k+2(k+1)+4$

$= (k+1)^2 + 5(k+1)$

(b) $n=1: \sum_{1}^{1}\left(2\cdot 3^{1-1}\right) = 3^1 - 1$

$n=k: \sum_{1}^{k}\left(2\cdot 3^{i-1}\right) = 3^k - 1$

$\Rightarrow n=k+1: \sum_{1}^{k+1}\left(2\cdot 3^{i-1}\right) = \sum_{1}^{k}\left(2\cdot 3^{i-1}\right) + 2\cdot 3^k = 3^k - 1 + 2\cdot 3^k$

$= 3\cdot 3^k - 1 = 3^{k+1} - 1$

(c) $n=1:\sum_{1}^{1}\frac{1}{(2i-1)(2i+1)}=\frac{1}{(2-1)(2+1)}=\frac{1}{3}=\frac{1}{2+1}$

$n=k:\sum_{1}^{k}\frac{1}{(2i-1)(2i+1)}=\frac{k}{2k+1}$

$$\Rightarrow n=k+1:\sum_{1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\sum_{1}^{k}\frac{1}{(2i-1)(2i+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$

Chapter 5 practice questions

1. **(a)** $a_2=\sqrt[3]{8-(1)^3}=\sqrt[3]{7};\ a_3=\sqrt[3]{8-\left(\sqrt[3]{7}\right)^3}=1;\ a_4=\sqrt[3]{8-(1)^3}=\sqrt[3]{7}$

$\{a_n\}=\{1,\sqrt[3]{7},1,\sqrt[3]{7},\ldots\}$

(b) $a_2=\sqrt[3]{8-(2)^3}=0;\ a_3=\sqrt[3]{8-(0)^3}=2;\ a_4=\sqrt[3]{8-(2)^3}=0$

$\{a_n\}=\{2,0,2,0,\ldots\}$

2. $n=1,\ 5^1+9^1+2=16$ is divisible by 4.

$n=k:\ 5^k+9^k+2=4m$

$n=k+1: 5^{k+1}+9^{k+1}+2=5\cdot 5^k+9\cdot 9^k+2$, now add and subtract $4\cdot 5^k$ and simplify:

$5\cdot 5^k+\ 9\cdot 9^k+\ 2=5\cdot 5^k+4\cdot 5^k+\ 9\cdot 9^k-4\cdot 5^k+\ 2$

$$=9\cdot 5^k+\ 9\cdot 9^k-4\cdot 5^k+\ 2=9\left(5^k+\ 9\cdot 9^k\right)-4\cdot 5^k+\ 2$$

$$=9(4m-2)-4\cdot 5^k+\ 2=36m-16--4\cdot 5^k=4\left(9m-4-5^k\right)$$

This last number is a multiple of 4.

3. $S_n = 4n^2 - 2n \Rightarrow S_{n-1} = 4(n-1)^2 - 2(n-1)$

$u_n = S_n - S_{n-1} = 8n - 6$

$\Rightarrow u_2 = 10, u_m = 8m - 6, u_{32} = 250$

Geometric sequence:

$$\frac{u_{32}}{u_m} = \frac{u_m}{u_2} \Rightarrow \frac{250}{8m-6} = \frac{8m-6}{10} \Rightarrow (8m-6)^2 = 50^2$$

$$\Rightarrow 8m - 6 = \pm 50 \Rightarrow m = 7,\ \xcancel{m = -\frac{11}{2}}$$

4. We use MI:

$n = 1$: $1^3 = \dfrac{1^2(1+1)^2}{4} = 1$, statement is true for $n = 1$.

Assume true for $n = k$: $1 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Now, for $n = k + 1$:

$$\begin{aligned}1 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k^2+4k+4)}{4}\\ &= \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$

5. We use MI:

For $n = 0$, $(5^0 - 1) = 0 \Rightarrow 24 \mid (5^0 - 1)$

Assume true for $n = k$: $24 \mid (5^{2k} - 1) \Rightarrow 5^{2k} = 24a + 1$

For $n = k + 1$, $5^{2(k+1)} - 1 = 5^{2k} \cdot 5^2 - 1$, and by substituting $5^{2k} = 24a + 1$ and simplifying:

$$\begin{aligned}5^{2(k+1)} - 1 &= 5^{2k} \cdot 5^2 - 1 = (24a + 1) \cdot 5^2 - 1\\ &= 24a \cdot 25 + 25 - 1 = 24(25a + 1)\\ &\Rightarrow 24 \mid (5^{2(k+1)} - 1)\end{aligned}$$

6. We use MI:

For $n = 1$: $F_1^2 = F_1F_{1+1} \Rightarrow 1^2 = 1\cdot 1$, which is a true statement.

Assume true for $n = k$: $F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 = F_kF_{k+1}$

Now for $n = k + 1$:

$$\begin{aligned} F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 + F_{k+1}^2 &= F_kF_{k+1} + F_{k+1}^2 \\ &= F_{k+1}\left(F_k + F_{k+1}\right) = F_{k+1}F_{k+2} \end{aligned}$$

7. We use MI:

For $n = 1$: $\frac{1}{2!} = 1 - \frac{1}{(1+1)!} = \frac{1}{2}$, which is true.

Assume true for $n = k$: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$

Now for $n = k + 1$:

$$\begin{aligned} \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} &= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\ &= 1 - \frac{k+2-k-1}{(k+2)!} = 1 - \frac{1}{(k+2)!} \end{aligned}$$

Exercise 6.1

1.

(a) $60° = \frac{60}{180}\pi = \frac{\pi}{3}$

(b) $150° = \frac{150}{180}\pi = \frac{5\pi}{6}$

(c) $-270° = -\frac{270}{180}\pi = -\frac{3\pi}{2}$

(d) $36° = \frac{36}{180}\pi = \frac{\pi}{5}$

(e) $135° = \frac{135}{180}\pi = \frac{3\pi}{4}$

(f) $50° = \frac{50}{180}\pi = \frac{5\pi}{18}$

(g) $-45° = -\frac{45}{180}\pi = -\frac{\pi}{4}$

(h) $400° = \frac{400}{180}\pi = \frac{20\pi}{9}$

2.

(a) $\frac{3\pi}{4} = \frac{3}{4}\cdot 180° = 135°$

(b) $\frac{7\pi}{2} = \frac{7}{2}\cdot 180° = 630°$

(c) $2 = 2\cdot\frac{180°}{\pi} = \frac{360°}{\pi} \approx 115°$

(d) $\frac{7\pi}{6} = \frac{7}{6}\cdot 180° = 210°$

(e) $-2.5 = -2.5\cdot\frac{180°}{\pi} = -\frac{450°}{\pi} \approx -143°$

(f) $\frac{5\pi}{3} = \frac{5}{3}\cdot 180° = 300°$

(g) $\frac{\pi}{12} = \frac{1}{12}\cdot 180° = 15°$

(h) $1.57 = 1.57\cdot\frac{180°}{\pi} = \frac{282.6°}{\pi} \approx 89.95 \approx 90.0°$

3.

(a) e.g. $30° + 360° = 390°$ and $30° - 360° = -330°$

(b) e.g. $\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$ and $\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

(c) e.g. $175° + 360° = 535°$ and $175° - 360° = -185°$

(d) e.g. $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$ and $-\frac{\pi}{6} - 2\pi = -\frac{13\pi}{6}$

(e) e.g. $\frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$ and $\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$

(f) e.g. $3.25 + 2\pi \approx 9.53$ and $3.25 - 2\pi = -3.03$

4. When the angle is expressed in radians,

(a) $s=\theta r$, so $s=\dfrac{120}{180}\pi\cdot 6\approx 12.6$ cm

(b) $s=\theta r$, so $s=\dfrac{70}{180}\pi\cdot 12\approx 14.7$ cm

5. When the angle is expressed in radians, $s=\theta r$, so $\theta=\dfrac{s}{r}=\dfrac{12}{8}=1.5$ rad or $\theta=1.5\dfrac{180°}{\pi}\approx 85.9°$

6. When the angle is expressed in radians, $s=\theta r$, so $r=\dfrac{s}{\theta}=\dfrac{15}{\dfrac{2\pi}{3}}=\dfrac{45}{2\pi}\approx 7.16$

7. When the angle is expressed in radians,

(a) $A=\dfrac{1}{2}\theta r^2=\dfrac{1}{2}100°\cdot\dfrac{\pi}{180°}\cdot 4^2\approx 13.96\approx 14.0\,\text{cm}^2$

(b) $A=\dfrac{1}{2}\theta r^2=\dfrac{1}{2}\cdot\dfrac{5\pi}{6}\cdot 10^2\approx 130.9\approx 131\,\text{cm}^2$

8. In radian measure, $\alpha=\dfrac{s}{r}=\dfrac{60}{20}=3$ rad, or $\alpha=3\cdot\dfrac{180°}{\pi}\approx 172°$

9. $s=\theta r=2\cdot 16=32$ cm

10. In radian measure, $A=\dfrac{1}{2}\theta r^2\rightarrow r^2=\dfrac{2A}{\theta}\rightarrow r=\sqrt{\dfrac{2A}{\theta}}$

We therefore have $r=\sqrt{\dfrac{2\cdot 24}{60^{\circ}\dfrac{\pi}{180^{\circ}}}}=\sqrt{\dfrac{2\cdot 24}{\dfrac{\pi}{3}}}\approx 6.77$ cm

11.

(a) Since one complete revolution corresponds to an angle of 2π rad, the angular velocity is $\omega=1.5\cdot 2\pi=3\pi$ rad s^{-1}.

(b) Assuming no slipping, the contact point of the wheel with the ground is not moving, and the centre of the wheel (and therefore the bicycle) moves with respect to it with speed

$v=\omega r=3\pi\cdot\dfrac{0.70}{2}\approx 3.30\,\text{m s}^{-1}$ or $3.30\cdot 3.6\approx 11.9$ km h^{-1}

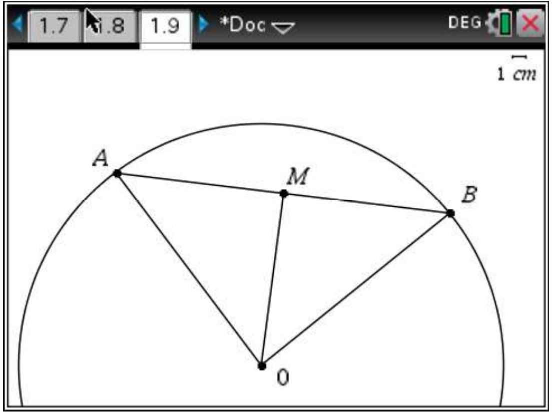
12. First of all we change units so everything is expressed using metres.

$v=25\ \text{km h}^{-1}=25\cdot\dfrac{1000}{60\cdot 60}\ \text{m s}^{-1}\approx 6.94\ \text{m s}^{-1}$, and $r=\dfrac{70\text{ cm}}{2}=0.35$ metres

Since $v=\omega r$, we have $\omega=\dfrac{v}{r}=\dfrac{6.94}{0.35}\approx 19.8\,\text{rad s}^{-1}$

13. The angle swept by the point in T seconds is given by $\theta=\omega T$ radians. The arc length – and therefore the distance – covered by the point in this second is given by $s=\theta r=\omega Tr$ cm. The linear speed is given by distance travelled divided by time taken, so we have $v=\dfrac{s}{T}=\dfrac{\omega rT}{T}=\omega r$.

14. First of all we find the angle at the centre AOB.



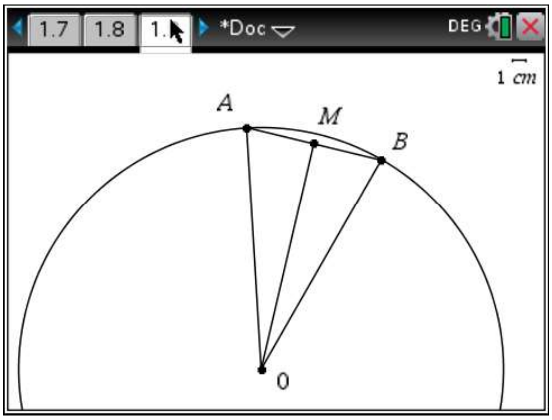
Triangle MOB, where M is the midpoint of AB, is right-angled, so

$$\sin MOB = \frac{MB}{OB} = \frac{\frac{26}{2}}{20} = 0.65 \rightarrow MOB = \sin^{-1} 0.65 \approx 0.7076 \text{ rad. It follows that}$$

$AOB = 2MOB \approx 1.415$ rad, and the length of arc AB is $s = AOB \cdot OB \approx 1.415 \cdot 20 = 28.3$ cm

15. The total area swept by the pipe is given by $A = \pi R^2 = \pi \cdot 400^2 \approx 502\,655 \text{ m}^2$. This area is swept in 24 hours, so in one hour the area covered is $a = \dfrac{A}{24} \approx 20944 \text{ m}^2$.

16. A $\frac{1}{64}th$ portion of the polygon and its circumscribed circle is shown below (diagram not to scale).



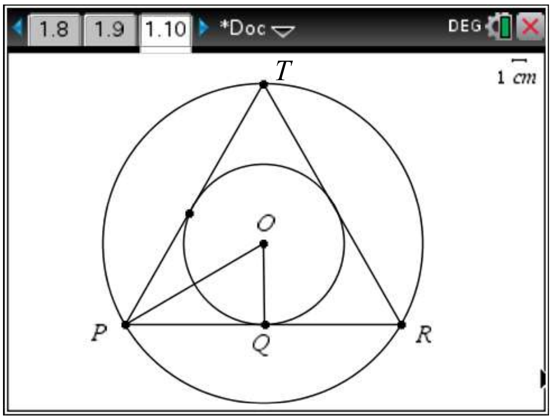
(a) The angle at the centre AOB is $\frac{1}{64}th$ of a round angle, so $AOB = \dfrac{2\pi}{64} = \dfrac{\pi}{32}$ and

$MOB = \dfrac{AOB}{2} = \dfrac{\pi}{64}$, where M is the midpoint of AB.

We have $\sin MOB = \dfrac{MB}{OB} \rightarrow OB = \dfrac{MB}{\sin MOB} = \dfrac{\frac{3}{2}}{\sin\frac{\pi}{64}} \approx 30.6$ cm

(b) The circumference has length $2\pi r \approx 192.0771$ cm, while the polygon has perimeter $64 \cdot 3 = 192$ cm. The difference is $d = 0.0771$ cm!

17. From the values of the areas of the inscribed and circumscribed circles shown below,



it follows that the radius OP of the circumscribed circle is double the radius OQ of the inscribed circle. In fact, from $Area = \pi r^2$ we have $OP = \sqrt{\dfrac{200\pi}{\pi}} = \sqrt{200} = 2\sqrt{50}$ and $OQ = \sqrt{\dfrac{50\pi}{\pi}} = \sqrt{50}$. It follows from Pythagoras that $PQ = \sqrt{OP^2 - OQ^2} = \sqrt{200-50} = \sqrt{150}$ and $PR = 2\sqrt{150}$. The height QT of the equilateral triangle is given by $OP + OQ = 2\sqrt{50} + \sqrt{50} = 3\sqrt{50}$, so the area of the triangle is $A = \dfrac{1}{2}PR \cdot QT = \dfrac{1}{2} \cdot 2\sqrt{150} \cdot 3\sqrt{50} = \sqrt{3 \cdot 50} \cdot 3\sqrt{50} = 150\sqrt{3}\ \text{cm}^2$.

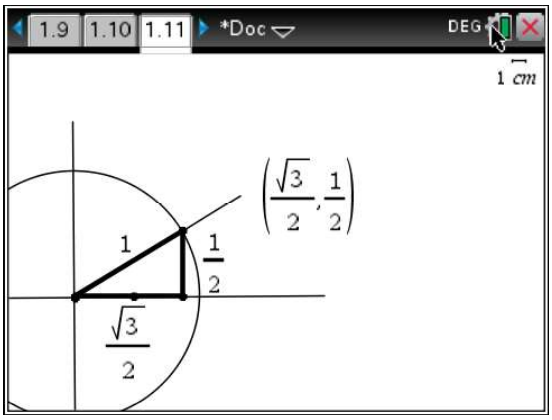
18. The area A of the segment is the difference between one quarter of the area C of the circle and the area T of the triangle, where $C = \pi r^2$ and $T = \dfrac{1}{2} r \cdot r = \dfrac{1}{2} r^2$. It follows that $T = \dfrac{C}{2\pi}$, so that $A = \dfrac{C}{4} - T = \dfrac{C}{4} - \dfrac{C}{2\pi}$. Solving for C gives $4\pi A = C(\pi - 2) \rightarrow C = \dfrac{4\pi A}{\pi - 2}$

Exercise 6.2

1.

Copying the triangle onto the unit circle, we have the following diagram for $t = \dfrac{\pi}{6}$

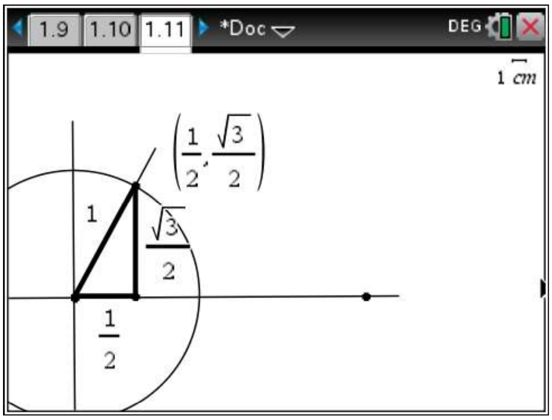
(arc length on unit circle = angle in radians, and $\dfrac{\pi}{6} = 30°$)



Here the hypotenuse is the radius of the unit circle, so its length is 1.

It follows that the horizontal leg has length $\frac{\sqrt{3}}{2}$ and the vertical leg has length $\frac{1}{2}$. Given the orientation of the triangle, these are also the coordinates of the terminal point, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

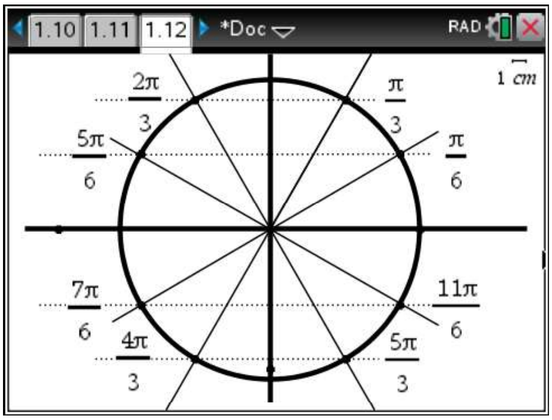
Repeating the process when the arc length is $\frac{\pi}{3}$ gives the following diagram



from which we find the coordinates as $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Equivalently, the coordinates when $t = \frac{\pi}{6}$ are given by $x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $y = \sin\frac{\pi}{6} = \frac{1}{2}$, so $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. The coordinates when $t = \frac{\pi}{3}$ are given by $x = \cos\frac{\pi}{3} = \frac{1}{2}$ and $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

(a)



The coordinates of the points are:

$$\frac{2\pi}{3}\to\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\quad \frac{5\pi}{6}\to\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right),\quad \frac{7\pi}{6}\to\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

$$\frac{4\pi}{3}\to\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\quad \frac{5\pi}{3}\to\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\quad \frac{11\pi}{6}\to\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

2.

(a) $\cos 50° \approx 0.64$

(b) $\sin 80° \approx 0.98$

(c) $\cos 1 \approx 0.54$

(d) $\sin 0.5 \approx 0.48$

(e) $\tan 70° = \frac{\sin 70°}{\cos 70°} \approx \frac{0.94}{0.34} = 2.76$

(f) $\cos 1.5 \approx 0.071$

(g) $\sin 20° \approx 0.34$

(h) $\tan 1 = \frac{\sin 1}{\cos 1} \approx \frac{0.84}{0.54} = 1.56$

3.

(a) First quadrant, $\left(\cos\frac{\pi}{6},\sin\frac{\pi}{6}\right)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

(b) Fourth quadrant, $\left(\cos\frac{5\pi}{3},\sin\frac{5\pi}{3}\right)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

(c) Fourth quadrant, $\left(\cos\frac{7\pi}{4},\sin\frac{7\pi}{4}\right)=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

(d) Axis between third and fourth quadrant, $\left(\cos\frac{3\pi}{2},\sin\frac{3\pi}{2}\right)=(0,-1)$

(e) Second quadrant, $(\cos 2,\sin 2)\approx(-0.416,0.909)$

(f) Fourth quadrant, $\left(\cos\left(-\frac{\pi}{4}\right),\sin\left(-\frac{\pi}{4}\right)\right)=\left(\cos\frac{\pi}{4},-\sin\frac{\pi}{4}\right)=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

(g) Fourth quadrant, $(\cos(-1),\sin(-1))\approx(0.540,-0.841)$

(h) Second quadrant, $\left(\cos\left(-\frac{5\pi}{4}\right),\sin\left(-\frac{5\pi}{4}\right)\right)=\left(\cos\frac{5\pi}{4},-\sin\frac{5\pi}{4}\right)=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

(i) Third quadrant, $(\cos 3.52,\sin 3.52)\approx(-0.929,-0.369)$

4.

(a) $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2},\cos\frac{\pi}{3}=\frac{1}{2},\tan\frac{\pi}{3}=\sqrt{3}$

(b) $\sin\frac{5\pi}{6}=\frac{1}{2},\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2},\tan\frac{5\pi}{6}=-\frac{\sqrt{3}}{3}$

(c) $\sin\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2},\cos\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2},\tan\left(-\frac{3\pi}{4}\right)=1$

(d) $\sin\frac{\pi}{2}=1, \cos\frac{\pi}{2}=0, \tan\frac{\pi}{2}=\text{undefined}$

(e) $\sin\left(-\frac{4\pi}{3}\right)=\frac{\sqrt{3}}{2}, \cos\left(-\frac{4\pi}{3}\right)=-\frac{1}{2}, \tan\left(-\frac{4\pi}{3}\right)=-\sqrt{3}$

(f) $\sin 3\pi=0, \cos 3\pi=-1, \tan 3\pi=0$

(g) $\sin\frac{3\pi}{2}=-1, \cos\frac{3\pi}{2}=0, \tan\frac{3\pi}{2}=\text{undefined}$

(h) $\sin\left(-\frac{7\pi}{6}\right)=\frac{1}{2}, \cos\left(-\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}, \tan\left(-\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{3}$

(i) since $1.25=\frac{5}{4}$, we have $\sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}, \cos\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}, \tan\frac{5\pi}{4}=1$

5.

(a) $\sin\frac{13\pi}{6}=\sin\left(\frac{13\pi}{6}-2\pi\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

$\cos\frac{13\pi}{6}=\cos\left(\frac{13\pi}{6}-2\pi\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

(b) $\sin\frac{10\pi}{3}=\sin\left(\frac{10\pi}{3}-2\pi\right)=\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2}$

$\cos\frac{10\pi}{3}=\cos\left(\frac{10\pi}{3}-2\pi\right)=\cos\frac{4\pi}{3}=-\frac{1}{2}$

(c) $\sin\frac{15\pi}{4}=\sin\left(\frac{15\pi}{4}-2\pi\right)=\sin\frac{7\pi}{4}=-\frac{\sqrt{2}}{2}$

$\cos\frac{15\pi}{4}=\cos\left(\frac{15\pi}{4}-2\pi\right)=\cos\frac{7\pi}{4}=-\frac{\sqrt{2}}{2}$

(d) $\sin\frac{17\pi}{6}=\sin\left(\frac{17\pi}{6}-2\pi\right)=\sin\frac{5\pi}{6}=\frac{1}{2}$

$\cos\frac{17\pi}{6}=\cos\left(\frac{17\pi}{6}-2\pi\right)=\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$

6.

(a) $\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$

(b) $\sin 315°=-\frac{\sqrt{2}}{2}$

(c) $\tan\frac{3\pi}{2}=\text{ undefined}$

(d) $\sec\frac{5\pi}{3}=\left(\cos\frac{5\pi}{3}\right)^{-1}=\left(\frac{1}{2}\right)^{-1}=2$

(e) $\csc 240°=(\sin 240°)^{-1}=\left(-\frac{\sqrt{3}}{2}\right)^{-1}=-\frac{2}{\sqrt{3}}=-\frac{2\sqrt{3}}{3}$

7.

(a) $\sin 2.5\approx 0.598$

(b) $\cot 120^\circ = \left(\tan 120^\circ\right)^{-1} = \left(-\sqrt{3}\right)^{-1} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

(c) $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

(d) $\sec 6 \approx 1.04$

(e) $\tan\pi = 0$

8.

(a) Since y-coordinate of point on unit circle is positive, either quadrant I or II

(b) Since y-coordinate of point on unit circle is positive and x-coordinate is negative, quadrant II

(c) y-coordinate of point on unit circle is negative. Tangent is positive, so x-coordinate must be negative as well (the ratio of two negative numbers is positive), so quadrant III

(d) x-coordinate of point on unit circle is negative. Tangent is negative, so y-coordinate must be positive (the ratio of a positive and a negative number is negative), so quadrant II

(e) Since x-coordinate of point on unit circle is positive, either quadrant I or IV

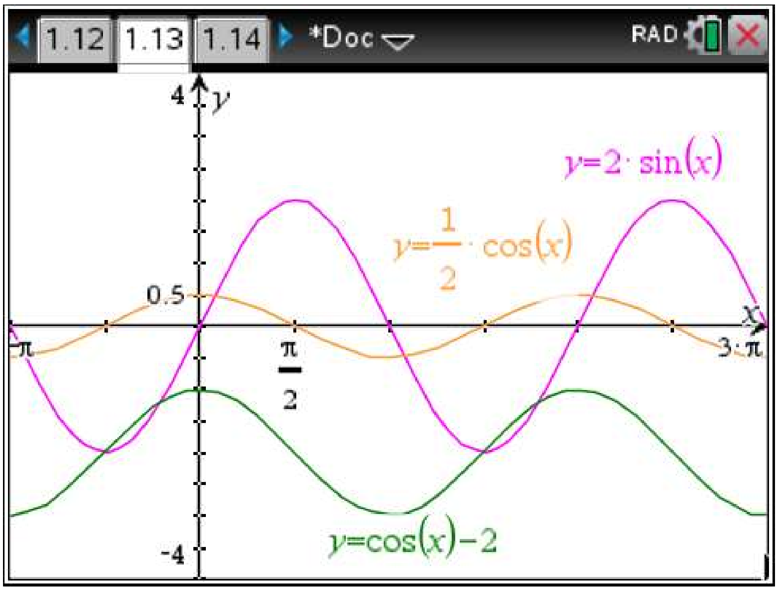
(f) Since $\sec\theta = \frac{1}{\cos\theta}$, x-coordinate of point on unit circle is positive. Tangent is positive, so y-coordinate must be positive (the ratio of two positive numbers is positive), so quadrant I

(g) x-coordinate of point on unit circle is positive. Since $\csc\theta = \frac{1}{\sin\theta}$, the y-coordinate is negative, so quadrant IV

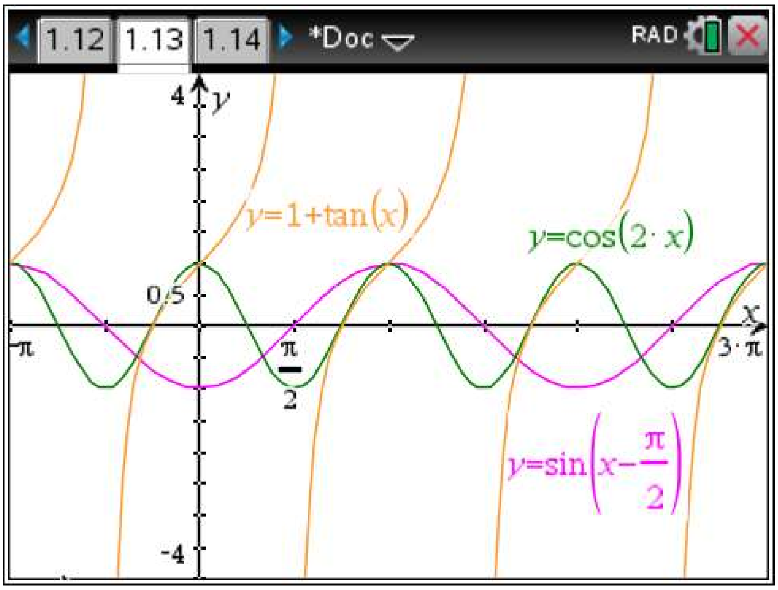
(h) since $\cot\theta = \frac{1}{\tan\theta}$, the x- and y-coordinate of point of unit circle must have different sign (the ratio of two numbers with different sign is negative), so quadrant II or IV

Exercise 6.3

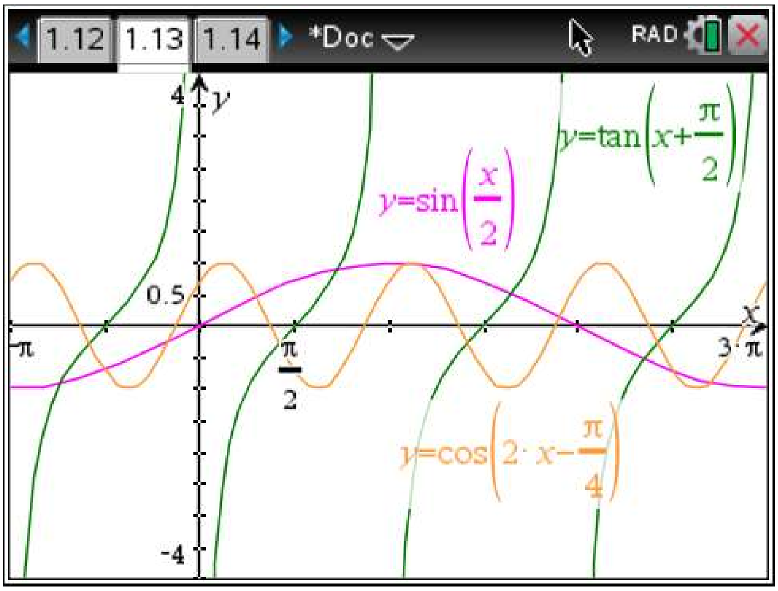
1. **(a)** **(b)** **(c)**



(d) (e) (f)


1.12
1.13
1.14
*Doc
RAD
4
y
$y=1+\tan(x)$
$y=\cos(2\cdot x)$
0.5
x
-π
π/2
3·π
$y=\sin\left(x-\frac{\pi}{2}\right)$
-4

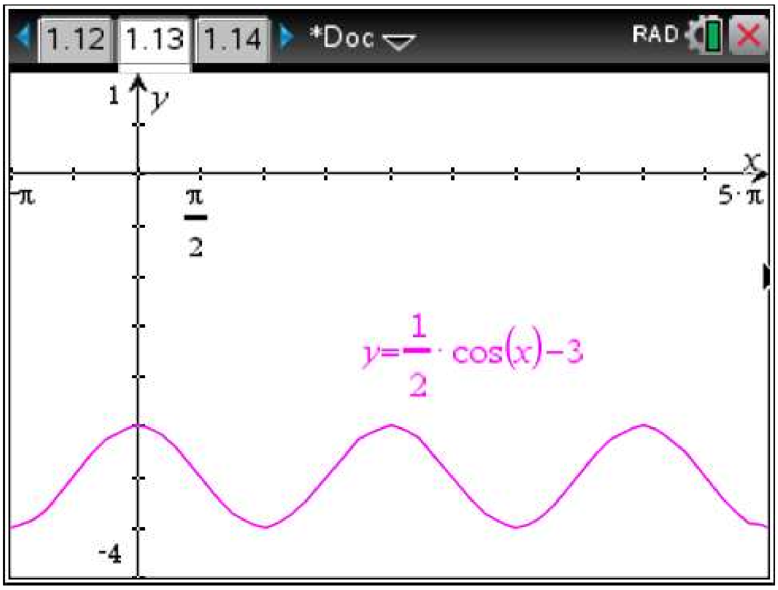
(g) (h) (i)


1.12
1.13
1.14
*Doc
RAD
4
y
$y=\tan\left(x+\frac{\pi}{2}\right)$
$y=\sin\left(\frac{x}{2}\right)$
0.5
x
-π
π/2
3·π
$y=\cos\left(2\cdot x-\frac{\pi}{4}\right)$
-4

2. Detailed graphs in book.

(a)

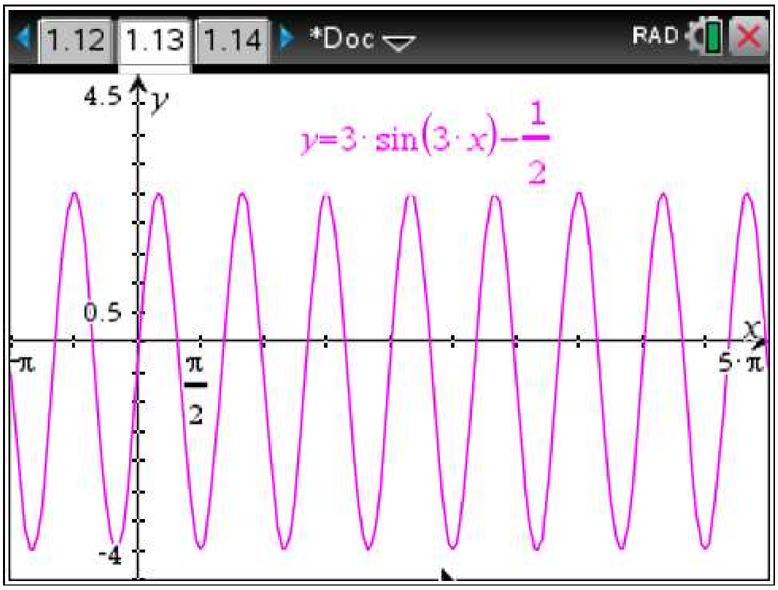
i. $A=\frac{1}{2}, T=2\pi$



ii. domain $=\mathbb{R}$, range $=[-3.5,-2.5]$

(b)

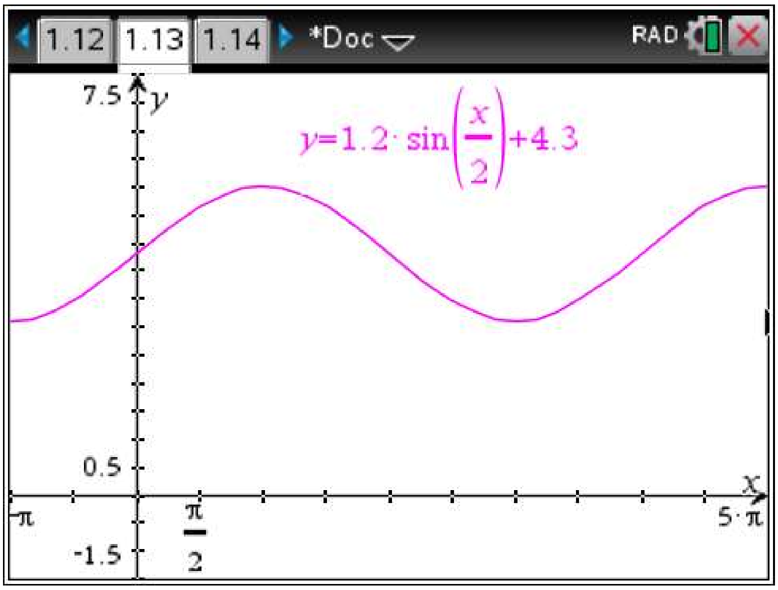
i. $A=3, T=\frac{2\pi}{3}$



ii. domain $=\mathbb{R}$, range $=[-3.5,2.5]$

(c)

i. $A=1.2, T=4\pi$



ii. $\text{domain}=\mathbb{R}$, $\text{range}=[4.3-1.2, 4.3+1.2]=[3.1, 5.5]$

3. The meaning of the parameters A and B in the equation of the trigonometric function is $A=$ amplitude and $B=$ vertical translation (or $y=B$ is the equation of the principal axis).

Therefore, we have $A=\frac{y_{\max}-y_{\min}}{2}$ and $B=\frac{y_{\max}+y_{\min}}{2}$, where $y_{\max}$ and $y_{\min}$ are the y-coordinates of the maximum and minimum points on the curve, respectively. So, we have:

(a) $A=\frac{10-4}{2}=3$ and $B=\frac{10+4}{2}=7$

(b) $A=\frac{8.6-3.2}{2}=2.7$ and $B=\frac{8.6+3.2}{2}=5.9$

4. The meaning of the parameters A and B in the equation of the trigonometric function is $A=$ amplitude and $B=$ vertical translation (or $y=B$ is the equation of the principal axis).

$A=\frac{y_{\max}-y_{\min}}{2}$ and $B=\frac{y_{\max}+y_{\min}}{2}$, where $y_{\max}$ and $y_{\min}$ are the y-coordinates of the maximum and minimum points on the curve, respectively. So, we have $A=\frac{6.2-2.4}{2}=1.9$ and

$B=\frac{6.2+2.4}{2}=4.3$

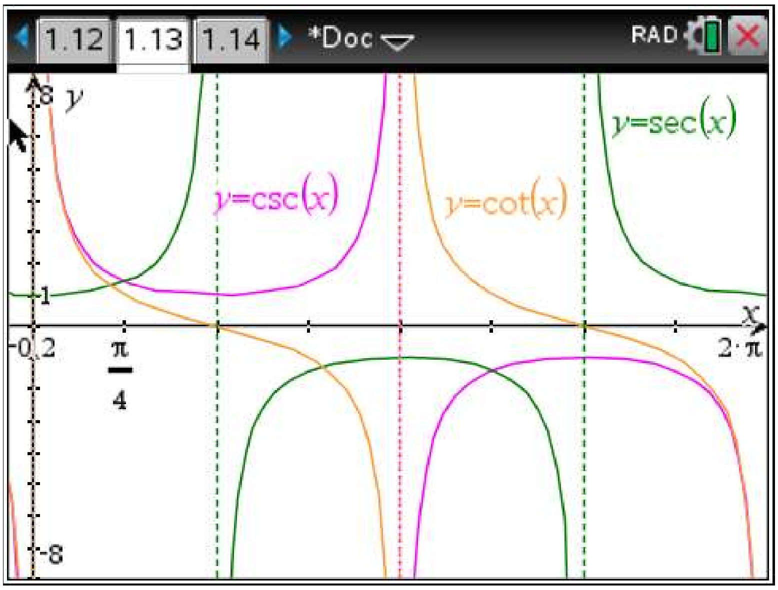
5.

(a) The parameter p is the amplitude of the trigonometric function, so from the graph we have $p=8$.

(b) The parameter q relates to the period T of the trigonometric function as $T=\frac{2\pi}{q}$. From the graph we have $T=\frac{\pi}{3}$ since there are three full oscillations from $x=0$ to $x=\pi$. This gives us $q=\frac{2\pi}{T}=\frac{2\pi}{\frac{\pi}{3}}=6$.

6. Detailed graphs in book.

(a)



(b) For the secant function, domain $=\left\{x\in\mathbb{R}, x\neq(2k+1)\frac{\pi}{2}, k\in\mathbb{Z}\right\}$ since there are vertical asymptotes at all odd multiples of $\frac{\pi}{2}$ (in fact, $\cos\left((2k+1)\frac{\pi}{2}\right)=0, k\in\mathbb{Z}$) and range $=\left]-\infty,-1\right]\cup\left[1,+\infty\right[$.

For the cosecant function, domain $=\{x\in\mathbb{R}, x\neq k\pi, k\in\mathbb{Z}\}$ since there are vertical asymptotes at all multiples of π (in fact, $\sin(k\pi)=0, k\in\mathbb{Z}$) and range $=\left]-\infty,-1\right]\cup\left[1,+\infty\right[$.

For the cotangent function, domain $=\{x\in\mathbb{R}, x\neq k\pi, k\in\mathbb{Z}\}$ since there are vertical asymptotes at all multiples of π (in fact, $\sin(k\pi)=0, k\in\mathbb{Z}$) and range $=\mathbb{R}$.

7.

(a) The meaning of the parameters a and c in the equation of the trigonometric function is $a =$ amplitude and $c =$ vertical translation (or $y = c$ is the equation of the principal axis).

Therefore, we have $a = \frac{y_{\max} - y_{\min}}{2}$ and $c = \frac{y_{\max} + y_{\min}}{2}$, where $y_{\max}$ and $y_{\min}$ are the y-coordinates of the maximum and minimum points on the curve, respectively. So, we have $a = \frac{1-(-3)}{2} = 2$ and $c = \frac{1+(-3)}{2} = -1$. The parameter b relates to the period T of the trigonometric function as $T = \frac{2\pi}{b}$. From the graph we have $T = \frac{2\pi}{3}$. This gives us $b = \frac{2\pi}{T} = \frac{2\pi}{\frac{2\pi}{3}} = 3$.

(b) Setting up the equation $y = 0$, we have $2\sin 3x - 1 = 0 \rightarrow \sin 3x = \frac{1}{2}$. This gives $3x = \sin^{-1}\frac{1}{2} + 2k\pi$ or $3x = \pi - \sin^{-1}\frac{1}{2} + 2k\pi$, $k \in \mathbb{Z}$. Finally, we have $x = \frac{1}{3}\left(\frac{\pi}{6} + 2k\pi\right) = \frac{\pi}{18} + \frac{2k\pi}{3}$ or $x = \frac{1}{3}\left(\pi - \frac{\pi}{6} + 2k\pi\right) = \frac{5\pi}{18} + \frac{2k\pi}{3}$.

Point P lies between $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so its x-coordinate is $\frac{5\pi}{18}$.

8. The meaning of the parameters a and c in the equation of the trigonometric function is $a =$ amplitude and $c =$ vertical translation (or $y = c$ is the equation of the principal axis).

Therefore, we have $a = \frac{y_{\max} - y_{\min}}{2}$ and $c = \frac{y_{\max} + y_{\min}}{2}$, where $y_{\max}$ and $y_{\min}$ are the y-coordinates of the maximum and minimum points on the curve, respectively. So, we have $a = \frac{2-(-4)}{2} = 3$ and $c = \frac{2+(-4)}{2} = -1$. The parameter b relates to a horizontal shift, and it is most easily found by requesting that $y\left(\frac{3\pi}{3}\right) = 2$. This gives $3\sin\left(\frac{3\pi}{4} + b\right) - 1 = 2$, or $\sin\left(\frac{3\pi}{4} + b\right) = 1$, from which derive $\frac{3\pi}{4} + b = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$.

Solving for b we have $b = \frac{\pi}{2} - \frac{3\pi}{4} + 2k\pi = -\frac{\pi}{4} + 2k\pi$: there are multiple values for b since horizontal shifts of a multiple of 2π do not affect the graph of the trigonometric function. Possible values for b could come from $k = 0$ $\left(b = -\frac{\pi}{4}\right)$, $k = 1$ $\left(b = \frac{7\pi}{4}\right)$, etc.

Exercise 6.4

1. In this question we only need to consider solutions between 0 and 2π, therefore we will write down the general solution and then limit ourselves to the given domain. This is especially necessary for equations involving the inverse sine and the inverse tangent, since the range of these two functions includes negative angles. In all answers below, $k \in \mathbb{Z}$.

(a) $\cos x=\frac{1}{2}\rightarrow x=\cos^{-1}\frac{1}{2}+2k\pi$ or $x=2\pi-\cos^{-1}\frac{1}{2}+2k\pi$, giving $x=\frac{\pi}{3}$ or $x=\frac{5\pi}{3}$

(b) $2\sin x+1=0\rightarrow\sin x=-\frac{1}{2}\rightarrow x=\sin^{-1}\left(-\frac{1}{2}\right)+2k\pi$ or $x=\pi-\sin^{-1}\left(-\frac{1}{2}\right)+2k\pi$, giving $x=-\frac{\pi}{6}+2k\pi$ or $x=\frac{7\pi}{6}+2k\pi$. Solutions in the given domain are therefore $x=\frac{11\pi}{6}$ from $k=1$ and $x=\frac{7\pi}{6}$ from $k=0$

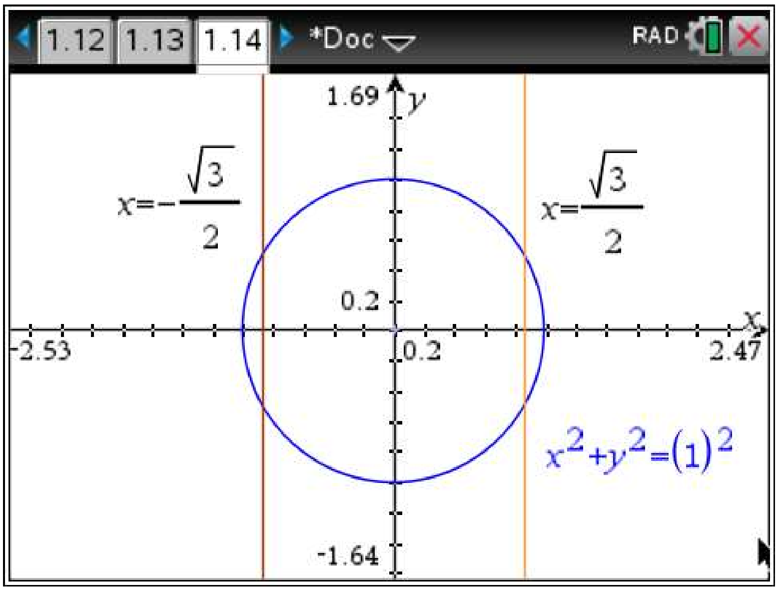
(c) $1-\cot x=0\rightarrow\cot x=1\rightarrow\tan x=1$, giving $x=\tan^{-1}1+k\pi$. Solutions in the given range are $x=\frac{\pi}{4}$ or $x=\frac{5\pi}{4}$

(d) $\sqrt{3}=2\sin x\rightarrow\sin x=\frac{\sqrt{3}}{2}$, giving $x=\sin^{-1}\frac{\sqrt{3}}{2}+2k\pi$ or $x=\pi-\sin^{-1}\frac{\sqrt{3}}{2}+2k\pi$. Solutions in the given domain are $x=\frac{\pi}{3}(k=0)$ or $x=\frac{2\pi}{3}(k=0)$

(e) $2\sin^2x=1\rightarrow\sin^2x=\frac{1}{2}\rightarrow\sin x=\pm\frac{\sqrt{2}}{2}$, giving $x=\sin^{-1}\frac{\sqrt{2}}{2}+2k\pi$, $x=\pi-\sin^{-1}\frac{\sqrt{2}}{2}+2k\pi$, $x=\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)+2k\pi$ or $x=\pi-\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)+2k\pi$. Solutions in the given domain are $x=\frac{\pi}{4}(k=0)$, $x=\frac{3\pi}{4}(k=0)$, $x=\frac{7\pi}{4}(k=1)$ or $x=\frac{5\pi}{4}(k=1)$

(f) As an alternative method, we solve the next equation using the unit circle.

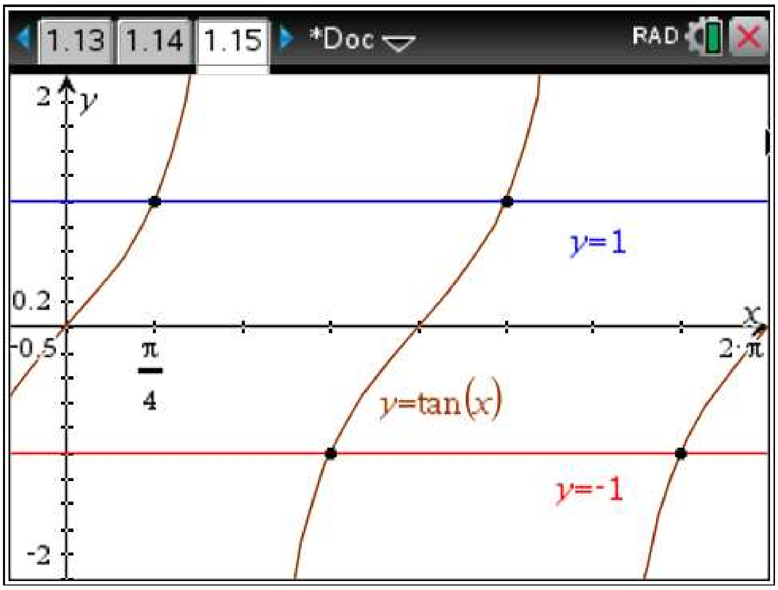
$4\cos^2\theta=3\rightarrow\cos\theta=\pm\frac{\sqrt{3}}{2}$, giving the graph below where $x=\cos\theta$.



The four intersections between the unit circle and $x=\cos\theta=\pm\frac{\sqrt{3}}{2}$ give the solutions in the given domain, $\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

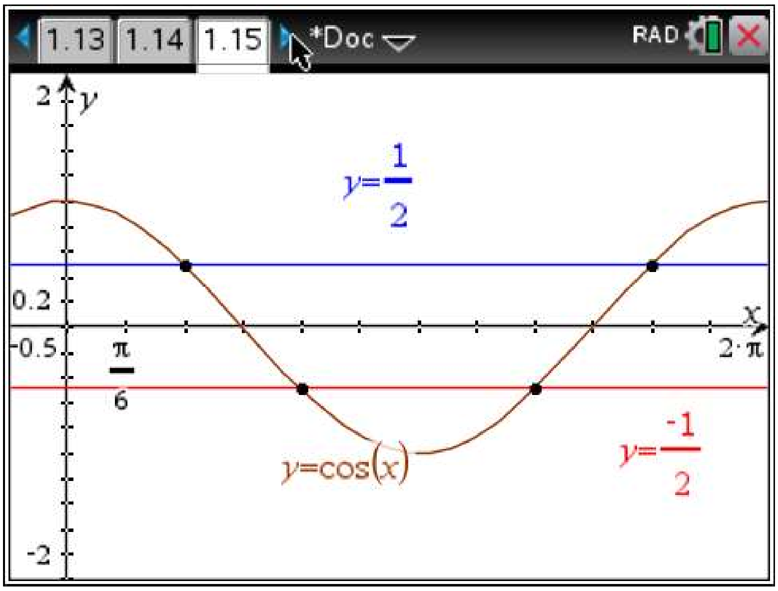
(g) As a further method, we solve the next equation using the graph of the function $f(x) = \tan x$ and its symmetries.

$\tan^2 x - 1 = 0 \rightarrow \tan^2 x = 1 \rightarrow \tan x = \pm 1$, giving the graph below:



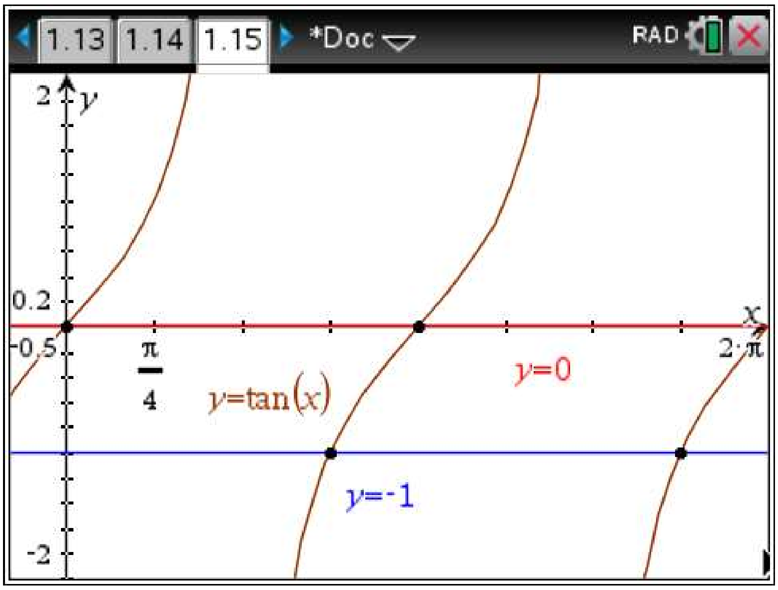
The four intersections between the graphs of $y = \tan x$ and $y = \pm 1$ give the solutions in the given domain, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(h) $4\cos^2 x = 1 \rightarrow \cos^2 x = \frac{1}{4} \rightarrow \cos x = \pm\frac{1}{2}$, giving the graph below:



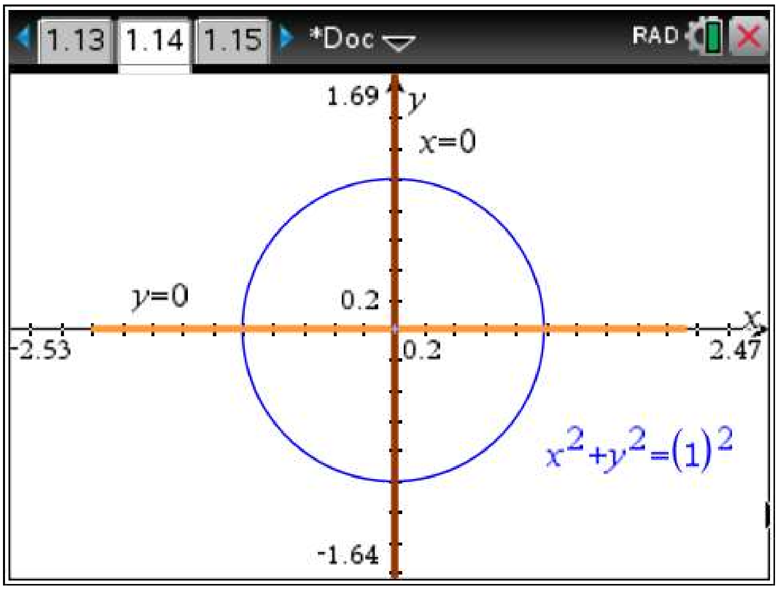
The four intersections between the graphs of $y = \cos x$ and $y = \pm\frac{1}{2}$ give the solutions in the given domain, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(i) $\tan x(\tan x + 1) = 0 \rightarrow \tan x = 0$ or $\tan x + 1 = 0 \rightarrow \tan x = -1$, giving the graph below:



The four intersections between the graphs of $y = \tan x$ and $y = 0, y = -1$ give the solutions in the given domain, $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$

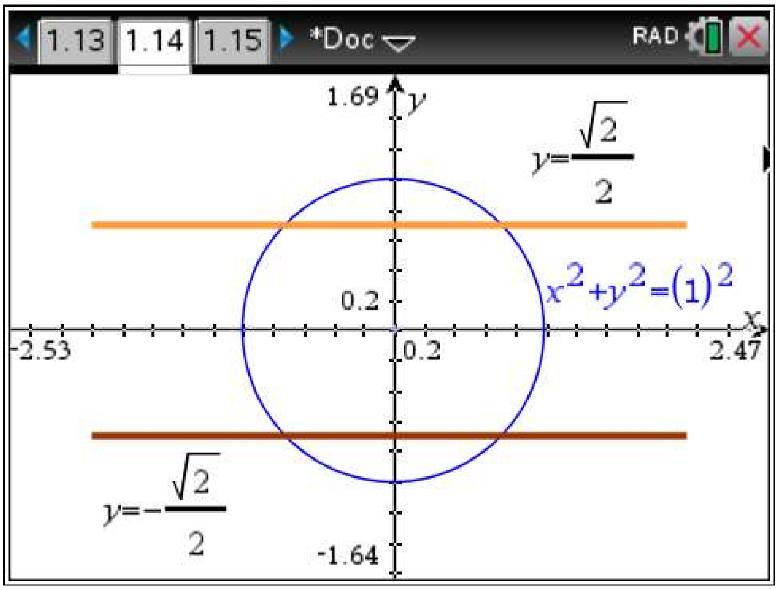
(j) $\sin\theta\cos\theta = 0 \rightarrow \sin\theta = 0$ or $\cos\theta = 0$. Using the unit circle with $x = \cos\theta$ and $y = \sin\theta$ we obtain the following diagram:



The two intersections between the unit circle and $x = \cos\theta = 0$ give $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while the two intersections between the unit circle and $y = \sin\theta = 0$ give $\theta = 0, \pi$

(k) $5 - \sec x = 3 \rightarrow \sec x = 2 \rightarrow \cos x = \frac{1}{2}$. Using any of the methods above, we obtain $x = \frac{\pi}{3}, \frac{5\pi}{3}$

(l) $\csc^2\theta = 2 \rightarrow \csc\theta = \pm\sqrt{2} \rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. Using the unit circle with $y = \sin\theta$ we obtain the following diagram:



The four intersections between the unit circle and $y = \sin\theta = \pm\frac{\sqrt{2}}{2}$ give the solutions in the given domain, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

2.

(a) $x_1 = \sin^{-1} 0.4 \approx 0.412$

$x_2 = \pi - \sin^{-1} 0.4 \approx 2.73$ since angles that add up to π have equal sine values

(b) $3\cos x + 1 = 0 \rightarrow \cos x = -\frac{1}{3}$

$x_1 = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91$

$x_2 = 2\pi - \cos^{-1}\left(-\frac{1}{3}\right) \approx 4.37$ since angles that add up to 2π have equal cosine values

(c) $x_1 = \tan^{-1} 2 \approx 1.11$

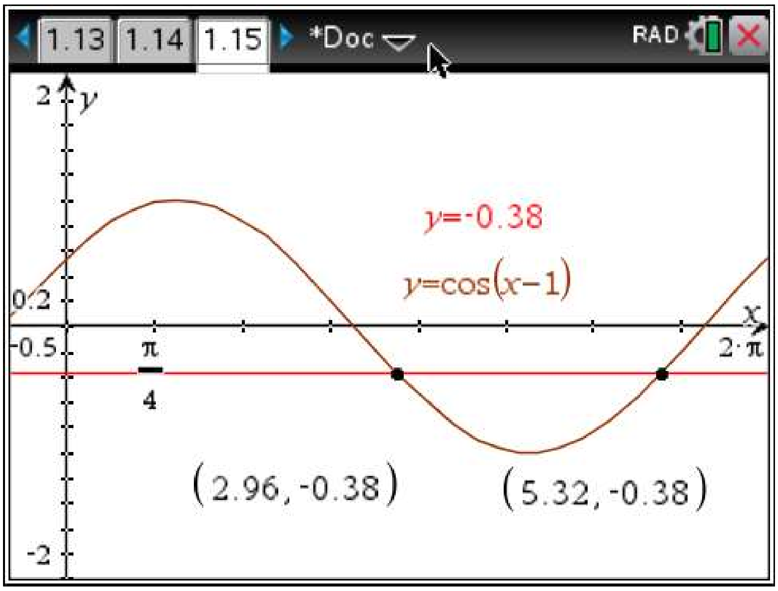
$x_2 = \pi + \tan^{-1} 2 \approx 4.25$ since the period of the tangent function is π

(d) $\sec 2x = 3.46 \rightarrow \frac{1}{\cos 2x} = 3.46 \rightarrow \cos 2x = \frac{1}{3.46}$

$2x = \cos^{-1}\frac{1}{3.46} + 2k\pi, k \in \mathbb{Z} \rightarrow 2x \approx 1.28, 7.56, \ldots$ where we wrote explicitly only solutions for $k = 0,1$ that when divided by two give values in the given domain $0 \le x < 2\pi$. So, $x_{1,2} \approx 0.639, 3.78$.

$2x = 2\pi - \cos^{-1}\frac{1}{3.46} + 2k\pi, k \in \mathbb{Z} \rightarrow 2x \approx 5.01, 11.29, \ldots$ where we wrote explicitly only solutions for $k = 0,1$ that when divided by two give values in the given domain $0 \le x < 2\pi$. So, $x_{3,4} \approx 2.50, 5.64$.

(e) Graphing $y = \cos(x-1)$ and $y = -0.38$ and looking for intersections in the given domain gives $x = 2.96, 5.32$ as shown below.



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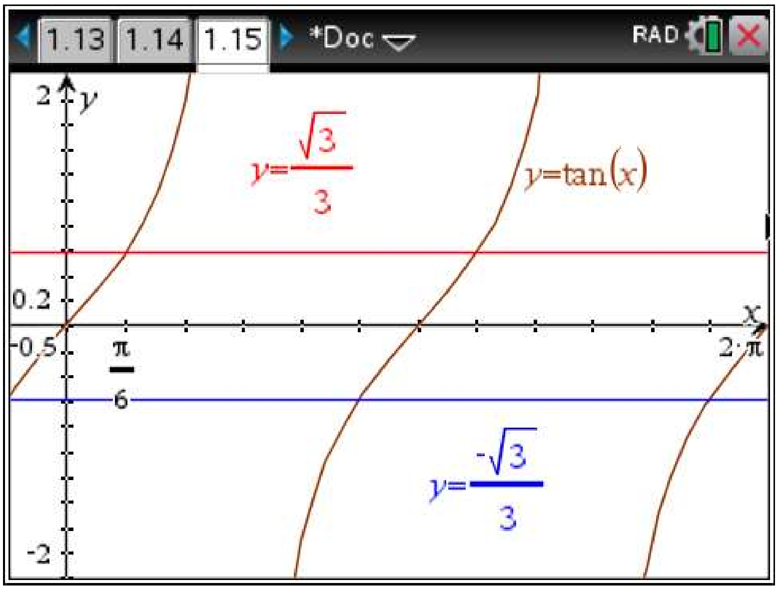
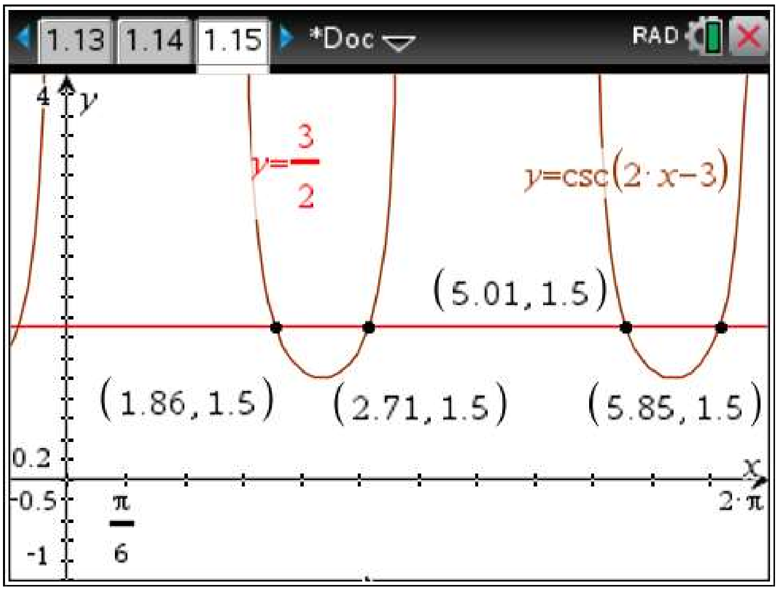
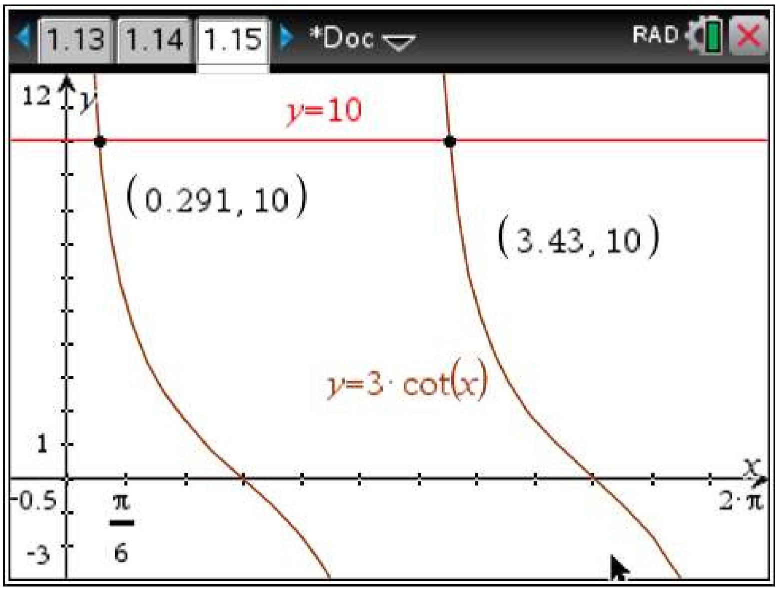
(f) $3\tan^2 x = 1 \to \tan x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$. Graphing $y = \tan x$ and $y = \pm\frac{\sqrt{3}}{3}$ and looking for intersections with the help of symmetries gives $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ as shown below.



(g) Graphing both hand sides of $\csc(2x-3) = \frac{3}{2}$ separately and looking for intersections gives $x = 1.86, 2.71, 5.01, 5.85$ as shown below.



(h) Graphing $y = 3\cot x$ and $y = 10$ and looking for intersections gives $x = 0.291, 3.43$ as shown below.



3.

(a) Here we look for integer values of k that make $x = \frac{\pi}{2} + k\pi$ larger than -3π and smaller than 3π. Since $0 < \frac{\pi}{2} < \pi$, we can start from $k = -3$ but we have to stop before $k = 3$. This gives $k = -3, -2, -1, 0, 1, 2$ and correspondingly $x = \frac{\pi}{2} - 3\pi, \frac{\pi}{2} - 2\pi, \frac{\pi}{2} - \pi, \frac{\pi}{2}, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 2\pi$ or explicitly $x = -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

(b) As an alternative, we can solve the inequality $-2\pi \le \frac{\pi}{6} + 2k\pi \le 2\pi$ for k. This gives

$$-2\pi \le \frac{\pi}{6} + 2k\pi \le 2\pi$$
$$-2\pi - \frac{\pi}{6} \le 2k\pi \le 2\pi - \frac{\pi}{6}$$
$$-\frac{13\pi}{6} \le 2k\pi \le \frac{11\pi}{6}$$
$$-\frac{13\pi}{6} \cdot \frac{1}{2\pi} \le k \le \frac{11\pi}{6} \cdot \frac{1}{2\pi}$$

$-\frac{13}{12} \le k \le \frac{11}{12}$, and the integers that satisfy this inequality are $k = -1, 0$.

Correspondingly, we have $x = \frac{\pi}{6} - 2\pi, \frac{\pi}{6}$ so $x = -\frac{11\pi}{6}, \frac{\pi}{6}$

(c) Here, $0 < \frac{7\pi}{12} < \pi$, and since we are adding multiples of π, the only acceptable values are $k = 0, 1$, and correspondingly $x = \frac{7\pi}{12}, \frac{7\pi}{12} + \pi$. Finally, $x = \frac{7\pi}{12}, \frac{19\pi}{12}$

(d) As a last method, we can simply enter integer values of *k*, as long as the condition $0 \le x < 4\pi$ is met. The first such value is $k = -1$, which gives $x = 0$, and from that value we keep adding $\frac{\pi}{4}$, obtaining $x = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4}, \frac{8\pi}{4}, \frac{9\pi}{4}, \frac{10\pi}{4}, \frac{11\pi}{4}, \frac{12\pi}{4}, \frac{13\pi}{4}, \frac{14\pi}{4}, \frac{15\pi}{4}$

which simplifies to $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} 2\pi, \frac{9\pi}{4}, \frac{5\pi}{2}, \frac{11\pi}{4}, 3\pi, \frac{13\pi}{4}, \frac{7\pi}{2}, \frac{15\pi}{4}$

4.

(a) $\cos\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$. When solving, we have to remember that angles that add up to 2π have equal cosine values, so we have two families of solutions:

$x - \frac{\pi}{6} = \cos^{-1}\left(-\frac{1}{2}\right) + 2k\pi$ or $x - \frac{\pi}{6} = 2\pi - \cos^{-1}\left(-\frac{1}{2}\right) + 2k\pi$, so

$x - \frac{\pi}{6} = -\frac{\pi}{3} + 2k\pi$ or $x - \frac{\pi}{6} = 2\pi - \left(-\frac{\pi}{3}\right) + 2k\pi$

$x = -\frac{\pi}{3} + \frac{\pi}{6} + 2k\pi$ or $x = 2\pi + \frac{\pi}{3} + \frac{\pi}{6} + 2k\pi$

$x = -\frac{\pi}{6} + 2k\pi$ or $x = \frac{\pi}{2} + (2k+1)\pi$

We now have to choose the values of $k \in \mathbb{Z}$ that satisfy $0 \le x < 2\pi$. We have $x = \frac{5\pi}{6}$ (from $k = 1$) or $x = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$ (from $k = 0$).

(b) $\tan(\theta + \pi) = 1$

$$\theta + \pi = \tan^{-1} 1$$
$$\theta = \tan^{-1} 1 - \pi$$

$\theta=\frac{\pi}{4}+k\pi-\pi$. We now have to choose the values of $k\in\mathbb{Z}$ that satisfy $-\pi\le\theta<\pi$.

We have $\theta=-\frac{3\pi}{4}$ (from $k=0$) or $\theta=\frac{\pi}{4}$ (from $k=1$).

(c) $\sin 2x=\frac{\sqrt{3}}{2}$. When solving, we have to remember that angles that add up to $180°$ have equal sine values, so we have two families of solutions:

$2x=\sin^{-1}\frac{\sqrt{3}}{2}+k\cdot 360°$ or $2x=180°-\sin^{-1}\frac{\sqrt{3}}{2}+k\cdot 360°$, so

$2x=60°+k\cdot 360°$ or $2x=180°-60^{\circ}+k\cdot 360°$

$x=30°+k\cdot 180°$ or $x=60°+k\cdot 180°$. We now have to choose the values of $k\in\mathbb{Z}$ that satisfy $0\le x<360°$. We have $x=30°,210°$ (from $k=0,1$) or $x=60°,240°$ (from $k=0,1$).

(d) $\sin^2\left(\alpha+\frac{\pi}{2}\right)=\frac{3}{4}$

$\sin\left(\alpha+\frac{\pi}{2}\right)=\pm\frac{\sqrt{3}}{2}$. When solving, we have to remember that angles that add up to π have equal sine values, so we have two families of solutions:

$\alpha+\frac{\pi}{2}=\sin^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)+2k\pi$ or $\alpha+\frac{\pi}{2}=\pi-\sin^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)+2k\pi$, so

$\alpha=-\frac{\pi}{2}\pm\frac{\pi}{3}+2k\pi$ or $\alpha=\frac{\pi}{2}\pm\frac{\pi}{3}+2k\pi$. We now have to choose the values of $k\in\mathbb{Z}$ that satisfy $-\frac{\pi}{2}\le\theta<\frac{\pi}{2}$.

$\alpha=-\frac{\pi}{2}+\frac{\pi}{3}=-\frac{\pi}{6}$ from $k=0$, or $\alpha=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$ from $k=0$.

(e) $2\cos^2\theta-5\cos\theta-3=0$. Setting $y=\cos\theta$, this becomes $2y^2-5y-3=0$, giving

$y=\frac{5\pm\sqrt{25+24}}{4}=\frac{5\pm 7}{4}=3$ or $-\frac{1}{2}$

This gives two equations, $\cos\theta=3$ and $\cos\theta=-\frac{1}{2}$. The former has no solutions, since 3 is outside the range of the cosine function, while the latter gives $\theta=\frac{2\pi}{3}$ or $\theta=\frac{4\pi}{3}$.

(f) $3\tan x=2\cos x$. First of all, we observe that in order for this equation to be satisfied, the angle x has to be either in the first quadrant, where both tangent and cosine are positive, or in the second quadrant, where both are negative. Recalling the definition of tangent, we have

$3\times\frac{\sin x}{\cos x}=2\cos x$. Using Pythagoras, $\sin^2 x=1-\cos^2 x$, we have

$3\times\frac{\pm\sqrt{1-\cos^2 x}}{\cos x}=2\cos x$

$\pm 3\sqrt{1-\cos^2 x}=2\cos^2 x$. Setting $t=\cos^2 x$, we have $\pm 3\sqrt{1-t}=2t$.

Squaring both sides gives $9(1-t)=4t^2$

$4t^2+9t-9=0$ or $t=\frac{-9\pm\sqrt{81+144}}{8}=\frac{-9\pm 15}{8}=-3$ or $\frac{3}{4}$, but only $\frac{3}{4}$ is an acceptable solution since $t = \cos^2 x > 0$. This gives $\cos^2 x = \frac{3}{4} \to \cos x = \pm\frac{\sqrt{3}}{2}$. The solutions to these two equations are $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$ from $\cos x = \frac{\sqrt{3}}{2}$, and $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$ from $\cos x = -\frac{\sqrt{3}}{2}$, but recalling that the angle x must be either in the first or in the second quadrant we have $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

(g) $2\cos(3x + 24°) = \sqrt{2}$

$\cos(3x + 24°) = \frac{\sqrt{2}}{2}$, so that $3x + 24° = 45° + k \cdot 360°$ or $3x + 24° = 315° + k \cdot 360°$.
Solving for x gives $3x = 21° + k \cdot 360°$ or $3x = 291° + k \cdot 360°$, and finally
or $x = 97° + k \cdot 120°$. The solutions in the domain $0° \le x < 360°$ are $x = 7°, 127°, 247°$
or $x = 97°, 217°, 337°$

(h) $9\sec^2\theta = 12$

$\sec^2\theta = \frac{12}{9} = \frac{4}{3}$

$\sec\theta = \pm\frac{2}{\sqrt{3}}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$, whose solutions are $x = \frac{\pi}{6}, \frac{11\pi}{6}$ from $\cos x = \frac{\sqrt{3}}{2}$ and $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$ from $\cos x = -\frac{\sqrt{3}}{2}$. The solutions in the domain $0 \le \theta \le \pi$ are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

5. The condition in the question amounts to the equation $N(t) = 90$, giving $74 + 42\sin\left(\frac{\pi}{12}t\right) = 90$.

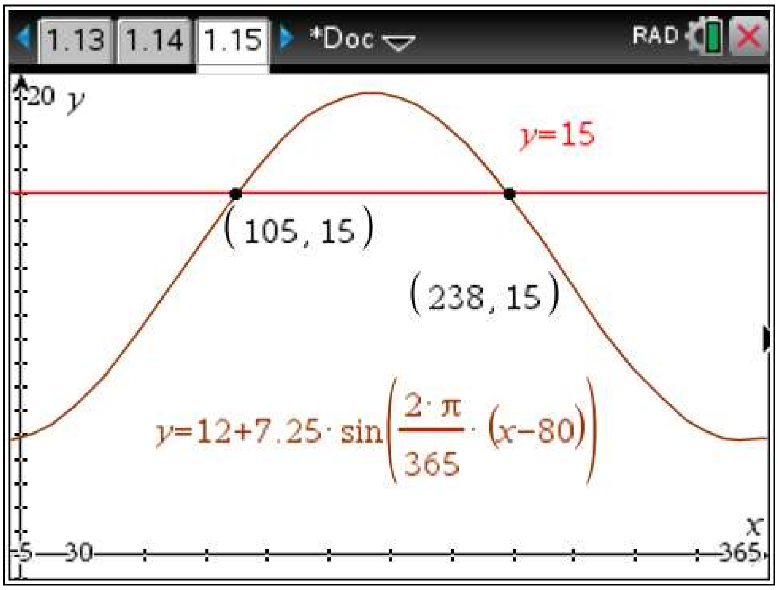
$42\sin\left(\frac{\pi}{12}t\right) = 16$

$\sin\frac{\pi}{12}t = \frac{16}{42} = \frac{8}{21}$. Since we are looking for the first time when $N = 90$, we consider the smallest solution $\frac{\pi}{12}t = \sin^{-1}\frac{8}{21}$ giving $t = \frac{12}{\pi}\sin^{-1}\frac{8}{21} \approx 1.49 \approx 1.5$ hours (assuming the argument of the sine function is in radians).

6.

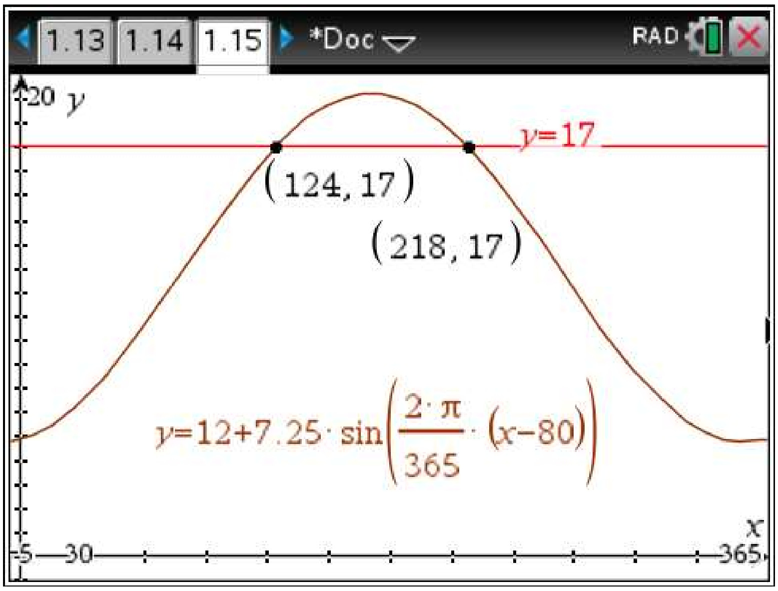
(a) This happens when $H = 12$, so when $\sin\left(\frac{2\pi}{365}(D - 80)\right) = 0$. A sine function equals zero when its argument is equal to integer multiples of π, so we have $\frac{2\pi}{365}(D - 80) = 0$, which gives $D = 80$, or $\frac{2\pi}{365}(D - 80) = \pi$, which gives $D - 80 = \frac{365}{2} = 182.5 \to D = 262.5$.
The days are therefore the 80th and, approximately, the 263rd of the year, so March 21st and September 20th.

(b) Setting up the equation $H=15$ and solving $12+7.26\sin\left(\frac{2\pi}{365}(D-80)\right)=15$ gives:



$D=105$ and $D=238$, so April 15th and August 26th.

(c) Requesting that $H>17$ and graphing the solutions to $H=17$ produces the following graph:



from which the number of days with more than 17 hours of daylight is $218-124=94$

7.

(a) $2\cos^2 x+\cos x=0$

$\cos x(2\cos x+1)=0\rightarrow\cos x=0$ or $2\cos x+1=0\rightarrow\cos x=-\frac{1}{2}$. In the given domain, these two equations have solutions $x=\frac{\pi}{2},\frac{3\pi}{2}$ or $x=\frac{2\pi}{3},\frac{4\pi}{3}$.

(b) $2\sin^2\theta-\sin\theta-1=0$. Solving for the variable $\sin\theta$ gives $\sin\theta=\frac{1\pm\sqrt{1+8}}{4}=\frac{1\pm3}{4}=1$ or $-\frac{1}{2}$.

In the given domain, these two equations have solutions $\theta=\frac{\pi}{2}$ or $\theta=\frac{7\pi}{6},\frac{11\pi}{6}$.

(c) $\tan^2 x-\tan x=2\rightarrow\tan^2 x-\tan x-2=0$. Solving for the variable $\tan x$ gives $\tan x=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm3}{2}=2$ or -1. In the given domain, these two equations have solutions $x=\tan^{-1}2\approx63.4°$ or $x=-45°$

(d) $3\cos^2 x-6\cos x=2\rightarrow3\cos^2 x-6\cos x-2=0$. Solving for the variable $\cos x$ gives $\cos x=\frac{6\pm\sqrt{36+24}}{6}=\frac{6\pm2\sqrt{15}}{6}=1\pm\frac{\sqrt{15}}{3}$. Since $1+\frac{\sqrt{15}}{3}>1$, the equation $\cos x=1+\frac{\sqrt{15}}{3}$ has no solutions. In the given domain, the remaining equation $\cos x=1-\frac{\sqrt{15}}{3}$ has solutions $x=\cos^{-1}\left(1-\frac{\sqrt{15}}{3}\right)\approx1.87$ or $x=-\cos^{-1}\left(1-\frac{\sqrt{15}}{3}\right)\approx-1.87$

(e) $2\sin\beta=3\cos\beta\rightarrow\frac{\sin\beta}{\cos\beta}=\frac{3}{2}\rightarrow\tan\beta=\frac{3}{2}$ (provided $\cos\beta\neq0$). Solving for β gives $\beta=\tan^{-1}\frac{3}{2}\approx56.3°$. In the given domain this is the only solution.

(f) $\sin^2 x=\cos^2 x\rightarrow\frac{\sin^2 x}{\cos^2 x}=1\rightarrow\tan^2 x=1\rightarrow\tan x=\pm1$ (provided $\cos x\neq0$). Solving $\tan x=\pm1$ in the given domain gives $x=\frac{\pi}{4},\frac{3\pi}{4}$

(g) $\sec^2 x+2\sec x+4=0$. Solving for the variable $\sec x$ gives $\sec x=\frac{-2\pm\sqrt{4-16}}{2}$, therefore there are no real solutions.

(h) $\sin x\tan x=3\sin x$. Here it is mandatory NOT to DIVIDE both hand sides by $\sin x$, as this would lose us solutions. Instead, we FACTOR $\sin x$ and apply the null factor law, obtaining $\sin x(\tan x-3)=0\rightarrow\sin x=0$ or $\tan x-3=0\rightarrow\tan x=3$. Solving these two equations in the given domain gives $x=0°,180°$ or $x=\tan^{-1}3\approx71.6°$ or $x=180°+\tan^{-1}3\approx251.6°\approx252°$.

Exercise 6.5

1. In all of the following exercises, the key is expressing the given angle as a sum or difference of angles whose sine and cosine are known.

(a) $\cos\frac{7\pi}{12}=\cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}=\frac{\sqrt{2}-\sqrt{6}}{4}$

(b) $\sin165°=\sin(135°+30°)=\sin135°\cos30°+\sin30°\cos135°=\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$

(c) $\tan\frac{\pi}{12}=\tan\left(\frac{4\pi}{12}-\frac{3\pi}{12}\right)=\tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\dfrac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}=\dfrac{\sqrt{3}-1}{1+\sqrt{3}\cdot 1}=\dfrac{\sqrt{3}-1}{1+\sqrt{3}}$

$=\dfrac{\sqrt{3}-1}{1+\sqrt{3}}\cdot\dfrac{1-\sqrt{3}}{1-\sqrt{3}}=\dfrac{\sqrt{3}-1-3+\sqrt{3}}{1-3}=\dfrac{2\sqrt{3}-4}{-2}=2-\sqrt{3}$

(d) $\sin\left(-\frac{5\pi}{12}\right)=\sin\left(\frac{7\pi}{12}-\frac{12\pi}{12}\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{4}-\pi\right)=\sin\left(\frac{\pi}{4}-\frac{2\pi}{3}\right)$

$=\sin\frac{\pi}{4}\cos\frac{2\pi}{3}-\sin\frac{2\pi}{3}\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}=\frac{-\sqrt{2}-\sqrt{6}}{4}$

(e) $\cos 255°=\cos(225°+30°)=\cos 225°\cos 30°-\sin 225°\sin 30°=-\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\frac{1}{2}$

$=\frac{-\sqrt{6}+\sqrt{2}}{4}=\frac{\sqrt{2}-\sqrt{6}}{4}$

(f) $\cot 75°=\dfrac{1}{\tan 75°}=\dfrac{1}{\tan(30°+45°)}=\dfrac{1-\tan 30°\tan 45°}{\tan 30°+\tan 45°}=\dfrac{1-\frac{\sqrt{3}}{3}\cdot 1}{\frac{\sqrt{3}}{3}+1}$

$=\dfrac{3-\sqrt{3}}{3+\sqrt{3}}=\dfrac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\dfrac{3-\sqrt{3}}{3-\sqrt{3}}=\dfrac{9-6\sqrt{3}+3}{9-3}=\dfrac{12-6\sqrt{3}}{6}=2-\sqrt{3}$

2.

(a) $\cos\left(\frac{\pi}{12}\right)=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}+\frac{1}{2}\frac{\sqrt{2}}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$

(b) $\cos 2x=2\cos^2 x-1$, so letting $2x=\theta$, we have:

$\cos\theta=2\cos^2\frac{\theta}{2}-1\to\cos\theta+1=2\cos^2\frac{\theta}{2}\to\frac{\cos\theta+1}{2}=\cos^2\frac{\theta}{2}$ and finally $\cos\frac{\theta}{2}=\pm\sqrt{\frac{\cos\theta+1}{2}}$.

Applying this to $\theta=\frac{\pi}{12}\to\frac{\theta}{2}=\frac{\pi}{24}$ we have

$\cos\frac{\pi}{24}=\sqrt{\dfrac{\cos\frac{\pi}{12}+1}{2}}=\sqrt{\dfrac{\frac{\sqrt{6}+\sqrt{2}}{4}+1}{2}}=\sqrt{\dfrac{\frac{\sqrt{6}+\sqrt{2}+4}{4}}{2}}=\sqrt{\dfrac{\sqrt{6}+\sqrt{2}+4}{8}}$, where we have chosen the positive sign since $\frac{\pi}{24}$ is in the first quadrant where the cosine is positive.

3.

(a) $\tan\left(\frac{\pi}{2}-\theta\right)=\dfrac{\tan\frac{\pi}{2}-\tan\theta}{1+\tan\frac{\pi}{2}\tan\theta}$. Since $\tan\frac{\pi}{2}$ is undefined, we consider an angle α instead, and consider what happens when $\alpha\to\frac{\pi}{2}$. $\tan(\alpha-\theta)=\dfrac{\tan\alpha-\tan\theta}{1+\tan\alpha\tan\theta}=\dfrac{1-\frac{\tan\theta}{\tan\alpha}}{\frac{1}{\tan\alpha}+\tan\theta}$, where we have divided both numerator and denominator by $\tan\alpha$. If we now let $\alpha\to\frac{\pi}{2}$, we have

$\tan\alpha \to \infty$, so

$$\tan(\alpha-\theta) \to \frac{1-0}{0+\tan\theta} = \frac{1}{\tan\theta} = \cot\theta$$

The same identity can be proved more directly using symmetry identities:

$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{\sin\frac{\pi}{2}\cos\theta - \sin\theta\cos\frac{\pi}{2}}{\cos\frac{\pi}{2}\cos\theta + \sin\theta\sin\frac{\pi}{2}} = \frac{\cos\theta - 0}{0+\sin\theta} = \cot\theta$$

(b) $\sin\left(\frac{\pi}{2}-\theta\right) = \sin\frac{\pi}{2}\cos\theta - \sin\theta\cos\frac{\pi}{2} = \cos\theta - 0 = \cos\theta$

(c) $\csc\left(\frac{\pi}{2}-\theta\right) = \frac{1}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1}{\sin\frac{\pi}{2}\cos\theta - \sin\theta\cos\frac{\pi}{2}} = \frac{1}{\cos\theta} = \sec\theta$.

4.

(a) $\cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

Since x is in the first quadrant we choose $\cos x = \frac{4}{5}$

(b) $\cos 2x = 1-2\sin^2 x = 1-2\left(\frac{3}{5}\right)^2 = 1-\frac{18}{25} = \frac{7}{25}$

(c) $\sin 2x = 2\sin x\cos x = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$

5.

(a) $\sin x = \pm\sqrt{1-\cos^2 x} = \pm\sqrt{1-\left(-\frac{2}{3}\right)^2} = \pm\sqrt{1-\frac{4}{9}} = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$

Since x is in the second quadrant, we choose $\sin x = \frac{\sqrt{5}}{3}$

(b) $\sin 2x = 2\sin x\cos x = 2\cdot\frac{\sqrt{5}}{3}\cdot\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}$

(c) $\cos 2x = 2\cos^2 x - 1 = 2\left(-\frac{2}{3}\right)^2 - 1 = 2\cdot\frac{4}{9} - 1 = -\frac{1}{9}$

6.

(a) $\sin\theta = \frac{2}{3} \to \cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\frac{\sqrt{5}}{3}$. We then choose $\cos\theta = -\frac{\sqrt{5}}{3}$ since θ is in the second quadrant. We then have:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\left(-\frac{\sqrt{5}}{3}\right) = -\frac{4\sqrt{5}}{9}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{2}{3}\right)^2 = 1 - \frac{8}{9} = \frac{1}{9}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{4\sqrt{5}}{9}}{\frac{1}{9}} = -4\sqrt{5}$$

(b) $\cos\theta = -\frac{4}{5} \rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\left(-\frac{4}{5}\right)^2} = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$.

We choose $\sin\theta = -\frac{3}{5}$ since θ is in the third quadrant. We then have:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(-\frac{3}{5}\right)\cdot\left(-\frac{4}{5}\right) = \frac{24}{25}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{4}{5}\right)^2 - 1 = \frac{32}{25} - 1 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{24}{25}}{\frac{7}{25}} = \frac{24}{7}$$

(c) $\tan\theta = 2 \rightarrow \frac{\sin\theta}{\cos\theta} = 2 \rightarrow \sin\theta = 2\cos\theta$. Together with $\sin^2\theta + \cos^2\theta = 1$, this gives $(2\cos\theta)^2 + \cos^2\theta = 1 \rightarrow \cos^2\theta = \frac{1}{5} \rightarrow \cos\theta = \pm\frac{\sqrt{5}}{5}$. We then chose $\cos\theta = \frac{\sqrt{5}}{5}$ since θ is in the first quadrant, so that $\sin\theta = 2\cos\theta = \frac{2\sqrt{5}}{5}$. We then have:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2\sqrt{5}}{5}\cdot\frac{\sqrt{5}}{5} = \frac{20}{25} = \frac{4}{5}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{\sqrt{5}}{5}\right)^2 - 1 = \frac{10}{25} - 1 = -\frac{15}{25} = -\frac{3}{5}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

(d) $\sec\theta = -4 \rightarrow \cos\theta = -\frac{1}{4}$. If follows that

$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\left(-\frac{1}{4}\right)^2} = \pm\sqrt{1-\frac{1}{16}} = \pm\frac{\sqrt{15}}{4}$. We then choose $\sin\theta = -\frac{\sqrt{15}}{4}$ since $\csc\theta < 0$ implies $\sin\theta = \frac{1}{\csc\theta} < 0$. We then have:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(-\frac{\sqrt{15}}{4}\right)\cdot\left(\frac{1}{4}\right) = -\frac{2\sqrt{15}}{16} = -\frac{\sqrt{15}}{8}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{1}{4}\right)^2 - 1 = \frac{1}{8} - 1 = -\frac{7}{8}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{\sqrt{15}}{8}}{-\frac{7}{8}} = \frac{\sqrt{15}}{7}$$

7.

(a) $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x = -1\cdot\cos x + 0\cdot\sin x = -\cos x$

(b) $\sin\left(x - \frac{\pi}{2}\right) = \sin x\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\cos x = \sin x\cdot 0 - 1\cdot\cos x = -\cos x$

(c) $\tan(x+\pi) = \frac{\tan x + \tan\pi}{1 - \tan x\tan\pi} = \frac{\tan x + 0}{1 - \tan x\cdot 0} = \tan x$

(d) $\cos\left(x + \frac{\pi}{2}\right) = \cos x\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\sin x = \cos x\cdot 0 - 1\cdot\sin x = -\sin x$

8.

(a) $\sec\theta + \sin\theta = \frac{1}{\cos\theta} + \sin\theta = \frac{1 + \sin\theta\cos\theta}{\cos\theta}$

(b) $\frac{\sec\theta\csc\theta}{\tan\theta\sin\theta} = \frac{\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}\cdot\sin\theta} = \frac{1}{\cos\theta\sin\theta}\cdot\frac{\cos\theta}{\sin^2\theta} = \frac{1}{\sin^3\theta}$

(c) $\frac{\sec\theta + \csc\theta}{2} = \frac{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}{2} = \frac{\frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}}{2} = \frac{\sin\theta + \cos\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta\cos\theta}{\sin 2\theta}$

(d) $\frac{1}{\cos^2\theta} + \frac{1}{\cot^2\theta} = \frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1 + \sin^2\theta}{\cos^2\theta}$

9.

(a) $\cos\theta - \cos\theta\sin^2\theta = \cos\theta\left(1 - \sin^2\theta\right) = \cos\theta\cos^2\theta = \cos^3\theta$

(b) $\frac{1 - \cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} = 1$

(c) $\cos 2\theta + \sin^2\theta = 1 - 2\sin^2\theta + \sin^2\theta = 1 - \sin^2\theta = \cos^2\theta$

(d) $\frac{\sin^2\theta}{\cos^2\theta} + \frac{1}{\cot^2\theta} = \tan^2\theta + \tan^2\theta = 2\tan^2\theta$

(e) $\sin(\alpha + \beta) + \sin(\alpha - \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha + \sin\alpha\cos\beta - \sin\beta\cos\alpha = 2\sin\alpha\cos\beta$

(f) $\frac{1 + \cos 2A}{2} = \frac{1 + 2\cos^2 A - 1}{2} = \cos^2 A$

(g) $\cos(\alpha + \beta) + \cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta = 2\cos\alpha\cos\beta$

(h) $2\cos^2\theta - \cos 2\theta = 2\cos^2\theta - \left(2\cos^2\theta - 1\right) = 1$

10. To prove the following identities, we manipulate both sides until they become equal.

(a)

$$\frac{\cos 2\theta}{\cos\theta+\sin\theta}=\cos\theta-\sin\theta$$

$$\frac{\cos^2\theta-\sin^2\theta}{\cos\theta+\sin\theta}=\cos\theta-\sin\theta$$

$$\frac{(\cos\theta+\sin\theta)(\cos\theta\sin\theta)}{\cos\theta+\sin\theta}=\cos\theta-\sin\theta$$

$$\cos\theta-\sin\theta=\cos\theta-\sin\theta$$

(b)

$$(1-\cos\alpha)(1+\sec\alpha)=\sin\alpha\tan\alpha$$

$$1-\cos\alpha+\sec\alpha-\cos\alpha\sec\alpha=\sin\alpha\frac{\sin\alpha}{\cos\alpha}$$

$$1-\cos\alpha+\frac{1}{\cos\alpha}-\frac{\cos\alpha}{\cos\alpha}=\frac{\sin^2\alpha}{\cos\alpha}$$

$$-\cos\alpha+\frac{1}{\cos\alpha}=\frac{\sin^2\alpha}{\cos\alpha}$$

$$\frac{-\cos^2\alpha+1}{\cos\alpha}=\frac{\sin^2\alpha}{\cos\alpha}$$

$$\frac{\sin^2\alpha}{\cos\alpha}=\frac{\sin^2\alpha}{\cos\alpha}$$

(c)

$$\frac{1-\tan^2 x}{1+\tan^2 x}=\cos 2x$$

$$\frac{1-\dfrac{\sin^2 x}{\cos^2 x}}{1+\dfrac{\sin^2 x}{\cos^2 x}}=\cos^2 x-\sin^2 x$$

$$\frac{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x+\sin^2 x}{\cos^2 x}}=\cos^2 x-\sin^2 x$$

$$\frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}=\cos^2 x-\sin^2 x$$

$$\frac{\cos^2 x-\sin^2 x}{1}=\cos^2 x-\sin^2 x$$

$$\cos^2 x-\sin^2 x=\cos^2 x-\sin^2 x$$

(d)

$$\cos^4\theta - \sin^4\theta = \cos 2\theta$$
$$\left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right) = \cos^2\theta - \sin^2\theta$$
$$1\cdot\left(\cos^2\theta - \sin^2\theta\right) = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta - \sin^2\theta = \cos^2\theta - \sin^2\theta$$

(e)

$$\cot\theta - \tan\theta = 2\cot 2\theta$$
$$\frac{1}{\tan\theta} - \tan\theta = 2\frac{1}{\tan 2\theta}$$
$$\frac{1-\tan^2\theta}{\tan\theta} = 2\frac{1-\tan^2\theta}{2\tan\theta}$$
$$\frac{1-\tan^2\theta}{\tan\theta} = \frac{1-\tan^2\theta}{\tan\theta}$$

(f)

$$\frac{\cos\beta - \sin\beta}{\cos\beta + \sin\beta} = \frac{\cos 2\beta}{1+\sin 2\beta}$$
$$\frac{\cos\beta - \sin\beta}{\cos\beta + \sin\beta}\cdot\frac{\cos\beta + \sin\beta}{\cos\beta + \sin\beta} = \frac{\cos^2\beta - \sin^2\beta}{1+2\sin\beta\cos\beta}$$
$$\frac{\cos^2\beta - \sin^2\beta}{\cos^2\beta + 2\sin\beta\cos\beta + \sin^2\beta} = \frac{\cos^2\beta - \sin^2\beta}{1+2\sin\beta\cos\beta}$$
$$\frac{\cos^2\beta - \sin^2\beta}{1+2\sin\beta\cos\beta} = \frac{\cos^2\beta - \sin^2\beta}{1+2\sin\beta\cos\beta}$$

(g)

$$\frac{1}{\sec\theta(1-\sin\theta)} = \sec\theta + \tan\theta$$
$$\frac{\cos\theta}{1-\sin\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$
$$\frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} = \frac{1+\sin\theta}{\cos\theta}$$
$$\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} = \frac{1+\sin\theta}{\cos\theta}$$
$$\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} = \frac{1+\sin\theta}{\cos\theta}$$
$$\frac{1+\sin\theta}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

(h)

$$(\tan A-\sec A)^2=\frac{1-\sin A}{1+\sin A}$$

$$\left(\frac{\sin A}{\cos A}-\frac{1}{\cos A}\right)^2=\frac{1-\sin A}{1+\sin A}\cdot\frac{1-\sin A}{1-\sin A}$$

$$\left(\frac{\sin A-1}{\cos A}\right)^2=\frac{(1-\sin A)^2}{1-\sin^2 A}$$

$$\frac{(\sin A-1)^2}{\cos^2 A}=\frac{(1-\sin A)^2}{\cos^2 A}$$

$$\frac{(1-\sin A)^2}{\cos^2 A}=\frac{(1-\sin A)^2}{\cos^2 A}$$

(i)

$$\frac{\tan 2x\tan x}{\tan 2x-\tan x}=\sin 2x$$

$$\frac{\frac{2\tan x}{1-\tan^2 x}\tan x}{\frac{2\tan x}{1-\tan^2 x}-\tan x}=2\sin x\cos x$$

$$\frac{\frac{2\tan^2 x}{1-\tan^2 x}}{\frac{2\tan x-\tan x+\tan^3 x}{1-\tan^2 x}}=2\sin x\cos x$$

$$\frac{2\tan^2 x}{\tan x+\tan^3 x}=2\sin x\cos x$$

$$\frac{2\tan x}{1+\tan^2 x}=2\sin x\cos x$$

$$\frac{2\frac{\sin x}{\cos x}}{\sec^2 x}=2\sin x\cos x$$

$$2\frac{\sin x}{\cos x}\cos^2 x=2\sin x\cos x$$

$$2\sin x\cos x=2\sin x\cos x$$

(j)

$$\frac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 1} = \tan\theta$$

$$\frac{2\sin\theta\cos\theta - \left(1 - 2\sin^2\theta\right) + 1}{2\sin\theta\cos\theta + \left(2\cos^2\theta - 1\right) + 1} = \tan\theta$$

$$\frac{2\sin\theta\cos\theta + 2\sin^2\theta}{2\sin\theta\cos\theta + 2\cos^2\theta} = \tan\theta$$

$$\frac{2\sin\theta\left(\cos\theta + \sin\theta\right)}{2\cos\theta\left(\sin\theta + \cos\theta\right)} = \tan\theta$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

(k)

$$\frac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha - \frac{\sin\alpha}{1+\cos\alpha}$$

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{2}{\sin\alpha} - \frac{\sin\alpha}{1+\cos\alpha}$$

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{2\left(1+\cos\alpha\right) - \sin^2\alpha}{\sin\alpha\left(1+\cos\alpha\right)}$$

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{2 + 2\cos\alpha - \left(1 - \cos^2\alpha\right)}{\sin\alpha\left(1+\cos\alpha\right)}$$

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{1 + 2\cos\alpha + \cos^2\alpha}{\sin\alpha\left(1+\cos\alpha\right)}$$

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{\left(1+\cos\alpha\right)^2}{\sin\alpha\left(1+\cos\alpha\right)}$$

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{1+\cos\alpha}{\sin\alpha}$$

(l)

$$\frac{1+\cos\beta}{\sin\beta} + \frac{\sin\beta}{1+\cos\beta} = 2\csc\beta$$

$$\frac{\left(1+\cos\beta\right)^2 + \sin^2\beta}{\sin\beta\left(1+\cos\beta\right)} = \frac{2}{\sin\beta}$$

$$\frac{1 + 2\cos\beta + \cos^2\beta + \sin^2\beta}{\sin\beta\left(1+\cos\beta\right)} = \frac{2}{\sin\beta}$$

$$\frac{1 + 2\cos\beta + 1}{\sin\beta\left(1+\cos\beta\right)} = \frac{2}{\sin\beta}$$

$$\frac{2\left(1+\cos\beta\right)}{\sin\beta\left(1+\cos\beta\right)} = \frac{2}{\sin\beta}$$

$$\frac{2}{\sin\beta} = \frac{2}{\sin\beta}$$

(m)

$$\frac{\cot x-1}{1-\tan x}=\frac{\csc x}{\sec x}$$

$$\frac{\dfrac{\cos x}{\sin x}-1}{1-\dfrac{\sin x}{\cos x}}=\frac{\cos x}{\sin x}$$

$$\frac{\dfrac{\cos x-\sin x}{\sin x}}{\dfrac{\cos x-\sin x}{\cos x}}=\frac{\cos x}{\sin x}$$

$$\frac{\cos x-\sin x}{\sin x}\cdot\frac{\cos x}{\cos x-\sin x}=\frac{\cos x}{\sin x}$$

$$\frac{\cos x}{\sin x}=\frac{\cos x}{\sin x}$$

(n)

$$\sin\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{2}$$

$$\sin^2\left(\frac{2\theta}{2}\right)=\frac{1-\cos(2\theta)}{2}$$

$$\sin^2\theta=\frac{1-\cos 2\theta}{2}$$

$$\sin^2\theta=\frac{1-\left(1-2\sin^2\theta\right)}{2}$$

$$\sin^2\theta=\frac{2\sin^2\theta}{2}$$

$$\sin^2\theta=\sin^2\theta$$

11. Considering the two right-angled triangles in the diagram, we have $\theta=\tan^{-1}\frac{7}{x}-\tan^{-1}\frac{2}{x}$. Taking the tangent of both sides we have $\tan\theta=\tan\left(\tan^{-1}\frac{7}{x}-\tan^{-1}\frac{2}{x}\right)$, and the compound angle formula for the tangent gives us

$$\tan\theta=\frac{\tan\left(\tan^{-1}\dfrac{7}{x}\right)-\tan\left(\tan^{-1}\dfrac{2}{x}\right)}{1+\tan\left(\tan^{-1}\dfrac{7}{x}\right)\cdot\tan\left(\tan^{-1}\dfrac{2}{x}\right)}=\frac{\dfrac{7}{x}-\dfrac{2}{x}}{1+\dfrac{7}{x}\cdot\dfrac{2}{x}}=\frac{\dfrac{5}{x}}{\dfrac{x^2+14}{x^2}}=\frac{5x}{x^2+14}$$

12.

(a)

$$2\sin^2 x-\cos x=1\rightarrow 2\left(1-\cos^2 x\right)-\cos x-1=0$$

$$2-2\cos^2 x-\cos x-1=0\rightarrow 2\cos^2+\cos x-1=0$$

$\cos x=\dfrac{-1\pm\sqrt{1+8}}{4}=\dfrac{-1\pm 3}{4}=\dfrac{1}{2}$ or -1, which in the given domain gives:

$x=\dfrac{\pi}{3},\dfrac{5\pi}{3}$ or $x=\pi$

(b)

$$\sec^2 x=8\cos x\rightarrow\frac{1}{\cos^2 x}=8\cos x$$

$$\cos^3 x=\frac{1}{8}\rightarrow\cos x=\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$$

In the given domain, this gives $x=-\dfrac{\pi}{3},\dfrac{\pi}{3}$

(c)

$$2\cos x+\sin 2x=0\rightarrow 2\cos x+2\sin x\cos x=0$$

$$2\cos x\left(1+\sin x\right)=0$$

This gives $2\cos x=0$ or $1+\sin x=0\rightarrow\sin x=-1$. The solutions to these equations in the given domain are $x=-90°,90°$ or $x=-90°$, so finally $x=-90°,90°$

(d)

$$2\sin x=\cos 2x$$

$$2\sin x=1-2\sin^2 x$$

$$2\sin^2 x+2\sin x-1=0$$

$\sin x=\dfrac{-2\pm\sqrt{4+8}}{4}=\dfrac{-2\pm 2\sqrt{3}}{4}=\dfrac{-1\pm\sqrt{3}}{2}$. The equation $\sin x=\dfrac{-1-\sqrt{3}}{2}$ has no solutions since $\dfrac{-1-\sqrt{3}}{2}<-1$, while $\sin x=\dfrac{-1+\sqrt{3}}{2}$ gives $x=\sin^{-1}\dfrac{-1+\sqrt{3}}{2}\approx 0.375$ or $x=\pi-\sin^{-1}\dfrac{-1+\sqrt{3}}{2}\approx 2.77$

(e)

$$\cos 2x=\sin^2 x$$

$$\cos^2 x-\sin^2 x=\sin x$$

$$\cos^2 x=2\sin^2 x$$

$$\frac{\sin^2 x}{\cos^2 x}=\frac{1}{2}$$

$$\tan^2 x=\frac{1}{2}\rightarrow\tan x=\pm\sqrt{\frac{1}{2}}$$

Solving for x in the given domain gives $x=\tan^{-1}\sqrt{\dfrac{1}{2}}\approx 0.615$, $x=\pi+\tan^{-1}\sqrt{\dfrac{1}{2}}\approx 3.76$,

$x=\tan^{-1}\left(-\sqrt{\dfrac{1}{2}}\right)+\pi\approx 2.53$, $x=\pi+\tan^{-1}\left(-\sqrt{\dfrac{1}{2}}\right)+\pi\approx 5.67$

(f)

$2\sin x\cos x+1=0$

$\sin 2x=-1$

$2x=\sin^{-1}(-1)+2k\pi=\frac{3\pi}{2}+2k\pi$

$x=\frac{3\pi}{4}+k\pi$

The solutions in the given domain are $x=\frac{3\pi}{4},\frac{7\pi}{4}$

(g)

$\cos^2 x-\sin^2 x=-\frac{1}{2}$

$\cos 2x=-\frac{1}{2}$

$2x=\cos^{-1}\left(-\frac{1}{2}\right)+2k\pi$ or $2x=2\pi-\cos^{-1}\left(-\frac{1}{2}\right)+2k\pi$, so

$2x=\frac{2\pi}{3}+2k\pi$ or $2x=2\pi-\frac{2\pi}{3}+2k\pi=\frac{4\pi}{3}+2k\pi$ and finally

$x=\frac{\pi}{3}+k\pi$ or $x=\frac{2\pi}{3}+k\pi$

The solutions in the given domain are $x=\frac{\pi}{3}$ or $x=\frac{2\pi}{3}$

(h)

$\sec^2 x-\tan x-1=0$

$\frac{1}{\cos^2 x}-\frac{\sin x}{\cos x}-1=0$

$\frac{1-\sin x\cos x-\cos^2 x}{\cos^2 x}=0$

Considering the numerator only, and rearranging it, we have:

$1-\cos^2 x-\sin x\cos x=0$

$\sin^2 x-\sin x\cos x=0$

$\sin x(\sin x-\cos x)=0$

$\sin x=0$ or $\sin x-\cos x=0\rightarrow\sin x=\cos x\rightarrow\frac{\sin x}{\cos x}=1\rightarrow\tan x=1$

The solutions to these two equations in the given domain are

$x=0,\pi$ or $x=\frac{\pi}{4},\frac{5\pi}{4}$

(i)

$$\tan 2x + \tan x = 0$$

$$\frac{2\tan x}{1-\tan^2 x} + \tan x = 0$$

$$\tan x\left(\frac{2}{1-\tan^2} + 1\right) = 0$$

$$\tan x\left(\frac{2+1-\tan^2 x}{1-\tan^2 x}\right) = 0$$

which gives $\tan x = 0$ or $\frac{2+1-\tan^2 x}{1-\tan^2 x} = 0 \to 3-\tan^2 x = 0 \to \tan x = \pm\sqrt{3}$

The solutions to these three equations in the given domain are

$x = 0, \pi$ or $x = \frac{\pi}{3}, \frac{4\pi}{3}$, or $x = \frac{2\pi}{3}, \frac{5\pi}{3}$ respectively.

(j)

$$2\sin 2x\cos 3x + \cos 3x = 0$$

$$\cos 3x\left(2\sin 2x + 1\right) = 0$$

This gives $\cos 3x = 0$ or $2\sin 2x + 1 = 0 \to \sin 2x = -\frac{1}{2}$

The solutions to the first equation are:
$3x = 90° + k\cdot 360°$ or $3x = 270° + k\cdot 360°$, giving
$x = 30° + k\cdot 120°$ or $x = 90° + k\cdot 120°$. The values in the given domain are $x = 30°, 90°, 150°$.
The solutions to the second equation are:
$2x = 210° + k\cdot 360°$ or $2x = 330° + k\cdot 360°$, giving
$x = 105° + k\cdot 180°$ or $x = 165° + k\cdot 180°$.
The values in the given domain are $x = 105°$ or $x = 165°$

13.

$$\sin 3x = \sin\left(2x + x\right) = \sin 2x\cos x + \sin x\cos 2x = 2\sin x\cos^2 x + \sin x\left(1-2\sin^2 x\right)$$

$$= 2\sin x\left(1-\sin^2 x\right) + \sin x - 2\sin^3 x = 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x$$

14.

(a) Squaring the expression as suggested, we obtain
$\left(\sin^2 x + \cos^2 x\right)^2 = \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x$. On the other hand,
$\left(\sin^2 x + \cos^2 x\right)^2 = 1^2 = 1$, so we have $\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 1$.
Solving for $\sin^4 x + \cos^4 x$, we have $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x\cos^2 x$. Working on the right hand side gives $1 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\left(2\sin x\cos x\right)^2 = 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}\cdot 2\sin^2 2x$. From the double angle formulae, we have $\cos 2\theta = 1 - 2\sin^2\theta$, from which $2\sin^2\theta = 1 - \cos 2\theta$ and $2\sin^2 2x = 1 - \cos 4x$. Replacing this last identity, we have
$1 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{4}\left(1 - \cos 4x\right) = 1 - \frac{1}{4} + \frac{1}{4}\cos 4x = \frac{1}{4}\left(\cos 4x + 3\right)$. So, we have proved that
$\sin^4 x + \cos^4 x = 1 - 2\sin^2 x\cos^2 x = \frac{1}{4}\left(\cos 4x + 3\right)$.

(b) The equation $\sin^4 x+\cos^4 x=\frac{1}{2}$ becomes $\frac{1}{4}(\cos 4x+3)=\frac{1}{2}$, so we have

$\cos 4x+3=2\to\cos 4x=-1$. This gives $4x=\pi+2k\pi\to x=\frac{\pi}{4}+k\frac{\pi}{2}$.

The solutions in the given domain are $x=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

Exercise 6.6

1. Recall that the range of the arcsine and arctangent functions is the first or fourth quadrant, while the range of the arccosine is the first or second quadrant.

(a) The angle in the first or fourth quadrant whose sine is 1 is $\frac{\pi}{2}$

(b) The angle in the first or second quadrant whose cosine is $\frac{1}{\sqrt{2}}$ is $\frac{\pi}{4}$

(c) The angle in the first or fourth quadrant whose tangent is $-\sqrt{3}$ is $-\frac{\pi}{3}$

(d) The angle in the first or second quadrant whose cosine is $-\frac{1}{2}$ is $\frac{2\pi}{3}$

(e) The angle in the first or fourth quadrant whose tangent is 0 is 0

(f) The angle in the first or fourth quadrant whose sine is $-\frac{\sqrt{3}}{2}$ is $-\frac{\pi}{3}$

2.

(a) The angle in the first or fourth quadrant that has the same sine as $\frac{2\pi}{3}$ is $\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

(b) The angle $\frac{3}{2}$ is in the first quadrant, so it is the range of the arccosine, so $\cos^{-1}\left(\cos\frac{3}{2}\right)=\frac{3}{2}$

(c) $\tan(\arctan 12)=12$

(d) The expression $\arccos\frac{2\pi}{3}$ is meaningless since $\frac{2\pi}{3}>1$ and the domain of the arccosine is numbers between -1 and 1 inclusive.

(e) The angle in the first or fourth quadrant that has the same tangent as $-\frac{3\pi}{4}$ is $\pi+\left(-\frac{3\pi}{4}\right)=\frac{\pi}{4}$

(f) The expression $\arcsin\pi$ is meaningless since $\pi>1$ and the domain of the arcsine is numbers between -1 and 1 inclusive.

(g) We define $y=\sin\arctan\frac{3}{4}$ and $x=\cos\arctan\frac{3}{4}$. It follows that $\frac{y}{x}=\frac{3}{4}$, since y and x are the sine and cosine of the arc whose tangent is $\frac{3}{4}$. According to Pythagoras' theorem, we also have $x^2+y^2=1$. Combining the two equations and solving for y we have:

$x=\frac{4}{3}y\to\frac{16}{9}y^2+y^2=1$

$y^2\left(1+\frac{16}{9}\right)=1$

$y^2=\frac{9}{25}$

$y=\pm\frac{3}{5}$, but we choose $y=\frac{3}{5}$ since the angle $\arctan\frac{3}{4}$ is in the first quadrant.

(h) Here we are looking for the cosine of the angle whose sine is $\frac{7}{25}$. This is simply given by Pythagoras' theorem with $x=\cos\arcsin\frac{7}{25}$, $y=\sin\arcsin\frac{7}{25}=\frac{7}{25}$, so

$x^2+\left(\frac{7}{25}\right)^2=1\rightarrow x^2=1-\frac{49}{625}=\frac{576}{625}$

$x=\pm\frac{24}{25}$, but we choose $x=\frac{24}{25}$ since the range of the arcsine function is the first or fourth quadrant where the cosine is positive.

(i) The domain of the arcsine function is $-1\le x\le 1$, but $\tan\frac{\pi}{3}=\sqrt{3}>1$, so the expression is undefined.

(j) $\tan^{-1}\left(2\sin\frac{\pi}{3}\right)=\tan^{-1}\left(2\cdot\frac{\sqrt{3}}{2}\right)=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$

(k) We define $x=\cos\arctan\frac{1}{2}$ and $y=\sin\arctan\frac{1}{2}$. It follows that $\frac{y}{x}=\frac{1}{2}$, since y and x are the sine and cosine of the arc whose tangent is $\frac{1}{2}$. According to Pythagoras' theorem, we also have $x^2+y^2=1$. Combining the two equations and solving for x we have:

$y=\frac{1}{2}x\rightarrow x^2+\frac{1}{4}x^2=1$

$x^2\left(1+\frac{1}{4}\right)=1$

$x^2=\frac{4}{5}$

$x=\pm\sqrt{\frac{4}{5}}=\pm\frac{2}{\sqrt{5}}=\pm\frac{2\sqrt{5}}{5}$, but we choose $x=\frac{2\sqrt{5}}{5}$ since the range of the arctangent function is the first or fourth quadrant where the cosine is positive.

(l) Here we are looking for the cosine of the angle whose sine is 0.6. This is simply given by Pythagoras' theorem with $x=\cos\left(\sin^{-1}0.6\right)$, $y=\sin\left(\sin^{-1}0.6\right)=0.6$, so

$x^2+0.6^2=1\rightarrow x^2=1-0.36=0.64$

$x=\pm 0.8$, but we choose $x=0.8=\frac{4}{5}$ since the range of the arcsine function is the first or fourth quadrant where the cosine is positive.

(m) We first use the compound angle formula

$$\sin\left(\arccos\frac{3}{5}+\arctan\frac{5}{12}\right)=\sin\arccos\frac{3}{5}\cos\arctan\frac{5}{12}+\cos\arccos\frac{3}{5}\sin\arctan\frac{5}{12}.$$

We then calculate the four pieces separately.

$\sin\arccos\frac{3}{5}$: setting $y=\sin\arccos\frac{3}{5}$ and $x=\cos\arccos\frac{3}{5}=\frac{3}{5}$, we have $x^2+y^2=1\to\left(\frac{3}{5}\right)^2+y^2=1\to y=\pm\sqrt{\frac{16}{25}}=\pm\frac{4}{5}$. We choose $y=\frac{4}{5}$ since the range of the arccosine function is the first or second quadrant where the sine is positive. $\cos\arctan\frac{5}{12}$: setting $x=\cos\arctan\frac{5}{12}$ and $y=\sin\arctan\frac{5}{12}$, we have $\frac{y}{x}=\frac{\sin\arctan\frac{5}{12}}{\cos\arctan\frac{5}{12}}=\tan\arctan\frac{5}{12}=\frac{5}{12}$ and $x^2+y^2=1$. This gives $x^2+\left(\frac{5}{12}x\right)^2=1\to x^2\left(\frac{169}{144}\right)=1\to x=\pm\frac{12}{13}$. We choose $x=\frac{12}{13}$ since the range of the arctangent function is the first or fourth quadrant where the cosine is positive. This also gives us $y=\sin\arctan\frac{5}{12}=\frac{5}{12}\cdot\frac{12}{13}=\frac{5}{13}$

We finally have $\cos\arccos\frac{3}{5}=\frac{3}{5}$. Replacing these four values in the first formula, we have

$$\sin\left(\arccos\frac{3}{5}+\arctan\frac{5}{12}\right)=\sin\arccos\frac{3}{5}\cos\arctan\frac{5}{12}+\cos\arccos\frac{3}{5}\sin\arctan\frac{5}{12}$$
$$=\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}$$

(n) We first use the compound angle formula $\cos\left(\tan^{-1}3+\sin^{-1}\frac{1}{3}\right)=\cos\tan^{-1}3\cdot\cos\sin^{-1}\frac{1}{3}-\sin\tan^{-1}3\cdot\sin\sin^{-1}\frac{1}{3}$. We then calculate the four pieces separately.

$\cos\tan^{-1}3$: setting $x=\cos\tan^{-1}3$ and $y=\sin\tan^{-1}3$, we have $\frac{y}{x}=3$ and $x^2+y^2=1$. These give $x^2+(3x)^2=1\to10x^2=1\to x^2=\frac{1}{10}$, so that $x=\pm\frac{1}{\sqrt{10}}$. We choose $x=\frac{1}{\sqrt{10}}$ since the range of the arctangent function is the first or fourth quadrant where the cosine is positive. This also gives us $y=\sin\tan^{-1}3=3x=\frac{3}{\sqrt{10}}$

$\cos\sin^{-1}\frac{1}{3}$: setting $x=\cos\sin^{-1}\frac{1}{3}$ and $y=\sin\sin^{-1}\frac{1}{3}=\frac{1}{3}$, we have $x^2+y^2=1\to x^2+\left(\frac{1}{3}\right)^2=1\to x^2=\frac{8}{9}$ which gives $x=\pm\frac{\sqrt{8}}{3}=\pm\frac{2\sqrt{2}}{3}$. We choose $x=\frac{2\sqrt{2}}{3}$ since the range of the arcsine function is the first or fourth quadrant where the cosine is positive. Finally, $\sin\sin^{-1}\frac{1}{3}=\frac{1}{3}$.

Replacing these values gives

$$\cos\left(\tan^{-1}3+\sin^{-1}\frac{1}{3}\right)=\cos\tan^{-1}3\cdot\cos\sin^{-1}\frac{1}{3}-\sin\tan^{-1}3\cdot\sin\sin^{-1}\frac{1}{3}$$

$$=\frac{1}{\sqrt{10}}\cdot\frac{2\sqrt{2}}{3}-\frac{3}{\sqrt{10}}\cdot\frac{1}{3}=\frac{2\sqrt{2}-3}{3\sqrt{10}}=\frac{\left(2\sqrt{2}-3\right)\sqrt{10}}{30}=\frac{2\sqrt{20}-3\sqrt{10}}{30}$$

3. Here we make repeated use of Pythagoras' theorem.

(a) $\cos(\arcsin x)$: setting $X=\cos(\arcsin x)$ and $Y=\sin(\arcsin x)=x$ we have $X^2+Y^2=1\rightarrow X^2+x^2=1\rightarrow X^2=1-x^2$, from which we have $X=\cos(\arcsin x)=\pm\sqrt{1-x^2}$. We then choose $\cos(\arcsin x)=\sqrt{1-x^2}$ since the range of the arcsine function is the first or fourth quadrant where the cosine is positive.

(b) $\tan(\arccos x)=\dfrac{\sin(\arccos x)}{\cos(\arccos x)}=\dfrac{\sin(\arccos x)}{x}$. To rewrite $\sin(\arccos x)$, we set $X=\cos(\arccos x)=x$ and $Y=\sin(\arccos x)$ so that we have $X^2+Y^2=1\rightarrow x^2+Y^2=1\rightarrow Y^2=1-x^2$, from which we have $Y=\sin(\arccos x)=\pm\sqrt{1-x^2}$. We then choose $\sin(\arccos x)=\sqrt{1-x^2}$ since the range of the arccosine function is the first or second quadrant where the sine is positive. Finally, we have $\tan(\arccos x)=\dfrac{\sqrt{1-x^2}}{x}$.

(c) $\cos(\tan^{-1}x)$: setting $X=\cos(\tan^{-1}x)$ and $Y=\tan(\tan^{-1}x)$, we have $\dfrac{Y}{X}=\tan(\tan^{-1}x)=x$ and $X^2+Y^2=1$. This gives $X^2+\left(xX\right)^2=1\rightarrow X^2\left(1+x^2\right)=1\rightarrow X^2=\dfrac{1}{1+x^2}\rightarrow X=\pm\dfrac{1}{\sqrt{1+x^2}}$.

We choose $X=\dfrac{1}{\sqrt{1+x^2}}$ since the range of the arctangent function is the first or fourth quadrant where the cosine is positive. We therefore have $\cos(\tan^{-1}x)=\dfrac{1}{\sqrt{1+x^2}}$.

(d) $\sin\left(2\cos^{-1}x\right)=2\sin(\cos^{-1}x)\cos(\cos^{-1}x)$, using the double angle formula. This gives, according to **(b)**, $\sin\left(2\cos^{-1}x\right)=2\sin(\cos^{-1}x)\cos(\cos^{-1}x)=2\sqrt{1-x^2}\cdot x=2x\sqrt{1-x^2}$.

(e) $\tan\left(\dfrac{1}{2}\arccos x\right)$: here it is convenient to recall the half-angle formulae that can be derived from the double angle formulae for the cosine (see Exercise 6.5, question 10). These state $\sin\dfrac{\theta}{2}=\pm\sqrt{\dfrac{1-\cos\theta}{2}}$ and $\cos\dfrac{\theta}{2}=\pm\sqrt{\dfrac{1+\cos\theta}{2}}$, so that we can rewrite

$$\tan\left(\frac{1}{2}\arccos x\right)=\frac{\sin\frac{1}{2}\arccos x}{\cos\frac{1}{2}\arccos x}=\frac{\sqrt{\frac{1-\cos\arccos x}{2}}}{\sqrt{\frac{1+\cos\arccos x}{2}}}=\frac{\sqrt{\frac{1-x}{2}}}{\sqrt{\frac{1+x}{2}}}=\sqrt{\frac{1-x}{1+x}}$$

. In the half-angle formulae, the positive roots were chosen because the angle $\dfrac{1}{2}\arccos x$ is necessarily in the first quadrant where both cosine and sine are positive.

(f) $\sin\left(\arcsin x + 2\arctan x\right) = \sin(\arcsin x)\cdot\cos(2\arctan x) + \cos(\arcsin x)\cdot\sin(2\arctan x)$

$= x\left(2\cos^2(\arctan x) - 1\right) + \sqrt{1-x^2}\cdot 2\sin(\arctan x)\cdot\cos(\arctan x)$, using first the compound angle formula for the sine and then the double angle formulae for the cosine and the sine and $\sin(\arcsin x) = x$, $\cos(\arcsin x) = \sqrt{1-x^2}$. From **(c)** we recall $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ and $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$, so by replacing we have

$$\sin\left(\arcsin x + 2\arctan x\right) = x\left(2\frac{1}{1+x^2} - 1\right) + \sqrt{1-x^2}\cdot 2\frac{x}{1+x^2}$$

$$= x\left(\frac{2-1-x^2}{1+x^2}\right) + \sqrt{1-x^2}\cdot 2\frac{x}{1+x^2} = \frac{x - x^3 + 2x\sqrt{1-x^2}}{1+x^2}$$

4. Applying the cosine function to the left-hand side, we have

$$\cos\left(\arcsin\frac{4}{5} + \arcsin\frac{5}{13}\right) = \cos\arcsin\frac{4}{5}\cos\arcsin\frac{5}{13} - \sin\arcsin\frac{4}{5}\sin\arcsin\frac{5}{13}$$

Recalling that $\cos\arcsin x = \sin\arccos x = \sqrt{1-x^2}$ and $\sin\arcsin x = x$, we have

$$\sqrt{1-\left(\frac{4}{5}\right)^2}\cdot\sqrt{1-\left(\frac{5}{13}\right)^2} - \frac{4}{5}\cdot\frac{5}{13} = \sqrt{\frac{9}{25}}\cdot\sqrt{\frac{144}{169}} - \frac{4}{13} = \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{13} = \frac{36-20}{65} = \frac{16}{65}$$

This proves that $\arcsin\frac{4}{5} + \arcsin\frac{5}{13} = \arccos\frac{16}{65}$

5. Applying the tangent function to the left hand side and using the compound angle formula for the tangent, we have

$\tan\left(\arctan\frac{1}{2} + \arctan\frac{1}{3}\right) = \frac{\tan\arctan\frac{1}{2} + \tan\arctan\frac{1}{3}}{1 - \tan\arctan\frac{1}{2}\tan\arctan\frac{1}{3}} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$, where we have used $\tan\arctan x = x$. Since $\arctan 1 = \frac{\pi}{4}$, this proves that $\tan\left(\arctan\frac{1}{2} + \arctan\frac{1}{3}\right) = \frac{\pi}{4}$. The proof can be equivalently demonstrated taking either the sine of the cosine of both hand sides.

6. $\tan^{-1} x + \tan^{-1}(1-x) = \tan^{-1}\frac{4}{3}$

$\tan\left(\tan^{-1} x + \tan^{-1}(1-x)\right) = \tan\tan^{-1}\frac{4}{3}$, taking the tangent of both sides

$\frac{\tan\tan^{-1} x + \tan\tan^{-1}(1-x)}{1 - \tan\tan^{-1} x\tan\tan^{-1}(1-x)} = \frac{4}{3}$, using the compound angle formula for the tangent

$\frac{x + (1-x)}{1 - x(1-x)} = \frac{4}{3}$, recalling that $\tan\tan^{-1} x = x$

$$\frac{1}{1-x+x^2} = \frac{4}{3}$$

$$\frac{3-4\left(x^2-x+1\right)}{3\left(x^2-x+1\right)}=0$$

$3-4x^2+4x-4=0$

$4x^2-4x+1=0$

$(2x-1)^2=0\rightarrow 2x-1=0\rightarrow x=\frac{1}{2}$

7.

(a)

$5\cos 2x=2$

$\cos 2x=\frac{2}{5}$

$2x=\cos^{-1}\frac{2}{5}+2k\pi$ or $2x=2\pi-\cos^{-1}\frac{2}{5}+2k\pi$. This gives

$x=\frac{1}{2}\cos^{-1}\frac{2}{5}+k\pi$ or $x=\pi-\frac{1}{2}\cos^{-1}\frac{2}{5}+k\pi$, which in the given domain gives

$x\approx 0.580$ or $x\approx 2.56$ (both from $k=0$).

(b)

$\tan\frac{x}{2}=2$

$\frac{x}{2}=\tan^{-1}2+k\pi$

$x=2\tan^{-1}2+2k\pi$, which in the given domain gives $x\approx 2.21$ from $k=0$.

(c)

$2\cos x-\sin x=0$

$2-\frac{\sin x}{\cos x}=0$

$\tan x=2$

$x=\tan^{-1}2+k\pi$, which in the given domain gives $x\approx 1.11, 4.25$

(d)

$3\sec^2 x = 2\tan x + 4$

$3\left(1+\tan^2 x\right) - 2\tan x - 4 = 0$, using one of the Pythagorean identities

$3\tan^2 x - 2\tan x - 1 = 0$

$\tan x = \frac{2 \pm \sqrt{4+12}}{6} = \frac{2 \pm 4}{6} = 1 \text{ or } -\frac{1}{3}$

These give $x = \tan^{-1} 1 + k\pi$ or $x = \tan^{-1}\left(-\frac{1}{3}\right) + k\pi$, or

$x = \frac{\pi}{4} + k\pi$ or $x \approx -0.322 + k\pi$, which in the given domain gives

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ or $x \approx 2.82, 5.96$

(e)

$2\tan^2 x - 3\tan x + 1 = 0$

$\tan x = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = 1 \text{ or } \frac{1}{2}$. These give

$x = \tan^{-1} 1 + k\pi$ or $x = \tan^{-1}\frac{1}{2} + k\pi$, or

$x = \frac{\pi}{4} + k\pi$ or $x \approx 0.464 + k\pi$, which in the given domain gives

$x = \frac{\pi}{4}$ or $x \approx 0.464$

(f)

$\tan x \csc x = 5$

$\frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} = 5$

$\frac{1}{\cos x} = 5$

$\cos x = \frac{1}{5}$

$x = \cos^{-1}\frac{1}{5} + 2k\pi$ or $x = 2\pi - \cos^{1}\frac{1}{5} + 2k\pi$, or

$x \approx 1.37$ or $x \approx 4.91$

(g)

$\tan 2x + 3\tan x = 0$

$\frac{2\tan x}{1-\tan^2 x} + 3\tan x = 0$

$\tan x\left(\frac{2}{1-\tan^2 x} + 3\right) = 0$

$\tan x\left(\frac{2+3-3\tan^2 x}{1-\tan^2 x}\right) = 0$, which gives either $\tan x = 0$ or $\frac{2+3-3\tan^2 x}{1-\tan^2 x} = 0$.

The first equation gives $x = \tan^{-1} 0 + k\pi$, which in the given domain gives $x = \pi, 2\pi$.

The second equation gives $5 - 3\tan^2 x = 0 \rightarrow \tan^2 x = \frac{5}{3}$, which gives either $\tan x = \sqrt{\frac{5}{3}}$, therefore $x=\tan^{-1}\sqrt{\frac{5}{3}}+k\pi \approx 0.912, 4.05$ in the given domain, or

$\tan x=-\sqrt{\frac{5}{3}}$, therefore $x=\tan^{-1}\left(-\sqrt{\frac{5}{3}}\right)+k\pi \approx 2.23, 5.37$ in the given domain.

(h)

$$2\cos^2 x-3\sin 2x=2$$
$$2\cos^2 x-3\sin 2x=2\left(\cos^2 x+\sin^2 x\right)$$
$$2\cos^2 x-3\sin 2x=2\cos^2 x+2\sin^2 x$$
$$-3\sin 2x=2\sin^2 x$$
$$-6\sin x\cos x-2\sin^2 x=0$$

$2\sin x\left(3\cos x+\sin x\right)=0$, which gives either $2\sin x=0$ or $3\cos x+\sin x=0$.

The first equation gives $x=k\pi$, which in the given domain gives $x=0,\pi$.

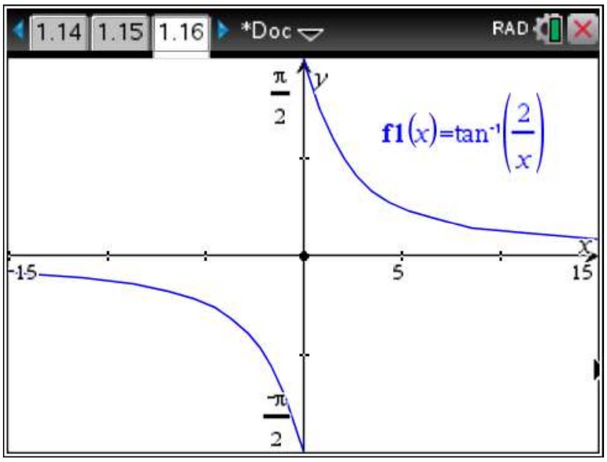
The second equation gives

$$3+\frac{\sin x}{\cos x}=0$$

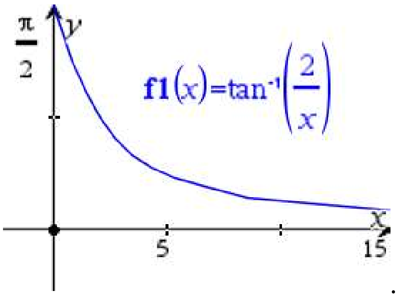
$\tan x=-3$, which gives $x=\tan^{-1}\left(-3\right)+k\pi$. In the given domain, this gives $x\approx 1.89$

8. From the given graph, we have $2=d\tan\theta$, so solving for the angle we have $\theta=\tan^{-1}\frac{2}{d}$.

A complete graph of this function is shown below.

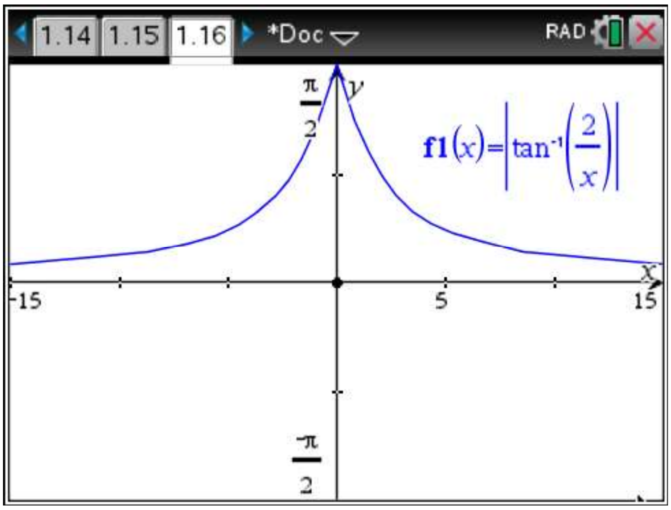


The function as it is now makes sense only for positive values of *d*, since *d* is defined as a distance and therefore it must be positive, so:



.

If d is allowed to take negative values, the interpretation of which would be distance *above P*, then the meaning of the negative angles would be angles that open *counterclockwise* with respect to the line joining the lighthouse with P. A possible way to remove the ambiguity about the sign of θ would be to consider $\theta = \left|\tan^{-1}\frac{2}{d}\right|$, whose graph is shown below:



Chapter 6 practice questions

1.

(a) The length after 2 seconds is $L(2)=110+25\cos(2\pi\cdot 2)=110+25=135$ cm.

(b) The minimum length is when the cosine function takes the value -1. This gives $L_{\min}=110-25=85$ cm.

(c) This occurs the first time the cosine function takes the value -1, so we set up the equation $\cos(2\pi t)=-1$, which gives $2\pi t=\pi \rightarrow t=0.5$ seconds.

(d) The period T of the function $\cos(bt)$ is given by $T=\frac{2\pi}{b}$, so in our case $T=\frac{2\pi}{2\pi}=1$ second.

2.

$$2\sin^2 x-\cos x-1=0$$
$$2(1-\cos^2 x)-\cos x-1=0$$
$$2\cos^2 x+\cos x-1=0$$

Solving for $\cos x$, we get $\cos x=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm 3}{4}=\frac{1}{2}$ or -1, which gives

$x=\frac{\pi}{3},\frac{5\pi}{3}$ and $x=\pi$.

3. Calling r the radius of the circle, the perimeter of the shaded section is given by $p=2r+(2\pi-\theta)r$. Solving for θ, we obtain

$$p=2r+2\pi r-\theta r\rightarrow \theta r=2r(1+\pi)-p\rightarrow\theta=\frac{2r(1+\pi)-p}{r}=2(1+\pi)-\frac{p}{r}.$$

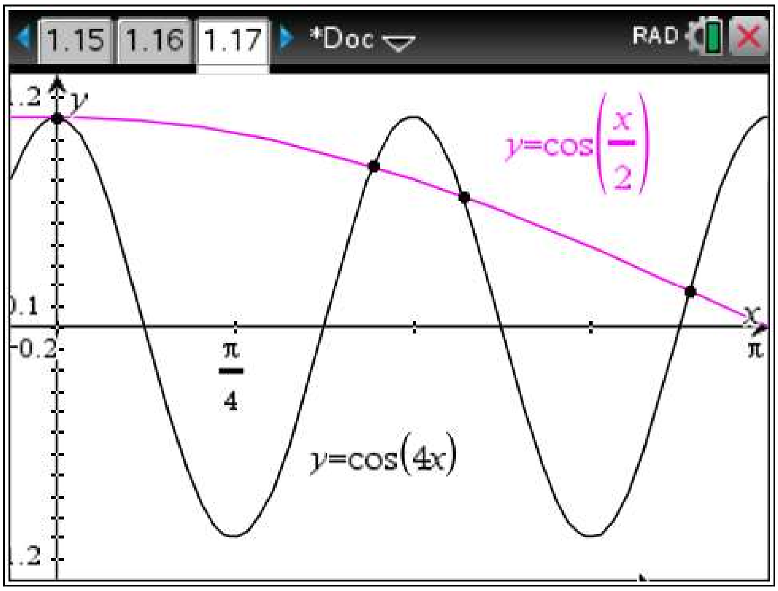
Replacing values, $\theta=2(1+\pi)-\frac{36}{6}\approx 2.28$

4.

(a)

- **i)** The function *f* is a cosine with a horizontal compression, so it is unaffected in its vertical properties and its minimum value is -1.
- **ii)** The function *g* is a cosine with a horizontal stretch by a factor of two, so its period is twice the period of the cosine, $2\cdot 2\pi = 4\pi$

(b) Graphing both functions on a GDC, we obtain:



which shows that in the given domain there are four intersections, therefore four solutions to the equation $f(x) = g(x)$.

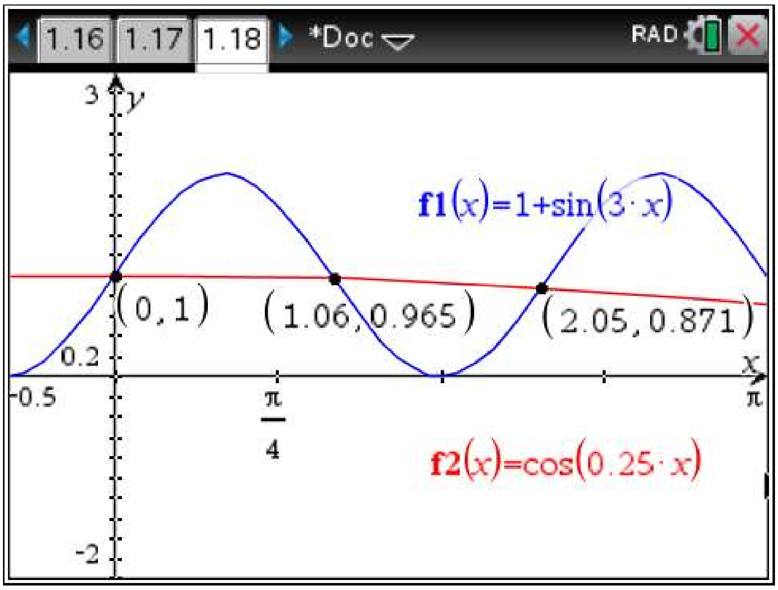
5.

(a) p plays the role of the principal axis of the oscillation, and in this case, it is the height of the hub from the ground. From $p = \dfrac{d_{\text{max}} + d_{\text{min}}}{2}$ we obtain $p = \dfrac{64+6}{2} = 35$ cm.

(b) q plays the role of the amplitude of the oscillation, and in this case, it is the distance between the highest position of the reflector and the hub, $q = d_{\text{max}} - p = 64 - 35 = 29$ cm.

(c) m plays the role of the period of the oscillation, and it is equal to twice the time interval between a maximum and the next minimum, so $m = 2(0.75 - 0.5) = 0.5$ seconds.

6. To solve $1+\sin 3x=\cos(0.25x)$, we graph both sides on a GDC for the given domain and look for intersections:



There are three intersections, therefore the solutions are $x=0, x\approx 1.06, 2.05$

7.

(a) $2\cos^2 x+5\cos x+2=0$

$\cos x=\frac{-5\pm\sqrt{25-16}}{4}=\frac{-5\pm 3}{4}=-\frac{1}{2}$ or -2. The first value gives

$\cos x=-\frac{1}{2}\to x=\cos^{-1}\left(-\frac{1}{2}\right)+2k\pi$ or $x=2\pi-\cos^{-1}\left(-\frac{1}{2}\right)+2k\pi$, so

$x=\frac{2\pi}{3}+2k\pi$ or $x=\frac{4\pi}{3}+2k\pi$. The values in the given domain are $x=\frac{2\pi}{3},\frac{4\pi}{3}$.

The second value gives $\cos x=-2$, which has no solutions since -2 is outside the range of the cosine function.

(b) $\sin 2x-\cos x=0$

$2\sin x\cos x-\cos x=0$

$\cos x(2\sin x-1)=0$ which gives $\cos x=0$ or $2\sin x-1=0\to\sin x=\frac{1}{2}$

In the given domain, the first equation gives $x=\frac{\pi}{2},\frac{3\pi}{2}$; the second equation gives $x=\frac{\pi}{6},\frac{5\pi}{6}$

8. Since $\frac{\pi}{2}<x<\pi$, x is in the second quadrant where the sine is positive. We also have $\pi<2x<2\pi$ which means that $2x$ is either in the third of fourth quadrant, where the sine is negative.

(a) $\sin^2 x+\cos^2 x=1\to\sin x=\pm\sqrt{1-\cos^2 x}$, but we choose the positive solution according to the discussion above: $\sin x=\sqrt{1-\frac{8}{9}}=\sqrt{\frac{1}{9}}=\frac{1}{3}$

(b) $\cos 2x = 2\cos^2 x - 1 = 2\cdot\frac{8}{9} - 1 = \frac{16-9}{9} = \frac{7}{9}$

(c) $\sin 2x = 2\sin x\cos x = 2\left(\frac{1}{3}\right)\left(\pm\sqrt{\cos^2 x}\right) = 2\left(\frac{1}{3}\right)\left(\pm\frac{\sqrt{8}}{3}\right) = \pm\frac{4\sqrt{2}}{9}$. We then choose the negative solution according to the discussion above: $\sin 2x = -\frac{4\sqrt{2}}{9}$

9.

(a) We look for a function of the form $d(t) = a\sin\left(b(t-c)\right) + p$, where $a = \frac{d_{\max} - d_{\min}}{2}$ is the amplitude, $p = \frac{d_{\max} + d_{\min}}{2}$ is the principal axis, c is a horizontal shift, and b relates to the period T as $b = \frac{2\pi}{T}$. This gives $a = \frac{5.8-2.6}{2} = 1.6$ and $p = \frac{5.8+2.6}{2} = 4.2$. The period of the oscillation is found considering that the time interval between the first maximum and the first minimum, so 5.5 hours, is one half of the period, giving $T = 11$ hours. This gives $b = \frac{2\pi}{11}$.

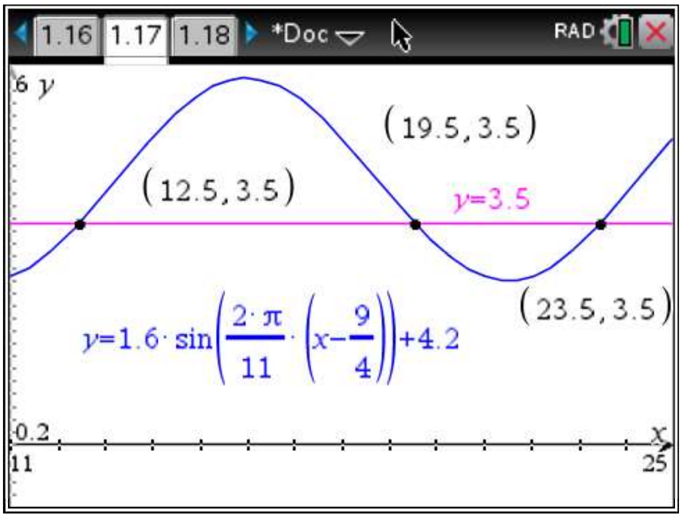
So far, the function is $d(t) = 1.6\sin\left(\frac{2\pi}{11}(t-c)\right) + 4.2$. In order to find the value of c, we set the condition that the tide is at a maximum 5 hours after midnight, so $\sin\left(\frac{2\pi}{11}(5-c)\right) = 1$.

This gives $\frac{2\pi}{11}(5-c) = \frac{\pi}{2} \to 5 - c = \frac{11}{4} \to c = 5 - \frac{11}{4} = \frac{9}{4}$. The function is given by

$$d(t) = 1.6\sin\left(\frac{2\pi}{11}\left(t - \frac{9}{4}\right)\right) + 4.2$$

(b) This is given by $d(12) = 1.6\sin\left(\frac{2\pi}{11}\left(12 - \frac{9}{4}\right)\right) + 4.2 \approx 3.15$ metres.

(c) Setting the inequality $d(t) > 3.5$ and graphing both sides in the given domain gives:



The boat can dock safely from 12.5 hours after midnight, so around 12.30 pm, to 19.5 hours after midnight, so around 7.30 pm, and then the cycle repeats every eleven hours.

10. $\tan^2 x + 2\tan x - 3 = 0$. Solving this quadratic equation for $\tan x$ gives $\tan x = \frac{-2 \pm \sqrt{4+12}}{2} = \frac{-2 \pm 4}{2} = 1$ or -3. The first solution gives $\tan x = 1 \to x = \frac{\pi}{4}$ in the given domain, while the second solution gives $\tan x = -3 \to x = \tan^{-1}(-3) + k\pi$. In the given domain, this gives $x = \tan^{-1}(-3) + \pi \approx 1.89$

11.

(a) The length of arc *ABC* is given by $s = \theta r = \frac{3}{2} \cdot 10 = 15$ cm

(b) The shaded region subtends an angle at the centre $\phi = 2\pi - \frac{3}{2}$. The area of the shaded region is given $A = \frac{1}{2}\phi r^2 = \frac{1}{2}\left(2\pi - \frac{3}{2}\right) \cdot 10^2 \approx 239\ \text{cm}^2$

12. The solutions of the equation $f(x) = k$ are the intersections between the graph of $f(x)$ and the horizontal line $y = k$. Since the function $f(x)$ oscillates around the *x*-axis with amplitude $\frac{5}{2}$, there will be no solutions for $k < -\frac{5}{2}$ or for $k > \frac{5}{2}$.

13. The two points given enable us to state $y(0) = 1$ and $y\left(\frac{3\pi}{2}\right) = 3$. The first condition gives

$$k + a \cdot \sin 0 = 1$$
$$k + a \cdot 0 = 1$$

so that $k = 1$. The second condition gives

$$1 + a \sin \frac{3\pi}{2} = 3$$
$$1 + a \cdot (-1) = 3$$
$$a = -2$$

14. The equation $2\tan^2\alpha - 5\sec\alpha - 10 = 0$ can be rewritten as $2(\sec^2\alpha - 1) - 5\sec\alpha - 10 = 0$ using the identity $1 + \tan^2 x = \sec^2 x$. So the equation becomes $2\sec^2\alpha - 5\sec\alpha - 12 = 0$, which is a quadratic equation in the variable $\sec\alpha$. This gives $\sec\alpha = \frac{5 \pm \sqrt{25+96}}{4} = \frac{5 \pm 11}{4} = 4$ or $-\frac{3}{2}$. Since the angle α is in the second quadrant where the cosine and hence the secant are negative, the only acceptable solution is $\sec\alpha = -\frac{3}{2}$.

15. Using the compound angle formulae and the definition of sine, cosine and tangent in right-angled triangles, we have

(a) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{15}{17} \cdot \frac{8}{10} + \frac{8}{17} \cdot \frac{6}{10} = \frac{60+24}{85} = \frac{84}{85}$

(b) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{8}{17} \cdot \frac{8}{10} - \frac{15}{17} \cdot \frac{6}{10} = \frac{32-45}{85} = -\frac{13}{85}$

(c) $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\dfrac{\frac{84}{85}}{-\frac{13}{85}}=-\dfrac{84}{13}$. Using the compound angle formula for the tangent

$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ with $\tan\alpha=\dfrac{15}{8}$ and $\tan\beta=\dfrac{6}{8}=\dfrac{3}{4}$ gives the same result.

16. First, we calculate the length of the hypotenuse as $\sqrt{1^2+2^2}=\sqrt{5}$. Then, using the double angle formulae, the compound angle formulae and the definition of sine and cosine in a right-angled triangle, we have

$\sin 2p=2\sin p\cos p=2\dfrac{1}{\sqrt{5}}\dfrac{2}{\sqrt{5}}=\dfrac{4}{5}$ and

$$\sin 3p=\sin(2p+p)=\sin 2p\cos p+\cos 2p\sin p=2\sin p\cos^2 p+\left(1-2\sin^2 p\right)\sin p$$

$$=\sin p\left(2\cos^2 p+1-2\sin^2 p\right)=\frac{1}{\sqrt{5}}\left(2\cdot\frac{4}{5}+1-2\cdot\frac{1}{5}\right)=\frac{11}{5\sqrt{5}}=\frac{11\sqrt{5}}{25}$$

17.

(a) Setting $y=\sin B$ and $x=\cos B$, we have $\dfrac{y}{x}=\dfrac{\sin B}{\cos B}=\tan B=-\dfrac{5}{12}$ and $x^2+y^2=\cos^2 B+\sin^2 B=1$. Combining these two conditions into one equation for y, we have $\left(-\dfrac{12}{5}y\right)^2+y^2=1\rightarrow\dfrac{169}{25}y^2=1\rightarrow y=\pm\dfrac{5}{13}$. Since B is obtuse, $\sin B$ is positive and its value is $\sin B=\dfrac{5}{13}$

(b) Same as above, with $x=\cos B=-\dfrac{12}{5}y=-\dfrac{12}{5}\cdot\dfrac{5}{13}=-\dfrac{12}{13}$

(c) Using double angle formulae, $\sin 2B=2\sin B\cos B=2\cdot\dfrac{5}{13}\cdot\left(-\dfrac{12}{13}\right)=-\dfrac{120}{169}$

(d) Using double angle formulae, $\cos 2B=1-2\sin^2 B=1-2\cdot\left(\dfrac{5}{13}\right)^2=1-\dfrac{50}{169}=\dfrac{119}{169}$

18. Using double angle formulae, we have $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{3}{4}$. This is a quadratic equation in $\tan\theta$, in fact rearranging it we obtain:

$$\frac{2\tan\theta}{1-\tan^2\theta}-\frac{3}{4}=0$$

$$\frac{8\tan\theta-3\left(1-\tan^2\theta\right)}{4\left(1-\tan^2\theta\right)}=0$$

$$8\tan\theta-3\left(1-\tan^2\theta\right)=0$$

$$3\tan^2\theta+8\tan\theta-3=0$$

$$\tan\theta=\frac{-8\pm\sqrt{64+36}}{6}=\frac{-8\pm10}{6}=\frac{1}{3}\text{ or }-3$$

19. Given that $\sin(x-\alpha)=k\sin(x+\alpha)$, we can state that $\sin x\cos\alpha-\sin\alpha\cos x=k(\sin x\cos\alpha+\sin\alpha\cos x)$. Dividing both hand sides by $\cos x$ we have $\tan x\cos\alpha-\sin\alpha=k(\tan x\cos\alpha+\sin\alpha)$. Solving for $\tan x$, we have:

$$\tan x\cos\alpha(1-k)=\sin\alpha(1+k)\to\tan x=\frac{\sin\alpha(1+k)}{\cos\alpha(1-k)}=\tan\alpha\frac{1+k}{1-k}$$

20. We have $\tan^2 2\theta=1\to\tan 2\theta=\pm1$

The first solution gives

$$\tan 2\theta=1\to 2\theta=\tan^{-1}1+k\pi\to\theta=\frac{\tan^{-1}1}{2}+k\frac{\pi}{2}=\frac{\pi}{8}+k\frac{\pi}{2}$$

The solutions in the given domain are $\theta=-\frac{3\pi}{8},\frac{\pi}{8}$

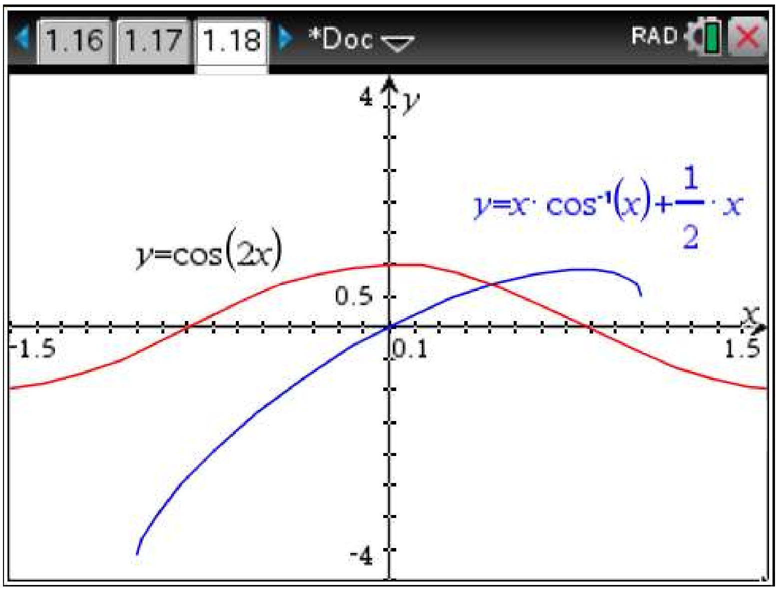
The second solution gives

$$\tan 2\theta=-1\to 2\theta=\tan^{-1}(-1)+k\pi\to\theta=\frac{\tan^{-1}(-1)}{2}+k\frac{\pi}{2}=-\frac{\pi}{8}+k\frac{\pi}{2}$$

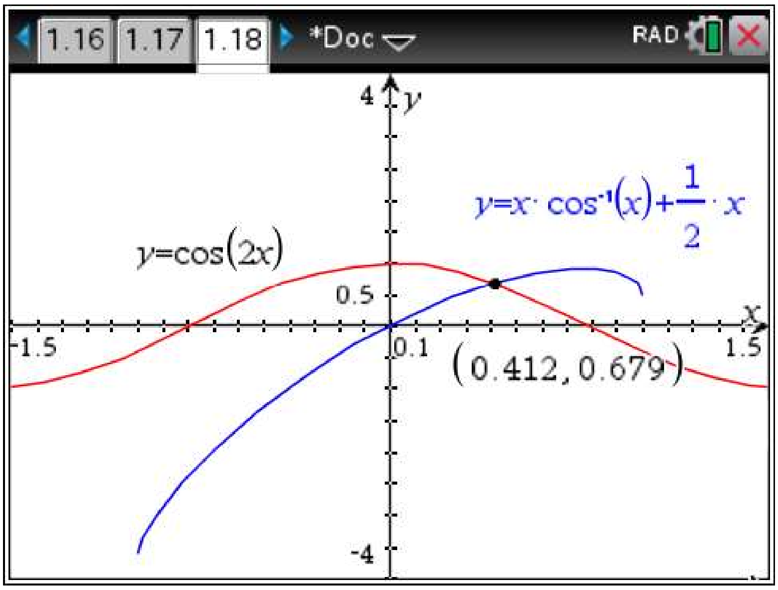
The solutions in the given domain are $\theta=-\frac{\pi}{8},\frac{3\pi}{8}$

21.

(a) Graphing both functions in the given domain, we obtain:



(b) Looking for the intersection of the two curves, we obtain:



The solution to the equation $f(x)=g(x)$ is $x\approx 0.412$

(c) The range of g over the given domain is determined by the maximum and minimum values of g. The maximum is given by $g(0)=1$, while the minima occur at the boundaries, $g(1)=g(-1)=\cos 2=\cos(-2)$. The range is therefore $\cos 2\le y\le 1$

22. Here we can find two expressions for the length of AC, taking advantage of the fact that it belongs to two different right-angled triangles. By equating these two expressions, we will obtain an equation for $D\hat{A}C$.

From triangle ABC, $AC=\dfrac{5}{\tan B\hat{A}C}$. From triangle ACD, $AC=\dfrac{2}{\tan D\hat{A}C}$.

Using the information in the question, and calling $x=D\hat{A}C$, we have:

$$\frac{5}{\tan 2x}=\frac{2}{\tan x}$$

$$2\tan 2x-5\tan x=0$$

$$2\frac{2\tan x}{1-\tan^2 x}-5\tan x=0$$

$\tan x\left(\dfrac{4}{1-\tan^2 x}-5\right)=0$. Since $\tan x$ cannot be equal to zero in this problem, the equation reduces to:

$$\frac{4}{1-\tan^2 x}-5=0$$

$$\frac{4-5+5\tan^2 x}{1-\tan^2 x}=0$$

$5\tan^2 x=1\rightarrow\tan^2 x=\dfrac{1}{5}\rightarrow\tan x=\pm\dfrac{\sqrt{5}}{5}$. Since x is an angle in a triangle, we choose $\tan x=\dfrac{\sqrt{5}}{5}$ which gives $x=\tan^{-1}\dfrac{\sqrt{5}}{5}\approx 24.1°$

23. The request that the water depth be 10 cm means that the water surface has equation $y = -16 + 10 = -6$. The width w of the water surface is the distance between the intersections of the channel boundary with the water surface, so the distance between the solutions to $16\sec\frac{\pi x}{36} - 32 = -6$. Solving this equation gives:

$$16\sec\frac{\pi x}{36} = 26$$

$$\sec\frac{\pi x}{36} = \frac{26}{16} = \frac{13}{8}$$

$$\cos\frac{\pi x}{36} = \frac{8}{13}$$

$\frac{\pi x}{36} = \pm\arccos\frac{8}{13} \rightarrow x = \pm\frac{36}{\pi}\arccos\frac{8}{13}$. The distance between these solutions is

$$w = +\frac{36}{\pi}\arccos\frac{8}{13} - \left(-\frac{36}{\pi}\arccos\frac{8}{13}\right) = \frac{72}{\pi}\arccos\frac{8}{13}$$

Exercise 7.1

1. **(a)** Define $A(3,\ -1,\ 5),\ B(-4,\ 0,\ 2),\ C(2,\ 2,\ -1)$

Then: $AB=\sqrt{(-4-3)^2+(0-(-1))^2+(2-5)^2}=\sqrt{49+1+9}=\sqrt{59}$

$$AC=\sqrt{(2-3)^2+(2-(-1))^2+(-1-5)^2}=\sqrt{1+9+36}=\sqrt{46}$$

$$BC=\sqrt{(2-(-4))^2+(2-0)^2+(-1-2)^2}=\sqrt{36+4+9}=\sqrt{49}$$

The triangle ABC is scalene.

(b) Define $A(-2,\ 4,\ -3),\ B(4,-3,\ -2),\ C(-3,\ -2,\ 4)$

Then: $AB=\sqrt{(4-(-2))^2+(-3-4)^2+(-2-(-3))^2}=\sqrt{36+49+1}=\sqrt{86}$

$$AC=\sqrt{(-3-(-2))^2+(-2-4)^2+(4-(-3))^2}=\sqrt{1+36+49}=\sqrt{86}$$

$$BC=\sqrt{(-3-4)^2+(-2-(-3))^2+(4-(-2))^2}=\sqrt{49+1+36}=\sqrt{86}$$

The triangle ABC is equilateral.

(c) Define $A(4,\ 5,\ 0),\ B(2,\ 6,\ 2),\ C(2,\ 3,\ -1)$

Then: $AB=\sqrt{(2-4)^2+(6-5)^2+(2-0)^2}=\sqrt{4+1+4}=\sqrt{9}=3$

$$AC=\sqrt{(2-4)^2+(3-5)^2+(-1-0)^2}=\sqrt{4+4+1}=\sqrt{9}=3$$

$$BC=\sqrt{(2-2)^2+(3-6)^2+(-1-2)^2}=\sqrt{0+9+9}=\sqrt{18}=3\sqrt{2}$$

The triangle ABC is isosceles.

(d) Define $A(a,\ b,\ c),\ B(b,\ c,\ a),\ C(c,\ a,\ b)$

Then: $AB=\sqrt{(b-a)^2+(c-b)^2+(a-c)^2}=\sqrt{(a-b)^2+(a-c)^2+(b-c)^2}$

$$AC=\sqrt{(c-a)^2+(a-b)^2+(b-c)^2}=\sqrt{(a-b)^2+(a-c)^2+(b-c)^2}$$

$$BC=\sqrt{(c-b)^2+(a-c)^2+(b-a)^2}=\sqrt{(a-b)^2+(a-c)^2+(b-c)^2}$$

The triangle ABC is equilateral.

2. A point on the y-axis has coordinates $(0,\ y,\ 0)$. The distance of this point to the point $A(1,\ 2,\ 3)$ is $\sqrt{(1-0)^2+(2-y)^2+(3-0)^2}=\sqrt{10}$. It follows that $1+(2-y)^2+9=10$ or $(2-y)^2=0$. Therefore, the point on the y-axis has coordinates $(0,\ 2,\ 0)$.

3. **(a)** Let $A(-1,\ 2,\ 3)$, $B(1,\ 4,\ 5)$, $C(5,\ 4,\ 0)$. Then:

$$AB=\sqrt{(1-(-1))^2+(4-2)^2+(5-3)^2}=\sqrt{4+4+4}=\sqrt{12}=2\sqrt{3}$$

$$AC=\sqrt{(5-(-1))^2+(4-2)^2+(0-3)^2}=\sqrt{36+4+9}=\sqrt{49}=7$$

$$BC=\sqrt{(5-1)^2+(4-4)^2+(0-5)^2}=\sqrt{16+0+25}=\sqrt{41}$$

Now, $AB+AC=2\sqrt{3}+7=10.5$ and $BC=\sqrt{41}=6.40$ (to 3 s. f.) $\Rightarrow AB+AC>BC$

$AB+BC=2\sqrt{3}+\sqrt{41}=9.87$ and $AC=7$ (to 3 s. f.) $\Rightarrow AB+BC>AC$

$AC+BC=7+\sqrt{41}=13.4$ and $AB=2\sqrt{3}=3.46$ (to 3 s. f.) $\Rightarrow AC+BC>AB$

The triangle inequality is satisfied, so the points A, B and C are the vertices of a triangle.

(b) Let $P(2,\ -3,\ 3)$, $Q(1,\ 2,\ 4)$, $R(3,\ -8,\ 2)$. Then:

$$PQ=\sqrt{(1-2)^2+(2-(-3))^2+(4-3)^2}=\sqrt{1+25+1}=\sqrt{27}=3\sqrt{3}$$

$$PR=\sqrt{(3-2)^2+(-8-(-3))^2+(2-3)^2}=\sqrt{1+25+1}=\sqrt{27}=3\sqrt{3}$$

$$QR=\sqrt{(3-1)^2+(-8-2)^2+(2-4)^2}=\sqrt{4+100+4}=\sqrt{108}=6\sqrt{3}$$

Now, since $PQ+PR=QR$, the triangle inequality is not satisfied, so the points P, Q and R are not the vertices of a triangle.

4. Let $A(0,\ 7,\ 10)$, $B(-1,\ 6,\ 6)$, $C(-4,\ 9,\ 6)$. Then:

$$AB=\sqrt{(-1-0)^2+(6-7)^2+(6-10)^2}=\sqrt{1+1+16}=\sqrt{18}=3\sqrt{2}$$

$$AC=\sqrt{(-4-0)^2+(9-7)^2+(6-10)^2}=\sqrt{16+4+16}=\sqrt{36}=6$$

$$BC=\sqrt{(-4-(-1))^2+(9-6)^2+(6-6)^2}=\sqrt{9+9+0}=\sqrt{18}=3\sqrt{2}$$

It is clearly seen that $AB=BC$. Now: $(AB)^2+(BC)^2=(3\sqrt{2})^2+(3\sqrt{2})^2=18+18=36$ and $(AC)^2=6^2=36\Rightarrow(AB)^2+(BC)^2=(AC)^2$. Therefore, the triangle ABC is a right-angled isosceles triangle.

5. **(a)** Let $A(0, -1, -7)$, $B(2, 1, -9)$, $C(6, 5, -13)$. Then:

$$AB = \sqrt{(2-0)^2 + (1-(-1))^2 + (-9-(-7))^2} = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$AC = \sqrt{(6-0)^2 + (5-(-1))^2 + (-13-(-7))^2} = \sqrt{36+36+36} = \sqrt{108} = 6\sqrt{3}$$

$$BC = \sqrt{(6-2)^2 + (5-1)^2 + (-13-(-9))^2} = \sqrt{16+16+16} = \sqrt{48} = 4\sqrt{3}$$

As $AC = AB + BC$, the points A, B and C are collinear.

(b) Let $A(-2, 0, 4)$, $B(5, -1, 1)$, $C(4, -6, 3)$. Then:

$$AB = \sqrt{(5-(-2))^2 + (-1-0)^2 + (1-4)^2} = \sqrt{9+1+9} = \sqrt{19}$$

$$AC = \sqrt{(4-(-2))^2 + (-6-0)^2 + (3-4)^2} = \sqrt{4+36+1} = \sqrt{41}$$

$$BC = \sqrt{(4-5)^2 + (-6-(-1))^2 + (3-1)^2} = \sqrt{1+25+4} = \sqrt{30}$$

As $AC \neq AB + BC$ and $AB \neq AC + BC$ and $BC \neq AC + AB$,
the points A, B and C are not collinear.

(c) Let $A(1, 8, -4)$, $B(-3, 5, -1)$, $C(2, 7, 2)$. Then:

$$AB = \sqrt{(-3-1)^2 + (5-8)^2 + (-1-(-4))^2} = \sqrt{16+9+9} = \sqrt{34}$$

$$AC = \sqrt{(2-1)^2 + (7-8)^2 + (2-(-4))^2} = \sqrt{1+1+36} = \sqrt{38}$$

$$BC = \sqrt{(2-(-3))^2 + (7-5)^2 + (2-(-1))^2} = \sqrt{25+4+9} = \sqrt{38}$$

As $AC \neq AB + BC$, $AB \neq AC + BC$, and $BC \neq AC + AB$,
the points A, B and C are not collinear.

(d) Let $A(2, 3, 4)$, $B(-1, 2, -3)$, $C(-4, 1, -10)$. Then:

$$AB = \sqrt{(-1-2)^2 + (2-3)^2 + (-3-4)^2} = \sqrt{9+1+49} = \sqrt{59}$$

$$AC = \sqrt{(-4-2)^2 + (1-3)^2 + (-10-4)^2} = \sqrt{36+4+196} = \sqrt{236} = 2\sqrt{59}$$

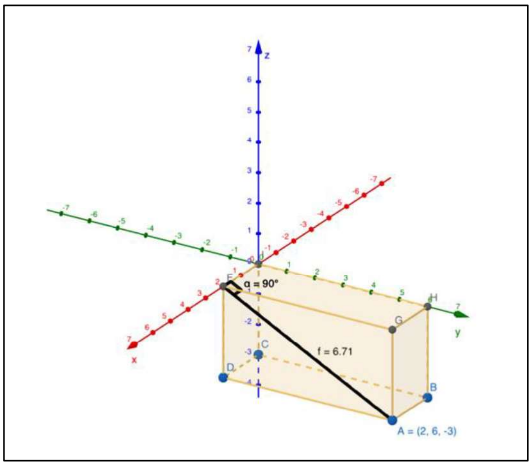
$$BC = \sqrt{(-4-(-1))^2 + (1-2)^2 + (-10-(-3))^2} = \sqrt{9+1+49} = \sqrt{59}$$

As $AC = AB + BC$, the points A, B and C are collinear.

6. **(a)** Let $A(2,\ 6,\ -3)$

(i) The origin of the coordinate system has coordinates $O(0,\ 0,\ 0)$, so the distance $OA = \sqrt{(2-0)^2 + (6-0)^2 + (-3-0)^2} = \sqrt{4+36+9} = \sqrt{49} = 7$

(ii)



The distance from the point $A(2,\ 6,\ -3)$ to the x-axis is equal to the length of the perpendicular segment joining the point A to the x-axis. This segment lies in the plane perpendicular to the xz-plane passing through the point A. The second endpoint of this segment lying on the x-axis has coordinates $(2,\ 0,\ 0)$. The distance, therefore, is:

$$\sqrt{(2-2)^2 + (0-6)^2 + (0-(-3))^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5} = 6.71 \text{ (to 3 s. f.)}$$

(iii) Similarly, the distance from the point A to the y-axis is equal to the length of the perpendicular segment joining the point to the y-axis. Since the segment lies in the plane perpendicular to the yz-plane, its second endpoint has coordinates $(0,\ 6,\ 0)$. The distance from the point A to the y-axis is therefore

$$\sqrt{(0-2)^2 + (6-6)^2 + (0-(-3))^2} = \sqrt{4+0+9} = \sqrt{13} = 3.61 \text{ (to 3 s. f.)}$$

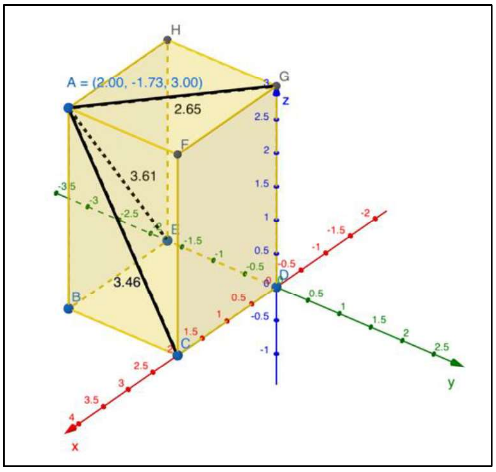
(iv) The distance from the point A to the z-axis is equal to the length of the perpendicular segment joining the point to the z-axis. Since the segment lies in the plane perpendicular to the xz-plane, its second endpoint has coordinates $(0,\ 0,\ -3)$. The distance from the point A to the y-axis is therefore

$$\sqrt{(0-2)^2 + (0-6)^2 + (-3-(-3))^2} = \sqrt{4+36+0} = \sqrt{40} = 2\sqrt{10} = 6.32 \text{ (to 3 s. f.)}$$

(b) Let $A\left(2, -\sqrt{3}, 3\right)$

(i) The origin of the coordinate system has coordinates $O\left(0, 0, 0\right)$, so the distance $OA = \sqrt{(2-0)^2 + \left(-\sqrt{3}-0\right)^2 + (3-0)^2} = \sqrt{4+3+9} = \sqrt{16} = 4$

(ii)



The distance from the point $A\left(2, -\sqrt{3}, 3\right)$ to the x-axis is equal to the length of the perpendicular segment joining the point A to the x-axis. This segment lies in the plane perpendicular to the xz-plane passing through the point A. The second endpoint of this segment lying on the x-axis has coordinates $\left(2, 0, 0\right)$. The distance, therefore, is:

$$\sqrt{(2-2)^2 + \left(0-\left(-\sqrt{3}\right)\right)^2 + (0-3)^2} = \sqrt{0+3+9} = \sqrt{12} = 2\sqrt{3} = 3.46 \text{ (to 3 s. f.)}$$

(iii) Similarly, the distance from the point A to the y-axis is equal to the length of the perpendicular segment joining the point to the y-axis. Since the segment lies in the plane perpendicular to the yz-plane, its second endpoint has coordinates $\left(0, -\sqrt{3}, 0\right)$. The distance from the point A to the y-axis is

$$\sqrt{(0-2)^2 + \left(-\sqrt{3}-\left(-\sqrt{3}\right)\right)^2 + (0-3)^2} = \sqrt{4+0+9} = \sqrt{13} = 3.61 \text{ (to 3 s. f.)}$$

(iv) The distance from the point A to the z-axis is equal to the length of the perpendicular segment joining the point to the z-axis. Since the segment lies in the plane perpendicular to the xz-plane, its second endpoint has coordinates $\left(0, 0, 3\right)$. The distance from the point A to the y-axis is therefore

$$\sqrt{(0-2)^2 + \left(0-\left(-\sqrt{3}\right)\right)^2 + (3-3)^2} = \sqrt{4+3+0} = \sqrt{7} = 2.65 \text{ (to 3 s. f.)}$$

7. The diagonals of the parallelogram *PQRS* bisect each other. Thus, the diagonals *PR* and *QS* must have the same midpoint *M*:

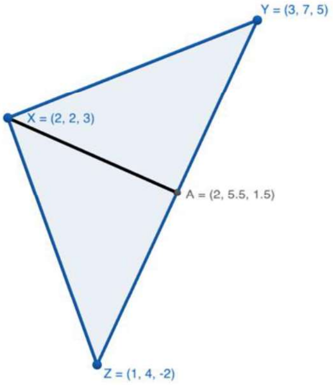
$$M\left(\frac{-2+6}{2}, \frac{2+(-2)}{2}, \frac{4+4}{2}\right) = M(2, 0, 8)$$

Let $S(x, y, z)$. Then

$$M\left(\frac{2+x}{2}, \frac{4+y}{2}, \frac{-8+z}{2}\right) = M(2, 0, 4) \Leftrightarrow \frac{2+x}{2} = 2, \quad \frac{4+y}{2} = 0, \quad \frac{-8+z}{2} = 4$$

Therefore $x = 2,\ y = -4,\ z = 16$ and $S(2, -4, 16)$

8.



(a) $XY = \sqrt{(3-2)^2 + (7-2)^2 + (5-3)^2} = \sqrt{1+25+4} = \sqrt{30}$

$XZ = \sqrt{(1-2)^2 + (4-2)^2 + (-2-3)^2} = \sqrt{1+4+25} = \sqrt{30}$

$YZ = \sqrt{(1-3)^2 + (4-7)^2 + (-2-5)^2} = \sqrt{4+9+49} = \sqrt{62}$

Since $XY = XZ$, the triangle *XYZ* is isosceles.

(b) The midpoint of *YZ*, $A\left(\frac{1+3}{2}, \frac{4+7}{2}, \frac{-2+5}{2}\right) = A\left(2, \frac{11}{2}, \frac{3}{2}\right)$.

The segment *XA* is perpendicular to the segment *YZ* and its length is

$$XA = \sqrt{(2-2)^2 + \left(\frac{11}{2}-2\right)^2 + \left(\frac{3}{2}-3\right)^2} = \sqrt{0+\frac{49}{4}+\frac{9}{4}} = \sqrt{\frac{58}{4}} = \frac{\sqrt{58}}{2}$$

Therefore, the area of the triangle *XYZ*:

$$A_{XYZ} = \frac{1}{2}(YZ)(XA) = \frac{1}{2}\left(\sqrt{62}\right)\left(\frac{\sqrt{58}}{2}\right) = \frac{1}{2}\left(\sqrt{2}\sqrt{31}\right)\left(\frac{\sqrt{2}\sqrt{29}}{2}\right) = \frac{1}{2}\sqrt{31}\sqrt{29} = \frac{\sqrt{899}}{2}$$

9. The length of segment $AB = \sqrt{(6-2)^2 + (1-(-7))^2 + (2-(-4))^2} = \sqrt{16+64+36} = 2\sqrt{29}$

The radius of the sphere: $r = \frac{1}{2}AB = \frac{1}{2}\left(2\sqrt{29}\right) = \sqrt{29}$

The surface area $S = 4\pi\left(\sqrt{29}\right)^2 = 116\pi$ units^2,

the volume $V = \frac{4}{3}\pi\left(\sqrt{29}\right)^3 = \frac{4}{3}\pi\left(29\sqrt{29}\right) = \frac{116}{3}\pi\sqrt{29}$ units^3.

10. The length of the longest piece of straight wire that can be placed completely inside the box is less than or equal to the length of the diagonal of the box.

$$\text{Diagonal} = \sqrt{62^2 + 44^2 + 20^2} = \sqrt{3844 + 1936 + 400} = \sqrt{6180} = 78.6 \text{ (to 3 s. f.)}$$

Therefore, the longest piece of straight wire that can be placed completely inside the box is equal to 78 cm.

11. The surface area of the cone is $S_1 = \pi r^2\sqrt{r^2 + h_1^2}$, the surface area of the cylinder is $S_2 = 2\pi r^2 h_2$, and the surface area of the hemisphere is $S_3 = 2\pi r^2$, where $r = 3$ cm, $h_1 = 5$ cm, $h_2 = 16\text{ cm} - 5\text{ cm} - 3\text{ cm} = 8\text{ cm}$.

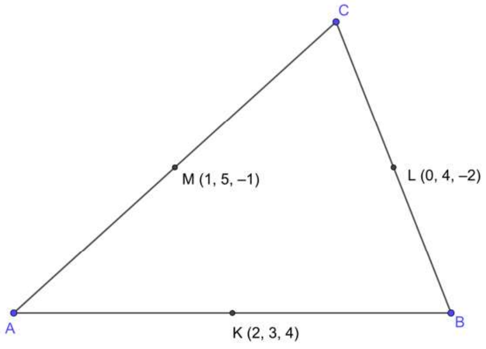
Therefore, the surface area of the solid is:

$$S = \pi(3)\left(\sqrt{3^2+5^2}\right) + 2\pi(3)(8) + 2\pi\left(3^2\right) = 3\pi\sqrt{34} + 48\pi + 18\pi = 66\pi + 3\pi\sqrt{34} = 262\text{ cm}^2 \text{(to 3 s. f.)}$$

The volume of the cone is $V_1 = \frac{1}{3}\pi r^2 h_1$, the volume of the cylinder is $V_2 = \pi r^2 h_2$ and the volume of the hemisphere is $V_3 = \frac{2}{3}\pi r^3$. Therefore, the volume of the solid is

$$V = \frac{1}{3}\pi\left(3^2\right)(5) + \pi\left(3^2\right)(8) + \frac{2}{3}\pi\left(3^3\right) = 15\pi + 72\pi + 18\pi = 105\pi = 330\text{ cm}^3 \text{ (to 3 s. f.)}$$

12. Draw the triangle.



Let $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$

When we join the midpoints of any two sides, the segment will be parallel to the third side. Thus, we have *CMKL* is a parallelogram. Then segment *ML* and segment *CK* have the same midpoint. The midpoint of *ML* is $\left(\frac{1}{2},\frac{9}{2},-\frac{3}{2}\right)$, so, we can write

$$\frac{x_C+2}{2}=\frac{1}{2}\Rightarrow x_C=-1 \text{ and } \frac{y_C+3}{2}=\frac{9}{2}\Rightarrow y_C=6 \text{ and } \frac{z_C+4}{2}=-\frac{3}{2}\Rightarrow z_C=-7$$

Similarly, the midpoint of *KL* is $\left(1,\frac{7}{2},1\right)$, and it is the midpoint of *BM*

$$\frac{x_B+1}{2}=1\Rightarrow x_B=1 \text{ and } \frac{y_B+5}{2}=\frac{7}{2}\Rightarrow y_B=2 \text{ and } \frac{z_B-1}{2}=1\Rightarrow z_B=3$$

Similarly, $A(3,4,5)$

13. **(a)** Volume of the sphere: $V_s=\frac{4}{3}\pi r^3$, volume of the cylinder

$V_c=\pi r^2h=\pi r^2(2r)=2\pi r^3$ The ratio:

$$\frac{V_s}{V_c}=\frac{\frac{4}{3}\pi r^3}{2\pi r^3}=\frac{2}{3}$$

(b) Surface of the sphere: $S_s=4\pi r^2$, surface of the cylinder

$S_c=2\pi r^2+2\pi rh=2\pi r^2+2\pi r(2r)=6\pi r^2$. The ratio: $\frac{S_s}{S_c}=\frac{4\pi r^2}{6\pi r^2}=\frac{2}{3}$

14. Surface area of the building consists of surface area of the cone and surface area of the cylinder, excluding the bases of both solids.

$$S=\pi rl+2\pi rh=\pi r(l+2h)=\pi(6)(15+2(50))=690\pi=2170\text{ m}^2$$

Volume of the building consists of volume of the cone and volume of the cylinder.

$$V=\frac{1}{3}\pi r^2h_c+\pi r^2h$$

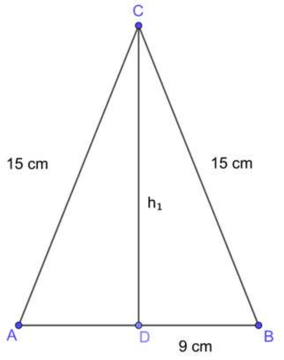
$$h_c=\sqrt{l^2-r^2}$$

$$V=\frac{1}{3}\pi r^2\sqrt{l^2-r^2}+\pi r^2h=\frac{1}{3}\pi(6^2)\sqrt{15^2-6^2}+\pi(6^2)(50)=6170\text{ m}^3$$

15. Volume of the spike consists of volume of the cube and volume of the pyramid:

$$V = 18^3 + \frac{1}{3}(18^2)h_p$$

We need to calculate the height of the pyramid.



In the triangle *BCD* (the side wall of the pyramid):

$$h_1 = \sqrt{15^2 - 9^2} = \sqrt{225 - 81} = \sqrt{144} = 12 \text{ cm}$$

Therefore, the height of the pyramid:

$$h_p = \sqrt{12^2 - 9^2} = \sqrt{144 - 81} = \sqrt{63} = 3\sqrt{7} \text{ cm}$$

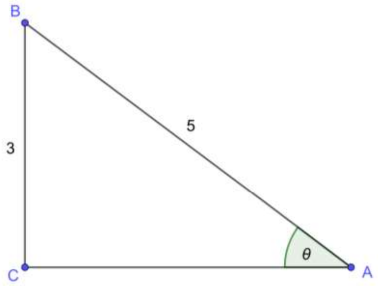
Volume of the spike: $V = 18^3 + \frac{1}{3}(18^2)3\sqrt{7} = 5832 + 324\sqrt{7} \text{ m}^3$

Surface area of the spike: $S = 5(18^2) + 4\left(\frac{1}{2}\right)(18)(12) = 2052 \text{ m}^2$

Exercise 7.2

1. **(a)** $\sin\theta = \frac{3}{5}$

(i)

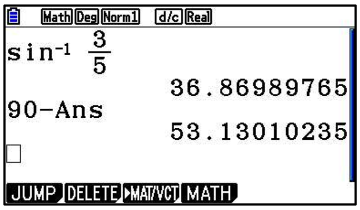


$AC = \sqrt{5^2 - 3^2} = 4$

(ii)

$$\cos\theta = \frac{AC}{AB} = \frac{4}{5}$$
$$\tan\theta = \frac{BC}{AC} = \frac{3}{4}$$
$$\cot\theta = \frac{AC}{BC} = \frac{4}{3}$$
$$\sec\theta = \frac{AB}{AC} = \frac{5}{4}$$
$$\csc\theta = \frac{AB}{BC} = \frac{5}{3}$$

(iii)

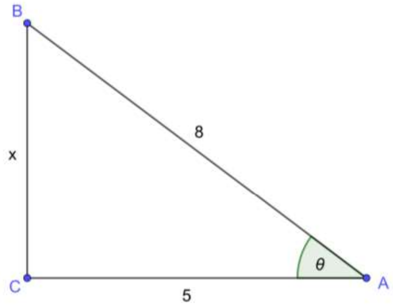


```
sin⁻¹ 3/5
                36.86989765
90-Ans
                53.13010235
```

$\theta = 36.9°$ or $\theta = 53.1°$

(b) $\cos\theta = \frac{5}{8}$

(i)



$x = \sqrt{8^2 - 5^2} = \sqrt{64-25} = \sqrt{39}$

(ii)

$$\sin\theta = \frac{BC}{AB} = \frac{\sqrt{39}}{8}$$

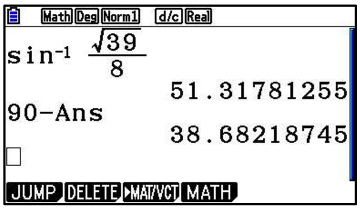
$$\tan\theta = \frac{BC}{AC} = \frac{\sqrt{39}}{5}$$

$$\cot\theta = \frac{AC}{BC} = \frac{5}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$$

$$\sec\theta = \frac{AB}{AC} = \frac{8}{5}$$

$$\csc\theta = \frac{AB}{BC} = \frac{8}{\sqrt{39}} = \frac{8\sqrt{39}}{39}$$

(iii)

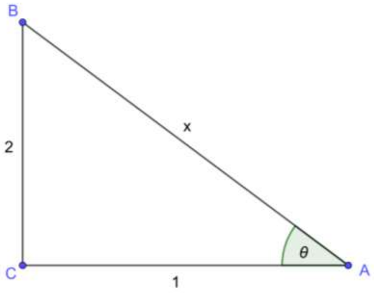


$\sin^{-1}\frac{\sqrt{39}}{8}$
51.31781255
90−Ans
38.68218745

$\theta = 51.3°$ or $\theta = 38.7°$

(c) $\tan\theta = 2$

(i)



$x = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$

(ii)

$$\sin\theta = \frac{BC}{AB} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

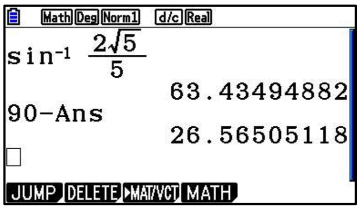
$$\cos\theta = \frac{AC}{AB} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cot\theta = \frac{AC}{BC} = \frac{1}{2}$$

$$\sec\theta = \frac{AB}{AC} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

$$\csc\theta = \frac{AB}{BC} = \frac{\sqrt{5}}{2}$$

(iii)

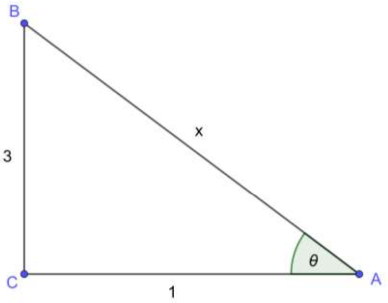


$\sin^{-1}\frac{2\sqrt{5}}{5}$
63.43494882
90−Ans
26.56505118

$\theta = 63.4°$ or $\theta = 26.6°$

(d) $\cot\theta = \dfrac{1}{3}$

(i)



$$x = \sqrt{3^2 + 1^2} = \sqrt{10}$$

(ii)

$$\sin\theta = \frac{BC}{AB} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

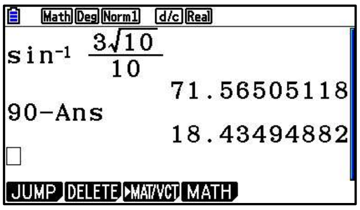
$$\cos\theta = \frac{AC}{AB} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\tan\theta = \frac{BC}{AC} = \frac{3}{1} = 3$$

$$\sec\theta = \frac{AB}{AC} = \frac{\sqrt{10}}{1} = \sqrt{10}$$

$$\csc\theta = \frac{AB}{BC} = \frac{\sqrt{10}}{3}$$

(iii)

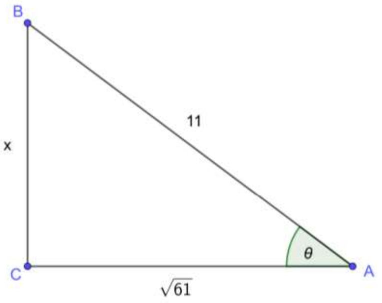


```
sin⁻¹ 3√10/10
                71.56505118
90-Ans
                18.43494882
```

$\theta = 71.6°$ or $\theta = 18.4°$

(e) $\sec\theta = \dfrac{11}{\sqrt{61}}$

(i)



$$x = \sqrt{11^2 - \left(\sqrt{61}\right)^2} = \sqrt{60} = 2\sqrt{15}$$

(ii)

$$\sin\theta = \frac{BC}{AB} = \frac{2\sqrt{15}}{11}$$

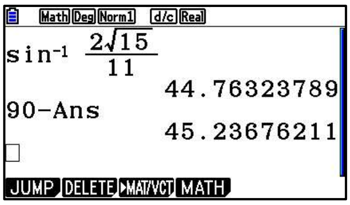
$$\cos\theta = \frac{AC}{AB} = \frac{\sqrt{61}}{11}$$

$$\tan\theta = \frac{BC}{AC} = \frac{2\sqrt{15}}{\sqrt{61}} = \frac{2\sqrt{915}}{61}$$

$$\cot\theta = \frac{AC}{BC} = \frac{\sqrt{61}}{2\sqrt{15}} = \frac{\sqrt{915}}{30}$$

$$\csc\theta = \frac{AB}{BC} = \frac{11}{2\sqrt{15}} = \frac{11\sqrt{15}}{30}$$

(iii)

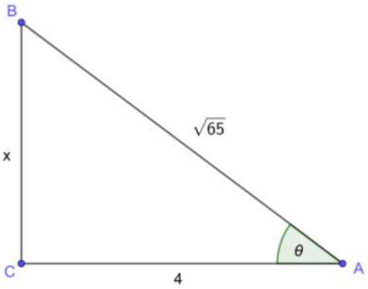


```
sin⁻¹ 2√15/11
                44.76323789
90-Ans
                45.23676211
```

$\theta = 44.8°$ or $\theta = 45.2°$

(f) $\cos\theta = \dfrac{4\sqrt{65}}{65} = \dfrac{4}{\sqrt{65}}$

(i)



$x = \sqrt{\left(\sqrt{65}\right)^2 - 4^2} = \sqrt{65-16} = 7$

(ii)

$$\sin\theta = \frac{BC}{AB} = \frac{7}{\sqrt{65}} = \frac{7\sqrt{65}}{65}$$

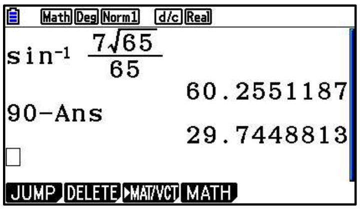
$$\tan\theta = \frac{BC}{AC} = \frac{7}{4}$$

$$\cot\theta = \frac{AC}{BC} = \frac{4}{7}$$

$$\sec\theta = \frac{AB}{AC} = \frac{\sqrt{65}}{4}$$

$$\csc\theta = \frac{AB}{BC} = \frac{\sqrt{65}}{7}$$

(iii)



```
sin⁻¹ 7√65/65
                60.2551187
90−Ans
                29.7448813
```

$\theta = 60.3°$ or $\theta = 29.7°$

2. **(a)** $\cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60°$ or $\theta = \dfrac{60°}{180°}\pi = \dfrac{\pi}{3}$

(b) $\sin\theta = \dfrac{\sqrt{2}}{2} \Rightarrow \theta = 45°$ or $\theta = \dfrac{45°}{180°}\pi = \dfrac{\pi}{4}$

(c) $\tan\theta = \sqrt{3} \Rightarrow \theta = 60°$ or $\theta = \dfrac{60°}{180°}\pi = \dfrac{\pi}{3}$

(d) $\csc\theta = \dfrac{2\sqrt{3}}{3} \Rightarrow \sin\theta = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 60°$ or $\theta = \dfrac{60°}{180°}\pi = \dfrac{\pi}{3}$

(e) $\cot\theta = 1 \Rightarrow \theta = 45°$ or $\theta = \dfrac{45°}{180°}\pi = \dfrac{\pi}{4}$

(f) $\cos\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 30°$ or $\theta = \dfrac{30°}{180°}\pi = \dfrac{\pi}{6}$

3. **(a)** $\cos 60° = \dfrac{50}{y} \Rightarrow y = \dfrac{50}{\cos 60°} = \dfrac{50}{\frac{1}{2}} = 100$, $\sin 60° = \dfrac{x}{y} \Rightarrow x = y\sin 60° = \dfrac{100\sqrt{3}}{2} = 50\sqrt{3}$

(b) $\sin 55° = \dfrac{y}{15} \Rightarrow y = 15\sin 55° = 12.3$, $\cos 55° = \dfrac{x}{15} \Rightarrow x = 15\cos 55° = 8.60$

(c) $\sin 40° = \dfrac{x}{32} \Rightarrow x = 32\sin 40° = 20.6$, $\cos 40° = \dfrac{y}{32} \Rightarrow y = 32\cos 40° = 24.5$

(d) $\tan 53° = \dfrac{y}{225} \Rightarrow y = 225\tan 53° = 299$, $\cos 53° = \dfrac{225}{x} \Rightarrow x = \dfrac{225}{\cos 53°} = 374$

(e) The triangle is a right-angled isosceles triangle, so $x = 18,\ y = 18\sqrt{2}$

(f) $\sin 30° = \dfrac{100}{x} \Rightarrow x = \dfrac{100}{\sin 30°} = \dfrac{100}{\frac{1}{2}} = 200$, $\cot 30° = \dfrac{y}{100} \Rightarrow y = 100\cot 30° = 100\sqrt{3}$

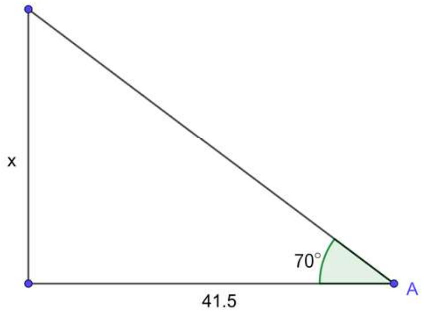
4. **(a)** $\tan\alpha = \dfrac{\sqrt{300}}{10} = \dfrac{10\sqrt{3}}{10} = \sqrt{3} \Rightarrow \alpha = 60°$, $\beta = 90° - \alpha = 90° - 60° = 30°$

(b) $\sin\beta = \dfrac{15}{39} = \dfrac{5}{13} \Rightarrow \beta = 22.6°$, $\alpha = 90° - \beta = 90° - 22.6° = 67.4°$

(c) $\tan\alpha = \dfrac{44}{121} = \dfrac{4}{11} \Rightarrow \alpha = 20.0°$, $\beta = 90° - \alpha = 90° - 20.0° = 70.0°$

(d) $\sin\alpha = \dfrac{\sqrt{7}}{\sqrt{28}} = \sqrt{\dfrac{7}{28}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2} \Rightarrow \alpha = 30°$, $\beta = 90° - \alpha = 90° - 30° = 60°$

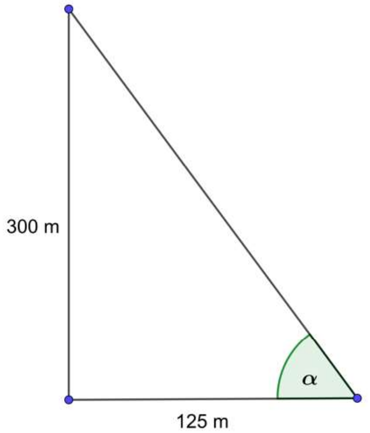
5.



Let the length of the tree be *x*.

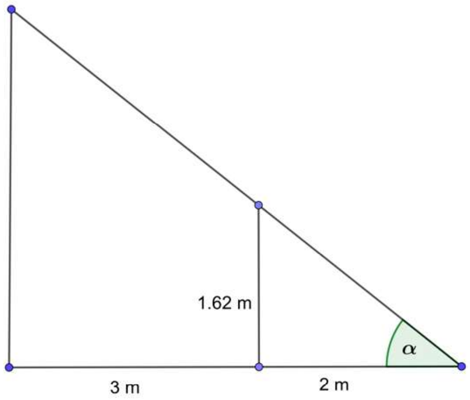
$\tan 70° = \dfrac{x}{41.5} \Rightarrow x = 41.5\tan 70° = 114\ \text{m}$

6.



$\tan\alpha = \dfrac{300}{125} = 2.4 \ \Rightarrow\ \alpha = 67.4°$ (to 3 s. f.)

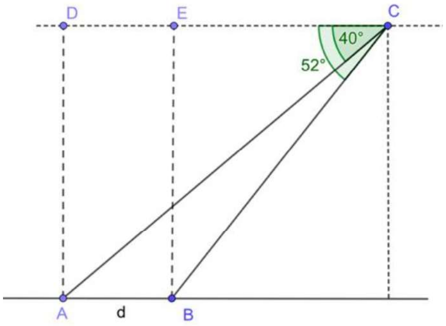
7.



$$\tan\alpha = \frac{1.62}{2} = 0.81$$

$$\tan\alpha = \frac{x}{5} \Rightarrow x = 5\tan\alpha = 5(0.81) = 4.05\text{ m}$$

8.



In the triangle *ACD*:

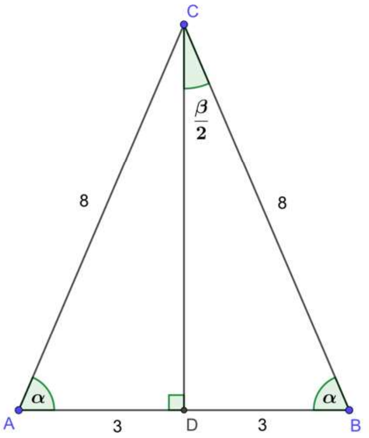
$$\cot 40° = \frac{CD}{AD} \Rightarrow CD = AD\cot 40° = 10\,000\cot 40°$$

$$\cot 52° = \frac{CE}{BE} \Rightarrow CE = BE\cot 52° = 10\,000\cot 52°$$

In the triangle *BCE*:

$$d = CD - CE = 10\,000(\cot 40° - \cot 52°) = 4105\text{ m}$$

9.



The triangle is isosceles, so the two angles at the base have the same measure.

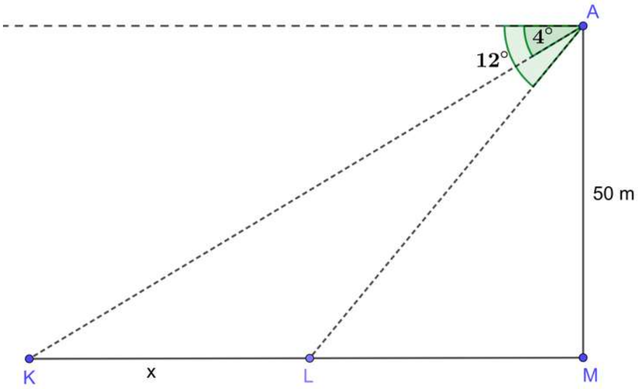
In the triangle *BCD*:

$$\cos\alpha = \frac{3}{8} \Rightarrow \alpha = 68.0° \text{ (to 3 s. f.)}$$

and $\frac{1}{2}\beta = 90° - \alpha = 90° - 68.0° = 22.0° \Rightarrow \beta = 44.0°$

Therefore, the angles are $68.0°$, $68.0°$ and $44.0°$

10.



In the triangle KMA:

$$|\measuredangle KAM| = 90° - 4° = 86°$$

$$\tan 86° = \frac{KM}{AM} = \frac{KM}{50} \Rightarrow KM = 50\tan 86°$$

In the triangle LMA:

$$|\measuredangle LAM| = 90° - 12° = 78°$$

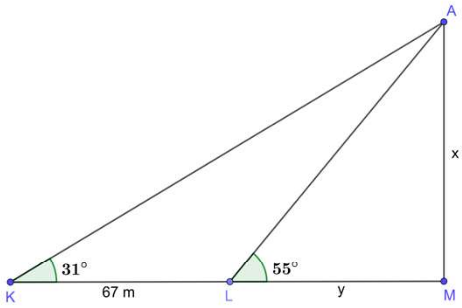
$$\tan 78° = \frac{LM}{AM} = \frac{LM}{50} \Rightarrow LM = 50\tan 78°$$

$$x = KM - LM = 50(\tan 86° - \tan 78°) = 479.8 \text{ m}$$

The boat moved 479.8 m in 5 minutes. Therefore, the speed of the boat was:

$$v = \frac{479.8 \text{ m}}{5 \text{ min}} = \frac{0.4798 \text{ km}}{\frac{1}{12} \text{ h}} = 5.76 \text{ kmh}^{-1}$$

11.



In triangle ABC: $\tan 31° = \dfrac{x}{67+y}$. In triangle BCD: $\tan 55° = \dfrac{x}{y} \Rightarrow y = \dfrac{x}{\tan 55°}$.

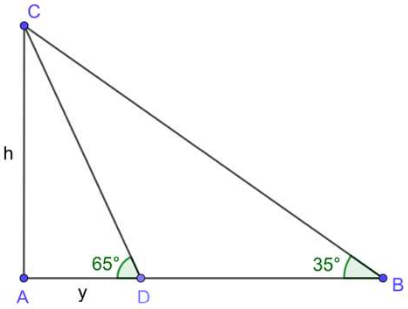
Now, $(67+y)\tan 31° = x$ and $y = \dfrac{x}{\tan 55°}$, therefore,

$$\left(67+\frac{x}{\tan 55°}\right)\tan 31° = x \Rightarrow 67\tan 31° + \frac{\tan 31°}{\tan 55°}x = x \Rightarrow 67\tan 31° \tan 55° + x\tan 31° = x\tan 55°$$

$$67\tan 31° \tan 55° = x\left(\tan 55° - \tan 31°\right) \Rightarrow x = \frac{\tan 31° \tan 55°}{\tan 55° - \tan 31°}(67)$$

$x = 69.5$ (to 3 s. f.)

12.



In triangle ABC: $\tan 35° = \dfrac{h}{25+y}$. In triangle ADC: $\tan 65° = \dfrac{h}{y} \Rightarrow h = y\tan 65°$.

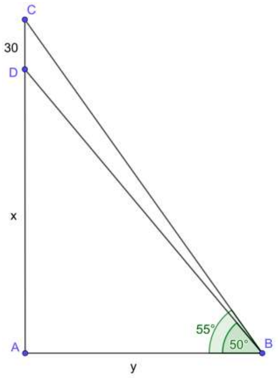
Now, $25\tan 35° + y\tan 35° = y\tan 65° \Rightarrow y\left(\tan 65° - \tan 35°\right) = 25\tan 35°$

$$y = \frac{25\tan 35°}{\tan 65° - \tan 35°}$$

In triangle ADC: $\cos 65° = \dfrac{y}{x} \Rightarrow x = \dfrac{y}{\cos 65°}$

$$x = \frac{25\tan 35°}{\cos 65°\left(\tan 65° - \tan 35°\right)} = 28.7 \text{ (to 3 s. f.)}$$

13.



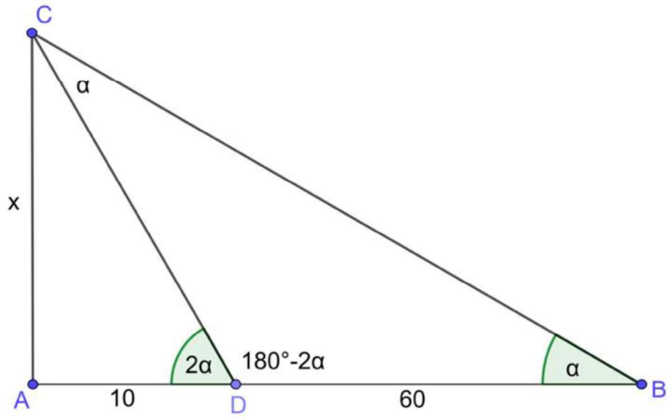
In triangle *ABC*: $\tan 55° = \dfrac{30+x}{y}$. In triangle *ABD*: $\tan 50° = \dfrac{x}{y} \Leftrightarrow y = \dfrac{x}{\tan 50°}$

Now, $y\tan 55° = 30 + x$

$$\frac{\tan 55°}{\tan 50°}x - x = 30$$

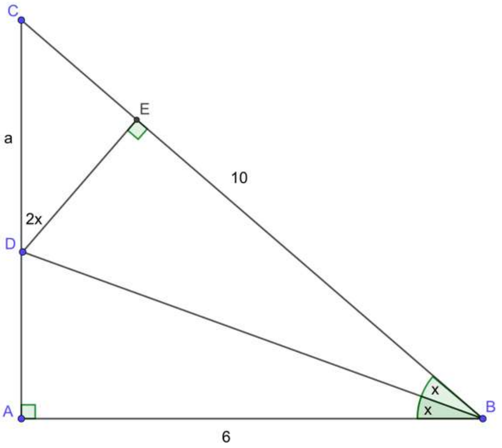
$$\left(\frac{\tan 55°}{\tan 50°} - 1\right)x = 30 \Rightarrow \frac{\tan 55° - \tan 50°}{\tan 50°}x = 30 \Rightarrow x = \frac{30\tan 50°}{\tan 55° - \tan 50°} = 151 \text{ m (to 3 s. f.)}$$

14.



In triangle *ABD*: $|\measuredangle ADB| = 180° - 2\alpha$ and $|\measuredangle DAB| = \alpha$. It follows that $|\measuredangle DBA| = \alpha$ and the triangle *ABD* is an isosceles triangle, where *BD* = *AD* = 60.

In triangle *BCD*:



$$x^2+10^2=60^2 \Rightarrow x^2=3600-100=3500 \Rightarrow x=10\sqrt{35}=59.2 \text{ (to 3 s. f.)}$$

15. In triangle *ABC*: $AC=\sqrt{10^2-6^2}=\sqrt{64}=8$. In triangle *BDE*: $\cos x=\dfrac{EB}{BD}$.

In triangle *ABD*: $\cos x=\dfrac{6}{BD}$. Comparing, we get $\dfrac{EB}{BD}=\dfrac{6}{BD}\Rightarrow EB=6$ and $CE=4$.

In triangle *CDE*: $\sin 2x=\dfrac{4}{a}$. In triangle *ABC*: $\sin 2x=\dfrac{8}{10}=\dfrac{4}{5}$.

Therefore, $\dfrac{4}{a}=\dfrac{4}{5}\Rightarrow a=5$

In triangle *CDE*: $(DE)^2=a^2-(CE)^2=25-16=9$ and in triangle *BED*:

$$(DE)^2+(EB)^2=(BD)^2$$

$$(BD)^2=9+36=45\Rightarrow BD=\sqrt{45}=3\sqrt{5}$$

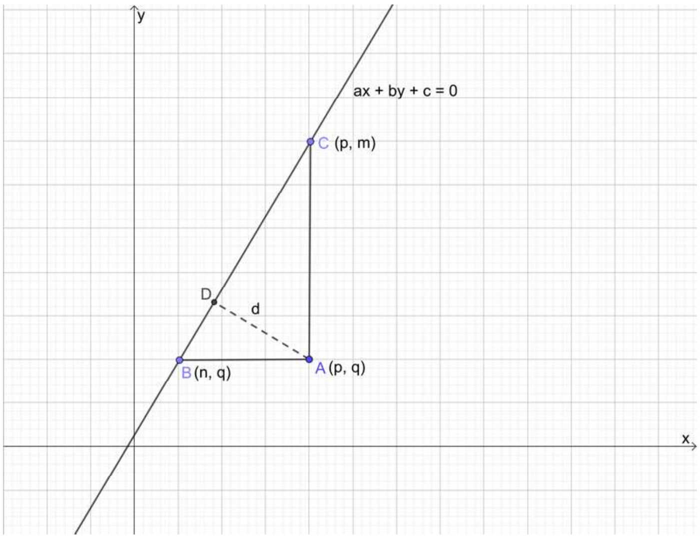
16. $|\measuredangle DEA|=90°+2x$

$$\cos|\measuredangle DEA|=\cos(90°+2x)=-\sin 2x=-2\sin x\cos x$$

From the picture: $\sin x=\dfrac{1}{\sqrt{10}}$, $\cos x=\dfrac{3}{\sqrt{10}}$

Therefore $\cos|\measuredangle DEA|=-2\left(\dfrac{1}{\sqrt{10}}\right)\left(\dfrac{3}{\sqrt{10}}\right)=-\dfrac{6}{10}=-0.6$

17.



The coordinates of point *B* and point *C* satisfy the equation of the line. We can write:

$an+bq+c=0 \Rightarrow n=-\frac{bq+c}{a}$ and $ap+bm+c=0 \Rightarrow m=-\frac{ap+c}{b}$. Therefore,

$B\left(-\frac{bq+c}{a},\ q\right)$ and $C\left(p,\ -\frac{ap+c}{b}\right)$

Area of triangle *ABC*:

$$A=\frac{1}{2}\left(p+\frac{bq+c}{a}\right)\left(q+\frac{ap+c}{b}\right)=\frac{1}{2}\left(\frac{ap+bq+c}{a}\right)\left(\frac{ap+bq+c}{b}\right)=\frac{(ap+bq+c)^2}{2ab}$$

but also $A=\frac{1}{2}(BC)(d)$

$$BC=\sqrt{\left(p+\frac{bq+c}{a}\right)^2+\sqrt{\left(q+\frac{ap+c}{b}\right)^2}}=\sqrt{\frac{(ap+bq+c)^2}{a^2}+\frac{(ap+bq+c)^2}{b^2}}\ .$$

Therefore, the area of triangle *ABC* can be calculated as

$A=\frac{1}{2}d\sqrt{\frac{(ap+bq+c)^2}{a^2}+\frac{(ap+bq+c)^2}{b^2}}$. Comparing we get

$$\frac{(ap+bq+c)^2}{2ab}=\frac{1}{2}d\sqrt{\frac{(ap+bq+c)^2}{a^2}+\frac{(ap+bq+c)^2}{b^2}}$$

$$\frac{(ap+bq+c)^2}{ab}=d\sqrt{(ap+bq+c)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

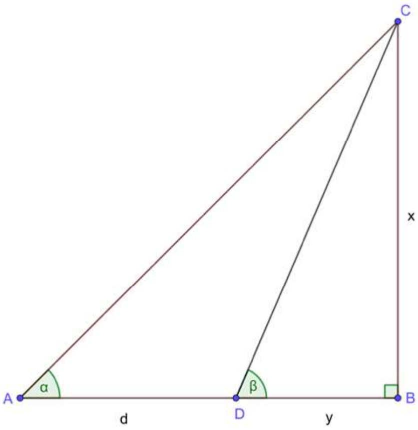
$$\frac{(ap+bq+c)^2}{ab}=d\sqrt{\frac{a^2+b^2}{a^2b^2}(ap+bq+c)^2}$$

$$\frac{(ap+bq+c)^2}{ab}=d\left|ap+bq+c\right|\frac{\sqrt{a^2+b^2}}{ab}$$

$$(ap+bq+c)^2=d\left|ap+bq+c\right|\sqrt{a^2+b^2}$$

$$d=\frac{(ap+bq+c)^2}{\left|ap+bq+c\right|\sqrt{a^2+b^2}}=\frac{\left|ap+bq+c\right|^2}{\left|ap+bq+c\right|\sqrt{a^2+b^2}}=\frac{\left|ap+bq+c\right|}{\sqrt{a^2+b^2}}$$

18.



In triangle ABC: $\tan\alpha=\frac{x}{d+y}$. In triangle BCD: $\tan\beta=\frac{x}{y}\Rightarrow y=\frac{x}{\tan\beta}$.

Now, $d\tan\alpha+y\tan\alpha=x$

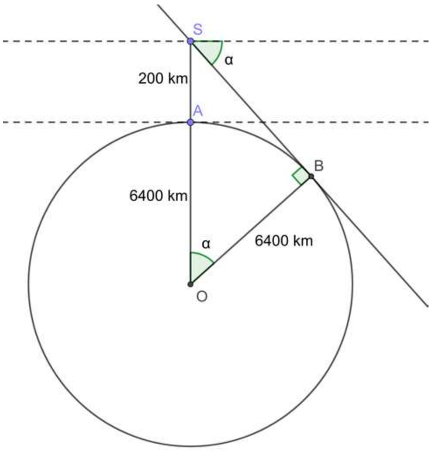
$$x=d\tan\alpha-\frac{x}{\tan\beta}\tan\alpha$$

$$x\tan\beta-x\tan\alpha=d\tan\alpha\tan\beta\Rightarrow x(\tan\beta-\tan\alpha)=d\tan\alpha\tan\beta$$

$x=\frac{\tan\alpha\tan\beta}{\tan\beta-\tan\alpha}d$. Multiplying numerator and denominator by $\frac{1}{\tan\alpha\tan\beta}$:

$$x=\frac{1}{\frac{1}{\tan\alpha}-\frac{1}{\tan\beta}}d\Rightarrow x=\frac{d}{\cot\alpha-\cot\beta}$$

19.



In triangle BOS, the angle $|\measuredangle OSB| = 90° - \alpha$ and $|\measuredangle SOB| = \alpha$.

Therefore, $\cos\alpha = \frac{OB}{OS} = \frac{6400}{6400+200} = \frac{6400}{6600} = 0.970$ (to 3 s. f)

and the angle of depression $\alpha = 14°$

Exercise 7.3

1. (a) Calculating the distance from the given point to the origin of the coordinate system we have:

$$r = \sqrt{12^2 + 9^2} = \sqrt{144+81} = \sqrt{225} = 15$$

Now, by definition: $\sin\theta = \frac{y}{r} = \frac{9}{15} = \frac{3}{5}$, $\cos\theta = \frac{x}{r} = \frac{12}{15} = \frac{4}{5}$, $\tan\theta = \frac{y}{x} = \frac{9}{12} = \frac{3}{4}$,

$\cot\theta = \frac{x}{y} = \frac{12}{9} = \frac{4}{3}$, $\csc\theta = \frac{r}{y} = \frac{15}{9} = \frac{5}{3}$, and $\sec\theta = \frac{r}{x} = \frac{15}{12} = \frac{5}{4}$

(b) Calculating the distance from the given point to the origin of the coordinate system we have:

$$r = \sqrt{(-35)^2 + 12^2} = \sqrt{1225+144} = \sqrt{1369} = 37$$

Now, by definition: $\sin\theta = \frac{y}{r} = \frac{12}{37}$, $\cos\theta = \frac{x}{r} = -\frac{35}{37}$, $\tan\theta = \frac{y}{x} = -\frac{12}{35}$,

$\cot\theta = \frac{x}{y} = -\frac{35}{12}$, $\csc\theta = \frac{r}{y} = \frac{37}{12}$, and $\sec\theta = \frac{r}{x} = -\frac{37}{35}$

(c) Calculating the distance from the given point to the origin of the coordinate system we have:

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

Now, by definition: $\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$, $\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$,

$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$, $\cot\theta = \frac{x}{y} = \frac{1}{-1} = -1$, $\csc\theta = \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$,

and $\sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$

(d) Calculating the distance from the given point to the origin of the coordinate system we have:

$$r = \sqrt{\left(-\sqrt{75}\right)^2 + (-5)^2} = \sqrt{75+25} = \sqrt{100} = 10$$

Now, by definition: $\sin\theta = \frac{y}{r} = \frac{-5}{10} = -\frac{1}{2}$, $\cos\theta = \frac{x}{r} = \frac{-\sqrt{75}}{10} = -\frac{5\sqrt{3}}{10} = -\frac{\sqrt{3}}{2}$,

$\tan\theta = \frac{y}{x} = \frac{-5}{-\sqrt{75}} = \frac{5}{5\sqrt{3}} = \frac{\sqrt{3}}{3}$, $\cot\theta = \frac{x}{y} = \frac{-\sqrt{75}}{-5} = \frac{5\sqrt{3}}{5} = \sqrt{3}$, $\csc\theta = \frac{r}{y} = \frac{10}{-5} = -2$,

and $\sec\theta = \frac{r}{x} = \frac{10}{-\sqrt{75}} = -\frac{10}{5\sqrt{3}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

2. **(a)** $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

$\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

$\csc 120° = \frac{1}{\sin 120°} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$, $\sec 120° = \frac{1}{\cos 120°} = -\frac{2}{1} = -2$

$\tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$

$\cot 120° = \tan(180° - 60°) = -\cot 60° = -\frac{\sqrt{3}}{3}$

(b) $\sin 135° = \sin(180° - 45°) = \sin 45° = \frac{\sqrt{2}}{2}$

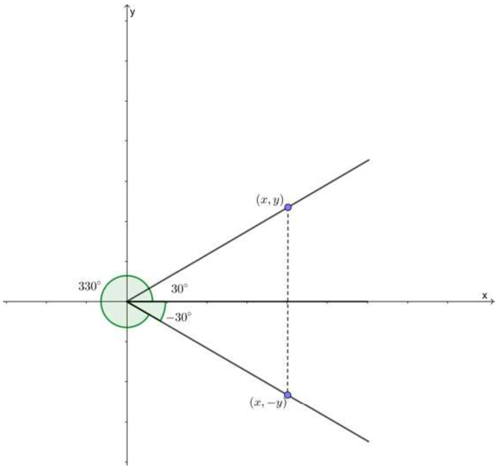
$\cos 135° = \cos(180° - 45°) = -\cos 45° = -\frac{\sqrt{2}}{2}$

$\csc 135° = \frac{1}{\sin 135°} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$, $\sec 135° = \frac{1}{\cos 135°} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

$\tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

$\cot 135° = \cot(180° - 45°) = -\cot 45° = -1$

(c)



$$\sin 330° = \frac{-y}{r} = -\frac{y}{r} = -\sin 30° = -\frac{1}{2}$$

$$\cos 330° = \frac{x}{r} = \cos 30° = \frac{\sqrt{3}}{2}$$

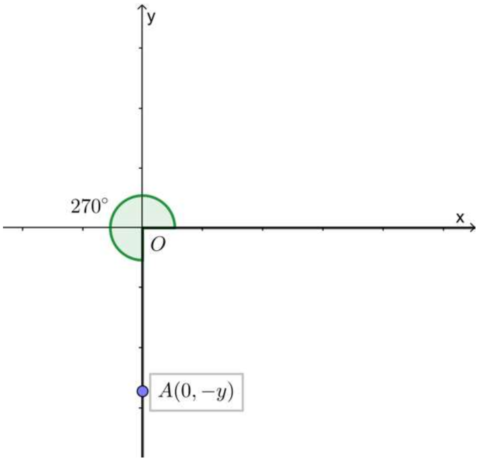
$$\csc 330° = \frac{1}{\sin 330°} = -2$$

$$\sec 330° = \frac{1}{\cos 330°} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan 330° = \frac{-y}{x} = -\frac{y}{x} = -\tan 30° = -\frac{\sqrt{3}}{3}$$

$$\cot 330° = \frac{x}{-y} = -\frac{x}{y} = -\cot 30° = -\sqrt{3}$$

(d)



$$\sin 270° = \frac{-y}{y} = -1$$

$$\cos 270° = \frac{x}{-y} = \frac{0}{-y} = 0,$$

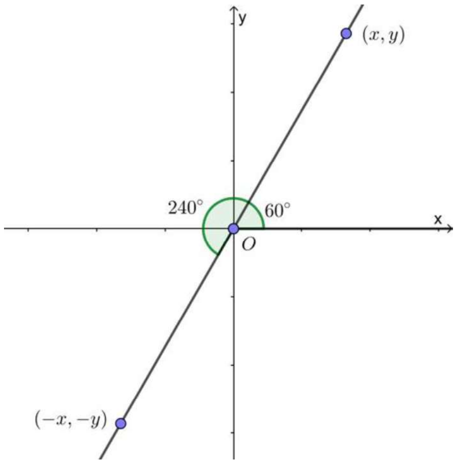
$$\csc 270° = \frac{1}{\sin 270°} = -1$$

$$\sec 270° = \frac{1}{\cos 270°} \quad - \text{ undefined}$$

$$\tan 270° = \frac{-y}{x} = \frac{-y}{0} \quad - \text{ undefined}$$

$$\cot 270° = \frac{0}{-y} = 0$$

(e)



$$\sin 240° = \frac{-y}{r} = -\frac{y}{r} = -\sin 60° = -\frac{\sqrt{3}}{2}$$

$$\cos 240° = \frac{-x}{r} = -\frac{x}{r} = -\cos 60° = -\frac{1}{2}$$

$$\csc 240° = \frac{1}{\sin 240°} = -\frac{2\sqrt{3}}{3}$$

$$\sec 240° = \frac{1}{\cos 240°} = -2$$

$$\tan 240° = \frac{-y}{-x} = \frac{y}{x} = \tan 60° = \sqrt{3}$$

$$\cot 240° = \frac{-x}{-y} = \frac{x}{y} = \cot 60° = \frac{\sqrt{3}}{3}$$

(f) $\sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$,

$\csc\frac{5\pi}{4} = \frac{1}{\sin\frac{5\pi}{4}} = -\sqrt{2}$, $\sec\frac{5\pi}{4} = \frac{1}{\cos\frac{5\pi}{4}} = -\sqrt{2}$,

$\tan\frac{5\pi}{4} = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$, $\cot\frac{5\pi}{4} = \cot\left(\pi + \frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1$

(g) $\sin\frac{\pi}{6} = \sin 30° = \frac{1}{2}$, $\cos\frac{\pi}{6} = \cos 30° = \frac{\sqrt{3}}{2}$, $\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = 2$,

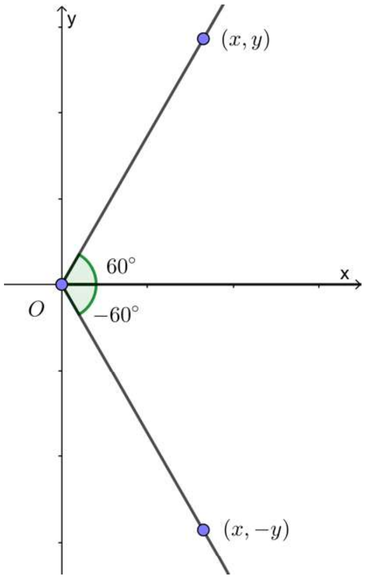
$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2\sqrt{3}}{3}$, $\tan\frac{\pi}{6} = \tan 30° = \frac{\sqrt{3}}{3}$, $\cot\frac{\pi}{6} = \cot 30° = \sqrt{3}$

(h) $\sin\frac{7\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{7\pi}{6} = \cos\left(\pi + \frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$,

$\csc\frac{7\pi}{6} = \frac{1}{\sin\frac{7\pi}{6}} = \frac{1}{\sin\frac{\pi}{6}} = 2$, $\sec\frac{7\pi}{6} = \frac{1}{\cos\frac{7\pi}{6}} = \frac{1}{-\cos\frac{\pi}{6}} = -\frac{2\sqrt{3}}{3}$,

$$\tan\frac{7\pi}{6}=\tan\left(\pi+\frac{\pi}{6}\right)=\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3},\ \cot\frac{7\pi}{6}=\cot\left(\pi+\frac{\pi}{6}\right)=\cot\frac{\pi}{6}=\sqrt{3}$$

(i)



$$\sin(-60°)=\frac{-y}{r}=-\frac{y}{r}=-\sin 60°=-\frac{\sqrt{3}}{2}$$

$$\cos(-60°)=\frac{x}{r}=\cos 60°=\frac{1}{2}$$

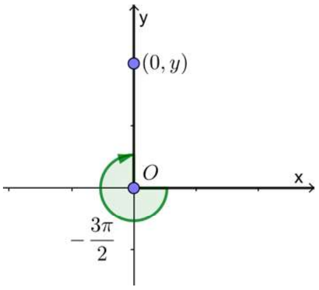
$$\csc(-60°)=\frac{1}{\sin(-60°)}=-\frac{2\sqrt{3}}{3}$$

$$\sec(-60°)=\frac{1}{\cos(-60°)}=2$$

$$\tan(-60°)=\frac{-y}{x}=-\frac{y}{x}=-\tan 60°=-\sqrt{3}$$

$$\cot(-60°)=\frac{x}{-y}=-\frac{x}{y}=-\cot 60°=-\frac{\sqrt{3}}{3}$$

(j)



The distance of the point $(0,\ y)$ to the origin of the coordinate system is equal to r, therefore:

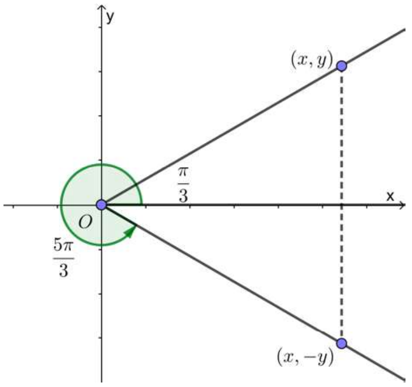
$$\sin\left(-\frac{3\pi}{2}\right)=\frac{y}{y}=1,\ \cos\left(-\frac{3\pi}{2}\right)=\frac{0}{y}=0$$

$$\csc\left(-\frac{3\pi}{2}\right)=\frac{1}{\sin\left(-\frac{3\pi}{2}\right)}=1$$

$$\sec\left(-\frac{3\pi}{2}\right)=\frac{1}{\cos\left(-\frac{3\pi}{2}\right)}=\frac{1}{0}\ \text{– undefined}$$

$$\tan\left(-\frac{3\pi}{2}\right)=\frac{y}{0}\ \text{– undefined},\ \cot\left(-\frac{3\pi}{2}\right)=\frac{0}{y}=0$$

(k)



$$\sin\left(\frac{5\pi}{3}\right)=\frac{-y}{r}=-\frac{y}{r}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{5\pi}{3}\right)=\frac{x}{r}=\cos\frac{\pi}{3}=\frac{1}{2}$$

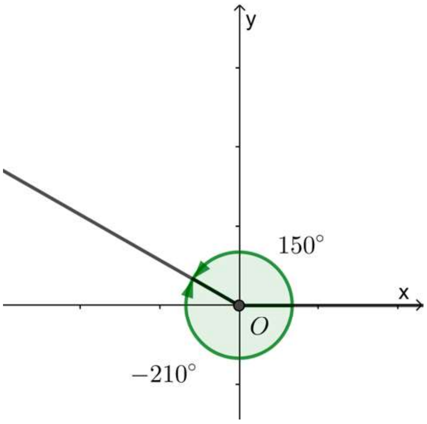
$$\csc\left(\frac{5\pi}{3}\right)=\frac{1}{\sin\left(\frac{5\pi}{3}\right)}=-\frac{2\sqrt{3}}{3}$$

$$\sec\left(\frac{5\pi}{3}\right)=\frac{1}{\cos\left(\frac{5\pi}{3}\right)}=2$$

$$\tan\left(\frac{5\pi}{3}\right)=\frac{-y}{x}=-\frac{y}{x}=-\tan\frac{\pi}{3}=-\sqrt{3}$$

$$\cot\left(\frac{5\pi}{3}\right)=\frac{x}{-y}=-\frac{x}{y}=-\cot\frac{\pi}{3}=-\frac{\sqrt{3}}{3}$$

(l)



$$\sin(-210°)=\sin 150°=\sin(180°-30°)=\sin 30°=\frac{1}{2}$$

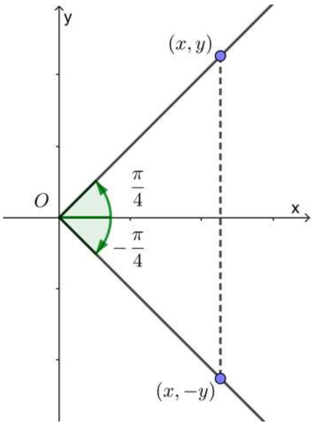
$$\cos(-210°)=\cos 150°=\cos(180°-30°)=-\cos 30°=-\frac{\sqrt{3}}{2}$$

$$\csc(-210°)=\frac{1}{\sin(-210°)}=2\ \sec(-210°)=\frac{1}{\cos(-210°)}=-\frac{2\sqrt{3}}{3}$$

$$\tan(-210°)=\tan 150°=\tan(180°-30°)=-\tan 30°=-\frac{\sqrt{3}}{3}$$

$$\cot(-210°)=\cot 150°=\cot(180°-30°)=-\cot 30°=-\sqrt{3}$$

(m)



$$\sin\left(-\frac{\pi}{4}\right)=\frac{-y}{r}=-\frac{y}{r}=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{\pi}{4}\right)=\frac{x}{r}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\csc\left(-\frac{\pi}{4}\right)=\frac{1}{\sin\left(-\frac{\pi}{4}\right)}=-\frac{2}{\sqrt{2}}=-\frac{2\sqrt{2}}{2}=-\sqrt{2}$$

$$\sec\left(-\frac{\pi}{4}\right)=\frac{1}{\cos\left(-\frac{\pi}{4}\right)}=\sqrt{2}$$

$$\tan\left(-\frac{\pi}{4}\right)=\frac{-y}{x}=-\frac{y}{x}=-\tan\frac{\pi}{4}=-1$$

$$\cot\left(-\frac{\pi}{4}\right)=\frac{x}{-y}=-\frac{x}{y}=-\cot\frac{\pi}{4}=-1$$

(n) The distance of the point $(-x,\ 0)$ to the origin of the coordinate system is equal to x, therefore:

$\sin\pi=\frac{0}{x}=0$, $\cos\pi=\frac{-x}{x}=-1$,

$\csc\pi=\frac{1}{\sin\pi}$ – undefined , $\sec\pi=\frac{1}{\cos\pi}=1$,

$\tan\pi=\frac{0}{-x}=0$, $\cot\pi=\frac{-x}{0}$ – undefined .

(o) The terminal side of the 4.25π angle forms an angle $\frac{\pi}{4}$ radians with the x-axis

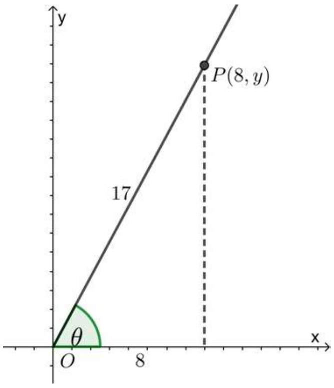
$$\left(4.25\pi=2(2\pi)+\frac{\pi}{4}\right)$$

Therefore: $\sin(4.25\pi)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, $\cos(4,25\pi)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$,

$\csc(4.25\pi) = \dfrac{1}{\sin\dfrac{\pi}{4}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$, $\sec(4.25\pi) = \dfrac{1}{\cos\dfrac{\pi}{4}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$,

$\tan(4.25\pi) = \tan\dfrac{\pi}{4} = 1$, $\cot(4.25\pi) = \cot\dfrac{\pi}{4} = 1$.

3.



Since θ is an angle whose terminal side lies in the first quadrant of the coordinate system, there must be a point on the terminal side that is 17 units from the origin. Given that $\cos\theta = \dfrac{8}{17}$ we can assume the first coordinate of the point is equal to 8. Then $y = \sqrt{17^2 - 8^2} = \sqrt{289 - 64} = \sqrt{225} = 15$, so the coordinates of the point P are $(8, 15)$. Now:

$\sin\theta = \dfrac{y}{r} = \dfrac{15}{17}$, $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{17}{8}$, $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{17}{15}$,

$\tan\theta = \dfrac{y}{x} = \dfrac{15}{8}$, $\cot\theta = \dfrac{x}{y} = \dfrac{8}{15}$.

Alternatively, we can use the Pythagorean identity: $\sin\theta = \sqrt{1 - \cos^2\theta}$

4. If $\tan\theta = -\dfrac{6}{5}$ and $\sin\theta < 0$, then the terminal side of the angle θ lies in the 4th quadrant of the coordinate system. We can assume that the coordinates of P on the terminal side are $(5, -6)$. Then: $r = \sqrt{5^2 + (-6)^2} = \sqrt{25 + 36} = \sqrt{61}$, so $\sin\theta = \dfrac{y}{r} = \dfrac{-6}{\sqrt{61}} = -\dfrac{6\sqrt{61}}{61}$ and

$\cos\theta = \dfrac{x}{r} = \dfrac{5}{\sqrt{61}} = \dfrac{5\sqrt{61}}{61}$.

5. If $\sin\theta = 0$ and $\cos\theta < 0$, then $\theta = 180°$. Therefore, $\cos\theta = \cos 180° = -1$,

$\csc\theta = \dfrac{1}{\sin 180°}$ – undefined, $\sec\theta = \dfrac{1}{\cos 180°} = -1$, $\tan\theta = \tan 180° = 0$,

$\cot\theta = \dfrac{1}{\tan 180°}$ – undefined.

6. If $\sec\theta = 2$ and $\dfrac{3\pi}{2} < \theta < 2\pi$, then $\cos\theta = \dfrac{1}{2}$. Therefore,

$\sin\theta = -\sqrt{1 - \cos^2\theta} = -\sqrt{1 - \left(\dfrac{1}{2}\right)^2} = -\sqrt{\dfrac{3}{4}} = -\dfrac{\sqrt{3}}{2}$ (the minus sign because the terminal side of the angle θ is in the 4th quadrant of the coordinate system),

$\csc\theta = \dfrac{1}{\sin\theta} = -\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\dfrac{\sqrt{3}}{2}}{\dfrac{1}{2}} = -\sqrt{3}$, $\cot\theta = \dfrac{1}{\tan\theta} = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$.

7. **(a)** The same sine ratio have angles in the 1st and 2nd quadrants.

(i) $\sin 150° = \sin(180° - 30°) = \sin 30°$, therefore $\alpha = 30°$,

(ii) $\sin 95° = \sin(180° - 85°) = \sin 85°$, therefore $\alpha = 85°$.

(b) The same cosine ratio have angles in the 1st and 4th quadrants.

(i) $\cos 315° = \cos(360° - 45°) = \cos 45°$, therefore $\alpha = 45°$,

(ii) $\cos 353° = \cos(360° - 7°) = \cos 7°$, therefore $\alpha = 7°$.

(c) The same tangent ratio have angles in the 1st and 3rd quadrants.
Using the identity $\tan(180° + \theta) = \tan\theta$, we have:

(i) $\tan 240° = \tan(180° + 60°) = \tan 60°$, therefore $\alpha = 60°$,

(ii) $\tan 200° = \tan(180° + 20°) = \tan 20°$, therefore $\alpha = 20°$.

8. **(a)** Area $= \frac{1}{2}(6)(4)\sin 60° = 12\sin 60° = 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3}$

(b) Area $= \frac{1}{2}(8)(23)\sin 105° = 92\sin 105° = 88.9$ (to 3 s.f.)

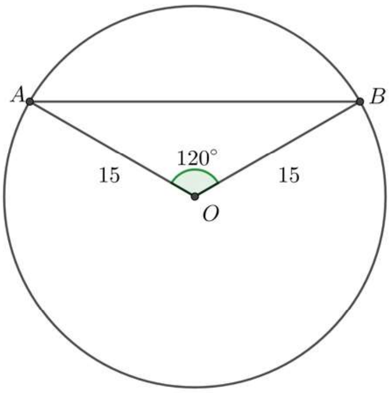
(c) Area $= \frac{1}{2}(30)(90)\sin 45° = 1350\sin 45° = 1350\left(\frac{\sqrt{2}}{2}\right) = 675\sqrt{2}$

9. Using the formula $Area = \frac{1}{2}(AB)(AC)\sin A$ we have

$$43 = \frac{1}{2}(12)(15)\sin A$$

$$43 = 90\sin A \Rightarrow \sin A = \frac{43}{90} \Rightarrow A = 28.5° \text{ (to 3 s.f.)}$$

10.



(a) Area of sector $A_s = \frac{1}{2}r^2\theta$, where θ is in radians. The degree measure of 120° is equivalent to $\frac{120°}{360°}(2\pi) = \frac{2\pi}{3}$. Therefore,

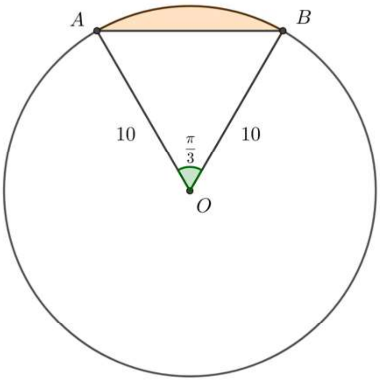
$$A_s = \frac{1}{2}(15)^2\left(\frac{2\pi}{3}\right) = \frac{225}{3}\pi = 236 \text{ cm}^2 \text{ (to 3 s. f.)}$$

(b) Area of triangle *AOB*:

$$A_t = \frac{1}{2}(OA)^2 \sin 120° = \frac{1}{2}(15)^2 \sin 60°$$

$$= \frac{1}{2}(225)\left(\frac{\sqrt{3}}{2}\right) = \frac{225}{4}\sqrt{3} = 97.4 \text{ cm}^2 \text{ (to 3 s.f.)}$$

11.



(a) Area of the shaded region $A = A_{\text{sector}} - A_{\text{triangle}}$

$$A = \frac{1}{2}r^2\left(\frac{\pi}{3}\right) - \frac{1}{2}r^2\sin\frac{\pi}{3} = \frac{1}{2}r^2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$$

$$A = \frac{1}{2}(10)^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = 9.06 \text{ cm}^2 \text{ (to 3 s.f.)}$$

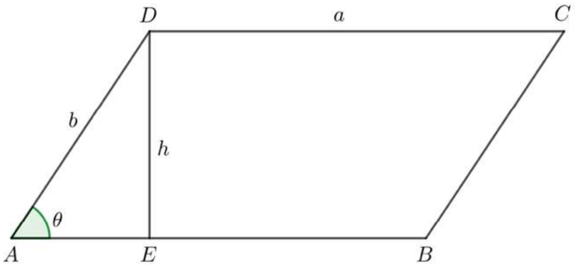
(b) First we need to convert measure of the central angle to radians: 135° is equivalent to $\frac{135°}{360°}(2\pi) = \frac{3\pi}{4}$. Now,

$$A = A_{\text{sector}} - A_{\text{triangle}} = \frac{1}{2}r^2\left(\frac{3\pi}{4}\right) - \frac{1}{2}r^2\sin\frac{3\pi}{4}$$

$$= \frac{1}{2}r^2\left(\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right)$$

$$= \frac{1}{2}(12)^2\left(\frac{3\pi}{4} - \frac{\sqrt{2}}{2}\right) = 119 \text{ cm}^2 \text{ (to 3 s. f.)}$$

12.



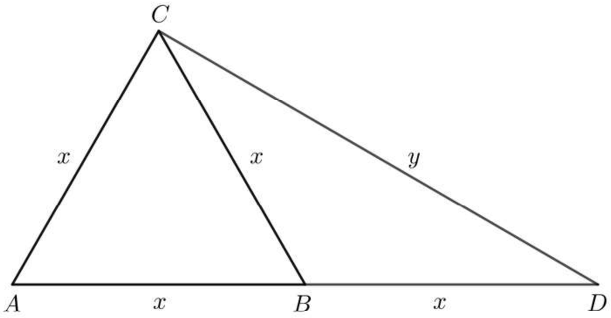
Let $DE = h$. Area of the parallelogram $A_p = ah$.

In the triangle *AED*:

$\sin\theta = \frac{h}{b} \Rightarrow h = b\sin\theta$. Therefore,

$A_p = ab\sin\theta$.

13.



The triangle *ABC* is equilateral, so $A\hat{B}C = 60°$ and $C\hat{B}D = 180° - A\hat{B}C = 180° - 60° = 120°$.

In triangle *BCD*, angles $B\hat{C}D$ and $B\hat{D}C$ are 30° each. So, triangle *ACD* is right-angled and hence

$$y = 2x\cos 30 = 3x\frac{\sqrt{3}}{2} = x\sqrt{3}$$

Alternatively, in triangle *BCD* by the cosine rule:

$$y^2 = x^2 + x^2 - 2(x)(x)\cos C\hat{B}D = 2x^2(1-\cos 120°) = 2x^2(1-\cos(180°-60°)) = 2x^2(1+\cos 60°)$$

$$= 2x^2(1+\cos 60^{\circ}) = 2x^2\left(1+\frac{1}{2}\right) = 2x^2\left(\frac{3}{2}\right) = 3x^2$$

Therefore, $y = \sqrt{3x^2} = x\sqrt{3}$

14. Area of the triangle *FGH* is the sum of areas of triangles *FGJ* and *JGH*:

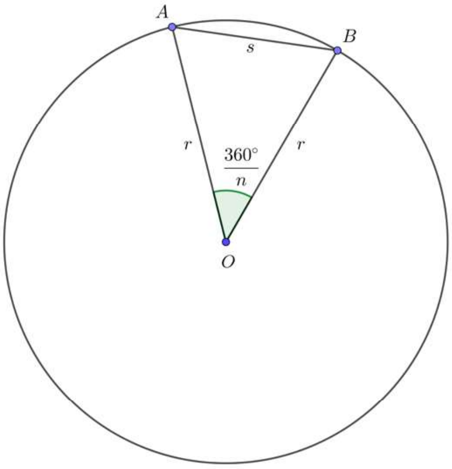
$$A_{FGH} = A_{FGJ} + A_{JGH}$$

$$A_{FGJ} = \frac{1}{2}hx\sin\theta,\ A_{JGH} = \frac{1}{2}xf\sin\theta \text{ and } A_{FGH} = \frac{1}{2}hx\sin\theta + \frac{1}{2}xf\sin\theta = \frac{1}{2}x(h+f)\sin\theta$$

But also $A_{FGH} = \frac{1}{2}hf\sin 2\theta$. Comparing: $\frac{1}{2}x(h+f)\sin\theta = \frac{1}{2}hf\sin 2\theta$ and

$$x = \frac{hf\sin 2\theta}{(h+f)\sin\theta} = \frac{2hf\sin\theta\cos\theta}{(h+f)\sin\theta} = \frac{2hf}{(h+f)}\cos\theta$$

15. In the triangle *OAB* by the cosine rule:



$$s^2 = r^2 + r^2 - 2(r)(r)\cos\left(\frac{360°}{n}\right) = 2r^2\left(1 - \cos\left(\frac{360°}{n}\right)\right)$$

Using the formula for the cosine of a double angle:

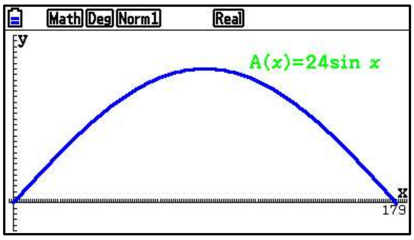
$\cos\left(\frac{360°}{n}\right) = 1 - 2\sin^2\left(\frac{180°}{n}\right)$, we get

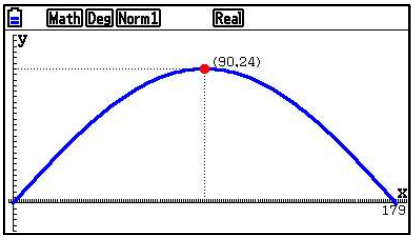
$$s^2 = 2r^2\left(1 - 1 + 2\sin^2\left(\frac{180°}{n}\right)\right) = 4r^2\sin^2\left(\frac{180°}{n}\right)$$

Therefore, $s = 2r\sin\left(\frac{180°}{n}\right)$

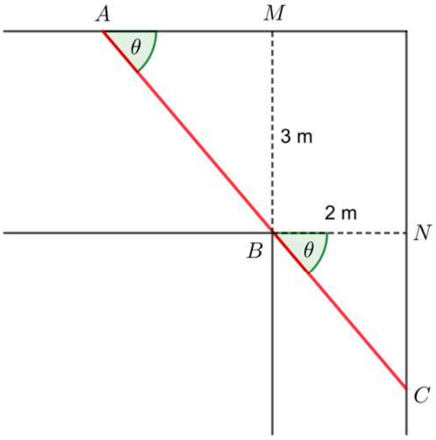
16. (a) Area of the triangle $A = \frac{1}{2}(6)(8)\sin x = 24\sin x$

(b) Since the angle x is one of the internal angles in the triangle, the domain of the function $A(x)$ is the interval $]0°, 180°[$. The function describes area of the triangle, so it must assume only positive values. Since its value depends on the value of $\sin x$, the maximum of the function occurs when $\sin x = 1$. Therefore, the range of the function is $]0, 24]$.





(c) By GDC, the maximum value of the function occurs when $x = 90°$. The coordinates of the maximum point are $(90°, 24)$. The triangle that corresponds to the maximum value is a right-angled triangle since $\sin x$ assumes its maximum value when $x = 90°$.



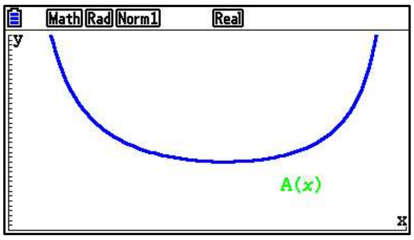
17. (a) The length of the rod, $L = AB + BC$

In the triangle *AMB*: $\sin\theta = \dfrac{BM}{AB} \Rightarrow AB = \dfrac{3}{\sin\theta} = 3\csc\theta$

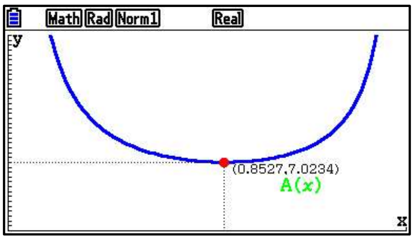
In the triangle *BCN*: $\cos\theta = \dfrac{BN}{BC} \Rightarrow BC = \dfrac{2}{\cos\theta} = 2\sec\theta$

Therefore, $L(\theta) = 3\csc\theta + 2\sec\theta$

(b)



(c)



In fact, $L(\theta)$ is equal to the length of the hypotenuse of the large right-angled triangle. This hypotenuse has to pass through the fixed point *B*. The minimum length of *AC* of 7.02 m (3 s.f.) happens when $\theta = 0.8527$

Since $\sin\theta = \dfrac{3}{AB}$, as θ gets smaller than 0.8527, the length of *AB* will get larger and tends to infinity as θ approaches 0. Similarly, when θ approaches 90, $\cos\theta = \dfrac{2}{BC}$ will approach 0 making the length of *BC* infinite. Thus, the rod can have any length as long as it is not longer than 7.02 because then, it will not fit around that turn.

18. From the diagram it follows, that $AC = AB + r = 383500 + r$

In the triangle ACD: $\frac{r}{AC} = \sin(0.2591°)$. We have:

$$\frac{r}{383500 + r} = \sin(0.2591°)$$
$$\Rightarrow\ r = (383500 + r)\sin(0.2591°)$$
$$\Rightarrow\ r - r\sin(0.2591°) = 383500\sin(0.2591°)$$

Simplifying:

$$r(1 - \sin(0.2591°)) = 383500\sin(0.2591°)\ \Rightarrow\ r = \frac{\sin(0.2591°)}{1 - \sin(0.2591°)}383500$$

$r = 1740$ km

19. (a) $\sin\theta = x \Rightarrow \cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - x^2}$. Therefore,
$\sec\theta = \frac{1}{\cos\theta} = \pm\frac{1}{\sqrt{1 - x^2}},\ -1 < x < 1$.

(b) $\tan\beta = y \Leftrightarrow \frac{\sin\beta}{\cos\beta} = y\ \Rightarrow\ \cos\beta = \frac{\sin\beta}{y}$. Since $\sin^2\beta + \cos^2\beta = 1$, then
$\sin^2\beta + \frac{\sin^2\beta}{y^2} = 1$. We have:

$$\sin^2\beta\left(1 + \frac{1}{y^2}\right) = 1\ \Leftrightarrow\ \frac{y^2 + 1}{y^2}\sin^2\beta = 1\ \Rightarrow\ \sin^2\beta = \frac{y^2}{y^2 + 1}\ \text{ and}$$
$$\sin\beta = \pm\frac{y}{\sqrt{y^2 + 1}} = \pm\frac{y\sqrt{y^2 + 1}}{y^2 + 1}.$$

20. In triangle OAP: $\cos\theta = \frac{OA}{OP} = \frac{OA}{1} = OA$

In triangle OPB: $\tan\theta = \frac{PB}{OP} = \frac{PB}{1} = PB$

In triangle OPC: $O\hat{C}P = \theta$ and $\cot\theta = \frac{CP}{OP} = \frac{CP}{1} = CP$. Also
$\sin\theta = \frac{OP}{OC} = \frac{1}{OC}\ \Rightarrow\ OC = \frac{1}{\sin\theta} = \csc\theta$

In triangle OPB: $\cos\theta = \frac{OP}{OB} = \frac{1}{OB}\ \Rightarrow\ OB = \frac{1}{\cos\theta} = \sec\theta$

21. (a) Let $A(1,\ 4)$ and $B(-1,\ 2)$. Then the gradient of the line passing through the points A and B, $m=\frac{y_A-y_B}{x_A-x_B}=\frac{2-4}{-1-1}=\frac{-2}{-2}=1$. Therefore, the angle that the line makes with the positive direction of the x-axis, $\theta=\tan^{-1}1=45°$

(b) Let $A(-3,\ 1)$ and $B(6,\ -5)$. Then the gradient of the line passing through the points A and B, $m=\frac{y_A-y_B}{x_A-x_B}=\frac{-5-1}{6-(-3)}=\frac{-6}{9}=-\frac{2}{3}$. Therefore, the angle that the line makes with the positive direction of the x-axis,

$\theta=\tan^{-1}\left(-\frac{2}{3}\right)=-33.7°\ (\text{or } 180°-33.7°=146°)\ (\text{to 3 s.f.})$.

(c) Let $A\left(2,\ \frac{1}{2}\right)$ and $B(-4,\ -10)$. Then the gradient of the line passing through the points A and B, $m=\frac{y_A-y_B}{x_A-x_B}=\frac{-10-\frac{1}{2}}{-4-2}=\frac{-\frac{21}{2}}{-6}=\frac{21}{12}=\frac{7}{4}$. Therefore, the angle that the line makes with the positive direction of the x-axis, $\theta=\tan^{-1}\left(\frac{7}{4}\right)=60.3°$ (to 3 s.f.)

22. (a) If $y=-2x$, then the gradient $m_1=-2\ \Rightarrow\ \tan\alpha=-2\ \Rightarrow\ \alpha=\tan^{-1}(-2)$. Likewise, if $y=x$, then the gradient $m_2=1\ \Rightarrow\ \tan\beta=1\ \Rightarrow\ \beta=\tan^{-1}(1)=45°$. Then the angle θ between the lines:

$\theta=\left|\tan^{-1}(m_1)-\tan^{-1}(m_2)\right|=\left|\tan^{-1}(-2)-\tan^{-1}(1)\right|=108.4°$. Therefore, the acute angle between the lines is $180°-108.4°=71.6°$

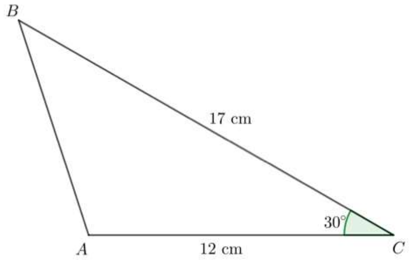
(b) If $y=-3x+5$, then the gradient $m_1=-3\ \Rightarrow\ \tan\alpha=-3\ \Rightarrow\ \alpha=\tan^{-1}(-3)$. Likewise, if $y=2x$, then the gradient $m_2=2\ \Rightarrow\ \tan\beta=2\ \Rightarrow\ \beta=\tan^{-1}(2)$. Then the angle θ between the lines:

$\theta=\left|\tan^{-1}(m_1)-\tan^{-1}(m_2)\right|=\left|\tan^{-1}(-3)-\tan^{-1}(2)\right|=135°$. Therefore, the acute angle between the lines is $180°-135°=45°$

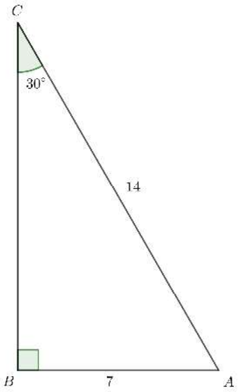
Exercise 7.4

1. **(a)** Three angles are known – infinite number of triangles.

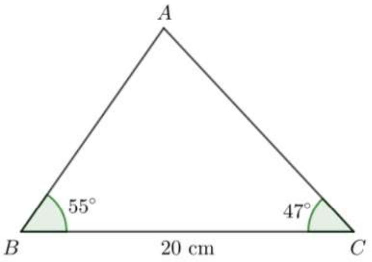
(b) Two sides and their included angle are known – one unique triangle:



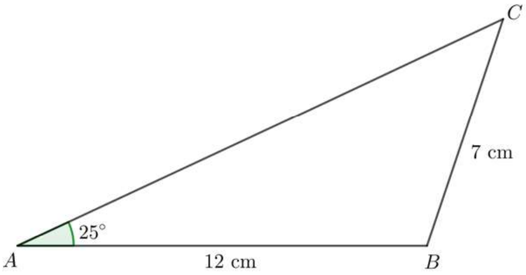
(c) Since $AB = AC\sin 30°$ – one right-angled triangle:

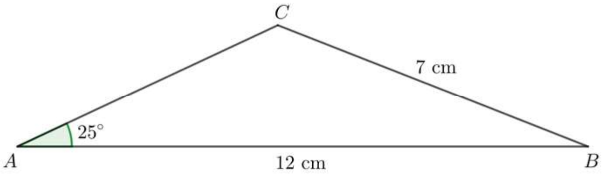


(d) One side and two angles – one unique triangle:

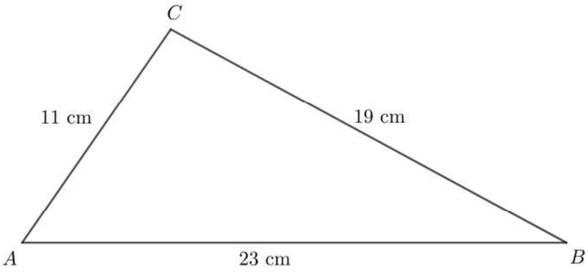


(e) Since $BC = 7$, $AB\sin 25° = 12\sin 25° = 5.07$ and $AB\sin 25° < BC < AB$, then there are two triangles possible:

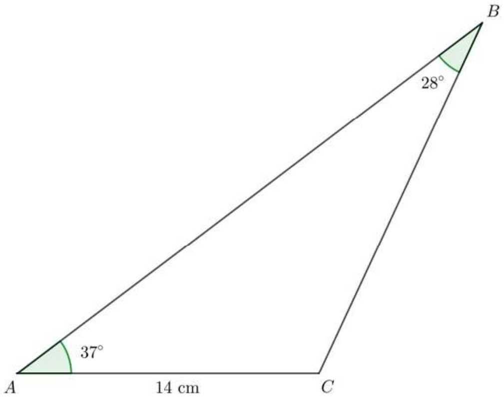




(f) In the triangle ABC: $BC < AC < AB$ and $AC + BC > AB$. There is one unique triangle that satisfies given conditions:



2. (a)

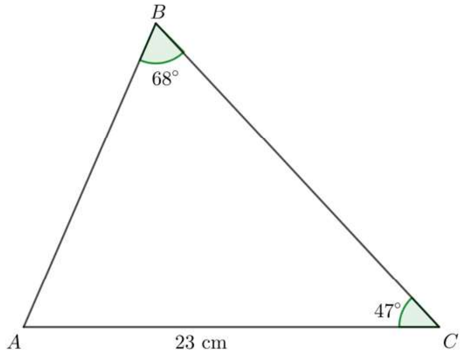


By the sine rule: $\dfrac{BC}{\sin 37°} = \dfrac{AC}{\sin 28°} \Rightarrow BC = \dfrac{AC\sin 37°}{\sin 28°} = \dfrac{14\sin 37°}{\sin 28°} = 17.9$

$$A\hat{C}B = 180° - (28° + 37°) = 115°$$

$$\frac{AB}{\sin 115°} = \frac{AC}{\sin 28°} \Rightarrow AB = \frac{14\sin 115°}{\sin 28°} = 27.0$$

(b)

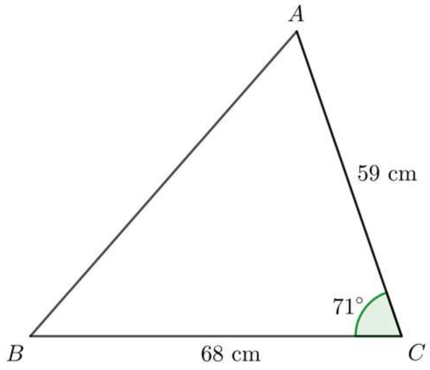


$B\hat{A}C = 180° - (68° + 47°) = 65°$. By the sine rule:

$$\frac{AB}{\sin 47°} = \frac{AC}{\sin 68°} \Rightarrow AB = \frac{AC \sin 47°}{\sin 68°} = \frac{23 \sin 47°}{\sin 68°} = 18.1$$

$$\frac{BC}{\sin 65°} = \frac{AC}{\sin 68°} \Rightarrow BC = \frac{AC \sin 65°}{\sin 68°} = \frac{23 \sin 65°}{\sin 68°} = 22.5$$

(c)



By the cosine rule: $(AB)^2 = (BC)^2 + (AC)^2 - 2(BC)(AC)\cos\hat{C}$

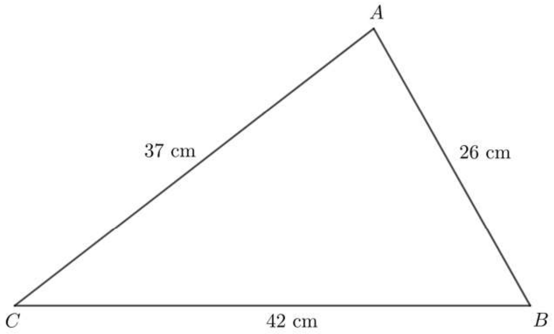
$AB = \sqrt{(68)^2 + (59)^2 - 2(68)(59)\cos 71°} = 74.1$ cm (to 3 s. f.)

By the sine rule: $\frac{AC}{\sin\hat{B}} = \frac{AB}{\sin 71°} \Rightarrow \sin\hat{B} = \frac{AC \sin 71°}{AB}$, so

$\sin\hat{B} = \frac{59 \sin 71°}{74.1} = 0.7528 \Rightarrow \hat{B} = 48.8°$ (or $131.2°$ which is impossible since $\hat{C} = 71°$).

Then $B\hat{A}C = 180° - (71° + 48.8°) = 60.2°$ (to 3 s. f.)

(d)



By the cosine rule:

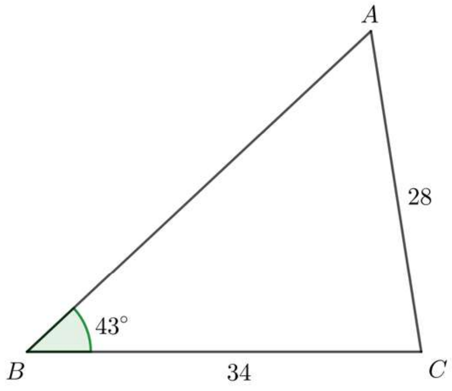
$$(AB)^2=(BC)^2+(AC)^2-2(BC)(AC)\cos\hat{C} \Rightarrow \cos\hat{C}=\frac{(BC)^2+(AC)^2-(AB)^2}{2(BC)(AC)}$$

$\cos\hat{C}=\dfrac{(42)^2+(37)^2-(26)^2}{2(42)(37)}=0.7905$. Therefore, $A\hat{C}B=37.8°$ (to 3 s.f.)

By the sine rule: $\dfrac{AB}{\sin\hat{C}}=\dfrac{AC}{\sin\hat{B}} \Rightarrow \sin\hat{B}=\dfrac{AC\sin\hat{C}}{AB}=\dfrac{37\sin 37.5°}{26}=0.8715$.

Then $A\hat{B}C=60.6°$ and $C\hat{A}B=180°-(60.6°+37.8°)=81.6°$ (to 3 s.f.)

(e)



By the sine rule: $\dfrac{AC}{\sin A\hat{B}C}=\dfrac{BC}{\sin B\hat{A}C} \Rightarrow \sin B\hat{A}C=\dfrac{BC\sin A\hat{B}C}{AC}=\dfrac{34\sin 43°}{28}=0.8281$

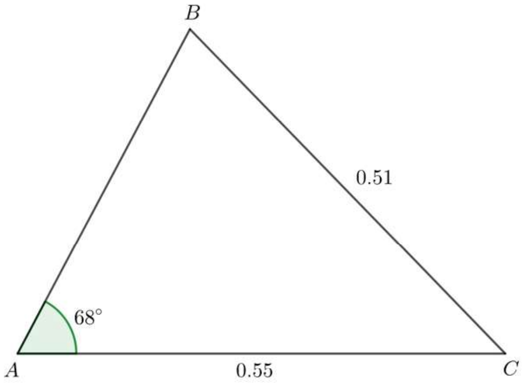
Then $B\hat{A}C=55.9°$ or $B\hat{A}C=124.1°$

Since $124.1°+43°=167.1°<180°$, there are two cases.

Case 1: Let $B\hat{A}C = 55.9°$. Then $A\hat{C}B = 180° - (55.9° + 43°) = 81.1°$ and by the sine rule: $\frac{AB}{\sin A\hat{C}B} = \frac{AC}{\sin A\hat{B}C} \Rightarrow AB = \frac{AC \sin A\hat{C}B}{\sin A\hat{B}C} = \frac{28 \sin 81.1°}{\sin 43°} = 40.6$. Therefore, $AB = 40.6,\ B\hat{A}C = 55.9°,\ A\hat{C}B = 81.1°$ (to 3 s.f.)

Case 2: Let $B\hat{A}C = 124.1°$. Then $A\hat{C}B = 180° - (124.1° + 43°) = 12.9°$ and by the sine rule: $\frac{AB}{\sin A\hat{C}B} = \frac{AC}{\sin A\hat{B}C} \Rightarrow AB = \frac{AC \sin A\hat{C}B}{\sin A\hat{B}C} = \frac{28 \sin 12.9°}{\sin 43°} = 9.17$. Therefore, $AB = 9.17,\ B\hat{A}C = 124.1°,\ A\hat{C}B = 12.9°$ (to 3 s.f.)

(f)



By the sine rule:

$$\frac{BC}{\sin B\hat{A}C} = \frac{AC}{\sin A\hat{B}C} \Rightarrow \sin A\hat{B}C = \frac{AC \sin B\hat{A}C}{BC} = \frac{0.55 \sin 62°}{0.51} = 0.9522$$

Then $A\hat{B}C = 72.2°$ or $A\hat{B}C = 107.8°$
Since $107.8° + 62° = 169.8° < 180°$, there are two cases.

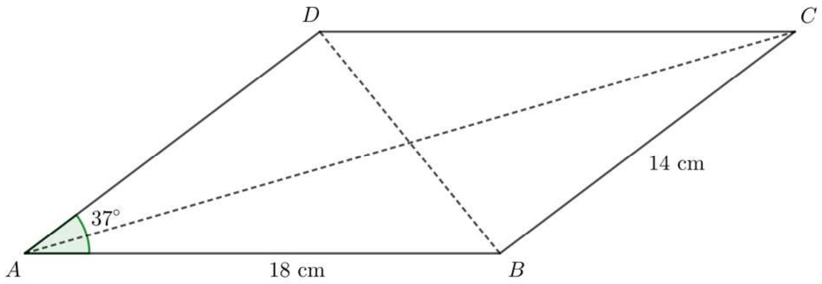
Case 1: Let $A\hat{B}C = 72.2°$. Then $A\hat{C}B = 180° - (62° + 72.2°) = 45.8°$ and by the sine rule: $\frac{AB}{\sin A\hat{C}B} = \frac{BC}{\sin B\hat{A}C} \Rightarrow AB = \frac{BC \sin A\hat{C}B}{\sin B\hat{A}C} = \frac{0.51 \sin 45.8°}{\sin 62°} = 0.414$

Therefore, $AB = 0.414,\ ABC = 72.2°,\ A\hat{C}B = 45.8°$ (to 3 s.f.)

Case 2: Let $A\hat{B}C = 107.8°$. Then $A\hat{C}B = 180° - (107.8° + 62°) = 10.2°$ and by the sine rule: $AB = \frac{BC \sin A\hat{C}B}{\sin B\hat{A}C} = \frac{0.51 \sin 10.2°}{\sin 62°} = 0.102$

Therefore, $AB = 0.102,\ A\hat{B}C = 107.8°,\ A\hat{C}B = 10.2°$ (to 3 s.f.)

3.



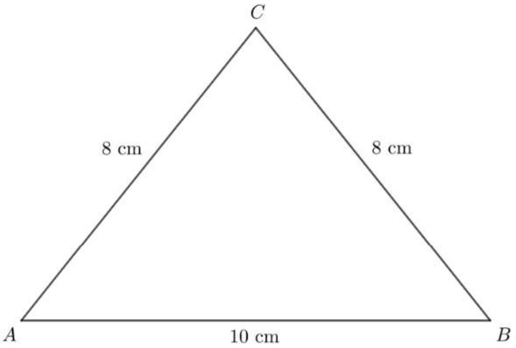
Since $D\hat{A}B = B\hat{C}D = 37°$, then $A\hat{B}C = A\hat{D}C = 143°$
In the triangle *ABD,* by the cosine rule:

$$(BD)^2 = (AB)^2 + (AD)^2 - 2(AB)(AD)\cos D\hat{A}B$$

Now, $BD = \sqrt{(18)^2 + (14)^2 - 2(18)(14)\cos 37°} = 10.8$ cm (to 3 s.f.)

In the triangle *ABC,* by the cosine rule:
$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos D\hat{A}B$. Then
$AC = \sqrt{(18)^2 + (14)^2 - 2(18)(14)\cos 143°} = 30.4$ cm (to 3 s.f.)

4. By the cosine rule:



$$(AB)^2 = (AC)^2 + (BC)^2 - 2(AC)(BC)\cos A\hat{C}B$$

$$\cos A\hat{C}B = \frac{(AC)^2 + (BC)^2 - (AB)^2}{2(AC)(BC)}$$

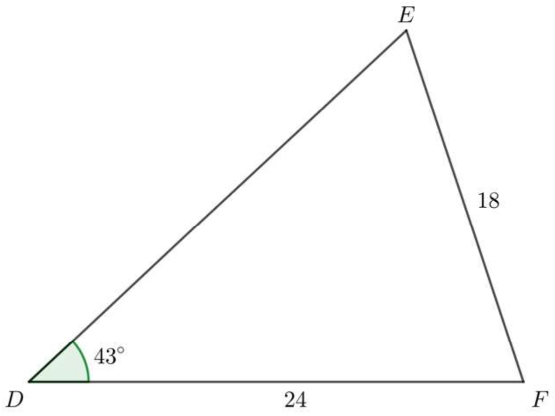
$$\cos A\hat{C}B = \frac{(8)^2 + (8)^2 - (10)^2}{2(8)(8)} = 0.21875$$

$$A\hat{C}B = 77.4°$$

Then $C\hat{A}B = C\hat{B}A = \frac{1}{2}(180° - A\hat{C}B) = \frac{1}{2}(180° - 77.4°) = 51.3°$

Therefore $C\hat{A}B = 51.3°,\ C\hat{B}A = 51.3°,\ A\hat{C}B = 77.4°$ (to 3 s.f.)

5. By the sine rule:



$$\frac{DF}{\sin D\hat{E}F} = \frac{EF}{\sin E\hat{D}F} \Rightarrow \sin D\hat{E}F = \frac{DF \sin E\hat{D}F}{EF}$$

$$\sin D\hat{E}F = \frac{24 \sin 43°}{18} = 0.90933$$

$D\hat{E}F = 65.4°$ or $D\hat{E}F = 114.6°$

Therefore, $D\hat{F}E = 180° - (43° + 65.4°) = 71.6°$

or $D\hat{F}E = 180° - (43° + 114.6°) = 22.4°$

6. The smallest angle lies opposite the shortest side of a triangle. Then by the cosine rule:

$$4^2 = 6^2 + 9^2 - 2(6)(9)\cos\alpha$$

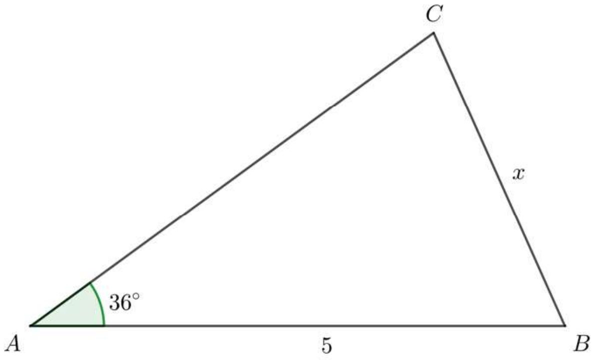
$$\cos\alpha = \frac{6^2 + 9^2 - 4^2}{2(6)(9)} = 0.9352 \Rightarrow \alpha = 20.7° \text{ (to 3 s.f.)}$$

7. Area $= \frac{1}{2}(RP)(RQ)\sin P\hat{R}Q$. We know, that $P\hat{Q}R = 180° - (78° + 40°) = 62°$.

By the sine rule: $\frac{RP}{\sin P\hat{Q}R} = \frac{RQ}{\sin R\hat{P}Q} \Rightarrow RP = \frac{RQ \sin P\hat{Q}R}{\sin R\hat{P}Q} = \frac{15 \sin 62°}{\sin 40°}$

Now, Area $= \frac{1}{2}\left(\frac{15 \sin 62°}{\sin 40°}\right)(15)\sin 78° = 151 \text{ cm}^2$ (to 3 s.f.)

8. **(a)**



Length of the perpendicular from B to AC: $x = 5\sin 36° = 2.94$

(i) one triangle if $BC \geq 5$ or $BC = 2.94$

(ii) two triangles if $2.94 < BC < 5$

(iii) no triangle if $BC < 2.94$

(b) Length of the perpendicular from B to AC: $x = 10\sin 60° = 5\sqrt{3}$

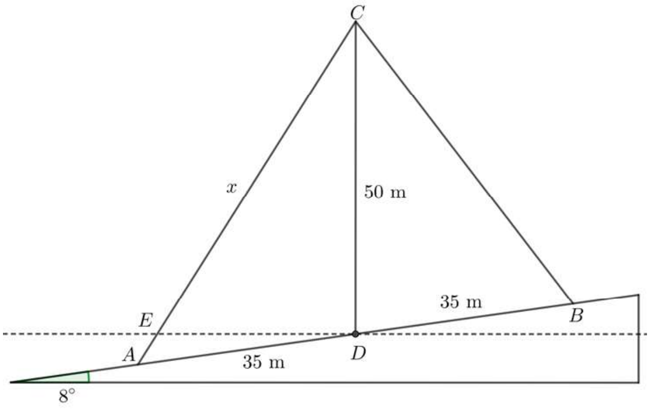
(i) one triangle if $BC \geq 10$ or $BC = 5\sqrt{3}$

(ii) two triangles if $5\sqrt{3} < BC < 10$

(iii) no triangle if $BC < 5\sqrt{3}$

9. In the triangle EAD: $A\hat{D}E = 8°$

Therefore, $A\hat{D}C = 90° + 8° = 98°$ and by the cosine rule:
$x^2 = 35^2 + 50^2 - 2(35)(50)\cos 98°$



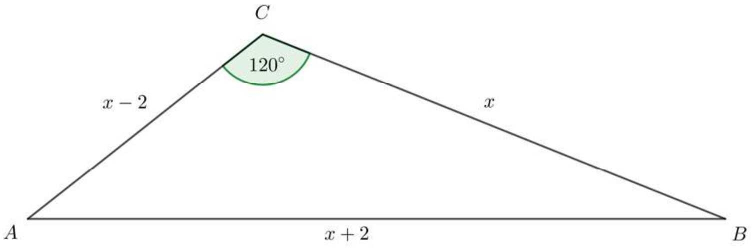
$$x = \sqrt{35^2 + 50^2 - 2(35)(50)\cos 98°} = 64.9 \text{ m}$$

In the triangle BCD: $C\hat{D}B = 90° - 8° = 82°$

By the cosine rule: $y^2 = 35^2 + 50^2 - 2(35)(50)\cos 82°$

$$y = \sqrt{35^2 + 50^2 - 2(35)(50)\cos 82°} = 56.9 \text{ m}$$

10.



(a) $x > 0$ since it is a side of a triangle, hence $x + 2$ is the largest side.

By the cosine rule: $(x+2)^2 = x^2 + (x-2)^2 - 2x(x-2)\cos 120°$. Simplifying:

$$x^2 + 4x + 4 = x^2 + x^2 - 4x + 4 + 2x(x-2)\left(\frac{1}{2}\right)$$

$$4x = x^2 - 4x + x^2 - 2x$$

$$2x^2 - 10x = 0 \Rightarrow x(x-5) = 0 \;\Rightarrow\; x = 5$$

(b) Since $x = 5$, then $AC = 3$ and $BC = 5$. Area of the triangle:
$A_t = \frac{1}{2}(AC)(BC)\sin A\hat{C}B$. We have

$$A_t = \frac{1}{2}(5)(3)\sin 120° = \frac{15}{2}\sin(180° - 60°) = \frac{15}{2}\sin 60° = \frac{15}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{15\sqrt{3}}{4}$$

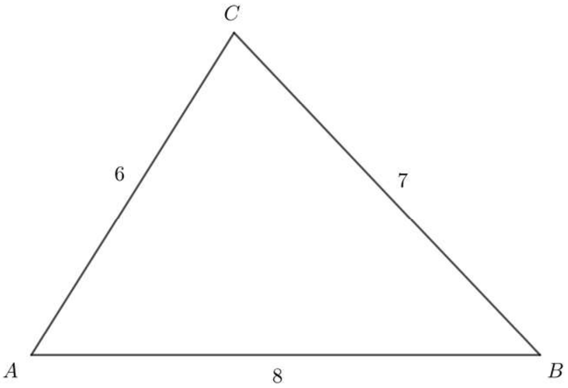
(c) $\sin\hat{C} = \sin 120° = \frac{\sqrt{3}}{2}$. By the sine rule:

$$\frac{\sin\hat{B}}{x} = \frac{\sin\hat{C}}{x+2} \Rightarrow \sin\hat{B} = \frac{x\sin\hat{C}}{x+2} = \frac{5\left(\frac{\sqrt{3}}{2}\right)}{5+2} = \frac{5\sqrt{3}}{14} \text{ and}$$

$$\frac{\sin\hat{A}}{x-2} = \frac{\sin\hat{C}}{x+2} \Rightarrow \sin\hat{A} = \frac{(x-2)\sin\hat{C}}{x+2} = \frac{3\left(\frac{\sqrt{3}}{2}\right)}{7} = \frac{3\sqrt{3}}{14}$$

Now, $\sin\hat{A} + \sin\hat{B} + \sin\hat{C} = \frac{3\sqrt{3}}{14} + \frac{5\sqrt{3}}{14} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{14} + \frac{5\sqrt{3}}{14} + \frac{7\sqrt{3}}{14} = \frac{15\sqrt{3}}{14}$

11. By the cosine rule:



$$(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos\hat{A}$$

$$\cos\hat{A} = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)}$$

$$\cos\hat{A} = \frac{8^2 + 6^2 - 7^2}{2(8)(6)} = \frac{17}{32}$$

$$\sin\hat{A} = \sqrt{1-\cos^2\hat{A}} = \sqrt{1-\left(\frac{17}{32}\right)^2} = \frac{7\sqrt{15}}{32}$$

Therefore, the area of the triangle $A_t = \frac{1}{2}(AB)(AC)\sin\hat{A} = \frac{1}{2}(8)(6)\left(\frac{7\sqrt{15}}{32}\right) = \frac{21\sqrt{15}}{4}$

12. **(a)** If $c^2 > a^2 + b^2$, then the triangle is obtuse.

(b) If $c^2 < a^2 + b^2$, then the triangle is acute.

(c) By the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos\hat{C} \Rightarrow \cos\hat{C} = \frac{a^2 + b^2 - c^2}{2ab}$. The denominator is positive for all $a, b > 0$.

Now, if $c^2 > a^2 + b^2$, then the numerator $a^2 + b^2 - c^2 < 0$ and $\cos\hat{C} < 0$. It follows that the angle $\hat{C}$ is obtuse. If $c^2 < a^2 + b^2$ then the numerator $a^2 + b^2 - c^2 > 0$ and $\cos\hat{C} > 0$. It follows that the angle $\hat{C}$ is acute.

13. Let $DF = x$. By the cosine rule:

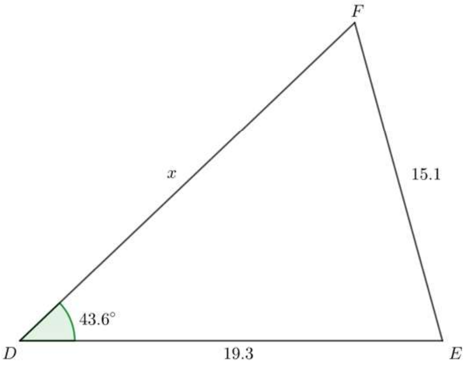
$$(15.1)^2 = x^2 + (19.3)^2 - 2x(19.3)\cos 43.6°$$

$$x^2 - 2x(19.3)\cos 43.6° + (19.3)^2 - (15.1)^2 = 0$$

$$x^2 - (38.6\cos 43.6°)x + 144.48 = 0$$

$$x = \frac{38.6\cos 43.6° \pm \sqrt{(38.6\cos 43.6°)^2 - 4(1)(144.48)}}{2}$$

$x = 6.84$ or $x = 21.1$



14. (a) Area $= \frac{1}{2}(WZ)(YZ)\sin\theta \Rightarrow 112 = \frac{1}{2}(WZ)(20)\left(\frac{4}{5}\right)$

$8(WZ) = 112 \Leftrightarrow WZ = 14$ cm

(b) If $\sin\theta = \frac{4}{5}$, then $\cos\theta = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$. In triangle *WYZ,* by the cosine rule:

$$(WY)^2 = (WZ)^2 + (YZ)^2 - 2(WZ)(YZ)\cos\theta$$

$$(WY)^2 = 14^2 + 20^2 - 2(14)(20)\left(\frac{3}{5}\right) = 260 \Rightarrow WY = 2\sqrt{65} \text{ cm}$$

(c) In triangle *WXY,* by the cosine rule: $(WY)^2 = (WX)^2 + (XY)^2 - 2(WX)(XY)\cos\hat{X}$.
We have:

$$260 = x^2 + 9x^2 - 2(x)(3x)\cos 120° \Rightarrow 260 = 10x^2 - 6x^2\cos(180° - 60°) \Rightarrow 260 = 10x^2 + 6x^2\cos 60°$$

$$260 = 10x^2 + 6x^2\left(\frac{1}{2}\right) \Rightarrow 260 = 10x^2 + 3x^2 \Rightarrow 260 = 13x^2 \Rightarrow x^2 = 20 \Rightarrow x = \sqrt{20} = 2\sqrt{5} \text{ cm}$$

(d) As can be seen on the diagram: $X\hat{Y}Z = X\hat{Y}W + W\hat{Y}Z$. In triangle *WXY,* by the sine rule:

$$\frac{WX}{\sin X\hat{Y}W} = \frac{WY}{\sin W\hat{X}Y} \Leftrightarrow \sin X\hat{Y}W = \frac{WX\sin W\hat{X}Y}{WY} = \frac{2\sqrt{5}\sin 120°}{2\sqrt{65}} = \frac{\sqrt{39}}{26}$$

$$X\hat{Y}W = \sin^{-1}\left(\frac{\sqrt{39}}{26}\right) = 13.9°$$

In the triangle *WYZ,* by the sine rule:

$$\frac{WZ}{\sin W\hat{Y}Z} = \frac{WY}{\sin W\hat{Z}Y} \Leftrightarrow \sin W\hat{Y}Z = \frac{WZ\sin W\hat{Z}Y}{WY} = \frac{14\left(\frac{4}{5}\right)}{2\sqrt{65}} = \frac{28}{5\sqrt{65}} = \frac{28\sqrt{65}}{325}$$

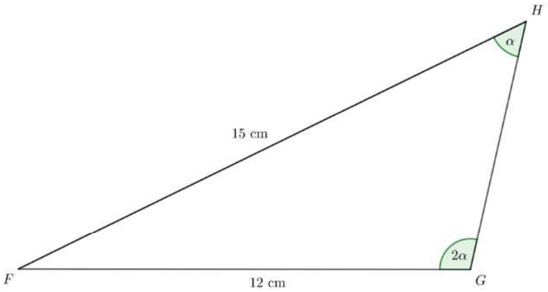
$$W\hat{Y}Z = \sin^{-1}\left(\frac{28\sqrt{65}}{325}\right)$$

Therefore $X\hat{Y}Z = \sin^{-1}\left(\frac{\sqrt{39}}{26}\right) + \sin^{-1}\left(\frac{28\sqrt{65}}{325}\right) = 57.9^\circ$

15. By the sine rule: $\frac{FG}{\sin\alpha} = \frac{FH}{\sin 2\alpha}$

Then $\frac{FG}{\sin\alpha} = \frac{FH}{2\sin\alpha\cos\alpha}$

$$\frac{FG}{\sin\alpha} = \frac{FH}{2\sin\alpha\cos\alpha}$$



$$\cos\alpha = \frac{FH}{2FG}$$

$$\cos\alpha = \frac{15}{2(12)} = \frac{5}{8}$$

$$\alpha = \cos^{-1}\left(\frac{5}{8}\right) = 51.3°$$

16. (a) In the triangle PQS: $\cos\hat{P} = \frac{PS}{PQ} \Rightarrow PS = PQ\cos\hat{P} = r\cos\hat{P}$

As $PR = PS + RS$, we can write $RS = PR - PS = q - r\cos\hat{P}$

(b) $(QR)^2 = (RS)^2 + (QS)^2$

$$p^2 = \left(q - r\cos\hat{P}\right)^2 + (QS)^2$$

In the triangle PQS: $(PQ)^2 = (PS)^2 + (QS)^2 \Rightarrow (QS)^2 = r^2 - r^2\cos^2\hat{P}$. Therefore,

$$p^2 = q^2 - 2qr\cos\hat{P} + r^2\cos^2\hat{P} + r^2 - r^2\cos^2\hat{P} = q^2 + r^2 - 2qr\cos\hat{P}$$

(c) Let $P\hat{Q}R = 60°$. Then, by the cosine rule, we have:

$$q^2 = p^2 + r^2 - 2pr\cos 60°$$

$$q^2 = p^2 + r^2 - 2pr\left(\frac{1}{2}\right)$$

$$q^2 = p^2 + r^2 - pr \Leftrightarrow p^2 - pr + r^2 - q^2 = 0$$

We have a quadratic equation in variable *p*. Solving:

$$\Delta = r^2 - 4(1)\left(r^2 - q^2\right) = r^2 - 4r^2 + 4q^2 = 4q^2 - 3r^2$$

$$p=\frac{r\pm\sqrt{4q^2-3r^2}}{2}=\frac{1}{2}\left(r\pm\sqrt{4q^2-3r^2}\right)$$

17. This can be a Paper 3 type of question.

(a) We can write the expression $A=\frac{1}{2}ab\sin C$ as $2A=ab\sin C$ and from $c^2=a^2+b^2-2ab\cos C$, we have
$2ab\cos C=a^2+b^2-c^2$, or after squaring both sides:

$$(2ab)^2\cos^2 C=\left(a^2+b^2-c^2\right)^2$$

Also, from $2A=ab\sin C$ follows that $4A^2=(ab)^2\sin^2 C$. Now,

$$4(ab)^2\left(1-\sin^2 C\right)=\left(a^2+b^2-c^2\right)^2$$

$$4(ab)^2-4(ab)^2\sin^2 C=\left(a^2+b^2-c^2\right)^2$$

$$4a^2b^2-4\left(4A^2\right)=\left(a^2+b^2-c^2\right)^2$$

$$16A^2=4a^2b^2-\left(a^2+b^2-c^2\right)^2$$

(b) $s=\frac{a+b+c}{2}\Leftrightarrow a+b+c=2s$

We can write:

$$16A^2=4a^2b^2-\left(a^2+b^2-c^2\right)^2$$

$$16A^2=\left(2ab-a^2-b^2+c^2\right)\left(2ab+a^2+b^2-c^2\right)$$

$$16A^2=-\left(a^2-2ab+b^2-c^2\right)\left(+a^2+2ab+b^2-c^2\right)$$

$$16A^2=-\left[(a-b)^2-c^2\right]\left[(a+b)^2-c^2\right]$$

$$16A^2=-(a-b-c)(a-b+c)(a+b-c)(a+b+c)$$

$$16A^2=(a+b+c)(b+c-a)(a+c-b)(a+b-c)$$

Using the fact that $b+c=2s-a$, $a+c=2s-b$, $a+b=2s-c$ we can write

$$16A^2=2s(2s-a-a)(2s-b-b)(2s-c-c)$$

$$16A^2=2s(2s-2a)(2s-2b)(2s-2c)$$

(c) Continuing previous calculations:

$$16A^2 = 2s(2s-2a)(2s-2b)(2s-2c)$$

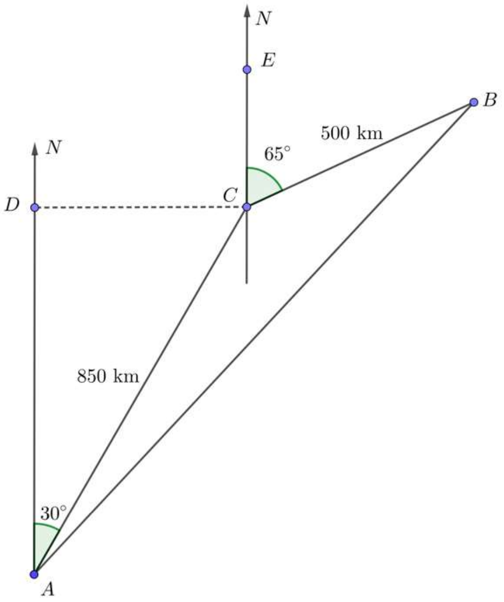
$$16A^2 = 2s(2)(s-a)(2)(s-b)(2)(s-c)$$

$$16A^2 = 16s(s-a)(s-b)(s-c) \Leftrightarrow A^2 = s(s-a)(s-b)(s-c)$$

Then $A = \sqrt{s(s-a)(s-b)(s-c)}$

18. **(a)** In the triangle ACD:

$D\hat{A}C = 30°,\ A\hat{D}C = 90° \Rightarrow A\hat{C}D = 60°$



We can write: $A\hat{C}D + D\hat{C}E + E\hat{C}B + A\hat{C}B = 360°$

$$60° + 90° + 65° + A\hat{C}B = 360° \Rightarrow A\hat{C}B = 145°$$

By the cosine rule:

$$(AB)^2 = (AC)^2 + (BC)^2 - 2(AC)(BC)\cos A\hat{C}B$$

$$AB = \sqrt{850^2 + 500^2 - 2(850)(500)\cos 145°} = 1291.8 \text{ km}$$

(b) The bearing from A to B equals $C\hat{A}D + C\hat{A}B$. In the triangle ABC by the sine rule:

$$\frac{BC}{\sin C\hat{A}B} = \frac{AB}{\sin A\hat{C}B} \Leftrightarrow \sin C\hat{A}B = \frac{BC \sin A\hat{C}B}{AB}$$

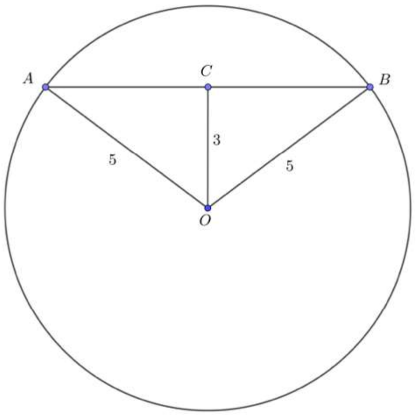
$$C\hat{A}B = \sin^{-1}\left(\frac{BC\sin A\hat{C}B}{AB}\right) = \sin^{-1}\left(\frac{500\sin 145°}{1291.8}\right)$$

$$C\hat{A}B = 12.8°$$

Therefore, the bearing from A to B is equal to $30° + 12.8° = 42.8°$

Chapter 7 practice questions

1.



The triangle AOB is an isosceles triangle where $OA = OB = 5$. The shortest distance from the chord AB to the centre of the circle is equal to the length of the altitude of the triangle. In the triangle BOC: $BC = \sqrt{(OB)^2 - (OC)^2} = \sqrt{5^2 - 3^2} = 4$. Therefore, AB = 8. Area of the triangle AOB: $A = \frac{1}{2}(AB)(OC) = \frac{1}{2}(8)(3) = 12$. But also: $A = \frac{1}{2}(5)^2 \sin A\hat{O}B = \frac{25}{2}\sin A\hat{O}B$. It follows that $\frac{25}{2}\sin A\hat{O}B = 12 \Leftrightarrow \sin A\hat{O}B = \frac{24}{25}$

2. If $\tan\theta = \frac{3}{7}$, then $\frac{\sin\theta}{\cos\theta} = \frac{3}{7} \Leftrightarrow \sin\theta = \frac{3}{7}\cos\theta$. We know that for any angle θ, $\sin^2\theta + \cos^2\theta = 1$. Substituting for $\sin\theta$ we get:

$$\cos^2\theta + \left(\frac{3}{7}\cos\theta\right)^2 = 1$$

$$\cos^2\theta + \frac{9}{49}\cos^2\theta = 1 \Leftrightarrow \frac{58}{49}\cos^2\theta = 1 \Leftrightarrow \cos^2\theta = \frac{49}{58}$$

Since θ is an angle in the right-angled triangle, then $\cos\theta = \sqrt{\frac{49}{58}} = \frac{7\sqrt{58}}{58}$ and

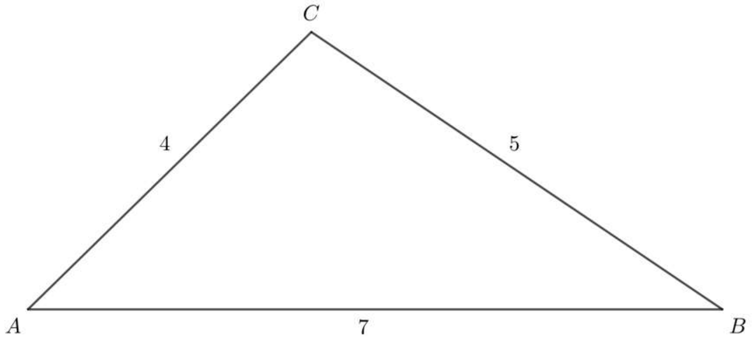
$$\sin\theta = \frac{3}{7}\left(\frac{7\sqrt{58}}{58}\right) = \frac{3\sqrt{58}}{58}$$

Now:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3\sqrt{58}}{58}\right)\left(\frac{7\sqrt{58}}{58}\right) = \frac{21}{29} \quad \text{and}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{7\sqrt{58}}{58}\right)^2 - \left(\frac{3\sqrt{58}}{58}\right)^2 = \frac{20}{29}$$

3.



The largest angle lies opposite the longest side. By the cosine rule:

$$(AB)^2 = (AC)^2 + (BC)^2 - 2(AC)(BC)\cos A\hat{C}B \Rightarrow \cos A\hat{C}B = \frac{(AC)^2 + (BC)^2 - (AB)^2}{2(AC)(BC)}$$

Using the data given:

$$\cos A\hat{C}B = \frac{4^2 + 5^2 - 7^2}{2(4)(5)} = -\frac{1}{5} \Rightarrow A\hat{C}B = 101.5°$$

4. If $\sin A = \frac{5}{13}$ and the angle A is obtuse, then $\cos A = -\sqrt{1-\sin^2 A} = -\sqrt{1-\left(\frac{5}{13}\right)^2} = -\frac{12}{13}$

Therefore, $\sin 2A = 2\sin A\cos A = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = -\frac{120}{169}$

5. **(a)** In triangle BQP: $\tan P\hat{B}Q = \frac{PQ}{BQ} \Rightarrow PQ = BQ\tan P\hat{B}Q = 40\tan 36° = 29.1$ m (to 3 s.f.)

(b) In triangle ABQ: $B\hat{Q}A = 180° - \left(Q\hat{B}A + B\hat{A}Q\right) = 180° - (30° + 70°) = 80°$, and by the sine rule:

$$\frac{AB}{\sin B\hat{Q}A} = \frac{BQ}{\sin Q\hat{A}B} \Rightarrow AB = \frac{BQ\sin B\hat{Q}A}{\sin Q\hat{A}B} = \frac{40\sin 80°}{\sin 70°} = 41.9 \text{ m (to 3 s.f.)}$$

6. In triangle *ABC* using the cosine rule:

$$\cos B\hat{A}C = \frac{(AB)^2+(AC)^2-(BC)^2}{2(AB)(AC)} = \frac{48^2+32^2-56^2}{2(48)(32)} = \frac{1}{16}$$

Then $B\hat{A}C = \cos^{-1}\left(\frac{1}{16}\right) = 86.4°$ (to 3 s.f.)

7. **(a)** The smallest angle *A* is opposite the shortest side. Using the cosine rule:

$$\cos A = \frac{8^2+7^2-5^2}{2(8)(7)} = \frac{11}{14} \Rightarrow A = 38.2° \text{ (to 3 s.f.)}$$

(b) Area of the triangle = $\frac{1}{2}(8)(7)\sin A = 28\sin 38.2° = 17.3 \text{ cm}^2$ (to 3 s.f.)

8. **(a)** In triangle 2, by the sine rule: $\frac{AB}{\sin A\hat{C}B} = \frac{AC}{\sin A\hat{B}C} \Rightarrow \sin A\hat{C}B = \frac{AB\sin A\hat{B}C}{AC}$

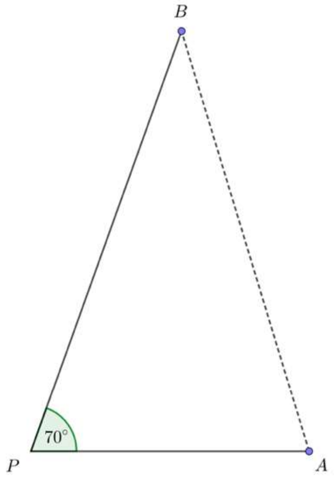
$$A\hat{C}B = \sin^{-1}\left(\frac{AB\sin A\hat{B}C}{AC}\right) = \sin^{-1}\left(\frac{20\sin 50°}{17}\right) = 64.3° \text{ or } A\hat{C}B = 116°$$

From the picture it follows that $A\hat{C}B = 116°$

(b) In triangle 1, $A\hat{C}B = 64.3°$ and $B\hat{A}C = 180° - (50° + 64.3°) = 65.7°$. Therefore,

$$\text{Area} = \frac{1}{2}(AB)(AC)\sin B\hat{A}C = \frac{1}{2}(20)(17)\sin 65.7° = 155 \text{ cm}^2 \text{ (to 3 s.f.)}$$

9.



After 2.5 hours boat *A* will be $20 \text{ kmh}^{-1}(2.5 \text{ h}) = 50 \text{ km}$ away from point *P*. Boat *B* will be $32 \text{ kmh}^{-1}(2.5 \text{ h}) = 80 \text{ km}$ away from the point *P*. The distance between the boats *A* and *B*:

By the Cosine Rule in triangle *APB*:

$$AB = \sqrt{(PA)^2+(PB)^2-2(PA)(PB)\cos A\hat{P}B}$$

$$= \sqrt{50^2+80^2-2(50)(80)\cos 70°} = 78.5 \text{ km}$$

10. By the sine rule in triangle *JKL*: $\frac{KL}{\sin K\hat{J}L}=\frac{JL}{\sin J\hat{K}L} \Rightarrow \sin J\hat{K}L=\frac{JL\sin K\hat{J}L}{KL}$.

$$J\hat{K}L=\sin^{-1}\left(\frac{JL\sin K\hat{J}L}{KL}\right)=\sin^{-1}\left(\frac{25\sin 51°}{38}\right)=31°$$

11. In triangle *ABC*: $B\hat{A}C=180°-\left(A\hat{B}C+A\hat{C}B\right)=180°-(60°+40°)=80°$

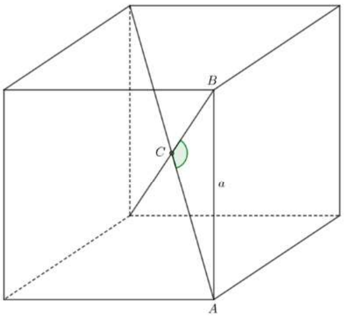
(a) By the sine rule: $\frac{AB}{\sin A\hat{C}B}=\frac{BC}{\sin B\hat{A}C} \Rightarrow AB=\frac{BC\sin A\hat{C}B}{\sin B\hat{A}C}=\frac{5\sin 40^{\circ}}{\sin 80^{\circ}}=3.26\text{ cm}$

(b) Area $A=\frac{1}{2}(AB)(BC)\sin A\hat{B}C=\frac{1}{2}(3.26)(5)\sin 60°=7.07\text{ cm}^2$ (to 3 s.f.)

12. Let a denote the length of the side edge of the cube. The diagonal of the cube has length $d=\sqrt{a^2+a^2+a^2}=\sqrt{3a^2}=a\sqrt{3}$

Applying the cosine rule in triangle *ABC* we

have: $\cos A\hat{C}B=\frac{(BC)^2+(AC)^2-(AB)^2}{2(BC)(AC)}$



$$=\frac{\left(\frac{a\sqrt{3}}{2}\right)^2+\left(\frac{a\sqrt{3}}{2}\right)^2-(a)^2}{2\left(\frac{a\sqrt{3}}{2}\right)\left(\frac{a\sqrt{3}}{2}\right)}=\frac{\frac{3}{4}a^2+\frac{3}{4}a^2-a^2}{\frac{3}{2}a^2}=\frac{\frac{1}{2}a^2}{\frac{3}{2}a^2}=\frac{1}{3}$$

Then $A\hat{C}B=\cos^{-1}\left(\frac{1}{3}\right)=70.5°$ (to 3 s.f.)

Alternatively, look at the rectangle formed by *AB* and the opposite edge. Two sides are a each, while the other sides are $a\sqrt{2}$ each. Thus the area of the rectangle is $a^2\sqrt{2}$. Area of triangle *ABC* is therefore $\frac{1}{4}a^2\sqrt{2}$. But the area of the triangle is also $\frac{1}{2}\frac{a\sqrt{3}}{2}\cdot\frac{a\sqrt{3}}{2}\sin A\hat{C}B$. Equating the two quantities will give us

$$\sin A\hat{C}B=\frac{2}{\sqrt{3}} \Rightarrow A\hat{C}B\approx 70.5°$$

13. (a) $$BC=\sqrt{(AB)^2+(AC)^2-2(AB)(AC)\cos B\hat{A}C}$$
$$=\sqrt{65^2+104^2-2(65)(104)\cos 60°}=91\text{ m}$$

(b) Area $A=\frac{1}{2}(AC)(AB)\sin B\hat{A}C=\frac{1}{2}(104)(65)\sin 60°=1690\sqrt{3}\text{ m}^2$

(c) **(i)** $A_1=\frac{1}{2}(AD)(AB)\sin B\hat{A}D=\frac{1}{2}(x)(65)\sin 30°=\frac{65}{2}x\left(\frac{1}{2}\right)=\frac{65x}{4}$

(ii) $A_2=\frac{1}{2}(AC)(AD)\sin C\hat{A}D=\frac{1}{2}(104)(x)\sin 30°=52x\left(\frac{1}{2}\right)=26x$

(iii) $A=A_1+A_2 \Leftrightarrow 1690\sqrt{3}=\frac{65x}{4}+26x \Leftrightarrow \frac{169}{4}x=1690\sqrt{3} \Leftrightarrow x=40\sqrt{3}$

(d) **(i)** $A\hat{D}C$ and $A\hat{D}B$ are supplementary angles. Therefore, $A\hat{D}C=180°-A\hat{D}B$ and $\sin A\hat{D}C=\sin\left(180°-A\hat{D}B\right)=\sin A\hat{D}B$

(ii) By the sine rule:

$$\frac{BD}{\sin D\hat{A}B}=\frac{AB}{\sin A\hat{D}B} \Rightarrow BD=\frac{AB\sin D\hat{A}B}{\sin A\hat{D}B}=\frac{AB\sin 30°}{\sin A\hat{D}B}=\frac{AB}{2\sin A\hat{D}B}$$ and

$$\frac{DC}{\sin C\hat{A}D}=\frac{AC}{\sin A\hat{D}C} \Rightarrow DC=\frac{AC\sin C\hat{A}D}{\sin A\hat{D}C}=\frac{AC\sin 30°}{\sin A\hat{D}C}=\frac{AC}{2\sin A\hat{D}C}$$. The ratio

$$\frac{BD}{DC}=\frac{\frac{AB}{2\sin A\hat{D}B}}{\frac{AC}{2\sin A\hat{D}C}}=\frac{AB}{AC}$$ (since $\sin A\hat{D}C=\sin A\hat{D}B$). Therefore, $\frac{BD}{DC}=\frac{65}{104}=\frac{5}{8}$

Note: This is a known result from geometry that the angle bisector divides the opposite side of an angle in the ratio of its adjacent sides.

14. (a) By the sine rule: $\frac{QR}{\sin R\hat{P}Q}=\frac{PR}{\sin P\hat{Q}R} \Rightarrow \frac{x-2}{\sin 30°}=\frac{x}{\sin 45°}$

$$(x-2)\sin 45°=x\sin 30° \Rightarrow (x-2)\frac{\sqrt{2}}{2}=\frac{1}{2}x$$

$$(x-2)\sqrt{2}=x \Rightarrow x\left(\sqrt{2}-1\right)=2\sqrt{2}$$

$$x=\frac{2\sqrt{2}}{\sqrt{2}-1}=\frac{2\sqrt{2}\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}=4+2\sqrt{2}$$

(b) Angle $P\hat{R}Q = 180° - \left(R\hat{P}Q + P\hat{Q}R\right) = 180° - (30° + 45°) = 105°$.

Since $P\hat{R}Q = 180° - (30° + 45°)$, then

$$\sin P\hat{R}Q = \sin(30° + 45°) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Area of triangle *PQR*:

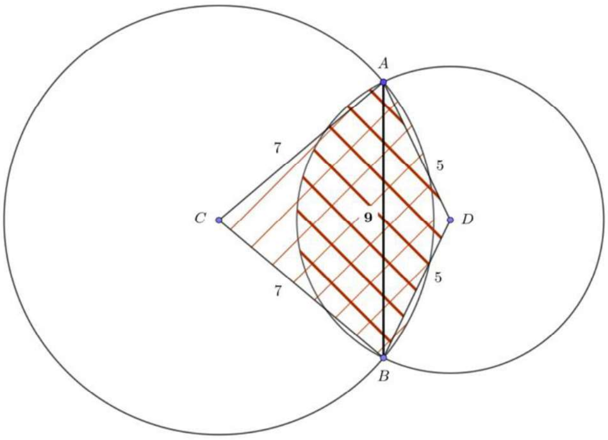
$$A = \frac{1}{2}x(x-2)\sin P\hat{Q}R = \frac{1}{2}\left(4+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\frac{\sqrt{6}+\sqrt{2}}{4}$$
$$= \frac{1}{2}\left(2+\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(\sqrt{6}+\sqrt{2}\right) = \frac{1}{2}\left(2+2\sqrt{2}+\sqrt{2}+2\right)\left(\sqrt{6}+\sqrt{2}\right) = \frac{1}{2}\left(4+3\sqrt{2}\right)\left(\sqrt{6}+\sqrt{2}\right)$$
$$= \frac{1}{2}\left(4\sqrt{6}+4\sqrt{2}+6\sqrt{3}+6\right) = 2\sqrt{6}+2\sqrt{2}+3\sqrt{3}+3$$

Alternatively, if $x = 4+2\sqrt{2}$, then the height of the triangle is half of that $2+\sqrt{2}$

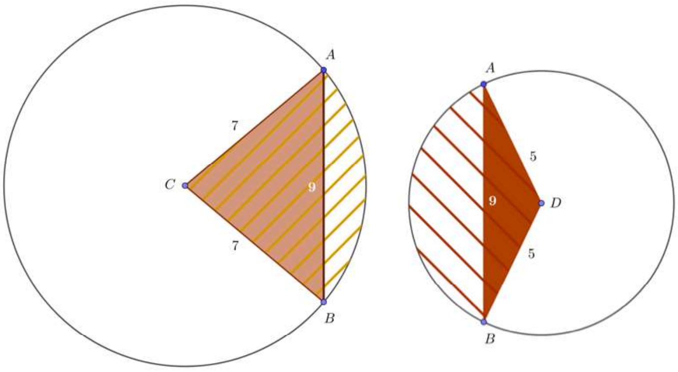
The left part of the base is $x\cos 30 = \left(4+2\sqrt{2}\right)\frac{\sqrt{3}}{2} = 2\sqrt{3}+\sqrt{6}$ and the right part is $2+\sqrt{2}$.

Thus, the area of the triangle is $\frac{1}{2}\left(2+\sqrt{2}\right)\left(2+\sqrt{2}+\sqrt{6}+2\sqrt{3}\right) = 3+2\sqrt{2}+2\sqrt{6}+3\sqrt{3}$

15. Relative position of the two circles is shown on the following diagram:



For better clarity, the diagram can be split in two parts:



The shaded area to be calculated is equal to the sum of areas of the two segments shown.

Area of the larger segment: $A_1 = A_{ABC} - A_{\Delta ABC}$, where A_{ABC} is the area of the sector *ABC*.
Similarly, $A_2 = A_{ABD} - A_{\Delta ABD}$, where A_{ABD} is the area of the sector *ABD*.

In triangle *ABC*:

$$\cos A\hat{C}B = \frac{(AC)^2+(BC)^2-(AB)^2}{2(AC)(BC)} = \frac{7^2+7^2-9^2}{2(7)(7)} = \frac{17}{98} \Rightarrow A\hat{C}B = \cos^{-1}\left(\frac{17}{98}\right) = 1.396 \text{ rad}$$

Now, $A_{ABC} = \frac{1}{2}(AC)^2\left(A\hat{C}B\right) = \frac{1}{2}(7)^2(1.396) = 34.2 \text{ cm}^2$

$$A_{\Delta ABC} = \frac{1}{2}(AC)^2 \sin A\hat{C}B = \frac{1}{2}(7)^2 \sin(1.396) = 24.1 \text{ cm}^2$$

$$A_1 = 34.2 \text{ cm}^2 - 24.1 \text{ cm}^2 = 10.1 \text{ cm}^2$$

In triangle *ABD*:

$$\cos A\hat{D}B = \frac{(AD)^2+(BD)^2-(AB)^2}{2(AD)(BD)} = \frac{5^2+5^2-9^2}{2(5)(5)} = -\frac{31}{50} \Rightarrow A\hat{D}B = \cos^{-1}\left(-\frac{31}{50}\right) = 2.240 \text{ rad}$$

Now, $A_{ABD} = \frac{1}{2}(AD)^2\left(A\hat{D}B\right) = \frac{1}{2}(5)^2(2.240) = 28 \text{ cm}^2$

$$A_{\Delta ABD} = \frac{1}{2}(AD)^2 \sin A\hat{D}B = \frac{1}{2}(5)^2 \sin(2.240) = 9.80 \text{ cm}^2$$

$$A_2 = 28 \text{ cm}^2 - 9.80 \text{ cm}^2 = 18.2 \text{ cm}^2$$

Therefore, the shaded area $A = A_1 + A_2 = 10.1 \text{ cm}^2 + 18.2 \text{ cm}^2 = 28.3 \text{ cm}^2$

16. (a) Let $J\hat{L}K = \alpha$. Then $\theta + \alpha + 60° = 180°$. Both θ and α must be greater than 0, thus $\theta = 180° - 60° - \alpha = 120 - \alpha \Rightarrow 0° < \theta < 120°$

(b) Area of triangle *JKL*:

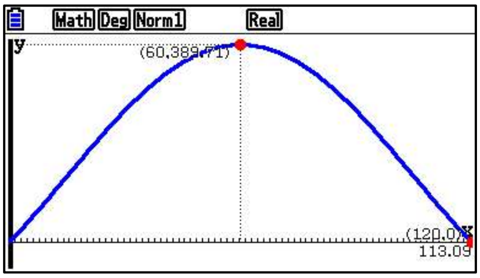
$$A = \frac{1}{2}(KL)(JL)\sin[180° - (\theta + 60°)] = \frac{1}{2}(KL)(30)\sin(\theta + 60°) = 15(KL)\sin(\theta + 60°)$$

By the sine rule:

$$\frac{KL}{\sin K\hat{J}L} = \frac{JL}{\sin J\hat{K}L} \Rightarrow KL = \frac{JL \sin K\hat{J}L}{\sin J\hat{K}L} = \frac{30\sin\theta}{\sin 60°} = \frac{30\sin\theta}{\frac{\sqrt{3}}{2}} = 20\sqrt{3}\sin\theta$$

and $A = 15\left(20\sqrt{3}\sin\theta\right)\sin(\theta + 60°) = 300\sqrt{3}\sin\theta\sin(\theta + 60°)$

(c) By GDC:



The maximum area occurs for $\theta = 60°$

17. (a) In triangle *BMC*: $(BM)^2 = (BC)^2 - (CM)^2 = 17^2 - 15^2 = 64 \Rightarrow BM = 8$ cm

Area of triangle *ABC*: $A = \frac{1}{2}(AC)(BM) = \frac{1}{2}(30)(8) = 120 \text{ cm}^2$

(b) Area of triangle *ABC* can be also written as

$$A_{\Delta ABC} = \frac{1}{2}(AB)(BC)\sin A\hat{B}C = \frac{1}{2}(17)(17)\sin A\hat{B}C = \frac{289}{2}\sin A\hat{B}C$$

Comparing both formulas: $\frac{289}{2}\sin A\hat{B}C = 120 \Rightarrow \sin A\hat{B}C = \frac{240}{289}$. It follows that $A\hat{B}C = 56.1°$ or $A\hat{B}C = 123.8°$. Since the angle $A\hat{B}C$ is obtuse, then $A\hat{B}C = 123.8°$ or, in radians, $A\hat{B}C = 123.8°\left(\frac{\pi}{180°}\right) = 2.16$ (to 3 s.f.)

(c) The area of the shaded region *R*:

$$A = \frac{1}{2}\pi(AM)^2 - \left(\frac{1}{2}(AB)^2(A\hat{B}C) - A_{\Delta ABC}\right) = \frac{1}{2}\pi 15^2 - \left(\frac{1}{2}17^2(2.16) - 120\right) = 161 \text{ cm}^2$$

(to 3 s.f.)

18. (a) By the cosine rule: $L^2 = 1^2 + 1^2 - 2(1)(1)\cos\alpha = 2 - 2\cos\alpha$.
It follows that $L = \sqrt{2 - 2\cos\alpha}$

(b) Using the formula $\cos 2\theta = 1 - 2\sin^2\theta$, let's take $\theta = \dfrac{\alpha}{2}$. Then

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \Rightarrow 2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha \text{ and } \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

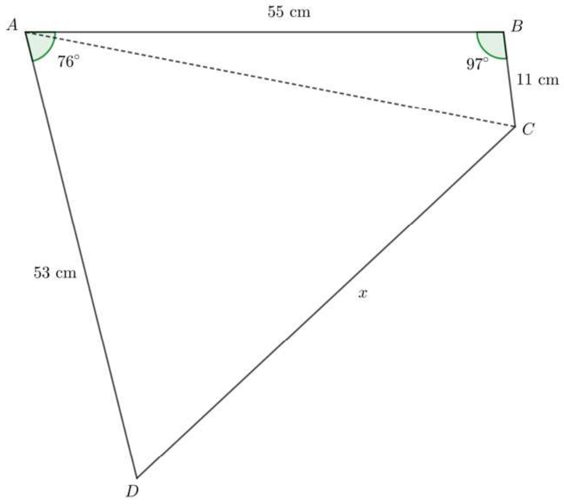
(c) As can be seen in the diagram, θ is an acute angle. Therefore, $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{2}}$.

We know that $L = \sqrt{2 - 2\cos\alpha}$. Using the result from **(b)**:

$$L = \sqrt{2 - 2\left(1 - 2\sin^2\frac{\alpha}{2}\right)} = \sqrt{2 - 2 + 4\sin^2\frac{\alpha}{2}} = \sqrt{4\sin^2\frac{\alpha}{2}} = 2\sin\frac{\alpha}{2}$$

19. By the sine rule: $\dfrac{b}{\sin 2\theta} = \dfrac{a}{\sin\theta} \Leftrightarrow \dfrac{b}{2\sin\theta\cos\theta} = \dfrac{a}{\sin\theta} \Leftrightarrow \dfrac{b}{2\cos\theta} = a \Leftrightarrow \cos\theta = \dfrac{b}{2a}$

20. In triangle *ABC,* we use the cosine rule.



$$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos A\hat{B}C$$

$$AC = \sqrt{55^2 + 11^2 - 2(55)(11)\cos 97°} = 57.4 \text{ cm}$$

In triangle *ABC* by the sine rule: $\dfrac{BC}{\sin C\hat{A}B} = \dfrac{AC}{\sin A\hat{B}C} \Rightarrow \sin C\hat{A}B = \dfrac{BC\sin A\hat{B}C}{AC}$

$$C\hat{A}B = \sin^{-1}\left(\frac{BC\sin A\hat{B}C}{AC}\right) = \sin^{-1}\left(\frac{11\sin 97°}{57.4}\right) = 11.0°$$

Therefore, $D\hat{A}C = D\hat{A}B - C\hat{A}B = 76° - 11.0° = 65.0°$

In triangle ACD by the cosine rule: $x^2 = (AD)^2 + (AC)^2 - 2(AD)(AC)\cos D\hat{A}C$

$$x = \sqrt{53^2 + (57.4)^2 - 2(53)(57.4)\cos 65°} = 59.5 \text{ cm}$$

21. In triangle ABC: $\tan A\hat{C}B = \frac{AB}{BC} \Rightarrow BC = \frac{AB}{\tan A\hat{C}B} = \frac{12}{\tan 45°} = 12 \text{ cm}$

In triangle ABD: $\tan A\hat{D}B = \frac{AB}{BD} \Rightarrow BD = \frac{AB}{\tan A\hat{D}B} = \frac{12}{\tan 60°} = \frac{12}{\sqrt{3}} = 4\sqrt{3} \text{ cm}$

In triangle BDC applying the cosine rule:

$$\cos C\hat{B}D = \frac{(BC)^2 + (BD)^2 - (CD)^2}{2(BC)(BD)} = \frac{12^2 + (4\sqrt{3})^2 - 10^2}{2(12)(4\sqrt{3})} = \frac{23\sqrt{3}}{72}$$

$$\sin C\hat{B}D = \sqrt{1 - \cos^2 C\hat{B}D} = \sqrt{1 - \frac{529}{1728}} = 0.8330$$

Now:

Area of triangle ABC: $A_1 = \frac{1}{2}(BC)(AB) = \frac{1}{2}(12)(12) = 72 \text{ cm}^2$

Area of triangle ABD: $A_2 = \frac{1}{2}(BD)(AB) = \frac{1}{2}(4\sqrt{3})(12) = 24\sqrt{3} \text{ cm}^2$

Area of triangle BDC: $A_3 = \frac{1}{2}(BD)(BC)\sin C\hat{B}D = \frac{1}{2}(4\sqrt{3})(12)(0.8330) = 34.6 \text{ cm}^2$ (to 3 s.f.)

In triangle ACD: $AC = 12\sqrt{2}$ cm, $AD = \frac{12}{\sin 60°} = \frac{12}{\frac{\sqrt{3}}{2}} = \frac{24}{\sqrt{3}} = 8\sqrt{3}$ cm, $CD = 10$ cm

Then,

$$\cos C\hat{A}D = \frac{(AC)^2 + (AD)^2 - (CD)^2}{2(AC)(AD)} = \frac{(12\sqrt{2})^2 + (8\sqrt{3})^2 - 10^2}{2(12\sqrt{2})(8\sqrt{3})} = 0.8080 \Rightarrow C\hat{A}D = 36.1°$$

Area of triangle ACD: $A_4 = \frac{1}{2}(AC)(AD)\sin C\hat{A}D = \frac{1}{2}(12\sqrt{2})(8\sqrt{3})(\sin 36.1°) = 69.3 \text{ cm}^2$

22. Calculate lengths of the segments *EF*, *DF* and *DE*. These segments are diagonals of the respective rectangles, so:

$$EF = \sqrt{6^2+3^2} = \sqrt{45} = 3\sqrt{5} \text{ cm}$$

$$DF = \sqrt{4^2+3^2} = \sqrt{25} = 5 \text{ cm}$$

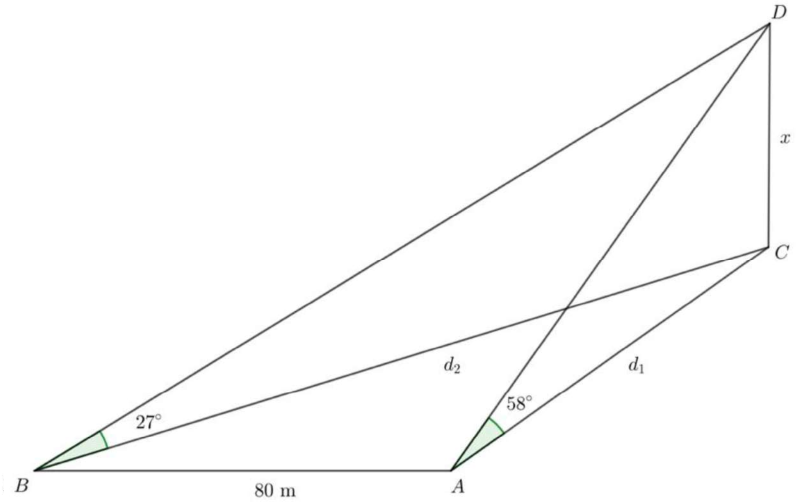
$$DE = \sqrt{6^2+4^2} = \sqrt{52} = 2\sqrt{13} \text{ cm}$$

Using the cosine rule in the triangle *DEF*:

$$\cos D\hat{E}F = \frac{(DE)^2+(EF)^2-(DF)^2}{2(DE)(EF)} = \frac{52+45-25}{2\left(2\sqrt{13}\right)\left(3\sqrt{5}\right)} = \frac{6\sqrt{65}}{65}$$

$$\text{Therefore, } D\hat{E}F = \cos^{-1}\left(\frac{6\sqrt{65}}{65}\right) = 41.9° \text{ (to 3 s.f.)}$$

23.



$$\text{In triangle } ACD\text{: } \tan 58° = \frac{x}{AC} \Rightarrow AC = \frac{x}{\tan 58°}$$

$$\text{In triangle } BCD\text{: } \tan 27° = \frac{x}{BC} \Rightarrow BC = \frac{x}{\tan 27°}$$

Triangle *ABC* is a right-angled triangle, so

$$(AC)^2 + (AB)^2 = (BC)^2$$

$$\left(\frac{x}{\tan 58°}\right)^2 + (80)^2 = \left(\frac{x}{\tan 27°}\right)^2$$

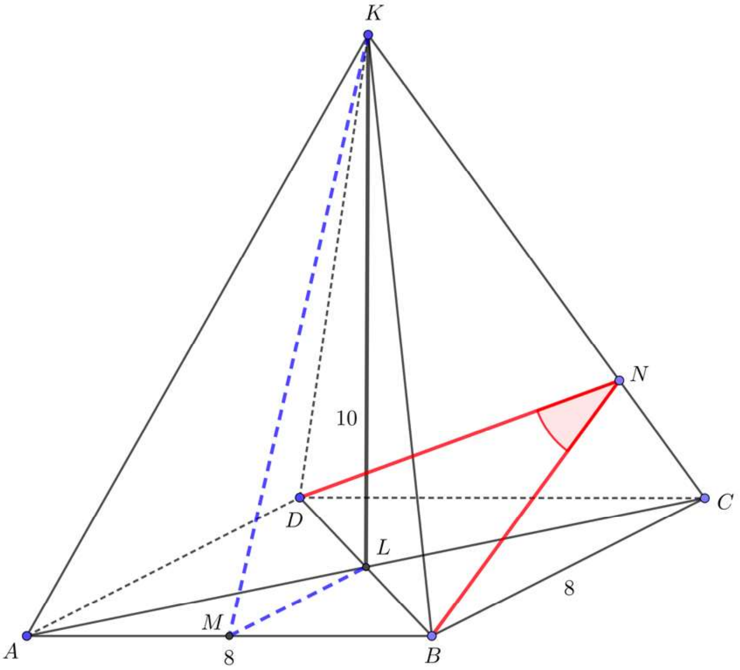
$$\frac{x^2}{\tan^2 58°} + 6400 = \frac{x^2}{\tan^2 27°}$$

$$\frac{x^2}{\tan^2 27°}-\frac{x^2}{\tan^2 58°}=6400 \Rightarrow \frac{\tan^2 58°-\tan^2 27°}{\left(\tan^2 27°\right)\left(\tan^2 58°\right)}x^2=6400$$

$$x^2=\frac{6400\left(\tan^2 27°\right)\left(\tan^2 58°\right)}{\tan^2 58°-\tan^2 27°}$$

$$x=\frac{80\tan 27°\tan 58°}{\sqrt{\tan^2 58°-\tan^2 27°}}=43.0 \text{ m}$$

24.



The angle between two adjacent lateral faces is the angle between *BN* and *DN* (marked red in the diagram above).

Method 1

We will not give details of some ‘obvious’ calculations.
Remember that *ABCD* is a square.

In triangle *KLB*: $KB=\sqrt{10^2+\left(4\sqrt{2}\right)^2}=\sqrt{132}=KC$
These are the edges of the pyramid.

In triangle *KBC* and using the law of cosines:

$$KB^2=BC^2+KC^2-2BC\cdot KC\cos B\hat{C}K \Rightarrow 132=64+132-2\cdot 8\cdot\sqrt{132}\cos B\hat{C}K$$

$$\Rightarrow \cos B\hat{C}K=\frac{2}{\sqrt{33}} \Rightarrow B\hat{C}K\approx 69.63$$

In triangle BCN, $BN = 8\sin B\hat{C}N = 8\sin 69.63 \approx 7.5 = DN$

In triangle DBN and using the law of cosines:

$$\cos\hat{N} = \frac{7.5^2 + 7.5^2 - \left(8\sqrt{2}\right)^2}{2\times 7.5\times 7.5} \Rightarrow \hat{N} \approx 97.9°$$

Method 2

Area of a lateral face $A_L = \frac{1}{2}(AB)(KM)$

In the right-angled triangle KLM:

$$KM = \sqrt{(KL)^2 + (LM)^2} = \sqrt{(KL)^2 + \left(\frac{1}{2}BC\right)^2} = \sqrt{(KL)^2 + \frac{1}{4}(BC)^2}$$

Therefore, $A_L = \frac{1}{2}(8)\left(\sqrt{10^2 + \frac{1}{4}(8)^2}\right) = 8\sqrt{29}$

But also, $A_L = \frac{1}{2}(BN)(KC) = \frac{1}{2}(BN)\sqrt{(KL)^2 + (LC)^2}$

In triangle ABC: $AC = 8\sqrt{2}$ and $LC = \frac{1}{2}AC = \frac{1}{2}\left(8\sqrt{2}\right) = 4\sqrt{2}$. Then

$$A_L = \frac{1}{2}(BN)\sqrt{(KL)^2 + (LC)^2} = \frac{1}{2}(BN)\sqrt{10^2 + \left(4\sqrt{2}\right)^2} = \frac{1}{2}(BN)\sqrt{132} = (BN)\sqrt{33}$$

Comparing:

$$(BN)\sqrt{33} = 8\sqrt{29} \Rightarrow BN = \frac{8\sqrt{29}}{\sqrt{33}} = DN$$

In triangle BDN: $\cos B\hat{N}D = \dfrac{(BN)^2 + (DN)^2 - (BD)^2}{2(BN)(DN)} = \dfrac{2\left(\frac{8\sqrt{29}}{\sqrt{33}}\right)^2 - \left(8\sqrt{2}\right)^2}{2\left(\frac{8\sqrt{29}}{\sqrt{33}}\right)^2} = -\dfrac{4}{29}$

Then $B\hat{N}D = \cos^{-1}\left(-\frac{4}{29}\right) = 97.9°$ (to 3 s.f.)

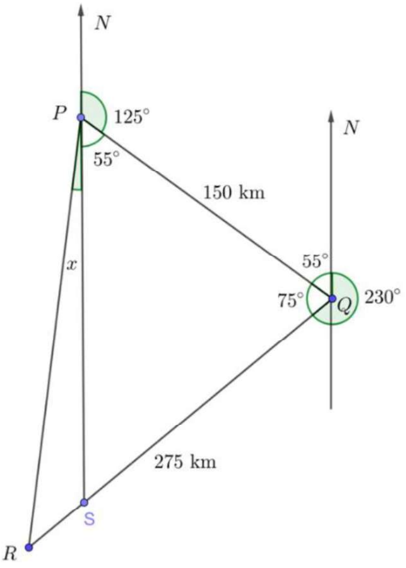
25. (a) General equation of the straight line is $L_1 : y = ax + b$. Since the line L_1 passes through the origin, then $b = 0$ and $L_1 : y = ax$. The line L_1 makes an angle of $30°$ with the positive x-axis, so $a = \tan 30° = \frac{\sqrt{3}}{3}$. Therefore, the equation of the line $L_1 : y = \frac{\sqrt{3}}{3}x$.

(b) The equation of the line $L_2 : x + 2y = 6$ can be written as $L_2 : y = -\frac{1}{2}x + 3$.

Let the angle between the lines L_1 and L_2 be α. Then:

$$\alpha = \left|\tan^{-1}\left(-\frac{1}{2}\right) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right| = |-26.6° - 30°| = |-56.6°| = 56.6° \text{ (to 3 s.f.)}$$

26.



In triangle PRQ:

$$PR = \sqrt{(PQ)^2 + (QR)^2 - 2(PQ)(QR)\cos P\hat{Q}R} = \sqrt{150^2 + 275^2 - 2(150)(275)\cos 75°} = 277 \text{ km}$$

By the sine rule: $\frac{PR}{\sin P\hat{Q}R} = \frac{QR}{\sin R\hat{P}Q}$. Then:

$$\frac{277}{\sin 75°} = \frac{275}{\sin(55° + x)} \Rightarrow \sin(55° + x) = \frac{275\sin 75°}{277}$$

It follows that

$$\sin(55° + x) = 0.9590 \Rightarrow 55° + x = \sin^{-1}(0.9590) \Rightarrow x = \sin^{-1}(0.9590) - 55° = 73.5° - 55° = 18.5°$$

The length of the flight from R to P is 277 km, the bearing is $18.5°$

Exercise 8.1

1. **(a)** Since $\sqrt{-4}=2\mathrm{i}$, $5+\sqrt{-4}=5+2\mathrm{i}$

(b) Since $\sqrt{-7}=\sqrt{7}\mathrm{i}$, $7-\sqrt{-7}=7-\sqrt{7}\mathrm{i}$

(c) $-6=-6+0\cdot\mathrm{i}$

(d) $-\sqrt{49}=-7=-7+0\cdot\mathrm{i}$

(e) $\sqrt{-81}=9i=0+9\mathrm{i}$

(f) $-\sqrt{\dfrac{-25}{16}}=-\dfrac{5}{4}\mathrm{i}=0-\dfrac{5}{4}\mathrm{i}$

2. **(a)** $-1-\mathrm{i}$

(b) $-3+4\mathrm{i}-2+5\mathrm{i}=-5+9\mathrm{i}$

(c) $(-3+4\mathrm{i})(2-5\mathrm{i})=-6+8\mathrm{i}+15\mathrm{i}+20=14+23\mathrm{i}$

(d) $3\mathrm{i}-2+4\mathrm{i}=-2+7\mathrm{i}$

(e) $(2-7\mathrm{i})(3+4\mathrm{i})=6-21\mathrm{i}+8\mathrm{i}+28=34-13\mathrm{i}$

(f) $(1+\mathrm{i})(2-3\mathrm{i})=2+2\mathrm{i}-3\mathrm{i}+3=5-\mathrm{i}$

(g) $\dfrac{3+2\mathrm{i}}{2+5\mathrm{i}}\cdot\dfrac{2-5\mathrm{i}}{2-5\mathrm{i}}=\dfrac{16-11\mathrm{i}}{4+25}=\dfrac{16-11\mathrm{i}}{29}=\dfrac{16}{29}-\dfrac{11}{29}\mathrm{i}$

(h) $\dfrac{2-\mathrm{i}}{3+2\mathrm{i}}\cdot\dfrac{3-2\mathrm{i}}{3-2\mathrm{i}}=\dfrac{4-7\mathrm{i}}{9+4}=\dfrac{4-7\mathrm{i}}{13}=\dfrac{4}{13}-\dfrac{7}{13}\mathrm{i}$

(i) 1

(j) $\dfrac{4}{9}+\dfrac{1}{4}=\dfrac{25}{36}$

(k) $\dfrac{\dfrac{2}{3}-\dfrac{1}{2}\mathrm{i}}{\dfrac{1}{3}+\dfrac{1}{2}\mathrm{i}}=\dfrac{\dfrac{4-3\mathrm{i}}{6}}{\dfrac{2+3\mathrm{i}}{6}}=\dfrac{-1-18\mathrm{i}}{4+9}=\dfrac{-1-18\mathrm{i}}{13}=\dfrac{-1}{13}-\dfrac{18}{13}\mathrm{i}$

(l) $8-\mathrm{i}$

(m) $\dfrac{-\mathrm{i}(3-7\mathrm{i})}{1^2}=-7-3\mathrm{i}$

(n) $4+10\mathrm{i}$

(o) $\dfrac{13(5+12\mathrm{i})}{25+144}=\dfrac{65+156\mathrm{i}}{169}=\dfrac{5+12\mathrm{i}}{13}=\dfrac{5}{13}+\dfrac{12}{13}\mathrm{i}$

(p) $\dfrac{12\mathrm{i}(3-4\mathrm{i})}{9+16}=\dfrac{48+36\mathrm{i}}{25}=\dfrac{48}{25}+\dfrac{36}{25}\mathrm{i}$

(q) $2+9\mathrm{i}$

(r) 68

(s) $$\frac{(39-52\mathrm{i})(24-10\mathrm{i})}{24^2+10^2}=\frac{416-1638\mathrm{i}}{676}=\frac{8}{13}-\frac{63}{26}\mathrm{i}$$

(t) $$\frac{1}{7-4\mathrm{i}}=\frac{7+4\mathrm{i}}{49+16}=\frac{7+4\mathrm{i}}{65}=\frac{7}{65}+\frac{4}{65}\mathrm{i}$$

(u) $$\frac{1}{5-12\mathrm{i}}=\frac{5+12\mathrm{i}}{25+144}=\frac{5+12\mathrm{i}}{169}=\frac{5}{169}+\frac{12}{169}\mathrm{i}$$

(v) $$\frac{3}{3-4\mathrm{i}}+\frac{2}{6+8\mathrm{i}}=\frac{3(3+4\mathrm{i})}{9+16}+\frac{2(6-8\mathrm{i})}{36+64}=\frac{9+12\mathrm{i}}{25}+\frac{12-16\mathrm{i}}{100}$$
$$=\frac{9+12\mathrm{i}}{25}+\frac{3-4\mathrm{i}}{25}=\frac{12+8\mathrm{i}}{25}=\frac{12}{25}+\frac{8}{25}\mathrm{i}$$

(w) $$\frac{54-19\mathrm{i}}{5-12\mathrm{i}}=\frac{(54-19\mathrm{i})(5+12\mathrm{i})}{25+144}=\frac{498+553\mathrm{i}}{169}=\frac{498}{169}+\frac{553}{169}\mathrm{i}$$

(x) $$\frac{5-12\mathrm{i}}{3+4\mathrm{i}}=\frac{(5-12\mathrm{i})(3-4\mathrm{i})}{9+16}=\frac{-33-56\mathrm{i}}{25}=\frac{-33}{25}-\frac{56}{25}\mathrm{i}$$

3. $$(2+3\mathrm{i})z=7+\mathrm{i}\Rightarrow z=\frac{7+\mathrm{i}}{2+3\mathrm{i}}=\frac{17-19\mathrm{i}}{13}$$

Alternatively, we can find z by solving a system of equations:

$$(2+3\mathrm{i})(a+b\mathrm{i})=7+\mathrm{i}\Rightarrow 2a-3b+(3a+2b)\mathrm{i}=7+\mathrm{i}\Rightarrow$$

$$\begin{array}{l}2a-3b=7\\3a+2b=1\end{array}\Rightarrow\begin{array}{l}a=\dfrac{17}{13}\\b=-\dfrac{19}{13}\end{array}$$

However, it is easier to find z using division, as we did.

4. $2x-y+(xy+2)\mathrm{i}=1+3\mathrm{i}$, using the fact that complex numbers are equal if their real parts as well as imaginary parts are equal, we get a system of two equations:

$$\begin{array}{l}2x-y=1\\xy+2=3\end{array}$$

From the first equation: $y=2x-1$; hence: $x(2x-1)+2=3\Rightarrow 2x^2-x-1=0\Rightarrow x_1=-\frac{1}{2},\ x_2=1$ and $y_1=-2,\ y_2=1$

The solutions are: $x_1=-\frac{1}{2},\ y_1=-2$ or $x_2=1,\ y_2=1$

5. **(a)**
$$\left(1+\sqrt{3}\mathrm{i}\right)^3 = 1^3 + 3\cdot 1^2\cdot\sqrt{3}\mathrm{i} + 3\cdot 1\cdot\left(\sqrt{3}\mathrm{i}\right)^2 + \left(\sqrt{3}\mathrm{i}\right)^3$$
$$= 1 + 3\sqrt{3}\mathrm{i} + 3\cdot 3(-1) + 3\sqrt{3}(-\mathrm{i})$$
$$= 1 - 9 + 3\sqrt{3}\mathrm{i} - 3\sqrt{3}\mathrm{i} = -8$$

(b) First, write the number in the form: $\left(1+\sqrt{3}\mathrm{i}\right)^{6n} = \left(\left(\left(1+\sqrt{3}\mathrm{i}\right)^3\right)^2\right)^n$

Now, use the fact established in **(a)** that $\left(1+\sqrt{3}\mathrm{i}\right)^3 = -8$ to carry out the calculations:

$$\left(1+\sqrt{3}\mathrm{i}\right)^{6n} = \left(\left(\left(1+\sqrt{3}\mathrm{i}\right)^3\right)^2\right)^n = \left((-8)^2\right)^n = (8)^{2n}$$

(c) Use the result in **(b)**:
$$\left(1+\sqrt{3}\mathrm{i}\right)^{48} = \left(1+\sqrt{3}\mathrm{i}\right)^{6\cdot 8} = 8^{2\cdot 8} = 8^{16}$$

Caution: you may be tempted to use a GDC to evaluate high powers of complex numbers. Some GDCs do not have the capacity to perform such an operation.

6. **(a)**
$$\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^2 = \left(-\sqrt{2}\right)^2 + 2\cdot\left(-\sqrt{2}\right)\sqrt{2}\mathrm{i} + \left(\sqrt{2}\mathrm{i}\right)^2$$
$$= 2 - 4\mathrm{i} + 2(-1) = -4\mathrm{i}$$

(b) First, write the number in the form: $\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{4k} = \left(\left(\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^2\right)^2\right)^n$

Now, use the fact established in **(a)** that $\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^2 = -4\mathrm{i}$ to carry out the requested

calculations: $\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{4k} = \left(\left(\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^2\right)^2\right)^k = \left((-4\mathrm{i})^2\right)^k = (-16)^k$

(c) Use the result in **(b)**:
$$\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{46} = \left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{44+2} = \left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^{4\cdot 11}\left(-\sqrt{2}+\mathrm{i}\sqrt{2}\right)^2$$
$$= (-16)^{11}(-4\mathrm{i}) = 16^{11}\cdot 4\mathrm{i} = 4^{23}\mathrm{i} = 2^{46}\mathrm{i}$$

7. Let $z = x + y\mathrm{i}$. Then $z + 4\mathrm{i} = x + (y+4)\mathrm{i}$ and $z + \mathrm{i} = x + (y+1)\mathrm{i}$

Thus, $|z+4\mathrm{i}| = 2|z+\mathrm{i}| \Rightarrow \sqrt{x^2+(y+4)^2} = 2\sqrt{x^2+(y+1)^2}$

$x^2 + (y+4)^2 = 4\left(x^2+(y+1)^2\right) \Rightarrow x^2 + y^2 + 8y + 16 = 4x^2 + 4y^2 + 8y + 4$

$3x^2 + 3y^2 = 12 \Rightarrow x^2 + y^2 = 4$

Therefore, $\sqrt{x^2+y^2} = 2 \Rightarrow |z| = 2$

8. First, simplify the fraction and then add 3:

$$\frac{2\mathrm{i}}{2-\mathrm{i}\sqrt{2}}\cdot\frac{2+\mathrm{i}\sqrt{2}}{2+\mathrm{i}\sqrt{2}}=\frac{4\mathrm{i}-2\sqrt{2}}{4+2}=\frac{2\mathrm{i}-\sqrt{2}}{3}$$

$$\Rightarrow z=3+\frac{2\mathrm{i}-\sqrt{2}}{3}=\frac{9-\sqrt{2}+2\mathrm{i}}{3}=\frac{9-\sqrt{2}}{3}+\frac{2}{3}\mathrm{i}$$

9. $(x+\mathrm{i}y)(4-7\mathrm{i})=3+2\mathrm{i}\Rightarrow x+\mathrm{i}y=\dfrac{3+2\mathrm{i}}{4-7\mathrm{i}}$

Now simplify the RHS:

$$\frac{3+2\mathrm{i}}{4-7\mathrm{i}}\cdot\frac{4+7\mathrm{i}}{4+7\mathrm{i}}=\frac{12+21\mathrm{i}+8\mathrm{i}-14}{16+49}=\frac{-2+29\mathrm{i}}{65}$$

Therefore, $x=-\dfrac{2}{65},y=\dfrac{29}{65}$

10. $\mathrm{i}(z+1)=3z-2\Rightarrow-3z+\mathrm{i}z=-2-\mathrm{i}\Rightarrow(-3+\mathrm{i})z=-2-\mathrm{i}$

Now, divide by $-3+\mathrm{i}$ and simplify: $z=\dfrac{-2-\mathrm{i}}{-3+\mathrm{i}}=\dfrac{1}{2}+\dfrac{1}{2}\mathrm{i}$

11. $\dfrac{2-\mathrm{i}}{1+2\mathrm{i}}\sqrt{z}=2-3\mathrm{i}\Rightarrow\sqrt{z}=\dfrac{(2-3\mathrm{i})(1+2\mathrm{i})}{2-\mathrm{i}}=\dfrac{8+\mathrm{i}}{2-\mathrm{i}}=3+2\mathrm{i}$

Thus, $z=(3+2\mathrm{i})^2=5+12\mathrm{i}$

12. $(x+\mathrm{i}y)^2=3-4\mathrm{i}\Rightarrow x^2-y^2+2xy\mathrm{i}=3-4\mathrm{i}$

Two complex numbers are equal if their real parts are equal and imaginary parts are equal. We get a system of equations to be solved:

$$x^2-y^2=3$$
$$2xy=-4$$

Solve for y in the second equation and substitute in the first:

$$y=-\frac{2}{x}\Rightarrow x^2-\left(-\frac{2}{x}\right)^2=3\Rightarrow x^2-\frac{4}{x^2}=3\Rightarrow x^4-3x^2-4=0\Rightarrow x_1^2=4,\ \cancel{x_2^2=-1}$$

Therefore, we have two solutions: $(x,y)=(2,-1)$ and $(x,y)=(-2,1)$

13. **(a)** $\left(x^2-y^2\right)+2xy\mathrm{i}=-8+6\mathrm{i}\Rightarrow\begin{cases}x^2-y^2=-8\\ 2xy=6\end{cases}$

Solve for y in the second equation and substitute in the first:

$$x^2-\left(\frac{3}{x}\right)^2=-8\Rightarrow x^4+8x^2-9=0\Rightarrow x_1^2=1,\ \cancel{x_2^2=-9}$$

So, either $x=1$ and $y=3$, or $x=-1$ and $y=-3$

(b) Solving a quadratic equation with complex coefficients:

$$z=\frac{-1+\mathrm{i}\pm\sqrt{(1-\mathrm{i})^2-4(2-2\mathrm{i})}}{2}=\frac{-1+\mathrm{i}\pm\sqrt{1-2\mathrm{i}-1-8+8\mathrm{i}}}{2}$$

$$=\frac{-1+\mathrm{i}\pm\sqrt{-8+6\mathrm{i}}}{2}$$

Since $\sqrt{-8+6\mathrm{i}}=\pm(1+3\mathrm{i})$, then z equals either $z_1=\dfrac{-1+\mathrm{i}+(1+3\mathrm{i})}{2}=2\mathrm{i}$, or

$$z_2=\frac{-1+\mathrm{i}-(1+3\mathrm{i})}{2}=-1-\mathrm{i}$$

14. Let $z=x+\mathrm{i}y$.

$$z^3=27\mathrm{i}\Rightarrow x^3+3x^2y\mathrm{i}-3xy^2-\mathrm{i}y^3=27\mathrm{i}$$

$$\Rightarrow\left(x^3-3xy^2\right)+\mathrm{i}\left(3x^2y-y^3\right)=27\mathrm{i}$$

Equate real and imaginary parts:

$$\begin{cases}x^3-3xy^2=0\\ 3x^2y-y^3=27\end{cases}\Rightarrow\begin{cases}x\left(x^2-3y^2\right)=0\\ 3x^2y-y^3=27\end{cases}$$

From the first equation, $x=0$, or $x^2=3y^2$

If we substitute these values into the second equation we get:

$$x=0\Rightarrow-y^3=27\Rightarrow y=-3\Rightarrow z=3\mathrm{i}$$

$$x^2=3y^2\Rightarrow3\left(3y^2\right)y-y^3=27\Rightarrow8y^3=27\Rightarrow y^3=\frac{27}{8}\Rightarrow y=\frac{3}{2}$$

Thus, $x^2=3\left(\dfrac{3}{2}\right)^2\Rightarrow x=\pm\dfrac{3}{2}\sqrt{3}$, and therefore, the solutions are:

$$z=\frac{3\sqrt{3}}{2}+\frac{3}{2}\mathrm{i}\text{ or }z=-\frac{3\sqrt{3}}{2}+\frac{3}{2}\mathrm{i}$$

15. Since the polynomial has real coefficients, then $\frac{1}{2}-2\mathrm{i}$ is also a zero.

Thus $\left(x-\left(\frac{1}{2}-2\mathrm{i}\right)\right)$ and $\left(x-\left(\frac{1}{2}+2\mathrm{i}\right)\right)$ are factors of the polynomial, and so is their product $\left(x-\left(\frac{1}{2}-2\mathrm{i}\right)\right)\left(x-\left(\frac{1}{2}+2\mathrm{i}\right)\right)=x^2-x+\frac{17}{4}$. So, the polynomial is of the form

$$f(x)=4\left(x^2-x+\frac{17}{4}\right)(x-c)=\left(4x^2-4x+17\right)(x-c)=4x^3+\cdots-17c$$

By comparing the constant terms, we have $-17c=-51\Rightarrow c=3$

16. Since a polynomial has real coefficients, $3-\mathrm{i}\sqrt{2}$ is also a zero. Hence, using the factor theorem, we have: $a\left(x-\frac{1}{2}\right)(x+1)\left(x-3-\mathrm{i}\sqrt{2}\right)\left(x-3+\mathrm{i}\sqrt{2}\right)=a\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)\left(x^2-6x+11\right)$

Since we need integer coefficients, we let a = 2 (or any multiple of 2).
After multiplication we have:

$$2\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)\left(x^2-6x+11\right)=\left(2x^2+x-1\right)\left(x^2-6x+11\right)$$
$$=2x^4-11x^3+15x^2+17x-11$$

17. Since a polynomial has real coefficients, $1-\mathrm{i}\sqrt{3}$ is also a zero. Hence, using the factor theorem, we have: $a(x+2)^2\left(x-1-\mathrm{i}\sqrt{3}\right)\left(x-1+\mathrm{i}\sqrt{3}\right)=a\left(x^2+4x+4\right)\left(x^2-2x+1+3\right)$

After multiplication we have: $\left(x^2+4x+4\right)\left(x^2-2x+4\right)=x^4+2x^3+8x+16$

So, we let $a=1$ and the polynomial is: $f(x)=x^4+2x^3+8x+16$

18. Since the polynomial has real coefficients, then $5-2\mathrm{i}$ is also a zero. Hence,
$f(x)=(x-(5+2\mathrm{i}))(x-(5-2\mathrm{i}))(x-c)=\left(x^2-10x+25+4\right)(x-c)$

To determine c, we can check for the constant term: $87=29\cdot(-c)\Rightarrow c=-3$

Hence, the other zeros are $5-2\mathrm{i}$ and -3

19. Since the polynomial has real coefficients, then $1+\mathrm{i}\sqrt{3}$ is also a zero. Hence,
$f(x)=3\left(x-\left(1+\mathrm{i}\sqrt{3}\right)\right)\left(x-\left(1-\mathrm{i}\sqrt{3}\right)\right)(x-c)=3\left(x^2-2x+1+3\right)(x-c)$

To determine c, we can check for the constant term: $8=3\cdot 4\cdot(-c)\Rightarrow c=-\frac{2}{3}$

Hence, the other zeros are $1+\mathrm{i}\sqrt{3}$ and $-\frac{2}{3}$

20. Let $z = x + \mathrm{i}y$. Then, $a + b\mathrm{i} = \dfrac{x + y\mathrm{i}}{x - y\mathrm{i}} = \dfrac{x + y\mathrm{i}}{x - y\mathrm{i}} \cdot \dfrac{x + y\mathrm{i}}{x + y\mathrm{i}} = \dfrac{x^2 - y^2}{x^2 + y^2} + \dfrac{2xy}{x^2 + y^2}\mathrm{i}$

$$|a + b\mathrm{i}| = \sqrt{\left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2 + \left(\frac{2xy}{x^2 + y^2}\right)^2} = \sqrt{\frac{x^4 - 2x^2y^2 + y^4 + 4x^2y^2}{\left(x^2 + y^2\right)^2}} = \sqrt{\frac{x^4 + 2x^2y^2 + y^4}{\left(x^2 + y^2\right)^2}} = 1$$

Note: Knowledge acquired in the next section will enable you to easily show this result without resorting to lengthy calculations:

Given that the number and its conjugate have the same modulus, then $|a + b\mathrm{i}| = \dfrac{|z|}{|z^*|} = 1$

21. **(a)** Using the binomial theorem, we have:

$\left(k + \mathrm{i}\right)^4 = k^4 + 4 \cdot k^3\mathrm{i} + 6k^2\left(-1\right) + 4k\left(-\mathrm{i}\right) + 1 = k^4 - 6k^2 + 1 + \mathrm{i}\left(4k^3 - 4k\right)$.

Therefore, the number is real if $4k^3 - 4k = 0 \Rightarrow 4k\left(k^2 - 1\right) = 0 \Rightarrow k = 0$, or $k = \pm 1$

(b) Using the calculation from **(a)**, we have:

$k^4 - 6k^2 + 1 = 0 \Rightarrow k^2 = \dfrac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$

Since both numbers are positive, $k = \pm\sqrt{3 \pm 2\sqrt{2}}$

22. $\begin{cases} \mathrm{i}z_1 + 2z_2 = 3 - \mathrm{i} \\ 2z_1 + \left(2 + \mathrm{i}\right)z_2 = 7 + 2\mathrm{i} \end{cases}$

Multiplying the first equation by -2, and the second by i, and adding the resulting equations:

$$\begin{cases} -\mathrm{i}2z_1 - 4z_2 = -6 + 2\mathrm{i} \\ 2\mathrm{i}z_1 + \left(2\mathrm{i} - 1\right)z_2 = 7\mathrm{i} - 2 \end{cases} \Rightarrow \left(-5 + 2\mathrm{i}\right)z_2 = -8 + 9\mathrm{i} \Rightarrow z_2 = \frac{-8 + 9\mathrm{i}}{-5 + 2\mathrm{i}} = 2 - \mathrm{i}$$

Substituting z_2 in the first equation, we have: $\mathrm{i}z_1 + 2\left(2 - \mathrm{i}\right) = 3 - \mathrm{i} \Rightarrow \mathrm{i}z_1 = -1 + \mathrm{i} \Rightarrow z_1 = 1 + \mathrm{i}$

Hence, the solutions are $z_1 = 1 + \mathrm{i}$ and $z_2 = 2 - \mathrm{i}$

23. $\begin{cases} \mathrm{i}z_1 - \left(1 + \mathrm{i}\right)z_2 = 3 \\ \left(2 + \mathrm{i}\right)z_1 + \mathrm{i}z_2 = 4 \end{cases}$

Multiplying the first equation by i, and the second by $1 + \mathrm{i}$, and adding the resulting equations:

$$\begin{cases} -z_1 - \mathrm{i}\left(1 + \mathrm{i}\right)z_2 = 3\mathrm{i} \\ \left(2 + \mathrm{i}\right)\left(1 + \mathrm{i}\right)z_1 + \mathrm{i}\left(1 + \mathrm{i}\right)z_2 = 4\left(1 + \mathrm{i}\right) \end{cases}$$

$$\Rightarrow -z_1 + \left(1 + 3\mathrm{i}\right)z_1 = 3\mathrm{i} + 4 + 4\mathrm{i} \Rightarrow 3\mathrm{i}z_1 = 4 + 7\mathrm{i} \Rightarrow z_1 = \frac{7 - 4\mathrm{i}}{3}$$

Substituting z_1 in the second equation, we have:

$(2+\mathrm{i})\frac{7-4\mathrm{i}}{3}+\mathrm{i}z_2=4\Rightarrow \mathrm{i}z_2=4-6+\frac{\mathrm{i}}{3}\Rightarrow z_2=\frac{1}{3}+2\mathrm{i}$

Hence, the solutions are $z_1=\frac{7-4\mathrm{i}}{3}$ and $z_2=\frac{1}{3}+2\mathrm{i}$

Exercise 8.2

1. **(a)** $r=\sqrt{2^2+2^2}=2\sqrt{2}$, $\tan\theta=\frac{2}{2}=1$, and, since the number is in the first quadrant, $\theta=\frac{\pi}{4}$. Hence, $z=2\sqrt{2}\text{cis}\frac{\pi}{4}$

(b) $r=\sqrt{\sqrt{3}^2+1^2}=2$, $\tan\theta=\frac{1}{\sqrt{3}}$, and, since the number is in the first quadrant, $\theta=\frac{\pi}{6}$

Hence, $z=2\,\text{cis}\frac{\pi}{6}$

(c) $r=\sqrt{2^2+(-2)^2}=2\sqrt{2}$, $\tan\theta=\frac{-2}{2}=-1$, and, since the number is in the fourth quadrant, $\theta=\frac{7\pi}{4}$. Hence, $z=2\sqrt{2}\,\text{cis}\frac{7\pi}{4}$

(d) $r=\sqrt{\sqrt{6}^2+\left(-\sqrt{2}\right)^2}=2\sqrt{2}$, $\tan\theta=\frac{-\sqrt{2}}{\sqrt{6}}=-\frac{1}{\sqrt{3}}$, and, since the number is in the fourth quadrant, $\theta=\frac{11\pi}{6}$. Hence, $z=2\sqrt{2}\,\text{cis}\frac{11\pi}{6}$

(e) $r=\sqrt{2^2+\left(-2\sqrt{3}\right)^2}=4$, $\tan\theta=\frac{-2\sqrt{3}}{2}=-\sqrt{3}$, and, since the number is in the fourth quadrant, $\theta=\frac{5\pi}{3}$. Hence, $z=4\,\text{cis}\frac{5\pi}{3}$

(f) $r=\sqrt{(-3)^2+3^2}=3\sqrt{2}$, $\tan\theta=\frac{3}{-3}=-1$, and, since the number is in the second quadrant, $\theta=\frac{3\pi}{4}$. Hence, $z=3\sqrt{2}\,\text{cis}\frac{3\pi}{4}$

(g) $r=\sqrt{4^2+0^2}=4$, $\tan\theta=\frac{4}{0}$ is not defined, and, since the number is on the positive y-axis, $\theta=\frac{\pi}{2}$. Hence, $z=4\,\text{cis}\frac{\pi}{2}$

Note: From the geometric interpretation, we can see that the distance from the origin is 4 and the angle is $\frac{\pi}{2}$

(h) $r=\sqrt{\left(-3\sqrt{3}\right)^2+\left(-3\right)^2}=6$, $\tan\theta=\frac{-3}{-3\sqrt{3}}=\frac{1}{\sqrt{3}}$, and, since the number is in the third quadrant, $\theta=\frac{7\pi}{6}$. Hence, $z=6\operatorname{cis}\frac{7\pi}{6}$

(i) $r=\sqrt{1^2+1^2}=\sqrt{2}$, $\tan\theta=\frac{1}{1}=1$, and, since the number is in the first quadrant, $\theta=\frac{\pi}{4}$

Hence, $z=\sqrt{2}\operatorname{cis}\frac{\pi}{4}$

(j) $r=\sqrt{\left(-15\right)^2+0^2}=15$, $\tan\theta=\frac{0}{15}=0$, and, since the number is on the negative x-axis, $\theta=\pi$. Hence, $z=15\operatorname{cis}\pi$

Note: From the geometric interpretation, we can see that the distance from the origin is 15 and the angle is π.

(k) $\left(4+3\text{i}\right)^{-1}=\frac{1}{4+3\text{i}}=\frac{1}{5\operatorname{cis}\left(\arctan\left(\frac{3}{4}\right)\right)}=\frac{1}{5}\operatorname{cis}\left(2\pi-\arctan\left(\frac{3}{4}\right)\right)=\frac{1}{5}\operatorname{cis}\left(5.64\right)$

Alternatively, $\left(4+3\text{i}\right)^{-1}=\frac{1}{4+3\text{i}}=\frac{4-3\text{i}}{25}$

$r=\sqrt{\left(\frac{4}{25}\right)^2+\left(-\frac{3}{25}\right)^2}=\frac{1}{5}$, $\tan\theta=\frac{-\frac{3}{25}}{\frac{4}{25}}=-\frac{3}{4}$, and, since the number is in the fourth quadrant, $\theta=\tan^{-1}\left(-\frac{3}{4}\right)+2\pi\approx 5.64$. Hence, $z=\frac{1}{5}\operatorname{cis}5.64$

(l) $\text{i}\left(3+3\text{i}\right)=-3+3\text{i}$

$r=\sqrt{\left(-3\right)^2+3^2}=3\sqrt{2}$, $\tan\theta=\frac{3}{-3}=-1$, and, since the number is in the second quadrant, $\theta=\frac{3\pi}{4}$. Hence, $z=3\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$.

(m) $r=\sqrt{\pi^2+0^2}=\pi$, $\tan\theta=\frac{0}{\pi}=0$, and, since the number is on the positive x-axis, $\theta=0$. Hence, $z=\pi\operatorname{cis}0$.

Note: From the geometric interpretation, we can see that the distance from the origin is π and the angle is 0.

(n) $r=\sqrt{0^2+\text{e}^2}=\text{e}$, $\tan\theta$ not defined, and, since the number is on the positive y-axis, $\theta=\frac{\pi}{2}$. Hence, $z=\text{e}\operatorname{cis}\frac{\pi}{2}$.

Note: From the geometric interpretation, we can see that the distance from the origin is e and the angle is $\frac{\pi}{2}$.

2. **(a)** $z_1z_2 = \operatorname{cis}\left(\frac{\pi}{2}+\frac{\pi}{3}\right) = \operatorname{cis}\frac{5\pi}{6} = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$\frac{z_1}{z_2} = \operatorname{cis}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \operatorname{cis}\frac{\pi}{6} = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

(b) $z_1z_2 = \operatorname{cis}\left(\frac{5\pi}{6}+\frac{7\pi}{6}\right) = \operatorname{cis}(2\pi) = \cos 2\pi + \mathrm{i}\sin 2\pi = 1 + 0\mathrm{i} = 1$

$\frac{z_1}{z_2} = \operatorname{cis}\left(\frac{5\pi}{6}-\frac{7\pi}{6}\right) = \operatorname{cis}\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

(c) $z_1z_2 = \operatorname{cis}\left(\frac{\pi}{6}+\frac{2\pi}{3}\right) = \operatorname{cis}\left(\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$\frac{z_1}{z_2} = \operatorname{cis}\left(\frac{\pi}{6}-\frac{2\pi}{3}\right) = \operatorname{cis}\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \mathrm{i}\sin\frac{\pi}{2} = 0 - 1\mathrm{i} = -\mathrm{i}$

(d) $z_1z_2 = \operatorname{cis}\left(\frac{13\pi}{12}+\frac{5\pi}{12}\right) = \operatorname{cis}\left(\frac{3\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{2}\right) = 0 - 1\cdot\mathrm{i} = -\mathrm{i}$

$\frac{z_1}{z_2} = \operatorname{cis}\left(\frac{13\pi}{12}-\frac{5\pi}{12}\right) = \operatorname{cis}\left(\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

(e) $z_1z_2 = 3\cdot\frac{2}{3}\operatorname{cis}\left(\frac{3\pi}{4}+\frac{4\pi}{3}\right) = 2\operatorname{cis}\left(\frac{25\pi}{12}\right) = 2\operatorname{cis}\left(\frac{\pi}{12}\right) = 2\cos\left(\frac{\pi}{12}\right) + 2\mathrm{i}\sin\left(\frac{\pi}{12}\right)$

By considering that $\frac{\pi}{12} = \frac{4\pi-3\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ and using trigonometric identities, this can be reduced to $z_1z_2 = \frac{\sqrt{6}+\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{2}}{2}\mathrm{i}$

$\frac{z_1}{z_2} = \frac{3}{\frac{2}{3}}\operatorname{cis}\left(\frac{3\pi}{4}-\frac{4\pi}{3}\right) = \frac{9}{2}\operatorname{cis}\left(-\frac{7\pi}{12}\right) = \frac{9}{2}\operatorname{cis}\left(\frac{17\pi}{12}\right)$

Using trigonometric identities for $\cos\left(\frac{3\pi}{4}-\frac{4\pi}{3}\right)$ and $\sin\left(\frac{3\pi}{4}-\frac{4\pi}{3}\right)$ this can be reduced to $\frac{z_1}{z_2} = \frac{9}{8}\left(-\sqrt{6}+\sqrt{2}\right) - \frac{9}{8}\left(\sqrt{6}+\sqrt{2}\right)\mathrm{i}$

(f) $z_1z_2 = 3\sqrt{2}\cdot 2\operatorname{cis}\left(\frac{5\pi}{4}+\frac{5\pi}{3}\right) = 6\sqrt{2}\operatorname{cis}\left(\frac{35\pi}{12}\right) = 6\sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right)$

This can be reduced as in previous questions to:

$z_1z_2 = -\frac{6\sqrt{3}+6}{2} + \frac{6\sqrt{3}-6}{2}\mathrm{i} = -3\sqrt{3} - 3 + \left(3\sqrt{3}-3\right)\mathrm{i}$

$\frac{z_1}{z_2} = \frac{3\sqrt{2}}{2}\operatorname{cis}\left(\frac{5\pi}{4}-\frac{5\pi}{3}\right) = \frac{3\sqrt{2}}{2}\operatorname{cis}\left(-\frac{5\pi}{12}\right) = \frac{3\sqrt{2}}{2}\operatorname{cis}\left(\frac{19\pi}{12}\right)$

This can be reduced as in previous questions to:

$$\frac{z_1}{z_2}=\frac{3\sqrt{3}-3}{4}-\frac{3\sqrt{3}+3}{4}\mathrm{i}$$

(g) $z_1z_2=\mathrm{cis}\left(135°+90°\right)=\mathrm{cis}\left(225°\right)=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}=-\frac{\sqrt{2}}{2}\left(1+\mathrm{i}\right)$

$$\frac{z_1}{z_2}=\mathrm{cis}\left(135°-90°\right)=\mathrm{cis}\left(45°\right)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}=\frac{\sqrt{2}}{2}\left(1+\mathrm{i}\right)$$

(h) $z_1z_2=3\cdot 2\,\mathrm{cis}\left(120°+240°\right)=6\,\mathrm{cis}\left(360°\right)=6-0\cdot\mathrm{i}=6$

$$\frac{z_1}{z_2}=\frac{3}{2}\mathrm{cis}\left(120°-240°\right)=\frac{3}{2}\mathrm{cis}\left(-120°\right)=-\frac{3}{4}-\frac{3\sqrt{3}}{4}\mathrm{i}$$

(i) $z_1z_2=\frac{5}{8}\cdot\frac{\sqrt{3}}{2}\mathrm{cis}\left(225°-30°\right)=\frac{5\sqrt{3}}{16}\mathrm{cis}\left(195°\right)$

Also, 195 = 240 – 45, so this can be reduced to $z_1z_2=\frac{-5\sqrt{6}-15\sqrt{2}}{64}+\mathrm{i}\frac{5\sqrt{6}-15\sqrt{2}}{64}$

$$\frac{z_1}{z_2}=\frac{\frac{5}{8}}{\frac{\sqrt{3}}{2}}\mathrm{cis}\left(225°+30°\right)=\frac{5\sqrt{3}}{12}\mathrm{cis}\left(-105°\right)=\frac{5\sqrt{3}}{12}\mathrm{cis}\left(-60°-45°\right)$$

and this can be simplified to $\frac{z_1}{z_2}=\frac{5\sqrt{6}-15\sqrt{2}}{48}-\mathrm{i}\frac{5\sqrt{6}+15\sqrt{2}}{48}$

(j) $z_1=3\sqrt{2}\,\mathrm{cis}\,315°$, $z_2=2\,\mathrm{cis}\,300°$, and hence:

$$z_1z_2=3\sqrt{2}\cdot 2\,\mathrm{cis}\left(315°+300°\right)=6\sqrt{2}\,\mathrm{cis}\left(615°\right)=6\sqrt{2}\,\mathrm{cis}\left(-105°\right)$$

which can be simplified to $z_1z_2=-3\sqrt{3}+3+\mathrm{i}\left(3\sqrt{3}+3\right)$

$$\frac{z_1}{z_2}=\frac{3\sqrt{2}}{2}\mathrm{cis}(315°-300°)=\frac{3\sqrt{2}}{2}\mathrm{cis}(15°)$$

and finally, $\frac{z_1}{z_2}=\frac{3\sqrt{2}}{2}\mathrm{cis}(45°-30°)=\frac{3\sqrt{3}+3}{4}+\frac{\mathrm{i}\left(3\sqrt{3}-3\right)}{4}$

3. **(a)**. For z_1: $r=\sqrt{\sqrt{3}^2+1^2}=2$, $\tan\theta=\frac{1}{\sqrt{3}}$, and, since the number is in the first quadrant, $\theta=\frac{\pi}{6}$. Hence, $z_1=2\mathrm{cis}\frac{\pi}{6}$.

For z_2: $r=\sqrt{2^2+\left(2\sqrt{3}\right)^2}=4$, $\tan\theta=\frac{-2\sqrt{3}}{2}=-\sqrt{3}$, and, since the number is in the fourth quadrant, $\theta=\frac{5\pi}{3}$. Hence, $z_2=4\mathrm{cis}\frac{5\pi}{3}\left(\text{or } 4\mathrm{cis}\frac{-\pi}{3}\right)$.

$$\frac{1}{z_1}=\frac{1}{2}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{z_2}=\frac{1}{4}\text{cis}\left(-\frac{5\pi}{3}\right)=\frac{1}{4}\text{cis}\left(\frac{\pi}{3}\right)$$

$$z_1z_2=2\cdot 4\,\text{cis}\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)=8\,\text{cis}\left(\frac{11\pi}{6}\right)\left(\text{or } 8\text{cis}\frac{-\pi}{6}\right)$$

$$\frac{z_1}{z_2}=\frac{2}{4}\text{cis}\left(\frac{\pi}{6}-\frac{5\pi}{3}\right)=\frac{1}{2}\text{cis}\left(-\frac{3\pi}{2}\right)=\frac{1}{2}\text{cis}\frac{\pi}{2}$$

(b) For z_1: $r=\sqrt{8}=2\sqrt{2}$, $\tan\theta=\frac{1}{\sqrt{3}}$, $\theta=\frac{\pi}{6}$; hence, $z_1=2\sqrt{2}\text{cis}\frac{\pi}{6}$.

For z_2: $r=\sqrt{48}=4\sqrt{3}$, $\tan\theta=-\sqrt{3}$, $\theta=\frac{5\pi}{3}$; hence,

$$z_2=4\sqrt{3}\text{cis}\frac{5\pi}{3}\left(\text{or } 4\sqrt{3}\text{cis}\frac{-\pi}{3}\right)$$

$$\frac{1}{z_1}=\frac{1}{2\sqrt{2}}\text{cis}\left(-\frac{\pi}{6}\right)=\frac{\sqrt{2}}{4}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{z_2}=\frac{1}{4\sqrt{3}}\text{cis}\left(-\frac{5\pi}{3}\right)=\frac{\sqrt{3}}{12}\text{cis}\left(\frac{\pi}{3}\right)$$

$$z_1z_2=8\sqrt{6}\,\text{cis}\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)=8\sqrt{6}\,\text{cis}\left(\frac{11\pi}{6}\right)\left(\text{or } 8\sqrt{6}\text{cis}\frac{-\pi}{6}\right)$$

$$\frac{z_1}{z_2}=\frac{\sqrt{2}}{2\sqrt{3}}\text{cis}\left(\frac{\pi}{6}-\frac{5\pi}{3}\right)=\frac{\sqrt{6}}{6}\text{cis}\left(-\frac{3\pi}{2}\right)=\frac{\sqrt{6}}{6}\text{cis}\frac{\pi}{2}$$

(c) For z_1: $r=8$, $\tan\theta=\frac{1}{\sqrt{3}}$, $\theta=\frac{\pi}{6}$; hence, $z_1=8\text{cis}\frac{\pi}{6}$

For z_2: $r=3\sqrt{2}$, $\tan\theta=1$, $\theta=\frac{5\pi}{4}$; hence, $z_2=3\sqrt{2}\text{cis}\frac{5\pi}{4}\left(\text{or } 3\sqrt{2}\text{cis}\frac{-3\pi}{4}\right)$

$$\frac{1}{z_1}=\frac{1}{8}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{z_2}=\frac{1}{3\sqrt{2}}\text{cis}\left(-\frac{5\pi}{4}\right)=\frac{\sqrt{2}}{6}\text{cis}\left(\frac{3\pi}{4}\right)$$

$$z_1z_2=24\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}+\frac{5\pi}{4}\right)=24\sqrt{2}\,\text{cis}\left(\frac{17\pi}{12}\right)\left(\text{or } 24\sqrt{2}\text{cis}\frac{-7\pi}{12}\right)$$

$$\frac{z_1}{z_2}=\frac{8}{3\sqrt{2}}\text{cis}\left(\frac{\pi}{6}-\frac{5\pi}{4}\right)=\frac{4\sqrt{2}}{3}\text{cis}\left(-\frac{13\pi}{12}\right)=\frac{4\sqrt{2}}{3}\text{cis}\left(\frac{11\pi}{12}\right)$$

(d) For z_1: $r=\sqrt{3}$, $\tan\theta$ is not defined, $\theta=\frac{\pi}{2}$; hence, $z_1=\sqrt{3}\text{cis}\frac{\pi}{2}$.

For z_2: $r=2\sqrt{2}$, $\tan\theta=\sqrt{3}$, $\theta=\frac{4\pi}{3}$; hence, $z_2=2\sqrt{2}\text{cis}\frac{4\pi}{3}\left(\text{or } 2\sqrt{2}\text{cis}\frac{-2\pi}{3}\right)$.

$$\frac{1}{z_1}=\frac{1}{\sqrt{3}}\text{cis}\left(-\frac{\pi}{2}\right)=\frac{\sqrt{3}}{3}\text{cis}\left(-\frac{\pi}{2}\right)$$

$$\frac{1}{z_2}=\frac{1}{2\sqrt{2}}\text{cis}\left(-\frac{4\pi}{3}\right)=\frac{\sqrt{2}}{4}\text{cis}\left(\frac{2\pi}{3}\right)$$

$$z_1z_2=2\sqrt{6}\,\text{cis}\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)=2\sqrt{6}\,\text{cis}\left(\frac{11\pi}{6}\right)\left(\text{or } 2\sqrt{6}\text{cis}\frac{-\pi}{6}\right)$$

$$\frac{z_1}{z_2}=\frac{\sqrt{3}}{2\sqrt{2}}\text{cis}\left(\frac{\pi}{2}-\frac{4\pi}{3}\right)=\frac{\sqrt{6}}{4}\text{cis}\left(-\frac{5\pi}{6}\right)=\frac{\sqrt{6}}{4}\text{cis}\left(\frac{7\pi}{6}\right)$$

(e) For z_1: $r=\sqrt{10}$, $\tan\theta=1$, $\theta=\frac{\pi}{4}$; hence, $z_1=\sqrt{10}\text{cis}\frac{\pi}{4}$

For z_2: $r=2\sqrt{2}$, $\tan\theta$ is not defined, $\theta=\frac{\pi}{2}$; hence, $z_2=2\sqrt{2}\text{cis}\frac{\pi}{2}$

$$\frac{1}{z_1}=\frac{1}{\sqrt{10}}\text{cis}\left(-\frac{\pi}{4}\right)=\frac{\sqrt{10}}{10}\text{cis}\left(-\frac{\pi}{4}\right)$$

$$\frac{1}{z_2}=\frac{1}{2\sqrt{2}}\text{cis}\left(-\frac{\pi}{2}\right)=\frac{\sqrt{2}}{4}\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z_1z_2=4\sqrt{5}\,\text{cis}\left(\frac{\pi}{4}+\frac{\pi}{2}\right)=4\sqrt{5}\,\text{cis}\left(\frac{3\pi}{4}\right)$$

$$\frac{z_1}{z_2}=\frac{\sqrt{10}}{2\sqrt{2}}\text{cis}\left(\frac{\pi}{4}-\frac{\pi}{2}\right)=\frac{\sqrt{5}}{2}\text{cis}\left(-\frac{\pi}{4}\right)$$

(f) For z_1: $r=2$, $\tan\theta=\sqrt{3}$, $\theta=\frac{\pi}{3}$; hence, $z_1=2\text{cis}\frac{\pi}{3}$

For z_2: $r=2\sqrt{3}$, $\tan\theta=0$, $\theta=0$; hence, $z_2=2\sqrt{3}\text{cis}0$

$$\frac{1}{z_1}=\frac{1}{2}\text{cis}\left(-\frac{\pi}{3}\right)$$

$$\frac{1}{z_2}=\frac{1}{2\sqrt{3}}\text{cis}(-0)=\frac{\sqrt{3}}{6}\text{cis}(0)$$

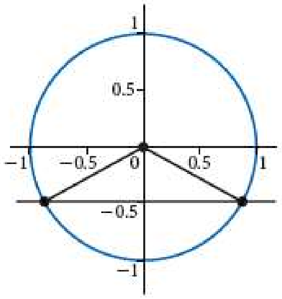
$$z_1z_2=4\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}+0\right)=4\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right)$$

$$\frac{z_1}{z_2}=\frac{2}{2\sqrt{3}}\text{cis}\left(\frac{\pi}{3}-0\right)=\frac{\sqrt{3}}{3}\text{cis}\left(\frac{\pi}{3}\right)$$

4. **(a)** Letting $z = x + y\text{i}$, we have:

$$\sqrt{x^2 + (y-1)^2} = \sqrt{x^2 + (y+2)^2} \Rightarrow y^2 - 2y + 1 = y^2 + 4y + 4 \Rightarrow y = -\frac{1}{2}$$

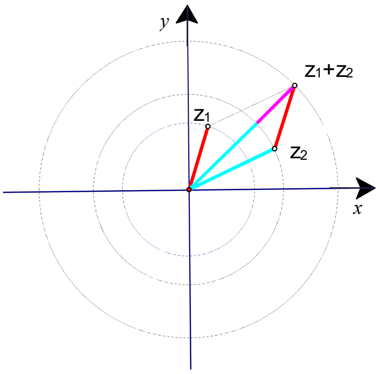
(b) **(i)** The points are on the unit circle and their *y*-coordinates are both $-\frac{1}{2}$

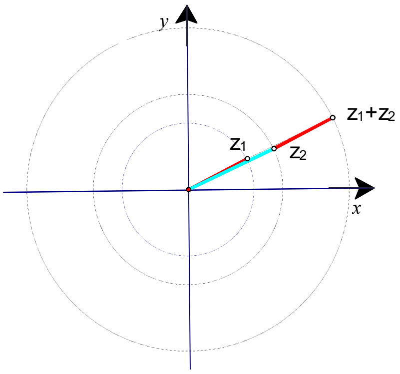


(ii) For z_1: $\sin\theta = -\frac{1}{2}$, and, since the number is in the fourth quadrant, $\theta = \frac{11\pi}{6}$ (or $-\frac{\pi}{6}$). Hence, $\arg(z_1) = \frac{11\pi}{6}$.

For z_2: $\sin\theta = -\frac{1}{2}$, and, since the number is in the third quadrant, $\theta = \frac{7\pi}{6}$ (or $-\frac{5\pi}{6}$). Hence, $\arg(z_2) = \frac{7\pi}{6}$.

5. The diagram below shows the inequality. The book answers have another version of the drawing.





In the above diagrams, $|z_1|$ is represented by the red line segment(s), and $|z_2|$ by the blue line segment(s). It is obvious that the line segment which represents $|z_1 + z_2|$ (blue + pink) is shorter than the 'blue + red' segment. They are the same if z_1 and z_2 are on the same line (on the same side of the origin), as shown on the second diagram.

6. $$z=\sqrt{3}\left(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)=\frac{-\sqrt{3}+3\mathrm{i}}{2} \text{ and } z^2=3\left(\cos\frac{4\pi}{3}+\mathrm{i}\sin\frac{4\pi}{3}\right)=\frac{-3-3\sqrt{3}\mathrm{i}}{2}$$

(a) $$\frac{3}{\sqrt{3}+z}=\frac{3}{\sqrt{3}+\frac{-\sqrt{3}+3\mathrm{i}}{2}}=\frac{3}{\frac{\sqrt{3}+3\mathrm{i}}{2}}=\frac{6\left(\sqrt{3}-3\mathrm{i}\right)}{3+9}=\frac{\sqrt{3}}{2}-\frac{3\mathrm{i}}{2}$$

(b) $$\frac{2z}{3+z^2}=\frac{2\left(\frac{-\sqrt{3}+3\mathrm{i}}{2}\right)}{3+\frac{-3-3\sqrt{3}\mathrm{i}}{2}}=\frac{-\sqrt{3}+3\mathrm{i}}{\frac{3-3\sqrt{3}\mathrm{i}}{2}}=\frac{-\sqrt{3}\cancel{\left(1-\sqrt{3}\mathrm{i}\right)}}{\frac{3\cancel{\left(1-\sqrt{3}\mathrm{i}\right)}}{2}}=\frac{-2\sqrt{3}}{3}$$

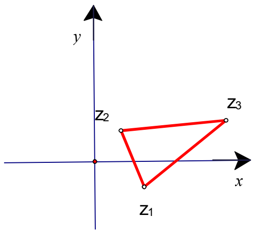
(c) $$\frac{3-z^2}{3+z^2}=\frac{3-\frac{-3-3\sqrt{3}\mathrm{i}}{2}}{3+\frac{-3-3\sqrt{3}\mathrm{i}}{2}}=\frac{\frac{9+3\sqrt{3}\mathrm{i}}{2}}{\frac{3-3\sqrt{3}\mathrm{i}}{2}}=\frac{3\sqrt{3}\left(\sqrt{3}+\mathrm{i}\right)}{3\left(1-\sqrt{3}\mathrm{i}\right)}=\frac{\sqrt{3}\cdot 4\mathrm{i}}{1+3}=\sqrt{3}\mathrm{i}$$

7. For z_1: $|z_1|=\sqrt{12+4}=4$, $\tan\theta=-\frac{1}{\sqrt{3}}$; hence, $\arg z_1=-\frac{\pi}{6}$

For z_2: $|z_2|=\sqrt{4+4}=2\sqrt{2}$, $\tan\theta=1$; hence, $\arg z_2=\frac{\pi}{4}$

Since $z_3=z_1z_2$, $|z_3|=8\sqrt{2}$, and $\arg z_3=-\frac{\pi}{6}+\frac{\pi}{4}=\frac{\pi}{12}$

8. There are several methods available. Some will give you numerical approximate answers using a GDC and others may require knowledge from Chapter 9, or reference to the matrix chapter online.



Below are two such methods.

Method I

We can find the side lengths and the angles of the triangle:

$$|z_1z_2|=\sqrt{\left(2-2\sqrt{3}\right)^2+4^2}=\sqrt{32-8\sqrt{3}}$$

$$|z_1z_3|=\sqrt{\left(2\sqrt{3}+4\right)^2+\left(4\sqrt{3}-2\right)^2}=\sqrt{80}$$

$$|z_2z_3| = \sqrt{\left(4\sqrt{3}+2\right)^2 + \left(4\sqrt{3}-6\right)^2} = \sqrt{136-32\sqrt{3}}$$

Angle θ between sides $|z_1z_2|$ and $|z_1z_3|$ is:

$$\cos\theta = \frac{|z_1z_2|^2 + |z_1z_3|^2 - |z_2z_3|^2}{2|z_1z_2||z_1z_3|} \Rightarrow \theta = 76.6689...°$$

Hence, the area is: $A = \frac{1}{2}|z_1z_2||z_1z_3|\sin\theta \approx 18.5$

Method II

There is a formula for the area of a triangle using the coordinates of the vertices. If the vertices are: $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$, then the area is:

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ 2\sqrt{3} & -2 & 1 \\ 4\left(1+\sqrt{3}\right) & 4\left(\sqrt{3}-1\right) & 1 \end{vmatrix} = 22 - 2\sqrt{3} \approx 18.5$$

9. Let $z = x + \mathrm{i}y$

(a) $|z| = 3 \Rightarrow \sqrt{x^2+y^2} = 3 \Rightarrow x^2 + y^2 = 9$

The set of points is the circle with centre (0, 0) and radius 3.

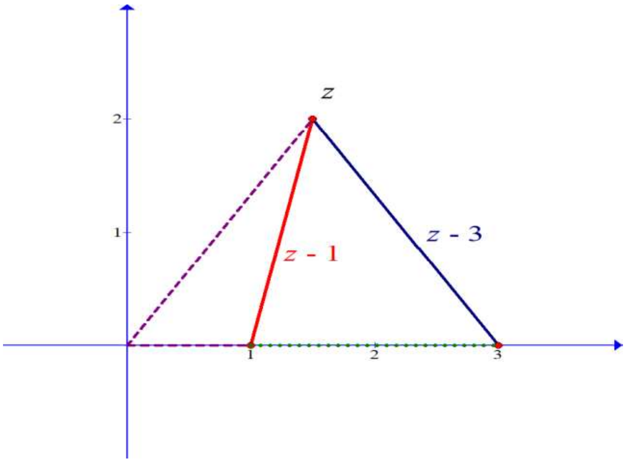
(b) $z^* = x - y\mathrm{i}$; then $x - y\mathrm{i} = -x - y\mathrm{i} \Rightarrow x = 0$. The set of points is the y-axis.

(c) $x + y\mathrm{i} + x - y\mathrm{i} = 8 \Rightarrow 2x = 8 \Rightarrow x = 4$. The set of points is the line $x = 4$.

(d) $\sqrt{(x-3)^2 + y^2} = 2 \Rightarrow \left(x-3\right)^2 + y^2 = 4$.

The set of points is the circle with centre (3, 0) and radius 2.

(e) We can look at this from a geometric perspective. In the diagram below, $z - 1$ is shown in red, and $z - 3$ in blue. The points representing 1, 3, and z, form a triangle. $|z-1| + |z-3|$ represent the sum of two sides of a triangle. Hence in all cases except one, $|z-1| + |z-3| > 2$. The only exception is when z lies between 1 and 3, i.e., there is no triangle. In that case $|z-1| + |z-3| = 2$



It is much more involved to solve this algebraically:

$$\sqrt{(x-1)^2+y^2}+\sqrt{(x-3)^2+y^2}=2 \Rightarrow \sqrt{(x-1)^2+y^2}=2-\sqrt{(x-3)^2+y^2}$$

Square both sides and simplify:

$$4x-12=-4\sqrt{(x-3)^2+y^2}$$

$$x-3=-\sqrt{(x-3)^2+y^2}$$

Since the right side of the equation is negative, then $x-3<0 \Rightarrow x<3$

Square both sides again: $(x-3)^2=(x-3)^2+y^2 \Rightarrow y=0$

Substitute back $y = 0$ into the original equation:

$$\sqrt{(x-1)^2}+\sqrt{(x-3)^2}=2 \Rightarrow |x-1|+|x-3|=2 \Rightarrow 1\le x\le 3$$

The set of points is $\{(x,y), 1\le x\le 3, y=0\}$; hence, the line segment between (1, 0) and (3, 0).

10. Let $z=x+y\mathrm{i}$

(a) $\sqrt{x^2+y^2}\le 3$

The set of points is the disk with centre (0, 0) and radius 3.

(b) $\sqrt{x^2+(y-3)^2}\ge 2$

The solution is the set of points outside the disk with centre (0, 3) and radius 2.

Exercise 8.3

1. **(a)** $z=4e^{-i\frac{2\pi}{3}}=4\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)=4\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-2-i2\sqrt{3}$

(b) $z=3e^{2\pi i}=3\left(\cos(2\pi)+i\sin(2\pi)\right)=3(1+i\cdot 0)=3$

(c) $z=3e^{0.5\pi i}=3\left(\cos(0.5\pi)+i\sin(0.5\pi)\right)=3(0+i\cdot 1)=3i$

(d) $z=4\operatorname{cis}\left(\frac{7\pi}{12}\right)=4\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)$

$=4\left(\frac{\sqrt{2}-\sqrt{6}}{4}+i\frac{\sqrt{2}+\sqrt{6}}{4}\right)=\sqrt{2}-\sqrt{6}+i\left(\sqrt{2}+\sqrt{6}\right)$

(e) $z=13e^{\frac{\pi i}{3}}=13\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)=13\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{13}{2}+\frac{13\sqrt{3}}{2}i$

(f) $z=3e^{1+\frac{\pi}{3}i}=3e\times e^{\frac{\pi}{3}i}=3e\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)=3e\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{3e}{2}+\frac{3e\sqrt{3}}{2}i$

2. **(a)** $r=\sqrt{8}=2\sqrt{2}$, $\tan\theta=1$, the first quadrant, $\theta=\frac{\pi}{4}$; hence, $z=2\sqrt{2}e^{\frac{\pi}{4}i}$

(b) $r=2$, $\tan\theta=\frac{1}{\sqrt{3}}$, the first quadrant, $\theta=\frac{\pi}{6}$; hence, $z=2e^{\frac{\pi}{6}i}$

(c) $r=\sqrt{8}=2\sqrt{2}$, $\tan\theta=-\frac{1}{\sqrt{3}}$, the fourth quadrant, $\theta=-\frac{\pi}{6}$; hence, $z=2\sqrt{2}e^{-\frac{\pi}{6}i}$

(d) $r=4$, $\tan\theta=-\sqrt{3}$, the fourth quadrant, $\theta=-\frac{\pi}{3}$; hence, $z=4e^{-\frac{\pi}{3}i}$

(e) $r=\sqrt{18}=3\sqrt{2}$, $\tan\theta=-1$, the second quadrant, $\theta=\frac{3\pi}{4}$; hence, $z=3\sqrt{2}e^{\frac{3\pi}{4}i}$

(f) $r=4$, $\tan\theta$ is not defined, positive y-axis, $\theta=\frac{\pi}{2}$; hence, $z=4e^{\frac{\pi}{2}i}$

(g) $r=6$, $\tan\theta=\frac{1}{\sqrt{3}}$, the third quadrant, $\theta=\frac{7\pi}{6}$; hence, $z=6e^{\frac{7\pi}{6}i}$

(h) $z=-3+3i$, $r=\sqrt{18}=3\sqrt{2}$, $\tan\theta=-1$, the second quadrant, $\theta=\frac{3\pi}{4}$; hence,

$z=3\sqrt{2}e^{\frac{3\pi}{4}i}$

(i) $r = \pi$, $\tan\theta = 0$, positive x-axis, $\theta = 0$, hence, $z = \pi e^{0\cdot i}$ $(= \pi e^{2\pi i})$

(j) $r = e$, $\tan\theta$ is not defined, positive y-axis, $\theta = \dfrac{\pi}{2}$; hence, $z = e \cdot e^{\frac{\pi}{2}i} = e^{1+\frac{\pi}{2}i}$

3. **(a)** $r = \sqrt{2}$, $\tan\theta = 1$, the first quadrant, $\theta = \dfrac{\pi}{4}$; hence, $1 + i = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)$

$$(1+i)^{10} = \left(\sqrt{2}\right)^{10} \text{cis}\left(10 \cdot \frac{\pi}{4}\right) = 2^5 \text{cis}\left(\frac{\pi}{2}\right) = 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 32i$$

(b) $r = 2$, $\tan\theta = -\dfrac{1}{\sqrt{3}}$, the fourth quadrant, $\theta = -\dfrac{\pi}{6}$; hence, $\sqrt{3} - i = 2\,\text{cis}\left(-\dfrac{\pi}{6}\right)$

$$\left(\sqrt{3} - i\right)^6 = (2)^6 \text{cis}\left(6 \cdot \frac{-\pi}{6}\right) = 64\,\text{cis}(-\pi) = 64\left(\cos(-\pi) + i\sin(-\pi)\right) = -64$$

(c) $r = 6$, $\tan\theta = \sqrt{3}$, the first quadrant, $\theta = \dfrac{\pi}{3}$; hence, $3 + 3i\sqrt{3} = 6\,\text{cis}\left(\dfrac{\pi}{3}\right)$

$$\left(3 + 3i\sqrt{3}\right)^9 = (6)^9 \text{cis}\left(9 \cdot \frac{\pi}{3}\right) = 10\,077\,696\,\text{cis}(3\pi)$$
$$= 10\,077\,696\left(\cos(\pi) + i\sin(\pi)\right) = -10\,077\,696$$

(d) $r = 2\sqrt{2}$, $\tan\theta = -1$, the fourth quadrant, $\theta = -\dfrac{\pi}{4}$; hence, $2 - 2i = 2\sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$

$$(2-2i)^{12} = \left(2\sqrt{2}\right)^{12} \text{cis}\left(12 \cdot \frac{-\pi}{4}\right) = 262144\,\text{cis}(-3\pi)$$
$$= 262144\left(\cos(\pi) + i\sin(\pi)\right) = -262144$$

(e) $r = \sqrt{6}$, $\tan\theta = -1$, the fourth quadrant, $\theta = -\dfrac{\pi}{4}$; hence, $\sqrt{3} - i\sqrt{3} = \sqrt{6}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$

$$\left(\sqrt{3} - i\sqrt{3}\right)^8 = \sqrt{6}^8 \text{cis}\left(8 \cdot \frac{-\pi}{4}\right) = 1296\,\text{cis}(-2\pi) = 1296\left(\cos(0) + i\sin(0)\right) = 1296$$

(f) $r = \sqrt{18} = 3\sqrt{2}$, $\tan\theta = -1$, the second quadrant, $\theta = \dfrac{3\pi}{4}$; hence, $z = 3\sqrt{2}\,\text{cis}\left(\dfrac{3\pi}{4}\right)$

$$(-3+3i)^7 = \left(3\sqrt{2}\right)^7 \text{cis}\left(7 \cdot \frac{3\pi}{4}\right) = 17\,496\sqrt{2}\,\text{cis}\left(\frac{5\pi}{4}\right)$$
$$= 17\,496\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$$
$$= 17\,496\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 17496(-1-i)$$

(g) $r=\sqrt{6}$, $\tan\theta=-1$, the fourth quadrant, $\theta=-\frac{\pi}{4}$; hence, $\sqrt{3}-\mathrm{i}\sqrt{3}=\sqrt{6}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$

$$\left(\sqrt{3}-\mathrm{i}\sqrt{3}\right)^{-8}=\sqrt{6}^{-8}\,\mathrm{cis}\left(-8\cdot\frac{-\pi}{4}\right)=\frac{1}{1296}\mathrm{cis}(2\pi)$$
$$=\frac{1}{1296}\left(\cos(0)+\mathrm{i}\sin(0)\right)=\frac{1}{1296}$$

(h) $r=6$, $\tan\theta=\frac{1}{\sqrt{3}}$, the third quadrant, $\theta=\frac{7\pi}{6}$; hence, $-3\sqrt{3}-3\mathrm{i}=6\,\mathrm{cis}\left(\frac{7\pi}{6}\right)$

$$\left(-3\sqrt{3}-3\mathrm{i}\right)^{-7}=(6)^{-7}\,\mathrm{cis}\left(-7\cdot\frac{7\pi}{6}\right)=\frac{1}{279936}\mathrm{cis}\left(-\frac{\pi}{6}\right)$$
$$=\frac{1}{279936}\left(\frac{\sqrt{3}}{2}-\mathrm{i}\frac{1}{2}\right)=\frac{1}{559872}\left(\sqrt{3}-\mathrm{i}\right)$$

(i) $r=2$, $\tan\theta=\frac{1}{\sqrt{3}}$, the first quadrant, $\theta=\frac{\pi}{6}$; hence, $\sqrt{3}+\mathrm{i}=2\,\mathrm{cis}\left(\frac{\pi}{6}\right)$

$$2\left(\sqrt{3}+\mathrm{i}\right)^{7}=2(2)^{7}\,\mathrm{cis}\left(7\cdot\frac{\pi}{6}\right)=256\,\mathrm{cis}\left(\frac{7\pi}{6}\right)=256\left(-\frac{\sqrt{3}}{2}-\mathrm{i}\frac{1}{2}\right)=-128\sqrt{3}-128\mathrm{i}$$

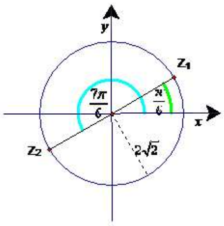
4. To find the nth roots of a complex number $z=r\,\mathrm{cis}\,\theta$, we use the formula developed in the chapter: $z=\sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)$ or $z=\sqrt[n]{r}e^{\mathrm{i}\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}$

(a) $r=8$, $\tan\theta=\sqrt{3}$, the first quadrant, $\theta=\frac{\pi}{3}$; hence, $4+4\mathrm{i}\sqrt{3}=8\,\mathrm{cis}\left(\frac{\pi}{3}\right)$

$$z=\sqrt{8}\,\mathrm{cis}\left(\frac{\frac{\pi}{3}}{2}+\frac{2k\pi}{2}\right)=2\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{6}+k\pi\right);\ k=0,1$$

$$z_1=2\sqrt{2}\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)=2\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)=\sqrt{6}+\mathrm{i}\sqrt{2}$$

$$z_2=2\sqrt{2}\left(\cos\frac{7\pi}{6}+\mathrm{i}\sin\frac{7\pi}{6}\right)=2\sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=-\sqrt{6}-\mathrm{i}\sqrt{2}$$



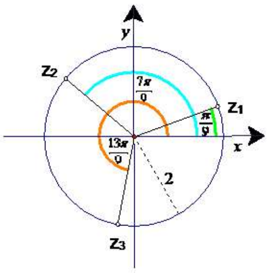
(b) $r=8$, $\tan\theta=\sqrt{3}$, the first quadrant, $\theta=\frac{\pi}{3}$; hence, $4+4\mathrm{i}\sqrt{3}=8\operatorname{cis}\left(\frac{\pi}{3}\right)$

$$z=\sqrt[3]{8}\operatorname{cis}\left(\frac{\frac{\pi}{3}}{3}+\frac{2k\pi}{3}\right)=2\operatorname{cis}\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right);\ k=0,1,2$$

$$z_1=2\operatorname{cis}\left(\frac{\pi}{9}\right)=2\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}$$

$$z_2=2\operatorname{cis}\left(\frac{\pi}{9}+\frac{2\pi}{3}\right)=2\operatorname{cis}\left(\frac{7\pi}{9}\right)=2\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}$$

$$z_3=2\operatorname{cis}\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)=2\operatorname{cis}\left(\frac{13\pi}{9}\right)=2\mathrm{e}^{\mathrm{i}\frac{13\pi}{9}}$$



(c) $r=1$, $\tan\theta=0$, on the negative x-axis, $\theta=\pi$; hence, $-1=1\operatorname{cis}(\pi)$

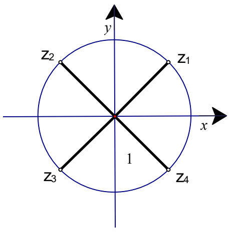
$$z=\sqrt[4]{1}\operatorname{cis}\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)=\operatorname{cis}\left(\frac{\pi}{4}+\frac{k\pi}{2}\right);\ k=0,1,2,3$$

$$z_1=\operatorname{cis}\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}$$

$$z_2=\operatorname{cis}\left(\frac{3\pi}{4}\right)=\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}$$

$$z_3=\operatorname{cis}\left(\frac{5\pi}{4}\right)=\cos\frac{5\pi}{4}+\mathrm{i}\sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}-\mathrm{i}\frac{\sqrt{2}}{2}$$

$$z_4=\operatorname{cis}\left(\frac{7\pi}{4}\right)=\cos\frac{7\pi}{4}+\mathrm{i}\sin\frac{7\pi}{4}=\frac{\sqrt{2}}{2}-\mathrm{i}\frac{\sqrt{2}}{2}$$



(d) $r=1$, $\tan\theta$ is not defined, on the positive y-axis, $\theta=\frac{\pi}{2}$; hence, $\mathrm{i}=1\operatorname{cis}\left(\frac{\pi}{2}\right)$

$$z=\sqrt[4]{1}\operatorname{cis}\left(\frac{\frac{\pi}{2}}{6}+\frac{2k\pi}{6}\right)=\operatorname{cis}\left(\frac{\pi}{12}+\frac{k\pi}{3}\right);\ k=0,1,2,3,4,5$$

$$z_1=\operatorname{cis}\left(\frac{\pi}{12}\right)=\frac{\sqrt{2}+\sqrt{6}}{4}+\mathrm{i}\frac{\sqrt{6}-\sqrt{2}}{4}$$

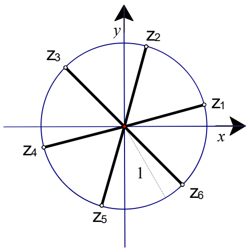
$$z_2=\operatorname{cis}\left(\frac{5\pi}{12}\right)=\frac{-\sqrt{2}+\sqrt{6}}{4}+\mathrm{i}\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$z_3=\operatorname{cis}\left(\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}$$

$$z_4=\operatorname{cis}\left(\frac{13\pi}{12}\right)=-\frac{\sqrt{2}+\sqrt{6}}{4}+\mathrm{i}\frac{-\sqrt{6}+\sqrt{2}}{4}$$

$$z_5=\operatorname{cis}\left(\frac{17\pi}{12}\right)=\frac{\sqrt{2}-\sqrt{6}}{4}-\mathrm{i}\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$z_6=\operatorname{cis}\left(\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2}-\mathrm{i}\frac{\sqrt{2}}{2}$$



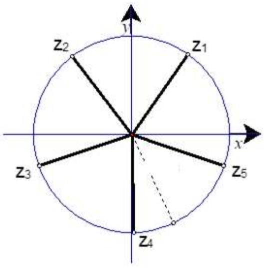
(e) $r=\sqrt{324}=18$, $\tan\theta=\sqrt{3}$, the third quadrant, $\theta=\arctan\sqrt{3}+\pi=\frac{4\pi}{3}$; hence,

$$-9-9\mathrm{i}\sqrt{3}=18\operatorname{cis}\left(\frac{4\pi}{3}\right)=18\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}$$

$$z=\sqrt[5]{18}\operatorname{cis}\left(\frac{4\pi}{15}+\frac{2k\pi}{5}\right)=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{2k\pi}{5}\right)};\ k=0,1,2,3,4$$

$$z_1=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}\right)};\ z_2=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{2\pi}{5}\right)}=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{10\pi}{15}\right)};\ z_3=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{4\pi}{5}\right)}=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{16\pi}{15}\right)}$$

$$z_4=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{6\pi}{5}\right)}=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{22\pi}{15}\right)};\ z_5=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{8\pi}{5}\right)}=\sqrt[5]{18}\mathrm{e}^{\mathrm{i}\left(\frac{28\pi}{15}\right)}$$



5. **(a)** $z^5 - 32 = 0 \Rightarrow z^5 = 32$; hence, we have to find the fifth roots of 32.

$r = 32$, $\tan\theta = 0$, the positive x-axis, $\theta = 0$; hence, $32 = 32\mathrm{e}^{\mathrm{i}\cdot 0}$

$$z = \sqrt[5]{32}\mathrm{e}^{\mathrm{i}\left(\frac{0}{5}+\frac{2k\pi}{5}\right)} = 2\mathrm{e}^{\mathrm{i}\frac{2k\pi}{5}};\ k = 0,1,2,3,4$$

$$z_1 = 2\mathrm{e}^{\mathrm{i}\cdot 0} = 2,\ z_2 = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}},\ z_3 = 2\mathrm{e}^{\mathrm{i}\frac{4\pi}{5}},\ z_4 = 2\mathrm{e}^{\mathrm{i}\frac{6\pi}{5}},\ z_5 = 2\mathrm{e}^{\mathrm{i}\frac{8\pi}{5}}$$

(b) $z^8 + \mathrm{i} = 0 \Rightarrow z^8 = -\mathrm{i}$; hence, we have to find the eighth roots of $-\mathrm{i}$.

$r = 1$, $\tan\theta$ is not defined, the negative y-axis, $\theta = \frac{3\pi}{2}$; hence, $-\mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}$ or $-\mathrm{i} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$

$$z = \sqrt[8]{1}\mathrm{e}^{\mathrm{i}\left(\frac{\frac{3\pi}{2}}{8}+\frac{2k\pi}{8}\right)} = \mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{16}+\frac{k\pi}{4}\right)};\ k = 0,1,2,3,4,5,6,7$$

$$z_1 = \mathrm{e}^{\mathrm{i}\frac{3\pi}{16}},\ z_2 = \mathrm{e}^{\mathrm{i}\frac{7\pi}{16}},\ z_3 = \mathrm{e}^{\mathrm{i}\frac{11\pi}{16}},\ z_4 = \mathrm{e}^{\mathrm{i}\frac{15\pi}{16}},$$

$$z_5 = \mathrm{e}^{\mathrm{i}\frac{19\pi}{16}},\ z_6 = \mathrm{e}^{\mathrm{i}\frac{23\pi}{16}},\ z_7 = \mathrm{e}^{\mathrm{i}\frac{27\pi}{16}},\ z_8 = \mathrm{e}^{\mathrm{i}\frac{31\pi}{16}}$$

(c) $z^3 + 4\sqrt{3} - 4\mathrm{i} = 0 \Rightarrow z^3 = -4\sqrt{3} + 4\mathrm{i}$; hence, we must find the third roots of $4\sqrt{3} - 4\mathrm{i}$.

$r = 8$, $\tan\theta = -\frac{1}{\sqrt{3}}$, the second quadrant, $\theta = \frac{5\pi}{6}$; hence, $4\sqrt{3} - 4\mathrm{i} = 8\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$

$$z = \sqrt[3]{8}\mathrm{e}^{\mathrm{i}\left(\frac{\frac{5\pi}{6}}{3}+\frac{2k\pi}{3}\right)} = 2e^{\mathrm{i}\left(\frac{5\pi}{18}+\frac{2k\pi}{3}\right)};\ \ k = 0,1,2$$

$$z_1 = 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{18}},\ z_2 = 2\mathrm{e}^{\mathrm{i}\frac{17\pi}{18}},\ z_3 = 2\mathrm{e}^{\mathrm{i}\frac{29\pi}{18}}$$

(d) $z^4 - 16 = 0 \Rightarrow z^4 = 16$; hence, we have to find the fourth roots of 16.

$r = 16$, $\tan\theta = 0$, the positive x-axis, $\theta = 0$; hence, $16 = 16\mathrm{e}^{\mathrm{i}\cdot 0}$

$$z = \sqrt[4]{16}\mathrm{e}^{\mathrm{i}\left(\frac{0}{4}+\frac{2k\pi}{4}\right)} = 2\mathrm{e}^{\mathrm{i}\frac{k\pi}{2}};\ k = 0,1,2,3$$

$$z_1 = 2\mathrm{e}^{\mathrm{i}0} = 2,\ z_2 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = 2\mathrm{i},\ z_3 = 2\mathrm{e}^{\mathrm{i}\pi} = -2,\ z_4 = 2\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} = -2\mathrm{i}$$

(e) $z^5+128=128\mathrm{i}\Rightarrow z^5=-128+128\mathrm{i}$; hence, we must find the fifth roots of $-128+128\mathrm{i}$

$r=128\sqrt{2}=2^{\frac{15}{2}}$, $\tan\theta=-1$, the second quadrant, $\theta=\dfrac{3\pi}{4}$; hence, $-128+128\mathrm{i}=2^{\frac{15}{2}}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$

$$z=\sqrt[5]{2^{\frac{15}{2}}}\,\mathrm{e}^{\mathrm{i}\left(\frac{\frac{3\pi}{4}}{5}+\frac{2k\pi}{5}\right)}=2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{20}+\frac{2k\pi}{5}\right)}=\sqrt{8}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{20}+\frac{2k\pi}{5}\right)};\quad k=0,1,2,3,4$$

$$z_1=\sqrt{8}\mathrm{e}^{\mathrm{i}\frac{3\pi}{20}},\ z_2=\sqrt{8}\mathrm{e}^{\mathrm{i}\frac{11\pi}{20}},\ z_3=\sqrt{8}\mathrm{e}^{\mathrm{i}\frac{19\pi}{20}},\ z_4=\sqrt{8}\mathrm{e}^{\mathrm{i}\frac{27\pi}{20}},$$

$$z_5=\sqrt{8}\mathrm{e}^{\mathrm{i}\frac{7\pi}{4}}=2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\mathrm{i}\right)=2-2\mathrm{i}$$

(f) $z^6-64\mathrm{i}=0\Rightarrow z^6=64\mathrm{i}$; hence, we have to find the sixth roots of $64\mathrm{i}$.

$r=64$, $\tan\theta$ is not defined, the positive y-axis, $\theta=\dfrac{\pi}{2}$; hence, $64\mathrm{i}=64\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

$$z=\sqrt[6]{64}\mathrm{e}^{\mathrm{i}\left(\frac{\frac{\pi}{2}}{6}+\frac{2k\pi}{6}\right)}=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{12}+\frac{k\pi}{3}\right)};\ k=0,1,2,3,4,5$$

$$z_1=2\mathrm{e}^{\mathrm{i}\frac{\pi}{12}},\ z_2=2\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}},\ z_3=2\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}=2\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)=-\sqrt{2}+\sqrt{2}\mathrm{i},\ z_4=2\mathrm{e}^{\mathrm{i}\frac{13\pi}{12}},$$

$$z_5=2\mathrm{e}^{\mathrm{i}\frac{17\pi}{12}},\ z_6=2\mathrm{e}^{\mathrm{i}\frac{7\pi}{4}}=2\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\mathrm{i}\right)=\sqrt{2}-\sqrt{2}\mathrm{i}$$

6. **(a)** $\mathrm{cis}(9\beta)\mathrm{cis}(-5\beta)=\mathrm{cis}(9\beta-5\beta)=\mathrm{cis}(4\beta)=\cos(4\beta)+\mathrm{i}\sin(4\beta)$

(b) $\dfrac{\mathrm{cis}(6\beta)\mathrm{cis}(4\beta)}{\mathrm{cis}(3\beta)}=\mathrm{cis}(6\beta+4\beta-3\beta)=\mathrm{cis}(7\beta)=\cos(7\beta)+\mathrm{i}\sin(7\beta)$

(c) $(\mathrm{cis}(9\beta))^{\frac{1}{3}}=\mathrm{cis}\left(\dfrac{9\beta}{3}\right)=\mathrm{cis}(3\beta)=\cos(3\beta)+\sin(3\beta)$

(d) $\sqrt[n]{\mathrm{cis}(2n\beta)}=\mathrm{cis}\left(\dfrac{2n\beta}{n}\right)=\mathrm{cis}(2\beta)=\cos(2\beta)+\mathrm{i}\sin(2\beta)$

7. $\cos(\alpha+\beta)=\mathrm{Re}\left(\mathrm{e}^{(\alpha+\beta)\mathrm{i}}\right)=\mathrm{Re}\left(\mathrm{e}^{\alpha\mathrm{i}}\mathrm{e}^{\beta\mathrm{i}}\right)$

Hence, we have to find $\mathrm{e}^{\alpha\mathrm{i}}\mathrm{e}^{\beta\mathrm{i}}$:

$\mathrm{e}^{\alpha\mathrm{i}}\mathrm{e}^{\beta\mathrm{i}}=(\cos\alpha+\mathrm{i}\sin\alpha)(\cos\beta+\mathrm{i}\sin\beta)$

$=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\mathrm{i}(\cos\alpha\sin\beta+\sin\alpha\cos\beta)$

The real part of $\mathrm{e}^{\alpha\mathrm{i}}\mathrm{e}^{\beta\mathrm{i}}$ is $\cos\alpha\cos\beta-\sin\alpha\sin\beta$, so $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

8. **(a)** $\cos(4\alpha)=\mathrm{Re}\left(\mathrm{e}^{(4\alpha)\mathrm{i}}\right)=\mathrm{Re}\left(\left(\mathrm{e}^{\alpha\mathrm{i}}\right)^4\right)$

Hence, we have to find $\left(\mathrm{e}^{\alpha\mathrm{i}}\right)^4$: (Using the binomial theorem)

$$\begin{aligned}\left(\mathrm{e}^{\alpha\mathrm{i}}\right)^4&=\left(\cos(\alpha)+\mathrm{i}\sin(\alpha)\right)^4\\&=\cos^4\alpha+4\mathrm{i}\cos^3\alpha\sin\alpha+6\mathrm{i}^2\cos^2\alpha\sin^2\alpha+4\mathrm{i}^3\cos\alpha\sin^3\alpha+\mathrm{i}^4\sin^4\alpha\end{aligned}$$

The real part of the number is:

$$\begin{aligned}\cos^4\alpha+6\mathrm{i}^2\cos^2\alpha\sin^2\alpha+\mathrm{i}^4\sin^4\alpha&=\cos^4\alpha-6\cos^2\alpha\sin^2\alpha+\sin^4\alpha\\&=\cos^4\alpha+6\cos^2\alpha\left(1-\cos^2\alpha\right)+\left(1-\cos^2\alpha\right)^2\\&=\cos^4\alpha-6\cos^2\alpha+6\cos^4\alpha+1-2\cos^2\alpha+\cos^4\alpha\\&=8\cos^4\alpha-8\cos^2\alpha+1\end{aligned}$$

therefore: $\cos(4\alpha)=8\cos^4\alpha-8\cos^2\alpha+1$

(b) $\cos(5\alpha)=\mathrm{Re}\left(\mathrm{e}^{(5\alpha)\mathrm{i}}\right)=\mathrm{Re}\left(\left(\mathrm{e}^{\alpha\mathrm{i}}\right)^5\right)$

Hence, we have to find $\left(\mathrm{e}^{\alpha\mathrm{i}}\right)^5$: (Using the binomial theorem.)

$$\begin{aligned}\left(\mathrm{e}^{\alpha\mathrm{i}}\right)^5&=\left(\cos(\alpha)+\mathrm{i}\sin(\alpha)\right)^5\\&=\cos^5\alpha+5\cos^4\alpha\sin(\alpha\mathrm{i})+10\cos^3\alpha\sin^2(\alpha\mathrm{i})^2\\&\quad+10\cos^2\alpha\sin^3(\alpha\mathrm{i})^3+5\cos\alpha\sin^4(\alpha\mathrm{i})^4+\sin^5(\alpha\mathrm{i})^5\end{aligned}$$

The real part of the number is:

$$\begin{aligned}&\cos^5\alpha-10\cos^3\alpha\sin^2\alpha+5\cos\alpha\sin^4\alpha\\&=\cos^5\alpha-10\cos^3\alpha(1-\cos^2\alpha)+5\cos\alpha(1-\cos^2\alpha)^2\\&=\cos^5\alpha-10\cos^3\alpha+10\cos^5\alpha+5\cos\alpha-10\cos^3\alpha+5\cos^4\alpha\\&=16\cos^5\alpha-20\cos^3\alpha+5\cos\alpha\end{aligned}$$

Therefore, $\cos(5\alpha)=16\cos^5\alpha-20\cos^3\alpha+5\cos\alpha$

(c) Using the formula from **(a)**, $\cos(4\alpha)=8\cos^4\alpha-8\cos^2\alpha+1$, and the double angle formula, $\cos 2\alpha=2\cos^2\alpha-1\Rightarrow 8\cos^2 a=4\left(\cos 2\alpha-1\right)$, we have:

$\cos(4\alpha)=8\cos^4\alpha-4\left(\cos 2\alpha-1\right)+1\Rightarrow\cos(4\alpha)+4\cos 2\alpha+3=8\cos^4\alpha$. Hence:

$$\cos^4\alpha=\frac{1}{8}\left(\cos(4\alpha)+4\cos 2\alpha+3\right)$$

9. **(a)** Since $\frac{1}{z} = \cos(-2\alpha) + \mathrm{i}\sin(-2\alpha)$, we have:

$$z + \frac{1}{z} = \left(\cos(2\alpha) + \mathrm{i}\sin(2\alpha)\right) + \left(\cos(-2\alpha) + \mathrm{i}\sin(-2\alpha)\right).$$

Using the even/odd property we have: $\cos(-2\alpha) = \cos(2\alpha)$ and $\sin(-2\alpha) = -\sin(2\alpha)$. Hence,

$$z + \frac{1}{z} = \cos(2\alpha) + \mathrm{i}\sin(2\alpha) + \cos(2\alpha) - \mathrm{i}\sin(2\alpha) = 2\cos(2\alpha)$$

$$\begin{aligned} z - \frac{1}{z} &= \left(\cos(2\alpha) + \mathrm{i}\sin(2\alpha)\right) - \left(\cos(-2\alpha) + \mathrm{i}\sin(-2\alpha)\right) \\ &= \cos(2\alpha) + \mathrm{i}\sin(2\alpha) - \cos(2\alpha) + \mathrm{i}\sin(2\alpha) = 2\mathrm{i}\sin(2\alpha) \end{aligned}$$

(b)

$$\begin{aligned} z^n + \frac{1}{z^n} &= \left(\cos(2n\alpha) + \mathrm{i}\sin(2n\alpha)\right) + \left(\cos(-2n\alpha) + \mathrm{i}\sin(-2n\alpha)\right) \\ &= \cos(2n\alpha) + \mathrm{i}\sin(2n\alpha) + \cos(2n\alpha) - \mathrm{i}\sin(2n\alpha) = 2\cos(2n\alpha) \end{aligned}$$

Hence, $z^n + \frac{1}{z^n} = 2\cos(2n\alpha) \Rightarrow \cos(2n\alpha) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

$$\begin{aligned} z^n - \frac{1}{z^n} &= \left(\cos(2n\alpha) + \mathrm{i}\sin(2n\alpha)\right) - \left(\cos(-2n\alpha) + \mathrm{i}\sin(-2n\alpha)\right) \\ &= \cos(2n\alpha) + \mathrm{i}\sin(2n\alpha) - \cos(2n\alpha) + \mathrm{i}\sin(2n\alpha) = 2\mathrm{i}\sin(2n\alpha) \end{aligned}$$

So, $z^n - \frac{1}{z^n} = 2\mathrm{i} \times \sin(2n\alpha) \Rightarrow \sin(2n\alpha) = \frac{1}{2\mathrm{i}}\left(z^n - \frac{1}{z^n}\right)$.

10. $(1+3w)(1+3w^2) = 1 + 3w + 3w^2 + 9\underbrace{w^3}_{1} = 10 + 3w(1+w)$

Since: $w(1+w) = \frac{w(1+w)(1-w)}{1-w} = \frac{w - \overbrace{w^3}^{1}}{1-w} = \frac{w-1}{1-w} = -1$, we have:

$(1+3w)(1+3w^2) = 10 + 3(-1) = 7$

Note: We can establish the formula $w + w^2 = -1$ using the values of the cube roots of 1:

1, $w = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$, $w^2 = \cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$. Hence,

$$w + w^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = -1$$

11. **(a)** For the fourth roots of $1 = 1\text{cis}(0)$:

$$z = \text{cis}\left(\frac{2k\pi}{4}\right) = \text{cis}\left(\frac{k\pi}{2}\right), k = 0,1,2,3$$

$$z_1 = \text{cis}(0) = 1,\ z_2 = \text{cis}\left(\frac{\pi}{2}\right) = \beta,\ z_3 = \text{cis}(\pi) = \left(\text{cis}\left(\frac{\pi}{2}\right)\right)^2 = \beta^2,$$

$$z_4 = \text{cis}\left(\frac{3\pi}{2}\right) = \left(\text{cis}\left(\frac{\pi}{2}\right)\right)^3 = \beta^3$$

Therefore, we can denote the fourth roots as $1, \beta, \beta^2, \beta^3$.

(b) We can solve the task using the values: $\beta = \text{i},\ \beta^2 = -1,\ \beta^3 = -\text{i}$, so:

$$(1+\beta)(1+\beta^2+\beta^3) = (1+\text{i})(1-1-\text{i}) = (1+\text{i})(-\text{i}) = 1-\text{i}.$$

(c) $\beta + \beta^2 + \beta^3 = i - 1 - i = -1$

12. **(a)** For the fifth roots of $1 = 1\text{cis}(0)$:

$$z = \text{cis}\left(\frac{2k\pi}{5}\right), k = 0,1,2,3,4$$

$$z_1 = \text{cis}(0) = 1,\ z_2 = \text{cis}\left(\frac{2\pi}{5}\right),\ z_3 = \text{cis}\left(\frac{4\pi}{5}\right) = \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^2$$

$$z_4 = \text{cis}\left(\frac{6\pi}{5}\right) = \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^3 \quad z_5 = \text{cis}\left(\frac{8\pi}{5}\right) = \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^4$$

Therefore, we can denote the fifth roots as $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

(b) First multiply and then substitute the values from **(a)**

$$(1+\alpha)(1+\alpha^4) = 1 + \alpha + \alpha^4 + \alpha^5 = 2 + \alpha + \alpha^4$$

$$= 2 + \cos\frac{2\pi}{5} + \text{i}\sin\frac{2\pi}{5} + \cos\left(-\frac{2\pi}{5}\right) + \text{i}\sin\left(-\frac{2\pi}{5}\right)$$

$$= 2 + 2\cos\frac{2\pi}{5} = 2 + \frac{-1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$$

(c) $$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = \frac{(1+\alpha+\alpha^2+\alpha^3+\alpha^4)(1-\alpha)}{1-\alpha} = \frac{1-\alpha^5}{1-\alpha} = \frac{1-1}{1-\alpha} = 0$$

13. $1+\mathrm{i}\sqrt{3}=2\,\mathrm{cis}\left(\frac{\pi}{3}\right)\Rightarrow\left(1+\mathrm{i}\sqrt{3}\right)^n=2^n\,\mathrm{cis}\left(\frac{n\pi}{3}\right)$

$1-\mathrm{i}\sqrt{3}=2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\Rightarrow\left(1-\mathrm{i}\sqrt{3}\right)^n=2^n\,\mathrm{cis}\left(-\frac{n\pi}{3}\right)$

Hence,

$$\begin{aligned}\left(1+\mathrm{i}\sqrt{3}\right)^n+\left(1-\mathrm{i}\sqrt{3}\right)^n&=2^n\left(\cos\left(\frac{n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{n\pi}{3}\right)+\cos\left(-\frac{n\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{n\pi}{3}\right)\right)\\&=2^n\left(\cos\left(\frac{n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{n\pi}{3}\right)+\cos\left(\frac{n\pi}{3}\right)-\mathrm{i}\sin\left(\frac{n\pi}{3}\right)\right)\\&=2^{n+1}\cos\left(\frac{n\pi}{3}\right)\end{aligned}$$

Therefore, the number is real.

For $n=18$, the value is: $2^{19}\cos\left(\frac{18\pi}{3}\right)=2^{19}=524288$

14. Since $\arg\left(2a+3\mathrm{i}\right)^3=135°$, then $\arg\left(2a+3\mathrm{i}\right)=\frac{135°}{3}=45°$. Therefore, $\tan\theta=1=\frac{3}{2a}\Rightarrow a=\frac{3}{2}$.

Chapter 8 practice questions

1. **Method I**

$\left(1-\mathrm{i}\right)z=1-3\mathrm{i}\Rightarrow z=\frac{1-3\mathrm{i}}{1-\mathrm{i}}=\frac{\left(1-3\mathrm{i}\right)\left(1+\mathrm{i}\right)}{1+1}=\frac{1+3+\mathrm{i}\left(1-3\right)}{2}=2-\mathrm{i}$. Hence, $x=2, y=-1$

Method II

Another method you are already familiar with is to equate the real and imaginary parts of two complex numbers. In this case, we have to solve system of equations:

$\left(1-\mathrm{i}\right)\left(x+y\mathrm{i}\right)=1-3\mathrm{i}\Rightarrow$
$x+y+\mathrm{i}\left(y-x\right)=1-3\mathrm{i}$

Hence, $x+y=1$

$$\begin{aligned}-x+y&=-3\\ \Rightarrow\ \begin{matrix}x=2\\y=-1\end{matrix}\end{aligned}$$

2. **(a)** w is a cubic root of 1 other than 1:

- If $w = \text{cis}\left(\frac{2\pi}{3}\right)$, then

  $$1 + w + w^2 = 1 + \text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(\frac{4\pi}{3}\right)$$
  $$= 1 + \cos\left(\frac{2\pi}{3}\right) + \text{i}\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + \text{i}\sin\left(\frac{4\pi}{3}\right)$$
  $$= 1 - \frac{1}{2} + \text{i}\frac{\sqrt{3}}{2} - \frac{1}{2} - \text{i}\frac{\sqrt{3}}{2} = 0$$

- If $w = \text{cis}\left(\frac{4\pi}{3}\right)$, then

  $$1 + w + w^2 = 1 + \text{cis}\left(\frac{4\pi}{3}\right) + \text{cis}\left(\frac{8\pi}{3}\right) = 1 + \text{cis}\left(\frac{4\pi}{3}\right) + \text{cis}\left(\frac{2\pi}{3}\right).$$

  This is the same as in the previous case; hence, the value is 0.

(b) $$\left(wx + w^2y\right)\left(w^2x + wy\right) = \overbrace{w^3}^{=1}x^2 + \overbrace{w^4}^{=w}xy + w^2xy + \overbrace{w^3}^{=1}y^2 = x^2 + y^2 + \left(w + w^2\right)xy$$

But, since $1 + w + w^2 = 0$, then $w + w^2 = -1$, and thus,

$$\left(wx + w^2y\right)\left(w^2x + wy\right) = x^2 + y^2 + \left(w + w^2\right)xy = x^2 + y^2 - xy$$

3. **(a)** $(1 + \text{i})^2 = 1 + 2\text{i} + \text{i}^2 = 1 + 2\text{i} - 1 = 2\text{i}$

(b) Let $P(n)$ be the statement: $(1 + \text{i})^{4n} = (-4)^n$.

The basis step must be $P(1)$.

$(1 + \text{i})^4 = \left((1 + \text{i})^2\right) = (2\text{i})^2 = 2^2\text{i}^2 = -4 = (-4)^1$; hence, $P(1)$ is true.

Next, assume that for some $k \in \mathbb{N}^+$, $P(k)$ is true.

$P(k) : (1 + \text{i})^{4k} = (-4)^k$

Now, $(1 + \text{i})^{4(k+1)} = (1 + \text{i})^{4k+4} = (1 + \text{i})^k (1 + \text{i})^4 = (-4)^k(-4) = (-4)^{k+1}$.

Therefore $P(k + 1)$ is true whenever $P(k)$ is true and by mathematical induction, $P(n)$ must be true for all $k \in \mathbb{N}^+$

(c) $(1 + \text{i})^{32} = (1 + \text{i})^{4\times 8} = (-4)^8 = 65\,536$

4. **(a)** For z_1: $|z_1| = \sqrt{\frac{6}{4}+\frac{2}{4}} = \sqrt{2}$, $\tan\theta = -\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}} = -\frac{1}{\sqrt{3}}$, the fourth quadrant, $\theta = -\frac{\pi}{6}$;

hence, $z_1 = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

For z_2: $|z_2| = \sqrt{1+1} = \sqrt{2}$, $\tan\theta = -\frac{1}{1} = -1$, the fourth quadrant, $\theta = -\frac{\pi}{4}$; hence,

$$z_2 = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

(b) In polar form:

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)} = 1\left(\cos\left(-\frac{\pi}{6}+\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)\right)$$

c)

$$= \cos\left(\frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{\pi}{12}\right)$$

(c) In standard $a + b\mathrm{i}$ form:

$$\frac{z_1}{z_2} = \frac{\frac{\sqrt{6}-\mathrm{i}\sqrt{2}}{2}}{1-\mathrm{i}} \cdot \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{\frac{\sqrt{6}-\sqrt{2}\mathrm{i}^2+\sqrt{6}\mathrm{i}-\mathrm{i}\sqrt{2}}{2}}{1+1} = \frac{\sqrt{6}+\sqrt{2}+\mathrm{i}\left(\sqrt{6}-\sqrt{2}\right)}{4}$$

Hence, $a = \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$, $b = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

5. Using de Moivre's theorem:

$$\left(\frac{z_1}{z_3}\right)^4 = \left(\frac{a}{b}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right)\right)^4 = \left(\frac{a}{b}\left(\cos\left(\frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)\right)^4$$

$$= \frac{a^4}{b^4}\left(\cos\left(\frac{\pi}{12}\cdot 4\right) + \mathrm{i}\sin\left(\frac{\pi}{12}\cdot 4\right)\right)$$

$$= \frac{a^4}{b^4}\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right) = \frac{a^4}{b^4}\left(\frac{1}{2} - \frac{\mathrm{i}\sqrt{3}}{2}\right)$$

6. Let $z = x + y\mathrm{i}$. Then:

$$\sqrt{(x+16)^2 + y^2} = 4\sqrt{(x+1)^2 + y^2}$$

$$\Rightarrow (x+16)^2 + y^2 = 16\left((x+1)^2 + y^2\right)$$

$$= x^2 + 32x + 256 + y^2 = 16x^2 + 32x + 16 + 16y^2$$

$$= 15x^2 + 15y^2 = 240$$

This implies that $x^2 + y^2 = 16,$ and consequently $|z| = \sqrt{x^2 + y^2} = \sqrt{16} = 4$

7. $a + b\mathrm{i} = \dfrac{5-\mathrm{i}}{2-\mathrm{i}} = \dfrac{5-\mathrm{i}}{2-\mathrm{i}} \cdot \dfrac{2+\mathrm{i}}{2+\mathrm{i}} = \dfrac{10+1+\mathrm{i}(5-2)}{4+1} = \dfrac{11+3\mathrm{i}}{5}$

Therefore, $a = \dfrac{11}{5},\ b = \dfrac{3}{5}$

8. $\arg(x+\mathrm{i})^2 = \dfrac{\pi}{3} \Rightarrow \arg\left(x^2 - 1 + 2x\mathrm{i}\right) = \dfrac{\pi}{3} \Rightarrow \tan\dfrac{\pi}{3} = \dfrac{2x}{x^2-1} = \sqrt{3}$

$\sqrt{3}x^2 - 2x - \sqrt{3} = \left(x\sqrt{3} + 1\right)\left(x - \sqrt{3}\right) = 0 \Rightarrow x = \sqrt{3},$ or $x = -\dfrac{1}{\sqrt{3}}$

Since x is positive, the solution is $x = \sqrt{3}$

9. $\mathrm{i}(z+2) = 1 - 2z \Rightarrow z(2+\mathrm{i}) = 1 - 2\mathrm{i}$

$\Rightarrow z = \dfrac{1-2\mathrm{i}}{2+\mathrm{i}} = \dfrac{1-2\mathrm{i}}{2+\mathrm{i}} \cdot \dfrac{2-\mathrm{i}}{2-\mathrm{i}} = \dfrac{-5\mathrm{i}}{5} = -\mathrm{i}$, i.e., $a = 0,\ b = -1$

10. **(a)** $z^5 - 1 = (z-1)\left(z^4 + z^3 + z^2 + z + 1\right)$

(b) $z^5 = 1 = \operatorname{cis} 0$

Hence, the zeros are the fifth roots of unity.

$z = \operatorname{cis}\left(\pm\dfrac{2k\pi}{5}\right),\ k = 0, 1, 2, 3, 4$

$$1, \operatorname{cis}\left(\frac{2\pi}{5}\right), \operatorname{cis}\left(\frac{4\pi}{5}\right), \underbrace{\operatorname{cis}\left(\frac{6\pi}{5}\right)}_{\operatorname{cis}\left(-\frac{4\pi}{5}\right)}, \underbrace{\operatorname{cis}\left(\frac{8\pi}{5}\right)}_{\operatorname{cis}\left(-\frac{2\pi}{5}\right)}.$$

The solutions are: $1, \operatorname{cis}\left(\pm\dfrac{2\pi}{5}\right), \operatorname{cis}\left(\pm\dfrac{4\pi}{5}\right)$

(c) We use the factor theorem:

$$\left(z-\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\operatorname{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$=\left(z-\cos\left(\frac{2\pi}{5}\right)-\mathrm{i}\sin\left(\frac{2\pi}{5}\right)\right)\left(z-\cos\left(-\frac{2\pi}{5}\right)-\mathrm{i}\sin\left(-\frac{2\pi}{5}\right)\right)$$

$$=\left(z-\cos\left(\frac{2\pi}{5}\right)-\mathrm{i}\sin\left(\frac{2\pi}{5}\right)\right)\left(z-\cos\left(\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{2\pi}{5}\right)\right)$$

$$=\left(z-\cos\left(\frac{2\pi}{5}\right)\right)^2+\sin^2\left(\frac{2\pi}{5}\right)=z^2-2\cos\left(\frac{2\pi}{5}\right)+\cos^2\left(\frac{2\pi}{5}\right)+\sin^2\left(\frac{2\pi}{5}\right)$$

$$=z^2-2\cos\left(\frac{2\pi}{5}\right)+1$$

In the same way:

$$\left(z-\operatorname{cis}\left(\frac{4\pi}{5}\right)\right)\left(z-\operatorname{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$=\left(z-\cos\left(\frac{4\pi}{5}\right)-\mathrm{i}\sin\left(\frac{4\pi}{5}\right)\right)\left(z-\cos\left(-\frac{4\pi}{5}\right)-\mathrm{i}\sin\left(-\frac{4\pi}{5}\right)\right)$$

$$=\left(z-\cos\left(\frac{4\pi}{5}\right)\right)^2+\sin^2\left(\frac{4\pi}{5}\right)$$

$$==z^2-2\cos\left(\frac{4\pi}{5}\right)z+1$$

Hence: $z^4+z^3+z^2+z+1=\left(z^2-2\cos\left(\frac{2\pi}{5}\right)z+1\right)\left(z^2-2\cos\left(\frac{4\pi}{5}\right)z+1\right)$

11. **(a)** $|8i|=8, \tan\theta$ is not defined, positive y-axis, $\theta=\frac{\pi}{2}, 8\mathrm{i}=8\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$

(b) **(i)** Let z be a cube root of 8i. Then:

$$z=\sqrt[3]{8}\operatorname{cis}\left(\frac{\frac{\pi}{2}}{3}+\frac{2k\pi}{3}\right)=2\operatorname{cis}\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right), k=0,1,2$$

For $k=0$, the number is in the first quadrant: $z=2\operatorname{cis}\frac{\pi}{6}$

(ii) $z=2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)=\sqrt{3}+\mathrm{i}$

12. **(a)** All the numbers are of modulus 1; hence, their product and quotient are of modulus 1 and thus $|z| = 1$.

$$z = \frac{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^3 \left(\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^8}{\left(\operatorname{cis}\left(-\frac{\pi}{24}\right)\right)^8} = \frac{\operatorname{cis}\left(3\cdot\frac{\pi}{3}\right)\operatorname{cis}\left(8\cdot\frac{\pi}{4}\right)}{\operatorname{cis}\left(8\cdot-\frac{\pi}{24}\right)}$$

$$= \frac{\operatorname{cis}(\pi)\operatorname{cis}(2\pi)}{\operatorname{cis}\left(-\frac{\pi}{3}\right)} = \frac{\operatorname{cis}(\pi)\operatorname{cis}(0)}{\operatorname{cis}\left(-\frac{\pi}{3}\right)}$$

Hence, $\arg z = \pi + 0 + \frac{\pi}{3} = \frac{4\pi}{3}$

(b) Since $z = \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)$, then

$$z^3 = \left(\cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right)^3 = \underbrace{\cos(4\pi)}_{=1} + \mathrm{i}\underbrace{\sin(4\pi)}_{=0} = 1$$

Hence, z is a cube root of 1.

(c) First expand the expression and then substitute the polar form to simplify calculations.

$$(1+2z)(2+z^2) = 2 + 4z + z^2 + 2\underbrace{z^3}_{=1} = 4 + 4z + z^2$$

$$= 4 + 4\operatorname{cis}\left(\frac{4\pi}{3}\right) + \left(\operatorname{cis}\left(\frac{4\pi}{3}\right)\right)^2 = 4 + 4\operatorname{cis}\left(\frac{4\pi}{3}\right) + \operatorname{cis}\left(\frac{8\pi}{3}\right)$$

$$= 4 + 4\left(-\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right)$$

$$= 4 - 2 + 2\sqrt{3}\mathrm{i} - \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} = \frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}$$

13. $\sqrt{z} = \frac{2}{1-\mathrm{i}} + 1 - 4\mathrm{i} = \frac{-1-5\mathrm{i}}{1-\mathrm{i}} \cdot \frac{1+\mathrm{i}}{1+\mathrm{i}} = 2 - 3\mathrm{i}$

$z = (2-3\mathrm{i})^2 = 4 - 12\mathrm{i} - 9 = -5 - 12\mathrm{i}$

14. **(a)** Let $P(n)$ be the statement: $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$

The basis step is $P(1)$ and it is true, because both sides are $\cos\theta + \mathrm{i}\sin\theta$.

Next, assume that $P(k)$ is true for some $k \in \mathbb{N}^+$.

$P(k): (\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$

Now,

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} &= (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)\\ &= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)\\ &= \cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta)\\ &= \cos(k\theta + \theta) + \mathrm{i}(\sin(k\theta + \theta)) = \cos((k+1)\theta) + \mathrm{i}\sin((k+1)\theta)\end{aligned}$$

Therefore, $P(k+1)$ is true and by mathematical induction $P(n)$ is true for all $k \in \mathbb{N}^+$.

(b) **(i)** Using de Moivre's theorem:

$$\frac{1}{z} = z^{-1} = (\cos(\theta) + \mathrm{i}\sin(\theta))^{-1} = \cos(-\theta) + \mathrm{i}\sin(-\theta)$$

(ii) $$z^{-n} = (z^{-1})^n = (\cos(-\theta) + \mathrm{i}\sin(-\theta))^n = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$$

$$z^n = (\cos(\theta) + \mathrm{i}\sin(\theta))^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$$

$$\begin{aligned}z^n + z^{-n} &= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta)\\ &= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta) = 2\cos(n\theta)\end{aligned}$$

(c) **(i)** $$\begin{aligned}(z + z^{-1})^5 &= z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5}\\ &= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}\end{aligned}$$

(ii) Using the result from **(i)**, we have:

$$(z + z^{-1})^5 = (2\cos\theta)^5 = 32\cos^5\theta$$

and

$$\begin{aligned}&(z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})\\ &\quad = 2\cos(5\theta) + 5\cdot 2\cos(3\theta) + 10\cdot 2\cos(\theta)\end{aligned}$$

Thus, $32\cos^5\theta = 2(\cos(5\theta) + 5\cos(3\theta) + 10\cos(\theta))$

Therefore, $\cos^5\theta = \frac{1}{16}(\cos(5\theta) + 5\cos(3\theta) + 10\cos(\theta))$, and $a = 1, b = 5, c = 10$

15. $2p + 2\mathrm{i}q = q - \mathrm{i}p - 2(1-\mathrm{i}) \Rightarrow 2p + 2\mathrm{i}q = -2 + q + \mathrm{i}(2-p)$

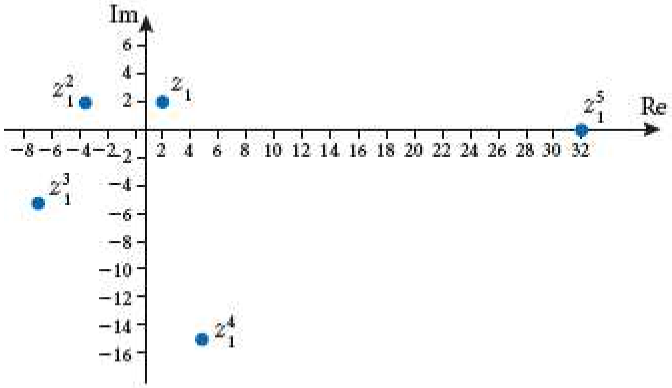
Hence: $$\left.\begin{aligned}2p &= -2 + q\\ 2q &= 2 - p\end{aligned}\right\} \Rightarrow \begin{aligned}2p - q &= -2\\ p + 2q &= 2\end{aligned} \Rightarrow p = -\frac{2}{5},\ q = \frac{6}{5}$$

16. **(a)** $z_1^5 = 2^5\left(\cos\left(5\cdot\frac{2\pi}{5}\right) + \text{i}\sin\left(5\cdot\frac{2\pi}{5}\right)\right) = 32\left(\cos(2\pi) + \text{i}\sin(2\pi)\right) = 32$

(b) $z_1^2 = 4\left(\cos\left(\frac{4\pi}{5}\right) + \text{i}\sin\left(\frac{4\pi}{5}\right)\right)$; $z_1^3 = 8\left(\cos\left(\frac{6\pi}{5}\right) + \text{i}\sin\left(\frac{6\pi}{5}\right)\right)$

$z_1^4 = 16\left(\cos\left(\frac{8\pi}{5}\right) + \text{i}\sin\left(\frac{8\pi}{5}\right)\right)$; $z_1^5 = 32\left(\cos(2\pi) + \text{i}\sin(2\pi)\right) = 32$

(c)



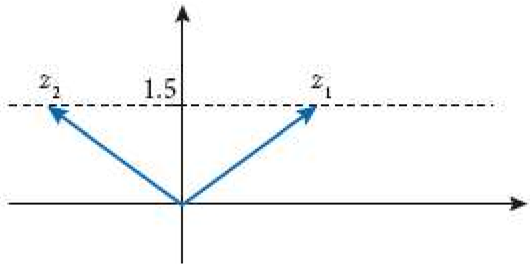
(d) The transformation is a combination (in any order) of an enlargement of scale factor 2, with the origin as the centre, and an anti-clockwise rotation of $\frac{2\pi}{5}$, again with the origin as the centre.

17. **(a)** Let $z = a + b\text{i}$. Then:

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + (b-3)^2} \Rightarrow a^2 + b^2 = a^2 + b^2 - 6b + 9$$

$$\Rightarrow 6b - 9 = 0 \Rightarrow b = \frac{3}{2}$$

(b) **(i)**



(ii) Since $\arg z_1 = \theta$ and $\sin\theta = \frac{1.5}{3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

(iii) $\arg z_2 = \pi - \arg z_1 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(c) $$\arg\left(\frac{z_1^k z_2}{2\mathrm{i}}\right)=\arg\left(z_1^k\right)+\arg\left(z_2\right)-\arg(\mathrm{i})=\frac{k\pi}{6}+\frac{5\pi}{6}-\frac{\pi}{2}=\frac{k\pi}{6}+\frac{\pi}{3}$$

Hence: $\frac{k\pi}{6}+\frac{\pi}{3}=\pi\Rightarrow k=4$

18. $2a+b+\mathrm{i}(2-ab)=7-\mathrm{i}\Rightarrow \begin{matrix}2a+b=7\\ 2-ab=-1\end{matrix}$

Substituting $b=7-2a$ (from the first equation) into the second equation:

$2-a(7-2a)=-1\Rightarrow 2a^2-7a+3=0\Rightarrow a_1=\frac{1}{2},\ a_2=3$

Since $a,b\in\mathbb{Z}$, the solution is $a=3,\ b=1$

19. **(a)** See practice question **14 (b)**

(b) Use binomial expansion:

$$\left(z+\frac{1}{z}\right)^4=z^4+4z^3\frac{1}{z}+6z^2\frac{1}{z^2}+4z\frac{1}{z^3}+\frac{1}{z^4}=z^4+4z^2+6+4\frac{1}{z^2}+\frac{1}{z^4}$$
$$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$$

Since $\left(z+\frac{1}{z}\right)^4=\left(2\cos(\theta)\right)^4$, $z^4+\frac{1}{z^4}=2\cos(4\theta)$, $z^2+\frac{1}{z^2}=2\cos(2\theta)$, we have:

$$2^4\cos^4(\theta)=2\cos(4\theta)+4\cdot2\cos(2\theta)+6\Rightarrow 8\cos^4(\theta)=\cos(4\theta)+4\cos(2\theta)+3$$

Hence, $\cos^4(\theta)=\frac{1}{8}\left(\cos(4\theta)+4\cos(2\theta)+3\right)$

20. **(a)** $$z=\frac{\frac{1}{2}\mathrm{e}^{2\mathrm{i}\theta}}{e^{\mathrm{i}\theta}}=\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$$

(b) $|z|=\frac{1}{2}$, so it is less than 1.

(c) Using the formula: $S_\infty=\frac{u_1}{1-r}=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\right)}$

(d) **(i)** $$S_\infty=\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}\left(\cos\theta+\mathrm{i}\sin\theta\right)}$$

(ii) Change to polar form and use result in **(i)**

$$\begin{aligned}S_\infty &= \mathrm{e}^{\mathrm{i}\theta}+\frac{1}{2}\mathrm{e}^{2\mathrm{i}\theta}+\frac{1}{4}\mathrm{e}^{3\mathrm{i}\theta}+\cdots\\ &=\left(\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos 3\theta+\ldots\right)+\mathrm{i}\left(\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{4}\sin 3\theta+\ldots\right)\\ &=\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\end{aligned}$$

Taking the real parts, we have:

$$\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos 3\theta+\ldots=\mathrm{Re}\left(\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\right)$$

Now, evaluate the RHS and simplify:

$$\begin{aligned}\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}&=\frac{\cos\theta+\mathrm{i}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)-\frac{1}{2}\mathrm{i}\sin\theta}\cdot\frac{\left(1-\frac{1}{2}\cos\theta\right)+\frac{1}{2}\mathrm{i}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)+\frac{1}{2}\mathrm{i}\sin\theta}\\ &=\frac{\cos\theta-\frac{1}{2}+\mathrm{i}\sin\theta}{\frac{5}{4}-\cos\theta}\end{aligned}$$

The real part of this number is

$$\frac{\cos\theta-\frac{1}{1}}{\frac{5}{4}-\cos\theta}=\frac{4\cos\theta-2}{5-4\cos\theta}$$

21. **Method I**

If $-3+2\mathrm{i}$ is a root, then $-3-2\mathrm{i}$ is another root; therefore:

$$\begin{aligned}P(z)&=(z+2)(z+3-2\mathrm{i})(z+3+2\mathrm{i})\\ &=(z+2)\left((z+3)^2-(2\mathrm{i})^2\right)=(z+2)\left(z^2+6z+13\right)=z^3+8z^2+25z+26\end{aligned}$$

So, $a=8, b=25, c=26$

Method II

$$P(-2)=0\Rightarrow -8+4a-2b+c=0$$

$$P(-3+2\mathrm{i})=0\Rightarrow 9+46\mathrm{i}+a(5-12\mathrm{i})+b(-3+2\mathrm{i})+c=0$$

$$\Rightarrow(9+5a-3b+c)+(46-12a+2b)\mathrm{i}=0$$

Hence, we have to solve the system of equations:

$$\left.\begin{array}{l} 4a-2b+c=8 \\ 5a-3b+c=-9 \\ -12a+2b=-46 \end{array}\right\} \Rightarrow a=8,\ b=25,\ c=26$$

22. Let $z=a+b\mathrm{i}$:

$$\sqrt{a^2+b^2}=2\sqrt{5}\Rightarrow a^2+b^2=20$$

$$\frac{25}{a+b\mathrm{i}}-\frac{15}{a-b\mathrm{i}}=1-8\mathrm{i}\Rightarrow\frac{25\left(a-b\mathrm{i}\right)-15\left(a+b\mathrm{i}\right)}{\left(a+b\mathrm{i}\right)\left(a-b\mathrm{i}\right)}=1-8\mathrm{i}$$

$$\frac{10a-40bi}{a^2+b^2}=1-8\mathrm{i}\Rightarrow\frac{10a-40bi}{20}=1-8\mathrm{i}$$

$10a-40bi=20-160\mathrm{i}\Rightarrow a=2,\ b=4$ (by equating real and imaginary parts)

Hence, $z=2+4\mathrm{i}$

23. $\begin{cases} \mathrm{i}z_1+2z_2=3 \\ z_1+\left(1-\mathrm{i}\right)z_2=4 \end{cases}$, multiply the second equation by $-\mathrm{i}$ and add the equations:

$$\underline{\begin{cases} \mathrm{i}z_1+2z_2=3 \\ -\mathrm{i}z_1-\left(1+\mathrm{i}\right)z_2=-4\mathrm{i} \end{cases}}$$

$$(2-1-\mathrm{i})z_2=3-4\mathrm{i}\Rightarrow z_2=\frac{3-4\mathrm{i}}{1-\mathrm{i}}=\frac{7}{2}-\frac{1}{2}\mathrm{i}$$

Substituting in the second equation: $z_1+3-4\mathrm{i}=4\Rightarrow z_1=1+4\mathrm{i}$

24. **(a)** $z_{1,2}=\dfrac{4\pm\sqrt{16-32}}{2}=\dfrac{4\pm4\mathrm{i}}{2}=2\pm2\mathrm{i}\Rightarrow z_1=2+2\mathrm{i},\ z_2=2-2\mathrm{i}$

$\left|z_1\right|=\sqrt{4+4}=2\sqrt{2}$, $\tan\theta=1$, in the first quadrant, $\theta=\dfrac{\pi}{4}$; hence, $z_1=2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

$\left|z_2\right|=\sqrt{4+4}=2\sqrt{2}$, $\tan\theta=-1$, in the fourth quadrant, $\theta=-\dfrac{\pi}{4}$; hence,

$z_2=2\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$

(b) $$\frac{z_1^{\,4}}{z_2^{\,2}}=\frac{\left(2\sqrt{2}\right)^4\mathrm{e}^{\mathrm{i}\frac{4\pi}{4}}}{\left(2\sqrt{2}\right)^2\mathrm{e}^{\mathrm{i}\frac{-2\pi}{4}}}=8\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}=8\left(\cos\frac{3\pi}{2}+\mathrm{i}\sin\frac{3\pi}{2}\right)=-8\mathrm{i}$$

(c) $$z_1^{\,4}=\left(2\sqrt{2}\right)^4\mathrm{e}^{\mathrm{i}\frac{4\pi}{4}}=64\mathrm{e}^{\mathrm{i}\pi}=-64$$

$$z_2^{\,4}=\left(2\sqrt{2}\right)^4\mathrm{e}^{\mathrm{i}\frac{-4\pi}{4}}=64\mathrm{e}^{-\mathrm{i}\pi}=-64$$ Thus, they are the same.

(d) $$\frac{z_1}{z_2}+\frac{z_2}{z_1}=\frac{2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}}+\frac{2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}}{2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)}+\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)}=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}+\mathrm{e}^{\mathrm{i}\frac{-\pi}{2}}=\mathrm{i}-\mathrm{i}=0$$

(e) $$z_1^{\,n}=\left(2\sqrt{2}\right)^n\mathrm{e}^{\mathrm{i}\frac{n\pi}{4}}=2^{\frac{3n}{2}}\mathrm{e}^{\mathrm{i}\frac{n\pi}{4}}$$

The number is real if $\frac{n\pi}{4}=k\pi\Rightarrow n=4k,\ k\in\mathbb{Z}$

25. **(a)** $$z^7=\left(\cos\frac{2\pi}{7}+\mathrm{i}\sin\frac{2\pi}{7}\right)^7=\cos\frac{7\cdot 2\pi}{7}+\mathrm{i}\sin\frac{7\cdot 2\pi}{7}=\underbrace{\cos 2\pi}_{=1}+\mathrm{i}\underbrace{\sin 2\pi}_{=0}=1,$$

hence $z^7-1=0$

(b) $$\left(z-1\right)\left(z^6+z^5+z^4+z^3+z^2+z+1\right)$$

$$=z^7+z^6+z^5+z^4+z^3+z^2+z-z^6-z^5-z^4-z^3-z^2-z-1=z^7-1$$

Using the result from **(a)**, we have $0=z^7-1=\left(z-1\right)\left(z^6+z^5+z^4+z^3+z^2+z+1\right)$

Since $z\neq 1$, then $z-1\neq 0$ and hence $z^6+z^5+z^4+z^3+z^2+z+1=0$

(c) Using the result from **(b)**, we have:

$$0=z^6+z^5+z^4+z^3+z^2+z+1=\mathrm{cis}\left(\frac{12\pi}{7}\right)+\mathrm{cis}\left(\frac{10\pi}{7}\right)+\cdots+\mathrm{cis}\left(\frac{2\pi}{7}\right)+1$$

$$\Rightarrow \mathrm{cis}\left(\frac{12\pi}{7}\right)+\mathrm{cis}\left(\frac{10\pi}{7}\right)+\cdots+\mathrm{cis}\left(\frac{2\pi}{7}\right)=-1$$

This implies that $\mathrm{Re}\left(\mathrm{cis}\left(\frac{12\pi}{7}\right)+\mathrm{cis}\left(\frac{10\pi}{7}\right)+\cdots+\mathrm{cis}\left(\frac{2\pi}{7}\right)\right)=-1$

Using the fact that real parts of two equal complex numbers must be equal:

$$\mathrm{Re}\left(\mathrm{cis}\left(\frac{12\pi}{7}\right)+\mathrm{cis}\left(\frac{10\pi}{7}\right)+\cdots+\left(\frac{2\pi}{7}\right)\right)=\cos\left(\frac{12\pi}{7}\right)+\cos\left(\frac{10\pi}{7}\right)+\cdots+\cos\left(\frac{2\pi}{7}\right)$$

But since $\cos\left(\frac{12\pi}{7}\right)=\cos\left(\frac{2\pi}{7}\right),\cos\left(\frac{10\pi}{7}\right)=\cos\left(\frac{4\pi}{7}\right)$, and $\cos\left(\frac{8\pi}{7}\right)=\cos\left(\frac{6\pi}{7}\right)$

we have:

$$\cos\left(\frac{12\pi}{7}\right)+\cos\left(\frac{10\pi}{7}\right)+\cos\left(\frac{8\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)$$

$$=2\left(\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)\right)$$

Finally:

$$2\left(\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)\right)=-1$$

$$\Rightarrow \cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)=-\frac{1}{2}$$

26. **(a)** $27z^3+8=0 \Rightarrow z^3=-\frac{8}{27}=\frac{8}{27}\operatorname{cis}\pi$

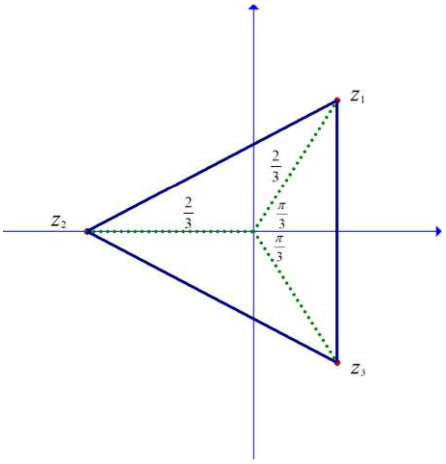
We need to find the cubic roots of $\frac{8}{27}\operatorname{cis}\pi$

$|z|=\sqrt[3]{\frac{8}{27}}=\frac{2}{3}$; $\arg(z)=\frac{\pi}{3}+\frac{2k\pi}{3}$, $k=0, 1, 2$

$z_1=\frac{2}{3}\operatorname{cis}\frac{\pi}{3}$; $z_2=\frac{2}{3}\operatorname{cis}\pi$; $z_3=\frac{2}{3}\operatorname{cis}\frac{5\pi}{3}$

(b) There several ways of finding the area. One is demonstrated in the diagram below. This is a sketch for the three roots in an Argand diagram.

The triangle is made up of three isosceles congruent triangles with a vertex angle of $\frac{2\pi}{3}$ and sides of $\frac{2}{3}$. Thus the area of the triangle is 3 times the area of each. Using the law of sines, we have: $\text{Area}=3\times\left(\frac{1}{2}\times\frac{2}{3}\times\frac{2}{3}\times\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{3}$



27. **(a)** $2+3\mathrm{i}$ must satisfy the equation, thus,

$$(2+3\mathrm{i})^4-4(2+3\mathrm{i})^3+17(2+3\mathrm{i})^2-16(2+3\mathrm{i})+52$$
$$=-119-120\mathrm{i}+184-36\mathrm{i}-85+204\mathrm{i}-32-48\mathrm{i}+52=0$$

(b) If $2+3\mathrm{i}$ is a root, then $2-3\mathrm{i}$ is also a root. This implies that one of factors is $(z-2+3\mathrm{i})(z-2-3\mathrm{i})=z^2-4z+13$

The other factor, z^2+4, can be found either by inspection or long division.

Therefore, the other roots are $\pm 2i$.

28. **(a)** $(a+b\mathrm{i})^2=\mathrm{i}\Rightarrow a^2-b^2+2ab\mathrm{i}=\mathrm{i}\Rightarrow a^2-b^2=0$ and $2ab=1$

Solving the simultaneous equations gives $a=b=\pm\dfrac{1}{\sqrt{2}}\Rightarrow z=\pm\dfrac{1+\mathrm{i}}{\sqrt{2}}$

(b) Using the quadratic equation, $z=\dfrac{3+\mathrm{i}\pm\sqrt{(3+\mathrm{i})^2-4(2+\mathrm{i})}}{2}=\dfrac{3+\mathrm{i}\pm\sqrt{2\mathrm{i}}}{2}$

Using the result from **(a)**

$$z=\frac{3+\mathrm{i}\pm\sqrt{2\mathrm{i}}}{2}=\frac{3+\mathrm{i}\pm\sqrt{2}\times\left(\pm\dfrac{1+\mathrm{i}}{\sqrt{2}}\right)}{2}=\frac{3+\mathrm{i}\pm(1+\mathrm{i})}{2}=2+\mathrm{i}\text{ or }1$$

29. Evaluating $(\cos\theta+\mathrm{i}\sin\theta)^n$ using de Moivre's formula, and using the binomial theorem and equating either real parts or imaginary parts will enable us to find expressions for $\cos n\theta$ or $\sin n\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^4=\cos 4\theta+\mathrm{i}\sin 4\theta$$
$$=\cos^4\theta+4\mathrm{i}\cos^3\theta\sin\theta-6\cos^2\theta\sin^2\theta-4\mathrm{i}\cos\theta\sin^3\theta+\sin^4\theta$$

$\Rightarrow \cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$

$\Rightarrow \sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$

Similarly, for $\cos 5\theta$ we will need the real part of $(\cos\theta+\mathrm{i}\sin\theta)^5$

Thus, $\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

Also, for $\sin 6\theta$ we will need the imaginary part of $(\cos\theta+\mathrm{i}\sin\theta)^6$

Thus, $\sin 6\theta=6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta$

30. $\mathrm{e}^{\ln 2(1+\mathrm{i})}=\mathrm{e}^{\ln 2}\mathrm{e}^{\mathrm{i}\ln 2}=\mathrm{e}^{\ln 2}\left(\cos(\ln 2)+\mathrm{i}\sin(\ln 2)\right)=2\left(\cos(\ln 2)+\mathrm{i}\sin(\ln 2)\right)$

$\Rightarrow$ real part $=2\cos(\ln 2)$ and imaginary part $=2\sin(\ln 2)$

Exercise 9.1

1. **(a)** $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}x_B-x_A\\y_B-y_A\\z_B-z_A\end{pmatrix}=\begin{pmatrix}1-\left(-\frac{3}{2}\right)\\-\frac{5}{2}-\left(-\frac{1}{2}\right)\\1-1\end{pmatrix}=\begin{pmatrix}\frac{5}{2}\\-2\\0\end{pmatrix}$

(b) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}x_B-x_A\\y_B-y_A\\z_B-z_A\end{pmatrix}=\begin{pmatrix}1-(-2)\\\sqrt{3}-(-\sqrt{3})\\-\frac{1}{2}-\left(-\frac{1}{2}\right)\end{pmatrix}=\begin{pmatrix}3\\2\sqrt{3}\\0\end{pmatrix}$

(c) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}x_B-x_A\\y_B-y_A\\z_B-z_A\end{pmatrix}=\begin{pmatrix}1-2\\-1-(-3)\\3-5\end{pmatrix}=\begin{pmatrix}-1\\2\\-2\end{pmatrix}$

(d) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}x_B-x_A\\y_B-y_A\\z_B-z_A\end{pmatrix}=\begin{pmatrix}-a-a\\-2a-(-a)\\a-2a\end{pmatrix}=\begin{pmatrix}-2a\\-a\\-a\end{pmatrix}$

2. **(a)** Given $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}x_Q-x_P\\y_Q-y_P\\z_Q-z_P\end{pmatrix}=\begin{pmatrix}1\\-\frac{5}{2}\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_Q-\left(-\frac{3}{2}\right)\\y_Q-\left(-\frac{1}{2}\right)\\z_Q-1\end{pmatrix}=\begin{pmatrix}1\\-\frac{5}{2}\\1\end{pmatrix}$

$\Rightarrow x_Q-\left(-\frac{3}{2}\right)=1\Rightarrow x_Q=1-\frac{3}{2}=-\frac{1}{2}$; $y_Q-\left(-\frac{1}{2}\right)=-\frac{5}{2}\Rightarrow y_Q=-\frac{5}{2}-\frac{1}{2}=-3$

$z_Q-1=1\Rightarrow z_Q=1+1=2$. So, $Q\left(-\frac{1}{2},-3,2\right)$

(b) Given $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}x_Q-x_P\\y_Q-y_P\\z_Q-z_P\end{pmatrix}=\begin{pmatrix}-\frac{3}{2}\\-\frac{1}{2}\\1\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-x_P \\ -\frac{5}{2}-y_P \\ 1-z_P \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \Rightarrow 1-x_P = -\frac{3}{2} \Rightarrow x_P = 1+\frac{3}{2} = \frac{5}{2}$$

$-\frac{5}{2}-y_P = -\frac{1}{2} \Rightarrow y_P = -\frac{5}{2}+\frac{1}{2} = -2$; $1-z_P = 1 \Rightarrow z_P = 1-1 = 0$. So, $P\left(\frac{5}{2}, -2, 0\right)$

(c) Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \\ z_Q - z_P \end{pmatrix} = \begin{pmatrix} -a \\ -2a \\ a \end{pmatrix}$

$\Rightarrow \left(x_Q - a, y_Q - (-2a), z_Q - 2a\right) = (-1, -2a, a) \Rightarrow x_Q - a = -a \Rightarrow x_Q = 0$;

$y_Q - (-2a) = -2a \Rightarrow y_Q = -2a - 2a = -4a$; $z_Q - 2a = a \Rightarrow z_Q = a + 2a = 3a$.

So, $Q(0, -4a, 3a)$

3. **(a)** For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. If the two vectors are parallel, then one of them is a scalar multiple of the other, for example, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$$\overrightarrow{AM} = \begin{pmatrix} x_M - x_A \\ y_M - y_A \\ z_M - z_A \end{pmatrix} = \begin{pmatrix} x-0 \\ y-0 \\ z-5 \end{pmatrix}; \quad \overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1-0 \\ 0-5 \end{pmatrix}$$

Therefore: $\begin{pmatrix} x \\ y \\ z-5 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z-5 \end{pmatrix} = \begin{pmatrix} t \\ t \\ -5t \end{pmatrix} \Rightarrow x = t, y = t, z-5 = -5t$

So, $M(t, t, 5-5t)$, where $t \in \mathbb{R}$

Note: We can find M if $\overrightarrow{BM} = t\,\overrightarrow{AB}$. Then we have:

$$\overrightarrow{BM} = \begin{pmatrix} x_M - x_B \\ y_M - y_B \\ z_M - z_b \end{pmatrix} = \begin{pmatrix} x-1 \\ y-1 \\ z-0 \end{pmatrix}, \ \overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

Therefore: $\begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ -5t \end{pmatrix} \Rightarrow x = 1+t,\ y = 1+t,\ z = -5t$

So, $M(1+t, 1+t, -5t)$, where $t \in \mathbb{R}$

Both conditions describe the same set of points; for example, we can obtain point $M(0,0,5)$ by putting $t=0$ in $M(t,t,5-5t)$, or $t=-1$ in $M(1+t,1+t,-5t)$; or $M(2,2,-5)$ by putting $t=2$ in $M(t,t,5-5t)$, or $t=1$ in $M(1+t,1+t,-5t)$.

(b) For points *M*, *A* and *B* to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$.

So, let's say, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$$\overrightarrow{AM} = \begin{pmatrix} x_M - x_A \\ y_M - y_A \\ z_M - z_A \end{pmatrix} = \begin{pmatrix} x-(-1) \\ y-0 \\ z-1 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} = \begin{pmatrix} 3-(-1) \\ 5-0 \\ -2-1 \end{pmatrix}$$

Therefore: $\begin{pmatrix} x+1 \\ y \\ z-1 \end{pmatrix} = t\begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x+1 \\ y \\ z-1 \end{pmatrix} = \begin{pmatrix} 4t \\ 5t \\ -3t \end{pmatrix} \Rightarrow x = -1+4t, y = 5t, z = 1-3t$

So, $M(-1+4t, 5t, 1-3t)$, where $t \in \mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will have $\overrightarrow{BM} = \begin{pmatrix} x-3 \\ y-5 \\ z+2 \end{pmatrix}$;

therefore, from $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will find $x = 3+4t, y = 5+5t, z = -2-3t$.

(c) For points *M*, *A* and *B* to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$.

So, let's say, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$$\overrightarrow{AM} = \begin{pmatrix} x_M - x_A \\ y_M - y_A \\ z_M - z_A \end{pmatrix} = \begin{pmatrix} x-2 \\ y-3 \\ z-4 \end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}x_B-x_A\\y_B-y_A\\z_B-z_A\end{pmatrix}=\begin{pmatrix}-2-2\\-3-3\\5-4\end{pmatrix}$$

Therefore: $\begin{pmatrix}x-2\\y-3\\z-4\end{pmatrix}=t\begin{pmatrix}-4\\-6\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x-2\\y-3\\z-4\end{pmatrix}=\begin{pmatrix}-4t\\-6t\\t\end{pmatrix}\Rightarrow x=2-4t, y=3-6t, z=4+t$

So, $M\left(2-4t,3-6t,4+t\right)$, where $t\in\mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM}=t\,\overrightarrow{AB}$, we will have $\overrightarrow{BM}=\begin{pmatrix}x+2\\y+3\\z-5\end{pmatrix}$;

therefore, from $\overrightarrow{BM}=t\,\overrightarrow{AB}$, we will find $x=-2-4t, y=-3-6t, z=5+t$.

4. If A is the midpoint of $[BC]$, then a relationship between them can be $\overrightarrow{BC}=2\overrightarrow{BA}$.
Let $C(x, y, z)$.

(a) $\overrightarrow{BC}=2\overrightarrow{BA}\Rightarrow\begin{pmatrix}x+1\\y\\z-1\end{pmatrix}=2\begin{pmatrix}4\\-4\\-1\end{pmatrix}=\begin{pmatrix}8\\-8\\-2\end{pmatrix}\Rightarrow C\left(7,-8,-1\right)$

(b) $\overrightarrow{BA}=\begin{pmatrix}-1-(-1)\\3-\frac{1}{2}\\5-\frac{1}{3}\end{pmatrix}=\begin{pmatrix}0\\\frac{5}{2}\\\frac{14}{3}\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}x+1\\y-\frac{1}{2}\\z-\frac{1}{3}\end{pmatrix}$. Therefore:

$$\overrightarrow{BC}=\begin{pmatrix}x+1\\y-\frac{1}{2}\\z-\frac{1}{3}\end{pmatrix}=2\overrightarrow{BA}=\begin{pmatrix}0\\5\\\frac{28}{3}\end{pmatrix}$$

$\Rightarrow x=-1,\ y=5+\frac{1}{2}=\frac{11}{2},\ z=\frac{1}{3}+\frac{28}{3}=\frac{29}{3}$;

so, $C\left(-1,\frac{11}{2},\frac{29}{3}\right)$.

(c) $\overrightarrow{BA}=\begin{pmatrix}1-a\\2-2a\\-1-b\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}x-a\\y-2a\\z-b\end{pmatrix}$. Therefore:

$$\overrightarrow{BC}=\begin{pmatrix}x-a\\y-2a\\z-b\end{pmatrix}=2\overrightarrow{BA}=\begin{pmatrix}2-2a\\4-4a\\-2-2b\end{pmatrix};$$

$\Rightarrow x=2-a,\ y=4-2a,\ z=-b-2;$

so, $C(2-a,4-2a,-b-2)$

5. Let $G(x,y,z)$:

(a) $\vec{0}=\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\begin{pmatrix}-1-x\\-1-y\\-1-z\end{pmatrix}+\begin{pmatrix}-1-x\\2-y\\-1-z\end{pmatrix}+\begin{pmatrix}1-x\\2-y\\3-z\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}-1-3x\\3-3y\\1-3z\end{pmatrix}\Rightarrow x=-\frac{1}{3},y=1,z=\frac{1}{3}$$

So, $G\left(-\frac{1}{3},1,\frac{1}{3}\right)$

(b) $\vec{0}=\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\begin{pmatrix}2-x\\-3-y\\1-z\end{pmatrix}+\begin{pmatrix}1-x\\-2-y\\-5-z\end{pmatrix}+\begin{pmatrix}0-x\\0-y\\1-z\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}3-3x\\-5-3y\\-3-3z\end{pmatrix}\Rightarrow x=1,y=-\frac{5}{3},z=-1$$

So, $G\left(1,-\frac{5}{3},-1\right)$

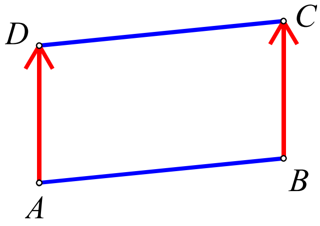
(c) $$\vec{0} = \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \begin{pmatrix} a-x \\ 2a-y \\ 3a-z \end{pmatrix} + \begin{pmatrix} b-x \\ 2b-y \\ 3b-z \end{pmatrix} + \begin{pmatrix} c-x \\ 2c-y \\ 3c-z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a+b+c-3x \\ 2a+2b+2c-3y \\ 3a-3b-3c-3z \end{pmatrix}$$

$$\Rightarrow x = \frac{a+b+c}{3}, y = \frac{2a+2b+2c}{3}, z = a+b+c$$

So, $G\left(\frac{a+b+c}{3}, \frac{2a+2b+2c}{3}, a+b+c\right)$

6. The relationship between points $A, B, C,$ and D of parallelogram $ABCD$ can be expressed using different vector relationships; for example, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$, $\overrightarrow{BA} = \overrightarrow{CD}, \cdots$. Here, we will use $\overrightarrow{AD} = \overrightarrow{BC}$. Let $D(x, y, z)$.



(a) $$\overrightarrow{BC} = \begin{pmatrix} -\sqrt{3}-1 \\ 2-3 \\ -5-0 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}-1 \\ -1 \\ -5 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} x-\sqrt{3} \\ y-2 \\ z-(-1) \end{pmatrix} = \begin{pmatrix} x-\sqrt{3} \\ y-2 \\ z+1 \end{pmatrix}$$

Therefore: $-\sqrt{3}-1 = x-\sqrt{3} \Rightarrow x = -1$, $y-2 = -1 \Rightarrow y = 1$, $z+1 = -5 \Rightarrow z = -6$

So, $D(-1, 1, -6)$

(b) $$\overrightarrow{BC} = \begin{pmatrix} -2\sqrt{2}-3\sqrt{2} \\ \sqrt{3}-\left(-\sqrt{3}\right) \\ -3\sqrt{5}-\sqrt{5} \end{pmatrix} = \begin{pmatrix} -5\sqrt{2} \\ 2\sqrt{3} \\ -4\sqrt{5} \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} x-\sqrt{2} \\ y-\sqrt{3} \\ z-\sqrt{5} \end{pmatrix}$$

Therefore: $-5\sqrt{2} = x-\sqrt{2} \Rightarrow x = -4\sqrt{2}$, $y-\sqrt{3} = 2\sqrt{3} \Rightarrow y = 3\sqrt{3}$, $z-\sqrt{5} = -4\sqrt{5} \Rightarrow z = -3\sqrt{5}$. So, $D\left(-4\sqrt{2}, 3\sqrt{3}, -3\sqrt{5}\right)$

(c) $\overrightarrow{BC}=\begin{pmatrix}\frac{7}{2}-\frac{1}{2}\\ \frac{1}{3}-\frac{2}{3}\\ 1-5\end{pmatrix}=\begin{pmatrix}3\\ -\frac{1}{3}\\ -4\end{pmatrix}$, $\overrightarrow{AD}=\begin{pmatrix}x+\frac{1}{2}\\ y-\frac{1}{3}\\ z-0\end{pmatrix}=\begin{pmatrix}x+\frac{1}{2}\\ y-\frac{1}{3}\\ z\end{pmatrix}$

Therefore: $3=x+\frac{1}{2}\Rightarrow x=\frac{5}{2}$, $y-\frac{1}{3}=-\frac{1}{3}\Rightarrow y=0$, $z=-4$

So, $D\left(\frac{5}{2},0,-4\right)$.

7. Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ have the same direction if, for $t>0$, $\boldsymbol{v}=t\boldsymbol{w}$. Therefore:

$$\begin{pmatrix}m-2\\ m+n\\ -2m+n\end{pmatrix}=t\begin{pmatrix}2\\ 4\\ -6\end{pmatrix}$$

$$\begin{cases}m-2=2t\\ m+n=4t\\ -2m+n=-6t\end{cases}\Rightarrow\begin{cases}m-2t=2\\ m+n-4t=0\\ -2m+n+6t=0\end{cases}$$

Solve this system of equations by any method of your choice
Therefore: $m=5$, $n=1$.

8. **(a)** The length of the vector $\boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ is $\sqrt{2^2+2^2+(-1)^2}=\sqrt{9}=3$, so the unit vector is $\frac{1}{3}(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})=\frac{2}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}-\frac{1}{3}\boldsymbol{k}$.

(b) The length of the vector $\boldsymbol{v}=6\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$ is $\sqrt{6^2+(-4)^2+2^2}=\sqrt{56}=2\sqrt{14}$, so the unit vector is $\frac{1}{2\sqrt{14}}(6\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k})=\frac{3}{\sqrt{14}}\boldsymbol{i}-\frac{2}{\sqrt{14}}\boldsymbol{j}+\frac{1}{\sqrt{14}}\boldsymbol{k}$.

(c) The length of the vector $\boldsymbol{v}=2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ is $\sqrt{2^2+(-1)^2+(-2)^2}=\sqrt{9}=3$, so the unit vector is $\frac{1}{3}(2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})=\frac{2}{3}\boldsymbol{i}-\frac{1}{3}\boldsymbol{j}-\frac{2}{3}\boldsymbol{k}$.

9. **(a)** The length of the vector $\boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ is $\sqrt{2^2+2^2+(-1)^2}=\sqrt{9}=3$, so the unit vector in its direction is $\frac{1}{3}(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})$, and the vector of magnitude 2 is:

$\frac{2}{3}(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})$.

(b) The length of the vector $v = 6\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$ is $\sqrt{6^2 + (-4)^2 + 2^2} = \sqrt{56} = 2\sqrt{14}$, so the unit vector in its direction is $\frac{1}{2\sqrt{14}}(6\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k})$, and the vector of magnitude 4 is:

$$\frac{4}{2\sqrt{14}}(6\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}) = \frac{2}{\sqrt{14}}(6\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}).$$

(c) The length of the vector $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ is $\sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$, so the unit vector in its direction is $\frac{1}{3}(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$, and the vector of magnitude 2 is:

$$\frac{5}{3}(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}).$$

10. **(a)** $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) + (2\boldsymbol{i} + \boldsymbol{j}) = 3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$

$$|\boldsymbol{u} + \boldsymbol{v}| = |3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$$

(b) $|\boldsymbol{u}| + |\boldsymbol{v}| = |\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}| + |2\boldsymbol{i} + \boldsymbol{j}| = \sqrt{1^2 + 3^2 + (-2)^2} + \sqrt{2^2 + 1^2 + 0^2} = \sqrt{14} + \sqrt{5}$

(c) $-3\boldsymbol{u} = -3(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) = -3\boldsymbol{i} - 9\boldsymbol{j} + 6\boldsymbol{k}$ $\qquad 3\boldsymbol{v} = 3(2\boldsymbol{i} + \boldsymbol{j}) = 6\boldsymbol{i} + 3\boldsymbol{j}$

$$|-3\boldsymbol{u}| + |3\boldsymbol{v}| = \sqrt{126} + \sqrt{45} = 3\sqrt{14} + 3\sqrt{5}$$

(d) $\frac{1}{|\boldsymbol{u}|}\boldsymbol{u} = \frac{1}{\sqrt{14}}\boldsymbol{u} = \frac{1}{\sqrt{14}}(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) = \frac{1}{\sqrt{14}}\boldsymbol{i} + \frac{3}{\sqrt{14}}\boldsymbol{j} - \frac{2}{\sqrt{14}}\boldsymbol{k}$

(e) $\left|\frac{1}{|\boldsymbol{u}|}\boldsymbol{u}\right| = \left|\frac{1}{\sqrt{14}}\boldsymbol{i} + \frac{3}{\sqrt{14}}\boldsymbol{j} - \frac{2}{\sqrt{14}}\boldsymbol{k}\right| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2 + \left(\frac{-2}{\sqrt{14}}\right)^2} = \sqrt{\frac{1+9+4}{14}} = 1$

11. **(a)** Using $B(x, y, z)$ for the terminal point and $A(-1, 2, -3)$ for the initial point:

$$\overrightarrow{AB} = \begin{pmatrix} x-(-1) \\ y-2 \\ z-(-3) \end{pmatrix} = \begin{pmatrix} x+1 \\ y-2 \\ z+3 \end{pmatrix}, \ \overrightarrow{AB} = \boldsymbol{w}$$

Therefore: $\begin{pmatrix} x+1 \\ y-2 \\ z+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \Rightarrow x = 3, \ y = 4, \ z = -5$

So, the terminal point is $(3, 4, -5)$

(b) Using $B(x,y,z)$ for the terminal point and $A(-2,1,4)$ for the initial point:

$$\overrightarrow{AB}=\begin{pmatrix}x-(-2)\\y-1\\z-4\end{pmatrix}=\begin{pmatrix}x+2\\y-1\\z-4\end{pmatrix},\ \overrightarrow{AB}=\boldsymbol{v}$$

Therefore: $\begin{pmatrix}x+2\\y-1\\z-4\end{pmatrix}=\begin{pmatrix}2\\-3\\1\end{pmatrix}\Rightarrow x=0,\ y=-2,\ z=5$.

So, the terminal point is $(0,-2,5)$

12. **(a)** A vector opposite in direction and a third the magnitude of $\boldsymbol{u}$ is $-\frac{1}{3}\boldsymbol{u}$.

Therefore: $-\frac{1}{3}\boldsymbol{u}=-\frac{1}{3}\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}1\\-\frac{4}{3}\end{pmatrix}$

(b) A vector in the same direction as $\boldsymbol{w}$ and whose magnitude equals 12 is 12 times a unit vector in the direction of $\boldsymbol{w}$. Therefore, the vector is of form:

$$12\frac{1}{|\boldsymbol{w}|}\boldsymbol{w}=12\frac{1}{\sqrt{4^2+2^2+(-2)^2}}\begin{pmatrix}4\\2\\-2\end{pmatrix}=\frac{12}{\sqrt{24}}\begin{pmatrix}4\\2\\-2\end{pmatrix}=\sqrt{6}\begin{pmatrix}4\\2\\-2\end{pmatrix}$$

(c) If vectors are parallel, then one can be represented as $\boldsymbol{t}$ times the other. Therefore: $x\boldsymbol{i}+y\boldsymbol{j}-2\boldsymbol{k}=t(\boldsymbol{i}-4\boldsymbol{j}+3\boldsymbol{k})$. From the $z-$coordinate, we can find the value of t: $-2=3t\Rightarrow t=-\frac{2}{3}$. So, $x=-\frac{2}{3}\cdot 1=-\frac{2}{3}$ and $y=-\frac{2}{3}\cdot(-4)=\frac{8}{3}$, and the vector is: $-\frac{2}{3}\boldsymbol{i}+\frac{8}{3}\boldsymbol{j}-2\boldsymbol{k}$.

13. Let $\boldsymbol{u}$ be the vector from the vertex A to the midpoint of side BC; so, $u=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}$.

Let $\boldsymbol{v}$ be the vector from the vertex B to the midpoint of side AC;so, $\boldsymbol{v}=\overrightarrow{BA}+\frac{1}{2}\overrightarrow{AC}$.

Let $\boldsymbol{w}$ be the vector from the vertex C to the midpoint of side AB; so, $\boldsymbol{w}=\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}$.

Adding the vectors:

$$\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}+\overrightarrow{BA}+\frac{1}{2}\overrightarrow{AC}+\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}\overrightarrow{BC}+\left(\frac{1}{2}\overrightarrow{AC}+\overrightarrow{CA}\right)+\frac{1}{2}\overrightarrow{AB}$$

$$=\frac{1}{2}\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}\left(\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}\right)=\frac{1}{2}\cdot\vec{0}=\vec{0}$$

14. The length of the vector $\boldsymbol{v}=t\boldsymbol{i}-2t\boldsymbol{j}+3t\boldsymbol{k}$ is $\sqrt{t^2+(-2t)^2+(3t)^2}=\sqrt{14t^2}=|t|\sqrt{14}$

Hence, $|t|\sqrt{14}=1\Rightarrow|t|=\dfrac{1}{\sqrt{14}}\Rightarrow t=\pm\dfrac{\sqrt{14}}{14}$

15. The length of the vector $\boldsymbol{v}=2\boldsymbol{i}-2t\boldsymbol{j}+3t\boldsymbol{k}$ is $\sqrt{2^2+(-2t)^2+(3t)^2}=\sqrt{4+13t^2}$

Hence, $\sqrt{4+13t^2}=1\Rightarrow4+13t^2=1\Rightarrow13t^2=-3$, so there is no solution.

16. The length of the vector $\boldsymbol{v}=0.5\boldsymbol{i}-t\boldsymbol{j}+1.5t\boldsymbol{k}$ is $\sqrt{0.5^2+(-t)^2+(1.5t)^2}=\sqrt{0.25+3.25t^2}$

Hence, $\sqrt{0.25+3.25t^2}=1\Rightarrow0.25+3.25t^2=1\Rightarrow t^2=\dfrac{3}{13}\Rightarrow t=\pm\sqrt{\dfrac{3}{13}}$

17. **(a)** $$\boldsymbol{a}=\begin{pmatrix}8\\0\\0\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}8\\8\\0\end{pmatrix},\boldsymbol{c}=\begin{pmatrix}0\\8\\0\end{pmatrix},\boldsymbol{d}=\begin{pmatrix}0\\0\\8\end{pmatrix},\boldsymbol{e}=\begin{pmatrix}8\\0\\8\end{pmatrix}$$

$$\boldsymbol{f}=\begin{pmatrix}8\\8\\8\end{pmatrix},\boldsymbol{g}=\begin{pmatrix}0\\8\\8\end{pmatrix}$$

(b) $$\boldsymbol{l}=\frac{1}{2}(\boldsymbol{e}+\boldsymbol{f})=\begin{pmatrix}8\\4\\8\end{pmatrix},\boldsymbol{m}=\begin{pmatrix}4\\8\\8\end{pmatrix},\boldsymbol{n}=\begin{pmatrix}8\\8\\4\end{pmatrix}$$

(c) $$\overrightarrow{LM}+\overrightarrow{MN}+\overrightarrow{NL}=(\boldsymbol{m}-\boldsymbol{l})+(\boldsymbol{n}-\boldsymbol{m})+(\boldsymbol{l}-\boldsymbol{n})=\vec{0}$$

Note: We can verify the statement using coordinates:

$$\overrightarrow{LM}+\overrightarrow{MN}+\overrightarrow{NL}=\begin{pmatrix}4-8\\8-4\\8-8\end{pmatrix}+\begin{pmatrix}8-4\\8-8\\4-8\end{pmatrix}+\begin{pmatrix}8-8\\4-8\\8-4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=\vec{0}$$

18. **(a)** $c = \overrightarrow{OE} + \overrightarrow{OA} = \begin{pmatrix} 8 \\ 0 \\ 12 \end{pmatrix}, d = \overrightarrow{OE} + \overrightarrow{OB} = \begin{pmatrix} 0 \\ 10 \\ 12 \end{pmatrix}$

(b) $f = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}, g = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OD}\right) = \begin{pmatrix} 4 \\ 5 \\ 12 \end{pmatrix}$

(c) $\overrightarrow{AG} = \overrightarrow{OG} - \overrightarrow{OA} = \begin{pmatrix} 4-8 \\ 5-0 \\ 12-0 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 12 \end{pmatrix}$

$$\overrightarrow{FD} = \overrightarrow{OD} - \overrightarrow{OF} = \begin{pmatrix} 0-4 \\ 10-5 \\ 12-0 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 12 \end{pmatrix}$$

The vectors are the same because they connect a vertex and the midpoint of the parallel side in a parallelogram.

19. $\left|\alpha \boldsymbol{i} + (\alpha - 1)\boldsymbol{j} + (\alpha + 1)\boldsymbol{k}\right| = \sqrt{\alpha^2 + (\alpha-1)^2 + (\alpha+1)^2} = \sqrt{3\alpha^2 + 2}$

Hence, we have to solve the equation: $\sqrt{3\alpha^2 + 2} = 2 \Rightarrow 3\alpha^2 + 2 = 4 \Rightarrow \alpha = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$

20. We have to solve a vector equation: $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha - \beta - 3\mu \\ \alpha + 3\beta \\ \alpha + 2\beta + \mu \end{pmatrix} \Rightarrow \begin{aligned} \alpha - \beta - 3\mu &= 4 \\ \alpha + 3\beta &= -1 \\ \alpha + 2\beta + \mu &= 1 \end{aligned}$$

A system of three equations which you can solve by a method of your choice

Therefore, $\alpha = \frac{26}{7}, \ \beta = -\frac{11}{7}, \ \mu = \frac{3}{7}$

21. We have to solve a vector equation: $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \alpha + 3\beta \\ 2\beta + \mu \\ \alpha + \mu \end{pmatrix} \Rightarrow \begin{aligned} \alpha + 3\beta &= -1 \\ 2\beta + \mu &= 1 \\ \alpha + \mu &= 5 \end{aligned}$$

Therefore, $\alpha = 2,\ \beta = -1,\ \mu = 3$

22. We have to solve a vector equation: $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha + 3\beta + 4\mu \\ -\alpha - \mu \\ \beta + \mu \end{pmatrix} \Rightarrow \begin{array}{r} \alpha + 3\beta + 4\mu = 2 \\ -\alpha - \mu = 1 \\ \beta + \mu = -1 \end{array}$$

An inconsistent system of equations. Hence, there are no such scalars α, β, μ.

23. **(a)** ***u–v*** and ***u+v*** are diagonals of a parallelogram. So, the parallelogram has diagonals of the same length; hence, it is a rectangle.

(b) $$\boldsymbol{u} - \boldsymbol{v} = \begin{pmatrix} v_1 - u_1 \\ v_2 - u_2 \\ v_3 - u_3 \end{pmatrix} \Rightarrow |\boldsymbol{u} - \boldsymbol{v}| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}$$

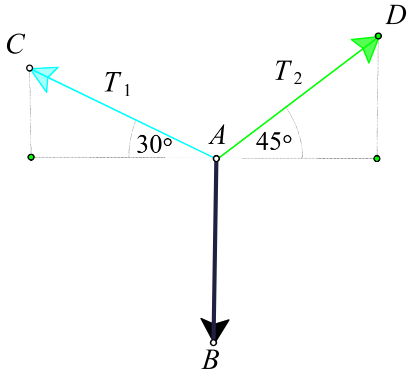
$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{pmatrix} \Rightarrow |\boldsymbol{u} + \boldsymbol{v}| = \sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2 + (v_3 + u_3)^2}$$

Hence,

$$\sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2} = \sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2 + (v_3 + u_3)^2}$$
$$\Rightarrow (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 = (v_1 + u_1)^2 + (v_2 + u_2)^2 + (v_3 + u_3)^2$$
$$\Rightarrow v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2$$
$$= v_1^2 + 2v_1u_1 + u_1^2 + v_2^2 + 2v_2u_2 + u_2^2 + v_3^2 + 2v_3u_3 + u_3^2$$

So: $0 = 4v_1u_1 + 4v_2u_2 + 4v_3u_3 \Rightarrow v_1u_1 + v_2u_2 + v_3u_3 = 0$

24. A summary of the information is shown below:



If the traffic light is in equilibrium: $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}=\vec{0}$

We will express the vectors in component form:

$\overrightarrow{AB}=\begin{pmatrix}0\\-125\end{pmatrix}$, $\overrightarrow{AC}$ is parallel with the unit vector $\begin{pmatrix}-\cos 30^\circ\\ \sin 30^\circ\end{pmatrix}=\begin{pmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$ and its magnitude is T_1; hence, $\overrightarrow{AC}=T_1\begin{pmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$

$\overrightarrow{AD}$ is parallel with the unit vector $\begin{pmatrix}\cos 45^\circ\\ \sin 45^\circ\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$ and its magnitude is T_2; hence,

$$\overrightarrow{AD}=T_2\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$$

Now, we have: $\begin{pmatrix}0\\-125\end{pmatrix}+T_1\begin{pmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}+T_2\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}=\vec{0}\Rightarrow\begin{pmatrix}-T_1\frac{\sqrt{3}}{2}+T_2\frac{\sqrt{2}}{2}\\ -125+T_1\frac{1}{2}+T_2\frac{\sqrt{2}}{2}\end{pmatrix}=\vec{0}$

So, $-T_1\frac{\sqrt{3}}{2}+T_2\frac{\sqrt{2}}{2}=0\Rightarrow -T_1\sqrt{3}+T_2\sqrt{2}\Rightarrow T_1=T_2\frac{\sqrt{2}}{\sqrt{3}}$

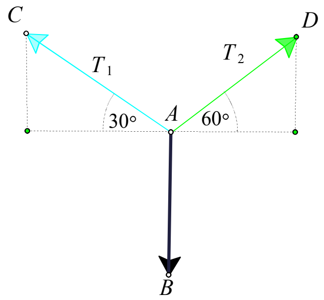
$$-125+T_1\frac{1}{2}+T_2\frac{\sqrt{2}}{2}=0\Rightarrow -125+T_2\frac{\sqrt{2}}{\sqrt{3}}\frac{1}{2}+T_2\frac{\sqrt{2}}{2}=0\Rightarrow$$

$$\Rightarrow -250\sqrt{3}+\sqrt{2}T_2+\sqrt{6}T_2=0\Rightarrow T_2=\frac{250\sqrt{3}}{\sqrt{2}+\sqrt{6}}=\frac{125\left(3\sqrt{2}-\sqrt{6}\right)}{2}$$

And: $T_1=T_2\frac{\sqrt{2}}{\sqrt{3}}=\frac{125\left(3\sqrt{2}-\sqrt{6}\right)}{2}\frac{\sqrt{2}}{\sqrt{3}}=\frac{125\sqrt{2}\sqrt{3}\left(\sqrt{3}-1\right)\sqrt{2}}{2\sqrt{3}}=125\left(\sqrt{3}-1\right)$

Therefore, the cable tensions are $T_1=125\left(\sqrt{3}-1\right)\text{N}$ and $T_2=\frac{125\left(3\sqrt{2}-\sqrt{6}\right)}{2}\text{N}$

25. A summary of the information is shown below:



For vectors it holds: $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}=\vec{0}$

We will express the vectors in component form:

$\overrightarrow{AB}=\begin{pmatrix}0\\-300\end{pmatrix}$, $\overrightarrow{AC}$ is parallel with the unit vector $\begin{pmatrix}-\cos 30^\circ\\ \sin 30^\circ\end{pmatrix}=\begin{pmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$

and its magnitude is T_1; hence, $\overrightarrow{AC}=T_1\begin{pmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$

$\overrightarrow{AD}$ is parallel with the unit vector $\begin{pmatrix}\cos 60^\circ\\ \sin 60^\circ\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$ and its magnitude is T_2.

Hence $\overrightarrow{AD}=T_2\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$

Now, we have: $\begin{pmatrix} 0 \\ -300 \end{pmatrix} + T_1 \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} + T_2 \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} -T_1\frac{\sqrt{3}}{2} + T_2\frac{1}{2} \\ -125 + T_1\frac{1}{2} + T_2\frac{\sqrt{3}}{2} \end{pmatrix} = \vec{0}$

So, $-T_1\frac{\sqrt{3}}{2} + T_2\frac{1}{2} = 0 \Rightarrow -T_1\sqrt{3} + T_2 \Rightarrow T_2 = T_1\sqrt{3}$

$-300 + T_1\frac{1}{2} + T_2\frac{\sqrt{3}}{2} = 0 \Rightarrow -300 + T_1\frac{1}{2} + T_1\sqrt{3}\frac{\sqrt{3}}{2} = 0 \Rightarrow$

$\Rightarrow -300 + 2T_1 = 0 \Rightarrow T_1 = 150$

And: $T_2 = T_1\sqrt{3} = 150\sqrt{3}$

Therefore, the cable tensions are $T_1 = 150$ N and $T_2 = 150\sqrt{3}$ N.

Exercise 9.2

1. **(a)** $\boldsymbol{u} \times \boldsymbol{v} = 3 \cdot 2 + (-2)\cdot(-1) + 4\cdot(-6) = -16$

$$\cos\theta = \frac{-16}{\sqrt{3^2 + (-2)^2 + 4^2}\sqrt{2^2 + (-1)^2 + (-6)^2}} = \frac{-16}{\sqrt{29}\sqrt{41}} \Rightarrow \theta \approx 117.65^\circ$$

(b) $\boldsymbol{u} \cdot \boldsymbol{v} = 2\cdot(-1) + (-6)\cdot 3 + 0 \cdot 5 = -20$

$$\cos\theta = \frac{-20}{\sqrt{2^2 + (-6)^2 + 0^2}\sqrt{(-1)^2 + (3)^2 + 5^2}} = \frac{-20}{\sqrt{40}\sqrt{35}} \Rightarrow \theta \approx 122.31^\circ$$

(c) $\boldsymbol{u} \cdot \boldsymbol{v} = 3 \cdot 5 + (-1)\cdot 2 = 13$

$$\cos\theta = \frac{13}{\sqrt{3^2 + (-1)^2}\sqrt{5^2 + 2^2}} = \frac{13}{\sqrt{10}\sqrt{29}} \Rightarrow \theta \approx 40.24^\circ$$

(d) $\boldsymbol{u} \cdot \boldsymbol{v} = 1 \cdot 0 + (-3)\cdot 5 + 0 \cdot 2 = -15$

$$\cos\theta = \frac{-15}{\sqrt{1^2 + (-3)^2 + 0^2}\sqrt{0^2 + 5^2 + (-2)^2}} = \frac{-15}{\sqrt{10}\sqrt{29}} \Rightarrow \theta \approx 151.74^\circ$$

2. **(a)** $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}||\boldsymbol{v}|\cos\theta=3\cdot4\cdot\cos\frac{\pi}{3}=3\cdot4\cdot\frac{1}{2}=6$

(b) $\boldsymbol{u}\cdot\boldsymbol{v}=|\boldsymbol{u}||\boldsymbol{v}|\cos\theta=3\cdot4\cdot\cos\frac{2\pi}{3}=3\cdot4\cdot\frac{-1}{2}=-6$

3. **(a)** $\boldsymbol{u}\cdot\boldsymbol{v}=2\cdot(-1)+(-6)\cdot3+4\cdot5=0$. The dot product is zero; hence, the vectors are orthogonal.

(b) $\boldsymbol{u}\cdot\boldsymbol{v}=3\cdot5+(-7)\cdot2=1$. The dot product is positive; hence, the angle is acute.

(c) $\boldsymbol{u}\cdot\boldsymbol{v}=1\cdot0+(-3)\cdot6+6\cdot3=0$. The dot product is zero; hence, the vectors are orthogonal.

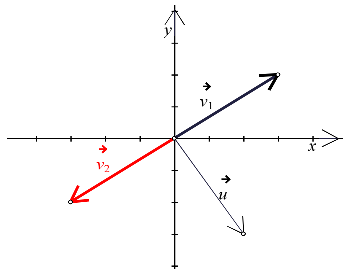
4. **(a)** $\boldsymbol{v}\cdot\boldsymbol{u}=-y\cdot x+x\cdot y=0$. The dot product is zero; hence, $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u}$.

$\boldsymbol{w}\cdot\boldsymbol{u}=y\cdot x+(-x)\cdot y=0$. The dot product is zero; hence, $\boldsymbol{w}$ is orthogonal to $\boldsymbol{u}$.

Pay attention to the relationship between the coordinates of a two-dimensional vector and a vector that is perpendicular to it.

(b) The vectors perpendicular to $\boldsymbol{u}=2\boldsymbol{i}-3\boldsymbol{j}$ are $3\boldsymbol{i}+2\boldsymbol{j}$ and $-3\boldsymbol{i}-2\boldsymbol{j}$.

Unit vectors in the direction of those vectors are: $\boldsymbol{v}_1=\frac{1}{\sqrt{13}}(3\boldsymbol{i}+2\boldsymbol{j})$ and $\boldsymbol{v}_2=-\frac{1}{\sqrt{13}}(3\boldsymbol{i}+2\boldsymbol{j})$.



5. **(a)** **(i)** $|\boldsymbol{v}|=\sqrt{2^2+(-3)^2+1^2}=\sqrt{14}$

$\boldsymbol{i}\cdot\boldsymbol{v}=1\cdot2+0\cdot(-3)+0\cdot1=2\Rightarrow\cos\alpha=\frac{2}{\sqrt{14}}$

$\boldsymbol{j}\cdot\boldsymbol{v}=0\cdot2+1\cdot(-3)+0\cdot1=-3\Rightarrow\cos\beta=\frac{-3}{\sqrt{14}}$

$\boldsymbol{k}\cdot\boldsymbol{v}=0\cdot2+0\cdot(-3)+1\cdot1=1\Rightarrow\cos\gamma=\frac{1}{\sqrt{14}}$

(ii) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{2}{\sqrt{14}}\right)^2+\left(\frac{-3}{\sqrt{14}}\right)^2+\left(\frac{1}{\sqrt{14}}\right)^2=\frac{4+9+1}{14}=1$

(iii) $\alpha=\cos^{-1}\left(\frac{2}{\sqrt{14}}\right)=57.6884...^\circ\approx58^\circ$

$\beta=\cos^{-1}\left(\frac{-3}{\sqrt{14}}\right)=143.30077...^\circ\approx143^\circ$

$\gamma=\cos^{-1}\left(\frac{1}{\sqrt{14}}\right)=74.4986...^\circ\approx74^\circ$

(b) **(i)** $|\boldsymbol{v}|=\sqrt{1^2+(-2)^2+1^2}=\sqrt{6}$

$\boldsymbol{i}\cdot\boldsymbol{v}=1\cdot1+0\cdot(-2)+0\cdot1=1\Rightarrow\cos\alpha=\frac{1}{\sqrt{6}}$

$\boldsymbol{j}\cdot\boldsymbol{v}=0\cdot1+1\cdot(-2)+0\cdot1=-2\Rightarrow\cos\beta=\frac{-2}{\sqrt{6}}$

$\boldsymbol{k}\cdot\boldsymbol{v}=0\cdot1+0\cdot(-2)+1\cdot1=1\Rightarrow\cos\gamma=\frac{1}{\sqrt{6}}$

(ii) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{1}{\sqrt{6}}\right)^2+\left(\frac{-2}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2=\frac{1+4+1}{6}=1$

(iii) $\alpha=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)=65.9051...^\circ\approx66^\circ$, $\beta=\cos^{-1}\left(\frac{-2}{\sqrt{6}}\right)=144.7356...^\circ\approx145^\circ$

$\gamma=\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)=65.9051...^\circ\approx66^\circ$

(c) **(i)** $|\boldsymbol{v}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{14}$

$\boldsymbol{i}\cdot\boldsymbol{v}=1\cdot3+0\cdot(-2)+0\cdot1=3\Rightarrow\cos\alpha=\frac{3}{\sqrt{14}}$

$\boldsymbol{j}\cdot\boldsymbol{v}=0\cdot3+1\cdot(-2)+0\cdot1=-2\Rightarrow\cos\beta=\frac{-2}{\sqrt{14}}$

$\boldsymbol{k}\cdot\boldsymbol{v}=0\cdot3+0\cdot(-2)+1\cdot1=1\Rightarrow\cos\gamma=\frac{1}{\sqrt{14}}$

(ii) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-2}{\sqrt{14}}\right)^2+\left(\frac{1}{\sqrt{14}}\right)^2=\frac{9+4+1}{14}=1$

(iii) $\alpha = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.6992...^{\circ} \approx 37^{\circ}$

$\beta = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right) = 122.3115...^{\circ} \approx 122^{\circ}$

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.4986...^{\circ} \approx 74^{\circ}$

(d) **(i)** $|\boldsymbol{v}| = \sqrt{3^2 + 0^2 + (-4)^2} = 5$

$\boldsymbol{i} \cdot \boldsymbol{v} = 1 \cdot 3 + 0 \cdot 0 + 0 \cdot (-4) = 3 \Rightarrow \cos\alpha = \frac{3}{5}$

$\boldsymbol{j} \cdot \boldsymbol{v} = 0 \cdot 3 + 1 \cdot 0 + 0 \cdot (-4) = 0 \Rightarrow \cos\beta = 0$

$\boldsymbol{k} \cdot \boldsymbol{v} = 0 \cdot 3 + 0 \cdot 0 + 1 \cdot (-4) = -4 \Rightarrow \cos\gamma = \frac{-4}{5}$

(ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{3}{5}\right)^2 + 0^2 + \left(\frac{-4}{5}\right)^2 = \frac{9+16}{25} = 1$

(iii) $\alpha = \cos^{-1}\left(\frac{3}{5}\right) = 53.1301...^{\circ} \approx 53^{\circ}$, $\beta = \cos^{-1} 0 = 90^{\circ}$, $\gamma = \cos^{-1}\left(\frac{-4}{5}\right) = 143.1301...^{\circ} \approx 143^{\circ}$

6. $\begin{pmatrix} \cos\frac{\pi}{3} \\ \cos\frac{\pi}{4} \\ \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix}$

7. $3\begin{pmatrix} \cos\frac{\pi}{4} \\ \cos\frac{\pi}{4} \\ \cos\frac{\pi}{2} \end{pmatrix} = 3\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \\ 0 \end{pmatrix}$

8. **(a)** $\boldsymbol{u} \cdot \boldsymbol{v} = 3 \cdot (m-2) + 5 \cdot (m+3) + 0 \cdot 0 = 8m + 9$

Vectors are perpendicular if their dot product is zero; therefore:

$8m + 9 = 0 \Rightarrow m = -\frac{9}{8}$

(b) $\boldsymbol{u}\cdot\boldsymbol{v}=(2m)\cdot(m-1)+(m-1)\cdot m+(m+1)\cdot(m-1)=4m^2-3m-1$

Vectors are perpendicular if their dot product is zero; therefore:

$4m^2-3m-1=0\Rightarrow m=1,\ m=-\frac{1}{4}$

9. $\boldsymbol{u}\cdot\boldsymbol{w}=\boldsymbol{u}\cdot(\boldsymbol{u}+m\boldsymbol{v})=\boldsymbol{u}\cdot\boldsymbol{u}+m\boldsymbol{u}\cdot\boldsymbol{v}=(-3)^2+1^2+2^2+m((-3)\cdot1+1\cdot2+2\cdot1)=14+m$

Vectors are orthogonal if their dot product is zero; therefore:
$14+m=0\Rightarrow m=-14$

10. **(a)** $\cos\theta=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}=\frac{(-2)\cdot6+5\cdot(-3)+4\cdot0}{\sqrt{(-2)^2+5^2+4^2}\sqrt{6^2+(-3)^2+0^2}}=\frac{-27}{\sqrt{45}\sqrt{45}}=-\frac{27}{45}$

$\Rightarrow\theta=\cos^{-1}\left(-\frac{27}{45}\right)=126.8698...^\circ\approx127^\circ$

(b) $\boldsymbol{u}+\boldsymbol{v}=(-2,5,4)+(6,-3,0)=(4,2,4)$

$\cos\theta=\frac{\boldsymbol{u}\cdot(\boldsymbol{u}+\boldsymbol{v})}{|\boldsymbol{u}||\boldsymbol{u}+\boldsymbol{v}|}=\frac{(-2)\cdot4+5\cdot2+4\cdot4}{\sqrt{45}\sqrt{4^2+2^2+4^2}}=\frac{18}{\sqrt{45}\sqrt{36}}=\frac{1}{\sqrt{5}}$

$\Rightarrow\theta=\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)=63.4349...^\circ\approx63^\circ$

(c) $\cos\theta=\frac{\boldsymbol{v}\cdot(\boldsymbol{u}+\boldsymbol{v})}{|\boldsymbol{v}||\boldsymbol{u}+\boldsymbol{v}|}=\frac{6\cdot4+(-3)\cdot2+0\cdot4}{\sqrt{45}\cdot6}=\frac{18}{\sqrt{45}\cdot6}=\frac{1}{\sqrt{5}}$

$\Rightarrow\theta=\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)=63.4349...^\circ\approx63^\circ$

11. $\overrightarrow{AB}=(3-1,5-2,-2-(-3))=(2,3,1)$

$\overrightarrow{AC}=(m-1,1-2,-10m-(-3))=(m-1,-1,-10m+3)$

(a) The points A, B and C are collinear if $\overrightarrow{AC}$ is parallel to $\overrightarrow{AB}$. Given this collinearity, $\overrightarrow{AC}=t\,\overrightarrow{AB}\Rightarrow(2,3,1)=t(m-1,-1,-10m+3)$. From the second set of coordinates we can determine *t*: $3=-t\Rightarrow t=-3$. Hence:

$2=-3\cdot(m-1)\Rightarrow m=\frac{1}{3}$. Checking with the third set of coordinates:

$1=-3\left(-10\cdot\frac{1}{3}+3\right)\Rightarrow1=-3\cdot\frac{-1}{3}$; so, it fits and for $m=\frac{1}{3}$ the points are collinear.

(b) $\overrightarrow{AC}\cdot\overrightarrow{AB} \Rightarrow 2\cdot(m-1)+3\cdot(-1)+1\cdot(-10m+3)=-8m-2$

Vectors are perpendicular if their dot product is zero; therefore:

$\overrightarrow{AC}\cdot\overrightarrow{AB}=0 \Rightarrow -8m-2=0 \Rightarrow m=-\frac{1}{4}$

12. The vector equation of the line is an equation of the form: $\boldsymbol{r}=\boldsymbol{r}_0+t\boldsymbol{v}$, where $\boldsymbol{r}_0$ is the position vector of any point on the line and the direction vector $\boldsymbol{v}$ is a vector parallel to the line.

For the median through A, we can take the position vector of point A for $\boldsymbol{r}_0$ and the vector from A to the midpoint of BC for $\boldsymbol{v}$. So:

$$\boldsymbol{r}_0=\begin{pmatrix}4\\-2\\-1\end{pmatrix},\quad m_{BC}=\left(\frac{3+3}{2},\frac{-5+1}{2},\frac{-1+2}{2}\right)=\left(3,-2,\frac{1}{2}\right)$$

$$\boldsymbol{v}=\begin{pmatrix}3-4\\-2-(-2)\\\frac{1}{2}-(-1)\end{pmatrix}=\begin{pmatrix}-1\\0\\\frac{3}{2}\end{pmatrix}$$. Therefore:

$$m_A:\ \boldsymbol{r}=\begin{pmatrix}4\\-2\\-1\end{pmatrix}+m\begin{pmatrix}-1\\0\\\frac{3}{2}\end{pmatrix}$$

For the median through B, we can take the position vector of point B for $\boldsymbol{r}_0$ and the vector from B to the midpoint of AC for $\boldsymbol{v}$. So:

$$\boldsymbol{r}_0=\begin{pmatrix}3\\-5\\-1\end{pmatrix},\quad m_{AC}=\left(\frac{4+3}{2},\frac{-2+1}{2},\frac{-1+2}{2}\right)=\left(\frac{7}{2},-\frac{1}{2},\frac{1}{2}\right)$$

$$\boldsymbol{v}=\begin{pmatrix}\frac{7}{2}-3\\-\frac{1}{2}-(-5)\\\frac{1}{2}-(-1)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\\frac{9}{2}\\\frac{3}{2}\end{pmatrix}$$

Therefore, $m_B:\ \boldsymbol{r}=\begin{pmatrix}3\\-5\\-1\end{pmatrix}+n\begin{pmatrix}\frac{1}{2}\\\frac{9}{2}\\\frac{3}{2}\end{pmatrix}$

For the median through C, we can take the position vector of point C for $\boldsymbol{r}_0$ and the vector from C to the midpoint of AB for $\boldsymbol{v}$. So:

$\boldsymbol{r}_0=(3,1,2)$, $m_{AB}=\left(\frac{4+3}{2},\frac{-2+(-5)}{2},\frac{-1+(-1)}{2}\right)=\left(\frac{7}{2},-\frac{7}{2},-1\right)$

$$\boldsymbol{v}=\begin{pmatrix}\frac{7}{2}-3\\-\frac{7}{2}-1\\-1-2\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-\frac{9}{2}\\-3\end{pmatrix}$$

Therefore, $m_C:\ \boldsymbol{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+k\begin{pmatrix}\frac{1}{2}\\-\frac{9}{2}\\-3\end{pmatrix}$

The centroid is the point where all the medians meet. We will find the intersection of two lines, and then check that this point is also on the third line.

If m_A and m_B intersect, then:

$\begin{pmatrix}4\\-2\\-1\end{pmatrix}+m\begin{pmatrix}-1\\0\\\frac{3}{2}\end{pmatrix}=\begin{pmatrix}3\\-5\\-1\end{pmatrix}+n\begin{pmatrix}\frac{1}{2}\\\frac{9}{2}\\\frac{3}{2}\end{pmatrix}$. Therefore, we have:

$$\left\{\begin{array}{l} 4-m=3+\frac{1}{2}n \\ \left.\begin{array}{l} -2+0m=-5+\frac{9}{2}n\Rightarrow n=\frac{2}{3} \\ -1+\frac{3}{2}m=-1+\frac{3}{2}n \end{array}\right\}\Rightarrow m=\frac{2}{3} \end{array}\right.$$

Putting $m=n=\frac{2}{3}$ we can see that it fits the equation, so the point of intersection of m_A and m_B is: $(4,-2,-1)+\frac{2}{3}\left(-1,0,\frac{3}{2}\right)=\left(\frac{10}{3},-2,0\right)$. Now, we have to check that $\left(\frac{10}{3},-2,0\right)$ is on the third line as well: $\left(\frac{10}{3},-2,0\right)=(3,1,2)+\frac{2}{3}\cdot\left(\frac{1}{2},-\frac{9}{2},-3\right)$. Hence, we can see that the centroid is $\left(\frac{10}{3},-2,0\right)$.

Note: For the centroid, it holds that: $\left(\frac{10}{3},-2,0\right)=\left(\frac{4+3+3}{3},\frac{-2-5+1}{3},\frac{-1-1+2}{3}\right)$.

The formula $\left(\frac{x_A+x_B+x_C}{3},\frac{y_A+y_B+y_C}{3},\frac{z_A+z_B+z_C}{3}\right)$ holds in general.

13. **(a)** $\overrightarrow{AB}=\begin{pmatrix}-3-1\\2-2\\1-3\end{pmatrix}=\begin{pmatrix}-4\\0\\-2\end{pmatrix}$; $\left|\overrightarrow{AB}\right|=\sqrt{20}$; $\overrightarrow{AC}=\begin{pmatrix}1-1\\-4-2\\3-3\end{pmatrix}=\begin{pmatrix}0\\-6\\0\end{pmatrix}$; $\left|\overrightarrow{AC}\right|=6$

$\overrightarrow{AD}=\begin{pmatrix}3-1\\2-2\\-3-3\end{pmatrix}=\begin{pmatrix}2\\0\\-6\end{pmatrix}$; $\left|\overrightarrow{AD}\right|=\sqrt{40}$; $\overrightarrow{BC}=\begin{pmatrix}1-(-3)\\-4-2\\3-1\end{pmatrix}=\begin{pmatrix}4\\-6\\2\end{pmatrix}$; $\left|\overrightarrow{BC}\right|=\sqrt{56}$

$\overrightarrow{BD}=\begin{pmatrix}3-(-3)\\2-2\\-3-1\end{pmatrix}=\begin{pmatrix}6\\0\\-4\end{pmatrix}$; $\left|\overrightarrow{BD}\right|=\sqrt{52}$; $\overrightarrow{CD}=\begin{pmatrix}3-1\\2-(-4)\\-3-3\end{pmatrix}=\begin{pmatrix}2\\6\\-6\end{pmatrix}$; $\left|\overrightarrow{CD}\right|=\sqrt{76}$

We will calculate the angles by finding the dot product and using the cosine angle formula:

$\overrightarrow{AB}\cdot\overrightarrow{AC}=0\Rightarrow \text{angle}=90^{\circ}$

$\overrightarrow{AB}\cdot\overrightarrow{AD}=-4\Rightarrow \cos^{-1}\left(\frac{-4}{\sqrt{20}\sqrt{40}}\right)=98.1301^{\circ}\Rightarrow \text{angle}=180^{\circ}-98.1301^{\circ}\approx 82^{\circ}$

$$\overrightarrow{AB}\cdot\overrightarrow{BD}=-16\Rightarrow\cos^{-1}\left(\frac{-16}{\sqrt{20}\sqrt{52}}\right)=119.7448^{\circ}\Rightarrow\text{angle}=180^{\circ}-119.7448^{\circ}\approx 60^{\circ}$$

$$\overrightarrow{AB}\cdot\overrightarrow{BC}=-20\Rightarrow\cos^{-1}\left(\frac{-20}{\sqrt{20}\sqrt{56}}\right)=126.6992^{\circ}\Rightarrow\text{angle}=180^{\circ}-126.6992^{\circ}\approx 53^{\circ}$$

$$\overrightarrow{AC}\cdot\overrightarrow{AD}=0\Rightarrow\text{angle}=90^{\circ}$$

$$\overrightarrow{AC}\cdot\overrightarrow{BC}=36\Rightarrow\cos^{-1}\left(\frac{36}{6\sqrt{56}}\right)=36.6992^{\circ}\Rightarrow\text{angle}\approx 37^{\circ}$$

$$\overrightarrow{AC}\cdot\overrightarrow{CD}=-36\Rightarrow\cos^{-1}\left(\frac{-36}{6\sqrt{76}}\right)=133.4915^{\circ}\Rightarrow\text{angle}=180^{\circ}-133.4915^{\circ}\approx 47^{\circ}$$

$$\overrightarrow{AD}\cdot\overrightarrow{BD}=36\Rightarrow\cos^{-1}\left(\frac{36}{\sqrt{40}\sqrt{52}}\right)=37.8749^{\circ}\Rightarrow\text{angle}\approx 38^{\circ}$$

$$\overrightarrow{AD}\cdot\overrightarrow{CD}=40\Rightarrow\cos^{-1}\left(\frac{40}{\sqrt{40}\sqrt{76}}\right)=43.4915^{\circ}\Rightarrow\text{angle}\approx 43^{\circ}$$

$$\overrightarrow{BC}\cdot\overrightarrow{BD}=16\Rightarrow\cos^{-1}\left(\frac{16}{\sqrt{56}\sqrt{52}}\right)=72.7525^{\circ}\Rightarrow\text{angle}\approx 73^{\circ}$$

$$\overrightarrow{BC}\cdot\overrightarrow{CD}=-40\Rightarrow\cos^{-1}\left(\frac{-40}{\sqrt{56}\sqrt{76}}\right)=127.8168^{\circ}\Rightarrow\text{angle}=180^{\circ}-127.8168^{\circ}\approx 52^{\circ}$$

$$\overrightarrow{BD}\cdot\overrightarrow{CD}=36\Rightarrow\cos^{-1}\left(\frac{36}{\sqrt{52}\sqrt{76}}\right)=55.0643^{\circ}\Rightarrow\text{angle}\approx 55^{\circ}$$

(b) We will use the law of sines for this part.

$$\Delta ABC=\frac{1}{2}AB\cdot AC\cdot\sin B\hat{A}C=\frac{1}{2}\sqrt{20}\cdot 6\cdot 1=3\sqrt{20}\approx 13.416$$

$$\Delta ABD=\frac{1}{2}AB\cdot AD\cdot\sin D\hat{A}B=\frac{1}{2}\sqrt{20}\cdot\sqrt{40}\cdot\sin 82\approx 14.005$$

$$\Delta ACD=\frac{1}{2}AC\cdot AD\cdot\sin D\hat{A}C=\frac{1}{2}6\cdot\sqrt{40}\cdot 1\approx 18.974$$

$$\Delta CBD=\frac{1}{2}BC\cdot BD\cdot\sin C\hat{B}D=\frac{1}{2}\sqrt{56}\cdot\sqrt{52}\cdot\sin 73\approx 25.803$$

Thus, the surface area = 72.2

(c) $\overrightarrow{DC}=\begin{pmatrix}-2\\-6\\6\end{pmatrix}\Rightarrow\left|\overrightarrow{DC}\right|=\sqrt{76}$; Thus, the angles it makes with the axes are:

$$\cos^{-1}\left(\frac{-2}{\sqrt{76}}\right)\approx 103.26°,\cos^{-1}\left(\frac{-6}{\sqrt{76}}\right)\approx 133.49°,\cos^{-1}\left(\frac{6}{\sqrt{76}}\right)\approx 46.51°$$

(d) Since $\overrightarrow{DA}-\overrightarrow{DB}=\overrightarrow{BA}$ and we already showed that $\overrightarrow{AB}\cdot\overrightarrow{AC}=0$, thus $\left(\overrightarrow{DA}-\overrightarrow{DB}\right)\cdot\ \overrightarrow{AC}=0$. Alternatively, you can use coordinates of the corresponding vectors.

$$\overrightarrow{DA}-\overrightarrow{DB}=\begin{pmatrix}-2\\0\\6\end{pmatrix}-\begin{pmatrix}-6\\0\\4\end{pmatrix}=\begin{pmatrix}4\\0\\2\end{pmatrix}\text{ and }\overrightarrow{AC}=\begin{pmatrix}0\\-6\\0\end{pmatrix}\Rightarrow\left(\overrightarrow{DA}-\overrightarrow{DB}\right)\cdot\overrightarrow{AC}=\begin{pmatrix}4\\0\\2\end{pmatrix}\cdot\begin{pmatrix}0\\-6\\0\end{pmatrix}=0$$

14. $$\cos\theta=\frac{\begin{pmatrix}3\\-k\\-1\end{pmatrix}\begin{pmatrix}1\\-3\\k\end{pmatrix}}{\sqrt{9+k^2+1}\sqrt{1+9+k^2}}=\frac{3+3k-k}{k^2+10}=\cos\frac{\pi}{3}=\frac{1}{2}$$

Therefore: $\frac{1}{2}=\frac{3+2k}{k^2+10}\Rightarrow k^2+10=6+4k\Rightarrow k^2-4k+4=0\Rightarrow k=2$

15. $$\cos\theta=\frac{\begin{pmatrix}k\\1\\1\end{pmatrix}\begin{pmatrix}1\\k\\1\end{pmatrix}}{\sqrt{k^2+1+1}\sqrt{1+k^2+1}}=\frac{k+k+1}{k^2+2}=\cos\frac{\pi}{3}=\frac{1}{2}$$

Therefore: $\frac{1}{2}=\frac{1+2k}{k^2+2}\Rightarrow k^2+2=2+4k\Rightarrow k^2-4k=0\Rightarrow k=0,k=4$

So, $k=0$ or $k=4$.

16. $$\begin{pmatrix}2\\x\\y\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=6+x-y=0;\ \begin{pmatrix}2\\x\\y\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\2\end{pmatrix}=8-x+2y=0$$

Hence, we have to solve the system of equations:

$$\begin{cases}6+x-y=0\\8-x+2y=0\end{cases}$$

Adding the equations: $14+y=0\Rightarrow y=-14$ and $x=-20$

17. Two vectors are parallel if one of them is a scalar multiple of the other, $\boldsymbol{u}=t\boldsymbol{v}$. If one vector is a scalar multiple of another, then its components are also multiples of the components of the other vector. Put differently, the components of the two vectors are proportional.

$\frac{1-x}{2-x}=\frac{2x-2}{1+x}=\frac{3+x}{1+x}$;

Take the first equality and solve for x:

$\frac{1-x}{2-x}=\frac{2x-2}{1+x}\Rightarrow x^2-6x+5=0\Rightarrow x=5 \text{ or } x=1.$

Substituting each value into the equalities, we get only $x = 5$ to satisfy both of them.

18. **(a)** $A\hat{B}C$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=\begin{pmatrix}2\\3\\1\end{pmatrix}-\begin{pmatrix}3\\5\\4\end{pmatrix}=\begin{pmatrix}-1\\-2\\-3\end{pmatrix}$$

$$\cos A\hat{B}C=\frac{\begin{pmatrix}-1\\-2\\-3\end{pmatrix}\begin{pmatrix}-1\\4\\0\end{pmatrix}}{\sqrt{1+4+9}\sqrt{1+16}}=\frac{1-8+0}{\sqrt{14\cdot 17}}\Rightarrow A\hat{B}C=\cos^{-1}\frac{-7}{\sqrt{14\cdot 17}}\approx 117^{\circ}$$

(b) $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{BA}+\overrightarrow{BC}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}-1\\4\\0\end{pmatrix}=\begin{pmatrix}0\\6\\3\end{pmatrix}$

$B\hat{A}C$ is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\cos B\hat{A}C=\frac{\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}0\\6\\3\end{pmatrix}}{\sqrt{1+4+9}\sqrt{36+9}}=\frac{0+12+9}{\sqrt{14\cdot 45}}\Rightarrow B\hat{A}C=\cos^{-1}\frac{21}{\sqrt{14\cdot 45}}\approx 33^{\circ}$$

19. **(a)** Vectors are orthogonal if their dot product is zero; therefore:

$$\begin{pmatrix}b\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\b\\1\end{pmatrix}=b+3b+2=4b+2=0\Rightarrow b=-\frac{1}{2}$$

(b) $\begin{pmatrix}4\\-2\\7\end{pmatrix}\cdot\begin{pmatrix}b^2\\b\\0\end{pmatrix}=4b^2-2b+7\cdot 0=4b^2-2b=0\Rightarrow b=0,\ \frac{1}{2}$

$4b^2-2b=0\Rightarrow b=0,\ \frac{1}{2}$

For $b=0$, the vector $\begin{pmatrix} b^2 \\ b \\ 0 \end{pmatrix}$ is a zero vector. A zero vector has no direction; therefore, it is not orthogonal to any vector. Hence, the vectors are only orthogonal for $b=\frac{1}{2}$.

(c) Similar to the two questions above:

$2b^2-11b+15=0 \Rightarrow b=\frac{5}{2}$ or $b=3$

(d) Similarly, $12+20-2b^2=0 \Rightarrow b^2=16 \Rightarrow b=\pm 4$

20. To determine the angle between two vectors, we are going to find their dot product:

$(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{p}-\boldsymbol{q})=\boldsymbol{p}^2-\boldsymbol{q}^2$.

Since, for any vector: $\boldsymbol{v}^2=|\boldsymbol{v}||\boldsymbol{v}|\cos 0^\circ=|\boldsymbol{v}|^2\cdot 1=|\boldsymbol{v}|^2$ we have:

$(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{p}-\boldsymbol{q})=\boldsymbol{p}^2-\boldsymbol{q}^2=|\boldsymbol{p}|^2-|\boldsymbol{q}|^2=0$; therefore, the vectors are perpendicular.

21. We can find the *z*-component by transforming 300 m/min into km/h:

$300\text{ m/min}=0.3\text{ km/min}=0.3\times 60\text{ km/h}=18\text{ km/h}$

Since the heading is $45°$ northwest, the velocity vector in the *xy*-plane is parallel to the vector $(-1,1)$. The unit vector in this direction is $\frac{1}{\sqrt{2}}(-1,1)$. Since the airspeed is 200 km/h, and its vertical component is 18 km/h, then the speed of the *xy*-component is:

$\sqrt{200^2-18^2}=\sqrt{39676}=199.188\approx 199$ km/h. So, the velocity vector in the *xy*-plane is:

$\frac{\sqrt{39676}}{\sqrt{2}}(-1,1)=\sqrt{19838}(-1,1)\approx(-140.8,140.8)$. Hence, the velocity vector is $(-140.8,\ 140.8,\ 18)$.

Note: If the 200 km h^{-1} is interpreted as the horizontal airspeed as seen by an observer on the ground, then the *xy*-component would be $200\times\frac{1}{\sqrt{2}}(-1,1)=(-141.4,141.4)$, and hence the velocity vector would be $(-141.4,\ 141.4,\ 18)$.

22. Vectors are perpendicular if their dot product is zero; therefore:

$2t+4t-10-t=0 \Rightarrow t=2$

23. Vectors are perpendicular if their dot product is zero; therefore:

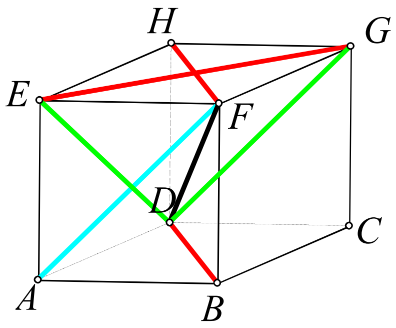
$$t+3t+2=0 \Rightarrow t=-\frac{1}{2}$$

24. Vectors are perpendicular if their dot product is zero; therefore:

$$4t^2-2t+0=0 \Rightarrow t=0,\ t=\frac{1}{2}$$

Note: For $t=0$, the second vector is a zero-vector, but, according to the definition a zero vector is perpendicular to all vectors.

25. Some of the diagonals are shown in the diagram below. Using the properties of symmetry, we can see that most of the angles are the same.



We will use the component form of the diagonals (considering D to be at the origin):

$$\overrightarrow{DF}=\begin{pmatrix}a\\a\\a\end{pmatrix},\ \overrightarrow{AF}=\overrightarrow{DG}=\begin{pmatrix}0\\a\\a\end{pmatrix},\ \overrightarrow{DE}=\overrightarrow{CF}=\begin{pmatrix}a\\0\\a\end{pmatrix},\ \overrightarrow{DB}=\overrightarrow{HF}=\begin{pmatrix}a\\a\\0\end{pmatrix},\ \overrightarrow{HA}=\overrightarrow{GB}=\begin{pmatrix}a\\0\\-a\end{pmatrix}$$

$$\overrightarrow{AC}=\overrightarrow{EG}=\begin{pmatrix}-a\\a\\0\end{pmatrix},\ \overrightarrow{EB}=\overrightarrow{HC}=\begin{pmatrix}0\\a\\-a\end{pmatrix}$$

$$\cos\left(\overrightarrow{DF},\overrightarrow{AF}\right)=\frac{2a^2}{\sqrt{3a^2}\sqrt{2a^2}}=\frac{2}{\sqrt{6}}$$

$$\cos\left(\overrightarrow{DF},\overrightarrow{DE}\right)=\frac{2a^2}{\sqrt{3a^2}\sqrt{2a^2}}=\frac{2}{\sqrt{6}}$$

$$\cos\left(\overrightarrow{DF},\overrightarrow{DB}\right)=\frac{2a^2}{\sqrt{3a^2}\sqrt{2a^2}}=\frac{2}{\sqrt{6}}$$

$$\cos\left(\overrightarrow{DF},\overrightarrow{HA}\right)=\frac{0}{\sqrt{3a^2}\sqrt{2a^2}}=0$$

$$\cos\left(\overrightarrow{DF}, \overrightarrow{AC}\right) = \frac{0}{\sqrt{3a^2}\sqrt{2a^2}} = 0$$

$$\cos\left(\overrightarrow{DF}, \overrightarrow{EB}\right) = \frac{0}{\sqrt{3a^2}\sqrt{2a^2}} = 0$$

Hence, the angle is either 90° or $\cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$.

26. To simplify the notation, denote the vector by $\boldsymbol{v} = |\boldsymbol{a}|\boldsymbol{b} + |\boldsymbol{b}|\boldsymbol{a}$. We will determine the angles between $\boldsymbol{v}$ and $\boldsymbol{a}$, and $\boldsymbol{v}$ and $\boldsymbol{b}$.

Using the fact that $\boldsymbol{a}^2 = |\boldsymbol{a}|^2$ and $\boldsymbol{b}^2 = |\boldsymbol{b}|^2$, we have:

$$\cos\alpha = \frac{\left(|\boldsymbol{a}|\boldsymbol{b} + |\boldsymbol{b}|\boldsymbol{a}\right)\boldsymbol{a}}{|\boldsymbol{v}||\boldsymbol{a}|} = \frac{|\boldsymbol{a}|\boldsymbol{ba} + |\boldsymbol{b}|\boldsymbol{a}^2}{|\boldsymbol{v}||\boldsymbol{a}|} = \frac{\boldsymbol{ba}}{|\boldsymbol{v}|} + \frac{|\boldsymbol{b}||\boldsymbol{a}|}{|\boldsymbol{v}|}$$

$$\cos\beta = \frac{\left(|\boldsymbol{a}|\boldsymbol{b} + |\boldsymbol{b}|\boldsymbol{a}\right)\boldsymbol{b}}{|\boldsymbol{v}||\boldsymbol{b}|} = \frac{|\boldsymbol{a}|\boldsymbol{b}^2 + |\boldsymbol{b}|\boldsymbol{ab}}{|\boldsymbol{v}||\boldsymbol{b}|} = \frac{|\boldsymbol{b}||\boldsymbol{a}|}{|\boldsymbol{v}|} + \frac{\boldsymbol{ab}}{|\boldsymbol{v}|}$$

Since, the cosines of the angles are the same, and the angles are from 0° to 180°, then $\alpha = \beta$, and vector $\boldsymbol{v}$ is the angle bisector.

27. The scalar product should be 0, so: $2 - m + n = 0 \Rightarrow m = 2 + n$.

If the magnitudes are equal, then: $\sqrt{1 + m^2 + 1} = \sqrt{4 + 1 + n^2} \Rightarrow m^2 = 3 + n^2$

Hence, $(2+n)^2 = 3 + n^2 \Rightarrow 4n = -1 \Rightarrow n = -\frac{1}{4}$ and $m = 2 - \frac{1}{4} = \frac{7}{4}$

28. For a set of three angles to be direction angles for a vector, the squares of their cosines must add up to 1.

$$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{6} + \cos^2\frac{2\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{3}{4} + \frac{1}{4} = \frac{3}{2} \neq 1;$$

hence, they cannot be the direction angles of one vector.

29. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \cos^2\gamma = \frac{3}{4} + \cos^2\gamma = 1 \Rightarrow \cos\gamma = \pm\frac{1}{2}$

Hence, $\gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$.

30. If $\alpha=\beta=\gamma\Rightarrow 3\cos^2\alpha=1\Rightarrow\cos\alpha=\pm\sqrt{\frac{1}{3}}$

Hence, the angles are $\cos^{-1}\left(\pm\frac{\sqrt{3}}{3}\right)$

31. We can write $\boldsymbol{u}=\begin{pmatrix}|\boldsymbol{u}|\cos\alpha\\|\boldsymbol{u}|\cos\beta\\|\boldsymbol{u}|\cos\gamma\end{pmatrix}\Rightarrow-\boldsymbol{u}=\begin{pmatrix}|\boldsymbol{u}|(-\cos\alpha)\\|\boldsymbol{u}|(-\cos\beta)\\|\boldsymbol{u}|(-\cos\gamma)\end{pmatrix}=\begin{pmatrix}|\boldsymbol{u}|\cos(\pi-\alpha)\\|\boldsymbol{u}|\cos(\pi-\beta)\\|\boldsymbol{u}|\cos(\pi-\gamma)\end{pmatrix}$.

Hence, the direction vectors are: $\pi-\alpha,\ \pi-\beta,\pi-\gamma$.

32. Let the vector be: $\boldsymbol{u}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. Then, $x^2+y^2+z^2=1$ and $\begin{cases}x+2y+z=0\\3x-4y+2z=0\end{cases}$

Solution of the system two equation with two unknowns is: $x=-\frac{4}{5}z, y=-\frac{1}{10}z$; hence:

$\frac{16}{25}z^2+\frac{1}{100}z^2+z^2=1\Rightarrow z^2=\frac{100}{165}$. So, the vectors are:

$$\boldsymbol{u}=\pm\frac{10}{\sqrt{165}}\begin{pmatrix}-\frac{4}{5}\\-\frac{1}{10}\\1\end{pmatrix}=\pm\frac{1}{\sqrt{165}}\begin{pmatrix}-8\\-1\\10\end{pmatrix}$$

Exercise 9.3

In calculating $\boldsymbol{u}\times\boldsymbol{v}$, you can use the formula given in the formula booklet, or the equivalent using determinants given in the book. We will use the determinant approach for convenience.

1. **(a)** $\boldsymbol{i}\times(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&0&0\\1&1&1\end{vmatrix}=-\boldsymbol{j}+\boldsymbol{k}$

(b) $$\boldsymbol{i}\times\boldsymbol{i}+\boldsymbol{i}\times\boldsymbol{j}+\boldsymbol{i}\times\boldsymbol{k}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&0&0\\1&0&0\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&0&0\\0&1&0\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&0&0\\0&0&1\end{vmatrix}==\boldsymbol{k}-\boldsymbol{j}$$

The results are the same.

2. **(a)** $$\boldsymbol{j}\times\left(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\right)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&1&0\\1&1&1\end{vmatrix}=\boldsymbol{i}-\boldsymbol{k}$$

(b) $$\boldsymbol{j}\times\boldsymbol{i}+\boldsymbol{j}\times\boldsymbol{j}+\boldsymbol{j}\times\boldsymbol{k}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&1&0\\1&0&0\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&1&0\\0&1&0\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&1&0\\0&0&1\end{vmatrix}=-\boldsymbol{k}+\boldsymbol{i}$$

The results are the same.

3. **(a)** $$\boldsymbol{k}\times\left(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\right)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&0&1\\1&1&1\end{vmatrix}=-\boldsymbol{i}+\boldsymbol{j}$$

(b) $$\boldsymbol{k}\times\boldsymbol{i}+\boldsymbol{k}\times\boldsymbol{j}+\boldsymbol{k}\times\boldsymbol{k}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&0&1\\1&0&0\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&0&1\\0&1&0\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\0&0&1\\0&0&1\end{vmatrix}=\boldsymbol{j}-\boldsymbol{i}$$

The results are the same.

4. $$\boldsymbol{u}\times\left(\boldsymbol{v}+\boldsymbol{w}\right)=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\u_1&u_2&u_3\\v_1+w_1&v_2+w_2&v_3+w_3\end{vmatrix}=\begin{pmatrix}u_2(v_3+w_3)-u_3(v_2+w_2)\\-u_1(v_3+w_3)+u_3(v_1+w_1)\\u_1(v_2+w_2)-u_2(v_1+w_1)\end{pmatrix}$$

$$\boldsymbol{u}\times\boldsymbol{v}+\boldsymbol{u}\times\boldsymbol{w}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}+\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\u_1&u_2&u_3\\w_1&w_2&w_3\end{vmatrix}=\begin{pmatrix}u_2v_3-u_3v_2\\-u_1v_3+u_3v_1\\u_1v_2-u_2v_1\end{pmatrix}+\begin{pmatrix}u_2w_3-u_3w_2\\-u_1w_3+u_3w_1\\u_1w_2-u_2w_1\end{pmatrix}$$

$$=\begin{pmatrix}u_2(v_3+w_3)-u_3(v_2+w_2)\\-u_1(v_3+w_3)+u_3(v_1+w_1)\\u_1(v_2+w_2)-u_2(v_1+w_1)\end{pmatrix}$$

Hence, $\boldsymbol{u}\times\left(\boldsymbol{v}+\boldsymbol{w}\right)=\boldsymbol{u}\times\left(\boldsymbol{v}+\boldsymbol{w}\right)$.

$$\boldsymbol{u}\times(\boldsymbol{v}-\boldsymbol{w})=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ v_1-w_1 & v_2-w_2 & v_3-w_3\end{vmatrix}=\begin{pmatrix}u_2(v_3-w_3)-u_3(v_2-w_2)\\ -u_1(v_3-w_3)+u_3(v_1-w_1)\\ u_1(v_2-w_2)-u_2(v_1-w_1)\end{pmatrix}$$

$$\boldsymbol{u}\times\boldsymbol{v}-\boldsymbol{u}\times\boldsymbol{w}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}-\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ w_1 & w_2 & w_3\end{vmatrix}=\begin{pmatrix}u_2v_3-u_3v_2\\ -u_1v_3+u_3v_1\\ u_1v_2-u_2v_1\end{pmatrix}-\begin{pmatrix}u_2w_3-u_3w_2\\ -u_1w_3+u_3w_1\\ u_1w_2-u_2w_1\end{pmatrix}$$

$$=\begin{pmatrix}u_2(v_3-w_3)-u_3(v_2-w_2)\\ -u_1(v_3-w_3)+u_3(v_1-w_1)\\ u_1(v_2-w_2)-u_2(v_1-w_1)\end{pmatrix}$$

Hence, $\boldsymbol{u}\times(\boldsymbol{v}-\boldsymbol{w})=\boldsymbol{u}\times(\boldsymbol{v}-\boldsymbol{w})$.

5. **(a)** $\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 3 & -2\\ -3 & 2 & 3\end{vmatrix}=13\boldsymbol{i}+13\boldsymbol{k}$

For orthogonal vectors, the scalar product must be zero:

$(13\boldsymbol{i}+13\boldsymbol{k})(2\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k})=26+0-26=0$

$(13\boldsymbol{i}+13\boldsymbol{k})(-3\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})=-39+0+39=0$

(b) $\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 4 & 3 & 0\\ 0 & -2 & 2\end{vmatrix}=6\boldsymbol{i}-8\boldsymbol{j}-8\boldsymbol{k}$

$(6\boldsymbol{i}-8\boldsymbol{j}-8\boldsymbol{k})(4\boldsymbol{i}+3\boldsymbol{j})=24-24+0=0$

$(6\boldsymbol{i}-8\boldsymbol{j}-8\boldsymbol{k})(-2\boldsymbol{j}+2\boldsymbol{k})=0+16-16=0$

(c) $\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & -1\\ 4 & 1 & -3\end{vmatrix}=\begin{pmatrix}-5\\ -1\\ -7\end{pmatrix}$

$\begin{pmatrix}-5\\ -1\\ -7\end{pmatrix}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}=-5-2+7=0$

$\begin{pmatrix}-5\\ -1\\ -7\end{pmatrix}\begin{pmatrix}4\\ 1\\ -3\end{pmatrix}=-20-1+21=0$

(d) $$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 5 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = \boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$$

$$(\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})(5\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}) = 5+1-6=0$$

$$(\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})(3\boldsymbol{i}+\boldsymbol{k}) = 3+0-3=0$$

6. **(a)** $$\begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 3 \\ m & 2 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \begin{pmatrix} -4 \\ -3+3m \\ 6-2m \end{pmatrix} = 3m-2m^2-12+6m+1 = -2m^2+9m-11$$

(b) $$\begin{pmatrix} m \\ 2 \\ 1 \end{pmatrix} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & m \\ 3 & 2 & 3 \end{vmatrix} = \begin{pmatrix} m \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3-2m \\ -6+3m \\ 1 \end{pmatrix} = 3m-2m^2-12+6m+1 = -2m^2+9m-11$$

(c) $$\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ m & 2 & 1 \\ 2 & 1 & m \end{vmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2m-1 \\ -m^2+2 \\ m-4 \end{pmatrix} = 6m-3-2m^2+4+3m-12 = -2m^2+9m-11$$

7. **(a)** $$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w}) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} -28 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} -40 \\ -115 \\ 30 \end{pmatrix}$$

(b) $$(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w} = \begin{pmatrix} -8 \\ -20 \\ 6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -150 \\ 60 \\ 0 \end{pmatrix}$$

(c) $$(\boldsymbol{u}\times\boldsymbol{v})\times(\boldsymbol{v}\times\boldsymbol{w}) = \begin{pmatrix} -8 \\ -20 \\ 6 \end{pmatrix} \times \begin{pmatrix} -28 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} -80 \\ -160 \\ -640 \end{pmatrix}$$

(d) $$(\boldsymbol{v}\times\boldsymbol{w})\times(\boldsymbol{u}\times\boldsymbol{v}) = \begin{pmatrix} -28 \\ 10 \\ 1 \end{pmatrix} \times \begin{pmatrix} -8 \\ -20 \\ 6 \end{pmatrix} = \begin{pmatrix} 80 \\ 160 \\ 640 \end{pmatrix}$$

(e) $$(\boldsymbol{uw})\boldsymbol{v}-(\boldsymbol{uv})\boldsymbol{w}=(6+0+24)\begin{pmatrix}1\\2\\8\end{pmatrix}-(3+0+32)\begin{pmatrix}2\\5\\6\end{pmatrix}=\begin{pmatrix}30\\60\\240\end{pmatrix}-\begin{pmatrix}70\\175\\210\end{pmatrix}=\begin{pmatrix}-40\\-115\\30\end{pmatrix}$$

(f) $$(\boldsymbol{wu})\boldsymbol{v}-(\boldsymbol{wv})\boldsymbol{u}=(6+0+24)\begin{pmatrix}1\\2\\8\end{pmatrix}-(2+10+48)\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}30\\60\\240\end{pmatrix}-\begin{pmatrix}180\\0\\240\end{pmatrix}=\begin{pmatrix}-150\\60\\0\end{pmatrix}$$

8. $$\boldsymbol{u}\times\boldsymbol{v}=\begin{pmatrix}-6\\4\\1\end{pmatrix}\times\begin{pmatrix}3\\1\\5\end{pmatrix}=\begin{pmatrix}19\\33\\-18\end{pmatrix},\ |\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{19^2+33^2+18^2}=\sqrt{1774}$$

There are two unit vectors: $\pm\frac{\sqrt{1774}}{1774}\begin{pmatrix}19\\33\\-18\end{pmatrix}$

9. We need a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}1\\1\\-3\end{pmatrix}\times\begin{pmatrix}-1\\3\\-1\end{pmatrix}=\begin{pmatrix}8\\4\\4\end{pmatrix}$$

Vector $\overrightarrow{AB}\times\overrightarrow{AC}$ is parallel to $\begin{pmatrix}2\\1\\1\end{pmatrix}$ whose magnitude is $\sqrt{4+1+1}=\sqrt{6}$.

There are two unit vectors: $\pm\frac{\sqrt{6}}{6}\begin{pmatrix}2\\1\\1\end{pmatrix}$

10. **(a)** $$\boldsymbol{u}\times\boldsymbol{v}=\begin{pmatrix}2\\0\\3\end{pmatrix}\times\begin{pmatrix}1\\4\\2\end{pmatrix}=\begin{pmatrix}-12\\-1\\8\end{pmatrix},\ Area=|\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{12^2+1^2+8^2}=\sqrt{209}$$

(b) $$\boldsymbol{u}\times\boldsymbol{v}=\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}0\\3\\-1\end{pmatrix}=\begin{pmatrix}-7\\3\\9\end{pmatrix},\ Area=|\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{7^2+3^2+9^2}=\sqrt{139}$$

11. Denote the points as: $A(2,-1,1),\ B(5,1,4),\ C(0,1,1),\ D(3,3,4).$

Vectors $\overrightarrow{AB}=\begin{pmatrix}5-2\\1+1\\4-1\end{pmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}$ and $\overrightarrow{DC}=\begin{pmatrix}3-0\\3-1\\4-1\end{pmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}$ are the same; hence, $ABCD$ is a parallelogram. Since $Area=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$, we have to find the vector product of those vectors:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}3\\2\\3\end{pmatrix}\times\begin{pmatrix}-2\\2\\0\end{pmatrix}=\begin{pmatrix}-6\\-6\\10\end{pmatrix}$$

$$Area=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{36+36+100}=\sqrt{172}=2\sqrt{43}$$

12. The points are coplanar if the vectors $\overrightarrow{PQ},\overrightarrow{PR},\overrightarrow{PS}$ are coplanar. That means their scalar triple product must be zero.

$$\overrightarrow{PQ}=\begin{pmatrix}1\\1\\-1\end{pmatrix},\ \overrightarrow{PR}=\begin{pmatrix}2\\3\\-2\end{pmatrix},\ \overrightarrow{PS}=\begin{pmatrix}4\\5\\-4\end{pmatrix}$$

$$\begin{vmatrix}1&1&-1\\2&3&-2\\4&5&-4\end{vmatrix}=0$$

13. We will find the scalar triple product of the vectors:

$$\overrightarrow{AB}=\begin{pmatrix}3-m\\1\\m+2\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}2-m\\-3\\0\end{pmatrix},\ \overrightarrow{AD}=\begin{pmatrix}4-m\\5\\6\end{pmatrix}$$

$$\begin{vmatrix}3-m&1&m+2\\2-m&-3&0\\4-m&5&6\end{vmatrix}=0\Rightarrow$$

$$(3-m)\cdot(-18)-(2-m)6+(m+2)(10-5m+12-3m)=0$$

$$\Rightarrow -8m^2+30m-22=0\Rightarrow m_1=1,\ m_2=\frac{11}{4}$$

14. **(a)** The area of the triangle is half the area of the parallelogram formed with $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-1\\-5\\2\end{pmatrix}\times\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}-13\\5\\6\end{pmatrix}$$

$$A=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{169+25+36}=\frac{\sqrt{230}}{2}$$

(b) The area of the triangle is half the area of the parallelogram formed with $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-1\\4\\8\end{pmatrix}\times\begin{pmatrix}3\\0\\10\end{pmatrix}=\begin{pmatrix}40\\34\\-12\end{pmatrix}$$

$$Area=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{40^2+34^2+12^2}=\frac{\sqrt{2900}}{2}=5\sqrt{29}$$

15. Check the scalar triple product on page 401 of the book.

(a) $$\begin{vmatrix}3&-2&2\\5&2&-2\\1&2&6\end{vmatrix}=128$$

(b) $$\begin{vmatrix}2&-1&3\\1&4&3\\-3&2&-2\end{vmatrix}=21$$

(c) $$\begin{vmatrix}3&2&1\\1&-3&1\\5&1&2\end{vmatrix}=1$$

16. **(a)** Since $volume=\left|\boldsymbol{u}(\boldsymbol{v}\times\boldsymbol{w})\right|$, we have to find the scalar triple product:

$$\begin{vmatrix}3&-5&3\\1&5&-1\\3&2&-3\end{vmatrix}=-78$$

So, $volume=78$

(b) Since $volume = \left|\boldsymbol{u}(\boldsymbol{v} \times \boldsymbol{w})\right|$, we have to find the scalar triple product:

$$\begin{vmatrix} 4 & 2 & 3 \\ 5 & 6 & 2 \\ 2 & 3 & 5 \end{vmatrix} = 63$$

So, $volume = 63$

17. **(a)** Vectors are coplanar if their scalar triple product is zero:

$$\begin{vmatrix} 2 & -1 & 2 \\ 4 & 1 & -1 \\ 6 & -3 & 1 \end{vmatrix} = -30 \neq 0$$

So, they are not coplanar.

(b) Vectors are coplanar if their scalar triple product is zero:

$$\begin{vmatrix} 4 & -2 & -1 \\ 9 & -6 & -1 \\ 6 & -6 & 1 \end{vmatrix} = 0$$

So, they are coplanar.

18. **(a)** Vectors are coplanar if their scalar triple product is zero:

$$\begin{vmatrix} 1 & m & 1 \\ 3 & 0 & m \\ 5 & -4 & 0 \end{vmatrix} = 4m - m(-5m) + (-12) = 5m^2 + 4m - 12$$

Hence, $5m^2 + 4m - 12 = 0 \Rightarrow m_1 = -2,\ m_2 = \frac{6}{5}$

(b) Vectors are coplanar if their scalar triple product is zero:

$$\begin{vmatrix} 2 & -3 & 2m \\ m & -3 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 2(3) + 3(-2m - 1) + 2m(3m + 3) = 6m^2 + 3$$

Hence, $6m^2 + 3 = 0$ has no solution, so they cannot be coplanar.

19. **(a)** Since $volume = \left|\boldsymbol{u}\left(\boldsymbol{v}\times\boldsymbol{w}\right)\right|$, we have to find the scalar triple product:

$$\begin{vmatrix} 1 & 4 & 2 \\ -3 & 2 & 1 \\ -1 & 1 & 4 \end{vmatrix} = 49$$

So, $volume = 49$

(b) $$\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}\times\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -14 \end{pmatrix}$$

The area is: $area = \left|\boldsymbol{u}\times\boldsymbol{v}\right| = \sqrt{0+7^2+14^2} = \sqrt{245} = 7\sqrt{5}$

(c) Since $volume = base \times h \Rightarrow h = \dfrac{volume}{base} = \dfrac{49}{7\sqrt{5}} = \dfrac{7\sqrt{5}}{5}$

(d) $$\cos\left(\boldsymbol{w},\boldsymbol{u}\times\boldsymbol{v}\right) = \frac{\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}\begin{pmatrix} 0 \\ 7 \\ -14 \end{pmatrix}}{\sqrt{18}\cdot 7\sqrt{5}} = \frac{-49}{21\sqrt{10}} = -\frac{7\sqrt{10}}{30}$$

So, the acute angle between the vector $\boldsymbol{w}$ and a vector perpendicular to the plane determined by $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\cos^{-1}\left(\dfrac{7\sqrt{10}}{30}\right)$. Hence, the angle between $\boldsymbol{w}$ and the plane is $90^\circ - \cos^{-1}\left(\dfrac{7\sqrt{10}}{30}\right)$.

20. **(a)** $$V_{\text{tetrahedron}} = \frac{1}{3}\left(7\sqrt{5}\right)\left(\frac{7\sqrt{5}}{5}\right) = \frac{49}{3} = \frac{1}{3}V_{\text{parallelepiped}}$$

Hence, $V_{\text{tetrahedron}} = \dfrac{1}{3}\left|\boldsymbol{u}\left(\boldsymbol{v}\times\boldsymbol{w}\right)\right|$

(b) $$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix},\ \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},\ \overrightarrow{AD} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}$$

$$\begin{vmatrix} 3 & -1 & -3 \\ 2 & -2 & 1 \\ 4 & -4 & 3 \end{vmatrix} = -4$$

Hence, $V = \dfrac{4}{3}$

21. From the definitions, we have: $\boldsymbol{uv} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$, $|\boldsymbol{u}\times\boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta$, so:

$|\boldsymbol{u}||\boldsymbol{v}|\cos\theta = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta \Rightarrow \cos\theta = \sin\theta$. Since $0 \le \theta \le 180^\circ$, it follows that $\theta = 45^\circ$.

22. We will transform the right side of the formula:

$$\sqrt{|\boldsymbol{u}|^2|\boldsymbol{v}|^2 - (\boldsymbol{uv})^2} = \sqrt{|\boldsymbol{u}|^2|\boldsymbol{v}|^2 - |\boldsymbol{u}|^2|\boldsymbol{v}|^2\cos^2\theta} = \sqrt{|\boldsymbol{u}|^2|\boldsymbol{v}|^2\left(1-\cos^2\theta\right)}$$

$$= \sqrt{|\boldsymbol{u}|^2|\boldsymbol{v}|^2\sin^2\theta} = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta = |\boldsymbol{u}\times\boldsymbol{v}|$$

23. We will transform the right side of the formula:

$$\frac{\left|\overrightarrow{AP}\times\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|} = \frac{\left|\overrightarrow{AP}\right|\left|\overrightarrow{AB}\right|\sin\theta}{\left|\overrightarrow{AB}\right|} = \left|\overrightarrow{AP}\right|\sin\theta = d$$

24. **(a)** In this case, the distance will be: $d = \dfrac{\left|\overrightarrow{BA}\times\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$

$$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ -22 \\ -8 \end{pmatrix}$$

Hence, $d = \dfrac{\sqrt{4^2+22^2+8^2}}{\sqrt{3^2+2^2+4^2}} = \dfrac{\sqrt{564}}{\sqrt{29}} = \sqrt{\dfrac{564}{29}}$

(b) In this case, the distance will be: $d = \dfrac{\left|\overrightarrow{BA}\times\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$

$$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

Hence, $d = \dfrac{6}{\sqrt{2^2+1^2}} = \dfrac{6}{\sqrt{5}} = \dfrac{6\sqrt{5}}{5}$

(c) In this case, the distance will be: $d = \dfrac{\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

Hence, $d = \dfrac{\sqrt{2^2 + 2^2 + 2^2}}{\sqrt{2^2 + 2^2}} = \dfrac{\sqrt{12}}{\sqrt{8}} = \sqrt{\dfrac{3}{2}}$

25. We will use the distributive property of the vector product and the fact that the vector product of parallel vectors is a zero-vector.

$$(\boldsymbol{u} + \boldsymbol{v}) \times (\boldsymbol{v} - \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{v} - \boldsymbol{u} \times \boldsymbol{u} + \boldsymbol{v} \times \boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{v} - \vec{0} + \vec{0} + \boldsymbol{u} \times \boldsymbol{v} = 2(\boldsymbol{u} \times \boldsymbol{v})$$

26. We will use the distributive property of the vector product and the fact that the vector product of parallel vectors is a zero-vector.

$$\begin{aligned}(2\boldsymbol{u} + 3\boldsymbol{v}) \times (4\boldsymbol{v} - 5\boldsymbol{u}) &= 8\boldsymbol{u} \times \boldsymbol{v} - 10\boldsymbol{u} \times \boldsymbol{u} + 12\boldsymbol{v} \times \boldsymbol{v} - 15\boldsymbol{v} \times \boldsymbol{u} \\ &= 8\boldsymbol{u} \times \boldsymbol{v} - \vec{0} + \vec{0} + 15\boldsymbol{u} \times \boldsymbol{v} = 23(\boldsymbol{u} \times \boldsymbol{v})\end{aligned}$$

27. We will use the distributive property of the vector product and the fact that the vector product of parallel vectors is a zero-vector.

$$\begin{aligned}(m\boldsymbol{u} + n\boldsymbol{v}) \times (p\boldsymbol{v} - q\boldsymbol{u}) &= mp(\boldsymbol{u} \times \boldsymbol{v}) - mq(\boldsymbol{u} \times \boldsymbol{u}) + np(\boldsymbol{v} \times \boldsymbol{v}) - nq(\boldsymbol{v} \times \boldsymbol{u}) \\ &= mp(\boldsymbol{u} \times \boldsymbol{v}) - \vec{0} + \vec{0} + nq(\boldsymbol{u} \times \boldsymbol{v}) = (mp + nq)(\boldsymbol{u} \times \boldsymbol{v})\end{aligned}$$

28. **(a)** $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -a \\ b \\ 0 \end{pmatrix} \times \begin{pmatrix} -a \\ 0 \\ c \end{pmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \begin{pmatrix} bc \\ ac \\ ab \end{pmatrix}$

Hence, $o = \dfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \dfrac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

(b) The faces are right-angled triangles, so we can find the area using the half product of the legs:

$A_1 = \dfrac{ab}{2}$, $A_2 = \dfrac{ac}{2}$, $A_3 = \dfrac{bc}{2}$

(c) Hence,

$$(A_1)^2+(A_2)^2+(A_3)^2=\frac{a^2b^2}{4}+\frac{a^2c^2}{4}+\frac{b^2c^2}{4}=\frac{a^2b^2+a^2c^2+b^2c^2}{4}$$

$$=\left(\frac{\sqrt{a^2b^2+a^2c^2+b^2c^2}}{2}\right)^2=o^2$$

29.
$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 3 \\ x & y & z \end{vmatrix}=\begin{pmatrix} 2z-3y \\ z+3x \\ -y-2x \end{pmatrix}$$

So, $$\begin{pmatrix} 2z-3y \\ z+3x \\ -y-2x \end{pmatrix}=\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}\Rightarrow \begin{matrix} -3y+2z=1 \\ 3x+z=5 \\ -2x-y=-3 \end{matrix}\Rightarrow x=\frac{5}{3}-\frac{1}{3}z, y=-\frac{1}{3}+\frac{2}{3}z, z=z$$

30.
$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 3 \\ x & y & z \end{vmatrix}=\begin{pmatrix} 2z-3y \\ z+3x \\ -y-2x \end{pmatrix}$$

So, $$\begin{pmatrix} 2z-3y \\ z+3x \\ -y-2x \end{pmatrix}=\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}\Rightarrow \begin{matrix} -3y+2z=1 \\ 3x+z=5 \\ -2x-y=0 \end{matrix}\qquad\Rightarrow\begin{cases} x+\frac{1}{3}z=0 \\ y-\frac{2}{3}z=0 \\ 0=1 \end{cases}$$

Trying to reduce the resulting system of equations, we notice that it is inconsistent. Hence, there is no such vector.

Exercise 9.4

1. **(a)** In the equation $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u}$, vector $\boldsymbol{r}_0 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, and $\boldsymbol{u} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$, so the vector equation of the line is $\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$. For the parametric equations:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1+t \\ 0+5t \\ 2-4t \end{pmatrix}$; therefore, the parametric equations are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1+t \\ 5t \\ 2-4t \end{pmatrix}$.

The cartesian equations are $\dfrac{x+1}{1} = \dfrac{y}{5} = \dfrac{z-2}{-4}$

(b) Substituting $\boldsymbol{r}_0 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ into $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u}$, we get the vector equation of the line: $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$. The parametric equations are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+2t \\ -1+5t \\ 2-t \end{pmatrix}$.

The cartesian equations are $\dfrac{x-3}{2} = \dfrac{y+1}{5} = \dfrac{z-2}{-1}$.

(c) Substituting $\boldsymbol{r}_0 = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$ into $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u}$, we get the vector equation of the line: $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$. The parametric equations are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+3t \\ -2+5t \\ 6-11t \end{pmatrix}$.

The cartesian equations are $\dfrac{x-1}{3} = \dfrac{y+2}{5} = \dfrac{z-6}{-11}$.

2. **(a)** In the equation $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u}$, vector $\boldsymbol{r}_0 = \boldsymbol{r}_A = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$, and $\boldsymbol{u} = \overrightarrow{AB} = \begin{pmatrix} 7+1 \\ 5-4 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$,

so the vector equation of the line: $\boldsymbol{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$

Note: For $\boldsymbol{r}_0$ we can use $\boldsymbol{r}_B$ and for $\boldsymbol{u}$ we can use $\overrightarrow{BA}$, or any vector parallel to this vector. Therefore, it is possible to find different, but correct, equations of the line.

(b) Substituting $\boldsymbol{r}_0 = \boldsymbol{r}_A = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$ and $\boldsymbol{u} = \overrightarrow{AB} = \begin{pmatrix} 0-4 \\ -2-2 \\ 1+3 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$ into $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u}$,

we get the vector equation of the line: $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$

Note: Since $\boldsymbol{u}$ is parallel to $\overrightarrow{AB}$, we can use $\boldsymbol{u} = -\frac{1}{4}\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, and the vector

equation of the line would be $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(c) Substituting $\boldsymbol{r}_0 = \boldsymbol{r}_A = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ and $\boldsymbol{u} = \overrightarrow{AB} = \begin{pmatrix} 5-1 \\ 1-3 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ into $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u}$,

we get the vector equation of the line: $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$.

3. **(a)** In the equation $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{b}$, we substitute $\boldsymbol{a}=\boldsymbol{r}_A=\begin{pmatrix}3\\-2\end{pmatrix}$ and

$\boldsymbol{u}=\overrightarrow{AB}=\begin{pmatrix}5-3\\1+2\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$, so the equation of the line is $\boldsymbol{r}=\begin{pmatrix}3\\-2\end{pmatrix}+t\begin{pmatrix}2\\3\end{pmatrix}$.

(b) In the equation $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{b}$, we substitute $\boldsymbol{a}=\boldsymbol{r}_A=\begin{pmatrix}0\\-2\end{pmatrix}$ and

$\boldsymbol{u}=\overrightarrow{AB}=\begin{pmatrix}5-0\\0+2\end{pmatrix}=\begin{pmatrix}5\\2\end{pmatrix}$, so the equation of the line is $\boldsymbol{r}=\begin{pmatrix}0\\-2\end{pmatrix}+t\begin{pmatrix}5\\2\end{pmatrix}$.

4. **Method 1**

To determine the equation of the line in the required form, we need to find two points on the line $\boldsymbol{r}=(2,1)+t(3,-2)$. One point is $(2,1)$; another point we can find by letting, for example, $t=1$; therefore, the point is $(5,-1)$. The equation of the line through those two points is: $\frac{y-1}{x-2}=\frac{-1-1}{5-2}\Rightarrow 3(y-1)=-2(x-2)\Rightarrow 2x+3y=7$

Method 2

We can write the equation of the line $\boldsymbol{r}=\begin{pmatrix}2\\1\end{pmatrix}+t\begin{pmatrix}3\\-2\end{pmatrix}$ in parametric form:

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2+3t\\1-2t\end{pmatrix}$. From the first row: $x=2+3t\Rightarrow t=\frac{x-2}{3}$. Substituting t into the second row, we get: $y=1-2t=1-2\frac{x-2}{3}\Rightarrow 3(y-1)=-2(x-2)\Rightarrow 2x+3y=7$

5. In the equation $\boldsymbol{r}=\boldsymbol{r}_0+t\boldsymbol{u}$, the vector $\boldsymbol{r}_0=\boldsymbol{r}_A=2\boldsymbol{i}-3\boldsymbol{j}$ and $\boldsymbol{u}$ can be the same as the direction vector of the given line; therefore, $\boldsymbol{u}=4\boldsymbol{i}-3\boldsymbol{j}$. So: $\boldsymbol{r}=2\boldsymbol{i}-3\boldsymbol{j}+\lambda(4\boldsymbol{i}-3\boldsymbol{j})$

6. In the equation $\boldsymbol{r}=\boldsymbol{r}_0+t\boldsymbol{u}$, the vector $\boldsymbol{r}_0=\boldsymbol{r}_A=\begin{pmatrix}-2\\1\\4\end{pmatrix}$ and $\boldsymbol{u}=\begin{pmatrix}3\\-4\\7\end{pmatrix}$. So, we have:

$$\boldsymbol{r}=\begin{pmatrix}-2\\1\\4\end{pmatrix}+t\begin{pmatrix}3\\-4\\7\end{pmatrix}$$

7. **(a)** The lines are not parallel since the direction vectors $\begin{pmatrix}1\\3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\4\\2\end{pmatrix}$ are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\boldsymbol{r} = \begin{pmatrix}2\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\3\\1\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + s\begin{pmatrix}1\\4\\2\end{pmatrix}$, for some values of t and s. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

$x_0 = 2+t = 2+s$

$y_0 = 2+3t = 3+4s$

$z_0 = 3+t = 4+2s$

From the first equation we see that $t = s$. Substituting into the second equation: $2+3t = 3+4t \Rightarrow t = -1 \Rightarrow t = s = -1$. Finally, substituting these values into the third equation: $3-1 = 4-2 \Rightarrow -2 = -2$. Hence, the lines intersect, and the point of intersection is: $(2,2,3)+(-1)(1,3,1) = (1,-1,2)$.

(b) The lines are not parallel since the direction vectors $\begin{pmatrix}4\\1\\0\end{pmatrix}$ and $\begin{pmatrix}12\\6\\3\end{pmatrix}$ are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\boldsymbol{r} = \begin{pmatrix}-1\\3\\1\end{pmatrix} + t\begin{pmatrix}4\\1\\0\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}-13\\1\\2\end{pmatrix} + s\begin{pmatrix}12\\6\\3\end{pmatrix}$, for some values of t and s. So:

$x_0 = -1+4t = -13+12s$

$y_0 = 3+t = 1+6s$

$z_0 = 1 + 0 \cdot t = 2 + 3s$

From the last equation, we see that $s = -\frac{1}{3}$. Substituting into the second equation: $3 + t = 1 + 6\left(-\frac{1}{3}\right) \Rightarrow t = -4$. Finally, substituting these values into the first equation: $-1 - 16 = -13 - 4 \Rightarrow -17 = -17$. Hence, the lines intersect, and the point of intersection is: $\boldsymbol{r} = (-1, 3, 1) + (-4)(4, 1, 0) = (-17, -1, 1)$.

(c) The lines are not parallel since the direction vectors $\begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, for some values of t and s. So:

$x_0 = 1 + 7t = 4 - s$

$y_0 = 3 + t = 6 + 0 \cdot s$

$z_0 = 5 - 3 \cdot t = 7 + 2s$

From the second equation, we can see that $t = 3$. Substituting into the first equation: $1 + 21 = 4 - s \Rightarrow s = -18$. Finally, substituting these values into the last equation: $5 - 3 \cdot 3 = 7 + 2(-18) \Rightarrow -4 \neq -29$. Hence, the lines do not intersect; they are skew.

(d) The lines have parallel direction vectors $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$, since $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$.

To check whether the lines coincide, we examine the point $(3,4,6)$, which is on the first line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, and see whether it lies on the second line also.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$$

So: $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} \Rightarrow \begin{matrix} 3 = 5 - 4s \Rightarrow s = \frac{1}{2} \\ 4 = -2 + 2s \Rightarrow s = 3 \\ 6 = 7 - 2s \Rightarrow \end{matrix}$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

8. **(a)** A direction vector is: $\begin{pmatrix} 3-2 \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; hence, the equations are:

$\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+t \\ -1+3t \end{pmatrix}$

(b) We know a point and a direction vector, so the equations are:

$\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-3t \\ -1+7t \end{pmatrix}$

(c) For the direction vector, we can use any vector perpendicular to $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$.

So we use vector $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ as the direction vector of the line, since $\begin{pmatrix} -3 \\ 7 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = -21 + 21 = 0$. Therefore, the equations are: $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+7t \\ -1+3t \end{pmatrix}$.

(d) We know a point and a direction vector, so the equations are:

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ 2-4t \end{pmatrix}$$

9. **(a)** Substituting the point $\left(0, \frac{11}{2}, \frac{9}{2}\right)$ into the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ \frac{11}{2} \\ \frac{9}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} 0 = 3-2t \\ \frac{11}{2} = 4+t \\ \frac{9}{2} = 6-t \end{cases}$$

From the first equation, we can see that $t = \frac{3}{2}$. Check this value in the second equation: $\frac{11}{2} = 4 + \frac{3}{2} \Rightarrow \frac{11}{2} = \frac{11}{2}$, and in the third equation: $\frac{9}{2} = 6 - \frac{3}{2} \Rightarrow \frac{9}{2} = \frac{9}{2}$.

So, the point is on the line when $t = \frac{3}{2}$.

(b) To check whether the point is on the line, we have to determine whether or not the system of equations has a solution:

$$\begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} -1 = 3-2t \\ 4 = 4+t \\ 6 = 6-t \end{cases}$$

From the last two equations, we can see that $t = 0$, but this will not satisfy the first equation; hence, there is no solution to the system and the point does not lie on the line.

(c) We have to solve the system of equations:

$$\begin{pmatrix} \frac{1-2m}{2} \\ 2m \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} \frac{1-2m}{2} = 3-2t \\ 2m = 4+t \\ 3 = 6-t \end{cases}$$

From the last equation, we can see that $t=3$. Substituting into the second equation: $2m=7 \Rightarrow m=\frac{7}{2}$. Check using the first equation:

$\frac{1-2\frac{7}{2}}{2}=3-6 \Rightarrow \frac{-6}{2}=-3$. Therefore, the point will be on the line when $m=\frac{7}{2}$.

10. **(a)** **(i)** The starting position is when $t=0$, so the point is $(3,-4)$

(ii) The velocity vector is $\boldsymbol{v}=\begin{pmatrix}7\\24\end{pmatrix}$

(iii) The speed is $|\boldsymbol{v}|=\sqrt{7^2+24^2}=25$

(b) **(i)** The starting position is when $t=0$, so the point is $(-3,1)$

(ii) The velocity vector is $\boldsymbol{v}=\begin{pmatrix}5\\-12\end{pmatrix}$

(iii) The speed is $|\boldsymbol{v}|=\sqrt{5^2+(-12)^2}=13$

(c) **(i)** The starting position is when $t=0$, so the point is $(5,-2)$

(ii) The velocity vector is $\boldsymbol{v}=(24,-7)$

(iii) The speed is $|\boldsymbol{v}|=\sqrt{24^2+(-7)^2}=25$

11. **(a)** The direction of the velocity vector is given by the unit vector:

$$\frac{1}{\sqrt{(-3)^2+4^2}}\begin{pmatrix}-3\\4\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix}$$

So, the velocity vector is: $160\cdot\frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix}=32\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}-96\\128\end{pmatrix}$

(b) The direction of the velocity vector is given by the unit vector:

$$\frac{1}{\sqrt{12^2+(-5)^2}}\begin{pmatrix}12\\-5\end{pmatrix}=\frac{1}{13}\begin{pmatrix}12\\-5\end{pmatrix}$$

So, the velocity vector is: $170\cdot\frac{1}{13}\begin{pmatrix}12\\-5\end{pmatrix}=\begin{pmatrix}\frac{2040}{13}\\-\frac{850}{13}\end{pmatrix}$

12. **(a)** The car is travelling from the point $(3,2)$ to $(7,5)$, so the direction vector of the velocity vector is given by the unit vector: $\frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$, where $\boldsymbol{v}=\begin{pmatrix}7-3\\5-2\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$.

Therefore, the unit vector is: $\frac{1}{\sqrt{4^2+3^2}}\begin{pmatrix}4\\3\end{pmatrix}=\frac{1}{5}\begin{pmatrix}4\\3\end{pmatrix}$; and the velocity vector is: $30\cdot\frac{1}{5}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}24\\18\end{pmatrix}$.

(b) The starting point is $(3,2)$ and the direction vector of the line is $\begin{pmatrix}24\\18\end{pmatrix}$, so the equation of the position of the car after *t* hours is $\boldsymbol{r}=\begin{pmatrix}3\\2\end{pmatrix}+t\begin{pmatrix}24\\18\end{pmatrix}$

(c) We have to determine the parameter of the point with position vector $\begin{pmatrix}7\\5\end{pmatrix}$

$$\begin{pmatrix}7\\5\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}+t\begin{pmatrix}24\\18\end{pmatrix}\Rightarrow\begin{pmatrix}4\\3\end{pmatrix}=t\begin{pmatrix}24\\18\end{pmatrix}\Rightarrow t=\frac{1}{6}$$

Therefore, in $\frac{1}{6}$ of an hour, i.e. in 10 minutes, the car will reach the traffic light.

13. **(a)** To be perpendicular to the vectors, both dot products must be zero.

$$\begin{pmatrix}1\\a\\b\end{pmatrix}\begin{pmatrix}1\\-3\\2\end{pmatrix}=1-3a+2b=0\text{, and }\begin{pmatrix}1\\a\\b\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix}=-2+a-b=0$$

So, we have to solve the system: $\begin{cases}-3a+2b=-1\\ \quad a-\ b=2\end{cases}\Rightarrow a=-3,\ b=-5$

(b) $$\cos\theta=\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{v}||\boldsymbol{w}|}=\frac{\begin{pmatrix}1\\-3\\2\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix}}{\sqrt{1^2+(-3)^2+2^2}\sqrt{(-2)^2+1^2+(-1)^2}}=\frac{-7}{2\sqrt{21}}=-\frac{\sqrt{21}}{6}$$

(c) Using the Pythagorean identity for sine, $\sin^2\theta = 1-\cos^2\theta,$ and the fact that sine is positive for angles from $0^\circ - 180^\circ$ we have:

$$\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\frac{21}{36}} = \sqrt{\frac{15}{36}} = \frac{\sqrt{15}}{6}$$

Area of triangle OPQ is: $A = \frac{1}{2}|OP||OQ|\sin P\hat{O}Q = \frac{1}{2}|\boldsymbol{v}||\boldsymbol{w}|\sin(\angle\boldsymbol{v},\boldsymbol{w})$

So, $Area = \frac{1}{2}\sqrt{14}\sqrt{6}\frac{\sqrt{15}}{6} = \frac{\sqrt{35}}{2}$

14. **(a)** First, we have to determine vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \begin{pmatrix} -1-(-1) \\ 3-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 0-(-1) \\ -1-2 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}}{\sqrt{0^2+1^2+2^2}\sqrt{1^2+(-3)^2+(-2)^2}} = \frac{0-3-4}{\sqrt{5}\sqrt{14}} = \frac{-7}{\sqrt{5}\sqrt{14}}$$

Therefore, $\theta = \cos^{-1}\frac{-7}{\sqrt{5}\sqrt{14}} \Rightarrow \theta \approx 147°$

(b) The area of the triangle is: $A = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta = \frac{1}{2}\sqrt{5}\sqrt{14}\sin\theta \approx 2.29$

(c) **(i)** Line L_1 goes through the point $(2,-1,0)$ and its direction vector is

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, so its equation is: $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Line L_2 goes through the point $(-1,1,1)$ and its direction vector is

$\overrightarrow{AC} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$, so its equation is: $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$

(ii) We have to solve the system of equations:

$$\begin{pmatrix}2\\-1\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}+s\begin{pmatrix}1\\-3\\-2\end{pmatrix}\Rightarrow\begin{matrix}2=-1+s\\-1+t=1-3s\\2t=1-2s\end{matrix}$$

From the first equation, we have $s=3$, and from the second $t=-7$. Substituting these values into the third equation:

$2(-7)=1-2\cdot3\Rightarrow-14\neq-5$. So, there is no point of intersection.

15. **(a)** Let the direction vector be a vector parallel to $\overrightarrow{AB}$:

$$\overrightarrow{AB}=\begin{pmatrix}6-1\\-7-3\\8-(-17)\end{pmatrix}=\begin{pmatrix}5\\-10\\25\end{pmatrix}=5\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

Thus, we can use the vector $\begin{pmatrix}1\\-2\\5\end{pmatrix}$ as the direction vector.

Therefore, the parametric equations of the line are: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+t\\3-2t\\-17+5t\end{pmatrix}$

(b) If point P is on the line, then vector $\overrightarrow{OP}=\begin{pmatrix}1+t\\3-2t\\-17+5t\end{pmatrix}$. If $\overrightarrow{OP}$ is perpendicular to the line, then $\overrightarrow{OP}$ and the direction vector of the line are perpendicular, and their scalar product is zero.

$$0=\overrightarrow{OP}\begin{pmatrix}1\\2\\5\end{pmatrix}=\begin{pmatrix}1+t\\3-2t\\-17+5t\end{pmatrix}\begin{pmatrix}1\\-2\\5\end{pmatrix}=1+t+-6+4t+-85+25t=30t-90\Rightarrow t=3$$

So, $\overrightarrow{OP}=\begin{pmatrix}1+3\\3-2\cdot3\\-17+5\cdot3\end{pmatrix}=\begin{pmatrix}4\\-3\\-2\end{pmatrix}$ and $P(4,-3,-2)$

16. **(a)** We will find a point and a vector on the line.

Let $y=0$, then $x=\frac{p}{m}$; hence, point $\left(\frac{p}{m},0\right)$ is on the line.

Let $x=0$, then $y=\frac{p}{n}$; hence, point $\left(0,\frac{p}{n}\right)$ is on the line.

Therefore, the vector $\begin{pmatrix}\frac{p}{m}-0\\0-\frac{p}{n}\end{pmatrix}=\begin{pmatrix}\frac{p}{m}\\-\frac{p}{n}\end{pmatrix}$ is parallel to the line. This vector is parallel to the vector $\frac{mn}{p}\cdot\begin{pmatrix}\frac{p}{m}\\-\frac{p}{n}\end{pmatrix}=\begin{pmatrix}n\\-m\end{pmatrix}$. So, a vector equation of the line is:

$$\boldsymbol{r}=\begin{pmatrix}\frac{p}{m}\\0\end{pmatrix}+t\begin{pmatrix}n\\-m\end{pmatrix}.$$

(b) **(i)** We already have one point on the line. To determine another point on the line $\boldsymbol{r}=\begin{pmatrix}x_0\\y_0\end{pmatrix}+t\begin{pmatrix}a\\b\end{pmatrix}$ we can let, for example, $t=1$; therefore, the point is: (x_0+a,y_0+b). The equation of the line through those two points is: $\frac{y-y_0}{x-x_0}=\frac{y_0+b-y_0}{x_0+a-x_0}\Rightarrow a(y-y_0)=b(x-x_0)$. Hence, an equation of the line is: $bx-ay=bx_0-ay_0$.

(ii) We will write the equation of the line in slope-intercept form: $ay=bx-bx_0+ay_0\Rightarrow y=\frac{b}{a}x-\frac{b}{a}x_0+y_0$. Hence, the slope of the line is $\frac{b}{a}$

17. Parameterisation of a segment: $\boldsymbol{r}(t)=(1-t)\overrightarrow{OA}+t\overrightarrow{OB},\ 0\le t\le 1$

(a) $\boldsymbol{r}(t)=(1-t)\begin{pmatrix}0\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\1\\3\end{pmatrix},\ 0\le t\le 1$; hence, $\boldsymbol{r}(t)=t\begin{pmatrix}1\\1\\3\end{pmatrix},\ 0\le t\le 1$

(b) $\boldsymbol{r}(t)=(1-t)\begin{pmatrix}-1\\0\\1\end{pmatrix}+t\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}-1+t+t\\t\\1-t-2t\end{pmatrix}$, $0\le t\le 1$; hence,

$$\boldsymbol{r}(t)=\begin{pmatrix}-1+2t\\t\\1-3t\end{pmatrix},\ 0\le t\le 1$$

(c) $\boldsymbol{r}(t)=(1-t)\begin{pmatrix}1\\0\\-1\end{pmatrix}+t\begin{pmatrix}0\\3\\0\end{pmatrix}=\begin{pmatrix}1-t\\3t\\-1+t\end{pmatrix}$, $0\le t\le 1$; hence, $\boldsymbol{r}(t)=\begin{pmatrix}1-t\\3t\\-1+t\end{pmatrix}$, $0\le t\le 1$

18. A direction vector of the parallel line is $2\boldsymbol{k}$; hence, a vector equation of the line whose equation we have to find is: $\boldsymbol{r}=(2\boldsymbol{j}+3\boldsymbol{k})+t(2\boldsymbol{k})$. The parametric equations are:

$$\begin{cases}x = 0\\ y = 2\\ z=3+2t\end{cases}$$

Note: We can write a vector equation in the form: $\boldsymbol{r}=2\boldsymbol{j}+(3+2t)\boldsymbol{k}$

19. A direction vector of the parallel line is $2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}$; hence a vector equation of the line whose equation we have to find is: $\boldsymbol{r}=(\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})+t(2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k})$. The parametric equations are: $\begin{cases}x=1+2t\\ y=2-3t\\ z=-1+t\end{cases}$

Note: We can write a vector equation in the form: $\boldsymbol{r}=(1+2t)\boldsymbol{i}+(2-3t)\boldsymbol{j}+(-1+t)\boldsymbol{k}$

20. A direction vector of the line is $x_0\boldsymbol{i}+y_0\boldsymbol{j}+z_0\boldsymbol{k}$; hence, a vector equation of the line is $\boldsymbol{r}=\boldsymbol{0}+t(x_0\boldsymbol{i}+y_0\boldsymbol{j}+z_0\boldsymbol{k})$, and the parametric equations are: $\begin{cases}x=tx_0\\ y=ty_0\\ z=tz_0\end{cases}$

21. **(a)** A direction vector of the line is a vector perpendicular to the xz-plane; hence, $\boldsymbol{j}$. Therefore, a vector equation of the line is $\boldsymbol{r}=(3\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k})+t\boldsymbol{j}$, and the parametric equations are: $\begin{cases} x=3 \\ y=2+t \\ z=-3 \end{cases}$

(b) A direction vector of the line is a vector perpendicular to the yz-plane; hence, $\boldsymbol{i}$. Therefore, a vector equation of the line is $\boldsymbol{r}=(3\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k})+t\boldsymbol{i}$, and the parametric equations are: $\begin{cases} x=3+t \\ y=2 \\ z=-3 \end{cases}$

22. A direction vector of the line is $x_0\boldsymbol{i}+y_0\boldsymbol{j}+z_0\boldsymbol{k}$. Hence, the symmetric (Cartesian) equations of the lines are: $\dfrac{x-0}{x_0}=\dfrac{y-0}{y_0}=\dfrac{z-0}{z_0}\Rightarrow\dfrac{x}{x_0}=\dfrac{y}{y_0}=\dfrac{z}{z_0}$

Note: We took the origin as a point on the line. We can take point A, then the equation will be: $\dfrac{x-x_0}{x_0}=\dfrac{y-y_0}{y_0}=\dfrac{z-z_0}{z_0}$

23. **(a)** The lines are not parallel since the direction vectors $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases} x=3+t \\ y=1-t \\ z=5+2t \end{cases}$, $\begin{cases} x=1 \\ y=4+\lambda \\ z=2+\lambda \end{cases}$

and solve the system:

$$\begin{cases} 3+t=1 \\ 1-t=4+\lambda \\ 5+2t=2+\lambda \end{cases}$$

From the first equation, we can see that $t=-2$.

From the second, $1+2=4+\lambda \Rightarrow \lambda=-1$; and, finally, substituting those values into the third equation, we have: $5+2(-2)=2+(-1)\Rightarrow 1=1$. Hence, the lines intersect, and the point of intersection is: $\begin{cases} x=3-2=1 \\ y=1+2=3 \\ z=5-4=1 \end{cases}$ $(1,3,1)$

Note: If it is a Paper 2 question, we can solve the system using matrices, or any other GDC specific method available. First, transform the system of equations:

$$\begin{cases} 3+t=1 \\ 1-t=4+\lambda \\ 5+2t=2+l \end{cases} \Rightarrow \begin{matrix} t=-2 \\ -t-\lambda=3 \\ 2t-\lambda=-3 \end{matrix}$$

and then use a GDC. Substitute back into one of the equations to find the intersection point. Notice that it is easier to solve the system without a GDC since, while preparing the system for the GDC, we would find the solutions.

(b) The lines are parallel since for the direction vectors $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$ it holds:

$\begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$. To check whether the lines coincide, we examine the point $(-1,2,1)$, which is on the first line, and see whether it also lies on the second line

$$\begin{cases} x=2-2m \\ y=-1+6m \\ z=-4m \end{cases}$$

So, $\begin{cases} -1=2-2m \\ 2=-1+6m \\ 1=-4m \end{cases} \Rightarrow \begin{cases} 3=2m \Rightarrow m=\frac{3}{2} \\ 3=6m \Rightarrow m=\frac{3}{6} \\ 1=-4m \end{cases}$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

(c) The lines are not parallel since the direction vectors $\begin{pmatrix}2\\4\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\2\end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases}x=3+2t\\y=-1+4t\\z=2-t\end{cases}$, $\begin{cases}x=3+2\lambda\\y=2+\lambda\\z=-2+2\lambda\end{cases}$ and solve the system:

$$\begin{cases}3+2t=3+2\lambda\\-1+4t=2+\lambda\\2-t=-2+2\lambda\end{cases}$$

From the first equation, we can see that $t=\lambda$. From the second, $-1+4t=2+t\Rightarrow t=1$;
and, finally, substituting those values into the third equation, we have: $2-1=-2+2\Rightarrow 1\neq 0$. Hence, the lines do not intersect. They are skew.

(d) The lines are not parallel since the direction vectors $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases}x=1-t\\y=1+3t\\z=-4+2t\end{cases}$, $\begin{cases}x=1-\lambda\\y=-1-\lambda\\z=2\lambda\end{cases}$

and solve the system:

$$\begin{cases}1-t=1-\lambda\\1+3t=-1-\lambda\\-4+2t=2\lambda\end{cases}$$

From the first equation, we can see that $t=\lambda$. From the second, $1+3t=-1-t\Rightarrow t=-\frac{1}{2}$; and, finally, substituting those values into the third equation, we have: $-4+2\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)\Rightarrow -5\neq -1$. Hence, the lines do not intersect. They are skew.

(e) The lines are parallel since for the direction vectors $\begin{pmatrix}-6\\9\\-3\end{pmatrix}$ and $\begin{pmatrix}2\\-3\\1\end{pmatrix}$, it holds:

$\begin{pmatrix}-6\\9\\-3\end{pmatrix}=-3\begin{pmatrix}2\\-3\\1\end{pmatrix}$. To check if the lines coincide, we examine the point $(1,2,0)$, which is on the first line, and see whether it also lies on the second line.

$$\begin{cases}x=2+2m\\y=3-3m\\z=m\end{cases}$$

So, $\begin{cases}1=2+2m\\2=3-3m\Rightarrow\\0=m\end{cases}\begin{cases}-1=2m\Rightarrow m=-\dfrac{1}{2}\\ \\0=m\end{cases}$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

(f) The lines are not parallel since the direction vectors $\begin{pmatrix}5\\1\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-3\\-4\end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases}x=2+5t\\y=1+t\\z=2+3t\end{cases}$, $\begin{cases}x=-4+3\lambda\\y=7-3\lambda\\z=10-4\lambda\end{cases}$

and solve the system:

$$\begin{cases}2+5t=-4+3\lambda\\1+t=7-3\lambda\\2+3t=10-4\lambda\end{cases}$$

From the second equation, we can see that $t = 6 - 3\lambda$. From the first, $2 + 5(6 - 3\lambda) = -4 + 3\lambda \Rightarrow 18\lambda = 36 \Rightarrow \lambda = 2$; and, finally, substituting the values $\lambda = 2,\ t = 6 - 3\cdot 2 = 0$ into the third equation, we have: $2 + 3\cdot 0 = 10 - 4\cdot 2 \Rightarrow 2 = 2$.

Hence, the lines intersect, and the point of intersection is: $\begin{cases} x = 2 + 5\cdot 0 \\ y = 1 + 0 \\ z = 2 + 3\cdot 0 \end{cases}$ $(2,1,2)$

(g) The lines are not parallel since the direction vectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -9 \\ 6 \end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines. We have to change the name of the parameter in one of the equations and then solve the system:

$$\begin{cases} 1 + t = 2 + 2\lambda \\ 2 - 2t = 5 - 9\lambda \\ t + 5 = 2 + 6\lambda \end{cases}$$

From the first equation, we can see that $t = 1 + 2\lambda$. From the third, $1 + 2\lambda + 5 = 2 + 6\lambda \Rightarrow 4\lambda = 4 \Rightarrow \lambda = 1$; and, finally, substituting the values $\lambda = 1,\ t = 1 + 2\cdot 1 = 3$ into the second equation, we have: $2 - 2\cdot 3 = 5 - 9\cdot 1 \Rightarrow -4 = -4$. Hence, the lines intersect, and the point of intersection is: $\begin{cases} x = 1 + 3 \\ y = 2 - 2\cdot 3 \\ z = 3 + 5 \end{cases}$ $(4,-4,8)$

24. The parametric equations of the line are: $\begin{cases} x = 2 - 3t \\ y = 3 + t \\ z = 1 + t \end{cases}$

The distance from the origin to a point on the line is: $d^2 = (2 - 3t)^2 + (3 + t)^2 + (1 + t)^2 = 14 - 4t + 11t^2$. Since this is a parabola that opens upwards, the distance is a minimum when

$$\frac{d\left(14 - 4t + 11t^2\right)}{dt} = 0 \Rightarrow -4 + 22t = 0 \Rightarrow t = \frac{2}{11}$$

Hence, the point on the line with $t=\frac{2}{11}$ is the closest to the origin.

$$\begin{cases} x=2-3\cdot\frac{2}{11}=\frac{16}{11} \\ y=3+\frac{2}{11}=\frac{35}{11} \Rightarrow A\left(\frac{16}{11},\frac{35}{11},\frac{13}{11}\right) \\ z=1+\frac{2}{11}=\frac{13}{11} \end{cases}$$

25. The parametric equations of the line are: $\begin{cases} x=t \\ y=4-3t \\ z=5+t \end{cases}$

The distance from the origin to a point on the line is:

$d^2=t^2+(4-3t)^2+(5+t)^2=41-14t+11t^2$. The distance is a minimum when

$$\frac{d\left(41-14t+11t^2\right)}{dt}=0\Rightarrow -14+22t=0\Rightarrow t=\frac{7}{11}$$

Hence, the point on the line with $t=\frac{7}{11}$ is the closest to the origin.

$$\begin{cases} x=\frac{7}{11} \\ y=4-3\cdot\frac{7}{11}=\frac{23}{11}\Rightarrow A\left(\frac{7}{11},\frac{23}{11},\frac{62}{11}\right) \\ z=5+\frac{7}{11}=\frac{62}{11} \end{cases}$$

26. The parametric equations of the line are: $\begin{cases} x=5+t \\ y=2-3t \\ z=1+t \end{cases}$

The distance from the point $(-1,4,1)$ to a point on the line is:

$d^2=(5+t+1)^2+(2-3t-4)^2+(1+t-1)^2=40+24t+11t^2$.

The distance is a minimum when

$$\frac{d\left(40+24t+11t^2\right)}{dt}=0\Rightarrow 24+22t=0\Rightarrow t=-\frac{12}{11}$$

Hence, the point on the line with $t=-\frac{12}{11}$ is the closest to the origin.

$$\begin{cases} x = 5 - \dfrac{12}{11} = \dfrac{43}{11} \\ y = 2 - 3\cdot\left(-\dfrac{12}{11}\right) = \dfrac{58}{11} \Rightarrow A\left(\dfrac{43}{11}, \dfrac{58}{11}, -\dfrac{1}{11}\right) \\ z = 1 - \dfrac{12}{11} = -\dfrac{1}{11} \end{cases}$$

Exercise 9.5

1. For A: $3\cdot 3 + 2\cdot(-2) - 3\cdot(-1) = 8 \neq 11$; hence, A does not lie in the plane.

For B: $3\cdot 2 + 2\cdot 1 - 3\cdot(-1) = 11$; hence, B lies in the plane.

For C: $3\cdot 1 + 2\cdot 4 - 3\cdot 0 = 11$; hence, C lies in the plane.

2. For A: $(\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})(3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 3 + 6 - 1 = 8 \neq -6$; hence, A does not lie in the plane.

For B: $(\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) = 2 - 3 - 2 = -3 \neq -6$; hence, B does not lie in the plane.

For C: $(\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})(\boldsymbol{i} + 4\boldsymbol{j} + 0\boldsymbol{k}) = 1 - 12 = -11 \neq -6$; hence, C does not lie in the plane.

3. **(a)** A Cartesian equation for the plane is:

$$2(x-3) - 4(y+2) + 3(z-4) = 0 \Rightarrow 2x - 4y + 3z = 6 + 8 + 12$$

$$\Rightarrow 2x - 4y + 3z = 26$$

A vector equation for the plane is:

$$\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 26$$

(b) A Cartesian equation for the plane is:

$$2(x+3) + 0(y-2) + 3(z-1) = 0 \Rightarrow 2x + 3z = -6 + 3$$

$$\Rightarrow 2x + 3z = -3$$

A vector equation for the plane is:

$$\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -3$$

(c) A Cartesian equation for the plane is:

$$0(x-0)+0(y-3)+3(z-1)=0\Rightarrow 3z=3$$

A vector equation for the plane is:

$$\begin{pmatrix}0\\0\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\3\end{pmatrix}\begin{pmatrix}0\\3\\1\end{pmatrix}\Rightarrow\begin{pmatrix}0\\0\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3$$

(d) A vector perpendicular to the plane is the same as a vector perpendicular to the parallel plane; hence, it is $\begin{pmatrix}5\\1\\-2\end{pmatrix}$

So, a Cartesian equation for the plane is:

$$5(x-3)+1(y+2)-2(z-4)=0\Rightarrow 5x+y-2z=15-2-8$$

$$\Rightarrow 5x+y-2z=5$$

A vector equation for the plane is:

$$\begin{pmatrix}5\\1\\-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\1\\-2\end{pmatrix}\begin{pmatrix}3\\-2\\4\end{pmatrix}\Rightarrow\begin{pmatrix}5\\1\\-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=5$$

(e) A vector perpendicular to the plane is the same as a vector perpendicular to the parallel plane; hence, it is $\begin{pmatrix}0\\1\\-2\end{pmatrix}$.

So, a Cartesian equation for the plane is:

$$0(x-3)+1(y-0)-2(z-1)=0\Rightarrow y-2z=-2$$

A vector equation for the plane is:

$$\begin{pmatrix}0\\1\\-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\1\\-2\end{pmatrix}\begin{pmatrix}3\\0\\1\end{pmatrix}\Rightarrow\begin{pmatrix}0\\1\\-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-2$$

(f) The plane is parallel to a direction vector of the line $\begin{pmatrix}2\\1\\2\end{pmatrix}$ and the vector

$\begin{pmatrix}3\\-2\\4\end{pmatrix}-\begin{pmatrix}1\\-2\\5\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$. So, a parametric equation for the plane is:

$$\boldsymbol{r}=\begin{pmatrix}3\\-2\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}1\\-6\\2\end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}1\\-6\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-6\\2\end{pmatrix}\begin{pmatrix}3\\-2\\4\end{pmatrix}\Rightarrow\begin{pmatrix}1\\-6\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=23$$

A Cartesian equation for the plane is: $x-6y+2z=23$.

Note: We can find other vectors parallel to the plane, or other points on the plane; hence, we can obtain different parametric and vector equations of the plane.

(g) The plane is parallel to the direction vectors of the lines $\begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\2\\2\end{pmatrix}$. So, a parametric equation for the plane is:

$$\boldsymbol{r}=\begin{pmatrix}1\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix}+\mu\begin{pmatrix}3\\2\\2\end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}2\\1\\2\end{pmatrix}\times\begin{pmatrix}3\\2\\2\end{pmatrix}=\begin{pmatrix}-2\\2\\1\end{pmatrix}$.

Therefore, a vector equation for the plane is:

$$\begin{pmatrix}-2\\2\\1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\2\\1\end{pmatrix}\begin{pmatrix}1\\-2\\5\end{pmatrix}\Rightarrow\begin{pmatrix}-2\\2\\1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1$$

A Cartesian equation for the plane is: $-2x+2y+z=-1$.

(h) The plane is parallel to a direction vector of the line $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and the vector

$\begin{pmatrix}1\\-3\\2\end{pmatrix}-\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\-5\\3\end{pmatrix}$. So, a parametric equation for the plane is:

$$\boldsymbol{r}=\begin{pmatrix}1\\-3\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-5\\3\end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}2\\1\\3\end{pmatrix}\times\begin{pmatrix}1\\-5\\3\end{pmatrix}=\begin{pmatrix}18\\-3\\-11\end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}18\\-3\\-11\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}18\\-3\\-11\end{pmatrix}\begin{pmatrix}1\\-3\\2\end{pmatrix}\Rightarrow\begin{pmatrix}18\\-3\\-11\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=5$$

A Cartesian equation for the plane is: $18x-3y-11z=5$.

(i) Vector $\overrightarrow{OM}=\begin{pmatrix}p\\q\\r\end{pmatrix}$; hence, a Cartesian equation for the plane is:

$$p(x-p)+q(y-q)+r(z-r)=0\Rightarrow px+qy+rz=p^2+q^2+r^2$$

A vector equation for the plane is:

$$\begin{pmatrix}p\\q\\r\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}p\\q\\r\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}\Rightarrow\begin{pmatrix}p\\q\\r\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=p^2+q^2+r^2$$

(j) The plane is parallel to vectors $\begin{pmatrix} 3-1 \\ -1-2 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7-3 \\ 0+1 \\ -2-0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

So, a parametric equation for the plane is:

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 14 \end{pmatrix} = 2\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 14$$

A Cartesian equation for the plane is: $4x - 2y + 7z = 14$.

(k) The plane is parallel to vectors $\begin{pmatrix} 3-2 \\ -1+2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0-3 \\ 1+1 \\ 5-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$.

So, a parametric equation for the plane is:

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -17 \\ 5 \end{pmatrix} = -1\begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -8$$

A Cartesian equation for the plane is: $8x + 17y - 5z = -8$.

(l) The plane is parallel to a direction vector of the line $\begin{pmatrix}1\\1\\3\end{pmatrix}$ and the vector

$\begin{pmatrix}2\\-1\\5\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$. So, a parametric equation for the plane is:

$$\boldsymbol{r}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\1\\2\end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}1\\-2\\3\end{pmatrix}\Rightarrow\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3$$

A Cartesian equation for the plane is: $x-y=3$.

(m) The plane is parallel to a direction vector of the line $\begin{pmatrix}3\\2\\4\end{pmatrix}$ and the vector

$\begin{pmatrix}-3\\4\\0\end{pmatrix}-\begin{pmatrix}1\\-1\\5\end{pmatrix}=\begin{pmatrix}-4\\5\\-5\end{pmatrix}$. So, a parametric equation for the plane is:

$$\boldsymbol{r}=\begin{pmatrix}1\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}-4\\5\\-5\end{pmatrix}+\mu\begin{pmatrix}3\\2\\4\end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}-4\\5\\-5\end{pmatrix}\times\begin{pmatrix}3\\2\\4\end{pmatrix}=\begin{pmatrix}30\\1\\-23\end{pmatrix}$.

Therefore, a vector equation for the plane is:

$$\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}1\\-1\\5\end{pmatrix}\Rightarrow\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-86$$

A Cartesian equation for the plane is: $30x+y-23z=-86$.

(n) The plane is parallel to the direction vectors of the lines $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

So, a parametric equation for the plane is:

$$\boldsymbol{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1$$

A Cartesian equation for the plane is: $x-z=1$.

4. **(a)**. The angle between the normals is given by:

$$\cos\theta_1=\frac{\begin{pmatrix}3\\4\\-1\end{pmatrix}\begin{pmatrix}1\\-2\\0\end{pmatrix}}{\sqrt{9+16+1}\sqrt{1+4}}=\frac{-5}{\sqrt{26\cdot 5}}$$

Since the angle between the planes is by definition the acute angle between the planes, the angle between the planes is: $\cos^{-1}\dfrac{5}{\sqrt{26\cdot 5}}\Rightarrow\theta=63.98....^{\circ}\approx 64.0^{\circ}$

(b) The angle between the normals is given by:

$$\cos\theta_1 = \frac{\begin{pmatrix}4\\-7\\1\end{pmatrix}\begin{pmatrix}3\\2\\2\end{pmatrix}}{\sqrt{16+49+1}\sqrt{9+4+4}} = \frac{0}{\sqrt{66\cdot 17}}$$

Hence, the angle between the planes is: $\cos^{-1} 0 \Rightarrow \theta = 90^\circ$

(c) The angle between the normals is given by

$$\cos\theta_1 = \frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}}{\sqrt{1}\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

Hence, the angle between the planes is: $\cos^{-1}\dfrac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$

(d) For the angle between the normal and the direction line, it holds:

$$\cos\theta_1 = \frac{\begin{pmatrix}1\\-2\\2\end{pmatrix}\begin{pmatrix}-6\\3\\-2\end{pmatrix}}{\sqrt{1+4+4}\sqrt{36+9+4}} = \frac{-16}{3\cdot 7}$$

Hence, the angle between the plane and the line is:

$$\theta = \sin^{-1}\frac{16}{21} \Rightarrow \theta = 49.6324... \approx 49.6^\circ$$

(e) For the angle between the normal and the direction line, it holds:

$$\cos\theta_1 = \frac{\begin{pmatrix}3\\0\\-1\end{pmatrix}\begin{pmatrix}-1\\2\\-1\end{pmatrix}}{\sqrt{9+1}\sqrt{1+4+1}} = \frac{-2}{\sqrt{60}}$$

Hence, the angle between the plane and the line is:

$$\theta = \sin^{-1}\frac{2}{\sqrt{60}} \Rightarrow \theta = 14.9632... \approx 15.0^\circ$$

(f) The angle between the normals is given by:

$$\cos\theta_1 = \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{1+1+1}\sqrt{1}} = \frac{1}{\sqrt{3}}$$

Hence, the angle between the planes is: $\theta = \cos^{-1}\dfrac{1}{\sqrt{3}} \Rightarrow \theta = 54.7356...^\circ \approx 54.7^\circ$

5. **(a)** Parametric equations of the line are:

$$\begin{cases} x = 5 + l \\ y = -3l \\ z = -2 + 4l \end{cases}$$

A Cartesian equation of the plane is: $x - 3y + 2z = -35$

For the intersection, we have:

$$(5+l) - 3(-3l) + 2(-2+4l) = -35 \Rightarrow 18l = -36 \Rightarrow l = -2$$

Hence, the point is: $\begin{cases} x = 5 - 2 = 3 \\ y = -3(-2) = 6 \\ z = -2 + 4(-2) = -10 \end{cases} \Rightarrow (3, 6, -10)$

(b) Parametric equations of the line are:

$$\begin{cases} x = 2 \\ y = 4 - 3\mu \\ z = 3\mu \end{cases}$$

For the intersection, we have:

$$4(2) - 2(4-3\mu) + 3(3\mu) - 30 = 0 \Rightarrow 15\mu = 30 \Rightarrow \mu = 2$$

Hence, the point is: $\begin{cases} x = 2 \\ y = 4 - 3(2) = -2 \Rightarrow (2, -2, 6) \\ z = 3(2) = 6 \end{cases}$

(c) The direction vector of the line and the normal of the plane are perpendicular (since $\begin{pmatrix}1\\5\\3\end{pmatrix}\begin{pmatrix}2\\-4\\6\end{pmatrix}=2-20+18=0$); hence, the line and the plane are either parallel or the line is in the plane. Since the point $(3,4,6)$ is not on the plane, they are parallel and there is no intersection.

Note: If we solve the system for intersection, we will have:

Parametric equations of the line are:

$$\begin{cases}x=3+t\\y=4+5t\\z=6+3t\end{cases}$$

A Cartesian equation of the plane is: $2x-4y+6z=5$

For the intersection, we have: $2(3+t)-4(4+5t)+6(6+3t)=5\Rightarrow 26=5$; hence, there is no intersection.

(d) The direction vector of the line and the normal of the plane are perpendicular (since $\begin{pmatrix}1\\-\frac{1}{3}\\-\frac{5}{3}\end{pmatrix}\begin{pmatrix}3\\-1\\2\end{pmatrix}=3+\frac{1}{3}-\frac{10}{3}=0$); hence, the line and the plane are either parallel or the line is in the plane. The point $(0,4,5)$ is on the plane, so the line is in the plane. Note: If we solve the system for intersection, we will have:

For the intersection, it holds: $3(t)-\left(4-\frac{1}{3}t\right)+2\left(5-\frac{5}{3}t\right)=6\Rightarrow -4+10=6$; hence, all points from the line are on the plane.

6. **(a)** Solving the system:

$$\begin{cases}x=10\\x+y+z=3\Rightarrow 10+y+z=3\Rightarrow y=-7-z\end{cases}$$

Hence, parametric equations of the line of intersection are:

$\begin{cases} x = 10 \\ y = -7 - t \\ z = t \end{cases}$ and the vector equation is $\boldsymbol{r} = \begin{pmatrix} 10 \\ -7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

(b) We have to solve the system:

$$\begin{cases} 2x - y + z = 5 \\ x + y - z = 4 \end{cases}$$

Adding the equations, we have: $3x = 9 \Rightarrow x = 3$; therefore:

$$\begin{cases} -y + z = -1 \\ y - z = 1 \end{cases} \Rightarrow y = 1 + z$$

Hence, parametric equations of the line of intersection are:

$\begin{cases} x = 3 \\ y = 1 + t \\ z = t \end{cases}$ and the vector equation is $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(c) A Cartesian equation of the first plane is: $x - y - 2z = 1$, so the planes are parallel and there is no intersection.

Note: If we solve the system for intersection, we will have:

$\begin{cases} x - y - 2z = 1 \\ x - y - 2z = 5 \end{cases}$, and this is obviously inconsistent.

(d) A vector perpendicular to the first plane is:

$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 8 \end{pmatrix} = 8\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$; hence, a Cartesian equation of the plane is:

$$2(x-1) + (y-0) + (z-2) = 0 \Rightarrow 2x + y + z = 4.$$

Now, we have to solve the system:

$$\begin{cases} 2x + y + z = 4 \\ 3x - y - z = 3 \end{cases}.$$

Adding the equations, we have: $5x=7\Rightarrow x=\frac{7}{5}$; therefore:

$$\begin{cases} 2\left(\frac{7}{5}\right)+y+z=4\Rightarrow y+z=\frac{6}{5} \\ 3\left(\frac{7}{5}\right)-y-z=3\Rightarrow -y-z=-\frac{6}{5} \end{cases} \Rightarrow y=\frac{6}{5}-z$$

Hence, parametric equations of the line of intersection are:

$$\begin{cases} x=\frac{7}{5} \\ y=\frac{6}{5}-t \\ z=t \end{cases} \text{ and the vector equation is } \boldsymbol{r}=\begin{pmatrix} \frac{7}{5} \\ \frac{6}{5} \\ 0 \end{pmatrix}+t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

7. A direction vector of the line is perpendicular to the normals of both planes:

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\times\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}=\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}=3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Hence, the normal to the required plane is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, and a vector equation for the plane is:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\Rightarrow\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$$

A Cartesian equation of the plane is: $x-y+z=0$

8. A normal to the plane is perpendicular to the vector $\overrightarrow{AB}$ and the direction vector of the given plane:

$$\begin{pmatrix} 3-1 \\ 2-2 \\ 1-3 \end{pmatrix}\times\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}\times\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}=\begin{pmatrix} -2 \\ -12 \\ -2 \end{pmatrix}=-2\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}.$$

Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}\times\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\Rightarrow\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=16$$

A Cartesian equation of the plane is: $x+6y+z=16$

9. A Cartesian equation of the line is:

$\frac{x-1}{3-1}=\frac{y-2}{1-2}=\frac{z-5}{1-5}\Rightarrow\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-5}{-4}$; hence, parametric equations of the line are:

$$\begin{cases}x=1+2t\\ y=2-t\\ z=5-4t\end{cases}$$

The distance from $(2,-1,5)$ to a point on the line is:

$d^2=(1+2t-2)^2+(2-t+1)^2+(5-4t-5)^2=21t^2-10t+10$.

The distance is a minimum when $\frac{d\left(21t^2-10t+10\right)}{dt}=0\Rightarrow 42t-10=0\Rightarrow t=\frac{5}{21}$

Hence, the point on the line with $t=\frac{5}{21}$ is the closest to the origin.

$$\begin{cases}x=1+2\left(\frac{5}{21}\right)\\ y=2-\left(\frac{5}{21}\right)\\ z=5-4\left(\frac{5}{21}\right)\end{cases}\Rightarrow A\left(\frac{31}{21},\frac{37}{21},\frac{85}{21}\right)$$

10. A normal to the plane is perpendicular to the direction vectors of both lines:

$\begin{pmatrix}1\\-2\\1\end{pmatrix}\times\begin{pmatrix}3\\2\\4\end{pmatrix}=\begin{pmatrix}-10\\-1\\8\end{pmatrix}=-\begin{pmatrix}10\\1\\-8\end{pmatrix}$. Therefore, the plane contains the point $(-1,2,3)$ and is perpendicular to the vector $\begin{pmatrix}10\\1\\-8\end{pmatrix}$; hence, its vector equation is:

$\begin{pmatrix}10\\1\\-8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}10\\1\\-8\end{pmatrix}\begin{pmatrix}-1\\2\\3\end{pmatrix}\Rightarrow\begin{pmatrix}10\\1\\-8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-32$, and Cartesian:

$10x+y-8z=-32$

11. The plane contains points of the first line, so the point $(1,1,2)$ is on the plane.

A normal to the plane is perpendicular to the direction vectors of both lines:

$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5$, and Cartesian: $4x - 3y + 2z = 5$

12. The equation $\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$ can be written in the form $BCx + ACy + ABz = ABC$

and this is a Cartesian equation of the plane whose normal is vector $\begin{pmatrix} BC \\ AC \\ AB \end{pmatrix}$ and contains the point $(A,0,0)$.

Note: The plane contains the points $(A,0,0)$, $(0,B,0)$ and $(0,0,C)$.

13. It is easier to write parametric equations of the plane, since we know one point and two non-parallel direction vectors:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

Note: If we need another form of the line, we can proceed as follows:
The normals of both planes are parallel to our plane. Hence, their vector product is normal to our plane:

$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$\begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -42$, and Cartesian: $9x + 22y + 12z = -42$

14. It is easier to write parametric equations of the plane, since we know one point and two non-parallel direction vectors (a direction vector of the line and normal of the plane):

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\0\end{pmatrix}+r\begin{pmatrix}1\\-2\\1\end{pmatrix}+s\begin{pmatrix}2\\-3\\4\end{pmatrix}$$

Note: If we need another form of the line, we can proceed as follows:
The normal of the plane and the direction vector of the line are parallel to our plane.
Hence, their vector product is normal to our plane:

$\begin{pmatrix}2\\-3\\4\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}5\\2\\-1\end{pmatrix}$. Therefore, a vector equation for the plane is:

$\begin{pmatrix}5\\2\\-1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\2\\-1\end{pmatrix}\begin{pmatrix}2\\3\\0\end{pmatrix}\Rightarrow\begin{pmatrix}5\\2\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=16$, and Cartesian:

$5x+2y-z=16$

Chapter 9 practice questions

1. **(a)** $\boldsymbol{u}+2\boldsymbol{v}=(-\boldsymbol{i}+2\boldsymbol{j})+2(3\boldsymbol{i}+5\boldsymbol{j})=-\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{i}+10\boldsymbol{j}=5\boldsymbol{i}+12\boldsymbol{j}$

(b) The unit vector in the direction of $\boldsymbol{u}+2\boldsymbol{v}$ is: $\frac{1}{\sqrt{5^2+12^2}}(5\boldsymbol{i}+12\boldsymbol{j})=\frac{1}{13}(5\boldsymbol{i}+12\boldsymbol{j})$

So, $\boldsymbol{w}=26\cdot\frac{1}{13}(5\boldsymbol{i}+12\boldsymbol{j})=2(5\boldsymbol{i}+12\boldsymbol{j})=10\boldsymbol{i}+24\boldsymbol{j}$

2. **(a)** $\left|\overrightarrow{OA}\right|=\sqrt{6^2+0^2}=6$, so A lies on the circle.

$\left|\overrightarrow{OB}\right|=\sqrt{(-6)^2+0^2}=6$, so B lies on the circle.

$\left|\overrightarrow{OC}\right|=\sqrt{5^2+\sqrt{11}^2}=6$, so C lies on the circle.

(b) $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}5\\\sqrt{11}\end{pmatrix}-\begin{pmatrix}6\\0\end{pmatrix}=\begin{pmatrix}-1\\\sqrt{11}\end{pmatrix}$

(c) **Method 1** Using a scalar product

$$\cos O\hat{A}C = \frac{\overrightarrow{AO}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AO}\right|\left|\overrightarrow{AC}\right|} = \frac{\begin{pmatrix}-6\\0\end{pmatrix}\begin{pmatrix}-1\\\sqrt{11}\end{pmatrix}}{6\sqrt{(-1)^2+\sqrt{11}^2}} = \frac{6}{6\sqrt{12}} = \frac{\sqrt{3}}{6}$$

Method 2 Using a cosine rule in triangle OAC

In triangle OAC, SSS is given: $OA = OC = 6$, and $AC = \left|\begin{pmatrix}-1\\\sqrt{11}\end{pmatrix}\right| = \sqrt{12}$;

Hence, $\cos O\hat{A}C = \dfrac{6^2+\left(\sqrt{12}\right)^2-6^2}{2\cdot\sqrt{12}} = \dfrac{12}{2\sqrt{12}} = \dfrac{\sqrt{3}}{6}$

(d) **Method 1** Using the result from **(c)**

Using the Pythagorean identity for sine, $\sin^2\theta = 1-\cos^2\theta$, and the fact that sine is positive for angles from $0^\circ - 180^\circ$ we have:

$\sin O\hat{A}C = \sqrt{1-\left(\dfrac{\sqrt{3}}{6}\right)^2} = \sqrt{1-\dfrac{1}{12}} = \sqrt{\dfrac{11}{12}}$. Hence,

$$A = \frac{1}{2}|AB||AC|\sin\hat{A} = \frac{1}{2}12\cdot\sqrt{12}\sqrt{\frac{11}{12}} = 6\sqrt{11}$$

Method 2 Finding the area using side and height dimensions

In triangle ABC, side $|AB| = 12$; the height on this side is the second coordinate of point C, so: $A = \dfrac{1}{2}12\cdot\sqrt{11} = 6\sqrt{11}$.

3. $\boldsymbol{u}+\boldsymbol{v} = 4\boldsymbol{i}+3\boldsymbol{j}$

Then, $a(4\boldsymbol{i}+3\boldsymbol{j}) = 8\boldsymbol{i}+(b-2)\boldsymbol{j} \Rightarrow \begin{matrix}4a=8\\3a=b-2\end{matrix} \Rightarrow \begin{matrix}a=2\\6+2=b\Rightarrow b=8\end{matrix}$

4. **(a)** The speed of T: $\left|\begin{pmatrix}18\\24\end{pmatrix}\right| = \sqrt{18^2+24^2} = 30\,\text{km/h}$

The speed of C: $\left|\begin{pmatrix}36\\-16\end{pmatrix}\right| = \sqrt{36^2+(-16)^2} = \sqrt{1552} \approx 39.4\text{ km/h}$

(b) **(i)** After half an hour, the vehicles have covered halve the distance:

$$\frac{1}{2}\begin{pmatrix}18\\24\end{pmatrix}=\begin{pmatrix}9\\12\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}36\\-16\end{pmatrix}=\begin{pmatrix}18\\-8\end{pmatrix}$$

(ii) The vector joining their positions at 06:30 is $\begin{pmatrix}9-18\\12-(-8)\end{pmatrix}=\begin{pmatrix}-9\\20\end{pmatrix}$; hence,

the distance between the vehicles is: $\left|\begin{pmatrix}-9\\20\end{pmatrix}\right|=\sqrt{9^2+20^2}=\sqrt{481}\approx 21.9$ km.

(c) T must continue until its position vector is $\begin{pmatrix}18\\k\end{pmatrix}$, so until $k=24$. At that point, its position is $\begin{pmatrix}18\\24\end{pmatrix}$. To reach this position, it must travel for a total of one hour.

Hence, the crew starts work at 07:00.

(d) The southern (C) crew lays: $800\cdot 5=4000$ m of cable.
The northern (T) crew lays: $800\cdot 4.5=3600$ m of cable.
Their starting points were $24-(-8)=32$ km apart; hence, they are now $32-3.6-4=24.4$ km apart.

(e) The position vector of T at 11:30 is $\begin{pmatrix}18\\24-3.6\end{pmatrix}=\begin{pmatrix}18\\20.4\end{pmatrix}$

The distance to base camp is: $\left|\begin{pmatrix}18\\20.4\end{pmatrix}\right|=\sqrt{18^2+20.4^2}=\sqrt{740.16}\approx 27.2$ km.

The time needed to cover this distance is: $\frac{27.2}{30}\cdot 60=54.4\approx 54$ minutes.

5. **(a)** **(i)** Initially, Aircraft 1 is at position $\boldsymbol{r}=\begin{pmatrix}16\\12\end{pmatrix}$; hence, its distance from the origin is: $\left|\begin{pmatrix}16\\12\end{pmatrix}\right|=\sqrt{16^2+12^2}=20$ km

(ii) The velocity vector is $\boldsymbol{v}=\begin{pmatrix}12\\-5\end{pmatrix}$; hence, its speed is:

$$\left|\begin{pmatrix}12\\-5\end{pmatrix}\right|=\sqrt{12^2+(-5)^2}=13\text{ km/min.}$$

(b) $$\boldsymbol{r}=\begin{pmatrix}16\\12\end{pmatrix}+t\begin{pmatrix}12\\-5\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}16+12t\\12-5t\end{pmatrix}\Rightarrow\begin{matrix}x=16+12t\\y=12-5t\end{matrix}$$

From the first equation, we have $t=\frac{x-16}{12}$. Substituting into the second equation:

$$y=12-5\frac{x-16}{12}=\frac{144-5x+80}{12}\Rightarrow 12y=224-5x\Rightarrow 5x+12y=224$$

Note: If we multiply the vector equation of the line by the vector perpendicular to the direction vector, we can find the result quite quickly.

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}16\\12\end{pmatrix}+t\begin{pmatrix}12\\-5\end{pmatrix}\Rightarrow\begin{pmatrix}5\\12\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\12\end{pmatrix}\begin{pmatrix}16\\12\end{pmatrix}+t\begin{pmatrix}5\\12\end{pmatrix}\begin{pmatrix}12\\-5\end{pmatrix}$$. Now we have:

$$5x+12y=5\cdot16+12\cdot12+t\cdot0\Rightarrow 5x+12y=224$$

(c) We have to determine the angle between the direction vectors:

$\begin{pmatrix}12\\-5\end{pmatrix}\cdot\begin{pmatrix}2.5\\6\end{pmatrix}=0$; hence, the angle between the paths of the aircrafts is 90°

(d) **(i)** $$\boldsymbol{r}=\begin{pmatrix}23\\-5\end{pmatrix}+t\begin{pmatrix}2.5\\6\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}23+2.5t\\-5+6t\end{pmatrix}\Rightarrow\begin{matrix}x=23+2.5t\Rightarrow t=\frac{x-23}{2.5}\\y=-5+6t\Rightarrow t=\frac{y+5}{6}\end{matrix}$$

Hence, $\frac{x-23}{2.5}=\frac{y+5}{6}$

Multiplying by 30:

$$30\frac{x-23}{2.5}=30\frac{y+5}{6}\Rightarrow 12(x-23)=5(y+5)\Rightarrow 12x-5y=301$$

Note: We could also have used the method from **(b)**.

(ii) $$\begin{cases}5x+12y=224\\12x-5y=301\end{cases}\Rightarrow 169x=4732\Rightarrow x=28,\ y=\frac{12\cdot28-301}{5}=7$$

Hence, the paths cross at the point $(28,7)$.

(e) We will determine the time at which each of the planes is at $(28,7)$.

For Aircraft 1:

$$\begin{pmatrix}28\\7\end{pmatrix}=\begin{pmatrix}16+12t\\12-5t\end{pmatrix}\Rightarrow\begin{matrix}28-16=12t\\7-12=-5t\end{matrix}\Rightarrow t=1$$

For Aircraft 2:

$$\begin{pmatrix} 28 \\ 7 \end{pmatrix} = \begin{pmatrix} 23+2.5t \\ -5+6t \end{pmatrix} \Rightarrow \begin{matrix} 28-23=2.5t \\ 7+5=6t \end{matrix} \Rightarrow t=2$$

So, the planes are not at the point where the two paths cross at the same time, i.e. the planes do not collide.

6. **Method 1**

If (x, y) is a point on the line, then the vector $\begin{pmatrix} x-4 \\ y+1 \end{pmatrix}$ is the vector on the line, and it is perpendicular to the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Hence, their dot product is zero:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} x-4 \\ y+1 \end{pmatrix} = 0$$

So, $0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} x-4 \\ y+1 \end{pmatrix} = 2(x-4)+3(y+1) = 2x-8+3y+3 = 2x+3y-5$ and the equation of the line is: $2x+3y=5$

Method 2

If vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is perpendicular to the line, then the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$,or $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, is a direction vector of the line. So, a vector equation of the line is $\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Now, we have to transform the equation into Cartesian form:

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4+3t \\ -1-2t \end{pmatrix} \Rightarrow \begin{cases} x = 4+3t \\ y = -1-2t \end{cases} \Rightarrow \begin{cases} 2x = 8+6t \\ 3y = -3-6t \end{cases} \Rightarrow 2x+3y=5$$

7. **(a)** At 13:00 $t=1$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + 1\cdot\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 6 \\ 20 \end{pmatrix}$

(b) **(i)** The velocity vector is: $\begin{pmatrix} x \\ y \end{pmatrix}_{t=1} - \begin{pmatrix} x \\ y \end{pmatrix}_{t=0} = \begin{pmatrix} 6 \\ 20 \end{pmatrix} - \begin{pmatrix} 0 \\ 28 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$

(ii) The speed is the magnitude of the velocity vector; therefore:

$$\left|\begin{pmatrix} 6 \\ -8 \end{pmatrix}\right| = \sqrt{6^2 + (-8)^2} = 10\,\text{km/h}$$

(c)
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 6t \\ 28-8t \end{pmatrix}$$
$$\Rightarrow \begin{cases} x = 6t & /\cdot 4 \\ y = 28-8t & /\cdot 3 \end{cases} \Rightarrow \begin{cases} 4x = 24t \\ 3y = 84-24t \end{cases} \Rightarrow 4x+3y=84$$

(d) The two ships will collide if the point $(18,4)$ is on the line. So:
$$\begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ -8 \end{pmatrix} \Rightarrow \begin{matrix} 18 = 6t \\ 4 = 28-8t \end{matrix} \Rightarrow t = 3$$
Therefore, the ships will collide at $t = 12+3 = 15:00$ hours.

(e)
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 4 \end{pmatrix} + (t-1) \cdot \begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 18+5t-5 \\ 4+12t-12 \end{pmatrix} = \begin{pmatrix} 13+5t \\ -8+12t \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix} + t \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

(f) At $t=3$, *Aristides* is at $\begin{pmatrix} 18 \\ 4 \end{pmatrix}$ and *Boadicea* is at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix} + 3 \cdot \begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 28 \\ 28 \end{pmatrix}$.

Therefore, their distance vector is: $\begin{pmatrix} 28 \\ 28 \end{pmatrix} - \begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 24 \end{pmatrix}$; hence, the ships are $\sqrt{10^2+24^2} = \sqrt{676} = 26\,\text{km}$ apart.

8. **(a)**
$$\begin{pmatrix} 2x \\ x-3 \end{pmatrix} \begin{pmatrix} x+1 \\ 5 \end{pmatrix} = 0 \Rightarrow 2x^2+2x+5x-15=0 \Rightarrow 2x^2+7x-15=0$$

(b)
$$2x^2+7x-15=0 \Rightarrow (2x-3)(x+5) \Rightarrow x = \frac{3}{2}, x = -5$$

9. **(a)** **(i)**
$$\overrightarrow{OA} = \begin{pmatrix} 240 \\ 70 \end{pmatrix} \Rightarrow \left|\overrightarrow{OA}\right| = \sqrt{240^2+70^2} = 250$$

So, the unit vector is: $\frac{1}{250}\begin{pmatrix} 240 \\ 70 \end{pmatrix} = \begin{pmatrix} \frac{24}{25} \\ \frac{7}{25} \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$

(ii)
$$\boldsymbol{v} = 300\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 288 \\ 84 \end{pmatrix}$$

(iii) $t = \frac{250}{300} = \frac{5}{6}$ hour, or 50 minutes

(b) $\overrightarrow{AB} = \begin{pmatrix} 480-240 \\ 250-70 \end{pmatrix} = \begin{pmatrix} 240 \\ 180 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 240 \\ 70 \end{pmatrix}\begin{pmatrix} 240 \\ 180 \end{pmatrix}}{\left|\begin{pmatrix} 240 \\ 70 \end{pmatrix}\right|\left|\begin{pmatrix} 240 \\ 180 \end{pmatrix}\right|} = \frac{70200}{250\cdot 300} = 0.936$$

So, $\theta = \cos^{-1} 0.936 \approx 20.609^\circ = 20.6^\circ$

(c) **(i)** $\overrightarrow{AX} = \begin{pmatrix} 339-240 \\ 238-70 \end{pmatrix} = \begin{pmatrix} 99 \\ 168 \end{pmatrix}$

(ii) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}\begin{pmatrix} 240 \\ 180 \end{pmatrix} = -3\cdot 240 + 4\cdot 180 = 0$; hence, $\boldsymbol{n} \perp \overrightarrow{AB}$.

(iii) The scalar projection of $\overrightarrow{AX}$ in the direction of $\boldsymbol{n}$ is

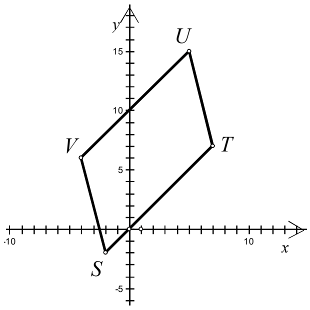
$$\frac{1}{5}\begin{pmatrix} 99 \\ 168 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{-297+672}{5} = 75;$$

hence, the distance XY is 75 km.

(d) Using Pythagoras' theorem, we can find the distance from A to Y using the distances AX and XY. So, $AX = \sqrt{99^2 + 168^2} = \sqrt{38025} = 195$; hence,

$AY = \sqrt{195^2 - 75^2} = \sqrt{32400} = 180\text{km}$

10. Refer to the diagram below.



(a) $\overrightarrow{ST} = \begin{pmatrix} 7-(-2) \\ 7-(-2) \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$, and, since $STUV$ is a parallelogram, $\overrightarrow{VU} = \overrightarrow{ST} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$

$\overrightarrow{VU} = \boldsymbol{u} - \boldsymbol{v} = \begin{pmatrix}9\\9\end{pmatrix} \Rightarrow \begin{pmatrix}5\\15\end{pmatrix} - \boldsymbol{v} = \begin{pmatrix}9\\9\end{pmatrix} \Rightarrow \begin{pmatrix}5\\15\end{pmatrix} - \begin{pmatrix}9\\9\end{pmatrix} = \boldsymbol{v} \Rightarrow \begin{pmatrix}-4\\6\end{pmatrix} = \boldsymbol{v}$, and the coordinates of V are: $(-4,6)$.

(b) The line contains the point $(-4,6)$ and the direction vector is parallel to $\begin{pmatrix}9\\9\end{pmatrix}$.

So, for the direction vector, we can use the vector $\begin{pmatrix}1\\1\end{pmatrix}$. So, $\boldsymbol{r} = \begin{pmatrix}-4\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\1\end{pmatrix}$.

Note: We can also use the direction vector $\begin{pmatrix}1\\1\end{pmatrix}$ and initial point $(5,15)$.

(c) $\begin{pmatrix}1\\11\end{pmatrix} = \begin{pmatrix}-4\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \begin{matrix}1 = -4 + \lambda \Rightarrow \lambda = 5\\ 11 = 6 + \lambda \Rightarrow \lambda = 5\end{matrix}$

So, the point is on the line when $\lambda = 5$

(d) **(i)** $\overrightarrow{EW} = \begin{pmatrix}a-1\\17-11\end{pmatrix} = \begin{pmatrix}a-1\\6\end{pmatrix}$

$\left|\overrightarrow{EW}\right| = \sqrt{(a-1)^2 + 36} = \sqrt{a^2 - 2a + 37} = 2\sqrt{13} \Rightarrow a^2 - 3a + 37 = 52$

So: $a^2 - 2a - 15 = 0 \Rightarrow a_1 = -3,\ a_2 = 5$

(ii) For $a = -3$: $\overrightarrow{EW} = \begin{pmatrix}-4\\6\end{pmatrix}$, $\overrightarrow{ET} = \begin{pmatrix}7-1\\7-11\end{pmatrix} = \begin{pmatrix}6\\-4\end{pmatrix}$

$$\text{So, } \cos\theta = \frac{\begin{pmatrix}-4\\6\end{pmatrix}\begin{pmatrix}6\\-4\end{pmatrix}}{\left|\begin{pmatrix}-4\\6\end{pmatrix}\right|\left|\begin{pmatrix}6\\-4\end{pmatrix}\right|} = \frac{-24-24}{\sqrt{16+36}\sqrt{16+36}} = \frac{-48}{52} = -\frac{12}{13},$$

and $\theta = \cos^{-1}\left(-\frac{12}{13}\right) \approx 157.38^\circ \approx 157^\circ$

11. The coordinates of the point of intersection should satisfy both equations.

So: $\begin{pmatrix}5+3\lambda\\1-2\lambda\end{pmatrix} = \begin{pmatrix}-2+4t\\2+t\end{pmatrix} \Rightarrow \begin{cases}3\lambda - 4t = -7\\ t = -2\lambda - 1\end{cases} \Rightarrow 3\lambda + 8\lambda + 4 = -7 \Rightarrow \lambda = -1,\ t = 1$

Therefore, the position vector of point is $\overrightarrow{OP} = \begin{pmatrix}5-3\\1+2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$

Note: We can transform the vector equations to Cartesian form ($2x+3y=13$, and $x-4y=-10$), and then solve the system.

12. **(a)** $\overrightarrow{OR}=\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}10\\1\end{pmatrix}-\begin{pmatrix}7\\3\end{pmatrix}=\begin{pmatrix}3\\-2\end{pmatrix}$

(b) $\cos O\hat{P}Q=\frac{\overrightarrow{PO}\overrightarrow{PQ}}{\left|\overrightarrow{PO}\right|\left|\overrightarrow{PQ}\right|}=\frac{\begin{pmatrix}-7\\-3\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix}}{\sqrt{49+9}\sqrt{9+4}}=\frac{-21+6}{\sqrt{58}\sqrt{13}}=\frac{-15}{\sqrt{754}}$

(c) **(i)** Since $P\hat{Q}R+O\hat{P}Q=180^{\circ}$, $\cos PQR=\cos\left(180^{\circ}-O\hat{P}Q\right)=-\cos O\hat{P}Q$.

(ii) Using the Pythagorean identity for sine and the fact that the sine of angles in a triangle is always positive, we have:

$$\sin P\hat{Q}R=\sqrt{1-\cos^2 P\hat{Q}R}=\sqrt{1-\cos^2 O\hat{P}Q}$$
$$=\sqrt{1-\frac{15^2}{754}}=\sqrt{\frac{529}{754}}=\frac{23}{\sqrt{754}}$$

(iii) Area of the parallelogram:

$$Area=\left|\overrightarrow{OR}\right|\left|\overrightarrow{OP}\right|\sin\theta=\left|\begin{pmatrix}3\\-2\end{pmatrix}\right|\left|\begin{pmatrix}7\\3\end{pmatrix}\right|=\sqrt{13}\sqrt{53}\frac{23}{\sqrt{13\cdot 53}}=23\text{ square units}$$

13. **(a)** $\overrightarrow{OB}=\begin{pmatrix}-1\\7\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}8\\9\end{pmatrix}$

(b) To find D, we have to find the vector of the side of the parallelogram: $\overrightarrow{AD}=\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}8\\9\end{pmatrix}-\begin{pmatrix}-1\\7\end{pmatrix}=\begin{pmatrix}9\\2\end{pmatrix}$. Now, we can find the position vector of D: $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}9\\2\end{pmatrix}=\begin{pmatrix}11\\4\end{pmatrix}$. Hence, $d=11$.

(c) $\overrightarrow{BD}=\begin{pmatrix}11\\4\end{pmatrix}-\begin{pmatrix}-1\\7\end{pmatrix}=\begin{pmatrix}12\\-3\end{pmatrix}$

(d) **(i)** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}+t\overrightarrow{BD}=\begin{pmatrix}-1\\7\end{pmatrix}+t\begin{pmatrix}12\\-3\end{pmatrix}$

Note: For the direction vector, we can use $\begin{pmatrix}4\\-1\end{pmatrix}$. Then the equation would be $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}+t\begin{pmatrix}4\\-1\end{pmatrix}$.

(ii) At point B, $t=0$. We can see that $\begin{pmatrix}-1\\7\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}+0\cdot\begin{pmatrix}12\\-3\end{pmatrix}$

(e) $$\begin{pmatrix}7\\5\end{pmatrix}=\begin{pmatrix}-1\\7\end{pmatrix}+t\begin{pmatrix}12\\-3\end{pmatrix}\Rightarrow\begin{matrix}7=-1+12t\Rightarrow t=\dfrac{8}{12}\\5=7-3t\Rightarrow t=\dfrac{-2}{-3}\end{matrix}\Rightarrow t=\frac{2}{3}$$

(f) $\overrightarrow{CP}=\overrightarrow{OP}-\overrightarrow{OC}=\begin{pmatrix}7\\5\end{pmatrix}-\begin{pmatrix}8\\9\end{pmatrix}=\begin{pmatrix}-1\\-4\end{pmatrix}$

$\overrightarrow{CP}\cdot\overrightarrow{BD}=\begin{pmatrix}-1\\-4\end{pmatrix}\begin{pmatrix}12\\-3\end{pmatrix}=-12+12=0$; hence, $\overrightarrow{CP}\perp\overrightarrow{BD}$.

14. **(a)** **(i)** $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(-5\boldsymbol{i}-5\boldsymbol{j})-(\boldsymbol{i}-3\boldsymbol{j})=-6\boldsymbol{i}-2\boldsymbol{j}$

(ii) $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=\overrightarrow{OA}+\overrightarrow{BC}=(4\boldsymbol{i}+2\boldsymbol{j})+(-6\boldsymbol{i}-2\boldsymbol{j})=-2\boldsymbol{i}$

(b) $\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=-2\boldsymbol{i}-(\boldsymbol{i}-3\boldsymbol{j})=-3\boldsymbol{i}+3\boldsymbol{j}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(-5\boldsymbol{i}-5\boldsymbol{j})-(4\boldsymbol{i}+2\boldsymbol{j})=-9\boldsymbol{i}-7\boldsymbol{j}$

$$\cos\theta=\frac{\overrightarrow{BD}\cdot\overrightarrow{AC}}{\left|\overrightarrow{BD}\right|\left|\overrightarrow{AC}\right|}=\frac{27-21}{\sqrt{9+9}\sqrt{81+49}}=\frac{6}{\sqrt{18}\sqrt{130}}=\frac{6}{\sqrt{2340}}$$

$\Rightarrow\theta\approx82.87^\circ\approx82.9^\circ$

(c) $\boldsymbol{r}=\boldsymbol{i}-3\boldsymbol{j}+t(2\boldsymbol{i}+7\boldsymbol{j})$

(d) We have to solve the vector equation: $\boldsymbol{i}-3\boldsymbol{j}+t(2\boldsymbol{i}+7\boldsymbol{j})=4\boldsymbol{i}+2\boldsymbol{j}+s(\boldsymbol{i}+4\boldsymbol{j})$.

Hence, $\begin{matrix}1+2t=4+s\\-3+7t=2+4s\end{matrix}\Rightarrow s=-3+2t\Rightarrow-3+7t=2-12+8t\Rightarrow t=7,\ s=11$

So, the position vector of the intersection is: $\boldsymbol{r}=\boldsymbol{i}-3\boldsymbol{j}+7\cdot(2\boldsymbol{i}+7\boldsymbol{j})=15\boldsymbol{i}+46\boldsymbol{j}$.

15. **(a)** The unit vector in the direction of $\begin{pmatrix}3\\4\\0\end{pmatrix}$ is: $\dfrac{1}{\sqrt{3^2+4^2+0^2}}\begin{pmatrix}3\\4\\0\end{pmatrix}=\dfrac{1}{5}\begin{pmatrix}3\\4\\0\end{pmatrix}$, so the velocity vector of the balloon is: $\dfrac{18}{5}\begin{pmatrix}3\\4\\0\end{pmatrix}$. Therefore, the equation of the path of the balloon is the same as the equation of the line through $(0,0,5)$ with direction vector $\dfrac{18}{5}\begin{pmatrix}3\\4\\0\end{pmatrix}$: $\boldsymbol{b}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\5\end{pmatrix}+t\cdot\dfrac{18}{5}\begin{pmatrix}3\\4\\0\end{pmatrix}$.

(b) **(i)** $t=0\Rightarrow\begin{pmatrix}49\\32\\0\end{pmatrix}$

(ii) The velocity vector is $\begin{pmatrix}-48\\-24\\6\end{pmatrix}$, so the speed is:

$$\left|\begin{pmatrix}-48\\-24\\6\end{pmatrix}\right|=\sqrt{48^2+24^2+6^2}=54\,\text{km/h}$$

(c) **(i)** At R: $\begin{pmatrix}0\\0\\5\end{pmatrix}+t\cdot\dfrac{18}{5}\begin{pmatrix}3\\4\\0\end{pmatrix}=\begin{pmatrix}49\\32\\0\end{pmatrix}+t\begin{pmatrix}-48\\-24\\6\end{pmatrix}\Rightarrow\begin{aligned}t\frac{18\cdot3}{5}&=49-48t\\t\frac{18\cdot4}{5}&=32-24t\\5&=6t\end{aligned}$

$t=\dfrac{5}{6}$ hour (50 minutes) satisfies all three equations.

(ii) Substituting $t=\dfrac{5}{6}$ into the expression for h (or b):

$$\boldsymbol{h}=\begin{pmatrix}49\\32\\0\end{pmatrix}+\frac{5}{6}\begin{pmatrix}-48\\-24\\6\end{pmatrix}=\begin{pmatrix}49-40\\32-20\\0+5\end{pmatrix}=\begin{pmatrix}9\\12\\5\end{pmatrix}.$$ Hence, $R(9,12,5)$.

16. **(a)** **(i)** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 200 \\ 400 \end{pmatrix} - \begin{pmatrix} -600 \\ -200 \end{pmatrix} = \begin{pmatrix} 800 \\ 600 \end{pmatrix}$

(ii) $\left|\overrightarrow{AB}\right| = \sqrt{800^2 + 600^2} = 1000$; hence, the unit vector is: $\frac{1}{1000}\begin{pmatrix} 800 \\ 600 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$

(b) **(i)** $\boldsymbol{v} = 250\begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 200 \\ 150 \end{pmatrix}$

(ii) At 13:00 $t = 1$, so: $\begin{pmatrix} -600 \\ -200 \end{pmatrix} + 1 \cdot \begin{pmatrix} 200 \\ 150 \end{pmatrix} = \begin{pmatrix} -400 \\ -50 \end{pmatrix}$

(iii) The distance from A to B is 1000 km, and, since the velocity of the aircraft is 250km/h, the time is $\frac{1000}{250} = 4$ hours; hence, the aircraft is flying over town B at 16:00.

(c) **Method 1** Evaluating the time needed

Time taken to travel from A to B to C is 9 hours $\left(\frac{81}{9}\text{ hours}\right)$. The warning light will go on after 16 000 litres of fuel have been used.

Time taken to use 16 000 litres $= \frac{16\,000}{1800} = \frac{80}{9}$. Hence, $\frac{1}{9}$ hour remains and the distance to town C is $\frac{1}{9}250 \approx 27.8$ km

Method 2 Evaluating the distances needed

The distance from A to B to C is 2250 km. The distance covered with 16 000 litres of fuel is: $\frac{16000}{1800} \times 250 \approx 2222.22$ km. So, the distance to town C is

$2250 - 2222.22 \approx 27.8$ km

Method 3 Evaluating fuel usage

Fuel used from A to B = $1800 \times 4 = 7200$ litres.

Fuel remaining until the light goes on = 16 000 – 7200 = 8800 litres.

Number of hours before the warning light goes on: $\frac{8800}{1800} = 4\frac{8}{9}$ hours; therefore, the time remaining is $\frac{1}{9}$ hour, and the distance to town C is: $\frac{1}{9}250 \approx 27.8$ km.

17. **(a)** The vectors are perpendicular if their scalar product is zero.
So, first, find the vectors:

$$\overrightarrow{QR}=\begin{pmatrix}1-3\\0-3\\2c-5\end{pmatrix}=\begin{pmatrix}-2\\-3\\2c-5\end{pmatrix},\ \overrightarrow{PR}=\begin{pmatrix}1-4\\0-1\\2c+1\end{pmatrix}=\begin{pmatrix}-3\\-1\\2c+1\end{pmatrix}$$

$$\overrightarrow{QR}\cdot\overrightarrow{PR}=\begin{pmatrix}-2\\-3\\2c-5\end{pmatrix}\begin{pmatrix}-3\\-1\\2c+1\end{pmatrix}=6+3+(2c-5)(2c+1)=4c^2-8c+4$$

The vectors are perpendicular if: $4c^2-8c+4=0\Rightarrow 4\left(c-1\right)^2=0\Rightarrow c=1$

(b) $$\overrightarrow{PR}=\begin{pmatrix}-3\\-1\\2\left(1\right)+1\end{pmatrix}=\begin{pmatrix}-3\\-1\\3\end{pmatrix},\ \overrightarrow{PS}=\begin{pmatrix}1-4\\1-1\\2+1\end{pmatrix}=\begin{pmatrix}-3\\0\\3\end{pmatrix}$$

$$\overrightarrow{PS}\times\overrightarrow{PR}=\begin{pmatrix}-3\\0\\3\end{pmatrix}\times\begin{pmatrix}-3\\-1\\3\end{pmatrix}=\begin{pmatrix}3\\0\\3\end{pmatrix}$$

(c) A vector equation of the line is:

$$\boldsymbol{r}=\begin{pmatrix}3\\3\\5\end{pmatrix}+t\begin{pmatrix}-3\\-1\\3\end{pmatrix}=\begin{pmatrix}3-3t\\3-t\\5+3t\end{pmatrix},\ t\in\mathbb{R}$$

(d) We need one more direction vector (which is not parallel to the direction vector of the line) to determine a normal to the plane. We will take a point on the line and point S : $\overrightarrow{SQ}=\begin{pmatrix}3-1\\3-1\\5-2\end{pmatrix}=\begin{pmatrix}2\\2\\3\end{pmatrix}$. Hence, the normal will be:

$$\begin{pmatrix}2\\2\\3\end{pmatrix}\times\begin{pmatrix}-3\\-1\\3\end{pmatrix}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&2&3\\-3&-1&3\end{vmatrix}=\begin{pmatrix}9\\-15\\4\end{pmatrix}$$

Therefore, the equation will be:

$$9\left(x-1\right)-15\left(y-1\right)+4\left(z-2\right)=0\Rightarrow 9x-15y+4z=2$$

Note: We have a point and two vectors in the plane, so we can write parametric equations of the plane:

$$\boldsymbol{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-3\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}2\\2\\3\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$$

(e) **Method 1**

Shortest distance is: $\dfrac{\left|\overrightarrow{PQ}\cdot\boldsymbol{n}\right|}{|\boldsymbol{n}|}$

Since $\overrightarrow{PQ}=\begin{pmatrix}3-4\\3-1\\5+1\end{pmatrix}=\begin{pmatrix}-1\\2\\6\end{pmatrix}$, we have:

$$\frac{\left|\overrightarrow{PQ}\boldsymbol{n}\right|}{|\boldsymbol{n}|}=\frac{\left|\begin{pmatrix}-1\\2\\6\end{pmatrix}\begin{pmatrix}9\\-15\\4\end{pmatrix}\right|}{\sqrt{81+225+16}}=\frac{15}{\sqrt{322}}$$

Method 2

Use the distance formula for a point (x_0,y_0,z_0) and a plane $ax+by+cz+d=0$:

$$d=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Hence, for the point $P(4,1,-1)$ and the plane $9x-15y+4z-2=0$, the distance is:

$$d=\frac{|9(4)-15(1)+4(-1)-2|}{\sqrt{9^2+15^2+4^2}}=\frac{15}{\sqrt{322}}$$

18. **(a)** $\overrightarrow{AB}=\begin{pmatrix}0-1\\-1-2\\2-1\end{pmatrix}=\begin{pmatrix}-1\\-3\\1\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}1-0\\0+1\\2-2\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$

(b) $\overrightarrow{AB}\times\overrightarrow{BC}=\begin{pmatrix}-1\\-3\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-1&-3&1\\1&1&0\end{vmatrix}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$

(c) We can use the formula for the area of a triangle: $A=\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$. Hence:

$$Area=\frac{1}{2}\left|\begin{pmatrix}-1\\1\\2\end{pmatrix}\right|=\frac{1}{2}\sqrt{1+1+4}=\frac{\sqrt{6}}{2}$$

(d) A normal to the plane is $\boldsymbol{n}=\overrightarrow{AB}\times\overrightarrow{BC}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$. Since point A is on the plane,

the equation is: $-1(x-1)+1(y-2)+2(z-1)=0\Rightarrow -x+y+2z=3$

(e) The normal $\boldsymbol{n}$ is parallel to the required line. Hence,

$$\begin{aligned}x&=2-t\\y&=-1+t\quad\text{, where } t\in\mathbb{R}\\z&=-6+2t\end{aligned}$$

(f) The distance formula for a point (x_0,y_0,z_0) and a plane $ax+by+cz+d=0$ is:

$$d=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Hence, for the point $(2,-1-6)$ and the plane $-x+y+2z-3=0$, the distance is:

$$d=\frac{|(-1)2+1(-1)+2(-6)-3|}{\sqrt{1+1+4}}=\frac{18}{\sqrt{6}}=3\sqrt{6}$$

(g) Since $|\boldsymbol{n}|=\sqrt{1+1+4}=\sqrt{6}$, a unit vector in the direction of $\boldsymbol{n}$ is: $\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}$

(h) First, we will find the point of intersection of the plane and the line through D perpendicular to the plane. Hence, we have to find the intersection of the plane P and the line from part **(e)**.

Since $x = 2 - t,\ y = -1 + t,\ z = -6 + 2t$, we have:

$$-(2-t)+(-1+t)+2(-6+2t)=3 \Rightarrow 6t=18 \Rightarrow t=3$$

So, the point of intersection is $(-1,2,0)$. This point is the midpoint between points D and E. Hence: $(-1,2,0)=\left(\frac{x_E+2}{2},\frac{y_E-1}{2},\frac{z_E-6}{2}\right)\Rightarrow E(-4,5,6)$

19. **(a)** $$\boldsymbol{u}\times\boldsymbol{v}=\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&2&3\\2&-1&2\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}2&3\\-1&2\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}1&3\\2&2\end{vmatrix}+k\begin{vmatrix}1&2\\2&-1\end{vmatrix}=\begin{pmatrix}7\\4\\-5\end{pmatrix}$$

(b) **Method 1**

$$\boldsymbol{w}=\begin{pmatrix}\lambda+2\mu\\2\lambda-\mu\\3\lambda+2\mu\end{pmatrix}$$

The line of intersection of the planes is parallel to $\boldsymbol{u}\times\boldsymbol{v}$. So,

$$\boldsymbol{w}(\boldsymbol{u}\times\boldsymbol{v})=\begin{pmatrix}\lambda+2\mu\\2\lambda-\mu\\3\lambda+2\mu\end{pmatrix}\begin{pmatrix}7\\4\\-5\end{pmatrix}=7\lambda+14\mu+8\lambda-4\mu-15\lambda-10\mu=0 \text{ (for all } \lambda,\mu\text{)}.$$

Hence, $\boldsymbol{w}$ is perpendicular to the line of intersection.

Method 2

The line of intersection is perpendicular to the normals of both planes; hence, to vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, it will be perpendicular to the plane containing those two vectors, that is, to all vectors of the form $\lambda\boldsymbol{u}+\mu\boldsymbol{v}=\boldsymbol{w}$.

Method 3

The line of intersection is perpendicular to the normals of both planes; hence, to vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, for a direction vector $\boldsymbol{d}$ of the line, it holds:

$$\begin{cases}\boldsymbol{d}\cdot\boldsymbol{u}=0\\\boldsymbol{d}\cdot\boldsymbol{v}=0\end{cases}\Rightarrow\boldsymbol{d}(\lambda\boldsymbol{u}+\mu\boldsymbol{v})=\lambda\boldsymbol{d}\cdot\boldsymbol{u}+\mu\boldsymbol{d}\cdot\boldsymbol{v}=0$$

and $\boldsymbol{d}$ is perpendicular to $\boldsymbol{w}$.

20. **(a)** $$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{OB}=\begin{pmatrix}2+2\\1-1\\-2-1\end{pmatrix}=\begin{pmatrix}4\\0\\-3\end{pmatrix}\Rightarrow P(4,0,-3)$$

$$\overrightarrow{OQ}=\overrightarrow{OA}+\overrightarrow{OC}=\begin{pmatrix}2+1\\1+2\\-2+2\end{pmatrix}=\begin{pmatrix}3\\3\\0\end{pmatrix}\Rightarrow Q(3,3,0)$$

$$\overrightarrow{OR}=\overrightarrow{OB}+\overrightarrow{OC}=\begin{pmatrix}2+1\\-1+2\\-1+2\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}\Rightarrow Q(3,1,1)$$

$$\overrightarrow{OS}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\begin{pmatrix}2+2+1\\1-1+2\\-2-1+2\end{pmatrix}=\begin{pmatrix}5\\2\\-1\end{pmatrix}\Rightarrow S(5,2,-1)$$

(b) $\overrightarrow{OA}\times\overrightarrow{OB}=\begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}2\\-1\\-1\end{pmatrix}=\begin{pmatrix}-3\\-2\\-4\end{pmatrix}=-\begin{pmatrix}3\\2\\4\end{pmatrix}$; hence, the equation of the plane is:

$$3(x-2)+2(y-1)+4(z+2)=0\Rightarrow 3x+2y+4z=0$$

Note: Parametric equations of the plane are:

$$\boldsymbol{r}=\begin{pmatrix}1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}4\\0\\3\end{pmatrix}+\mu\begin{pmatrix}3\\3\\0\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$$

(c) $$V=\left|\left(\overrightarrow{OA}\times\overrightarrow{OB}\right)\cdot\overrightarrow{OC}\right|=\left|\begin{pmatrix}-3\\-2\\-4\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|=|-3-4-8|=15$$

21. **(a)** $$\overrightarrow{AB}=\begin{pmatrix}-1+1\\3-2\\5-3\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}0+1\\-1-2\\1-3\end{pmatrix}=\begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

$$\cos\theta=\frac{\begin{pmatrix}0\\1\\2\end{pmatrix}\begin{pmatrix}1\\-3\\-2\end{pmatrix}}{\sqrt{1+4}\sqrt{1+9+4}}=\frac{-7}{\sqrt{5}\sqrt{14}}\Rightarrow\theta\approx 146.789^\circ\approx 147^\circ$$

(b) **Method 1**

$$Area=\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta\approx\frac{1}{2}\sqrt{5}\sqrt{14}\sin 146.789^\circ\approx 2.29\ \text{units}^2$$

Method 2

$$Area=\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta\sin\theta$$

Using the Pythagorean identity for sine and cosine, we have:

$$\sin\theta=+\sqrt{1-\left(\frac{-7}{\sqrt{5}\sqrt{14}}\right)^2}=\sqrt{1-\frac{7}{10}}=\sqrt{\frac{3}{10}}$$

$$Area=\frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta=\frac{1}{2}\sqrt{5}\sqrt{14}\frac{\sqrt{3}}{\sqrt{10}}=\frac{\sqrt{21}}{2}$$

Method 3

$$Area=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\left|\begin{pmatrix}0\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\-3\\-2\end{pmatrix}\right|=\frac{1}{2}\left|\begin{pmatrix}4\\2\\-1\end{pmatrix}\right|=\frac{1}{2}\sqrt{16+4+1}=\frac{\sqrt{21}}{2}$$

(c) **(i)** For $l_1:\boldsymbol{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\2\end{pmatrix}\Rightarrow\begin{cases}x=2\\y=-1+t,\ t\in\mathbb{R}\\z=2t\end{cases}$

For $l_2:\boldsymbol{r}=\begin{pmatrix}-1\\1\\1\end{pmatrix}+s\begin{pmatrix}1\\-3\\-2\end{pmatrix}\Rightarrow\begin{cases}x=-1+s\\y=1-3s,\ s\in\mathbb{R}\\z=1-2s\end{cases}$

(ii) The lines are not parallel, because the direction vectors are not parallel. Hence, we have to solve the system:

$$2=-1+s\Rightarrow s=3$$
$$-1+t=1-3s$$
$$2t=1-2s$$

From the first equation, $s=3$, and substituting into the second equation: $t=2-3(3)=-7$ and third equation: $2(-7)=1-2(3)\Rightarrow-14\neq-5$.

Therefore, the system has no solution and the lines do not intersect.

(d) The shortest distance is given by $\frac{|(\boldsymbol{e}-\boldsymbol{d})(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}{|(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}$, where $\boldsymbol{d}$ and $\boldsymbol{e}$ are position vectors of the points on the lines, and $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are direction vectors of the lines.

$$\boldsymbol{l}_1\times\boldsymbol{l}_2 = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}4\\2\\-1\end{pmatrix}$$

Hence, $$\frac{|(\boldsymbol{e}-\boldsymbol{d})\cdot(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}{|(\boldsymbol{l}_1\times\boldsymbol{l}_2)|} = \frac{\left|\left(\begin{pmatrix}-1\\1\\1\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}4\\2\\-1\end{pmatrix}\right|}{\sqrt{16+4+1}} = \frac{\left|\begin{pmatrix}-3\\2\\1\end{pmatrix}\begin{pmatrix}4\\2\\-1\end{pmatrix}\right|}{\sqrt{21}} = \frac{9}{\sqrt{21}}$$

22. **(a)** **Method 1** Use matrices and their properties:

$$\begin{pmatrix}1&3&-2\\2&1&3\\3&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-6\\7\\6\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&3&-2\\2&1&3\\3&-1&1\end{pmatrix}^{-1}\begin{pmatrix}-6\\7\\6\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

Method 2 Use row operations to solve a system of three equations (first and last steps shown).

$$\begin{array}{rrrl} x & +3y & -2z & =-6\\ & -5y & +7z & =19\\ & -10y & +7z & =24 \end{array} \quad \cdots \quad \Rightarrow \quad \begin{array}{rrrl} x & & & =1\\ & y & & =-1\\ & & z & =2 \end{array}$$

(b) $$\boldsymbol{v}=\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&3&-2\\2&1&3\end{vmatrix}=\begin{pmatrix}11\\-7\\-5\end{pmatrix}$$

(c) $$\boldsymbol{u}=m\begin{pmatrix}1\\3\\-2\end{pmatrix}+n\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}m+2n\\3m+n\\-2m+3n\end{pmatrix}$$

$$\boldsymbol{vu}=\begin{pmatrix}11\\-7\\-5\end{pmatrix}\begin{pmatrix}m+2n\\3m+n\\-2m+3n\end{pmatrix}=11m+22n-21m-7n+10m-15n=0$$

(d) The line is perpendicular to vector $\boldsymbol{v}$ and to the vector $\begin{pmatrix}3\\-1\\1\end{pmatrix}$.

So, a direction vector of the line is:

$$\begin{pmatrix}11\\-7\\-5\end{pmatrix}\times\begin{pmatrix}3\\-1\\1\end{pmatrix}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\11 & -7 & -5\\3 & -1 & 1\end{vmatrix}=\begin{pmatrix}-12\\-26\\10\end{pmatrix}=-2\begin{pmatrix}6\\13\\-5\end{pmatrix},$$

and a vector equation of the line is:

$$\boldsymbol{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}6\\13\\-5\end{pmatrix}$$

23. **(a)** **(i)** $\overrightarrow{AB}=\begin{pmatrix}1-1\\2-3\\4-1\end{pmatrix}=\begin{pmatrix}0\\-1\\3\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}2-1\\3-3\\6-1\end{pmatrix}=\begin{pmatrix}1\\0\\5\end{pmatrix}$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}0\\-1\\3\end{pmatrix}\times\begin{pmatrix}1\\0\\5\end{pmatrix}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\0 & -1 & 3\\1 & 0 & 5\end{vmatrix}=\begin{pmatrix}-5\\3\\1\end{pmatrix}$$

(ii) $Area=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\left|\begin{pmatrix}-5\\3\\1\end{pmatrix}\right|=\frac{1}{2}\sqrt{25+9+1}=\frac{\sqrt{35}}{2}$

(b) **(i)** The plane contains the point A and its normal is $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-5\\3\\1\end{pmatrix}$;

hence, for the Cartesian equation, it holds:

$-5(x-1)+3(y-3)+1(z-1)=0\Rightarrow -5x+3y+z=-5+9+1$

The equation is: $-5x+3y+z=5$

(ii) The line contains the point D and its direction vector is $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-5\\3\\1\end{pmatrix}$;

hence, the Cartesian equation of the line is: $\frac{x-5}{-5}=\frac{y+2}{3}=\frac{z-1}{1}$

(c) We first write the equation of the line in parametric form, then solve the system:

$x = 5 - 5t$

$y = -2 + 3t$

$z = 1 + t$

$-5(5-5t) + 3(-2+3t) + (1+t) = 5$

$\Rightarrow -25 + 25t - 6 + 9t + 1 + t = 5 \Rightarrow 35t = 35 \Rightarrow t = 1$

The point is: $\left.\begin{aligned} x &= 5 - 5(1) = 0 \\ y &= -2 + 3(1) = 1 \\ z &= 1 + (1) = 2 \end{aligned}\right\} \Rightarrow (0, 1, 2)$

(d) The distance is the same as the distance between points D and P:

$d = \sqrt{(5-0)^2 + (-2-1)^2 + (1-2)^2} = \sqrt{25+9+1} = \sqrt{35}$

24. **(a)** The line contains the point A and its direction vector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; hence, the Cartesian equation of the line is:

$\frac{x-2}{1} = \frac{y-5}{1} = \frac{z+1}{1}$

(b) We first write the equation of the line in parametric form, then solve the system:

$x = 2 + t$

$y = 5 + t$

$z = -1 + t$

$1(2+t) + 1(5+t) + 1(-1+t) - 1 = 0$

$\Rightarrow 2 + t + 5 + t - 1 + t - 1 = 0 \Rightarrow 3t = -5 \Rightarrow t = -\frac{5}{3}$

The point is: $\left.\begin{aligned} x &= 2 + \left(-\frac{5}{3}\right) \\ y &= 5 + \left(-\frac{5}{3}\right) \\ z &= -1 + \left(-\frac{5}{3}\right) \end{aligned}\right\} \Rightarrow \left(\frac{1}{3}, \frac{10}{3}, -\frac{8}{3}\right)$

(c) **Method 1**

Denote the image point A'. Then, the point of intersection $\left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)$ of the line and the plane is the midpoint of AA'. Hence,

$$\left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)=\left(\frac{2+x'}{2},\frac{5+y'}{2},\frac{-1+z'}{2}\right)\Rightarrow$$

$$\frac{1}{3}=\frac{2+x'}{2}\Rightarrow x'=\frac{2}{3}-2=-\frac{4}{3}$$

$$\frac{10}{3}=\frac{5+y'}{2}\Rightarrow y'=\frac{20}{3}-5=\frac{5}{3}$$

$$-\frac{8}{3}=\frac{-1+z'}{2}\Rightarrow z'=-\frac{16}{3}+1=-\frac{13}{3}$$

$$A'\left(-\frac{4}{3},\frac{5}{3},-\frac{13}{3}\right)$$

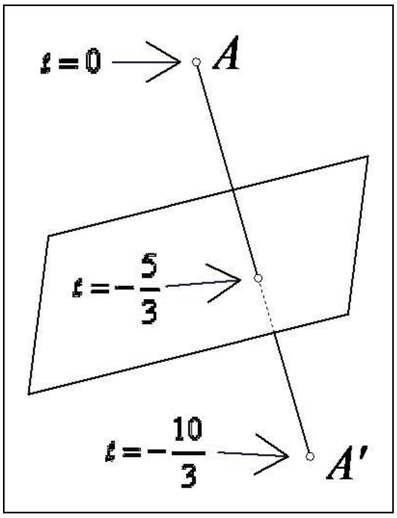
Method 2

Parameters of the points of the line are:

$t=0$ for A

$t=-\frac{5}{3}$ for the intersection; hence, $t=-\frac{10}{3}$ for the reflected point.

$$\text{Thus: }\left.\begin{array}{l} x'=2+\left(-\frac{10}{3}\right) \\ y'=5+\left(-\frac{10}{3}\right) \\ z-=-1+\left(-\frac{10}{3}\right)\end{array}\right\}\Rightarrow\left(-\frac{4}{3},\frac{5}{3},-\frac{13}{3}\right)$$



(d) We have: $\overrightarrow{AB}=\begin{pmatrix}2-2\\0-5\\6+1\end{pmatrix}=\begin{pmatrix}0\\-5\\7\end{pmatrix}$, and a direction vector of the line $\boldsymbol{d}=\begin{pmatrix}1\\1\\1\end{pmatrix}$. Hence:

$$d=\frac{\left|\overrightarrow{AB}\times\boldsymbol{d}\right|}{|\boldsymbol{d}|}=\frac{\left|\begin{pmatrix}0\\-5\\7\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{1+1+1}}=\frac{\begin{pmatrix}-12\\7\\5\end{pmatrix}}{\sqrt{3}}=\frac{\sqrt{218}}{\sqrt{3}}\left(=\frac{\sqrt{654}}{3}\right)$$

25. **(a)** The plane contains the point P and its normal is $\begin{pmatrix}3\\-4\\1\end{pmatrix}$; hence, the Cartesian equation of the plane is:

$3(x-1)-4(y-2)+1(z-11)=0\Rightarrow 3x-4y+z=6$

(b) **(i)** $(1)+3(2)-(11)=1+6-11=-4$; hence, P lies in π_2

(ii) The intersection of the planes contains the point P and its direction vector is the vector product of the normal:

$\begin{pmatrix}3\\-4\\1\end{pmatrix}\times\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}1\\4\\13\end{pmatrix}$. Hence, a vector equation of the line is:

$$\boldsymbol{r}=\begin{pmatrix}1\\2\\11\end{pmatrix}+t\begin{pmatrix}1\\4\\13\end{pmatrix},\ t\in\mathbb{R}$$

(c) The angle between the normals is:

$$\cos\theta_1=\frac{\begin{pmatrix}3\\-4\\1\end{pmatrix}\begin{pmatrix}1\\3\\-1\end{pmatrix}}{\sqrt{9+16+1}\sqrt{1+9+1}}=\frac{-10}{\sqrt{26\cdot11}}$$

Hence, the angle between the planes is: $\cos\theta=\frac{10}{\sqrt{26\cdot11}}\Rightarrow\theta\approx53.7498^{\circ}\approx53.7^{\circ}$

26. **(a)** $\frac{x+2}{3}=\frac{y}{1}=\frac{z-9}{-2}=\mu \Rightarrow x=-2+3\mu, y=\mu, z=9-2\mu$;

hence: $M(-2+3\mu,\mu,9-2\mu)$

(b) **(i)** $\frac{x-4}{3}=\frac{y}{1}=\frac{z+3}{-2}$

(ii) $\overrightarrow{PM}=\begin{pmatrix}-2+3\mu-4\\ \mu-0\\ 9-2\mu+3\end{pmatrix}=\begin{pmatrix}3\mu-6\\ \mu\\ -2\mu+12\end{pmatrix}$

(c) **(i)** $\overrightarrow{PM}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}=0\Rightarrow\begin{pmatrix}3\mu-6\\ \mu\\ -2\mu+12\end{pmatrix}\begin{pmatrix}3\\1\\-2\end{pmatrix}=0$

$\Rightarrow 9\mu-18+\mu+4\mu-24=0\Rightarrow 14\mu=42\Rightarrow \mu=3$

(ii) The distance between the lines is equal to $\left|\overrightarrow{PM}\right|$, where $\mu=3$:

$\overrightarrow{PM}=\begin{pmatrix}3\mu-6\\ \mu\\ -2\mu+12\end{pmatrix}=\begin{pmatrix}3\\3\\6\end{pmatrix}$; hence, the distance is:

$d=\sqrt{9+9+36}=\sqrt{54}\left(=3\sqrt{6}\right)$

(d) A normal to the plane equals: $\begin{pmatrix}3\\3\\6\end{pmatrix}\times\begin{pmatrix}3\\1\\-2\end{pmatrix}=\begin{pmatrix}-12\\24\\-6\end{pmatrix}=-6\begin{pmatrix}2\\-4\\1\end{pmatrix}$;

hence, the Cartesian equation of the plane is:

$2(x-4)-4(y-0)+1(z+3)=0\Rightarrow 2x-4y+z=5$

(e) The line is on π_1 (from **(d)**).

Testing the line on π_2: $(-2+3\mu)-5(\mu)-(9-2\mu)=-2+3\mu-5\mu-9+2\mu=-11$.

Therefore, the line is in both planes; hence, l_1 is the line of intersection.

Alternatively, solve the system:

$$\begin{cases}2x-4y+z=5\\ x-5y-z=-11\end{cases}$$

So, the intersection is the line:

$$\boldsymbol{r}=\begin{pmatrix}11.5\\4.5\\0\end{pmatrix}+t\begin{pmatrix}-1.5\\-0.5\\1\end{pmatrix},\ t\in\mathbb{R}$$

A direction vector of the line is: $-2\cdot\begin{pmatrix}-1.5\\-0.5\\1\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$, and, for $t=9$, the position vector of a point on the line is: $\begin{pmatrix}11.5\\4.5\\0\end{pmatrix}+9\begin{pmatrix}-1.5\\-0.5\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\9\end{pmatrix}$, which is line l_1.

27. **(a)** $L_1:\ x=2+t,\ y=2+3t,\ z=3+t$ and $L_2:\ x=2+s,\ y=3+4s,\ z=4+2s$

Hence, at the point of intersection:

$$\left.\begin{array}{l}x=2+t=2+s\\ y=2+3t=3+4s\\ z=3+t=4+2s\end{array}\right\}\Rightarrow\left\{\begin{array}{l}t-s=0\\ 3t-4s=1\\ t-2s=1\end{array}\right.$$

This is a system of three equations which can be solved using a method of your choice. Hence, $t=s=-1$ and the point of intersection is $(1,-1,2)$.

(b) The normal to the plane is perpendicular to both direction vectors, hence:

$$\begin{pmatrix}1\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\4\\2\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

Since, the plane contains the intersection point $(1,-1,2)$, the Cartesian equation of the plane is:

$$2(x-1)-1(y+1)+1(z-2)=0\Rightarrow 2x-y+z=5$$

(c) The midpoint M of $[PQ]$ is: $M=\left(\frac{1+3}{2},\frac{-1+4}{2},\frac{2+3}{2}\right)=\left(2,\frac{3}{2},\frac{5}{2}\right)$.

The vector $\overrightarrow{MS}$ is parallel to the normal to the plane π, so $\overrightarrow{MS}=t\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}2t\\-t\\t\end{pmatrix}$;

hence, $S\left(2t+2,-t+\frac{3}{2},t+\frac{5}{2}\right)$.

Now, $\left|\overrightarrow{PS}\right| = 3 \Rightarrow$

$$\sqrt{(2t+2-1)^2+\left(-t+\frac{3}{2}+1\right)^2+\left(t+\frac{5}{2}-2\right)^2} = \sqrt{(2t+1)^2+\left(-t+\frac{5}{2}\right)^2+\left(t+\frac{1}{2}\right)^2}$$

$$= \sqrt{6t^2+\frac{15}{2}} = 3 \Rightarrow t^2 = \frac{1}{4} \Rightarrow t = \pm\frac{1}{2}$$

So, the possible solutions for S are:

$$S_1\left(1+2, -\frac{1}{2}+\frac{3}{2}, \frac{1}{2}+\frac{5}{2}\right) = (3,1,3) \text{ or } S_2\left(-1+2, \frac{1}{2}+\frac{3}{2}, -\frac{1}{2}+\frac{5}{2}\right) = (1,2,2)$$

Note: We used the fact that $\left|\overrightarrow{PS}\right| = 3$. The line L is the symmetry line of the segment $[PQ]$; hence, $\left|\overrightarrow{QS}\right|$ should be 3. That means that we will have the same equations if we use $\left|\overrightarrow{QS}\right| = 3$.

28. **(a)** **(i)** Points on each line satisfy the following systems of equations:

$$L_1 : \begin{cases} x = 2-2\lambda+\mu \\ y = 1+\lambda-3\mu \\ z = 1+8\lambda-9\mu \end{cases} ; \quad L_2 : \begin{cases} x = 2+s+t \\ y = 0+2s+t \\ z = 1+s+t \end{cases}$$

Hence, at the points of intersection:

$2-2\lambda+\mu = 2+s+t$

$1+\lambda-3\mu = 2s+t$

$1+8\lambda-9\mu = 1+s+t$

Subtracting the third equation from the first, we have:

$1-10\lambda+10\mu = 1 \Rightarrow \lambda = \mu$

(ii) If $\lambda = \mu$ for points on the plane L_1, then those points are on the line:

$x = 2-2\lambda+\lambda = 2-\lambda$

$y = 1+\lambda-3\lambda = 1-2\lambda$

$z = 1+8\lambda-9\lambda = 1-\lambda$

Thus, a vector equation is: $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$

(b) The point $(2,0,-1)$ from the line is on the plane. Hence, the Cartesian equation is:
$3(x-2)-2(y)+(z+1)=0 \Rightarrow 3x-2y+z=5$

(c) Planes π_1 and π_2 intersect at the line $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$, so we find the intersection of this line and the plane π_3. We write the equation of the line in parametric form, and then substitute into the equation of π_3:

$x=2-\lambda$
$y=1-2\lambda$
$z=1-\lambda$

$3(2-\lambda)-2(1-2\lambda)+(1-\lambda)=5 \Rightarrow 5=5$

The equation is satisfied by any real value of λ; hence, the plane π_3 contains the line, and intersection of the three planes is the line $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$

29. **(a)** The angle between the two planes is the acute angle between their normals.

$$\cos\theta=\frac{\begin{pmatrix}4\\1\\1\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-1\end{pmatrix}}{\sqrt{18}\sqrt{26}}=\frac{18}{3\sqrt{2}\sqrt{2\times13}}=\frac{3}{\sqrt{13}}=\sqrt{\frac{9}{13}}$$

(b) **(i)** The line of intersection between the planes is orthogonal to both of their normals. Thus, its direction vector is parallel to their vector product.

$$\boldsymbol{n}_1\times\boldsymbol{n}_2=\begin{pmatrix}4\\1\\1\end{pmatrix}\times\begin{pmatrix}4\\3\\-1\end{pmatrix}=\begin{pmatrix}-4\\8\\8\end{pmatrix}=4\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

(ii) Substitute the coordinates of the point in each of the equations:
$4(1)+0+4=8$ and $4(1)+3(0)-4=0$

(iii) $\boldsymbol{r}=\begin{pmatrix}1\\0\\4\end{pmatrix}+t\begin{pmatrix}-1\\2\\3\end{pmatrix}$

(c) If B is on π_1 then $4a+b+1=8\Rightarrow b=7-4a$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}a-1\\7-4a\\-3\end{pmatrix}$, and if it is perpendicular L, then

$$\begin{pmatrix}a-1\\7-4a\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\2\end{pmatrix}=-9a+9=0\Rightarrow a=1\Rightarrow b=3$$

(d) With values found in **(c)**, $\overrightarrow{AB}=\begin{pmatrix}1-1\\7-4\\-3\end{pmatrix}=\begin{pmatrix}0\\3\\-3\end{pmatrix}\Rightarrow\left|\overrightarrow{AB}\right|=\sqrt{9+9}=3\sqrt{2}$

(e) With P as described, we have an isosceles right-angled triangle with right angle at A. Thus, $\left|\overrightarrow{AB}\right|=\left|\overrightarrow{AP}\right|\Rightarrow\left|\overrightarrow{AP}\right|=3\sqrt{2}$

Since P is on L, then its coordinates are $(1-t,2t,4+2t)$, and $\overrightarrow{AP}=\begin{pmatrix}-t\\2t\\2t\end{pmatrix}$.

So, $\left|\overrightarrow{AP}\right|=\sqrt{t^2+4t^2+4t^2}=3\sqrt{2}\Rightarrow t=\pm\sqrt{2}$, and the possible positions are $\left(1-\sqrt{2},2\sqrt{2},4+2\sqrt{2}\right)$ or $\left(1+\sqrt{2},-2\sqrt{2},4-2\sqrt{2}\right)$

30. **(a)** For L to be perpendicular to π, it has to be parallel to the normal vector.

That is $\begin{pmatrix}p\\2\\1\end{pmatrix}=k\begin{pmatrix}1\\1\\3\end{pmatrix}$ for some value k, which will give an inconsistent system with no solution: $\begin{cases}p=k\\2=k\\1=3k\end{cases}$. Therefore, L cannot be perpendicular to π.

(b) For L to lie in π then point $(2,q,1)$ must be in the plane.

$2+q+3=9\Rightarrow q=4$

Also, for L to lie in π then its direction vector and the normal to the plane must be orthogonal:

$$\begin{pmatrix}p\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix}=0\Rightarrow p=-5$$

(c) **(i)** If the angle between L and π is $\arcsin\frac{1}{\sqrt{11}}$, then the angle between L and the normal must be the complement of this angle. This means that the cosine of the angle between the normal and L is the same as

$$\sin\left(\arcsin\frac{1}{\sqrt{11}}\right)=\frac{1}{\sqrt{11}}$$

Thus, $\dfrac{\begin{pmatrix}p\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix}}{\sqrt{p^2+5}\sqrt{11}}=\dfrac{1}{\sqrt{11}}\Rightarrow\dfrac{p+5}{\sqrt{p^2+5}}=1\Rightarrow p=-2$

(ii) The equation of L is now: $\dfrac{x-2}{-2}=\dfrac{y-q}{2}=z-1\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2-2t\\q+2t\\1+t\end{pmatrix}$

If the intersection is at $z=-1$, then $1+t=-1\Rightarrow t=-2$.

Now substitute the coordinates of any point on L with the value of t just found into the equation of the plane:

$2+4+q-4-3=9\Rightarrow q=10.$

Exercise 10.1

Answers may differ slightly from one person to another. This depends on GDC, software, or arrangement of data. General patterns will still look similar.

1. **(a)** The experimental unit would be a student. A sensible population would be all the students in a certain school (a large one), city, district, or country. The sample could be students from one class, or even a smaller group of students. The variable is qualitative as it describes a characteristic of a student (female or male) rather than a numerical quantity.

(b) The experimental unit would be a final exam taken by a 10th-grade student. A sensible population would be all the exams taken by 10th-grade students in a certain school (a large one), city, district, or country. The sample could be the exams of students from one class, or even a smaller group of students. The variable is quantitative since we are counting the number of errors.

(c) The experimental unit would be a new-born child. A sensible population would be all the new-born children in a certain city, district, or country. The sample could be new-born children born in the same hospital or born on the same day. The variable is quantitative since we are measuring height.

(d) The experimental unit would be a child aged less than 14. A sensible population would be all the children aged less than 14 who live in a certain city, district, and country. The sample could be all the children aged less than 14 who live in the same building, block, or street. The variable is qualitative as it describes a characteristic of a child (blue eyes, brown eyes, green eyes, and so on) rather than a numerical quantity.

(e) The experimental unit would be a working person. A sensible population would be all the working people in a certain city, district, or country. The sample could be people working in the same company, or people living in the same part of the city. The variable is quantitative since we measure the time it takes them to travel to work.

(f) The experimental unit would be a country leader. A sensible population would be all the country leaders worldwide. The sample could be all the country leaders within a certain geographical region, or a continent, or even the leaders of the same country throughout history. The variable is qualitative as it describes a characteristic of a leader (excellent, good, fair, or poor) rather than a numerical quantity.

(g) The experimental unit would be a student. A sensible population would be all the countries of origin of students at an international school, or a group of international schools within a certain country or a geographical region. The sample could be all the countries of origin of students from one grade of an international school. The variable is qualitative as it describes a characteristic of a student through their country of origin (Austria, Germany, Italy, Croatia, and so on) rather than a numerical quantity.

2. **Note:** Answers for this question are not unique.

(a) Skewed to the right, since there are many players who don't score at all, and there are a few who are top scorers.

(b) It should be symmetric since the weights will be grouped around one particular weight. (Later on, we will find out that this weight is called the mean weight.)

(c) Again, skewed to the right, since there are a few students who travel a lot and visit many countries.

(d) In this case, we would again expect a distribution skewed to the right, because some students do receive a lot of emails (especially those who use social networking sites).

3. **(a)** Quantitative, because we can measure the time taken to finish the essay.

(b) Quantitative, because we can count the number of students in each section.

(c) Qualitative, since the rating has descriptors rather than a numerical quantity.

(d) Qualitative, since the country of origin is a name rather than a numerical quantity.

4. **(a)** Discrete, since we can count the exact number of students from each country.

(b) The weight of exam papers can be measured and therefore it is continuous.

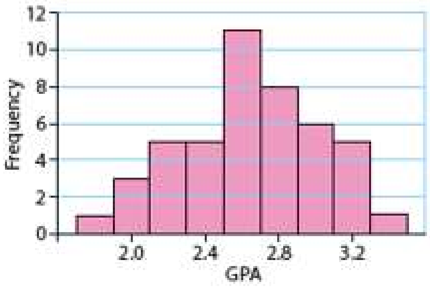
(c) Time can be measured and therefore it is continuous.

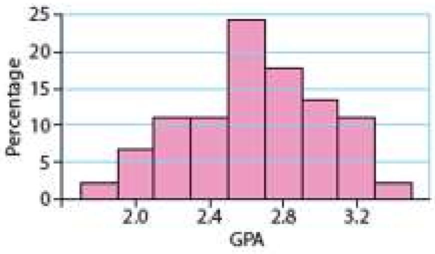
(d) Discrete, since the number of customers must be counted.

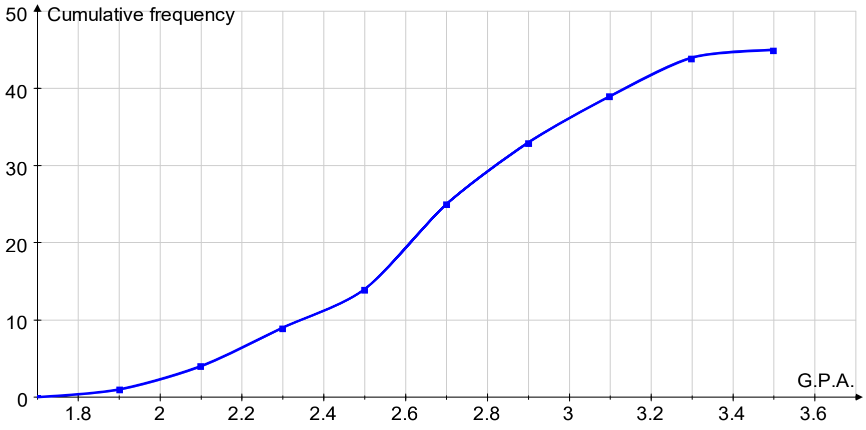
(e) Time can be measured and therefore it is continuous. We always measure time to a certain degree of accuracy.

(f) The amount of sugar, as a mass, can be measured and therefore it is continuous. On the other hand, if we count the grains of sugar without breaking them apart, then we can say that it is discrete.

5. This analysis is done by grouping the grades into classes of length 0.2, since the range of the data is not very large. (Use a GDC or software.)



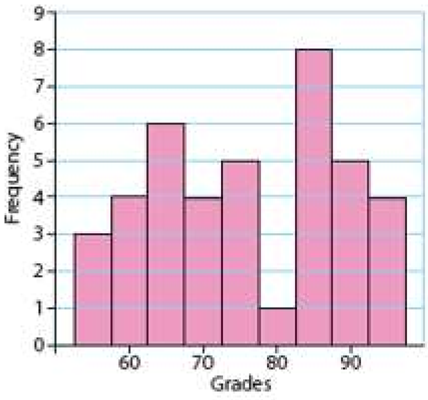




The data looks relatively symmetric, with no apparent outliers.

6. We will group the data into classes of length 5 (not unique – for example, you may start at 50 and end at 100) and count the frequencies for each interval.

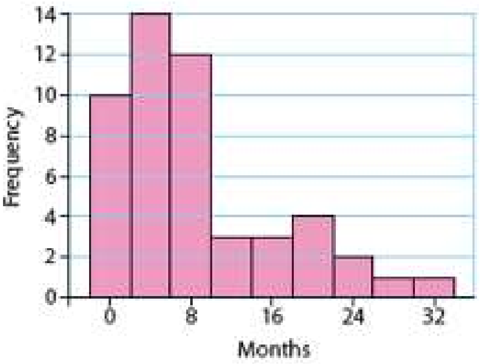
Interval	Midpoint	Frequency
52.5–57.5	55	3
57.5 –62.5	60	4
62.5 –67.5	65	6
67.5 –72.5	70	4
72.5 –77.5	75	5
77.5 –82.5	80	1
82.5 –87.5	85	8
87.5 –92.5	90	5
92.5 –97.5	95	4



We can say that this distribution is almost bimodal, where one group has a mode of 65 and the other group has a mode of 85. This may indicate that the class is split into two ability groups.

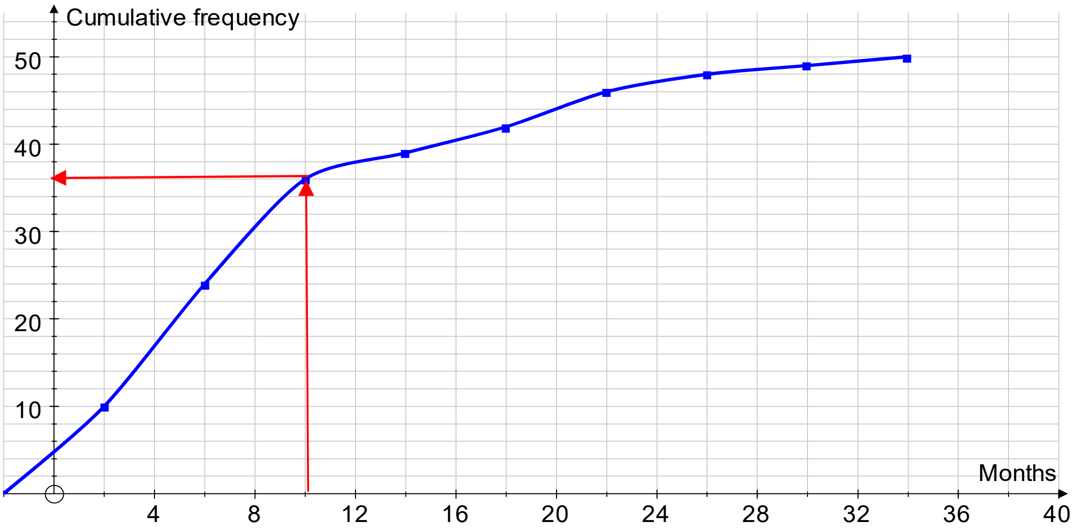
7. **(a)** This set of data has a very large range and therefore we are going to group it into suitable intervals. The midpoint of each interval is shown in the table.

Midpoint	Frequency
0	10
4	14
8	12
12	3
16	3
20	4
24	2
28	1
32	1



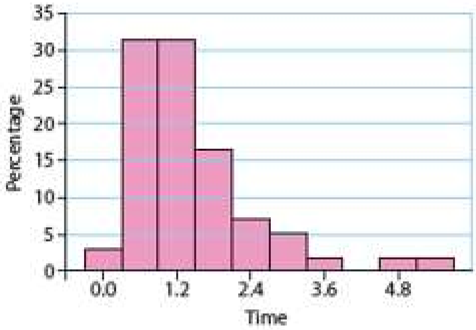
(b) The data is not symmetric but skewed to the right.

(c)

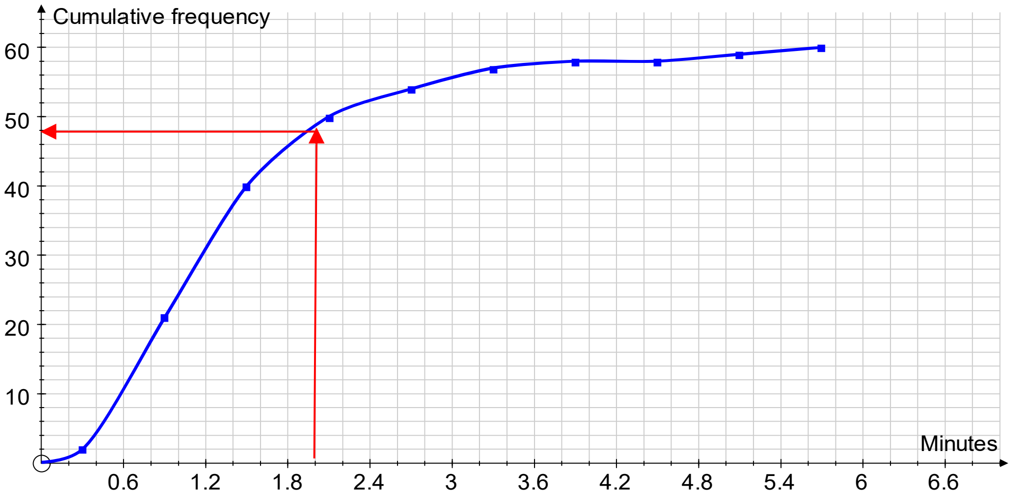


From the diagram, we notice that there are 36 young drivers who will lose their licence; therefore, 72% may lose their licence.

8. **(a)** We will use classes of length 0.6, having 0 as the first midpoint of the interval.



(b)

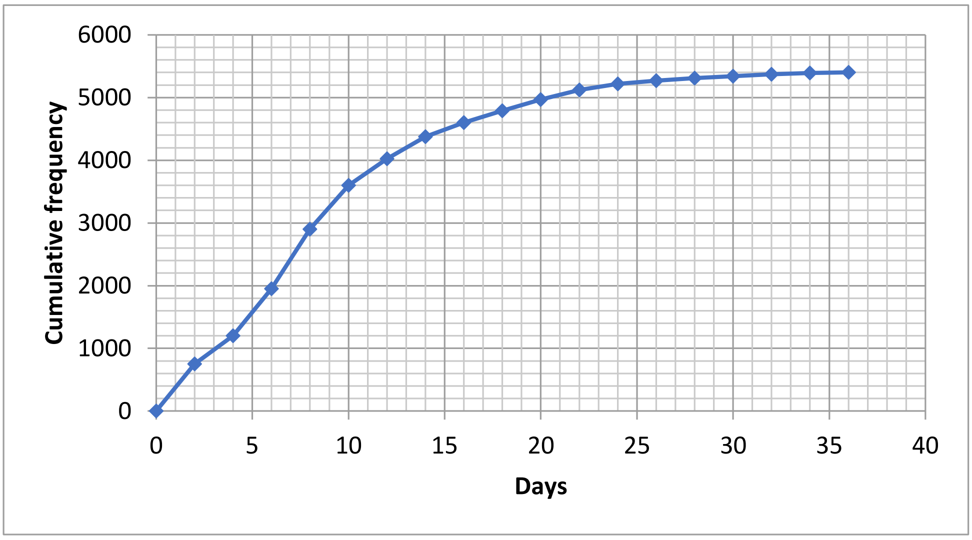


Using the graph, we can say that there are approximately 49 customers who need to wait up to 2 minutes; therefore, 11 customers will need to wait longer than that.

9. **(a)** The data is skewed to the right, with the modal value as 6–8 days spent at hospital. A very few patients will stay longer than 20 days.

(b) Due to the poor scale on the frequency diagram, it is going to be difficult to estimate the frequencies. Our estimates are given in the table:

Interval	Frequency	Cumulative frequency
0–2	750	750
2–4	450	1200
4–6	750	1950
6–8	950	2900
8–10	700	3600
10–12	425	4025
12–14	350	4375
14–16	225	4600
16–18	190	4790
18–20	180	4970
20–22	150	5120
22–24	100	5220
24–26	50	5270
26–28	40	5310
28–30	30	5340
30–32	30	5370
32–34	20	5390
34–36	10	5400



(c) Using the table or graph, we can estimate the percentage of patients who stayed less than 6 days as: $p = \dfrac{1950}{5400} = 0.36111\ldots \approx 36\%$

10. **(a)** From the frequency graph, we can see that the right-most time recorded is 40 minutes. Therefore, the longest time is 40 minutes.

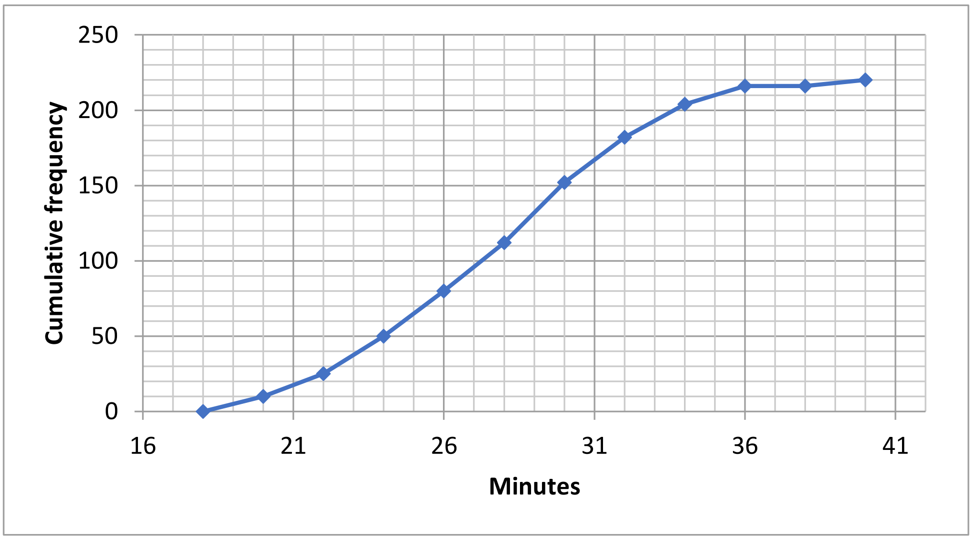
(b) In order to solve this problem, we need the cumulative frequencies.

Interval	Frequency	Cumulative frequency
18–20	10	10
20–22	15	25
22–24	25	50
24–26	30	80
26–28	32	112
28–30	40	152
30–32	30	182
32–34	22	204
34–36	12	216
36–38	0	216
38–40	4	220

The percentage of time spent exercising more than 30 minutes is:

$$p = 1 - \frac{152}{220} = 0.30909... \approx 31\%$$

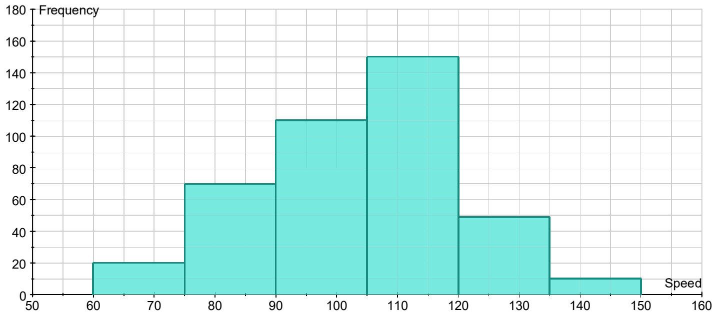
(c)



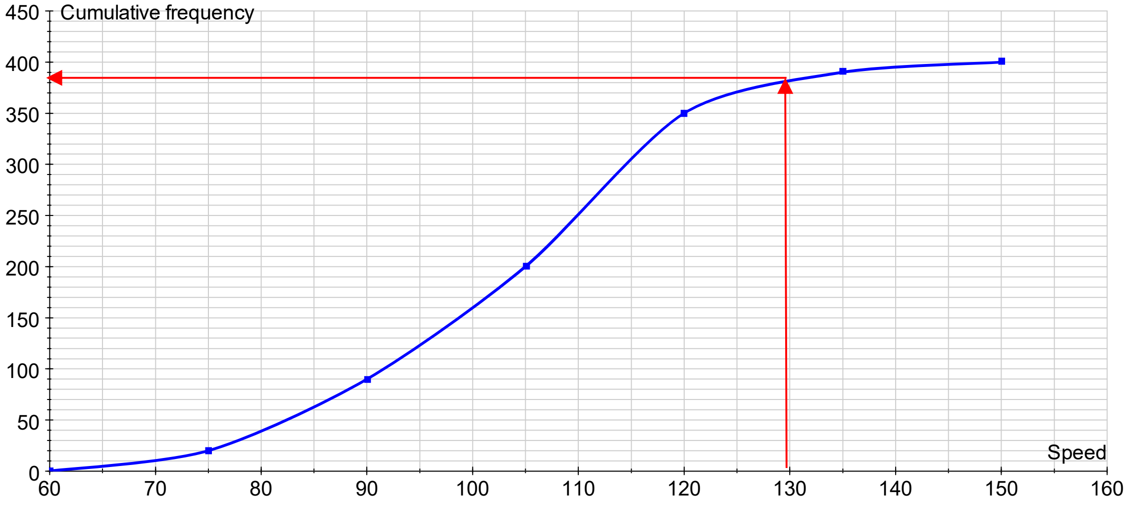
11. **(a)** The frequency table is given below.

Speed	Frequency
60 ≤ speed < 75	20
75 ≤ speed < 90	70
90 ≤ speed < 105	110
105 ≤ speed < 120	150
120 ≤ speed < 135	40
135 ≤ speed < 150	10

(b)

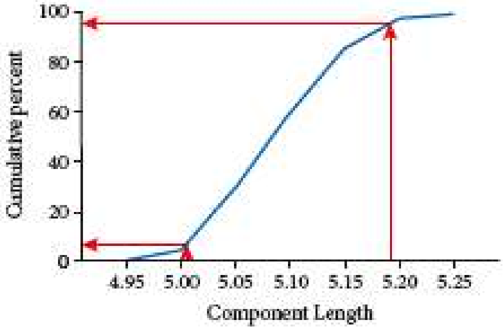


(c) To draw the cumulative frequency graph, we take the endpoints of the intervals and calculate the corresponding cumulative frequencies: 20, 90, 200, 350, 390 and 400.



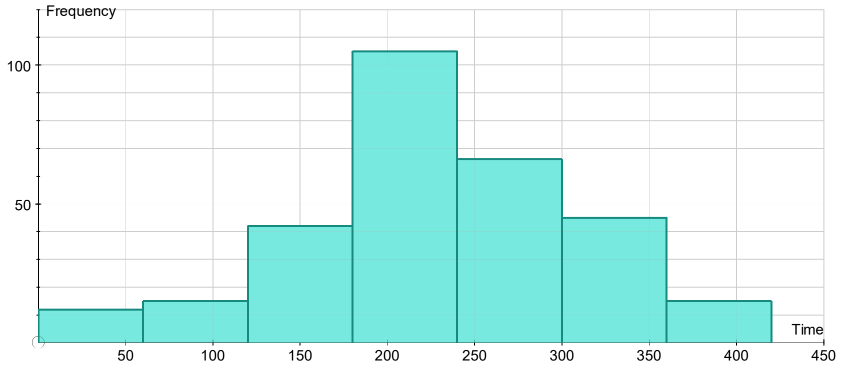
(d) To estimate the number of drivers exceeding the speed limit, we draw a vertical line from 130 km/h on the cumulative frequency diagram. Our estimation is 375; therefore, since there are 400 cars, 25 cars (or 6.2% of the cars) were exceeding the speed limit.

12. **(a)** To draw the relative cumulative frequency graph, we take the endpoints of the intervals and calculate the corresponding cumulative relative frequencies, which are: 16, 116, 239, 343, 391 and 400. Dividing by 400 changes them to relative. The graph shows them as percent.

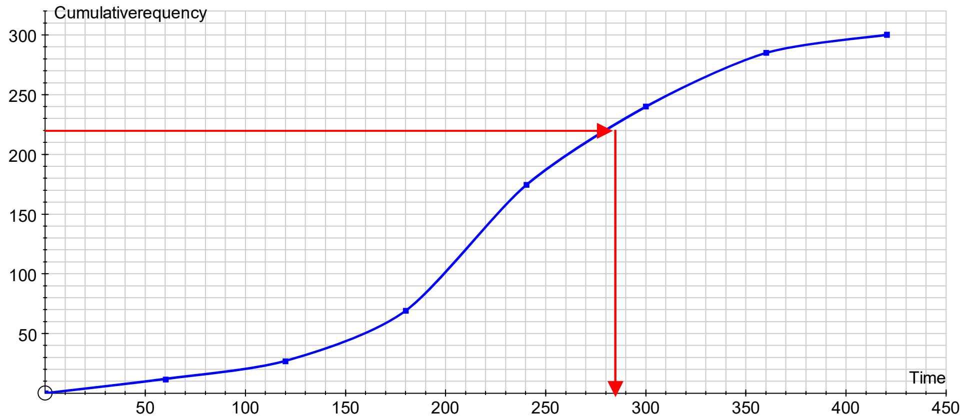


(b) We need to draw two vertical lines, from 5.01 and 5.18, on the diagram above. Our estimate for the percentage of components of length up to 5.01 mm is 5% and up to 5.18 mm is 95%. Therefore, about 10% components will be scrapped.

13. **(a)**



(b) To draw the cumulative frequency graph, we take the endpoints of the intervals and calculate the corresponding cumulative frequencies: 12, 27, 69, 174, 240, 285 and 300.



(c) There are 300 customers, and 25% of 300 is 75. To find the waiting time that is exceeded by 75 customers, we draw a horizontal line at the cumulative frequency of 225 (see diagram above). Our estimate is 285 seconds (4 minutes 45 seconds).

Exercise 10.2

Note: In most of the cases, GDC/software can give the required answers.

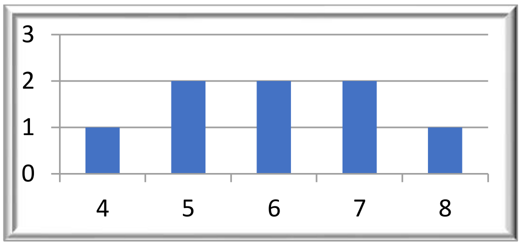
1. **(a)** $\overline{x} = \frac{5+4+7+8+6+6+5+7}{8} = \frac{48}{8} = 6$

(b) In order to find the median, we need to list the observations in order of magnitude. Since there are eight observations, we need to take the two middle ones, which are the fourth and the fifth observations, and take their average.

4, 5, 5, 6, 6, 7, 7, 8

Since the middle observations are both 6, the median is 6.

(c) The data is symmetric with respect to the mean value. A bar graph/histogram demonstrates the fact.



2. **(a)** $\bar{x} = \frac{5+7+8+6+12+7+8+11+4+10}{10} = \frac{78}{10} = 7.8$

(b) 4, 5, 6, 7, 7, 8, 8, 10, 11, 12

There are ten observations, so we need to identify the fifth and sixth observations. The fifth observation is 7 and the sixth is 8; therefore, the median is 7.5

(c) This set of data is bimodal, with 7 and 8 as the modes, since these two observations have the highest frequency.

3.

Number of cars (x_i)	0	1	2	3	Σ
Number of households (f_i)	12	24	8	6	50
$x_i \times f_i$	0	24	16	18	58

We calculate the mean value by using the formula $\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{58}{50} = 1.16$

There are 50 pieces of data, so to find the median value we have to identify the 25th and 26th observations, when listed in order of magnitude, and then take their average. From the table, we notice that all the observations from the 13th until the 36th are 1; therefore, the median is 1.

The median is the measure that best describes this data since the data is skewed to the right and as such the mean is more influenced by it.

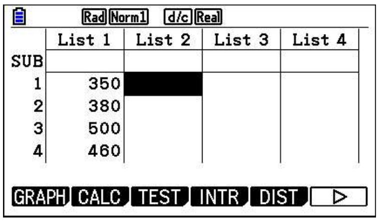
4. We list the revenues in order of magnitude: 242137, …, 265172, 311870, …, 500343.

Since there are 10 observations, the median value is the average of the 5th and 6th observations: $\bar{x} = \frac{265172 + 311870}{2} = 288521$ millions of dollars.

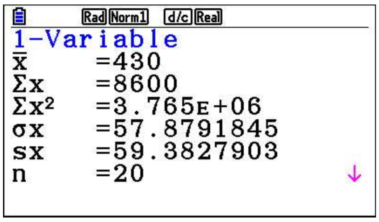
The mean is the total sum of revenues divided by 10: $\bar{x} = \frac{3070359}{10} = 307035.9$ millions of dollars. In this case, the median is more appropriate as there are extreme values.

5. For this question we will use a calculator since there are many observations.

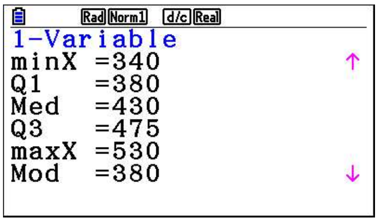
First, we input all the observations into a list and then, from the list menu, we can use the descriptive statistics features of the GDC.



```
     List 1  List 2  List 3  List 4
SUB
  1     350
  2     380
  3     500
  4     460
```



```
1-Variable
x̄     =430
Σx    =8600
Σx²   =3.765E+06
σx    =57.8791845
sx    =59.3827903
n     =20
```



```
1-Variable
minX =340
Q1   =380
Med  =430
Q3   =475
maxX =530
Mod  =380
```

As both measures are equal, and the data looks relatively symmetric, either measure looks good.

6. **(a)** $\bar{x} = \frac{4460}{90} = 49.56$, correct to the nearest cent.

(b) $\bar{y} = \frac{4460 + 74 + 60}{90 + 2} = 49.93,$ correct to the nearest cent.

7. We consider all the bags, measure their total weight, and divide it by the total number of bags.

This year's total = $2.15 \times 144 = 309.6$; last year's total = $1.80 \times 56 = 100.80$, thus:

$$\bar{x} = \frac{2.15 \times 144 + 1.80 \times 56}{144 + 56} = 2.052$$

So, the mean weight of a bag of potatoes is 2.052 g

8. **(a)** $\bar{x} = \frac{\sum x_i}{25} = \frac{749}{25} = 29.96$

(b)

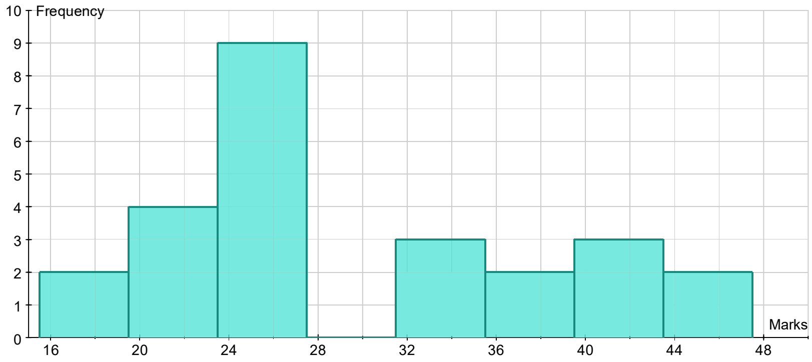
1 | 8 9
2 | 0 2 3 3 4 4 5 6 6 6 $\underline{\underline{7}}$ 7 7
3 | 3 4 5 6 8
4 | 0 2 2 6 6

Since there are 25 observations, we have to find the 13th $\left(\frac{n+1}{2}\right)$th observation.

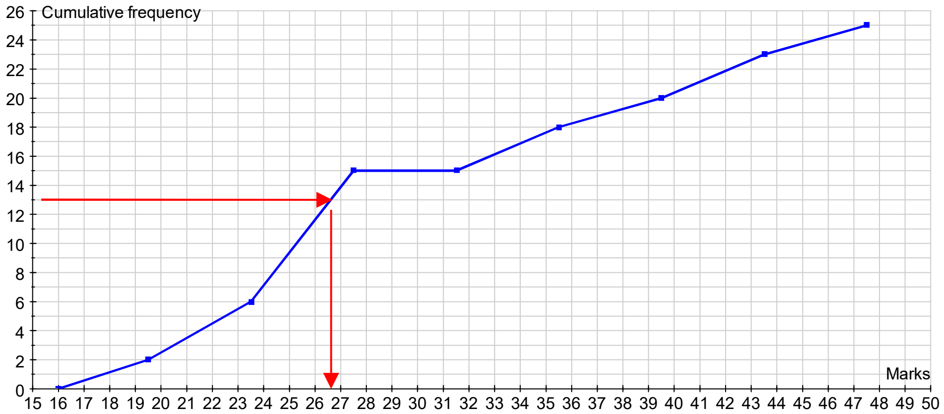
Looking at the stem plot, the 13th observation is 27; therefore, the median is 27.

(c) To draw a histogram, we need to group the data into suitable intervals.
We will use intervals of length 4, starting from 16. (Or use GDC/software.)

Grades	Frequency
16–19	2
20–23	4
24–27	9
28–31	0
32–35	3
36–39	2
40–43	3
44–47	2



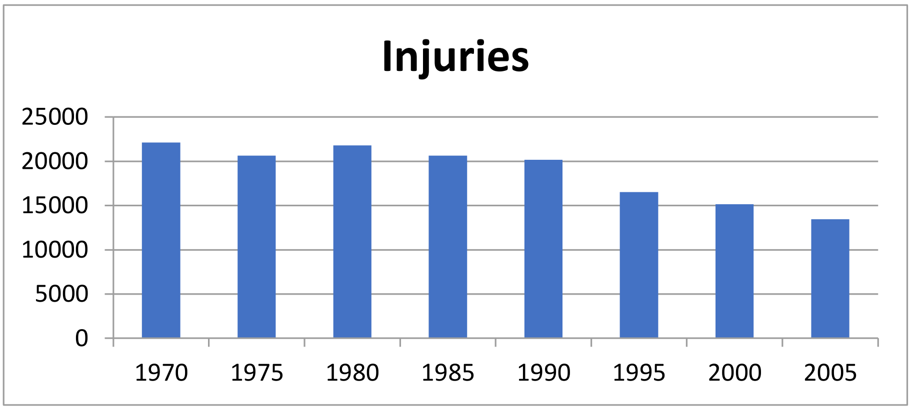
(d)



An estimate for the median which corresponds to a cumulative frequency of 13 is approximately 27.

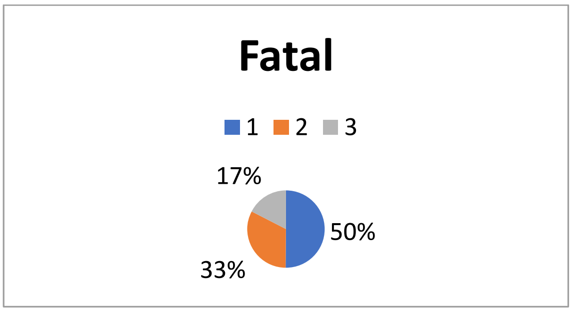
9. **(a)**

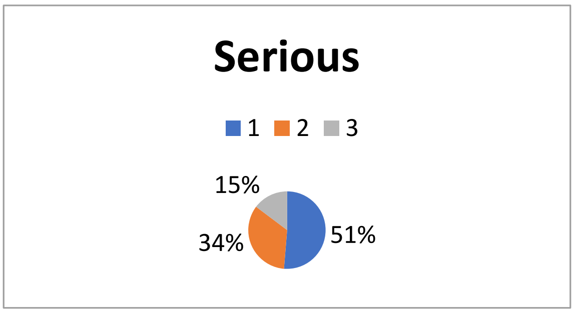
Year	Fatal	Serious	Slight	Total
1970	758	7860	13515	22133
1975	699	6912	13041	20652
1980	644	7218	13926	21788
1985	550	6507	13587	20644
1990	491	5237	14443	20171
1995	361	4071	12102	16534
2000	297	3007	11825	15129
2005	264	2250	10922	13436

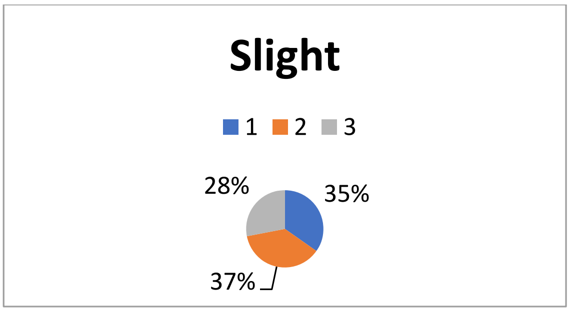


From the bar graph, we notice that the number of injuries is decreasing year on year.

(b) Key: 1 denotes year 1970, 2 denotes year 1990, and 3 denotes year 2005

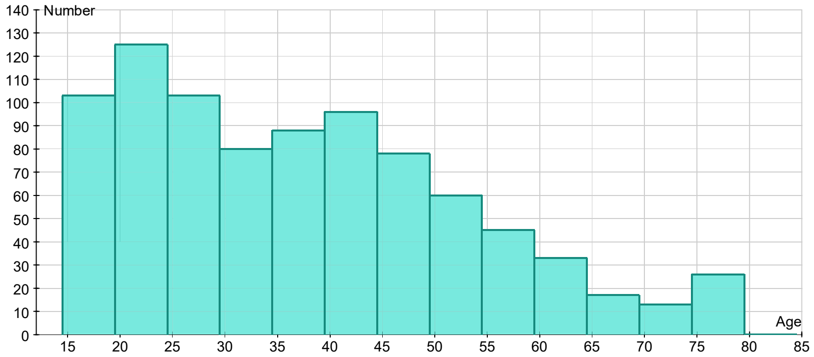






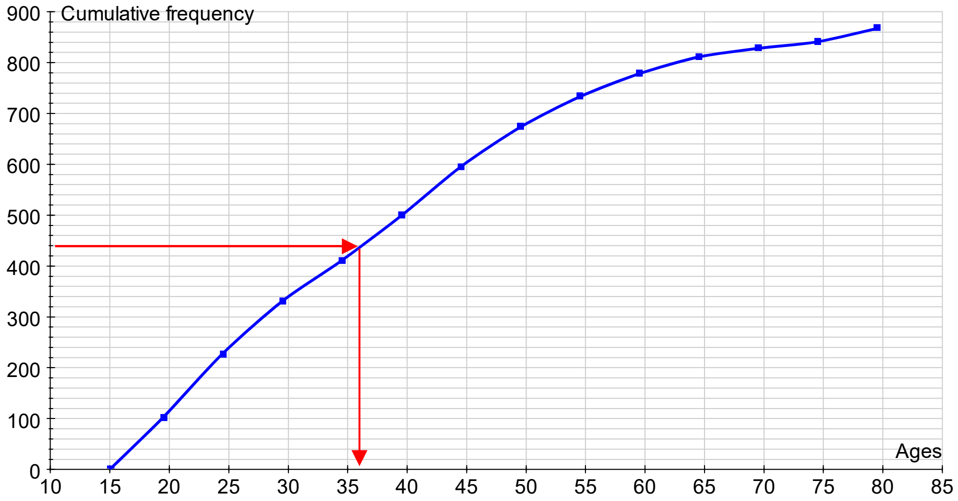
Alternatively, pie charts can be also produced for each year as given in the answers to the question in the book itself.

10. **(a)** Use of GDC/software is advisable



(b) To estimate the mean value, we need to use the midpoints of the intervals (17, 22, 27, ...,77) and the corresponding frequencies: $\bar{x} = \dfrac{\sum x_i f_i}{\sum f_i} = 37.61$

(c)

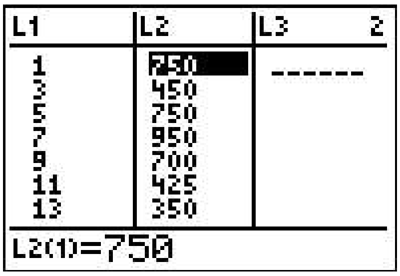


Since there were a total of 867 casualties, we need to draw a horizontal line at 433.5 to find an estimate of the median of the data. Our estimate is about 37 years of age.

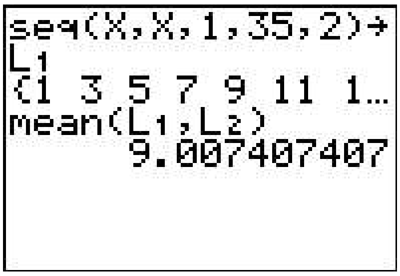
To answer questions 11–15, we use our graphs from the previous exercise, together with GDC/software.

11. Since there are 5400 patients, we need to draw a horizontal line at 2700 (on the cumulative frequency diagram) to estimate the median. Our estimate is 7.5 days.

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies.



```
L1      L2      L3    2
1       750     ------
3       450
5       750
7       950
9       700
11      425
13      350
L2(1)=750
```

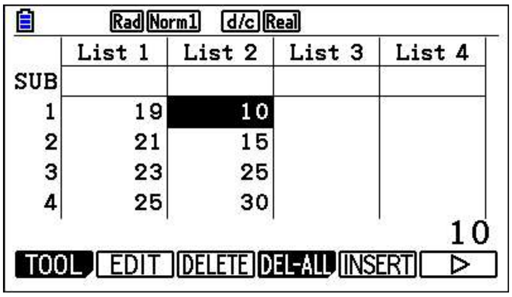


```
seq(X,X,1,35,2)→
L1
{1 3 5 7 9 11 1…
mean(L1,L2)
         9.007407407
```

So, the estimate of the mean is 9.01 days (correct to 3 significant figures).

12. Since there are 220 recordings, we need to draw a horizontal line at 110 (on the cumulative frequency diagram) to estimate the median. Our estimate is 28 minutes.

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies.



	List 1	List 2	List 3	List 4
SUB				
1	19	10		
2	21	15		
3	23	25		
4	25	30		

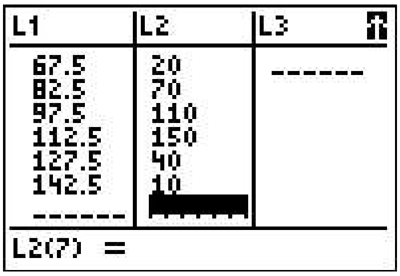
```
10
```

```
Mean(List 1,List 2)
            27.66363636
Median(List 1,List 2▷
                     27
```

So, the estimate of the mean is 27.7 minutes (correct to three significant figures). The median according to calculations is 27, slightly different from the estimate using the graph.

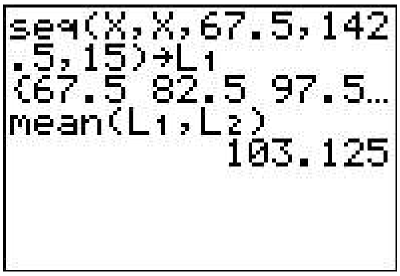
13. There are 400 cars and one of the cumulative frequencies is exactly 200; therefore, the median value is 105 km/h; (105, 200) is a point on the cumulative frequency diagram.

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies.



L1	L2	L3
67.5	20	------
82.5	70	
97.5	110	
112.5	150	
127.5	40	
142.5	10	

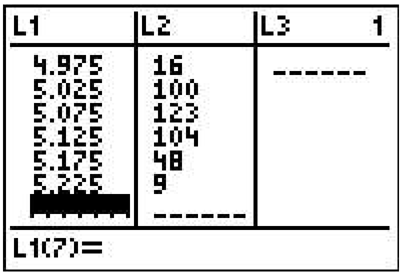
```
L2(7) =
```



```
seq(X,X,67.5,142
.5,15)→L1
{67.5 82.5 97.5…
mean(L1,L2)
            103.125
```

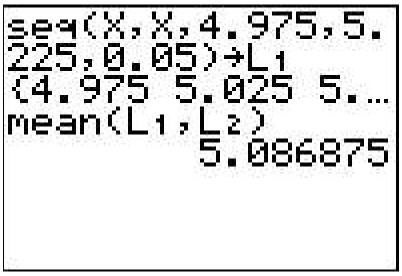
So, the estimate of the mean is 103 km/h (correct to three significant figures).

14. An estimate for the median (which corresponds to a cumulative frequency of 200) is 5.08.
To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies.



L1	L2	L3 1
4.975	16	------
5.025	100	
5.075	123	
5.125	104	
5.175	48	
5.225	9	

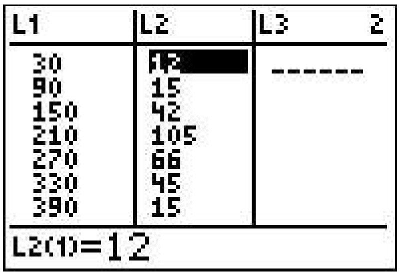
```
L1(7)=
```



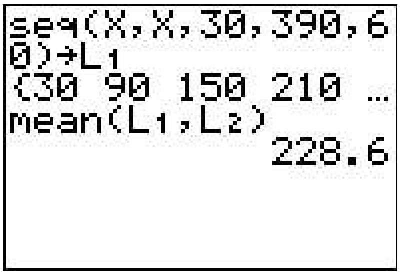
```
seq(X,X,4.975,5.
225,0.05)→L1
{4.975 5.025 5.…
mean(L1,L2)
           5.086875
```

So, the estimate of the mean length is 5.09 (correct to three significant figures).

15. An estimate for the median (which corresponds to a cumulative frequency of 150) is 225 seconds. To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies.



```
L1     L2     L3      2
30     12     ------
90     15
150    42
210    105
270    66
330    45
390    15
L2(1)=12
```



```
seq(X,X,30,390,6
0)→L1
{30 90 150 210 …
mean(L1,L2)
            228.6
```

So, the estimate of the mean waiting time is 229 seconds (correct to three significant figures).

16. **(a)** $$\sum_{i=1}^{40} x_i = 1664 \Rightarrow \bar{x} = \frac{1664}{40} = 41.6$$

(b) $$\sum_{i=1}^{20} (x_i - 20) = 1664 \Rightarrow \bar{x} - 20 = \frac{1664}{20} \Rightarrow \bar{x} = 83.2 + 20 = 103.2$$

17. **(a)** $$\sum_{i=1}^{60} (x_i + 12) = 4404 \Rightarrow \bar{x} + 12 = \frac{4404}{60} \Rightarrow \bar{x} = 73.4 - 12 = 61.4$$

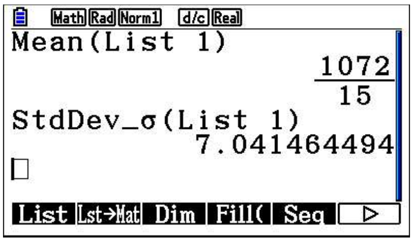
(b) Average score of the whole group of 100 students $= \dfrac{61.4 \times 60 + 67.4 \times 40}{100} = 63.8$

Exercise 10.3

Note: In this part, answers may differ slightly from the answers in the book because of uses of different GDCs or software and expected variation in the accuracy of estimates from graphs.

1. We use a GDC. Since there are only 15 patients, we will use a simple list with 15 elements.

(a)



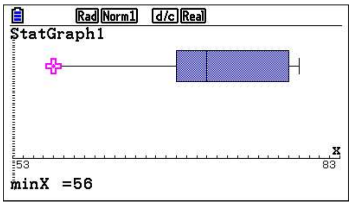
```
Math Rad Norm1  d/c Real
Mean(List 1)
                 1072
                 ────
                   15
StdDev_σ(List 1)
          7.041464494
```

Therefore, the mean pulse of the 15 patients is $\bar{x} = \frac{1072}{15} \approx 71.5$ (to 3 s.f.)

For the standard deviation, recall that for descriptive statistics, the syllabus uses σ and not *s*.

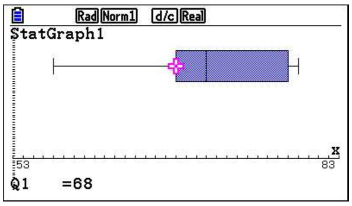
Thus, standard deviation is 7.04 (to 3 s.f.)

(b) We use a GDC to plot the box-whisker diagram.



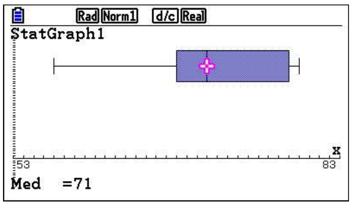
StatGraph1

minX =56



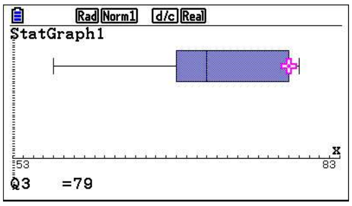
StatGraph1

Q1 =68



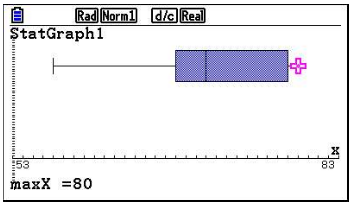
StatGraph1

Med =71



StatGraph1

Q3 =79



StatGraph1

maxX =80

(c) Since we can see that the data is skewed to the left, we need to check whether there are any outliers to the left. $\text{IQR} = \text{Q}_3 - \text{Q}_1 = 79 - 68 = 11$. The outliers lie 1.5×IQR from the lower or upper quartile, so, in this problem, we calculate:
$\text{Q}_1 - 1.5 \times \text{IQR} = 68 - 1.5 \times 11 = 51.5 < 56 = x_{\min}$; therefore, there are no outliers.

2. **(a)** For this question, we used a spreadsheet. We input the data into a column and used the functions to find the following:

Mean value $\bar{x} = 162.6$; standard deviation $s_{n-1} = 23.35 \Leftrightarrow s_n = 23.12$

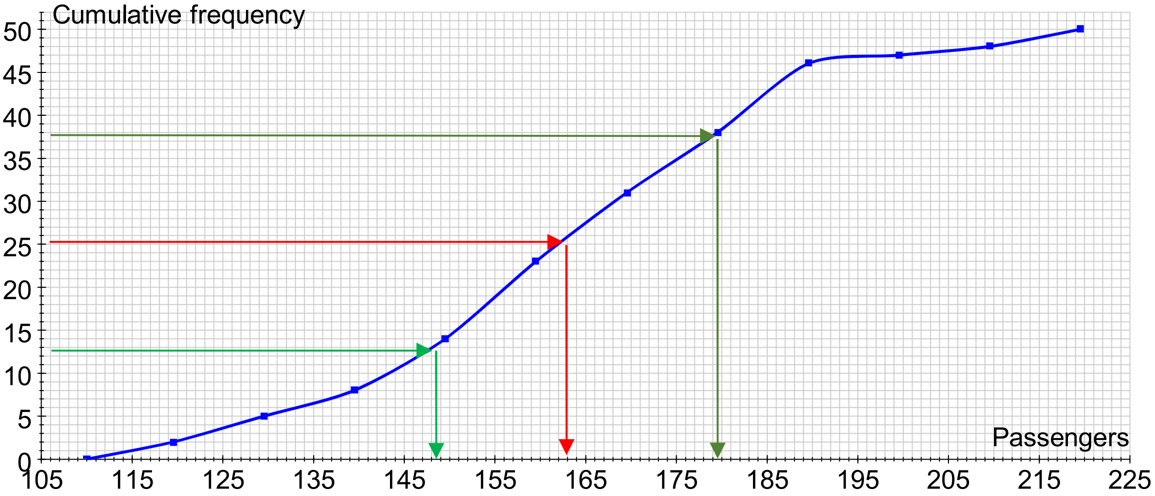
(b)

11	7 9
12	5 6 7
13	0 8 9
14	1 2 3 6 7 9
15	0 3 3 4 4 5 6 8 9
16	0 2 3 3 4 5 6 8
17	1 3 4 4 7 8 9
18	0 2 2 5 5 7 79
19	8
20	9
21	0 8

Since there are 50 observations, we need to find the 25th and 26th observations and take their average. By counting the observations, we find the 25th and 26th observations are 162 and 163 respectively; therefore, the median is $\frac{162+163}{2}=162.5$

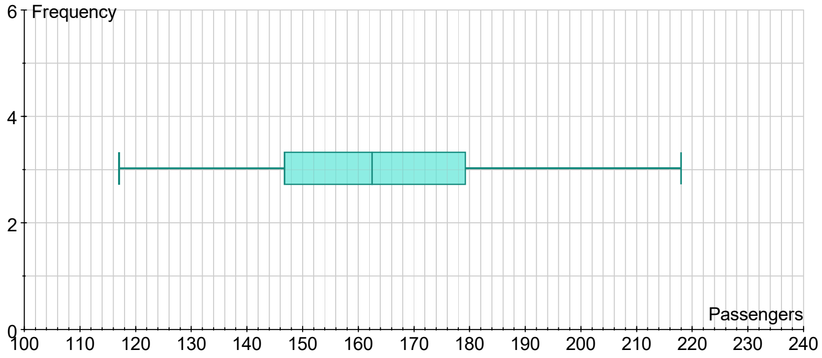
(c) To draw the cumulative frequency curve, we must group the data into suitable intervals.

Interval	Frequency
110–119	2
120–129	3
130–139	3
140–149	6
150–159	9
160–169	8
170–179	7
180–189	8
190–199	1
200–209	1
210–219	2



The first and third quartiles correspond to the cumulative frequencies of 12.5 and 37.5 respectively; therefore, an estimate for Q_1 is 148 and Q_3 is 179. An estimate for the median (which corresponds to a cumulative frequency of 25) is 163.

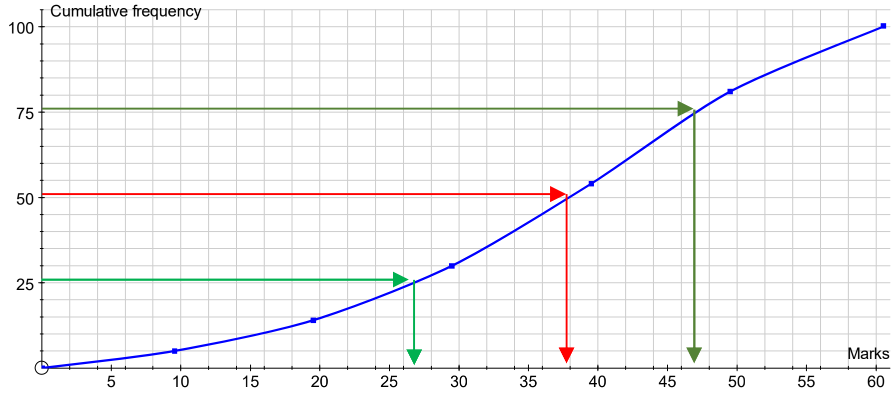
It is also possible to use the raw data that we entered into the spreadsheet to draw the box diagram. In that case, the measures are slightly more accurate.



(d) $\text{IQR} = Q_3 - Q_1 = 179 - 148 = 31$. If there are any outliers, they would be outside of the interval $\left[Q_1 - 1.5 \times \text{IQR}, Q_3 + 1.5 \times \text{IQR}\right] \Rightarrow \left[101.5, 225.5\right]$, which includes the whole range of the number of passengers; therefore, there are no outliers.

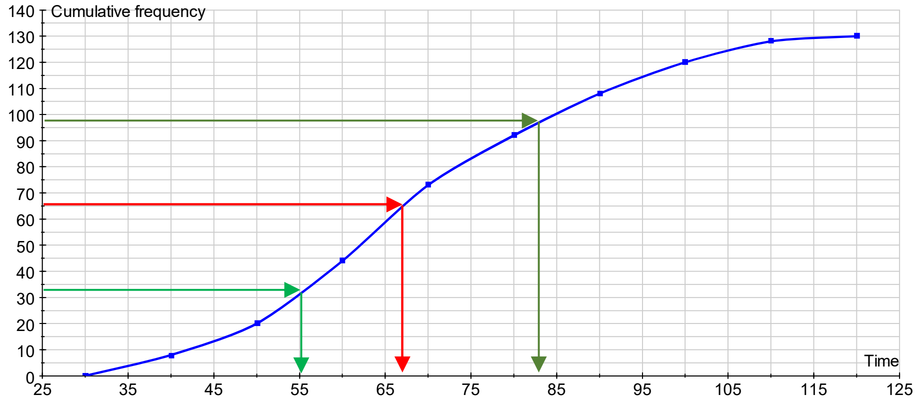
(e) The empirical rule states that the whole range should lie within three standard deviations of the mean value $\Rightarrow 162.6 \pm 3 \times 23.12 = \left[93.24, 231.96\right]$; therefore, there are no outliers.

3. **(a)** Set up a cumulative frequency table as done before. Then use spreadsheet/GDC to graph.



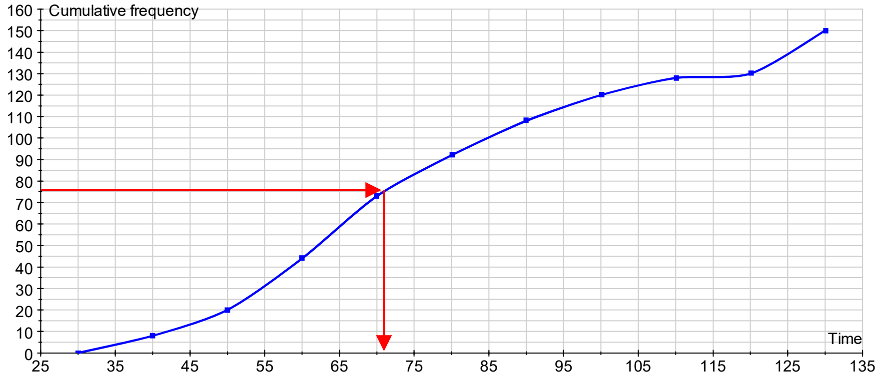
(b) We draw horizontal lines at 50, 25 and 75 to find estimates for the median and quartiles. The estimates are as follows: the median is 38, the lower quartile is 27, and the upper quartile is 47.

4. **(a)**



(b) We draw horizontal lines at 65, 32.5 and 97.5 to find estimates for the median and quartiles. The estimates are as follows: the median is 67, the lower quartile is 55, and the upper quartile is 83. Therefore, $\mathrm{IQR} = \mathrm{Q}_3 - \mathrm{Q}_1 = 83 - 55 = 28$

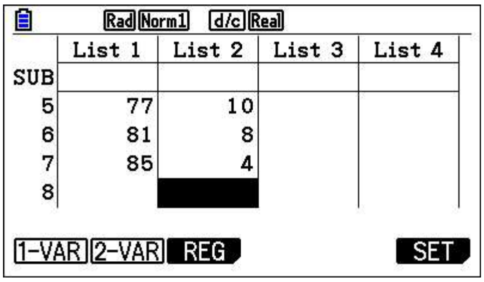
(c)



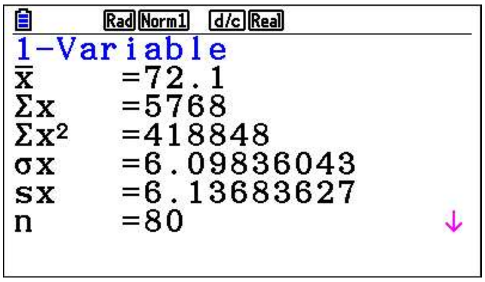
The horizontal line at 75 gives us an estimate of 71 minutes for the median finishing time for all 130 students.

5. $\bar{x} = \dfrac{26\times 22 + 84\times 32}{110} = \dfrac{326}{11} \approx 29.6$

6. **(a)** To find the mean and standard deviation of the given data, we need to use the midpoints of the intervals.



	List 1	List 2	List 3	List 4
SUB				
5	77	10		
6	81	8		
7	85	4		
8				



```
1-Variable
x̄     =72.1
Σx    =5768
Σx²   =418848
σx    =6.09836043
sx    =6.13683627
n     =80
```

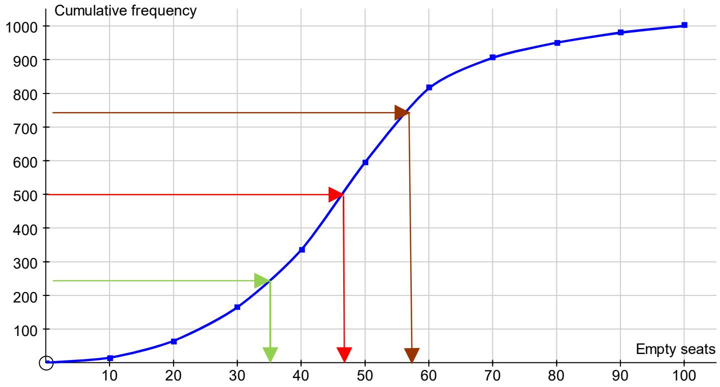
The mean value is 72.1 and the standard deviation is 6.10 (correct to 3 s.f.)

(b) The mean will shift by 13 points and thus will become 85.1. Since we are adding a constant to each term, the standard deviation will not change.

7. **(a)**

Number of empty seats	$x \le 10$	$x \le 20$	$x \le 30$	$x \le 40$	$x \le 50$	$x \le 60$	$x \le 70$	$x \le 80$	$x \le 90$	$x \le 100$
Days	15	65	165	335	595	815	905	950	980	1000

(b)



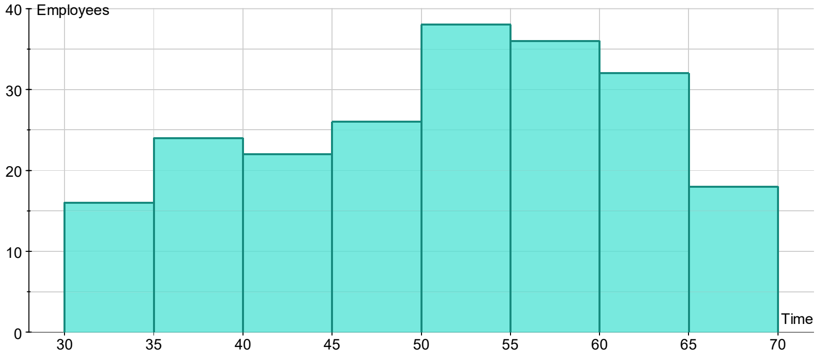
(c) **(i)** To find an estimate of the median number of empty seats, we need to look at the cumulative frequency of 500 (see diagram above). The estimate is 47.

(ii) For the first and third quartiles, we look at the cumulative frequencies of 250 and 750 respectively. Hence, an estimate for the first quartile is 35, while an estimate for the third quartile is 58. The IQR is the difference between the third and first quartiles, which is 23.

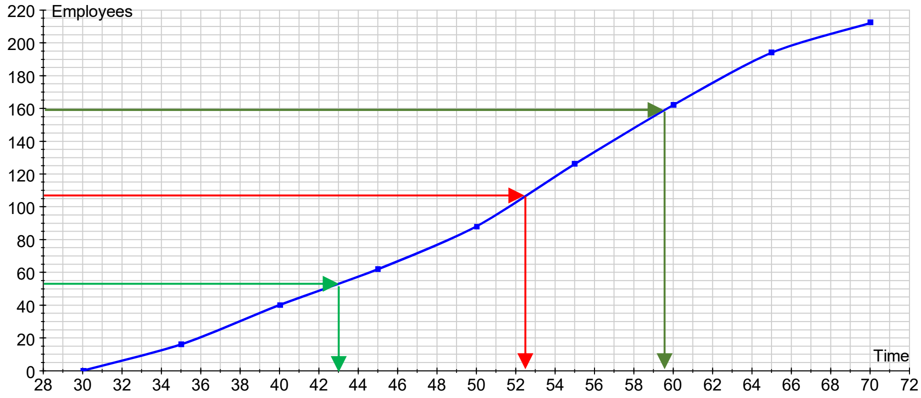
(iii) From the previous estimates, we can see that the number of bumper days was about 250.

(iv) The highest 15% corresponds to the cumulative frequency of 850. An estimate for the number of empty seats for that cumulative frequency is 65.

8. **(a)**

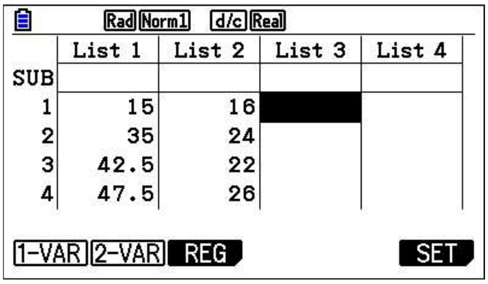


(b)

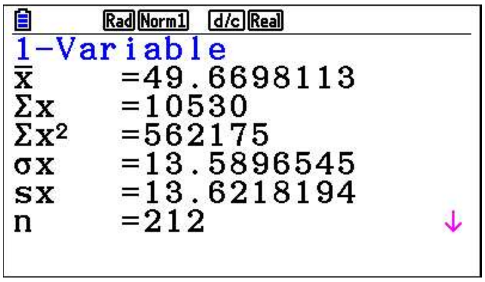


We draw horizontal lines at 106, 53 and 159 to find estimates for the median and quartiles. The estimates are as follows: the median is approximately 53, the lower quartile is 43, and the upper quartile is 59.5. Therefore, $\text{IQR} = 59.5 - 43 = 16.5$

(c) To estimate the mean and standard deviation, we will use the midpoints of the intervals.



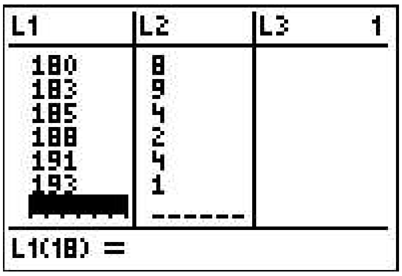
```
     List 1  List 2  List 3  List 4
SUB
  1      15      16
  2      35      24
  3    42.5      22
  4    47.5      26
```



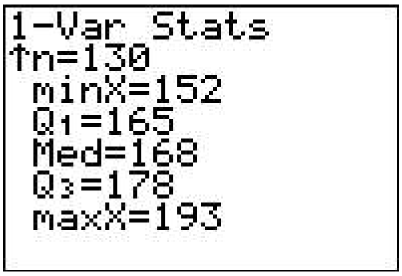
```
1-Variable
x̄     =49.6698113
Σx    =10530
Σx²   =562175
σx    =13.5896545
sx    =13.6218194
n     =212
```

The mean value is 49.7 and the standard deviation is 13.6

9. **(a)** We can solve this problem by using a calculator.



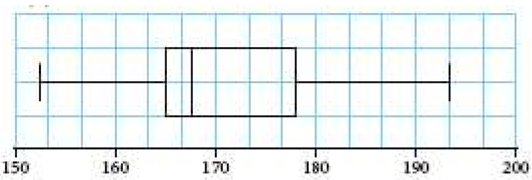
```
L1      L2      L3     1
 180    8
 183    9
 185    4
 188    2
 191    4
 193    1
        ------
L1(18) =
```

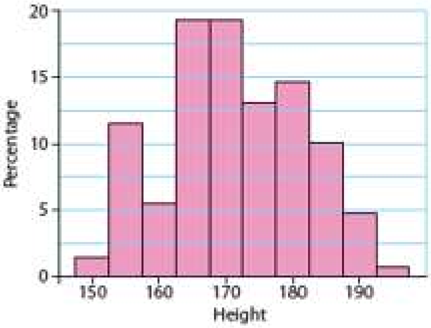


```
1-Var Stats
↑n=130
 minX=152
 Q1=165
 Med=168
 Q3=178
 maxX=193
```

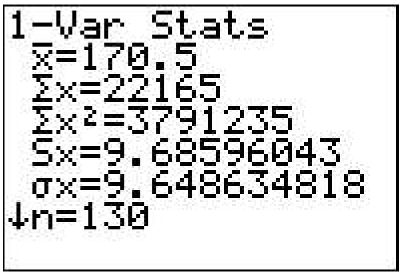
The minimum value is 152 and the maximum value is 193. The lower and upper quartiles are 165 and 178 respectively, whilst the median value is 168.

(b)





(c)

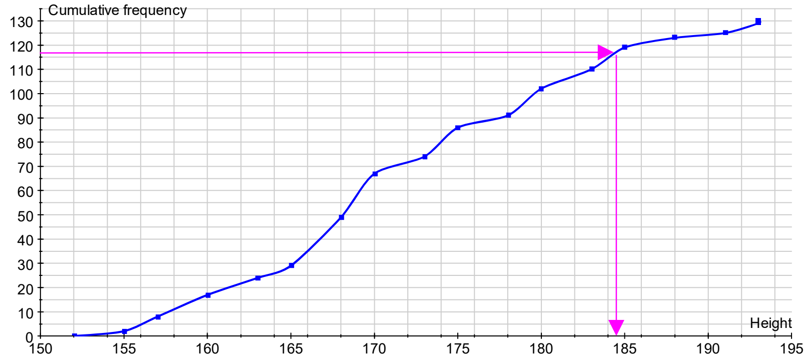


```
1-Var Stats
 x̄=170.5
 Σx=22165
 Σx²=3791235
 Sx=9.68596043
 σx=9.648634818
↓n=130
```

The mean value is 170.5 and the standard deviation is 9.65, correct to 3 s.f.

(d) The heights are widely spread from very short to very tall players. Heights are slightly skewed to the right, bimodal at 165 and 170, with no apparent outliers. There is a very small range from the first quartile to the median. 25% of all the players have heights within those 3 cm, from 165–168 cm.

(e)



The 90th percentile of 130 players corresponds to a cumulative frequency of 117. Using the graph, we estimate the 90th percentile as 184.5 cm.

(f) $\bar{x} = \dfrac{22165 + 182 \times 10}{130 + 10} = \dfrac{23985}{140} \approx 171$, correct to the nearest cm.

10. **(a)** The mean is not going to change since the added observation has the same value as the mean of the previous observations. Therefore, the new mean is also 12.

(b) $\bar{x} = \dfrac{9 \times 11 + 21}{10} = 12$; the new mean is 12.

(c) $21 = \dfrac{9 \times 11 + x}{10} \Rightarrow 210 = 99 + x \Rightarrow x = 111$; therefore, the last observation is 111.

11. If the mean of all 10 data points is 30, then $\sum x = 30 \times 10 = 300$

(a) If the value of 25 was incorrectly entered as 15, that means the total sum should increase by 10; therefore, the correct mean value is: $\bar{x} = \dfrac{310}{10} = 31$

(b) Since the added value is greater than the mean value, the mean is going to increase. The new mean is: $\bar{x} = \dfrac{310 + 32}{10 + 1} = \dfrac{342}{11} \approx 31.1$

12. You can take the size of the sample to be any number n:

$$\bar{x} = \frac{\frac{n}{2}\times 20 + \frac{n}{6}\times 40 + \frac{n}{3}\times 60}{n} = \frac{110}{3} \approx 36.7$$

13. $$\frac{7+10+12+17+21+x+y}{7} = 12 \Rightarrow 67+x+y=84 \Rightarrow x+y=17$$

$$\frac{7^2+10^2+12^2+17^2+21^2+x^2+y^2}{7} - 12^2 = \frac{172}{7} \Rightarrow \frac{15+x^2+y^2}{7} = \frac{172}{7} \Rightarrow x^2+y^2=157$$

To solve the simultaneous equations, we will use the substitution method by expressing the variable y (from the first equation) in terms of x.

$$\begin{cases} x+y=17 \\ x^2+y^2=157 \end{cases} \Rightarrow \begin{cases} y=17-x \\ x^2+(17-x)^2=157 \end{cases} \Rightarrow \begin{cases} y=17-x \\ 2x^2-34x+289=157 \end{cases} \Rightarrow \begin{cases} y=17-x \\ x^2-17x+66=0 \end{cases}$$

$$\Rightarrow \begin{cases} y=17-x \\ x=6 \text{ or } x=11 \end{cases} \Rightarrow \begin{cases} y=11 \text{ or } \cancel{y=6} \\ x=6 \text{ or } \cancel{x=11} \end{cases}$$

Note that we have discarded one solution because of the condition that $x<y$.

14. $$\sum_{i=1}^{25} x_i = 278 \Rightarrow \bar{x} = \frac{278}{25} = 11.12$$

$$\sum_{i=1}^{25} x_i^{\,2} = 3682 \Rightarrow s_n^{\,2} = \frac{3682}{25} - 11.12^2 = 23.6256$$

To answer questions 15–19, we use tables and graphs from the previous exercises, together with GDC/software. In each case, enter the data (class midpoint to represent each class) into lists and use the menu of your GDC to calculate the standard deviation. For IQR, reading from the cumulative frequency graph to find the first quartile, third quartile and subtract them will give you the estimate, but the GDC output usually differs from that answer because the GDC is assuming all data in each class to have one value – the class midpoint. So, the IQR from that reading may be less accurate than reading from the graph.

15. Estimates of the upper and lower quartiles are 12.5 and 4.5 respectively; therefore, $\text{IQR}=8$. The standard deviation is 6.63 (to 3 s.f.).

16. Estimates of the upper and lower quartiles are 30.5 and 24.5 respectively; therefore, $\text{IQR}=6$. The standard deviation is 4.46 (to 3 s.f.)

17. Estimates: $\text{IQR}=23$. The standard deviation is 16.7 (to 3 s.f.)

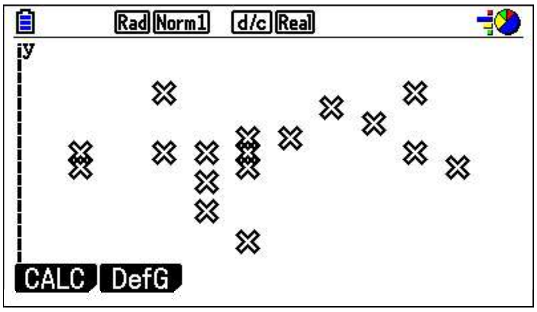
18. Estimates: IQR $= 0.1$. The standard deviation is 0.0569 (to 3 s.f.)

19. Estimates of the upper and lower quartiles are 280 and 180 respectively; therefore, IQR $= 100$ (or 60 from GDC). The standard deviation is 82.1 (to 3 s.f.)

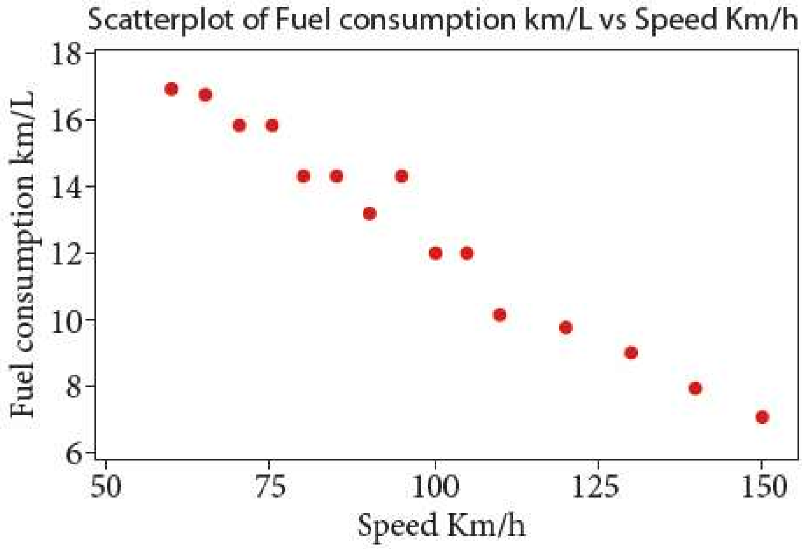
Exercise 10.4

1. The scatter plot below shows a weak positive linear relationship. The correlation coefficient is 0.26 which confirms the weakness of the relationship.

The regression equation is: $y = 6.56 + 0.29x$. For every change of 1 unit in the x-values, the y-values will change, on average, by 0.29



2. **(a)**

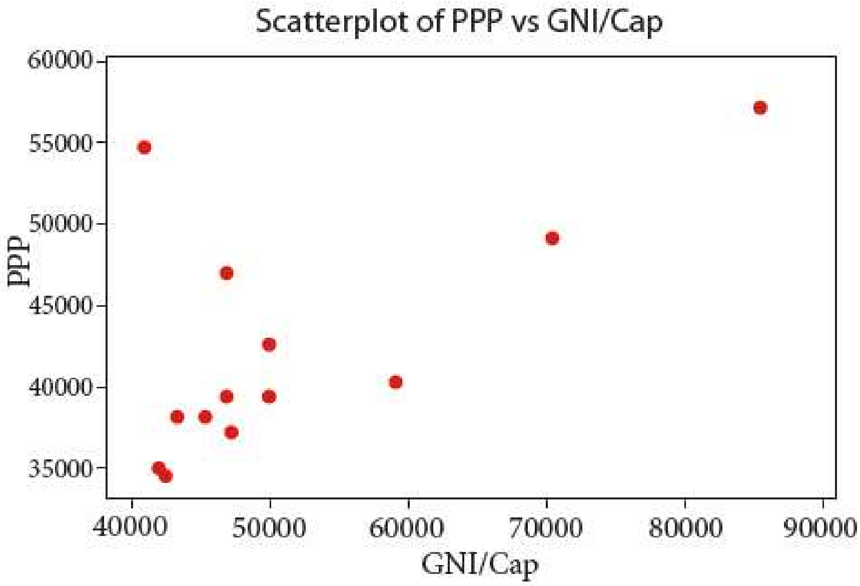


(b) We chose speed as the explanatory variable because the car must first run to cause any fuel consumption. Hence the speed helps explain the fuel consumption. The relationship appears to be negatively sloped because the consumption is measured by the distance travelled per litre of fuel.

(c) The relationship appears to be a relatively strong negative one without any apparent outliers. The correlation coefficient is –0.986 which is very close to –1, and is thus a very strong relationship.

(d) The regression equation is: Fuel cons.km/L = 24.1 – 0.116 × Speed Km/h. For every increase of 1 km/h in speed, the average number of km per liter will decrease by 0.116 km/L. i.e. consumption will increase.

3. **(a)**

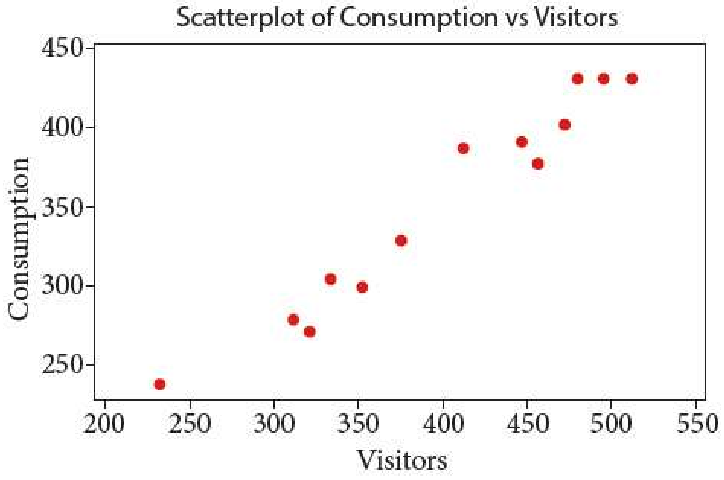


(b) The relationship appears to be a positive one except for an outlier which can be traced to be Singapore. We chose the explanatory variable to be the Income because the income level dictates how willing are people to pay for goods.

(c) The relationship is relatively strong (weakened by Singapore’s numbers). The correlation coefficient is 0.621. If we remove Singapore’s data, then it becomes 0.886

(d) The regression equation is: PPP = 24383 + 0.351 GNI/cap. For every increase of $1 in GNI/cap, the PPP will increase, on average by $0.351

4. **(a)**



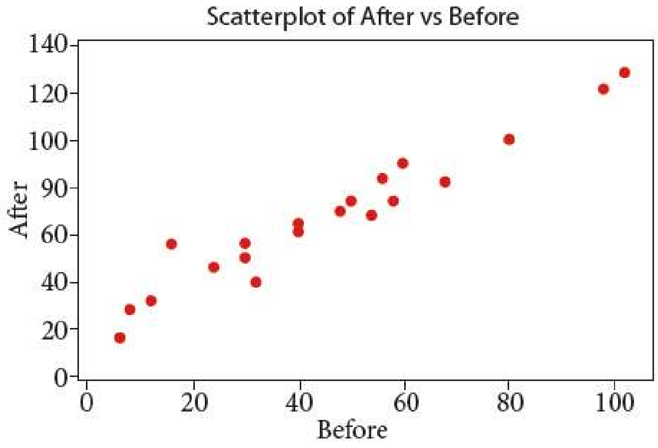
(b) There is obviously a positive relationship between the number of visitors and consumption. As the number of visitors increases the consumption will also increase.

(c) The relationship seems to be strong and there is an absence of outliers. The correlation coefficient is 0.978 which is very close to 1.

(d) The regression equation is: Consumption = 40.0 + 0.777 × Visitors. For every increase of 1 visitor, we expect, on average, that consumption will increase by 0.777

5. The answers for this question are found at the end of each question from 1 to 4.

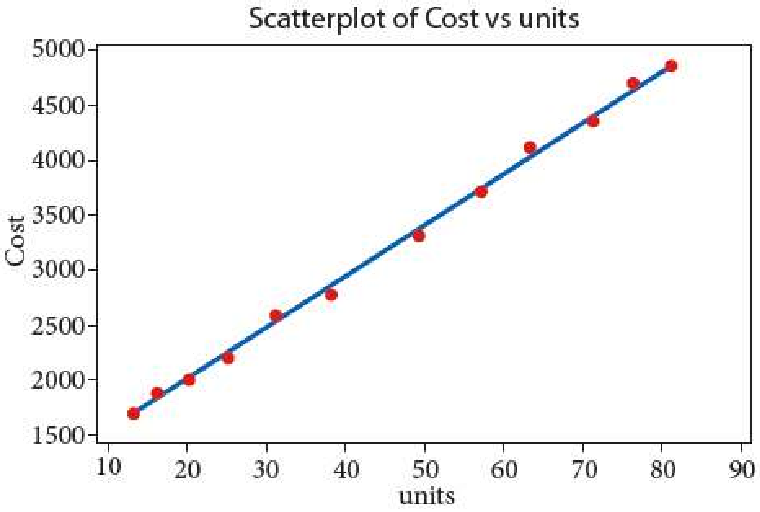
6.



The scatter plot shows a strong positive relationship. The higher the 'Before' score the higher the 'After' score. The regression equation is: After = 20.2 + 1.03 × Before.

This means that, on average, for every change of 1 mark on the 'Before' test, the 'After' test is expected to change by 1.03. The correlation coefficient is 0.97 indicating a very strong linear relationship. For a student with 60 score on the 'Before' test, the model predicts, on average, a score of 81.90 on the 'After' test.

7. **(a)**



(b) The regression equation is: Cost = 1066 + 47.1 × units.

(c) For every increase of 1000 units in production, the cost, on average, will increase by 47100 euros. The correlation coefficient is 0.999, which is almost perfect association. This is a strong linear relationship.

(d) Let number of 1000 units be x, then:

$$\text{Cost} = 1066 + 47.1x$$

$$\frac{\text{Cost}}{x} = \frac{1066}{x} + 47.1 = \text{ cost per unit}$$

If the cost is 105, then

$$105 = \frac{1066}{x} + 47.1 \Rightarrow x = 18.411$$

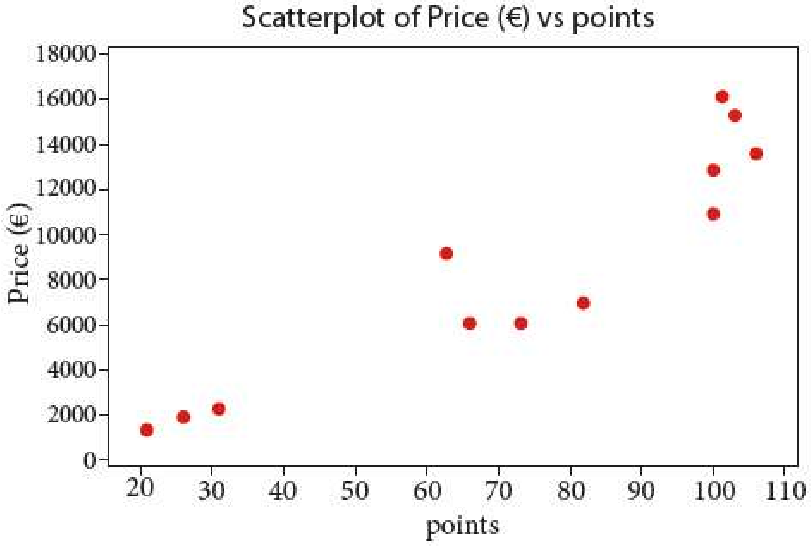
Thus, the number of units will be 18 411

8. **(a)** $r = 0.493$. This is a relatively weak correlation between the two scores.

(b) The regression equation is: Economics = 2.07 + 0.649 × Physics

(c) Economics = 2.07 + 0.649 (4) = 4.7 (which can be rounded up to 5)

9. **(a)**



This appears to be a positively sloped trend.

(b) The regression equation is: Price (€) = – 2689 + 154 × points

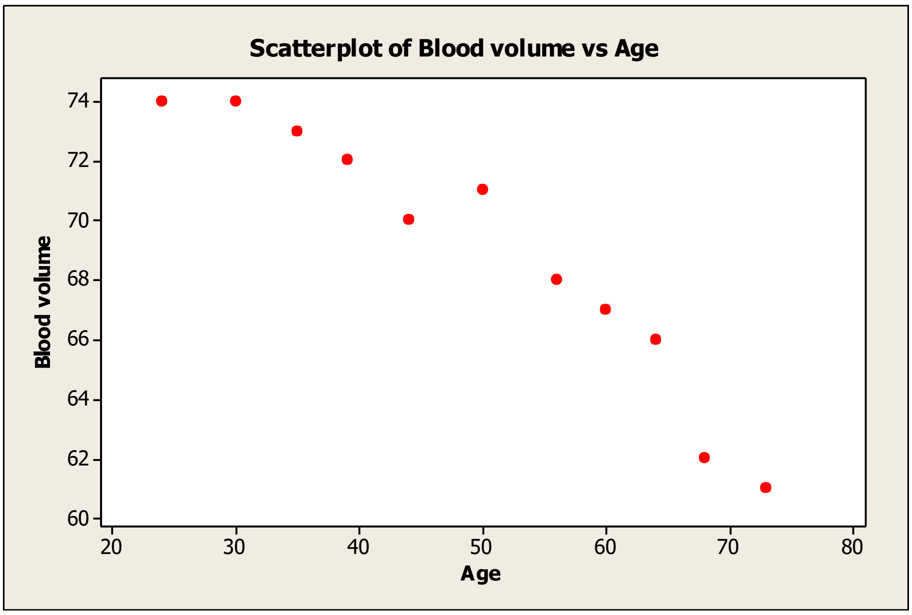
(c) The intercept is meaningless, as zero is not in the domain of the explanatory variable. On average, for every increase of 1 point, we expect the price to increase by 154 euros.

(d) $r = 0.93$ indicating a strong association between points and price.

(e) The average price of a 63-point diamond is predicted to be
– 2689 + 154 (63) = 7013 euros.

(f) Residual = 9117 – 7013 = 2104

10. **(a)**

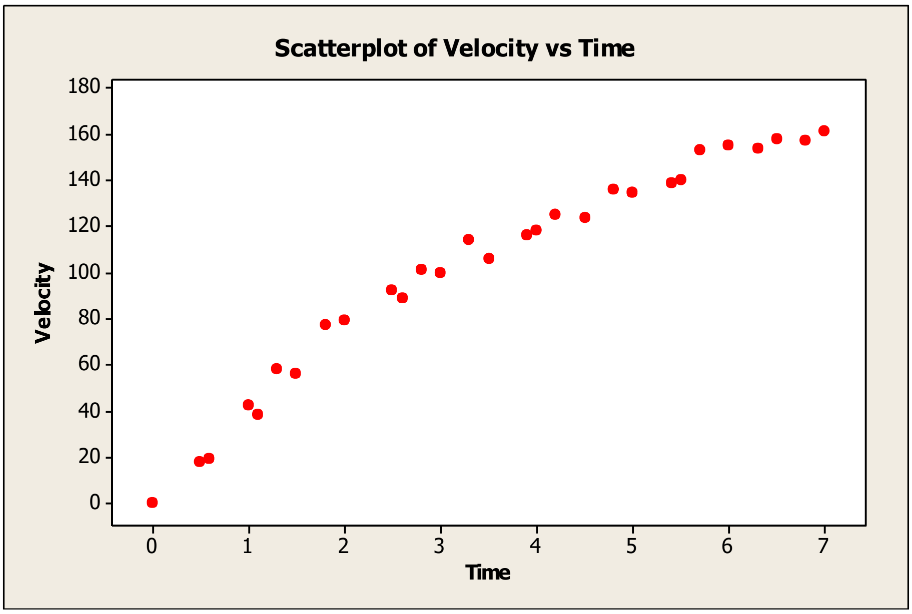


(b) $r = -0.958$. There is a strong negative correlation between the stroke volume and age of patients.

(c) The regression equation is: Blood volume = 82.1 – 0.268 × Age. On average for an increase of 1 year, we expect blood volume to be decreasing by 0.268 ml per stroke. The interpretation of the intercept of 82.1 does not make sense in this situation.

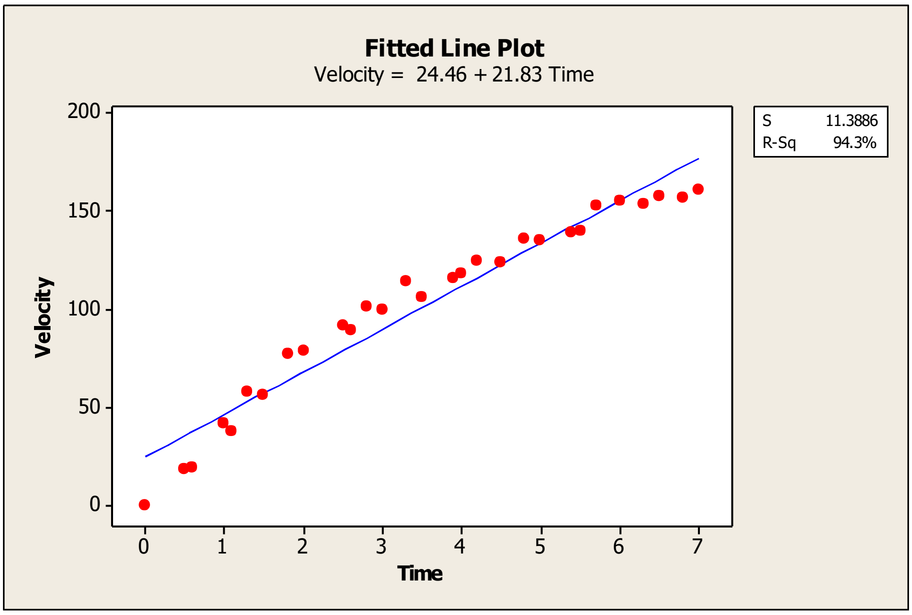
(d) On average, 45-year-olds may have 70 stroke volume. Using the model to predict the 90-year old volume is not advisable as it is an extrapolation of 17 years beyond the range of collected data.

11. **(a)**



Apparently, there is a strong association between time and speed as expected. However, it appears that there is a break point around 3 seconds.

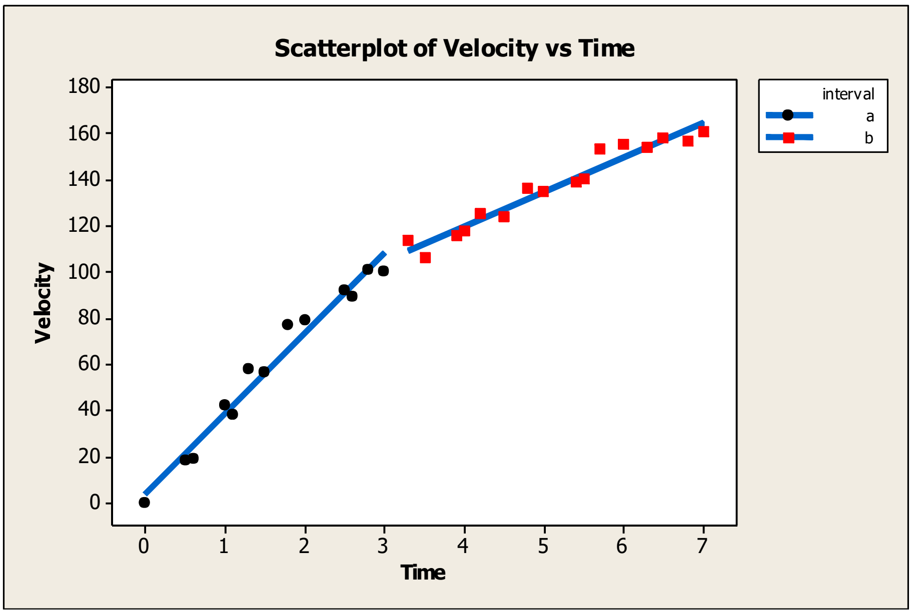
(b) The regression equation is: Velocity = 24.5 + 21.8 × Time



Apparently, the data do not follow a linear model through the whole range. There is a clear deviation from the line at both ends.

(c) $r = 0.97$, which is a strong association indication. However, this number may not be of great validity since the data does not appear to be linear.

(d) By splitting the data, we can clearly see that the new model fits the data better.
The data clearly has two phases, one before 3 seconds and the other after 3 seconds.



Chapter 10 practice questions

1. **(a)** $$\sum_{i=1}^{30} y_i = 360 \Rightarrow \mu = \frac{360}{30} = 12$$

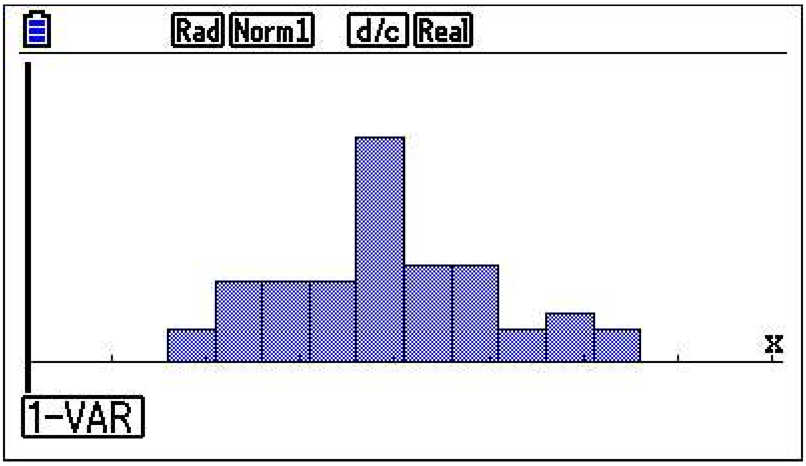
(b) $$\sum_{i=1}^{30} (y_i - \mu)^2 = 925 \Rightarrow s_n = \sqrt{\frac{925}{30}} \approx \sqrt{30.83} \approx 5.55$$

2. $$\mu = \frac{\sum x_i f_i}{\sum f_i} \Rightarrow 34 = \frac{10\times 1 + 20\times 2 + 30\times 5 + 40\times n + 50\times 3}{11+n} \Rightarrow 34 = \frac{350+40n}{11+n}$$

$$\Rightarrow 374 + 34n = 350 + 40n \Rightarrow 24 = 6n \Rightarrow n = 4$$

3. **(a)** Use a GDC/software for the histogram.

Time	1.6	2.1	2.6	3.1	3.6	4.1	4.6	5.1	5.6	6.1	6.6
Frequency	2	5	5	5	14	6	6	2	3	2	0

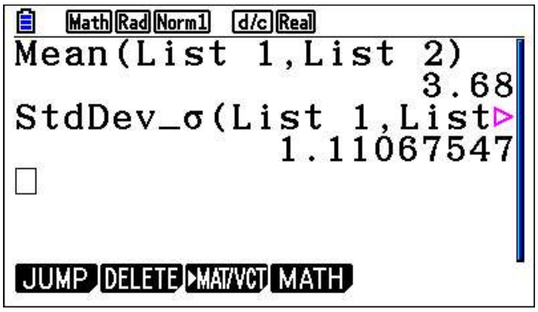


(b) There are 7 out of 50 measurements that are greater than or equal to 5.1

Therefore, the fraction of the measurements less than 5.1 is: $\frac{43}{50} = 0.86 = 86\%$

(c) There are 50 pieces of data, so to determine the median we need to find the 25th and 26th observations. We notice that these two observations are within the interval 3.6–4.1; therefore, the median is approximately 3.9

(d) Using GDC with data entered in **(a)**, we get:

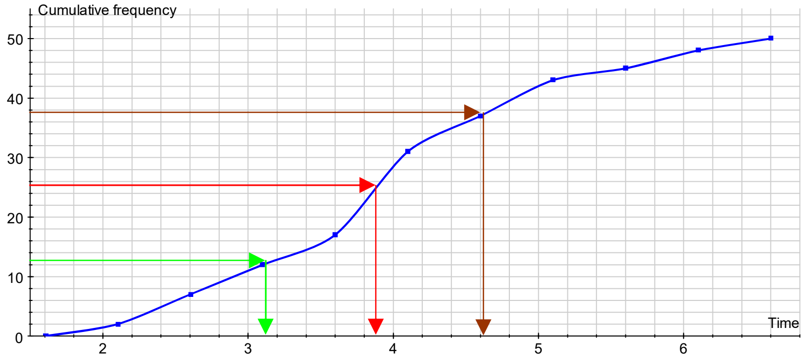


```
Mean(List 1,List 2)
                 3.68
StdDev_σ(List 1,List▷
           1.11067547
□
```

The mean value is 3.68, whilst the standard deviation is 1.11, correct to 3 s.f.

(e) A cumulative frequency table is given below. A graph is given as we need it for part **(f)**.

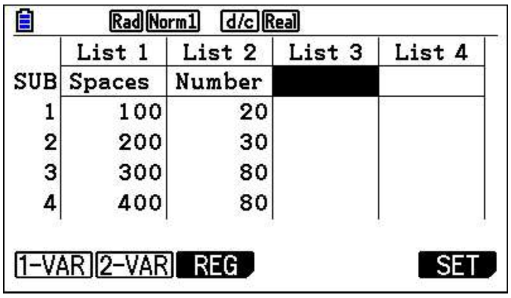
Time	1.6	2.1	2.6	3.1	3.6	4.1	4.6	5.1	5.6	6.1	6.6
Cu. Freq.	2	7	12	17	31	37	43	45	47	50	0



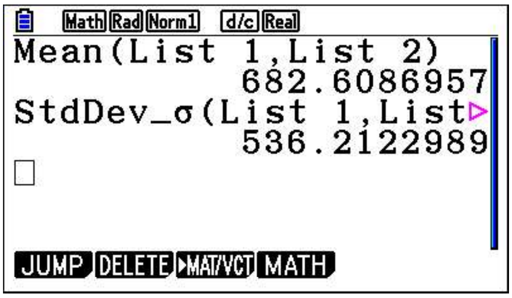
(f) Estimates for the minimum and maximum values are 1.6 and 6.6 respectively. The first and third quartiles correspond to the cumulative frequencies of 12.5 and 37.5 respectively; therefore, an estimate for the first quartile is 3.15 and the third quartile is 4.65. An estimate for the median (which corresponds to a cumulative frequency of 25) is 3.9

4. **(a)** The median and the IQR would best represent the data, since the data is skewed to the right and there are a few outliers on the right.

(b) First, we read the frequencies from the histogram and input them into the frequency distribution table on our GDC.



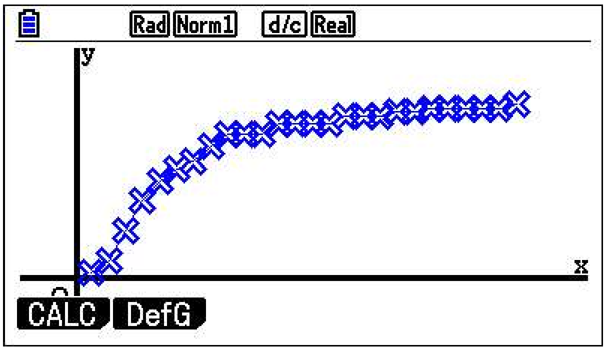
```
        Rad Norm1  d/c Real
    List 1   List 2   List 3   List 4
SUB Spaces   Number
  1     100      20
  2     200      30
  3     300      80
  4     400      80
1-VAR 2-VAR REG              SET
```



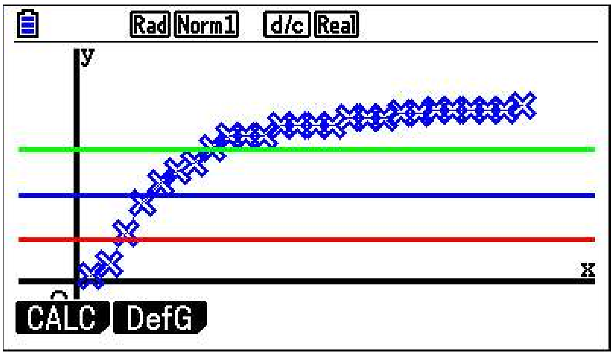
```
   Math Rad Norm1  d/c Real
Mean(List 1,List 2)
          682.6086957
StdDev_σ(List 1,List▷
          536.2122989
☐
JUMP DELETE ▸MAT/VCT MATH
```

The mean value is 682.6 and the standard deviation is 536.2

(c) Since we have grouped data, the endpoints of our intervals will be 150, 250, 350, ..., and so on. On a calculator, we can use the adding a number to the list feature. Alternatively, we can calculate the cumulative frequencies from the list menu.



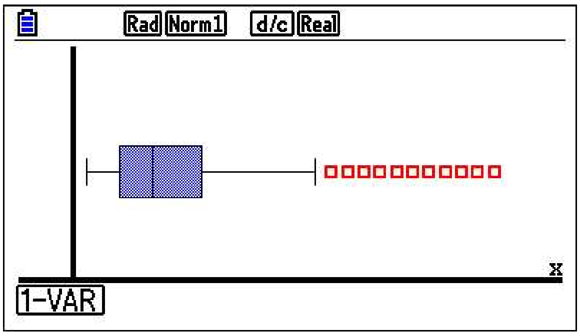
(d) There are 460 cities, so to estimate the median, we draw a horizontal line from 230. To find the lower and upper quartiles, we need to draw horizontal lines from 115 and 345 respectively.



We may use the "Zoom" command on a GDC to get a good estimate.

So, the median is about 500. The first quartile is about 330 and the third quartile is about 830. Therefore, the IQR is about 830 – 330 = 500

(e) There are a few outliers to the right. The outliers are those points which are over $Q_3 + 1.5 \times \text{IQR}$, i.e. $830+1.5\times 500=1580$, which gives us 50 cities from the histogram. We can also use a box-plot as shown:



(f) The data is skewed to the right with quite a few outliers to the right (1600 and above). The data is also bimodal, with the modal values being 300 and 400. After the class with 900 places, there are a few gaps which makes the spread of the data between the third quartile and the maximum very large.

5. **(a)** It appears that Spain produces both the most expensive (estimated €152 per case) and the cheapest (estimated €55 per case) red wine.

(b) Red wines are generally more expensive in France as we can see that the median price is the highest; the minimum value in France is also the highest, but the upper 50% of wines are also within a very small range of approximately €10 per case. This is also clear because the lower quartile in France is higher than the medians in both other countries. This indicates that at least 75% of the French wines are more expensive than at least 50% in each of the other countries' wines.

(c) It appears that the wines are, on average, more expensive in France, where the prices are skewed towards the higher end. In Spain, you can find a higher percentage of cheaper wine than in the other two countries, but you also find the most expensive wines on the market; so, Spain has the widest range of prices. Italy seems to have the most symmetric distribution of wine prices.

6. Given the large size of the data, it is advisable to use software/GDC for calculations.

(a) The mean value of the data is 52.6 and the standard deviation is 7.60, both given correct to the three significant figures.

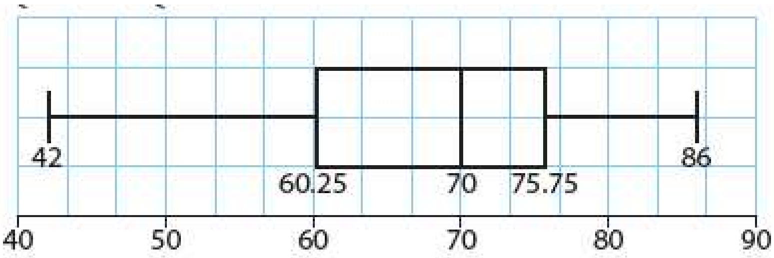
(b) The median value is 51.3. The upper and lower quartiles are 49.9 and 52.6 respectively; therefore, the IQR is 2.65, all correct to three significant figures.

(c) Apparently, the data is skewed to the right with a clear outlier of 112.72. This outlier pulled the value of the mean to the right and increased the spread of the data.
The median and IQR are not influenced by the extreme value.

7. **(a)** The distribution does not appear to be symmetric as the mean is less than the median, the lower whisker is longer than the upper one and the distance between Q_1 and the median is larger than the distance between the median and Q_3. The data are left skewed.

(b) There are no outliers as $Q_1 - 1.5 \times \text{IQR} = 37 < 42$, and $Q_3 + 1.5 \times \text{IQR} = 99 > 86$

(c)



(d) See **(a)**

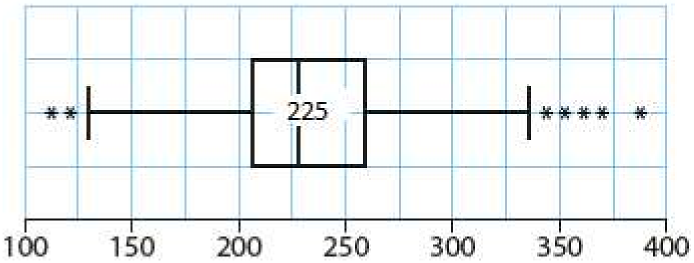
8. **(a)** Using the given cumulative graph, we draw a horizontal line at 50%. The cholesterol level corresponding to the point of intersection is approximately 225.

(b) Again, by drawing horizontal lines at 25%, 75% for the quartiles, and 10% and 90% for the percentiles, we can make the following estimates:

$Q_1 \approx 210$, $Q_3 \approx 260$. 10th percentile ≈ 185, and 90th percentile ≈ 300

(c) IQR = $Q_3 - Q_1 = 260 - 210 = 50$. The number of patients in the middle 50% is 50% of the 2000 subjects, which is 1000.

(d) It appears that the minimum level is 100 and the maximum is 400; the quartiles are mentioned above and the median is 225. To decide whether we have outliers, we calculate the length of the whiskers. The left one goes as far as $210 - 1.5 \times 50 = 135$ and the right whisker goes to $260 + 1.5 \times 50 = 335$. Thus, on the left side we have a few outliers below 135 as the minimum is 100, and some outliers on the right side since the maximum of 400 is way beyond 335. So, an approximate boxplot of the data is shown below.



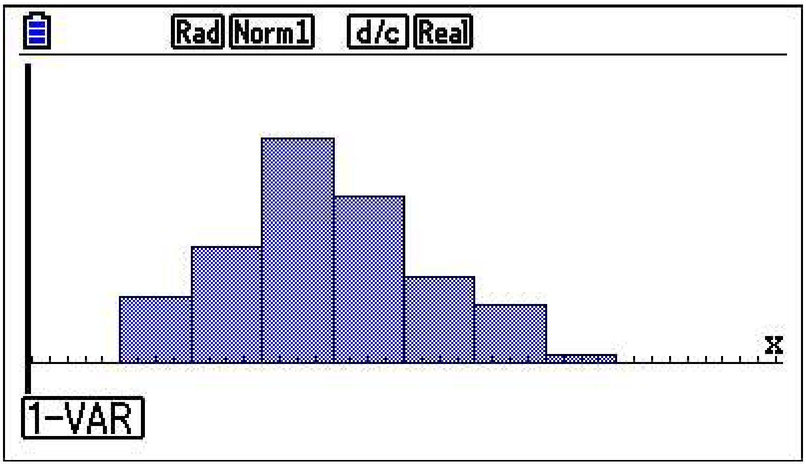
(e) In addition to what was mentioned in part **(d)**, we notice that the data is skewed to the right a bit, with more outliers on the right side, since the outliers lie outside of the interval 135 to 335. From the cumulative frequency graph, we see that there are almost 100 patients who have a cholesterol level greater than 335 mg/dl and only a few patients with a level less than 135 mg/dl.

9. **(a)**

Speed	Frequency
26-30	9
31-34	16
35-38	31
39-42	23
43-46	12
47-50	8
51-54	1

(b) Histograms for this data are not unique. It depends on the choice of class width.

Here is a sample GDC output:



Data is relatively symmetric with possible outlier at 55. The mode is approximately 37

(c) We will show the work by using the following table of calculations

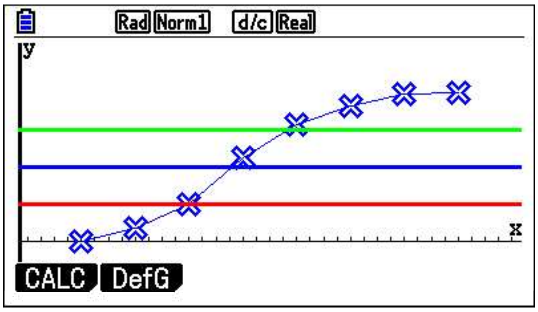
x_i	f_i	$x_i \times f_i$	$x_i^2 \times f_i$
28	9	252	7056
32.5	16	520	16900
36.5	31	1131.5	41299.75
40.5	23	931.5	37725.75
44.5	12	534	23763
48.5	8	388	18818
52.5	1	52.5	2756.25
	100	3809.5	148318.8
	$\mu =$	38.1	
		$s =$	5.65

Here we used the formulas: $\mu = \frac{\sum x_i f(x_i)}{n}$ and $s_n^2 = \frac{\sum x_i^2 f(x_i)}{n} - \overline{x}^2$

(d)

Speed	Cumulative frequency
30	9
34	25
38	56
42	79
46	91
50	99
54	100

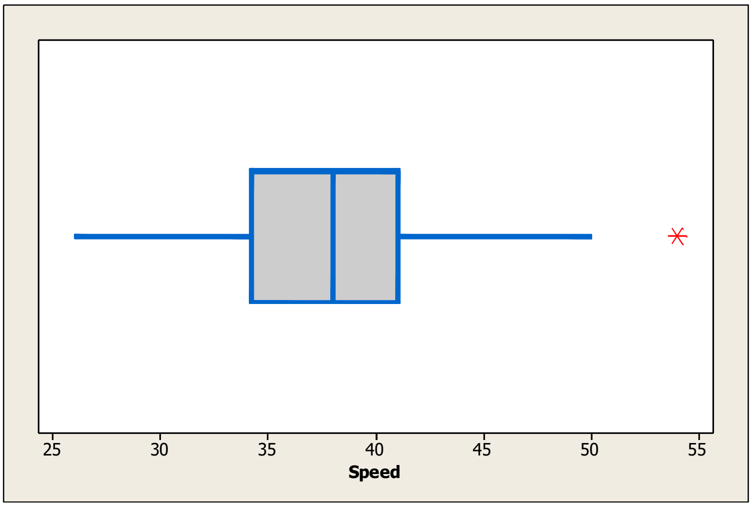
(e) In order to estimate the median and quartiles, we draw the cumulative frequency graph and draw horizontal lines at 50, 25 and 75. We use a GDC here.



An estimate of the median is 37. Q_1 is 34.5 and Q_3 is 42; therefore, the IQR is 7.5

(f) Since Q_3 is 42 and 1.5 × IQR is 11.25, and the upper whisker goes up to 53.25 (which is smaller than the largest observation) we suspect that there may a possible outlier, given that our values are only estimates that may also differ from the real values. On the lower end, the whisker goes down to 23.25, which is less than the minimum observation, and hence, there are no outliers here.

A box-plot may be used.



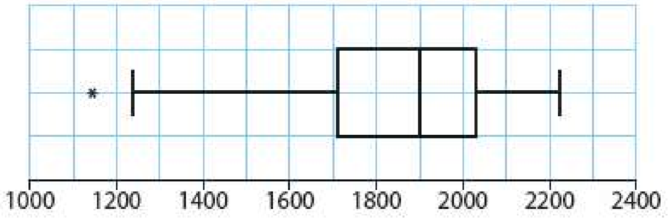
Using the raw data, we confirm that 54 is, in fact, an outlier.

10. **(a)** Using software/GDC the required values are:

Mean = 1846.9, median = 1898.6, standard deviation = 233.8,
Q_1 = 1711.8, Q_3 = 2031.3, IQR = 319.5

(b) Q_1 – 1.5 × IQR = 1232.55 > minimum, so there is an outlier on the left.
Q_3 + 1.5× IQR = 2510.55 > maximum, so there is no outlier on the right.

(c) Use a GDC/software:



(d) $\mu \pm s_n = 1846.9 \pm 233.8 \Rightarrow [1613.1, 2080.7]$

(e) Germany will definitely be an outlier since $Q_3 + 1.5 \times \text{IQR} = 2510.55 < 2758$. Therefore, it will influence the mean and standard deviation. However, the median, the first and third quartiles, and the IQR will not change much.

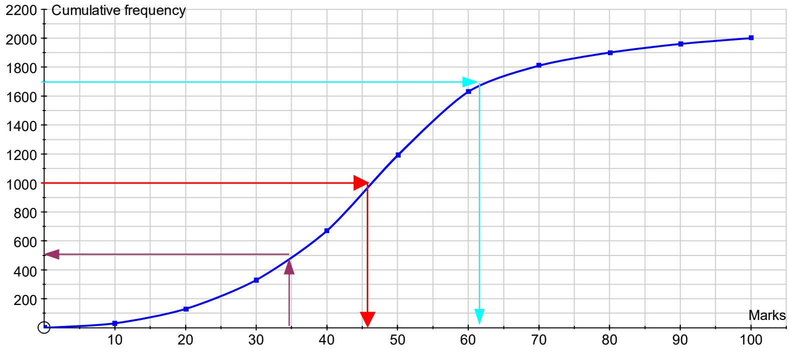
11. **(a)** $\sum_{i=1}^{90} x_i = 4460 \Rightarrow \mu = \frac{4460}{90} \approx 49.6 \text{ minutes}$

(b) $\mu_{\text{New}} = \frac{4460 + 35 + 39 + 28 + 32}{90 + 4} = \frac{4594}{94} \approx 48.9 \text{ minutes}$

12. **(a)**

Marks	≤ 10	≤ 20	≤ 30	≤ 40	≤ 50	≤ 60	≤ 70	≤ 80	≤ 90	≤ 100
Number of candidates	30	130	330	670	1190	1630	1810	1900	1960	2000

(b) We used a spreadsheet to produce this graph:



(c) **(i)** By looking at the graph, we estimate that the median score (which corresponds to the cumulative frequency of 1000) is 47.

(ii) We draw a vertical line from 35 on the Marks axis and reach the cumulative frequency curve at the point at which the cumulative frequency is about 500. Therefore, 500 candidates had to retake the exam.

(iii) The highest scoring 15% corresponds to the highest 300 results; therefore, we draw a horizontal line from 1700 on the cumulative frequency axis and reach the curve at the point at which the number of marks is about 61. Hence, a distinction will be awarded if 61 or more marks are scored on the test.

13. $$\mu = \frac{n_1\mu_1 + n_2\mu_2}{n_1 + n_2} = \frac{72\times179 + 28\times162}{72+28} = \frac{17424}{100} \approx 174\,\text{cm}$$

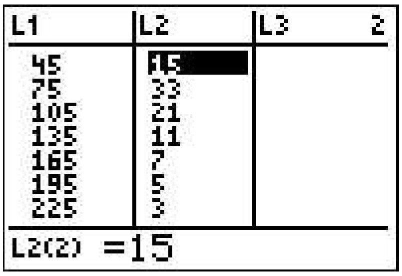
14. **(a)** $$\sum_{i=1}^{25} x_i = 300 \Rightarrow \mu = \frac{\sum_{i=1}^{25} x_i}{25} = \frac{300}{25} = 12$$

(b) $$\sum_{i=1}^{25}(x_i-\mu)^2 = 625 \Rightarrow s_n = \sqrt{\frac{\sum_{i=1}^{25}(x_i-\mu)^2}{25}} = \sqrt{\frac{625}{25}} = 5$$

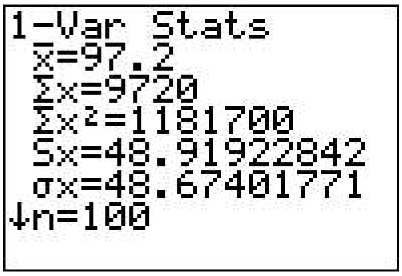
15. $$\bar{x} = 34 \Rightarrow \bar{x} = \frac{\sum x_i f(x_i)}{n} = \frac{10\times1+20\times2+30\times5+40\times k+50\times3}{11+k} = 34$$

$$\Rightarrow 350+40k = 374+34k \Rightarrow 6k = 24 \Rightarrow k = 4$$

16. **(a)** Similar to what has been done earlier, to calculate an estimate for the mean, we will take the midpoints of the intervals (15, 45, 75, and so on) and the corresponding frequencies.



```
L1       L2       L3     2
 45      15
 75      33
 105     21
 135     11
 165     7
 195     5
 225     3
L2(2) =15
```



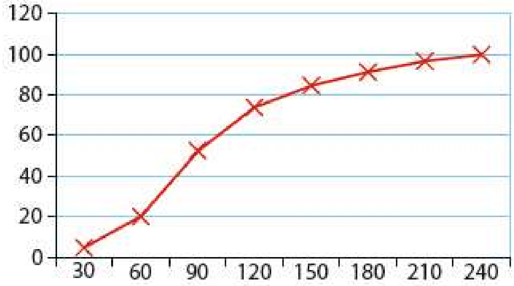
```
1-Var Stats
 x̄=97.2
 Σx=9720
 Σx²=1181700
 Sx=48.91922842
 σx=48.67401771
↓n=100
```

So, an estimate for the mean of the waiting times is 97.2 seconds.

(b)

Time	≤ 30	≤ 60	≤ 90	≤ 120	≤ 150	≤ 180	≤ 210	≤ 240
Cum freq	5	20	53	74	85	92	97	100

(c)



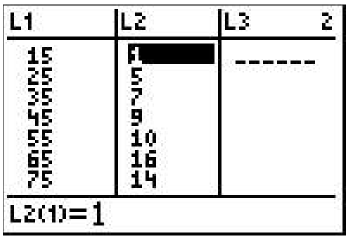
(d) To find the three estimations asked for, we need to draw a horizontal line at 50 for the median, and at 25 and 75 for the quartiles. An estimation of the median value is 88, while the lower and upper quartiles are 66 and 124 respectively.

17. **(a)** **(i)** The bar between 50 and 60 has a height of 10, so the number of plants in that range is 10.

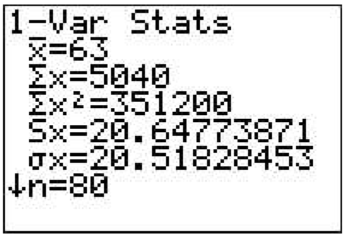
(ii) There are two bars here that add up to 14 + 10 = 24, so the number of plants in that range is 24.

(b) Reading frequencies as the heights of the bars, and representing every class with its midpoint, we create a frequency distribution table and use a GDC to do the calculations. (Alternatively, you can use the formulas too.)

x_i	15	25	35	45	55	65	75	85	95
f_i	1	5	7	9	10	16	14	10	8



```
L1      L2      L3      2
 15     1       -------
 25     5
 35     7
 45     9
 55     10
 65     16
 75     14
L2(1)=1
```



```
1-Var Stats
 x̄=63
 Σx=5040
 Σx²=351200
 Sx=20.64773871
 σx=20.51828453
↓n=80
```

So, an estimate for the mean is 63, and the standard deviation is 20.5

(c) The data is skewed to the left; therefore, the mean is pulled to the left with the extreme small values. Thus, the median must be larger than the mean.

(d) Since there are 80 plants, we know that the median is about 40. If we draw the first part of the cumulative frequency graph and draw a horizontal line from 40 to find the point of intersection with the graph, our estimate is approximately 65 cm.

18. (a) Again, we will use the midpoints of the intervals (82.5, 87.5, 92.5, ..., and so on) and the corresponding frequencies. We put the two lists into a GDC and obtain an estimation of the standard deviation. An estimate for the standard deviation of the weights is 7.405

(b)

Weight	Number of packets
$w \le 85$	5
$w \le 90$	15
$w \le 95$	30
$w \le 100$	56
$w \le 105$	69
$w \le 110$	76
$w \le 115$	80

(c) By drawing horizontal lines at 40 for the media and at 20, and 60 for the quartiles, the estimates will be:

(i) median ≈ 97

(ii) $Q_3 \approx 101$

(d) This is the total deviation from the mean; it has to be zero.

Also,

$$(W_1 - \overline{W}) + \cdots + (W_{80} - \overline{W}) = (W_1 + \cdots + W_{80}) - 80 \times \overline{W}$$
$$= (W_1 \cdots + W_{80}) - \cancel{80} \times \frac{W_1 + \cdots + W_{80}}{\cancel{80}} = 0$$

(e) There are 71 packets that satisfy the condition $85 < W \le 110$. There are 20 packets that satisfy the condition $100 < W \le 110$. Therefore, the probability is:

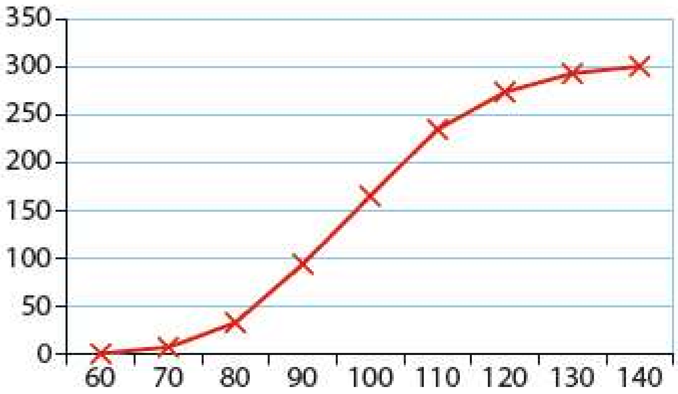
$P(E) = \frac{20}{71} \approx 0.282$, correct to three significant figures.

19. **(a)** Again, we will use the midpoints of the intervals (65, 75, 85, ..., and so on) and the corresponding frequencies. We put the two lists into a GDC and obtain an estimation of the mean. An estimate for the mean speed is 98.2 km h^{-1}.

(b) **(i)** To find the value of *m,* we can either add 70 (the frequency of the speed interval 90–100) to the previous cumulative frequency, 95; or subtract 71 (the frequency of the speed interval 100–110) from the next cumulative frequency, 236. In both cases we get the same value: $m = 165$

In a similar manner, we find the value of *n*: $n = 236 + 39 = 275$

(ii) Using a spreadsheet, we get the following graph:



(c) **(i)** We draw a vertical line from $v = 105$ until we reach the cumulative frequency curve. This gives us an estimate for the cumulative frequency of 200. So, there are 100 cars that will exceed the speed of 105 km h^{-1} and $P = \dfrac{100}{300} \approx 33.3\%$

(ii) If 15% of the cars exceed this speed, then 85% do not exceed that speed. 85% of 300 is 255, so we draw a horizontal line from $y = 255$ until we reach the cumulative frequency curve. This gives us an estimate of a speed of 115 km h^{-1}.

20. **(a)** **(i)** We take a horizontal line across from 100 on the vertical axis until it touches the graph. From that point, we take a vertical line down to the horizontal axis and read the value. An estimate for the median fare is \$24.

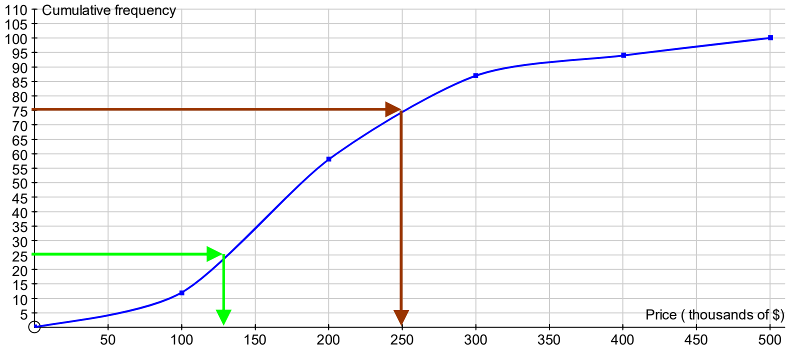
(ii) We take a vertical line up from 35 on the horizontal axis until we reach the graph. From that point, we take a horizontal line across to the vertical axis and read the value. An estimate for the number of cabs in which the fare taken is \$35 or less is 158 cabs.

(b) 40% of the cabs is $0.4\times 200=80.$ So, we take a horizontal line from 80 on the vertical axis until we reach the graph and then we estimate the *x*-coordinate, which is 22. Therefore, the fare is \$22. To find the number of kilometres, we need to divide the fare by 0.55 (which is the fare per kilometre for distance travelled). Therefore, the distance travelled is 40 km.

(c) If the distance travelled is 90 km, the driver will earn $90\times 0.55=49.5$ dollars. We will use the graph to estimate the number of cabs that will earn less than 49.5 dollars – there are about 185, and therefore there are 16 cabs that will earn more than that. So, the percentage of the cabs that travel more than 90 km is: $\dfrac{15}{200}=0.075=7.5\%$

21. Since the three numbers are given in order of magnitude $a<b<c,$ we know that the middle one (the median) is 11. Given that the range is 10, we know that the difference between the minimum and maximum value is: $c-a=10$. Since the mean value is 9, we can establish another equation in terms of *a* and *c*: $\dfrac{a+11+c}{3}=9\Rightarrow a+c=16$. Solving these two equations (using the elimination method), we get: $2a=6\Rightarrow a=3$

22. **(a)**



(b) By using horizontal lines at 25 and 75, we estimate the values of Q_1 and Q_3 as \$130,000 and \$250,000. Hence, the IQR is \$120,000. (The answer here is different from the book because of the different accuracy of the graphs used.)

(c) To find the frequencies *m* and *n*, we need to subtract two successive cumulative frequencies.

$$f_i=c_i-c_{i-1}\Rightarrow\ m=94-87=7,\, n=100-94=6$$

(d) We take the midpoints of the intervals in the first list and the corresponding frequencies in the second list, and then we use a GDC.

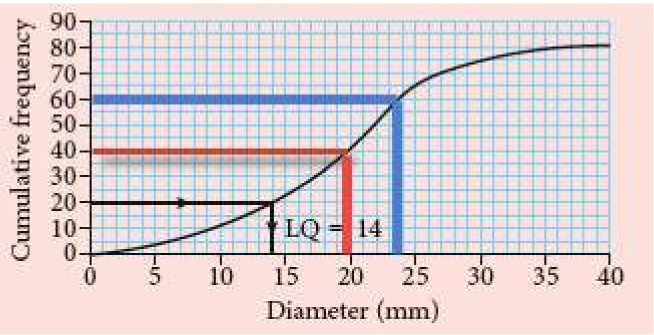
So, an estimate of the mean selling price is \$199,000.

(e) **(i)** An estimate of the cumulative frequency for \$350,000 is 90; therefore, there are about 10 houses that can be described as Luxury houses.

(ii) Out of 10 Luxury houses, six were sold for \$400,000; therefore, the probability that both selected houses have a selling price more than \$400,000 is:

$$\mathrm{P}(E)=\frac{6}{10}\times\frac{5}{9}=\frac{1}{3}\approx 0.33$$

23. **(a)** **(i)** To mark the median, we draw the horizontal line $y=40$ until it hits the graph, at which point we draw a vertical line down to the Diameter axis. An estimate for the median is 20 mm.



(ii) To mark the upper quartile, we draw the horizontal line $y=60$ until it hits the graph, at which point we draw a vertical line down to the Diameter axis. An estimate for the upper quartile is 24 mm.

(b) The interquartile range is: $\text{IQR}=24-14=10\,\text{mm}$

24. **(a)** In this question, we can accept an error of ± 2 students.

We need to read the cumulative frequencies at the endpoints of the intervals and then subtract the successive ones to obtain the frequencies.

For 40, the cumulative frequency is 72, so the corresponding frequency is $72-22=50$

For 60, the cumulative frequency is 142, so the corresponding frequency is $142-72=70$

For 80, the cumulative frequency is 180, so the corresponding frequency is $180-142=38$

Mark	[0,20[	[20,40[	[40,60[	[60,80[	[80,100[
Number of students	22	50	70	38	20

(b) 40% of 200 students is 80, and then we draw a horizontal line and estimate the *x*-coordinate of the point. Our estimation is 43%

25. **(a)** To find the median height, we draw a horizontal line $y=60$ until it hits the graph. We then estimate the *x*-coordinate of the point of intersection. We estimate 183 cm.

(b) For the lower and upper quartiles, we draw two horizontal lines $y=30$ and $y=90$. Then we estimate the *x*-coordinates of the points of intersection. Therefore:
$\text{Q}_1=175, \text{Q}_3=189 \Rightarrow \text{IQR}=189-175=14$

26. Since the modal value is 11, we know that $c=d=11$. Given that the range is 8, we can find the value of *a*: $11-a=8 \Rightarrow a=3$. Finally, given that the mean value is 8, we can find the remaining number *b*: $\frac{3+b+11+11}{4}=8 \Rightarrow 25+b=32 \Rightarrow b=7$

27. **(a)** We draw a vertical line $x=40$ until it hits the graph. We then estimate the *y*-coordinate of the point of intersection, i.e. 100. So, the number of students who scored 40 marks or less is 100.

(b) There are 800 students, so the middle 50% is between 200 and 600 students. For a cumulative frequency of 200, the estimated mark is 55; whilst for 600, the estimated mark is 75. Hence, we say that the middle 50% of test results lie between 55 and 75 marks: $a=55, b=75$.

28. Use the mean formula to find the first equation relating the unknowns:

$$13=\frac{x+y+90}{8} \Rightarrow 104=x+y+90 \Rightarrow x+y=14$$

Use the sum of the squares of the known observations and the variance formula to find the second equation relating the squares of the unknowns:

$$21=\frac{x^2+y^2+1404}{8}-13^2 \Rightarrow 190=\frac{x^2+y^2+1404}{8} \Rightarrow x^2+y^2=116$$

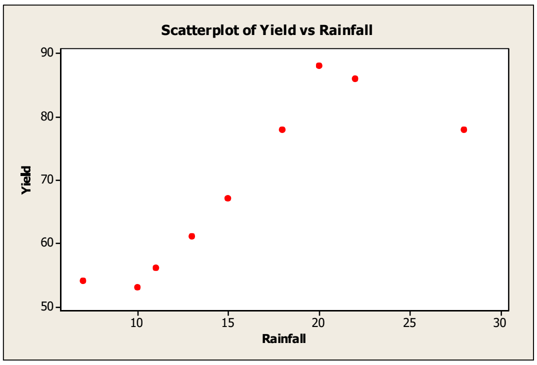
Now, use substitution to solve the simultaneous equations.

$$\begin{cases} x+y=14 \\ x^2+y^2=116 \end{cases} \Rightarrow \begin{cases} y=14-x \\ x^2+(14-x)^2=116 \end{cases} \Rightarrow \begin{cases} y=14-x \\ x^2-14x-40=0 \end{cases}$$

$$\Rightarrow \begin{cases} y=14-x \\ x=4 \text{ or } x=10 \end{cases} \Rightarrow \begin{cases} y=10 \text{ or } \cancel{y=4} \\ x=4 \text{ or } \cancel{x=10} \end{cases}$$

Since $x<y$, we can discard the second solution. A GDC can also be used.

29. **(a)**



The scatter plot shows an apparently linear association with two possible outliers: (7, 54) and (28, 78). Apart from these, the association seems to be relatively strong.

(b) $r=$ 0.853, which is a relatively strong positive linear relationship.

(c) Yield = 40.5 + 1.78 × Rainfall. On average, a change of 1 cm in rainfall corresponds to a change of 1.78 kg change in crop. The intercept is not useful in this case since 0 is not in the domain of the explanatory variable.

(d) Since a rainfall of 19 cm lies within the domain of the explanatory variable, a prediction is appropriate:

Yield = 40.5 + 1.78 (19) = 74.32

So, on average, the yield per tree in this region is expected to be about 74 kg.

(e) Let the gradient of the regression of x on y be n, and that of the regression of y on x be m.

We know that $mn=r^2 \Rightarrow n=\dfrac{r^2}{m}=\dfrac{0.853^2}{1.78}=0.4088$

The angle between 2 lines is given by

$\tan^{-1} m-\tan^{-1} n=\tan^{-1}1.78-\tan^{-1}0.4088\approx 38°$

Note: Some answers may differ from one student to another because of variations in graph accuracy.

Exercise 11.1

1. **(a)** S = {left-handed, right-handed}

(b) $S=\{h\in\mathbb{R}:50<h<250\}$, where height ($h$) is in centimetres.

(c) $S=\{t\in\mathbb{R}:0<h<240\}$, if we decide that the night starts at 20:00 and we don't study after midnight.

2. S ={(1, h), (2, h), (3, h), (4, h), (5, h), (6, h), (1, t), (2, t), (3, t), (4, t), (5, t), (6, t)}

3. **(a)** S = {2♣, 3♣, 4♣, 5♣, 6♣, 7♣, 8♣, 9♣, 10♣, J♣, Q♣, K♣, A♣, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠, A♠, 2♦, 3♦, 4♦, 5♦, 6♦, 7♦, 8♦, 9♦, 10♦, J♦, Q♦, K♦, A♦, 2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, J♥, Q♥, K♥, A♥}

(b) We need to list all the possible pairs from a deck of 52 cards, which is many more than the listing in part **(a)**, so we are just going to initiate a possible listing.

S = {(2♣, 3♣), (2♣, 4♣), ...(2♣, A♣), (2♣, 2♠), (2♣, 3♠), ...(2♣, A♠), (2♣, 2♦), (2♣, 3♦), ...(2♣, A♦), (2♣, 2♥), (2♣, 3♥), ...(2♣, A♥), (3♣, 4♣), (3♣, 5♣), ... (3♣, A♣), (3♣, 2♠), (3♣, 3♠), ...(3♣, A♠), (3♣, 2♦), (3♣, 3♦), ...(3♣, A♦), (3♣, 2♥), (3♣, 3♥), ...(3♣, A♥), ...(K♥, A♥), …}

(c) In the first experiment there are 52 outcomes, as there are 52 cards in the deck. In the second experiment there are $\frac{52\times 51}{2}=\frac{2652}{2}=1326$ outcomes, since the order of the pair doesn't matter.

4. **(a)** Since Tim tossed 20 coins 10 times, there are 200 possible outcomes. The sum of all the number of heads that appeared in the experiment is the number of favourable outcomes.

$$11+9+10+8+13+9+6+7+10+11=94\Rightarrow \mathrm{P}(H)=\frac{94}{200}=\frac{47}{100}=0.47$$

(b) Tim should expect any number between 0 and 20.

(c) If he tossed 20 coins 1000 times, we would expect heads to be obtained exactly half the time; therefore, 10 000 heads.

5. **(a)** $S=\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),...,(3,1),(3,2),(3,3),(3,4),...,(4,4)\}$

(b) We need to look at each pair from **(a)** and add 1 to the sum of the components.
So, $S=\{3,4,5,6,7,8,9\}$.

6. **(a)** Since we are replacing the first ball drawn, there are three different colours possible for the first ball drawn **and** three different colours possible for the second ball drawn.
S ={(b, b), (b, g), (b, y), (g, b), (g, g), (g, y), (y, b), (y, g), (y, y)}

(b) A ={(y, b), (y, g), (y, y)}

(c) B = {(b, b), (g, g), (y, y)}

7. **(a)** Since we do not replace the first ball drawn, there are only two colours possible for the second ball drawn.

S ={(b, g), (b, y), (g, b), (g, y), (y, b), (y, g)}

(b) A = {(y, b), (y, g)}

(c) $B = \varnothing$

8. **(a)** S ={(h, h, h), (h, h, t), (h, t, h), (t, h, h), (h, t, t), (t, h, t), (t, t, h), (t, t, t)}

(b) A ={(h, h, h), (h, h, t), (h, t, h), (t, h, h)}

9. Let H = Hungary, I = Italy, b = boat, d = drive, f = fly.

Go on vacation: S ={(I, f), (I, d), (I, t), (H, d), (H, b)}

Fly to destination: S ={(I, f)}

10. **(a)** S ={(0, g), (0, f), (0, s), (0, c), (1, g), (1, f), (1, s), (1, c)}

(b) A ={(0, s), (0, c)}

(c) B ={(0, g), (0, f), (1, g), (1, f)}

Note: In this case we are not concerned as to whether or not the patient is insured.

(d) C ={(1, g), (1, f), (1, s), (1, c)}

11. The study is investigating three different characteristics. There are 2 classifications for gender, 3 classifications for drinking habits, and 4 classifications for marital status; so, there are $2\times3\times4=24$ different classifications for a person in this study.

(a)

$$S=\{(G_1,K_1,M_1),...,(G_1,K_1,M_4),(G_1,K_2,M_1),(G_1,K_2,M_2),...,(G_1,K_2,M_4),\\(G_1,K_3,M_1),...,(G_1,K_3,M_4),(G_2,K_1,M_1),...,(G_2,K_1,M_4),(G_2,K_2,M_1),...,\\(G_2,K_2,M_4),(G_2,K_3,M_1),...,(G_2,K_3,M_4)\}$$

(b) Set *A* is defined as 'the person is a male'; therefore, it consists of all the triplets containing G_2.

$$A=\{(G_2,K_1,M_1),...,(G_2,K_1,M_4),(G_2,K_2,M_1),(G_2,K_2,M_2),...,(G_2,K_2,M_4),\\(G_2,K_3,M_1),...,(G_2,K_3,M_4)\}$$

Set *B* is defined as 'the person drinks'; therefore, it consists of all the triplets containing K_2 or K_3.

$$B=\{(G_1,K_2,M_1),...,(G_1,K_2,M_4),(G_1,K_3,M_1),...,(G_1,K_3,M_4),\\(G_2,K_2,M_1),...,(G_2,K_2,M_4),(G_2,K_3,M_1),...,(G_2,K_3,M_4)\}$$

Set *B* can also be described as the set that consists of all the triplets not containing K_1.

Set *C* is defined as 'the person is single'; it consists of all triplets containing M_2.

$$C=\{(G_1,K_1,M_2)(G_1,K_2,M_2),(G_1,K_3,M_2),(G_2,K_1,M_2),(G_2,K_2,M_2),(G_2,K_3,M_2)\}$$

(c)

(i) Set $A\cup B$ can be described as the set that consists of male persons or persons who drink.

(ii) Set $A\cap C$ can be described as the set that consists of single male persons.

(iii) Set C' can be described as the set that consists of non-single persons.

(iv) Set $A\cap B\cap C$ can be described as the set that consists of single male persons who drink.

(v) The set $A'\cap B$ can be described as the set that consists of female persons who drink.

12. **(a)** Since we are taking four cars at a time, we will be recording quadruplets. For example, (L, L, L, L), (R, R, R, S), (L, S, R, S), and so on. Every car leaving the highway has three options; therefore, there are $3^4=81$ different quadruplets.

(b) If all cars go in the same direction, we have (L, L, L, L), (R, R, R, R) and (S, S, S, S).

(c) If only two cars turn right, the remaining two cars will either turn left or go straight: (R, R, L, L), (R, R, S, S), (R, R, L, S), and now we have to find the remaining permutations. For example, let us take the first quadruplet and its permutations: (R, L, R, L), (R, L, L, R), (L, R, L, R), (L, R, R, L), (L, L, R, R).

The same pattern works for the second quadruplet. For these two quadruplets, there are 12 different permutations altogether. The last quadruplet has more permutations since we have three possible ways of leaving the highway. There are 12 possibilities, so the remaining quadruplets are:

(R, R, S, L), (R, L, R, S), (R, S, R, L), (R, L, S, R), (R, S, L, R), (L, R, R, S), (S, R, R, L), (L, R, S, R), (S, R, L, R), (L, S, R, R), (S, L, R, R).

Therefore, there are a total of 24 outcomes where only two cars turn right.

(d) Only two cars going in the same direction contains the previous part, and there are two more ways of the cars going in the same direction: L, L and S, S.

So, altogether, there are $3 \times 24 = 72$ different outcomes.

13. Since we have to look at three different components, we will be recording triplets. The first component, size of the vehicle, has three different classifications, whereas the remaining two components have just two different outcomes. Therefore, there are $3 \times 2 \times 2 = 12$ different triplets.

(a) $U = \{(\text{T,SY,O}),(\text{T,SY,F}),(\text{T,SN,O}),(\text{T,SN,F}),(\text{B,SY,O}),(\text{B,SY,F}),$
$(\text{B,SN,O}),(\text{B,SN,F}),(\text{C,SY,O}),(\text{C,SY,F}),(\text{C,SN,O}),(\text{C,SN,F})\}$

(b) $\text{SY}=\{(\text{T,SY,O}),(\text{T,SY,F}),(\text{B,SY,O}),(\text{B,SY,F}),(\text{C,SY,O}),(\text{C,SY,F})\}$

(c) $\text{C}=\{(\text{C,SY,O}),(\text{C,SY,F}),(\text{C,SN,O}),(\text{C,SN,F})\}$

(d) $\text{C}\cap\text{SY}=\{(\text{C,SY,O}),(\text{C,SY,F})\}$

$\text{C}'=\{(\text{T,SY,O}),(\text{T,SY,F}),(\text{T,SN,O}),(\text{T,SN,F}),(\text{B,SY,O}),$
$(\text{B,SY,F}),(\text{B,SN,O}),(\text{B,SN,F})\}$

$\text{C}\cup\text{SY}=\{(\text{T,SY,O}),(\text{T,SY,F}),(\text{B,SY,O}),(\text{B,SY,F}),(\text{C,SY,O}),(\text{C,SY,F}),$
$(\text{C,SN,O}),(\text{C,SN,F})\}$

14. **(a)** Since there are three components and each can work or not, there are $2\times2\times2=8$ different outcomes.

$U=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$

(b) $X=\{(0,1,1),(1,0,1),(1,1,0)\}$

(c) $Y=\{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$

(d) $Z=\{(1,0,1),(1,1,0),(1,1,1)\}$

(e) $Z'=\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1)\}$

$X\cup Z=\{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$

$X\cap Z=\{(1,0,1),(1,1,0)\}$, $Y\cup Z=\{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}=Y$

$Y\cap Z=\{(1,0,1),(1,1,0),(1,1,1)\}=Z$

15. **(a)** $U=\{1,2,31,32,41,42,51,52,341,342,431,432,351,352,531,532,$

$451,452,541,542,3451,3452,3541,3542,4351,4352,$

$4531,4532,5341,5342,5431,5432\}$

There are 32 different outcomes in total.

(b) $A=\{31,32,41,42,51,52\}$. There are six possible outcomes.

(c) $B=\{31,32,41,42,51,52,341,342,431,432,351,352,531,532,451,452,541,542,$

$3451,3452,3541,3542,4351,4352,4531,4532,5341,5342,5431,5432\}$.

There are 30 possible outcomes.

(d) $C=\{1,31,41,51,341,431,351,531,451,541,3451,3541,4351,4531,5341,5431\}$.
There are 16 possible outcomes.

Exercise 11.2

1. **(a)** There are six multiples of 3 from 1 to 20. So, $\mathrm{P}(A)=\frac{6}{20}=\frac{3}{10}$

Note: The number of multiples can be obtained by using the greatest integer function:

$$\left\lfloor \frac{20}{3} \right\rfloor = \lfloor 6.67 \rfloor = 6$$

(b) We will use the complementary event that the number is a multiple of 4.

There are five multiples of 4 from 1 to 20. So, $\mathrm{P}(B)=1-\mathrm{P}(B')=1-\frac{5}{20}=1-\frac{1}{4}=\frac{3}{4}$

2. **(a)** $\mathrm{P}(A')=1-\mathrm{P}(A)=1-0.37=0.63$

(b) $\mathrm{P}(A\cup A')=\mathrm{P}(S)=1$ *or* $\mathrm{P}(A\cup A')=\mathrm{P}(A)+\mathrm{P}(A')=0.37+0.63=1$

3. **(a)** **(i)** There is one ace of hearts in a deck of cards, so: $\mathrm{P}(A_i)=\frac{1}{52}$

(ii) There is one ace of hearts and 13 spades in a deck, so: $\mathrm{P}(A_{ii})=\frac{1+13}{52}=\frac{14}{52}=\frac{7}{26}$

(iii) The ace of hearts is already included in the 13 hearts in a deck, so we only need to add the three remaining aces. So, $\mathrm{P}(A_{iii})=\frac{13+3}{52}=\frac{16}{52}=\frac{4}{13}$

(iv) There are 12 face cards. We will use the probability of the complementary event:

$$\mathrm{P}(A_{iv})=1-\mathrm{P}(A_{iv}')=1-\frac{12}{52}=1-\frac{3}{13}=\frac{10}{13}$$

(b) As the drawn card is not replaced, there are 51 cards remaining in the deck.

(i) $\mathrm{P}(B_i)=\frac{1}{51}$

(ii) $\mathrm{P}(B_{ii})=1-\mathrm{P}(A_{i1}')=1-\frac{12}{51}=1-\frac{4}{17}=\frac{13}{17}$

(c) As the drawn card is replaced in the deck, there are no influences on the drawing of the next card. Therefore, the probability is the same as in **(a)**.

(i) $\mathrm{P}(C_i)=\frac{1}{52}$ **(ii)** $\mathrm{P}(C_{ii})=1-\mathrm{P}(C_{ii}')=1-\frac{12}{52}=1-\frac{3}{13}=\frac{10}{13}$

4. The total number of students is 30. Looking at the table, we obtain:

(a) $\mathrm{P}(A)=\frac{4+12+8}{30}=\frac{24}{30}=\frac{4}{5}$

(b) $\mathrm{P}(B)=\frac{8+3}{30}=\frac{11}{30}$

(c) All of the students studied less than 6 hours; therefore, $\mathrm{P}(C)=1$

5. There are 12 different possible outcomes: 6 possible outcomes for the dice and 2 possible outcomes for the coin, so by the counting principle we obtain 12.

(a) There are three outcomes that are greater than 3, i.e. 4, 5 or 6, and since it doesn't matter whether a head or a tail is obtained we get: $\mathrm{P}(A)=\frac{6}{12}=\frac{1}{2}$

(b) Obtaining a head and a 6 is just one possible outcome out of 12; therefore, $\mathrm{P}(B)=\frac{1}{12}$

6. Let the probability of any other number than 1 appear be x, so the probability of 1 is $2x$.
The sum of all the probabilities is 1; therefore, $\sum p_i=1\Rightarrow 7x=1\Rightarrow x=\frac{1}{7}$

(a) $\mathrm{P}(A)=\frac{1}{7}$

(b) The odd numbers are 1, 3 and 5, so $\mathrm{P}(B)=\frac{2+1+1}{7}=\frac{4}{7}$

7. **(a)** **(i)** There are 6 possible outcomes for the first dice and 6 possible outcomes for the second dice; therefore, there are a total of 36 possible outcomes.

$$S=\{(1,1),(1,2),...,(1,6),(2,1),...,(2,6),(3,1),...,(3,6),$$
$$(4,1),...,(4,6),(5,1),...,(5,6),(6,1),...,(6,6)\}$$

(ii) There are six possible pairs with equal numbers; therefore, $\mathrm{P}(A)=\frac{6}{36}=\frac{1}{6}$

(iii) Looking at the sample space, we notice that there are eight such outcomes:

$$B=\{(1,3),(2,4),(3,1),(3,5),(4,2),(4,6),(5,3),(6,4)\}$$
$$\Rightarrow \mathrm{P}(B)=\frac{8}{36}=\frac{2}{9}$$

(iv) This event is complementary to the event in part **(ii)**, so $\mathrm{P}(C)=1-\frac{1}{6}=\frac{5}{6}$

(b) The probability distribution for the sum of the numbers that appear is shown in the table.

X (sum)	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(i) There is no sum equal to 1 and therefore $\text{P}(D)=0$

(ii) Looking at the table, we can read that $\text{P}(E)=\frac{4}{36}=\frac{1}{9}$

(iii) Looking at the table, we can read that $\text{P}(F)=\frac{5}{36}$

(iv) The largest sum is 12 and therefore $\text{P}(G)=0$

8. **(a)** Since the sum of all the probabilities is equal to 1,
$\text{P}(\text{AB}|\text{USA})=1-(0.43+0.41+0.12)=1-0.96=0.04$

(b) Since the events are mutually exclusive, the probability of their union is the sum of their probabilities. Therefore, $\text{P}(\text{O}\cup\text{B}|\text{USA})=0.43+0.12=0.55$

(c) Since we have to independently select two people, their probabilities should multiply;

therefore, $\text{P}(\text{China})=\text{P}(\text{O}|\text{USA})\times\text{P}(\text{O}|\text{China})=0.43\times0.36=0.1548=0.155\,(3\text{ s.f.})$

(d) Since we have to independently select three people, their probabilities should multiply:

$\text{P}(\text{O})=\text{P}(\text{O}|\text{USA})\times\text{P}(\text{O}|\text{China})\times\text{P}(\text{O}|\text{Russia})=0.43\times0.36\times0.39=0.060372=0.0604$, correct to three significant figures.

(e) First, we need to find the probability of type B in Russia:
$\text{P}(B|\text{Russia})=1-(0.39+0.34+0.09)=1-0.82=0.18$

Similarly, as in **(d)**, we need to calculate the probability of only one blood type:

$$\text{P}(\text{A})=\text{P}(\text{A}|\text{USA})\times\text{P}(\text{A}|\text{China})\times\text{P}(\text{A}|\text{Russia})=0.41\times0.27\times0.34=0.037638=0.0376$$

$$\text{P}(\text{B})=\text{P}(\text{B}|\text{USA})\times\text{P}(\text{B}|\text{China})\times\text{P}(\text{B}|\text{Russia})=0.12\times0.26\times0.18=0.05616=0.00562$$

$$\text{P}(\text{AB})=\text{P}(\text{AB}|\text{USA})\times\text{P}(\text{AB}|\text{China})\times\text{P}(\text{AB}|\text{Russia})=0.04\times0.11\times0.09=0.000396$$

$$\text{P}(\text{S})=\text{P}(\text{O})+\text{P}(\text{A})+\text{P}(\text{B})+\text{P}(\text{AB})=0.104022=0.104\ \ (3\text{ s.f.})$$

9. **(a)** Yes, since the sum of all the probabilities is 1.

(b) No, since four mutually exclusive events are given, and their sum exceeds 1.

(c) No. There is the same number of cards in each suit, but by looking at the probability distribution we notice that one heart and one diamond are missing, and there are two extra spades.

10. **(a)** Since the sum of all the probabilities is 1,
$\mathrm{P}(\text{Other}) = 1-(0.58+0.24+0.12) = 1-0.94 = 0.06$

(b) We need to use the complementary event, so $\mathrm{P}(\text{not German}) = 1-0.58 = 0.42$

(c) $\mathrm{P}(\mathrm{GG}) = 0.58\times 0.58 = 0.3364 = 0.336\ (3\text{ s.f.})$

(d) The two Swiss that we select could have German, French, or Italian as their mother tongue. Therefore:
$\mathrm{P}(\mathrm{GG})+\mathrm{P}(\mathrm{FF})+\mathrm{P}(\mathrm{II})+\mathrm{P}(\mathrm{OO}) = 0.58^2+0.24^2+0.12^2+0.06^2 = 0.412$

11. **(a)** We use the probability of the complementary event, so:
$\mathrm{P}(\mathrm{A}) = 1-(0.165+0.142+0.075+0.081+0.209+0.145) = 1-0.817 = 0.183$

(b) Again, the complementary event will be used:
$\mathrm{P}(\mathrm{B}) = 1-(0.165+0.145) = 1-0.31 = 0.69$

12. $$f(x) = \frac{{}_nC_{x+1}}{{}_nC_x} = \frac{\dfrac{\cancel{n!}}{(x+1)\cancel{!}\,\cancel{(n-x-1)!}}}{\dfrac{\cancel{n!}}{\cancel{x!}\,(n-x)\cancel{!}}} = \frac{n-x}{x+1} < 1 \Rightarrow n-x < x+1 \Rightarrow x > \frac{n-1}{2}$$

13. **(a)** $${}_nC_2 = 190 \Rightarrow \frac{n(n-1)}{2} = 190 \Rightarrow n^2-n = 380 \Rightarrow (n-20)(n+19) = 0$$
Since n must be a positive integer, the only possible solution is $n=20$

(b) We know the symmetric property of binomial coefficients, ${}_nC_r = {}_nC_{n-r}$

${}_nC_4 = {}_nC_8 \Rightarrow n = 4+8 = 12$. Alternatively:

$${}_nC_4 = {}_nC_8 \Rightarrow \frac{n!}{4!(n-4)!} = \frac{n!}{8!(n-8)!} \Rightarrow 4!(n-4)! = 8!(n-8)!$$
$$\Rightarrow \cancel{4!}(n-4)(n-5)(n-6)(n-7)\cancel{(n-8)}! = 8\times 7\times 6\times 5\times \cancel{4!}\cancel{(n-8)}!$$
$$\Rightarrow (n-4)(n-5)(n-6)(n-7) = 8\times 7\times 6\times 5 \Rightarrow n = 12$$

14. There are 36 different outcomes that we will present as ordered pairs. The first component will represent the outcome of the white dice, and the second component will represent the red dice.

$$U=\{(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),(3,1),\cdots,(3,6),\\ (4,1),\cdots,(4,6),(5,1),\cdots,(5,6),(6,1),\cdots,(6,6)\}$$

(a) By inspection, we can see that the sum is greater than 8 on ten different outcomes; therefore, $\mathrm{P}(\mathrm{A})=\frac{10}{36}=\frac{5}{18}$

(b) A number greater than 4 means that 5 or 6 will appear on the first dice.

There are 12 such possible outcomes; therefore, $\mathrm{P}(\mathrm{B})=\frac{12}{36}=\frac{1}{3}$

(c) At most a total of 5 means that the sum could be 2, 3, 4 or 5. By inspection, we can see that again there are 10 possible pairs; therefore, $\mathrm{P}(\mathrm{C})=\frac{10}{36}=\frac{5}{18}$. We see that it is the same as the probability of obtaining a sum greater than 8, which is true due to the symmetric property of the outcomes.

15. **(a)** There are 9 books on the shelf altogether. We select 3 books, and one of the books must be the only thesaurus on the shelf. Since we don't care which of the remaining books will be selected, we have to select 2 out of 8 books.

$$\mathrm{P}(\mathrm{A})=\frac{{}_8C_2\times 1}{{}_9C_3}=\frac{28}{84}=\frac{1}{3}$$

(b) We need to select 3 books from the 9 books on the shelf. There are ${}_5C_2$ ways to select two novels and 3 ways to select one science book (from three).

$$\mathrm{P}(\mathrm{B})=\frac{{}_5C_2\,{}_3C_1}{{}_9C_3}=\frac{10\times 3}{84}=\frac{5}{14}$$

16. **(a)** We have to select 5 cards (from 52) and we need 3 kings (from 4). That means the remaining 2 drawn cards can be any of the 48 non-king cards.

$$\mathrm{P}(\mathrm{A})=\frac{{}_4C_3\,{}_{48}C_2}{{}_{52}C_5}=\frac{94}{54145}\approx 0.00174$$

(b) Again, we need to select 5 cards altogether. We have to select 4 hearts (from 13) and 1 diamond (from 13).

$$\mathrm{P}(\mathrm{B})=\frac{{}_{13}C_4\,{}_{13}C_1}{{}_{52}C_5}=\frac{143}{39984}\approx 0.00358$$

17. **(a)** We have to select 6 students from a class of 22. We would like to have 1 out of the 12 boys and 5 out of the10 girls.

$$\mathrm{P}(\mathrm{A})=\frac{{}_{12}C_1\,{}_{10}C_5}{{}_{22}C_6}=\frac{144}{3553}\approx 0.0405$$

(b) It could be 4, 5 or 6 boys and 2, 1 or no girls respectively in the team of 6 students.

$$\mathrm{P}(\mathrm{B})=\frac{{}_{12}C_4\,{}_{10}C_2}{{}_{22}C_6}+\frac{{}_{12}C_5\,{}_{10}C_1}{{}_{22}C_6}+\frac{{}_{12}C_6\,{}_{10}C_0}{{}_{22}C_6}=\frac{495\times 45+792\times 10+924}{74613}=\frac{943}{2261}\approx 0.417$$

18. In questions where conditions are given, we need to fulfil the conditions first and then we have to see what the possibilities of the remaining elements are. We select 6 people from a group of 15.

(a) We need to select both married couples, that is, all 4 from the group of 4, and then we look at the remaining 11 people from which we need to select a further 2 people.

$$\mathrm{P}(\mathrm{A})=\frac{{}_{4}C_4\,{}_{11}C_2}{{}_{15}C_6}=\frac{1\times 55}{5005}=\frac{1}{91}\approx 0.0110$$

(b) If we select the three youngest members in the group, we can select any of the remaining members of the group for the final three places.

$$\mathrm{P}(\mathrm{B})=\frac{{}_{3}C_3\,{}_{12}C_3}{{}_{15}C_6}=\frac{1\times 220}{5005}=\frac{4}{91}\approx 0.0440$$

19. **(a)** $\mathrm{P}(\mathrm{A})=\frac{{}_{10}C_3\,{}_{15}C_3}{{}_{25}C_6}=\frac{120\times 455}{177100}=\frac{78}{253}\approx 0.308$

(b) At least 3 means 3, 4, 5 or 6. We will use the complementary event since it has fewer calculations. The complementary event is at most 2, which is 0, 1 or 2 colour laser printers.

$$\mathrm{P}(\mathrm{B})=1-\mathrm{P}(\mathrm{B}')=1-\frac{{}_{10}C_0\,{}_{15}C_6}{{}_{25}C_6}+\frac{{}_{10}C_1\,{}_{15}C_5}{{}_{25}C_6}+\frac{{}_{10}C_2\,{}_{15}C_4}{{}_{25}C_6}$$

$$=1-\frac{1\times 5005+10\times 3003+45\times 1365}{177100}=1-\frac{689}{1265}\approx 0.445$$

20. **(a)** Since there are 30 buses and we need to select 6 for inspection, there are

${}_{30}C_6=593775$ ways to select six buses.

(b) Half means that 3 buses have cracks on the instrument panel.

$$\mathrm{P}(\mathrm{B})=\frac{{}_{10}C_3\,{}_{20}C_3}{{}_{30}C_6}=\frac{608}{2639}\approx 0.230$$

(c) At least half means 3, 4, 5 or 6. Again, we will calculate the probability by using the complementary event, which is 0, 1 or 2 buses have cracks.

$$\mathrm{P}(\mathrm{C})=1-\mathrm{P}(\mathrm{C'})=1-\frac{{}_{10}C_0\,{}_{20}C_6}{{}_{30}C_6}+\frac{{}_{10}C_1\,{}_{20}C_5}{{}_{30}C_6}+\frac{{}_{10}C_2\,{}_{20}C_4}{{}_{30}C_6}$$
$$=1-\frac{38760+155040+218025}{593775}=\frac{2426}{7917}\approx 0.306$$

(d) At most half means 0, 1, 2 or 3. Again, we will calculate the probability by using the complementary event, which is 4, 5 or 6 buses have cracks.

$$\mathrm{P}(\mathrm{D})=1-\mathrm{P}(\mathrm{D'})=1-\left(\frac{{}_{10}C_4\,{}_{20}C_2}{{}_{30}C_6}+\frac{{}_{10}C_5\,{}_{20}C_1}{{}_{30}C_6}+\frac{{}_{10}C_6\,{}_{20}C_0}{{}_{30}C_6}\right)$$
$$=1-\frac{39900+5040+210}{593775}=\frac{1045}{1131}\approx 0.924$$

21. There are 67 workers in the factory altogether and we have to select 9.

(a) $$\mathrm{P}(\mathrm{A})=\frac{{}_{30}C_9}{{}_{67}C_9}=\frac{10005}{29900492}\approx 0.000335$$

(b) The same shift means either from the day, evening or morning shift.

$$\mathrm{P}(\mathrm{B})=\frac{{}_{30}C_9}{{}_{67}C_9}+\frac{{}_{22}C_9}{{}_{67}C_9}+\frac{{}_{15}C_9}{{}_{67}C_9}=\frac{269265}{777412792}\approx 0.000346$$

(c) We will calculate the probability that at least two of the shifts are represented by considering the complementary event, which is that only one of the shifts is represented, as in part **(b)**.

$$\mathrm{P}(\mathrm{C})=1-\mathrm{P}(\mathrm{B})=1-\frac{269265}{777412792}\approx 0.9997$$

(d) The probability that at least one of the shifts is unrepresented means that one shift is not selected, either from the day, evening or morning shift. That is, when a shift is not represented, all 9 are selected from the two other shifts.

$$\mathrm{P}(\mathrm{D})=\overset{\text{morning}}{\frac{{}_{52}C_9}{{}_{67}C_9}}+\overset{\text{evening}}{\frac{{}_{45}C_9}{{}_{67}C_9}}+\overset{\text{day}}{\frac{{}_{37}C_9}{{}_{67}C_9}}=\frac{468942155}{42757703560}\approx 0.1097$$

22. **(a)** Since we have to select 2 out of these 8 chips, there are ${}^{8}P_2 = 56$ different outcomes. A sum of 7 is obtained in the following pairs: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4) and (4, 3); therefore, the probability is $\text{P}(\text{A}) = \frac{6}{56} = \frac{3}{28} \approx 0.107$

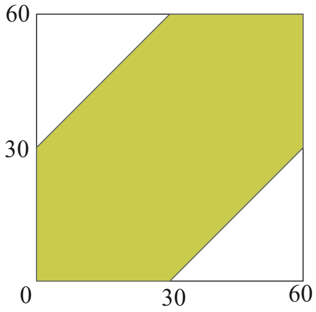
(b) Since we have to select 2 out of these 20 chips, there are ${}^{20}P_2 = 380$ different outcomes. If we take the smaller number first, we notice that there are 17 possible outcomes, because 18, 19 and 20 don't have numbers in the box that differ by 3. If we start with the larger numbers, there are also 17 ways. So, the probability is $\text{P}(\text{B}) = \frac{34}{380} \approx 0.0895$

(c) We have 380 different outcomes and we are going to use the complementary event to find the required probability. Again, we take the smaller number first and need to find all of those pairs in which the numbers differ by 1, 2, or 3. As in the previous part, we will count the number of pairs that differ by 1, which is 19×2, as before. (20 doesn't have such a number in the box). The number of pairs in which the numbers differ by 2 is 18×2. (19 and 20 don't have such numbers in the box.). The number of pairs in which the numbers differ by 3 is 17×2. (18, 19 and 20 don't have such numbers in the box.)

$$\text{P}(\text{B}) = 1 - \frac{38+36+34}{380} = \frac{272}{380} \approx 0.716$$

Note: It is impossible to draw two equal numbers.

23. To solve this problem, we are going to use geometric probability. Let x be Tim's time of arrival and y be Val's time of arrival. For ease of calculations, we are going to measure the time from 20:00, with time in minutes on both axes. Now, we can conclude that the range of both variables will be $0 \le x, y \le 60$. In order to have dinner together, they have to arrive at the restaurant within 30 minutes of each other; therefore, they have to satisfy the inequality $|x - y| \le 30$. The shaded area represents the favourable outcomes.

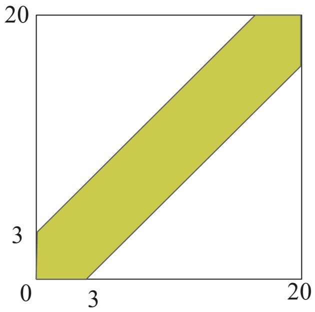


We notice that all the possible outcomes are represented by a square with an area of 60^2, while the unshaded part consists of 2 triangles, which, when put together, make a square of side 30.

The area of the shaded region is the area of the large square minus the unshaded area:

$$P(D)=\frac{60^2-30^2}{60^2}=\frac{3}{4}=0.75$$

24. Let x represent the time of the tram and y the time of the bus at the station. Since both stay for 3 minutes in the station, we are going to use minutes as units on both axes. Now, we can conclude that the range of both variables will be $0 \le x, y \le 20$. In order to be at the station at the same time, they have to arrive at the station within 3 minutes of each other; thus, they have to satisfy the inequality $|x-y| \le 3$. The shaded area represents the favourable outcomes.



We notice that all the possible outcomes are represented by a square with an area of 20^2, whereas the unshaded part consists of 2 triangles, which, when put together, make a square of side 17. The area of the shaded region is the area of the large square minus the unshaded area.

Thus $P(D)=\frac{20^2-17^2}{20^2}=\frac{111}{400}\approx 0.278$

25. We have to select 6 from 30 slips. There are ${}_8C_2$ ways of selecting a film, ${}_{10}C_2$ and ${}_{12}C_2$ of the others:

(a) $P(A)=\frac{{}_8C_2 \times {}_{10}C_2 \times {}_{12}C_2}{{}_{30}C_6}=\frac{264}{1885}\approx 0.140$

(b) $P(B)=\frac{{}_8C_6 + {}_{10}C_6 + {}_{12}C_6}{{}_{30}C_6}=\frac{166}{84825}\approx 0.00196$

(c) Selecting only films and songs means that we have $8+12=20$ slips for the favourable event. Hence, $P(C)=\frac{{}_{20}C_6}{{}_{30}C_6}=\frac{2584}{39585}\approx 0.0653$

26. There are 1000 cubes altogether. Eight cubes are painted green at three faces (vertices of the original cube). There are eight cubes painted green at two sides per edge. There are 12 edges, so there are 96 such cubes. There are 64 cubes on each of six faces that have only a green face; therefore, there are 384 such cubes.

(a) $$\text{P}(\text{A})=\frac{96}{1000}=\frac{12}{125}$$

(b) $$\text{P}(\text{B})=\frac{8}{1000}=\frac{1}{125}$$

(c) From the above explanation, we notice that there are $8+96+384=488$ cubes with at least one face painted. So, to find the probability that no face is coloured, we use the complement which is the cube with at least one face is green:

$$\text{P}(\text{C})=1-\text{P}(\text{C}')\Rightarrow\text{P}(\text{C})=1-\frac{488}{100}=\frac{64}{125}$$

Exercise 11.3

1. Using the addition rule:

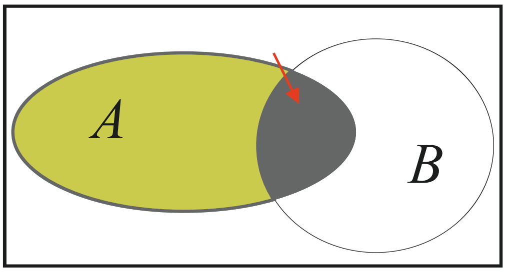
$$\text{P}(A\cup B)=\text{P}(A)+\text{P}(B)-\text{P}(A\cap B)\Rightarrow\text{P}(B)=\text{P}(A\cup B)+\text{P}(A\cap B)-\text{P}(A)$$

$$\Rightarrow\text{P}(B)=\frac{4}{5}+\frac{3}{10}-\frac{3}{4}=\frac{7}{20}$$

2. **(a)** $$\text{P}(B)=\text{P}(A\cup B)+\text{P}(A\cap B)-\text{P}(A)\Rightarrow\text{P}(B)=\frac{9}{10}+\frac{3}{10}-\frac{7}{10}=\frac{1}{2}$$

(b) First, we need to write the event in a different form: $B'\cap A=A\setminus(A\cap B)$, and since $A\cap B\subseteq A$ we can now calculate the probability:

$$\text{P}(B'\cap A)=\text{P}(A\setminus(A\cap B))=\text{P}(A)-\text{P}(A\cap B)=\frac{7}{10}-\frac{3}{10}=\frac{4}{10}=\frac{2}{5}$$



(c) Similarly, $\mathrm{P}(B\cap A')=\mathrm{P}(B\setminus(A\cap B))=\mathrm{P}(B)-\mathrm{P}(A\cap B)=\frac{5}{10}-\frac{3}{10}=\frac{1}{5}$

(d) By using a Venn diagram, we can spot that the intersection of the complements of two sets can be written as the complement of the union of the sets. Hence:

$$B'\cap A'=(A\cup B)'\Rightarrow \mathrm{P}(B'\cap A')=\mathrm{P}((A\cup B)')=1-\mathrm{P}(A\cup B)=\frac{1}{10}$$

(e) Using the conditional probability formula: $\mathrm{P}(B|A')=\frac{\mathrm{P}(B\cap A')}{\mathrm{P}(A')}=\frac{\frac{1}{5}}{\frac{3}{10}}=\frac{2}{3}$

3. Given the addition rule, we can calculate the intersection.

$$\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cup B)\Rightarrow \mathrm{P}(A\cap B)=\frac{1}{3}+\frac{2}{9}-\frac{4}{9}=\frac{1}{9}$$

It is obvious that the events are not mutually exclusive since $\mathrm{P}(A\cap B)=\frac{1}{9}\neq 0$,

and also, they are not independent since $\mathrm{P}(A)\times\mathrm{P}(B)=\frac{1}{3}\times\frac{2}{9}=\frac{2}{27}\neq\frac{1}{9}=\mathrm{P}(A\cap B)$

4. Since the events are independent, we can use the multiplication rule of independent events to find the probability of event *B*. $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)\Rightarrow \mathrm{P}(B)=\frac{\frac{3}{10}}{\frac{3}{7}}=\frac{7}{10}$

From the addition rule, we obtain:

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)\Rightarrow \mathrm{P}(A\cup B)=\frac{3}{7}+\frac{7}{10}-\frac{3}{10}=\frac{58}{70}=\frac{29}{35}$$

5. In order to pass the test without having to wait 6 months, the new driver has to pass either the first time or fail the first attempt and pass the second time.

$$\mathrm{P}(A)=\overset{\text{pass 1st attempt}}{0.6}+\overset{\text{fail 1st attempt}}{0.4}\times\overset{\text{pass 2nd attempt}}{0.75}=0.9$$

6. **(a)** Using the complementary event, we get

$$\mathrm{P}(\mathrm{O}^-{}')=1-\mathrm{P}(\mathrm{O}^-)\Rightarrow \mathrm{P}(\mathrm{O}^-{}')=1-0.08=0.92\ (92\%)$$

(b) **(i)** $\mathrm{P}(\mathrm{O}^-\mathrm{O}^-)=0.08\times0.08=0.0064=0.64\%$

(ii) The complementary event of at least one of them has O⁻ blood is none of them has O⁻ blood.

$$\mathrm{P}\left(\mathrm{O}^{-}\,'\mathrm{O}^{-}\,'\right)=0.92\times 0.92=0.8464\Rightarrow \mathrm{P}\left(B\right)=1-0.8464=0.1536$$

(iii) If only one of them has O⁻, that means that the other is not O⁻, so:

$$\mathrm{P}\left(\mathrm{O}^{-}\mathrm{O}^{-}\,'\right)+\mathrm{P}\left(\mathrm{O}^{-}\,'\mathrm{O}^{-}\right)=2\times 0.08\times 0.92=0.1472=14.72\%$$

(c) Using the complementary event that none has O⁻ blood:

$$\mathrm{P}\left(C\right)=1-\mathrm{P}\left(8\mathrm{O}^{-}\,'\right)=1-0.92^{8}=0.486781=48.7\%\ (3\text{ s.f.})$$

7. **(a)** 10 different digits can be used to make each four-digit number; as such, there are

$10^{4}=10\,000$ different possible PIN numbers.

(b) Since the first digit cannot be zero, there are 9 digits that can be in the ten-thousand's position, while each of the other digits can be any of the 10 numerals. Therefore, there are 9×10^{3} numbers without a zero as a starting number. So, the probability is:

$$\mathrm{P}\left(B\right)=\frac{9000}{10000}=\frac{9}{10}$$

(c) The complementary event is that the code doesn't contain a zero; therefore,

$$\mathrm{P}\left(C\right)=1-\frac{9^{4}}{10^{4}}=\frac{10000-6561}{10000}=\frac{3439}{10000}$$

(d) Using the conditional probability formula: $\mathrm{P}\left(D|C\right)=\dfrac{\mathrm{P}\left(D\cap C\right)}{\mathrm{P}\left(C\right)}=\dfrac{\frac{1000}{10000}}{\frac{3439}{10000}}=\dfrac{1000}{3439}$

8. **(a)** We need to use the complementary event that no red ball is drawn:

$$\mathrm{P}\left(A\right)=1-\mathrm{P}\left(BB\right)=1-\frac{2}{8}\times\frac{2}{8}=\frac{64-4}{64}=\frac{60}{64}=\frac{15}{16}$$

(b) Using the conditional probability formula: $\mathrm{P}\left(B|A\right)=\dfrac{\mathrm{P}\left(A\cap B\right)}{\mathrm{P}\left(B\right)}=\dfrac{\frac{2}{8}\times\frac{6}{8}+\frac{6}{8}\times\frac{6}{8}}{\frac{15}{16}}=\dfrac{\frac{48}{64}}{\frac{15}{16}}=\dfrac{4}{5}$

(c) $\mathrm{P}\left(C|A\right)=\dfrac{\mathrm{P}\left(A\cap C\right)}{\mathrm{P}\left(A\right)}=\dfrac{\frac{2}{8}\times\frac{6}{8}}{\frac{15}{16}}=\dfrac{1}{5}$

9. **(a)** $U=\{(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots(5,1),\cdots,(5,6),(6,1),\cdots,(6,6)\}$

(b)

x	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
P(x)	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

(c) **(i)** There are 11 pairs with at least one dice showing 6, so $\mathrm{P}(C)=\frac{11}{36}$

(ii) For a sum of at most 10, we will use a complementary event that is 11 or 12.

Looking at the table, we obtain: $\mathrm{P}(D)=1-\left(\frac{1}{18}+\frac{1}{36}\right)=1-\frac{1}{12}=\frac{11}{12}$

(iii) Using the addition formula:

$$\mathrm{P}(X\cup Y)=\mathrm{P}(X)+\mathrm{P}(Y)-\mathrm{P}(X\cap Y)$$

$$\Rightarrow \mathrm{P}(X\cup Y)=\frac{11}{36}+\frac{3}{36}-\frac{2}{36}=\frac{12}{36}=\frac{1}{3}$$

(iv) Using the conditional probability formula:

$$\mathrm{P}(Y|X)=\frac{\mathrm{P}(X\cap Y)}{\mathrm{P}(X)}\Rightarrow \mathrm{P}(Y|X)=\frac{\frac{2}{\cancel{36}}}{\frac{3}{\cancel{36}}}=\frac{2}{3}$$

The reason is that $X\cap Y=\{(6,4),(4,6)\}$ and $X=\{(6,4),(4,6),(5,5)\}$

10. **(a)** There are 1500 students altogether, 700 of which are female, so: $\mathrm{P}(A)=\frac{700}{1500}=\frac{7}{15}$

(b) $\mathrm{P}(B)=\frac{220}{1500}=\frac{11}{75}$

(c) Now the sample space contains only female students, so: $\mathrm{P}(C)=\frac{180}{700}=\frac{9}{35}$

(d) In this problem, we need to use the addition formula. Let T be the event of selecting a student from grade 12, while F is an event of selecting a female student.

$$\mathrm{P}(T\cup F)=\mathrm{P}(T)+\mathrm{P}(F)-\mathrm{P}(T\cap F)\Rightarrow \mathrm{P}(T\cup F)=\frac{400+700-180}{1500}=\frac{46}{75}$$

(e) There are 400 grade 12 students altogether, 220 of which are male, so:

$$\mathrm{P}(E)=\frac{220}{400}=\frac{11}{20}$$

(f) In order for these two events to be independent, we need to see if the probability of the intersection of the two events is equal to the product of the probabilities of each individual event. Let's focus on grade 12 (G) and female (F).

$$\mathrm{P}(G)=\frac{400}{1500}=\frac{4}{15},\ \mathrm{P}(F)=\frac{7}{15}\Rightarrow \mathrm{P}(G)\times\mathrm{P}(F)=\frac{4}{15}\times\frac{7}{15}=\frac{28}{225}$$

By looking at the table we notice that $\mathrm{P}(G\cap F)=\frac{180}{1500}=\frac{9}{75}=\frac{27}{225}\neq\frac{28}{225}$

Therefore, the events are not independent.

11. **(a)** Using the table, we can read off the results:

(i) $\mathrm{P}(\mathit{Need})=0.41+0.15=0.56$

(ii) $\mathrm{P}(\mathit{Need}\cap \mathit{No\ Use})=0.15$

(b) In this case, we use the conditional probability formula:

$$\mathrm{P}(\mathit{No\ Use}\,|\,\mathit{Need})=\frac{0.15}{0.56}=\frac{15}{56}$$

(c) A way of showing independence is by considering whether $\mathrm{P}(\mathit{Use}\,|\,\mathit{Need})=\mathrm{P}(\mathit{Use})$.

If we use the probabilities from the table, we obtain:

$$\mathrm{P}(\mathit{Use})=0.41+0.04=.045,\ \mathrm{P}(\mathit{Need})=0.56$$

$$\mathrm{P}(\mathit{Use}\,|\,\mathit{Need})=\frac{\mathrm{P}(\mathit{Use}\cap\mathit{Need})}{\mathrm{P}(\mathit{Need})}=\frac{0.41}{0.56}=\frac{41}{56}\neq\frac{45}{100}=\mathrm{P}(\mathit{Use})$$

Hence, the events are not independent.

12. If the events are mutually exclusive, their intersection is an empty set and its probability is zero: $\mathrm{P}(A\cap B)=\mathrm{P}(\varnothing)=0$

If they are independent, then the probability of their intersection is the product of their probabilities: $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$, and $\mathrm{P}(A\,|\,B)=\mathrm{P}(A)$

To find the probability of the union, we need to use the addition formula: $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$

The conditional probability uses the formula $\mathrm{P}(A\,|\,B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$

P(A)	P(B)	Conditions for events A and B	P($A \cap B$)	P($A \cup B$)	P($A\|B$)
0.3	**0.4**	Mutually exclusive	0	0.7	0
0.3	**0.4**	Independent	0.12	0.58	0.3
0.1	**0.5**	Mutually exclusive	0	**0.6**	0
0.2	**0.5**	Independent	**0.1**	0.6	0.2

Note: Numbers in bold were given.

13. **(a)** Since the condition is that the chosen student is doing Economics SL, we have to use the conditional probability formula:

$$\mathrm{P}\left(Physics\,|\,Economics\right)=\frac{n\left(Physics\cap Economics\right)}{n\left(Economics\right)}=\frac{12}{40}=\frac{3}{10}$$

(b) If doing Physics and Economics are independent, then by definition

$$\mathrm{P}\left(Physics|Economics\right)=\frac{3}{10}=\mathrm{P}\left(Physics\right)=\frac{30}{100}$$

Alternatively, we can use the multiplication rule:

$$\mathrm{P}\left(Economics\right)\times\mathrm{P}\left(Physics\right)=\frac{40}{100}\times\frac{30}{100}=\frac{3}{25}=\frac{12}{100}=\mathrm{P}\left(Economics\cap Physics\right)$$

So the events are independent.

14. **(a)** We use the addition formula: $\mathrm{P}\left(M\cup V\right)=0.21+0.57-0.13=0.65=65\%$

(b) We use the complementary event: $\mathrm{P}\left(\left(M\cup V\right)'\right)=1-0.65=0.35=35\%$

(c) Exactly one acceptable card means that the person cannot have both cards.

$$\mathrm{P}\left(\left(M\cup V\right)\setminus\left(M\cap V\right)\right)=0.65-0.13=0.52=52\%$$

15. Let S be the set of patients taking a swimming class and A be the set of patients taking an aerobics class. The following probabilities are given:

$$\mathrm{P}\left(S\cup A\right)=\frac{132}{300}=\frac{11}{25},\mathrm{P}\left(S\right)=\frac{78}{300}=\frac{13}{50},\mathrm{P}\left(A\right)=\frac{84}{300}=\frac{7}{25}$$

(a) $\mathrm{P}\left(\left(S\cup A\right)'\right)=1-\mathrm{P}\left(S\cup A\right)=1-\frac{11}{25}=\frac{14}{25}$

(b) $\mathrm{P}\left(S\cap A\right)=\mathrm{P}\left(S\right)+\mathrm{P}\left(A\right)-\mathrm{P}\left(S\cup A\right)\Rightarrow\mathrm{P}\left(S\cap A\right)=\frac{13}{50}+\frac{7}{25}-\frac{11}{25}=\frac{1}{10}$

16. **(a)** In each attempt, the probability of rolling a two is 1 out of 6.

$$\mathrm{P}(A)=\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{216}$$

(b) We calculate the probability of the event 'at least one two is rolled' by using the complementary event, which is 'no twos are rolled'.

$$\mathrm{P}(B)=1-\mathrm{P}(B')=1-\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=1-\frac{125}{216}=\frac{91}{216}$$

(c) Exactly one two can be rolled in three different ways. A two can be rolled at the first, second or third rolling; therefore, the probability is $\mathrm{P}(C)=3\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{25}{72}$

17. **(a)** She needs to miss the centre with her first shot and then hit the centre with her second shot; therefore, the probability is $\mathrm{P}(A)=0.7\times0.3=0.21=21\%$

(b) She can hit the centre with her first, second or third shot; therefore, the probability is:

$$\mathrm{P}(B)=\overset{\text{number of ways}}{3}\times\overset{\text{hit}}{0.3}\times\overset{\text{twice no hit}}{0.7^2}=0.441=44.1\%$$

(c) We will calculate the probability of 'at least once' by using the complementary event, which is 'no hit in the centre'.

$$\mathrm{P}(C)=1-\mathrm{P}(C')=1-\overset{\text{no hit in 3 attempts}}{0.7^3}=1-0.343=0.657=65.7\%$$

18. Since each dice has 12 different outcomes, rolling two dice has $12\times12=144$ outcomes.

(a) We calculate the probability of 'at least one 12 shows' by using the complementary event, which is 'no 12 shows on either dice'.

$$\mathrm{P}(A)=1-\mathrm{P}(A')=1-\frac{11}{12}\times\frac{11}{12}=1-\frac{121}{144}=\frac{23}{144}$$

(b) A sum of 12 can be achieved as follows: (1, 11), (2, 10), (3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4), (9, 3), (10, 2), (11, 1). Therefore, the probability is $\mathrm{P}(B)=\frac{11}{144}$

(c) A total score of at least 20 can be achieved as follows: (8, 12), (9, 11), (9, 12), (10, 10), (10, 11), (10, 12), (11, 9), (11, 10), (11, 11), (11, 12), (12, 8), (12, 9), (12, 10), (12, 11), (12, 12). Therefore, the probability is $\mathrm{P}(C)=\frac{15}{144}=\frac{5}{48}$

(d) If 12 shows on a dice, there are 23 different outcomes. From these 23 outcomes, we look for those that have a sum of at least 20. Looking at the previous part, we notice nine such pairs, so $\mathrm{P}(D)=\frac{9}{23}$

19. **(a)** The event 'at least one of the numbers is a 10 and the sum is at most 15' is satisfied as follows: (10, 1), (10, 2), (10, 3), (10, 4), (10, 5), together with these pairs in reverse order.

$$\mathrm{P}(A\cap B)=\frac{10}{144}=\frac{5}{72}$$

(b) The event is 'at least one number is 10 or the sum is at most 15'. At least one 10 can be obtained in 23 different ways, while the sum of at most 15 can be obtained in 99 ways (144 – 45, where the 45 outcomes represent a sum of greater than 15). By using the addition rule, and the result from the previous part, we get:

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=\frac{23+99-10}{144}=\frac{112}{144}=\frac{7}{9}$$

(c) This is the complementary event of the event in part **(a)**. We need to list all the remaining pairs that are not listed in part **(a)**. The probability is:

$$\mathrm{P}((A\cap B)')=1-\mathrm{P}(A\cap B)=1-\frac{5}{72}=\frac{67}{72}$$

(d) This is the complementary event of the event in part **(b)**. We need to list all the remaining pairs that are not listed in part **(b)**. The probability is:

$$\mathrm{P}((A\cup B)')=1-\mathrm{P}(A\cup B)=1-\frac{7}{9}=\frac{2}{9}$$

(e) By using De Morgan's laws, the result is the same as that in part **(c)**.

(f) By using De Morgan's laws, the result is the same as that in part **(d)**.

(g) By using the symmetrical difference property, we obtain

$(A'\cap B)\cup(A\cap B')=(A\cup B)-(A\cap B)$, and, since $A\cap B\subseteq A\cup B$, we calculate the probability in the following way.

$$\mathrm{P}((A'\cap B)\cup(A\cap B'))=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)=\frac{7}{9}-\frac{5}{72}=\frac{51}{72}=\frac{17}{24}$$

20. **(a)** We can use the addition rule:

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)\Rightarrow\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cup B)$$

Since we know that $\mathrm{P}(A\cup B)\le 1$, we can conclude that $\mathrm{P}(A\cap B)\ge\mathrm{P}(A)+\mathrm{P}(B)-1$

(b) We can use the addition rule twice:

$$\begin{aligned}\mathrm{P}((A\cup B)\cup C)&=\mathrm{P}(A\cup B)+\mathrm{P}(C)-\mathrm{P}((A\cup B)\cap C)\\&=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)+\mathrm{P}(C)-\mathrm{P}((A\cup B)\cap C)\end{aligned}$$

The last probability can be handled separately by using the distribution properties of set operations and the addition rule once again.

$$\begin{aligned}\mathrm{P}((A\cup B)\cap C)&=\mathrm{P}((A\cap C)\cup(B\cap C))\\&=\mathrm{P}(A\cap C)+\mathrm{P}(B\cap C)-\mathrm{P}((A\cap C)\cap(B\cap C))\end{aligned}$$

Now we can finish the proof.

$$\begin{aligned}\mathrm{P}((A\cup B)\cup C)&=\mathrm{P}(A\cup B)+\mathrm{P}(C)-\mathrm{P}((A\cup B)\cap C)\\&=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)+\mathrm{P}(C)-\mathrm{P}((A\cup B)\cap C)\\&=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)+\mathrm{P}(C)-(\mathrm{P}(A\cap C)+\mathrm{P}(B\cap C)-\mathrm{P}(A\cap B\cap C))\\&=\mathrm{P}(A)+\mathrm{P}(B)+\mathrm{P}(C)-\mathrm{P}(A\cap B)-\mathrm{P}(A\cap C)-\mathrm{P}(B\cap C)+\mathrm{P}(A\cap B\cap C)\end{aligned}$$

Note: $(A\cap C)\cap(B\cap C)=A\cap B\cap C$

21. When we roll three fair 6-sided dice, there are 216 possible outcomes.

(a) Triples can be rolled in 6 different ways; therefore, $\mathrm{P}(A)=\frac{6}{216}=\frac{1}{36}$

(b) A sum of 8 or less can be rolled in 56 different ways and only 2 are triples. The possible combinations containing a triple are those with 1 or 2 since a triple with 3 adds to more than 8. Then there are 10 possible combinations with two equal numbers and each combination will appear 3 times, for example, (1, 1, 2), (1, 1, 3)...(2, 2, 3)...(2, 2, 4), (2, 3, 3). At the end there are 4 combinations, each with different numbers, for example, (1, 2, 3), (1, 2, 4), (1, 2, 5) and (1, 3, 4); and each of these combinations appear 6 times. So, we have a total of 56 different combinations.

$$\mathrm{P}(B)=\frac{2}{56}=\frac{1}{28}$$

(c) We will calculate the probability of 'at least one six' by using the complementary event, which is 'no six will appear'.

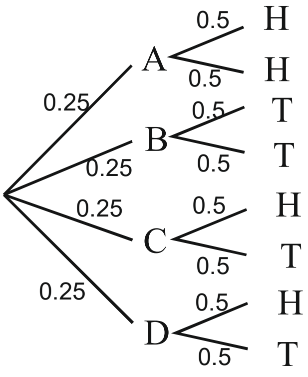
$$\mathrm{P}(C)=1-\mathrm{P}(C')=1-\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=1-\frac{125}{216}=\frac{91}{216}$$

Note: The result is the same as that in question 16 **(b)** since there is no difference in the appearance of a 2 or 6 on a dice.

(d) If all three dice have different numbers, there are ${}_6C_3=20$ different combinations and each will appear $3!=6$ times; therefore, we have 120 different ways. Again we will use the complementary event, 'no six will appear'.

$$\mathrm{P}(D)=1-\mathrm{P}(D')=1-\frac{{}_5C_3\times 3!}{120}=1-\frac{60}{120}=\frac{1}{2}$$

22. Let the coins be denoted as A with two heads, B with two tails, and C and D are normal. Let H be the event of getting heads, and T be the event of getting tails. A tree diagram can help.



To choose a coin at random means that the probability of choosing any coin is 0.25

The event T of the experiment where the outcome is tails contains the following mutually exclusive events: $B\cap T, C\cap T,$ and $D\cap T$. Thus,

$$\mathrm{P}(T)=\mathrm{P}(B\cap T)+\mathrm{P}(C\cap T)+\mathrm{P}(D\cap T)$$
$$=0.25\times(0.5+0.5)+0.25\times0.5+0.25\times0.5=0.5$$

The subset $H\cap T$ containing heads in the opposite face contains the events $C\cap T$ and $D\cap T$

with $\mathrm{P}(H\cap T)=0.25\times0.5+0.25\times0.5=0.25$

Now, the question can be stated as $\mathrm{P}(H|T)=\frac{\mathrm{P}(H\cap T)}{\mathrm{P}(T)}=\frac{0.25}{0.5}=0.5$

23. There are five different ways of rolling a sum of 6: (1, 5), (2, 4), (3, 3), (4, 2) and (5, 1).

So, the probability that either of the players will roll that sum is $\frac{5}{36}$

(a) $$\mathrm{P}(A)=\overset{K:\text{ no sum }6}{\frac{31}{36}}\times\overset{G:\text{ no sum }6}{\frac{31}{36}}\times\overset{K:\text{ sum }6}{\frac{5}{36}}=\frac{4805}{46656}\approx 0.103$$

(b) $$\mathrm{P}(B)=\overset{K:\text{ no sum }6}{\frac{31}{36}}\times\overset{G:\text{ no sum }6}{\frac{31}{36}}\times\overset{K:\text{ no sum }6}{\frac{31}{36}}\times\overset{G:\text{ sum }6}{\frac{5}{36}}=\frac{148955}{1679616}\approx 0.0887$$

(c) Kassanthra wins if:

she wins on her first roll: $\frac{5}{36}$

or, she wins on her second roll: $\overset{K:\text{ no sum }6}{\frac{31}{36}}\times\overset{G:\text{ no sum }6}{\frac{31}{36}}\times\overset{K:\text{ sum }6}{\frac{5}{36}}=\frac{5}{36}\times\left(\frac{31}{36}\right)^2$

or, she wins on her third roll:

$$\overset{K:\text{ no sum }6}{\frac{31}{36}}\times\overset{G:\text{ no sum }6}{\frac{31}{36}}\times\overset{K:\text{ no sum }6}{\frac{31}{36}}\times\overset{G:\text{ no sum }6}{\frac{31}{36}}\times\overset{K:\text{ sum }6}{\frac{5}{36}}=\frac{5}{36}\times\left(\frac{31}{36}\right)^4$$

and so on.

Thus, to calculate the probability that Kassanthra wins, we need to find the sum of an infinite geometric sequence with a first term of $a=\frac{5}{36}$ and a common ratio $r=\left(\frac{31}{36}\right)^2$

Therefore, P(Kassanthra wins) = $\frac{a}{1-r}=\frac{\frac{5}{36}}{1-\left(\frac{31}{36}\right)^2}=\frac{36}{67}\approx 0.537$

24. **(a)** The day has no effect on the observation, so we just need to find the probability that more than two requests will be made using complements:

$$\mathrm{P}(A)=1-(0.1+0.3+0.5)=0.1=10\%$$

(b) Since the days are independent, we need to multiply the probabilities:

$$\mathrm{P}(B)=0.1^5=0.00001$$

25. The class has 11 students altogether and we need to select 4 students at random.

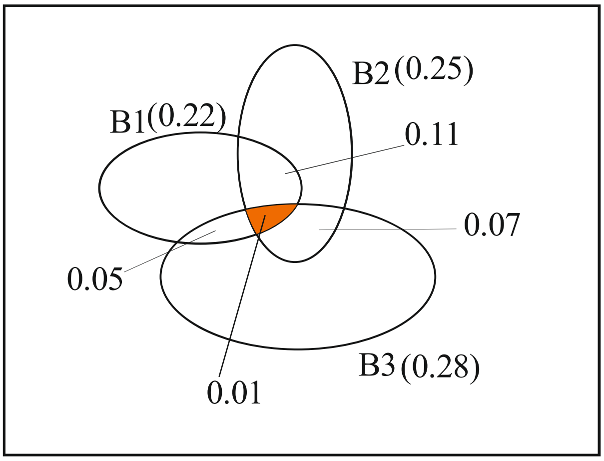
(a) We will calculate the probability of 'at least one boy' by using the complementary event, which is 'no boy is selected': $\mathrm{P}(A)=1-\frac{{}_6C_4}{{}_{11}C_4}=1-\frac{15}{330}=\frac{21}{22}$

(b) There must be either 3 or 4 girls: $\mathrm{P}(B)=\frac{{}_6C_3\cdot{}_5C_1+{}_6C_4}{{}_{11}C_4}=\frac{100+15}{330}=\frac{23}{66}$

(c) Let C be the event that the boys are in the majority. The conditional probability formula gives us: $\mathrm{P}(C|A)=\frac{\mathrm{P}(C\cap A)}{\mathrm{P}(A)}$. First, we will find the probability of the numerator.

$$\mathrm{P}(C\cap A)=\frac{{}_5C_3\cdot{}_6C_1+{}_5C_4}{{}_{11}C_4}=\frac{13}{66}\Rightarrow\mathrm{P}(C|A)=\frac{\frac{13}{\cancel{66}\,3}}{\frac{21}{\cancel{22}\,1}}=\frac{13}{63}$$

26. A Venn diagram may help identify the calculations involved.



(a) $\mathrm{P}(B_1\cup B_2)=\mathrm{P}(B_1)+\mathrm{P}(B_2)-\mathrm{P}(B_1\cap B_2)=0.22+0.25-0.11=0.36$

(b) Use De Morgan's law: $\mathrm{P}(B_1'\cap B_2')=\mathrm{P}\left((B_1\cup B_2)'\right)=1-\mathrm{P}(B_1\cup B_2)=1-0.36=0.64$

(c) Notice that the part of B_3 that does not intersect B_1 and B_2 is a subset of $(B_1 \cup B_2)'$ and therefore we only need to add $B_1 \cap B_3$ and $B_2 \cap B_3$, making sure not to double count the event $B_1 \cap B_2 \cap B_3$ shown in colour. Thus:

$$\begin{aligned}\mathrm{P}(B_1' \cap B_2') \cup B_3 &= \mathrm{P}\left(B_1' \cap B_2'\right) + \mathrm{P}(B_1 \cap B_3) + \mathrm{P}(B_2 \cap B_3) - \mathrm{P}(B_1 \cap B_2 \cap B_3)\\ &= 0.64 + 0.05 + .07 - 0.01 = 0.75\end{aligned}$$

There are also alternative approaches to this.

(d) Notice that this event consists of B_3 which does not contain B_1 and B_2. Thus:

$$\begin{aligned}\mathrm{P}\left(B_1' \cap B_2' \cap B_3\right) &= \mathrm{P}(B_3) - \mathrm{P}(B_1 \cap B_3) - \mathrm{P}(B_2 \cap B_3) + \mathrm{P}(B_1 \cap B_2 \cap B_3)\\ &= 0.28 - 0.05 - .07 + 0.01 = 0.17\end{aligned}$$

(e) $$\mathrm{P}(B_2 \cap B_3 | B_1) = \frac{\mathrm{P}(B_1 \cap B_2 \cap B_3)}{\mathrm{P}(B_1)} = \frac{0.01}{0.22} = \frac{1}{22} \approx 0.0455$$

(f) $$\mathrm{P}(B_2 \cup B_3 | B_1) = \frac{\mathrm{P}(B_2 \cup B_3 \cap B_1)}{\mathrm{P}(B_1)}$$

First, we need to find the probability of the numerator:

$$\begin{aligned}\mathrm{P}(B_2 \cup B_3 \cap B_1) &= \mathrm{P}(B_1 \cap B_2) + \mathrm{P}(B_1 \cap B_3) - \mathrm{P}(B_1 \cap B_2 \cap B_3)\\ &= 0.11 + 0.05 - 0.01 = 0.15\end{aligned}$$

Therefore, $\mathrm{P}(B_2 \cup B_3 | B_1) = \dfrac{0.15}{0.22} = \dfrac{15}{22}$

27. Let N and D be the sets of all the joints found to be faulty by Nick and David respectively. Given the information in the question, we can find the probabilities:

$\mathrm{P}(N) = \dfrac{1448}{20000}$, $\mathrm{P}(D) = \dfrac{1502}{20000}$ and $\mathrm{P}(N \cup D) = \dfrac{2390}{20000}$

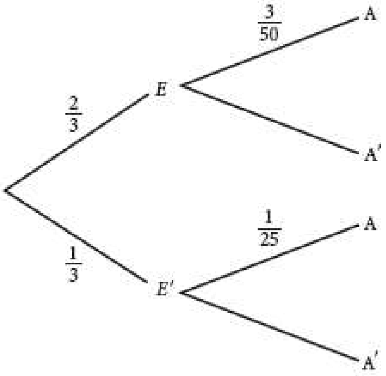
(a) $$\mathrm{P}\left((N \cup D)'\right) = 1 - \mathrm{P}(N \cup D) = 1 - \frac{239}{2000} = \frac{1761}{2000} = 0.8805$$

(b) $$\mathrm{P}(N \cap D) = \mathrm{P}(N) + \mathrm{P}(D) - \mathrm{P}(N \cup D) = \frac{1448 + 1502 - 2390}{20000} = \frac{560}{20000}$$

$$\mathrm{P}(D \cap N') = \mathrm{P}(D) - \mathrm{P}(D \cap N) = \frac{1502 - 560}{20000} = \frac{942}{20000} = \frac{471}{10000} = 0.0471$$

Exercise 11.4

1. **(a)**



(b) $\mathrm{P}(A)=\frac{2}{3}\times\frac{3}{50}+\frac{1}{3}\times\frac{1}{25}=\frac{4}{75}$

(c) $\mathrm{P}(E|A)=\frac{\mathrm{P}(E\cap A)}{\mathrm{P}(A)}=\frac{\frac{1}{25}}{\frac{4}{75}}=\frac{3}{4}$

2. Let C be the event that a person is diagnosed with cancer, and E_1 and E_2 the events that the person does have and doesn't have the disease respectively. (A tree diagram can help!)

(a) This is what we call total probability – a person is diagnosed with cancer if he/she has it or not.

$$\mathrm{P}(C)=\frac{26}{100\,000}\times 0.78+\frac{99\,974}{100\,000}\times 0.06=0.0601872\approx 0.0602$$

(b) This is a case of Bayes' theorem.

$$\mathrm{P}(E_1|C)=\frac{0.00026\times 0.78}{0.0601872}=0.0033694872\approx 0.00337$$

3. Let S be the event that a driver is spotted speeding, and E and W the events that the driver used the east or west entrance.

(a) $\mathrm{P}(S)=0.4\times 0.4+0.6\times 0.6=0.52$

(b) $\mathrm{P}(W|S)=\frac{0.6\times 0.6}{0.52}=0.692$ (correct to 3 s.f.)

4. Let G be the event that a drawn ball is green, and E_1, E_2 and E_3 the events that the ball is drawn from box 1, 2 and 3 respectively.

(a) $$\mathrm{P}(G)=\frac{1}{3}\times\frac{4}{20}+\frac{1}{3}\times\frac{8}{16}+\frac{1}{3}\times\frac{6}{20}=\frac{1}{3}\times\frac{2+5+3}{10}=\frac{1}{3}$$

(b) $$\mathrm{P}(E_2|G)=\frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}}=\frac{1}{2}$$

5. Let H be the event that the selected coin lands on heads, and E_1 and E_2 the events that the selected coin is biased and unbiased respectively.

(a) $$\mathrm{P}(H)=0.5\times0.6+0.5\times0.5=0.55$$

(b) $$\mathrm{P}(E_2|H')=\frac{0.5\times0.5}{0.45}=\frac{5}{9}\approx0.556$$

6. Let C be the event that the question was correctly answered, and E_1 and E_2 the events that the student was well-prepared and unprepared respectively.

$$\mathrm{P}(E_1|C)=\frac{0.7\times0.6}{\underset{\text{well prepared}}{0.7}\times\underset{\text{correct answer}}{0.6}+\underset{\text{not prepared}}{0.3}\times\underset{\text{guessing answer}}{0.2}}=\frac{0.42}{0.48}=\frac{7}{8}=0.875$$

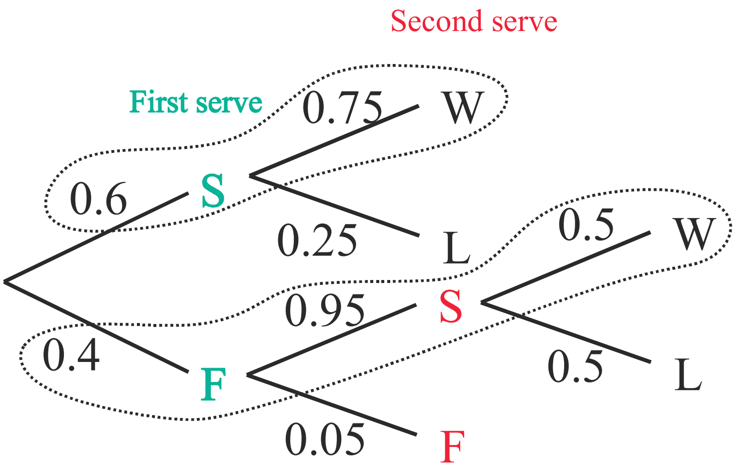
7. Let T be the event that Nigel gets to his morning class on time, and E_1 and E_2 the events that the alarm clock was set and was not set respectively.

(a) $$\mathrm{P}(T)=0.85\times0.9+0.15\times0.6=0.855=85.5\%$$

(b) $$\mathrm{P}(E_2|T)=\frac{0.15\times0.6}{0.855}=\frac{2}{19}\approx0.105=10.5\%$$

8. We are going to use a probability tree. Let F represent a successful first serve, S a successful second serve, and W and L winning and losing the point respectively.

A tree diagram will help visualise the situation:



(a) Cheung wins either from the first serve or from the second as shown.

$$\mathrm{P}(\mathrm{W}) = 0.6 \times 0.75 + 0.4 \times 0.95 \times 0.5 = \frac{16}{25} = 0.64$$

(b) $$\mathrm{P}(\mathrm{First} \mid \mathrm{W}) = \frac{P(\mathrm{First} \cap \mathrm{W})}{P(\mathrm{W})} = \frac{0.6 \times 0.75}{0.64} = \frac{45}{64} = 0.703125 \approx 0.703$$

9. Let F be the event that the first of February is a fine day, and S the event that the second of February is a fine day.

(a) $$\mathrm{P}(S) = 0.75 \times 0.8 + 0.25 \times 0.4 = 0.7$$

(b) $$\mathrm{P}(F \mid S') = \frac{\mathrm{P}(F \cap S')}{\mathrm{P}(S')} = \frac{0.75 \times 0.2}{0.3} = \frac{1}{2} = 0.5$$

10. Let A be the event that the person has arthritis, and P that the test is positive.

$$\mathrm{P}(A|P) = \frac{\mathrm{P}(A \cap P)}{\mathrm{P}(P)} = \frac{0.33 \times 0.87}{0.33 \times 0.87 + 0.67 \times 0.04} \approx 0.915$$

11. Let H represent students in the HL class, and L and A students studying locally and abroad respectively.

(a) $$\mathrm{P}(H \cap L) = 0.05 \times 0.72 = 0.036$$

(b) From the table, we know that: $\mathrm{P}(L)=0.67 \Rightarrow \mathrm{P}(H\cap L)+\mathrm{P}(H'\cap L)=0.67$

Therefore, $\mathrm{P}(H'\cap L)=0.67-\mathrm{P}(H\cap L)=0.67-0.036=0.634$

On the other hand: $\mathrm{P}(H'\cap L)=\mathrm{P}(H')\times\mathrm{P}(L\,|\,H')\Rightarrow\mathrm{P}(L\,|\,H')=\dfrac{0.634}{0.95}\approx 0.667$

(c) $\mathrm{P}(H\,|\,L)=\dfrac{\mathrm{P}(H\cap L)}{\mathrm{P}(L)}=\dfrac{0.036}{0.67}\approx 0.0537$

(d) $\mathrm{P}(H\,|\,A)=\dfrac{\mathrm{P}(H)\times\mathrm{P}(A\,|\,H)}{\mathrm{P}(A)}=\dfrac{0.05\times 0.28}{0.33}=\dfrac{0.014}{0.33}\approx 0.0424$

12. Let U represent athletes who are users, and P and N a positive and negative test respectively.

$$\mathrm{P}(U\,|\,P)=\frac{\mathrm{P}(U\cap P)}{\mathrm{P}(P)}=\frac{0.1\times 0.5}{0.1\times 0.5+0.9\times 0.09}\approx 0.382$$

13. Let A, R and S represent the estimates made by Antonio, Richard and Sarah respectively, and E an error in the estimation. We would like to find the largest from $\mathrm{P}(A\,|\,E),\mathrm{P}(R\,|E)$ *and* $\mathrm{P}(S\,|\,E)$.
We notice that all three expressions have the same denominator, and therefore the largest one will be the one with the largest numerator. So, we calculate the following:
$\mathrm{P}(A\cap E)=0.3\times 0.03=0.009,\mathrm{P}(R\cap E)=0.2\times 0.02=0.004$ and
$\mathrm{P}(S\cap E)=0.5\times 0.01=0.005$
Thus, Antonio is probably responsible for most of the serious errors.

14. Let A represent the event that an aircraft is present, and S that there is a signal.

(a) $\mathrm{P}(S)=\mathrm{P}(A)\times\mathrm{P}(S\,|\,A)+\mathrm{P}(A')\times\mathrm{P}\left(S\,|\,A'\right)=0.05\times 0.99+0.95\times 0.1=0.1445$

(b) $\mathrm{P}(A\,|\,S')=\dfrac{\mathrm{P}(A)\times\mathrm{P}(S'\,|\,A)}{\mathrm{P}(S')}=\dfrac{0.05\times 0.01}{1-0.1445}\approx 0.000584$

15. Let N, E and M represent a game against a novice, an experienced player and a master player respectively, and W the probability of winning the game.

(a) $$\mathrm{P}(W)=\mathrm{P}(N)\times\mathrm{P}(W|N)+\mathrm{P}(E)\times\mathrm{P}(W|E)+\mathrm{P}(M)\times\mathrm{P}(W|M)$$

$$=0.5\times0.5+0.25\times0.4+0.25\times0.3=0.425$$

(b) $$\mathrm{P}(M\,|\,W)=\frac{\mathrm{P}(M)\times\mathrm{P}(W\,|M)}{\mathrm{P}(W)}=\frac{0.25\times0.3}{0.425}\approx0.176$$

16. **(a)** Let A be the event of passing the test:

$$\mathrm{P}(A)=\overset{\text{pass 1st}}{0.8}+\overset{\text{fail 1st}}{0.2}\times\overset{\text{pass 2nd}}{0.5}+\overset{\text{fail 1st}}{0.2}\times\overset{\text{fail 2nd}}{0.5}\times\overset{\text{pass 3rd}}{0.3}=0.93$$

(b) $$\mathrm{P}(\text{2nd}|A)=\frac{0.2\times0.1}{0.93}\approx0.108$$

17. **(a)** $$\mathrm{P}(F)=\frac{56+26+18}{250}=\frac{100}{250}=\frac{2}{5}=0.4,\quad \mathrm{P}(F\cap T)=\frac{56}{250}=\frac{28}{125}=0.224$$

$$\mathrm{P}(F\cup A')=\frac{84+52+56+18+26}{250}=\frac{236}{250}=\frac{118}{125}=0.944$$

$$\mathrm{P}(F'|A)=\frac{\mathrm{P}(F'\cap A)}{\mathrm{P}(A)}=\frac{\frac{14}{\cancel{250}}}{\frac{14+26}{\cancel{250}}}=\frac{14}{40}=\frac{7}{20}=0.35$$

(b) T is independent of F since $\mathrm{P}(F)=0.4=\mathrm{P}(F\,|\,T)=\frac{56}{140}$

The event M is mutually exclusive to F since the probability of their intersection is zero.

(c) Let C be the event of owning a car.

(i) $$\mathrm{P}(C)=\mathrm{P}(T)\times\mathrm{P}(C\,|\,T)+\mathrm{P}(A)\times\mathrm{P}(C\,|\,A)+\mathrm{P}(S)\times\mathrm{P}(C\,|\,S)$$

$$\Rightarrow\mathrm{P}(C)=\frac{14\cancel{0}}{25\cancel{0}}\times\frac{9}{10}+\frac{4\cancel{0}}{25\cancel{0}}\times\frac{8}{10}+\frac{7\cancel{0}}{25\cancel{0}}\times\frac{3}{10}=\frac{126+32+21}{250}=\frac{179}{250}=0.716$$

(ii) $$\mathrm{P}(T|C)=\frac{\mathrm{P}(T)\times\mathrm{P}(C|T)}{\mathrm{P}(C)}=\frac{\frac{126}{250}}{\frac{179}{250}}=\frac{126}{179}\approx0.704$$

18. Let H, M and L represent high risk, medium risk and low risk drivers respectively, and A those drivers who will have an accident.

(a) $\mathrm{P}(H \cap A) = \mathrm{P}(H) \times \mathrm{P}(A|H) = 0.2 \times 0.06 = 0.012$

(b) $\mathrm{P}(A) = \mathrm{P}(H) \times \mathrm{P}(A \mid H) + \mathrm{P}(M) \times \mathrm{P}(A \mid M) + \mathrm{P}(L) \times \mathrm{P}(A \mid L)$

$$= 0.2 \times 0.06 + 0.5 \times 0.03 + 0.3 \times 0.01 = 0.012 + 0.015 + 0.003 = 0.03$$

(c) $\mathrm{P}(H|A) = \dfrac{\mathrm{P}(H) \times \mathrm{P}(A|H)}{\mathrm{P}(A)} = \dfrac{0.012}{0.03} = 0.4$

Chapter 11 practice questions

1. **(a)** Since the events are independent:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) \Rightarrow 0.18 = k \times (k + 0.3) \Rightarrow k^2 + 0.3k - 0.18 = 0$$

$$\Rightarrow (k + 0.6)(k - 0.3) = 0 \Rightarrow k = -0.6 \text{ or } k = 0.3$$

Algebraically we get two solutions, but only 0.3 can be a probability value, since probability cannot be negative.

(b) Using the addition formula:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.3 + 0.6 - 0.18 = 0.72$$

(c) Since the events are independent, the complementary events are independent too.

$$\mathrm{P}(A'|B') = \mathrm{P}(A') \Rightarrow \mathrm{P}(A'|B') = 1 - 0.3 = 0.7$$

2. **(a)** Since the tests are taken independently, we multiply the probabilities:

$$\mathrm{P}(A) = 0.02 \times 0.02 = 0.0004$$

(b) Using the complementary event: $\mathrm{P}(B) = 1 - \mathrm{P}(A) = 1 - 0.0004 = 0.9996$

(c) $\mathrm{P}(C) = 0.02 \times 0.02 = 0.0004$

3. Since they work independently of each other, we need to multiply the probabilities and then use the complementary event.

$$\mathrm{P}(A') = 1 - \mathrm{P}(A) = 1 - 0.002 \times 0.01 = 0.99998$$

4. **(a)** **(i)** Using the addition formula:

$$\mathrm{P}(S \cup F) = \mathrm{P}(S) + \mathrm{P}(F) - \mathrm{P}(S \cap F) = \frac{120}{200} + \frac{60}{200} - \frac{10}{200} = \frac{17}{20}$$

(ii) 'Either but not both' means that we exclude the intersection from the union, so:

$$\mathrm{P}(S \cup F) - \mathrm{P}(S \cap F) = \frac{170}{200} - \frac{10}{200} = \frac{160}{200} = \frac{4}{5}$$

(iii) Does not take French or Spanish is the complementary event of the union, so:

$$\mathrm{P}\left((S \cup F)'\right) = 1 - \mathrm{P}(S \cup F) = 1 - \frac{17}{20} = \frac{3}{20}$$

(b) Using the conditional probability formula: $\mathrm{P}(F|S) = \dfrac{\mathrm{P}(F \cap S)}{\mathrm{P}(S)} = \dfrac{\frac{10}{200}}{\frac{120}{200}} = \dfrac{1}{12}$

5. It would be a good idea to find the total sums first. There are 126, 84 and 160 disks produced after one run on machines I, II and III respectively. There are a total of 20 defective and 350 non-defective disks. That means there are 370 disks produced in total. Let D denote defective and ND non-defective.

(a) **(i)** $\mathrm{P}(\mathrm{I}) = \dfrac{126}{370} = \dfrac{63}{185}$

(ii) $\mathrm{P}(\mathrm{D} \cap \mathrm{II}) = \dfrac{4}{370} = \dfrac{2}{185}$

(iii) We need to use the addition formula:

$$\mathrm{P}(\mathrm{ND} \cup \mathrm{I}) = \frac{126 + 350 - 120}{370} = \frac{356}{370} = \frac{178}{185}$$

(iv) Since this is a conditional probability, our sample space is defective disks and the favourable outcomes are the defective items produced by machine I.

$$\mathrm{P}(\mathrm{I}|\mathrm{D}) = \frac{6}{20} = \frac{3}{10}$$

(b) If the quality is independent of the machine, then, for example $\mathrm{P}(\mathrm{I}|\mathrm{D})=\mathrm{P}(\mathrm{I})$. However, by comparing our answers to **(i)** and **(iv)**, this is not true. So, the quality and machine used are not independent. Alternatively, you can use the multiplication rule where you notice for example that $\mathrm{P}(\mathrm{D}\cap\mathrm{II})=\frac{2}{185}\neq\mathrm{P}(\mathrm{D})\times\mathrm{P}(\mathrm{II})=\frac{20}{370}\cdot\frac{84}{370}=\frac{84}{6845}$

6. **(a)** There are 126 envelopes which satisfy your wish, so: $\mathrm{P}(A)=\frac{126}{200}=\frac{63}{100}$

(b) There are 68 red envelopes without a prize, so: $\mathrm{P}(B)=\frac{68}{70}=\frac{34}{35}$

7. **(a)** $\mathrm{P}(A|B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}\Rightarrow 0.3=\frac{0.18}{\mathrm{P}(B)}\Rightarrow\mathrm{P}(B)=\frac{0.18}{0.3}=0.6$

(b) The events are independent since $\mathrm{P}(B)=0.6=\mathrm{P}(B|A)$

(c) Given that A and B are independent, then A' and B are independent too, so:

$$\mathrm{P}(B\cap A')=\mathrm{P}(B)\times\mathrm{P}(A')=0.6\times(1-0.3)=0.42$$

8. **(a)** Since we know that there are 74 students who took the test, the number of boys who failed is $74-(32+16+12)=14$. There are 6 girls who are too young to take the test and, since there are 10 students altogether that are too young to take the test, the number of boys who are too young is 10 – 6 = 4. Since the total numbers of boys and girls are 70 and 50 respectively, we calculate all of those who were training but did not take the test as: $70-(32+14+4)=20$, and $50-(16+12+6)=16$.

	Boys	Girls
Passed the ski test	32	16
Failed the ski test	14	12
Training, but did not take the test yet	20	16
Too young to take the test	4	6

(b) **(i)** $\mathrm{P}(\mathrm{Test})=\frac{74}{120}=\frac{37}{60}$

(ii) $\mathrm{P}(\mathrm{Test}\,|\,\mathrm{Girl})=\frac{16+12}{50}=\frac{28}{50}=\frac{14}{25}$

(iii) These two events are independent, so we multiply the probabilities:

$$\mathrm{P}\left((\mathrm{passed}\,|\,\mathrm{Boy})\cap(\mathrm{passed}\,|\,\mathrm{Girl})\right)=\frac{32}{70}\times\frac{16}{50}=\frac{128}{875}$$

9. **(a)** $\mathrm{P}(A\cap B)=\mathrm{P}(A|B)\times\mathrm{P}(B)=\frac{1}{4}\times\frac{3}{8}=\frac{3}{32}$

(b) $\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-2\times\mathrm{P}(A\cap B)=\frac{9}{16}+\frac{3}{8}-2\times\frac{3}{32}=\frac{3}{4}$

(c) $\mathrm{P}\left((A\cup B)'\right)=1-\mathrm{P}(A\cup B)=1-\frac{27}{32}=\frac{5}{32}$

10. **(a)** Probability that she will miss both serves is $\mathrm{P}(A)=0.4\times0.05=0.02=2\%$

(b) To win a point, she will make the first serve and win the point, or she will miss the first serve, make the second serve and win the point, so:
$\mathrm{P}(B)=0.6\times0.75+0.4\times0.95\times0.5=0.64=64\%$

11. **(a)** $\mathrm{P}(X\cap Y)=\mathrm{P}(X)+\mathrm{P}(Y)-\mathrm{P}(X\cup Y)=0.6+0.8-1=0.4$

(b) $\mathrm{P}(X'\cup Y')=\mathrm{P}\left((X\cap Y)'\right)=1-\mathrm{P}(X\cap Y)=1-0.4=0.6$

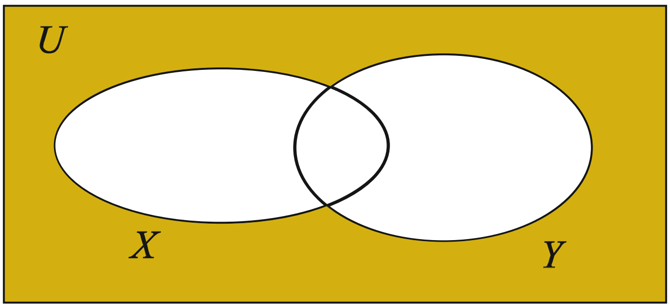
12. Using the given information, here is a complete table.

	Men	**Women**	**Total**
News	13	25	38
Sport	**33**	**29**	62
Total	**46**	54	**100**

(a) $\mathrm{P}(news)=\frac{38}{100}=\frac{19}{50}$

(b) $\mathrm{P}(news|man)=\frac{13}{46}$

13. **(a)**



(b) **(i)** Since $n(X\cup Y)=n(U)-n\left((X\cup Y)'\right)=36-21=15,$ then

$n(X\cap Y)=n(X)+n(Y)-n(X\cup Y)=11+6-15=2.$

(ii) $\mathrm{P}(X\cap Y)=\dfrac{2}{36}=\dfrac{1}{18}$

(c) Events X and Y are not mutually exclusive since there are two elements in the intersection of the two sets: $X\cap Y\neq\varnothing$.

14. **(a)** There are 90 females, so there are 110 males. If 60 were unemployed, then 140 were employed. If 20 males were unemployed, then 40 females were unemployed, and so on.

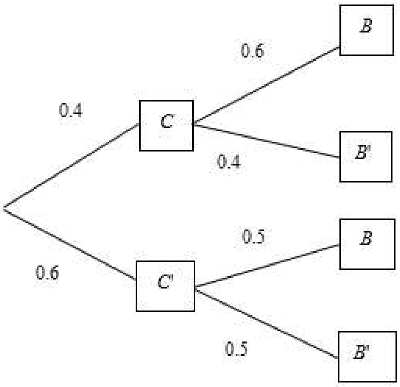
	Males	**Females**	**Totals**
Unemployed	20	40	60
Employed	90	50	140
Totals	110	90	200

(b) **(i)** $\mathrm{P}(U\cap F)=\dfrac{40}{200}=\dfrac{1}{5}$

(ii) $\mathrm{P}(M|E)=\dfrac{90}{140}=\dfrac{9}{14}$

15. There are three possible combinations of different colours: WR, WG, or RG. In total, there are 26 chips in the bag. The WR chips can be chosen in $10\times10=100$ ways, the WG in 60 ways and the RG in 60 ways. In total there 220 ways of choosing 2 chips with different colours. There are ${}_{26}C_2=325$ ways of choosing any 2 chips, and thus, $\mathrm{P}(A)=\dfrac{220}{325}=\dfrac{44}{65}$

16. **(a)**



The student can take Chemistry and Biology or not take Chemistry but take Biology.

So, $\mathrm{P}(B)=0.4\times0.6+0.6\times0.5=0.54$

(b) This is a case of Bayes' theorem: $\mathrm{P}(C|B)=\dfrac{\mathrm{P}(C\cap B)}{\mathrm{P}(B)}=\dfrac{0.4\times0.6}{0.54}=\dfrac{4}{9}$

17. **(a)** The probability distribution table for the sum of the 2 numbers on a pair of dice is:

s	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
P(*S*)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\mathrm{P}(S\le7)=\frac{1+2+\dots+6}{36}=\frac{21}{36}=\frac{7}{12}$$

(b) There are 11 possible pairs when at least one dice shows a 3: six pairs with 3 showing on the first dice and six pairs with 3 showing on the second dice, but the pair (3, 3) should only be counted once. So, $\mathrm{P}(B)=\dfrac{11}{36}$

(c) Now the event from **(a)** becomes a sample space and we need to find the favourable pairs (out of 21 pairs) found in **(b)**. The pairs that satisfy the condition are (1, 3), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4) and (4, 3).

$$\mathrm{P}(B\,|\,S\le7)=\frac{7}{21}=\frac{1}{3}$$

18. **(a)** $\mathrm{P}(A\cap B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cup B)=\frac{3}{11}+\frac{4}{11}-\frac{6}{11}=\frac{1}{11}$

(b) $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)=\frac{3}{11}\times\frac{4}{11}=\frac{12}{121}$

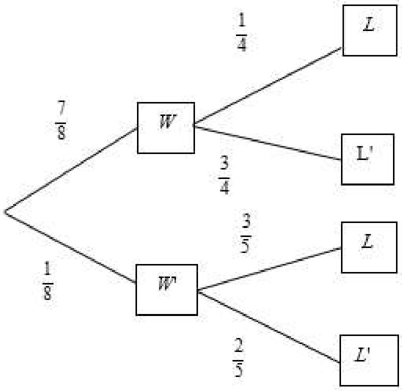
19. **(a)** Since the probability of A didn't change when B occurred, $\mathrm{P}(A|B)=\mathrm{P}(A)$.

Therefore, the events are independent.

(b) If $\mathrm{P}(A\cap B)=0\Rightarrow A\cap B=\varnothing$; therefore, the events are mutually exclusive.

(c) Given that $\mathrm{P}(A\cap B)=\mathrm{P}(A)\Rightarrow A\cap B=A\Rightarrow A\subseteq B$, neither.

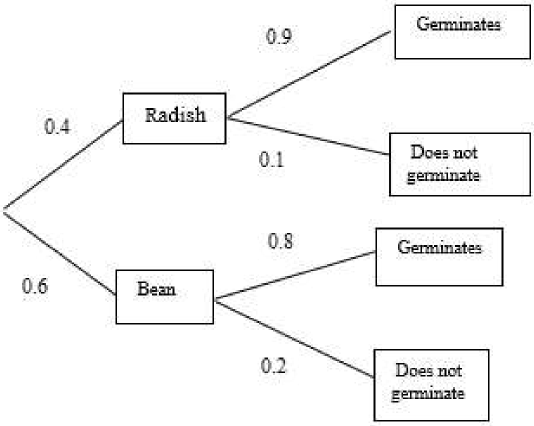
20. **(a)**



(b) $\mathrm{P}(L)=\frac{7}{8}\times\frac{1}{4}+\frac{1}{8}\times\frac{3}{5}=\frac{7}{32}+\frac{3}{40}=\frac{35+12}{160}=\frac{47}{160}$

(c) $\mathrm{P}(W|L)=\frac{\mathrm{P}(W\cap L)}{\mathrm{P}(L)}=\frac{\frac{7}{32}}{\frac{47}{160}}=\frac{35}{47}$

21. **(a)**



(b) **(i)** $\mathrm{P}(R \cap G) = 0.4 \times 0.9 = 0.36$

(ii) $\mathrm{P}(G) = \overset{\text{radish}}{0.4} \times \overset{\text{germinates}}{0.9} + \overset{\textit{beans}}{0.6} \times \overset{\text{germinates}}{0.8} = 0.36 + 0.48 = 0.84$

(iii) $\mathrm{P}(R|G) = \dfrac{\mathrm{P}(R \cap G)}{\mathrm{P}(G)} = \dfrac{0.36}{0.84} = \dfrac{3}{7} \approx 0.429$

22. **(a)** **(i)** $\mathrm{P}(A) = \dfrac{80}{210} = \dfrac{8}{21}$

(ii) $\mathrm{P}(A \cap B) = \dfrac{35}{210} = \dfrac{1}{6}$

(iii) To confirm whether the events are independent we first find $\mathrm{P}(B) = \dfrac{100}{210} = \dfrac{10}{21}$.

Now we look at the product of the probabilities:

$\mathrm{P}(A) \times \mathrm{P}(B) = \dfrac{8}{21} \times \dfrac{10}{21} = \dfrac{80}{441} \neq \dfrac{1}{6} = \mathrm{P}(A \cap B)$. Hence, they are not independent.

Alternatively, we can look at the conditional probability.

(b) $\mathrm{P}(Y_1|H) = \dfrac{50}{85} = \dfrac{10}{17}$

(c) We can select a student from Year 1 in 110 ways and a student from Year 2 in 100 ways.

We can select 2 students from the college in ${}_{210}C_2$ ways.

Therefore: $\mathrm{P}(C) = \dfrac{100 \times 110}{{}_{210}C_2} = \dfrac{11000}{21945} = \dfrac{200}{399}$

23. Let O be the event that an odd number is chosen and E an even number is chosen. If an odd and an even number have to be selected in any order, we can say that the chosen slips can be odd **and** even **or** even **and** odd.

$$\mathrm{P}(OE)+\mathrm{P}(EO)=\mathrm{P}(O)\times\mathrm{P}(E)+\mathrm{P}(E)\times\mathrm{P}(O)=\frac{3}{9}\times\frac{4}{8}+\frac{4}{9}\times\frac{3}{8}=\frac{1}{3}$$

24. $\mathrm{P}(X\cap Y)+\mathrm{P}(X\cap Y')=\mathrm{P}(X)\Rightarrow\mathrm{P}(X)=0.3+0.3=0.6$. Since the events are independent, we can use the multiplication law. $\mathrm{P}(X\cap Y)=\mathrm{P}(X)\times\mathrm{P}(Y)\Rightarrow\mathrm{P}(Y)=\dfrac{\mathrm{P}(X\cap Y)}{\mathrm{P}(X)}=\dfrac{0.3}{0.6}=0.5$

$$\mathrm{P}(X\cup Y)=\mathrm{P}(X)+\mathrm{P}(Y)-\mathrm{P}(X\cap Y)=0.6+0.5-0.3=0.8$$

25. Let R be the event that it is raining, and L that Sophia is late.

$$\mathrm{P}(R|L)=\frac{\mathrm{P}(R)\times\mathrm{P}(L|R)}{\mathrm{P}(L)}=\frac{\frac{1}{4}\times\frac{2}{3}}{\frac{1}{4}\times\frac{2}{3}+\frac{3}{4}\times\frac{1}{5}}=\frac{10}{19}$$

26. Let F be the event that the umbrella is left at the first friend's place, and S that the umbrella is left at the second friend's place.

$$\mathrm{P}(S|F\cup S)=\frac{\frac{2}{3}\times\frac{1}{3}}{\frac{1}{3}+\frac{2}{3}\times\frac{1}{3}}=\frac{\frac{2}{9}}{\frac{5}{9}}=\frac{2}{5}$$

27. **(a)** **(i)** $$\mathrm{P}(L_1)=\overset{\substack{Cath\\ no\ win}}{\frac{5}{6}}\times\overset{\substack{Lucy\\ win}}{\frac{1}{6}}=\frac{5}{36}$$

(ii) $$\mathrm{P}(C_2)=\overset{\substack{Cath\\ no\ win}}{\frac{5}{6}}\times\overset{\substack{Lucy\\ no\ win}}{\frac{5}{6}}\times\overset{\substack{Cath\\ win}}{\frac{1}{6}}=\frac{25}{216}$$

(iii) $$\mathrm{P}(C_n)=\left(\frac{5}{6}\times\frac{5}{6}\right)^{n-1}\times\frac{1}{6}=\frac{1}{6}\left(\frac{25}{36}\right)^{n-1}=\frac{1}{6}\left(\frac{5}{6}\right)^{2n-2}$$

(b) We have an infinite geometric series:

$$p=\frac{1}{6}+\frac{25}{36}\times\frac{1}{6}+\left(\frac{25}{36}\right)^2\frac{1}{6}+\left(\frac{25}{36}\right)^3\frac{1}{6}+\ldots$$

$$=\frac{1}{6}+\frac{25}{36}\overbrace{\left(\frac{1}{6}+\frac{25}{36}\times\frac{1}{6}+\left(\frac{25}{36}\right)^2\frac{1}{6}+\left(\frac{25}{36}\right)^3\frac{1}{6}+\ldots\right)}^{p}=\frac{1}{6}+\frac{25}{36}\times p$$

(c) First, we will calculate the probability that Catherine wins the game:

$$p=\frac{1}{6}+\frac{25}{36}\times p\Rightarrow p-\frac{25}{36}p=\frac{1}{6}\Rightarrow\frac{11}{36}p=\frac{1}{6}\Rightarrow p=\frac{1}{6}\times\frac{36}{11}\Rightarrow p=\frac{6}{11}$$

The probability that Lucy wins the game is complementary to the above, so:
$\mathrm{P}(\text{Lucy wins})=1-\frac{6}{11}=\frac{5}{11}$. Alternatively, we can look at the process of Lucy winning:

$$\overset{\textit{Cath no win}}{\frac{5}{6}}\times\overset{\textit{Lucy win}}{\frac{1}{6}}+\overset{\textit{Cath no win}}{\frac{5}{6}}\times\overset{\textit{Lucy no win}}{\frac{5}{6}}\times\overset{\textit{Cath no win}}{\frac{5}{6}}\times\overset{\textit{Lucy win}}{\frac{1}{6}}+\cdots$$

Which is an infinite geometric series with first term $\frac{5}{6}\times\frac{1}{6}$ and common ratio $\left(\frac{5}{6}\right)^2$

(d) If Catherine wins more games than Lucy, that means that she has to win 4, 5 or 6 times.

We also need to find the number of sequences with that number of wins. For example, if Catherine wins four out of six games played, she can do that in ${}_6C_4$ different ways (we are using the binomial coefficients).

$$\mathrm{P}(D)={}_6C_4\overset{\text{Cath wins 4 times}}{\left(\frac{6}{11}\right)^4}\overset{\text{Lucy wins twice}}{\left(\frac{5}{11}\right)^2}+{}_6C_5\overset{\text{Cath wins 5 times}}{\left(\frac{6}{11}\right)^5}\overset{\text{Lucy wins once}}{\left(\frac{5}{11}\right)}+\overset{\text{Cath wins 6 times}}{\left(\frac{6}{11}\right)^6}\approx 0.432$$

28. (a) If the first two selected balls are white, then 1 white and 22 red balls remain in the box.

Therefore, the probability that the next ball will be red is $\frac{22}{23}\approx 0.957$

(b) There are three different selections that will give the result of exactly two red balls: RRW, WGR or WRR. So, the probability is calculated as follows:

$$\mathrm{P}(B)=3\times\frac{22}{25}\times\frac{21}{24}\times\frac{3}{23}=\frac{693}{2300}\approx 0.3013$$

Alternatively

$$\mathrm{P}(B)=\frac{{}_{22}C_2\times{}_{3}C_1}{{}_{25}C_3}=\frac{231\times 3}{2300}\approx 0.3013$$

29. Let F be the event that the chosen day is Monday, and B that Roberto catches the 08:00 bus.

(a) $\mathrm{P}(B)=\mathrm{P}(F)\times\mathrm{P}(B|F)+\mathrm{P}(F')\times\mathrm{P}(B|F')=0.2\times0.66+0.8\times0.75=0.732$

(b) $\mathrm{P}(F|B)=\dfrac{\mathrm{P}(F)\times\mathrm{P}(B|F)}{\mathrm{P}(B)}=\dfrac{0.132}{0.732}=\dfrac{11}{61}\approx 0.180$

30. **(a)** $\mathrm{P}(E_1)=\dfrac{4}{6}=\dfrac{2}{3}$

(b) $\mathrm{P}(H_1)=\overset{\substack{\textit{Eric}\\ \textit{no win}}}{\dfrac{2}{6}}\times\dfrac{4}{6}=\dfrac{2}{9}$

(c) We will have an infinite geometric series:

$$\mathrm{P}(E)=\overset{\substack{\textit{Eric}\\ \textit{win}}}{\frac{2}{3}}+\overset{\substack{\textit{Eric}\\ \textit{no win}}}{\frac{1}{3}}\times\overset{\substack{\textit{Harriet}\\ \textit{no win}}}{\frac{1}{3}}\times\overset{\substack{\textit{Eric}\\ \textit{win}}}{\frac{2}{3}}+\left(\frac{1}{3}\times\frac{1}{3}\right)^2\times\frac{2}{3}+\left(\frac{1}{3}\times\frac{1}{3}\right)^3\times\frac{2}{3}+\ldots$$

$$=\frac{2}{3}\left(1+\frac{1}{9}+\left(\frac{1}{9}\right)^2+\left(\frac{1}{9}\right)^3+\ldots\right)=\frac{2}{3}\times\frac{1}{1-\frac{1}{9}}=\frac{2}{3}\times\frac{9}{8}=\frac{3}{4}$$

31. **(a)** $\mathrm{P}(RR|A)=\dfrac{2}{5}\times\dfrac{1}{4}=\dfrac{1}{10}$

(b)

$$\mathrm{P}(B)=\frac{4}{n+4}\times\frac{3}{n+3}=\frac{12}{(n+4)(n+3)}=\frac{2}{15}$$
$$\Rightarrow (n+4)(n+3)=90$$

This leads to a quadratic equation $n^2+7n-78=0$.

$\Rightarrow n=-13$ or $n=6$. Discard $n=-13$ because the number of balls cannot be negative.

(c)

$$\mathrm{P}(RR)=\overset{\text{Bag A}}{\frac{1}{3}}\times\overset{\text{Two red}}{\frac{1}{10}}+\overset{\text{Bag B}}{\frac{2}{3}}\times\overset{\text{Two red}}{\frac{2}{15}}=\frac{3+8}{90}=\frac{11}{90}$$

(d) Recall that 1 or 6 means that we draw from bag A. Let *RR* be the event that two red balls are drawn, and *A* that the balls are drawn from bag A.

$$\mathrm{P}(A|RR)=\frac{\mathrm{P}(A)\times\mathrm{P}(RR|A)}{\mathrm{P}(RR)}=\frac{\frac{1}{3}\times\frac{1}{10}}{\frac{11}{90}}=\frac{3}{11}$$

32. **(a)**

$$\mathrm{E}(X)=\sum xP(x)=1\times\frac{1}{6}+2\times\frac{2}{6}+3\times\frac{3}{6}=\frac{7}{3}$$

(b) **(i)** To get a sum of 5, the numbers would be either $A=\{1, 2, 2\}$ which can be arranged in 3 different ways, or $B=\{1, 1, 3\}$ which can also be arranged in 3 different ways.

$$\mathrm{P}(A\cup B)=3\left(\frac{1}{6}\cdot\frac{2}{6}\cdot\frac{2}{6}\right)+3\left(\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{3}{6}\right)=\frac{7}{72}$$

(ii) To get a median of 1, the numbers would be either $C=\{1, 1, 2\}$ that can be arranged in 3 different ways, $D=\{1, 1, 3\}$ which could also be arranged in 3 different ways, or $E=\{1, 1, 1\}$.

$$\mathrm{P}(C\cup D\cup E)=3\left(\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{2}{6}\right)+3\left(\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{3}{6}\right)+\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{2}{27}$$

(c) The probability of selecting a 2 is $\frac{1}{3}$ and the probability of not selecting a 2 is $\frac{2}{3}$. Also for each number of 2s selected, say 3, there will be ${}_{10}C_3$ ways of doing that. Thus, the probability in question is

$$\overbrace{{}_{10}C_0}^{choices}\overbrace{\left(\frac{1}{3}\right)^0}^{\#\ of\ 2s}\overbrace{\left(\frac{2}{3}\right)^{10}}^{\#\ of\ no-2s}+{}_{10}C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^9+{}_{10}C_0\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^8+{}_{10}C_0\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7\approx 0.559$$

(d) Let the number of balls be *n*. The complement of having at least one 2 is having no 2.

Thus, if the probability of taking at least one 2 is greater than 0.95, then the probability of having no 2 must be smaller than 0.05.

$$_nC_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n < 0.05 \Rightarrow \left(\frac{2}{3}\right)^n < 0.05 \Rightarrow n\ln\left(\frac{2}{3}\right) < \ln 0.05$$

Now, since $\ln\left(\frac{2}{3}\right) < 0$, dividing by it will change the sense of the inequality, i.e.,

$n > \dfrac{\ln 0.05}{\ln\left(\frac{2}{3}\right)} \approx 7.39$. Since n is a whole number, the least possible number will be 8.

(e) Again, to get 1 is a binomial. If the expected value is 4.8, then $np_1 = 4.8 \Rightarrow p_1 = 0.6$. If the variance of the number of 2s is 1.5, then

Variance = $np_2\left(1-p_2\right) = 1.5 \Rightarrow 8p_2\left(1-p_2\right) = 1.5$

This is a quadratic equation with roots 0.75 or 0.25. We cannot use 0.75 since the probabilities for 1, 2, and 3 must not add up to more than 1. Thus, 2s constitute 25% of the balls in the bag. This leaves 15% for the 3s.

We need the number of each kind to be a whole number. Putting the probabilities in fraction form and using the same denominator gives us an idea:

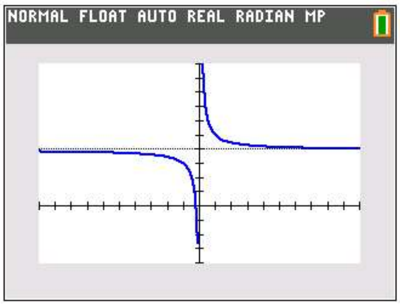
$$0.15 = \frac{3}{20}; 0.25 = \frac{5}{20}; 0.60 = \frac{12}{20}$$

The smallest number of balls in the bag that will work (trial and error) is 20.
This means the smallest possible number of balls with 3 is 3.

Exercise 12.1

1. **(a)** $\lim_{n\to\infty}\dfrac{1+4n}{n}=\lim_{n\to\infty}(\dfrac{1}{n}+\dfrac{4n}{n})=\lim_{n\to\infty}\dfrac{1}{n}+\lim_{n\to\infty}4=0+4=4$

n	$\dfrac{1+4n}{n}$
10	4.1
50	4.02
100	4.01
1000	4.001
10000	4.0001
1000000	4.000001



The dotted line included in the graph of $y=\dfrac{1+4x}{x}$ is the line $y=4$, the behavior of the graph towards the positive end confirms that the limit value is 4.

(b) The variable in the limit is h, so x is treated as a constant:

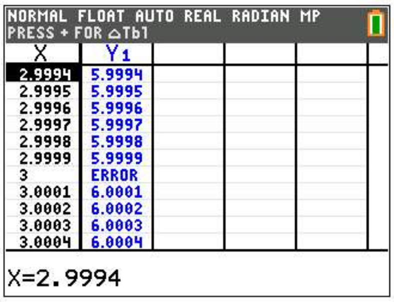
$$\lim_{h\to0}(3x^2+2hx+h^2)=\lim_{h\to0}3x^2+\lim_{h\to0}2hx+\lim_{h\to0}h^2=3x^2+0+0=3x^2$$

(c) The variable in the limit is d, so x is treated as a constant:

$$\lim_{d\to0}\frac{(x+d)^2-x^2}{d}=\lim_{d\to0}\frac{x^2+2xd+d^2-x^2}{d}=\lim_{d\to0}\frac{2xd+d^2}{d}=\lim_{d\to0}\frac{2xd}{d}+\lim_{d\to0}\frac{d^2}{d}$$

$$=\lim_{d\to0}2x+\lim_{d\to0}d=2x+0=2x$$

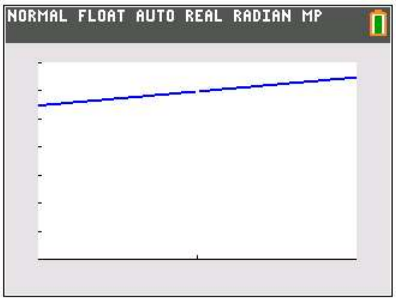
(d) $\lim_{x\to3}\dfrac{x^2-9}{x-3}=\lim_{x\to3}\dfrac{(x-3)(x+3)}{x-3}=\lim_{x\to3}(x+3)=3+3=6$



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl

X	Y₁
2.9994	5.9994
2.9995	5.9995
2.9996	5.9996
2.9997	5.9997
2.9998	5.9998
2.9999	5.9999
3	ERROR
3.0001	6.0001
3.0002	6.0002
3.0003	6.0003
3.0004	6.0004

X=2.9994



The graph has a hole at $(3, 6)$, as seen in the graph.

2. **(a)** Let $f(x)=\dfrac{3x+2}{x^2-3}$. The required values are presented in the table.

x	$f(x)$
10	0.329897
50	0.060873
100	0.030209
1000	0.003002
10000	0.000300
1000000	0.000003

The values of f decrease towards 0 as x becomes infinitely large, it follows that:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{3x+2}{x^2-3}=0$$

(b) Let $f(x)=\dfrac{5x-6}{2x+5}$. The required values are presented in the table.

x	$f(x)$
10	1.760000
50	2.323810
100	2.409756
1000	2.490773
10000	2.499075
1000000	2.499991

The values of f increase towards 2.5 as x becomes infinitely large, it follows that:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{5x-6}{2x+5}=\frac{5}{2}$$

(c) Let $f(x)=\dfrac{3x^2+2}{x-3}$. The required values are presented in the table.

The values of f increase without bound as x becomes infinitely large, it follows that:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{3x^2+2}{x-3}=\infty$$

x	$f(x)$
10	43
50	160
100	309
1000	3009
10000	30009
1000000	3000009

3. In many cases where substituting $x = a$ into the function results in the indeterminate case of $\frac{0}{0}$, this means both numerator and denominator have a common factor $x - a$. The task then is to extract this common factor and cancel it for values of $x \neq a$, and then re-evaluate the remaining expression for $x = a$.

(a) $\lim\limits_{x\to 4}\frac{x-4}{x^2-16} = \frac{0}{0}$, which means that the common factor is $x-4$, which has to be cancelled before we can evaluate the function.

$$\lim_{x\to 4}\frac{x-4}{x^2-16} = \lim_{x\to 4}\frac{1\cancel{x-4}}{(\cancel{x-4})(x+4)} = \lim_{x\to 4}\frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$$

(b) $$\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+1)} = \lim_{x\to 1}\frac{x+2}{x+1} = \frac{1+2}{1+1} = \frac{3}{2}$$

(c) When substituting $x = 0$ into the expression of the limit, the undefined case $\frac{0}{0}$ is obtained. This means that the expression of the limit needs to be manipulated further in order for the limit to be evaluated. The solution is to multiply both the numerator and the denominator by the conjugate of the numerator:

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0}\frac{(\sqrt{2+x})^2-(\sqrt{2})^2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x\to 0}\frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2+0}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

(d) $$\lim_{x\to\infty}\frac{x^3-1}{4x^3-3x+1} = \lim_{x\to\infty}\frac{x^3(1-\frac{1}{x^3})}{x^3(4-\frac{3}{x^2}+\frac{1}{x^3})} = \lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{4-\frac{3}{x^2}+\frac{1}{x^3}} = \frac{1-0}{4-0+0} = \frac{1}{4}$$

(e) To find this limit, another limit has to be used, namely $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

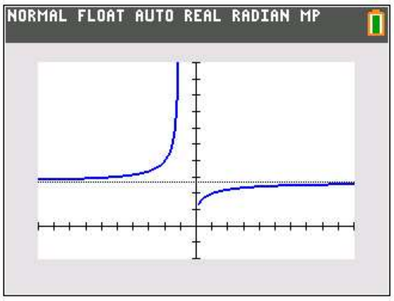
(You are allowed to use this result.)

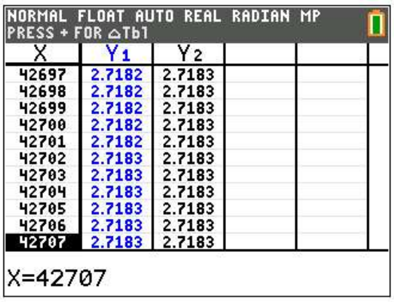
$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to 0}\frac{\sin x}{x}\times\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{1}{\cos x} = 1\times\frac{1}{\cos 0} = 1$$

(f) For solving this limit, the same limit as in part **(e)** has to be used:

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3x}{x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}3 = 1\cdot 3 = 3$$

4. Let $f(x)=\left(1+\dfrac{1}{x}\right)^x$. The graph of $y=f(x)$ and the corresponding table of values, obtained from a GDC, are shown below:





NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1	Y2
42697	2.7182	2.7183
42698	2.7182	2.7183
42699	2.7182	2.7183
42700	2.7182	2.7183
42701	2.7182	2.7183
42702	2.7183	2.7183
42703	2.7183	2.7183
42704	2.7183	2.7183
42705	2.7183	2.7183
42706	2.7183	2.7183
42707	2.7183	2.7183

X=42707

By inspection, the graph has a horizontal asymptote, as confirmed by the table of values. The value entered in Y_2 and displayed as 2.7183, is $\text{e}=2.7182818...$(The GDC can only display six characters in the table, so all values are rounded such that they fit.) It follows that the horizontal asymptote is $y=\text{e}$.

This means that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=\text{e}$, or, more general, $\lim\limits_{x\to x_0}\left(1+\dfrac{1}{g(x)}\right)^{g(x)}=\text{e}$, where x_0 is a value such that $\lim\limits_{x\to x_0}g(x)=\infty$.

5. If function f has a horizontal asymptote with equation $y=3$, then the values of f approach 3 when x becomes infinitely large either positively or negatively. It follows that:

(a) $\lim\limits_{x\to\infty}f(x)=3$

(b) $\lim\limits_{x\to-\infty}f(x)=3$

6. If function g has a vertical asymptote with equation $x=a$, then it has a denominator containing a factor of $(x-a)$. When the values of x approach value a, the values of $(x-a)$ decrease towards 0 (as $g(x)>0$). This causes the denominator of the function to have a factor which is almost 0, which in turn causes the values of g to increase without bound. It follows that $\lim\limits_{x\to a}g(x)=\infty$.

7. **(a)** $$\lim_{x\to\pm\infty}\frac{3x-1}{1+x}=\lim_{x\to\pm\infty}\frac{x(3-\frac{1}{x})}{x(\frac{1}{x}+1)}=\lim_{x\to\pm\infty}\frac{3-\frac{1}{x}}{\frac{1}{x}+1}=3$$

The value of the limit is finite, so the function has a horizontal asymptote, $y=3$.

When investigating the existence of vertical asymptotes, the denominator of the expression must be considered:

$$x+1=0\Rightarrow x=-1$$

The limit when $x\to-1$ has to be evaluated when the values of x increase and decrease towards -1:

$$\lim_{\substack{x\to-1\\x<-1}}\frac{3x-1}{1+x}=+\infty$$

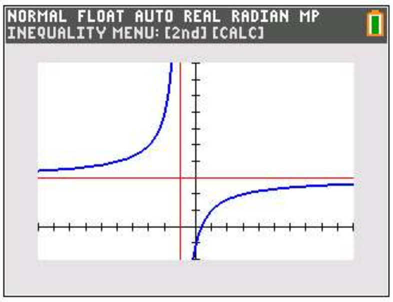
(The numerator is negative, while the denominator takes negative values increasing toward 0).

$$\lim_{\substack{x\to-1\\x>-1}}\frac{3x-1}{1+x}=-\infty$$

(Because the numerator is negative, while the denominator takes positive values decreasing toward 0).

Consequently, due to the fact that both limits are infinite, there is a vertical asymptote, $x=-1$.

A graph confirming these findings is shown below, where the asymptotes are drawn in red:



(b) $$\lim_{x\to\pm\infty}\frac{1}{(x-2)^2}=0$$

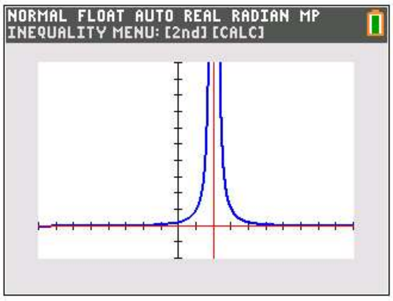
The function has a horizontal asymptote, $y=0$. $x-2=0\Rightarrow x=2$

To investigate the existence of a vertical asymptote, the limit of the function when $x \to 2$ has to be evaluated:

$$\lim_{\substack{x\to 2\\ x<2}} \frac{1}{(x-2)^2} = +\infty$$

Consequently, there is a vertical asymptote, $x = 2$

The graph is shown below:



(c) $$\lim_{x\to\pm\infty} \left(\frac{1}{x-a} + b\right) = \lim_{x\to\pm\infty} \frac{1}{x-a} + \lim_{x\to\pm\infty} b = 0 + b = b$$

$\Rightarrow$the function has a horizontal asymptote, $y = b$

$x - a = 0 \Rightarrow x = a$

The limit of the function when $x \to a$ has to be evaluated when the values of x increase and decrease towards a:

$$\lim_{\substack{x\to a\\ x<a}} \left(\frac{1}{x-a} + b\right) = \lim_{\substack{x\to a\\ x<a}} \left(\frac{1}{x-a}\right) + \lim_{\substack{x\to a\\ x<a}} b = -\infty + b = -\infty$$

$$\lim_{\substack{x\to a\\ x>a}} \left(\frac{1}{x-a} + b\right) = \lim_{\substack{x\to a\\ x>a}} \frac{1}{x-a} + \lim_{\substack{x\to a\\ x>a}} b = +\infty + b = +\infty$$

Consequently, due to the fact that both limits are infinite, there is a vertical asymptote, $x = a$.

(d) $$\lim_{x\to\pm\infty}\frac{2x^2-3}{x^2-9}=\lim_{x\to\pm\infty}\frac{x^2(2-\frac{3}{x^2})}{x^2(1-\frac{9}{x^2})}=\lim_{x\to\pm\infty}\frac{2-\frac{3}{x^2}}{1-\frac{9}{x^2}}=2$$

$\Rightarrow$the function has a horizontal asymptote, $y=2$

$x^2-9=0\Rightarrow x=\pm3$

The limits when $x\to\pm3$ have to be evaluated when the values of x increase and decrease towards 3 and –3, respectively:

$$\lim_{\substack{x\to3\\x<3}}\frac{2x^2-3}{x^2-9}=\lim_{\substack{x\to3\\x<3}}\frac{2x^2-3}{(x-3)(x+3)}=\lim_{\substack{x\to3\\x<3}}\frac{2x^2-3}{x+3}\cdot\lim_{\substack{x\to3\\x<3}}\frac{1}{x-3}=\frac{15}{6}\cdot-\infty=-\infty$$

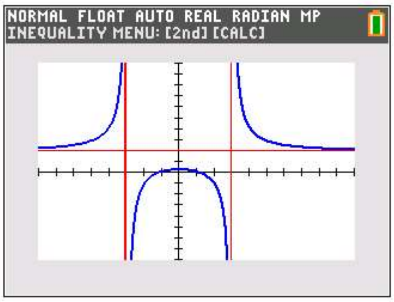
$$\lim_{\substack{x\to3\\x>3}}\frac{2x^2-3}{x^2-9}=\lim_{\substack{x\to3\\x>3}}\frac{2x^2-3}{(x-3)(x+3)}=\lim_{\substack{x\to3\\x>3}}\frac{2x^2-3}{x+3}\cdot\lim_{\substack{x\to3\\x>3}}\frac{1}{x-3}=\frac{15}{6}\cdot+\infty=+\infty$$

$$\lim_{\substack{x\to-3\\x<-3}}\frac{2x^2-3}{x^2-9}=\lim_{\substack{x\to-3\\x<-3}}\frac{2x^2-3}{(x-3)(x+3)}=\lim_{\substack{x\to-3\\x<-3}}\frac{2x^2-3}{x-3}\cdot\lim_{\substack{x\to-3\\x<-3}}\frac{1}{x+3}=\frac{15}{-6}\cdot-\infty=+\infty$$

$$\lim_{\substack{x\to-3\\x>-3}}\frac{2x^2-3}{x^2-9}=\lim_{\substack{x\to-3\\x>-3}}\frac{2x^2-3}{(x-3)(x+3)}=\lim_{\substack{x\to-3\\x>-3}}\frac{2x^2-3}{x-3}\cdot\lim_{\substack{x\to-3\\x>-3}}\frac{1}{x+3}=\frac{15}{-6}\cdot+\infty=-\infty$$

Consequently, there are two vertical asymptotes, $x=3$ and $x=-3$

The graph is shown below:



(e) $$\lim_{x\to\pm\infty}\frac{5-3x}{x^2-5x}=\lim_{x\to\pm\infty}\frac{x\left(\frac{5}{x}-3\right)}{x^2\left(1-\frac{5}{x}\right)}=\lim_{x\to\pm\infty}\frac{\frac{5}{x}-3}{x\left(1-\frac{5}{x}\right)}=\lim_{x\to\pm\infty}\frac{1}{x}\cdot\lim_{x\to\pm\infty}\frac{\frac{5}{x}-3}{1-\frac{5}{x}}=0\cdot(-3)=0$$

$\Rightarrow$the function has a horizontal asymptote, $y=0$

$x^2-5x=0\Rightarrow x(x-5)=0\Rightarrow x=0, x=5$

The limits when $x\to 0$ and when $x\to 5$ have to be evaluated when the values of x increase and decrease towards 0 and 5, respectively:

$$\lim_{\substack{x\to 0\\ x<0}}\frac{5-3x}{x^2-5x}=\lim_{\substack{x\to 0\\ x<0}}\frac{5-3x}{x(x-5)}=\lim_{\substack{x\to 0\\ x<0}}\frac{1}{x}\cdot\lim_{\substack{x\to 0\\ x<0}}\frac{5-3x}{x-5}=-\infty\cdot(-1)=+\infty$$

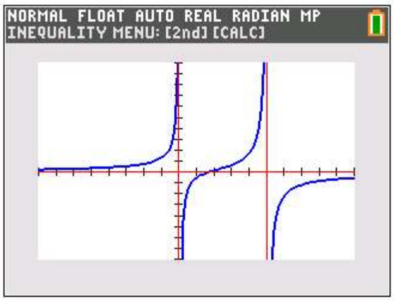
$$\lim_{\substack{x\to 0\\ x>0}}\frac{5-3x}{x^2-5x}=\lim_{\substack{x\to 0\\ x>0}}\frac{5-3x}{x(x-5)}=\lim_{\substack{x\to 0\\ x>0}}\frac{1}{x}\cdot\lim_{\substack{x\to 0\\ x>0}}\frac{5-3x}{x-5}=+\infty\cdot(-1)=-\infty$$

$$\lim_{\substack{x\to 5\\ x<5}}\frac{5-3x}{x^2-5x}=\lim_{\substack{x\to 5\\ x<5}}\frac{5-3x}{x(x-5)}=\lim_{\substack{x\to 5\\ x<5}}\frac{1}{x-5}\cdot\lim_{\substack{x\to 5\\ x<5}}\frac{5-3x}{x}=-\infty\cdot(-2)=+\infty$$

$$\lim_{\substack{x\to 5\\ x>5}}\frac{5-3x}{x^2-5x}=\lim_{\substack{x\to 5\\ x>5}}\frac{5-3x}{x(x-5)}=\lim_{\substack{x\to 5\\ x>5}}\frac{1}{x-5}\cdot\lim_{\substack{x\to 5\\ x>5}}\frac{5-3x}{x}=+\infty\cdot(-2)=-\infty$$

Consequently, there are two vertical asymptotes, $x=0, x=5$

The graph is shown below:



(f) $$\lim_{x\to\pm\infty}\frac{x^2-4}{x-4}=\lim_{x\to\pm\infty}\frac{x^2\left(1-\frac{4}{x^2}\right)}{x\left(1-\frac{4}{x}\right)}=\lim_{x\to\pm\infty}\frac{x\left(1-\frac{4}{x^2}\right)}{1-\frac{4}{x}}=\lim_{x\to\pm\infty}x\cdot\lim_{x\to\pm\infty}\frac{1-\frac{4}{x^2}}{1-\frac{4}{x}}=\pm\infty\cdot 1=\pm\infty$$

$\Rightarrow$the value of the limit is infinite, so the function does not have a horizontal asymptote. $x-4=0\Rightarrow x=4$

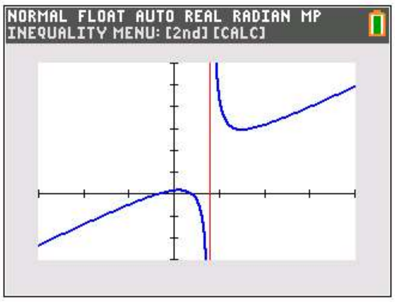
The limit when $x\to 4$ has to be evaluated when the values of x increase and decrease towards 4:

$$\lim_{\substack{x\to 4\\ x<4}}\frac{x^2-4}{x-4}=\lim_{\substack{x\to 4\\ x<4}}(x^2-4)\cdot\lim_{\substack{x\to 4\\ x<4}}\frac{1}{x-4}=12\cdot(-\infty)=-\infty$$

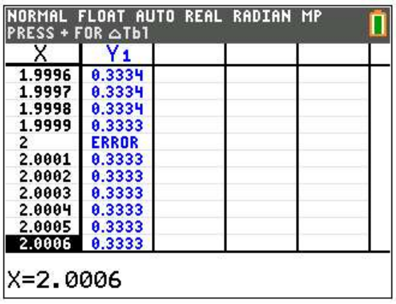
$$\lim_{\substack{x\to 4\\ x>4}}\frac{x^2-4}{x-4}=\lim_{\substack{x\to 4\\ x>4}}(x^2-4)\cdot\lim_{\substack{x\to 4\\ x>4}}\frac{1}{x-4}=12\cdot(+\infty)=+\infty$$

Consequently, due to the fact that both limits are infinite, there is a vertical asymptote, $x=4$.

The graph is shown below:



8. **(a)** **(i)** The values displayed by the GDC when given the function $f(x)=\frac{\sqrt{x^2+5}-3}{x^2-2x}$ suggest that $\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}\approx 0.3333$, or $\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}=\frac{1}{3}$, when taking into account the limitations of the calculator regarding rounding.



```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl
   X    | Y1
1.9996  | 0.3334
1.9997  | 0.3334
1.9998  | 0.3334
1.9999  | 0.3333
2       | ERROR
2.0001  | 0.3333
2.0002  | 0.3333
2.0003  | 0.3333
2.0004  | 0.3333
2.0005  | 0.3333
2.0006  | 0.3333
X=2.0006
```

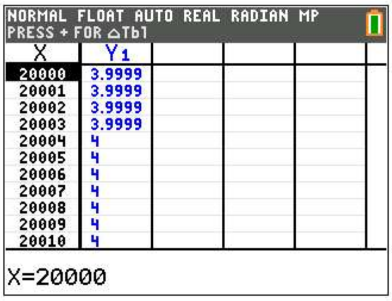
(ii)

$$\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}=\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}\cdot\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}=\lim_{x\to 2}\frac{x^2+5-9}{x(x-2)(\sqrt{x^2+5}+3)}$$

$$=\lim_{x\to 2}\frac{x^2-4}{x(x-2)\sqrt{x^2+5}+3}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x(x-2)(\sqrt{x^2+5}+3)}$$

$$=\lim_{x\to 2}\frac{x+2}{x(\sqrt{x^2+5}+3)}=\frac{2+2}{2(\sqrt{2^2+5}+3)}=\frac{4}{2\cdot 6}=\frac{1}{3}$$

(b) **(i)** Based on the values generated by a GDC for the function $f(x)=\dfrac{4x-1}{\sqrt{x^2+2}}$, the value of the limit can be estimated to be 4.



```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl
   X    | Y1
20000   | 3.9999
20001   | 3.9999
20002   | 3.9999
20003   | 3.9999
20004   | 4
20005   | 4
20006   | 4
20007   | 4
20008   | 4
20009   | 4
20010   | 4
X=20000
```

(ii)

$$\lim_{x\to+\infty}\frac{4x-1}{\sqrt{x^2+2}}=\lim_{x\to+\infty}\frac{x\left(4-\frac{1}{x}\right)}{\sqrt{x^2\left(1+\frac{2}{x^2}\right)}}=\lim_{x\to+\infty}\frac{x\left(4-\frac{1}{x}\right)}{x\sqrt{1+\frac{2}{x^2}}}=\lim_{x\to+\infty}\frac{4-\frac{1}{x}}{\sqrt{1+\frac{2}{x^2}}}=4$$

9. The variable in the limit is h, so x is treated as a constant. Multiply both the numerator and the denominator of the expression by the conjugate of the numerator, to remove the undefined case $\frac{0}{0}$:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

10. This is an indeterminate case of the form $\frac{0}{0}$.

$$\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\frac{x-x-h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\frac{-h}{hx(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=\frac{-1}{x(x+0)}=-\frac{1}{x^2}$$

Exercise 12.2

1. **(a)** From first principles:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1-(x+h)^2-(1-x^2)}{h}$$

$$=\lim_{h\to 0}\frac{1-x^2-2xh-h^2-1+x^2}{h}=\lim_{h\to 0}\frac{-2xh-h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(-2x-h)}{h}=\lim_{h\to 0}(-2x-h)=-2x-0=-2x$$

(b) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3+2-(x^3+2)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+2-x^3-2}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2+0+0=3x^2$$

(c) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

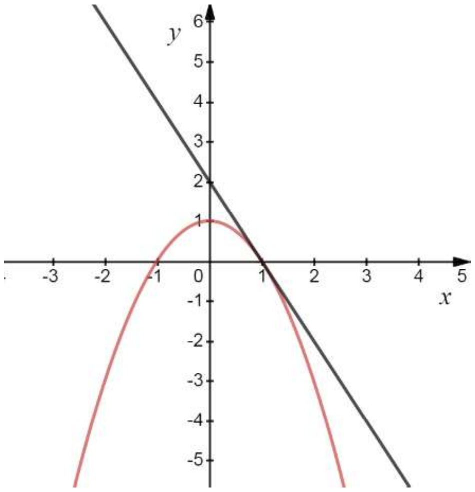
(d) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}}{h}=\lim_{h\to 0}\frac{-2xh-h^2}{x^2(x+h)^2}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{h(-2x-h)}{x^2(x+h)^2}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}=\frac{-2x-0}{x^2(x+0)^2}=\frac{-2x}{x^4}=-\frac{2}{x^3}$$

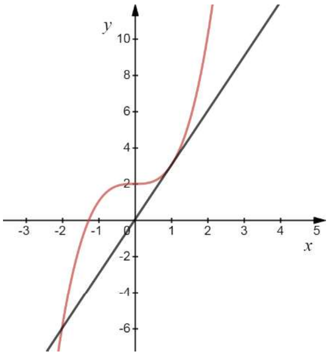
2. **(a)** The slope of the curve at the point where $x=1$ is $f'(1)=-2(1)=-2$.

The graph of the function and the required tangent are shown below:



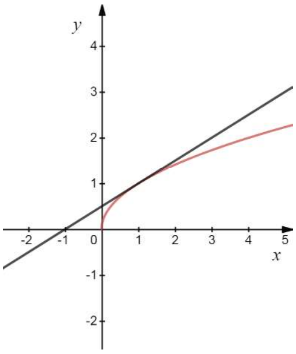
(b) The slope of the curve at the point where $x=1$ is $f'(1)=3\cdot 1^2=3$.

The graph is shown below:



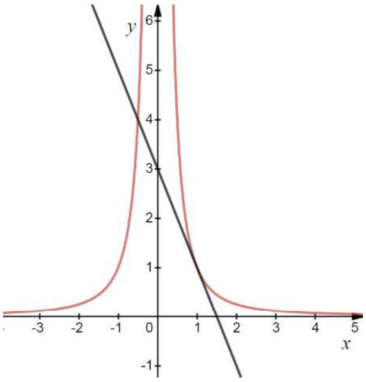
(c) The slope of the curve at the point where $x=1$ is $f'(1)=\dfrac{1}{2\sqrt{1}}=\dfrac{1}{2}$.

The graph is shown below:



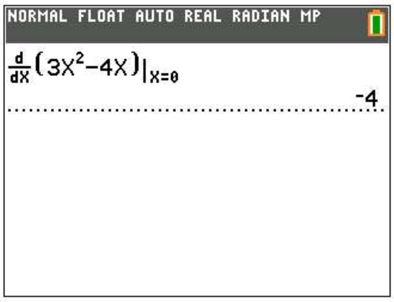
(d) The slope of the curve at the point where $x=1$ is $f'(1)=-\dfrac{2}{1^3}=-2$.

The graph is shown below:



3. **(a)** **(i)** $y' = 3 \cdot 2x - 4 \cdot 1 \Rightarrow y' = 6x - 4$

(ii) $y'(0) = 6(0) - 4 \Rightarrow y'(0) = -4$

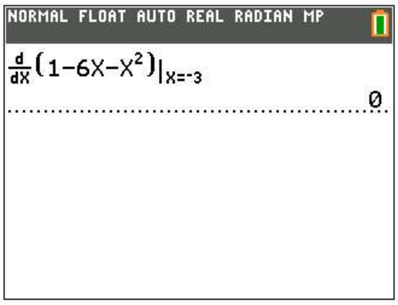


$\frac{d}{dX}(3X^2-4X)|_{X=0}$

$^-4$

(b) **(i)** $y' = 0 - 6 \cdot 1 - 2x \Rightarrow y' = -6 - 2x$

(ii) $y'(-3) = -6 - 2(-3) \Rightarrow y'(-3) = 0$

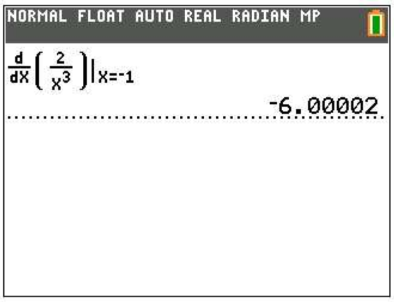


$\frac{d}{dX}(1-6X-X^2)|_{X=-3}$

0

(c) **(i)** $y = 2x^{-3} \Rightarrow y' = 2(-3)x^{-4} \Rightarrow y' = -\frac{6}{x^4}$

(ii) $y'(-1) = -\frac{6}{(-1)^4} \Rightarrow y'(-1) = -6$

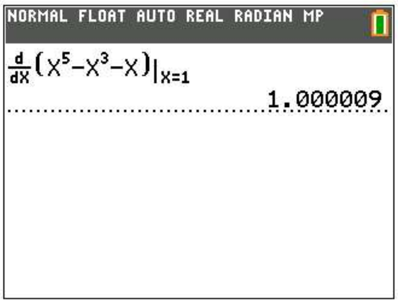


$\frac{d}{dX}\left(\frac{2}{X^3}\right)|_{X=-1}$

$^-6.00002$

(d) **(i)** $y' = 5x^4 - 3x^2 - 1$

(ii) $y'(1) = 5 \cdot 1^4 - 3 \cdot 1^2 - 1 \Rightarrow y'(1) = 1$



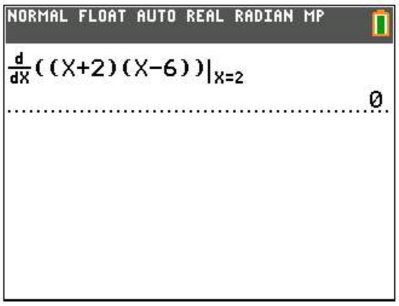
NORMAL FLOAT AUTO REAL RADIAN MP

$\frac{d}{dX}(X^5-X^3-X)|_{X=1}$

1.000009

(e) **(i)** $y = x^2 - 4x - 12 \Rightarrow y' = 2x - 4$

(ii) $y'(2) = 2 \cdot 2 - 4 \Rightarrow y' = 0$



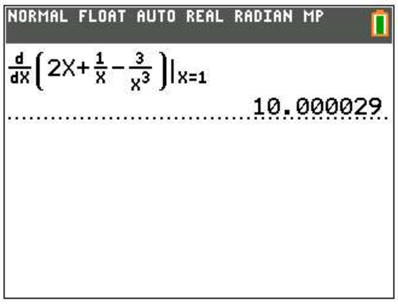
NORMAL FLOAT AUTO REAL RADIAN MP

$\frac{d}{dX}((X+2)(X-6))|_{X=2}$

0

(f) **(i)** $y = 2x + x^{-1} - 3x^{-3} \Rightarrow y' = 2 \cdot 1 + (-1)x^{-2} - 3(-3)x^{-4} \Rightarrow y' = 2 - \dfrac{1}{x^2} + \dfrac{9}{x^4}$

(ii) $y'(1) = 2 - \dfrac{1}{1^2} + \dfrac{9}{1^4} \Rightarrow y'(1) = 10$



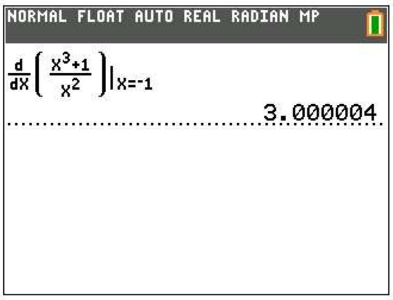
NORMAL FLOAT AUTO REAL RADIAN MP

$\frac{d}{dX}\left(2X+\frac{1}{X}-\frac{3}{X^3}\right)|_{X=1}$

10.000029

(g) **(i)** $y=\dfrac{x^3+1}{x^2}\Rightarrow y'=x+x^{-2}\Rightarrow y'=1+(-2)x^{-3}\Rightarrow y'=1-\dfrac{2}{x^3}$

(ii) $y'(-1)=1-\dfrac{2}{(-1)^3}\Rightarrow y'=3$



NORMAL FLOAT AUTO REAL RADIAN MP

$\frac{d}{dX}\left(\frac{X^3+1}{X^2}\right)|_{X=-1}$

3.000004

4. The point with coordinates $(2,-4)$ is on the curve representing the given function, this means its coordinates satisfy the equation $y=x^2+ax+b$:

$-4=2^2+a(2)+b\Rightarrow 2a+b=-8$

Next, the first derivative must be found:

$y=x^2+ax+b\Rightarrow y'=2x+a$

The slope of the curve at $x=2$ is -1:

$y'(2)=-1\Rightarrow 2\cdot 2+a=-1\Rightarrow a=-5$

To find *b, s*ubstitute the value of *a* into the first equation:

$2(-5)+b=-8\Rightarrow b=2$

5. In the following questions, first find the expression of the first derivative, then equate it to the given value to obtain an equation, and solve for *x*. Finally, substitute the *x*-value(s) into the original function to find the *y*-coordinate of the required point(s).

(a) $y=x^2+3x\Rightarrow y'=2x+3$

$3=2x+3\Rightarrow 2x=0\Rightarrow x=0\Rightarrow y=0^2+3\cdot 0\Rightarrow y=0$

The required point is $(0,0)$

(b) $y'=3x^2$

$12=3x^2\Rightarrow x^2=4\Rightarrow x=\pm 2\Rightarrow y=(\pm 2)^3\Rightarrow y=\pm 8$

The required points are $(2,8)$ and $(-2,-8)$

(c) $y' = 2x - 5$

$0 = 2x - 5 \Rightarrow 2x = 5 \Rightarrow x = \frac{5}{2} \Rightarrow y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1 \Rightarrow y = -\frac{21}{4}$

The required point is $\left(\frac{5}{2}, -\frac{21}{4}\right)$

(d) $y' = 2x - 3$

$-1 = 2x - 3 \Rightarrow 2x = 2 \Rightarrow x = 1 \Rightarrow y = 1^2 - 3 \cdot 1 \Rightarrow y = -2$

The required point is $(1, -2)$

6. **(a)** Between *A* and *B* is where the curve is the steepest, which means the slope of the curve is the greatest.

(b) **(i)** When the instantaneous rate of change is positive, the values of the function are increasing, so the points are *A*, *B* and *F*.

(ii) When the instantaneous rate of change is negative, the values of the function are decreasing, so the points are *D* and *E*.

(iii) The rate of change is 0 at a turning point, so the required point is *C*.

(c) *B* and *D,* and *E* and *F*, as the line segments joining the points in each pair have approximately the same slope.

7. The slope of the curve $y = x^2 - 4x + 6$ is given by the first derivative:

$y' = 2x - 4$

At the point where $x = 3$ the slope is:

$y'(3) = 2 \cdot 3 - 4 \Rightarrow y'(3) = 2$

The same applies for the second curve:

$y = 8x - 3x^2 \Rightarrow y' = 8 - 6x \Rightarrow y'(a) = 8 - 6a$

The two slopes are equal:

$8 - 6a = 2 \Rightarrow -6a = -6 \Rightarrow a = 1$

To find the value of *b*, substitute $a = 1$ into the equation of the second curve (the required point lies on this curve):

$b = 8(1) - 3 \cdot 1^2 \Rightarrow b = 5$

8. Find the first derivative of the given function:

$y=ax^3-2x^2-x+7\Rightarrow y'=3ax^2-4x-1$

Substitute $x=2$ into this equation to find the slope at the indicated point:

$y'(2)=3a\cdot 2^2-4\cdot 2-1\Rightarrow y'(2)=12a-9$

It is given that the slope is 3, so equate the expression for the slope to 3 and solve for *a*:

$3=12a-9\Rightarrow 12a=12\Rightarrow a=1$

9. It is known that two parallel lines have the same gradient, this means that, at the point (a,b) on the graph of the function, the gradient of the curve (which is the same as the gradient of the tangent line), must be the same as the gradient of the given line, namely 5. The gradient of the curve is given by the first derivative:

$y=x^2-x\Rightarrow y'=2x-1$

The gradient of the curve at $x=a$ is 5:

$5=2a-1\Rightarrow 2a=6\Rightarrow a=3$

The *y*-coordinate of the required point is calculated by substituting $a=\frac{1}{3}$ into the expression of the original function:

$b=3^2-3\Rightarrow b=6$

The required point is $(3,\ 6)$

10. **(a)** $f(x)=x^3+1,\ h=0.1\Rightarrow \dfrac{f(2+0.1)-f(2)}{0.1}=\dfrac{2.1^3+1-(2^3+1)}{0.1}=\dfrac{10.261-9}{0.1}=12.61$

(b) The expression $\dfrac{f(2+h)-f(2)}{h}$ gives the value of the first derivative of *f* at the point where $x=2, f'(2)$.

$f'(x)=3x^2\Rightarrow f'(2)=3\cdot 2^2\Rightarrow f'(2)=12$

The given expression approaches 12 when $x=2$ and $h=0.1$.

11. **(a)** $f(x)=ax^2+bx+c\Rightarrow f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$f'(x)=\lim_{h\to 0}\frac{a(x+h)^2+b(x+h)+c-(ax^2+bx+c)}{h}$$

$$=\lim_{h\to 0}\frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h}$$

$$=\lim_{h\to 0}\frac{2axh+ah^2+bh}{h}=\lim_{h\to 0}(2ax+ah+b)=2ax+b$$

(b) When using the expression found in **(a)**, the respective derivatives are:

$a=1, b=0, c=0 \Rightarrow f'(x)=2\cdot 1x+0 \Rightarrow f'(x)=2x$

$a=3, b=-4, c=2 \Rightarrow f'(x)=2\cdot 3x-4 \Rightarrow f'(x)=6x-4$

When calculating the derivatives directly, the following expressions are obtained:

$a=1, b=0, c=0 \Rightarrow f(x)=x^2 \Rightarrow f'(x)=2x$

$a=3, b=-4, c=2 \Rightarrow f(x)=3x^2-4x+2 \Rightarrow f'(x)=6x-4$

In both cases, the expressions found using direct differentiation are the same as the ones found by using first principles.

12. **(a)** When $t=1, C=2\sqrt{1^3}+17=19°C$ and when $t=4, C=2\sqrt{4^3}+17=33°C$.

The required average rate of change is the gradient of the line going through points $(1,19)$ and $(4,33)$.

$$\Rightarrow \text{average rate of change}=\frac{33-19}{4-1}=\frac{14}{3}=4.\dot{6} \text{ degrees Celsius per hour.}$$

(b) The first derivative of C has to be found:

$$C=2t^{\frac{3}{2}}+17 \Rightarrow C'=2\cdot\frac{3}{2}t^{\frac{1}{2}} \Rightarrow C'=3\sqrt{t}$$

(c) $3\sqrt{t}=\frac{14}{3} \Rightarrow \sqrt{t}=\frac{14}{9} \Rightarrow t=\frac{196}{81} \Rightarrow t=2.419... \Rightarrow t\approx 2.42$ hours

13. **(a)** The derivative of $h(x)$ is $h'(x)=\lim\limits_{h\to 0}\frac{h(x+h)-h(x)}{h}$

The derivative of $h(-x)$ can be found from first principles, remembering that the denominator is the difference between the two x-values:

$$h'(-x)=\lim_{h\to 0}\frac{h(-(x+h))-h(-x)}{-h}=\lim_{h\to 0}\frac{h((x+h))-h(x)}{-h}=-h'(x)$$

This means $h'(-x)=-h'(x)$, which shows that h' is odd.

(b) Use a similar method:

$$p'(-x)=\lim_{h\to 0}\frac{p(-(x+h))-p(-x)}{-h}=\lim_{h\to 0}\frac{-p((x+h))+p(x)}{-h}$$

$$=\lim_{h\to 0}\frac{p((x+h))-p(x)}{h}$$

This means $p'(-x)=p'(x)$, which shows that p' is even.

14. Let $L = \lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

The expression $\dfrac{\cos x - 1}{x}$ is undefined when $x \to 0$, as it results in $\dfrac{0}{0}$

One of the trigonometric identities for the cosine of the double angle has to be used, namely:

$$\cos x = 1 - 2\sin^2 \frac{x}{2} \Rightarrow \cos x - 1 = -2\sin^2 \frac{x}{2}$$

$$\Rightarrow L = \lim_{x \to 0} \frac{-2\sin^2 \frac{x}{2}}{x} = -\lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x}{2}}$$

The given limit $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ has to be used now:

$$\Rightarrow L = -\lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x}{2}} = -\lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \sin \frac{x}{2} \right) = -\lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right) \cdot \lim_{x \to 0} \left(\sin \frac{x}{2} \right) = -1 \times 0 = 0$$

$$\Rightarrow \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

15. **(a)** Applying first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x - x - h}{x(x+h)}}{h} = \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

(b) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{2+(x+h)}{3-(x+h)}-\dfrac{2+x}{3-x}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{(2+(x+h))(3-x)-(3-(x+h))(2+x)}{(3-x)(3-(x+h))}}{h}$$

$$=\lim_{h\to 0}\frac{6+3(x+h)-2x-x(x+h)-6+2(x+h)-3x+x(x+h)}{(3-x)(3-(x+h))}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{3(x+h)-2x+2(x+h)-3x}{(3-x)(3-(x+h))}\cdot\frac{1}{h}=\lim_{h\to 0}\frac{5h}{(3-x)(3-(x+h))}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{5}{(3-x)(3-(x+h))}=\frac{5}{(3-x)(3-(x+0))}=\frac{5}{(3-x)^2}$$

(c) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h+2}}-\dfrac{1}{\sqrt{x+2}}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{\sqrt{x+2}-\sqrt{x+h+2}}{\sqrt{x+2}\sqrt{x+h+2}}}{h}=\lim_{h\to 0}\frac{\sqrt{x+2}-\sqrt{x+h+2}}{\sqrt{x+2}\sqrt{x+h+2}}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+2}-\sqrt{x+h+2}}{\sqrt{x+2}\sqrt{x+h+2}}\cdot\frac{\sqrt{x+2}+\sqrt{x+h+2}}{\sqrt{x+2}+\sqrt{x+h+2}}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+2})^2-\left(\sqrt{x+h+2}\right)^2}{\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{x+2-x-h-2}{\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{-h}{\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}$$

$$=\frac{-1}{2\left(\sqrt{x+2}\right)^3}=-\frac{1}{2\sqrt{(x+2)^3}}$$

16. Let $f(x) = c$, where c is a constant, $c \in \mathbb{R}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Exercise 12.3

1. **(a)** The vertex is a stationary point; it follows that its *x*-coordinate can be found by solving the equation $y' = 0$:

$y = x^2 - 2x - 6 \Rightarrow y' = 2x - 2$

$0 = 2x - 2 \Rightarrow x = 1$

The corresponding *y*-coordinate is:

$y = 1^2 - 2\cdot 1 - 6 \Rightarrow y = -7 \Rightarrow$ the vertex is $(1, -7)$

(b) $y = 4x^2 + 12x + 17 \Rightarrow y' = 8x + 12$

$0 = 8x + 12 \Rightarrow x = -\frac{3}{2}$

$y = 4(-\frac{3}{2})^2 + 12(-\frac{3}{2}) + 17 \Rightarrow y = 8 \Rightarrow$ the vertex is $(-\frac{3}{2}, 8)$

(c) $y = -x^2 + 6x - 7 \Rightarrow y' = -2x + 6$

$0 = -2x + 6 \Rightarrow x = 3$

$y = -3^2 + 6\cdot 3 - 7 \Rightarrow y = 2 \Rightarrow$ the vertex is $(3, 2)$

2. **(a)** **(i)** $f(x) = x^2 - 5x + 6 \Rightarrow f'(x) = 2x - 5$

(ii) $f'(x) > 0 \Rightarrow 2x - 5 > 0 \Rightarrow x > \frac{5}{2} \Rightarrow f$ is increasing for $x > \frac{5}{2}$

(iii) $f'(x) < 0 \Rightarrow 2x - 5 < 0 \Rightarrow x < \frac{5}{2} \Rightarrow f$ is decreasing for $x < \frac{5}{2}$

(b) **(i)** $f(x) = 7 - 4x - 3x^2 \Rightarrow f'(x) = -4 - 6x$

(ii) $f'(x) > 0 \Rightarrow -4 - 6x > 0 \Rightarrow x < -\frac{2}{3} \Rightarrow f$ is increasing for $x < -\frac{2}{3}$

(iii) $f'(x) < 0 \Rightarrow -4 - 6x < 0 \Rightarrow x > -\frac{2}{3} \Rightarrow f$ is decreasing for $x > -\frac{2}{3}$

(c) **(i)** $f(x)=\frac{1}{3}x^3-x \Rightarrow f'(x)=x^2-1$

(ii) $f'(x)>0 \Rightarrow x^2-1>0 \Rightarrow |x|>1 \Rightarrow x<-1, x>1 \Rightarrow f$ is increasing for $x<-1, x>1$

(iii) $f'(x)<0 \Rightarrow x^2-1<0 \Rightarrow |x|<1 \Rightarrow -1<x<1 \Rightarrow f$ is decreasing for $-1<x<1$

(d) **(i)** $f(x)=x^4-4x^3 \Rightarrow f'(x)=4x^3-12x^2$

(ii) $f'(x)>0 \Rightarrow 4x^2(x-3)>0 \Rightarrow x-3>0 \Rightarrow x>3 \Rightarrow f$ is increasing for $x>3$

(iii) $f'(x)<0 \Rightarrow 4x^2(x-3)<0 \Rightarrow x-3<0 \Rightarrow x<3 \Rightarrow f$ is decreasing for $x<3$ but it has a stationary point at $x = 0$.

3. **(a)** **(i)** To find the stationary points, the equation $y'=0$ must be solved.

$y=2x^3+3x^2-72x+5 \Rightarrow y'=6x^2+6x-72$

$\Rightarrow 0=6x^2+6x-72=6(x^2+x-12)$

$\Rightarrow 6(x+4)(x-3)=0 \Rightarrow x=-4, x=3$

To find the y-coordinates, substitute the two x-values into the expression of the original function:

$x=-4 \Rightarrow y=2(-4)^3+3(-4)^2-72(-4)+5 \Rightarrow y=213$

$x=3 \Rightarrow y=2\cdot 3^3+3\cdot 3^2-72\cdot 3+5 \Rightarrow y=-130$

The stationary points are $(-4,\ 213)$ and $(3,\ 130)$

(ii) To determine the nature of the two stationary points, the second derivative is used:

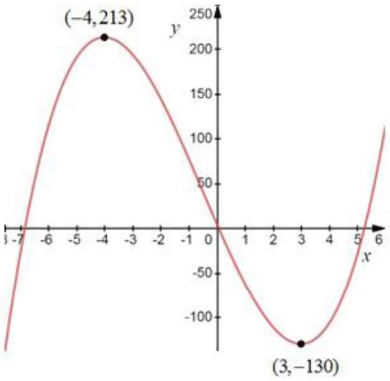
$\Rightarrow y'=6x^2+6x-72 \Rightarrow y''=12x+6$

Substitute the x-coordinates of the stationary points into the expression of y'', to determine the concavity of the function:

$y''(-4)=12(-4)+6=-42<0 \Rightarrow$ the graph of the function is concave down. So, there is a maximum at $(-4, 213)$.

$y''(3)=12\cdot 3+6=42>0 \Rightarrow$ the graph of the function is concave up, so there is a minimum at $(3,-130)$.

(iii) The graph is shown below:



(b) **(i)** $y=\frac{1}{6}x^3-5 \Rightarrow y'=\frac{1}{2}x^2 \Rightarrow 0=\frac{1}{2}x^2 \Rightarrow x=0$

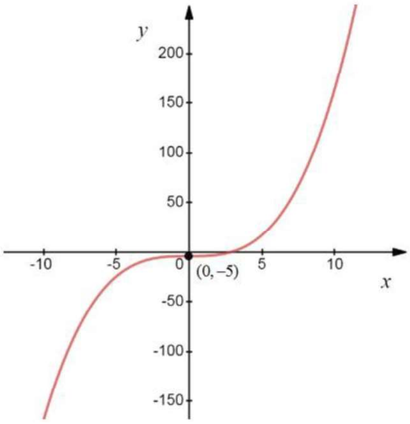
To find the *y*-coordinates, substitute this *x*-value into the expression of the original function:

$x=0 \Rightarrow y=\frac{1}{6}\cdot 0^3-5 \Rightarrow y=-5$

The stationary point is $(0,-5)$.

(ii) As the first derivative is always positive ($2x^2>0$ for all $x\in\mathbb{R}, x\neq 0$), the point where $x=0$ is neither a maximum nor a minimum is a point where the tangent to the graph is horizontal.

(iii) The graph is shown below:



(c) **(i)** $y = x(x-3)^2 \Rightarrow y = x^3 - 6x^2 + 9x \Rightarrow y' = 3x^2 - 12x + 9$

$\Rightarrow 0 = 3x^2 - 12x + 9 \Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow 3(x-1)(x-3) = 0 \Rightarrow x = 1, x = 3$

$x = 1 \Rightarrow y = 1(1-3)^2 \Rightarrow y = 4$

$x = 3 \Rightarrow y = 3(3-3)^2 \Rightarrow y = 0$

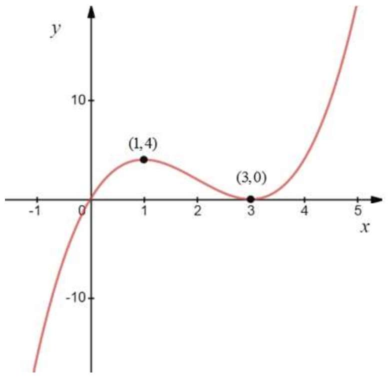
The stationary points are $(1,\ 4)$ and $(3,\ 0)$

(ii) $y' = 3x^2 - 12x + 9 \Rightarrow y'' = 6x - 12$

$y''(1) = 6 \cdot 1 - 12 = -6 < 0 \Rightarrow$ the graph of the function is concave down, so there is a maximum at $(1,\ 4)$.

$y''(3) = 6 \cdot 3 - 12 = 6 > 0 \Rightarrow$ the graph of the function is concave up, so there is a minimum at $(3,\ 0)$.

(iii) The graph is shown below:



(d) **(i)** $y = x^4 - 2x^3 - 5x^2 + 6 \Rightarrow y' = 4x^3 - 6x^2 - 10x$

$\Rightarrow 0 = 4x^3 - 6x^2 - 10x \Rightarrow 2x(2x^2 - 3x - 5) = 0 \Rightarrow 2x(2x-5)(x+1) = 0$

$\Rightarrow x = 0, x = \dfrac{5}{2}, x = -1$

$x = 0 \Rightarrow y = 0^4 - 2 \cdot 0^3 - 5 \cdot 0^2 + 6 \Rightarrow y = 6$

$x = \dfrac{5}{2} \Rightarrow y = \left(\dfrac{5}{2}\right)^4 - 2 \cdot \left(\dfrac{5}{2}\right)^3 - 5 \cdot \left(\dfrac{5}{2}\right)^2 + 6 \Rightarrow y = -\dfrac{279}{16}$

$x = -1 \Rightarrow y = (-1)^4 - 2 \cdot (-1)^3 - 5 \cdot (-1)^2 + 6 \Rightarrow y = 4$

The stationary points are $(0, 6)$, $\left(\dfrac{5}{2}, -\dfrac{279}{16}\right)$ and $(-1, 4)$

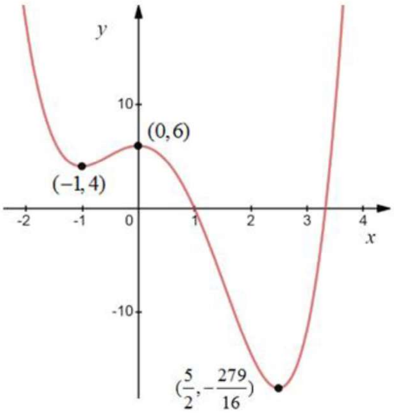
(ii) $y' = 4x^3 - 6x^2 - 10x \Rightarrow y'' = 12x^2 - 12x - 10$

$y''(0) = 12 \cdot 0^2 - 12 \cdot 0 - 10 = -10 < 0 \Rightarrow$ the graph of the function is concave down, so there is a maximum at $(0, 6)$.

$y''\left(\frac{5}{2}\right) = 12\left(\frac{5}{2}\right)^2 - 12\left(\frac{5}{2}\right) - 10 = 35 > 0 \Rightarrow$ the graph of the function is concave up, so there is a minimum at $\left(\frac{5}{2}, -\frac{279}{16}\right)$.

$y''(-1) = 12(-1)^2 - 12(-1) - 10 = 14 > 0 \Rightarrow$ the graph of the function is concave up, so there is a minimum at $(-1, 4)$.

(iii) The graph is shown below:



(e) **(i)** $y = x - x^{\frac{1}{2}} \Rightarrow y' = 1 - \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow y' = 1 - \frac{1}{2\sqrt{x}}$

$\Rightarrow 0 = 1 - \frac{1}{2\sqrt{x}} \Rightarrow 1 = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

$x = \frac{1}{4} \Rightarrow y = \frac{1}{4} - \sqrt{\frac{1}{4}} \Rightarrow y = -\frac{1}{4}$

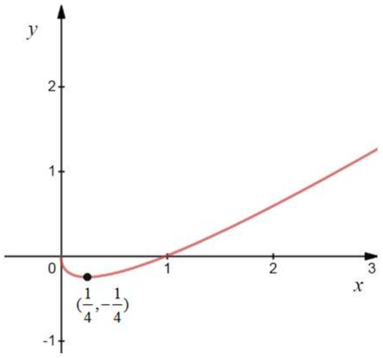
The stationary point is $\left(\frac{1}{4}, -\frac{1}{4}\right)$

(ii) $y'=1-\frac{1}{2}x^{-\frac{1}{2}} \Rightarrow y''=-\frac{1}{2}(-\frac{1}{2})x^{-\frac{3}{2}} \Rightarrow y''=\frac{1}{4\sqrt{x^3}}$

$y''\left(\frac{1}{4}\right)=\frac{1}{4\sqrt{\left(\frac{1}{4}\right)^3}}=2>0 \Rightarrow$ the graph of the function is concave up, so

there is a minimum at $\left(\frac{1}{4},-\frac{1}{4}\right)$.

(iii) The graph is shown below:

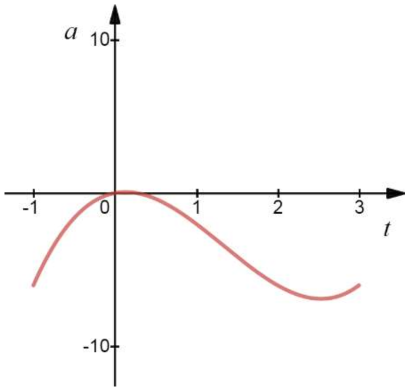


4. **(a)** $v(t)=s'(t),\ a(t)=s''(t)$

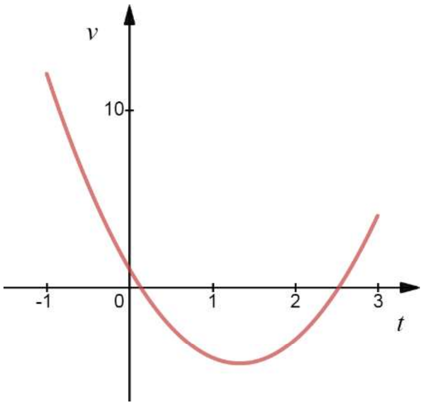
$s(t)=t^3-4t^2+t \Rightarrow s'(t)=3t^2-8t+1 \Rightarrow s''(t)=6t-8$

$\Rightarrow v(t)=3t^2-8t+1,\ a(t)=6t-8$

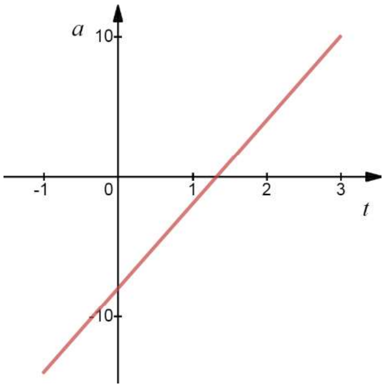
(b) The displacement–time graph is:



The velocity–time graph is:



The acceleration–time graph is:



(c) Displacement is a maximum when $s'(t)=0$

$\Rightarrow 3t^2-8t+1=0 \Rightarrow t=0.1314..., \ t=2.535... \Rightarrow t\approx 0.131, \ t\approx 2.54$

From graph: the maximum occurs when $t\approx 0.131$, the displacement at this time is:

$s(0.1314...)=0.1314...^3-4\cdot 0.1314...^2+0.1314... \Rightarrow s(0.1314...)=0.06460...$

$\Rightarrow s_{\max}\approx 0.0646$

(d) Velocity is a minimum when $v'(t)=0$:

$\Rightarrow 6t-8=0 \Rightarrow t=\frac{4}{3} \Rightarrow t=1.\bar{3}$

The minimum velocity is:

$v\left(\frac{4}{3}\right)=3\left(\frac{4}{3}\right)^2-8\left(\frac{4}{3}\right)+1 \Rightarrow v\left(\frac{4}{3}\right)=-\frac{13}{3} \Rightarrow v_{\min}=-4.\dot{3}$

(e) The object moves in the positive direction with decreasing velocity until it stops at $t \approx 0.131$, after which it accelerates in the opposite (negative) direction until it reaches its maximum velocity at $t = 1.\dot{3}$. The object continues to move in the same direction with decreasing velocity until it rests at $t \approx 2.54$, after which it turns and accelerates in the positive direction.

5. **(a)** **(i)** To find the stationary points, the equation $y' = 0$ must be solved.

$f(x) = x^3 - 12x \Rightarrow f'(x) = 3x^2 - 12$

$0 = 3x^2 - 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

To determine the nature of the two stationary points, the second derivative is used:

$f'(x) = 3x^2 - 12 \Rightarrow f''(x) = 6x$

$x = 2 \Rightarrow f''(2) = 6 \cdot 2 = 12 > 0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

$x = -2 \Rightarrow f''(-2) = 6(-2) = -12 < 0 \Rightarrow$ the curve is concave down, hence the point is a maximum point.

To find the *x*-coordinate of the point of inflection, solve the equation $y'' = 0$:

$f''(x) = 6x \Rightarrow 0 = 6x \Rightarrow x = 0$

(ii) The corresponding *y*-coordinates are:

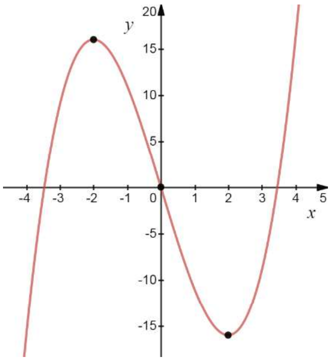
$x = 2 \Rightarrow y = 2^3 - 12 \cdot 2 \Rightarrow y = -16$

$x = -2 \Rightarrow y = (-2)^3 - 12 \cdot (-2) \Rightarrow y = 16$

$x = 0 \Rightarrow y = 0^3 - 12 \cdot 0 \Rightarrow y = 0$

The graph of the function has a maximum at $(-2, 16)$, a minimum at $(2, -16)$ and a point of inflection at $(0, 0)$.

The graph is shown below:



(b) **(i)** Solve $y' = 0$:

$$f(x) = \frac{1}{4}x^4 - 2x^2 \Rightarrow f'(x) = x^3 - 4x$$

$$0 = x^3 - 4x \Rightarrow x(x^2 - 4) = 0 \Rightarrow x = 0, x = \pm 2$$

Use the second derivative to determine the nature of the three stationary points:

$$f'(x) = x^3 - 4x \Rightarrow f''(x) = 3x^2 - 4$$

$x = 2 \Rightarrow f''(2) = 3 \cdot 2^2 - 4 = 8 > 0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

$x = -2 \Rightarrow f''(-2) = 3(-2)^2 - 4 = 8 > 0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

$x = 0 \Rightarrow f''(0) = 3 \cdot 0^2 - 4 = -4 < 0 \Rightarrow$ the curve is concave down, hence the point is a maximum point.

To find the x-coordinate of the point(s) of inflection, solve the equation $y'' = 0$:

$f''(x) = 3x^2 - 4 \Rightarrow 0 = 3x^2 - 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\frac{2}{\sqrt{3}} \Rightarrow x = \pm\frac{2\sqrt{3}}{3} \Rightarrow$ there are two points of inflection.

(ii) The corresponding y-coordinates are:

$$x = 2 \Rightarrow y = \frac{1}{4} \cdot 2^4 - 2 \cdot 2^2 \Rightarrow y = -4$$

$$x = -2 \Rightarrow y = \frac{1}{4}(-2)^4 - 2(-2)^2 \Rightarrow y = -4$$

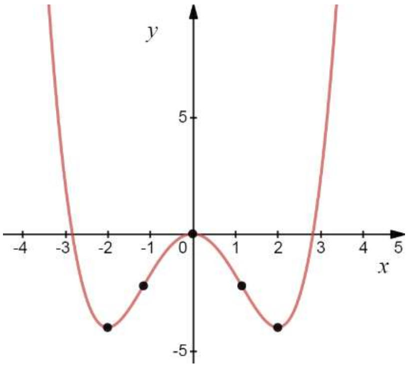
$$x = 0 \Rightarrow y = \frac{1}{4} \cdot 0^4 - 2 \cdot 0^2 \Rightarrow y = 0$$

$$x = \frac{2\sqrt{3}}{3} \Rightarrow y = \frac{1}{4}\left(\frac{2\sqrt{3}}{3}\right)^4 - 2\left(\frac{2\sqrt{3}}{3}\right)^2 \Rightarrow y = -\frac{20}{9}$$

$$x = -\frac{2\sqrt{3}}{3} \Rightarrow y = \frac{1}{4}\left(-\frac{2\sqrt{3}}{3}\right)^4 - 2\left(-\frac{2\sqrt{3}}{3}\right)^2 \Rightarrow y = -\frac{20}{9}$$

The graph of the function has a maximum at $(0,0)$, two minimum points, at $(2,-4)$ and $(-2,-4)$, and two points of inflection, at $\left(\frac{2\sqrt{3}}{3},-\frac{20}{9}\right)$ and $\left(-\frac{2\sqrt{3}}{3},-\frac{20}{9}\right)$.

The graph is shown below:



(c) **(i)** $f(x)=x+4x^{-1}\Rightarrow f'(x)=1+4(-1)x^{-2}\Rightarrow f'(x)=1-\frac{4}{x^2}$

Solve $y'=0$:

$0=1-\frac{4}{x^2}\Rightarrow 1=\frac{4}{x^2}\Rightarrow x^2=4\Rightarrow x=\pm 2$

Use the second derivative to determine the nature of the stationary points:

$f'(x)=1-4x^{-2}\Rightarrow f''(x)=-4(-2)x^{-3}\Rightarrow f''(x)=\frac{8}{x^3}$

$x=2\Rightarrow f''(2)=\frac{8}{2^3}=1>0\Rightarrow$ the curve is concave up, hence the point is a minimum point.

$x=-2\Rightarrow f''(-2)=\frac{8}{(-2)^3}=-1<0\Rightarrow$ the curve is concave down, hence the point is a maximum point.

Solve $y''=0$ to find the *x*-coordinate of the point(s) of inflection:

$f''(x)=\frac{8}{x^3}\Rightarrow 0=\frac{8}{x^3}\Rightarrow$ there are no points of inflection, as this equation has no solutions.

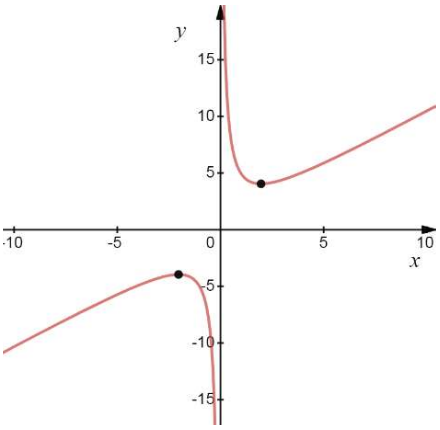
(ii) The corresponding y-coordinates are:

$$x = 2 \Rightarrow y = 2 + \frac{4}{2} \Rightarrow y = 4$$

$$x = -2 \Rightarrow y = -2 + \frac{4}{-2} \Rightarrow y = -4$$

The function has a minimum at $(2, 4)$, a maximum at $(-2, -4)$ and no points of inflection.

The graph is shown below:



(d) **(i)** $f(x) = -3x^5 + 5x^3 \Rightarrow f'(x) = -3 \cdot 5x^4 + 5 \cdot 3x^2 \Rightarrow f'(x) = -15x^4 + 15x^2$

Solve $y' = 0$:

$0 = -15x^4 + 15x^2 \Rightarrow 15x^2(-x^2 + 1) = 0 \Rightarrow x = 0, x^2 = 1 \Rightarrow x = 0, x = \pm 1$

Use the second derivative to determine the nature of the three stationary points:

$f'(x) = -15x^4 + 15x^2 \Rightarrow f''(x) = -15 \cdot 4x^3 + 15 \cdot 2x \Rightarrow f''(x) = -60x^3 + 30x$

$x = 0 \Rightarrow f''(0) = -60 \cdot 0^3 + 30 \cdot 0 = 0 \Rightarrow$ $x = 0$ is neither a maximum or a minimum.

$x = -1 \Rightarrow f''(-1) = -60(-1)^3 + 30(-1) = 30 > 0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

$x = 1 \Rightarrow f''(1) = -60 \cdot 1^3 + 30 \cdot 1 = -30 < 0 \Rightarrow$ the curve is concave down, hence the point is a maximum point.

Solve $y''=0$ to find the *x*-coordinate of the point(s) of inflection:

$f''(x)=-60x^3+30x \Rightarrow 0=-60x^3+30x \Rightarrow 30x(-2x^2+1)=0$

$\Rightarrow x=0, x^2=\frac{1}{2} \Rightarrow x=0, x=\pm\frac{1}{\sqrt{2}} \Rightarrow x=0, x=\pm\frac{\sqrt{2}}{2} \Rightarrow$ there are three points of inflection.

(ii) The corresponding *y*-coordinates are:

$x=1 \Rightarrow y=-3\cdot 1^5+5\cdot 1^3 \Rightarrow y=2$

$x=-1 \Rightarrow y=-3(-1)^5+5(-1)^3 \Rightarrow y=-2$

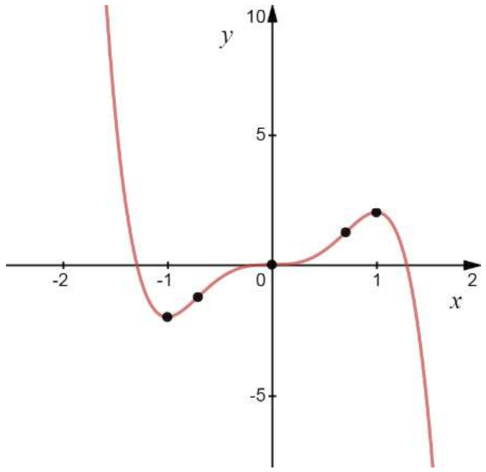
$x=0 \Rightarrow y=-3\cdot 0^5+5\cdot 0^3 \Rightarrow y=0$

$x=-\frac{\sqrt{2}}{2} \Rightarrow y=-3(-\frac{\sqrt{2}}{2})^5+5(-\frac{\sqrt{2}}{2})^3 \Rightarrow y=-\frac{7\sqrt{2}}{8}$

$x=\frac{\sqrt{2}}{2} \Rightarrow y=-3(\frac{\sqrt{2}}{2})^5+5(\frac{\sqrt{2}}{2})^3 \Rightarrow y=\frac{7\sqrt{2}}{8}$

The function has a maximum at $(1,2)$, a minimum at $(-1,-2)$ and inflection points at $(-\frac{\sqrt{2}}{2},-\frac{7\sqrt{2}}{8})$ and $(\frac{\sqrt{2}}{2},\frac{7\sqrt{2}}{8})$

The graph is shown below:



(e) **(i)** $f(x)=3x^4-4x^3-12x^2+5 \Rightarrow f'(x)=3\cdot 4x^3-4\cdot 3x^2-12\cdot 2x$

$\Rightarrow f'(x)=12x^3-12x^2-24x$

Solve $y'=0$:

$0=12x(x^2-x-2) \Rightarrow 12x(x+1)(x-2)=0 \Rightarrow x=0, x=-1, x=2$

Use the second derivative to determine the nature of the stationary points:

$f'(x)=12x^3-12x^2-24x \Rightarrow f''(x)=12\cdot 3x^2-12\cdot 2x-24$

$\Rightarrow f''(x)=36x^2-24x-24$

$x=0 \Rightarrow f''(0)=36\cdot 0^2-24\cdot 0-24=-24<0 \Rightarrow$ the curve is concave down, hence the point is a maximum point.

$x=-1 \Rightarrow f''(-1)=36(-1)^2-24(-1)-24=36>0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

$x=2 \Rightarrow f''(2)=36\cdot 2^2-24\cdot 2-24=72>0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

Solve $y''=0$ to find the x-coordinate of the point(s) of inflection:

$f''(x)=36x^2-24x-24 \Rightarrow 0=12(3x^2-2x-2)$

$\Rightarrow x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4\cdot 3\cdot(-2)}}{2\cdot 3} \Rightarrow x=\dfrac{2\pm\sqrt{28}}{6} \Rightarrow x=\dfrac{1\pm\sqrt{7}}{3} \Rightarrow$ there are two points of inflection.

(ii) The corresponding y-coordinates are:

$x=0 \Rightarrow y=3\cdot 0^4-4\cdot 0^3-12\cdot 0^2+5 \Rightarrow y=5$

$x=-1 \Rightarrow y=3(-1)^4-4(-1)^3-12(-1)^2+5 \Rightarrow y=0$

$x=2 \Rightarrow y=3\cdot 2^4-4\cdot 2^3-12\cdot 2^2+5 \Rightarrow y=-27$

$$x=\frac{1+\sqrt{7}}{3} \Rightarrow y=3\left(\frac{1+\sqrt{7}}{3}\right)^4-4\left(\frac{1+\sqrt{7}}{3}\right)^3-12\left(\frac{1+\sqrt{7}}{3}\right)^2+5$$

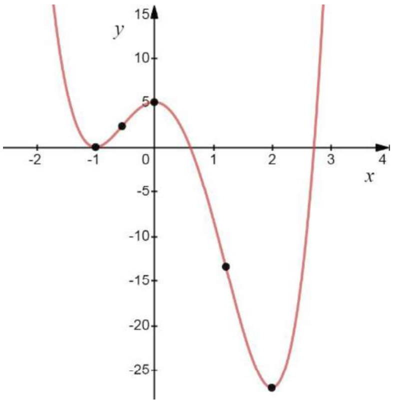
$$\Rightarrow y=-\frac{140+80\sqrt{7}}{27}$$

$$x=\frac{1-\sqrt{7}}{3} \Rightarrow y=3\left(\frac{1-\sqrt{7}}{3}\right)^4-4\left(\frac{1-\sqrt{7}}{3}\right)^3-12\left(\frac{1-\sqrt{7}}{3}\right)^2+5$$

$$\Rightarrow y=\frac{80\sqrt{7}-149}{27}$$

The function has a maximum at $(0,5)$, a minimum at $(-1,0)$ and $(2,-27)$, and inflection points at $\left(\frac{1+\sqrt{7}}{3},-\frac{140+80\sqrt{7}}{27}\right)$ (or $(1.22,-13.4)$) and $\left(\frac{1-\sqrt{7}}{3},\frac{80\sqrt{7}-149}{27}\right)$(or $(-0.549,2.32)$)

The graph is shown below:



6. **(a)** $v(t)=s'(t),\ a(t)=s''(t)$

$s(t)=t(8t^2-33t+27)=8t^3-33t^2+27t \Rightarrow s'(t)=24t^2-66t+27$

$\Rightarrow s''(t)=48t-66$

$\Rightarrow v(t)=24t^2-66t+27$, $a(t)=48t-66$

Substitute $t=0$ into expressions of the velocity and acceleration to obtain their initial values:

$v(0)=24(0^2)-66(0)+27 \Rightarrow v(0)=27\text{ ms}^{-1}$

$a(0)=48(0)-66 \Rightarrow a(0)=-66\text{ ms}^{-2}$

(b) $v(3)=24(3^2)-66(3)+27 \Rightarrow v(3)=45\text{ ms}^{-1}$

$a(3)=48(3)-66 \Rightarrow a(3)=78\text{ ms}^{-2}$

(c) The values of t for which the object changes direction are obtained by solving the equation $v(t)=0$, as the object must first come to rest before changing direction:

$$0=24t^2-66t+27 \Rightarrow 3(4t-9)(2t-1)=0 \Rightarrow t=\frac{9}{4},t=\frac{1}{2}$$

As the equation $v(t)=0$ is the same as $s'(t)=0$, its two solutions represent stationary points for the displacement.

(d) To find the required value for t, the equation $v'(t)=0$ (or $a(t)=0$) must be solved:

$$0=48t-66 \Rightarrow 48t=66 \Rightarrow t=\frac{11}{8}=1.375$$

(e) The acceleration is 0.

7. Find the expression of D':

$$D=3x+100x^{-1} \Rightarrow D'=3+100(-1)x^{-2} \Rightarrow D'=3-\frac{100}{x^2}$$

Solve the equation $D'=0$ to find the value of x which minimises the delivery cost:

$$0=3-\frac{100}{x^2} \Rightarrow 3=\frac{100}{x^2} \Rightarrow x^2=\frac{100}{3} \Rightarrow x=\pm\sqrt{\frac{100}{3}}=\pm 5.773...$$

But $x>0 \Rightarrow x=5.773... \Rightarrow x\approx 5.77$ tonnes

The corresponding delivery cost is:

$$D=3\cdot\sqrt{\frac{100}{3}}+\frac{100}{\sqrt{\frac{100}{3}}} \Rightarrow D=20\sqrt{3}=34.64...\text{ thousand} \Rightarrow D\approx \$34,600$$

To determine the nature of this stationary point, the second derivative is found:

$$D'=3-100x^{-2} \Rightarrow D''=-100(-2)x^{-3} \Rightarrow D''=\frac{200}{x^3}$$

Substitute $x=\sqrt{\frac{100}{3}}$ into the expression of D'' to analyse the concavity of the graph:

$D''(\sqrt{\frac{100}{3}})=\frac{200}{\sqrt{\frac{100}{3}}^3} \Rightarrow D''(\sqrt{\frac{100}{3}})=1.039...>0 \Rightarrow$ the curve is concave up, hence the point is a minimum point.

8. If the point $(-1,-8)$ is on the graph representing $y=x^4+ax^2+bx+c$, then its coordinates satisfy the equation of the curve:

$$-8=(-1)^4+a(-1)^2+b(-1)+c \Rightarrow a-b+c=-9$$

To use the other given information, y' and y'' must be found:

$$y'=4x^3+2ax+b,\quad y''=12x^2+2a$$

Equate the expressions of $y'(-1)$ and $y''(-1)$ to 6 to obtain two more equations:

$$6=4(-1)^3+2a(-1)+b \Rightarrow -2a+b=10$$

$$6=12(-1)^2+2a \Rightarrow 2a=-6 \Rightarrow a=-3$$

Substitute the value of a in the remaining equations to find the values of b and c:

$$-2(-3)+b=10 \Rightarrow b=4$$
$$-3-4+c=-9 \Rightarrow c=-2$$

9. To find the stationary points, the equation $y'=0$ must be solved.

Find the expression of y':

$$y=\frac{x^3+3x-1}{x^2} \Rightarrow y=x+3x^{-1}-x^{-2} \Rightarrow y'=1+3(-1)x^{-2}-(-2)x^{-3} \Rightarrow y'=1-\frac{3}{x^2}+\frac{2}{x^3}$$

$$\Rightarrow 0=1-\frac{3}{x^2}+\frac{2}{x^3} \Rightarrow \frac{x^3-3x+2}{x^3}=0 \Rightarrow \frac{(x-1)(x^2+x-2)}{x^3}=0 \Rightarrow \frac{(x-1)(x+2)(x-1)}{x^3}=0$$

$$\Rightarrow x=1, x=-2$$

To determine the nature of the two stationary points, the second derivative is used:

$$\Rightarrow y'=1-3x^{-2}+2x^{-3} \Rightarrow y''=-3(-2)x^{-3}+2(-3)x^{-4} \Rightarrow y''=\frac{6}{x^3}-\frac{6}{x^4}$$

Substitute the x-coordinates of the stationary points into the expression of y'', to determine the concavity of the function:

$y''(1)=\frac{6}{1^3}-\frac{6}{1^4}=0 \Rightarrow$ the point is neither a minimum nor a maximum (there is an inflection point at $x=1$, and the tangent at this point is horizontal)

$y''(2)=\frac{6}{(-2)^3}-\frac{6}{2^4}=-\frac{6}{8}-\frac{3}{8}=-\frac{9}{8}<0 \Rightarrow$ the function is concave down, so there is a maximum at $x=-2$.To find the y-coordinates, substitute the two x-values into the expression of the original function:

$$x=1 \Rightarrow y=\frac{1^3+3\cdot 1-1}{1^2} \Rightarrow y=3$$

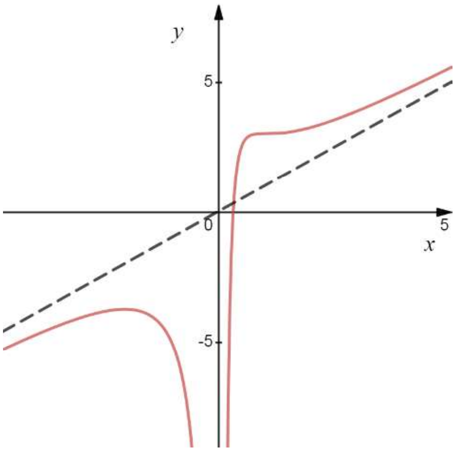
$$x=-2 \Rightarrow y=\frac{(-2)^3+3\cdot(-2)-1}{(-2)^2} \Rightarrow y=-\frac{15}{4}$$

There is a point of maximum at $\left(-2,-\frac{15}{4}\right)$ and a stationary point of inflection at $(1,3)$.

To determine the behavior of the function when $x\to\pm\infty$, the limit of the function must be evaluated:

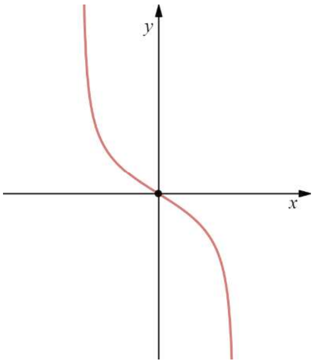
$$\lim_{x\to\pm\infty}\frac{x^3+3x-1}{x^2}=\lim_{x\to\pm\infty}\left(x+\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to\pm\infty}x+\lim_{x\to\pm\infty}\frac{1}{x}-\lim_{x\to\pm\infty}\frac{1}{x^2}=\pm\infty+0-0=\pm\infty$$

This means the values of the function will increase towards $+\infty$ when $x\to+\infty$ and will decrease towards $-\infty$ when $x\to-\infty$, in both cases without bound. It must be noticed that the values of the given function will approach $\pm\infty$ in the same manner as the values of the linear function $y = x$ (the graph has an oblique asymptote). The graph of the function is shown below:



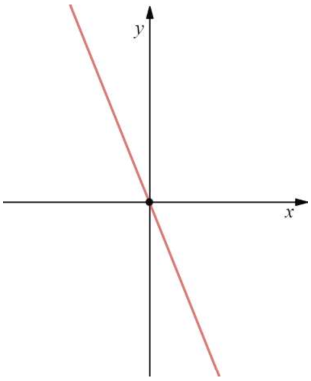
10. In each of the following questions, in order to be able to draw the graph of $y = f\,'(x)$, the gradient of the curve $y = f(x)$ has to be analysed.

(a) The graph has one maximum point at $x = 0 \Rightarrow f\,'(0) = 0$, which means that point $(0, 0)$ is a x-intercept for the graph of $y = f\,'(x)$. Mark this point on a new set of axes. Next, determine the gradient of the curve representing the function $y = f(x)$, going from left to right, keeping in mind that the values of the gradient are y values for the graph of $y = f\,'(x)$. At the left end the gradient is $+\infty$ (because the tangent to the graph of the function at this point is vertical), then it decreases until it reaches 0, then it becomes negative and, at the right end, it approaches $-\infty$.
The graph is shown below:



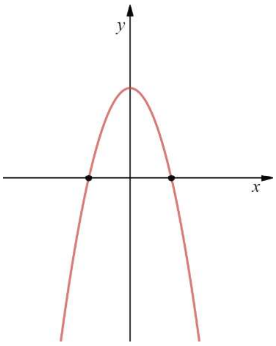
(b) The graph has one maximum point at $x = 0 \Rightarrow f\,'(0) = 0$

This means that point $(0,0)$ is a x-intercept for the graph of $y = f'(x)$. At the left end the gradient has a positive value, then it decreases until it becomes 0, then it becomes negative.
The graph is shown below:



(c) The graph has two stationary points, a minimum at $x = -a$, and a maximum at $x = a$. $\Rightarrow f'(a) = f'(-a) = 0$

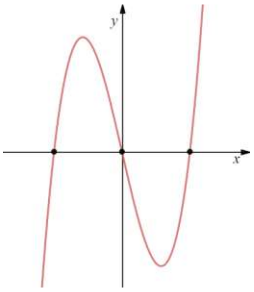
This means that points $(a,0)$ and $(-a,0)$ are the x-intercepts of the graph of $y = f'(x)$. At the left end of the given graph, the gradient is negative and increases until it becomes 0 at $x = -a$, then it becomes positive and increases up to point $(0,0)$, due to the symmetry of the graph of an odd function, after which it starts to decrease until it becomes 0 again at $x = a$. Finally, past $x = a$, the slope is negative. The graph is shown below:



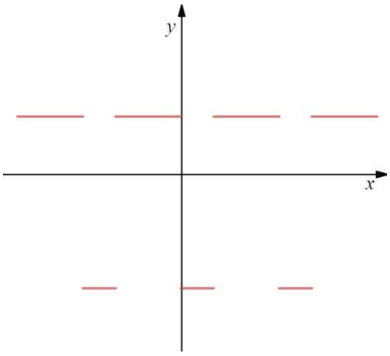
(d) The graph has three stationary points, minimum points at $x = -a$ and $x = a$, and a maximum at $x = 0$.

$\Rightarrow f'(a) = f'(-a) = f'(0) = 0$

This means that points $(-a, 0)$, $(0, 0)$ and $(a, 0)$ are the *x*-intercepts of the graph of $y = f'(x)$. To start with, the gradient is negative and increases until it reaches 0 at $x = -a$, then it becomes positive and increases up to a point, say $(-b, 0)$, where $b > 0, b < a$, after which it starts to decrease until it reaches 0 again, at $(0, 0)$. Past $x = 0$, the gradient turns negative and decreases up to the point where $x = b$, then it increases towards 0, which is reached at $x = a$. Finally, past $x = a$, the slope is positive. The graph is shown below:



(e) As the graph represents a periodic function, the points of minimum and maximum occur following a pattern related to the period of the function. The repeating part of the graph has two sections, each of them a straight line, the first one with a constant positive gradient, the second one with a constant negative one. The graph of the derivative will also be periodic, and it will be represented by repeating horizontal line segments:



Notice that at the extrema the derivative is not defined because of the nature of the original function.

11. A function *f* is increasing when its derivative, *f'*, is positive, and is decreasing when *f'* is negative.

(a) **(i)** By inspection, *f* is increasing for $1 < x < 5$ and is decreasing for $x < 1$ or $x > 5$.

(ii) The function has a minimum at $x = 1$, because $f'(1) = 0$, and f' changes its sign from negative to positive about $x = 1$. Similarly, the function has a maximum at $x = 5$ because $f'(1) = 0$, and f' changes its sign from positive to negative about $x = 5$.

(b) **(i)** *f* is increasing for $x < 1$ or $3 < x < 5$, and is decreasing for $1 < x < 3$ or $x > 5$.

(ii) The function has maximum points at $x = 1$ and $x = 5$, and a minimum at $x = 3$.

12. By inspection, there are three points on the given graph where the second derivative is 0, at $x \approx 0.5$, $x = 4$ and $x \approx 7.5$. Additionally, for a point of inflection to occur, there has to be a change in the sign of f'' about the point where f'' is 0. This only happens for $x \approx 0.5$ and $x \approx 7.5$, the values of f'' about $x = 4$ are always positive.

13. **(a)** $f(-2) = 8 \Rightarrow (-2, 8)$ is on the graph

(b) $f(0) = 4 \Rightarrow y = 4$ is the *y*-intercept of the graph

(c) $f(2) = 0 \Rightarrow x = 2$ is an *x*-intercept of the graph

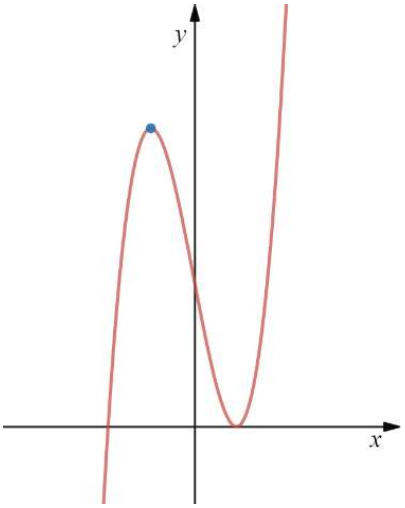
(d) $f'(2) = f'(-2) = 0 \Rightarrow$ there are stationary points at $x = \pm 2$

(e) The first derivative is positive for $|x| > 2$, this means the function is increasing for $x < -2$ or $x > 2$.

(f) The first derivative is negative for $|x| < 2$, this means the function is decreasing for $-2 < x < 2$.

(g) The second derivative is negative for $x < 0$, this means the graph is concave down on this interval.

(h) The second derivative is positive for $x > 0$, this means the graph is concave up on this interval. The graph is shown overleaf:



14. **(a)** To determine the direction in which the object moves, its velocity has to be analysed. As the velocity is the rate of change of displacement, the first derivative of *s* has to be found:

$v(t) = s'(t) \Rightarrow v(t) = -6t^2 + 30t - 24$

The intervals where the velocity is positive will give the set of values for *t* when the object is moving to the right. Similarly, when the velocity is negative, the object moves to the left. To find when the change of direction occurs, the equation $v(t) = 0$ must be solved:

$0 = -6t^2 + 30t - 24 \Rightarrow -6(t-4)(t-1) = 0 \Rightarrow t = 1, t = 4$

This means that the object is moving to the left when $0 \le t < 1$ or $t > 4$ (velocity takes negative values), and it moves to the right for $1 < t < 4$ (velocity is positive).

(b) The initial values are found by substituting $t = 0$ into the expressions of the velocity and acceleration, respectively:

(i) $v(0) = -6 \cdot 0^2 + 30 \cdot 0 - 24 \Rightarrow v(0) = -24$

(ii) $a(t) = v'(t) \Rightarrow a(t) = -12t + 30 \Rightarrow a(0) = 12 \cdot 0 + 30 \Rightarrow a(0) = 30$

(c) **(i)** The displacement is at a maximum at either $t = 1$ or $t = 4$. To decide which value will give the maximum point on the graph of the displacement, the second derivative, s'', will be used:

$s''(t) = a(t) \Rightarrow s''(t) = -12t + 30$

$s''(1) = -12 \cdot 1 + 30 = 18 > 0 \Rightarrow$ the graph is concave up,

so $t = 1$ is a minimum

$s''(4) = -12 \cdot 4 + 30 = -18 < 0 \Rightarrow$ the graph is concave down,

so $t = 4$ is a maximum

The maximum displacement is:

$$s(4) = -2\cdot 4^3 + 15\cdot 4^2 - 24\cdot 4 \Rightarrow s(4) = 16$$

(ii) The maximum velocity is obtained when $v' = 0$:

$$0 = -12t + 30 \Rightarrow t = \frac{5}{2}$$

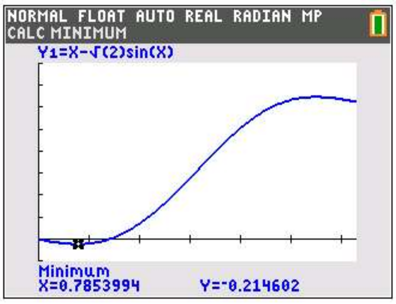
The maximum velocity is:

$$v\left(\frac{5}{2}\right) = -6\cdot\left(\frac{5}{2}\right)^2 + 30\cdot\left(\frac{5}{2}\right) - 24 \Rightarrow v\left(\frac{5}{2}\right) = 13.5$$

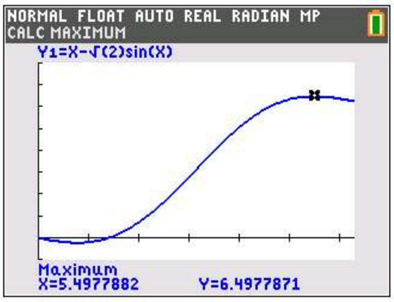
(d) The object's acceleration is 0 when $t = \frac{5}{2}$ (see part **(c)(ii)**).

At this point the object has achieved its greatest velocity.

15. **(a)** The minimum value of the function is $y \approx -0.215$, the maximum is $y \approx 6.50$



NORMAL FLOAT AUTO REAL RADIAN MP
CALC MINIMUM
Y1=X−√(2)sin(X)
Minimum
X=0.7853994 Y=−0.214602



NORMAL FLOAT AUTO REAL RADIAN MP
CALC MAXIMUM
Y1=X−√(2)sin(X)
Maximum
X=5.4977882 Y=6.4977871

(b) $$f(x) = x - \sqrt{2}\sin x \Rightarrow f'(x) = 1 - \sqrt{2}\cos x$$

$$f'(x) = 0 \Rightarrow 1 - \sqrt{2}\cos x = 0 \Rightarrow \cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}, x = \frac{7\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \sqrt{2}\sin\frac{\pi}{4} \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \sqrt{2}\cdot\frac{\sqrt{2}}{2} \Rightarrow f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1$$

$$f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} - \sqrt{2}\sin\frac{7\pi}{4} \Rightarrow f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} - \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) \Rightarrow f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} + 1$$

Exercise 12.4

1. In each of the questions in this exercise, follow the steps listed below:

i. find the y-coordinate of the point of tangency question by substituting the given x-coordinate into the expression of the function

ii. find y', the first derivative of the given function

iii. find m, the gradient (slope) of the tangent line, by substituting the given x-coordinate into the expression of y'

iv. substitute into the slope-intercept form of the equation of a line, $y-y_1=m(x-x_1)$

v. rearrange this equation into the required form, $y=mx+c$.

(a) $x=-3\Rightarrow y=(-3)^2+2(-3)+1\Rightarrow y=4\Rightarrow$ the point of tangency is $(-3,4)$.

$y=x^2+2x+1\Rightarrow y'=2x+2$

$\Rightarrow m=2(-3)+2\Rightarrow m=-4$

The equation of the tangent line is:

$y-4=-4(x-(-3))\Rightarrow y=-4x-12+4\Rightarrow y=-4x-8$

(b) $x=-\dfrac{2}{3}\Rightarrow y=\left(-\dfrac{2}{3}\right)^3+\left(-\dfrac{2}{3}\right)^2\Rightarrow y=\dfrac{4}{27}\Rightarrow$ the point of tangency is $\left(-\dfrac{2}{3},\dfrac{4}{27}\right)$.

$y=x^3+x^2\Rightarrow y'=3x^2+2x$

$\Rightarrow m=3\left(-\dfrac{2}{3}\right)^2+2\left(-\dfrac{2}{3}\right)\Rightarrow m=0$

The equation of the tangent line is:

$y-\dfrac{4}{27}=0\left(x-\left(-\dfrac{2}{3}\right)\right)\Rightarrow y=\dfrac{4}{27}$

(c) $x=0\Rightarrow y=3\cdot0^2-0+1\Rightarrow y=1\Rightarrow$ the point of tangency is $(0,\ 1)$

$y=3x^2-x+1\Rightarrow y'=6x-1$

$\Rightarrow m=6\cdot0-1\Rightarrow m=-1$

The equation of the tangent line is:

$y-1=-1(x-0)\Rightarrow y=-x+1$

(d) $x=\dfrac{1}{2}\Rightarrow y=2\cdot\dfrac{1}{2}+\dfrac{1}{\frac{1}{2}}\Rightarrow y=3\Rightarrow$ the point of tangency is $\left(\dfrac{1}{2},3\right)$.

$y=2x+x^{-1}\Rightarrow y'=2+(-1)x^{-2}\Rightarrow y'=2-\dfrac{1}{x^2}$

$$\Rightarrow m = 2 - \frac{1}{\left(\frac{1}{2}\right)^2} \Rightarrow m = -2$$

The equation of the tangent line is:

$$y - 3 = -2\left(x - \frac{1}{2}\right) \Rightarrow y = -2x + 4$$

2. The gradient of the normal line is the negative reciprocal of the gradient of the tangent line.

(a) $y - 4 = -\frac{1}{-4}(x-(-3)) \Rightarrow y = \frac{1}{4}x + \frac{3}{4} + 4 \Rightarrow y = \frac{1}{4}x + \frac{19}{4}$

(b) The line tangent to the curve at $\left(-\frac{2}{3}, \frac{4}{27}\right)$ is a horizontal line; this means the normal line at the same point is a vertical line, its equation is $x = -\frac{2}{3}$

(c) $y - 1 = -\frac{1}{-1}(x - 0) \Rightarrow y = x + 1$

(d) $y - 3 = -\frac{1}{-2}\left(x - \frac{1}{2}\right) \Rightarrow y = \frac{1}{2}x - \frac{1}{4} + 3 \Rightarrow y = \frac{1}{2}x + \frac{11}{4}$

3. The points of intersection between the curve and the *x*-axis have $y = 0$:

$0 = x^3 - 3x^2 + 2x \Rightarrow 0 = x(x-1)(x-2) \Rightarrow x = 0,\ x = 1,\ x = 2$

$y = x^3 - 3x^2 + 2x \Rightarrow y' = 3x^2 - 6x + 2$

When $x = 0$: $m = 3(0)^2 - 6\cdot 0 + 2 \Rightarrow m = 2$

The equation of the tangent line at $(0, 0)$ is:

$y - 0 = 2(x - 0) \Rightarrow y = 2x$

When $x = 1$: $m = 3(1)^2 - 6\cdot 1 + 2 \Rightarrow m = -1$

The equation of the tangent line at $(1, 0)$ is:

$y - 0 = -1(x - 1) \Rightarrow y = -x + 1$

When $x = 2$: $m = 3(2)^2 - 6\cdot 2 + 2 \Rightarrow m = 2$

The equation of the tangent line at $(2, 0)$ is:

$y - 0 = 2(x - 2) \Rightarrow y = 2x - 4$

4. $x-2y=1\Rightarrow 2y=x-1\Rightarrow y=\frac{1}{2}x-\frac{1}{2}\Rightarrow$ the gradient of the line is $m=\frac{1}{2}$

The gradient of the required tangent is the negative reciprocal of m, namely -2.

The first derivative of the given function is:

$y=x^2-2x\Rightarrow y'=2x-2$

The gradient of the curve is the same as the gradient of the tangent line, so an equation can be formed to find the x-coordinate of the tangency point:

$2x-2=-2\Rightarrow x=0$

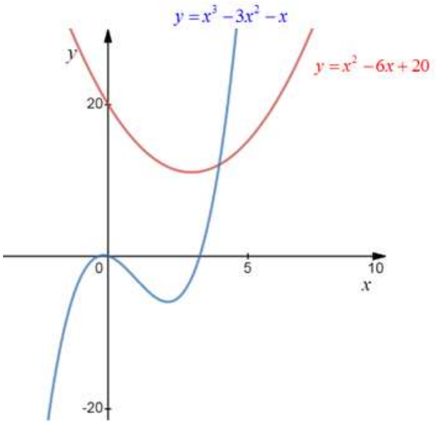
The y-coordinate is:

$\Rightarrow y=0^2-2\cdot 0\Rightarrow y=0\Rightarrow$ the point of tangency is $(0,0)$

The equation of the tangent line at $(0,0)$ is:

$y-0=-2(x-0)\Rightarrow y=-2x$

5. The two graphs are shown in the diagram below:



(a) Find the first derivative of both functions to obtain the expression of their gradients, then equate them and solve for x:

$y=x^2-6x+20\Rightarrow y'=2x-6$

$y=x^3-3x^2-x\Rightarrow y'=3x^2-6x-1$

$\Rightarrow 2x-6=3x^2-6x-1\Rightarrow 3x^2-8x+5=0\Rightarrow (3x-5)(x-1)=0\Rightarrow x=\frac{5}{3}, x=1$

But $x\in\mathbb{Z}\Rightarrow x=1$

(b) Let $f(x)=x^2-6x+20$ and $g(x)=x^3-3x^2-x$

$f(1)=1^2-6\cdot1+20\Rightarrow y=15\Rightarrow$ the point of tangency for $y=f(x)$ is $(1,15)$

$g(1)=1^3-3\cdot1^2-1\Rightarrow y=-3\Rightarrow$ the point of tangency for $y=g(x)$ is $(1,-3)$

The gradient of both tangent lines is:

$m=2\cdot1-6\Rightarrow m=-4$

It follows that the equation of the tangent line to the graph of $y=f(x)$ at $(1,15)$ is:

$y-15=-4(x-1)\Rightarrow y=-4x+19$

Similarly, the equation of the tangent line to the graph of $y=g(x)$ at $(1,-3)$ is:

$y-(-3)=-4(x-1)\Rightarrow y=-4x+1$

6. $x=-3\Rightarrow y=(-3)^2+4(-3)-2\Rightarrow y=-5\Rightarrow$ the point common to the normal and the curve is $(-3,-5)$. The gradient of the curve at $x=-3$ is:

$y=x^2+4x-2\Rightarrow y'=2x+4\Rightarrow y'(-3)=2(-3)+4\Rightarrow y'(-3)=-2$

The gradient of the normal line is the negative reciprocal of the gradient of the curve, so the equation of the normal at $(-3,-5)$ is:

$$y-(-5)=-\frac{1}{-2}(x-(-3))\Rightarrow y=\frac{1}{2}x+\frac{3}{2}-5\Rightarrow y=\frac{1}{2}x-\frac{7}{2}$$

To find where this line intersects the curve again, equate the expressions of the curve and the normal and solve for *x*:

$$\frac{1}{2}x-\frac{7}{2}=x^2+4x-2\Rightarrow x^2+\frac{7}{2}x+\frac{3}{2}=0\Rightarrow 2x^2+7x+3=0\Rightarrow(2x+1)(x+3)=0$$

$$\Rightarrow x=-\frac{1}{2},x=-3$$

The required point is where $x=-\frac{1}{2}$, the corresponding *y*-coordinate is:

$$y=(-\frac{1}{2})^2+4(-\frac{1}{2})-2\Rightarrow y=\frac{1}{4}-4\Rightarrow y=-\frac{15}{4}$$

The other point of intersection is $(-\frac{1}{2},-\frac{15}{4})$

7. Find the expression of the first derivative:

$$y=\frac{1-x^3}{x^4}\Rightarrow y=x^{-4}-x^{-1}\Rightarrow y'=-4x^{-5}-(-1)x^{-2}\Rightarrow y'=-\frac{4}{x^5}+\frac{1}{x^2}$$

The gradient of the tangent is found by substituting $x=1$ into the expression of y':

$$\Rightarrow m=-\frac{4}{1^5}+\frac{1}{1^2}\Rightarrow m=-3$$

The equation of the tangent is:
$y-0=-3(x-1)\Rightarrow y=-3x+3$
The slope of the normal line is the negative reciprocal of m, so the equation of the normal line is:

$$y-0=-\frac{1}{-3}(x-1)\Rightarrow y=\frac{1}{3}x-\frac{1}{3}$$

8. Based on the information given, the equation of the normal line is $y=x-4$.

$x=1\Rightarrow y=a\cdot 1^{\frac{1}{2}}+b\cdot 1\Rightarrow y=a+b\Rightarrow$ the point common to the normal and the curve is $(1,a+b)$.
The coordinates of this point must satisfy the equation of the normal:
$y=x-4\Rightarrow a+b=1-4\Rightarrow a+b=-3\Rightarrow a=-3-b$
The gradient of the curve at $x=1$ is:

$$y=ax^{\frac{1}{2}}+bx\Rightarrow y'=\frac{1}{2}ax^{-\frac{1}{2}}+b\Rightarrow y'(1)=\frac{1}{2}a\cdot 1^{-\frac{1}{2}}+b\Rightarrow y'(1)=\frac{a+2b}{2}$$

The slope of the normal line, n, is the negative reciprocal of the gradient of the curve, and is equal to 1:

$$n=-\frac{2}{a+2b}\Rightarrow 1=-\frac{2}{a+2b}\Rightarrow a+2b=-2$$

Substitute the expression of a into the last equation:
$a=-3-b\Rightarrow(-3-b)+2b=-2\Rightarrow b=1\Rightarrow a=-3-1\Rightarrow a=-4$

9. **(a)** The first derivative of the function is:

$$y=x^3+\frac{1}{2}x^2+1\Rightarrow y'=3x^2+x$$

The gradient of the tangent is found by substituting $x=-1$ into the expression of $y'\Rightarrow m=3(-1)^2+(-1)\Rightarrow m=2$
The equation of the tangent is:

$$y-\frac{1}{2}=2(x-(-1))\Rightarrow y=2x+\frac{5}{2}$$

(b) The gradient of the other tangent is the same as the gradient of the tangent line found in **(a)**. The gradient is given by the expression of y':

$$2=3x^2+x\Rightarrow 3x^2+x-2=0\Rightarrow(3x-2)(x+1)=0\Rightarrow x=\frac{2}{3},x=-1$$

The required x-coordinate is $x=\frac{2}{3}$, the corresponding y-coordinate is:

$$y=\left(\frac{2}{3}\right)^3+\frac{1}{2}\left(\frac{2}{3}\right)^2+1\Rightarrow y=\frac{41}{27}$$

10. $x=4 \Rightarrow y=\sqrt{4}\left(1-\sqrt{4}\right) \Rightarrow y=-2 \Rightarrow$ the point of tangency is $(4,-2)$.

Find the expression of the first derivative:

$$y=\sqrt{x}\left(1-\sqrt{x}\right) \Rightarrow y=\sqrt{x}-x \Rightarrow y'=\frac{1}{2\sqrt{x}}-1$$

The gradient of the tangent, *m*, is found by substituting $x=4$ into the expression of y':

$$\Rightarrow m=\frac{1}{2\sqrt{4}}-1 \Rightarrow m=-\frac{3}{4}$$

The equation of the tangent is:

$$y-(-2)=-\frac{3}{4}(x-4) \Rightarrow y=-\frac{3}{4}x+3-2 \Rightarrow y=-\frac{3}{4}x+1$$

The slope of the normal line is the negative reciprocal of *m*, so the equation of the normal line is:

$$y-(-2)=-\frac{1}{-\frac{3}{4}}(x-4) \Rightarrow y=\frac{4}{3}x-\frac{16}{3}-2 \Rightarrow y=\frac{4}{3}x-\frac{22}{3}$$

11. **(a)** $x=1 \Rightarrow y=(1+x)^2(5-x) \Rightarrow y=(1+1)^2(5-1) \Rightarrow y=16 \Rightarrow$ the point of tangency is $(1,16)$. The first derivative is:

$$y=(1+x)^2(5-x) \Rightarrow y=-x^3+3x^2+9x+5 \Rightarrow y'=-3x^2+6x+9$$

The gradient of the tangent, *m*, is found by substituting $x=1$ into the expression of $y' \Rightarrow m=-3\cdot 1^2+6\cdot 1+9 \Rightarrow m=12$

The equation of the tangent at point $(1,16)$ is:

$$y-16=12(x-1) \Rightarrow y=12x+4$$

To show that the two graphs do not meet again, equate the expression of the function and the one of the tangent and solve for x – the only solution for this equation should be the x-coordinate of the tangency point, $x=1$:

$$-x^3+3x^2+9x+5=12x+4 \Rightarrow x^3-3x^2+3x-1=0 \Rightarrow (x-1)^3=0 \Rightarrow x=1$$

(b) The gradient of the graph at the point where $x=0$ is:

$$y'(0)=-3\cdot 0^2+6\cdot 0+9 \Rightarrow y'(0)=9$$

The equation of the tangent at point $(0,5)$ is:

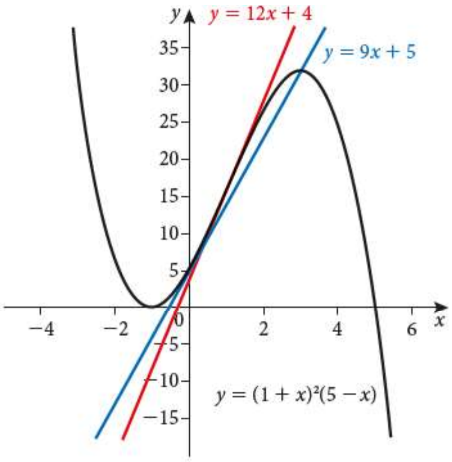
$$y-5=9(x-0) \Rightarrow y=9x+5$$

To show that the two graphs meet again, equate the expression of the function and the one of the tangent and solve for x – there should be at least one more solution additional to the solution given by the x-coordinate of the tangency point, $x=0$:

$$-x^3+3x^2+9x+5=9x+5 \Rightarrow x^3-3x^2=0 \Rightarrow x^2(x-3)=0 \Rightarrow x=0, x=3$$

The required x-coordinate is $x = 3$. The value of the gradient at this point is: $y'(3) = -3\cdot 3^2 + 6\cdot 3 + 9 \Rightarrow y'(3) = -27 + 27 \Rightarrow y'(3) = 0 \Rightarrow$ the point on the graph where $x = 3$ is a turning point.

(c) The three graphs are shown in the diagram below:



12. Let (a, b) be a point of tangency. As this point is on the graph of the given function, it follows that its coordinates satisfy the equation of the curve.

$\Rightarrow b = a^2 + a \Rightarrow$ the tangency point has coordinates $(a, a^2 + a)$.

The expression of the first derivative is:

$y = x^2 + x \Rightarrow y' = 2x + 1$

The gradient of the graph at the point where $x = a$ is:

$y'(a) = 2a + 1$

The equation of the tangent at point $(a,\ a^2 + a)$ is:

$y - (a^2 + a) = (2a+1)(x-a) \Rightarrow y = (2a+1)x + a^2 + a - a(2a+1)$

$\Rightarrow y = (2a+1)x + a^2 + a - 2a^2 - a \Rightarrow y = (2a+1)x - a^2$

The point $(2, -3)$ lies on the tangent, so its coordinates must satisfy the equation of this line:

$\Rightarrow -3 = (2a+1)2 - a^2 \Rightarrow a^2 - 4a - 5 = 0 \Rightarrow (a-5)(a+1) = 0 \Rightarrow a = 5, a = -1$

The two points of tangency are:

$a = 5 \Rightarrow b = 5^2 + 5 \Rightarrow b = 30 \Rightarrow (5, 30)$

$a = -1 \Rightarrow b = (-1)^2 + (-1) \Rightarrow b = 0 \Rightarrow (-1, 0)$

The gradients of the two lines are:

$a = 5 \Rightarrow y'(5) = 2 \cdot 5 + 1 \Rightarrow y'(5) = 11$

$a = -1 \Rightarrow y'(5) = 2 \cdot (-1) + 1 \Rightarrow y'(-1) = -1$

Consequently, the equations of the two tangent lines are:

At $(5, 30)$: $y - 30 = 11(x - 5) \Rightarrow y = 11x - 25$

At $(-1, 0)$: $y - 0 = -1(x - (-1)) \Rightarrow y = -x - 1$

13. Let (a, b) be a point of tangency.

$\Rightarrow b = 1 + (a-1)^2 \Rightarrow b = a^2 - 2a + 2$, the tangency point has coordinates $(a, a^2 - 2a + 2)$.

The expression of the first derivative is:

$y = x^2 - 2x + 2 \Rightarrow y' = 2x - 2$

The gradient of the graph at the point where $x = a$ is:

$y'(a) = 2a - 2$

The equation of the tangent at point $(a, a^2 - 2a + 2)$ is:

$y - (a^2 - 2a + 2) = (2a - 2)(x - a) \Rightarrow y = (2a - 2)x - a^2 + 2$

The origin lies on the tangent, so its coordinates must satisfy the equation of this line:

$\Rightarrow 0 = (2a - 2) \cdot 0 - a^2 + 2 \Rightarrow a^2 - 2 = 0 \Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2}$

The two points of tangency are:

$a = \sqrt{2} \Rightarrow b = \sqrt{2}^2 - 2\sqrt{2} + 2 \Rightarrow b = 4 - 2\sqrt{2} \Rightarrow (\sqrt{2}, 4 - 2\sqrt{2})$

$a = -\sqrt{2} \Rightarrow b = (-\sqrt{2})^2 - 2(-\sqrt{2}) + 2 \Rightarrow b = 4 + 2\sqrt{2} \Rightarrow (-\sqrt{2}, 4 + 2\sqrt{2})$

The gradients of the two lines are:

$a = \sqrt{2} \Rightarrow y'(\sqrt{2}) = 2\sqrt{2} - 2$

$a = -\sqrt{2} \Rightarrow y'(-\sqrt{2}) = -2\sqrt{2} - 2$

Consequently, the equations of the two tangent lines are:

At $(\sqrt{2}, 4 - 2\sqrt{2})$: $y - (4 - 2\sqrt{2}) = (2\sqrt{2} - 2)(x - \sqrt{2}) \Rightarrow y = (2\sqrt{2} - 2)x$

At $(-\sqrt{2}, 4 + 2\sqrt{2})$: $y - (4 + 2\sqrt{2}) = (-2\sqrt{2} - 2)(x - (-\sqrt{2}))$

$\Rightarrow y = (-2\sqrt{2} - 2)x + 4 + 2\sqrt{2} - 4 - 2\sqrt{2} \Rightarrow y = -(2\sqrt{2} + 2)x$

14. **(a)** $x=8\Rightarrow y=\sqrt[3]{8}\Rightarrow y=2\Rightarrow$ the point of tangency is $(8,2)$.

The first derivative is:

$$y=x^{\frac{1}{3}}\Rightarrow y'=\frac{1}{3}x^{-\frac{2}{3}}\Rightarrow y'=\frac{1}{3x^{\frac{2}{3}}}$$

The gradient of the tangent, *m*, is found by substituting $x=8$ into the expression of y':

$$\Rightarrow m=\frac{1}{3\cdot 8^{\frac{2}{3}}}\Rightarrow m=\frac{1}{3\cdot 4}\Rightarrow m=\frac{1}{12}$$

The equation of the tangent at point $(8,2)$ is:

$$y-2=\frac{1}{12}(x-8)\Rightarrow y=\frac{1}{12}x-\frac{2}{3}+2\Rightarrow y=\frac{1}{12}x+\frac{4}{3}$$

(b) Substitute $x=9$ into the equation of the tangent:

$$y=\frac{1}{12}\cdot 9+\frac{4}{3}\Rightarrow y=\frac{3}{4}+\frac{4}{3}\Rightarrow y=\frac{25}{12}\Rightarrow y=2.0833...\Rightarrow \sqrt[3]{9}\approx 2.08$$

15. $x=a\Rightarrow y=\frac{1}{\sqrt{a}}\Rightarrow$ the point of tangency is $\left(a,\frac{1}{\sqrt{a}}\right)$

The first derivative is:

$$y=x^{-\frac{1}{2}}\Rightarrow y'=-\frac{1}{2}x^{-\frac{3}{2}}\Rightarrow y'=-\frac{1}{2x^{\frac{3}{2}}}$$

The gradient of the tangent, *m*, is found by substituting $x=a$ into the expression of y':

$$\Rightarrow m=-\frac{1}{2a^{\frac{3}{2}}}\Rightarrow m=-\frac{1}{2a\sqrt{a}}$$

The equation of the tangent at point $\left(a,\frac{1}{\sqrt{a}}\right)$ is:

$$y-\frac{1}{\sqrt{a}}=-\frac{1}{2a\sqrt{a}}(x-a)\Rightarrow y=-\frac{1}{2a\sqrt{a}}x+\frac{1}{2\sqrt{a}}+\frac{1}{\sqrt{a}}\Rightarrow y=-\frac{1}{2a\sqrt{a}}x+\frac{3}{2\sqrt{a}}$$

16. Let $P(a,b)$ be the point of tangency.

$x=a\Rightarrow y=a^3\Rightarrow$ the point of tangency is $(a,\,a^3)$.

The first derivative is $y'=3x^2$.

The gradient of the tangent, *m*, is found by substituting $x=a$ into the expression of y'

$\Rightarrow m=3a^2$

The equation of the tangent at point (a,a^3) is: $y-a^3=3a^2(x-a)\Rightarrow y=3a^2x-2a^3$

To find where the two graphs meet again, equate the expression of the function and the one of the tangent and solve for x:

$x^3 = 3a^2x - 2a^3 \Rightarrow x^3 - 3a^2x + 2a^3 = 0$

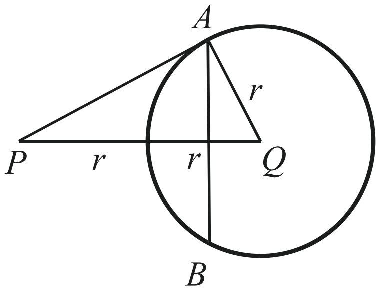
As $x = a$ is a solution to this equation, it follows that $(x-a)$ is a factor. Use synthetic (or long) division to factorise the cubic $\Rightarrow (x-a)(x^2+ax-2a^2) = 0$

The required x-coordinates can be found by solving the equation:

$x^2 + ax - 2a^2 = 0 \Rightarrow (x-a)(x+2a) = 0 \Rightarrow x = a, x = -2a$

Consequently, point Q has coordinates $x = -2a$ and $y = (-2a)^3 = -8a^3 \Rightarrow Q(-2a, -8a^3)$ or $Q(-2a, -8b)$.

17. **Method 1** A geometric solution

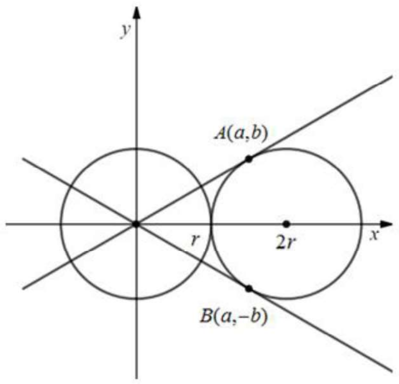


In triangle APQ with QA being the radius at the point of tangency and hence perpendicular to the tangent: $AP = \sqrt{4r^2 - r^2} = r\sqrt{3}$

Also, $\hat{P} = 30°$ because the side opposite to it is half the hypotenuse. It can be easily shown that triangle APB is equilateral, and thus $AB = AP = r\sqrt{3}$.

Method 2 Using calculus

Consider a set of axes with the origin $O(0,0)$ at the center of the first circle. This means that the center of the second circle is at point $(2r, 0)$, where r is the radius of the circles.



Let point *A* have coordinates (a,b), it follows that *B* has coordinates $(a,-b)$.

The distance *AB* is vertical, so $AB = b-(-b) \Rightarrow AB = 2b$.

The equation of the circle centered at $(2r,0)$ is $(x-2r)^2+y^2=r^2$

$$x=a \Rightarrow b=\pm\sqrt{r^2-(a-2r)^2} \Rightarrow b=\pm\sqrt{-a^2+4ar-3r} \Rightarrow A(a,\sqrt{-a^2+4ar-3r^2})$$

and $B(a,-\sqrt{-a^2+4ar-3r^2})$ are the two points of tangency.

To write the equation of the tangent at point *A*, start by finding the first derivative:

$$y=\sqrt{r^2-(x-2r)^2} \Rightarrow y=(-x^2+4xr-3r^2)^{\frac{1}{2}} \Rightarrow y'=\frac{1}{2}(-x^2+4xr-3r^2)^{-\frac{1}{2}}\cdot(-2x+4r)$$

$$\Rightarrow y'=\frac{2r-x}{\sqrt{-x^2+4xr-3r^2}}$$

The gradient of the tangent, *m*, is found by substituting $x=a$ into the expression of y':

$$\Rightarrow m=\frac{2r-a}{\sqrt{-a^2+4ar-3r^2}}$$

The gradient of the curve representing the top semicircle is the same as the gradient of the tangent line.

The gradient of the tangent line OA can be calculated as the gradient of the line joining the origin, $O(0,0)$, and $A(a,\sqrt{-a^2+4ar-3r^2})$:

$$m_{OA}=\frac{\sqrt{-a^2+4ar-3r^2}-0}{a-0} \Rightarrow m_{OA}=\frac{\sqrt{-a^2+4ar-3r^2}}{a}$$

But $m=m_{OA} \Rightarrow \dfrac{2r-a}{\sqrt{-a^2+4ar-3r^2}}=\dfrac{\sqrt{-a^2+4ar-3r^2}}{a} \Rightarrow -a^2+4ar-3r^2=2ar-a^2$

$$\Rightarrow 2ar-3r^2=0 \Rightarrow 2ar=3r^2 \Rightarrow a=\frac{3r}{2}$$

$$\Rightarrow b=\sqrt{-(\frac{3r}{2})^2+4(\frac{3r}{2})r-3r^2} \qquad \Rightarrow b=\sqrt{-\frac{9r^2}{4}+3r^2} \Rightarrow b=\sqrt{\frac{3r^2}{4}} \Rightarrow b=\frac{r\sqrt{3}}{2}$$

Consequently, the distance *AB* is: $AB=2\cdot\dfrac{r\sqrt{3}}{2} \Rightarrow AB=r\sqrt{3}$

18. Let $P(a,b)$ be the point of tangency.

$x=a \Rightarrow y=4-a^2 \Rightarrow$ the point of tangency is $(a,4-a^2)$.

The first derivative is $y'=-2x$.

The gradient of the tangent, *m*, is found by substituting $x=a$ into the expression of y':

$\Rightarrow m=-2a$

The equation of the tangent at point $(a,4-a^2)$ is:

$y-(4-a^2)=-2a(x-a)\Rightarrow y=-2ax+2a^2+4-a^2\Rightarrow y=-2ax+a^2+4$

If point $(1,2)$ lies on any of the tangent lines, then its coordinates should satisfy the equation of the tangent:

$2=-2a\cdot 1+a^2+4\Rightarrow a^2-2a+2=0\Rightarrow(a-1)^2+1=0\Rightarrow(a-1)^2=-1\Rightarrow a\notin\mathbb{R}$

This quadratic equation does not have any real solutions, so there is no line going through $(1,2)$ which is tangent to the graph of the given function.

Chapter 12 practice questions

1. **(a)** $f(x)=x^2\Rightarrow f'(x)=2x$

The gradient of *f* at $x=1.5$, *m*, is:

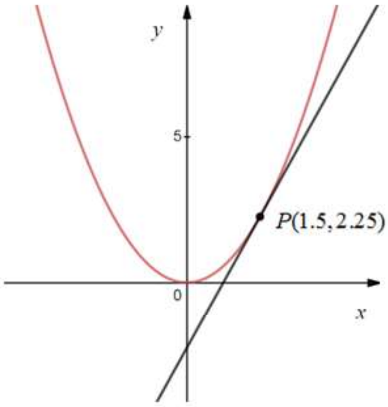
$m=f'(1.5)=2\cdot 1.5=3$

(b) $f(x)=x^2\Rightarrow y=1.5^2\Rightarrow y=2.25\Rightarrow$ the tangency point is $P(1.5,2.25)$

The equation of the tangent at this point is:

$y-2.25=3(x-1.5)\Rightarrow y=3x-4.5+2.25\Rightarrow y=3x-2.25\Rightarrow y=3x-\dfrac{9}{4}$

(c) The graph is shown below:



(d) Point *Q* is the *x*-intercept of the line, so $y=0$:

$0=3x-\dfrac{9}{4}\Rightarrow 3x=\dfrac{9}{4}\Rightarrow x=\dfrac{3}{4}\Rightarrow Q\left(\dfrac{3}{4},0\right)$

Point *R* is the *y*-intercept of the line, so $x=0$:

$y=3\cdot 0-\dfrac{9}{4}\Rightarrow y=-\dfrac{9}{4}\Rightarrow R\left(0,-\dfrac{9}{4}\right)$

(e) Find the coordinates of *M*, the midpoint of line segment *PR*:

$$x_M = \frac{x_P + x_R}{2} \Rightarrow x_M = \frac{1.5+0}{2} \Rightarrow x_M = 0.75 \Rightarrow x_M = \frac{3}{4}$$

$$y_M = \frac{y_P + y_R}{2} \Rightarrow y_M = \frac{2.25 - \frac{9}{4}}{2} \Rightarrow y_M = \frac{0}{2} \Rightarrow y_M = 0$$

$\Rightarrow M\left(\frac{3}{4}, 0\right)$ is the midpoint between *P* and *R*.

This point has the same coordinates as point *Q*, so *Q* is the midpoint of $[PR]$.

(f) The gradient of *f* at *x* = *a* is:
$f'(a) = 2a$

The equation of the tangent at point $S(a, a^2)$ is:

$$y - a^2 = 2a(x-a) \Rightarrow y = 2ax - 2a^2 + a^2 \Rightarrow y = 2ax - a^2$$

(g) Point *T* is the *x*-intercept of the line, so *y* = 0.

$$0 = 2ax - a^2 \Rightarrow 2ax = a^2 \Rightarrow x = \frac{a}{2} \Rightarrow T\left(\frac{a}{2}, 0\right), a \neq 0$$

Point *U* is the *y*-intercept of the line, so $x = 0$:

$$y = 2a \cdot 0 - a^2 \Rightarrow y = -a^2 \Rightarrow R(0, -a^2), a \neq 0$$

(h) Find the coordinates of *N*, the midpoint of line segment *SU*:

$$x_N = \frac{x_S + x_U}{2} \Rightarrow x_N = \frac{a+0}{2} \Rightarrow x_N = \frac{a}{2}$$

$$y_N = \frac{y_S + y_U}{2} \Rightarrow y_N = \frac{a^2 + (-a^2)}{2} \Rightarrow y_n = \frac{0}{2} \Rightarrow y_n = 0$$

$\Rightarrow N\left(\frac{a}{2}, 0\right)$ is the midpoint between *S* and *U*.

This point has the same coordinates as point *T*, so *T* is the midpoint of $[SU]$ for any $a \in \mathbb{R}, a \neq 0$.

2. The first derivative needs to be found:

$$y = Ax + B + Cx^{-1} \Rightarrow y' = A + 0 + C \cdot (-1)x^{-2} \Rightarrow y' = A - \frac{C}{x^2}$$

If the point $(1, 4)$ is a stationary point for the graph, then the first derivative of the function takes value 0 when $x = 1$:

$$y' = A - \frac{C}{x^2} \Rightarrow 0 = A - \frac{C}{1^2} \Rightarrow 0 = A - C \Rightarrow A = C$$

The equation of the graph is now $y = Ax + B + \frac{A}{x}$.

Also, if points $(1, 4)$ and $(-1, 0)$ are on the graph, then their coordinates must satisfy the equation of the curve:

$$4 = A(1) + B + \frac{A}{1} \Rightarrow 2A + B = 4$$

$$0 = A(-1) + B + \frac{A}{-1} \Rightarrow -2A + B = 0$$

Subtracting these two equations leads to:

$$4A = 4 \Rightarrow A = 1 \Rightarrow B = 4 - 2(1) \Rightarrow B = 2$$

The values of A, B and C are: $A = 1, B = 2, C = 1$

3. **(a)** Find the expression of the first derivative and set it equal to 0, then solve for x:

$$f(x) = 8x^{-1} + 2x \Rightarrow f'(x) = 8(-1)x^{-2} + 2 \Rightarrow f'(x) = -\frac{8}{x^2} + 2$$

$$0 = -\frac{8}{x^2} + 2 \Rightarrow \frac{8}{x^2} = 2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

But $x > 0$, so $x = 2$

To have a turning point at $x = 2$, there must be a change in the sign of f' about this point:

$$f'(x) = -\frac{8}{x^2} + 2 < 0 \Rightarrow x^2 < 4 \Rightarrow x < 2 \text{(recall that } x > 0\text{)}$$

Similarly, $f'(x) = -\frac{8}{x^2} + 2 > 0 \Rightarrow x^2 > 4 \Rightarrow x > 2$

Consequently, there is a change in the sign of f' about $x = 2$, so this point is a turning point for the graph of the function.

(b) The limit when $x \to 0$ has to be evaluated when the values of x decrease towards 0 ($x > 0$):

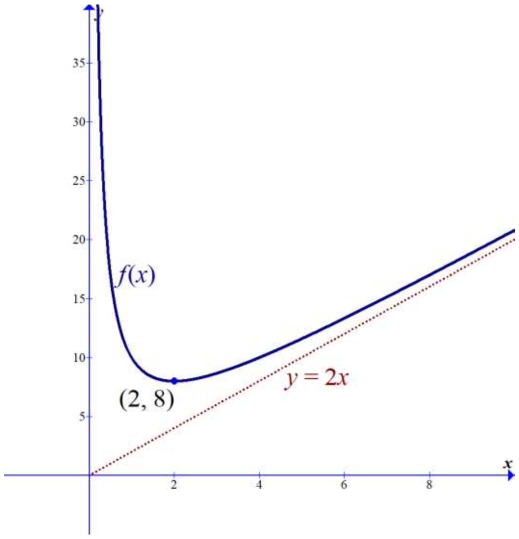
$$\lim_{\substack{x\to 0\\ x>0}} \left(\frac{8}{x} + 2x\right) = \lim_{\substack{x\to a\\ x>a}} \frac{8}{x} + \lim_{\substack{x\to a\\ x>a}} 2x = +\infty + 0 = +\infty$$

Consequently, the graph has a vertical asymptote, $x = 0$.

To determine the existence of a horizontal asymptote, the end behavior of the function will be investigated, by considering the limit of the function when $x \to \infty$:

$$\lim_{x\to\infty}\left(\frac{8}{x}+2x\right)=\lim_{x\to\pm\infty}\left(\frac{8}{x}\right)+\lim_{x\to\pm\infty}2x=0+\infty=\infty$$

This means the values of the function will increase towards $+\infty$ when $x\to+\infty$, so there is no horizontal asymptote. It must be noticed that the values of the given function will approach $\pm\infty$ in the same manner as the values of the linear function $y=2x$, consequently, the graph has an oblique asymptote). The graph of the function is shown below:



4. To find the stationary point, determine the expression of the first derivative and set it equal to 0, then solve for *x*:

$$y'=8x-\frac{1}{x^2}=0\Rightarrow\frac{8x^3-1}{x^2}=0\Rightarrow 8x^3-1=0\Rightarrow x=\frac{1}{2}$$

The corresponding *y*-coordinate is $y=4\left(\frac{1}{2}\right)^2+\frac{1}{\frac{1}{2}}\Rightarrow y=3\Rightarrow$ the stationary point is $\left(\frac{1}{2},3\right)$.

5. Find the expression of the first derivative: $y'=3ax^2-4x-1$

$y'(2)=3\Rightarrow 3=12a-9\Rightarrow a=1$.

6. **(a)** The equation of the tangent to the graph of $y=f(x)$ at the point where $x=2$ is:

$y-f(2)=f'(2)(x-2)\Rightarrow y-3=5(x-2)\Rightarrow y=5x-7$

(b) The equation of the normal to the graph of $y=f(x)$ at the point where $x=2$ is:

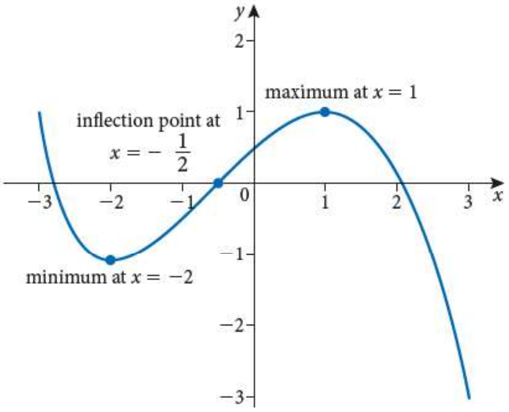
$$y-f(2)=-\frac{1}{f'(2)}(x-2)\Rightarrow y-3=-\frac{1}{5}(x-2)\Rightarrow y=-\frac{1}{5}x+\frac{17}{5}$$

7. **(a)** The maximum point is where $x=1$, because $g'(1)=0$, $g'(x)>0$ when $x<1$, and $g'(x)<0$, when $x>1$. That is, the values of the function increase for $-2<x<1$ and decrease for $1<x<3$.

(b) The function *g* is decreasing on the intervals where $g'(x)<0$, so *g* is decreasing for $-3<x<-2$ or $1<x<3$.

(c) The inflection point is where $x=-\frac{1}{2}$, because $g''\left(-\frac{1}{2}\right)=0$, $g''(-1)>0$ and $g''(0)<0$, meaning there is a change in the concavity of the graph of *g*, from concave up to concave down.

(d) The graph is shown below:



8. $f(1)=0 \Rightarrow 1^2-3b\cdot 1+(c+2)=0 \Rightarrow -3b+c=-3$

Find the expression of the first derivative:

$f(x)=x^2-3bx+(c+2) \Rightarrow f'(x)=2x-3b$

Use $f'(3)=0$ to form another equation:

$0=2\cdot 3-3b \Rightarrow 6-3b=0 \Rightarrow 3b=6 \Rightarrow b=2$

Substitute in the first equation to obtain the value of *c*: $-3\cdot 2+c=-3 \Rightarrow c=3$

9. Graph **(a)** is the graph of the derivative of f_4, because it represents a constant function. This type of function is the derivative of a linear function (in this case with a negative gradient as the horizontal line is below the *x*-axis).

Graph **(b)** is the graph of the derivative of f_3, because it has three *x*-intercepts, which correspond to three stationary points for the original graph. The only graph satisfying this condition is the graph of f_3.

Graph **(c)** is the graph of the derivative of f_1, because it has a *x*-intercept at the origin. Also, there is a change in the sign of the derivative about $x = 0$ (before $x = 0$ the values of the derivative are negative, while past $x = 0$ the derivative is positive), which means the original graph has a minimum at $x = 0$. The only graph satisfying this condition is the graph of f_1.

Graph **(d)** is the graph of the derivative of f_2, because this graph has a *x*-intercept at the origin and negative derivative values elsewhere, meaning the original function is decreasing for all *x* except $x = 0$, where it has a horizontal tangent. The only graph displaying this behaviour is the graph of f_1.

10. **(a)** When $x = 0 \Rightarrow y = f(0) = 1 + \sin 0 = 1$, when

$$x = \frac{\pi}{2} \Rightarrow y = f\left(\frac{\pi}{2}\right) = 1 + \sin\left(\frac{\pi}{2}\right) = 1 + 1 = 2$$

The average rate of change is the gradient of the line going through points $(0,1)$ and $\left(\frac{\pi}{2}, 2\right) \Rightarrow$ average rate of change $= \frac{2-1}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

(b) The first derivative of *f* has to be found:

$f(x) = 1 + \sin x \Rightarrow f'(x) = \cos x$

The instantaneous rate of change of *f* at $x = \frac{\pi}{4}$ is $f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

(c) $\cos x = \frac{2}{\pi} \Rightarrow x = \arccos\frac{2}{\pi} \Rightarrow x = 0.88068... \Rightarrow x \approx 0.881$

11. **(a)** **(i)** To find the equation of the vertical asymptote, equate the denominator of the rational function to 0 and solve for *x*. It follows that $x = 0$ is the equation of the vertical asymptote.

(ii) Evaluate the limit of the function when $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{3x-2}{x} = \lim_{x\to\pm\infty} \frac{3x}{x} - \lim_{x\to\pm\infty} \frac{2}{x} = 3 - 0 = 3$$

As the limit is finite, there is a horizontal asymptote, its equation is $y = 3$.

(b) $f(x) = \frac{3x-2}{x} = 3 - 2x^{-1} \Rightarrow f'(x) = 0 - 2(-1)x^{-2} = \frac{2}{x^2}$

(c) $f'(x) = \frac{2}{x^2} > 0$ for all $x \in \mathbb{R}, x \neq 0$, this means that the function is increasing for all $x \in \mathbb{R}, x \neq 0$

(d) The value of the first derivative at a stationary point is 0. As $f'(x) > 0$ for all $x \in \mathbb{R}, x \neq 0$, it will never take value 0, so there are no stationary point on the graph of *f*.

12. To find the stationary points, the equation $y' = 0$ must be solved.

$y' = 4x - 4x^3 = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0, x = \pm 1$

To determine the nature of the stationary points, the second derivative is used:

$y'' = 4 - 12x^2$

Substitute the *x*-coordinates of the stationary points into the expression of y'', to determine the concavity of the function:

$y''(0) = 4 - 12 \cdot 0^2 \Rightarrow y''(0) = 4 > 0 \Rightarrow$ the function is concave up, so there is a minimum at the point where $x = 0$.

$y''(1) = 4 - 12 \cdot 1^2 = -8 < 0 \Rightarrow$ the function is concave down, so there is a maximum at the point where $x = 1$.

$y''(-1) = 4 - 12 \cdot (-1)^2 = -8 < 0 \Rightarrow$ the function is concave down, so there is a maximum at the point where $x = -1$.

To find the *y*-coordinates, substitute the three *x*-values into the expression of the original function:

$x = 0 \Rightarrow y = y = 2 \cdot 0^2 - 0^4 \Rightarrow y = 0$; $x = 1 \Rightarrow y = 2 \cdot 1^2 - 1^4 \Rightarrow y = 1$;

$x = -1 \Rightarrow y = 2(-1)^2 - (-1)^4 \Rightarrow y = 1$

There are two points of maximum, at $(-1, 1)$ and $(1, 1)$, and a minimum point at $(0, 0)$.

13. The gradient of the curve at $x = 1$ is:

$$y = x^{\frac{1}{2}} + x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} \Rightarrow y'(1) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

The gradient of the normal line is the negative reciprocal of the gradient of the curve, so the equation of the normal at $(1, 2)$ is:

$$y - 2 = -\frac{6}{5}(x - 1) \Rightarrow y = -\frac{6}{5}x + \frac{6}{5} + 2 \Rightarrow y = -\frac{6}{5}x + \frac{16}{5}$$

Points $(a, 0)$ and $(0, b)$ satisfy the equation of the normal:

$$0 = -\frac{6}{5}a + \frac{16}{5} \Rightarrow a = \frac{16}{6} = \frac{8}{3}; \; b = \frac{6}{5} \cdot 0 + \frac{16}{5} \Rightarrow b = \frac{16}{5}$$

14. **(a)** The velocity is the rate of change of displacement, so the first derivative of *s* has to be found:

$$s(t)=10t-\frac{1}{2}t^2 \Rightarrow v(t)=s'(t)=10-t \Rightarrow v(0)=10-0 \Rightarrow v(0)=10 \text{ ms}^{-1}$$

(b) $0=10-t \Rightarrow t=10\text{ s}$

(c) Substitute $t=10$ into the expression of *s*: $s(10)=10\cdot 10-\frac{1}{2}\cdot 10^2 \Rightarrow s(10)=50\text{ m}$

15. **(a)** $s(t)=14t-4.9t^2 \Rightarrow s'(t)=14-9.8t$

$\Rightarrow s''(t)=-9.8$

$\Rightarrow v(t)=14-9.8t,\ a(t)=-9.8$

(b) Solve the equation $s'(t)=0$ to find the time needed to reach the maximum height:

$$0=14-9.8t \Rightarrow 9.8t=14 \Rightarrow t=\frac{14}{9.8} \Rightarrow t=1.428... \Rightarrow t\approx 1.43\text{ s}$$

The maximum height is:

$s(1.428...)=14\cdot 1.428...-4.9\cdot 1.428...^2 \Rightarrow s(1.428...)=10\text{ m}$

It takes approximately 1.43 seconds for the ball to reach the maximum height of 10 meters.

(c) The ball's velocity is 0 at the maximum height (at the maximum point $s'(t)=0$, which is the same as $v(t)=0$).

The acceleration is -9.8 ms^{-2}, as it is constant at all times.

16. To find the *x*-coordinate of the point of inflection, solve the equation $y''=0$.

$y=x^3+12x^2-x-12 \Rightarrow y'=3x^2+24x-1 \Rightarrow y''=6x+24$

$\Rightarrow 0=6x+24 \Rightarrow x=-4$

The corresponding *y*-coordinate is:

$y=(-4)^3+12\cdot(-4)^2-(-4)-12 \Rightarrow y=120$

The point of inflection is at $(-4,120)$

17. **(a)** $x=\frac{\pi}{3} \Rightarrow y=2\cos\frac{\pi}{3}-3 \Rightarrow y=2\cdot\frac{1}{2}-3 \Rightarrow y=-2$.

The point of tangency is $\left(\frac{\pi}{3},-2\right)$.

$y=2\cos x-3 \Rightarrow y'=2(-\sin x) \Rightarrow y'=-2\sin x$

$\Rightarrow m=-2\sin\frac{\pi}{3} \Rightarrow m=-2\cdot\frac{\sqrt{3}}{2} \Rightarrow m=-\sqrt{3}$

The equation of the tangent line is:

$$y-(-2)=-\sqrt{3}\left(x-\frac{\pi}{3}\right)\Rightarrow y=-\sqrt{3}x+\frac{\pi\sqrt{3}}{3}-2$$

(b) The gradient of the normal is the negative reciprocal of the gradient of the tangent, so the equation of the normal at $\left(\frac{\pi}{3},-2\right)$ is:

$$y-(-2)=-\frac{1}{-\sqrt{3}}\left(x-\frac{\pi}{3}\right)\Rightarrow y=\frac{\sqrt{3}}{3}x-\frac{\pi\sqrt{3}}{9}-2$$

18. **(a)** The total surface area of a cylinder, A, is given by the sum of the curved surface area and the area of the two circular bases:

$$A=2\pi rh+2\cdot\pi r^2\Rightarrow 54\pi=2\pi rh+2\pi r^2$$

Make h the subject:

$$2\pi rh=54\pi-2\pi r^2\Rightarrow h=\frac{2\pi(27-r^2)}{2\pi r}\Rightarrow h=\frac{27-r^2}{r}$$

The expression of the volume of the cylinder is:

$$V=\pi r^2h\Rightarrow V=\pi r^2\cdot\frac{27-r^2}{r}\Rightarrow V=\pi r(27-r^2)$$

(b) To find the value of the radius for which the maximum volume is obtained, solve for r the equation $V'=0$:

$$V=27\pi r-\pi r^3\Rightarrow V'=27\pi-3\pi r^2$$

$$0=27\pi-3\pi r^2\Rightarrow 27\pi=3\pi r^2\Rightarrow r^2=9\Rightarrow r=\pm3$$

But $r>0\Rightarrow r=3$

19. Point $(0,10)$ is on the curve $\Rightarrow 10=a(0)^2+b(0)+c\Rightarrow c=10$

The equation of the function becomes $y=ax^2+bx+10$

Point $(2,18)$ is also on the curve: $18=a\cdot2^2+b\cdot2+10\Rightarrow 4a+2b=8\Rightarrow 2a+b=4$

The curve has a maximum at $(2,18)$, so $y'(2)=0$:

$$y=ax^2+bx+10\Rightarrow y'=2ax+b\Rightarrow 0=2a\cdot2+b\Rightarrow 4a+b=0$$

Solve the simultaneous equations to obtain the values of a and b:

$$b=4-2a\Rightarrow 4a+4-2a=0\Rightarrow 2a=-4\Rightarrow a=-2$$

$$\Rightarrow b=4-2(-2)\Rightarrow b=8$$

20. **(a)** $x=-2\Rightarrow y=\frac{1}{2}x^2-5x+3\Rightarrow y=\frac{1}{2}(-2)^2-5(-2)+3\Rightarrow y=15\Rightarrow$ the point of tangency is $(-2,15)$.

$y'=x-5\Rightarrow m=-2-5\Rightarrow m=-7$

The equation of the tangent line is:

$y-15=-7(x-(-2))\Rightarrow y=-7x-14+15\Rightarrow y=-7x+1$

(b) The gradient of the normal is the negative reciprocal of the gradient of the tangent, so the equation of the normal at $(-2,15)$ is:

$$y-15=-\frac{1}{-7}(x-(-2))\Rightarrow y=\frac{1}{7}x+\frac{2}{7}+15\Rightarrow y=\frac{1}{7}x+\frac{107}{7}$$

21. **(a)** Solve the equation $y'=0$ to find the x-coordinates of the stationary points:

$$y'=4x^3-3x^2\Rightarrow y'=x^2(4x-3)=0\Rightarrow x=0,x=\frac{3}{4}$$

To determine the nature of the two stationary points, the second derivative is used:

$y''=12x^2-6x$

Substitute the x-coordinates of the stationary points into the expression of y'', to determine the concavity of the function:

$y''(0)=12\cdot0^2-6\cdot0=0\Rightarrow$ this is not a maximum nor minimum point (there is an inflection point at $x=0$, and the tangent at this point is horizontal)

$y''\left(\frac{3}{4}\right)=12\left(\frac{3}{4}\right)^2-6\left(\frac{3}{4}\right)=\frac{27}{4}-\frac{18}{4}=\frac{9}{4}>0\Rightarrow$ the function is concave up, so there is a minimum at $x=\frac{3}{4}$.

To find the y-coordinates, substitute the two x-values into the expression of the original function:

$x=0\Rightarrow y=0^4-0^3=0$

$$x=\frac{3}{4}\Rightarrow y=\left(\frac{3}{4}\right)^4-\left(\frac{3}{4}\right)^3\Rightarrow y=\frac{81}{256}-\frac{27}{64}\Rightarrow y=-\frac{27}{256}$$

There is a point of minimum (relative, and absolute, as it is the only one) at $M\left(\frac{3}{4},-\frac{27}{256}\right)$ and a stationary point of inflection at $(0,0)$.

(b) The domain is $x \in \mathbb{R}$. For the range, the end behavior of the function has to be investigated:

$$\lim_{x\to-\infty}(x^4 - x^3) = \lim_{x\to-\infty} x^3(x-1) = -\infty \cdot -\infty = +\infty$$

$$\lim_{x\to\infty}(x^4 - x^3) = \lim_{x\to\infty} x^3(x-1) = \infty \cdot \infty = +\infty$$

As the values of the function become infinitely large when $x \to \pm\infty$, and there is an absolute minimum at $\left(\frac{3}{4}, -\frac{27}{256}\right)$, it follows that the range is $y \geq -\frac{27}{256}$.

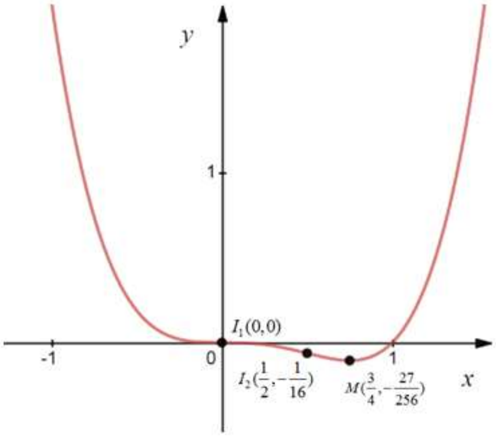
(c) To find the x-coordinate of the point of inflection, solve the equation $y'' = 0$.

$$0 = 12x^2 - 6x \Rightarrow 6x(2x-1) = 0 \Rightarrow x = 0, x = \frac{1}{2}$$

$$x = \frac{1}{2} \Rightarrow y = \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3 \Rightarrow y = -\frac{1}{16}$$

There are two points of inflection: $I_1(0,0)$ and $I_2\left(\frac{1}{2}, -\frac{1}{16}\right)$.

(d) The graph is shown below:



22. **(a)**
$$\lim_{x\to\infty}\frac{2-3x+5x^2}{8-3x^2} = \lim_{x\to\infty}\frac{x^2\left(\frac{2}{x^2}-\frac{3}{x}+5\right)}{x^2\left(\frac{8}{x^2}-3\right)} = \lim_{x\to\infty}\frac{\frac{2}{x^2}-\frac{3}{x}+5}{\frac{8}{x^2}-3} = \frac{0-0+5}{0-3} = -\frac{5}{3}$$

(b)
$$\lim_{x\to0}\frac{\sqrt{x+4}-2}{x} = \lim_{x\to0}\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x\to0}\frac{(\sqrt{x+4})^2-2^2}{x(\sqrt{x+4}+2)}$$

$$= \lim_{x\to0}\frac{x+4-4}{x(\sqrt{x+4}+2)} = \lim_{x\to0}\frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$$

(c) $$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=3$$

(d) $$\lim_{h\to 0}\frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}=\lim_{h\to 0}\frac{\sqrt{(x+h)+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{(x+h)+2}+\sqrt{x+2}}{\sqrt{(x+h)+2}+\sqrt{x+2}}$$

$$=\lim_{h\to 0}\frac{(\sqrt{(x+h)+2})^2-(\sqrt{x+2})^2}{h(\sqrt{(x+h)+2}+\sqrt{x+2})}=\lim_{h\to 0}\frac{(x+h)+2-(x+2)}{h(\sqrt{(x+h)+2}+\sqrt{x+2})}$$

$$=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h+2}+\sqrt{x+2}}=\frac{1}{2\sqrt{x+2}}$$

23. **(a)** $$f(x)=x^{\frac{3}{2}}-4x^{\frac{1}{2}}\Rightarrow f'(x)=\frac{3}{2}x^{\frac{1}{2}}-4\cdot\frac{1}{2}x^{-\frac{1}{2}}\Rightarrow f'(x)=\frac{3}{2}\sqrt{x}-\frac{2}{\sqrt{x}}$$

$$\Rightarrow f'(x)=\frac{3x-4}{2\sqrt{x}}$$

(b) $$f(x)=x^3-3\sin x\Rightarrow f'(x)=3x^2-3\cos x$$

(c) $$f(x)=\frac{1}{x}+\frac{x}{2}\Rightarrow f(x)=x^{-1}+\frac{1}{2}x\Rightarrow f'(x)=(-1)x^{-2}+\frac{1}{2}\Rightarrow f'(x)=-\frac{1}{x^2}+\frac{1}{2}$$

(d) $$f(x)=\frac{7}{3x^{13}}\Rightarrow f(x)=\frac{7}{3}x^{-13}\Rightarrow f'(x)=\frac{7}{3}\cdot(-13)x^{-14}\Rightarrow f'(x)=-\frac{91}{3x^{14}}$$

24. The point (p,q) is on the graph of the given function, it follows that its coordinates satisfy the equation of the curve:

$\Rightarrow q=p^3+p^2-9p-9\Rightarrow$ the tangency point has coordinates (p,p^3+p^2-9p-9).

The expression of the first derivative is:

$y=x^3+x^2-9x-9\Rightarrow y'=3x^2+2x-9$

The gradient of the graph at the point where $x=p$ is:

$y'(p)=3p^2+2p-9$

The equation of the tangent at point (p,p^3+p^2-9p-9) is:

$y-(p^3+p^2-9p-9)=(3p^2+2p-9)(x-p)$

$\Rightarrow y=(3p^2+2p-9)x-p(3p^2+2p-9)+(p^3+p^2-9p-9)$

$\Rightarrow y=(3p^2+2p-9)x-2p^3-p^2-9$

The given point $(4,-1)$ lies on the tangent, so its coordinates must satisfy the equation of this line:

$\Rightarrow -1=(3p^2+2p-9)4-2p^3-p^2-9\Rightarrow -2p^3+11p^2+8p-44=0$

The equation $-2p^3+11p^2+8p-44=0$ has three solutions (GDC): $x=\frac{11}{2}, x=2, x=-2$

The three points of tangency are:

$$p=\frac{11}{2}\Rightarrow q=\left(\frac{11}{2}\right)^3+\left(\frac{11}{2}\right)^2-9\left(\frac{11}{2}\right)-9\Rightarrow b=\frac{1105}{8}\Rightarrow\left(\frac{11}{2},\frac{1105}{8}\right)$$

$$p=-2\Rightarrow b=(-2)^3+(-2)^2-9(-2)-9\Rightarrow b=5\Rightarrow(-2,5)$$

$$p=2\Rightarrow b=2^3+2^2-9\cdot 2-9\Rightarrow b=-15\Rightarrow(2,-15)$$

25. The gradient of the normal is $-\frac{1}{12}$, so the gradient of the tangent is 12.

The first derivative of the function is:

$$y=x^3+\frac{1}{3}\Rightarrow y'=3x^2$$

At the point of tangency, the gradient of the function and the gradient of the tangent are equal:

$$\Rightarrow 3x^2=12\Rightarrow x^2=4\Rightarrow x=\pm 2$$

The corresponding y-coordinates are:

$$x=-2\Rightarrow y=(-2)^3+\frac{1}{3}\Rightarrow y=-8+\frac{1}{3}\Rightarrow y=-\frac{23}{3}$$

$$x=2\Rightarrow y=2^3+\frac{1}{3}\Rightarrow y=8+\frac{1}{3}\Rightarrow y=\frac{25}{3}$$

The value of c can be found by substituting the coordinates of the tangency points into the equation of the normal line:

$$\left(-2,-\frac{23}{3}\right)\Rightarrow -\frac{23}{3}=-\frac{1}{12}(-2)+c\Rightarrow c=-\frac{1}{6}-\frac{23}{3}\Rightarrow c=-\frac{47}{6}<0$$

$$\left(2,\frac{25}{3}\right)\Rightarrow \frac{25}{3}=-\frac{1}{12}\cdot 2+c\Rightarrow c=\frac{1}{6}+\frac{25}{3}\Rightarrow c=\frac{51}{6}\Rightarrow c=\frac{17}{2}>0$$

As $c\geq 0\Rightarrow c=\frac{17}{2}$

26. It is known that two parallel lines have the same gradient, this means that, at the point (a,b) on the graph of the function, the gradient of the curve (which is the same as the gradient of the tangent line), must be the same as the gradient of the given line, namely 3. The gradient of the curve is given by the first derivative:

$$y=\frac{1}{3}x^3-x\Rightarrow y'=\frac{1}{3}\cdot 3x^2-1\Rightarrow y'=x^2-1$$

The gradient of the curve at $x = a$ is 3:

$3 = a^2 - 1 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$

The y-coordinates of the required points are:

$$b = \frac{1}{3} \cdot 2^3 - 2 \Rightarrow b = \frac{8}{3} - 2 \Rightarrow b = \frac{2}{3}$$

$$b = \frac{1}{3} \cdot (-2)^3 - (-2) \Rightarrow b = -\frac{8}{3} + 2 \Rightarrow b = -\frac{2}{3}$$

The points where the tangent lines are parallel to line $y = 3x$ are $\left(2, \frac{2}{3}\right)$ and $\left(-2, -\frac{2}{3}\right)$.

27. The gradient of the curve at $x = 1$ is:

$y = x - x^2 \Rightarrow y' = 1 - 2x \Rightarrow y'(1) = 1 - 2 \cdot 1 \Rightarrow y'(1) = -1$

The gradient of the normal line is the negative reciprocal of the gradient of the curve, so the equation of the normal at $(1, 0)$ is:

$$y - 0 = -\frac{1}{-1}(x - 1) \Rightarrow y = x - 1$$

To find where this line intersects the curve again, equate the expressions of the curve and the normal and solve for x:

$x - 1 = x - x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

The required point is where $x = -1$, the corresponding y-coordinate is:

$y = (-1) - (-1)^2 \Rightarrow y = -2$

The coordinates of the point of intersection are $(-1, -2)$.

28.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h} \cdot \frac{\sqrt{(x+h)+2} + \sqrt{x+2}}{\sqrt{(x+h)+2} + \sqrt{x+2}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{(x+h)+2})^2 - (\sqrt{x+2})^2}{h(\sqrt{(x+h)+2} + \sqrt{x+2})} = \lim_{h \to 0} \frac{(x+h)+2-(x+2)}{h(\sqrt{(x+h)+2} + \sqrt{x+2})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})} = \frac{1}{\sqrt{x+0+2} + \sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

29. $v(t) = s'(t),\ a(t) = s''(t)$

$s(t) = t^3 - 9t^2 + 24t \Rightarrow s'(t) = 3t^2 - 18t + 24 \Rightarrow s''(t) = 6t - 18$

$\Rightarrow v(t) = 3t^2 - 18t + 24,\ a(t) = 6t - 18$

(a) $0 = 3t^2 - 18t + 24 \Rightarrow 3(t-2)(t-4) = 0 \Rightarrow t = 2, t = 4$

The displacement of the object is:

$t = 2 \Rightarrow s(2) = 2^3 - 9 \cdot 2^2 + 24 \cdot 2 \Rightarrow s(2) = 20$

$t = 4 \Rightarrow s(4) = 4^3 - 9 \cdot 4^2 + 24 \cdot 4 \Rightarrow s(4) = 16$

The object has velocity 0 when is at $(2, 20)$ or $(4, 16)$

(b) $0 = 6t - 18 \Rightarrow t = 3$

The displacement of the object is:

$t = 3 \Rightarrow s(3) = 3^3 - 9 \cdot 3^2 + 24 \cdot 3 \Rightarrow s(3) = 18$

The object has acceleration 0 when is at $(3, 18)$

30. **(a),(b)** In order for the particle to change direction, it must first come to rest, meaning its velocity should be 0. To determine when this happens, find the expression of velocity, v, by differentiating the expression of displacement, then set v equal to 0 and solve for t:

$s(t) = t + \sin t \Rightarrow s'(t) = 1 + \cos t \Rightarrow v(t) = 1 + \cos t$

$0 = 1 + \cos t \Rightarrow \cos t = -1 \Rightarrow t = \pi$

The range for the particle's velocity has to be considered:

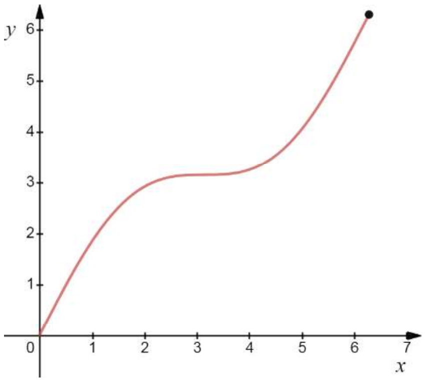
$0 \le t \le 2\pi \Rightarrow -1 \le \cos t \le 1 \Rightarrow 0 \le 1 + \cos t \le 2 \Rightarrow 0 \le v(t) \le 2$

This means that the velocity is always positive or 0, so the particle never changes its direction of movement, it merely stops at $t = \pi$, then it continues in the same direction. This also means that the particle is always on the same side of the origin.

(c) $a(t) = s''(t) \Rightarrow a(t) = -\sin t$

$0 = -\sin t \Rightarrow \sin t = 0 \Rightarrow t = 0, t = \pi, t = 2\pi$

(d) The graph of the displacement is shown below:



The maximum displacement is reached when $t = 2\pi$ seconds and it is:
$s(2\pi) = 2\pi + \sin 2\pi = 2\pi$ meters

31. If the point $(-1, 4)$ and $(3, -7)$ are on the graph representing $y = ax^3 + bx^2 + cx + d$, then their coordinates satisfy the equation of the curve:

$(-1, 4) \Rightarrow 4 = a(-1)^3 + b(-1)^2 + c(-1) + d \Rightarrow -a + b - c + d = 4$ **(1)**

$(3, -7) \Rightarrow -7 = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d \Rightarrow 27a + 9b + 3c + d = -7$ **(2)**

To use the other given information, y' and y'' must be found:

$y' = 3ax^2 + 2bx + c, \quad y'' = 6ax + 2b$

The graph has a turning point where $x = 2$, so $y'(2) = 0$:

$\Rightarrow 0 = 3a \cdot 2^2 + 2b \cdot 2 + c \Rightarrow 12a + 4b + c = 0$ **(3)**

The graph has an inflection point at $(-1, 4)$, so $y''(-1) = 0$:

$\Rightarrow 0 = 6a \cdot (-1) + 2b \Rightarrow -6a + 2b = 0 \Rightarrow 3a - b = 0$ **(4)**

We have a system of four equations, which you can use a GDC, or any other method of choice to solve.

$a = \frac{1}{4}; b = \frac{3}{4}; c = -6; d = -\frac{5}{2}$

The expression of the function is: $y = \frac{1}{4}x^3 + \frac{3}{4}x^2 - 6x - \frac{5}{2}$

The y-coordinate of the turning point is:

$y(2) = \frac{1}{4} \cdot 2^3 + \frac{3}{4} \cdot 2^2 - 6 \cdot 2 - \frac{5}{2} \Rightarrow y(2) = 2 + 3 - 12 - \frac{5}{2} \Rightarrow y(2) = -\frac{19}{2}$

32. Find the expression of y':

$y = 1 - 9x^{-2} + 18x^{-4} \Rightarrow y' = -9(-2)x^{-3} + 18(-4)x^{-5} \Rightarrow y' = 18x^{-3} - 72x^{-5} \Rightarrow y' = \frac{18}{x^3} - \frac{72}{x^5}$

To find the stationary points, the equation $y' = 0$ must be solved.

$0 = \frac{18x^2 - 72}{x^5} \Rightarrow 18x^2 - 72 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

To determine the nature of the stationary points, the second derivative is used:

$y' = 18x^{-3} - 72x^{-5} \Rightarrow y'' = 18(-3)x^{-4} - 72(-5)x^{-6} \Rightarrow y'' = \frac{-54}{x^4} + \frac{360}{x^6}$

Substitute the x-coordinates of the stationary points into the expression of y'', to determine the concavity of the function:

$y''(2) = \frac{-54}{2^4} + \frac{360}{2^6} \Rightarrow y''(2) = \frac{9}{4} > 0 \Rightarrow$ the function is concave up, so there is a minimum at the point where $x = 2$.

$y''(-2) = \frac{-54}{(-2)^4} + \frac{360}{(-2)^6} \Rightarrow y''(2) = \frac{9}{4} > 0 \Rightarrow$ the function is concave up, so there is a minimum at the point where $x = -2$.

To find the y-coordinates, substitute the two x-values into the expression of the original function:

$$x = 2 \Rightarrow y = 1 - \frac{9}{2^2} + \frac{18}{2^4} \Rightarrow y = 1 - \frac{9}{4} + \frac{9}{8} \Rightarrow y = -\frac{1}{8}$$

$$x = -2 \Rightarrow y = 1 - \frac{9}{(-2)^2} + \frac{18}{(-2)^4} \Rightarrow y = -\frac{1}{8}$$

There are two points of minimum, at $\left(-2, -\frac{1}{8}\right)$ and $\left(2, \frac{1}{8}\right)$.

33. **(a)** Find the expression of the first derivative:

$$y = x^{-1} \Rightarrow y = (-1)x^{-2} \Rightarrow y' = -\frac{1}{x^2}$$

The gradient of the tangent is found by substituting $x = 1$ into the expression of y':

$$\Rightarrow m = -\frac{1}{1^2} \Rightarrow m = -1$$

The equation of the tangent is:

$$y - 1 = -1(x - 1) \Rightarrow y = -x + 2$$

(b) $y = \cos x \Rightarrow y' = -\sin x$

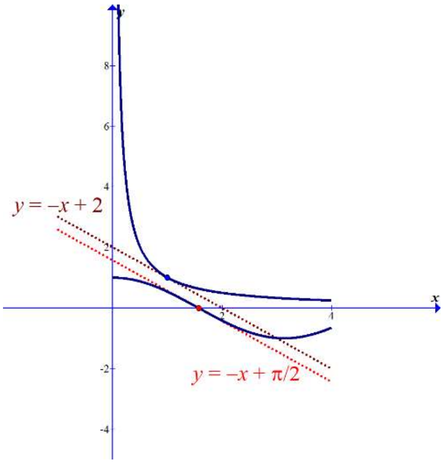
$$\Rightarrow m = -\sin\frac{\pi}{2} \Rightarrow m = -1$$

The equation of the tangent is:

$$y - 0 = -1(x - \frac{\pi}{2}) \Rightarrow y = -x + \frac{\pi}{2}$$

(c) $0 \le x \le \frac{\pi}{2} \Rightarrow -\frac{1}{x^2} < 0,\ -\sin x < 0 \Rightarrow y = \frac{1}{x}$ and $y = \cos x$ are decreasing (their derivatives are always negative on the given interval). This means that the two tangent lines do not intersect again the graphs they are tangent to.

A diagram helps clarify the argument.



The second derivative of $y=\frac{1}{x}$ is $y"=\frac{2}{x^3}>0$ on $0\le x\le\frac{\pi}{2}$, so the function is concave up. It follows that the *y*-coordinates of the points lying on the tangent to $y=\frac{1}{x}$ are always less than the *y*-coordinates of the points lying on $y=\frac{1}{x}$.

$$\Rightarrow -x+2<\frac{1}{x}$$

Similarly, the second derivative of $y=\cos x$ is $y"=-\cos x<0$ on $0\le x\le\frac{\pi}{2}$, so the function is concave down, so, for the same *x*-coordinate, the *y*-coordinates of the points lying on the tangent to $y=\cos x$ are always greater than the *y*-coordinates of the points lying on $y=\cos x$.

$$\Rightarrow -x+\frac{\pi}{2}>\cos x\Rightarrow \cos x<-x+\frac{\pi}{2}$$

The tangent lines are parallel as they have the same gradient. The *y*-coordinates of the points on $y=-x+2$ are always greater than the *y*-coordinates of the points on $y=-x+\frac{\pi}{2}$ for the same *x*-coordinate, as $2>\frac{\pi}{2}$.

Combine the three inequalities to obtain the required inequality:

$$\cos x<-x+\frac{\pi}{2}<-x+2<\frac{1}{x}\Rightarrow\cos x<\frac{1}{x}\text{ for all }0\le x\le\frac{\pi}{2}$$

34. Let (a,b) be the point of tangency.

$\Rightarrow b = a^3 - a + 2 \Rightarrow$ the tangency point has coordinates $(a, a^2 - a + 2)$

The expression of the first derivative is:

$y = x^3 - x + 2 \Rightarrow y' = 3x^2 - 1$

$\Rightarrow y'(a) = 3a^2 - 1$

The equation of the tangent at point $(a, a^3 - a + 2)$ is:

$y - (a^3 - a + 2) = (3a^2 - 1)(x - a) \Rightarrow y = (3a^2 - 1)x + a^3 - a + 2 - a(3a^2 - 1)$

$\Rightarrow y = (3a^2 - 1)x + a^2 - a + 2 - 2a^2 + a \Rightarrow y = (3a^2 - 1)x - 2a^3 + 2$

The tangent should pass through the origin, $(0,0)$:

$0 = (3a^2 - 1)\cdot 0 - 2a^3 + 2 \Rightarrow -2a^3 + 2 = 0 \Rightarrow a^3 = 1 \Rightarrow a = 1$

There is only one possible value for a, so there is only one tangent to $y = x^3 - x + 2$ passing through the origin: $y = (3a^2 - 1)x - 2a^3 + 2 = 2x$

The point of tangency is:

$a = 1 \Rightarrow b = 1^3 - 1 + 2 \Rightarrow b = 2 \Rightarrow (1,2)$

35. **(a)** $v(t) = s'(t) \Rightarrow v(t) = 50 - 20t$

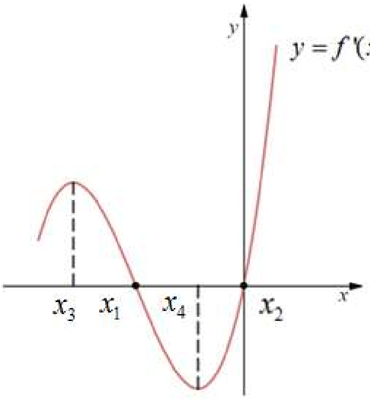
(b) The maximum displacement occurs when $s'(t) = 0$:

$$0 = 50 - 20t \Rightarrow t = \frac{5}{2}\,\text{s}$$

The maximum displacement is:

$$s\left(\frac{5}{2}\right) = 50\cdot\frac{5}{2} - 10\cdot\left(\frac{5}{2}\right)^2 + 1000 \Rightarrow s\left(\frac{5}{2}\right) = 1062.5\,\text{m}$$

36. Let x_1 and x_2 be the x-intercepts of the graph of $y = f'(x)$. The graph has a maximum at $x = x_3$ and a minimum at $x = x_4$. This information is shown in the diagram below:



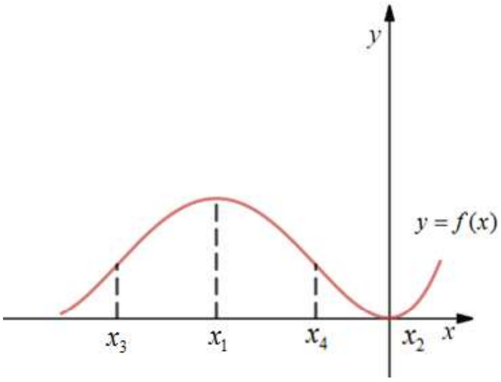
As $f'(x_1) = f'(x_2) = 0$, it follows that the points where $x = x_1$ and $x = x_2$ are stationary points for the graph of $y = f(x)$.

To decide their nature, the sign of the first derivative about each point needs to be investigated. The values of f' to the left of $x = x_1$ are positive (f is increasing), after this point they become negative (f is decreasing), this means that f has a maximum when $x = x_1$.

Similarly, the values of f' to the left of $x = x_2$ are negative (f is decreasing), past this point they are negative (f is increasing), this means that f has a minimum when $x = x_2$.

At the points where $x = x_3$ and $x = x_4$, the gradient of the graph of $y = f'(x)$ is 0, this means: $(f')'(x_3) = (f')'(x_4) = 0 \Rightarrow f''(x_3) = f''(x_4) = 0$. It follows that there are two points of inflection on the graph of $y = f(x)$.

The graph of $y = f(x)$ is shown below:



Exercise 13.1

1. **(a)**

$y=\left(3x-8\right)^{4}$

Let $f(u)=u^{4},\ g(x)=3x-8,$

then $f\,'(u)=4u^{3}$

Using the chain rule $\dfrac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot g\,'(x)=4(3x-8)^{3}\cdot 3=12(3x-8)^{3}$

(b)

$y=\sqrt{1-x}$

Let $f(u)=\sqrt{u}=u^{\frac{1}{2}},\ g(x)=1-x,$

then $f\,'(u)=\dfrac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule $\dfrac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot g\,'(x)=\dfrac{1}{2}\left(1-x\right)^{-\frac{1}{2}}\cdot(-1)=\dfrac{-1}{2\sqrt{1-x}}$

(c)

$y=\sin x\cos x$

Let $f(x)=\sin x,\ g(x)=\cos x$

Using the product rule

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)=f(x)\cdot g\,'(x)+g(x)\cdot f\,'(x)$$
$$=\sin x\cdot(-\sin x)+\cos x\cdot\cos x=-\sin^{2}x+\cos^{2}x=\cos^{2}x-\sin^{2}x$$

(d)

$y=2\sin\left(\dfrac{x}{2}\right)$

Let $f(u)=2\sin u,\ g(x)=\dfrac{x}{2},$

then $f\,'(u)=2\cos u$

Using the chain rule $\dfrac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot g\,'(x)=2\cos\dfrac{x}{2}\cdot\left(\dfrac{1}{2}\right)=\cos\dfrac{x}{2}$

(e)

$f(x)=\left(x^{2}+4\right)^{-2}$

Let $f(u)=u^{-2},\ g(x)=x^{2}+4,$

then $f\,'(u)=-2x^{-3}$

Using the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot g\,'(x)=-2\left(x^{2}+4\right)^{-3}\cdot 2x$$
$$=-4x\left(x^{2}+4\right)^{-3}=\frac{-4x}{\left(x^{2}+4\right)^{3}}$$

(f)

$f(x)=\frac{x+1}{x-1}$

Let $f(x)=x+1,\ g(x)=x-1,$

Applying the quotient rule: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=$

$\frac{(x-1)\cdot 1-(x+1)\cdot 1}{(x-1)^2}=\frac{x-1-x-1}{(x-1)^2}=\frac{-2}{(x-1)^2}$

(g)

$y=\frac{1}{\sqrt{x+2}}=(x+2)^{-\frac{1}{2}}$

Let $f(u)=u^{-\frac{1}{2}},\ g(x)=x+2,$

then $f'(u)=-\frac{1}{2}u^{-\frac{3}{2}}$

Applying the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=-\frac{1}{2}(x+2)^{-\frac{3}{2}}\cdot 1=\frac{-1}{2\sqrt{(x+2)^3}}$

(h)

$y=\cos^2 x$

Let $f(u)=u^2,\ g(x)=\cos x,$

then $f'(u)=2u$

Applying the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=2\cos x\cdot(-\sin x)=-2\sin x\cos x\ \ (=-\sin 2x)$

(i)

$y=x\sqrt{1-x}$

Let $f(x)=x,\ g(x)=\sqrt{1-x}=(1-x)^{\frac{1}{2}}$

Applying the product rule $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)=f(x)\cdot g'(x)+g(x)\cdot f'(x)=$

$\left[\text{applying also the chain rule for } \frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{1-x})\right]$

$=x\cdot\frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1)+(1-x)^{\frac{1}{2}}\cdot(1)=$

[applying the power rule]

$=\frac{-x}{2\sqrt{1-x}}+\sqrt{1-x}=$

[finding common denominator]

$=\frac{-x+2(1-x)}{2\sqrt{1-x}}=\frac{2-3x}{2\sqrt{1-x}}$

(j) $y=\dfrac{1}{3x^2-5x+7}=\left(3x^2-5x+7\right)^{-1}$

Let $f(u)=u^{-1},\ g(x)=3x^2-5x+7,$

then $f'(u)=-u^{-2}$

Applying the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=-\left(3x^2-5x+7\right)^{-2}\cdot(6x-5)=\frac{-6x+5}{\left(3x^2-5x+7\right)^2}$$

(k) $y=\sqrt[3]{2x+5}=(2x+5)^{\frac{1}{3}}$

Let $f(u)=u^{\frac{1}{3}},\ g(x)=2x+5,$

then $f'(u)=\frac{1}{3}u^{-\frac{2}{3}}$

Applying the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=\frac{1}{3}(2x+5)^{-\frac{2}{3}}\cdot 2=\frac{2}{3\sqrt[3]{(2x+5)^2}}$$

(l) $y=(2x-1)^3\left(x^4+1\right)$

Let $f(x)=(2x-1)^3,\ g(x)=\left(x^4+1\right)$

Applying the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\cdot g(x)\big)=f(x)\cdot g'(x)+g(x)\cdot f'(x)=(2x-1)^3\cdot 4x^3+\left(x^4+1\right)\cdot 3(2x-1)^2\cdot 2$$

Taking out the common factor of $(2x-1)^2$ and simplifying:

$$(2x-1)^2\left[(2x-1)\cdot 4x^3+6\left(x^4+1\right)\right]=(2x-1)^2\left(14x^4-4x^3+6\right)=2(2x-1)^2\left(7x^4-2x^3+3\right)$$

(m) $f(x)=\dfrac{\sin x}{x}$

Let $f(x)=\sin x,\ g(x)=x$

Applying the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{x\cos x-\sin x}{x^2}$$

(n)

$f(x)=\dfrac{x^2}{x+2}$

Let $f(x)=x^2,\ g(x)=x+2$

Applying the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{(x+2)\cdot 2x-x^2\cdot 1}{(x+2)^2}=\frac{x^2+4x}{(x+2)^2}$$

(o)

$$y=\sqrt[3]{x^2}\cdot\cos x=x^{\frac{2}{3}}\cdot\cos x$$

Let $f(x)=x^{\frac{2}{3}},\ g(x)=\cos x$

Applying the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)=f(x)\cdot g'(x)+g(x)\cdot f'(x)=x^{\frac{2}{3}}\cdot\left(-\sin x\right)+\cos x\cdot\frac{2}{3}x^{-\frac{1}{3}}$$

Applying the power rule:

$$\sqrt[3]{x^2}\cdot\left(-\sin x\right)+\frac{2\cos x}{3\sqrt[3]{x}}=\frac{2\cos x-3x\sin x}{3\sqrt[3]{x}}$$

2. **(a)** $y=\left(2x^2-1\right)^3,\ x=-1$

Let $f(u)=u^3,\ g(x)=2x^2-1,$

then $f'(u)=3u^2$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=3\left(2x^2-1\right)^2\cdot 4x=12x\left(2x^2-1\right)^2$$

at $x=-1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=12\cdot(-1)\left(2\cdot(-1)^2-1\right)^2=-12$$

Hence the slope of the tangent is –12.

Finding *y*-coordinate of the tangency point:

at $x=-1,\quad y=\left(2\cdot\left(-1\right)^2-1\right)^3=1$

So, the tangency point is: $\left(-1,1\right)$

Using the point-slope form for a linear equation gives:
is the tangent to the graph at $x=-1$

(b) $y=\sqrt{3x^2-2},\ x=3$

Let $f(u)=\sqrt{u}=u^{\frac{1}{2}},\ g(x)=3x^2-2,$

then $f'(u)=\frac{1}{2}u^{-\frac{1}{2}}$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=\frac{1}{2}\left(3x^2-2\right)^{-\frac{1}{2}}\cdot 6x=3x\cdot\left(3x^2-2\right)^{-\frac{1}{2}}$$

at $x=3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\cdot 3\cdot\left(3\cdot 3^2-2\right)^{-\frac{1}{2}}=\frac{9}{5}$$

Hence the slope of the tangent is $\frac{9}{5}$.

Finding *y*-coordinate of the tangency point:

at $x=3,\quad y=y=\sqrt{3\cdot 3^2-2}=5$

So, the tangency point is: $(3,5)$.

Using the point-slope form for a linear equation gives:

$y-5=\frac{9}{5}(x-3)\Rightarrow y=\frac{9}{5}x-\frac{2}{5}$ is the tangent to the graph at $x=3$.

(c)

$y=\sin 2x,\quad x=\pi$

Let $f(u)=\sin u,\ g(x)=2x,$

then $f'(u)=\cos u$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=2\cos 2x$$

at $x=\pi$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos 2\pi=2$$

Hence the slope of the tangent is 2.

Finding *y*-coordinate of the tangency point:

at $x=\pi$, $\quad y=\sin 2\pi=0$

So, the tangency point is: $(\pi,0)$

Using the point-slope form for a linear equation gives:

$y-0=2(x-\pi)\Rightarrow y=2x-2\pi$ is the tangent to the graph at $x=\pi$

(d)

$f(x)=\frac{x^3+1}{2x},\quad x=1$

Let $f(x)=x^3+1,\ g(x)=2x$

Applying the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{2x\cdot 3x^2-(x^3+1)\cdot 2}{4x^2}=\frac{2x^3-1}{2x^2}$$

at $x=1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cdot 1^3-1}{2\cdot 1^2}=\frac{1}{2}$$

Hence the slope of the tangent is $\frac{1}{2}$.

Finding *y*-coordinate of the tangency point:

at $x=1$, $\quad y=\frac{1^3+1}{2\cdot 1}=1$

So, the tangency point is: $(1,1)$

Using the point-slope form for a linear equation gives:

$y-1=\frac{1}{2}(x-1)\Rightarrow y=\frac{1}{2}x+\frac{1}{2}$ is the tangent to the graph at $x=1$

3. **(a)** $s(t)=\cos\left(t^2-1\right)$

$v(t)=s'(t)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\cos\left(t^2-1\right)\right)$

Let $s(u)=\cos u,\ g(t)=t^2-1,$

then $s'(u)=-\sin u$

Applying the chain rule:

$$\frac{\mathrm{d}s}{\mathrm{d}t}=s'(g(t))\cdot g'(t)=-\sin\left(t^2-1\right)\cdot 2t=-2t\sin\left(t^2-1\right)$$

(b) at $t=0$

$v(0)=-2\cdot 0\cdot\sin(-1)=0$

(c) $0<t<2.5$

Substituting 0 as $v(t)$:

$v(t)=0$

$-2t\sin\left(t^2-1\right)=0$

$\sin\left(t^2-1\right)=0$ or $t=0$ (outside the interval)

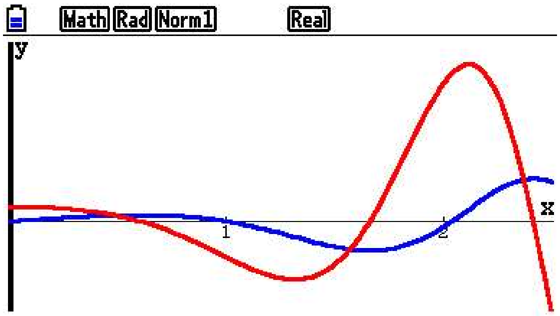
$t^2-1=0$ or $t^2-1=\pi$

$t=1$ or $t=\sqrt{\pi+1}$ (I choose positive solutions from the interval $0<t<2.5$)

(d)

$v(t)=-2t\sin\left(t^2-1\right)$ (the blue graph)

$a(t)=v'(t)=-2\sin\left(t^2-1\right)-4t^2\cos\left(t^2-1\right)$ (the red graph)



Accelerating to the right (both positive) then slowing down (positive velocity and negative acceleration), turning around, accelerating to the left (both negative), slowing down (negative velocity, positive acceleration), turning around again, then accelerating to the right (both positive), slowing down.

4. **(a)** $y=\dfrac{2}{x^2-8}=2\left(x^2-8\right)^{-1},\ (3,2)$

Let $f(u)=2u^{-1},\ g(x)=x^2-8,$

then $f'(u)=-2u^{-2}$

Applying the chain rule:

$$\frac{dy}{dx}=f'(g(x))\cdot g'(x)=-2\left(x^2-8\right)^{-2}\cdot 2x=\frac{-4x}{\left(x^2-8\right)^2}$$

at $x=3$:

$$\frac{dy}{dx}=\frac{-4\cdot 3}{\left(3^2-8\right)^2}=-12$$

Hence the slope of the tangent is -12 ang the slope of the normal is $\frac{1}{12}$.

Using the point-slope form for a linear equation gives:

(i) tangent: $y-2=-12(x-3)\Rightarrow y=-12x+38$

(ii) normal: $y-2=\frac{1}{12}(x-3)\Rightarrow y=\frac{1}{12}x+\frac{7}{4}$

(b) $y=\sqrt{1+4x},\quad \left(2,\frac{2}{3}\right)$

Let $f(u)=\sqrt{u}=u^{\frac{1}{2}},\ g(x)=1+4x,$

then $f'(u)=\frac{1}{2}u^{-\frac{1}{2}}$

Applying the chain rule:

$$\frac{dy}{dx}=f'(g(x))\cdot g'(x)=\frac{1}{2}(1+4x)^{-\frac{1}{2}}\cdot 4=\frac{2}{\sqrt{1+4x}}$$

at $x=2$:

$$\frac{dy}{dx}=\frac{2}{\sqrt{1+4\cdot 2}}=\frac{2}{3}$$

Hence the slope of the tangent is $\frac{2}{3}$ and the slope of the normal is $-\frac{3}{2}$.

Using the point-slope form for a linear equation gives:

(i) tangent: $y-\frac{2}{3}=\frac{2}{3}(x-2)\Rightarrow y=\frac{2}{3}x-\frac{2}{3}$

(ii) normal: $y-\frac{2}{3}=-\frac{3}{2}(x-2)\Rightarrow y=-\frac{3}{2}x+\frac{11}{3}$

(c) $f(x)=\frac{x}{x+1}$ at $\left(1,\frac{1}{2}\right)$

Let $f(x)=x,\ g(x)=x+1$

Applying the quotient rule:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{(x+1)\cdot 1-x\cdot 1}{(x+1)^2}=\frac{1}{(x+1)^2}$$

at $x=1$:

$$\frac{dy}{dx}=\frac{1}{(1+1)^2}=\frac{1}{4}$$

Hence the slope of the tangent is $\frac{1}{4}$ and the slope of the normal is -4.

Using the point-slope form for a linear equation gives:

(i) tangent: $y-\frac{1}{2}=\frac{1}{4}(x-1)\Rightarrow y=\frac{1}{4}x+\frac{1}{4}$

(ii) normal: $y-\frac{1}{2}=-4(x-1)\Rightarrow y=-4x+\frac{9}{2}$

5. $y=\sin\left(2x-\frac{\pi}{2}\right)$

Let $f(u)=\sin u,\ g(x)=2x-\frac{\pi}{2}$,

then $f\,'(u)=\cos u$

Applying the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot g\,'(x)=2\cos\left(2x-\frac{\pi}{2}\right)=2\sin 2x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(2\cos\left(2x-\frac{\pi}{2}\right)\right)$$

Let $f(u)=2\cos u,\ g(x)=2x-\frac{\pi}{2}$,

then $f\,'(u)=-2\sin u$

Applying the chain rule:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=f\,'(g(x))\cdot g\,'(x)=-2\sin\left(2x-\frac{\pi}{2}\right)\cdot 2=-4\sin\left(2x-\frac{\pi}{2}\right)=4\cos 2x$$

There is an inflection point at the point $x_0 \Leftrightarrow$ at x_0:

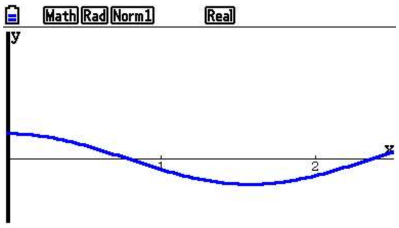
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is undefined and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes sign at x_0

$$0<x<\pi\Rightarrow-\frac{\pi}{2}<2x-\frac{\pi}{2}<\frac{3\pi}{2}$$

$$-4\sin\left(2x-\frac{\pi}{2}\right)=0$$

$$2x-\frac{\pi}{2}=0 \text{ or } 2x-\frac{\pi}{2}=\pi$$

$$x=\frac{\pi}{4} \text{ or } x=\frac{3\pi}{4}$$



$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4\sin\left(2x-\frac{\pi}{2}\right)$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ at $x=\frac{\pi}{4}$ or $x=\frac{3\pi}{4}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes sign at $x=\frac{\pi}{4}$ or $x=\frac{3\pi}{4}$ (from the graph)

at $x=\frac{\pi}{4}$ $\quad y=\sin\left(2\cdot\frac{\pi}{4}-\frac{\pi}{2}\right)=0$

at $x=\dfrac{3\pi}{4}$ $\quad y=\sin\left(2\cdot\dfrac{3\pi}{4}-\dfrac{\pi}{2}\right)=0$

Inflection points: $\left(\dfrac{\pi}{4},0\right)$, $\left(\dfrac{3\pi}{4},0\right)$

6. $y=x(x-4)^2$

(a) **(i)** x-intercepts where $y=0$

$$x(x-4)^2=0$$
$$x=0 \text{ or } x=4$$

(ii) Let $f(x)=x,\ g(x)=(x-4)^2$

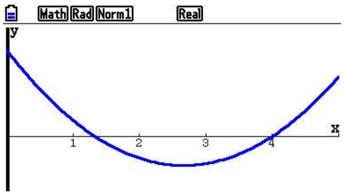
Applying the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)=f(x)\cdot g'(x)+g(x)\cdot f'(x)$$
$$=x\cdot 2(x-4)+(x-4)^2\cdot 1=(x-4)(3x-4)$$

Stationary points occur at the points where $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0\Leftrightarrow(x-4)(3x-4)=0\Leftrightarrow x=4 \text{ or } x=\frac{4}{3}$$

Using first derivative test to determine the nature of the stationary points:



$$\frac{\mathrm{d}y}{\mathrm{d}x}=(x-4)(3x-4)$$

First derivative changes sign from “+” to “–” at $x=\dfrac{4}{3}\Rightarrow$ there is a relative maximum at $x=\dfrac{4}{3}$

At $x=\dfrac{4}{3}$ $\quad y=\dfrac{4}{3}\left(\dfrac{4}{3}-4\right)^2=\dfrac{256}{27}$

First derivative changes the sign from “–” to “+” at $x=4\Rightarrow$ there is no relative maximum at $x=4$.

The coordinates of the maximum point: $\left(\dfrac{4}{3},\dfrac{256}{27}\right)$

(ii) There is an inflection point at the point $x_0\Leftrightarrow$ at x_0:

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ or is undefined and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes the sign at x_0

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(x-4)(3x-4)=3x^2-16x+16$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x-16=2(3x-8)$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(3x-8)=0\Leftrightarrow x=\frac{8}{3}$$

The second derivative changes the sing at $x=\frac{8}{3}$ (as a linear function) $\Rightarrow$ there is a point of inflection at $x=\frac{8}{3}$

At $x=\frac{8}{3}$: $y=\frac{8}{3}\cdot\left(\frac{8}{3}-4\right)^2=\frac{128}{27}$

$\left(\frac{8}{3},\frac{128}{27}\right)$ - the point of inflection

(b) $x=0$ (double root)

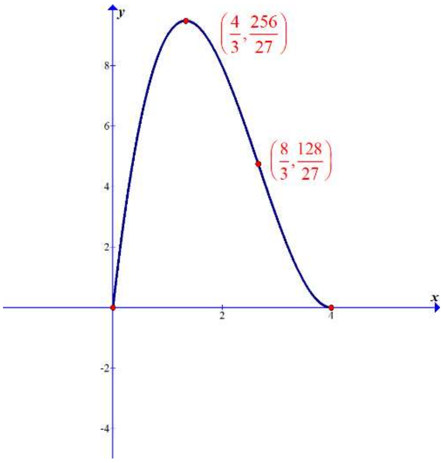
$x=4$ (double root; the graph reflects in the x-axis)

$\left(\frac{4}{3},\frac{256}{27}\right)$ - relative maximum

$\left(\frac{8}{3},\frac{128}{27}\right)$ - the point of inflection

Table:

	0	$\left(0,\frac{4}{3}\right)$	$\frac{4}{3}$	$\left(\frac{4}{3},\frac{8}{3}\right)$	$\frac{8}{3}$	$\left(\frac{8}{3},4\right)$	4
$f'(x)$	+	+	0	-	-	-	0
$f''(x)$	-	-	-	-	0	+	+
$f(x)$	0	↗	$\frac{256}{27}$	↘	$\frac{128}{27}$	↘	0



7. $f(x)=\frac{x^2-3x+4}{(x+1)^2}$

(a) Let $g(x)=x^2-3x+4,\ h(x)=(x+1)^2$

Applying the quotient rule:

$$f'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{h(x)}\right)=\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}=\frac{(x+1)^2(2x-3)-(x^2-3x+4)\cdot(x+1)}{(x+1)^4}$$

$$=\frac{(x+1)(5x-11)}{(x+1)^4}=\frac{5x-11}{(x+1)^3}$$

(b) Let $g(x)=5x-11,\ h(x)=(x+1)^3$

Applying the quotient rule:

$$f''(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{h(x)}\right)=\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}=\frac{(x+1)^3\cdot 5-(5x-11)\cdot 3(x+1)^2}{(x+1)^6}$$

$$=\frac{(x+1)^2(-10x+38)}{(x+1)^6}=\frac{-10x+38}{(x+1)^4}$$

(c) (1) $f''(3.8)=\dfrac{-38+38}{4.8^3}=0$

(2) f'' changes the sign at $x = 3.8$ (denominator is always positive, linear function in the numerator changes the sign at $x = 3.8$)

$$\left.\begin{array}{l}1)\\2)\end{array}\right\}\Rightarrow \text{ there is a point of inflection at } x = 3.8$$

8. $f(x)=\dfrac{x-a}{x+a}$

1st derivative:

Let $g(x)=x-a,\ h(x)=x+a$

Applying the quotient rule:

$$f'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{h(x)}\right)=\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}=\frac{(x+a)\cdot 1-(x-a)\cdot 1}{(x+a)^2}=\frac{2a}{(x+a)^2}$$

2nd derivative:

$$f'(x)=2a(x+a)^{-2}$$

Applying the chain rule:

$$f''(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(2a(x+a)^{-2}\right)=-4a(x+a)^{-3}$$

$$f''(x)=\frac{-4a}{(x+a)^3}$$

9. $y=\dfrac{1}{1-x}=-(x-1)^{-1}$

Derivatives:

1st: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=(-1)^2(x-1)^{-2}$

2nd: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=(-1)^2\cdot(-1)\cdot 2\cdot(x-1)^{-3}=(-1)^3\cdot 2!(x-2)^{-3}$

3rd: $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=(-1)^3\cdot(-1)\cdot 2!\cdot 3\cdot(x-1)^{-4}=(-1)^4\cdot 3!(x-2)^{-4}$

4th: $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)=(-1)^4\cdot(-1)\cdot 3!\cdot 4\cdot(x-1)^{-4}=(-1)^5\cdot 4!(x-2)^{-5}$

…

Recognizing that the *n*th derivative of the function is formed by:

- multiplying the previous by –*n*
- decreasing the power of *x*–1 by 1

we get the formula:

*n*th derivative: $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right)=(-1)^n\cdot(-1)\cdot(n-1)!\cdot n\cdot(x-1)^{-n-1}=\frac{(-1)^{n+1}\cdot n!}{(x-1)^{n+1}}$

10. $y=\frac{8}{4+x^2}$

(a) Relative extrema occur at the points where $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ or is undefined and $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes sign

$y=8\cdot\left(4+x^2\right)^{-1}$

Let $f(u)=8u^{-1},\ g(x)=4+x^2,$

then $f'(u)=-8u^{-2}$

Applying the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=-8\left(4+x^2\right)^{-2}\cdot 2x=\frac{-16x}{\left(4+x^2\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0\Leftrightarrow\frac{-16x}{\left(4+x^2\right)^2}=0\Leftrightarrow -16x=0\Leftrightarrow x=0$$

$$\left(4+x^2\right)^2>0\text{ when }x\in\mathbb{R}\Rightarrow\begin{cases}\frac{\mathrm{d}y}{\mathrm{d}x}>0\Leftrightarrow -16x>0\Leftrightarrow x<0\\ \frac{\mathrm{d}y}{\mathrm{d}x}<0\Leftrightarrow -16x<0\Leftrightarrow x>0\end{cases}\Rightarrow$$

At $x = 0$, the function changes its monotonicity from increasing to decreasing ⇒ there is a relative maximum at $x = 0$

$f(0)=\frac{8}{4+0}=2\Rightarrow$ there is a relative maximum at $(0,2)$

The inflection points occur at the points where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes sign

$\frac{\mathrm{d}y}{\mathrm{d}x}=-16x\left(4+x^2\right)^{-2}$

Let $f(x)=-16x,\ g(x)=\left(4+x^2\right)^{-2}$

Applying the product rule:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right)=f(x)\cdot g'(x)+g(x)\cdot f'(x)$$

Applying also the chain rule to $\frac{\mathrm{d}y}{\mathrm{d}x}\left(4+x^2\right)^{-2}$

$$-16x\cdot\left(-2\left(4+x^2\right)^{-3}\cdot 2x\right)+\left(4+x^2\right)^{-2}\cdot(-16)=\frac{64x^2}{\left(4+x^2\right)^3}+\frac{-16}{\left(4+x^2\right)^2}=\frac{48x^2-64}{\left(4+x^2\right)^3}$$

– $\dfrac{\mathrm{d}^2y}{dx^2}=0\Leftrightarrow\dfrac{48x^2-64}{\left(4+x^2\right)^3}=0\Leftrightarrow 48x^2-64=0\Leftrightarrow x^2=\dfrac{4}{3}\Leftrightarrow x=\dfrac{2\sqrt{3}}{3}$ or $x=-\dfrac{2\sqrt{3}}{3}$

$$\left(4+x^2\right)^3>0 \text{ when } x\in\mathbb{R}\Rightarrow\begin{cases}\dfrac{d^2y}{dx^2}>0\Leftrightarrow 48x^2-64>0\Leftrightarrow x<-\dfrac{2\sqrt{3}}{3} \ or \ x>\dfrac{2\sqrt{3}}{3}\\ \dfrac{d^2y}{dx^2}<0\Leftrightarrow 48x^2-64<0\Leftrightarrow-\dfrac{2\sqrt{3}}{3}<x<\dfrac{2\sqrt{3}}{3}\end{cases}\Rightarrow$$

the function changes its concavity at $x=-\dfrac{2\sqrt{3}}{3}$ and at $x=\dfrac{2\sqrt{3}}{3}$ $\Rightarrow$ there are points of inflection at $x=-\dfrac{2\sqrt{3}}{3}$ and $x=\dfrac{2\sqrt{3}}{3}$

$$f\left(-\frac{2\sqrt{3}}{3}\right)=\frac{8}{4+\left(-\frac{2\sqrt{3}}{3}\right)^2}=\frac{3}{2}$$

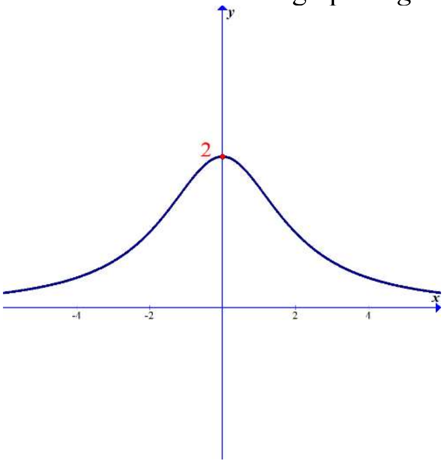
$$f\left(\frac{2\sqrt{3}}{3}\right)=\frac{8}{4+\left(\frac{2\sqrt{3}}{3}\right)^2}=\frac{3}{2}$$

Points of inflection: $\left(-\dfrac{2\sqrt{3}}{3},\dfrac{3}{2}\right);\left(\dfrac{2\sqrt{3}}{3},\dfrac{3}{2}\right)$

(b) Since $g(x)$ is the quotient of a positive constant and a positive expression, it will never be negative or zero. It is positive for all real numbers.

(c) Since the numerator is constant, then as $x\to\infty, 4+x^2\to 8\Rightarrow\lim\limits_{x\to\pm\infty}\dfrac{8}{4+x^2}=0$

(d) Below is a sketch of the graph of g.



11. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(cf(x)\right)=c\cdot\dfrac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)$

Let $\mathrm{g}(x)=c$, then $g'(x)=0$

Applying the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\big(g(x)\cdot f(x)\big)=g(x)\cdot f'(x)+f(x)\cdot g'(x)=cf'(x)+f(x)\cdot 0=cf'(x)=\text{RHS}$$

what was to be proved.

12. $y=x^4-6x^2$

Consecutive derivatives:

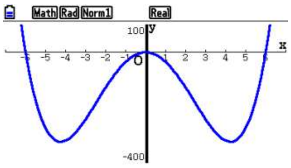
$$\frac{\mathrm{d}y}{\mathrm{d}x}=4x^3-12x=4x\left(x^2-3\right)=4x\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=12x^2-12=12(x-1)(x+1)$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=24x$$

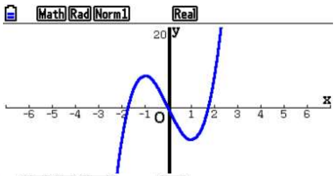
1) $y<0 \Leftrightarrow x^4-6x^2<0 \Leftrightarrow x^2\left(x-\sqrt{6}\right)\left(x+\sqrt{6}\right)<0 \Leftrightarrow$

$-\sqrt{6}<x<0$ or $0<x<\sqrt{6} \Rightarrow$ in the interval $0<x<1$ $y<0$



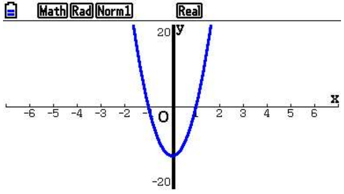
2) $\dfrac{\mathrm{d}y}{\mathrm{d}x}<0 \Leftrightarrow 4x\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)<0$

$x<-\sqrt{3}$ or $0<x<\sqrt{3} \Rightarrow$ in the interval $0<x<1$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}<0$



3) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}<0 \Leftrightarrow 12(x-1)(x+1)<0$

$-1<x<1 \Rightarrow$ in the interval $0<x<1$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$



4) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}>0 \Leftrightarrow 24x>0 \Leftrightarrow x>0 \Rightarrow$ in the interval $0<x<1$

$\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}>0$

Exercise 13.2

1. (a) $y=x^2\mathrm{e}^x$

Let $f(x)=x^2,\ g(x)=\mathrm{e}^x$

Applying the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\cdot g(x)\big)=f(x)\cdot g'(x)+g(x)\cdot f'(x)=x^2\mathrm{e}^x+\mathrm{e}^x\cdot 2x=x^2\mathrm{e}^x+2x\mathrm{e}^x$$

(b) $y=8^x$

Applying the formula of the derivative of an exponential function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=8^x\ln 8$$

(c) $y = \tan \mathrm{e}^x$

Let $f(u) = \tan u,\ g(x) = \mathrm{e}^x,$

then $f'(u) = \sec^2 u$

Applying the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot g'(x) = \sec^2 \mathrm{e}^x \cdot \mathrm{e}^x = \mathrm{e}^x \sec^2 \mathrm{e}^x$$

(d) $y = \dfrac{x}{1 + \cos x}$

Let $f(x) = x,\ g(x) = 1 + \cos x$

Applying the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{(1 + \cos x) \cdot 1 - x \cdot (-\sin x)}{(1 + \cos x)^2} = \frac{\cos x + x \sin x + 1}{(1 + \cos x)^2}$$

(e) $y = \dfrac{\mathrm{e}^x}{x}$

Let $f(x) = \mathrm{e}^x,\ g(x) = x$

Applying the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{x\mathrm{e}^x - \mathrm{e}^x \cdot 1}{x^2} = \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2}$$

(f) $y = \dfrac{1}{3}\sec^3 2x - \sec 2x$

(1) Let $f(u) = \dfrac{1}{3}x^3,\ g(x) = \sec 2x,$

then $f'(u) = x^2,\ \mathrm{g}'(x) = 2\sec 2x \tan 2x$

Applying the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\sec^3 2x\right) = f'(g(x)) \cdot g'(x) = \sec^2 2x \cdot 2\sec 2x \tan 2x = 2\sec^3 2x \tan 2x$$

(2) Let $f(u) = \sec u,\ g(x) = 2x,$

then $f'(u) = \sec u \tan u$

Applying the chain rule:

$$\frac{d}{dx}(\sec 2x) = f'(g(x)) \cdot g'(x) = 2\sec 2x \tan 2x$$

$(1),(2) \Rightarrow$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sec^3 2x \tan 2x - 2\sec 2x \tan 2x = 2\sec 2x \tan 2x\left(\sec^2 2x - 1\right)$$

$$\frac{dy}{dx}=2\sec 2x\tan 2x\left(\frac{1}{\cos^2 2x}-1\right)=2\sec 2x\tan 2x\left(\frac{1-\cos^2 2x}{\cos^2 2x}\right)$$

$$=2\sec 2x\tan 2x\left(\frac{\sin^2 2x}{\cos^2 2x}\right)=2\tan^3 2x\sec 2x$$

(g) $y=4^{-x}=\left(\frac{1}{4}\right)^x$

Applying the formula of the derivative of an exponential function:

$$\frac{dy}{dx}=\left(\frac{1}{4}\right)^x\ln\left(\frac{1}{4}\right)$$

(h) $y=\cos x\tan x=\cos x\cdot\frac{\sin x}{\cos x}=\sin x$

Applying the formula of the derivative of a sine function:

$$\frac{dy}{dx}=\cos x$$

(i) $y=\frac{x}{e^x-1}$

Let $f(x)=x,\ g(x)=e^x-1$

Applying the quotient rule:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{(e^x-1)\cdot 1-x\cdot e^x}{(e^x-1)^2}=\frac{-xe^x+e^x-1}{(e^x-1)^2}$$

(j) $y=4\cos(\sin 3x)$

Let $f(u)=4\cos u,\ g(x)=\sin 3x,$

then $f'(u)=-4\sin u\quad g'(x)=3\cos 3x$ (applying the chain rule)

Applying the chain rule:

$$\frac{dy}{dx}=f'(g(x))\cdot g'(x)=-4\sin(\sin 3x)\cdot 3\cos 3x=-12\sin 3x\cos 3x\qquad(=-6\sin 6x)$$

(k) $y=2^{x+1}=2\cdot 2^x$

Applying the formula of the derivative of an exponential function:

$$\frac{dy}{dx}=2\cdot 2^x\ln 2=2\ln 2\cdot 2^x$$

(l) $y=\frac{1}{\csc x-\sec x}=(\csc x-\sec x)^{-1}$

Let $f(u)=u^{-1},\ g(x)=\csc x-\sec x,$

then $f'(u)=-u^{-2}, g'(x)=-\cot x\csc x-\tan x\sec x$

Applying the chain rule:

$$\frac{dy}{dx}=f'(g(x))\cdot g'(x)=-(\csc x-\sec x)^{-2}\cdot(-\cot x\csc x-\tan x\sec x)$$

$$=\frac{\cot x\csc x+\tan x\sec x}{(\csc x-\sec x)^2}$$

$$=\frac{\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}+\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}}{\left(\frac{1}{\cos x}-\frac{1}{\sin x}\right)^2}=\frac{\cos^3 x+\sin^3 x}{\sin^2 x\cos^2 x}\cdot\frac{\sin^2 x\cos^2 x}{(\sin x-\cos x)^2}$$

$$=\frac{\cos^3 x+\sin^3 x}{(\sin x-\cos x)^2}$$

2. **(a)** $y=\sin x,\ \ x=\frac{\pi}{3}$

$\frac{dy}{dx}=\cos x$

at $x=\frac{\pi}{3}$:

$\frac{dy}{dx}=\cos\frac{\pi}{3}=\frac{1}{2}$

Hence the slope of the tangent is $\frac{1}{2}$.

Finding the y-coordinate of the tangency point:

at $x=-1$ $\quad y=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

So, the tangency point is: $\left(\frac{\pi}{3},\frac{\sqrt{3}}{2}\right)$.

Using the point-slope form for a linear equation gives:

$y-\frac{\sqrt{3}}{2}=\frac{1}{2}\left(x-\frac{\pi}{3}\right)\Rightarrow y=\frac{1}{2}x+\frac{3\sqrt{3}-\pi}{6}$ is the tangent to the graph at $x=-1$.

(b) $y=x+e^x,\ \ x=0$

$\frac{dy}{dx}=1+e^x$

at $x=0$:

$\frac{dy}{dx}=1+e^0=2$

Hence the slope of the tangent is 2.

Finding the y-coordinate of the tangency point:

at $x=0$ $\quad y=0+e^0=1$

So, the tangency point is: $(0,1)$.

Using the point-slope form for a linear equation gives:

$y-1=2(x-0)\Rightarrow y=2x+1$ is the tangent to the graph at $x=0$

(c) $y = 4\tan 2x,\ \ x = \dfrac{\pi}{8}$

Let $f(u) = 4\tan u,\ \ g(x) = 2x,$

then $f'(u) = 4\sec^2 u$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x))\cdot g'(x) = \left(4\sec^2 2x\right)\cdot 2 = 8\sec^2 2x$$

at $x = \dfrac{\pi}{8}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8\sec^2\left(2\cdot\frac{\pi}{8}\right) = 8\cdot\frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = 16$$

Hence the slope of the tangent is 16.

Finding the *y*-coordinate of the tangency point:

at $x = \dfrac{\pi}{8}$ $\quad y = 4\tan\left(2\cdot\dfrac{\pi}{8}\right) = 4$

So, the tangency point is: $\left(\dfrac{\pi}{8}, 4\right)$.

Using the point-slope form for a linear equation gives:

$y - 4 = 16\left(x - \dfrac{\pi}{8}\right) \Rightarrow y = 16x + 4 - 2\pi$ is the tangent to the graph at $x = \dfrac{\pi}{8}$

3. $g(x) = x + 2\cos x,\ \ 0 \le x \le 2\pi$

(a) Stationary points occur at the points where $g'(x) = 0$

$g'(x) = 1 - 2\sin x$

$g'(x) = 0 \Leftrightarrow 1 - 2\sin x = 0 \Leftrightarrow \sin x = \dfrac{1}{2} \Leftrightarrow x = \dfrac{\pi}{6}$ or $x = \dfrac{5\pi}{6}$, in the interval $0 \le x \le 2\pi$

$\Rightarrow$ there are stationary points at $x = \dfrac{\pi}{6}$ or $x = \dfrac{5\pi}{6}$

(b) The nature of the stationary points:
the 2nd derivative test:
$g''(x) = -2\cos x$
Considering the sign of the 2nd derivative at the stationary points:

$g''\left(\dfrac{\pi}{6}\right) = -2\cos\dfrac{\pi}{6} = -\sqrt{3} < 0 \Rightarrow$ there is a relative maximum at $x = \dfrac{\pi}{6}$

$g''\left(\dfrac{5\pi}{6}\right) = -2\cos\dfrac{5\pi}{6} = \sqrt{3} > 0 \Rightarrow$ there is a relative minimum at $x = \dfrac{5\pi}{6}$

4. $y = x - \mathrm{e}^x$

Stationary points occur at the points where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \mathrm{e}^x$$

$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Leftrightarrow 1-\mathrm{e}^x=0 \Leftrightarrow \mathrm{e}^x=1 \Leftrightarrow x=0$

$\Rightarrow$ there is a stationary point at $x=0$

Applying the 2nd derivative test to determine the nature of the stationary point:

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{e}^x$

Considering the sign of the 2nd derivative at the stationary point:

at $x=0$: $\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{e}^0=-1<0 \Rightarrow$ there is a relative maximum at $x=0$

at $x=0$: $y=0-\mathrm{e}^0=-1$

The point $(0,-1)$ is a relative maximum, the function is continuous in a set of real numbers and there are no other relative extremes $\Rightarrow$ The point $(0,-1)$ is an absolute maximum.

5. **(a)** $f(x)=4\sin x-\cos 2x$

Stationary points occur at the points where $f'(x)=0$

Applying the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}(\cos 2x)$:

Let $h(u)=\cos u,\ g(x)=2x,$

then $h'(u)=-\sin u$

$\frac{\mathrm{d}y}{\mathrm{d}x}=h'(g(x))\cdot g'(x)=-2\sin 2x$

$f'(x)=4\cos x+2\sin 2x$

$f'(x)=0 \Rightarrow 4\cos x+2\sin 2x=0 \Rightarrow 4\cos x+4\sin x\cos x=0$

$\Rightarrow 4\cos x(1+\sin x)=0$

$\Rightarrow (\cos x=0 \text{ or } \sin x=-1)$ and $0\le x\le 2\pi$

There are stationary points at $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$

Applying the 2nd derivative test to determine the nature of the stationary points:

Applying the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x)$:

Let $h(u)=\sin u,\ g(x)=2x,$

then $h'(u)=\cos u$

$\frac{\mathrm{d}y}{\mathrm{d}x}=h'(g(x))\cdot g'(x)=2\cos 2x$

$f''(x)=-4\sin x+4\cos 2x$

Considering the sign of the 2nd derivative at the stationary points:

$f''\left(\frac{\pi}{2}\right)=-4\sin\frac{\pi}{2}+4\cos\left(2\cdot\frac{\pi}{2}\right)=-8<0 \Rightarrow$ there is a relative maximum at $x=\frac{\pi}{2}$

$f''\left(\frac{3\pi}{2}\right)=-4\sin\frac{3\pi}{2}+4\cos\left(2\cdot\frac{3\pi}{2}\right)=0 \Rightarrow$ it is not possible to determine the nature using the 2nd derivative.

1st derivative test at $x = \frac{3\pi}{2}$:

$f'(\pi) = 4\cos\pi + 2\sin 2\pi = -4 < 0$

$f'(2\pi) = 4\cos 2\pi + 2\sin 4\pi = 4 > 0$

1st derivative changes the sign from "–" to "+"at $x = \frac{3\pi}{2} \Rightarrow$ there is a relative minimum at $x = \frac{3\pi}{2}$.

$f\left(\frac{\pi}{2}\right) = 4\sin\frac{\pi}{2} - \cos\left(2\cdot\frac{\pi}{2}\right) = 5$

$f\left(\frac{3\pi}{2}\right) = 4\sin\frac{3\pi}{2} - \cos\left(2\cdot\frac{3\pi}{2}\right) = -3$

$\left(\frac{\pi}{2}, 5\right)$ - relative maximum point

$\left(\frac{3\pi}{2}, -3\right)$ - relative minimum point

(b) $g(x) = \tan x(\tan x + 2)$

Stationary points occur at the points where $g'(x) = 0$

Let $f(x) = \tan x,\ h(x) = \tan x + 2$

Applying the product rule:

$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)\cdot h(x)) = f(x)\cdot h'(x) + h(x)\cdot f'(x)$

$= \tan x \sec^2 x + (\tan x + 2)\sec^2 x = 2\sec^2 x(\tan x + 1)$

$g'(x) = 0 \Rightarrow 2\sec^2 x(\tan x + 1) = 0$

$\Rightarrow (\sec^2 x = 0 \text{ or } \tan x + 1 = 0)$ and $0 \le x \le 2\pi$

$\Rightarrow \frac{1}{\cos x} = 0$ (contradiction) or $\tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

There are stationary points at $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

Applying the 2nd derivative test to determine the nature of the stationary point:

Let $f(x) = 2\sec^2 x,\ h(x) = \tan x + 1$

Applying the product rule:

$g''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)\cdot h(x)) = f(x)\cdot h'(x) + h(x)\cdot f'(x)$

$= 2\sec^2 x\cdot\sec^2 x + (\tan x + 1)\cdot 4\sec x\cdot\sec x\tan x$

$= 2\sec^2 x(\sec^2 x + 2\tan^2 x + 2\tan x)$

Considering the sign of the 2nd derivative at the stationary points:

$$g''\left(\frac{3\pi}{4}\right)=2\sec^2\frac{3\pi}{4}\left(\sec^2\frac{3\pi}{4}+2\tan^2\frac{3\pi}{4}+2\tan\frac{3\pi}{4}\right)$$

$$=2\cdot\frac{1}{\frac{1}{2}}\left(\frac{1}{\frac{1}{2}}+2\cdot1+2\cdot(-1)\right)=8>0$$

$\Rightarrow$ there is a relative minimum at $x=\frac{3\pi}{4}$

$$g''\left(\frac{7\pi}{4}\right)=2\sec^2\frac{7\pi}{4}\left(\sec^2\frac{7\pi}{4}+2\tan^2\frac{7\pi}{4}+2\tan\frac{7\pi}{4}\right)$$

$$=2\cdot\frac{1}{\frac{1}{2}}\left(\frac{1}{\frac{1}{2}}+2\cdot1+2\cdot(-1)\right)=8>0$$

$\Rightarrow$ there is a relative minimum at $x=\frac{7\pi}{4}$

2nd coordinates of the points:

$$g\left(\frac{3\pi}{4}\right)=\tan\frac{3\pi}{4}\left(\tan\frac{3\pi}{4}+2\right)=-1\cdot(-1+2)=-1$$

$$g\left(\frac{7\pi}{4}\right)=\tan\frac{7\pi}{4}\left(\tan\frac{7\pi}{4}+2\right)=-1\cdot(-1+2)=-1$$

$\left(\frac{3\pi}{4},-1\right);\left(\frac{7\pi}{4},-1\right)$ - relative minimum points

6. $y=3+\sin x,\ x=\frac{\pi}{2}$

$$\frac{dy}{dx}=\cos x$$

at $x=\frac{\pi}{2}$:

$\frac{dy}{dx}=\cos\frac{\pi}{2}=0\ \Rightarrow$ the tangent is a horizontal line $\Rightarrow$ the normal is a vertical line.

The vertical line going through the point with $\frac{\pi}{2}$ as its first coordinate $x=\frac{\pi}{2}$

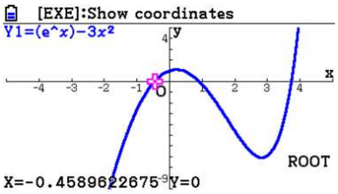
7. $f(x)=e^x-x^3$

(a) 1st derivative: $f'(x)=e^x-3x^2$

2nd derivative: $f''(x)=e^x-6x$

(b) $f'(x)=0\Leftrightarrow e^x-3x^2=0$

$x\approx-0.459$ or $x\approx0.910$ or $x\approx3.73$



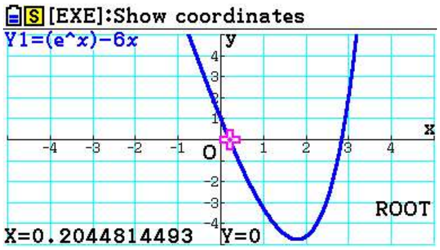
(c) Reading from the graph, considering the sign of the 1st derivative:

$f'(x)>0\Leftrightarrow-0.459<x<0.910$ or $x>3.73\Rightarrow$ f is increasing on $(-0.459,0.910);(3.73,+\infty)$

$f'(x)<0\Leftrightarrow x<-0.459$ or $0.910<x<3.73\Rightarrow$ f is decreasing on $(-\infty,-0.459);(0.910,3.73)$

(d) Applying the 1st derivative test:
$y = f'(x)$ changes sign from "–" to "+" at the points $x = –0.459$ and $x = 3.73$, therefore there are relative minimums at these points.
$y = f'(x)$ changes sign from "+" to "–" at the point $x = 0.910$, therefore there is a relative maximum at this point.

(e) Inflection points occur at the points where the second derivative is 0 or is undefined and the second derivative changes sign.



Exploring the graph of $f''(x) = \mathrm{e}^x - 6x$, we get:

$f''(x) = 0 \Leftrightarrow x \approx 0.204$ or $x \approx 2.83$

1) at $x \approx 0.204$ $f''(x) = 0$ and f'' changes sign from "+" to "–" so changes the concavity from up to down.

2) at $x \approx 2.83$ $f''(x) = 0$ and f'' changes sign from "–" to "+" so changes the concavity from down to up.

$\left.\begin{matrix}1)\\2)\end{matrix}\right\} \Rightarrow$ there are inflection points at $x \approx 0.204$ or $x \approx 2.83$

(f) Reading from the graph, exploring the sign of the second derivative:

$$\left.\begin{matrix} f''(x) > 0 \Leftrightarrow x \in (-\infty, 0.204) \cup (2.83, +\infty) \\ f''(x) < 0 \Leftrightarrow x \in (0.204, 2.83) \end{matrix}\right\} \Rightarrow$$

the function is concave up on $(-\infty, 0.204), (2.83, +\infty)$

the function is concave down on $(0.204, 2.83)$

8. $y = \mathrm{e}^{-x}$ $\quad y = \mathrm{e}^{-x}\cos x$

The common points are at the arguments where

$\mathrm{e}^{-x} = \mathrm{e}^{-x}\cos x$

$\mathrm{e}^{-x}(1 - \cos x) = 0 \Rightarrow \mathrm{e}^{-x} = 0$ (contradiction) or $1 - \cos x = 0 \Rightarrow \cos x = 1 \Leftrightarrow x = 2k\pi, k \in Z$

Therefore, the common points are at $x = 2k\pi, k \in Z$

At the points of intersection, the gradient of each curve are given by the value of each derivative.

$y = \mathrm{e}^{-x}$

Applying the chain rule to the function:

$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot g'(x) = -\mathrm{e}^{-x}$. At the points of intersection, the gradient is $-\mathrm{e}^{2k\pi}$

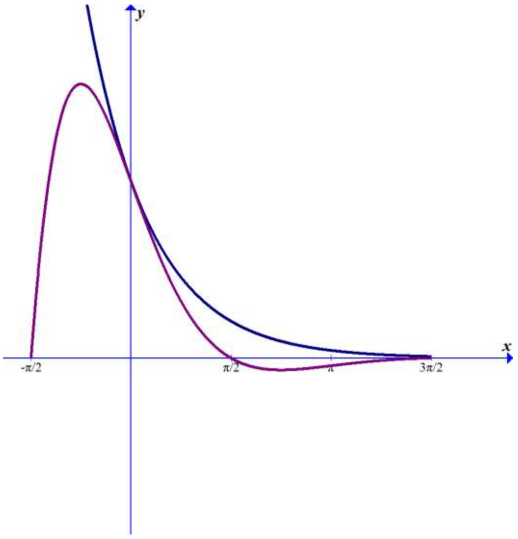
$y = \mathrm{e}^{-x}\cos x$

Applying the product rule:

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x)\right) = f(x) \cdot g'(x) + g(x) \cdot f'(x) = \mathrm{e}^{-x}(-\sin x) + \cos x \cdot \left(-\mathrm{e}^{-x}\right) = -\mathrm{e}^{-x}\left(\sin x + \cos x\right)$

At the points of intersection, the gradient is $-\mathrm{e}^{-2k\pi}\left(\sin 2k\pi + \cos 2k\pi\right) = -\mathrm{e}^{-2k\pi}(0 + 1) = -\mathrm{e}^{-2k\pi}$

At the intersection points, the gradients are the same and the graphs are tangent to each other.



9. $s(t) = 4\cos t - \cos 2t$

(a) Applying the derivatives to find the acceleration: $a = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$

Applying the chain rule to $\dfrac{\mathrm{d}}{\mathrm{d}t}(\cos 2t) = -2\sin 2t$

$\dfrac{\mathrm{d}s}{\mathrm{d}t}(= v(t)) = -4\sin t + 2\sin 2t$

Applying the chain rule to $\dfrac{\mathrm{d}}{\mathrm{d}t}(\sin 2t) = 2\cos 2t$

$a = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}s}{\mathrm{d}t}\right) = -4\cos t + 4\cos 2t$

The particle is at rest $\Leftrightarrow v(t) = 0 \Leftrightarrow \dfrac{\mathrm{d}s}{\mathrm{d}t} = 0$

$-4\sin t + 2\sin 2t = 0 \Leftrightarrow -4\sin t + 4\sin t\cos t = 0$

$4\sin t(-1 + \cos t) = 0 \Leftrightarrow \sin t = 0$ or $-1 + \cos t = 0 \Leftrightarrow t = k\pi$ or $t = 2k\pi, k \in Z \Rightarrow$

$T = \pi$ is the minimum positive argument for which the particle is at rest

Acceleration at $T = \pi$:

$a(\pi) = -4\cos\pi + 4\cos(2\cdot\pi) = 8\left[\dfrac{m}{s^2}\right]$

(b) maximum speed is at the point where

$a(t) = 0 \Leftrightarrow \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0$

$-4\cos t + 4\cos 2t = 0$

$-\cos t + 2\cos^2 t - 1 = 0$

$(\cos t - 1)(2\cos t + 1) = 0$

$\cos t = 1$ or $\cos t = -\frac{1}{2}$

$t = 2k\pi$ or $t = \pm\dfrac{2\pi}{3} + 2k\pi, k \in Z$

$0 < t < T(= \pi) \Rightarrow t = \dfrac{2\pi}{3}$

$$\left|v\left(\frac{2\pi}{3}\right)\right| = \left|-4\sin\frac{2\pi}{3} + 2\sin\left(\frac{4\pi}{3}\right)\right| = \left|-4\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2}\right| = 3\sqrt{3} \approx 5.20\left[\frac{m}{s}\right]$$

10. $y = \mathrm{e}^x$

Let $A = (a,b)$ be the tangency point, $b = \mathrm{e}^a \Rightarrow A = (a,\mathrm{e}^a)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$

at $x = a$:

$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^a$

Hence the slope of the tangent is e^a.

Using the point-slope form for a linear equation gives:

(1) $y - \mathrm{e}^a = \mathrm{e}^a(x - a) \Rightarrow y = \mathrm{e}^a x - a\mathrm{e}^a + \mathrm{e}^a$ is the tangent to the graph at $x = a$.

Substituting the coordinates of the origin $(0,0)$:

$0 = \mathrm{e}^a \cdot 0 - a\mathrm{e}^a + \mathrm{e}^a \Rightarrow \mathrm{e}^a(1-a) = 0$

$\mathrm{e}^a = 0$ (contradiction) or $a = 1 \Rightarrow$

the tangency point is $A = (1,\mathrm{e})$

substituting a=1 in equation (1):

$y = \mathrm{e}x - \mathrm{e} + \mathrm{e}$

$y = \mathrm{e}x$ is the tangent to the graph going through the origin.

11. $f(x) = 2^x$

(a) Applying the formula of the derivative of an exponential function:

$f'(x) = 2^x \ln 2$

(b) $(0,1)$ is the tangency point

$f'(0) = 2^0 \ln 2 = \ln 2$

Hence the slope of the tangent is $\ln 2$.

Using the point-slope form for a linear equation gives:

$y - 1 = \ln 2(x - 0) \Rightarrow y = x\ln 2 + 1$ is the tangent to the graph at $(0,1)$.

(c) Stationary points occur at the points where $f'(x) = 0$

$f'(x) = 0 \Leftrightarrow 2^x \ln 2 = 0$

No solutions ($2^x \ln 2$ is greater than 0 for each real number x)
therefore, there are no stationary points.

12. $h(x) = \frac{x^2 - 3}{\mathrm{e}^x}$

(a) *Let* $f(x) = x^2 - 3,\ g(x) = \mathrm{e}^x$

Applying the quotient rule:

$$h'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

$$=\frac{\mathrm{e}^x\cdot 2x-\left(x^2-3\right)\cdot\mathrm{e}^x}{\left(\mathrm{e}^x\right)^2}=\frac{\mathrm{e}^x\left(2x-x^2+3\right)}{\mathrm{e}^{2x}}=\frac{-x^2+2x+3}{\mathrm{e}^x}$$

Stationary points occur at the points where $h'(x)=0$

$$h'(x)=0\Leftrightarrow\frac{-x^2+2x+3}{\mathrm{e}^x}=0\Leftrightarrow -x^2+2x+3=0$$

$$-(x+1)(x-3)=0$$

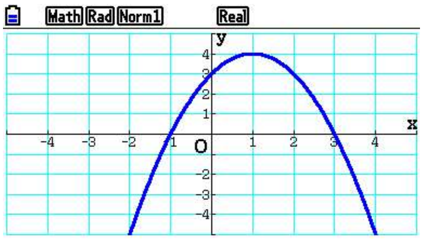
$$x=-1 \text{ or } x=3$$

$$h(-1)=\frac{(-1)^2-3}{\mathrm{e}^{-1}}=-2\mathrm{e}$$

$$h(3)=\frac{3^2-3}{\mathrm{e}^3}=\frac{6}{\mathrm{e}^3}$$

$$\Rightarrow(-1,-2\mathrm{e});\left(3,\frac{6}{\mathrm{e}^3}\right) \text{ are stationary points}$$

(b) Using 1st derivative test to explore the nature of the stationary points: $\mathrm{e}^x>0$ when $x\in\mathbb{R}$ $\Rightarrow$ the expression $-x^2+2x+3$ determines the sign of the derivative.



$$h'(x)>0\Leftrightarrow -x^2+2x+3>0$$

$$\Rightarrow -(x+1)(x-3)>0\Rightarrow x\in(-1,3)$$

$$h'(x)<0\Leftrightarrow x\in(-\infty,-1)\cup(3,+\infty)$$

$y=h'(x)$ changes the sign from "–" to "+" at the point $x=-1\Rightarrow$ $(-1,-2\mathrm{e})$ is a relative minimum $y=h'(x)$ changes the sign from "+" to "–" at the point $x=3$ $\Rightarrow\left(3,\frac{6}{\mathrm{e}^3}\right)$ is a relative maximum.

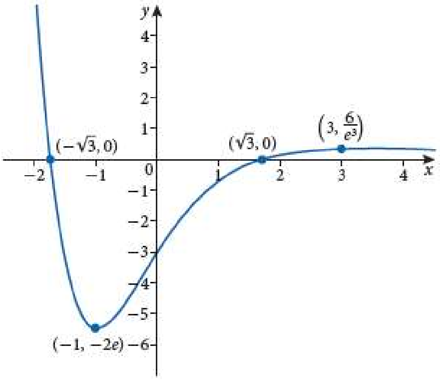
(c) $\lim\limits_{x\to\infty}h(x)=\lim\limits_{x\to\infty}\frac{x^2-3}{\mathrm{e}^x}=0$ (as e^x tends to infinity faster than x^2-3)

$\lim\limits_{x\to-\infty}h(x)=\lim\limits_{x\to-\infty}\left(x^2-3\right)\cdot\mathrm{e}^{-x}=+\infty$

(d) $\lim\limits_{x\to\infty}h(x)=0\Rightarrow y=0$ is a horizontal asymptote

(e) x-intercepts: $x^2=3\Leftrightarrow x=\sqrt{3}$ or $x=-\sqrt{3}$

y-intercept: $h(0)=\frac{-3}{\mathrm{e}^0}=-3$



13. $y = \sin x$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$ (using related angles)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(x+\frac{\pi}{2}\right)\right) = \cos\left(x+\frac{\pi}{2}\right) = \sin\left(x+\frac{\pi}{2}+\frac{\pi}{2}\right) = \sin(x+\pi)$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x+\pi)\right) = \cos(x+\pi) = \sin\left(x+\pi+\frac{\pi}{2}\right) = \sin\left(x+\frac{3\pi}{2}\right)$$

$$\Rightarrow a = \frac{\pi}{2}, b = \pi, c = \frac{3\pi}{2}$$

(b) The term which is added to x is increased by $\dfrac{\pi}{2}$ each step:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin\left(x+\frac{1\cdot\pi}{2}\right), \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sin\left(x+\frac{2\cdot\pi}{2}\right), \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \sin\left(x+\frac{3\cdot\pi}{2}\right)$$

$$\ldots, \frac{\mathrm{d}^{(n)}y}{\mathrm{d}x^{(n)}} = \sin\left(x+\frac{n\cdot\pi}{2}\right), n\in\mathbb{Z}$$

14. **(a)** $y = x\mathrm{e}^x$

Applying the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^x + \mathrm{e}^x\cdot 1 = (x+1)\mathrm{e}^x = x\mathrm{e}^x + \mathrm{e}^x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)\mathrm{e}^x\right)$$

Applying the product rule again:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (x+1)\mathrm{e}^x + \mathrm{e}^x\cdot 1 = (x+2)\mathrm{e}^x = x\mathrm{e}^x + 2\mathrm{e}^x$$

Similarly, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = (x+3)\mathrm{e}^x = x\mathrm{e}^x + 3\mathrm{e}^x$

Hence, we may notice that the general formula is:

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = (x+n)\mathrm{e}^x = x\mathrm{e}^x + n\mathrm{e}^x$$

(b) $y = xe^x$

$$\frac{d^{(n)}y}{dx^{(n)}} = (x+n)e^x, n \in \mathbb{Z}$$

Proof:

For $n=1$, we just showed that $\frac{dy}{dx} = (x+1)e^x$

Thus, the proposition is true for $n=1$

Assume that the proposition is true for $n=k$ so $\frac{d^{(k)}y}{dx^{(k)}} = (x+k)e^x, k \in \mathbb{Z}$

then

$$\frac{d^{(k+1)}y}{dx^{(k+1)}} = \frac{d}{dx}\left(\frac{d^{(k)}y}{dx^{(k)}}\right) = \frac{d}{dx}\left((x+k)e^x\right)$$

$$= (x+k)e^x + e^x \cdot 1 = (x+k+1)e^x = xe^x + (k+1)e^x$$

Given that the proposition is true for $n = k$, we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$, the proposition is true for all $n \in \mathbb{Z}$.

Exercise 13.3

1. **(a)** $x^2 + y^2 = 16$

Differentiate both sides implicitly:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(16)$$

Applying the chain rule: $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

$$2x + 2y\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

(b) $x^2y + xy^2 = 6$

Differentiate both sides implicitly:

$$\frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(6)$$

Applying the product rule and the chain rule:

$$x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) + x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) = 0$$

Applying the chain rule: $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$, $\frac{d}{dx}(y) = \frac{dy}{dx}\cdot 1 = \frac{dy}{dx}$

$$x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 = 0$$

$$(x^2 + 2xy)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx}=\frac{-2xy-y^2}{x^2+2xy}$$

(c) $x=\tan y$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(\tan y)$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}(\tan y)=\left(\sec^2 y\right)\frac{\mathrm{d}y}{dx}$,

$$1=\left(\sec^2 y\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sec^2 y}\Leftrightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\cos^2 y$$

(d) $x^2-3xy^2+y^3x-y^2=2$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-3xy^2+y^3x-y^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}(2)$$

Applying the product rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(3xy^2\right)=3x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)+y^2\frac{\mathrm{d}}{\mathrm{d}x}(3x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3x\right)=y^3\frac{\mathrm{d}}{\mathrm{d}x}(x)+x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)$:

$$2x-3x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)-y^2\frac{\mathrm{d}}{\mathrm{d}x}(3x)+y^3\frac{\mathrm{d}}{\mathrm{d}x}(x)+x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)-2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)=2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$2x-3x\cdot 2y\frac{\mathrm{d}y}{\mathrm{d}x}-3y^2+y^3+3xy^2\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\left(-6xy+3xy^2-2y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-2x+3y^2-y^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2x+3y^2-y^3}{-6xy+3xy^2-2y}$$

(e) $\frac{x}{y}-\frac{y}{x}=1$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{y}-\frac{y}{x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(1)$$

Applying the quotient rule for $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{y}\right)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right)$:

$$\frac{y\frac{\mathrm{d}}{\mathrm{d}x}(x)-x\frac{\mathrm{d}}{\mathrm{d}x}(y)}{y^2}-\frac{x\frac{\mathrm{d}}{\mathrm{d}x}(y)-y\frac{\mathrm{d}}{\mathrm{d}x}(x)}{x^2}=0$$

$$\frac{y-x\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2}-\frac{x\frac{\mathrm{d}y}{\mathrm{d}x}-y}{x^2}=0$$

Multiplying both sides by x^2y^2 :

$$x^2y-x^3\frac{\mathrm{d}y}{\mathrm{d}x}-xy^2\frac{\mathrm{d}y}{\mathrm{d}x}+y^3=0$$

$$\left(x^3+xy^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x^2y+y^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2y+y^3}{x^3+xy^2}$$

(f) $xy\sqrt{x+y}=1$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xy\sqrt{x+y}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(1)$$

Applying the product rule:

$$(xy)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x+y}\right)+\left(\sqrt{x+y}\right)\frac{\mathrm{d}}{\mathrm{d}x}(xy)=0$$

Applying the chain rule for $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x+y}\right)=\frac{1}{2\sqrt{x+y}}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ and the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}}{\mathrm{d}x}(y)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$:

$$\Rightarrow xy\cdot\frac{1}{2\sqrt{x+y}}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(\sqrt{x+y}\right)\cdot\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)=0$$

Multiply the whole equation with $2\sqrt{x+y}$ and simplify

$$\Rightarrow xy\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2(x+y)\cdot\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)=0$$

$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2y^2-3xy}{3xy+2x^2}$$

(g) $x+\sin y=xy$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+\sin y)=\frac{\mathrm{d}}{\mathrm{d}x}(xy)$$

Applying the chain rule for $\frac{\mathrm{d}}{\mathrm{d}x}(\sin y)=\cos y\frac{\mathrm{d}y}{\mathrm{d}x}$ and the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}y}{\mathrm{d}x}(y)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$, we get:

$$1+\cos y\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$$

$$(\cos y-x)\frac{\mathrm{d}y}{\mathrm{d}x}=y-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-1}{\cos y-x}$$

(h) $x^2y^3=x^4-y^4$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y^3\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4-y^4\right)$$

Applying the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y^3\right)=x^2\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)+y^3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)$:

$$x^2\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)+y^3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)=4x^3-\frac{\mathrm{d}}{\mathrm{d}x}\left(y^4\right)$$

Applying the chain rule for $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^4\right)=4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy^3=4x^3-4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(3x^2y^2+4y^3\right)\frac{\mathrm{d}y}{\mathrm{d}x}=4x^3-2xy^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4x^3-2xy^3}{3x^2y^2+4y^3}$$

(i) $xy+\mathrm{e}^y=0$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xy+\mathrm{e}^y\right)=\frac{\mathrm{d}}{\mathrm{d}x}(0)$$

Applying the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}}{\mathrm{d}x}(y)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ and the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^y\right)=\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}$,

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+y+\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\left(x+\mathrm{e}^y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-y}{x+\mathrm{e}^y}$$

(j) $(x+2)^2+(y+3)^2=25$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x+2)^2+(y+3)^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}(25)$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left((x+2)^2\right)=2(x+2)\cdot 1$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left((y+3)^2\right)=2(y+3)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$

$$2(x+2)\cdot 1+2(y+3)\cdot \frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x+2}{y+3}$$

(k) $x=\arctan x-y$

Differentiate both sides implicitly:

$$\frac{d}{dx}(x)=\frac{d}{dx}(\arctan x-y)$$

$$1=\frac{1}{1+x^2}-\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1}{1+x^2}-1$$

$$\frac{dy}{dx}=-\frac{x^2}{1+x^2}$$

(l) $y+\sqrt{xy}=3x^3$

Differentiate both sides implicitly:

$$\frac{d}{dx}\left(y+\sqrt{xy}\right)=\frac{d}{dx}\left(3x^3\right)$$

Applying the chain rule for: $\frac{d}{dx}\left(\sqrt{xy}\right)=\frac{1}{2\sqrt{xy}}\frac{d}{dx}(xy)$:

$$\frac{dy}{dx}+\frac{1}{2\sqrt{xy}}\frac{d}{dx}(xy)=9x^2$$

Applying the product rule for: $\frac{d}{dx}(xy)=x\frac{d}{dx}(y)+y\frac{d}{dx}(x)=x\frac{dy}{dx}+y$:

$$\frac{dy}{dx}+\frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx}+y\right)=9x^2$$

$$\frac{dy}{dx}+\frac{x}{2\sqrt{xy}}\frac{dy}{dx}+\frac{y}{2\sqrt{xy}}=9x^2$$

$$\left(1+\frac{x}{2\sqrt{xy}}\right)\frac{dy}{dx}=9x^2-\frac{y}{2\sqrt{xy}}$$

Finding the common denominator:

$$\left(\frac{2\sqrt{xy}+x}{2\sqrt{xy}}\right)\frac{dy}{dx}=\frac{18x^2\sqrt{xy}-y}{2\sqrt{xy}}$$

$$\frac{dy}{dx}=\frac{18x^2\sqrt{xy}-y}{x+2\sqrt{xy}}$$

2. **(a)** $x^3 - xy - 3y^2 = 0$ at $(2,-2)$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 - xy - 3y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}(0)$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(3y^2\right) = 6y\frac{\mathrm{d}y}{\mathrm{d}x}$ and the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}}{\mathrm{d}x}(y) + y\frac{\mathrm{d}}{\mathrm{d}x}(x) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$:

$$3x^2 - x\frac{\mathrm{d}y}{\mathrm{d}x} - y - 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(-x - 6y)\frac{\mathrm{d}y}{\mathrm{d}x} = y - 3x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - y}{x + 6y}$$

at $x = 2, y = -2$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\cdot 2^2 + 2}{2 + 6\cdot(-2)} = -\frac{7}{5}$$

Hence the slope of the tangent is $-\frac{7}{5}$ and the slope of the normal is $\frac{5}{7}$

Using the point-slope form for a linear equation gives:

$y + 2 = -\frac{7}{5}(x - 2) \Rightarrow y = -\frac{7}{5}x + \frac{4}{5}$ is the tangent to the graph at $(2,-2)$

$y + 2 = \frac{5}{7}(x - 2) \Rightarrow y = \frac{5}{7}x - \frac{24}{7}$ is the normal to the graph at $(2,-2)$

(b) $16x^4 + y^4 = 32$ at $(1,2)$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(16x^4 + y^4\right) = \frac{\mathrm{d}}{\mathrm{d}x}(32)$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^4\right) = 4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$

$$64x^3 + 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-16x^3}{y^3}$$

at $x = 1, y = 2$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-16\cdot 1^3}{2^3} = -2$$

Hence the slope of the tangent is -2 and the slope of the normal is $\frac{1}{2}$.

Using the point-slope form for a linear equation gives:

$y-2=-2(x-1)\Rightarrow y=-2x+4$ is the tangent to the graph at $(1,2)$

$y-2=\frac{1}{2}(x-1)\Rightarrow y=\frac{1}{2}x+\frac{3}{2}$ is the normal to the graph at $(1,2)$

(c) $2xy+\pi\sin y=2\pi$ at $\left(1,\frac{\pi}{2}\right)$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}(2xy+\pi\sin y)=\frac{\mathrm{d}}{\mathrm{d}x}(2\pi)$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}(\sin y)=\cos y\frac{\mathrm{d}y}{\mathrm{d}x}$ and the product rule for

$\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}}{\mathrm{d}x}(y)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$:

$$2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y+\pi\cos y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$(2x+\pi\cos y)\frac{\mathrm{d}y}{\mathrm{d}x}=-2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2y}{2x+\pi\cos y}$$

at $x=1, y=\frac{\pi}{2}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2\cdot\frac{\pi}{2}}{2\cdot1+\pi\cos\frac{\pi}{2}}=-\frac{\pi}{2}$$

Hence the slope of the tangent is $-\frac{\pi}{2}$ and the slope of the normal is $\frac{2}{\pi}$.

Using the point-slope form for a linear equation gives:

$y-\frac{\pi}{2}=-\frac{\pi}{2}(x-1)\Rightarrow y=-\frac{\pi}{2}x+\pi$ is the tangent to the graph at $\left(1,\frac{\pi}{2}\right)$

$y-\frac{\pi}{2}=\frac{2}{\pi}(x-1)\Rightarrow y=\frac{2}{\pi}x+\frac{\pi^2-4}{2\pi}$ is the normal to the graph at $\left(1,\frac{\pi}{2}\right)$

(d) $\sqrt[3]{xy}=14x+y$ at $(2,-32)$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt[3]{xy}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(14x+y)$$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt[3]{xy}\right)=\frac{1}{3}(xy)^{-\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}(xy)$:

$$\frac{1}{3}(xy)^{-\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}(xy)=14+\frac{\mathrm{d}y}{\mathrm{d}x}$$

Applying the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}}{\mathrm{d}x}(y)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$:

$$\frac{1}{3}(xy)^{-\frac{2}{3}}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)=14+\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{1}{3}x^{\frac{1}{3}}y^{-\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{3}}=14+\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$x^{\frac{1}{3}}y^{-\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}x}+x^{-\frac{2}{3}}y^{\frac{1}{3}}=42+3\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\left(x^{\frac{1}{3}}y^{-\frac{2}{3}}-3\right)\frac{\mathrm{d}y}{\mathrm{d}x}=42-x^{-\frac{2}{3}}y^{\frac{1}{3}}$$
$$\left(x^{\frac{1}{3}}y^{-\frac{2}{3}}-3\right)\frac{\mathrm{d}y}{\mathrm{d}x}=42-x^{-\frac{2}{3}}y^{\frac{1}{3}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{42-\sqrt[3]{\frac{y}{x^2}}}{\sqrt[3]{\frac{x}{y^2}}-3}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{42\sqrt[3]{x^2}-\sqrt[3]{y}}{\sqrt[3]{x^2}}\cdot\frac{\sqrt[3]{y^2}}{\sqrt[3]{x}-3\sqrt[3]{y^2}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{42\sqrt[3]{x^2y^2}-y}{x-3\sqrt[3]{x^2y^2}}$$
at $x=2, y=-32$:
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{42\sqrt[3]{2^2\cdot 32^2}+32}{2-3\sqrt[3]{2^2\cdot 32^2}}=\frac{42\cdot 16+32}{2-3\cdot 16}=-\frac{352}{23}$$

Hence the slope of the tangent is $-\frac{352}{23}$ and the slope of the normal is $\frac{23}{352}$.

Using the point-slope form for a linear equation gives:

$y+32=-\frac{352}{23}(x-2)\Rightarrow y=-\frac{352}{23}x-\frac{32}{23}$ is the tangent to the graph at $(2,-32)$

$y+32=\frac{23}{352}(x-2)\Rightarrow y=\frac{23}{352}x-\frac{5655}{176}$ is the normal to the graph at $(2,-32)$

3. $x^2+y^2=r^2$

The center of the circle: (0, 0)
Gradient of the line which passes through the points (x_1, y_1) and (0, 0):

$$m_1=\frac{y_1-0}{x_1-0}=\frac{y_1}{x_1}$$

Gradient of the tangent to the circle at the point (x_1, y_1):
Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+y^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(r^2\right)$$
$$\Rightarrow 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

At the point (x_1, y_1): $\dfrac{dy}{dx}=-\dfrac{x_1}{y_1}=m_2$

$m_1 \cdot m_2=\dfrac{y_1}{x_1}\cdot\left(-\dfrac{x_1}{y_1}\right)=-1 \Rightarrow$ the two lines are perpendicular

4. $x^2+xy+y^2=7$

(a) The curve intersects the *x*-axis $\Leftrightarrow y=0$

Substituting $y=0$, we get: $x^2=7 \Rightarrow x=\sqrt{7}$ or $x=-\sqrt{7}$

$(-\sqrt{7},0);(\sqrt{7},0)$ - the points of intersection with the *x*-axis

Differentiate both sides implicitly:

$$\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}(7)$$

$$\frac{d}{dx}(xy)=x\frac{d}{dx}(y)+y\frac{d}{dx}(x)=x\frac{dy}{dx}+y:$$

$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$(x+2y)\frac{dy}{dx}=-y-2x$$

$$\frac{dy}{dx}=\frac{-y-2x}{x+2y}$$

Gradient at $(-\sqrt{7},0)$:

$$\frac{dy}{dx}=\frac{-0+2\sqrt{7}}{-\sqrt{7}+2\cdot 0}=-2=m_1$$

Gradient at $(\sqrt{7},0)$:

$$\frac{dy}{dx}=\frac{-0-2\sqrt{7}}{\sqrt{7}+2\cdot 0}=-2=m_2$$

$m_1=m_2=-2 \Rightarrow$ the tangents are parallel

(b) The tangent is parallel to the *x*-axis $\Rightarrow \dfrac{dy}{dx}=0$

$$\frac{-y-2x}{x+2y}=0$$

$$y=-2x$$

Substituting to the equation of the curve:

$$x^2+x\cdot(-2x)+(-2x)^2=7$$

$$3x^2=7$$

$$x=\sqrt{\frac{7}{3}} \quad \text{or} \quad x=-\sqrt{\frac{7}{3}}$$

For $x=\sqrt{\frac{7}{3}}\ :\ y=-2\cdot\sqrt{\frac{7}{3}}=-2\sqrt{\frac{7}{3}}$

For $x=-\sqrt{\frac{7}{3}}\ :\ y=-2\cdot\left(-\sqrt{\frac{7}{3}}\right)=2\sqrt{\frac{7}{3}}$

The points where the tangent to the curve is parallel to the x-axis: $\left(\sqrt{\frac{7}{3}},-2\sqrt{\frac{7}{3}}\right)$, $\left(-\sqrt{\frac{7}{3}},2\sqrt{\frac{7}{3}}\right)$

(c) The tangent is parallel to y-axis, so the gradient does not exist (the denominator is 0)

$x+2y=0$

$y=-\frac{x}{2}$

Substituting to the equation of the curve:

$x^2+x\cdot\left(-\frac{x}{2}\right)+\left(-\frac{x}{2}\right)^2=7$

$\frac{3}{4}x^2=7$

$x=2\sqrt{\frac{7}{3}}$ or $x=-2\sqrt{\frac{7}{3}}$

For $x=2\sqrt{\frac{7}{3}}\ :\ y=-\frac{1}{2}\cdot 2\sqrt{\frac{7}{3}}=-\sqrt{\frac{7}{3}}$

For $x=-2\sqrt{\frac{7}{3}}\ :\ y=-\frac{1}{2}\cdot 2\cdot\left(-\sqrt{\frac{7}{3}}\right)=\sqrt{\frac{7}{3}}$

The points where the tangent to the curve is parallel to the y-axis: $\left(2\sqrt{\frac{7}{3}},-\sqrt{\frac{7}{3}}\right)$, $\left(-2\sqrt{\frac{7}{3}},\sqrt{\frac{7}{3}}\right)$

5. (a) $x^2+2xy-3y^2=0$ at $(1,1)$

Differentiate both sides implicitly:

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+2xy-3y^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}(0)$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)=2y\frac{\mathrm{d}y}{\mathrm{d}x}$ and the product rule for $\frac{\mathrm{d}}{\mathrm{d}x}(xy)=x\frac{\mathrm{d}}{\mathrm{d}x}(y)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$:

$2x+2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y-6y\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$(2x-6y)\frac{\mathrm{d}y}{\mathrm{d}x}=-2x-2y$

$$\frac{dy}{dx}=\frac{2x+2y}{6y-2x}$$

at $x=1, y=1$:

$$\frac{dy}{dx}=\frac{2\cdot1+2\cdot1}{6\cdot1-2\cdot1}=1$$

Hence the slope of the normal is -1.

Using the point-slope form for a linear equation gives:

$y-1=-1(x-1)\Rightarrow y=-x+2$ is the normal to the graph at $(1, 1)$

Substituting $y=-x+2$ to the equation of the curve:

$$x^2+2x\cdot(-x+2)-3(-x+2)^2=0$$

$$-x^2+5x-4=0$$

$x=1$ - the tangency point or $x=3$

Substituting $x=3$ to the equation of the tangent, we get the y-coordinate of the point:

$y=-3+2=-1$

The other point of intersection: $(3,-1)$.

6. (a) $4x^2+9y^2=36$

Differentiate both sides implicitly: $\frac{d}{dx}(4x^2+9y^2)=\frac{d}{dx}(36)$

Applying the chain rule for: $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$: $\quad 8x+18y\frac{dy}{dx}=0$

(1) $\frac{dy}{dx}=-\frac{4x}{9y}$

Differentiate both sides implicitly:

$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{4x}{9y}\right)$, Applying the quotient rule to $\frac{d}{dx}\left(-\frac{4x}{9y}\right)$:

$$\Rightarrow\quad \frac{d^2y}{dx^2}=\frac{-36y+36x\frac{dy}{dx}}{81y^2}$$

Substituting (1) as $\frac{dy}{dx}$: $\quad \frac{d^2y}{dx^2}=\frac{-36y+36x\cdot\left(-\frac{4x}{9y}\right)}{81y^2}=\frac{-36y-\frac{16x^2}{y}}{81y^2}$

Simplifying:

$$\frac{d^2y}{dx^2}=\frac{-36y^2-16x^2}{81y^3}$$

(b) $xy = 2x - 3y$

Differentiate both sides implicitly: $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \frac{\mathrm{d}}{\mathrm{d}x}(2x - 3y)$

Applying the chain rule for: $\frac{\mathrm{d}}{\mathrm{d}x}(3y) = 3 \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$ and the product rule for

$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}}{\mathrm{d}x}(y) + y\frac{\mathrm{d}}{\mathrm{d}x}(x) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2 - 3\frac{\mathrm{d}y}{\mathrm{d}x} \Leftrightarrow (x+3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - y$$

$$(1)\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-y}{x+3}$$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2-y}{x+3}\right)$$

Applying the quotient rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2-y}{x+3}\right)$ and the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}(y) = 1 \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(x+3)\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right) - (2-y)\cdot 1}{(x+3)^2} = \frac{(x+3)\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2 + y}{(x+3)^2}$$

Substituting (1) as $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(x+3)\left(-\frac{2-y}{x+3}\right) - 2 + y}{(x+3)^2} = \frac{-2 + y - 2 + y}{(x+3)^2}$$

Simplifying:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2y-4}{(x+3)^2}$$

7. $xy^3 = 1$

(a) Expressing y in terms of x:

$$y = \sqrt[3]{\frac{1}{x}} = x^{-\frac{1}{3}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}x^{-\frac{4}{3}} = \frac{-1}{x^{\frac{4}{3}}}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{3}x^{-\frac{4}{3}}\right) = \frac{4}{9}x^{-\frac{7}{3}} = \frac{4}{9x^{\frac{7}{3}}}$$

(b) Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xy^3\right)=\frac{\mathrm{d}}{\mathrm{d}x}(1)$$

The product rule for $\frac{\mathrm{d}}{\mathrm{d}x}\left(xy^3\right)=x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)+y\frac{\mathrm{d}}{\mathrm{d}x}(x)$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)+y^3\frac{\mathrm{d}}{\mathrm{d}x}(x)=0$$

Applying the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x}+y^3=0$$

(1) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{3x}$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{y}{3x}\right)$$

Applying the quotient rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{y}{3x}\right)$ and the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}(y)=1\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3x\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right)-(-y)\cdot 3}{(3x)^2}=\frac{3x\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right)+3y}{9x^2}$$

Substituting (1) as $\frac{\mathrm{d}y}{\mathrm{d}x}$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3x\left(\frac{y}{3x}\right)+3y}{9x^2}$

Simplifying: (2) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{4y}{9x^2}$

8. $x^2+y^2=\left(2x^2+2y^2-x\right)^2$ at $\left(0,\frac{1}{2}\right)$

Differentiate both sides implicitly: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+y^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(2x^2+2y^2-x\right)^2\right)$

Applying the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)=2y\frac{\mathrm{d}y}{\mathrm{d}x}$

and to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(2x^2+2y^2-x\right)^2\right)=2\left(2x^2+2y^2-x\right)\left(4x+4y\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)$:

$$2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\left(2x^2+2y^2-x\right)\left(4x+4y\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)$$

at $x=0, y=\frac{1}{2}$: $2\cdot 0+2\cdot\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}=2\left(2\cdot 0^2+2\cdot\left(\frac{1}{2}\right)^2-0\right)\left(4\cdot 0+4\cdot\frac{1}{2}\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x}=2\frac{\mathrm{d}y}{\mathrm{d}x}-1 \Leftrightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=1$ Hence the slope of the tangent is 1.

Using the point-slope form for a linear equation gives:
$y-\frac{1}{2}=1\cdot(x-0)\Rightarrow y=x+\frac{1}{2}$ is the tangent to the graph at $\left(0,\frac{1}{2}\right)$.

9. **(a)** $y=\ln\left(x^3+1\right)$

Applying the chain rule: Let $f(u)=\ln u,\ g(x)=x^3+1,$ then $f'(u)=\frac{1}{u}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{x^3+1}\cdot 3x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2}{x^3+1}$$

(b) $y=\ln(\sin x)$

Applying the chain rule: Let $f(u)=\ln u,\ g(x)=\sin x,$ then $f'(u)=\frac{1}{u}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{\sin x}\cdot\cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\cot x$$

(c) $y=\log_5\sqrt{x^2-1}=\log_5\left(x^2-1\right)^{\frac{1}{2}}$

Applying the chain rule:

Let $f(u)=\log_5 u,\ g(x)=\left(x^2-1\right)^{\frac{1}{2}},$ then $f'(u)=\frac{1}{u\ln 5}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{\left(x^2-1\right)^{\frac{1}{2}}\ln 5}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-1\right)^{\frac{1}{2}}$$

Applying the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-1\right)^{\frac{1}{2}}=\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}\cdot 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{\left(x^2-1\right)^{\frac{1}{2}}\ln 5}\cdot\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}\cdot 2x$$

Simplifying: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{\left(x^2-1\right)\ln 5}$

(d) $\ln\sqrt{\frac{1+x}{1-x}}=\ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

Applying the chain rule: Let $f(u)=\ln u,\ g(x)=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}},$ then $f'(u)=\frac{1}{u}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right)$$

Applying the quotient rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x}{1-x}\right)=\frac{(1-x)\cdot 1-(1+x)\cdot(-1)}{(1-x)^2}=\frac{2}{(1-x)^2}$ and

chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1+x}{1-x}\right)=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\cdot\frac{2}{(1-x)^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}}\cdot\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\cdot\frac{2}{(1-x)^2}=\frac{x-1}{x+1}\cdot\frac{1}{(1-x)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{x^2-1}$$

(e) $y=\sqrt{\log_{10}x}$

Applying the chain rule:

Let $f(u)=\sqrt{u}=u^{\frac{1}{2}},\ g(x)=\log_{10}x,$

then $f'(u)=\frac{1}{2}u^{-\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{2}(\log_{10}x)^{-\frac{1}{2}}\cdot\frac{1}{x\ln 10}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2x\ln 10\sqrt{\log x}}$$

(f) $\ln\left(\frac{a-x}{a+x}\right)$

Applying the chain rule:

Let $f(u)=\ln u,\ g(x)=\frac{a-x}{a+x},$ then $f'(u)=\frac{1}{u}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot f'(x)=\frac{1}{\frac{a-x}{a+x}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a-x}{a+x}\right)=$$

Applying the quotient rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a-x}{a+x}\right)=\frac{(a+x)\cdot(-1)-(a-x)\cdot(1)}{(a+x)^2}=\frac{-2a}{(a+x)^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\frac{a-x}{a+x}}\cdot\frac{-2a}{(a+x)^2}$$

Simplifying: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2a}{x^2-a}$

(g) $y = \ln\left(\mathrm{e}^{\cos x}\right)$

Applying the chain rule: Let $f(u) = \ln u,\ g(x) = \mathrm{e}^{\cos x},\ $ then $f'(u) = \frac{1}{u}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot f'(x) = \frac{1}{\mathrm{e}^{\cos x}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\cos x}\right)$

Applying the chain rule to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\cos x}\right) = \mathrm{e}^{\cos x} \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\mathrm{e}^{\cos x} \sin x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^{\cos x}} \cdot \left(-\mathrm{e}^{\cos x} \sin x\right)$, simplifying $\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$

(h) $y = \frac{1}{\log_3 x} = \left(\log_3 x\right)^{-1}$

Applying the chain rule: Let $f(u) = u^{-1},\ g(x) = \log_3 x,\ $ then $f'(u) = -u^{-2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot f'(x) = -\left(\log_3 x\right)^{-2} \cdot \frac{1}{x \ln 3}$

Simplifying, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x \ln 3 \left(\log_3 x\right)^2}$

(i) $y = x \ln x - x$

Let $f(x) = x,\ g(x) = \ln x$

Applying the product rule:

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x) - x\right) = f(x) \cdot g'(x) + g(x) \cdot f'(x) - 1 = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1$

Simplifying: $\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x$

(j) $y = \ln(ax) - \ln(b) \log_b x$

Method 1 $y = \ln(ax) - \ln(b) \log_b x = \ln a + \ln x - \ln b \cdot \frac{\ln x}{\ln b} = \ln a$, and the derivative of a constant is zero.

Method 2 Applying the chain rule: Let $f(u) = \ln u,\ g(x) = ax,\ $ then $f'(u) = \frac{1}{u}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot f'(x) - \ln(b) \cdot \frac{1}{x \ln(b)} = \frac{1}{ax} \cdot a - \frac{1}{x}$

Simplifying: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

10. $y = \log_2 x$ at $x = 8$

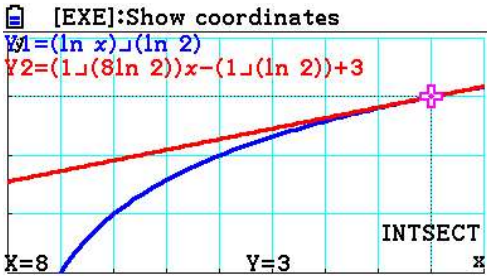
(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln 2}$, at $x = 8$: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{8 \ln 2}$, hence, the slope of the tangent is $\frac{1}{8 \ln 2}$.

Finding y-coordinate of the tangency point: at $x = 8,\ y = \log_2 8 = 3$. So the point of tangency $(8, 3)$.

Using the point-slope form for a linear equation gives:

$$y-3=\frac{1}{8\ln 2}(x-8)\Rightarrow y=\frac{1}{8\ln 2}x-\frac{1}{\ln 2}+3$$ is the tangent to the graph at $x=8$.

(b) We graph $y=\dfrac{\ln x}{\ln 2}$ and the tangent line:



```
[EXE]:Show coordinates
Y1=(ln x)⌟(ln 2)
Y2=(1⌟(8ln 2))x−(1⌟(ln 2))+3
INTSECT
X=8          Y=3
```

11. $$y=\sqrt{\frac{x^2-1}{x^2+1}}=\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}$$

Method 1

Applying the chain rule: Let $f(u)=u^{\frac{1}{2}}$, $g(x)=\dfrac{x^2-1}{x^2+1}$, then $f\,'(u)=\dfrac{1}{2}u^{-\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot f\,'(x)=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{-\frac{1}{2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$$

Applying the quotient rule to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{\left(x^2+1\right)\cdot 2x-\left(x^2-1\right)\cdot(2x)}{\left(x^2+1\right)^2}=\frac{2x\left(x^2+1-x^2+1\right)}{x^2+1^2}$$

$$=\frac{4x}{\left(x^2+1\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f\,'(g(x))\cdot f\,'(x)=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{-\frac{1}{2}}\cdot\frac{4x}{\left(x^2+1\right)^2}$$

$$=\frac{2x}{\sqrt{\left(x^2-1\right)\left(x^2+1\right)^3}}=\frac{2x}{\left(x^2-1\right)^{\frac{1}{2}}\left(x^2+1\right)^{\frac{3}{2}}}$$

Method 2

$$\ln y=\ln\sqrt{\frac{x^2-1}{x^2+1}}=\frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$\Rightarrow 2\ln y=\ln\left(\frac{x^2-1}{x^2+1}\right)=\ln\left(x^2-1\right)-\ln\left(x^2+1\right)$$

Differentiate both sides implicitly:

$$\frac{\mathrm{d}}{\mathrm{d}x}(2\ln y)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2-1\right)-\ln\left(x^2+1\right)\right)$$

$$\Rightarrow \frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{x^2-1}-\frac{2x}{x^2+1}=\frac{2x\left(x^2+1-x^2+1\right)}{x^4-1}=\frac{4x}{x^4-1}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy}{x^4-1}$$

Substituting: $y=\sqrt{\dfrac{x^2-1}{x^2+1}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x\sqrt{\dfrac{x^2-1}{x^2+1}}}{x^4-1}=\frac{2x\sqrt{x^2-1}}{\left(x^2-1\right)\left(x^2+1\right)\sqrt{x^2+1}}\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{\left(x^2-1\right)^{\frac{1}{2}}\left(x^2+1\right)^{\frac{3}{2}}}$$

12. $f(x)=x^2\ln\left(x^2\right)$

Inflection point is where $f''(x)=0$ or is undefined and $f''(x)$ changes sign

1st derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x^2\cdot\frac{2}{x}+\ln\left(x^2\right)\cdot 2x \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=2x\left(1+\ln\left(x^2\right)\right)$$

2nd derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(2x\left(1+\ln\left(x^2\right)\right)\right)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2x\cdot\frac{2}{x}+\left(1+\ln\left(x^2\right)\right)\cdot 2 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(3+\ln\left(x^2\right)\right)$$

$$f''(x)=0\Rightarrow 2\left(3+\ln\left(x^2\right)\right)=0$$

$$\ln\left(x^2\right)=-3\Leftrightarrow x^2=\mathrm{e}^{-3}\Leftrightarrow\left(\left(x=\sqrt{\frac{1}{\mathrm{e}^3}}\text{ or }x=-\sqrt{\frac{1}{\mathrm{e}^3}}\right)\text{ and }0<x<1\right)\Rightarrow x=\sqrt{\frac{1}{\mathrm{e}^3}}=\frac{1}{\mathrm{e}^{\frac{3}{2}}}$$

$$\left.\begin{array}{l} f''\left(\sqrt{\dfrac{1}{\mathrm{e}^4}}\right)=2\left(3+\ln\mathrm{e}^{-4}\right)=-2<0 \\ f''(1)=2(3+\ln 1)=6>0 \end{array}\right\}\Rightarrow \text{there is an inflection point at } x=\frac{1}{\mathrm{e}^{\frac{3}{2}}}$$

$$x=\frac{1}{\mathrm{e}^{\frac{3}{2}}}$$

13. $g(x)=\dfrac{\ln x}{x}$

(a) (1) Let $f(x)=\ln$, $h(x)=x$ Applying the quotient rule:

$$g'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{h(x)}\right)=\frac{h(x)f'(x)-f(x)h'(x)}{[h(x)]^2}=\frac{x\cdot\frac{1}{x}-(\ln x)\cdot 1}{x^2}=\frac{1-\ln x}{x^2}$$

(2) Let $f(x)=1-\ln x,\ h(x)=x^2$

Applying the quotient rule:

$$g''(x)=\frac{d}{dx}\left(\frac{f(x)}{h(x)}\right)=\frac{h(x)f'(x)-f(x)h'(x)}{[h(x)]^2}$$

$$=\frac{x^2\cdot\left(-\frac{1}{x}\right)-(1-\ln x)\cdot 2x}{x^4}=\frac{-3x+2x\ln x}{x^4}=\frac{-3+2\ln x}{x^3}$$

(b) The function is continuous on its domain $(0,+\infty)$

$$g'(x)=0\Rightarrow\frac{1-\ln x}{x^2}=0\Rightarrow\ln x=1\Rightarrow x=e$$

$g''(e)=\frac{-3+2\ln e}{e^3}=\frac{-3+2\ln e}{e^3}=\frac{-1}{e^3}<0\Rightarrow$ there is an absolute maximum at $x=e$ (the only extreme value in the continuous domain)

Maximum value is $g(e)=\frac{\ln e}{e}=\frac{1}{e}$

14. **(a)** $y=\arctan(x+1)$

Applying the chain rule to the function:

$$\frac{dy}{dx}=f'(g(x))\cdot g'(x)=\frac{1}{1+(x+1)^2}\cdot 1=\frac{1}{x^2+2x+2}$$

(b) $y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Applying the chain rule to the function:

$$\frac{dy}{dx}=f'(g(x))\cdot g'(x)=\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Applying the quotient rule to $\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)$ and the chain rule to $\frac{d}{dx}\left(\sqrt{1+x^2}\right)$

$$\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{\sqrt{1+x^2}\cdot 1-x\cdot\frac{1}{2\sqrt{1+x^2}}\cdot 2x}{1+x^2}=\frac{\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}}{1+x^2}=\frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}}\cdot\frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}=\frac{1}{\sqrt{\frac{1}{1+x^2}}}\cdot\frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}$$
$$=\frac{1}{\left(1+x^2\right)^{-\frac{1}{2}}}\cdot\frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}=\frac{1}{1+x^2}$$

(c) $y=\arccos\left(\frac{3}{x^2}\right)=\arccos\left(3x^{-2}\right)$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=\frac{-1}{\sqrt{1-\left(3x^{-2}\right)^2}}\cdot\left(-6x^{-3}\right)=\frac{6}{x^3\sqrt{1-9x^{-4}}}$$
$$=\frac{6}{x\sqrt{x^4-9x^{-4}\cdot x^4}}=\frac{6}{x\sqrt{x^4-9}}$$

(d) $\ln y=x\arctan x\ \Rightarrow\ (1)\ y=\mathrm{e}^{x\tan^{-1}x}$

Differentiate both sides implicitly:

$\frac{\mathrm{d}}{\mathrm{d}x}(\ln y)=\frac{\mathrm{d}}{\mathrm{d}x}(x\arctan x)$, but

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\arctan x)=x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x)+\arctan x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x)=x\cdot\frac{1}{1+x^2}+\arctan x\cdot 1$$
$$=\frac{x}{1+x^2}+\arctan x$$

Thus,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{1+x^2}+\arctan x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\tan^{-1}x+\frac{x}{1+x^2}\right)\cdot y$$

Substituting $y=\mathrm{e}^{x\tan^{-1}x}$ (from (1)): $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\tan^{-1}x+\frac{x}{1+x^2}\right)\cdot\mathrm{e}^{x\tan^{-1}x}$

15. $f(x)=\arcsin x+\arccos x$

$f'(x)=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}=0\ \Rightarrow$ the function f is constant.

16. **(a)** $y=\arctan\frac{x}{a}$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\left(\frac{1}{a}\right)=\frac{1}{a+\frac{x^2}{a}}=\frac{a}{a^2+x^2}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\frac{x}{a}\right)=\frac{a}{a^2+x^2}$$

(b) $y=\arcsin\dfrac{x}{a}$

Applying the chain rule to the function:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\frac{1}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$$

17. $y=4x\arctan 2x,\quad x=\dfrac{1}{2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4x\cdot\frac{1}{1+(2x)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x)+(\arctan 2x)\cdot 4=\frac{8x}{1+4x^2}+4\arctan 2x$$

at $x=\dfrac{1}{2}$: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{8\cdot\frac{1}{2}}{1+4\cdot\left(\frac{1}{2}\right)^2}+4\arctan\left(2\cdot\dfrac{1}{2}\right)=2+\pi$

Hence the slope of the tangent is $2+\pi$.

The y-coordinate of the tangency point:

at $x=\dfrac{1}{2}$ $\quad y=4\cdot\dfrac{1}{2}\cdot\arctan\left(2\cdot\dfrac{1}{2}\right)=2\cdot\dfrac{\pi}{4}=\dfrac{\pi}{2}$. So, the point of tangency is: $\left(\dfrac{1}{2},\dfrac{\pi}{2}\right)$.

Using the point-slope form for a linear equation gives:

$y-\dfrac{\pi}{2}=(2+\pi)\left(x-\dfrac{1}{2}\right)\Rightarrow y=(2+\pi)x-1$ is the tangent to the graph at $x=\dfrac{1}{2}$.

18. $f(x)=\arcsin(\cos x),\quad 0\le x<\pi$

(a) Let $f(u)=\arcsin u,\ g(x)=\cos x,\ \text{then } f'(u)=\dfrac{1}{\sqrt{1-u^2}}$

Applying the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=f'(g(x))\cdot g'(x)=\frac{1}{\sqrt{1-\cos^2 x}}\cdot-\sin x=\frac{-\sin x}{\sqrt{\sin^2 x}}=-1$$

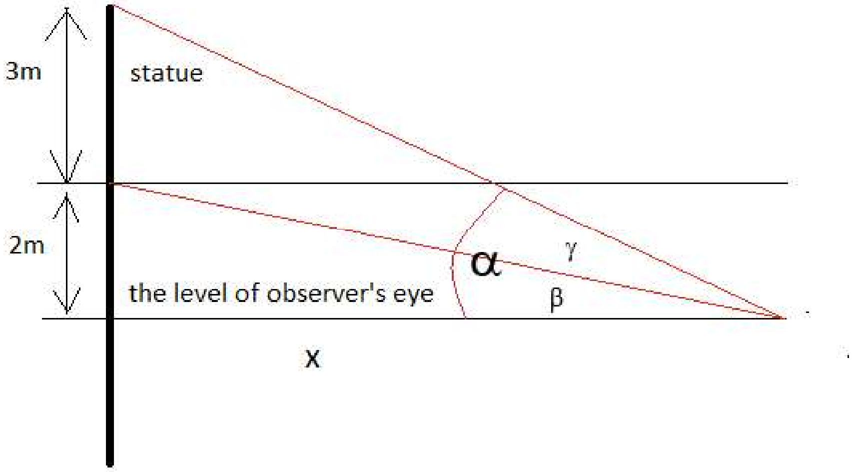
(as $0\le x<\pi$) $\Rightarrow$ the function has a constant gradient and hence it is linear

(b) $f(x)=\arcsin(\cos x)$

Applying the related angles:

$$f(x)=\arcsin\left(\sin\left(\frac{\pi}{2}-x\right)\right)\Rightarrow f(x)=-x+\frac{\pi}{2}$$

19.



$$\tan\beta = \frac{2}{x}, \tan\alpha = \frac{5}{x}$$

$$\tan\gamma = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\frac{5}{x} - \frac{2}{x}}{1 + \frac{5}{x}\cdot\frac{2}{x}} = \frac{3x}{x^2 + 10}$$

The angle subtended at the observer's eye by the statue is maximum if the function $y = \dfrac{3x}{x^2 + 10}$ is maximum

Let $f(x) = 3x,\ g(x) = x^2 + 10$

Applying the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{(x^2 + 10)\cdot 3 - 3x\cdot 2x}{(x^2 + 10)^2} = \frac{-3x^2 + 30}{(x^2 + 10)^2}$$

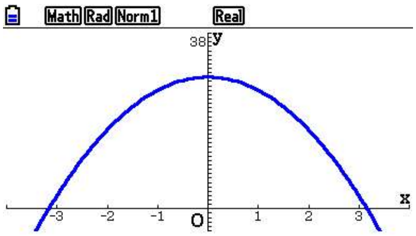
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{-3x^2 + 30}{(x^2 + 10)^2} = 0 \Rightarrow -3x^2 + 30 = 0$$

$\Rightarrow x = -\sqrt{10}$ or $x = \sqrt{10}$ and $x > 0 \Rightarrow x = \sqrt{10}$

$(x^2 + 10)^2 > 0$ when $x \in \mathbb{R} \Rightarrow$ the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ depends on the sign of the numerator.

The numerator: $-3\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right)$

$\Rightarrow$ the derivative changes its sign from "+" to "–"at $x = \sqrt{10}$ $\Rightarrow$there is a maximum value (the only maximum) at $x = \sqrt{10}$



The distance between the observer and the base of the column should be $x=\sqrt{10}\text{ m}\approx 3.16\text{ m}$

20. $s(t)=\arctan\sqrt{t}$

(a) $v(t)=\dfrac{\mathrm{d}s}{\mathrm{d}t}=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\arctan\sqrt{t}\right)$

Applying the chain rule to the function:

$$v(t)=f'(g(t))\cdot g'(t)=\frac{1}{1+\left(\sqrt{t}\right)^2}\cdot\frac{1}{2\sqrt{t}}=\frac{1}{2\sqrt{t}+2t\sqrt{t}}$$

$$v(1)=\frac{1}{2\sqrt{1}+2\cdot1\cdot\sqrt{1}}=\frac{1}{4}\left[\frac{m}{s}\right],\quad v(4)=\frac{1}{2\sqrt{4}+2\cdot4\cdot\sqrt{4}}=\frac{1}{20}\left[\frac{m}{s}\right]$$

(b) $a(t)=\dfrac{\mathrm{d}v}{\mathrm{d}t}=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\left(2t^{\frac{1}{2}}+2t^{\frac{3}{2}}\right)^{-1}\right)$

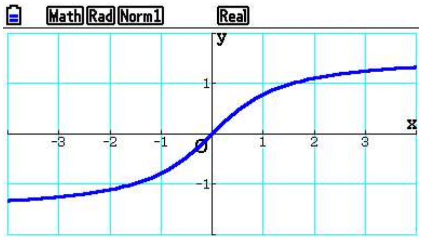
$$a(t)=-\left(2t^{\frac{1}{2}}+2t^{\frac{3}{2}}\right)^{-2}\cdot\left(t^{-\frac{1}{2}}+3t^{\frac{1}{2}}\right)=-\frac{1+3t}{\sqrt{t}\left(2\sqrt{t}+2t\sqrt{t}\right)^2}$$

$$a(1)=-\frac{1+3\cdot1}{\sqrt{1}\left(2\sqrt{1}+2\cdot1\sqrt{1}\right)^2}=-\frac{1}{4}\left[\frac{m}{s^2}\right],\quad a(4)=-\frac{1+3\cdot4}{\sqrt{4}\left(2\sqrt{4}+2\cdot4\sqrt{4}\right)^2}=-\frac{13}{800}\left[\frac{m}{s^2}\right]$$

(c) The particle is moving fast to the right and then gradually slows down while continuing to move to the right.

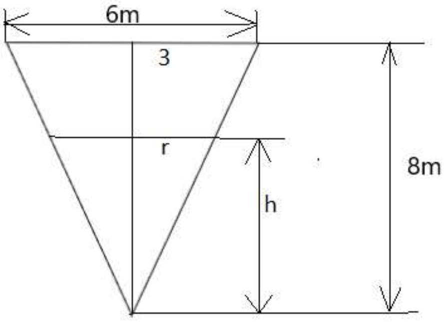
(d) Let $x=\sqrt{t}$, when $t\to\infty$ then $x\to\infty$

$$\lim_{x\to\infty}\arctan x=\frac{\pi}{2}\Rightarrow\lim_{t\to\infty}\arctan\sqrt{t}=\frac{\pi}{2}$$



Exercise 13.4

1.



We know the rate of change of the volume with respect to time: $\frac{\mathrm{d}V}{\mathrm{d}t}=-2m^3\,\text{min}^{-1}$, the height $h=8m$, the diameter of the top is 6m.

We need to find the rate of change of the level of the water with respect to time: $\frac{\mathrm{d}h}{\mathrm{d}t}$.

The formula of the volume of a cone gives us an equation that relates the variables: V, r and h.

$$V=\frac{1}{3}\pi r^2 h$$

(a) By using similar triangles, we get: $\frac{r}{h}=\frac{3}{8}\Rightarrow r=\frac{3}{8}h$ and substituting the result into the formula of the volume, we get: $V=\frac{1}{3}\pi\cdot\left(\frac{3}{8}h\right)^2\cdot h=\frac{3}{64}\pi h^3$

Differentiate both sides implicitly with respect to t:

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{9}{64}\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{64}{9\pi h^2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

Substitute $h=5,\frac{\mathrm{d}V}{\mathrm{d}t}=-2$:

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{64}{9\pi\cdot 5^2}\cdot(-2)=\frac{-128}{225\pi}\approx-0.181083\left[\text{m}\,\text{min}^{-1}\right]\approx-18.1\left[\text{cm}\,\text{min}^{-1}\right]$$

(b) By using similar triangles, we get: $\frac{r}{h}=\frac{3}{8}\Rightarrow h=\frac{8}{3}r$ and substituting the result into the formula of the volume, we get: $V=\frac{1}{3}\pi\cdot r^2\cdot\frac{8}{3}r=\frac{8}{9}\pi r^3$

Differentiate both sides implicitly with respect to t:

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{8}{3}\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{3}{8\pi r^2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

We use similar triangles when $h=5$: $\frac{r}{5}=\frac{3}{8}\Rightarrow r=\frac{15}{8}$

Substitute $r=\frac{15}{8},\frac{\mathrm{d}V}{\mathrm{d}t}=-2$:

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{3}{8\pi\cdot\left(\frac{15}{8}\right)^2}\cdot(-2)=-\frac{-48}{225\pi}\approx-0.0679\left[\text{m}\,\text{min}^{-1}\right]\approx-6.79\left[\text{cm}\,\text{min}^{-1}\right]$$

2. We know the rate of change of the volume with respect to time: $\frac{\mathrm{d}V}{\mathrm{d}t}=240\ \text{cm}^3\text{s}^{-1}$

We need to find the rate of change of the radius R respect to time: $\frac{\mathrm{d}R}{\mathrm{d}t}$.

The formula of the volume of the ball relates the variables V and R:

$$V=\frac{4}{3}\pi R^3$$

Differentiate both sides implicitly with respect to t: $\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi R^2\frac{\mathrm{d}R}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{1}{4\pi R^2}\frac{\mathrm{d}V}{\mathrm{d}t}$

(a) Substitute $R=8$: $\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{1}{4\pi\cdot 8^2}\cdot 240=\frac{15}{16\pi}\approx 0.299\left[\frac{\text{cm}}{\text{s}}\right]$

(b) After 5 seconds: $V=240\cdot 5=1200\left[\text{cm}^3\right]$ then $1200=\frac{4}{3}\pi R^3\Rightarrow R=\sqrt[3]{\frac{900}{\pi}}$

Substitute $R=\sqrt[3]{\frac{900}{\pi}}$, $\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{1}{4\pi\cdot\left(\sqrt[3]{\frac{900}{\pi}}\right)^2}\cdot 240\approx 0.439\left[\frac{\text{cm}}{\text{s}}\right]$

3. We know the rate of change of the radius with respect to time: $\frac{\mathrm{d}r}{\mathrm{d}t}=1\ \text{cm}\,\text{h}^{-1}$

We need to find the rate of change of the circumference with respect to time: $\frac{\mathrm{d}l}{\mathrm{d}t}$

(a) We use the formula of the circumference of the circle $l=2\pi r$

Differentiate both sides implicitly with respect to t: $\frac{\mathrm{d}l}{\mathrm{d}t}=2\pi\frac{\mathrm{d}r}{\mathrm{d}t}$

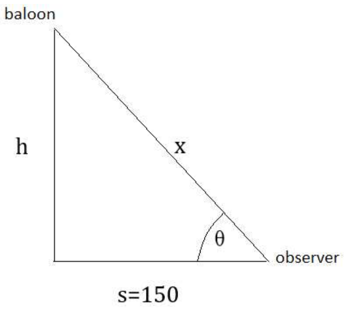
Substitute: $\frac{\mathrm{d}r}{\mathrm{d}t}=1$: $\frac{\mathrm{d}l}{\mathrm{d}t}=2\pi\frac{\text{cm}}{\text{h}}$

(b) We use the formula of the area of the circle $A=\pi r^2$

Differentiate both sides implicitly with respect to t: $\frac{\mathrm{d}A}{\mathrm{d}t}=2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$

Substitute: $r=4,\frac{dr}{dt}=1$: $\frac{\mathrm{d}l}{\mathrm{d}t}=2\pi\cdot 4=8\pi\frac{\text{cm}}{\text{h}}$

4. We know the rate of change of the height with respect to time: $\frac{\mathrm{d}h}{\mathrm{d}t}=50\frac{m}{\text{min}}$



We need to find the rate of change of the angle θ with respect to time: $\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

We use the trigonometric ratio: $\tan\theta=\frac{h}{150}$

Differentiate both sides implicitly with respect to t:

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{150}\frac{\mathrm{d}h}{\mathrm{d}t}\Rightarrow\qquad(1)\quad\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\cos^2\theta}{150}\frac{\mathrm{d}h}{\mathrm{d}t}$

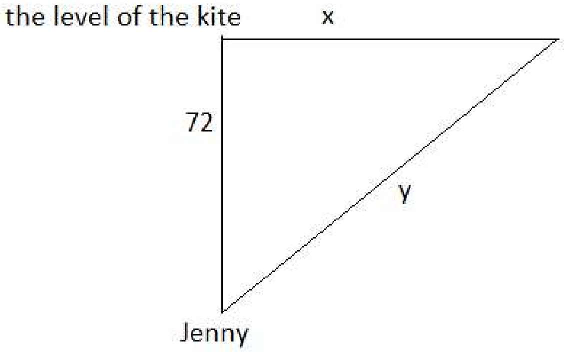
When $h=250$ $\tan\theta=\frac{250}{150}=\frac{5}{3}$

Using Pythagoras theorem and the trigonometric ratio:

$$\sec^2\theta = 1+\tan^2\theta = 1+\frac{25}{9} \Rightarrow \cos^2\theta = \frac{9}{34}$$

Substitute to (1): $\cos^2\theta = \frac{9}{34}$, $\frac{dh}{dt} = 50 \Rightarrow \frac{d\theta}{dt} = \frac{\frac{9}{34}}{150}\cdot 50 = \frac{3}{34} \approx 0.0882\frac{\text{rad}}{\text{min}}$

5. We know the rate of change of the horizontal distance x with respect to time: $\frac{dx}{dt} = 6\frac{m}{s}$



We need to find the rate of change of length of the string y with respect to time: $\frac{dy}{dt}$.

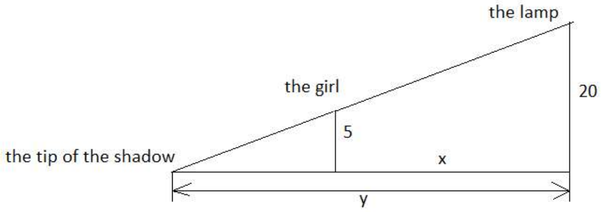
Using Pythagoras' theorem $y^2 = 72^2 + x^2$

Differentiate both sides implicitly with respect to t: $2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow$ (1) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

When $y = 120$, $x = \sqrt{120^2 - 72^2} = 96$

Substitute to (1): $x = 96, y = 120$, $\frac{dx}{dt} = 6$: $\frac{dy}{dt} = \frac{96}{120}\cdot 6 = \frac{24}{5} = 4.8\frac{\text{m}}{\text{s}}$

6. We know the rate of change of the horizontal distance x with respect to time: $\frac{dx}{dt} = 6\frac{\text{ft}}{\text{s}}$



We need to find the rate of change of the horizontal distance y with respect to time: $\frac{dy}{dt}$.

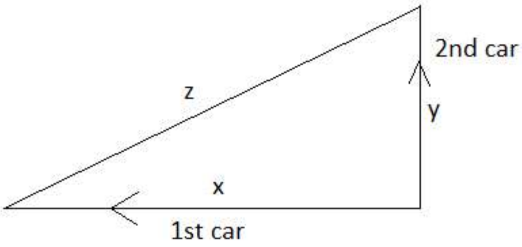
By using similar triangles, we get: $\frac{y}{y-x} = \frac{20}{5} \Rightarrow 3y = 4x$

Differentiate both sides implicitly with respect to t:

$$3\frac{dy}{dt} = 4\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{4}{3}\frac{dx}{dt}$$

Substitute $\frac{dx}{dt} = 6 \Rightarrow \frac{dy}{dt} = \frac{4}{3}\frac{dx}{dt} = \frac{4}{3}\cdot 6 = 8\frac{\text{ft}}{\text{s}}$

7. We know the rate of change of the horizontal distance x with respect to time $v_1 = \frac{\mathrm{d}x}{\mathrm{d}t} = 60\frac{\text{km}}{\text{h}}$ and the rate of change of the vertical distance y with respect to time $v_2 = \frac{\mathrm{d}y}{\mathrm{d}t} = 35\frac{\text{km}}{\text{h}}$



We need to find the rate of change of the distance z with respect to time, $\frac{\mathrm{d}z}{\mathrm{d}t}$ after 3 hours.

After 3 hours: $x = 60 \cdot 3 = 180\,\text{km}$, $y = 35 \cdot 3 = 105[km]$, $z = \sqrt{180^2 + 105^2} = 15\sqrt{193}$

Using Pythagoras' theorem: $z^2 = x^2 + y^2$

Differentiate both sides implicitly with respect to t: $2z\frac{\mathrm{d}z}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{x}{z}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{y}{z}\frac{\mathrm{d}y}{\mathrm{d}t}$

Substitute $x = 180, y = 105, z = 15\sqrt{193}$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 60$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 35$:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{180}{15\sqrt{193}} \cdot 60 + \frac{105}{15\sqrt{193}} \cdot 35 \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t} \approx 69.5\left[\frac{\text{km}}{\text{h}}\right]$$

8. $y = \sqrt{x^2 + 1}$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$

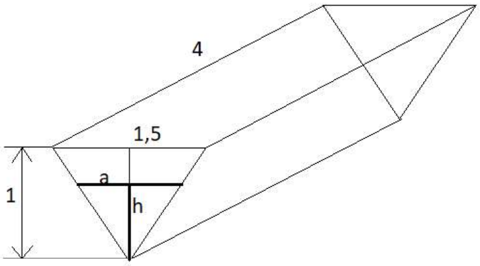
Square both sides of the equation: $y^2 = x^2 + 1$

Differentiate both sides implicitly with respect to t: $2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}$

When $x = 3$, $y = \sqrt{10}$

Substitute $x = 3$, $y = \sqrt{10}$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3}{\sqrt{10}} \cdot 4 = \frac{12}{\sqrt{10}} \approx 3.79$

9.



We know the rate of change of the amount of water with respect to time: $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.03\frac{\text{m}^3}{\text{s}}$ We need to find the rate of change of the level of the water h with respect to time: $\frac{\mathrm{d}h}{\mathrm{d}t}$.

We use the formula of the volume of the prism:

$$V = \frac{1}{2}ah \cdot 4 = 2ah$$

By using similar triangles, we get: $\frac{a}{1.5} = \frac{h}{1} \Rightarrow a = 1.5h$

Substitute in the formula of the volume: $V = 2\cdot 1.5h\cdot h \Rightarrow V = 3h^2$

Differentiate both sides implicitly with respect to t: $\frac{\mathrm{d}V}{\mathrm{d}t} = 6h\frac{\mathrm{d}h}{\mathrm{d}t} \Rightarrow \quad (1)\ \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{6h}\frac{\mathrm{d}V}{\mathrm{d}t}$

After 25 seconds: $V = 0.03\cdot 25 = 0.75\left[m^3\right] \Rightarrow 3h^2 = 0.75 \Rightarrow h = 0.5[m] \quad (h>0)$

Substitute $h = 0.5, \frac{\mathrm{d}V}{\mathrm{d}t} = 0.03$ in equation (1): $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{6\cdot 0.5}\cdot 0.03 \Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = 0.01\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$

10. We know the rate of change of radius of a sphere with respect to time: $\frac{\mathrm{d}R}{\mathrm{d}t} = 3\frac{\mathrm{mm}}{\mathrm{s}}$

We need to find the rate of change of the volume V with respect to time $\frac{\mathrm{d}V}{\mathrm{d}t}$.

We use the formula of the volume of the ball: $V = \frac{4}{3}\pi R^3$

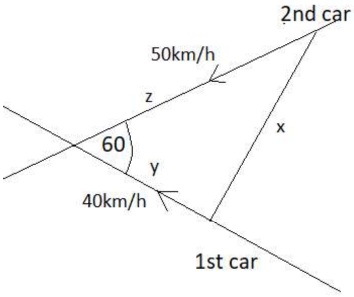
Differentiate both sides implicitly with respect to t: (1) $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi R^2\frac{\mathrm{d}R}{\mathrm{d}t}$

We use the formula of the surface of a surface to find the radius when the surface is $10\ \mathrm{mm}^2$:

$4\pi R^2 = 10 \Rightarrow R = \sqrt{\frac{5}{2\pi}} \quad (R>0)$

Substitute $R = \sqrt{\frac{5}{2\pi}}$ and $\frac{\mathrm{d}R}{\mathrm{d}t} = 3$ in (1): $\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi\left(\sqrt{\frac{5}{2\pi}}\right)^2\cdot 3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 30\left[\frac{\mathrm{mm}^3}{\mathrm{s}}\right]$

11.



We know the velocity of the first car : $v_1 = \frac{\mathrm{d}y}{\mathrm{d}t} = 40\frac{\mathrm{km}}{\mathrm{h}}$ and the velocity of the second car : $v_2 = \frac{\mathrm{d}z}{\mathrm{d}t} = 50\frac{\mathrm{km}}{\mathrm{h}}$ We need to find the rate of change of the distance between them with respect to time $\frac{\mathrm{d}x}{\mathrm{d}t}$.

Applying the cosine rule:
$x^2 = y^2 + z^2 - 2yz\cos 60° \Rightarrow x^2 = y^2 + z^2 - yz$

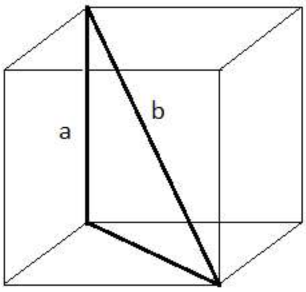
Differentiate both sides implicitly with respect to t: (1) $2x\frac{\mathrm{d}x}{\mathrm{d}t} = 2y\frac{\mathrm{d}y}{\mathrm{d}t} + 2z\frac{\mathrm{d}z}{\mathrm{d}t} - y\frac{\mathrm{d}z}{\mathrm{d}t} - z\frac{\mathrm{d}y}{\mathrm{d}t}$

When $y = z = 2$, the triangle is equilateral, so $x = 2$

Substitute $x = y = z = 2$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 40$, $\frac{\mathrm{d}z}{\mathrm{d}t} = 50$ to (1) : $2\cdot 2\frac{\mathrm{d}x}{\mathrm{d}t} = 2\cdot 2\cdot 40 + 2\cdot 2\cdot 50 - 2\cdot 50 - 2\cdot 40$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 45\left[\frac{\mathrm{km}}{\mathrm{h}}\right]$

12.



We know the rate of change of the diagonal of a cube with respect to time: $(1)\quad \frac{\mathrm{d}b}{\mathrm{d}t} = 8\frac{\text{cm}}{\text{s}}$. We need to find the rate of change of a side a with respect to time $\frac{\mathrm{d}a}{\mathrm{d}t}$

Using Pythagoras' theorem: $a^2 + \left(a\sqrt{2}\right)^2 = b^2 \Rightarrow b^2 = 3a^2$

Differentiate both sides implicitly with respect to *t*:

$$2b\frac{\mathrm{d}b}{\mathrm{d}t} = 6a\frac{\mathrm{d}a}{\mathrm{d}t} \Rightarrow \quad (2)\quad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{b}{3a}\frac{\mathrm{d}b}{\mathrm{d}t}$$

$$b^2 = 3a^2 \Rightarrow \quad (3)\ \ b = a\sqrt{3}\ \ (a,b>0)$$

Substitute (1) and (3) to (2): $\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{a\sqrt{3}}{3a}\cdot 8 \Rightarrow \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{8\sqrt{3}}{3} \approx 4.62\left[\frac{\text{cm}}{\text{s}}\right]$

13. Let *l* be the distance travelled around the circle, then,

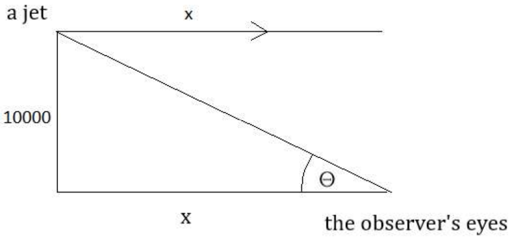
$$l = r\theta \Rightarrow \theta = \frac{l}{10} \Rightarrow \frac{d\theta}{dt} = \frac{1}{10}\frac{dl}{dt} = 0.3\,\text{units/sec}$$

If the vertical distance to the *x*-axis is 5, we can calculate:

$$\sin\theta = \frac{\cancel{5}1}{\cancel{10}2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6},\quad x = 10\cos\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = -10\sin\theta$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \left|\frac{\mathrm{d}x}{\mathrm{d}t}\left(\theta = \frac{\pi}{6}\right)\right| = \left|-\sin\frac{\pi}{6}\right|\times 10\times 0.3 = 1.5\,\text{units/sec}$$

14.



We know the rate of change of the angle θ with respect to time: $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{60}\frac{\text{rad}}{\text{s}}$ We need to find the speed of the jet $|v| = \left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|$

Applying trigonometric ratios: $\tan\theta = \frac{10\,000}{x}$

Differentiate both sides implicitly with respect to *t*:

$$(1)\quad \sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{10\,000}{x^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x^2}{10\,000}\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

When $\theta = \frac{\pi}{3}$: $\tan\frac{\pi}{3} = \frac{10000}{x} \Rightarrow x = \frac{10000}{\sqrt{3}}$

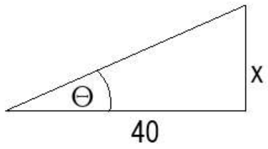
Substitute $\theta=\frac{\pi}{3}, x=\frac{10000}{\sqrt{3}}, \frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{60}$ to (1) :

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{\left(\frac{10000}{\sqrt{3}}\right)^2}{10000}\sec^2\frac{\pi}{3}\cdot\frac{1}{60}=-222.\bar{2}\left[\frac{\mathrm{m}}{\mathrm{s}}\right]=-800\left[\frac{\mathrm{km}}{\mathrm{h}}\right]$$

Speed: $\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|=800\left[\frac{\mathrm{km}}{\mathrm{h}}\right]$

15. **(a)** We know the rate of change of the horizontal distance x with respect to time (the velocity of the car): $\frac{\mathrm{d}x}{\mathrm{d}t}=288\frac{\mathrm{km}}{\mathrm{h}}=80\frac{\mathrm{m}}{\mathrm{s}}$ and the horizontal distance of the cameraman from the racing track: $40\ \mathrm{m}$.

We need to find the rate of change of the angle θ with respect to time $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when the car is directly in front of the camera $(\theta=0)$



Applying the trigonometric ratios: $\tan\theta=\frac{x}{40}$

Differentiate both sides implicitly: $\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{40}\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\cos^2\theta}{40}\frac{\mathrm{d}x}{\mathrm{d}t}$

Substitute $\frac{\mathrm{d}x}{\mathrm{d}t}=80,\ \theta=0:\ \frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\cos^2 0}{40}\cdot 80=2\left[\frac{\mathrm{rad}}{\mathrm{s}}\right]\approx 115\left[\frac{\mathrm{degrees}}{\mathrm{s}}\right]$

(b) We know the rate of change of the vertical distance x with respect to time (the velocity of the car): $\frac{\mathrm{d}y}{\mathrm{d}t}=288\frac{\mathrm{km}}{\mathrm{h}}=80\frac{\mathrm{m}}{\mathrm{s}}$ and the horizontal distance of the cameraman from the racing track: $40\ \mathrm{m}$. We need to find the rate of change of the angle θ with respect to time $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ a half second later.

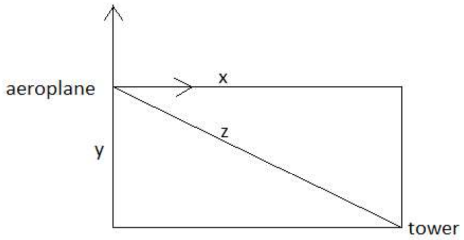
The distance driven by the car during 1 second: $80\ \mathrm{m}\Rightarrow$ the distance driven during $\frac{1}{2}$ of a second: $40\ \mathrm{m}\Rightarrow$ we need to find the rate of change of the angle θ with respect to time $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $x=40\ \mathrm{m}$ $\Rightarrow$ the triangle is right-angled and isosceles $\Rightarrow\ \theta=\frac{\pi}{4}$

Applying the same method as in **(a)**: $\tan\theta=\frac{x}{40}$

Differentiate both sides implicitly and transform: $\frac{d\theta}{dt}=\frac{\cos^2\theta}{40}\frac{dx}{dt}$

Substitute $\frac{\mathrm{d}x}{\mathrm{d}t}=80,\ \theta=\frac{\pi}{4}:\ \frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\cos^2\frac{\pi}{4}}{40}\cdot 80=1\left[\frac{\mathrm{rad}}{\mathrm{s}}\right]\approx 57\left[\frac{\mathrm{degrees}}{\mathrm{s}}\right]$

16.



We know the rate of change of the horizontal distance with respect to time: $\frac{\mathrm{d}x}{\mathrm{d}t} = -640\frac{\text{km}}{\text{h}}$ and the rate of change of the vertical distance with respect to time: $\frac{\mathrm{d}y}{\mathrm{d}t} = 180\frac{\text{m}}{\text{h}} = 10.8\frac{\text{km}}{\text{h}}$ We need to find the rate of change of the distance between the tower and the airplane with respect to time: $\frac{\mathrm{d}z}{\mathrm{d}t}$

Applying Pythagoras' theorem: $x^2 + y^2 = z^2$

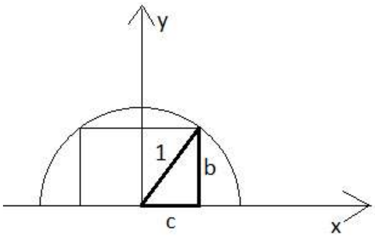
Differentiate both sides implicitly with respect to t: $2z\frac{\mathrm{d}z}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{x}{z}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{y}{z}\frac{\mathrm{d}y}{\mathrm{d}t}$

When $x = 6, y = 5\ :\ z = \sqrt{5^2 + 6^2} = \sqrt{61}$

Substitute $x = 6, y = 5, z = \sqrt{61}$, $\frac{\mathrm{d}x}{\mathrm{d}t} = -640, \frac{\mathrm{d}y}{\mathrm{d}t} = 10.8$:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{6}{\sqrt{61}}\cdot 640 + \frac{5}{\sqrt{61}}\cdot 10.8 \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}t} \approx -485\left[\frac{\text{km}}{\text{h}}\right]$$

Exercise 13.5



1. The area of the rectangle shown on the picture is $(1)\ \ A = 2c\cdot b$

From Pythagoras' theorem:

$c^2 + b^2 = 1 \Rightarrow\ \ (2)\ b = \sqrt{1-c^2}\quad (b > 0)$

Substitute (2) into (1) : $A(c) = 2c\sqrt{1-c^2},\ \ 0 < c < 1$

Differentiate the function with respect to c:

$$A'(c) = 2c\cdot\frac{\mathrm{d}}{\mathrm{d}c}\left(\sqrt{1-c^2}\right) + \sqrt{1-c^2}\cdot\frac{\mathrm{d}}{\mathrm{d}c}(2c)$$

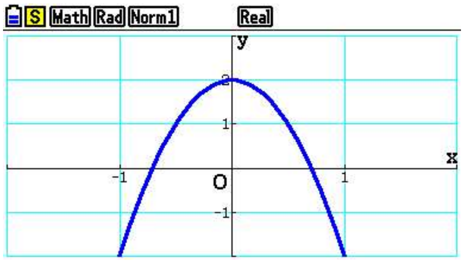
$$A'(c) = 2c\cdot\frac{-c}{\sqrt{1-c^2}} + \sqrt{1-c^2}\cdot 2 = \frac{-2c^2}{\sqrt{1-c^2}} + \frac{2-2c^2}{\sqrt{1-c^2}} = \frac{-4c^2+2}{\sqrt{1-c^2}}$$

$$A'(c) = 0 \Leftrightarrow \frac{-4c^2+2}{\sqrt{1-c^2}} = 0 \Leftrightarrow \left(\left(c = \frac{\sqrt{2}}{2} \vee c = -\frac{\sqrt{2}}{2}\right) \text{ and } 0 < c < 1\right) \Rightarrow c = \frac{\sqrt{2}}{2}$$

The critical point is $c = \frac{\sqrt{2}}{2}$

The denominator $\sqrt{1-c^2} > 0$ for each $c \in (0,1)$ $\Rightarrow$ the sign of the first derivative depends on the sign of the numerator.

Numerator: $-4\left(c-\frac{\sqrt{2}}{2}\right)\left(c+\frac{\sqrt{2}}{2}\right)$



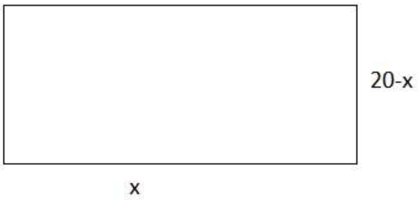
$$\left.\begin{array}{l} A'(c)>0 \Leftrightarrow 0<c<\frac{\sqrt{2}}{2} \\ A'(c)<0 \Leftrightarrow \frac{\sqrt{2}}{2}<c<1 \end{array}\right\} \Rightarrow \text{there is a relative}$$

maximum at $c=\frac{\sqrt{2}}{2}$ $A\left(\frac{\sqrt{2}}{2}\right)=2\cdot\frac{\sqrt{2}}{2}\cdot\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}=\sqrt{2}\cdot\frac{1}{\sqrt{2}}=1$

Then $b=\sqrt{1-c^2}=\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}=\frac{\sqrt{2}}{2}$, $2c=\sqrt{2}$

The dimensions: $\sqrt{2}$ by $\frac{\sqrt{2}}{2}$

2.



Let the rectangle have dimensions $x \times y$, where x is the fold into the base and y is the height of the cylinder.

The radius of the base will be: $2r\pi = x \Rightarrow r=\frac{x}{2\pi}$.

Therefore, the volume of the cylinder is:

$$V=r^2\pi h \Rightarrow V=\left(\frac{x}{2\pi}\right)^2\pi y=\frac{x^2}{4\pi^{\cancel{2}}}\cancel{\pi}y=\frac{1}{4\pi}x^2y\,.$$

Since the perimeter is equal to 40 cm, we can express the volume in terms of x only: $2x+2y=40 \Rightarrow x+y=20 \Rightarrow y=20-x$, and so $V(x)=\frac{1}{4\pi}x^2(20-x)$. To find the maximum volume, we need to differentiate the volume with respect to x and find the zero of the derivative:

$$V'(x)=\frac{1}{4\pi}\left(2x(20-x)+x^2\times(-1)\right)=\frac{1}{4\pi}x(40-3x) \Rightarrow V'(x)=0 \Rightarrow \frac{1}{4\pi}x(40-3x)=0$$

$x=0$ or $40-3x=0 \Rightarrow x=\frac{40}{3}$.

The first solution is not possible, so we take the second and calculate y:

$y=20-x \Rightarrow y=\frac{20}{3}$.

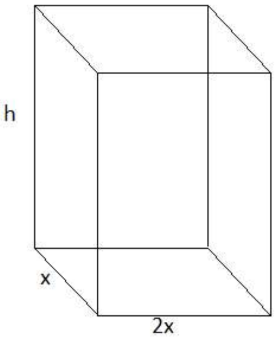
So, the dimensions of the rectangle are $\frac{40}{3}$ cm and $\frac{20}{3}$ cm

3. Any point on the graph has coordinates $(x,\sqrt{x})$; therefore, to find the distance to the given point, we will use the distance formula. To make the calculation simpler, we will look at the square of the distance and then, at the end, we will simply take the square root of the value we obtain.

$$g(x)=\left(x-\frac{3}{2}\right)^2+\left(\sqrt{x}-0\right)^2=x^2-3x+\frac{9}{4}+x=x^2-2x+\frac{9}{4}\Rightarrow g'(x)=2x-2$$

$$g'(x)=0\Rightarrow 2x-2=0\Rightarrow x=1\Rightarrow g(1)=1^2-2\times1+\frac{9}{4}=\frac{5}{4}\Rightarrow d=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$$

4. **(a)** The volume of the box: $1000=2x\cdot x\cdot h\Rightarrow h=\frac{500}{x^2}[\text{cm}]$



(b) $s(x)=2\cdot x\cdot 2x+2x\cdot h+2\cdot 2x\cdot h\Rightarrow s(x)=4x^2+6hx$

Substitute $h=\frac{500}{x^2}$:

$$s(x)=4x^2+6x\cdot\frac{500}{x^2}\Rightarrow s(x)=4x^2+\frac{3000}{x},x>0$$

(c) Differentiate with respect to *x*: $s'(x)=8x-\frac{3000}{x^2}$

$$s'(x)=0\Leftrightarrow 8x-\frac{3000}{x^2}=0\Leftrightarrow 8x^3=3000\Leftrightarrow x=5\sqrt[3]{3}\approx 7.21$$

2nd derivative test: $s''(x)=8+\frac{6000}{x^3}$

$$s''\left(5\sqrt[3]{3}\right)=8+\frac{6000}{\left(5\sqrt[3]{3}\right)^3}=24>0\ \Rightarrow \text{there is a relative minimum at } x=5\sqrt[3]{3}$$

Then $2x=2\cdot5\sqrt[3]{3}=10\sqrt[3]{3}\approx 14.4[\text{cm}],\ h=\frac{500}{\left(5\sqrt[3]{3}\right)^2}=\frac{20}{\sqrt[3]{9}}\approx 9.61[\text{cm}]$

Dimensions: 7.21 cm × 14.4 cm × 9.61 cm

5. Let $|AD|=2y$

The area of the rectangle: $100=2xy\Rightarrow y=\frac{50}{x}$ - the radius of the semicircle

The perimeter of the figure (two sides *AB* + the circle with the radius *y* (2 semicircles)):

$$l(x)=2x+2\pi\cdot\frac{50}{x}\Rightarrow l(x)=2x+\frac{100\pi}{x},x>0$$

Differentiate with respect to *x*: $l'(x)=2-\frac{100\pi}{x^2}$

$$l'(x)=0\Leftrightarrow 2-\frac{100\pi}{x^2}=0\Rightarrow x^2=50\pi$$

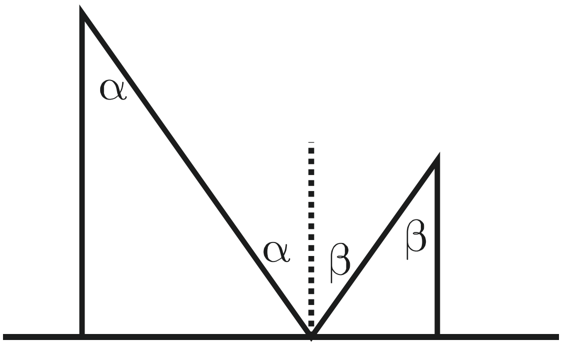
$$\Rightarrow\left(\left(x=5\sqrt{2\pi}\vee x=-5\sqrt{2\pi}\right)\text{ and } x>0\right)\Rightarrow x=5\sqrt{2\pi}$$

2nd derivative test: $l''(x)=\frac{200\pi}{x^3}$

$$l''\left(5\sqrt{2\pi}\right)=\frac{200\pi}{\left(5\sqrt{2\pi}\right)^3}>0\ \Rightarrow \text{there is a relative minimum at } x=5\sqrt{2\pi}$$

The perimeter of the figure is minimum when $x = 5\sqrt{2\pi} \approx 12.5\left[\text{cm}\right]$

6. Denote the angles at the vertical posts by α and β. Using the property that alternate interior angles have the same measure, we establish the relationship $\theta = \alpha + \beta$.



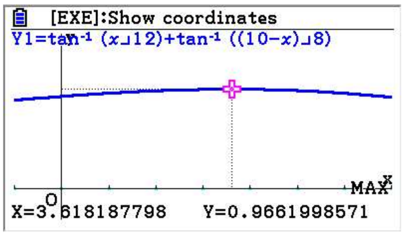
Now, using the right-angled triangle trigonometry formulae, both angles can be expressed in terms of x only:

$$\left.\begin{array}{r}\tan\alpha = \dfrac{x}{12}\\ \tan\beta = \dfrac{10-x}{8}\end{array}\right\} \Rightarrow \left.\begin{array}{r}\alpha = \arctan\left(\dfrac{x}{12}\right)\\ \beta = \arctan\left(\dfrac{10-x}{8}\right)\end{array}\right\} \Rightarrow \theta(x) = \arctan\left(\dfrac{x}{12}\right) + \arctan\left(\dfrac{10-x}{8}\right)$$

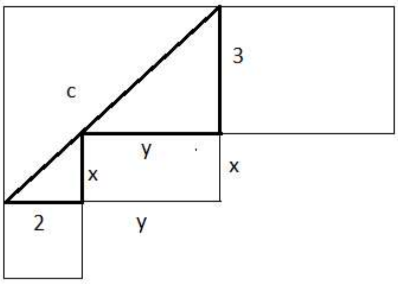
$\theta'(x) = \dfrac{12}{x^2+144} - \dfrac{8}{x^2-20x+164}$. Simplifying this expression will give

$$\theta'(x) = \frac{4\left(x^2-60x+204\right)}{\left(x^2+144\right)\left(x^2-20x+164\right)}$$

At this point a GDC must be used because the work will require intensive symbolic manipulation. $x \approx 3.62$ will give the maximum value for θ.


[EXE]:Show coordinates
Y1=tan⁻¹ (x⌟12)+tan⁻¹ ((10−x)⌟8)
X=3.618187798 Y=0.9661998571

7. **Method 1** As shown below, let c be the straight distance from the wide hallway to the narrow one. The ladder will pass the corner when it matches the minimum such distance.



Using similar triangles, we get:

$$\frac{2}{x}=\frac{y}{3}\Rightarrow y=\frac{6}{x}\Rightarrow c^2=\left(2+\frac{6}{x}\right)^2+(3+x)^2, x>0$$

Consider the function $f(x)=\left(2+\frac{6}{x}\right)^2+(3+x)^2, x>0$ and try to see the minimum value of this quantity.

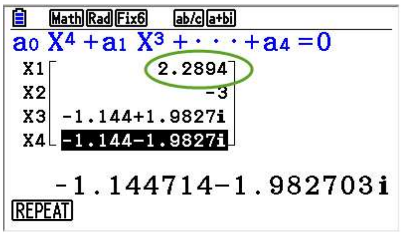
$$f(x)=x^2+6x+13+\frac{24}{x}+\frac{36}{x^2}$$

Differentiate with respect to x: $f'(x)=2x+6-\frac{24}{x^2}-\frac{72}{x^3}$

$$f'(x)=0\Rightarrow 2x+6-\frac{24}{x^2}-\frac{72}{x^3}=0\Rightarrow\frac{2x^4+6x^3-24x-72}{x^3}=0$$

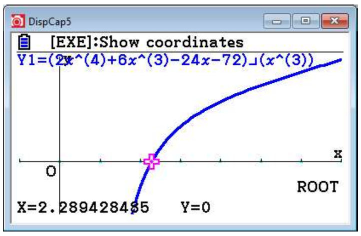
Solution of polynomial equation $2x^4+6x^3-24x-72=0$ will require a GDC:

$$\left((x\approx 2.2894 \text{ or } x=-3) \text{ and } x>0\right)\Rightarrow x\approx 2.29$$



```
Math Rad Fix6   ab/c a+bi
a0 X4 +a1 X3 +···+a4 =0
 X1   2.2894
 X2       -3
 X3 -1.144+1.9827i
 X4 -1.144-1.9827i

   -1.144714-1.982703i
REPEAT
```



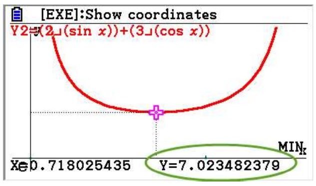
```
DispCap5
[EXE]:Show coordinates
Y1=(2x^(4)+6x^(3)-24x-72)⌟(x^(3))
ROOT
X=2.289428435     Y=0
```

Using the first derivative test, you will notice that $f'(x)$ is switching from negative values to positive values. This implies that $f(x)$ itself has a minimum at this point.
The value of this minimum is $f(2.29)\approx 7.02$. Thus, the longest ladder to pass this corner must be less than or at most equal to 7.02 m.

Method 2 (Seen in chapter 7)

Denote the angle between the 2-metre wide hallway and a straight line touching the corner by α. Then the angle between the 3-metre wide hallway and this line is: $180^\circ-(\alpha+90^\circ)=90^\circ-\alpha$. We split the total length of the line segment a and b, to correspond to the lengths in the 2-metre wide and 3-metre wide hallways respectively.
Now, from the corresponding right-angled triangles, we get $\sin\alpha=\frac{2}{a}\Rightarrow a=\frac{2}{\sin\alpha}$ and $\sin(90^\circ-\alpha)=\cos\alpha=\frac{3}{b}\Rightarrow b=\frac{3}{\cos\alpha}$. Therefore, the total length of the segment can be expressed in terms of α only: $l=a+b\Rightarrow l(\alpha)=\frac{2}{\sin\alpha}+\frac{3}{\cos\alpha}$

Again, we will use a GDC to find the answer.



```
[EXE]:Show coordinates
Y2=(2⌟(sin x))+(3⌟(cos x))
MIN
X=0.718025435    Y=7.023482379
```

So, the longest ladder that can be carried around the corner is 7.02 m.
Note: The minimum of this function is the maximum length of the ladder that can be carried around the corner of the hallway.

8. We know the rate of change of the distance in the sandy terrain of Erica with respect to time (the velocity v_1): $\frac{\mathrm{d}z}{\mathrm{d}t}=2\frac{km}{h}$ and the rate of change of the distance on road of Erica with respect to time (the velocity v_2): $\frac{\mathrm{d}y}{\mathrm{d}t}=5\frac{\mathrm{km}}{\mathrm{h}}$ – We need to find the distance *d*, such that the time *T* of Erica's walking is minimum.

$T=t_1+t_2$ (t_1 - the time during walk on sandy terrain, t_2 - the time during walk on the road)

$$t_1=\frac{z}{v_1}=\frac{z}{2}\ ,\ t_2=\frac{d}{v_2}=\frac{d}{5}\Rightarrow T=\frac{z}{2}+\frac{d}{5}$$

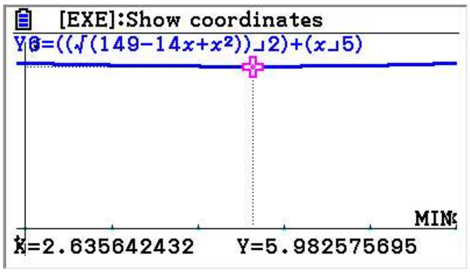
Express *z* in terms of *d*: $z=\sqrt{(7-d)^2+100}=\sqrt{149-14d+d^2}$, and substitute:

$$T=\frac{\sqrt{149-14d+d^2}}{2}+\frac{d}{5},0\le d\le 7$$

Differentiate with respect to *d* and find the value of *d* that minimises *T*.

Calculations are long and it is better to use a GDC for such cases:

$$\frac{\mathrm{d}T}{\mathrm{d}d}=0\Rightarrow d\approx 2.64$$



```
[EXE]:Show coordinates
Y6=((√(149-14x+x²))⌟2)+(x⌟5)
MIN
X=2.635642432      Y=5.982575695
```

So, point *A* is 2.64 km due west from the office.

9. $P=\left(x,\frac{8}{x^2+4}\right),x>0$ - is a point on the graph of the function $y=\frac{8}{x^2+4}$

The area of the rectangle: $A=2xy=2x\cdot\frac{8}{x^2+4}=\frac{16x}{x^2+4},x>0$

Differentiate with respect to *x*:

$$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{(x^2+4)\cdot 16-16x\cdot 2x}{(x^2+4)^2}=\frac{16x^2+64-32x^2}{(x^2+4)^2}=\frac{64-16x^2}{(x^2+4)^2},x>0$$

$\frac{\mathrm{d}A}{\mathrm{d}x}=0\Rightarrow\frac{64-16x^2}{(x^2+4)^2}=0\Rightarrow 64-16x^2=0\Rightarrow x=2$ since we take the positive value.

$\left(x^2+4\right)^2>0$ for all values; thus, the sign of the expression $64-16x^2$ determines the sign of the derivative.

The graph of $64-16x^2$ is a parabola that has a maximum of 64 at $x = 0$. It will intersect the x-axis at $x = 2$ and it will become negative. Therefore, the function A has a maximum when $x = 2$. At $x=2$: $A=\dfrac{16\cdot 2}{2^2+4}=4$ is therefore the maximum area of the rectangle.

10. Let t (in hours) be the time, x be the horizontal distance, y the vertical distance, z the distance between the ships

$$z^2=x^2+y^2$$

Substitute: $x=12t$, $y=10-16t$: $\quad z^2=\left(12t\right)^2+\left(10-16t\right)^2$

Differentiate both sides implicitly with respect to t:

$$2z\frac{\mathrm{d}z}{\mathrm{d}t}=2\cdot\left(12t\right)\cdot 12+2\left(10-16t\right)\cdot\left(-16\right)\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{-160+400t}{z}$$

Substitute $z=\sqrt{\left(12t\right)^2+\left(10-16t\right)^2}$: $\dfrac{\mathrm{d}z}{\mathrm{d}t}=\dfrac{-160+400t}{\sqrt{\left(12t\right)^2+\left(10-16t\right)^2}}$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=0\Rightarrow\frac{-160+400t}{\sqrt{\left(12t\right)^2+\left(10-16t\right)^2}}=0\Rightarrow -160+400t=0\Rightarrow t=\frac{2}{5}$$

$\sqrt{\left(12t\right)^2+\left(10-16t\right)^2}>0$ for every $t\in\mathbb{R}$ $\Rightarrow$ the sign of the expression $-160+400t$ determines the sign of the derivative.

$y=-160+400t$ is a linear function changes sign from negative to positive at $t=\dfrac{2}{5}$.

Thus, the minimum distance is $z=\sqrt{\left(12\cdot\dfrac{2}{5}\right)^2+\left(10-16\cdot\dfrac{2}{5}\right)^2}=6$

11. $V=r^2\pi h$ is the volume of the inscribed cylinder. If we look at the cross-section of the sphere, we can find the relationship between R, r and h.

$$R^2=r^2+\left(\frac{h}{2}\right)^2\Rightarrow h=2\sqrt{R^2-r^2}\Rightarrow V\left(r\right)=2r^2\pi\sqrt{R^2-r^2}\Rightarrow$$

$$V'\left(r\right)=4r\pi\sqrt{R^2-r^2}+2r^2\pi\frac{-\cancel{2}r}{\cancel{2}\sqrt{R^2-r^2}}=\frac{2r\pi\left(2R^2-2r^2-r^2\right)}{\sqrt{R^2-r^2}}=\frac{2r\pi\left(2R^2-3r^2\right)}{\sqrt{R^2-r^2}}$$

$$V'\left(r\right)=0\Rightarrow 2R^2-3r^2=0\Rightarrow r_{1,2}=\pm\sqrt{\frac{2}{3}R^2}=\frac{\pm\sqrt{6}R}{3}$$

Since the radius r of the base of the cylinder cannot be negative, we have only one solution:

$$r=\frac{\sqrt{6}R}{3}\Rightarrow h=2\sqrt{R^2-\frac{2}{3}R^2}=\frac{2\sqrt{3}R}{3}.$$

12. Let p be the distance between the points X and P, and t the time.

$$AP=\sqrt{a^2+p^2},\, t(p)=\frac{\sqrt{a^2+p^2}}{c}+\frac{b-p}{r}$$

$$\Rightarrow t'(p)=\frac{\cancel{2}p}{\cancel{2}c\sqrt{a^2+p^2}}-\frac{1}{r}=\frac{pr-c\sqrt{a^2+p^2}}{cr\sqrt{a^2+p^2}}$$

$$t'(p)=0\Rightarrow pr-c\sqrt{a^2+p^2}=0$$

$$\Rightarrow pr=c\sqrt{a^2+p^2}\Rightarrow p^2r^2=c^2a^2+c^2p^2$$

$p^2r^2-c^2p^2=a^2c^2\Rightarrow p^2=\dfrac{a^2c^2}{r^2-c^2}\Rightarrow p=\sqrt{\dfrac{a^2c^2}{r^2-c^2}}=\dfrac{ac}{\sqrt{r^2-c^2}}$, since p has to be positive.

13. This question is best done with a CAS.

Since the circumference of the base is: $2\pi\times 10-x=20\pi-x\Rightarrow r=\dfrac{20\pi-x}{2\pi}=10-\dfrac{x}{2\pi}$, we can also express the height h in terms of x.

$$h=\sqrt{100-r^2}=\sqrt{100-\left(10-\frac{x}{2\pi}\right)^2}=\sqrt{\cancel{100}\cancel{-100}+10\frac{x}{\pi}-\frac{x^2}{4\pi^2}}$$

$$=\sqrt{10\frac{x}{\pi}-\frac{x^2}{4\pi^2}}=\frac{\sqrt{40\pi x-x^2}}{2\pi}$$

Now, the volume can be expressed in terms of x only.

$$V=\frac{1}{3}r^2\pi h=\frac{1}{3}\left(10-\frac{x}{2\pi}\right)^2\cancel{\pi}\frac{\sqrt{40\pi x-x^2}}{2\cancel{\pi}}=\frac{(20\pi-x)^2\sqrt{40\pi x-x^2}}{24\pi^2}\Rightarrow$$

$$V'(x)=\frac{1}{24\pi^2}\left[\cancel{2}(20\pi-x)\times(-1)\times\sqrt{40\pi x-x^2}+(20\pi-x)^2\frac{\cancel{2}(20\pi-x)}{\cancel{2}\sqrt{40\pi x-x^2}}\right]$$

$$=\frac{(20\pi-x)}{24\pi^2\sqrt{40\pi x-x^2}}\left[2\left(x^2-40\pi x\right)+(20\pi-x)^2\right]$$

$$=\frac{(20\pi-x)}{24\pi^2\sqrt{40\pi x-x^2}}\left(2x^2-80\pi x+400\pi^2-40\pi+x^2\right)$$

$$=\frac{(20\pi-x)\left(3x^2-120\pi x+400\pi^2\right)}{24\pi^2\sqrt{40\pi x-x^2}}$$

$$V'(x)=0\Rightarrow 3x^2-120\pi x+400\pi^2=0$$

$$\Rightarrow x_{1,2}=\frac{120\pi\pm\sqrt{14\,400\pi^2-4800\pi^2}}{6}=20\pi\pm\frac{20\pi\sqrt{6}}{3}$$

$x=20\pi-\dfrac{20\pi\sqrt{6}}{3}\approx 11.5\text{ cm}\Rightarrow V=\dfrac{2000\pi\sqrt{3}}{27}\approx 403\text{ cm}^3$, to 3 significant figures.

Notice that the other possible solutions are discarded because the value of x exceeds the perimeter of the circle.

14. Set up the coordinate system in such a way that the origin is at the point R. The distance $P'Q'$ is a fixed positive number d and the distance $P'R$ is our variable x, $x > 0$. Then, the coordinates of the points are as follows: $P(-x,a), R(0,0), Q(d-x,b)$.

$$t_{Total}=t_1+t_2=\frac{PR}{u}+\frac{RQ}{v}=\frac{\sqrt{x^2+a^2}}{u}+\frac{\sqrt{(d-x)^2+b^2}}{v}$$

Since the ray travels in such a way that the time is a minimum, we can deduce that $\frac{\mathrm{d}t}{\mathrm{d}x}=0$.

$$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\cancel{2}x}{u\times\cancel{2}\sqrt{x^2+a^2}}+\frac{\cancel{2}(d-x)\times(-1)}{v\times\cancel{2}\sqrt{(d-x)^2+b^2}}=0\Rightarrow\frac{1}{u}\times\frac{x}{\sqrt{x^2+a^2}}=\frac{1}{u}\times\frac{(d-x)}{\sqrt{(d-x)^2+b^2}}$$

By looking at the triangles $PP'R$ and $RQ'Q$, we can establish the following relationships.

$$\sin\alpha=\frac{P'R}{PR}=\frac{x}{\sqrt{x^2+a^2}}\text{ and }\sin\beta=\frac{RQ'}{RQ}=\frac{d-x}{\sqrt{(d-x)^2+b^2}}.$$

Therefore, we obtain the formula:

$$\frac{1}{u}\times\sin\alpha=\frac{1}{u}\times\sin\beta\Rightarrow\frac{\sin\alpha}{\sin\beta}=\frac{u}{v}$$

Exercise 13.6

1. **(a)** $\lim_{x\to0}\left(\frac{1-\cos x}{x^2}\right)$

Substituting $x=0$ into the rational expression gives: $\frac{1-1}{0^2}=\frac{0}{0}\Rightarrow$ the limit is of the indeterminate form $\frac{0}{0}$ and we can apply l'Hopital's rule:

$$\lim_{x\to0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to0}\left(\frac{\sin x}{2x}\right)$$

This is also of the indeterminate form $\frac{0}{0}$ and we can apply l'Hopital's rule again:

$$\lim_{x\to0}\left(\frac{\sin x}{2x}\right)=\lim_{x\to0}\left(\frac{\cos x}{2}\right)=\frac{1}{2}$$

(b) $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x^2+3}-2}\right)$ Substituting $x=1$ into the rational expression gives: $\frac{1-1}{2-2}=\frac{0}{0}$ and we can apply l'Hopital's rule:

$$\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x^2+3}-2}\right)=\lim_{x\to 1}\left(\frac{1}{\frac{2x}{2\sqrt{x^2+3}}}\right)=\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}}{x}\right)=2$$

(c) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}$

Apply l'Hopital's rule, as substituting $x=1$ we get the indeterminate form $\frac{0}{0}$:

$$\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x}+\ln x}=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\frac{0}{0}$$

Applying l'Hopital's rule for the second time:

$$\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\lim_{x\to 1}\frac{1}{1+x\cdot\frac{1}{x}+\ln x}=\lim_{x\to 1}\frac{1}{2+\ln x}=\frac{1}{2}$$

2. **(a)** $\lim_{x\to 1}\left(\frac{x^2-1}{x^2-4x+3}\right)$ Substituting $x=1$, we get $\frac{0}{0}$, so

Method 1

We may factorise the numerator and the denominator and reduce the fraction to find the limit:

$$\lim_{x\to 1}\left(\frac{(x-1)(x+1)}{(x-1)(x-3)}\right)=\lim_{x\to 1}\left(\frac{x+1}{x-3}\right)=-1$$

Method 2

Apply l'Hopital's rule:

$$\lim_{x\to 1}\left(\frac{x^2-1}{x^2-4x+3}\right)=\lim_{x\to 1}\left(\frac{2x}{2x-4}\right)=-1$$

(b) $\lim_{x\to 0}\frac{\sqrt[3]{1-x}-1}{x}$ Apply l'Hopital's rule, as substituting $x=0$ we get the indeterminate form $\frac{0}{0}$:

$$\lim_{x\to 0}\frac{\sqrt[3]{1-x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{3}(1-x)^{-\frac{2}{3}}\cdot(-1)}{1}=-\frac{1}{3}$$

(c) $\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right)$ Apply l'Hopital's rule three times, as substituting $x=0$ we get the indeterminate form $\frac{0}{0}$:

$$\lim_{x\to0}\left(\frac{x-\sin x}{x^3}\right)=\lim_{x\to0}\left(\frac{1-\cos x}{3x^2}\right)=\lim_{x\to0}\left(\frac{\sin x}{6x}\right)=\lim_{x\to0}\left(\frac{\cos x}{6}\right)=\frac{1}{6}$$

(d) $\lim\limits_{x\to0}\left(\dfrac{\frac{1}{x}-\frac{1}{\tan x}}{x}\right)=\lim\limits_{x\to0}\left(\dfrac{\tan x-x}{x^2\tan x}\right)$ Apply l'Hopital's rule, as substituting $x=0$ we get the indeterminate form $\frac{0}{0}$:

$$\lim_{x\to0}\left(\frac{\tan x-x}{x^2\tan x}\right)=\lim_{x\to0}\left(\frac{\frac{1}{\cos^2 x}-1}{x^2\cdot\frac{1}{\cos^2 x}+(\tan x)\cdot 2x}\right)$$

$$=\lim_{x\to0}\left(\frac{\frac{1-\cos^2 x}{\cos^2 x}}{\frac{x^2}{\cos^2 x}+\frac{\sin x}{\cos x}\cdot 2x}\right)=\lim_{x\to0}\left(\frac{\sin^2 x}{x^2+2x\cdot\sin x\cos x}\right)$$

This is still the indeterminate form $\frac{0}{0}$.

Note that $\sin 2x=2\sin x\cos x$, and apply l'Hopital's rule again:

$$\lim_{x\to0}\left(\frac{\sin^2 x}{x^2+x\sin 2x}\right)=\lim_{x\to0}\left(\frac{2\sin x\cos x}{2x+2x\cos 2x+\sin 2x}\right)$$

$$=\lim_{x\to0}\left(\frac{\sin 2x}{2x+2x\cos 2x+\sin 2x}\right)$$

$$=\lim_{x\to0}\left(\frac{2\cos 2x}{2-4x\sin 2x+2\cos 2x+2\cos 2x}\right)$$

$$=\frac{2}{2-0+2+2}=\frac{1}{3}$$

(e) $\lim\limits_{x\to\infty}\left(\dfrac{\ln(x+1)}{\log_2 x}\right)$ Apply l'Hopital's rule, as when $x\to\infty$ we get the indeterminate form $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty}\left(\frac{\ln(x+1)}{\log_2 x}\right)=\lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}=\lim_{x\to\infty}\frac{x\ln 2}{x+1}$$

Apply l'Hopital's rule again as when $x\to\infty$ we get the indeterminate form $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty}\frac{x\ln 2}{x+1}=\lim_{x\to\infty}\frac{\ln 2}{1}=\ln 2$$

(f) $\lim\limits_{x\to0}\left(\dfrac{\ln\left(1+x^2\right)}{\ln\left(1-x^2\right)}\right)$ Apply l'Hopital's rule, as substituting $x=0$ we get the indeterminate form $\dfrac{0}{0}$ and transforming the expression:

$$\lim_{x\to0}\left(\frac{\ln\left(1+x^2\right)}{\ln\left(1-x^2\right)}\right)=\lim_{x\to0}\frac{\frac{2x}{1+x^2}}{\frac{-2x}{1-x^2}}=\lim_{x\to0}\frac{x^2-1}{1+x^2}=-1$$

(g) $\lim\limits_{x\to0}\left(\dfrac{2+x^2-2\cos x}{\mathrm{e}^x+\mathrm{e}^{-x}-2\cos x}\right)$ Apply l'Hopital's rule twice, as substituting $x=0$ we get the indeterminate form $\dfrac{0}{0}$:

$$\lim_{x\to0}\left(\frac{2+x^2-2\cos x}{\mathrm{e}^x+\mathrm{e}^{-x}-2\cos x}\right)=\lim_{x\to0}\left(\frac{2x+2\sin x}{\mathrm{e}^x-\mathrm{e}^{-x}+2\sin x}\right)=\lim_{x\to0}\left(\frac{2+2\cos x}{\mathrm{e}^x+\mathrm{e}^{-x}+2\cos x}\right)=\frac{4}{4}=1$$

3. $\lim\limits_{x\to\infty}\left(1+x\right)^{\frac{1}{x}}$. as $x\to\infty$ we get the indeterminate form ∞^0.

We can symbolically manipulate the expression to reduce it to a form where l'Hopital's rule can apply.

$$\left(1+x\right)^{\frac{1}{x}}=\mathrm{e}^{\ln\left(1+x\right)^{\frac{1}{x}}}=\mathrm{e}^{\frac{1}{x}\ln\left(1+x\right)}$$

Now, $\lim\limits_{x\to\infty}\left(1+x\right)^{\frac{1}{x}}=\lim\limits_{x\to\infty}\mathrm{e}^{\frac{1}{x}\ln\left(1+x\right)}=\mathrm{e}^{\lim\limits_{x\to\infty}\frac{1}{x}\ln\left(1+x\right)}$, but,

$\lim\limits_{x\to\infty}\dfrac{1}{x}\ln\left(1+x\right)=\lim\limits_{x\to\infty}\dfrac{\ln\left(1+x\right)}{x}$, which when $x\to\infty$ is of the indeterminate form $\dfrac{\infty}{\infty}$.

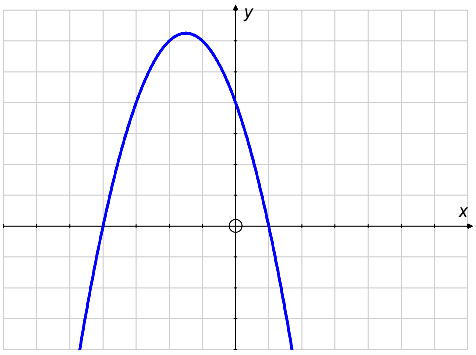
$\lim\limits_{x\to\infty}\dfrac{\ln\left(1+x\right)}{x}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{1+x}}{1}=\lim\limits_{x\to\infty}\dfrac{1}{1+x}=0$, and therefore,

$$\lim_{x\to\infty}\left(1+x\right)^{\frac{1}{x}}=\mathrm{e}^{\lim\limits_{x\to\infty}\frac{1}{x}\ln\left(1+x\right)}=\mathrm{e}^0=1$$

Chapter 13 practice questions

1. The important points of the first derivative are zeros, where the original function reaches minimum or maximum, $x=-4$ and $x=1$, and the maximum point, at the midpoint. The intervals of the positive and negative values of the first derivative are to be established by the increasing/decreasing intervals of the original function. So:

$$f'(x)>0, -4\le x\le 1, \text{ and } f'(x)<0, x\le -4 \text{ or } x\ge 1$$



2. **(a)** Given the product form of the function, the values a and b are zeroes that can be easily read from the graph. Therefore, **(i)** $a=-4$ and **(ii)** $b=2$.

(b) **(i)** We can use the product rule, but we have to be careful since there are three factors.

$$f'(x)=-(x+4)(x-2)-x(x-2)-x(x+4)$$

$$=-\left(x^2+2x-8+x^2-2x+x^2+4x\right)=8-4x-3x^2$$

(ii) $f'(x)=0\Rightarrow 8-4x-3x^2=0\Rightarrow 3x^2+4x-8=0$

$$x=\frac{-4\pm\sqrt{16+96}}{6}=\frac{-2\pm2\sqrt{7}}{3}\Rightarrow x=\frac{-2-2\sqrt{7}}{3} \text{ or } x=\frac{-2+2\sqrt{7}}{3}$$

(iii) D is relative maximum and the 1st derivative changes sign from positive to negative at $x=\dfrac{-2+2\sqrt{7}}{3}$

x-coordinate of the point D is $\dfrac{-2+2\sqrt{7}}{3}$

$$f\left(\frac{-2+2\sqrt{7}}{3}\right)=-\left(\frac{-2+2\sqrt{7}}{3}\right)^3\cdots=\frac{16\left(-10+7\sqrt{7}\right)}{27}$$

Note: This is very close to (1, 5) as it may look on the graph.

(c) **(i)** $m = f'(0) = 8 \Rightarrow$ Equation of tangent: $y = 8x$

(ii) $-x(x+4)(x-2) = 8x \Rightarrow x\left(\cancel{8} + x^2 + 2x - \cancel{8}\right) = 0 \Rightarrow x^2(x+2) = 0$.

Since the point differs from the origin, we can conclude that the x-coordinate of the second point is $x = -2$.

3. **(a)** **(i)** When $t = 0 : v(0) = 66 - 66e^{-0.15\times 0} = 66 - 66\times 1 = 0$

(ii) When $t = 10 : v(10) = 66 - 66e^{-0.15\times 10} = 66\left(1 - e^{-1.5}\right) \approx 51.3\,\text{m/s}$

(b) **(i)** $a(t) = v'(t) = -66e^{-0.15t} \times (-0.15) = 9.9e^{-0.15t}$

(ii) $a(0) = 9.9e^{-0.15\times 0} = 9.9\,\text{m/s}^2$

(c) **(i)** $\lim\limits_{t\to\infty}\left(66 - 66e^{-0.15t}\right) = 66$

(ii) $\lim\limits_{t\to\infty}\left(9.9e^{-0.15t}\right) = 0$

(iii) Since the velocity is constant (66 m/s) the acceleration must be zero.

4. **(a)** $y' = 3x^2 + 14x + 8 \Rightarrow y' = 0 \Rightarrow 3x^2 + 14x + 8 = 0 \Rightarrow (3x+2)(x+4) = 0$

$\Rightarrow x = -\dfrac{2}{3}$ or $x = -4$

x	$x < -4$	-4	$-4 < x < -\dfrac{2}{3}$	$-\dfrac{2}{3}$	$x > -\dfrac{2}{3}$
$f'(x)$	positive	0	negative	0	positive
$f(x)$	increases	maximum	decreases	minimum	increases

To find the exact coordinates we can either use synthetic division or simple substitution: $f(-4) = 13; f\left(-\dfrac{2}{3}\right) = -\dfrac{149}{27}$.

So, the maximum point is $(-4, 13)$ and the minimum point is $\left(-\dfrac{2}{3}, -\dfrac{149}{27}\right)$

(b) $y'' = 6x + 14 \Rightarrow 6x + 14 = 0 \Rightarrow x = -\dfrac{14}{6} = -\dfrac{7}{3}$

$$f\left(-\frac{7}{3}\right) = \left(-\frac{7}{3}\right)^3 + 7\times\left(-\frac{7}{3}\right)^2 + 8\times\left(-\frac{7}{3}\right) - 3$$

$$= -\frac{343}{27} + \frac{343}{9} - \frac{56}{3} - 3 = \frac{101}{27}$$

So, the point of inflection is $\left(-\dfrac{7}{3}, \dfrac{101}{27}\right)$

5. **(a)** **(i)** $g(x)=2+\mathrm{e}^{-3x}\Rightarrow g'(x)=\mathrm{e}^{-3x}\times-3=-3\mathrm{e}^{-3x}$

(ii) Since $-3\mathrm{e}^{-3x}<0$ for all real values of x, we can conclude that the function always decreases.

(b) **(i)** $g(x)=2+\mathrm{e}^{-3x}\Rightarrow g\left(-\frac{1}{3}\right)=2+\mathrm{e}^{-3\times\left(-\frac{1}{3}\right)}=2+\mathrm{e}$

(ii) $g'(x)=-3\mathrm{e}^{-3x}\Rightarrow g'\left(-\frac{1}{3}\right)=-3\mathrm{e}^{-3\times\left(-\frac{1}{3}\right)}=-3\mathrm{e}$

(c) $y=-3\mathrm{e}\left(x+\frac{1}{3}\right)+2+\mathrm{e}\Rightarrow y=-3\mathrm{e}x-\mathrm{e}+2+\mathrm{e}=-3\mathrm{e}x+2$

6. **(a)** Instead of quotient form, we are going to write it in product form:

$$\begin{aligned}f(x)&=\left(2x^2-13x+20\right)(x-1)^{-2}\Rightarrow f'(x)\\&=(4x-13)(x-1)^{-2}+\left(2x^2-13x+20\right)\times(-2)(x-1)^{-3}\\&=(x-1)^{-3}\left((4x-13)(x-1)+\left(2x^2-13x+20\right)\times(-2)\right)\\&=\frac{4x^2-17x+13-4x^2+26x-40}{(x-1)^3}=\frac{9x-27}{(x-1)^3},\ x\neq1\end{aligned}$$

(b) We know that a minimum point has the first derivative equal to zero, and therefore $f'(x)=0\Rightarrow\frac{9x-27}{(x-1)^3}=0\Rightarrow9x-27=0\Rightarrow x=\frac{27}{9}=3$. Also, the second derivative must be positive, so

$$f''(x)=\frac{72-18x}{(x-1)^4}\Rightarrow f''(3)=\frac{72-18\times3}{(3-1)^4}=\frac{72-54}{16}=\frac{18}{16}=\frac{9}{8}>0.$$

Therefore, $f(3)$ is a minimum.

(c) For the point of inflection, the second derivative must be equal to zero.

$$f''(x)=0\Rightarrow\frac{72-18x}{(x-1)^4}=0\Rightarrow72-18x=0\Rightarrow x=\frac{72}{18}=4$$

$$y=f(4)=\frac{2\times4^2-13\times4+20}{(4-1)^2}=0\Rightarrow I(4,0)$$

7. In all these questions, you need to be careful to apply the chain rule.

(a) We are going to rewrite this expression in a form with an integer exponent.

$$y=(2x+3)^{-2}\Rightarrow y'=-2\times(2x+3)^{-3}\times2=\frac{-4}{(2x+3)^3},x\neq-\frac{3}{2}$$

(b) $y=\mathrm{e}^{\sin(5x)}\Rightarrow y'=\mathrm{e}^{\sin(5x)}\times\cos(5x)\times5=5\cos(5x)\mathrm{e}^{\sin(5x)}$

(c) $y=\tan^2\left(x^2\right)\Rightarrow y'=2\tan\left(x^2\right)\sec^2\left(x^2\right)\times 2x=4x\tan\left(x^2\right)\sec^2\left(x^2\right)$

An equivalent form is $y'=\dfrac{4x\sin\left(x^2\right)}{\cos^3\left(x^2\right)}$

8. $y=Ax+B+\dfrac{C}{x}\Rightarrow y'=A-\dfrac{C}{x^2}\Rightarrow y'=0\Rightarrow A=\dfrac{C}{x^2}\Rightarrow x^2=\dfrac{C}{A}\Rightarrow x=\pm\sqrt{\dfrac{C}{A}}$

By observing the given stationary points, we can establish a relationship between *A* and *C*: $x=\pm1\Rightarrow\dfrac{C}{A}=1\Rightarrow C=A$. Now we need to use the fact that points *P* and *Q* lie on the curve itself, and therefore their coordinates satisfy the equation of the curve.

$$\left.\begin{array}{l} P(1,4)\Rightarrow 4=A\times1+B+\dfrac{A}{1}\Rightarrow 2A+B=4 \\ Q(-1,0)\Rightarrow 0=A\times(-1)+B+\dfrac{A}{-1}\Rightarrow -2A+B=0 \end{array}\right\}\Rightarrow 2B=4\Rightarrow B=2\Rightarrow A=C=1$$

9. $x^3+y^3=2\Rightarrow 3x^2+3y^2y'=0\Rightarrow y'=-\dfrac{3x^2}{3y^2}=-\dfrac{x^2}{y^2}\Rightarrow y'(1,1)=-\dfrac{1^2}{1^2}=-1$

$3x^2=-3y^2y'\Rightarrow 6x=-3\left(2yy'\times y'+y^2y"\right)\Rightarrow 2x+2y\left(y'\right)^2=-y^2y''$

$y''=-\dfrac{2x+2y\left(y'\right)^2}{y^2}\Rightarrow y''(1,1)=-\dfrac{2\times1+2\times1\times(-1)^2}{1^2}=-4$

10. **(a)** $y=\dfrac{x}{\mathrm{e}^x-1}\Rightarrow y'=\dfrac{1\times\left(\mathrm{e}^x-1\right)-x\times\mathrm{e}^x}{\left(\mathrm{e}^x-1\right)^2}=\dfrac{(1-x)\mathrm{e}^x-1}{\left(\mathrm{e}^x-1\right)^2}$

(b) $y=\mathrm{e}^x\sin 2x\Rightarrow y'=\mathrm{e}^x\sin 2x+\mathrm{e}^x\cos 2x\times 2=\mathrm{e}^x\left(\sin 2x+2\cos 2x\right)$

(c) $y'=2x\ln x+2x\ln 3+\left(x^2-1\right)\cdot\dfrac{1}{x}=2x\ln x+2x\ln 3+x-\dfrac{1}{x}$

11. $y=x^2-4x\Rightarrow y'=2x-4\Rightarrow m_N=-\dfrac{1}{y'(3)}=-\dfrac{1}{2},$

So, equation of normal is $y=-\dfrac{1}{2}(x-3)-3\Rightarrow y=-\dfrac{1}{2}x-\dfrac{3}{2}$

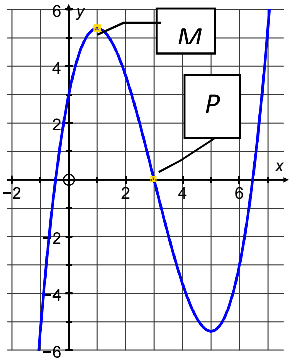
$x=0\Rightarrow y=-\dfrac{1}{2}\times0-\dfrac{3}{2}=-\dfrac{3}{2},P\left(0,-\dfrac{3}{2}\right),$

$y=0\Rightarrow 0=-\dfrac{1}{2}x-\dfrac{3}{2}\Rightarrow x=-3,Q(-3,0)$

12. **(a)** The *x*-coordinate of *P* can be read from both graphs. In the first derivative graph, we can see that the curve has a minimum point at the point where $x=3$, which means that the rate of change of the gradient is changing direction, while in the second graph, we can see that the line has a zero at the point where $x=3$.

(b) Since point M is a maximum point, we know that the first derivative must be equal to zero; therefore, by observing the first graph, we have two possible values: $x=1$ or $x=5$. Now, by looking at the second graph, we can see that $x=1 \Rightarrow y''(1)<0$ and we have a maximum point. For the second point where $x=5 \Rightarrow y''(5)>0$, we have a local minimum.

(c) Here is a sketch showing the points M and P.



13. $x^2+xy+y^2-3y=10 \Rightarrow 2x+y+xy'+2yy'-3y'=0 \Rightarrow 2x+y=y'(3-x-2y)$

$$\Rightarrow y'=\frac{2x+y}{3-x-2y}, \quad m_N=-\frac{1}{y'(2,3)}=-\frac{3-2-2\times 3}{2\times 2+3}=\frac{5}{7}$$

Thus, the equation of normal is $y=\frac{5}{7}(x-2)+3 \Rightarrow y=\frac{5}{7}x+\frac{11}{7}$

14. $V=r^2\pi h \Rightarrow 128\pi=r^2\pi h \Rightarrow h=\frac{128}{r^2}$

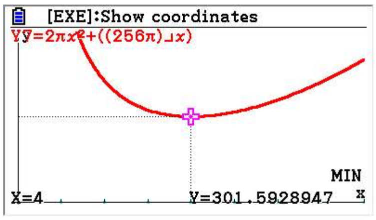
$$S=2r^2\pi+2r\pi h \Rightarrow S(r)=2r^2\pi+2r\pi\frac{128}{r^2}=2r^2\pi+\frac{256\pi}{r}$$

$$S'(r)=4r\pi-\frac{256\pi}{r^2}=4\pi\left(r-\frac{64}{r^2}\right) \Rightarrow S'(r)=0 \Rightarrow r-\frac{64}{r^2}=0$$

$$\Rightarrow r=\frac{64}{r^2} \Rightarrow r^3=64 \Rightarrow r=\sqrt[3]{64}=4 \Rightarrow h=\frac{128}{4^2}=8$$

So, the radius is 4 cm and the height is 8 cm.

If we want to use a GDC, then we simply input the surface area function and use a ‘minimum’ feature on the GDC.



[EXE]:Show coordinates

Y3=2πx²+((256π)⌟x)

MIN

X=4 Y=301.5928947

So, the minimum surface area occurs when the radius is 4 cm and the height is 8 cm.

15. Let's focus on the vertex of the rectangle in the first quadrant. The coordinates are $(x, y)=(x, 12-x^2)$. Now, the dimensions of the rectangle and its area are:

$$l=2x, w=y \Rightarrow A=lw=2xy \Rightarrow A(x)=2x(12-x^2)=24x-2x^3$$

To find the maximum possible area, we need to find the zero of the first derivative.

$A'(x)=24-6x^2 \Rightarrow A'(x)=0 \Rightarrow 24-6x^2=0 \Rightarrow 24=6x^2 \Rightarrow x^2=4 \Rightarrow x=2$, since the vertex is in the first quadrant. Again, we can verify that we have a maximum point since the second derivative test gives us a negative value:
$A''(x)=-12x \Rightarrow A''(2)=-12\times 2=-24<0$.

So, the dimensions are $l=2\times 2=4$ and $w=12-2^2=8$, and that gives us the maximum possible area of 32.

16. **(a)** The first derivative is negative when the function is decreasing, while the second derivative is negative when the function is concave down. By looking for these features, we identify point *E*.

(b) The first derivative is negative when the function is decreasing, while the second derivative is positive when the function is concave up. By looking for these features, we identify point *A*.

(c) The first derivative is positive when the function is increasing, while the second derivative is negative when the function is concave down. By looking for these features, we identify point *C*.

17.
$$y=\frac{2x-1}{x+2} \Rightarrow y'=\frac{2(x+2)-(2x-1)\times 1}{(x+2)^2}=\frac{\cancel{2x}+4\cancel{-2x}+1}{(x+2)^2}=\frac{5}{(x+2)^2}$$

$$y'(-3)=\frac{5}{(-3+2)^2}=5 \Rightarrow m_n=-\frac{1}{5}$$

Normal: $y=-\frac{1}{5}(x+3)+7 \Rightarrow y=-\frac{1}{5}x-\frac{3}{5}+7 \Rightarrow y=-\frac{1}{5}x+\frac{32}{5}$

18.
$$y=\ln(4x-3) \Rightarrow y'=\frac{4}{4x-3} \Rightarrow y'(1)=\frac{4}{4\times 1-3}=4$$

(a) Tangent: $m_T=y'(1)=4 \Rightarrow y=4(x-1)+0 \Rightarrow y=4x-4$

(b) Normal: $m_N=-\frac{1}{y'(1)}=-\frac{1}{4} \Rightarrow y=-\frac{1}{4}(x-1)+0 \Rightarrow y=-\frac{1}{4}x+\frac{1}{4}$

19.
$$y=x^2\ln x \Rightarrow y'=2x\ln x+x^2\times\frac{1}{x}=x(2\ln x+1) \Rightarrow y''=2\ln x+1+x\times\frac{2}{x}=2\ln x+3$$

(a) $y'=0 \Rightarrow x(2\ln x+1)=0 \Rightarrow \ln x=-\frac{1}{2} \Rightarrow x=\mathrm{e}^{-\frac{1}{2}}=\frac{1}{\mathrm{e}^{\frac{1}{2}}}$ or $\frac{1}{\sqrt{\mathrm{e}}}$

$$y=\left(\mathrm{e}^{-\frac{1}{2}}\right)^2\ln \mathrm{e}^{-\frac{1}{2}}=-\frac{1}{2\mathrm{e}},\quad P\left(\frac{1}{\sqrt{\mathrm{e}}},-\frac{1}{2\mathrm{e}}\right)$$

Notice that the domain of the function is $x>0$; therefore, we discarded the solution $x=0$.

$y''\left(\mathrm{e}^{-\frac{1}{2}}\right)=2\ln \mathrm{e}^{-\frac{1}{2}}+3=-1+3=2>0$; therefore, the point P is a minimum.

$\lim\limits_{x\to 0^-} y=\lim\limits_{x\to 0^-} x^2\ln x=0$ and $\lim\limits_{x\to\infty} y=\lim\limits_{x\to\infty} x^2\ln x=\infty$; therefore, the point P is the absolute minimum.

(b) $y''=0\Rightarrow 2\ln x+3=0\Rightarrow x=\mathrm{e}^{-\frac{3}{2}}=\frac{1}{\mathrm{e}^{\frac{3}{2}}}=\frac{1}{\mathrm{e}\sqrt{\mathrm{e}}}$

$$y=\left(\mathrm{e}^{-\frac{3}{2}}\right)^2\ln \mathrm{e}^{-\frac{3}{2}}=-\frac{3}{2\mathrm{e}^3},\ I\left(\frac{1}{\mathrm{e}\sqrt{\mathrm{e}}},-\frac{3}{2\mathrm{e}^3}\right)$$

20. $f(x)=x^2+\frac{a}{x}\Rightarrow f'(x)=2x-\frac{a}{x^2}\Rightarrow f''(x)=2+\frac{2a}{x^3}$

(a) $f'(2)=0\Rightarrow 2\times 2-\frac{a}{2^2}=0\Rightarrow a=16,\ f''(2)=2+\frac{2\times 16}{8}=6>0$

(b) $f'(-3)=0\Rightarrow 2\times(-3)-\frac{a}{(-3)^2}=0\Rightarrow a=-54$

$$f''(-3)=2+\frac{2\times(-54)}{-27}=6>0$$

(c) $f'(x)=2x-\frac{a}{x^2}=0\Rightarrow 2x^3=a\Rightarrow x=\sqrt[3]{\frac{a}{2}}$

$$f''\left(\sqrt[3]{\frac{a}{2}}\right)=2+\frac{2a}{\left(\sqrt[3]{\frac{a}{2}}\right)^3}=2+\frac{2a}{\frac{a}{2}}=6>0$$

Since the second derivative is always positive, the stationary point cannot be a maximum.

21. A line $y=mx+l$ that passes through (3, 2) satisfies the following equation:

$2=m\times 3+l\Rightarrow l=2-3m,\quad y=mx+2-3m$

$x=0\Rightarrow y=2-3m,\quad y=0\Rightarrow 0=mx+2-3m\Rightarrow x=\frac{3m-2}{m}$

Therefore, the area of the triangle is given by the expression:

$$A(m)=\frac{1}{2}(2-3m)\frac{3m-2}{m}=-\frac{(3m-2)^2}{2m}$$

$$A'(m)=-\frac{2(3m-2)\times3\times2m-(3m-2)^2\times2}{4m^2}$$

$$=-\frac{2(3m-2)(6m-3m+2)}{4m^2}=-\frac{2(3m-2)(3m+2)}{4m^2}$$

$$\Rightarrow A'(m)=0\Rightarrow(3m-2)(3m+2)=0\Rightarrow m=\frac{2}{3}\text{ or }m=-\frac{2}{3}$$

We can discard the first solution since, for $m=\frac{2}{3}$, the line will pass through the origin and so the triangle doesn't exist.

$$m=-\frac{2}{3}\Rightarrow y=mx+2-3m=-\frac{2}{3}x+2-3\left(-\frac{2}{3}\right)=-\frac{2}{3}x+4$$

22. $y=x\tan x,\quad x=\frac{\pi}{4}\Rightarrow y=\frac{\pi}{4}\tan\frac{\pi}{4}=\frac{\pi}{4},\ P\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

$$y'(x)=\tan x+x\sec^2x\Rightarrow y'\left(\frac{\pi}{4}\right)=\tan\frac{\pi}{4}+\frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)=1+\frac{\pi}{4}\times2=1+\frac{\pi}{2}$$

Tangent: $m_T=y'\left(\frac{\pi}{4}\right)=1+\frac{\pi}{2}\Rightarrow y=\left(1+\frac{\pi}{2}\right)\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}\Rightarrow y=\left(1+\frac{\pi}{2}\right)x-\frac{\pi^2}{8}$

Normal: $m_N=-\frac{1}{y'\left(\frac{\pi}{4}\right)}=-\frac{1}{1+\frac{\pi}{2}}=-\frac{2}{2+\pi}\Rightarrow y=\left(-\frac{2}{2+\pi}\right)\left(x-\frac{\pi}{4}\right)+\frac{\pi}{4}$

$$y=\left(-\frac{2}{2+\pi}\right)x+\frac{2\pi}{4(2+\pi)}+\frac{\pi}{4}\Rightarrow y=\left(-\frac{2}{2+\pi}\right)x+\frac{4\pi+\pi^2}{4(2+\pi)}$$

23. **(a)** $f(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Rightarrow f'(x)=\frac{e^{-\frac{x^2}{2}}\times\frac{-2x}{2}}{\sqrt{2\pi}}=-\frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$

$$f''(x)=-\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}-x\frac{e^{-\frac{x^2}{2}}\times\frac{-2x}{2}}{\sqrt{2\pi}}=\frac{e^{-\frac{x^2}{2}}\left(x^2-1\right)}{\sqrt{2\pi}}$$

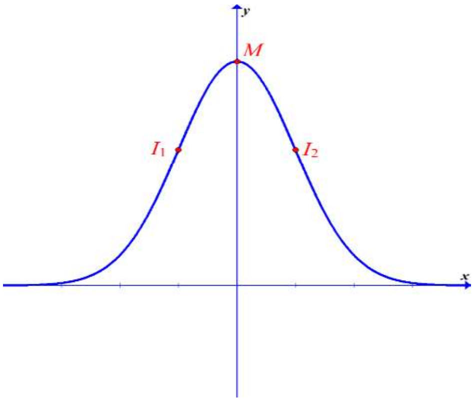
$$f'(x)=0\Rightarrow x=0,y=f(0)=\frac{e^0}{\sqrt{2\pi}}=\frac{1}{\sqrt{2\pi}}\Rightarrow M\left(0,\frac{1}{\sqrt{2\pi}}\right)$$

$$f''(x)=0\Rightarrow x^2-1=0\Rightarrow x=\pm1,y=-\frac{e^{-\frac{(\pm1)^2}{2}}}{\sqrt{2\pi}}=\frac{1}{\sqrt{2e\pi}}$$

$$\Rightarrow I_1\left(-1,\frac{1}{\sqrt{2e\pi}}\right),I_2\left(1,\frac{1}{\sqrt{2e\pi}}\right)$$

(b) We notice that the function *f* is even, and therefore symmetrical with respect to the *y*-axis. Also, $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = 0$; therefore, the *x*-axis is a horizontal asymptote.

(c) Sketch below



24. (a) $$f(x) = 2\ln(x^2+3) - x \Rightarrow f'(x) = 2\frac{2x}{x^2+3} - 1 = \frac{4x}{x^2+3} - 1 \left[= \frac{-x^2+4x-3}{x^2+3} \right]$$

$$f'(x) = 0 \Rightarrow \frac{-x^2+4x-3}{x^2+3} = 0 \Rightarrow -(x-3)(x-1) = 0$$

$x = 3, y = 2\ln 12 - 3$ or $x = 1, y = 2\ln 4 - 1 = 4\ln 2 - 1$

We notice that the denominator is always positive; therefore, the sign of the first derivative depends on the numerator, which is a quadratic expression that has a negative leading coefficient.

Use a table, or 'test-point' method to conclude that

$x < 1$ or $x > 3, f'(x) < 0$ and $1 < x < 3, f'(x) > 0$

To conclude: $(1,\ 4\ln 2 - 1)$ is a minimum point and $(3,\ 2\ln 12 - 3)$ is a maximum point.

(b) $$f''(x) = \frac{4(x^2+3) - 4x \times 2x}{(x^2+3)^2} = \frac{12-4x^2}{(x^2+3)^2}$$

$$\Rightarrow f''(x) = 0 \Rightarrow 12 - 4x^2 = 0 \Rightarrow x = \pm\sqrt{3}$$

Again, since the denominator is always positive, we can conclude that the sign depends on the numerator only and the numerator is a quadratic expression which changes its sign at the zeros. Therefore, we can conclude that the *x*-coordinates we found are those of inflection points.

25. $f(x)=\dfrac{2x}{18+0.015x^2}\Rightarrow f'(x)=\dfrac{2\left(18+0.015x^2\right)-2x\times 0.3x}{\left(18+0.015x^2\right)^2}=\dfrac{2\left(18-0.015x^2\right)}{\left(18+0.015x^2\right)^2}$

$f'(x)=0\Rightarrow 18-0.015x^2=0\Rightarrow x^2=\dfrac{18}{0.015}=1200$

$\Rightarrow x=\sqrt{1200}=20\sqrt{3}\approx 34.6\,\text{km/h r}$

26. $\lim\limits_{x\to\pi^-}\dfrac{\sin x}{1-\cos x}=\dfrac{0}{1-(-1)}=0$ and so it is not an indeterminate form. Thus, the argument is false. l'Hopital's rule does not apply in this case.

27. $y=\arccos\left(1-2x^2\right)\Rightarrow y'=\dfrac{-1}{\sqrt{1-\left(1-2x^2\right)^2}}\times(-4x)=\dfrac{4x}{\sqrt{1-\left(1-4x^2+4x^4\right)}}$

$$=\frac{4x}{2|x|\sqrt{1-x^2}}=\begin{cases}\dfrac{-2}{\sqrt{1-x^2}}, & -1<x<0\\ \dfrac{2}{\sqrt{1-x^2}}, & 0\le x<1\end{cases}$$

Notice that we had to restrict the domain of the derivative because of the expression in the denominator.

28. $3x^2+4y^2=7\Rightarrow 6x+8yy'=0\Rightarrow y'=-\dfrac{6x}{8y}=-\dfrac{3x}{4y}$

$x=1\Rightarrow 3\times 1^2+4y^2=7\Rightarrow 4y^2=4\Rightarrow y^2=1\Rightarrow y=1$ because $y>0$

$y'(1,1)=-\dfrac{3\times 1}{4\times 1}=-\dfrac{3}{4}$

29. Since the normal has the slope of -2, we need to find a point with a gradient of $\dfrac{1}{2}$.

$y=\arctan(x-1)\Rightarrow y'=\dfrac{1}{1+(x-1)^2}=\dfrac{1}{2-2x+x^2},\quad y'(x)=\dfrac{1}{2}\Rightarrow 2-2x+x^2=2$

$\Rightarrow x^2-2x=0\Rightarrow x(x-2)=0\Rightarrow x=2$ since the condition $x>0$

$x=2\Rightarrow y=\arctan(1)=\dfrac{\pi}{4}\Rightarrow\dfrac{\pi}{4}=-2\times 2+c\Rightarrow c=\dfrac{\pi}{4}+4\approx 4.79$ (3 s.f.)

30. Given that the line $y=16x-9$ is tangent at (1,7), we can conclude that $y'(1)=16$.

$y=2x^3+ax^2+bx-9\Rightarrow y'=6x^2+2ax+b,\quad y'(1)=6+2a+b=16\Rightarrow 2a+b=10$

The second equation is by the given point at the curve.

$y(1)=2+a+b-9=7\Rightarrow a+b=14$, so we have a pair of simultaneous equations to solve.

$$\begin{cases}2a+b=10\\ a+b=14\end{cases}\Rightarrow\begin{cases}b=10-2a\\ a+10-2a=14\end{cases}\Rightarrow\begin{cases}b=10-2a\\ -4=a\end{cases}\Rightarrow\begin{cases}b=18\\ a=-4\end{cases}$$

31. $y=\sin(kx)-kx\cos(kx)\Rightarrow y'=\cos(kx)\times k-\cos(kx)\times k+kx\sin(kx)\times k=k^2x\sin(kx)$

32. $xy^3+2x^2y=3\Rightarrow y^3+x\times 3y^2y'+4xy+2x^2\times y'=0\Rightarrow y'\left(3xy^2+2x^2\right)=-\left(y^3+4xy\right)$

$$\Rightarrow y'=-\frac{y^3+4xy}{3xy^2+2x^2},\ y'(1,1)=-\frac{1+4}{3+2}=-1\Rightarrow \text{Tangent}: y=-1(x-1)+1\Rightarrow y=-x+2$$

33. **(a)** **(i)** $$f'(x)=\frac{(2x-1)\left(x^2+x+1\right)-\left(x^2-x+1\right)(2x+1)}{\left(x^2+x+1\right)^2}=\frac{2x^2-2}{\left(x^2+x+1\right)^2}$$

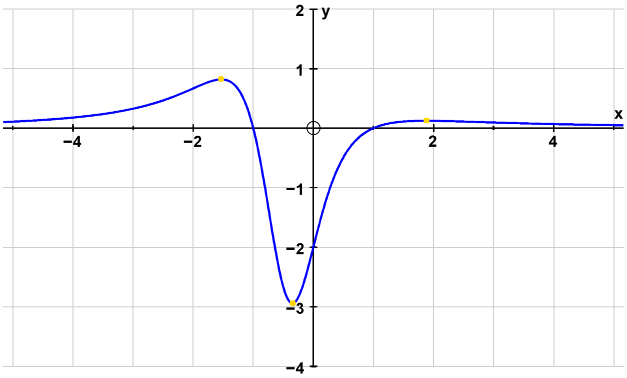
(ii) If the tangents are parallel to the x-axis, then the gradient is zero.

$f'(x)=0\Rightarrow x_{1,2}=\pm1; x=-1$

$$\Rightarrow y=\frac{1+1+1}{1-1+1}=3, x=1\Rightarrow y=\frac{1-1+1}{1+1+1}=\frac{1}{3}$$

So, the points are $A(-1,3)$ and $B\left(1,\frac{1}{3}\right)$

(b) **(i, ii)** Sketch below



On the graph of the first derivative, the stationary points are points where the second derivative is zero; thus, the points of inflection. A calculator gives us the following:

$x=-1.53$ or $x=-0.347$ or $x=1.88$

(c) **(i)** We notice that the denominator is never equal to zero, and therefore the domain of the function is the whole set of real numbers. Also, the function has a horizontal asymptote.

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{\frac{x^2-x+1}{x^2}}{\frac{x^2+x+1}{x^2}}=\lim_{x\to\pm\infty}\frac{1-\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x}+\frac{1}{x^2}}=1$$

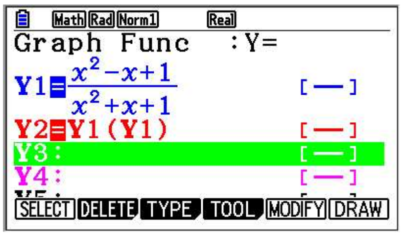
Since the minimum point is below the asymptote and the maximum point is above the asymptote of this continuous function, we can say that the range is $\left[\frac{1}{3},3\right]$.

(ii) $(f \circ f)(x) = f(f(x))$

To understand how we can find the range of this composition, let us call the functions as f_1 and $f_2 \Rightarrow (f \circ f)(x) = f_2(f_1(x))$

Since the range of $f(x)$ is $\left[\frac{1}{3}, 3\right]$, after applying f_1 the restricted domain of f_2 is reduced to $\left[\frac{1}{3}, 3\right]$. Now, in this interval, extreme values of the function happen at a stationary point or end of an interval. Since the only stationary point in this interval corresponds to $x = 1$, the potential extreme value of the function is $f_2(1) = \frac{1}{3}$. This corresponds to $(f \circ f)(0) = f(1) = \frac{1}{3}$. The other two points to check are the endpoints of the interval, namely $\frac{1}{3}$ and 3, i.e., $f_2(3) = \frac{7}{13}$, or $f_2\left(\frac{1}{3}\right) = \frac{7}{13}$. Thus, the minimum of this function is $\frac{1}{3}$ and the maximum is $\frac{7}{13}$. So, the range of $(f \circ f)(x)$ is $\left[\frac{1}{3}, \frac{7}{13}\right]$.

Note: We could have found the expression for the composition and then tried to find the range. You can confirm all these findings by sketching the graphs of f and its composite.



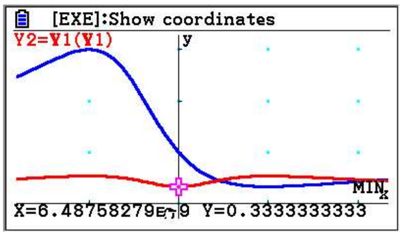
Graph Func :Y=

Y1=$\frac{x^2-x+1}{x^2+x+1}$ [—]

Y2=Y1(Y1) [—]

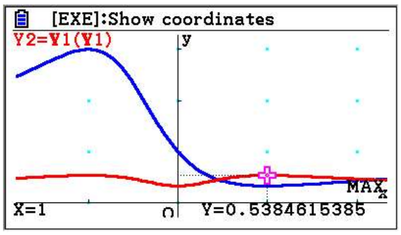
Y3: [—]

Y4: [—]



[EXE]:Show coordinates

X=6.48758279E-9 Y=0.3333333333



[EXE]:Show coordinates

X=1 Y=0.5384615385

34. $\frac{\mathrm{d}V}{\mathrm{d}t} = 8\,\text{cm}^3\text{s}^{-1}, \quad V = \frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4}{3}\pi \cdot 3r^2 \frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow 8 = 4\pi \cdot 2^2 \frac{\mathrm{d}r}{\mathrm{d}t}$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4 \times 2^2 \pi} \times 8 = \frac{1}{2\pi}\,\text{cms}^{-1}$$

35. **(a)** In this question, we need to rewrite the expression in the form of a product for ease of calculation of the first and second derivative.

(i) $f(x)=\frac{x^2}{2^x}=x^2\times 2^{-x}\Rightarrow f'(x)=2x\times 2^{-x}+x^2\times 2^{-x}\ln 2\times(-1)=\frac{2x-x^2\ln 2}{2^x}$

(ii) $f'(x)=\left(2x-x^2\ln 2\right)\times 2^{-x}$

$$f''(x)=(2-2x\ln 2)\times 2^{-x}+\left(2x-x^2\ln 2\right)\times 2^{-x}\ln 2\times(-1)$$

$$=\frac{2-4x\ln 2+x^2\ln^2 2}{2^x}$$

(b) **(i)** $f'(x)=0\Rightarrow\frac{2x-x^2\ln 2}{2^x}=0\Rightarrow x(2-x\ln 2)=0\Rightarrow x=\frac{2}{\ln 2}$

We discarded the second solution because of the domain: $x>0$.

(ii) $$f''\left(\frac{2}{\ln 2}\right)=\frac{2-4\left(\frac{2}{\ln 2}\right)\ln 2+\left(\frac{2}{\ln 2}\right)^2\ln^2 2}{2^{\left(\frac{2}{\ln 2}\right)}}=\frac{2-8+4}{2^{\left(\frac{2}{\ln 2}\right)}}=\frac{-2}{2^{\left(\frac{2}{\ln 2}\right)}}<0$$

Therefore, we have a maximum value of the function *f*. We could have tested the nature of the stationary point by using the sign of the first derivative. We notice that the denominator is always positive and that the numerator is a quadratic function with a negative quadratic coefficient; therefore, at $x=\frac{2}{\ln 2}$ it changes sign from positive to negative, which yields the maximum value.

(c) $f''(x)=0\Rightarrow\frac{2-4x\ln 2+x^2\ln^2 2}{2^x}=0\Rightarrow 2-4x\ln 2+x^2\ln^2 2=0\Rightarrow$

$$x_{1,2}=\frac{4\ln 2\pm\sqrt{16\ln^2 2-4\times\ln^2 2\times 2}}{2\ln^2 2}=\frac{4\ln 2\pm\sqrt{8\ln^2 2}}{2\ln^2 2}$$

$$=\frac{4\ln 2\pm 2\ln 2\sqrt{2}}{2\ln^2 2}=\frac{2\ln 2\left(2\pm\sqrt{2}\right)}{2\ln^2 2}$$

$$=\frac{2\pm\sqrt{2}}{\ln 2}\Rightarrow x=\frac{2-\sqrt{2}}{\ln 2}\approx 0.845\text{ or }x=\frac{2+\sqrt{2}}{\ln 2}\approx 4.93$$

36. $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{60}\text{rad/s},\quad \tan\theta=\frac{3000}{x}\Rightarrow x=3000\cot\theta,$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=-3000\csc^2\theta\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}t}=-3000\csc^2\left(\frac{\pi}{3}\right)\times\frac{1}{60}$$

$$=-3000\times\frac{4}{3}\times\frac{1}{60}=-\frac{200}{3}$$

So, the speed is $\frac{200}{3}\,\text{m/s} = 240\,\text{km/h r}$

37. **(a)** $f(x) = a\left(b + \mathrm{e}^{-cx}\right)^{-1}$

$$f'(x) = a \times (-1)\left(b + \mathrm{e}^{-cx}\right)^{-2} \times \mathrm{e}^{-cx} \times (-c) = ac\mathrm{e}^{-cx}\left(b + \mathrm{e}^{-cx}\right)^{-2}$$

$$f''(x) = ac\left[\mathrm{e}^{-cx} \times (-c)\left(b + \mathrm{e}^{-cx}\right)^{-2} + \mathrm{e}^{-cx} \times (-2)\left(b + \mathrm{e}^{-cx}\right)^{-3} \times \mathrm{e}^{-cx} \times (-c)\right]$$

$$= \frac{-ac^2\mathrm{e}^{-cx}}{\left(b + \mathrm{e}^{-cx}\right)^3}\left(b + \mathrm{e}^{-cx} - 2\mathrm{e}^{-cx}\right) = \frac{-ac^2\mathrm{e}^{-cx}\left(b - \mathrm{e}^{-cx}\right)}{\left(b + \mathrm{e}^{-cx}\right)^3} = \frac{ac^2\mathrm{e}^{-cx}\left(\mathrm{e}^{-cx} - b\right)}{\left(b + \mathrm{e}^{-cx}\right)^3}$$

(b) $ac^2\mathrm{e}^{-cx} \neq 0$, thus,

$$f''(x) = 0 \Rightarrow \frac{ac^2\mathrm{e}^{-cx}\left(\mathrm{e}^{-cx} - b\right)}{\left(b + \mathrm{e}^{-cx}\right)^3} = 0 \Rightarrow \mathrm{e}^{-cx} - b = 0$$

$$\Rightarrow \mathrm{e}^{-cx} = b \Rightarrow -cx = \ln b \Rightarrow x = -\frac{\ln b}{c}$$

$$\Rightarrow y = \frac{a}{b + \mathrm{e}^{-c\times\left(-\frac{\ln b}{c}\right)}} = \frac{a}{2b}$$

Therefore, the point where the second derivative is zero is $\left(-\frac{\ln b}{c}, \frac{a}{2b}\right)$.

(c) This is a point of inflection because the second derivative changes its sign at that point. We notice that to the left of zero the expression is positive, while to the right of zero the expression is negative.

38. **(a)** $x = 1 \Rightarrow 2 \times 1^2 \times y + 3y^2 = 16 \Rightarrow 3y^2 + 2y - 16 = 0 \Rightarrow (3y + 8)(y - 2) = 0$

$\Rightarrow y = -\frac{8}{3}$ or $y = 2 \Rightarrow p = 2$ since the condition is $p > 0$.

(b) Differentiate implicitly

$$4xy + 2x^2y' + 6yy' = 0 \Rightarrow y'\left(2x^2 + 6y\right) = -4xy$$

$$\Rightarrow y' = -\frac{4xy}{2\left(x^2 + 3y\right)} = -\frac{2xy}{x^2 + 3y} \Rightarrow y'(1,2) = -\frac{2 \times 1 \times 2}{1^2 + 3 \times 2} = -\frac{4}{7}$$

39. Let $\measuredangle CAB = \alpha \Rightarrow \tan\alpha = \frac{h}{5} \Rightarrow \frac{1}{\cos^2\alpha}\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{1}{5}\cdot\frac{\mathrm{d}h}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \cos^2\alpha\cdot\frac{1}{5}\cdot\frac{\mathrm{d}h}{\mathrm{d}t}$

$\frac{\mathrm{d}h}{\mathrm{d}t} = 2\,\text{cm/s}$, and when the triangle is equilateral then: $\alpha = \frac{\pi}{3} \Rightarrow \cos\alpha = \frac{1}{2}$.

$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \left(\frac{1}{2}\right)^2\cdot\frac{1}{5}\cdot 2 = \frac{1}{10}$ radians per second.

40. Differentiate implicitly

$$3x^2 + 3y^2y' - 9y - 9xy' = 0 \Rightarrow 3y'\left(y^2 - 3x\right) = 3\left(3y - x^2\right)$$

$$\Rightarrow y' = \frac{3y - x^2}{y^2 - 3x},\quad m_N = -\frac{1}{y'(2,4)} = -\frac{4^2 - 3\times 2}{3\times 4 - 2^2} = -\frac{10}{8} = -\frac{5}{4}$$

$$\text{Normal}: y = -\frac{5}{4}(x-2) + 4 = -\frac{5}{4}x + \frac{13}{2}$$

41. $f(x) = \frac{x^5+2}{x}, x \neq 0 \Rightarrow f'(x) = \frac{5x^4\times x - \left(x^5+2\right)\times 1}{x^2} = \frac{4x^5 - 2}{x^2} = \left(4x^5-2\right)x^{-2}$

$$f''(x) = 20x^4\times x^{-2} + \left(4x^5-2\right)\times(-2)x^{-3} = \frac{20x^5 - 8x^5 + 4}{x^3} = \frac{4\left(3x^5+1\right)}{x^3}$$

Point of inflection:

$$f''(x) = 0 \Rightarrow \frac{4\left(3x^5+1\right)}{x^3} = 0 \Rightarrow 3x^5 + 1 = 0 \Rightarrow x = -\sqrt[5]{\frac{1}{3}},\quad y = \frac{-\frac{1}{3}+2}{-\sqrt[5]{\frac{1}{3}}} = -\frac{5\sqrt[5]{3}}{3}$$

Thus, the coordinates are: $\left(-\frac{1}{\sqrt[5]{3}}, -\frac{5\sqrt[5]{3}}{3}\right) \approx (-0.803, -2.08)$

42. **(a)** $n(t) = 650\mathrm{e}^{kt} \Rightarrow n(0) = 650$ is the initial number of bacteria.

Since the number of bacteria double every 20 minutes, there are 1300 bacteria after 20 minutes.

$$n(20) = 650\mathrm{e}^{k\times 20} = 1350 \Rightarrow \mathrm{e}^{20k} = 2 \Rightarrow 20k = \ln 2 \Rightarrow k = \frac{\ln 2}{20}$$

(b) $n(t) = 650\mathrm{e}^{\frac{\ln 2}{20}t} \Rightarrow \frac{\mathrm{d}n}{\mathrm{d}t} = 650\mathrm{e}^{\frac{\ln 2}{20}t}\times\frac{\ln 2}{20} = \frac{65\ln 2}{2}\mathrm{e}^{\frac{\ln 2}{20}t}$

$$\left.\frac{\mathrm{d}n}{\mathrm{d}t}\right|_{(t=90)} = \frac{65}{2}\times(\ln 2)\mathrm{e}^{\frac{\ln 2}{20}\times 90} = \frac{65\ln 2\times 16\sqrt{2}}{2} = 520\sqrt{2}\ln 2 \approx 510\,\text{bacteria/min}$$

43. $f\left(x\right)=\ln\left(3x+1\right),\ x>-\dfrac{1}{3}$

(a) $f'\left(x\right)=\dfrac{3}{3x+1}$

(b) $x=2\Rightarrow y=\ln 7,\quad m_N=-\dfrac{1}{f'\left(2\right)}=-\dfrac{7}{3}$

Normal: $y=-\dfrac{7}{3}\left(x-2\right)+\ln 7\Rightarrow y=-\dfrac{7}{3}x+\dfrac{14}{3}+\ln 7$

44. $y=x\arcsin x, x\in\left]-1,1\right[\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\arcsin x+\dfrac{x}{\sqrt{1-x^2}}=\arcsin x+x\left(1-x^2\right)^{-\frac{1}{2}}$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}+x\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-\frac{3}{2}}\times\left(-2x\right)=\frac{2\left(1-x^2\right)+x^2}{\left(1-x^2\right)^{\frac{3}{2}}}=\frac{2-x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$$

45. Differentiate implicitly

$$\mathrm{e}^{xy}\left(y+xy'\right)-2yy'\ln x-y^2\times\frac{1}{x}=0\Rightarrow y\mathrm{e}^{xy}-\frac{y^2}{x}=y'\left(2y\ln x-x\mathrm{e}^{xy}\right)$$

$$\Rightarrow y'=\frac{xy\mathrm{e}^{xy}-y^2}{x\left(2y\ln x-x\mathrm{e}^{xy}\right)}\Rightarrow y'(1,1)=\frac{\mathrm{e}-1}{-\mathrm{e}}=\frac{1-\mathrm{e}}{\mathrm{e}}$$

46. $f\left(x\right)=\dfrac{2x}{x^2+6}, x\geq b, b\in\mathbb{R}$

(a) $f'\left(x\right)=\dfrac{2\left(x^2+6\right)-2x\times 2x}{\left(x^2+6\right)^2}=\dfrac{2x^2+12-4x^2}{\left(x^2+6\right)^2}=\dfrac{12-2x^2}{\left(x^2+6\right)^2}$

(b) This function needs to be restricted to the interval where every value occurs only once, that is, when it is strictly increasing or decreasing. The sign of the derivative depends on the sign of $12-2x^2$ because the denominator is strictly positive.

$f'\left(x\right)=0\Rightarrow 12-2x^2=0\Rightarrow x=\pm\sqrt{6}\Rightarrow f'\left(x\right)$ switches from positive values to negative values, making the function strictly decrease from the maximum point until the horizontal asymptote, (which is the x-axis) after $b=\sqrt{6}$.

47. Differentiate implicitly

$$2x+y+xy'+2yy'=0\Rightarrow y'\left(x+2y\right)=-\left(2x+y\right)\Rightarrow y'=-\frac{2x+y}{x+2y}$$

(a) $y'\left(-1,k\right)=-\frac{2\times\left(-1\right)+k}{-1+2k}=\frac{2-k}{2k-1}$

(b) If the tangent is parallel to the x-axis, then the slope is equal to zero; therefore,

$$y'\left(-1,k\right)=0\Rightarrow\frac{2-k}{2k-1}=0\Rightarrow 2-k=0\Rightarrow k=2$$

48. Look at the diagram given with the problem.

(a) Let T_S and T_R be the swiming and running times.

Andre's swiming speed is $\frac{1}{5\sqrt{5}}$ km/min and his running speed is $\frac{1}{5}$ km/min.

$$T_S=\frac{AQ}{\frac{1}{5\sqrt{5}}}=5\sqrt{5}AQ,\text{ and } T_R=\frac{QY}{\frac{1}{5}}=5QY=5\left(2-x\right)$$

Now, since $AQ=\sqrt{4+x^2}$, then

$$T=T_S+T_R=5\sqrt{5}\sqrt{4+x^2}+5\left(2-x\right)=5\sqrt{5}\sqrt{4+x^2}+10-5x\text{ minutes}$$

(b) $\frac{\mathrm{d}T}{\mathrm{d}x}=5\sqrt{5}\frac{2x}{2\sqrt{4+x^2}}-5=\frac{5\sqrt{5}\,x}{\sqrt{4+x^2}}-5$

(c) **(i)** $\frac{\mathrm{d}T}{\mathrm{d}x}=0\Rightarrow\frac{5\sqrt{5}\,x}{\sqrt{4+x^2}}-5=0\Rightarrow 5\sqrt{5}\,x=5\sqrt{4+x^2}$

$$\Rightarrow 125x^2=100-25x^2\Rightarrow 100x^2=100\Rightarrow x=1$$

We have only one solution since the distance must be positive.

(ii) $x=1\Rightarrow T=5\sqrt{5}\sqrt{4+1}+10-5=30\text{ minutes}$

(iii) To simplify calculations, we use the product rule

$$\frac{\mathrm{d}T}{\mathrm{d}x}=5\sqrt{5}\,x\left(4+x^2\right)^{-\frac{1}{2}}-5$$

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2}=5\sqrt{5}\left(4+x^2\right)^{-\frac{1}{2}}+5\sqrt{5}\,x\times\left(-\frac{1}{2}\right)\left(4+x^2\right)^{-\frac{3}{2}}\times 2x$$

$$=\frac{5\sqrt{5}}{\left(4+x^2\right)^{\frac{3}{2}}}\left(4+x^2-x^2\right)=\frac{20\sqrt{5}}{\left(4+x^2\right)^{\frac{3}{2}}}$$

$$\frac{d^2T}{dx^2}(1)=\frac{20\sqrt{5}}{(4+1)^{\frac{3}{2}}}=\frac{20\sqrt{5}}{5\sqrt{5}}=4>0\,;$$

Therefore, the time found is a minimum.

49. In order to find the width, we need to find the intersection between the curve and the horizontal line $y=-6$ since the water depth is 10 cm.

$$16\sec\left(\frac{\pi x}{36}\right)-32=-6\Rightarrow \sec\left(\frac{\pi x}{36}\right)=\frac{13}{8}\Rightarrow \cos\left(\frac{\pi x}{36}\right)=\frac{8}{13}\Rightarrow x=\frac{36}{\pi}\arccos\left(\frac{8}{13}\right)$$

The width is twice as long, so: $w=\frac{72}{\pi}\arccos\left(\frac{8}{13}\right)$

Note: The form of answers to integration exercises may differ from one user to another. The answers usually differ by a constant. In all integration results, the constant term, mostly denoted by c, is a real number.

Exercise 14.1

1. **(a)** $\int (x+2)\,\mathrm{d}x = \frac{1}{2}x^2 + 2x + c,\ c \in \mathbb{R}$

As mentioned at the beginning, if we substitute u for $x+2$, then du = dx and the integration will result with $\int (x+2)\,\mathrm{d}x = \frac{1}{2}(x+2)^2 + c_1,\ c_1 \in \mathbb{R}$. The two answers can be consolidated by observing that $\frac{1}{2}(x+2)^2 + c_1 = \frac{1}{2}x^2 + 2x + 2 + c_1$.

By comparison you can see that $c = 2 + c_1$, which are constants.

(b) Direct application of the power rule:

$$\int (3t^2 - 2t + 1)\,\mathrm{d}x = 3\cdot\frac{1}{3}t^3 - 2\cdot\frac{1}{2}t^2 + t + c = t^3 - t^2 + t + c,\ c \in \mathbb{R}$$

(c) $$\int \left(\frac{1}{3} - \frac{2}{7}x^3\right)\mathrm{d}x = \frac{1}{3}x - \frac{2}{7}\cdot\frac{1}{4}x^4 + c = \frac{1}{3}x - \frac{1}{14}x^4 + c,\ c \in \mathbb{R}$$

(d) Simplify the integrand first and then apply power rule:

$$\int (t-1)(2t+3)\,\mathrm{d}t = \int (2t^2 + t - 3)\,\mathrm{d}t$$

$$= 2\cdot\frac{1}{3}t^3 + \frac{1}{2}t^2 - 3t + c = \frac{2}{3}t^3 + \frac{1}{2}t^2 - 3t + c,\ c \in \mathbb{R}$$

(e) $$\int \left(u^{\frac{2}{5}} - 4u^3\right)\mathrm{d}u = \frac{1}{\frac{2}{5}+1}u^{\frac{2}{5}+1} - 4\cdot\frac{1}{4}u^4 + c = \frac{5}{7}u^{\frac{7}{5}} - u^4 + c,\ c \in \mathbb{R}$$

(f) $$\int \left(2\sqrt{x} - \frac{3}{2\sqrt{x}}\right)\mathrm{d}x = \int \left(2x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$= 2\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{3}{2}\cdot\frac{1}{\frac{1}{2}}x^{\frac{1}{2}} + c = \frac{4}{3}x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + c,\ c \in \mathbb{R}$$

(g) $$\int(3\sin\theta+4\cos\theta)\,\mathrm{d}\theta=3(-\cos\theta)+4\sin\theta+c=-3\cos\theta+4\sin\theta+c,c\in\mathbb{R}$$

(h) $$\int(3t^2-2\sin t)\,\mathrm{d}t=3\cdot\frac{1}{3}t^3-2\cdot(-\cos t)+c=t^3+2\cos t+c,c\in\mathbb{R}$$

(i) $$\int\sqrt{x}(2x-5)\,\mathrm{d}x=\int\left(2x^{\frac{3}{2}}-5x^{\frac{1}{2}}\right)\mathrm{d}x=2\cdot\frac{1}{\frac{5}{2}}x^{\frac{5}{2}}-5\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}}+c=\frac{4}{5}x^{\frac{5}{2}}-\frac{10}{3}x^{\frac{3}{2}}+c,c\in\mathbb{R}$$

(j) $$\int(3\cos\theta-2\sec^2\theta)\,\mathrm{d}\theta=3\sin\theta-2\tan\theta+c,c\in\mathbb{R}$$

(k) $$\int\mathrm{e}^{3t-1}\,\mathrm{d}t=\int\frac{1}{3}\mathrm{e}^{3t-1}\,\mathrm{d}(3t-1)=\frac{1}{3}\mathrm{e}^{3t-1}+c,c\in\mathbb{R}$$

(l) $$\int\frac{2}{t}\,\mathrm{d}t=2\int\frac{1}{t}\,\mathrm{d}t=2\ln|t|+c,c\in\mathbb{R}$$

(m) $$\int\frac{t}{3t^2+5}\,\mathrm{d}t=\int\frac{1}{6}\frac{6t\,\mathrm{d}t}{3t^2+5}=\frac{1}{6}\int\frac{\mathrm{d}(3t^2+5)}{3t^2+5}=\frac{1}{6}\ln(3t^2+5)+c,c\in\mathbb{R}$$

(n) $$\int\mathrm{e}^{\sin\theta}\cos\theta\,\mathrm{d}\theta=\int\mathrm{e}^{\sin\theta}\,\mathrm{d}(\sin\theta)=\mathrm{e}^{\sin\theta}+c,\ c\in\mathbb{R}$$

(o) $$\int(3+2x)^2\,\mathrm{d}x=\int\frac{1}{2}(3+2x)^2\,\mathrm{d}(3+2x)=\frac{1}{2}\times\frac{1}{3}(3+2x)^3+c=\frac{1}{6}(3+2x)^3+c,\ c\in\mathbb{R}$$

2. **(a)** $$f'(x)=\int(4x-15x^2)\,\mathrm{d}x=4\times\frac{1}{2}x^2-15\times\frac{1}{3}x^3+c=2x^2-5x^3+c,\ c\in\mathbb{R},$$

Integrate again,

$$f(x)=\int(2x^2-5x^3+c)\,\mathrm{d}x=2\times\frac{1}{3}x^3-5\times\frac{1}{4}x^4+cx+k$$
$$=\frac{2}{3}x^3-\frac{5}{4}x^4+cx+k,\ c,k\in\mathbb{R}$$

(b) $$f'(x)=\int(1+3x^2-4x^3)\,\mathrm{d}x=x+3\times\frac{1}{3}x^3-4\times\frac{1}{4}x^4+c=x+x^3-x^4+c,\ c\in\mathbb{R}$$

With the initial condition that $f\,'(0)=2$, we can calculate the constant c:

$f'(0)=2 \Rightarrow 0+c=2 \Rightarrow c=2 \Rightarrow f(x)=2+x+x^3-x^4$. Hence,

$$f(x)=\int\left(2+x+x^3-x^4\right)\mathrm{d}x=2x+\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^5}{5}+k, k\in\mathbb{R}.$$

Also, with the initial condition $f(1)=2$, we can calculate the constant k:

$$f(1)=2+\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+k=2 \Rightarrow k=-\frac{11}{20} \Rightarrow f(x)=-\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^2}{2}+2x-\frac{11}{20}$$

(c) $$f\,'(t)=\int(8t-\sin t)\,\mathrm{d}t=8\times\frac{1}{2}t^2-(-\cos t)+c=4t^2+\cos t+c, c\in\mathbb{R}$$

$$f(t)=\int\left(4t^2+\cos t+c\right)\mathrm{d}t=\frac{4}{3}t^3+\sin t+ct+k; c,k\in\mathbb{R}$$

(d) $$f(x)=\int\left(12x^3-8x+7\right)\mathrm{d}x=12\times\frac{1}{4}x^4-8\times\frac{1}{2}x^2+7x+c=3x^4-4x^2+7x+c,\ c\in\mathbb{R}$$

With the initial condition $f(0)=3$, we can calculate the constant c:

$$f(0)=3 \Rightarrow 0+c=3 \Rightarrow c=3 \Rightarrow f(x)=3x^4-4x^2+7x+3$$

(e) $$f(\theta)=\int(2\cos\theta-\sin(2\theta))\,\mathrm{d}\theta=2\sin\theta-\int\sin(2\theta)\times\frac{1}{2}\mathrm{d}(2\theta)+c$$

$$=2\sin\theta-\frac{1}{2}(-\cos(2\theta))+c=2\sin\theta+\frac{1}{2}\cos(2\theta)+c, c\in\mathbb{R}$$

3. **(a)** Use the substitution $u=3x^2+7 \Rightarrow \mathrm{d}u=6x\,\mathrm{d}x \Rightarrow x\,\mathrm{d}x=\frac{1}{6}\mathrm{d}u$

$$\int x\left(3x^2+7\right)^5\mathrm{d}x=\int u^5\times\frac{1}{6}\mathrm{d}u=\frac{1}{6}\times\frac{u^6}{6}+c=\frac{\left(3x^2+7\right)^6}{36}+c$$

(b) Use the substitution $u=3x^2+5 \Rightarrow \mathrm{d}u=6\mathrm{d}x$

$$\int\frac{x}{\left(3x^2+5\right)^4}\mathrm{d}x=\int u^{-4}\times\frac{1}{6}\mathrm{d}u=\frac{1}{6}\times\frac{u^{-3}}{-3}+c=-\frac{1}{18\left(3x^2+5\right)^3}+c$$

(c) Use the substitution $u = 5x^3 + 2 \Rightarrow \mathrm{d}u = 15x^2\mathrm{d}x$

$$\int 2x^2\sqrt[4]{5x^3+2}\,\mathrm{d}x = \int u^{\frac{1}{4}} \times \frac{2}{15}\mathrm{d}u = \frac{2}{15} \times \frac{u^{\frac{5}{4}}}{\frac{5}{4}} + c = \frac{8\sqrt[4]{\left(5x^3+2\right)^5}}{75} + c$$

Note: in the rest of the exercises, substitutions are implied by the work and not given directly. Some 'obvious' intermediate steps are also not included.

(d) $$\int \frac{\left(3+2\sqrt{x}\right)^5}{\sqrt{x}}\mathrm{d}x = \int u^5\mathrm{d}u = \frac{\left(3+2\sqrt{x}\right)^6}{6} + c$$

(e) $$\int t^2\sqrt{2t^3-7}\,\mathrm{d}t = \int u^{\frac{1}{2}} \times \frac{1}{6}\mathrm{d}u = \frac{1}{6} \times \frac{\left(2t^3-7\right)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{\sqrt{\left(2t^3-7\right)^3}}{9} + c$$

(f) $$\int \left(2+\frac{3}{x}\right)^5\left(\frac{1}{x^2}\right)\mathrm{d}x = \int u^5 \times (-3)\,\mathrm{d}u = -\frac{\left(2+\frac{3}{x}\right)^6}{18} + c = -\frac{(2x+3)^6}{18x^6} + c$$

(g) $$\int \sin(7x-3)\,\mathrm{d}x = \int \sin u \times \frac{1}{7}\mathrm{d}u = \frac{1}{7} \times \left(-\cos(7x-3)\right) + c = -\frac{\cos(7x-3)}{7} + c$$

(h) $$\int \frac{\sin(2\theta-1)}{\cos(2\theta-1)+3}\mathrm{d}\theta = \int \frac{1}{u} \times \left(-\frac{1}{2}\right)\mathrm{d}u = -\frac{1}{2}\ln\left(\cos(2\theta-1)+3\right) + c$$

(i) $$\int \sec^2(5\theta-2)\,\mathrm{d}\theta = \int \sec^2 u \times \frac{1}{5}\mathrm{d}u = \frac{\tan(5\theta-2)}{5} + c$$

(j) $$\int \cos(\pi x+3)\,\mathrm{d}x = \int \cos u \times \frac{1}{\pi}\mathrm{d}u = \frac{1}{\pi}\sin(\pi x+3) + c$$

(k) $\int \sec 2t \tan 2t\,\mathrm{d}t = \int \sec u \tan u \times \frac{1}{2}\mathrm{d}u = \frac{1}{2} \times \sec 2t + c$, alternatively

$$\int \frac{\sin 2t}{\cos^2 2t}\mathrm{d}t = \int u^{-2} \times -\frac{1}{2}\mathrm{d}u = \frac{1}{2} \times \frac{1}{(\cos 2t)} + c$$

(l) $$\int x\mathrm{e}^{x^2+1}\mathrm{d}x=\int \mathrm{e}^u\times\frac{1}{2}\mathrm{d}u=\frac{1}{2}\mathrm{e}^{x^2+1}+c$$

(m) $$\int\sqrt{t}\mathrm{e}^{2t\sqrt{t}}\mathrm{d}t=\int t^{\frac{1}{2}}\mathrm{e}^{2t^{\frac{3}{2}}}\mathrm{d}t=\int\mathrm{e}^u\times\frac{1}{3}\mathrm{d}u=\frac{1}{3}\mathrm{e}^{2t^{\frac{3}{2}}}+c=\frac{1}{3}\mathrm{e}^{2t\sqrt{t}}+c$$

(n) $$\int\frac{2}{\theta}(\ln\theta)^2\,\mathrm{d}\theta=\int u^2\times 2\mathrm{d}u=\frac{2(\ln\theta)^3}{3}+c$$

4. **(a)** $$\int t\sqrt{3-5t^2}\,\mathrm{d}t=\int u^{\frac{1}{2}}\times\left(-\frac{1}{10}\right)\mathrm{d}u=-\frac{1}{10}\times\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c=-\frac{\sqrt{\left(3-5t^2\right)^3}}{15}+c$$

(b) $$\int\theta^2\sec^2\theta^3\mathrm{d}\theta=\int\sec^2 u\times\frac{1}{3}\mathrm{d}u=\frac{1}{3}\tan\theta^3+c$$

(c) $$\int\frac{\sin\sqrt{t}}{2\sqrt{t}}\mathrm{d}t=\int\sin u\times\mathrm{d}u=-\cos\left(\sqrt{t}\right)+c$$

(d) $$\int\tan^5 2t\sec^2 2t\,\mathrm{d}t=\int u^5\times\frac{1}{2}\mathrm{d}u=\frac{1}{2}\frac{u^6}{6}+c=\frac{\tan^6 2t}{12}+c$$

(e) $$\int\frac{\mathrm{d}x}{\sqrt{x}\left(\sqrt{x}+2\right)}=\int\frac{1}{u}\times 2\mathrm{d}u=2\ln u+c=2\ln\left(\sqrt{x}+2\right)+c$$

(f) $$\int\sec^5 2t\tan 2t\,\mathrm{d}t=\int(\sec 2t)^4\sec 2t\tan 2t\,\mathrm{d}t=\int u^4\times\frac{1}{2}\mathrm{d}u=\frac{\sec^5 2t}{10}+c$$

(g) $$\int\frac{x+3}{x^2+6x+7}\mathrm{d}x=\int\frac{1}{u}\times\frac{1}{2}\mathrm{d}u=\frac{1}{2}\ln\left|x^2+6x+7\right|+c$$

(h) $$\int\frac{k^3x^3}{\sqrt{a^2-a^4x^4}}\mathrm{d}x=\int u^{-\frac{1}{2}}\times\left(-\frac{k^3}{4a^4}\right)\mathrm{d}u=-\frac{k^3}{4a^4}\times\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c=-\frac{k^3\sqrt{a^2-a^4x^4}}{2a^4}+c$$

$$=-\frac{k^3|a|\sqrt{1-a^2x^4}}{2a^4}+c=-\frac{k^3\sqrt{1-a^2x^4}}{2|a|^3}+c$$

(i) In this question, we are using a slightly different method of substitution, since a direct one may lead to a more complex setup.

$$x-1=t\Rightarrow x=t+1\Rightarrow \mathrm{d}x=\mathrm{d}t$$

$$\int 3x\sqrt{x-1}\,\mathrm{d}x=\int 3(t+1)\sqrt{t}\mathrm{d}t=3\int\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)\mathrm{d}t=3\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}}+\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+c=\frac{2}{5}t^{\frac{3}{2}}(3t+5)+c$$

$$=\frac{2}{5}\sqrt{x-1}\left(3(x-1)^2+5(x-1)\right)+c=\frac{2}{5}\left(3x^2-x-2\right)\sqrt{x-1}+c$$

(j) $$\int\csc^2\pi t\,\mathrm{d}t=\int\csc^2 u\times\frac{1}{\pi}\mathrm{d}u=-\frac{1}{\pi}\cot(\pi t)+c$$

(k) $$\int\sqrt{1+\cos\theta}\sin\theta\,\mathrm{d}\theta=\int u^{\frac{1}{2}}\times(-\mathrm{d}u)=-\frac{(1+\cos\theta)^{\frac{3}{2}}}{\frac{3}{2}}+c=-\frac{2}{3}\sqrt{(1+\cos\theta)^3}+c$$

In questions **(l)** – **(n)** we use two-stage methods of substitution.

(l) Start with the substitution $1-t=u\Rightarrow 1-u=t\Rightarrow \mathrm{d}t=-\mathrm{d}u$

$$\Rightarrow\int t^2\sqrt{1-t}\,\mathrm{d}t=\int(1-u)^2\sqrt{u}(-\mathrm{d}u)=-\int\left(u^{\frac{1}{2}}-2u^{\frac{3}{2}}+u^{\frac{5}{2}}\right)\mathrm{d}u$$

$$=-\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+2\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{7}{2}}}{\frac{7}{2}}+c=\frac{2}{105}u^{\frac{1}{2}}\left(-35u+42u^2-15u^3\right)+c$$

$$=\frac{2\sqrt{1-t}}{105}\left(-35(1-t)+42(1-t)^2-15(1-t)^3\right)+c$$

$$=\frac{2\sqrt{1-t}}{105}(t-1)\left(15t^2+12t+8\right)+c$$

(m) Start with the substitution $2r-1=x \Rightarrow r=\frac{x+1}{2} \Rightarrow \mathrm{d}r=\frac{\mathrm{d}x}{2}$

$$\int \frac{r^2-1}{\sqrt{2r-1}}\,\mathrm{d}r=\int \frac{\left(\frac{x+1}{2}\right)^2-1}{\sqrt{x}}\frac{\mathrm{d}x}{2}=\frac{1}{8}\int\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$=\frac{1}{8}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-3\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)+c=\frac{1}{60}x^{\frac{1}{2}}\left(3x^2+10x-45\right)+c$$

$$=\frac{1}{60}\sqrt{2r-1}\left(3(2r-1)^2+10(2r-1)-45\right)+c$$

$$=\frac{\sqrt{2r-1}}{15}\left(3r^2+2r-13\right)+c$$

(n) Substitution: $\mathrm{e}^{x^2}+\mathrm{e}^{-x^2}=t \Rightarrow \left(2x\mathrm{e}^{x^2}-2x\mathrm{e}^{-x^2}\right)\mathrm{d}x=\mathrm{d}t$

$$\int\frac{\mathrm{e}^{x^2}-\mathrm{e}^{-x^2}}{\mathrm{e}^{x^2}+\mathrm{e}^{-x^2}}x\,\mathrm{d}x=\int\frac{1}{t}\frac{\mathrm{d}t}{2}=\frac{1}{2}\ln t+c=\frac{1}{2}\ln\left(\mathrm{e}^{x^2}+\mathrm{e}^{-x^2}\right)+c$$

Exercise 14.2

As a rule, we'll use an expression for *u* that is not too difficult to differentiate, and for *dv* an expression that is relatively easy to integrate. The new integral should be simpler than the original integral.

1. **(a)** To evaluate this integral we will use a direct method of substitution.

$$-x^3=t \Rightarrow -3x^2\mathrm{d}x=\mathrm{d}t \Rightarrow x^2\mathrm{d}x=-\frac{1}{3}\mathrm{d}t$$

$$\int x^2\,\mathrm{e}^{-x^3}\,\mathrm{d}x=\int \mathrm{e}^t\times\left(-\frac{1}{3}\mathrm{d}t\right)=-\frac{1}{3}\mathrm{e}^t+c=-\frac{1}{3}\mathrm{e}^{-x^3}+c$$

(b) We apply integration by parts twice here.

$$u = x^2 \qquad \mathrm{d}v = \mathrm{e}^{-x}\mathrm{d}x$$
$$\mathrm{d}u = 2x\mathrm{d}x \qquad v = -\mathrm{e}^{-x}$$
$$\Rightarrow \int x^2\mathrm{e}^{-x}\mathrm{d}x = -x^2\mathrm{e}^{-x} - \int -\mathrm{e}^{-x}2x\mathrm{d}x = -x^2\mathrm{e}^{-x} + 2\int x\mathrm{e}^{-x}\mathrm{d}x$$

Now, apply by parts to $2\int x\mathrm{e}^{-x}\mathrm{d}x$

$$\begin{array}{ll} u = x & \mathrm{d}v = \mathrm{e}^{-x}\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = -\mathrm{e}^{-x} \end{array} \Rightarrow 2\int x\mathrm{e}^{-x}\mathrm{d}x = 2\left(-x\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\mathrm{d}x\right) = -2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x}$$
$$\Rightarrow \int x^2\mathrm{e}^{-x}\mathrm{d}x = -x^2\mathrm{e}^{-x} - 2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x} + c = -\mathrm{e}^{-x}\left(x^2 + 2x + 2\right) + c$$

(c) Integration by parts – twice

$$u = x^2 \qquad \mathrm{d}v = \cos 3x\,\mathrm{d}x$$
$$\mathrm{d}u = 2x\mathrm{d}x \qquad v = \frac{1}{3}\sin 3x$$
$$\int x^2\cos 3x\,\mathrm{d}x = \frac{1}{3}x^2\sin 3x - \int \frac{2}{3}x\sin 3x\,\mathrm{d}x$$

Now apply by parts to $\int x\sin 3x\,\mathrm{d}x$

$$u = x \qquad \mathrm{d}v = \sin 3x\mathrm{d}x$$
$$\mathrm{d}u = \mathrm{d}x \qquad v = -\frac{1}{3}\cos 3x$$
$$\int x\sin 3x\,\mathrm{d}x = -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,\mathrm{d}x = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x$$

and hence

$$\begin{aligned}\int x^2\cos 3x\,\mathrm{d}x &= \frac{1}{3}x^2\sin 3x - \int \frac{2}{3}x\sin 3x\,\mathrm{d}x \\ &= \frac{1}{3}x^2\sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right) \\ &= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + c\end{aligned}$$

(d) Integration by parts

$$u = x^2 \qquad \mathrm{d}v = \sin ax\,\mathrm{d}x$$

$$\mathrm{d}u = 2x\mathrm{d}x \quad v = -\frac{1}{a}\cos ax$$

$$\int x^2 \sin ax\,\mathrm{d}x = -\frac{1}{a}x^2\cos ax + \frac{2}{a}\int x\cos ax\,\mathrm{d}x$$

We now apply by parts again to $\int x\cos ax\,\mathrm{d}x$

$$u = x \qquad \mathrm{d}v = \cos ax\,\mathrm{d}x$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{a}\sin ax$$

$$\int x\cos ax\,\mathrm{d}x = \frac{1}{a}x\sin ax - \int\frac{1}{a}\sin ax\,\mathrm{d}x = \frac{1}{a}x\sin ax + \frac{1}{a^2}\cos ax$$

$$\int x^2\sin ax\,\mathrm{d}x = -\frac{1}{a}x^2\cos ax + \frac{2}{a}\left(\frac{1}{a}x\sin ax + \frac{1}{a^2}\cos ax\right)$$

$$= -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + c$$

(e) We start integration by parts with the following substitution

$$u = \ln(\sin x) \qquad \mathrm{d}v = \cos x\,\mathrm{d}x$$

$$\mathrm{d}u = \frac{1}{\sin x}\cos x\,\mathrm{d}x \qquad v = \sin x$$

$$\int \cos x\ln(\sin x)\,\mathrm{d}x = \sin x\ln(\sin x) - \int \sin x\cdot\frac{1}{\sin x}\cos x\,\mathrm{d}x = \sin x\ln(\sin x) - \int\cos x\,\mathrm{d}x$$

$$= \sin x\ln(\sin x) - \sin x + c = \sin x\left(\ln(\sin x) - 1\right) + c$$

(f) Again, integration by parts

$$u = \ln x^2 = 2\ln x \quad \mathrm{d}v = x\mathrm{d}x$$

$$\mathrm{d}u = \frac{2}{x}\mathrm{d}x \qquad v = \frac{1}{2}x^2$$

$$\int x\ln x^2\mathrm{d}x = x^2\ln x - \int x\mathrm{d}x = x^2\ln x - \frac{x^2}{2} + c = \frac{x^2}{2}(2\ln x - 1) + c$$

(g) Again, integration by parts

$$u = \ln x \qquad dv = x^2 dx$$

$$du = \frac{1}{x}dx \qquad v = \frac{1}{3}x^3$$

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x} dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3} \times \frac{x^3}{3} + c = \frac{x^3}{9}(3\ln x - 1) + c$$

(h) We first use the distribution property and split the integrals. We can also apply by parts directly.

$$\int x^2 (e^x - 1) dx = \int x^2 e^x dx - \int x^2 dx = \int x^2 e^x dx - \frac{x^3}{3}$$

$$u = x^2 \qquad dv = e^x dx$$

$$du = 2x \, dx \qquad v = e^x$$

Using by parts on the first integral: $\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$

Now using integration by parts on $\int xe^x dx$ and substituting back:

$$u = x \qquad dv = e^x dx$$

$$du = dx \qquad v = e^x$$

$$\int x^2 e^x dx - \frac{x^3}{3} = x^2 e^x - 2\left(xe^x - \int e^x dx\right) - \frac{x^3}{3}$$

$$= x^2 e^x - 2xe^x + 2e^x - \frac{x^3}{3} + c = e^x\left(x^2 - 2x + 2\right) - \frac{x^3}{3} + c$$

(i) By parts

$$u = x \qquad dv = \cos \pi x \, dx$$

$$du = dx \qquad v = \frac{1}{\pi}\sin \pi x$$

$$\int x \cos \pi x \, dx = \frac{1}{\pi} x \sin \pi x - \frac{1}{\pi}\int \sin \pi x \, dx$$

$$= \frac{1}{\pi} x \sin \pi x - \frac{1}{\pi}\left(-\frac{1}{\pi}\cos \pi x\right) + c = \frac{1}{\pi^2}(\pi x \sin \pi x + \cos \pi x) + c$$

(j) By parts twice with a slight variation. Denote the original integral by *I*.

$$u = e^{3t} \qquad dv = \cos 2t\,dt$$

$$du = 3e^{3t}dt \qquad v = \frac{1}{2}\sin 2t$$

$$I = \int e^{3t}\cos 2t\,dt = \frac{1}{2}e^{3t}\sin 2t - \frac{3}{2}\int \sin 2t\,e^{3t}dt$$

Use integration by parts again.

$$u = e^{3t} \qquad dv = \sin 2t\,dt$$

$$du = 3e^{3t}dt \qquad v = -\frac{1}{2}\cos 2t$$

$$I = \frac{1}{2}e^{3t}\sin 2t - \frac{3}{2}\left(-\frac{1}{2}e^{3t}\cos 2t + \frac{3}{2}\int e^{3t}\cos 2t\,dt\right) = \frac{1}{2}e^{3t}\sin 2t + \frac{3}{4}e^{3t}\cos 2t - \frac{9}{4}I$$

Notice that we have obtained the original integral, so we will solve this equation for the integral.

$$I = \frac{1}{2}e^{3t}\sin 2t + \frac{3}{4}e^{3t}\cos 2t - \frac{9}{4}I \Rightarrow \frac{13}{4}I = \frac{1}{2}e^{3t}\sin 2t + \frac{3}{4}e^{3t}\cos 2t$$

$$\Rightarrow I = \int e^{3t}\cos 2t\,dt = \frac{1}{13}e^{3t}\left(2\sin 2t + 3\cos 2t\right) + c$$

(k) By parts first, then using direct substitution.

$$u = \arcsin x \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$$

$$\int \arcsin x dx = x\arcsin x - \int \frac{xdx}{\sqrt{1-x^2}} = x\arcsin x + \frac{1}{2}\cdot\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x\arcsin x + \sqrt{1-x^2} + c$$

(l) This exercise will need integration by parts three times.

$$u = x^3 \qquad dv = e^x dx$$

$$du = 3x^2 dx \qquad v = e^x$$

$$\int x^3 e^x dx = x^3 e^x - 3\int x^2 e^x dx$$

Now with the substitution:

$$u = x^2 \qquad dv = e^x dx$$
$$du = 2x dx \qquad v = e^x$$
$$x^3e^x - 3\int x^2e^x dx = x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right)$$
$$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right)$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c = e^x\left(x^3 - 3x^2 + 6x - 6\right) + c$$

(m) This exercise will need integration by parts twice. Call the first integral I.

$$u = e^{-2x} \qquad dv = \sin 2x\,dx$$
$$du = -2e^{-2x}dx \qquad v = -\frac{1}{2}\cos 2x$$
$$I = \int e^{-2x}\sin 2x\,dx = -\frac{1}{2}e^{-2x}\cos 2x - \int e^{-2x}\cos 2x\,dx$$

By parts again

$$u = e^{-2x} \qquad dv = \cos 2x\,dx$$
$$du = -2e^{-2x}dx \qquad v = \frac{1}{2}\sin 2x$$
$$\int e^{-2x}\cos 2x\,dx = \frac{1}{2}e^{-2x}\sin 2x + \int e^{-2x}\sin 2x\,dx$$
$$\Rightarrow I = -\frac{1}{2}e^{-2x}\cos 2x - \frac{1}{2}e^{-2x}\sin 2x - I$$
$$\Rightarrow 2I = -\frac{1}{2}e^{-2x}\left(\cos 2x + \sin 2x\right)$$
$$\Rightarrow \int e^{-2x}\sin 2x\,dx = -\frac{1}{4}e^{-2x}\left(\cos 2x + \sin 2x\right) + c$$

(n) By parts twice

$$u = \sin(\ln x) \qquad dv = dx$$
$$du = \cos(\ln x)\frac{dx}{x} \qquad v = x$$
$$I = \int \sin(\ln x)dx = x\sin(\ln x) - \int x\cos(\ln x)\frac{dx}{x} = x\sin(\ln x) - \int \cos(\ln x)dx$$

New substitution and result substituted back

$$u=\cos(\ln x) \qquad \mathrm{d}v=\mathrm{d}x$$

$$\mathrm{d}u=-\sin(\ln x)\frac{\mathrm{d}x}{x} \qquad v=x$$

$$I=x\sin(\ln x)-\left(x\cos(\ln x)+\int x\sin(\ln x)\frac{\mathrm{d}x}{x}\right)=x\sin(\ln x)-x\cos(\ln x)-I$$

$$\Rightarrow 2I=x\sin(\ln x)-x\cos(\ln x)$$

$$\Rightarrow I=\int\sin(\ln x)\mathrm{d}x=\frac{1}{2}x\big(\sin(\ln x)-\cos(\ln x)\big)+c$$

(o) By parts twice

$$u=\cos(\ln x) \qquad \mathrm{d}v=\mathrm{d}x$$

$$\mathrm{d}u=-\sin(\ln x)\frac{\mathrm{d}x}{x} \qquad v=x$$

$$\int\cos(\ln x)\mathrm{d}x=x\cos(\ln x)+\int\sin(\ln x)\mathrm{d}x$$

New substitution and result substituted back:

$$u=\sin(\ln x) \qquad \mathrm{d}v=\mathrm{d}x$$

$$\mathrm{d}u=\cos(\ln x)\frac{\mathrm{d}x}{x} \qquad v=x$$

$$\Rightarrow\int\cos(\ln x)\mathrm{d}x=x\cos(\ln x)+\left(x\sin(\ln x)-\int\cos(\ln x)\mathrm{d}x\right)$$

$$\Rightarrow 2\int\cos(\ln x)\mathrm{d}x=x\cos(\ln x)+x\sin(\ln x)$$

$$\Rightarrow\int\cos(\ln x)\mathrm{d}x=\frac{1}{2}x\big(\cos(\ln x)+\sin(\ln x)\big)+c$$

(p) By parts

$$u=\ln\left(x+x^2\right) \qquad \mathrm{d}v=\mathrm{d}x$$

$$\mathrm{d}u=\frac{1+2x}{x+x^2}\mathrm{d}x \qquad v=x$$

$$\int\ln\left(x+x^2\right)\mathrm{d}x=x\ln\left(x+x^2\right)-\int\frac{x+2x^2}{x+x^2}\mathrm{d}x=x\ln\left(x+x^2\right)-\int\frac{1+2x}{1+x}\mathrm{d}x$$

$$=x\ln\left(x+x^2\right)-\int\left(2-\frac{1}{1+x}\right)\mathrm{d}x=x\ln\left(x+x^2\right)-2x+\ln|1+x|+c$$

(q) By parts twice

$u = e^{kx} \qquad dv = \sin x\,dx$

$du = ke^{kx}dx \qquad v = -\cos x$

$$\int e^{kx}\sin x\,dx = -e^{kx}\cos x + k\int e^{kx}\cos x\,dx$$

By parts again

$u = e^{kx} \qquad dv = \cos x\,dx$

$du = ke^{kx}dx \qquad v = \sin x$

$$\int e^{kx}\sin x\,dx = -e^{kx}\cos x + k\left(e^{kx}\sin x - k\int e^{kx}\sin x\,dx\right)$$

$$= -e^{kx}\cos x + ke^{kx}\sin x - k^2\int e^{kx}\sin x\,dx$$

$$\Rightarrow \left(1+k^2\right)\int e^{kx}\sin x\,dx = e^{kx}\left(k\sin x - \cos x\right)$$

$$\Rightarrow \int e^{kx}\sin x\,dx = \frac{e^{kx}\left(k\sin x - \cos x\right)}{1+k^2} + c$$

(r) By parts

$u = x \qquad dv = \sec^2 x\,dx$

$du = dx \qquad v = \tan x$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}dx$$

$$= x\tan x - \int \frac{-d\left(\cos x\right)}{\cos x} = x\tan x + \ln\left|\cos x\right| + c$$

(s) By parts twice

$u = \sin 2x \qquad dv = \sin x\,dx$

$du = 2\cos 2x\,dx \qquad v = -\cos x$

$$\int \sin x\sin 2x\,dx = -\sin 2x\cos x + 2\int \cos x\cos 2x\,dx$$

By parts again

$u = \cos 2x \qquad dv = \cos x\,dx$

$du = -2\sin 2x\,dx \qquad v = \sin x$

$$\int \sin x\sin 2x\,dx = -\sin 2x\cos x + 2\left(\cos 2x\sin x + 2\int \sin x\sin 2x\,dx\right)$$

$$= -\sin 2x\cos x + 2\cos 2x\sin x + 4\int \sin x\sin 2x\,dx$$

$$\Rightarrow -3\int \sin x\sin 2x\,dx = -\sin 2x\cos x + 2\cos 2x\sin x$$

$$\Rightarrow \int \sin x\sin 2x\,dx = \frac{1}{3}\sin 2x\cos x - \frac{2}{3}\cos 2x\sin x + c$$

We obtained this solution by using integration by parts. A much simpler solution can be found by using trigonometric identities and simple substitution.

$$\int \sin x \overset{2\sin x\cos x}{\sin 2x}\,\mathrm{d}x = 2\int \sin^2 x \cos x\,\mathrm{d}x = 2\int u^2\,\mathrm{d}u = \frac{2}{3}\sin^3 x + c$$

Notice that the solutions of trigonometric integrals can be written in different forms, but we can show that all of these forms are mutually equivalent or that they differ only by a constant, and can be shown by using a series of trigonometric identities. Looking at the answer given in the book and the ones here, you can demonstrate that they are equal.

(t) By parts

$$u = \arctan x \quad \mathrm{d}v = x\,\mathrm{d}x$$

$$\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2} \quad v = \frac{1}{2}x^2$$

$$\int x\arctan x\,\mathrm{d}x = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int\frac{x^2\mathrm{d}x}{1+x^2} = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + c = \frac{1}{2}\left(\left(x^2+1\right)\arctan x - x\right) + c$$

Again, the form here is different from the book's answer, but they are equivalent.

(u) By parts

$$u = \ln x \quad \mathrm{d}v = \frac{\mathrm{d}x}{\sqrt{x}}$$

$$\mathrm{d}u = \frac{\mathrm{d}x}{x} \quad v = 2\sqrt{x}$$

$$\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x}\ln x - 2\int x^{\frac{1}{2}}\frac{\mathrm{d}x}{x} = 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\mathrm{d}x$$

$$= 2\sqrt{x}\ln x - 4\sqrt{x} + c = 2\sqrt{x}\left(\ln x - 2\right) + c$$

2.

u	dv	sign
	$\sin x\,dx$	
x^2	$-\cos x$	+
$2x$	$-\sin x$	–
2	$\cos x$	+

Notice that in the first column we have a sequence of derivatives of u and in the second column the sequence of antiderivatives of dv. The reason why the signs alternate is a direct consequence of the formula where the second integral always takes a minus.

$$\int x^2 \sin x\mathrm{d}x = -x^2 \cos x - \int -\cos x\, 2x\,\mathrm{d}x = -x^2 \cos x - \left(2x(-\sin x) - \int(-\sin x)2\mathrm{d}x\right)$$

$$= -x^2 \cos x + 2x\sin x - 2\cos x + c$$

3. **(a)**

	$\sin x$	
x^4	$-\cos x$	+
$4x^3$	$-\sin x$	–
$12x^2$	$\cos x$	+
$24x$	$\sin x$	–
24	$-\cos x$	+

$$\int x^4 \sin x\,\mathrm{d}x = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x\sin x - 24\cos x + c$$

(b)

	$\cos x$	
x^5	$\sin x$	+
$5x^4$	$-\cos x$	–
$20x^3$	$-\sin x$	+
$60x^2$	$\cos x$	–
$120x$	$\sin x$	+
120	$-\cos x$	–

$$\int x^5 \cos x\,\mathrm{d}x = x^5 \sin x + 5x^4 \cos x - 20x^3 \sin x - 60x^2 \cos x + 120x\sin x + 120\cos x + c$$

(c)

	e^x	
x^4	e^x	+
$4x^3$	e^x	–
$12x^2$	e^x	+
$24x$	e^x	–
24	e^x	+

$$\int x^4 e^x \, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c$$
$$= e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + c$$

4. The method used in question **2** cannot give the result because, in the second column, we have antiderivatives and the antiderivative of the natural logarithm is a more complex logarithmic integrand, which in fact needs integration by parts. $\int \ln x \, dx = x(\ln x - 1)$.
Finding an antiderivative would further complicate the integration.
Note: The logarithmic function is always taken as *u* since we need to differentiate it.

5. Use by parts

$$u = x^n \qquad dv = e^x dx$$
$$du = nx^{n-1}dx \qquad v = e^x$$
$$\int x^n e^x dx = x^n e^x - \int nx^{n-1} e^x dx = x^n e^x - n\int x^{n-1} e^x dx$$
$$\int x^4 e^x \, dx = x^4 e^x - 4\int x^3 e^x \, dx = x^4 e^x - 4\left(x^3 e^x - 3\int x^2 e^x \, dx\right)$$
$$= x^4 e^x - 4x^3 e^x + 12\left(x^2 e^x - 2\int xe^x \, dx\right) = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\left(xe^x - \int e^x \, dx\right)$$
$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c$$

6. By parts

$$u = \ln x \qquad dv = x^n dx$$
$$du = \frac{1}{x}dx \quad v = \frac{1}{n+1}x^{n+1}$$

$$\int x^n \ln x \, \mathrm{d}x = \frac{1}{n+1}x^{n+1}\ln x - \int \frac{1}{n+1}x^{n+1} \times \frac{1}{x}\mathrm{d}x$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1} \times \frac{x^{n+1}}{n+1} + c = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

7. By parts

$u = \mathrm{e}^{mx}$ $\quad \mathrm{d}v = \cos nx \, \mathrm{d}x$

$\mathrm{d}u = m\mathrm{e}^{mx}\mathrm{d}x$ $\quad v = \frac{1}{n}\sin nx$

$$\int \mathrm{e}^{mx} \cos nx \, \mathrm{d}x = \frac{1}{n}\mathrm{e}^{mx} \sin nx - \frac{m}{n}\int \sin nx \, \mathrm{e}^{mx}\mathrm{d}x$$

Another integration by parts step:

$u = \mathrm{e}^{mx}$ $\quad \mathrm{d}v = \sin nx \, \mathrm{d}x$

$\mathrm{d}u = m\mathrm{e}^{mx}\mathrm{d}x$ $\quad v = -\frac{1}{n}\cos nx$

$$\int \mathrm{e}^{mx} \cos nx \, \mathrm{d}x = \frac{1}{n}\mathrm{e}^{mx} \sin nx - \frac{m}{n}\left(-\frac{1}{n}\mathrm{e}^{mx} \cos nx + \frac{m}{n}\int \cos nx \, \mathrm{e}^{mx}\mathrm{d}x\right)$$

$$= \frac{1}{n}\mathrm{e}^{mx} \sin nx + \frac{m}{n^2}\mathrm{e}^{mx} \cos nx - \frac{m^2}{n^2}\int \cos nx \, \mathrm{e}^{mx}\mathrm{d}x$$

$$\Rightarrow \left(1 + \frac{m^2}{n^2}\right)\int \cos nx \, \mathrm{e}^{mx}\mathrm{d}x = \frac{\mathrm{e}^{mx}(n \sin nx + m \cos nx)}{n^2}$$

$$\Rightarrow \int \cos nx \, \mathrm{e}^{mx}\mathrm{d}x = \frac{\mathrm{e}^{mx}(n \sin nx + m \cos nx)}{\cancel{n^2}} \times \frac{\cancel{n^2}}{m^2 + n^2} + c = \frac{\mathrm{e}^{mx}(n \sin nx + m \cos nx)}{m^2 + n^2} + c$$

8. By parts

$u = \mathrm{e}^{mx}$ $\quad \mathrm{d}v = \sin nx \, \mathrm{d}x$

$\mathrm{d}u = m\mathrm{e}^{mx}\mathrm{d}x$ $\quad v = -\frac{1}{n}\cos nx$

$$\int \mathrm{e}^{mx} \sin nx \, \mathrm{d}x = -\frac{\mathrm{e}^{kx} \cos nx}{n} + \frac{m}{n}\int \mathrm{e}^{mx} \cos nx \, \mathrm{d}x$$

Another step of by parts

$u = \mathrm{e}^{mx}$ $\quad \mathrm{d}v = \cos nx \, \mathrm{d}x$

$\mathrm{d}u = m\mathrm{e}^{mx}\mathrm{d}x$ $\quad v = \frac{1}{n}\sin nx$

$$\int \mathrm{e}^{mx}\sin nx\,\mathrm{d}x = -\frac{\mathrm{e}^{kx}\cos nx}{n} + \frac{m}{n}\left(\frac{1}{n}\mathrm{e}^{mx}\sin nx - \frac{m}{n}\int \mathrm{e}^{mx}\sin nx\,\mathrm{d}x\right)$$

$$= -\frac{\mathrm{e}^{kx}\cos nx}{n} + \frac{m\mathrm{e}^{mx}\sin nx}{n^2} - \frac{m^2}{n^2}\int \mathrm{e}^{mx}\sin nx\,\mathrm{d}x$$

$$\Rightarrow \left(1+\frac{m^2}{n^2}\right)\int \mathrm{e}^{kx}\sin nx\,\mathrm{d}x = \frac{-n\mathrm{e}^{kx}\cos nx + m\mathrm{e}^{mx}\sin nx}{n^2}$$

$$\Rightarrow \int \mathrm{e}^{kx}\sin nx\,\mathrm{d}x = \left(\frac{-n\mathrm{e}^{mx}\cos nx + m\mathrm{e}^{mx}\sin nx}{\cancel{n^2}}\right)\frac{\cancel{n^2}}{m^2+n^2} + c = \frac{\mathrm{e}^{mx}\left(m\sin nx - n\cos nx\right)}{m^2+n^2} + c$$

Exercise 14.3

Note: as mentioned in earlier sections, answers may differ from those given in the book, but they will be equivalent.

1. **(a)** $\int \sin^3 t\cos^2 t\,\mathrm{d}t = \int \sin t\left(1-\cos^2 t\right)\cos^2 t\,\mathrm{d}t = \int \sin t\cos^2 t\,\mathrm{d}t - \int \sin t\cos^4 t\,\mathrm{d}t$

$$= \int \cos^2 t\left(-\mathrm{d}\left(\cos t\right)\right) - \int \cos^4 t\left(-\mathrm{d}\left(\cos t\right)\right) = \frac{\cos^5 t}{5} - \frac{\cos^3 t}{3} + c$$

(b) $\int \sin^3 t\cos^3 t\,\mathrm{d}t = \int \sin t\left(1-\cos^2 t\right)\cos^3 t\,\mathrm{d}t = \int \sin t\cos^3 t\,\mathrm{d}t - \int \sin t\cos^5 t\,\mathrm{d}t$

$$= -\int u^3\,\mathrm{d}u + \int u^5\,\mathrm{d}u = \frac{\cos^6 t}{6} - \frac{\cos^4 t}{4} + c$$

This answer differs from the answer given in the book by a constant.
Also, we could have done a similar transformation with the cosine expression:

$\int \sin^3 t\cos^3 t\,\mathrm{d}t = \int \sin^3 t\cos t\left(1-\sin^2 t\right)\mathrm{d}t = \int \sin^3 t\cos t\,\mathrm{d}t - \int \sin^5 t\cos t\,\mathrm{d}t$

$$= \int u^3\,\mathrm{d}u - \int u^5\,\mathrm{d}u = \frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + c$$

(c) $\int \sin^3 3\theta\cos 3\theta\,\mathrm{d}\theta = \int \sin^3 3\theta \times \frac{1}{3}\mathrm{d}\left(\sin 3\theta\right) = \frac{1}{3}\times\frac{\sin^4 3\theta}{4} + c = \frac{\sin^4 3\theta}{12} + c$

(d) Substitute x for $\frac{1}{t} \Rightarrow \mathrm{d}x = -\frac{\mathrm{d}t}{t^2}$

$\int \frac{1}{t^2}\sin^5\left(\frac{1}{t}\right)\cos^2\left(\frac{1}{t}\right)\mathrm{d}t = \int -\sin^5 x\cos^2 x\,\mathrm{d}x$, now use basic trigonometric identities, then let $\cos x = u \Rightarrow \sin x\mathrm{d}x = \mathrm{d}u$

$$\int -\sin^5 x\cos^2 x\,\mathrm{d}x = -\int \sin x\left(1-\cos^2 x\right)^2 \cos^2 x\,\mathrm{d}x$$
$$= \int u^2\,\mathrm{d}u - 2\int u^4\,\mathrm{d}u + \int u^6\,\mathrm{d}u$$
$$= \frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7} + c = \frac{\cos^3\left(\frac{1}{t}\right)}{3} - \frac{2\cos^5\left(\frac{1}{t}\right)}{5} + \frac{\cos^7\left(\frac{1}{t}\right)}{7} + c$$

(e)
$$\int \frac{\sin^3 x}{\cos^2 x}\,\mathrm{d}x = \int \frac{\sin^2 x}{\cos^2 x}\sin x\,\mathrm{d}x = \int \frac{1-\cos^2 x}{\cos^2 x}\sin x\,\mathrm{d}x = \int\left(\frac{1}{\cos^2 x} - 1\right)\sin x\,\mathrm{d}x$$
$$= -\frac{\left(\cos x\right)^{-1}}{-1} + \cos x + c = \sec x + \cos x + c$$

(f) Use the substitution $\tan 3x = t \Rightarrow \sec^2 3x \times 3\,\mathrm{d}x = \mathrm{d}t$

$$\int \tan^5 3x \sec^2 3x\,\mathrm{d}x = \int t^5 \times \frac{1}{3}\mathrm{d}t = \frac{1}{3}\times\frac{t^6}{6} + c = \frac{\tan^6 3x}{18} + c$$

(g) Use the substitution $\theta^2 = t \Rightarrow 2\theta\mathrm{d}\theta = \mathrm{d}t$, then let $\tan t = u \Rightarrow \sec^2 t\mathrm{d}t = \mathrm{d}u$

$$\int \theta\tan^3\theta^2\sec^4\theta^2\mathrm{d}\theta = \int \tan^3 t\sec^4 t\times\frac{1}{2}\mathrm{d}t = \frac{1}{2}\int\tan^3 t\left(\tan^2 t + 1\right)\sec^2 t\,\mathrm{d}t$$
$$= \frac{1}{2}\int\left(u^5 + u^3\right)\mathrm{d}u = \frac{1}{2}\left(\frac{\tan^6 t}{6} + \frac{\tan^4 t}{4}\right) + c = \frac{\tan^6\theta^2}{12} + \frac{\tan^4\theta^2}{8} + c$$

(h) Use the substitution $\sqrt{t} = x \Rightarrow \frac{1}{2\sqrt{t}}\mathrm{d}t = \mathrm{d}x$, use the identity $\tan^2 x = \sec^2 x - 1$

and let $u = \sec x \Rightarrow \mathrm{d}u = \sec x\tan x\mathrm{d}x$

$$\int\frac{1}{\sqrt{t}}\tan^3\sqrt{t}\sec^3\sqrt{t}\,\mathrm{d}t = \int\tan^3 x\sec^3 x\times 2\mathrm{d}x = 2\int\frac{\sin^3 x}{\cos^6 x}\mathrm{d}x$$
$$= 2\int\left(\tan^2 x\sec^2 x\right)\left(\tan x\sec x\right)\mathrm{d}x = 2\int\left(\left(\sec^2 x - 1\right)\sec^2 x\right)\left(\tan x\sec x\right)\mathrm{d}x$$
$$= 2\int\left(u^4 - u^2\right)du = \frac{2\sec^5\sqrt{t}}{5} - \frac{2\sec^3\sqrt{t}}{3} + c$$

(i) This needs some symbolic manipulation followed by a trigonometric substitution.

$$\int\tan^4 5t\,\mathrm{d}t = \int\left(\left(\tan^4 5t - 1\right) + 1\right)\mathrm{d}t = \int\left(\left(\tan^2 5t - 1\right)\left(\tan^2 5t + 1\right) + 1\right)\mathrm{d}t$$
$$= \int\left(\left(\tan^2 5t - 1\right)\sec^2 5t + 1\right)\mathrm{d}t = \int\tan^2 5t\sec^2 5t\,\mathrm{d}t - \int\sec^2 5t\,\mathrm{d}t + \int\mathrm{d}t$$
$$= \frac{1}{5}\times\frac{\tan^3 5t}{3} - \frac{\tan 5t}{5} + t + c = \frac{\tan^3 5t}{15} - \frac{\tan 5t}{5} + t + c$$

(j) Use the given hint.

$$\int \frac{\mathrm{d}t}{1+\sin t}=\int \frac{1-\sin t}{(1+\sin t)(1-\sin t)}\mathrm{d}t=\int \frac{1-\sin t}{1-\sin^2 t}\mathrm{d}t=\int \frac{1-\sin t}{\cos^2 t}\mathrm{d}t$$

$$=\int \frac{\mathrm{d}t}{\cos^2 t}-\int \frac{\sin t\,\mathrm{d}t}{\cos^2 t}=\tan t-(\cos t)^{-1}+c=\tan t-\sec t+c$$

(k) Use a similar procedure to the previous question

$$\int \frac{\mathrm{d}\theta}{1+\cos\theta}=\int \frac{1-\cos\theta}{(1+\cos\theta)(1-\cos\theta)}\mathrm{d}\theta=\int \frac{1-\cos\theta}{1-\cos^2\theta}\mathrm{d}\theta=\int \frac{1-\cos\theta}{\sin^2\theta}\mathrm{d}\theta$$

$$=\int \frac{\mathrm{d}\theta}{\sin^2\theta}-\int \frac{\cos\theta\mathrm{d}\theta}{\sin^2\theta}=-\cot\theta-(\sin\theta)^{-1}+c=-\cot\theta+\csc\theta+c$$

(l) Similar to the above

$$\int \frac{1+\sin t}{\cos t}\mathrm{d}t=\int \frac{(1+\sin t)(1-\sin t)}{\cos t(1-\sin t)}\mathrm{d}t=\int \frac{1-\sin^2 t}{\cos t(1-\sin t)}\mathrm{d}t$$

$$=\int \frac{\cos^2 t}{\cos t(1-\sin t)}\mathrm{d}t=\int \frac{\cos t\,dt}{1-\sin t}=-\ln|1-\sin t|+c$$

(m) Some symbolic manipulation to simplify the expression

$$\int \frac{\sin x-5\cos x}{\sin x+\cos x}\mathrm{d}x=\int \frac{-2(\sin x+\cos x)-3(\cos x-\sin x)}{\sin x+\cos x}\mathrm{d}x$$

$$=-2\int \mathrm{d}x-3\int \frac{\cos x-\sin x}{\sin x+\cos x}\mathrm{d}x=-2\int \mathrm{d}x-3\int \frac{\mathrm{d}u}{u}$$

$$=-2x-3\ln|\sin x+\cos x|+c$$

(n) Use the substitution $\sec\theta=t\Rightarrow \sec\theta\tan\theta\,\mathrm{d}\theta=\mathrm{d}t$

$$\int \frac{\sec\theta\tan\theta}{1+\sec^2\theta}\mathrm{d}\theta=\int \frac{\mathrm{d}t}{1+t^2}=\arctan t+c=\arctan(\sec\theta)+c$$

(o) Use the substitution $\arctan t=u\Rightarrow \frac{\mathrm{d}t}{1+t^2}=\mathrm{d}u$

$$\int \frac{\arctan t}{1+t^2}\mathrm{d}t=\int u\,\mathrm{d}u=\frac{\arctan^2 t}{2}+c$$

(p) Use the substitution $\arctan t=u\Rightarrow \frac{\mathrm{d}t}{1+t^2}=\mathrm{d}u$

$$\int \frac{1}{(1+t^2)\arctan t}\mathrm{d}t=\int \frac{\mathrm{d}u}{u}=\ln|\arctan t|+c$$

(q) Use the substitution $\ln x = t \Rightarrow \frac{1}{x}\mathrm{d}x = \mathrm{d}t$

$$\int \frac{\mathrm{d}x}{x\sqrt{1-\ln^2 x}} = \int \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin\left(\ln x\right) + c$$

(r) $\int \sin^3 x\,\mathrm{d}x = \int \sin x\left(1-\cos^2 x\right)\mathrm{d}x = \int \sin x\mathrm{d}x - \int \cos^2 x \sin x\,\mathrm{d}x = -\cos x + \frac{\cos^3 x}{3} + c$

(s) Let $\cos x = u \Rightarrow \sin x\mathrm{d}x = -\mathrm{d}u$

$$\int \frac{\sin^3 x}{\sqrt{\cos x}}\mathrm{d}x = \int \frac{\sin x\left(1-\cos^2 x\right)}{\sqrt{\cos x}}\mathrm{d}x = \int \frac{\sin x}{\sqrt{\cos x}}\mathrm{d}x - \int \sin x \cos^{\frac{3}{2}} x\,\mathrm{d}x$$

$$= \int u^{-\frac{1}{2}}\left(-\mathrm{d}u\right) - \int u^{\frac{3}{2}}\left(-\mathrm{d}u\right) = \frac{-u^{\frac{1}{2}}x}{\frac{1}{2}} + \frac{u^{\frac{5}{2}}x}{\frac{5}{2}} + c$$

$$= 2\sqrt{\cos x}\left(\frac{1}{5}\cos^2 x - 1\right) + c$$

(t) After a simple substitution, we can use the result from question **(r)**.

Let $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}\mathrm{d}x = \mathrm{d}t$

$$\int \frac{\sin^3 \sqrt{x}}{\sqrt{x}}\mathrm{d}x = \int \sin^3 t \times 2\mathrm{d}t = 2\left(-\cos t + \frac{\cos^3 t}{3}\right) + c = -2\cos\sqrt{x} + \frac{2\cos^3\sqrt{x}}{3} + c$$

(u) Let $\sin t = x \Rightarrow \cos t\,\mathrm{d}t = \mathrm{d}x$

$$\int \cos t \cos^3\left(\sin t\right)\mathrm{d}t = \int \cos^3\left(x\right)\mathrm{d}x = \int \cos x\left(1-\sin^2 x\right)\mathrm{d}x$$

$$= \int \cos x\,\mathrm{d}x - \int \sin^2 x \cos x\,\mathrm{d}x$$

$$= \sin x - \frac{\sin^3 x}{3} + c = \sin\left(\sin t\right) - \frac{\sin^3\left(\sin t\right)}{3} + c$$

(v) $\int \frac{\cos\theta + \sin 2\theta}{\sin\theta}\mathrm{d}\theta = \int \frac{\cos\theta + 2\sin\theta\cos\theta}{\sin\theta}\mathrm{d}\theta = \int \frac{\cos\theta\,\mathrm{d}\theta}{\sin\theta} + 2\int \cos\theta\,\mathrm{d}\theta$

$$= \ln\left|\sin\theta\right| + 2\sin\theta + c$$

2. **(a)** We will use integration by parts:

$u = t \qquad \mathrm{d}v = \sec t \tan t\,\mathrm{d}t$

$\mathrm{d}u = \mathrm{d}t \qquad v = \sec t$

$$\int t\sec t\tan t\,\mathrm{d}t = t\sec t - \int \sec t\,\mathrm{d}t$$

To find the second integral, refer to Example 14.14 in the book.

Therefore, the solution of our original integral is:

$$\int t\sec t\tan t\,\mathrm{d}t = t\sec t - \ln\left|\sec t + \tan t\right| + c$$

(b) Let $2-\sin x = t \Rightarrow -\cos x\,\mathrm{d}x = \mathrm{d}t$

$$\int \frac{\cos x}{2-\sin x}\mathrm{d}x = \int \frac{-\mathrm{d}t}{t} = -\ln|t| + c = -\ln|2-\sin x| + c$$

There is no need for the absolute value as the expression inside is always positive.

(c) Let $\mathrm{e}^{-2x} = t \Rightarrow -2\mathrm{e}^{-2x}\mathrm{d}x = \mathrm{d}t$

$$\int \mathrm{e}^{-2x}\tan\left(\mathrm{e}^{-2x}\right)\mathrm{d}x = -\frac{1}{2}\int \tan t\,\mathrm{d}t = -\frac{1}{2}\left(-\ln|\cos t|\right) + c = \frac{1}{2}\ln\left|\cos\left(\mathrm{e}^{-2x}\right)\right| + c$$

(d) Let $\sqrt{t} = x \Rightarrow \frac{1}{2\sqrt{t}}\mathrm{d}t = \mathrm{d}x$

$$\int \frac{\sec\left(\sqrt{t}\right)}{\sqrt{t}}\mathrm{d}t = \int 2\sec x\,\mathrm{d}x = 2\ln\left|\sec x + \tan x\right| + c = 2\ln\left|\sec\sqrt{t} + \tan\sqrt{t}\right| + c$$

(e) $$\int \frac{\mathrm{d}t}{1+\cos 2t} = \int \frac{\mathrm{d}t}{\cancel{1} + 2\cos^2 t - \cancel{1}} = \frac{1}{2}\int \frac{\mathrm{d}t}{\cos^2 t} = \frac{1}{2}\tan t + c$$

Note: In the following exercises, familiarity with trigonometric substitutions is an asset.

(f) Let $3x = \sin\theta \Rightarrow 3\,\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$

$$\int \sqrt{1-9x^2}\,\mathrm{d}x = \int \cos\theta \times \frac{1}{3}\cos\theta\,\mathrm{d}\theta = \frac{1}{3}\int \frac{1+\cos 2\theta}{2}\mathrm{d}\theta$$

$$== \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right) + c = \frac{\arcsin(3x)}{6} + \frac{\sin\left(2\arcsin(3x)\right)}{12} + c$$

$$= \frac{\arcsin(3x)}{6} + \frac{2\sin\left(\arcsin(3x)\right)\cos\left(\arcsin(3x)\right)}{12} + c$$

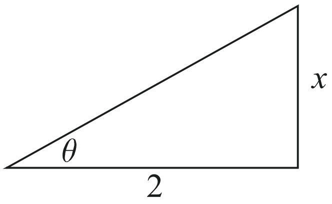
$$= \frac{1}{6}\arcsin(3x) + \frac{1}{6}3x \times \sqrt{1-9x^2} + c = \frac{1}{6}\arcsin(3x) + \frac{x\sqrt{1-9x^2}}{2} + c$$

(g) We will use a trigonometric substitution: $x = 2\tan\theta \Rightarrow \mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$

$$\int\frac{\mathrm{d}x}{\left(x^2+4\right)^{\frac{3}{2}}} = \int\frac{2\sec^2\theta\,\mathrm{d}\theta}{\left(4\left(\tan^2\theta+1\right)\right)^{\frac{3}{2}}} = \int\frac{2\sec^2\theta\,\mathrm{d}\theta}{8\sec^3\theta} = \frac{1}{4}\int\cos\theta\,\mathrm{d}\theta = \frac{1}{4}\sin\theta + c$$

The slight difficulty with such methods is going back to the original variable.

If $x = 2\tan\theta$, then $\theta = \arctan\left(\frac{x}{2}\right)$. Drawing a right-angled triangle can help.



The hypotenuse of this triangle is $\sqrt{4+x^2}$, thus,

$\frac{1}{4}\sin\theta + c = \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + c$, and finally, $\int\frac{\mathrm{d}x}{\left(x^2+4\right)^{\frac{3}{2}}} = \frac{x}{4\sqrt{4+x^2}} + c$

(h) We will use a trigonometric substitution: $t = 2\tan\theta \Rightarrow \mathrm{d}t = 2\sec^2\theta\,\mathrm{d}\theta$

$$\int\sqrt{4+t^2}\,\mathrm{d}t = \int 2\sqrt{1+\tan^2\theta}\times 2\sec^2\theta\,\mathrm{d}\theta = 4\int\sec^3\theta\,\mathrm{d}\theta$$

The integral $\int\sec^3\theta\,\mathrm{d}\theta$ is not a simple one, but we found it in Example 14.15.

$$\int\sec^3\theta\,\mathrm{d}\theta = \frac{\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|}{2} + c$$

Going back to the original value, if $t = 2\tan\theta$, then $\theta = \arctan\left(\frac{t}{2}\right)$.

Refer to the diagram in the previous exercise (replace x with t):

$\sec\theta = \frac{\sqrt{4+t^2}}{2}$; $\tan\theta = \frac{t}{2}$, and thus

$$4\int\sec^3\theta\,\mathrm{d}\theta = \frac{t\sqrt{4+t^2} + 4\ln\left|\frac{\sqrt{4+t^2}}{2} + \frac{t}{2}\right|}{2} + c = \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|\frac{\sqrt{4+t^2}+1}{4}\right| + c$$

$$= \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|\sqrt{4+t^2}+t\right| - 2\ln 4 + c$$

$$= \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|\sqrt{4+t^2}+t\right| + k$$

(i) We also use trigonometric substitution here

$$e^t = 2\tan\theta \Rightarrow e^t\,dt = 2\sec^2\theta\,d\theta$$

$$\int \frac{3e^t\,dt}{4+e^{2t}} = 3\int \frac{2\sec^2\theta\,d\theta}{4\left(1+\tan^2\theta\right)} = \frac{3}{2}\int d\theta = \frac{3}{2}\theta + c = \frac{3}{2}\arctan\left(\frac{e^t}{2}\right) + c$$

(j) $x = \frac{3}{2}\cos\theta \Rightarrow dx = -\frac{3}{2}\sin\theta\,d\theta$

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \int \frac{-\frac{3}{2}\sin\theta\,d\theta}{\sqrt{9-4\times\frac{9}{4}\cos^2\theta}} = \int \frac{-\frac{3}{2}\sin\theta\,d\theta}{3\sqrt{1-\cos^2\theta}} = -\frac{1}{2}\int d\theta$$

$$= -\frac{1}{2}\theta + c = -\frac{1}{2}\arccos\left(\frac{2x}{3}\right) + c$$

Note: If we had used the substitution $x = \frac{3}{2}\sin\theta$, then the result would have been $\frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + c$, since the two results differ only by a constant.

(k) $x = \frac{2}{3}\tan\theta \Rightarrow dx = \frac{2}{3}\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{4+9x^2}}\,dx = \int \frac{\frac{2}{3}\sec^2\theta\,d\theta}{\sqrt{4+9\times\frac{4}{9}\tan^2\theta}} = \int \frac{\frac{2}{3}\sec^2\theta\,d\theta}{2\sqrt{1+\tan^2\theta}} = \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\left(\ln\left|\sec\theta+\tan\theta\right|\right) = \frac{1}{3}\ln\left|\sec\left(\arctan\left(\frac{3x}{2}\right)\right) + \tan\left(\arctan\left(\frac{3x}{2}\right)\right)\right|$$

$$= \frac{1}{3}\ln\left|\sqrt{1+\left(\frac{3x}{2}\right)^2} + \frac{3x}{2}\right| + c = \frac{1}{3}\ln\left|\frac{\sqrt{4+9x^2}+3x}{2}\right| + c$$

(l) We use a two-stage substitution for simplicity.

$$\sin x = t \Rightarrow \cos x\,dx = dt\,;\ t = \tan\theta \Rightarrow dt = \sec^2\theta\,d\theta$$

$$\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int \frac{dt}{\sqrt{1+t^2}} = \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right| + c$$

$$= \ln\left|\sec\left(\arctan t\right) + \tan\left(\arctan t\right)\right| + c = \ln\left|\sqrt{1+t^2}+t\right| + c$$

$$= \ln\left|\sqrt{1+\sin^2 x} + \sin x\right| + c$$

(m) $4-x^2=t \Rightarrow -2x\,\mathrm{d}x=\mathrm{d}t$

$$\int \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x=\int \frac{-\frac{1}{2}\mathrm{d}t}{\sqrt{t}}=-\frac{1}{2}\times\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c=-\sqrt{4-x^2}+c$$

(n) $x^2+16=t \Rightarrow 2x\,\mathrm{d}x=\mathrm{d}t$

$$\int \frac{x}{x^2+16}\,\mathrm{d}x=\int \frac{\frac{1}{2}\mathrm{d}t}{t}=\frac{1}{2}\ln t+c=\frac{1}{2}\ln\left(x^2+16\right)+c$$

(o) $x=2\cos\theta \Rightarrow \mathrm{d}x=-2\sin\theta\,\mathrm{d}\theta$

$$\int \frac{\sqrt{4-x^2}}{x^2}\,\mathrm{d}x=\int \frac{2\sqrt{1-\cos^2\theta}\times(-2\sin\theta)\,\mathrm{d}\theta}{4\cos^2\theta}=-\int \frac{\sin^2\theta\,\mathrm{d}\theta}{\cos^2\theta}$$

$$=\int \frac{\cos^2\theta-1}{\cos^2\theta}\,\mathrm{d}\theta=\int \mathrm{d}\theta-\int \frac{1}{\cos^2\theta}\,\mathrm{d}\theta=\theta-\tan\theta+c$$

$$=\arccos\left(\frac{x}{2}\right)-\tan\left(\arccos\left(\frac{x}{2}\right)\right)+c$$

$$=\arccos\left(\frac{x}{2}\right)-\frac{\sqrt{1-\left(\cos\left(\arccos\left(\frac{x}{2}\right)\right)\right)^2}}{\cos\left(\arccos\left(\frac{x}{2}\right)\right)}+c$$

$$=\arccos\left(\frac{x}{2}\right)-\frac{\sqrt{4-x^2}}{x}+c$$

Note: If we had used the substitution $x=\frac{3}{2}\sin\theta$, then the result would have been

$-\arcsin\left(\frac{x}{2}\right)-\frac{\sqrt{4-x^2}}{x}+c$ since the two results differ by a constant.

(p) $x = 3\cos\theta \Rightarrow \mathrm{d}x = -3\sin\theta\,\mathrm{d}\theta$

$$\int \frac{\mathrm{d}x}{\left(9-x^2\right)^{\frac{3}{2}}} = \int \frac{-3\sin\theta\,\mathrm{d}\theta}{\left(9-9\cos^2\theta\right)^{\frac{3}{2}}} = \int \frac{-3\sin\theta\,\mathrm{d}\theta}{27\left(1-\cos^2\theta\right)^{\frac{3}{2}}}$$

$$= -\frac{1}{9}\int \frac{\sin\theta\,\mathrm{d}\theta}{\sin^3\theta} = -\frac{1}{9}\int \frac{\mathrm{d}\theta}{\sin^2\theta} = -\frac{1}{9}\left(-\cot\theta\right)+c$$

$$= \frac{1}{9}\cot\left(\arccos\left(\frac{x}{3}\right)\right)+c$$

$$= \frac{1}{9}\frac{\cos\left(\arccos\left(\frac{x}{3}\right)\right)}{\sqrt{1-\cos^2\left(\arccos\left(\frac{x}{3}\right)\right)}}+c = \frac{x}{9\sqrt{9-x^2}}+c$$

(q) $1+x^2 = t \Rightarrow 2x\,\mathrm{d}x = \mathrm{d}t$

$$\int x\sqrt{1+x^2}\,\mathrm{d}x = \int \sqrt{t}\times\frac{1}{2}\,\mathrm{d}t = \frac{1}{2}\times\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{1}{3}\left(1+x^2\right)\sqrt{1+x^2}+c$$

(r) $1+\mathrm{e}^{2x} = t \Rightarrow 2\mathrm{e}^{2x}\mathrm{d}x = \mathrm{d}t$

$$\int \mathrm{e}^{2x}\sqrt{1+\mathrm{e}^{2x}}\,\mathrm{d}x = \int \sqrt{t}\times\frac{1}{2}\,\mathrm{d}t = \frac{1}{2}\times\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{1}{3}\left(1+\mathrm{e}^{2x}\right)\sqrt{1+\mathrm{e}^{2x}}+c$$

(s) $\mathrm{e}^x = t \Rightarrow \mathrm{e}^x\mathrm{d}x = \mathrm{d}t$ followed by $t = \sin\theta \Rightarrow \mathrm{d}t = \cos\theta\,\mathrm{d}\theta$

$$\int \mathrm{e}^x\sqrt{1-\mathrm{e}^{2x}}\,\mathrm{d}x = \int \sqrt{1-t^2}\,\mathrm{d}t = \int \cos^2\theta\,\mathrm{d}\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = \frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+c$$

$$= \frac{1}{2}\arcsin t+\frac{1}{2}\sin\left(\arcsin t\right)\cos\left(\arcsin t\right)+c$$

$$= \frac{1}{2}\arcsin t+\frac{1}{2}t\sqrt{1-t^2}+c = \frac{1}{2}\left(\arcsin\left(\mathrm{e}^x\right)+\mathrm{e}^x\sqrt{1-\mathrm{e}^{2x}}\right)+c$$

(t) $e^x = t \Rightarrow e^x dx = dt$ followed by $t = 3\tan\theta \Rightarrow dt = 3\sec^2\theta\, d\theta$

$$\int \frac{e^x}{\sqrt{e^{2x}+9}}dx = \int \frac{dt}{\sqrt{t^2+9}} = \int \frac{3\sec^2\theta\, d\theta}{3\sqrt{\tan^2\theta+1}} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c = \ln\left|\sec\left(\arctan\frac{t}{3}\right) + \tan\left(\arctan\frac{t}{3}\right)\right|$$

$$= \ln\left|\sqrt{1+\left(\frac{t}{3}\right)^2} + \frac{t}{3}\right| = \ln\left|\frac{\sqrt{9+t^2}+t}{3}\right| = \ln\left(\frac{\sqrt{9+e^{2x}}+e^x}{3}\right) + c$$

(u) By parts

$u = \ln x \quad dv = \frac{dx}{\sqrt{x}}$

$du = \frac{1}{x}dx \quad v = 2\sqrt{x}$

$$\int \frac{\ln x}{\sqrt{x}}dx = 2\sqrt{x}\ln x - \int 2\sqrt{x} \times \frac{1}{x}dx = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}dx$$

$$= 2\sqrt{x}\ln x - 4\sqrt{x} + c = 2\sqrt{x}(\ln x - 2) + c$$

(v) $x + 2 = t \Rightarrow x = t - 2; dx = dt$

$$\int \frac{x^3}{(x+2)^2}dx = \int \frac{(t-2)^3}{t^2}dt = \int \frac{t^3 - 6t^2 + 12t - 8}{t^2}dt$$

$$= \int (t - 6 + 12t^{-1} - 8t^{-2})dt = \frac{t^2}{2} - 6t + 12\ln|t| + 8t^{-1} + c$$

$$= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c$$

3. Direct substitution: $x^2 + 9 = t \Rightarrow 2x dx = dt$

$$\int \frac{x}{x^2+9}dx = \int \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\ln t + c_1 = \frac{1}{2}\ln(x^2+9) + c_1$$

Trigonometric substitution: $x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$

$$\int \frac{x}{x^2+9}\,\mathrm{d}x = \int \frac{3\tan\theta}{9\tan^2\theta+9}3\sec^2\theta\,\mathrm{d}\theta = \int \frac{9\tan\theta\sec^2\theta\,\mathrm{d}\theta}{9\Big(\underbrace{\tan^2\theta+1}_{\sec^2\theta}\Big)} = \int \tan\theta\,\mathrm{d}\theta$$

$$= -\ln\left|\cos\theta\right| + c_2 = -\ln\left|\cos\left(\arctan\left(\frac{x}{3}\right)\right)\right| + c_2 = -\ln\left|\frac{3}{\sqrt{9+x^2}}\right| + c_2$$

$$= \ln\left(\sqrt{9+x^2}\right) - \ln 3 + c_2 = \frac{1}{2}\ln\left(9+x^2\right) - \ln 3 + c_2$$

Here we can see that the two solutions differ by a constant.

Note: evaluating $\arctan\left(\frac{x}{3}\right)$ is done by drawing a right-angled triangle with one side of x and the other 3 as has been done earlier.

4. Rewrite the numerator and use antidifferentiation:

$$\int \frac{x^2}{x^2+9}\,\mathrm{d}x = \int \frac{\left(x^2+9\right)-9}{x^2+9}\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{9}{x^2+9}\,\mathrm{d}x = x - 9\times\frac{1}{3}\arctan\left(\frac{x}{3}\right) + c_1$$

$$= x - 3\arctan\left(\frac{x}{3}\right) + c_1$$

Trigonometric substitution: $x = 3\tan\theta \Rightarrow \mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$

$$\int \frac{x^2}{x^2+9}\,\mathrm{d}x = \int \frac{9\tan^2\theta}{9\tan^2\theta+9}3\sec^2\theta\,\mathrm{d}\theta = \int \frac{27\tan^2\theta\sec^2\theta\,\mathrm{d}\theta}{9\Big(\underbrace{\tan^2\theta+1}_{\sec^2\theta}\Big)}$$

$$= 3\int \tan^2\theta\,\mathrm{d}\theta = 3\int\left(\sec^2\theta - 1\right)\mathrm{d}\theta = 3\left(\tan\theta - \theta\right) + c_2$$

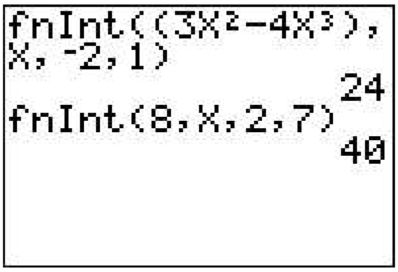
$$= 3\left(\tan\left(\arctan\left(\frac{x}{3}\right)\right) - \arctan\left(\frac{x}{3}\right)\right) + c_2 = x - 3\arctan\left(\frac{x}{3}\right) + c_2$$

Exercise 14.4

1. **(a)** $\int_{-2}^{1}\left(3x^2-4x^3\right)\mathrm{d}x=\left[x^3-x^4\right]_{-2}^{1}=\left(1^3-1^4\right)-\left(\left(-2\right)^3-\left(-2\right)^4\right)=24$

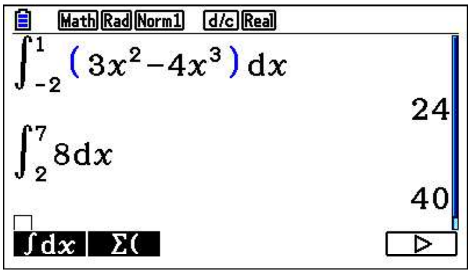
(b) $\int_{2}^{7}8\,\mathrm{d}x=\left[8x\right]_{2}^{7}=56-16=40$

Note: When answering such questions on a GDC-active exam, you can use your GDC to get the answer. On the exam, you need to just write the integral with the value. Here is an example.



```
fnInt((3X²-4X³),
X,-2,1)
                24
fnInt(8,X,2,7)
                40
```

, or



```
Math Rad Norm1 d/c Real
∫_{-2}^{1} (3x²-4x³) dx
                       24
∫_2^7 8dx
                       40
```

(c) $\int_{1}^{5}\frac{2}{t^3}\,\mathrm{d}t=\left[2\times\frac{t^{-2}}{-2}\right]_{1}^{5}=\left(-\frac{1}{25}\right)-\left(-\frac{1}{1}\right)=\frac{24}{25}$

(d) $\int_{2}^{2}\left(\cos t-\tan t\right)\mathrm{d}t=0,$ since the upper and lower limits are equal.

(e) Simplify the integrand before attempting to evaluate it.

$$\int_{1}^{7}\frac{2x^2-3x+5}{\sqrt{x}}\,\mathrm{d}x=\int_{1}^{7}\left(2x^{\frac{3}{2}}-3x^{\frac{1}{2}}+5x^{-\frac{1}{2}}\right)\mathrm{d}x=\left[2\frac{x^{\frac{5}{2}}}{\frac{5}{2}}-3\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+5\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{1}^{7}$$

$$=\left(\frac{4}{5}\times7^{\frac{5}{2}}-2\times7^{\frac{3}{2}}+10\times7^{\frac{1}{2}}\right)-\left(\frac{4}{5}-2+10\right)=\frac{176\sqrt{7}-44}{5}$$

(f) $\int_{0}^{\pi}\cos\theta\,\mathrm{d}\theta=\left[\sin\theta\right]_{0}^{\pi}=\sin\pi-\sin0=0$

You can get the result without integration if you notice that the graph is symmetric about the point $\left(\frac{\pi}{2},0\right)$ and the negative area will ‘cancel’ the positive one.

(g) $$\int_0^{\pi} \sin\theta\, \mathrm{d}\theta = \left[-\cos\theta\right]_0^{\pi} = -\cos\pi - \left(-\cos 0\right) = -\left(-1\right) - \left(-1\right) = 2$$

(h) $$\int_3^1 \left(5x^4 + 3x^2\right)\mathrm{d}x = \left[5\frac{x^5}{5} + 3\frac{x^3}{3}\right]_3^1 = \left(1^5 + 1^3\right) - \left(3^5 + 3^3\right) = -268$$

(i) $$\int_1^3 \frac{u^5 + 2}{u^2}\mathrm{d}u = \int_1^3 (u^3 + 2u^{-2})\mathrm{d}u = \left[\frac{u^4}{4} + 2\frac{u^{-1}}{-1}\right]_1^3 = \left(\frac{3^4}{4} - \frac{2}{3}\right) - \left(\frac{1}{4} - 2\right) = \frac{64}{3}$$

(j) $$\int_1^{\mathrm{e}} \frac{2\mathrm{d}x}{x} = \left[2\ln x\right]_1^{\mathrm{e}} = 2\ln\mathrm{e} - 2\ln(1) = 2$$

(k) $$\int_1^3 \frac{2x}{x^2 + 2}\mathrm{d}x = \int_1^3 \frac{\mathrm{d}\left(x^2 + 2\right)}{x^2 + 2} = \left[\ln\left(x^2 + 2\right)\right]_1^3 = \ln 11 - \ln 3 = \ln\left(\frac{11}{3}\right)$$

This integral could have been evaluated without going back to the original variable too.

$$\int_1^3 \frac{2x}{x^2 + 2}\mathrm{d}x = \int_3^{11} \frac{\mathrm{d}u}{u} = \left[\ln|u|\right]_3^{11} = \ln 11 - \ln 3 = \ln\left(\frac{11}{3}\right)$$

(l) $$\int_1^3 \left(2 - \sqrt{x}\right)^2 \mathrm{d}x = \int_1^3 \left(4 - 4\sqrt{x} + x\right)\mathrm{d}x = \left[4x - 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2}\right]_1^3 = \frac{44}{3} - 8\sqrt{3}$$

(m) $$\int_0^{\frac{\pi}{4}} 3\sec^2\theta\, \mathrm{d}\theta = \left[3\tan\theta\right]_0^{\frac{\pi}{4}} = 3\left(\tan\left(\frac{\pi}{4}\right) - \tan 0\right) = 3$$

(n) $$\int_0^1 \left(8x^7 + \sqrt{\pi}\right)\mathrm{d}x = \left[8\frac{x^8}{8} + \sqrt{\pi} \times x\right]_0^1 = 1 + \sqrt{\pi}$$

(o) Use the fact that $|3x| = \begin{cases} 3x, & x \geq 0 \\ -3x, & x < 0 \end{cases}$

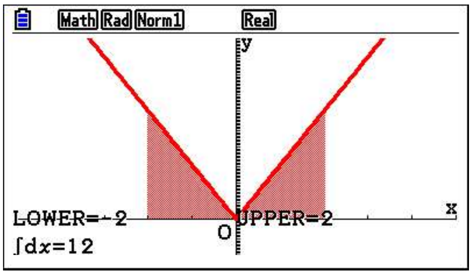
(i) $$\int_0^2 |3x|\,\mathrm{d}x = \int_0^2 3x\,\mathrm{d}x = \left[3\frac{x^2}{2}\right]_0^2 = 6$$

(ii) $\int_{-2}^{0}|3x|\,\mathrm{d}x=\int_{-2}^{0}-3x\,\mathrm{d}x=\left[-3\frac{x^2}{2}\right]_{-2}^{0}=0-(-6)=6$

(iii) Here, we simply split the definite integral into the two previous parts:

$\int_{-2}^{2}|3x|\,\mathrm{d}x=\int_{-2}^{0}|3x|\,\mathrm{d}x+\int_{0}^{2}|3x|\,\mathrm{d}x=6+6=12$

A GDC demonstration is shown here:



LOWER=-2
UPPER=2
∫dx=12

(p) $\int_{0}^{\frac{\pi}{2}}\sin 2x\,\mathrm{d}x=\left(-\frac{1}{2}\cos 2x\right)\Big]_{0}^{\frac{\pi}{2}}=-\frac{1}{2}\left(\underbrace{\cos\pi}_{-1}-\underbrace{\cos 0}_{1}\right)=1$

(q) $\int_{1}^{9}\frac{1}{\sqrt{x}}\,\mathrm{d}x=\left(2\sqrt{x}\right)\Big]_{1}^{9}=2\left(\sqrt{9}-\sqrt{1}\right)=4$

(r) $\int_{-2}^{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)\mathrm{d}x=\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)\Big]_{-2}^{2}=\left(\mathrm{e}^{2}+\mathrm{e}^{-2}\right)-\left(\mathrm{e}^{-2}+\mathrm{e}^{2}\right)=0$

If you recognise the symmetry about the origin, the result will be obvious, without the need for integration.

(s) $\int_{-1}^{1}\frac{\mathrm{d}x}{1+x^2}=\arctan x\Big]_{-1}^{1}=\arctan(1)-\arctan(-1)=\frac{\pi}{2}$

(t) $\int_{0}^{\frac{1}{2}}\frac{\mathrm{d}x}{\sqrt{1-x^2}}=\arcsin x\Big]_{0}^{\frac{1}{2}}=\arcsin\left(\frac{1}{2}\right)-\arcsin 0=\frac{\pi}{6}$

(u) $\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{4-x^2}}=\arcsin\left(\frac{x}{2}\right)\Big]_{-1}^{1}=\arcsin\left(\frac{1}{2}\right)-\arcsin\left(-\frac{1}{2}\right)=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}$

(v) $\int_{-2}^{0}\frac{\mathrm{d}x}{4+x^2}=\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big]_{-2}^{0}=\frac{1}{2}\arctan(0)-\frac{1}{2}\arctan(-1)=\frac{1}{2}\arctan(1)=\frac{\pi}{8}$

Note: In the integrals where we use substitution, if we change the limits of integration we do not need to go back to the original variable.

2. **(a)** Use substitution: $x^2+1=t;\ 2x\,\mathrm{d}x=\mathrm{d}t$

$$\int_0^4 \frac{x^3\mathrm{d}x}{\sqrt{x^2+1}}=\int_1^{17}\frac{(t-1)\frac{1}{2}\mathrm{d}t}{\sqrt{t}}=\frac{1}{2}\int_1^{17}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\mathrm{d}t=\frac{1}{2}\left(\frac{2}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)\Bigg]_1^{17}$$

$$=\left(\frac{1}{3}\times 17^{\frac{3}{2}}-\sqrt{17}\right)-\left(\frac{1}{3}\times 1^{\frac{3}{2}}-\sqrt{1}\right)=\frac{14\sqrt{17}+2}{3}$$

(b) Use substitution: $\pi\ln x=t;\ \dfrac{\pi}{x}\mathrm{d}x=\mathrm{d}t$

$$\int_1^{\sqrt{\mathrm{e}}}\frac{\sin(\pi\ln x)\,\mathrm{d}x}{x}=\int_0^{\frac{\pi}{2}}\frac{\sin t\,\mathrm{d}t}{\pi}=\frac{1}{\pi}(-\cos t)\Bigg]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\pi}\left(-\cos\left(\frac{\pi}{2}\right)\right)-\frac{1}{\pi}(-\cos 0)=\frac{1}{\pi}$$

Notice that when $x=1$, $t=0$, and when $x=\sqrt{\mathrm{e}}, t=\pi\ln\sqrt{\mathrm{e}}=\dfrac{\pi}{2}$

(c) Use substitution: $\ln t=u;\ \dfrac{1}{t}\mathrm{d}t=\mathrm{d}u$

$$\int_{\mathrm{e}}^{\mathrm{e}^2}\frac{\mathrm{d}t}{t\ln t}=\int_1^2\frac{\mathrm{d}u}{u}=\ln|u|\Bigg]_1^2=\ln(2)-\ln(1)=\ln 2$$

(d) Use substitution: $9-x^2=t;\ -2x\,\mathrm{d}x=\mathrm{d}t$

$$\int_{-1}^2 3x\sqrt{9-x^2}\,\mathrm{d}x=\int_8^5\frac{3\sqrt{t}\mathrm{d}t}{-2}=\frac{3}{2}\int_5^8\sqrt{t}\mathrm{d}t=\frac{3}{2}\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg]_5^8=16\sqrt{2}-5\sqrt{5}$$

(e) Use substitution: $3+\cos x=t;\ -\sin x\,\mathrm{d}x=\mathrm{d}t$

$$\int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{\sin x}{\sqrt{3+\cos x}}\mathrm{d}x=\int_{3+\frac{1}{2}}^{3-\frac{1}{2}}\frac{-\mathrm{d}t}{\sqrt{t}}=\int_{\frac{5}{2}}^{\frac{7}{2}}\frac{\mathrm{d}t}{\sqrt{t}}=\left(2\sqrt{t}\right)\Big]_{\frac{5}{2}}^{\frac{7}{2}}=\sqrt{14}-\sqrt{10}$$

(f) $\ln x = t, \frac{1}{x}\mathrm{d}x = \mathrm{d}t \Rightarrow \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{\ln x}{x}\mathrm{d}x = \int_1^2 t\,\mathrm{d}t = \left.\frac{t^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$

(g) Substitution: $\arctan x = t;\ \frac{1}{1+x^2}\mathrm{d}x = \mathrm{d}t$

$$\int_1^{\sqrt{3}} \frac{\sqrt{\arctan x}}{1+x^2}\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{t}\mathrm{d}t = \left.\left(\frac{2}{3}t\sqrt{t}\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{2}{3}\left(\frac{\pi}{3}\sqrt{\frac{\pi}{3}} - \frac{\pi}{4}\sqrt{\frac{\pi}{4}}\right)$$

$$= \frac{2\pi}{216}\left(8\sqrt{3\pi} - 9\sqrt{\pi}\right) = \frac{\pi\sqrt{\pi}}{108}\left(8\sqrt{3} - 9\right)$$

(h) Substitution: $\ln x = t;\ \frac{1}{x}\mathrm{d}x = \mathrm{d}t$

$$\int_1^{\sqrt{\mathrm{e}}} \frac{\mathrm{d}x}{x\sqrt{1-(\ln x)^2}} = \int_0^{\frac{1}{2}} \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arcsin t\Big]_0^{\frac{1}{2}} = \arcsin\left(\frac{1}{2}\right) - \arcsin(0) = \frac{\pi}{6}$$

(i) Substitution: $\mathrm{e}^{2x} + 9 = t;\ 2\mathrm{e}^{2x}\mathrm{d}x = \mathrm{d}t$

$$\int_{-\ln 2}^{\ln 2} \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+9}\mathrm{d}x = \int_{\frac{37}{4}}^{13} \frac{\frac{1}{2}\mathrm{d}t}{t} = \left.\left(\frac{1}{2}\ln|t|\right)\right]_{\frac{37}{4}}^{13} = \frac{1}{2}\left(\ln 13 - \ln\left(\frac{37}{4}\right)\right) = \frac{1}{2}\ln\left(\frac{52}{37}\right)$$

(j) $\mathrm{e}^{-2x} = t;\ -2\mathrm{e}^{-2x}\mathrm{d}x = \mathrm{d}t$

$$\int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{\mathrm{e}^{-2x}}{\sqrt{1-\mathrm{e}^{-4x}}}\mathrm{d}x = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{-\frac{1}{2}\mathrm{d}t}{\sqrt{1-t^2}} = \left.\left(-\frac{1}{2}\arcsin t\right)\right]_{\frac{1}{4}}^{\frac{3}{4}} = -\frac{1}{2}\left(\arcsin\left(\frac{3}{4}\right) - \arcsin\left(\frac{1}{4}\right)\right)$$

The value of this integral is the same as the one in the book. However, the form of the answers is different due to different substitutions. Try it!

(k) Substitution: $\tan x = t;\ \sec^2 x\,\mathrm{d}x = \mathrm{d}t$

$$\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sec^2 x\,\mathrm{d}x = \int_0^1 \sqrt{t}\mathrm{d}t = \left.\left(\frac{2}{3}t\sqrt{t}\right)\right]_0^1 = \frac{2}{3}$$

(l) Substitution: $x^2 = t;\ 2x\mathrm{d}x = \mathrm{d}t$

$$\int_0^{\sqrt{\pi}} 7x\cos\left(x^2\right)\mathrm{d}x = \int_0^{\pi}\frac{7}{2}\cos t\,\mathrm{d}t = \left(\frac{7}{2}\sin t\right)\Bigg]_0^{\pi} = \frac{7}{2}\left(\underbrace{\sin\pi}_{0} - \underbrace{\sin 0}_{0}\right) = 0$$

(m) Substitution: $\sqrt{x} = t;\ \dfrac{\mathrm{d}x}{2\sqrt{x}} = \mathrm{d}t$

$$\int_{\pi^2}^{4\pi^2}\frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x = \int_{\pi}^{2\pi} 2\sin t\,\mathrm{d}t = \left(-2\cos t\right)\Big]_{\pi}^{2\pi} = -2\left(\underbrace{\cos(2\pi)}_{1} - \underbrace{\cos(\pi)}_{-1}\right) = -4$$

(n) Substitution: $\dfrac{\sqrt{3}}{2}x^2 = \sin\theta \Rightarrow x\sqrt{3}\mathrm{d}x = \cos\theta\mathrm{d}\theta$

$$\int_0^1 \frac{\sqrt{3}x}{\sqrt{4-3x^4}}\mathrm{d}x = \int_0^{\frac{\pi}{3}}\frac{\cos\theta\mathrm{d}\theta}{2\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\frac{\pi}{3}}\mathrm{d}\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{3}} = \frac{\pi}{6}$$

(o) Either by substitution or by antidifferentiation:

$$\int_0^{\frac{2}{\sqrt{3}}}\frac{\mathrm{d}x}{9+4x^2} = \frac{1}{9}\int_0^{\frac{2}{\sqrt{3}}}\frac{\mathrm{d}x}{1+\left(\frac{2x}{3}\right)^2} = \left(\frac{1}{9}\times\frac{3}{2}\arctan\left(\frac{2x}{3}\right)\right)\Bigg]_0^{\frac{2}{\sqrt{3}}} = \frac{1}{6}\arctan\left(\frac{4\sqrt{3}}{9}\right)$$

(p) Substitution: $\dfrac{x^2}{\sqrt{3}} = t; \dfrac{2}{\sqrt{3}}x\,\mathrm{d}x = \mathrm{d}t$

$$\int_1^{\sqrt{2}}\frac{x\,\mathrm{d}x}{3+x^4} = \int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}}\frac{\frac{\sqrt{3}}{2}\mathrm{d}t}{3+3t^2} = \left(\frac{\sqrt{3}}{6}\arctan(t)\right)\Bigg]_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}}{6}\left(\arctan\frac{2}{\sqrt{3}} - \frac{\pi}{6}\right)$$

This answer is equivalent to the one given in the book.

(q) Substitution: $1-\sin 3t = x \Rightarrow -\cos 3t \times 3\mathrm{d}t = \mathrm{d}x$

$$\int_0^{\frac{\pi}{6}} (1-\sin 3t)\cos 3t\,\mathrm{d}t = \int_1^0 x\times\left(-\frac{1}{3}\right)\mathrm{d}x = \frac{1}{3}\int_0^1 x\,\mathrm{d}x = \frac{1}{3}\times\frac{x^2}{2}\Big]_0^1 = \frac{1}{6}$$

(r) Substitution: $\sin 2\theta = t;\ 2\cos 2\theta\,\mathrm{d}\theta = \mathrm{d}t$

$$\int_0^{\frac{\pi}{4}} \mathrm{e}^{\sin 2\theta}\cos 2\theta\,\mathrm{d}\theta = \int_0^1 \mathrm{e}^t\times\frac{1}{2}\mathrm{d}t = \frac{1}{2}\mathrm{e}^t\Big]_0^1 = \frac{\mathrm{e}-1}{2}$$

(s) Substitution: $\tan 2t = x;\ 2\sec^2 2t\,\mathrm{d}t = \mathrm{d}x$

$$\int_0^{\frac{\pi}{8}} \left(3+\mathrm{e}^{\tan 2t}\right)\sec^2 2t\,\mathrm{d}t = \int_0^1 \left(3+\mathrm{e}^x\right)\times\frac{1}{2}\mathrm{d}x = \frac{1}{2}\left(3x+\mathrm{e}^x\right)\Big]_0^1 = 1+\frac{\mathrm{e}}{2}$$

(t) Substitution: $\mathrm{e}^{t^2} = x \Rightarrow 2t\mathrm{e}^{t^2}\mathrm{d}t = \mathrm{d}x$

$$\int_0^{\sqrt{\ln\pi}} 4t\,\mathrm{e}^{t^2}\sin\left(e^{t^2}\right)\mathrm{d}t = 2\int_1^{\pi}\sin x\,\mathrm{d}x = 2\left(-\cos x\right)\Big]_1^{\pi} = 2+2\cos 1$$

3. **(a)** $$av(f) = \frac{1}{2-1}\int_1^2 x^4\mathrm{d}x = \frac{x^5}{5}\Big]_1^2 = \frac{32}{5}-\frac{1}{5} = \frac{31}{5}$$

(b) $$av(f) = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x = \frac{2}{\pi}\left(\sin x\right)\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right)-\sin 0\right) = \frac{2}{\pi}$$

(c) $$av(f) = \frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^2 x\,\mathrm{d}x = \frac{12}{\pi}\left(\tan x\right)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{12}{\pi}\left(\underbrace{\tan\left(\frac{\pi}{4}\right)}_{1}-\underbrace{\tan\left(\frac{\pi}{6}\right)}_{\frac{\sqrt{3}}{3}}\right) = \frac{12-4\sqrt{3}}{\pi}$$

(d) $$av(f) = \frac{1}{4-0}\int_0^4 \mathrm{e}^{-2x}\mathrm{d}x = \frac{1}{4}\times\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\Big]_0^4 = -\frac{1}{8}\left(\mathrm{e}^{-8}-\mathrm{e}^0\right) = \frac{\mathrm{e}^8-1}{8\mathrm{e}^8}$$

(e) Substitution: $\mathrm{e}^{3x} = t;\ 3\mathrm{e}^{3x}\mathrm{d}x = \mathrm{d}t$

$$av(f) = \frac{1}{0-\left(-\frac{\ln 3}{6}\right)}\int_{-\frac{\ln 3}{6}}^{0}\frac{\mathrm{e}^{3x}}{1+\mathrm{e}^{6x}}\,\mathrm{d}x = \frac{6}{\ln 3}\int_{\frac{1}{\sqrt{3}}}^{1}\frac{\frac{1}{3}\mathrm{d}t}{1+t^2}$$

$$= \frac{6}{\ln 3}\times\frac{1}{3}(\arctan t)\Big]_{\frac{1}{\sqrt{3}}}^{1} = \frac{2}{\ln 3}\left(\arctan 1 - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{2}{\ln 3}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\pi}{6\ln 3}$$

4. **(a)** $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{2}^{x}\frac{\sin t}{t}\,\mathrm{d}t = \frac{\sin x}{x}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{t}^{3}\frac{\sin x}{x}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{3}^{t}\left(-\frac{\sin x}{x}\right)\mathrm{d}x = -\frac{\sin t}{t}$

(c) We use the chain rule.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{0}\frac{\sin t}{t}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x^2}\left(-\frac{\sin t}{t}\right)\mathrm{d}t = -\frac{\sin x^2}{x^2}\cdot 2x = -\frac{2\sin x^2}{x}$$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{0}^{x^2}\frac{\sin u}{u}\,\mathrm{d}u = \frac{\sin x^2}{x^2}\cdot 2x = \frac{2\sin x^2}{x}$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{-\pi}^{t}\frac{\cos y}{1+y^2}\,\mathrm{d}y = \frac{\cos t}{1+t^2}$

(f) We use the chain rule at both integration limits.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{ax}^{bx}\frac{\mathrm{d}t}{5+t^4} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{ax}^{k}\frac{\mathrm{d}t}{5+t^4}+\int_{k}^{bx}\frac{\mathrm{d}t}{5+t^4}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{k}^{ax}-\frac{\mathrm{d}t}{5+t^4}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{k}^{bx}\frac{\mathrm{d}t}{5+t^4}\right)$$

$$= \frac{b}{5+(bx)^4}-\frac{a}{5+(ax)^4}$$

(g)
$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\sin\theta}^{\cos\theta}\frac{1}{1-x^2}\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\sin\theta}^{k}\frac{1}{1-x^2}\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{k}^{\cos\theta}\frac{1}{1-x^2}\,\mathrm{d}x$$
$$=\frac{-1}{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}\times\cos\theta+\frac{1}{\underbrace{1-\cos^2\theta}_{\sin^2\theta}}\times(-\sin\theta)=-\sec\theta-\csc\theta$$

(h) $$\frac{\mathrm{d}}{\mathrm{d}x}\int_5^{x^{\frac{1}{4}}}\mathrm{e}^{t^4+3t^2}\,\mathrm{d}t=\mathrm{e}^{x+3\sqrt{x}}\times\frac{1}{4x^{\frac{3}{4}}}=\frac{\mathrm{e}^{x+3\sqrt{x}}}{4x^{\frac{3}{4}}}$$

5. If $F(x)=\int_0^{2x-x^2}\cos\left(\frac{1}{1+t^2}\right)\mathrm{d}t$ has an extreme value, then its derivative with respect to the variable x has to be zero and the derivative has to be changing signs at that point.

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2x-x^2}\cos\left(\frac{1}{1+t^2}\right)\mathrm{d}t=\cos\left(\frac{1}{1+\left(2x-x^2\right)^2}\right)\times(2-2x)$$

We can see that the expression can equal zero, whenever $\cos\left(\frac{1}{1+\left(2x-x^2\right)^2}\right)=0$ or $x=1$.

However, $0<\frac{1}{1+\left(2x-x^2\right)^2}\leq 1$ because this fraction has a numerator that is less than or equal to the denominator. Thus $\cos\left(\frac{1}{1+\left(2x-x^2\right)^2}\right)>0$ for all values. This means that the derivative is equal to zero when $x = 1$ and is positive before that, but negative after that. That is, it has a maximum at this point.

6. **(a)** $$\int_0^k\frac{\mathrm{d}x}{3x+2}=\frac{1}{3}\ln(3x+2)\Big]_0^k=\frac{1}{3}\left(\ln(3k+2)-\ln 2\right)=\frac{1}{3}\ln\left(\frac{3k+2}{2}\right)$$

Notice that we did not take the absolute value of the natural logarithm since k must be positive. If k is negative, the interval $[k,0]$ contains a vertical asymptote at $x=-\frac{2}{3}$; therefore, the function cannot be integrated.

(b) $\frac{1}{3}\ln\left(\frac{3k+2}{2}\right)=1 \Rightarrow \ln\left(\frac{3k+2}{2}\right)=3 \Rightarrow \frac{3}{2}k+1=\mathrm{e}^3 \Rightarrow k=\frac{2\left(\mathrm{e}^3-1\right)}{3}$

7. To evaluate $\int_0^1 x^p\left(1-x\right)^q \,\mathrm{d}x$, we can use substitution. Substitute $u=1-x \Rightarrow \mathrm{d}x=-\mathrm{d}u$ and change the limits of integration to the new variable.
Remember that $u = 1$ when $x = 0$, and $u = 0$ when $x = 1$.

$$\int_0^1 x^p\left(1-x\right)^q \,\mathrm{d}x=-\int_1^0\left(1-u\right)^p u^q \mathrm{d}u=\int_0^1\left(1-u\right)^p u^q \mathrm{d}u$$

Now replace the variable name in the answer by x, and the result follows.

8. **(a)** We use the substitution method applied in question **7**:

$$1-x=t \Rightarrow -\mathrm{d}x=\mathrm{d}t$$

$$\int x\left(1-x\right)^k \mathrm{d}x=\int\left(1-t\right)t^k\left(-\mathrm{d}t\right)=\int\left(t^{k+1}-t^k\right)\mathrm{d}t=\frac{t^{k+2}}{k+2}-\frac{t^{k+1}}{k+1}+c$$
$$=\frac{\left(1-x\right)^{k+2}}{k+2}-\frac{\left(1-x\right)^{k+1}}{k+1}+c, \quad k\in\mathbb{N}, c\in\mathbb{R}$$

(b) $\int_0^1 x\left(1-x\right)^k \mathrm{d}x=\left.\frac{\left(1-x\right)^{k+2}}{k+2}-\frac{\left(1-x\right)^{k+1}}{k+1}\right]_0^1=0-\left(\frac{1}{k+2}-\frac{1}{k+1}\right)=\frac{1}{\left(k+2\right)\left(k+1\right)}$

9. **(a)** $F\left(3\right)=\int_3^3\sqrt{5t^2+2}\,\mathrm{d}t=0$

(b) $F'\left(x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\int_3^x\sqrt{5t^2+2}\,\mathrm{d}t=\sqrt{5x^2+2} \Rightarrow F'\left(3\right)=\sqrt{5\times3^2+2}=\sqrt{47}$

(c) $F''\left(x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{5x^2+2}\right)=\frac{10x}{2\sqrt{5x^2+2}} \Rightarrow F''\left(3\right)=\frac{10\times3}{2\sqrt{5\times3^2+2}}=\frac{15}{\sqrt{47}}=\frac{15\sqrt{47}}{47}$

10. If the function $f(x)$ is constant over the set of positive real numbers, then it is neither increasing nor decreasing. Thus, its derivative is equal to zero.

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{3x}\frac{\mathrm{d}t}{t}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{k}\frac{\mathrm{d}t}{t}+\int_k^{3x}\frac{\mathrm{d}t}{t}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_k^{3x}\frac{\mathrm{d}t}{t}-\int_k^{x}\frac{\mathrm{d}t}{t}\right)$$

$$=\frac{1}{3x}\times 3-\frac{1}{x}=0$$

Exercise 14.5

For each question, before integration is performed, the rational function will be decomposed using partial fractions, then the resulting function is integrated. All the questions in this section were decomposed in Exercise 2.6. We will use the results of section 2.6 and integrate the resulting functions.

1. **(a)** $\dfrac{5x+1}{x^2+x-2}\equiv\dfrac{3}{x+2}+\dfrac{2}{x-1}$, thus,

$$\int\frac{5x+1}{x^2+x-2}\mathrm{d}x=\int\frac{3}{x+2}\mathrm{d}x+\int\frac{2}{x-1}\mathrm{d}x=3\ln|x+2|+2\ln|x-1|\ +\ c$$

(b) $\dfrac{x+4}{x^2-2x}\equiv\dfrac{x+4}{x(x-2)}=\dfrac{3}{x-2}-\dfrac{2}{x}$

$$\int\frac{x+4}{x^2-2x}\mathrm{d}x=\int\frac{3}{x-2}\mathrm{d}x-\int\frac{2}{x}\mathrm{d}x=3\ln|x-2|-2\ln|x|+c$$

(c) $\dfrac{x+2}{x^2+4x+3}\equiv\dfrac{x+2}{(x+3)(x+1)}=\dfrac{1}{2(x+3)}+\dfrac{1}{2(x+1)}$

$$\int\frac{x+2}{x^2+4x+3}\mathrm{d}x=\int\frac{1}{2(x+3)}\mathrm{d}x+\int\frac{1}{2(x+1)}\mathrm{d}x$$

$$=\frac{1}{2}\ln|x+3|+\frac{1}{2}\ln|x+1|+c=\frac{1}{2}\ln\left|x^2+4x+3\right|+c$$

(d) $$\frac{5x^2+20x+6}{x^3+2x^2+x} \equiv \frac{5x^2+20x+6}{x(x+1)^2} = \frac{9}{(x+1)^2} - \frac{1}{x+1} + \frac{6}{x}$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,\mathrm{d}x = \int \frac{9}{(x+1)^2}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x + \int \frac{6}{x}\,\mathrm{d}x$$
$$= -\frac{9}{x+1} - \ln|x+1| + 6\ln|x| + c$$

(e) $$\frac{2x^2+x-12}{x^3+5x^2+6x} \equiv \frac{2x^2+x-12}{x(x+3)(x+2)} = \frac{1}{x+3} + \frac{3}{x+2} - \frac{2}{x}$$

$$\int \frac{2x^2+x-12}{x^3+5x^2+6x}\,\mathrm{d}x = \int \frac{1}{x+3}\,\mathrm{d}x + \int \frac{3}{x+2}\,\mathrm{d}x - \int \frac{2}{x}\,\mathrm{d}x$$
$$= \ln|x+3| + 3\ln|x+2| - 2\ln|x| + c$$

(f) $$\frac{4x^2+2x-1}{x^3+x^2} \equiv \frac{4x^2+2x-1}{x^2(x+1)} = \frac{1}{x+1} - \frac{1}{x^2} + \frac{3}{x}$$

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,\mathrm{d}x = \int \frac{1}{x+1}\,\mathrm{d}x - \int \frac{1}{x^2}\,\mathrm{d}x + \int \frac{3}{x}\,\mathrm{d}x$$
$$= \ln|x+1| + \frac{1}{x} + 3\ln|x| + c$$

The rest of the exercises were done in a similar fashion.

(g) $$\int \frac{3}{x^2+x-2}\,\mathrm{d}x = \int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{1}{x+2}\,\mathrm{d}x$$
$$= -\ln|x+2| + \ln|x-1| + c = \ln\left|\frac{x-1}{x+2}\right| + c$$

(h) $$\int \frac{5-x}{2x^2+x-1}\,\mathrm{d}x = \int \frac{3}{2x-1}\,\mathrm{d}x - \int \frac{2}{x+1}\,\mathrm{d}x$$
$$= \frac{3\ln|2x-1|}{2} - 2\ln|x+1| + c$$

(i)
$$\int \frac{3x+4}{(x+2)^2}\,\mathrm{d}x = \int \frac{3}{x+2}\,\mathrm{d}x - \int \frac{2}{(x+2)^2}\,\mathrm{d}x$$
$$= 3\ln|x+2| + \frac{2}{x+2} \ + \ c$$

(j)
$$\int \frac{12}{x^4-x^3-2x^2}\,\mathrm{d}x = \int \frac{1}{x-2}\,\mathrm{d}x - \int \frac{4}{x+1}\,\mathrm{d}x - \int \frac{6}{x^2}\,\mathrm{d}x + \int \frac{3}{x}\,\mathrm{d}x$$
$$= \ln|x-2| - 4\ln|x+1| + \frac{6}{x} + 3\ln|x| \ + \ c$$

(k)
$$\int \frac{2}{x^3+x}\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x - \int \frac{2x}{x^2+1}\,\mathrm{d}x$$
$$= 2\ln|x| - \ln(x^2+1) + c$$

(l)
$$\int \frac{x+2}{x^3+3x}\,\mathrm{d}x = \int \frac{2}{3x}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{(x^2+3)} - \int \frac{2x}{3(x^2+3)}\,\mathrm{d}x =$$
$$= \frac{2\ln|x|}{3} + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}x}{3}\right) - \frac{\ln(x^2+3)}{3} + c$$

(m)
$$\int \frac{3x+2}{x^3+6x}\,\mathrm{d}x = \int \frac{1}{3x}\,\mathrm{d}x + \int \frac{3}{x^2+6}\,\mathrm{d}x - \int \frac{x}{3(x^2+6)}\,\mathrm{d}x$$
$$= \frac{\ln|x|}{3} + \frac{\sqrt{6}}{2}\arctan\left(\frac{\sqrt{6}x}{6}\right) - \frac{\ln(x^2+6)}{6} + c$$

(n)
$$\int \frac{2x+3}{x^3+8x}\,\mathrm{d}x = \int \frac{3}{8x}\,\mathrm{d}x + \int \frac{2}{x^2+8}\,\mathrm{d}x - \int \frac{3x}{8(x^2+8)}\,\mathrm{d}x$$
$$= \frac{3}{8}\ln|x| + \frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}x}{4}\right) - \frac{3}{16}\ln(x^2+8) + c$$

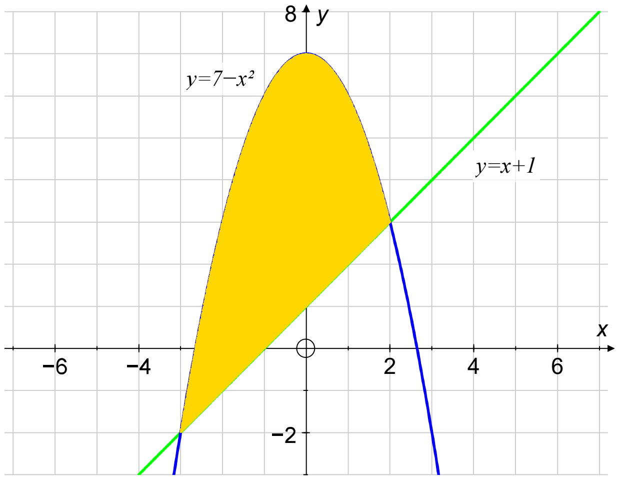
Exercise 14.6

1. We will alternate writing solutions with and without using a GDC.

(a) For this exercise, we will demonstrate two types of answers; the first is a Paper 1 type and the second is a Paper 2 type.

Examiner note: If this were a Paper 1 question:

First, we sketch the line and the parabola, and then shade the enclosed area.



Now we need to find the points of intersection by solving the system of equations.

$$\left.\begin{array}{r} y=x+1 \\ y=7-x^2 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=x+1 \\ x+1=7-x^2 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=x+1 \\ x^2+x-6=0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} y=x+1 \\ (x-2)(x+3)=0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} y_1=3, y_2=-2 \\ x_1=2, x_2=-3 \end{array}\right\} \Rightarrow (-3,-2) \text{ or } (2,3)$$

So, the integral that we need to calculate is:

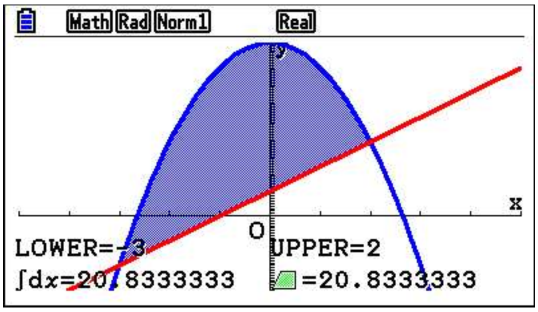
$$\int_{-3}^{2} \left|(7-x^2)-(x+1)\right| \mathrm{d}x = \int_{-3}^{2} (6-x-x^2)\,\mathrm{d}x = \left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$$

$$=\left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+9\right)=\frac{125}{6}$$

Examiner note: If this were a Paper 2 question, then you do not need to show every step of calculating intercepts and evaluating the integral. It is enough to write for example:

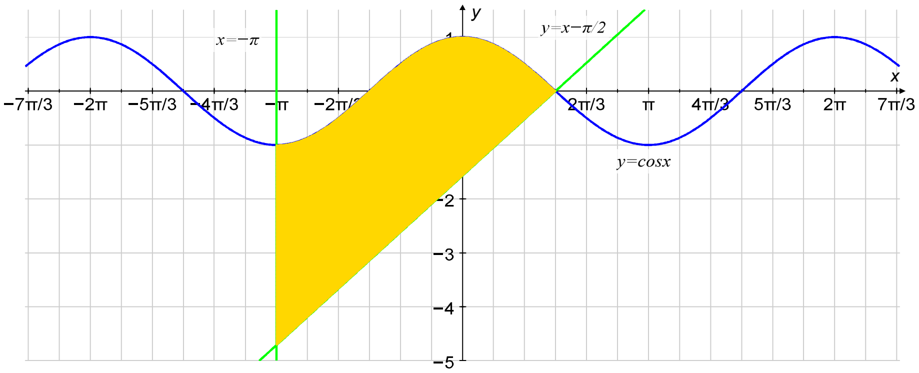
The area of the region is bounded by the graphs of the two functions and hence it can be found by subtracting the areas between each of the functions and the *x*-axis between the vertical lines at the *x*-coordinates of their points of intersection.

$$Area = \int_{-3}^{2} \left|\left(7-x^2\right)-\left(x+1\right)\right| \mathrm{d}x \approx 20.83$$



LOWER=-3
UPPER=2
∫dx=20.8333333
=20.8333333

(b) First, we sketch the cosine curve, the oblique line and the vertical line, and then shade the enclosed area.



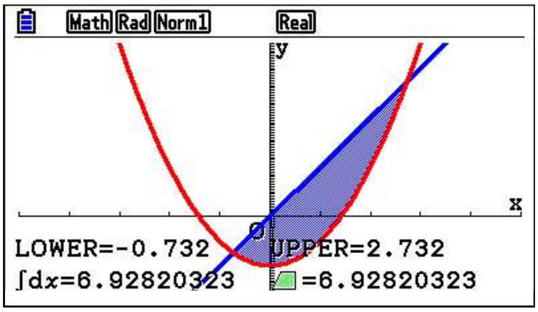
Now we have to find the point of intersection of the curve and the oblique line. By inspection, we can see that the point is $\left(\frac{\pi}{2}, 0\right)$; therefore, to find the area, we need to solve the following integral:

$$\int_{-\pi}^{\frac{\pi}{2}} \left|\cos x-\left(x-\frac{\pi}{2}\right)\right| \mathrm{d}x = \left[\sin x-\frac{x^2}{2}+\frac{\pi}{2}x\right]_{-\pi}^{\frac{\pi}{2}} = \left(1-\frac{\pi^2}{8}+\frac{\pi^2}{4}\right)-\left(0-\frac{\pi^2}{2}-\frac{\pi^2}{2}\right)$$

$$=1+\frac{\pi^2}{8}+\pi^2=1+\frac{9\pi^2}{8}$$

(c) The two functions intersect at points with x-coordinates $1-\sqrt{3}$ and $1+\sqrt{3}$

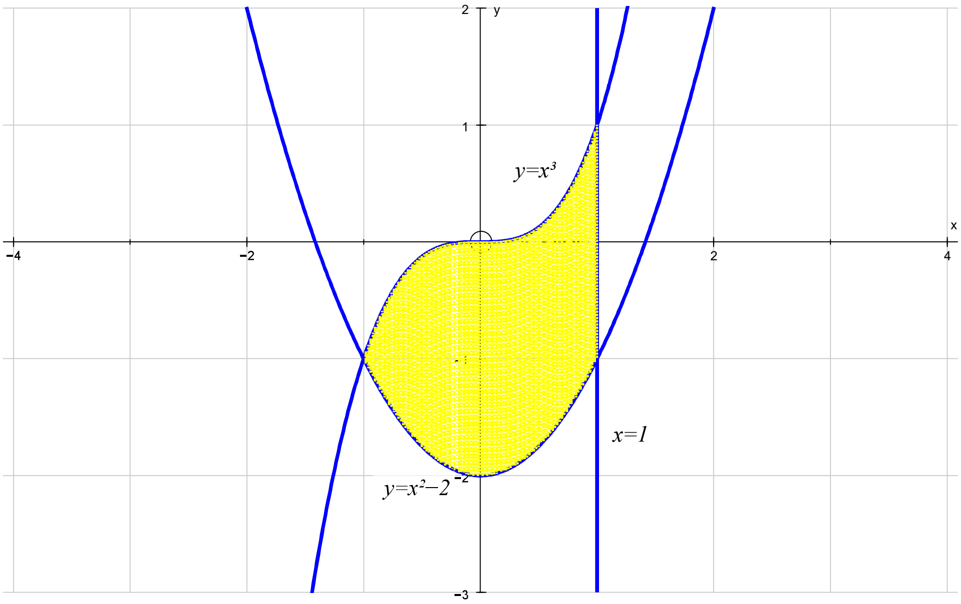
The area of the bounded region shown below is given by $\int_{1-\sqrt{3}}^{1+\sqrt{3}}\left|2x-\left(x^2-2\right)\right|\mathrm{d}x$



LOWER=-0.732 UPPER=2.732
∫dx=6.92820323 🟩=6.92820323

The required area is 6.93 units^2.

(d) Sketching all the curves and shading the enclosed area gives the following graph.



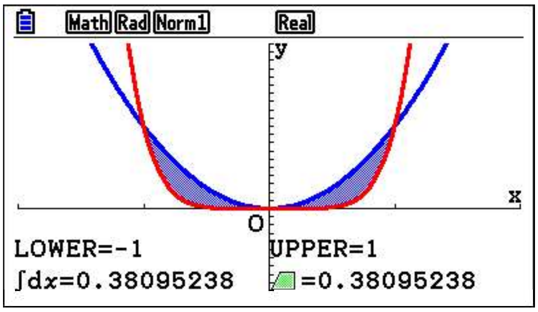
By inspection, we can see that the point of intersection of the curves is $(-1,-1)$. Therefore, the integral that we need to calculate to find the area is:

$$\int_{-1}^{1}\left|x^3-\left(x^2-2\right)\right|\mathrm{d}x=\int_{-1}^{1}\left(x^3-x^2+2\right)\mathrm{d}x=\left[\frac{x^4}{4}-\frac{x^3}{3}+2x\right]_{-1}^{1}=\frac{10}{3}$$

(e) The two functions intersect when $x^6 = x^2$. Thus,

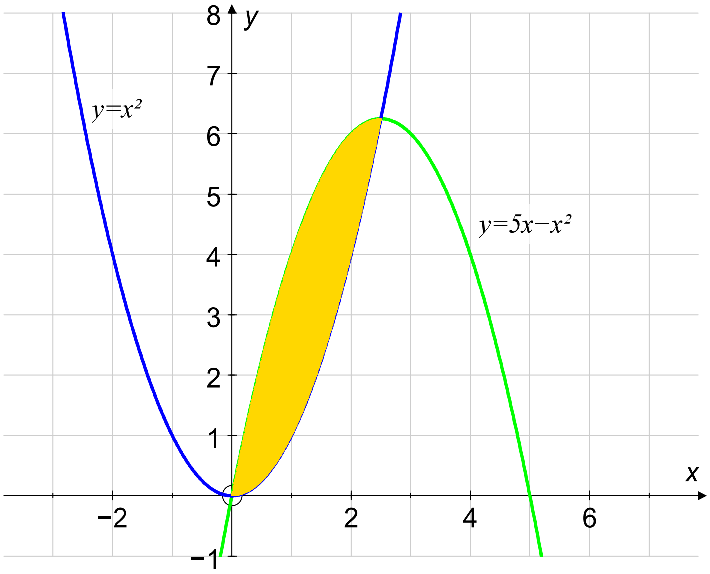
$x^6 - x^2 = x^2\left(x^2+1\right)\left(x^2-1\right)$ and the points of intersection are at the points with x-coordinates -1, 0 and 1. At $x = 0$, they are tangent to each other making the graph of $y = x^2$ above that of $y = x^6$. Thus the area is:

$$\int_{-1}^{1}\left|x^2 - x^6\right|\mathrm{d}x = \frac{8}{21} \approx 0.381$$



LOWER=-1
∫dx=0.38095238
UPPER=1
▨=0.38095238

(f) Sketching the curves and shading the enclosed area gives the following graph.



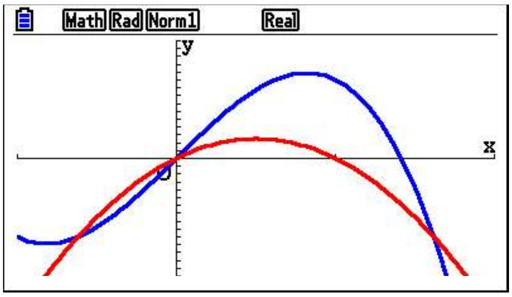
One point of intersection is obviously the origin, while the other looks like it has an x-coordinate of 2.5, but since we are not sure we will solve the simultaneous equations.

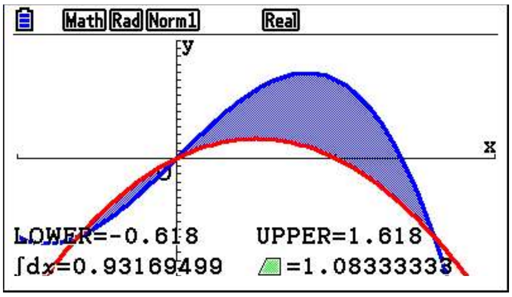
$$\left.\begin{array}{r} y = 5x - x^2 \\ y = x^2 \end{array}\right\} \Rightarrow x^2 = 5x - x^2 \Rightarrow 2x^2 - 5x = 0 \Rightarrow \left\{\begin{array}{l} x_1 = 0,\ x_2 = \dfrac{5}{2} \\ y_1 = 0,\ y_2 = \dfrac{25}{4} \end{array}\right.$$

So, the integral we have to calculate is:

$$\int_0^{\frac{5}{2}} \left|\left(5x - x^2\right) - x^2\right| \mathrm{d}x = \int_0^{\frac{5}{2}} \left(5x - 2x^2\right) \mathrm{d}x = \left[5 \times \frac{x^2}{2} - 2 \times \frac{x^3}{3}\right]_0^{\frac{5}{2}} = \frac{125}{24}$$

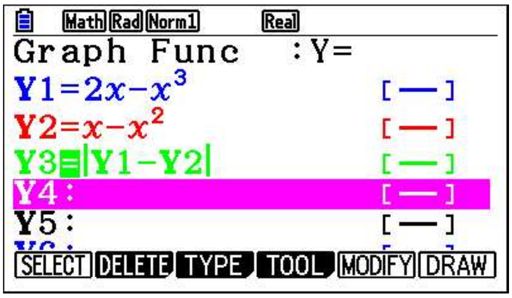
(g) There are two areas that are defined by the two curves, and the curves alternate between being the upper and lower graphs. In order to get both the areas enclosed by the curves shaded, we need to pay attention to the absolute value of the difference between their integrals. This can be done directly or some GDCs will also give the right answer. We demonstrate both cases below.



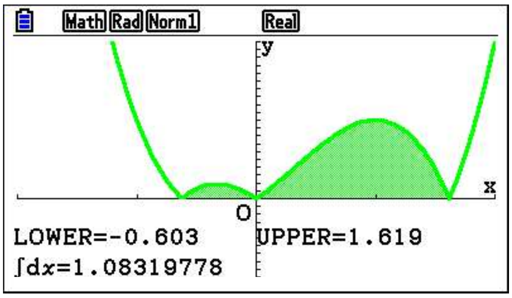


LOWER=-0.618 UPPER=1.618
∫dx=0.93169499 ▱=1.08333333

Writing the functions separately, then evaluating the absolute value of the difference:



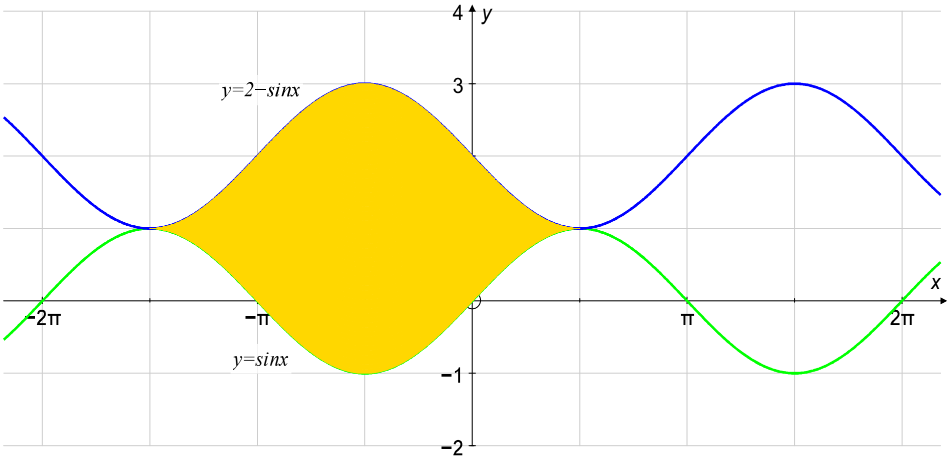
```
Graph Func   :Y=
Y1=2x-x³            [—]
Y2=x-x²             [—]
Y3=|Y1-Y2|          [—]
Y4:                 [—]
Y5:                 [—]
```



LOWER=-0.603 UPPER=1.619
∫dx=1.08319778

So, the required area is approximately 1.083 units^2.

(h) Sketching the curves and shading the enclosed area for one period only gives the following graph.

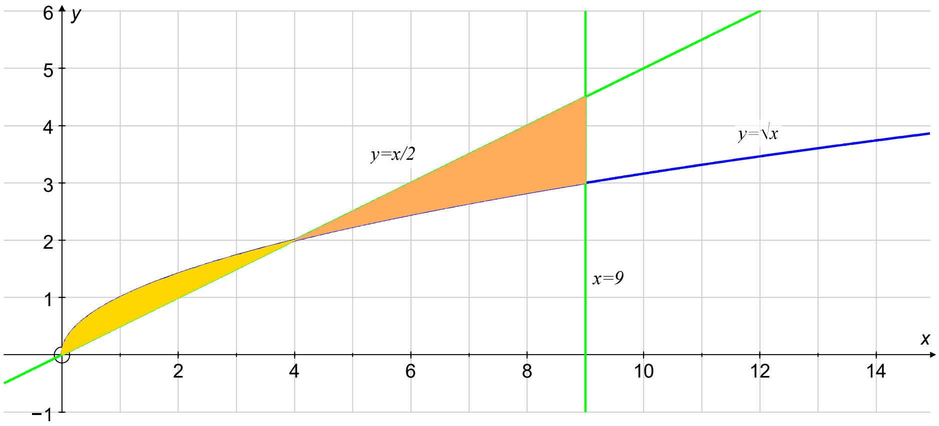


By inspection, we notice that the points of intersection are $\left(-\frac{3\pi}{2},1\right)$ and $\left(\frac{\pi}{2},1\right)$, so the integral we have to calculate is:

$$\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}\left|(2-\sin x)-\sin x\right|\mathrm{d}x=\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}(2-2\sin x)\mathrm{d}x=\left[2x+2\cos x\right]_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$=\left(2\times\frac{\pi}{2}+2\cos\frac{\pi}{2}\right)-\left(2\times\left(-\frac{3\pi}{2}\right)+2\cos\left(-\frac{3\pi}{2}\right)\right)=4\pi$$

(i) Sketching the curves and shading the enclosed area gives the following graph.

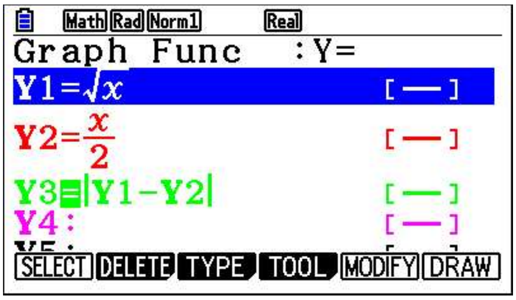


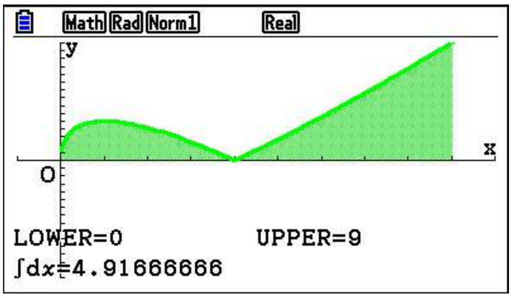
Again, we notice that there are two areas bounded by the curves and that the curves exchange positions of being upper and lower. By inspection, we can see that the points of intersection are the origin and (4, 2). Therefore, to find the total shaded area, we need to find the following integrals and add them up.

$$\text{Area} = \int_0^9 \left|\sqrt{x} - \frac{x}{2}\right| \mathrm{d}x = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) \mathrm{d}x + \int_4^9 \left(\frac{x}{2} - \sqrt{x}\right) \mathrm{d}x$$

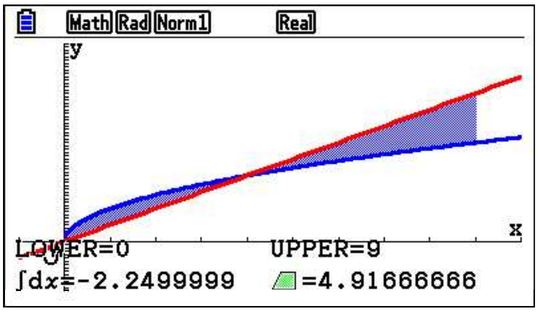
$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4 + \left[\frac{x^2}{4} - \frac{2}{3}x^{\frac{3}{2}}\right]_4^9 = \left(\frac{16}{3} - 4\right) + \left(\frac{81}{4} - 18\right) = \frac{4}{3} + \frac{43}{12} = \frac{59}{12} \approx 4.92$$

GDC work demonstration:

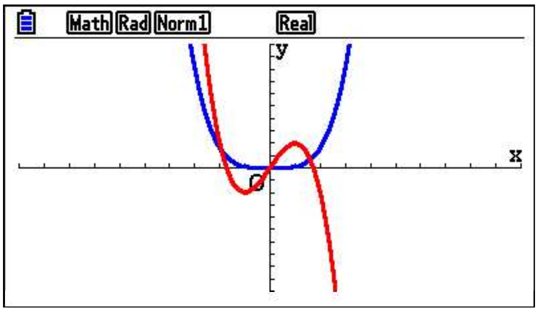

Graph Func :Y=
Y1=$\sqrt{x}$ [—]
Y2=$\frac{x}{2}$ [—]
Y3=|Y1−Y2| [—]
Y4: [—]


LOWER=0 UPPER=9
∫dx=4.91666666

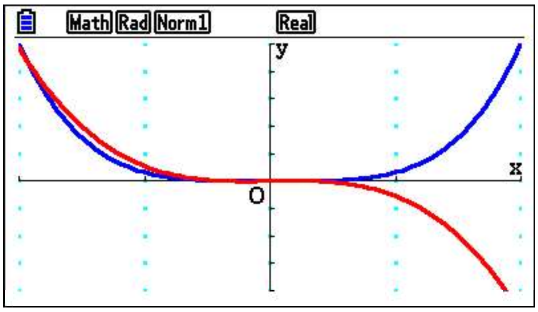
or


LOWER=0 UPPER=9
∫dx=-2.2499999 ▨=4.91666666

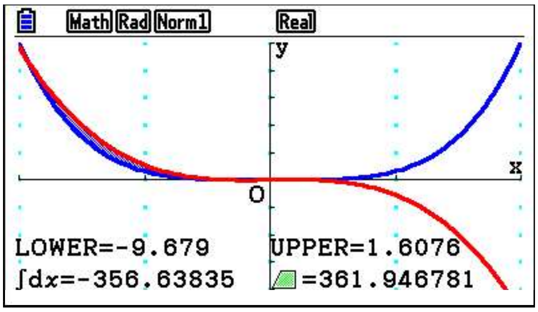
(j) This question cannot be done algebraically since we cannot solve the equation of the fourth degree for finding the boundaries of the integral. Sketch the graphs first.



In cases similar to this, you have to understand that the graph you see may not be the whole one. We know that, in general, a quartic function decreases faster than a cubic function. Therefore, apart from the three points of intersection that we can see on the graph, we expect to find one more to the left. To find the fourth point of intersection, we need to adjust the window.



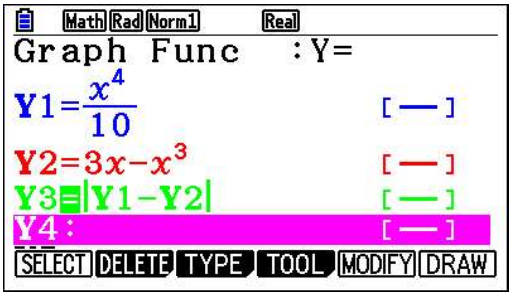
Apparently, there is another point of intersection on the left.
It is wise to have the GDC do the calculation of finding the area between curves.



```
LOWER=-9.679       UPPER=1.6076
∫dx=-356.63835     ▱=361.946781
```

Area is 361.95

Examiner note: By using the absolute value function, we can skip all the points in between and simply calculate the integral from the first point on the left side until the last point on the right side. The final answer in the IB exam would be given correct to 3 significant figures, 362, or otherwise as stated in the question.



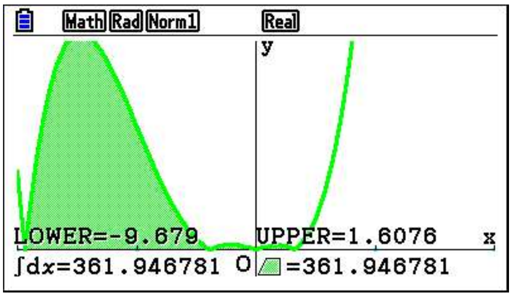
Graph Func :Y=

$Y1=\frac{x^4}{10}$ [—]

$Y2=3x-x^3$ [—]

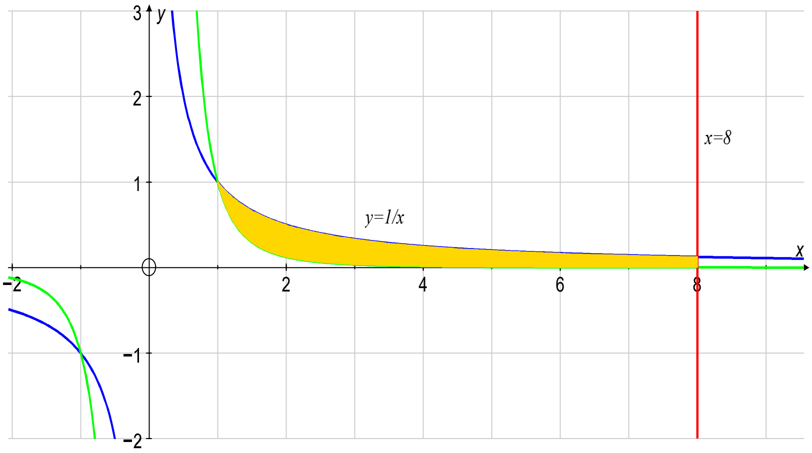
$Y3=|Y1-Y2|$ [—]

Y4: [—]



```
LOWER=-9.679       UPPER=1.6076
∫dx=361.946781     ▱=361.946781
```

(k) Sketching the curves and shading the enclosed area gives the following graph.

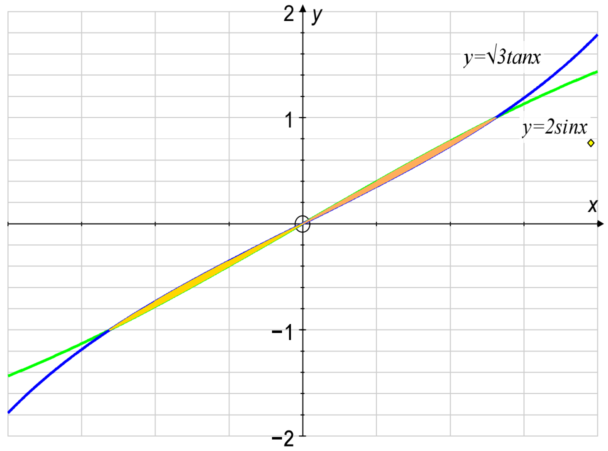


Points of intersection are given by

$\frac{1}{x}=\frac{1}{x^3}\Rightarrow x(x-1)(x+1)=0$ where the only two possible points are $(-1,-1)$ or $(1,1)$.

$$\text{Area} = \int_1^8\left|\frac{1}{x}-\frac{1}{x^3}\right|\mathrm{d}x=\left[\ln|x|+\frac{1}{2x^2}\right]_1^8=\left(\ln 8+\frac{1}{128}\right)-\left(\ln 1+\frac{1}{2}\right)=3\ln 2-\frac{63}{128}$$

(l) Sketching the curves for the restricted domain and shading the enclosed area gives the following graph.



To find the points of intersection, we need to solve the following equation:

$$2\sin x=\sqrt{3}\tan x\Rightarrow 2\sin x-\sqrt{3}\frac{\sin x}{\cos x}=0\Rightarrow \sin x\left(2-\frac{\sqrt{3}}{\cos x}\right)=0$$

$$\Rightarrow (\sin x=0\ )\text{or}\left(2-\frac{\sqrt{3}}{\cos x}=0\right)\Rightarrow(\sin x=0\ )\text{or}\left(\cos x=\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow x_1=0,\ x_2=-\frac{\pi}{6},\ x_3=\frac{\pi}{6}$$

We have two areas (one in the first quadrant and one in the third), bounded by the curves, and both functions are odd (symmetrical with respect to the origin); therefore, the enclosed areas must each have the same area. So, the final area is obtained by finding the area of one region and multiplying it by 2.

$$\int_0^{\frac{\pi}{6}}\left(2\sin x-\sqrt{3}\tan x\right)\mathrm{d}x=\left[-2\cos x+\sqrt{3}\ln|\cos x|\right]_0^{\frac{\pi}{6}}$$

$$=\left(-2\cos\left(\frac{\pi}{6}\right)+\sqrt{3}\ln\left(\cos\left(\frac{\pi}{6}\right)\right)\right)-\left(-2\cos 0+\sqrt{3}\ln(\cos 0)\right)$$

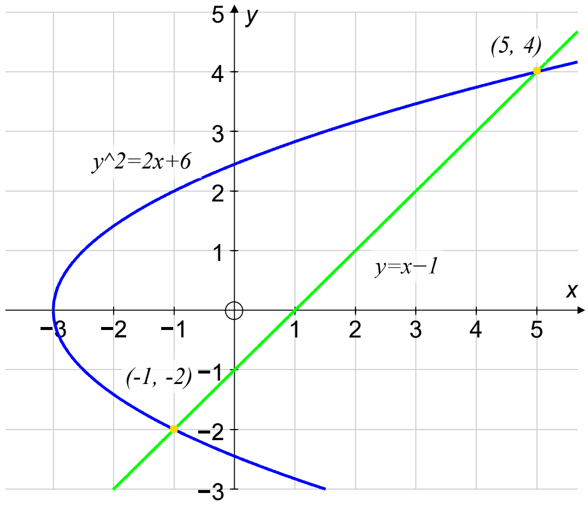
$$=-2\times\frac{\sqrt{3}}{2}+\sqrt{3}\ln\left(\frac{\sqrt{3}}{2}\right)+2=2-\sqrt{3}+\sqrt{3}\left(\frac{1}{2}\ln 3-\ln 2\right)$$

$$\text{Area}=2\times\left[2-\sqrt{3}+\sqrt{3}\left(\frac{1}{2}\ln 3-\ln 2\right)\right]=4-2\sqrt{3}+\sqrt{3}\left(\ln 3-2\ln 2\right)$$

Note: $\int\tan x\,\mathrm{d}x=\int\frac{\sin x}{\cos x}\mathrm{d}x=\int\frac{-\mathrm{d}(\cos x)}{\cos x}=-\ln|\cos x|+c$

In questions **(m)** – **(q)**, we will integrate with respect to y since, in both equations of the curves, the variable x is linear and therefore it is easier to express it in terms of y.

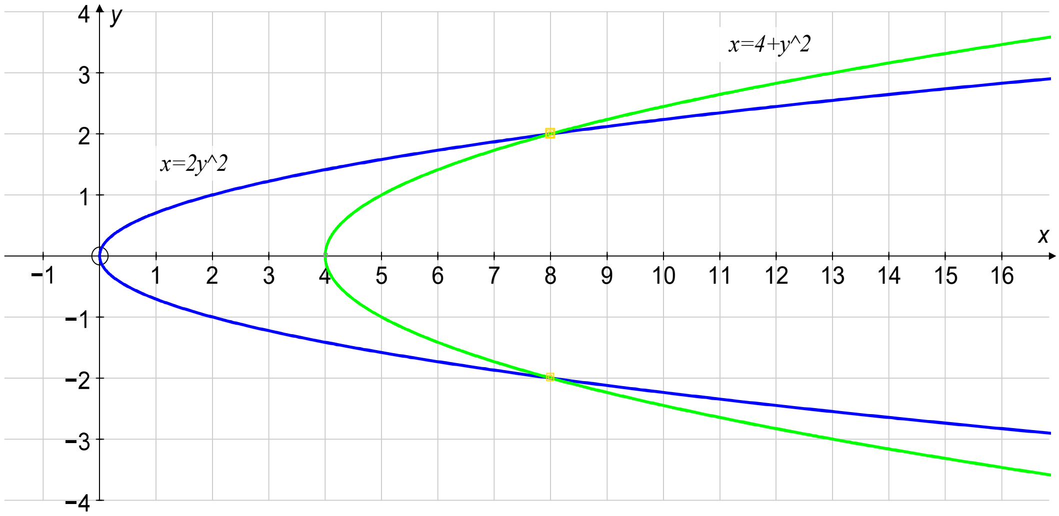
(m) We sketch the curves, find the points of intersection and swap the variable of integration.



$$\int_{-2}^{4}\left((y+1)-\left(\frac{1}{2}y^2-3\right)\right)\mathrm{d}y=\int_{-2}^{4}\left(y-\frac{1}{2}y^2+4\right)\mathrm{d}y=\left(\frac{y^2}{2}-\frac{y^3}{6}+4y\right)\Bigg]_{-2}^{4}$$

$$=\left(8-\frac{32}{3}+16\right)-\left(2+\frac{4}{3}-8\right)=18$$

(n) We sketch the curves, find the points of intersection and swap the variable of integration.

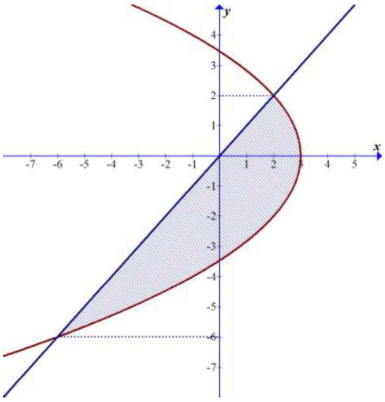


$$\int_{-2}^{2}\left(\left(4+y^2\right)-\left(2y^2\right)\right)\mathrm{d}y=\int_{-2}^{2}\left(4-y^2\right)\mathrm{d}y=\left(4y-\frac{y^3}{3}\right)\Bigg]_{-2}^{2}=\frac{32}{3}$$

Notice that since the functions are symmetrical with respect to the *x*-axis, we could have used an integral from 0 to 2 and multiplied it by 2.

(o) To use a GDC, we need to express *x* explicitly in terms of *y* and then use the finite integral feature on the calculator.

$$\begin{cases} 4x + y^2 = 12 \\ y = x \end{cases} \Rightarrow \begin{cases} x = 3 - \dfrac{y^2}{4} \\ x = y \end{cases}$$

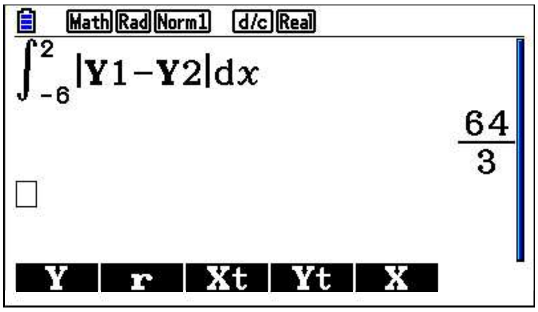


Next step is to partially solve the simultaneous equations to find the limits of integration. We will use *x* instead of *y* on the GDC.

Solving the system, we can either use GDC or directly. This system is simple:

$$\left.\begin{matrix} x = 3 - \dfrac{y^2}{4} \\ x = y \end{matrix}\right\} \Rightarrow y^2 + 4y - 12 = 0 \Rightarrow y = -6 \text{ or } y = 2. \text{ Thus,}$$

$$\text{Area} = \int_{-6}^{2} \left| 3 - \frac{y^2}{4} - y \right| \mathrm{d}y = \frac{64}{3}$$



$\int_{-6}^{2}|Y1-Y2|dx$

$\frac{64}{3}$

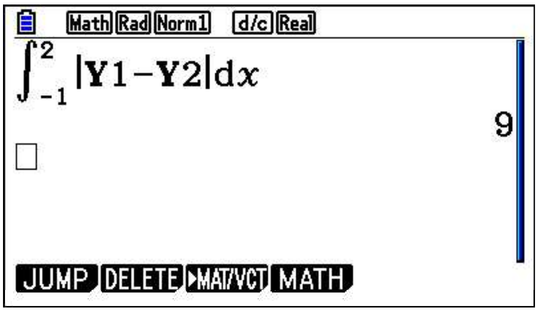
(p) To use a GDC, we need to express *x* explicitly in terms of *y* and then use the finite integral feature on the calculator.

$$\begin{cases} x-y=7 \\ x=2y^2-y+3 \end{cases} \Rightarrow \begin{cases} x=7+y \\ x=2y^2-y+3 \end{cases}$$

Next step is to partially solve the simultaneous equations to find the limits of integration. We will use *x* instead of *y* on the GDC.

Solving the system, we can either use GDC or directly.

$7+y=2y^2-y+3 \Rightarrow 2y^2-2y-4=0 \Rightarrow y^2-y-2=0 \Rightarrow y=-1$ or $y=2$

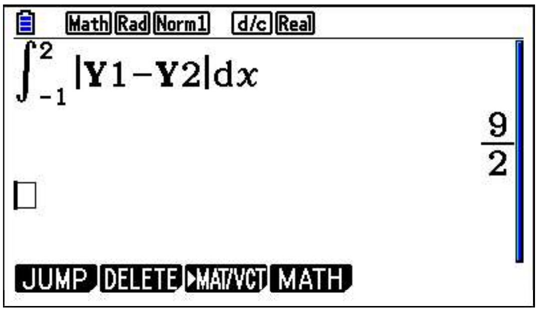


$\int_{-1}^{2} |Y1-Y2|dx$

9

(q) Since the variable *x* is already expressed explicitly in terms of *y*, we simply need to partially solve the simultaneous equations to find the borders of integration.

$y^2=2y^2-y-2 \Rightarrow y^2-y-2=0 \Rightarrow y=-1$ or $y=2$

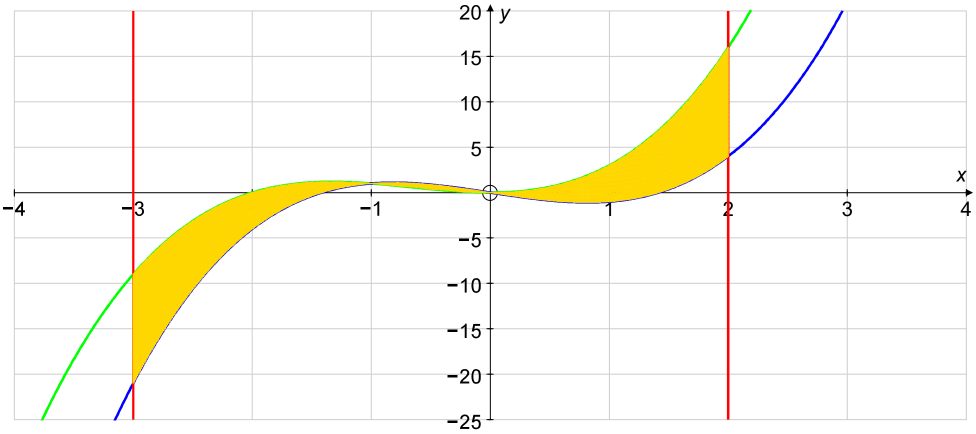


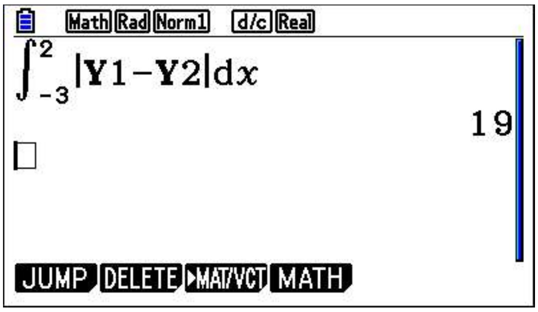
$\int_{-1}^{2} |Y1-Y2|dx$

$\frac{9}{2}$

Note: We don't have to simplify the expressions, and, if we are not sure which function is the upper one and which is the lower one, we don't need to spend too much time on graphing and identifying. We can simply use the absolute value of the difference of two functions; the result will always be positive and therefore it is the area between the curves.

(r) In this case, since the boundaries of integration are given by the vertical lines, we don’t even have to sketch the curves. Since there are multiple areas enclosed by the two curves (alternating upper and lower curves), we can simply apply the absolute value function.

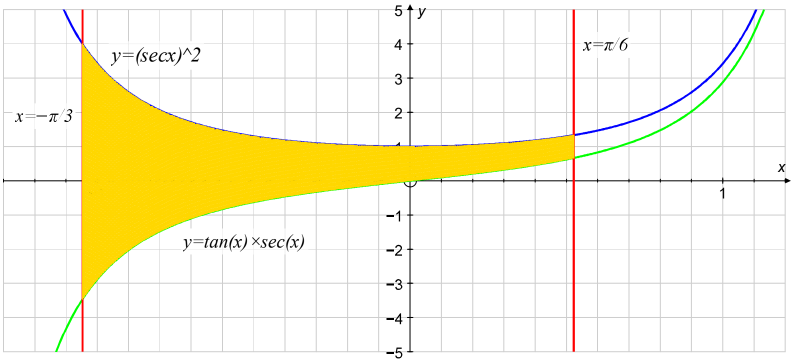



Math Rad Norm1 d/c Real

$\int_{-3}^{2} |Y1-Y2|dx$

19

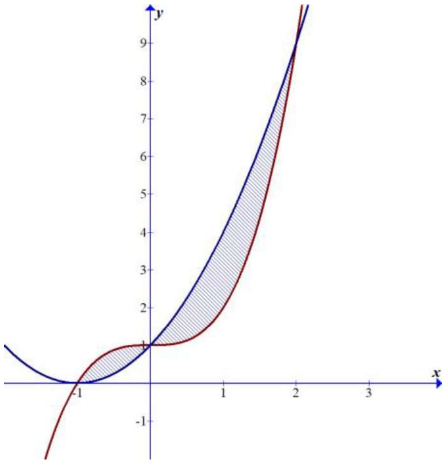
(s) The sketch of the region is given below. Make sure, when using a GDC that the mode of the angle is in radians.

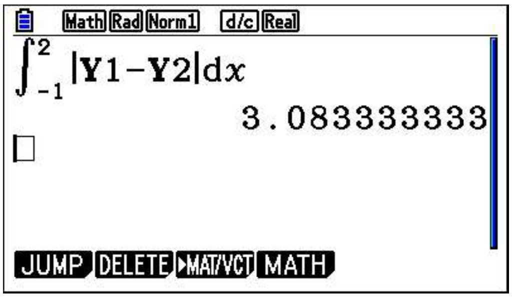


$$\text{Area} = \int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \left(\sec^2 x - \sec x \tan x\right) \mathrm{d}x = \left(\tan x - \sec x\right)\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{6}}$$

$$= \left(\tan\left(\frac{\pi}{6}\right) - \sec\left(\frac{\pi}{6}\right)\right) - \left(\tan\left(-\frac{\pi}{3}\right) - \sec\left(-\frac{\pi}{3}\right)\right) = \frac{2\sqrt{3}}{3} + 2$$

(t) The sketch of the region is given below.





$\int_{-1}^{2} |\text{Y1}-\text{Y2}|\mathrm{d}x$

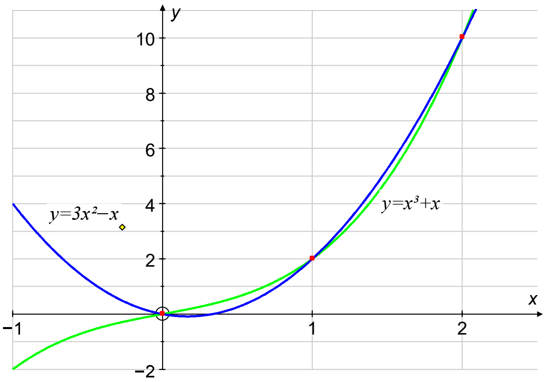
3.083333333

$$\text{Area} = \int_{-1}^{2} \left|\left(x^3+1\right) - \left(x+1\right)^2\right| \mathrm{d}x = 3.083$$

Even though the GDC does not offer an answer in fraction form, we can find the exact value by splitting the work into two intervals:

$$\text{Area} = \int_{-1}^{0} \left(\left(x^3+1\right) - \left(x+1\right)^2\right) \mathrm{d}x + \int_{0}^{2} \left(\left(x+1\right)^2 - \left(x^3+1\right)\right) \mathrm{d}x = \frac{37}{12}$$

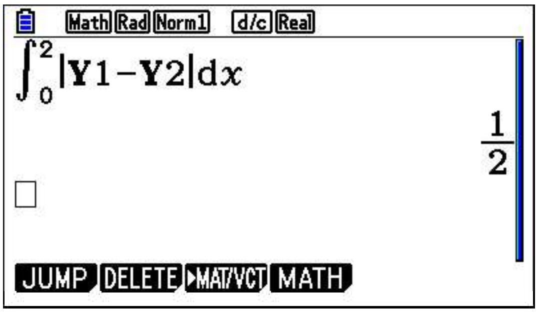
(u) First, we need to sketch the functions and find the points of intersection.



To find the points of intersection, we need to solve the system of simultaneous equations $\left.\begin{array}{r} y=x^3+x \\ y=3x^2-x \end{array}\right\} \Rightarrow x^3+x=3x^2-x \Rightarrow x(x-1)(x-2)=0$

Since there are two regions enclosed by the curves, we will need to split the integral into two integrals, exchanging the upper and the lower function.

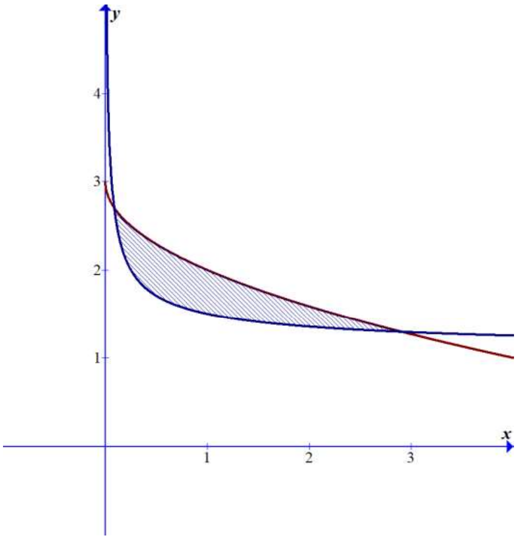
$$\begin{aligned} \text{Area} &= \int_0^1 \left(\left(x^3+x\right)-\left(3x^2-x\right)\right)\mathrm{d}x+\int_1^2 \left(\left(3x^2-x\right)-\left(x^3+x\right)\right)\mathrm{d}x \\ &= \int_0^1 \left(x^3-3x^2+2x\right)\mathrm{d}x+\int_1^2 \left(-x^3+3x^2-2x\right)\mathrm{d}x=\frac{1}{4}-1+1+(-4)+8-4+\frac{1}{4}-1+1=\frac{1}{2} \end{aligned}$$



$\int_0^2 |Y1-Y2|dx$

$\frac{1}{2}$

(v) The sketch of the region is shown below.

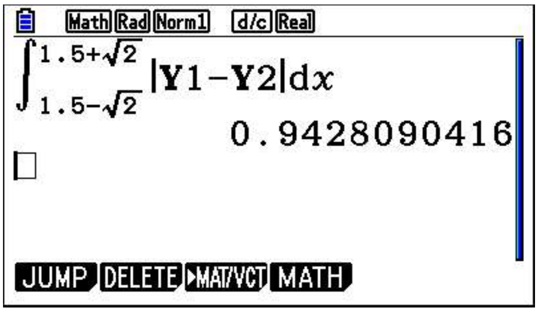


To find the intersections exactly, we solve the system of equations

$$3-\sqrt{x}=\frac{2\sqrt{x}+1}{2\sqrt{x}} \Rightarrow 2x-4\sqrt{x}+1=0 \Rightarrow x=\frac{3}{2}\pm\sqrt{2}$$

$$\text{Area} = \int_{\frac{3}{2}-\sqrt{2}}^{\frac{3}{2}+\sqrt{2}} \left| 3-\sqrt{x}-\frac{2\sqrt{x}+1}{2\sqrt{x}} \right| \mathrm{d}x = \frac{2\sqrt{2}}{3}$$

Using a GDC

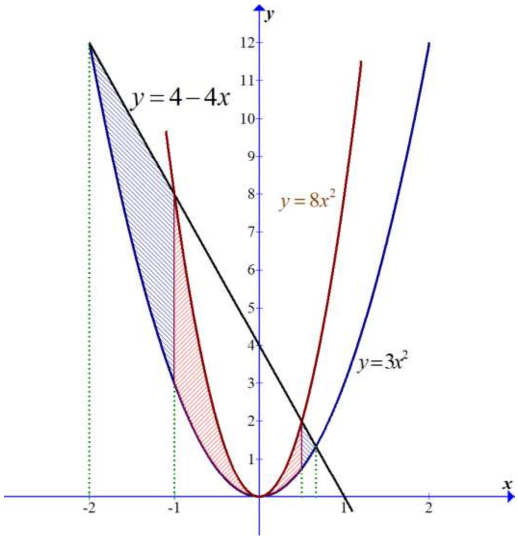


$\int_{1.5-\sqrt{2}}^{1.5+\sqrt{2}} |Y1-Y2| dx$

0.9428090416

This is approximately equal to the exact answer above.

2. In order to calculate the area, we split it into three different intervals as shown below.



The first one is between $y=4-4x$ and $y=3x^2$, the second between $y=8x^2$ and $y=3x^2$, and the third between $y=4-4x$ and $y=3x^2$.

We need to find the points of intersection of the curves so that we can establish the limits of integration by solving the corresponding systems of simultaneous equations.

$$\begin{cases} y = 3x^2 \\ y = 4-4x \end{cases} \Rightarrow 4-4x = 3x^2 \Rightarrow x_1 = -2, x_4 = \frac{2}{3}$$

$$\begin{cases} y = 8x^2 \\ y = 4-4x \end{cases} \Rightarrow 8x^2 + 4x - 4 = 0 \Rightarrow x_2 = -1, x_3 = \frac{1}{2}$$

Notice that in both cases we did not calculate the y-values since, for the integration, we simply need the x-coordinates of the points of intersection.

The area in question is therefore

$$\text{Area} = \int_{-2}^{-1}\left(4-4x-3x^2\right)\mathrm{d}x + \int_{-1}^{\frac{1}{2}}\left(8x^2-3x^2\right)\mathrm{d}x + \int_{\frac{1}{2}}^{\frac{2}{3}}\left(4-4x-3x^2\right)\mathrm{d}x$$

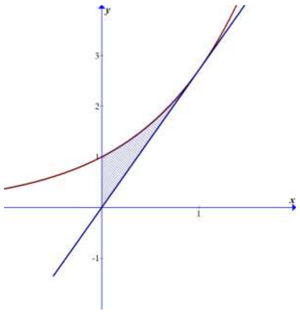
$$= \left(4x-2x^2-x^3\right)\Big]_{-2}^{-1} + \left(\frac{5x^3}{3}\right)\Bigg]_{-1}^{\frac{1}{2}} + \left(4x-2x^2-x^3\right)\Big]_{\frac{1}{2}}^{\frac{2}{3}} = \frac{269}{54}$$

3. First, we need to find the equation of the tangent at the point $(1, \mathrm{e})$.

$$y = \mathrm{e}^x \Rightarrow y' = \mathrm{e}^x \text{; } m = y'(1) = \mathrm{e} \Rightarrow \text{Tangent}: y = \mathrm{e}(x-1) + \mathrm{e} \Rightarrow y = \mathrm{e}x$$

Since $y'' = \mathrm{e}^x \Rightarrow y''(1) = \mathrm{e} > 0$, we can conclude that the curve is above the tangent; to find the area of the region enclosed by the curves, we calculate the following integral.

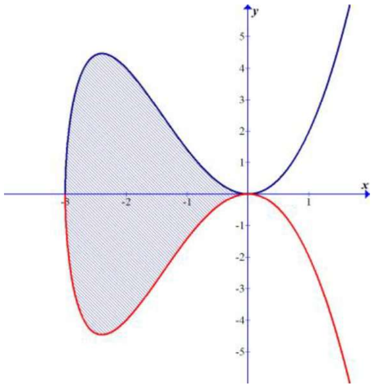
$$\int_0^1\left(\mathrm{e}^x - \mathrm{e}x\right)\mathrm{d}x = \left(\mathrm{e}^x - \mathrm{e}\frac{x^2}{2}\right)\Bigg]_0^1 = \left(\mathrm{e} - \frac{\mathrm{e}}{2}\right) - (1-0) = \frac{\mathrm{e}}{2} - 1$$



4. Since the implicitly defined function is symmetrical with respect to the *x*-axis (only even powers of *y*), we simply need to use one of the two branches (see graph below). Notice that the left-hand side is nonnegative, therefore $x+3 \geq 0 \Rightarrow x \geq -3$. Thus the function is zero when $x = -3$ and when $x = 0$ and creates the 'loop' in the graph.

$$y^2 = x^4 (x+3) \Rightarrow y = \pm x^2 \sqrt{x+3}$$

$$\text{Area} = 2\int_{-3}^{0} \left(x^2 \sqrt{x+3}\right) \mathrm{d}x$$



To evaluate the integral, we use substitution: $u = x+3 \Rightarrow \mathrm{d}u = \mathrm{d}x$

$$\text{Area} = 2\int_{-3}^{0} \left(x^2 \sqrt{x+3}\right) \mathrm{d}x = 2\int_{0}^{3} (u-3)^2 \sqrt{u}\, \mathrm{d}u = 2\int_{0}^{3} \left(u^{5/2} - 6u^{3/2} + 9u^{1/2}\right) \mathrm{d}u$$

$$= 2\left[\frac{2}{7}u^{7/2} - \frac{12}{5}u^{5/2} + 6u^{3/2}\right]_0^3 = \frac{288\sqrt{3}}{35}$$

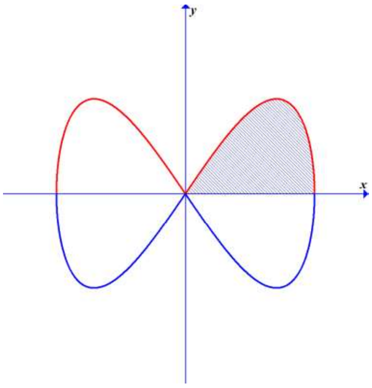
5. The implicitly defined function is symmetrical with respect to the *x*-axis (only even powers of *y*). To find the borders of integration, we solve the equation by setting $y = 0$.

$$0 = 2x^2 - 4x^4 \Rightarrow 2x^2 \left(1 - 2x^2\right) = 0 \Rightarrow x_1 = 0, x_2 = -\frac{\sqrt{2}}{2}, x_3 = \frac{\sqrt{2}}{2}$$

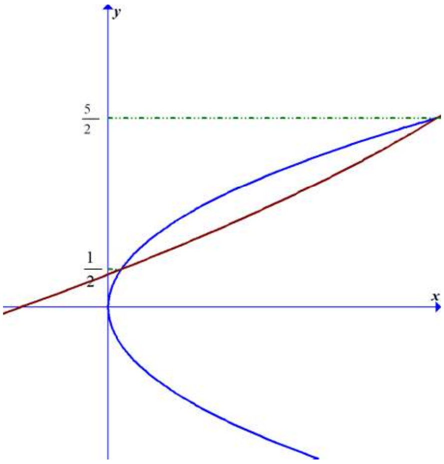
We also notice that the implicitly defined function is symmetrical with respect to the *y*-axis (only even powers of *x*); therefore, we can calculate a simpler integral and then multiply it by 4. That is, $\text{Area} = 4\int_{0}^{\frac{\sqrt{2}}{2}} \sqrt{2x^2 - 4x^4}\, \mathrm{d}x = 4\sqrt{2}\int_{0}^{\frac{\sqrt{2}}{2}} x\sqrt{1 - 2x^2}\, \mathrm{d}x$

To evaluate the integral, we use substitution $u = 1 - 2x^2 \Rightarrow \mathrm{d}u = -4x\mathrm{d}x$

$$4\sqrt{2}\int_0^{\frac{\sqrt{2}}{2}} x\sqrt{1-2x^2}\,\mathrm{d}x = 4\sqrt{2}\times\int_1^0\sqrt{u}\times\left(-\frac{1}{4}\mathrm{d}u\right) = 4\sqrt{2}\times\frac{1}{4}\int_0^1\sqrt{t}\,\mathrm{d}t = \sqrt{2}\times\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg]_0^1 = \frac{2\sqrt{2}}{3}$$



6. In this question, we can do all the calculations with respect to y since x is expressed as the subject in both equations.



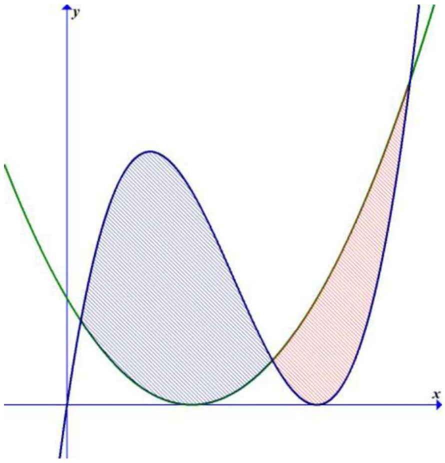
The y-coordinates of the points of intersection are found by solving

$$3y^2 = 12y - y^2 - 5 \Rightarrow 4y^2 - 12y + 5 = 0 \Rightarrow y_1 = \frac{1}{2}, y_2 = \frac{5}{2}$$

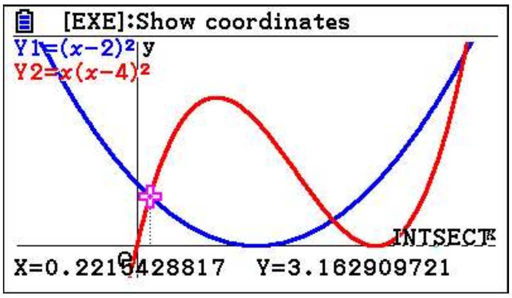
Therefore,

$$\text{Area} = \int_{1/2}^{5/2}\left|12y - y^2 - 5 - 3y^2\right|\mathrm{d}y = \left[6y^2 - \frac{4}{3}y^3 - 5y\right]_{1/2}^{5/2} = \frac{16}{3}$$

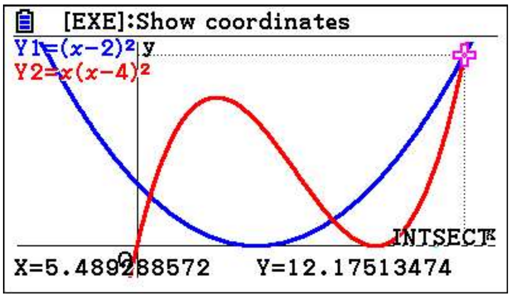
7. If we sketch the graphs, we get the following:



The x-coordinates of the points of intersection cannot be found exactly.
We will use a GDC for this. First, we find the points of intersection.

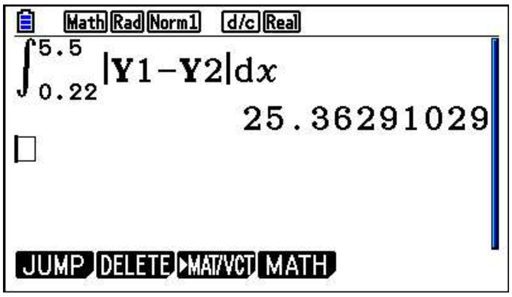


```
X=0.2215428817   Y=3.162909721
```



```
X=5.489288572   Y=12.17513474
```

Then we find the area.



$$\int_{0.22}^{5.5} |Y1-Y2|dx$$

```
25.36291029
```

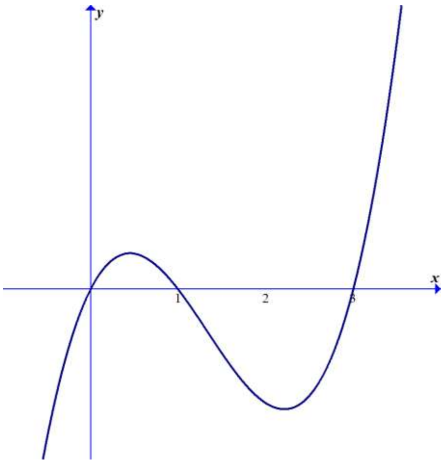
8. The area under this curve is given by $\int_0^m e^{2x}dx = \left(\frac{1}{2}e^{2x}\right)\Big]_0^m = \frac{1}{2}\left(e^{2m}-1\right)$

Now, if the area is 3 square units, then $\frac{1}{2}\left(e^{2m}-1\right) = 3 \Rightarrow e^{2m} - 1 = 6 \Rightarrow m = \frac{\ln 7}{2}$

9. First, we need to find the zeros of the function.

$$x^3-4x^2+3x=0\Rightarrow x(x-1)(x-3)=0\Rightarrow x_1=0, x_2=1, x_3=3$$

Sketch the graph:

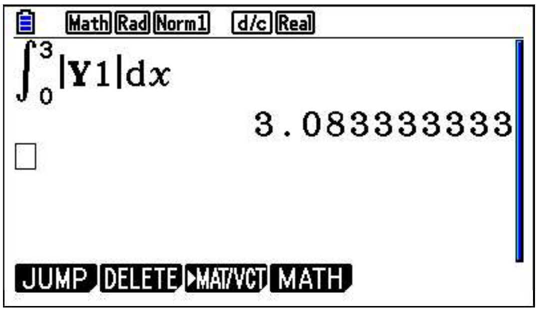


To calculate the area enclosed by the function and the *x*-axis, we need to take into account the fact that the region between the second and the third zero is below the *x*-axis and hence we need to take the opposite expression.

$$\text{Area}=\int_0^1\left(x^3-4x^2+3x\right)\mathrm{d}x+\int_1^3\left(-x^3+4x^2-3x\right)\mathrm{d}x$$

$$=\left(\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right)\Bigg]_0^1+\left(-\frac{x^4}{4}+\frac{4x^3}{3}-\frac{3x^2}{2}\right)\Bigg]_1^3=\frac{37}{12}$$

Using a GDC, we find the integral of the absolute value.



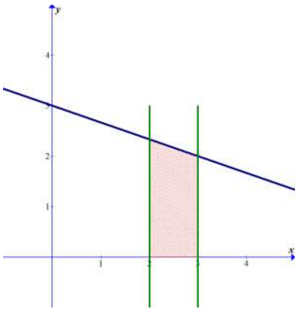
$\int_0^3 |\text{Y}1|\,\mathrm{d}x$

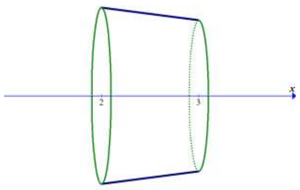
3.083333333

Exercise 14.7

1. We will sketch the region and a typical disk with the created volume in the first two questions. We will leave the drawings for you to complete in the rest of the exercises.

(a) We begin by sketching the lines and shading the region that is rotated about the *x*-axis.

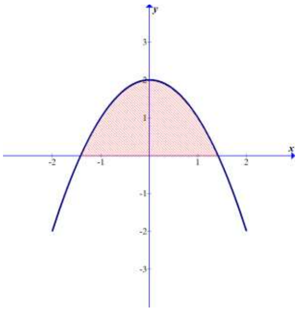


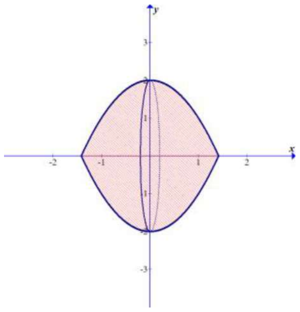


To find the volume, we need to evaluate the following integral:

$$V = \pi\int_2^3 \left(3-\frac{x}{3}\right)^2 \mathrm{d}x = \pi\int_2^3 \left(9-2x+\frac{x^2}{9}\right)\mathrm{d}x = \pi\left[9x - x^2 + \frac{x^3}{27}\right]_2^3 = \frac{127}{27}\pi$$

(b) We begin by sketching the parabola and shading the region between the *x*-axis ($y = 0$) and the parabola that is rotated about the *x*-axis. By inspection, we find that the parabola intersects the *x*-axis at the points $\left(-\sqrt{2},0\right)$ and $\left(\sqrt{2},0\right)$.

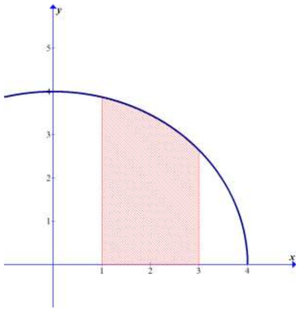




So, to find the volume of the solid, we need to evaluate the following integral:

$$V = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(2-x^2\right)^2 \mathrm{d}x = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(4-4x^2+x^4\right)\mathrm{d}x = \pi\left[4x-\frac{4}{3}x^3+\frac{x^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\sqrt{2}}{15}\pi$$

(c) We begin by sketching the curves and shading the region that is rotated about the *x*-axis.



So, to find the volume of the solid, we need to evaluate the following integral:

$$V = \pi\int_1^3 \left(\sqrt{16-x^2}\right)^2 \mathrm{d}x = \pi\int_1^3 \left(16-x^2\right)\mathrm{d}x = \pi\left[16x-\frac{x^3}{3}\right]_1^3 = \frac{70}{3}\pi$$

If you are answering a Paper 2 question of this type, you only need to write down

$V = \pi\int_1^3 \left(\sqrt{16-x^2}\right)^2 \mathrm{d}x = \frac{70}{3}\pi$, or just $V = \pi\int_1^3 \left(\sqrt{16-x^2}\right)^2 \mathrm{d}x \approx 23.3\pi \approx 73.3$

(d) $V = \pi\int_1^3 \frac{9}{x^2}\mathrm{d}x = \pi\left[-\frac{9}{x}\right]_1^3 = 6\pi$

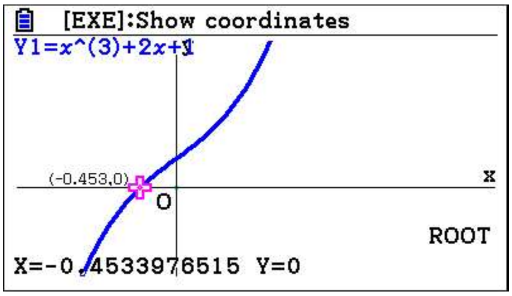
(e) $V = \pi\int_0^3 (3-x)^2\,\mathrm{d}x = \pi\int_0^3 \left(9-6x+x^2\right)\mathrm{d}x = \pi\left[9x-6\times\frac{x^2}{2}+\frac{x^3}{3}\right]_0^3 = 9\pi$

(f) $V = \pi\int_0^{\pi} \left(\sqrt{\sin x}\right)^2 \mathrm{d}x = \pi\int_0^{\pi} \sin x\,\mathrm{d}x = \pi\left[-\cos x\right]_0^{\pi} = 2\pi$

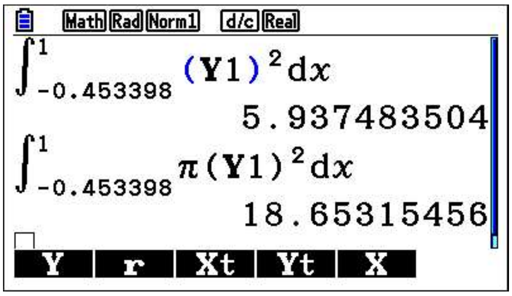
(g) $V = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\sqrt{\cos x}\right)^2 \mathrm{d}x = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \cos x\,\mathrm{d}x = \pi\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{3}} = \pi\left(\frac{\sqrt{3}}{2}+1\right)$

(h) $V = \pi\int_{-2}^{2} \left(4-x^2\right)^2 \mathrm{d}x = \pi\int_{-2}^{2} \left(16-8x^2+x^4\right)\mathrm{d}x = \pi\left[16x-8\times\frac{x^3}{3}+\frac{x^5}{5}\right]_{-2}^{2} = \frac{512\pi}{15}$

(i) This is apparently a GDC active question, as we need to find the point of intersection between the cubic curve and the x-axis.



```
[EXE]:Show coordinates
Y1=x^(3)+2x+1
(-0.453,0)
ROOT
X=-0.4533976515 Y=0
```



$$\int_{-0.453398}^{1} (Y1)^2 dx$$

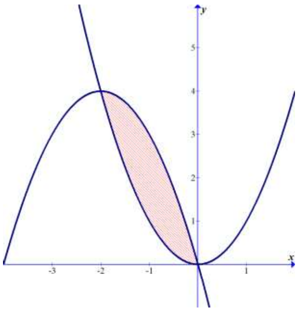
5.937483504

$$\int_{-0.453398}^{1} \pi (Y1)^2 dx$$

18.65315456

So, the volume is $\pi\int_{-0.4534}^{1}\left(x^3+2x+1\right)^2 \mathrm{d}x \approx 5.94\pi \approx 18.7$ correct to 3 s.f.

(j) We begin by sketching the parabolas and shading the region that is rotated about the x-axis. By inspection, we find that the points of intersection of the *curves* are $(-2,4)$ and $(0,0)$.



To find the volume, we need to evaluate the integrals of the upper function and the integral of the lower function, and then subtract them to obtain the answer. (This can be done in one step too.)

$$V_{upper}=\pi\int_{-2}^{0}\left(-4x-x^2\right)^2 \mathrm{d}x=\pi\int_{-2}^{0}\left(16x^2+8x^3+x^4\right)\mathrm{d}x=\pi\left(\frac{128}{3}-32+\frac{32}{5}\right)=\frac{256}{15}\pi$$

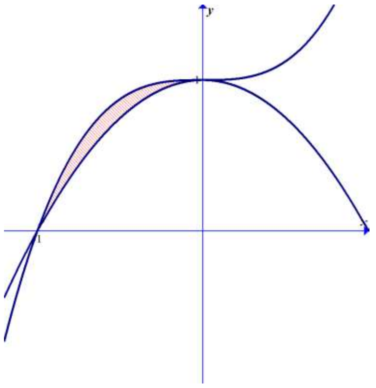
$$V_{lower}=\pi\int_{-2}^{0}\left(x^2\right)^2 \mathrm{d}x=\pi\int_{-2}^{0}x^4\mathrm{d}x=\pi\left[\frac{x^5}{5}\right]_{-2}^{0}=\frac{32\pi}{5}$$

So, the final volume is equal to the difference of those two volumes:

$$V=V_{upper}-V_{lower}=\frac{256}{15}\pi-\frac{32}{5}\pi=\frac{32}{3}\pi$$

(k) $$V=\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sec^2 x\,\mathrm{d}x=\pi\left(\tan x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}\right)=\pi\left(\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{4}\right)\right)=\pi\left(\sqrt{3}-1\right)$$

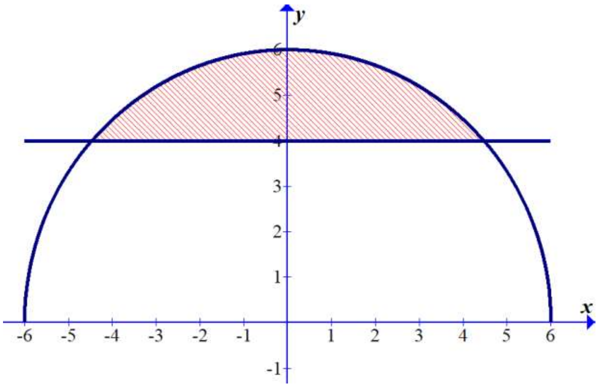
(l) We sketch the curve and shade the region that is rotated about the *x*-axis.



$$V=\pi\int_{-1}^{0}\left(\left(x^3+1\right)^2-\left(1-x^2\right)^2\right)\mathrm{d}x=\pi\int_{-1}^{0}\left(x^6-x^4+2x^3+2x^2\right)\mathrm{d}x$$

$$=\pi\left(\frac{x^7}{7}-\frac{x^5}{5}+\frac{x^4}{2}+\frac{2x^3}{3}\right)\Bigg]_{-1}^{0}=\frac{23\pi}{210}$$

(m) We sketch the curve and shade the region that is rotated about the *x*-axis.



Before we find the volume of revolution, we first need to find the limits of integration by solving the simultaneous equations.

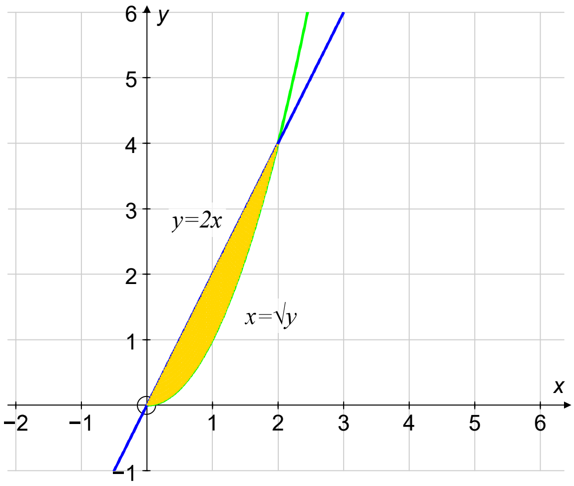
$$\begin{cases} y=\sqrt{36-x^2} \\ y=4 \end{cases}\Rightarrow 4=\sqrt{36-x^2}\Rightarrow x_1=-2\sqrt{5},\, x_2=2\sqrt{5}$$

Since both functions are even (symmetrical with respect to the *y*-axis), we can simply calculate the integral from 0 to $2\sqrt{5}$ and then multiply it by 2.

$$V = 2\pi\int_0^{2\sqrt{5}}\left(\left(\sqrt{36-x^2}\right)^2 - 4^2\right)\mathrm{d}x = 2\pi\int_0^{2\sqrt{5}}\left(20-x^2\right)\mathrm{d}x = 2\pi\left[\left(20x-\frac{x^3}{3}\right)\right]_0^{2\sqrt{5}}$$

$$= 2\pi\left(40\sqrt{5}-\frac{40\sqrt{5}}{3}\right) = \frac{160\pi\sqrt{5}}{3}$$

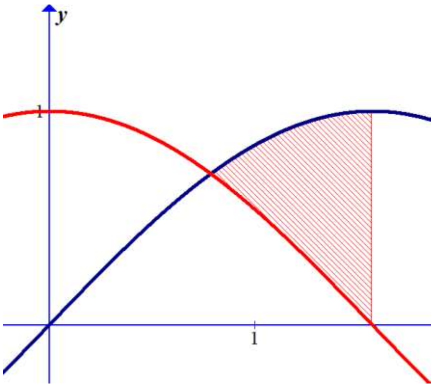
(n) We sketch the curve and shade the region that is rotated about the *x*-axis.



To ease our calculations, we notice that $x=\sqrt{y} \Leftrightarrow y=x^2, x\geq 0$. By inspection, we can find the limits of integration.

$$V = \pi\int_0^2\left((2x)^2-\left(x^2\right)^2\right)\mathrm{d}x = \pi\left[\left(\frac{4x^3}{3}-\frac{x^5}{5}\right)\right]_0^2 = \pi\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{64\pi}{15}$$

(o) Sketch the curves and shade the region that is rotated about the x-axis.



$$V=\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2 x-\cos^2 x\right)\mathrm{d}x=\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(-\cos 2x\right)\mathrm{d}x=\pi\left(-\frac{1}{2}\sin 2x\right)\Bigg]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{\pi}{2}$$

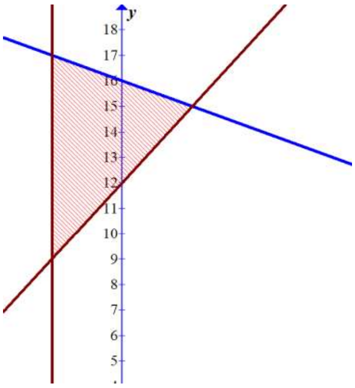
(p)

$$V=\pi\int_1^3\left(\left(2x^2+4\right)^2-x^2\right)\mathrm{d}x=\pi\int_1^3\left(4x^4+15x^2+16\right)\mathrm{d}x$$

$$=\pi\left(\frac{4x^5}{5}+5x^3+16x\right)\Bigg]_1^3=\frac{1778\pi}{5}$$

(q) $$V=\pi\int_1^3\left(\sqrt{x^4+1}\right)^2\mathrm{d}x=\pi\int_1^3\left(x^4+1\right)\mathrm{d}x=\pi\left(\frac{x^5}{5}+x\right)\Bigg]_1^3=\frac{252\pi}{5}$$

(r) We begin by sketching the curve, shading the region that is rotated about the x-axis, and find the point of intersection by inspection.

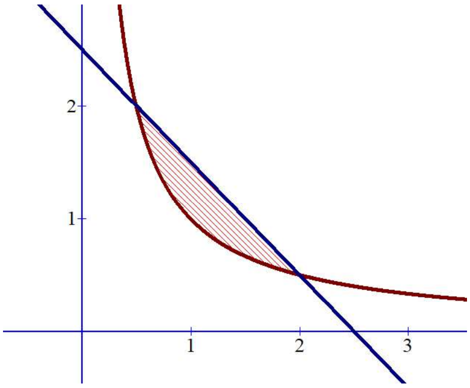


$$V=\pi\int_{-1}^{1}\left(\left(16-x\right)^{2}-\left(3x+12\right)^{2}\right)\mathrm{d}x=\pi\int_{-1}^{1}\left(-8x^{2}-104x+112\right)\mathrm{d}x$$

$$=\pi\left(-\frac{8x^{3}}{3}-57x^{2}+112x\right)\Bigg]_{-1}^{1}=\frac{656\pi}{3}$$

(s) We sketch the curve, shading the region that is rotated about the *x*-axis.
We can find the points of intersection by solving the simultaneous equations.

$$\begin{cases} y=\dfrac{1}{x} \\ y=\dfrac{5}{2}-x \end{cases}\Rightarrow\frac{1}{x}=\frac{5}{2}-x\Rightarrow 2x^{2}-5x+2=0\Rightarrow x_{1}=\frac{1}{2},x_{2}=2$$

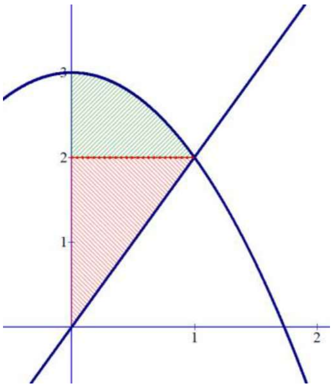


$$V=\pi\int_{\frac{1}{2}}^{2}\left(\left(\frac{5}{2}-x\right)^{2}-\left(\frac{1}{x}\right)^{2}\right)\mathrm{d}x=\pi\int_{\frac{1}{2}}^{2}\left(\frac{25}{4}-5x+x^{2}-\frac{1}{x^{2}}\right)\mathrm{d}x$$

$$=\pi\left(\frac{25}{4}x-\frac{5x^{2}}{2}+\frac{x^{3}}{3}+\frac{1}{x}\right)\Bigg]_{\frac{1}{2}}^{2}=\frac{9\pi}{8}$$

2. An alternative method of solving a rotation about the *y*-axis involves swapping the variables of integration from *x* to *y*, which means expressing *x* in terms of *y* to find the volume of revolution (disk method). So, in some parts of questions 2 and 3, we will use both methods to find the volume of revolution.

Note: The second method, known as the shell method, is described in the HL textbook. In most cases, the shell method is simpler. The IB syllabus (in this cycle), it is not required. Unless it is specifically requested that you use the 'disk' method, we recommend using the shell method.



(a)

$$V = \pi\int_0^1\left(\left(3-x^2\right)^2-\left(2x\right)^2\right)\mathrm{d}x = \pi\int_0^1\left(9-10x^2+x^4\right)\mathrm{d}x$$

$$= \pi\left(9x-\frac{10x^3}{3}+\frac{x^5}{5}\right)\Bigg]_0^1 = \frac{88\pi}{15}$$

(b) If we want to use the disk method, then we need to split the region into two regions and change the limits of integration from 0 to 2 for the first integral and from 2 to 3 for the second integral.

$$y = 3-x^2 \Rightarrow x = \sqrt{3-y}, \quad y = 2x \Rightarrow x = \frac{y}{2}$$

$$V = \pi\int_0^2\left(\frac{y^2}{4}\right)\mathrm{d}y + \pi\int_2^3\left(\sqrt{3-y}\right)^2\mathrm{d}y = \pi\left(\frac{y^3}{12}\right)\Bigg]_0^2 + \pi\left(3y-\frac{y^2}{2}\right)\Bigg]_2^3$$

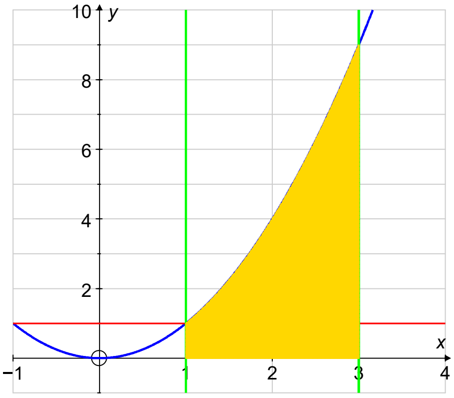
$$= \pi\left(\frac{2}{3}+9-\frac{9}{2}-6+2\right) = \frac{7\pi}{6}$$

The shell method:

$$V = 2\pi\int_0^1\left(x\left(3-x^2-2x\right)\right)\mathrm{d}x = 2\pi\int_0^1\left(3x-x^3-2x^2\right)\mathrm{d}x = 2\pi\left(\frac{3x^2}{2}-\frac{x^4}{4}-\frac{2x^3}{3}\right)\Bigg]_0^1$$

$$= 2\pi\left(\frac{3}{2}-\frac{1}{4}-\frac{2}{3}\right) = 2\pi\times\frac{18-3-8}{12} = \frac{7\pi}{6}$$

3. **(a)** For demonstration purposes, we will do the calculations here using both methods.



Disk method: We need to express x in terms of y: $y=x^2 \Rightarrow x=\sqrt{y}$

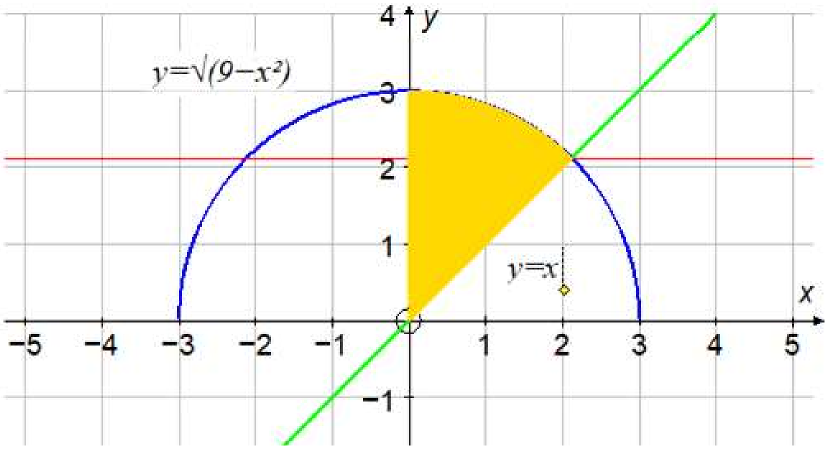
$$V=\pi\int_0^1\left(3^2-1^2\right)\mathrm{d}y+\pi\int_1^9\left(3^2-\left(\sqrt{y}\right)^2\right)\mathrm{d}y=\pi\left(\left(8y\right)\Big]_0^1+\left(9y-\frac{y^2}{2}\right)\Bigg]_1^9\right)$$

$$=\pi\left(8+81-\frac{81}{2}-9+\frac{1}{2}\right)=40\pi$$

The shell method:

$$V=2\pi\int_1^3\left(x\left(x^2\right)\right)\mathrm{d}x=2\pi\int_1^3 x^3\mathrm{d}x=2\pi\left(\frac{x^4}{4}\right)\Bigg]_1^3=2\pi\left(\frac{81}{4}-\frac{1}{4}\right)=2\pi\times 20=40\pi$$

(b) **Disk method**



Notice that these two curves intersect at the point with equal x- and y-coordinates:

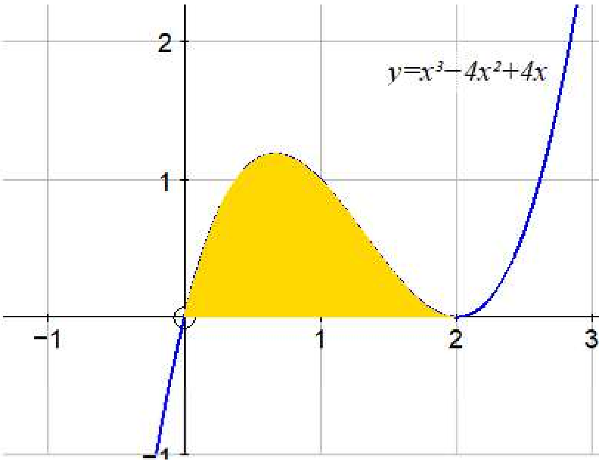
$\sqrt{9-x^2}=x \Rightarrow 9-x^2=x^2 \Rightarrow 9=2x^2 \Rightarrow x=\sqrt{\frac{9}{2}}=\frac{3\sqrt{2}}{2}$, and that x and y are symmetrical in the equation of the curve; therefore, $x=\sqrt{9-y^2}$.

We need to split the integration into two parts.

$$V=\pi\int_0^{\frac{3\sqrt{2}}{2}} y^2\,\mathrm{d}y+\pi\int_{\frac{3\sqrt{2}}{2}}^{3}\left(\sqrt{9-y^2}\right)^2\mathrm{d}y=\pi\left(\left[\left(\frac{y^3}{3}\right)\right]_0^{\frac{3\sqrt{2}}{2}}+\left[\left(9y-\frac{y^3}{3}\right)\right]_{\frac{3\sqrt{2}}{2}}^{3}\right)$$

$$=\pi\left(\frac{9\sqrt{2}}{4}+27-9-\frac{27\sqrt{2}}{2}+\frac{9\sqrt{2}}{4}\right)=9\pi\left(2-\sqrt{2}\right)$$

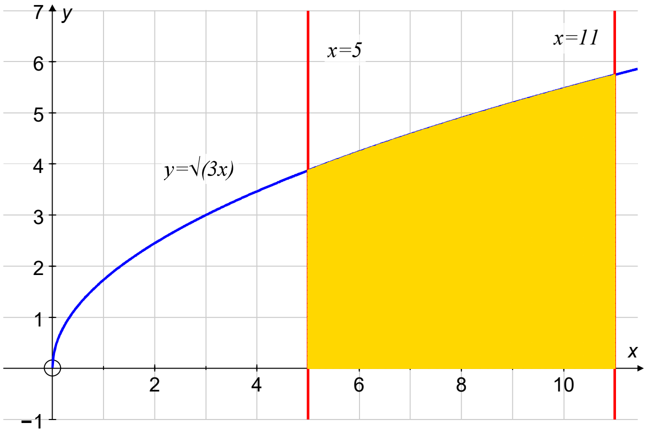
(c) This question cannot be done by swapping the variables, so we will use the shell method.



$$V=2\pi\int_0^2\left(x\times\left(x^3-4x^2+4x\right)\right)\mathrm{d}x=2\pi\int_0^2\left(x^4-4x^3+4x^2\right)\mathrm{d}x=2\pi\left[\left(\frac{x^5}{5}-x^4+\frac{4x^3}{3}\right)\right]_0^2$$

$$=2\pi\left(\frac{32}{5}-16+\frac{32}{3}\right)=\frac{32\pi}{15}$$

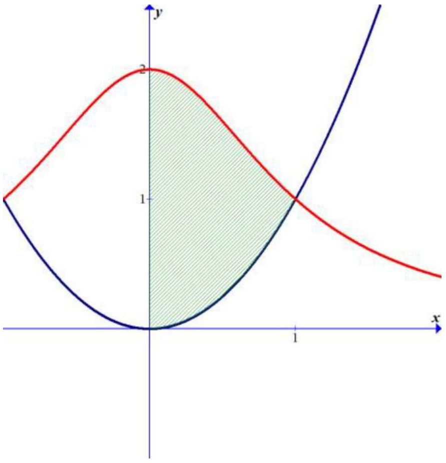
(d) The shell method may be simpler.



The shell method:

$$V = 2\pi\int_5^{11}\left(x\sqrt{3x}\right)\mathrm{d}x = 2\pi\sqrt{3}\int_5^{11} x^{\frac{3}{2}}\,\mathrm{d}x = 2\pi\sqrt{3}\left(\frac{2}{5}x^{\frac{5}{2}}\right)\Bigg]_5^{11} = \frac{484\pi\sqrt{33}}{5} - 20\pi\sqrt{15}$$

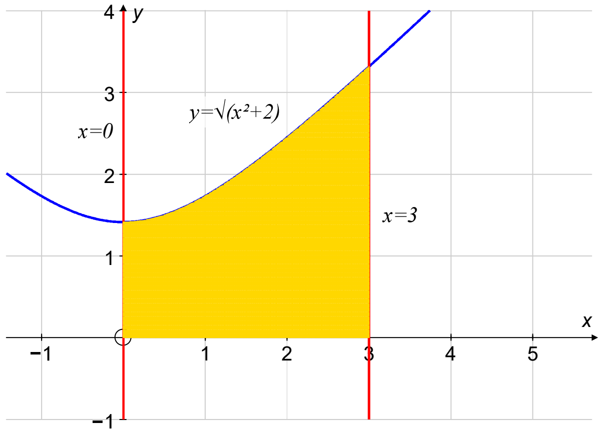
(e) Given the intensive work with expressing x in terms of y, we will use the shell method instead.
It is enough to rotate the region in the first quadrant to create the solid.



$$V = 2\pi\int_0^1\left(x\left(\frac{2}{1+x^2} - x^2\right)\right)\mathrm{d}x = 2\pi\int_0^1\left(\frac{2x}{1+x^2} - x^3\right)\mathrm{d}x = 2\pi\left(\ln\left(1+x^2\right) - \frac{x^4}{4}\right)\Bigg]_0^1$$

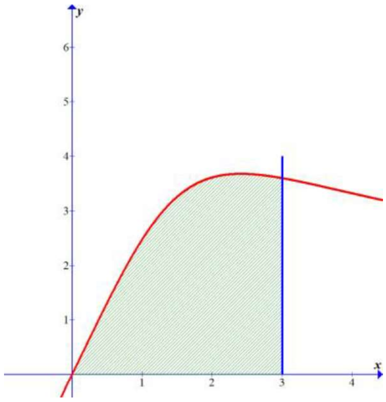
$$= 2\pi\left(\ln 2 - \frac{1}{4}\right) = 2\pi\ln 2 - \frac{\pi}{2}$$

(f) This one lends itself to the shell method too.



$$V = 2\pi\int_0^3 \left(x\sqrt{x^2+2}\right)\mathrm{d}x = 2\pi\int_2^{11}\sqrt{t}\times\frac{1}{2}\mathrm{d}t = \pi\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg]_2^{11} = \frac{2\pi}{3}\left(11\sqrt{11}-2\sqrt{2}\right)$$

(g) Again, this question cannot be done by swapping the variables, so we will use the shell method.



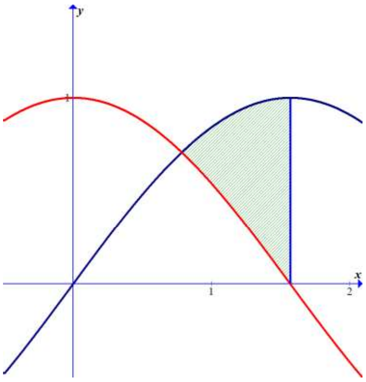
$$V = 2\pi\int_0^3\left(x\times\frac{7x}{\sqrt{x^3+7}}\right)\mathrm{d}x = 2\pi\int_0^3\left(\frac{7x^2}{\sqrt{x^3+7}}\right)\mathrm{d}x = 14\pi\int_7^{34}\frac{\frac{1}{3}dt}{\sqrt{t}} = \frac{14\pi}{3}\left(2\sqrt{t}\right)\Big]_7^{34}$$

$$= \frac{28\pi}{3}\left(\sqrt{34}-\sqrt{7}\right)$$

We used the substitution $t = x^3 + 7$ and $\mathrm{d}t = 3x^2\mathrm{d}x$ for the integral.

(h) Using the disk method will involve expressions containing powers of inverse trigonometric functions which will require lengthy calculations. We will use the shell method instead.

These two curves meet at the point $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$



$$V = 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(x\sin x - x\cos x)\,\mathrm{d}x$$

This requires finding the following integrals using integration by parts.

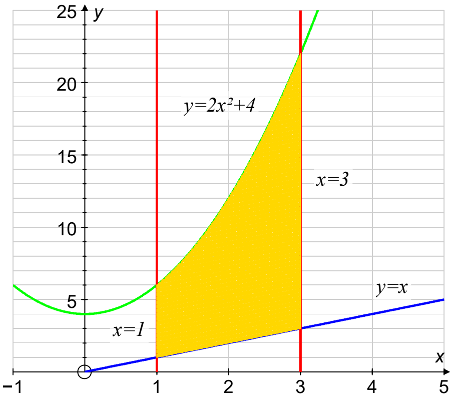
$\int x\sin x\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + c$, and

$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + c$

$$V = 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(x\sin x - x\cos x)\,\mathrm{d}x = 2\pi\left(-x\cos x + \sin x - x\sin x - \cos x\right)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 2\pi\left(1 - \frac{\pi}{2} + \frac{\pi\sqrt{2}}{4}\right) = \pi\left(2 - \pi + \frac{\pi\sqrt{2}}{2}\right)$$

(i) Using the disk method first. Points of intersection can be found by inspection and left for you to verify.



$$V=\pi\left(\int_1^3\left(y^2-1\right)\mathrm{d}y+\int_3^6\left(3^2-1^2\right)\mathrm{d}y+\int_6^{22}\left(9-\left(\sqrt{\frac{y}{2}-2}\right)^2\right)\right)\mathrm{d}y$$

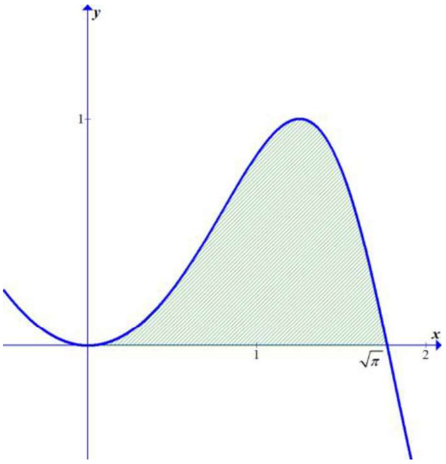
$$=\pi\left(\left(\frac{y^3}{3}-y\right)\Bigg]_1^3+\left(8y\right)\Big]_3^6+\left(11y-\frac{y^2}{4}\right)\Bigg]_6^{22}\right)=\frac{284\pi}{3}$$

The shell method:

$$V=2\pi\int_1^3\left(x\left(2x^2+4-x\right)\right)\mathrm{d}x=2\pi\int_1^3\left(2x^3+4x-x^2\right)\mathrm{d}x=2\pi\left(\frac{x^4}{2}+2x^2-\frac{x^3}{3}\right)\Bigg]_1^3$$

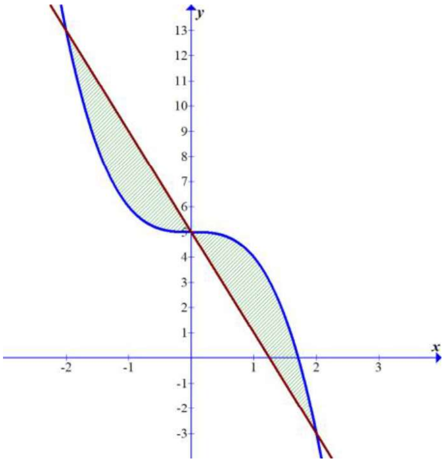
$$=2\pi\left(\frac{81}{2}+18-9-\frac{1}{2}-2+\frac{1}{3}\right)=\frac{284\pi}{3}$$

(j) This question cannot be done by swapping variables, so we use the shell method.



$$V=2\pi\int_0^{\sqrt{\pi}}\left(x\sin\left(x^2\right)\right)\mathrm{d}x=2\pi\int_0^{\pi}\sin t\times\frac{1}{2}\mathrm{d}t=\pi\left(-\cos t\right)\Big]_0^{\pi}=2\pi$$

(k) The shell method is more straightforward.
Due to symmetry of the graphs, it will be enough to evaluate the integral over the interval [0, 2] and multiply by 2.



$$V=4\pi\int_0^2\left(x\left(5-x^3-5+4x\right)\right)\mathrm{d}x=4\pi\int_0^2\left(4x^2-x^4\right)\mathrm{d}x=4\pi\left[\frac{4x^3}{3}-\frac{x^5}{5}\right]_0^2=\frac{256\pi}{15}$$

Exercise 14.8

1. **(a)** To find the displacement (net distance), we evaluate the following integral:

$$\text{Displacement}=\int_0^{10}\left(t^2-11t+24\right)\mathrm{d}t=\left[\frac{t^3}{3}-11\times\frac{t^2}{2}+24t\right]_0^{10}=\frac{1000}{3}-550+240=\frac{70}{3}\text{ m}$$

To find the total distance travelled, we find $\int_0^{10}\left|t^2-11t+24\right|\mathrm{d}t$

We perform two steps: first, find the zeroes of the parabola and identify where the velocity changes its direction. (from negative to positive and vice versa)

$v(t)=0\Rightarrow t^2-11t+24=0\Rightarrow t=3\text{ or }t=8$

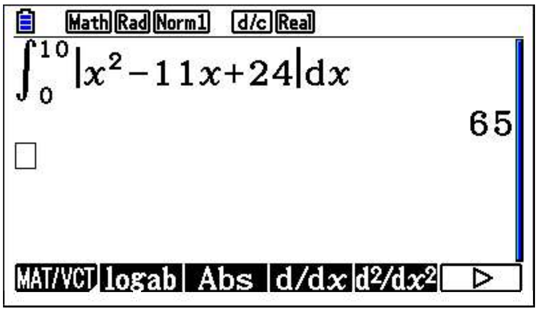
We notice that the particle changes its direction twice, so we need to split the integral into three different integrals.

$$\text{Total distance}=\int_0^3\left(t^2-11t+24\right)\mathrm{d}t+\int_3^8\left(-t^2+11t-24\right)\mathrm{d}t+\int_8^{10}\left(t^2-11t+24\right)\mathrm{d}t$$

$$=\left[\frac{t^3}{3}-11\times\frac{t^2}{2}+24t\right]_0^3+\left[-\frac{t^3}{3}+11\times\frac{t^2}{2}-24t\right]_3^8+\left[\frac{t^3}{3}-11\times\frac{t^2}{2}+24t\right]_8^{10}$$

$$=63-2\times\frac{32}{3}+\frac{70}{3}=65\text{ m}$$

If this were a Paper 2 question, then the use of a GDC will make it simpler.



$\int_0^{10} |x^2-11x+24| \mathrm{d}x$

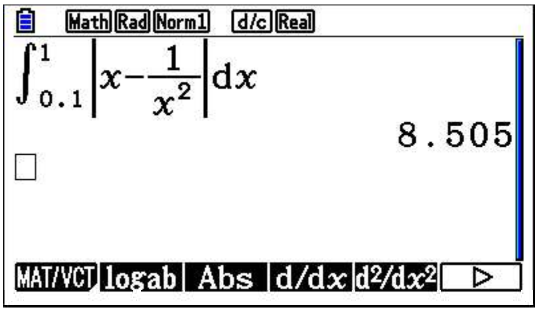
65

(b) To find the net distance, we evaluate the following integral:

$$\int_{0.1}^{1}\left(t-\frac{1}{t^2}\right)\mathrm{d}t=\left[\frac{t^2}{2}+\frac{1}{t}\right]_{0.1}^{1}=1.5-10.005=-8.505$$

We can see that the particle moves to the left and that the displacement is 8.505 m. However, since there are no zeroes within the given interval, the total distance travelled is the same, i.e. 8.505 m.

Again, if this were a Paper 2 question, then the use of a GDC will make it simpler.



$\int_{0.1}^{1} \left|x-\frac{1}{x^2}\right| \mathrm{d}x$

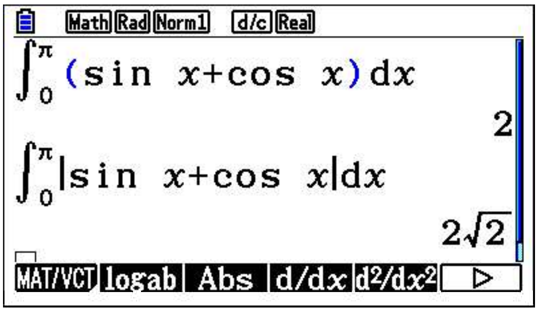
8.505

(c) We notice that the whole graph is above the x-axis within the given interval, and therefore the displacement and the total distance travelled are the same, i.e.

$$\int_0^{\frac{\pi}{2}}\sin(2t)\,\mathrm{d}t=-\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{2}}=\frac{1}{2}+\frac{1}{2}=1\text{ m}$$

(d) Net distance = $\int_0^{\pi}(\sin t+\cos t)\,\mathrm{d}t=-\cos t+\sin t\Big|_0^{\pi}=-(-1)+1=2\text{ m}$

Net and total distance – with a GDC



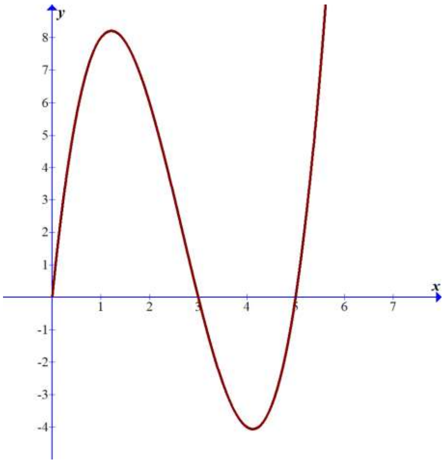
$\int_0^{\pi}(\sin x+\cos x)dx$

2

$\int_0^{\pi}|\sin x+\cos x|dx$

$2\sqrt{2}$

(e) By looking at the curve $v(t)=t^3-8t^2+15t=t(t-3)(t-5)$, we notice that within the interval]0, 6[there are two zeroes, 3 and 5; therefore, the particle changes its direction twice. To find the (net distance) displacement, we evaluate the integral within that interval; but to find the total distance travelled, we take the integral of the absolute value since the curve is below the *x*-axis in the interval]3, 5[.



$$\text{Displacement}=\int_0^6\left(t^3-8t^2+15t\right)dt=\left[\frac{t^4}{4}-8\times\frac{t^3}{3}+15\times\frac{t^2}{2}\right]_0^6=18\text{ m}$$

$$\text{Total distance}=\int_0^6\left|t^3-8t^2+15t\right|dt$$

$$=\int_0^3\left(t^3-8t^2+15t\right)dt+\int_3^5\left(-t^3+8t^2-15t\right)dt+\int_5^6\left(t^3-8t^2+15t\right)dt$$

$$=\left[\frac{t^4}{4}-\frac{8t^3}{3}+\frac{15t^2}{2}\right]_0^3+\left[-\frac{t^4}{4}+\frac{8t^3}{3}-\frac{15t^2}{2}\right]_3^5+\left[\frac{t^4}{4}-\frac{8t^3}{3}+\frac{15t^2}{2}\right]_5^6$$

$$=\frac{86}{3}\approx 28.7\text{ m}$$

(f) We notice that the function is always positive on the given interval and therefore net distance and total distance travelled are going to be the same.

$$\int_0^1\left(\sin\left(\frac{\pi t}{2}\right)+\cos\left(\frac{\pi t}{2}\right)\right)\mathrm{d}t=\left[\frac{2}{\pi}\left(-\cos\left(\frac{\pi t}{2}\right)\right)+\frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right)\right]_0^1$$
$$=\frac{2}{\pi}(0+1+1-0)=\frac{4}{\pi}\text{ m}$$

2. **(a)** $v(t)=\int a(t)\,\mathrm{d}t\Rightarrow v(t)=\int 3\,\mathrm{d}t=3t+c, c\in\mathbb{R}$

To find the value of the constant, we use the initial velocity:
$v(0)=0\Rightarrow 3\times 0+c=0\Rightarrow c=0$. So, the velocity function is: $v(t)=3t$.

Since there is no change in the direction, net and total distance are the same

$$\text{Distance}=\int_0^2 3t\,\mathrm{d}t=\left[3\frac{t^2}{2}\right]_0^2=\left(3\times\frac{2^2}{2}\right)-0=6\text{ m}$$

(b) $v(t)=\int a(t)\,\mathrm{d}t\Rightarrow v(t)=\int(2t-4)\,\mathrm{d}t=2\frac{t^2}{2}-4t+c=t^2-4t+c,\ c\in\mathbb{R}$.

Also, $v(0)=3\Rightarrow 0^2-4\times 0+c=3\Rightarrow c=3$

So, the velocity function is: $v(t)=t^2-4t+3$

$$\text{net distance}=\int_0^3\left(t^2-4t+3\right)\mathrm{d}t=\left[\frac{t^3}{3}-4\times\frac{t^2}{2}+3t\right]_0^3=(9-18+9)-0=0$$

We notice that the zeroes of the velocity parabola are 1 and 3; therefore, there is a change in the direction at 1 and we need to split the integral into two integrals to calculate the total distance travelled.

$$\text{Total distance travelled}=\int_0^3\left|t^2-4t+3\right|\mathrm{d}t=\int_0^1\left(t^2-4t+3\right)\mathrm{d}t+\int_1^3\left(-t^2+4t-3\right)\mathrm{d}t$$
$$=\left[\frac{t^3}{3}-4\times\frac{t^2}{2}+3t\right]_0^1+\left[-\frac{t^3}{3}+4\times\frac{t^2}{2}-3t\right]_1^3=\frac{8}{3}\approx 2.67\text{ m}$$

(c) $v(t)=\int a(t)\,\mathrm{d}t\Rightarrow v(t)=\int\sin t\,\mathrm{d}t=-\cos t+c$

Also, $v(0)=0\Rightarrow-\cos(0)+c=0\Rightarrow -1+c=0\Rightarrow c=1$

So, the velocity function is: $v(t)=1-\cos t$

The cosine value is always between –1 and 1 so we see that the velocity function is never negative. There is no change in direction and therefore the displacement and the total distance travelled are the same.

$$\int_0^{\frac{3\pi}{2}}(1-\cos t)\,\mathrm{d}t=[t-\sin t]_0^{\frac{3\pi}{2}}=\left(\frac{3\pi}{2}-\sin\left(\frac{3\pi}{2}\right)\right)-0=\frac{3\pi}{2}+1\approx 5.71\text{ m}$$

(d) $v(t)=\int a(t)\,\mathrm{d}t\Rightarrow v(t)=\int\frac{-1}{\sqrt{t+1}}\,\mathrm{d}t=-2\sqrt{t+1}+c,\ c\in\mathbb{R}$

Also, $v(0)=2\Rightarrow -2\sqrt{0+1}+c=2\Rightarrow -2+c=2\Rightarrow c=4$

So, the velocity function is: $v(t)=-2\sqrt{t+1}+4$

$$\text{Displacement}=\int_0^4\left(4-2\sqrt{t+1}\right)\mathrm{d}t=\left[4t-\frac{4}{3}(t+1)^{\frac{3}{2}}\right]_0^4=\frac{52-20\sqrt{5}}{3}\approx 2.43\text{ m}$$

By inspection, we can see that the zero is 3, which lies in the interval]0, 4[, so we split the integral into two integrals when calculating the total distance travelled.

$$\text{Total distance travelled}=\int_0^4\left|4-2\sqrt{t+1}\right|\mathrm{d}t=\int_0^3\left(4-2\sqrt{t+1}\right)\mathrm{d}t+\int_3^4\left(2\sqrt{t+1}-4\right)\mathrm{d}t$$

$$=\left[4t-\frac{4}{3}(t+1)^{\frac{3}{2}}\right]_0^3+\left[\frac{4}{3}(t+1)^{\frac{3}{2}}-4t\right]_3^4$$

$$=2\left(12-\frac{4}{3}\times 8\right)+\frac{4}{3}+\frac{20\sqrt{5}-48}{3}=\frac{20\sqrt{5}-36}{3}\approx 2.91\text{ m}$$

(e) $v(t)=\int a(t)\,\mathrm{d}t\Rightarrow v(t)=\int 6t-\frac{1}{(t+1)^3}\,\mathrm{d}t=3t^2+\frac{1}{2(t+1)^2}+c,\ c\in\mathbb{R}$

Also, $v(0)=2\Rightarrow 0+\frac{1}{2}+c=2\Rightarrow c=\frac{3}{2}$

So, the velocity function is: $v(t)=3t^2+\frac{1}{2(t+1)^2}+\frac{3}{2}$

The velocity function consists of a sum of positive terms and as such the velocity function is never negative. There is no change in direction and therefore net and the total distance travelled are the same.

$$\text{Distance}=\int_0^2\left(3t^2+\frac{1}{2(t+1)^2}+\frac{3}{2}\right)\mathrm{d}t=\left[t^3+\frac{(t+1)^{-1}}{2\times(-1)}+\frac{3t}{2}\right]_0^2$$

$$=\left[t^3-\frac{1}{2(t+1)}+\frac{3t}{2}\right]_0^2=\frac{68}{6}=\frac{34}{3}\approx 11.3\text{ m}$$

3. **(a)** $v = 9.8t + 5 \Rightarrow s(t) = \int (9.8t+5)\,\mathrm{d}t = 9.8 \times \frac{t^2}{2} + 5t + c = 4.9t^2 + 5t + c,\ c \in \mathbb{R}$

$s(0) = 10 \Rightarrow 10 = c$. The position of the object at time t is given by:

$s(t) = 4.9t^2 + 5t + 10$

(b) $v = 32t - 2 \Rightarrow s(t) = \int (32t-2)\,\mathrm{d}t = 32 \times \frac{t^2}{2} - 2t + c = 16t^2 - 2t + c,\ c \in \mathbb{R}$

$s(0.5) = 4 \Rightarrow 4 = 16 \times 0.25 - 2 \times 0.5 + c \Rightarrow c = 1$

The position of the object at time t is given by: $s(t) = 16t^2 - 2t + 1$

(c) $v = \sin(\pi t) \Rightarrow s(t) = \int \sin(\pi t)\,\mathrm{d}t = -\frac{1}{\pi}\cos(\pi t) + c,\ c \in \mathbb{R}$

$s(0) = 0 \Rightarrow 0 = -\frac{1}{\pi}\underset{1}{\cos 0} + c \Rightarrow c = \frac{1}{\pi}$

The position of the object at time t is given by: $s(t) = -\frac{1}{\pi}\cos(\pi t) + \frac{1}{\pi}$

(d) $v = \frac{1}{t+2}, t > -2, \quad s(t) = \int \frac{\mathrm{d}t}{t+2} = \ln(t+2) + c, t > -2$

$s(-1) = \frac{1}{2} \Rightarrow \frac{1}{2} = \ln(1) + c \Rightarrow c = \frac{1}{2}$

The position of the object at time t is given by: $s(t) = \ln(t+2) + \frac{1}{2}$

4. **(a)** $a = \mathrm{e}^t \Rightarrow v(t) = \int \mathrm{e}^t \mathrm{d}t = e^t + c,\ c \in \mathbb{R} \quad v(0) = 20 \Rightarrow 20 = \mathrm{e}^0 + c \Rightarrow c = 19$

$v(t) = \mathrm{e}^t + 19 \Rightarrow s(t) = \int (\mathrm{e}^t + 19)\,\mathrm{d}t = \mathrm{e}^t + 19t + k, k \in \mathbb{R} \quad s(0) = 5 \Rightarrow k = 4$

So, the position of the object at time t is given by: $s(t) = \mathrm{e}^t + 19t + 4$

(b) $a = 9.8 \Rightarrow v(t) = \int 9.8\,\mathrm{d}t = 9.8t + c, \quad v(0) = -3 \Rightarrow -3 = 0 + c \Rightarrow c = -3$

$v(t) = 9.8t - 3 \Rightarrow s(t) = \int (9.8t - 3)\,\mathrm{d}t = 9.8 \times \frac{t^2}{2} - 3t + k,\ s(0) = 0 \Rightarrow k = 0$

So, the position of the object at time t is given by: $s(t) = 4.9t^2 - 3t$

(c) $a = -4\sin 2t \Rightarrow v(t) = \int -4\sin 2t\,\mathrm{d}t = -4\left(-\frac{1}{2}\right)\cos 2t + c,\ v(0) = 2 \Rightarrow c = 0$

$v(t) = 2\cos 2t \Rightarrow s(t) = \int 2\cos 2t\,\mathrm{d}t = 2 \times \frac{1}{2}\sin 2t + k,\ s(0) = -3 \Rightarrow c = -3$

So, the position of the object at time t is given by: $s(t) = \sin(2t) - 3$

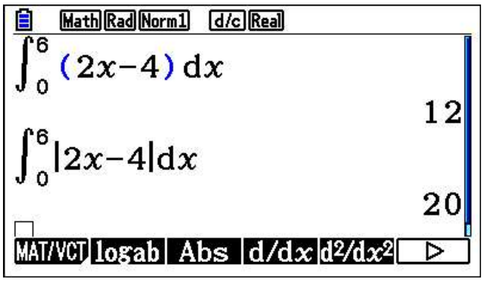
(d) $a=\frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right)\Rightarrow v(t)=\int\frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right)\mathrm{d}t=\frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)+c,\ v(0)=0\Rightarrow c=0$

$v(t)=\frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)\Rightarrow s(t)=\int\frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)\mathrm{d}t=-\cos\left(\frac{3t}{\pi}\right)+k,\ s(0)=-1\Rightarrow k=0$

So, the position of the object at time t is given by: $s(t)=-\cos\left(\frac{3t}{\pi}\right)$

In questions 5–8, we denote displacement of a point by s and total distance travelled by d.

5. **(a)** $v(t)=2t-4\Rightarrow s=\int_0^6(2t-4)\mathrm{d}t,\ d=\int_0^6|2t-4|\mathrm{d}t$



$\int_0^6(2x-4)\,dx$

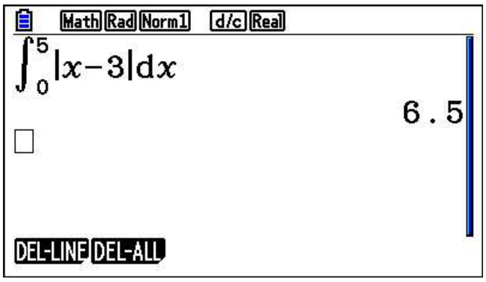
12

$\int_0^6|2x-4|\,dx$

20

The displacement is 12 m and the total distance travelled is 20 m.

(b) $v(t)=|t-3|\Rightarrow s=d=\int_0^5|t-3|\mathrm{d}t$

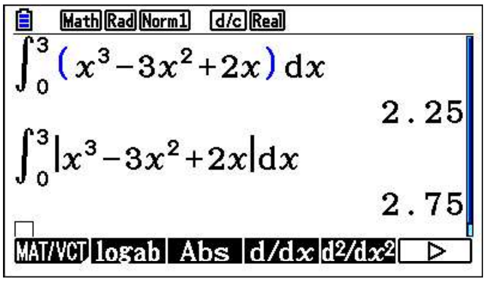


$\int_0^5|x-3|\,dx$

6.5

In this case, both the displacement and the total distance travelled are 6.5 m since the function is always positive.

(c) $v(t)=t^3-3t^2+2t\Rightarrow s=\int_0^3(t^3-3t^2+2t)\mathrm{d}t,\ d=\int_0^3|t^3-3t^2+2t|\mathrm{d}t$



$\int_0^3(x^3-3x^2+2x)\,dx$

2.25

$\int_0^3|x^3-3x^2+2x|\,dx$

2.75

So, the displacement is 2.25 m and the total distance travelled is 2.75 m.

(d) $$v(t)=\sqrt{t}-2 \Rightarrow s=\int_0^3\left(\sqrt{t}-2\right)\mathrm{d}t=\frac{2t^{3/2}}{3}-2t\Bigg|_0^3=2\sqrt{3}-6\approx -2.54$$

$$d=\int_0^3\left|\sqrt{t}-2\right|\mathrm{d}t=\int_0^3\left(2-\sqrt{t}\right)\mathrm{d}t=2t-\frac{2t^{3/2}}{3}\Bigg|_0^3=6-2\sqrt{3}\approx 2.54$$

Since the function is always negative on the given interval, the displacement is –2.54 m, whereas the total distance travelled is 2.54 m.

6. **(a)** $$a(t)=t-2\Rightarrow v(t)=\int(t-2)\,\mathrm{d}t=\frac{t^2}{2}-2t+c,\ v(0)=0\Rightarrow 0=c\Rightarrow v(t)=\frac{t^2}{2}-2t$$

$$s=\int_1^5\left(\frac{t^2}{2}-2t\right)\mathrm{d}t=\left(\frac{t^3}{6}-t^2\right)\Bigg]_1^5=\left(\frac{125}{6}-25\right)-\left(\frac{1}{6}-1\right)=-\frac{10}{3}\approx -3.33 \text{ m}$$

Since the velocity function is quadratic and changes from negative to positive values in the given interval, we need to split the integral into two integrals to calculate the distance travelled.

$$\frac{t^2}{2}-2t=0\Rightarrow t_1=0, t_2=4$$

$$d=\int_1^5\left|2t-\frac{t^2}{2}\right|\mathrm{d}t=\int_1^4\left(2t-\frac{t^2}{2}\right)\mathrm{d}t+\int_4^5\left(\frac{t^2}{2}-2t\right)\mathrm{d}t=\left(t^2-\frac{t^3}{6}\right)\Bigg]_1^4+\left(\frac{t^3}{6}-t^2\right)\Bigg]_4^5$$
$$=\left(16-\frac{32}{3}-1+\frac{1}{6}\right)+\left(\frac{125}{6}-25-\frac{32}{3}+16\right)=\frac{17}{3}\approx 5.67 \text{ m}$$

(b) $$a(t)=\frac{1}{\sqrt{5t+1}}\Rightarrow v(t)=\int\frac{\mathrm{d}t}{\sqrt{5t+1}}\Rightarrow v(t)=\frac{2}{5}\sqrt{5t+1}+c,\quad v(0)=2\Rightarrow c=\frac{8}{5}$$

Since the function $v(t)=\frac{2}{5}\sqrt{5t+1}+\frac{8}{5}$ is always positive, the displacement and the distance travelled are the same.

$$d=s=\int_0^3\left(\frac{2}{5}\sqrt{5t+1}+\frac{8}{5}\right)\mathrm{d}t=\left(\frac{4}{75}(5t+1)^{\frac{3}{2}}+\frac{8}{5}t\right)\Bigg]_0^3=\frac{84}{25}+\frac{24}{5}=\frac{204}{25}=8.16 \text{ m}$$

(c) $$a(t)=-2\Rightarrow v(t)=-2t+c\quad v(0)=3\Rightarrow c=3\Rightarrow v(t)=-2t+3$$

$$s=\int_1^4(-2t+3)\,\mathrm{d}t=\left(-t^2+3t\right)\Big]_1^4=-16+12+1-3=-6 \text{ m}$$

This is a linear function that has a zero, $\frac{3}{2}$, in the given interval, so, to calculate the total distance travelled, we need to split the integral into two integrals using only positive values.

$$d=\int_1^4\left|-2t+3\right|\mathrm{d}t=\int_1^{\frac{3}{2}}(-2t+3)\,\mathrm{d}t+\int_{\frac{3}{2}}^4(2t-3)\,\mathrm{d}t=\left(-t^2+3t\right)\Big]_1^{\frac{3}{2}}+\left(t^2-3t\right)\Big]_{\frac{3}{2}}^4$$

$$=-\frac{9}{4}+\frac{9}{2}+1-3+16-12-\frac{9}{4}+\frac{9}{2}=\frac{13}{2}=6.5\text{ m}$$

7. **(a)** $s=\int_1^3(9.8t-3)\,\mathrm{d}t=4.9t^2-3t\Big|_1^3=33.2$

(b) $s=\int_1^3(9.8t-3)\,\mathrm{d}t=33.2$

(c) $s=\int_1^3(9.8t-3)\,\mathrm{d}t=33.2$

Note: The displacement does not depend on the initial conditions since the displacement is the integral of the velocity function.

8. **(a)** $v(t)=s'(t)=50-20t$

(b) The maximum displacement takes place when the object stops and starts returning towards point $O\Rightarrow v(t)=0$.

$$v(t)=0\Rightarrow 50-20t=0\Rightarrow t=\frac{5}{2},\quad s\left(\frac{5}{2}\right)=\frac{2125}{2}=1062.5\text{ m}$$

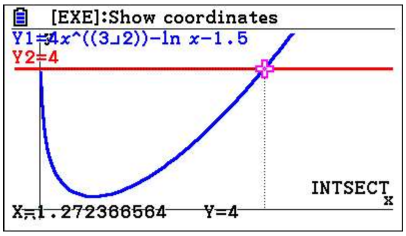
9. $$v(t)=\begin{cases}5t, & 0\le t<1\\ 6\sqrt{t}-\frac{1}{t}, & t\ge 1\end{cases}$$

For the distance to be 4 cm, we actually need to look at the integral of the velocity function. The first part of the integral between 0 and 1 is: $\int_0^1 5t\,\mathrm{d}t=\left(5\frac{t^2}{2}\right)\Big]_0^1=\frac{5}{2}$;

therefore, we need to find the time after 1 second in which the particle covers another 1.5 cm. We first need to find the correct displacement function so that the motion is continuous. Thus, our initial condition for the second part of the displacement function is that $s(1)=2.5$

$s(t)=\int\left(6\sqrt{t}-\frac{1}{t}\right)\mathrm{d}t=4t^{3/2}-\ln(t)+c$; $s(1)=\frac{5}{2}\Rightarrow c=-\frac{3}{2}$; thus, the displacement function after 1 second is $s(t)=4t^{3/2}-\ln(t)-\frac{3}{2}$

The solution can be estimated using GDC.



```
[EXE]:Show coordinates
Y1=4x^((3⌟2))-ln x-1.5
Y2=4
INTSECT
X=1.272366564      Y=4
```

So, the time at which the particle is 4 cm from the starting point, is 1.27 seconds.

10. The velocity of a projectile fired upwards is going to be influenced by gravity and hence the deceleration is going to slow the projectile down. The deceleration we are going to use is 9.81 ms^{-2}.

(a) $v(t)=49-9.81t \Rightarrow v(t)=0 \Rightarrow 49-9.81t=0 \Rightarrow t=\dfrac{49}{9.81}\approx 4.995$; i.e. 5 seconds.

(b) To find the maximum height, we need to find the height formula.

$$v(t)=49-9.81t \Rightarrow h(t)=\int(49-9.81t)\,\mathrm{d}t=49t-9.81\frac{t^2}{2}+c;$$

$$h(0)=150 \Rightarrow c=150 \Rightarrow h(t)=49t-9.81\frac{t^2}{2}+150$$

The maximum height is reached when the velocity is zero, i.e., at t = 5 seconds. Thus, the maximum height is $h(5)$ = 272.4 metres.

(c) Since the parabola is symmetrical with respect to the vertical axis of symmetry that passes through the vertex, we can say that the time taken to reach the maximum height will be doubled. So, the answer is approximately 10 seconds. Alternatively, you can solve the equation $h(t)$ = 150.

$$49t-9.81\frac{t^2}{2}+150=150 \Rightarrow 49t-9.81\frac{t^2}{2}=0 \Rightarrow t_1=0 \text{ or } t=\frac{98}{9.81}\approx 9.9898\approx 10 \text{ s}$$

(d) This is simply $v(10)=49-9.81\times 10\approx -49.1 \text{ ms}^{-1}$

The velocity will be approximately –49 ms^{-1}

(e) The projectile hits the ground when its height is zero.

$$49t-9.81\frac{t^2}{2}+150=0 \Rightarrow \ t\approx 12.447 \text{ s}$$

So, the projectile will take 12.4 seconds to hit the ground.

(f) This is simply $|v(12.447)|=|49-9.81\times 12.447|\approx 73.11 \text{ ms}^{-1}$

So, the speed at impact is 73.1 ms^{-1}

Chapter 14 practice questions

1. **(a)** The parameter p is equal to the amplitude; therefore, $p=3$

(b) $\int_0^{\frac{\pi}{2}} 3\cos x\,\mathrm{d}x=[3\sin x]_0^{\frac{\pi}{2}}=3\sin\left(\frac{\pi}{2}\right)-3\sin 0=3$

Examiner's note: Even though you might not know how to find the parameter in part **(a)**, it is advisable to proceed with part **(b)** and attempt to write the definite integral.

2. **(a)** $y=\mathrm{e}^{\frac{x}{2}}\Rightarrow y(0)=\mathrm{e}^{\frac{0}{2}}=1,$ therefore point P has the coordinates (0, 1)

(b) $V=\pi\int_0^{\ln 2}\left(\mathrm{e}^{\frac{x}{2}}\right)^2\mathrm{d}x=\pi\int_0^{\ln 2}\mathrm{e}^x\mathrm{d}x$

(c) $\pi\int_0^{\ln 2}\mathrm{e}^x\mathrm{d}x=\pi\left[\mathrm{e}^x\right]_0^{\ln 2}=\pi\left(\mathrm{e}^{\ln 2}-\mathrm{e}^0\right)=\pi(2-1)=\pi$

3. $\int_1^a\frac{1}{x}\mathrm{d}x=2\Rightarrow\left[\ln|x|\right]_1^a=2\Rightarrow\ln a-\ln 1=2\Rightarrow\ln a=2\Rightarrow a=\mathrm{e}^2$

4. **(a)** $y=\ln x\Rightarrow y'=\frac{1}{x}$. At the point (e, 1) the slope of the tangent is: $m=y'(\mathrm{e})=\frac{1}{\mathrm{e}}$.

The tangent can be found by using the formula for the tangent:
$y=f'(x_1)(x-x_1)+y_1$, where (x_1,y_1) is a particular point on the graph of the function.

$y=\frac{1}{\mathrm{e}}(x-\mathrm{e})+1\Rightarrow y=\frac{1}{\mathrm{e}}x-1+1\Rightarrow y=\frac{1}{\mathrm{e}}x$

Since (0, 0) satisfies this equation, then the origin is on this line.

(b) Direct application of derivative rules.

$(x\ln x-x)'=\ln x+x\times\frac{1}{x}-1=\ln x+1-1=\ln x$

(c) Notice that the shaded region can be split into two regions. The first region is a triangle bounded by the tangent line, the x-axis and the vertical line $x=1$. Since that is a right-angled triangle, the area is calculated as: $A_{\text{Triangle}}=\frac{1\times\frac{1}{\mathrm{e}}}{2}=\frac{1}{2\mathrm{e}}$

In order to find the area of the second region, we evaluate the following integral:

$$\int_1^{e}\left(\frac{1}{e}x-\ln x\right)dx=\left[\frac{1}{e}\times\frac{x^2}{2}-(x\ln x-x)\right]_1^{e}=\frac{1}{2}e-\frac{1}{2e}-1$$

Now, the total area is the sum of those two areas; therefore,

$$A=\frac{1}{2e}+\frac{1}{2}e-\frac{1}{2e}-1=\frac{1}{2}e-1$$

5. **(a)** **(i)** $s(t)=800+100t-4t^2\Rightarrow s(5)=800+100\times5-4\times5^2=1200\text{ m}$

Distance travelled = 1200 – 800 = 400 m

(ii) $v(t)=s'(t)\Rightarrow v(t)=100-8t\Rightarrow v(5)=100-8\times5=60\text{ m s}^{-1}$

(iii) $v(t)=36\Rightarrow 100-8t=36\Rightarrow 64=8t\Rightarrow t=8\text{ s}$

(iv) $s(8)=800+100\times8-4\times8^2=1344\text{ m}$

(b) First, we need to find the time at which the plane stops after touchdown:

$$v(t)=0\Rightarrow 100-8t=0\Rightarrow t=\frac{100}{8}=12.5\text{ s}$$

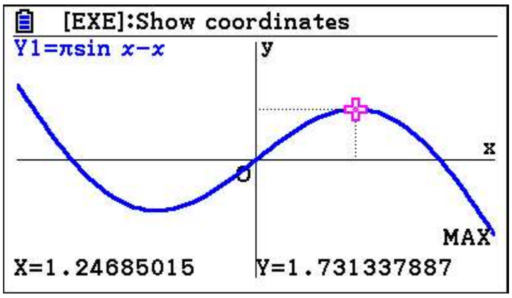
Now we need to find the distance the plane will travel after touchdown:

$$s\left(\frac{25}{2}\right)-s(0)=800+100\times\frac{25}{2}-4\times\left(\frac{25}{2}\right)^2-800=1250-625=625\text{ m}$$

We notice from part **(a)(iv)**, the remaining runway length is $2000-1344=656$ m; therefore, there is enough runway to stop the plane if it makes a touchdown before point *P*.

6. Note: Parts **(a)** and **(b)** cannot be solved without using a calculator.

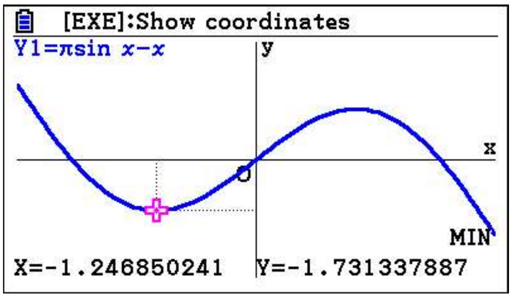
(a) To draw the function, we input the function into a GDC and use its features to make estimates. Samples giving the maximum and minimum are given below.



[EXE]:Show coordinates

Y1=πsin x−x

X=1.24685015 Y=1.731337887

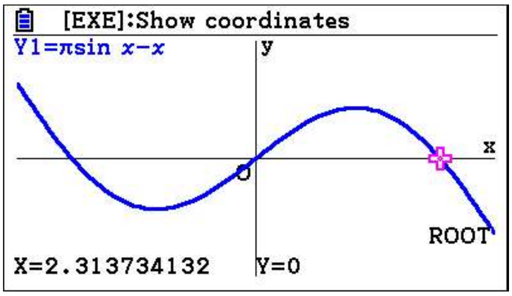


[EXE]:Show coordinates

Y1=πsin x−x

X=-1.246850241 Y=-1.731337887

(b) The solution is approximately $x = 2.314$ as shown:



(c) $\int(\pi \sin x - x)\,\mathrm{d}x = -\pi \cos x - \frac{x^2}{2} + c,\ \ c \in \mathbb{R}$

$$\text{Area} = \int_0^1 (\pi \sin x - x)\,\mathrm{d}x = \left[-\pi \cos x - \frac{x^2}{2}\right]_0^1 = \pi(1 - \cos 1) - \frac{1}{2} \approx 0.944$$

7. One method is to split the shaded region into two regions, R_1, the rectangle enclosed by the lines $x = 0, y = \frac{4}{3}, x = 1,$ and $y = 2$; and R_2, the region enclosed by the curves $x = 1, y = \frac{4}{3},$ and $y = 1 + \frac{1}{x}$. The area of the first region is: $A_1 = \left(2 - \frac{4}{3}\right) \times (1 - 0) = \frac{2}{3}$.

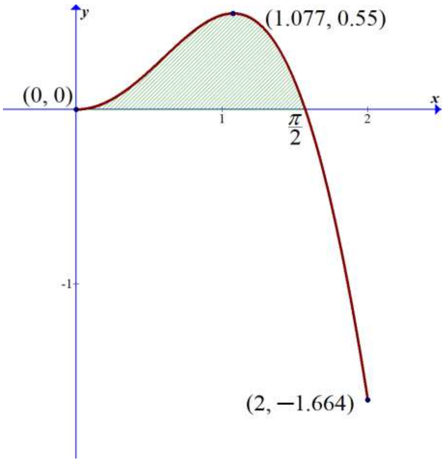
To find the area of the second region, we first find the point of intersection of the line $y = \frac{4}{3}$ and the curve $y = 1 + \frac{1}{x}$. By inspection or solving $1 + \frac{1}{x} = \frac{4}{3}$, we see that the point of intersection is $\left(3, \frac{4}{3}\right)$.

$$A_2 = \int_1^3 \left(1 + \frac{1}{x} - \frac{4}{3}\right)\mathrm{d}x = \int_1^3 \left(\frac{1}{x} - \frac{1}{3}\right)\mathrm{d}x = \left[\ln|x| - \frac{1}{3}x\right]_1^3 = \ln 3 - \frac{2}{3}$$

$$\Rightarrow A = A_1 + A_2 = \frac{2}{3} + \ln 3 - \frac{2}{3} = \ln 3$$

Alternatively, you can express x in terms of y and find the area between the curve and the y-axis.

8. For parts **(a) (i)**, **(ii)** and **(c)** we can use a GDC.



(a) **(i, ii)** See the diagram above.

(b) $x^2\cos x = 0, x > 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

(c) **(i)** See the diagram above.

(ii) $Area = \int_0^{\frac{\pi}{2}} x^2\cos x\,\mathrm{d}x$

(d)

$$\int_0^{\frac{\pi}{2}} x^2\cos x\,\mathrm{d}x = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\frac{\pi}{2}}$$

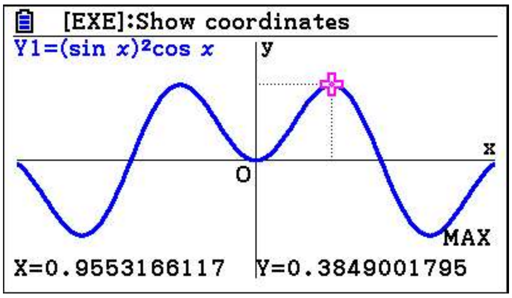
$$= \left(\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{\pi}{2}\right)\right) - 0 = \frac{\pi^2}{4} - 2 \approx 0.467$$

9. **(a)** Since $\sin^2 x$ is periodic with period π and $\cos x$ is periodic with period 2π, the period of this function is 2π.

Moreover, $f(x+2\pi) = \sin^2(x+2\pi)\cos(x+2\pi) = \sin^2(x)\cos(x) = f(x)$,

so the fundamental period of *f* is 2π.

(b) By looking at the graph, we estimate that the range would be $[-0.4, 0.4]$


[EXE]:Show coordinates

Y1=(sin x)²cos x

X=0.9553166117 Y=0.3849001795

(c) **(i)** $f'(x)=(2\sin x\cos x)\cos x+\sin^2 x(-\sin x)=2\sin x\cos^2 x-\sin^3 x$

(ii) $f'(x)=0\Rightarrow 2\sin x\cos^2 x-\sin^3 x=0\Rightarrow \sin x\left(2\cos^2 x-\sin^2 x\right)=0$

Since the value of sine cannot be equal to 0 at *A*, we can conclude that:

$2\cos^2 x-\sin^2 x=0\Rightarrow 2\cos^2 x-\left(1-\cos^2 x\right)=0$

$$\Rightarrow 3\cos^2 x-1=0\Rightarrow \cos x=\sqrt{\frac{1}{3}}$$

(iii) Point *A* is at the maximum of the function, and thus,

$$f(x)=\sin^2(x)\cos(x)=\left(1-\frac{1}{3}\right)\times\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{9}$$

(d) $f(x)=0\Rightarrow \sin x=0$ or $\cos x=0\Rightarrow x=0, x=\pi$ or $x=\frac{\pi}{2}$, so, the *x*-coordinate of point *B* is $\frac{\pi}{2}$.

(e) **(i)** $\int\sin^2(x)\cos(x)\,\mathrm{d}x=\int(\sin x)^2\mathrm{d}(\sin x)=\frac{\sin^3 x}{3}+c,\ c\in\mathbb{R}$

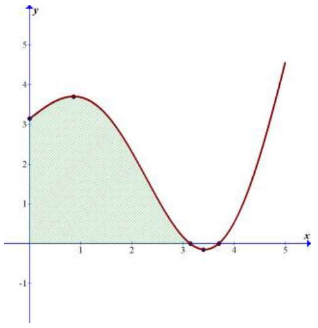
(ii) $\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x=\left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}}=\frac{\sin^3\left(\frac{\pi}{2}\right)}{3}-\frac{\sin^3 0}{3}=\frac{1}{3}$

(f) $f''(x)=0\Rightarrow 9\cos^3 x-7\cos x=0\Rightarrow \cos x\left(9\cos^2 x-7\right)=0$

Since the *x*-coordinate of *C* is less than $\frac{\pi}{2}$, the second factor must be equal to 0.

$9\cos^2 x-7=0\Rightarrow \cos^2 x=\frac{7}{9}\Rightarrow \cos x=\frac{\sqrt{7}}{3}\Rightarrow x=\arccos\left(\frac{\sqrt{7}}{3}\right)\approx 0.491$

10. For parts **(a) (i, ii)**, **(c)** and **(d)** a GDC can be used.



(a) See graph on the previous page.

(b) $x=\pi \Rightarrow \pi+\pi\cos\pi=\pi-\pi=0$

(c) From GDC output, $x = 3.69672$

(d) See graph on the previous page. Area $=\int_0^{\pi}(\pi+x\cos x)\,\mathrm{d}x$

(e) Area $=\int_0^{\pi}(\pi+x\cos x)\,\mathrm{d}x=\left[\pi x+x\sin x+\cos x\right]_0^{\pi}=\pi^2-2\approx 7.86960$

11. **(a)** **(i)** $p=g(x)-f(x)=(10x+2)-\left(1+\mathrm{e}^{2x}\right)=10x+1-\mathrm{e}^{2x}$

(ii) $p'=10-2\mathrm{e}^{2x}=0\Rightarrow 2\mathrm{e}^{2x}=10\Rightarrow \mathrm{e}^{2x}=5\Rightarrow x=\dfrac{\ln 5}{2}\approx 0.805\,(3\text{ s.f.})$

(b) **(i)** $x=1+\mathrm{e}^{2y}\Rightarrow \mathrm{e}^{2y}=x-1\Rightarrow 2y=\ln(x-1)\Rightarrow y=\dfrac{1}{2}\ln(x-1)$

(ii) $f^{-1}(x)=\ln\sqrt{x-1}\Rightarrow f^{-1}(5)=\ln\sqrt{5-1}=\ln 2$

(c) $V=\pi\int_0^{\ln 2}\left(1+\mathrm{e}^{2x}\right)^2\mathrm{d}x$

12. It is not possible to solve this question with a GDC.

$$V=\pi\int_0^a\left((ax+2)^2-\left(x^2+2\right)^2\right)\mathrm{d}x=\pi\int_0^a\left(a^2x^2+4ax+4-x^4-4x^2-4\right)\mathrm{d}x$$

$$=\pi\int_0^a\left(\left(a^2-4\right)x^2+4ax-x^4\right)\mathrm{d}x=\pi\left(\frac{\left(a^2-4\right)x^3}{3}+2ax^2-\frac{x^5}{5}\right)\Bigg]_0^a=\pi\left(\frac{a^5-4a^3}{3}+2a^3-\frac{a^5}{5}\right)$$

$$=\pi\left(\frac{2a^5}{15}+\frac{2a^3}{3}\right)$$

13. We use the suggested substitution $u=\dfrac{1}{2}x+1\Rightarrow x=2u-2;\mathrm{d}x=2\mathrm{d}u$

$$\int x\sqrt{\frac{1}{2}x+1}\,\mathrm{d}x=\int(2u-2)\sqrt{u}\times 2\mathrm{d}u=4\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)\mathrm{d}u$$

$$=4\left(\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right)+c=\frac{8}{5}\left(\frac{1}{2}x+1\right)^{\frac{5}{2}}-\frac{8}{3}\left(\frac{1}{2}x+1\right)^{\frac{3}{2}}+c,\ \ c\in\mathbb{R}$$

14. $a=\frac{\mathrm{d}v}{\mathrm{d}s}\times\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}s}\times v\Rightarrow a=\frac{3(2s-1)-2(3s+2)}{(2s-1)^2}\times\frac{3s+2}{2s-1}=\frac{-7(3s+2)}{(2s-1)^3}$

$s=2\Rightarrow a=\frac{-7(6+2)}{(4-1)^3}=-\frac{56}{27}$

15. $V=\pi\int_0^k\left(\mathrm{e}^x\right)^2\mathrm{d}x=\pi\int_0^k\mathrm{e}^{2x}\mathrm{d}x=\pi\times\left(\frac{1}{2}\mathrm{e}^{2x}\right)\Big]_0^k=\frac{\pi\left(\mathrm{e}^{2k}-1\right)}{2}$

16. $\int_1^k\left(1+\frac{1}{x^2}\right)\mathrm{d}x=\left(x-\frac{1}{x}\right)\Big]_1^k=\frac{k^2-1}{k}=\frac{3}{2}\Rightarrow 2k^2-3k-2=0\Rightarrow(2k+1)(k-2)=0$

$\cancel{k=-\frac{1}{2}}$ or $k=2$, since $k>1$

17. We are given that $a(t)=-\frac{1}{20}t+2,\ v(0)=0$

$v(t)=\int\left(-\frac{1}{20}t+2\right)\mathrm{d}t=-\frac{1}{40}t^2+2t+c,\ c\in\mathbb{R}\quad v(0)=0\Rightarrow c=0\Rightarrow v(t)=-\frac{1}{40}t^2+2t$

$d=\int_0^{60}\left|-\frac{1}{40}t^2+2t\right|\mathrm{d}t=\left(-\frac{t^3}{120}+t^2\right)\Big]_0^{60}=-1800+3600=1800\,\mathrm{m}$

18. First, we need to find the zeros of the parabola.

$y=a^2-x^2\Rightarrow y=(a-x)(a+x)\Rightarrow x_1=-a,x_2=a$

The area of the rectangle is $A_R=2ah$, where h is the height of the rectangle.

The area under the parabola is calculated by the following integral.

$A_P=\int_{-a}^{a}\left(a^2-x^2\right)\mathrm{d}x=\left(a^2x-\frac{x^3}{3}\right)\Big]_{-a}^{a}=\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)=\frac{4}{3}a^3$

Since the two areas must be the same, we can find the height of the rectangle:

$2ah=\frac{4}{3}a^3\Rightarrow h=\frac{2}{3}a^2$

So, the dimensions of the rectangle are: $2a$ by $\frac{2}{3}a^2$

19. **(a)** $f_k(x) = x\ln x - kx, x > 0 \Rightarrow f_k{}'(x) = \ln x + 1 - k, x > 0$

(b) If the function is increasing, the first derivative is positive; therefore:

$\ln x + 1 - k > 0 \Rightarrow \ln x > k - 1 \Rightarrow x > \mathrm{e}^{k-1},\ \ x \in \left]\mathrm{e}^{k-1}, +\infty\right[$

The question asks us to find the interval over which $f(x)$ is increasing; therefore, the value of k is 1 and the interval is: $x > \mathrm{e}^{-1} = \frac{1}{\mathrm{e}},\ \ x \in \left]\frac{1}{\mathrm{e}}, +\infty\right[$

(c) **(i)** $f_k{}'(x) = \ln x + 1 - k = 0 \Rightarrow \ln x = k - 1 \Rightarrow x = \mathrm{e}^{k-1}$

(ii) $f_k(x) = x\ln x - kx = 0 \Rightarrow x(\ln x - k) = 0 \Rightarrow x = 0$ or $\ln x - k = 0$

So, the other x-intercept is at: $\ln x - k = 0 \Rightarrow \ln x = k \Rightarrow x = \mathrm{e}^k$

(d) The area between the curve and the x-axis is given by $\int_0^{\mathrm{e}^k} |x\ln x - kx|\,\mathrm{d}x$

The integral can be evaluated partly using integration by parts and the rest is just the power rule:

$$\int x\ln x\,\mathrm{d}x = = \frac{x^2}{2}\ln x - \int\left(\frac{x^2}{2}\times\frac{1}{x}\right)\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

$$\int_0^{\mathrm{e}^k} |x\ln x - kx|\,\mathrm{d}x = \left|\left[\frac{x^2}{2}\ln x - \frac{x^2}{4} - k\frac{x^2}{2}\right]_0^{\mathrm{e}^k}\right| = \left|\frac{\mathrm{e}^{2k}}{4}\left(2\ln \mathrm{e}^k - 1 - 2k\right)\right| = \frac{\mathrm{e}^{2k}}{4}$$

So, the area enclosed by the curve and the x-axis is: $\frac{\mathrm{e}^{2k}}{4}$

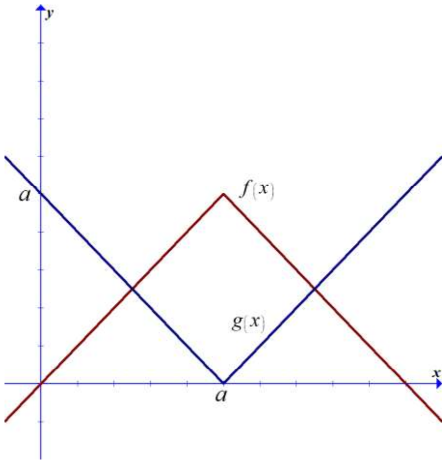
(e) $A\left(\mathrm{e}^k, 0\right), m = f_k'\left(\mathrm{e}^k\right) = \ln \mathrm{e}^k + 1 - k = 1 \Rightarrow T : y = 1\times\left(x - \mathrm{e}^k\right) + 0 \Rightarrow y = x - \mathrm{e}^k$

(f) The y-intercept is $-\mathrm{e}^k$, so the area of the triangle enclosed by the tangent and the coordinate axes is: $A = \frac{1}{2}\left|\mathrm{e}^k \times \left(-\mathrm{e}^k\right)\right| = \frac{\mathrm{e}^{2k}}{2} = 2\times\frac{\mathrm{e}^{2k}}{4}$, which is twice the area enclosed by the curve.

(g) $k = 1 \Rightarrow x_1 = \mathrm{e}, k = 2 \Rightarrow x_1 = \mathrm{e}^2, k = 3 \Rightarrow x_1 = \mathrm{e}^3, k = 4 \Rightarrow x_1 = \mathrm{e}^4, \ldots$

To verify the statement, we are going to take two consecutive x-intercepts, for k and $k + 1$: $\frac{x_{k+1}}{x_k} = \frac{\mathrm{e}^{k+1}}{\mathrm{e}^k} = \mathrm{e}$. The ratio is constant and therefore the intercepts form a geometric sequence with common ratio e.

20. If you sketch a graph of the functions, you can see that the area between them is a square with diagonal *a*.



Area of the square is $\frac{1}{2}a^2$. The value of a that gives an area of 12.5 is such that

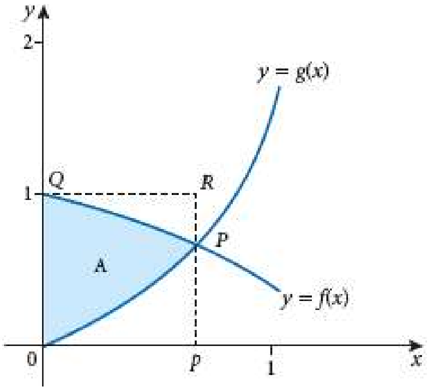
$$\frac{1}{2}a^2 = 12.5 \Rightarrow a = 5$$

21. $$kx = mv\frac{\mathrm{d}v}{\mathrm{d}x} \Rightarrow kx\,\mathrm{d}x = mv\,\mathrm{d}v \Rightarrow \int kx\,\mathrm{d}x = \int mv\,\mathrm{d}v \Rightarrow k\frac{x^2}{2} = m\frac{v^2}{2} + c,\ \ c \in \mathbb{R}$$

$$x = 0, v = v_0 \Rightarrow k\times 0 = m\frac{v_0^2}{2} + c \Rightarrow c = -\frac{mv_0^2}{2} \Rightarrow k\frac{x^2}{2} = m\frac{v^2}{2} - \frac{mv_0^2}{2}$$

$$x = 2 \Rightarrow k\frac{2^2}{2} = m\frac{v^2}{2} - \frac{mv_0^2}{2} \Rightarrow \frac{4k}{m} = v^2 - v_0^2 \Rightarrow v = \sqrt{\frac{4k}{m} + v_0^2}$$

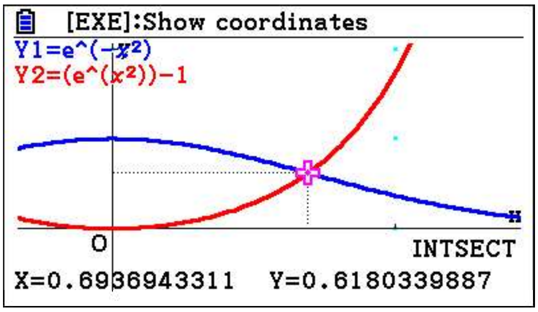
22. **(a)** Graph below:



(b) Consider triangle *OPQ* and rectangle *OPRQ* in the diagram. Notice that the rectangle has dimension 1 × p and that the triangle has a vertical base of length 1 and height p.

$$A_{TRIANGLE} < A_{REGION} < A_{RECTANGLE} \Rightarrow \frac{1}{2}p\times 1 < A_{REGION} < p\times 1 \Rightarrow \frac{p}{2} < A_{REGION} < p$$

(c) A GDC will give a good approximation.

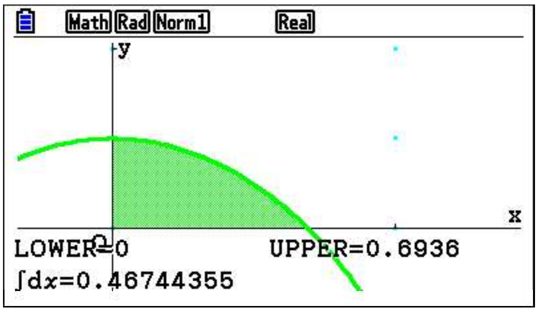


```
[EXE]:Show coordinates
Y1=e^(-x²)
Y2=(e^(x²))-1
INTSECT
X=0.6936943311   Y=0.6180339887
```

$p = 0.6937$ (correct to 4 decimal places)

(d) $A_{REGION} = \int_0^p \left(e^{-x^2} - \left(e^{x^2} - 1\right)\right)dx$

Again, a GDC will give the requested approximation. We chose to graph the difference of the functions and integrate.



```
LOWER=0         UPPER=0.6936
∫dx=0.46744355
```

So, the area of the region is 0.467 (correct to 3 significant figures).
Notice that the last two parts of the question cannot be done without using a calculator.

23. **(a)** Use the substitution $\begin{matrix} u = x & \mathrm{d}v = \cos 3x\,\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = \frac{1}{3}\sin 3x \end{matrix}$

$$\int x\cos 3x\,\mathrm{d}x = \frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\,\mathrm{d}x = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c,\ c \in \mathbb{R}$$

(b) **(i)**

$$A = \int_{\frac{\pi}{6}}^{\frac{3\pi}{6}} |x\cos 3x|\,\mathrm{d}x = \left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{3\pi}{6}}\right|$$

$$= \left|\left(\frac{\pi}{6}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi}{2}\right)\right) - \left(\frac{\pi}{18}\sin\left(\frac{\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{\pi}{2}\right)\right)\right| = \frac{2\pi}{9}$$

(ii)

$$\int_{\frac{3\pi}{6}}^{\frac{5\pi}{6}} |x\cos 3x|\,\mathrm{d}x = \left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{3\pi}{6}}^{\frac{5\pi}{6}}\right|$$

$$= \left|\left(\frac{5\pi}{18}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{5\pi}{2}\right)\right) - \left(\frac{\pi}{6}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi}{2}\right)\right)\right| = \frac{4\pi}{9}$$

(iii)

$$\int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}} |x\cos 3x|\,\mathrm{d}x = \left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}\right|$$

$$= \left|\left(\frac{7\pi}{18}\sin\left(\frac{7\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{7\pi}{2}\right)\right) - \left(\frac{5\pi}{18}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{9}\cos\left(\frac{5\pi}{2}\right)\right)\right| = \frac{6\pi}{9}$$

(c) The areas enclosed by the given boundaries form an arithmetic sequence with first term $u_1 = \frac{2\pi}{9}$ and common difference $d = \frac{2\pi}{9}$. Therefore, the sum of the first n terms is given by: $S_n = \frac{n}{2}\left(2\left(\frac{2\pi}{9}\right) + (n-1)\left(\frac{2\pi}{9}\right)\right) = \frac{n(n+1)\pi}{9}, n \in \mathbb{Z}^+$

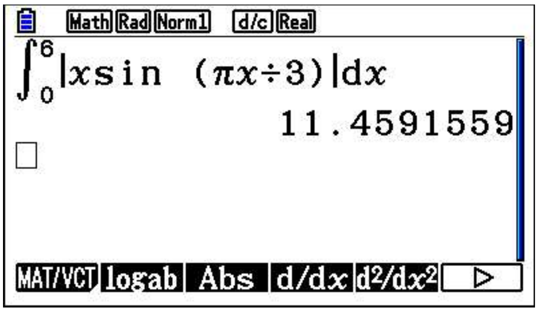
24. **(a)** $v(t) = 0 \Rightarrow t\sin\left(\frac{\pi}{3}t\right) = 0 \Rightarrow t = 0$ or $\frac{\pi}{3}t = k\pi \Rightarrow t = 3k, k \in \mathbb{Z}$

Using the restricted domain, we can calculate the values of t:
$t = 0$ or $t = 3$ or $t = 6$.

(b) **(i)** We use the absolute value of the integral for the total distance.

$$\text{Total distance travelled} = \int_0^6 \left| t\sin\left(\frac{\pi}{3}t\right)\text{d}t \right|$$

(ii) We use GDC for this part.



```
Math Rad Norm1  d/c Real
∫₀⁶|xsin (πx÷3)|dx
                11.4591559
```

So, the total distance travelled is 11.5 m (correct to 3 significant figures). Note: If not using a GDC, we should split the integral into two parts, from 0 to 3 and from 3 to 6, where the last one has a negative value and we take its opposite value. The anti-derivative can be found by using integration by parts.

25. **(a)**

$$\text{Distance travelled} = \int_0^1 v(t)\,\text{d}t = \int_0^1 \frac{1}{2+t^2}\text{d}t$$

$$= \left(\frac{1}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right)\right)\Bigg]_0^1 = \left(\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\right)\right) - \left(\frac{1}{\sqrt{2}}\arctan 0\right)$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\right) \approx 0.435 \text{ m}$$

(b) $a = \frac{\text{d}v}{\text{d}t} \Rightarrow a(t) = \frac{-2t}{\left(2+t^2\right)^2}$

26. **(a)** $y = 2x\sqrt{1+x^2} \Rightarrow \frac{\text{d}y}{\text{d}x} = 2\sqrt{1+x^2} + 2x \times \frac{2x}{2\sqrt{1+x^2}} = 2\sqrt{1+x^2} + \frac{2x^2}{\sqrt{1+x^2}}$

(b) $\int 2x\sqrt{1+x^2}\,\text{d}x = \int \sqrt{u}\,\text{d}u = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + c,\ c \in \mathbb{R}$

$$\int_0^k 2x\sqrt{1+x^2}\,\mathrm{d}x = \left(\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right)\Bigg|_0^k = \frac{2}{3}\left(1+k^2\right)^{\frac{3}{2}} - \frac{2}{3} \Rightarrow \frac{2}{3}\left(1+k^2\right)^{\frac{3}{2}} - \frac{2}{3} = 1$$

(c) $$\Rightarrow \left(1+k^2\right)^{\frac{3}{2}} = \frac{5}{2} \Rightarrow 1+k^2 = \left(\frac{5}{2}\right)^{\frac{2}{3}} \Rightarrow k^2 = \left(\frac{5}{2}\right)^{\frac{2}{3}} - 1$$

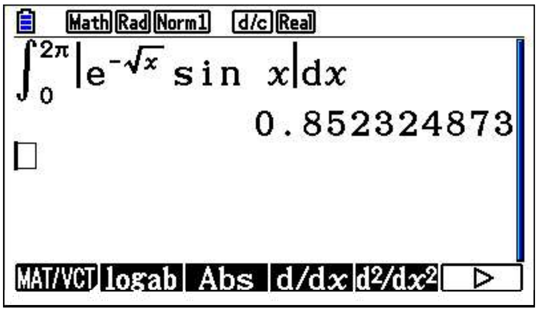
$$\Rightarrow k = \sqrt{\left(\frac{5}{2}\right)^{\frac{2}{3}} - 1} \approx 0.918$$

27. Total distance is given by the integral of the absolute value. $v(t) = 6t^2 - 6t, t \geq 0 \Rightarrow$

$$\text{distance} = \int_0^2 \left|6t^2 - 6t\right|\mathrm{d}t = \int_0^1 \left(6t - 6t^2\right)\mathrm{d}t + \int_1^2 \left(6t^2 - 6t\right)\mathrm{d}t$$

$$= \left(3t^2 - 2t^3\right)\Big]_0^1 + \left(2t^3 - 3t^2\right)\Big]_1^2 = (1-0) + (16-12-2+3) = 6\text{ m}$$

28. We will use a GDC in this question.



$$\int_0^{2\pi} \left|\mathrm{e}^{-\sqrt{x}} \sin x\right| \mathrm{d}x$$

0.852324873

The total distance travelled is 0.852 m (correct to 3 significant figures).

29. **(a)** $$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T-22) \Rightarrow \frac{\mathrm{d}T}{T-22} = k\mathrm{d}t \Rightarrow \int \frac{\mathrm{d}T}{T-22} = \int k\mathrm{d}t \Rightarrow$$

$$\Rightarrow \ln|T-22| = kt + c \Rightarrow T - 22 = \mathrm{e}^{kt+c}, c \in \mathbb{R} \Rightarrow T = 22 + A\mathrm{e}^{kt}, A \in \mathbb{R}^+$$

(b) To find the constants A and k, we need to solve the simultaneous equations given by the information.

(i) $T(0) = 100 \Rightarrow 22 + A\mathrm{e}^0 = 100 \Rightarrow A = 78$

$$T(15) = 70 \Rightarrow 22 + A\mathrm{e}^{15k} = 70 \Rightarrow 78\mathrm{e}^{15k} = 48 \Rightarrow \mathrm{e}^{15k} = \frac{8}{13}$$

$$\Rightarrow k = \frac{1}{15}\ln\left(\frac{8}{13}\right) \approx -0.324$$

(ii) $T = 22 + 78\mathrm{e}^{\frac{\ln\left(\frac{8}{13}\right)t}{15}} \Rightarrow 40 = 22 + 78\mathrm{e}^{\frac{\ln\left(\frac{8}{13}\right)t}{15}} \Rightarrow \frac{18}{78} = \mathrm{e}^{\frac{\ln\left(\frac{8}{13}\right)t}{15}}$

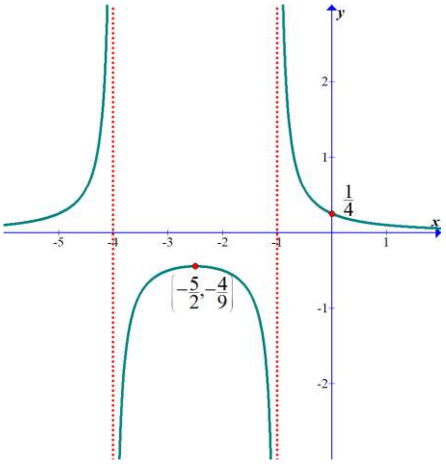
$$\Rightarrow \ln\left(\frac{3}{13}\right) = \frac{\ln\left(\frac{8}{13}\right)t}{15} \Rightarrow\Rightarrow \ln\left(\frac{3}{13}\right) = \frac{\ln\left(\frac{8}{13}\right)t}{15} \Rightarrow t = \frac{15\ln\left(\frac{3}{13}\right)}{\ln\left(\frac{8}{13}\right)} \approx 45.3$$

30. **(a)** The function can be simplified and written as follows using partial fractions

$$f(x) = \frac{1}{(x+4)(x+1)} = \frac{1}{3(x+1)} - \frac{1}{3(x+4)}$$

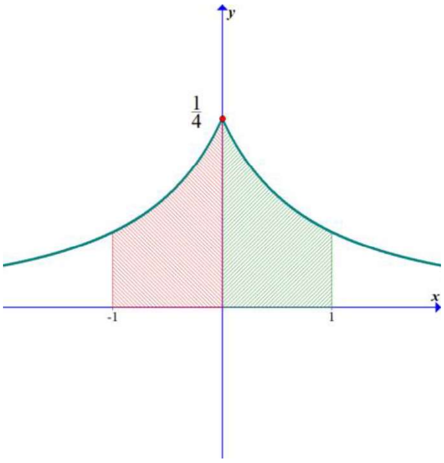
Since the numerator's degree is less than the denominator's, it has the *x*-axis as a horizontal asymptote. The vertical asymptotes are clearly $x = -1$ and $x = -4$.

When $x = 0, y = \frac{1}{4}$. Also $f'(x) = -\frac{2x+5}{(x+4)^2(x+1)^2} \Rightarrow \left(-\frac{5}{2}, -\frac{4}{9}\right)$ is an extreme value.



(b) $\int_0^1 \frac{\mathrm{d}x}{(x+4)(x+1)} = \frac{1}{3}\int_0^1 \frac{\mathrm{d}x}{(x+1)} - \frac{1}{3}\int_0^1 \frac{\mathrm{d}x}{(x+4)} = \frac{1}{3}\ln\left(\frac{x+1}{x+4}\right)\Bigg|_0^1 = \ln\sqrt[3]{\frac{8}{5}}$

(c) As is clear from the graph, the required area is twice the area of the function bounded by the *y*-axis, the curve and *x*-axis.



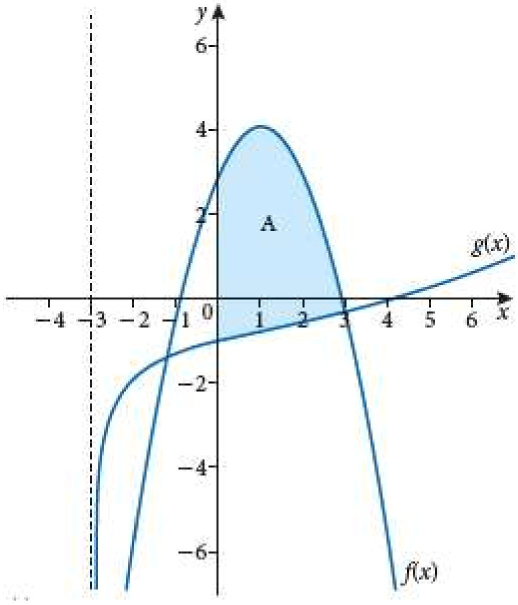
Therefore, the required area is $2\ln\sqrt[3]{\frac{8}{5}}$

31. Using the suggested substitution:

$$\int\frac{x^3}{(x+2)^2}\,\mathrm{d}x=\int\frac{(u-2)^3}{u^2}\,\mathrm{d}u=\int\frac{u^3-6u^2+12u-8}{u^2}\,\mathrm{d}u=\int\left(u-6+\frac{12}{u}-\frac{8}{u^2}\right)\mathrm{d}u$$

$$=\frac{u^2}{2}-6u+12\ln|u|+\frac{8}{u}+c=\frac{(x+2)^2}{2}-6(x+2)+12\ln|x+2|+\frac{8}{x+2}+c,\ c\in\mathbb{R}$$

32. **(a)** Graphs below.

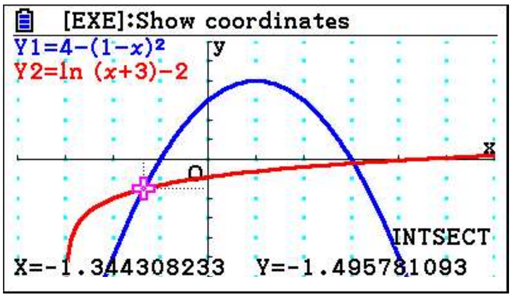


(b) **(i)** Only the logarithmic function $g(x)=\ln(x+3)-2$ has a vertical asymptote: $x=-3$

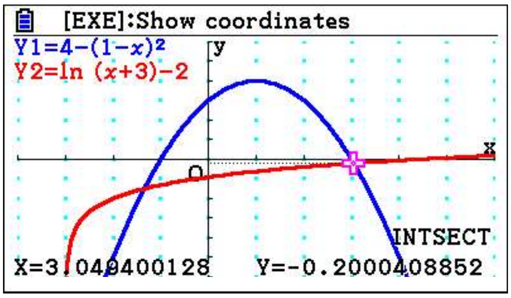
(ii) y-intercept: $x=0 \Rightarrow g(x)=\ln(3)-2\approx -0.901$

x-intercept: $y=0 \Rightarrow 0=\ln(x+3)-2 \Rightarrow x+3=\mathrm{e}^2 \Rightarrow x=\mathrm{e}^2-3\approx 4.39$

(c) We use a GDC to find the points of intersection.



```
[EXE]:Show coordinates
Y1=4-(1-x)²
Y2=ln (x+3)-2
INTSECT
X=-1.344308233   Y=-1.495781093
```



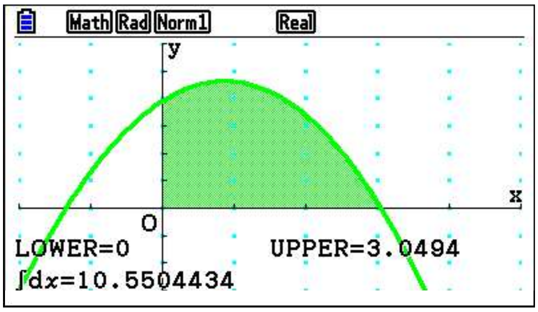
```
[EXE]:Show coordinates
Y1=4-(1-x)²
Y2=ln (x+3)-2
INTSECT
X=3.049400128   Y=-0.2000408852
```

$x = -1.34$, or $x = 3.05$

(d) **(i)** Refer to the diagram in part **(a)**.

(ii) $\int_0^{3.05}\left(\left(4-(1-x)^2\right)-\left(\ln(x+3)-2\right)\right)\mathrm{d}x$

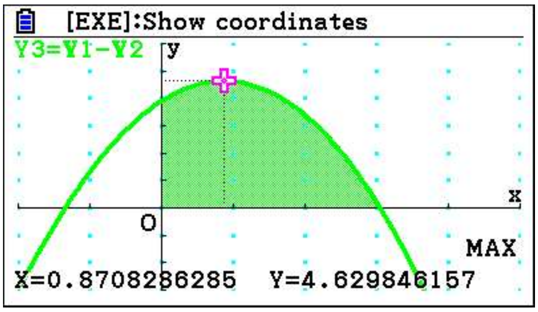
(iii) Use a GDC.



```
LOWER=0          UPPER=3.0494
∫dx=10.5504434
```

So, the shaded region has an area of 10.6 (correct to 3 significant figures).

(e) To find the maximum distance, we refer to the function: $h(x)=f(x)-g(x)$, which we created in part **(d)** to calculate the area.


[EXE]:Show coordinates

Y3=Y1−Y2

MAX

X=0.8708286285 Y=4.629846157

The maximum distance between $f(x)$ and $g(x)$ is 4.63 (correct to 3 s.f.)

33. **(a)** $x = \mathrm{e}^{\theta} \Rightarrow \mathrm{d}x = \mathrm{e}^{\theta}\mathrm{d}\theta \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \dfrac{y}{\mathrm{e}^{2\theta}+1} \Rightarrow \dfrac{x\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x^2+1} \Rightarrow \dfrac{\mathrm{d}y}{y} = \dfrac{\mathrm{d}x}{x\left(x^2+1\right)}$

(b) Use partial fractions first to simplify and then integrate:

$$\int \frac{\mathrm{d}x}{x\left(x^2+1\right)} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)\mathrm{d}x = \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + c,\ \ c \in \mathbb{R}$$

(c) Using the result from part **(a)**

$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x\left(x^2+1\right)} \Rightarrow \ln|y| = \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + c$$

$$\Rightarrow \ln|y| = \ln\left|\mathrm{e}^{\theta}\right| - \frac{1}{2}\ln\left(\mathrm{e}^{2\theta}+1\right) + c$$

Now, apply the initial conditions.

$$\theta = 0, y = \sqrt{2} \Rightarrow \frac{1}{2}\ln 2 = \ln 1 - \frac{1}{2}\ln 2 + c \Rightarrow c = \ln 2$$

$$\ln|y| = \ln\left|\mathrm{e}^{\theta}\right| - \frac{1}{2}\ln\left(\mathrm{e}^{2\theta}+1\right) + \ln 2 \Rightarrow \ln|y| = \ln\left(\frac{2\mathrm{e}^{\theta}}{\sqrt{\mathrm{e}^{2\theta}+1}}\right)$$

$$\Rightarrow y = \frac{2\mathrm{e}^{\theta}}{\sqrt{\mathrm{e}^{2\theta}+1}}$$

34. **(a)** To find the extreme points, we first find the derivative

$$f(x)=\frac{(\ln x)^2}{x}\Rightarrow f'(x)=\frac{2\ln x-(\ln x)^2}{x^2}=\frac{\ln x(2-\ln x)}{x^2}$$

$$f'(x)=0\Rightarrow \ln x(2-\ln x)=0\Rightarrow x=1 \text{ or } x=\mathrm{e}^2$$

Extreme points are: $(1,\ 0),\left(\mathrm{e}^2,\frac{4}{\mathrm{e}^2}\right)$

(b) $$\text{Area}=\int_1^{\mathrm{e}}\frac{(\ln x)^2}{x}=\frac{1}{3}(\ln x)^3\Big|_1^{\mathrm{e}}=\frac{1}{3}$$

(c) $$\text{Volume}=\pi\int_1^{\mathrm{e}}\frac{(\ln x)^4}{x^2}\mathrm{d}x$$

One way to evaluate this is first by substitution, and then apply integration by parts on the resulting expression.

First use the substitution $\ln x=u\Rightarrow\frac{\mathrm{d}x}{x}=\mathrm{d}u$ and $x=\mathrm{e}^u$. The integral now is

$$\int\frac{(\ln x)^4}{x^2}\mathrm{d}x=\int\frac{(\ln x)^4}{x}\frac{\mathrm{d}x}{x}=\int\frac{u^4}{\mathrm{e}^u}\mathrm{d}u=\int u^4\mathrm{e}^{-u}\mathrm{d}u$$

Now apply integration by parts on this 4 times.

$$\int u^4\mathrm{e}^{-u}\mathrm{d}u=-\mathrm{e}^{-u}\left(u^4+4u^3+12u^2+24u+24\right)$$

$$\Rightarrow\int\frac{(\ln x)^4}{x^2}\mathrm{d}x=-\frac{1}{x}\left((\ln x)^4+4(\ln x)^3+12(\ln x)^2+24\ln x+24\right)$$

Thus,

$$V=\pi\int_1^{\mathrm{e}}\frac{(\ln x)^4}{x^2}\mathrm{d}x=-\frac{\pi}{x}\left((\ln x)^4+4(\ln x)^3+12(\ln x)^2+24\ln x+24\right)\Big|_1^{\mathrm{e}}=\frac{\pi}{\mathrm{e}}(24\mathrm{e}-65)$$

35. **(a)** **(i)** We simplify the expression in the denominator and then take the reciprocal

$$4f(x)-2g(x)=2\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)-\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)=\mathrm{e}^{x}+3\mathrm{e}^{-x}$$

$$\Rightarrow\frac{1}{4f(x)-2g(x)}=\frac{1}{\mathrm{e}^{x}+3\mathrm{e}^{-x}}=\frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}+3}$$

(ii) Use the substitution $u=\mathrm{e}^{x}$ and integrate.

$$\int_{0}^{\ln 3}\frac{1}{4f(x)-2g(x)}\mathrm{d}x=\int_{0}^{\ln 3}\frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}+3}\mathrm{d}x=\int_{1}^{3}\frac{\mathrm{d}u}{u^{2}+3}=\frac{1}{3}\int_{1}^{3}\frac{\mathrm{d}u}{\left(\frac{u}{\sqrt{3}}\right)^{2}+1}$$

$$=\frac{\sqrt{3}}{3}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{1}^{3}=\frac{\sqrt{3}}{3}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{18}\pi$$

(b) $h(x)=nf(x)+g(x)=n\dfrac{\left(\mathrm{e}^{x}+\mathrm{e}^{-x}\right)}{2}+\dfrac{\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)}{2}=\dfrac{(n+1)\mathrm{e}^{x}+(n-1)\mathrm{e}^{-x}}{2}$

(i) $\dfrac{(n+1)\mathrm{e}^{2x}+(n-1)}{2\mathrm{e}^{x}}=k\Rightarrow(n+1)\mathrm{e}^{2x}-2k\mathrm{e}^{x}+(n-1)=0$

Use the quadratic formula

$$\mathrm{e}^{x}=\frac{k\pm\sqrt{k^{2}-n^{2}+1}}{n+1}\Rightarrow x=\ln\left(\frac{k\pm\sqrt{k^{2}-n^{2}+1}}{n+1}\right)$$

(ii) The quadratic equation will have two solutions when the expression $k^{2}-n^{2}+1>0$.

That is $k^{2}>n^{2}-1>0\Rightarrow k>\sqrt{n^{2}-1}$. Additionally, $k\pm\sqrt{k^{2}-n^{2}+1}$ must also be positive, i.e., $k>\sqrt{k^{2}-n^{2}+1}$.

(c) **(i)** Simplify $t(x)$:

$$t(x)=\frac{g(x)}{f(x)}\Rightarrow t'(x)=\frac{g'(x)f(x)-f'(x)g(x)}{f^{2}(x)}$$

But $g'(x)=\dfrac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}=f(x)$ and $f'(x)=\dfrac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}=g(x)$

Thus, $t'(x)=\dfrac{f(x)f(x)-g(x)g(x)}{f^{2}(x)}=\dfrac{f^{2}(x)-g^{2}(x)}{f^{2}(x)}$

(ii) Since $\mathrm{e}^{x}+\mathrm{e}^{-x}>\mathrm{e}^{x}-\mathrm{e}^{-x}$, then $f(x)>g(x)$, which in turn implies that $f^{2}(x)>g^{2}(x)\Rightarrow f^{2}(x)-g^{2}(x)>0\Rightarrow t'(x)>0$

In answers requiring GDC output, we will first give the answer, and the GDC screens will appear at the end of the exercise or part.

Exercise 15.1

1. **(a)** Discrete, since the number of words in a spelling test is finite.

(b) Continuous; the amount of water is measured in litres which is correct to a given accuracy but, in fact, it could be any value within the given intervals.

(c) Continuous; the amount of time is measured in minutes or seconds and it could be any value within a given interval of accuracy.

(d) Discrete; even though the number of bacteria could be very large, it is countable. You cannot have 234.23 bacteria, for example.

(e) Continuous; the amount of CO is measured as a volume that could be any value within a given interval of accuracy.

(f) Continuous, since the amount of vaccine is measured as a volume.

(g) Discrete, since the heart rate (per minute) is always measured as an exact number of heart beats within the given period of time.

(h) Continuous, since the pressure is a measure that can take on any value within the given interval of accuracy.

(i) Continuous, since the distance travelled is a measure that can take on any value within the given interval of accuracy.

(j) Discrete, since the scores in the league cannot be any other values than integers up to the single larger integer; therefore, the total score is finite and countable.

(k) Continuous, since the height can take on any value within the given interval of accuracy.

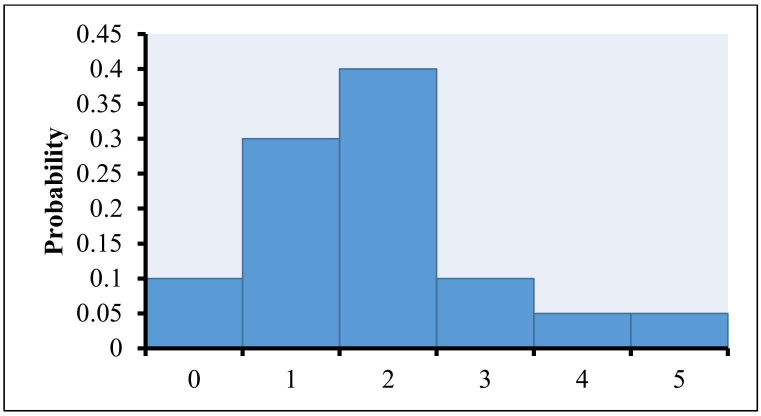
(l) Continuous, since the strength can take on any value within the given interval of accuracy.

(m) Discrete, since the number of overdue books cannot exceed the number of books in the library, that is, the number is finite and countable.

2. **(a)** Since the sum of all the probabilities should be 1, we get the following:

$$0.1+0.3+\mathrm{P}\left(2\right)+0.1+0.05+0.05=1 \Rightarrow \mathrm{P}(2)=0.4$$

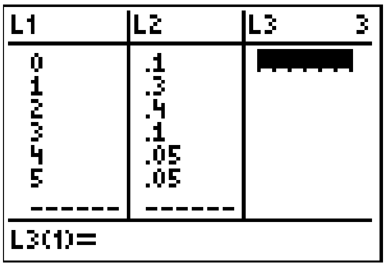
(b)



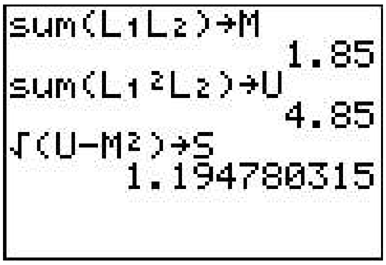
(c) $\mu = \sum y_i p_i = 0\times 0.1 + 1\times 0.3 + \cdots + 5\times 0.05 = 1.85$

$\sigma^2 = \sum y_i^2 p_i - \mu^2 = 0^2 \times 0.1 + \cdots + 5^2 \times 0.05 - 1.85^2 = 1.4275 \Rightarrow \sigma = 1.19$

Or, by using a calculator:



```
L1      L2      L3     3
 0      .1
 1      .3
 2      .4
 3      .1
 4      .05
 5      .05
------  ------
L3(1)=
```



```
sum(L₁L₂)→M
                1.85
sum(L₁²L₂)→U
                4.85
√(U-M²)→S
         1.194780315
```

(d) $\mu \pm \sigma = 1.85 \pm 1.19 \Rightarrow [0.66, 3.04]$ The interval will spread from the first bar up to the bar above 3.

$\mu \pm 2\sigma = 1.85 \pm 2\times 1.19 \Rightarrow [-0.54, 4.24]$ The interval will spread from the first bar up to the bar above 5.

(e) $\mu = \sum z_i p_i = 1\times 0.1 + 2\times 0.3 + \cdots + 6\times 0.05 = 2.85$

$\sigma^2 = \sum z_i^2 p_i - \mu^2 = 1^2 \times 0.1 + \cdots + 6^2 \times 0.05 - 2.85^2 = 1.4275 \Rightarrow \sigma = 1.19$

(f) Since we are adding a constant to each variable, the standard deviation will not change but the new mean will become 1.85 + *b*.

3. **(a)** Since the sum of all the probabilities should be 1, we get the following:

$0.14 + 0.11 + \mathrm{P}(15) + 0.26 + 0.23 = 1 \Rightarrow 0.74 + \mathrm{P}(15) = 1 \Rightarrow \mathrm{P}(15) = 0.26$

(b) $\mathrm{P}(x = 12 \text{ or } x = 20) = \mathrm{P}(12) + \mathrm{P}(20) = 0.14 + 0.23 = 0.37$

(c) $\mathrm{P}(x \le 18) = \mathrm{P}(12) + \mathrm{P}(13) + \mathrm{P}(15) + \mathrm{P}(18) = 0.14 + 0.11 + 0.26 + 0.26 = 0.77$

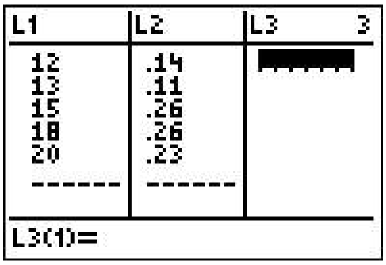
or we can use the complementary event; therefore,

$\mathrm{P}(x \le 18) = 1 - \mathrm{P}(20) = 1 - 0.23 = 0.77$

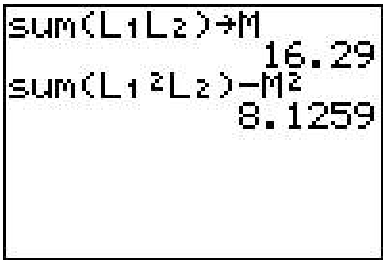
(d) $\mathrm{E}(X) = \sum x_i p_i = 12\times 0.14 + 13\times 0.11 + \cdots + 20\times 0.23 = 16.29$

(e) $\mathrm{V}(X)=\sum x_i^2 p_i-(\mathrm{E}(X))^2=12^2\times0.14+\cdots+20^2\times0.23-16.29^2=8.1259$

Or, we can solve both parts by using a calculator:



```
L1      L2      L3      3
12      .14
13      .11
15      .26
18      .26
20      .23
------  ------
L3(1)=
```



```
sum(L₁L₂)→M
                16.29
sum(L₁²L₂)-M²
               8.1259
```

(f) $\mathrm{E}(Y)=\sum y_i p_i=2\times0.14+2.5\times0.11+\cdots+6\times0.23=4.145$

$\mathrm{V}(Y)=\sum y_i^2 p_i-(\mathrm{E}(Y))^2=2^2\times0.14+\cdots+6^2\times0.23-4.145^2=2.031475$

(g) $\mathrm{E}(Y)=4.145=0.5\times16.29-4$; $\mathrm{V}(Y)=2.031475=0.5^2\times8.1259$

$\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$; $\mathrm{V}(aX+b)=a^2\mathrm{V}(X)$

4. **(a)** At least two patients means 2, 3, 4 or 5; but in this case, it would be easier to use the complementary event, which is 0 or 1; therefore, we get:
$\mathrm{P}(X\geq2)=1-\mathrm{P}(X\leq1)=1-(0.002+0.029)=1-0.031=0.969$

(b) The majority in a group of five would be 3, 4 or 5; therefore, we need to find:
$\mathrm{P}(X\leq2)=\mathrm{P}(X\leq1)+\mathrm{P}(2)=0.031+0.132=0.163$

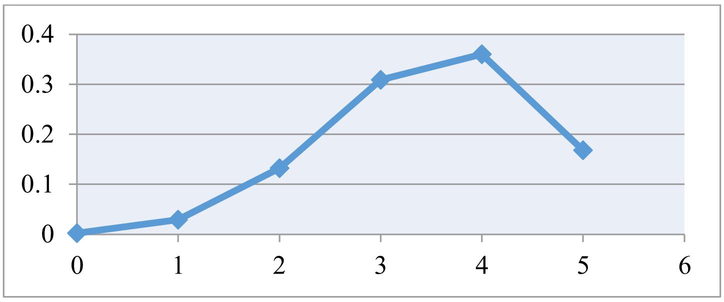
(c) $\mathrm{E}(X)=\sum x_i p_i=0\times0.002+1\times0.29+\cdots+5\times0.168=3.5$, which means that in the long run, on average, we expect between 3 and 4 out of every group of 5 will benefit from the treatment.

(d) $\sigma^2(X)=\sum x_i^2 p_i-(\mathrm{E}(X))^2=0^2\times0.002+\cdots+5^2\times0.168-3.5^2=1.048$

$\Rightarrow\sigma=\sqrt{1.048}=1.02372\approx1.02$

(e) $\mu\pm\sigma=3.5\pm1.02\Rightarrow[2.48,4.52]$ and $\mu\pm2\sigma=3.5\pm2\times1.02\Rightarrow[1.46,5.54]$

From the empirical rule, we know that $\mathrm{P}(\mu-\sigma\leq x\leq\mu+\sigma)\approx0.68$. Since the set of values is of a discrete nature, we can identify the border values as 3 and 4; so we calculate the probability of the given model, $\mathrm{P}(3\leq x\leq4)=0.669$, which is quite close to the result suggested by the empirical rule. The second probability from the empirical rule is $\mathrm{P}(\mu-2\sigma\leq x\leq\mu+2\sigma)\approx0.95$, and in the probability model we get $\mathrm{P}(2\leq x\leq5)=0.969$, which again is very close to the suggested result.



5. The probability distribution is:

x	12	14	16	18
P(*x*)	$6k$	$7k$	$8k$	$9k$

The sum of all the probabilities is 1, so we obtain the following:

$$6k+7k+8k+9k=1 \Rightarrow k=\frac{1}{30}$$

6. **(a)** Since the sum of all the probabilities should be 1, we get the following:

$$\frac{3}{20}+\frac{7}{30}+k+\frac{3}{10}+\frac{13}{60}=1 \Rightarrow k=\frac{1}{10}$$

(b) $\mathrm{P}(X>10)=\mathrm{P}(15)+\mathrm{P}(20)+\mathrm{P}(25)=\frac{1}{10}+\frac{3}{10}+\frac{13}{60}=\frac{37}{60}$

Or, again, we can use the complementary event with a calculation that does not involve the result from part **(a)**, so there is no possible mistake to carry through.

$$\mathrm{P}(X>10)=1-\left(\mathrm{P}(5)+\mathrm{P}(10)\right)=1-\left(\frac{3}{20}+\frac{7}{30}\right)=\frac{37}{60}$$

(c) $\mathrm{P}\left(5<X\le 20\right)=\mathrm{P}(10)+\mathrm{P}(15)+\mathrm{P}(20)=\frac{7}{30}+\frac{1}{10}+\frac{3}{10}=\frac{19}{30}$

Or, again, we can use the complementary event with a calculation that does not involve the result from part **(b)**, so there is no possible mistake to carry through.

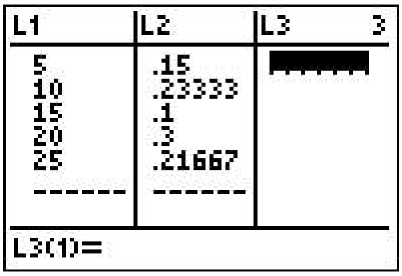
$$\mathrm{P}\left(5<X\le 20\right)=1-\left(\mathrm{P}(5)+\mathrm{P}(25)\right)=1-\left(\frac{3}{20}+\frac{13}{60}\right)=\frac{19}{30}$$

(d) $\mathrm{E}(X)=\sum x_i p_i=5\times\frac{3}{20}+10\times\frac{7}{30}+\cdots+25\times\frac{13}{60}=16$

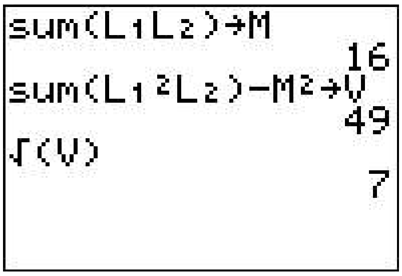
$$\mathrm{V}(X)=\sum x_i^2 p_i-(\mathrm{E}(X))^2=5^2\times\frac{3}{20}+10^2\times\frac{7}{30}+\cdots+25^2\times\frac{13}{60}-16^2=49$$

$$\Rightarrow \sigma=7$$

Or, by using a calculator:



```
L1      |L2      |L3      3
 5      | .15    |
 10     | .23333 |
 15     | .1     |
 20     | .3     |
 25     | .21667 |
 ------ | ------ |
L3(1)=
```



```
sum(L1L2)→M
                16
sum(L1²L2)-M²→V
                49
√(V)
                 7
```

(e) $\mathrm{E}(Y)=\frac{1}{5}\mathrm{E}(X)-1=\frac{1}{5}\times 16-1=\frac{11}{5}$

$\mathrm{V}(Y)=\left(\frac{1}{5}\right)^2;\ \mathrm{V}(X)=\left(\frac{1}{5}\right)^2\times 49=\frac{49}{25}$

7. **(a)** We substitute the values for the random variable Y to get the probability for each value in terms of k.

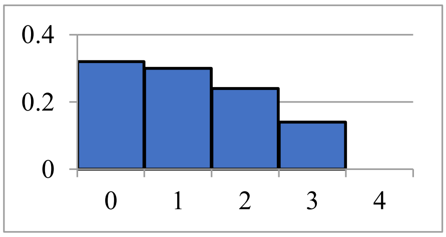
$\mathrm{P}(Y=0)=k(16-0)^2=16k,\ \mathrm{P}(Y=1)=15k,\ \mathrm{P}(Y=2)=12k,$

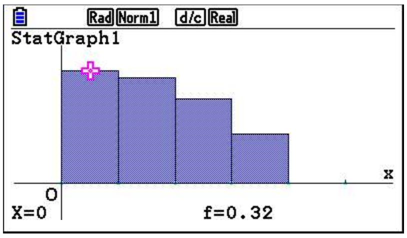
$\mathrm{P}(Y=3)=7k,\ \mathrm{P}(Y=4)=0$

Since the sum of all probabilities is 1, we obtain the following:

$16k+15k+12k+7k+0=1\Rightarrow k=\frac{1}{50}$

(b) Spreadsheet and GDC outputs shown:





(c) Direct addition of the probabilities corresponding to 1, 2, and 3, or using the complement are equally efficient:

$\mathrm{P}(1\le Y\le 3)=\frac{15}{50}+\frac{12}{50}+\frac{7}{50}=\frac{34}{50}=0.68$

(d) $\mathrm{E}(Y)=\sum y_i p_i=0\times\frac{16}{50}+1\times\frac{15}{50}+\cdots+4\times 0=\frac{60}{50}=1.2$

$\mathrm{V}(Y)=\sum y_i^2 p_i-(\mathrm{E}(Y))^2=0^2\times\frac{16}{50}+1^2\times\frac{12}{50}+\cdots+4^2\times 0-1.2^2=1.08$

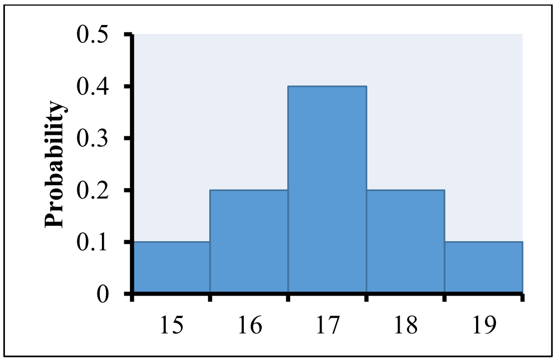
8. **(a)** Let's use a probability distribution table to record the details.

x	15	16	17	18	19
P(x)	0.1	0.2	0.4	$2p$	p

Since the sum of all the probabilities is 1, we obtain the following:

$$0.1+0.2+0.4+2p+p=1 \Rightarrow p=0.1$$

$$\mathrm{P}(X=18)=0.2, \mathrm{P}(X=19)=0.1$$

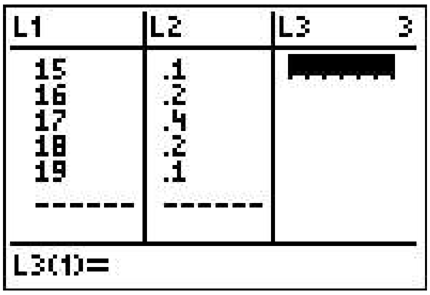


The distribution is symmetrical.

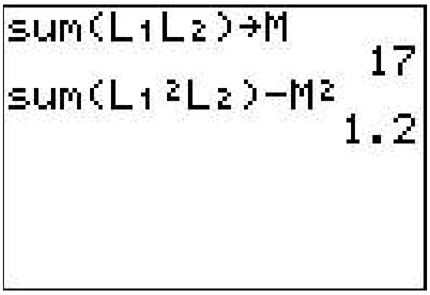
(b) $\mathrm{E}(X)=\sum x_i p_i = 15\times0.1+16\times0.2+\cdots+19\times0.1=17$

$\mathrm{V}(X)=\sum x_i^2 p_i-(\mathrm{E}(X))^2 = 15^2\times0.1+16^2\times0.2+\cdots+19^2\times0.1-17^2=1.2$

Or, by using a GDC:



```
L1      L2      L3      3
 15     .1
 16     .2
 17     .4
 18     .2
 19     .1
 ------ ------
L3(1)=
```



```
sum(L1L2)→M
                 17
sum(L1²L2)-M²
                1.2
```

9. **(a)** $\mathrm{E}(X)=\sum x_i p_i = 0\times0.1+1\times0.4+\cdots+5\times0.05=1.9$

$\mathrm{V}(X)=\sum x_i^2 p_i-(\mathrm{E}(X))^2 = 0^2\times0.1+1^2\times0.2+\cdots+5^2\times0.1-1.9^2=1.79$

Thus, the standard deviation $=\sqrt{1.79}=1.338$

(b) According to the empirical rule, 95% of the observations are within 2 standard deviations of the mean.

$\mu\pm2\sigma=1.79\pm2\times1.338=-0.89$ or $4.47\Rightarrow[-0.89, 4.47]$

Therefore, we would say that between 0 and 4 laptops are sold 95% of the time.

10. We use a pdf table to list all probabilities.

x	2	3	4	5	6	7
P(*x*)	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	k

The sum of all probabilities must be 1.

$$\sum \mathrm{P}(x) = \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{1024} + k = 1 \Rightarrow k = \frac{683}{1024} \approx 0.667$$

$$\mathrm{E}(X) = \sum x_i p_i = 2 \times \frac{1}{4} + 3 \times \frac{1}{16} + \cdots + 7 \times \frac{683}{1024} = \frac{5575}{1024} \approx 5.44$$

11. **(a)** $\sum \mathrm{P}(y) = 0.1 + 0.40 + k + (k-1)^2 = 1 \Rightarrow k^2 - k + 0.21 = 0$

Solving the quadratic equation, we get $k = 0.3$ or $k = 0.7$

(b) **(i)** For $k = 0.3$, $\mathrm{E}(X) = \sum x_i p_i = 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.49 \approx 2.18$

(ii) For $k = 0.7$, $\mathrm{E}(X) = \sum x_i p_i = 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.7 + 3 \times 0.09 \approx 1.78$

12. **(a)** Since we draw a ball three times, the number of red balls, X, can be 0, 1, 2 and 3. The probability that one drawn ball is red is $p = \frac{8}{8+4} = \frac{2}{3}$. Also, since the ball is returned to the box, these events are independent of each other.

Calculation of the corresponding probabilities:

$$\mathrm{P}(X=0) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}; \mathrm{P}(X=1) = {}^3C_1 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) = \frac{2}{9}$$

$$\mathrm{P}(X=2) = {}^3C_2 \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{9}; \mathrm{P}(X=3) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

x	0	1	2	3
P(*X* = *x*)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

(b) $\mathrm{E}(X) = \sum x_i p_i = 0 \times \frac{1}{27} + 1 \times \frac{2}{9} + \cdots + 3 \times \frac{8}{27} = \frac{18}{9} = 2$

13. **(a)** Substituting the values of Y into the pdf formula yields the following pdf table:

y	0	1	2	3	4
P($Y=y$)	$4k$	$3k$	$2k$	k	0

$$\sum \mathrm{P}(y) = 4k + 3k + 2k + k + 1 = 1 \Rightarrow k = \frac{1}{10}$$

(b) $\mathrm{P}(1 \le y < 3) = \mathrm{P}(y=1) + \mathrm{P}(y=2) = \frac{3}{10} + \frac{1}{5} = \frac{1}{2}$

14. Here is the PDF table for the distribution:

x	45	46	47	48	49	50	51	52	53	54	55
P(x)	0.05	0.08	0.12	0.15	0.25	0.20	0.05	0.04	0.03	0.02	0.01

(a) Below is the CDF for this distribution:

x	45	46	47	48	49	50	51	52	53	54	55
P($X \le x$)	0.05	0.13	0.25	0.40	0.65	0.85	0.90	0.94	0.97	0.99	1.00

(b) If all the ticket holders that show up are to be accommodated, no more than 50 passengers can show up. Therefore, we can use the cumulative distribution function directly from the table: $\mathrm{P}(X \le 50) = 0.85$

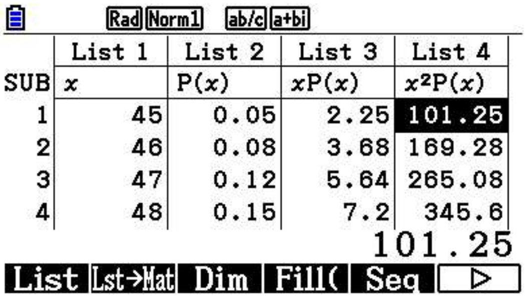
(c) This is the complementary event of the event in part **(b)**.
$\mathrm{P}(X \ge 51) = 1 - \mathrm{P}(X \le 50) = 1 - 0.85 = 0.15$

(d) We use the PDF table along with the expected value formula: $\mathrm{E}(X) = \sum x\mathrm{P}(x)$

We will demonstrate this type of calculation in this exercise. However, it is best done with a GDC or, if not on an exam, then with spreadsheet.

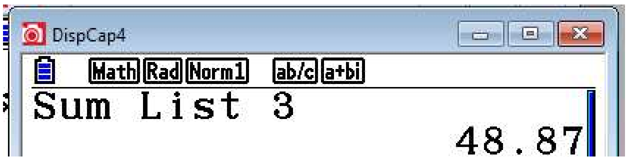
$$\mathrm{E}(X) = 45 \times 0.05 + 46 \times 0.08 + 47 \times 0.12 + \cdots + 54 \times 0.02 + 55 \times 0.01$$
$$= 2.25 + 3.68 + \cdots + 1.08 + 0.55 = 48.87$$

Arrange the data in lists as shown and then add the entries in the third list.



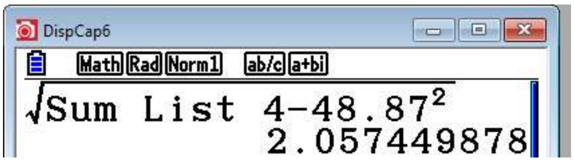
```
     List 1  List 2  List 3  List 4
SUB  x       P(x)    xP(x)   x²P(x)
  1      45    0.05    2.25  101.25
  2      46    0.08    3.68  169.28
  3      47    0.12    5.64  265.08
  4      48    0.15     7.2   345.6
                             101.25
```

There are more columns to cater for **(e)**.
The sum of List 3 is the expected value: 48.87



```
Sum List 3
                48.87
```

(e) When we have a distribution with a lot of possible outcomes, it is much easier to use a GDC. Below is the calculation for the standard deviation using the formula $\sigma = \sqrt{\sum x^2 \mathrm{P}(x) - \mu^2} \approx 2.06$



```
√Sum List 4-48.87²
          2.057449878
```

(f) We need to calculate $\mathrm{P}(\mu - \sigma \le X \le \mu + \sigma)$, thus

$$\mathrm{P}(48.78 - 2.06 \le X \le 48.78 + 2.06) = \mathrm{P}(47 \le X \le 51)$$

So, we need to calculate: $\mathrm{P}(47 \le X \le 51) = 0.12 + 0.15 + 0.25 \ + 0.20 + 0.05 = 0.77$

Alternatively, we can use the cumulative distribution function from the second table:

$\mathrm{P}(47 \le X \le 50) = \mathrm{P}(X \le 51) - \mathrm{P}(X \le 46) = 0.90 - \ 0.13 = 0.77$

15. Here is the PDF table for the distribution:

x	0	1	2	3	4	5	6
P(*x*)	0.08	0.15	0.22	0.27	0.20	0.05	0.03

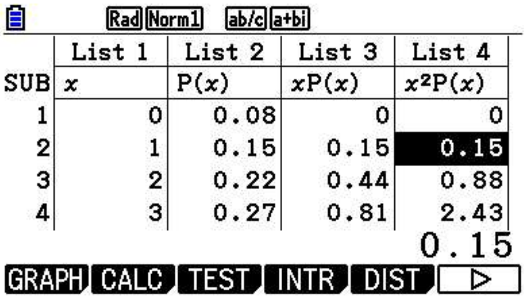
(a) Below is the CDF for this distribution:

x	0	1	2	3	4	5	6
$\mathrm{P}(X \le x)$	0.08	0.23	0.45	0.72	0.92	0.97	1.00

(b) Reading from CDF table: $\mathrm{P}(x \le 3) = 0.72$

(c) To obtain a free line means that there will be at least one free line.
Therefore, we can use the cumulative distribution function: $\mathrm{P}(X \le 5) = 0.97$

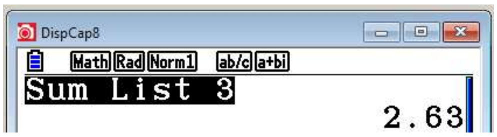
(d) We use the PDF table along with the expected value formula: $\mathrm{E}(X) = \sum x\mathrm{P}(x)$
It is best done with a GDC or, if not on an exam, then with spreadsheet.



	List 1	List 2	List 3	List 4
SUB	x	P(x)	xP(x)	x^2P(x)
1	0	0.08	0	0
2	1	0.15	0.15	0.15
3	2	0.22	0.44	0.88
4	3	0.27	0.81	2.43

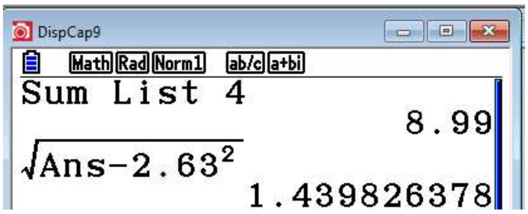
0.15

There are more columns to cater for **(e)**.
The sum of List 3 is the expected value: 2.63



```
Sum List 3
                2.63
```

(e) When we have a distribution with a lot of possible outcomes, it is much easier to use a GDC. Below is the calculation for the standard deviation using the formula $\sigma = \sqrt{\sum x^2 P(x) - \mu^2} \approx 1.44$:



```
Sum List 4
                 8.99
√Ans-2.63²
          1.439826378
```

16. **(a)** Since 90% of the batteries have an acceptable voltage: P(X = 1) = 0.9

(b) X = 2 means getting an unacceptable battery first followed by an acceptable one: $\text{P}(X = 2) = 0.1 \times 0.9 = 0.09$

(c) Similarly, for X = 3, we need two unacceptable followed by an acceptable one:

$\text{P}(X = 3) = 0.1 \times 0.1 \times 0.9 = 0.009$

(d) **(i)** The fourth tested battery, as well as the previous three batteries, should be unacceptable.

(ii) The fifth battery should be acceptable.
$\text{P}(X = 5) = 0.1 \times 0.1 \times 0.1 \times 0.1 \times 0.9 = 0.1^4 \times 0.9$

(e) To have $X = x$ means to have x – 1 unacceptable followed by an acceptable one.
That is, $\text{P}(X = x) = \underbrace{0.1 \times \cdots \times 0.1}_{x-1 \text{ times}} \times 0.9 = 0.1^{x-1} \times 0.9$

17. Denote acceptable with *a* and unacceptable with *u*.

(a) If the torch needs two batteries, just selecting one battery is not enough for the torch to work; therefore, P(x = 1) = 0

(b) Both batteries should be acceptable: $\mathrm{P}(X=2)=0.9^2=0.81$

(c) X = 3 can happen in one of two ways: *aua* or *uaa*

$$\mathrm{P}(X=3)=0.9\times0.1\times0.9+0.1\times0.9\times0.9=0.162$$

(d) **(i)** The fourth battery depends on the previous three batteries. If there is no acceptable battery in the previous three, the fourth battery should be acceptable. But, if there was one acceptable battery in the previous three, the fourth battery should be unacceptable.

(ii) The fifth battery must be acceptable in all cases.

(e) In the first $x-1$ tested batteries, there must be one acceptable battery and the last battery should be acceptable as well. One acceptable battery in the first $x-1$ can occur in ${}^{x-1}C_1=x-1$ different ways:

$$\mathrm{P}(X=x)=\underbrace{(x-1)\times0.1^{x-2}\times0.9\times}_{x-1\ u\text{ containing one }a}\underbrace{0.9}_{\text{last one }a}=(x-1)0.1^{x-2}\times0.9^2$$

18. Notice here that the score is not involved in the calculations because the question is about counters. Let X be the number of counters a player receives, thus:

$$\mathrm{E}(X)=\sum x\mathrm{P}(x)=\frac{1}{2}\times4+\frac{1}{5}\times5+\frac{1}{5}\times15+\frac{1}{10}\times n=3\Rightarrow n=30$$

19. **(a)** **(i)** There are four different ways of obtaining a sum of 9: (3, 6), (4, 5), (5, 4) and (6, 3). We know that there are 36 possible outcomes, thus:

$$\mathrm{P}(A=9)=\frac{4}{36}=\frac{1}{9}$$

(ii) Probability that Belle obtains a 9 is also $\frac{1}{9}$, and since they are independent of each other, $\mathrm{P}(A,B=9)=\frac{1}{9}\times\frac{1}{9}=\frac{1}{81}$

(b) **(i)** There are 11 different scores that Alan and Belle can obtain. We can list all of them and calculate the probability. For example, they can get a sum of 3 in two different ways, a 4 in three different ways, a 7 in six different ways, etc.

$$\mathrm{P}(A=B)=\underbrace{\left(\frac{1}{36}\right)^2}_{sum=2}+\underbrace{\left(\frac{2}{36}\right)^2}_{sum=3}+\cdots+\underbrace{\left(\frac{5}{36}\right)^2}_{sum=6}+\cdots+\underbrace{\left(\frac{1}{36}\right)^2}_{sum=12}=\frac{73}{648}$$

(ii) Since all the dice are fair, the probability that Alan's score exceeds Belle's score, and vice versa, are equal. Let's call these events x. The sum of all three events is 1.

$$\mathrm{P}(A=B)+\underbrace{\mathrm{P}(A>B)}_{x}+\underbrace{\mathrm{P}(A<B)}_{x}=1\Rightarrow 2x=\frac{575}{648}\Rightarrow x=\frac{575}{1296}$$

(c) **(i)** If the largest number score is 1, then all four dice must score 1:

$$\mathrm{P}(X\le 1)=\left(\frac{1}{6}\right)^4$$

If the largest number is less than or equal to 2, then a combination of 1 or 2 on each dice is favourable; therefore: $\mathrm{P}(x\le 2)=\left(\frac{2}{6}\right)^4$

Similarly, we can discuss all the remaining possibilities:

$\mathrm{P}(X\le 3)=\left(\frac{3}{6}\right)^4$, and for any number x, the possible outcomes should be any number 1, 2, …, x. Thus $\mathrm{P}(X\le x)=\left(\frac{x}{6}\right)^4$

(ii) To complete the table, we will use the equations relating the probability function and the cumulative distribution function.

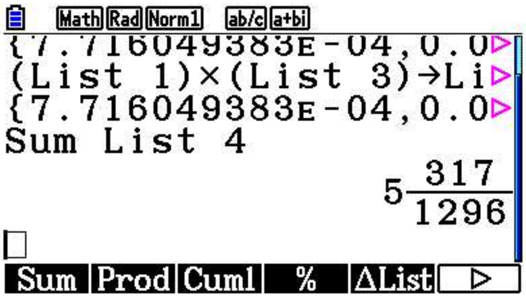
$$\mathrm{P}(X=k)=\mathrm{P}(X\le k)-\mathrm{P}(X\le k-1), k=2,\cdots,6 \text{ and } \mathrm{P}(X=1)=\left(\frac{1}{6}\right)^4$$

$$\mathrm{P}(X=3)=\mathrm{P}(X\le 3)-\mathrm{P}(X\le 2)=\left(\frac{3}{6}\right)^4-\left(\frac{2}{6}\right)^4=\frac{65}{1296}$$

$$\mathrm{P}(X=4)=\mathrm{P}(X\le 4)-\mathrm{P}(X\le 3)=\left(\frac{4}{6}\right)^4-\left(\frac{3}{6}\right)^4=\frac{175}{1296}$$

$$\mathrm{P}(X=5)=\mathrm{P}(X\le 5)-\mathrm{P}(X\le 4)=\left(\frac{5}{6}\right)^4-\left(\frac{4}{6}\right)^4=\frac{369}{1296}$$

(iii) This can be done on a GDC by using the List menu. In the first list we input the dice scores, while in the second list we input the probabilities by using the formula for the cumulative distribution function.



```
{7.716049383E-04,0.0▷
(List 1)×(List 3)→Li▷
{7.716049383E-04,0.0▷
Sum List 4
                 317
               5────
                1296
```

So, the expected value is $\frac{6797}{1296}=5\frac{317}{1296}\approx 5.245$

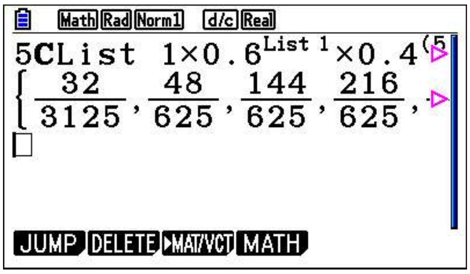
20. We use the short-cut formula for the variance: $s^2 = \dfrac{\sum x^2}{n} - \overline{x}^2$

$$s^2 = 6.9^2 = \frac{\sum_{1}^{10} x_i^2}{10} - \overline{x}^2 \Rightarrow \overline{x}^2 = 134.1 - 47.61 = 86.49 \Rightarrow \overline{x} = 9.3$$

Exercise 15.2

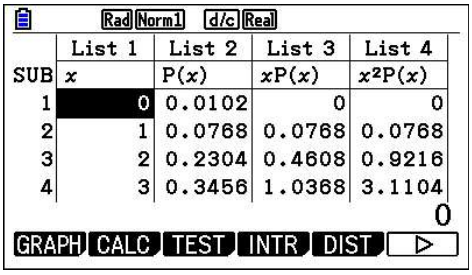
1. $X \sim \mathrm{B}(n = 5, p = 0.6)$

(a) An efficient way to calculate the probabilities given by the formula is by using a GDC or a spreadsheet. Some GDCs also give the exact answer in fraction form.



$5\mathrm{C}\,\text{List } 1 \times 0.6^{\text{List } 1} \times 0.4^{(5}$

$\left\{ \dfrac{32}{3125}, \dfrac{48}{625}, \dfrac{144}{625}, \dfrac{216}{625}, \right.$



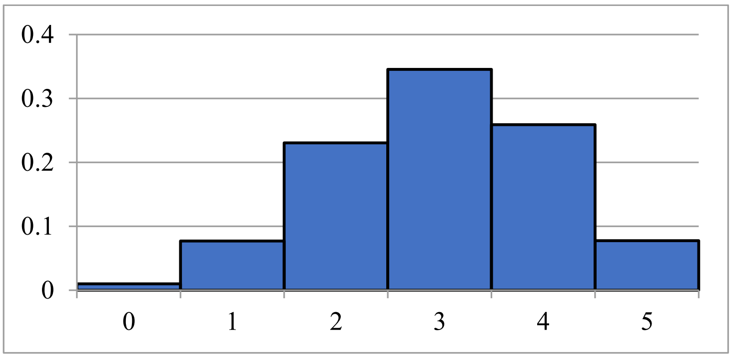
	List 1	List 2	List 3	List 4
SUB	x	P(x)	xP(x)	x^2P(x)
1	0	0.0102	0	0
2	1	0.0768	0.0768	0.0768
3	2	0.2304	0.4608	0.9216
4	3	0.3456	1.0368	3.1104

0

So, the table would be:

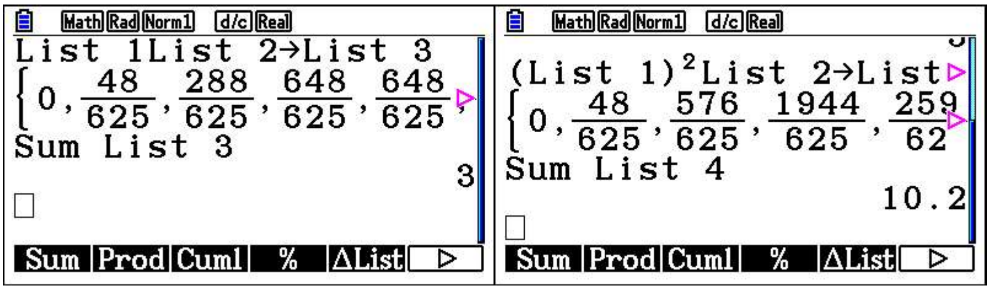
x	0	1	2	3	4	5
P(x)	$\dfrac{32}{3125}$	$\dfrac{48}{625}$	$\dfrac{144}{625}$	$\dfrac{216}{625}$	$\dfrac{162}{625}$	$\dfrac{243}{3125}$

(b) Here is a histogram of the distribution:



(c) **(i)** $\mu = np = 5 \times 0.6 = 3; \sigma = \sqrt{npq} = \sqrt{5 \times 0.6 \times 0.4} \approx 1.095$

(ii) Again, we can use a GDC for this.



```
List 1List 2→List 3
{0, 48/625, 288/625, 648/625, 648/625,
Sum List 3
                                     3
```

```
(List 1)²List 2→List
{0, 48/625, 576/625, 1944/625, 259/62
Sum List 4
                                  10.2
```

Since $\sigma = \sqrt{\sum x^2 p)x) - \mu 2}$, then $\sigma = \sqrt{10.2 - 9} \approx 1.095$

(d) $\mu \pm \sigma = 3 \pm 1.095 \overset{\text{approx.}}{\Longrightarrow} [2,4]; \mu \pm 2\sigma = 3 \pm 2(1.095) \overset{\text{approx.}}{\Longrightarrow} [1,5]$

The first includes the three middle bars and the second includes all bars of the graph.

(e) Now we need to calculate the following probabilities: $\mathrm{P}(2 \le X \le 4)$ and $\mathrm{P}(1 \le X \le 5)$.

We can either delegate the task to the GDC or simply add the entries from the table in **(a)**. We notice that the probability within one standard deviation, 0.8352, is much higher than the empirical one, 0.6827. The probability within two standard deviations, 0.98976, is fairly close to the empirical one, 0.95

2. Let x be the number of respondents in favour of the decision. $X \sim \mathrm{B}(n = 20, p = 0.6)$

(a) $\mathrm{P}(x = 5) = {}_{20}C_5 0.6^5 \times 0.4^{15} \approx 0.0012944935$

Using a GDC:

```
Binomial P.D
     p=1.2945E-03
```

(b) This means that all respondents are against the decision.

You can apply the binomial formula, but in essence it is straightforward.

$\mathrm{P}(X = 0) = 0.4^{20} = 0.00000001099$

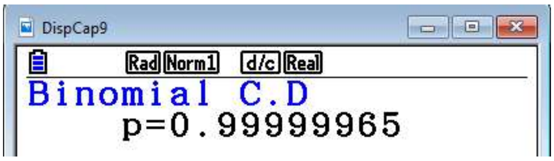
(c) This is the compliment of "none of the 20 supports the decision".

Thus, $\mathrm{P}(X \ge 1) = 1 - \mathrm{P}(X = 0) = 0.999999909005$

(d) Even though we can use the direct form, it is again easier to calculate the probability by using the complementary event; therefore,

$\mathrm{P}(X \ge 2) = 1 - (\mathrm{P}(X = 0) + \mathrm{P}(X = 1)) = 0.9999996592$

With a GDC, this could be achieved by using the cdf function.

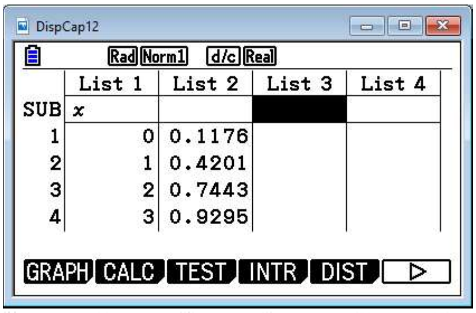


```
Binomial C.D
     p=0.99999965
```

(e) Since this is a binomial distribution, the mean and standard deviation are

$$\mu = np = 20 \times 0.6 = 12; \sigma = \sqrt{npq} = \sqrt{20 \times 0.6 \times 0.4} \approx 2.19$$

3. (a) In this part, we will use the binomial cumulative distribution function on a GDC.



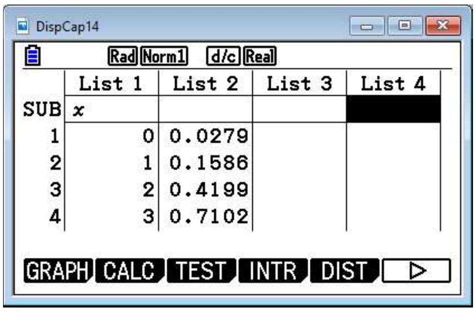
```
    List 1   List 2   List 3   List 4
SUB x
  1      0   0.1176
  2      1   0.4201
  3      2   0.7443
  4      3   0.9295
```

k	0	1	2	3	4	5	6
$p(x \leqslant k)$	0.11765	0.42018	0.74431	0.92953	0.98907	0.99927	1

(b) Here is the filled table which has been deduced from the cdf table above.

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3	0, 1, 2, 3	$p(x \leqslant 3)$	$p(x \leqslant 3)$	0.92953
At least 3	3, 4, 5, 6	$P(x \geqslant 3)$	$1 - p(x \leqslant 2)$	0.25569
More than 3	4, 5, 6	$p(x > 3)$	$1 - p(x \leqslant 3)$	0.07047
Fewer than 3	0, 1, 2	$p(x \leqslant 2)$	$p(x \leqslant 2)$	0.74431
Between 3 and 5 (inclusive)	3, 4, 5	$p(3 \leqslant x \leqslant 5)$	$p(x \leqslant 5) - p(x \leqslant 2)$	0.25496
Exactly 3	3	$P(x = 3)$	$P(x = 3)$	0.18522

4. (a) Again, we will use the binomial cumulative distribution function on a GDC.



```
    List 1   List 2   List 3   List 4
SUB x
  1      0   0.0279
  2      1   0.1586
  3      2   0.4199
  4      3   0.7102
```

k	0	1	2	3
$p(x \leqslant k)$	0.02799	0.15863	0.41990	0.71021
k	4	5	6	7
$p(x \leqslant k)$	0.90374	0.98116	0.99836	1

(b) Here is the filled table which has been deduced from the cdf table above.

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3	0, 1, 2, 3	$p(x \leqslant 3)$	$p(x \leqslant 3)$	0.71021
At least 3	3, 4, 5, 6, 7	$P(x \geqslant 3)$	$1 - p(x \leqslant 2)$	0.58010
More than 3	4, 5, 6, 7	$p(x > 3)$	$1 - p(x \leqslant 3)$	0.28979
Fewer than 3	0, 1, 2	$p(x \leqslant 2)$	$p(x \leqslant 2)$	0.41990
Between 3 and 5 (inclusive)	3, 4, 5	$p(3 \leqslant x \leqslant 5)$	$p(x \leqslant 5) -$ $p(x \leqslant 2)$	0.56125
Exactly 3	3	$P(x = 3)$	$P(x = 3)$	0.290304

5. **(a)** This is not a binomial distribution since we don’t have a sequence of several independent trials with equal probabilities.

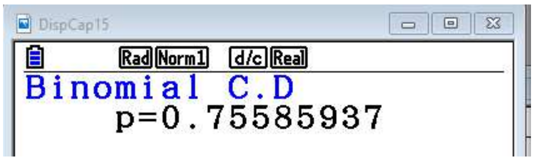
(b) If we choose the balls with replacement, then the trials become independent, so a sequence of three such trials with equal probabilities is a binomial distribution.

(c) $Y \sim \mathrm{B}\left(n = 3, p = \dfrac{5}{8}\right)$

y	0	1	2	3
$P(Y = y)$	0.05273	0.26367	0.43945	0.24414

(d) It is easier to calculate the complementary event, which is all three green balls.
$\mathrm{P}(Y \le 2) = 1 - \mathrm{P}(Y = 3) = 0.7559$

Using a GDC, you can use the binomial cdf menu.



```
Binomial C.D
    p=0.75585937
```

(e) $\mathrm{E}(Y) = np = 3 \times \dfrac{5}{8} = \dfrac{15}{8} = 1.875$

(f) $\mathrm{V}(Y) = npq = 3 \times \dfrac{5}{8} \times \dfrac{3}{8} = \dfrac{45}{64} = 0.703125$

(g) The complementary event of ‘some green balls’ would be ‘no green balls will be chosen’; therefore, we calculate the probability:

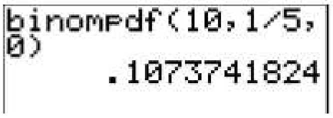
$\mathrm{P}(Y \ge 1) = 1 - \mathrm{P}(Y = 0) = 1 - 0.0527 = 0.9473$

6. Since Nick guesses every single question, the probability that he chooses the correct answer from five possible answers per question is $\frac{1}{5}$; therefore, the distribution is

$$X \sim \mathrm{B}\left(n = 10, p = \frac{1}{5}\right)$$

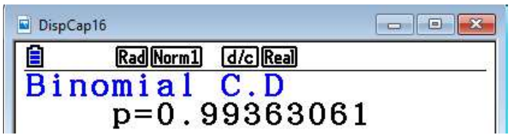
(a) This means that he answers every question incorrectly. That is,

$\mathrm{P}(X = 0) = \left(\frac{4}{5}\right)^{10} \approx 0.10737$, or use GDC



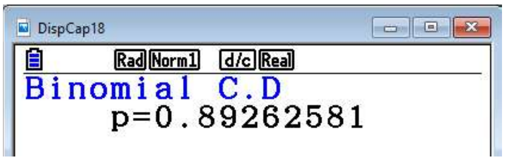
```
binompdf(10,1/5,
0)
        .1073741824
```

(b) We will use a GDC for this. The answer is $\mathrm{P}(X \le 5) \approx 0.99363$



```
Binomial C.D
     p=0.99363061
```

(c) $\mathrm{P}(X \ge 1) = 1 - \mathrm{P}(X = 0) = 0.89263$



```
Binomial C.D
     p=0.89262581
```

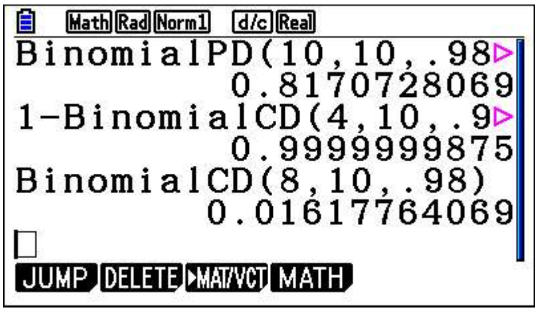
(d) $\mathrm{E}(X) = np = 10 \times \frac{1}{5} = 2$

7. There are 10 houses, and in each we have an alarm system that is 98% reliable, so the distribution is $X \sim \mathrm{B}(n = 10, p = 0.98)$

(a) $\mathrm{P}(X = 10) = 0.817073$

(b) $\mathrm{P}(X \ge 5) = 1 - \mathrm{P}(X \le 4) = 0.99999 \approx 1$

(c) $\mathrm{P}(X \le 8) = 0.016178$



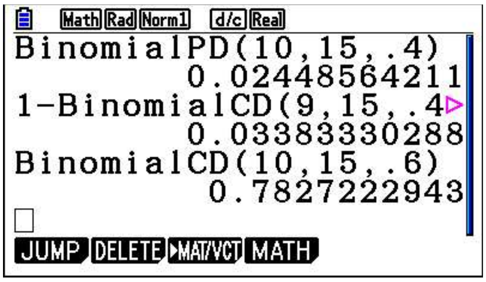
```
BinomialPD(10,10,.98▷
            0.8170728069
1-BinomialCD(4,10,.9▷
            0.9999999875
BinomialCD(8,10,.98)
           0.01617764069
```

8. Let X represent the number of readers over 30 years of age. $X \sim \mathrm{B}(15, 0.4)$

(a) $\mathrm{P}(X \geq 10) = 1 - \mathrm{P}(X \leq 9) = 0.033833$

(b) $\mathrm{P}(X = 10) = 0.024486$

(c) The distribution for the younger readers is $X \sim \mathrm{B}(15, 0.6)$
$\mathrm{P}(X \leq 10) = 0.78272$



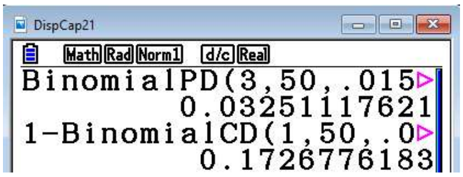
```
BinomialPD(10,15,.4)
            0.02448564211
1-BinomialCD(9,15,.4▷
            0.03383330288
BinomialCD(10,15,.6)
             0.7827222943
```

9. Let X represent the number of defective hard disks. $X \sim \mathrm{B}(50, 0.015)$

(a) $\mathrm{E}(X) = np = 50 \times 0.015 = 0.75$

(b) $\mathrm{P}(X = 3) \approx 0.0325112$

(c) This is the compliment of 'at most 1 hard disk is defective'.
$\mathrm{P}(X > 1) = 1 - \mathrm{P}(X \leq 1) = 0.17268$



```
BinomialPD(3,50,.015▷
            0.03251117621
1-BinomialCD(1,50,.0▷
             0.1726776183
```

10. Let X represent the number of metallic grey cars. $X \sim \mathrm{B}(20, 0.1)$

(a) $\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4) = 0.0431745$

(b) $\mathrm{P}(X \leq 6) = 0.99761$

(c) $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \leq 5) = 0.01125$

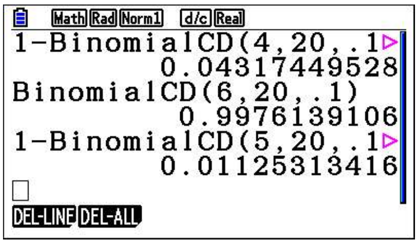
(d) $\mathrm{P}(4 \leq X \leq 6) = \mathrm{P}(X \leq 6) - \mathrm{P}(X \leq 3) = 0.13057$

(e) If more than 15 are not metallic grey, then at least 4 are metallic grey.
$\mathrm{P}(X \leq 4) = 0.95683$

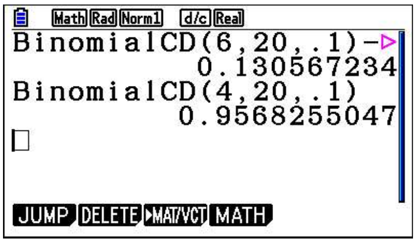
(f) $\mathrm{E}(X) = np = 100 \times 0.1 = 10$

(g) $\sigma = \sqrt{npq} = \sqrt{100 \times 0.1 \times 0.9} = 3$

(h) According to the empirical rule, $\text{P}(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$ and in the probability model we can calculate $10 - 2\times 3 \le X \le 10 + 2\times 3 \Rightarrow 4 \le X \le 16$. So, $a = 4$ and $b = 16$.



```
1-BinomialCD(4,20,.1▷
           0.04317449528
BinomialCD(6,20,.1)
            0.9976139106
1-BinomialCD(5,20,.1▷
           0.01125313416
```



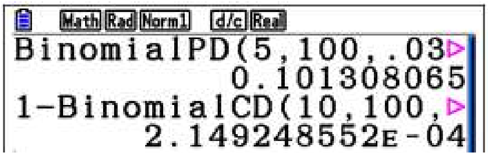
```
BinomialCD(6,20,.1)-▷
             0.130567234
BinomialCD(4,20,.1)
            0.9568255047
```

11. Let X represent the number of dogs with health insurance. $X \sim \text{B}(100, 0.03)$

(a) $\text{E}(X) = np = 100 \times 0.03 = 3$

(b) $\text{P}(X = 5) \approx 0.101308$

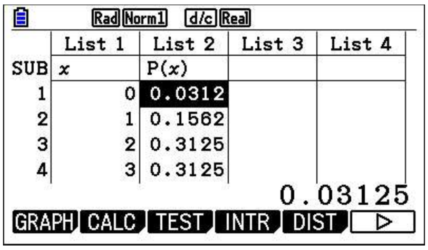
(c) $\text{P}(X \ge 11) = 1 - \text{P}(X \le 10) \approx 0.000214925$



```
BinomialPD(5,100,.03▷
          0.101308065
1-BinomialCD(10,100,▷
       2.149248552E-04
```

12. Let X represent the number of heads observed. $X \sim \text{B}(5, 0.5)$

(a) Using a GDC, the pdf is given below.



	List 1	List 2	List 3	List 4
SUB	x	P(x)		
1	0	0.0312		
2	1	0.1562		
3	2	0.3125		
4	3	0.3125		

0.03125

(b) From the table, we can read that $\text{P}(X = 0) \approx 0.0312$

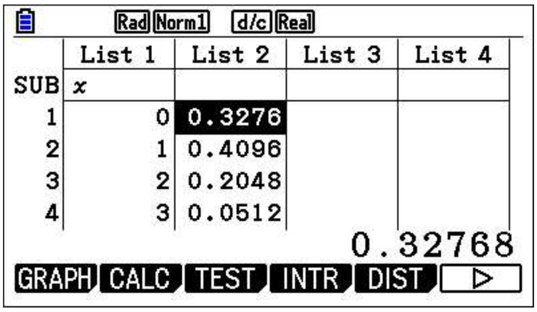
(c) From the table, we can read that $\text{P}(X = 5) \approx 0.0312$

(d) This is the compliment of ‘no heads showing’:
$\text{P}(X \ge 1) = 1 - \text{P}(X = 0) = 0.96875$

(e) There are several ways of answering this.
Since it is a fair coin then heads or tails happen. So, this is the same as **(d)**.
Or, if at least one tail is observed, then at most 4 heads are observed:
$\text{P}(X \le 4) = 1 - \text{P}(X = 5) = 0.96875$

(f) Since two heads are observed in every 10 tosses, the probability is 0.2

(a) GDC output:



	List 1	List 2	List 3	List 4
SUB	x			
1	0	0.3276		
2	1	0.4096		
3	2	0.2048		
4	3	0.0512		

0.32768

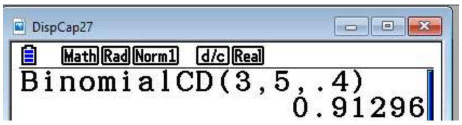
(b) From the table, we can read that $\text{P}(X=0)\approx 0.32768$

(c) From the table, we can read that $\text{P}(X=5)\approx 0.00032$

(d) This is the compliment of 'no heads showing':
$\text{P}(X\geq 1)=1-\text{P}(X=0)=0.67232$

(e) If at least one tail is observed, then at most 4 heads are observed:
$\text{P}(X\leq 4)=1-\text{P}(X=5)=0.99968$

13. Let X represent the number of days Alice watched the news. $X\sim \text{B}(5,0.4)$
$\text{P}(X\leq 3)\approx 0.91296$



```
BinomialCD(3,5,.4)
            0.91296
```

14. Let X represent the number of cells that fail within a year. $X\sim \text{B}(10,0.8)$

(a) $\text{P}(X=10)=(0.8)^{10}=0.10737$

(b) This is complementary to the event in part **(a)**.
$\text{P}(X\leq 9)=1-\text{P}(X=10)\approx 0.89263$

(c) $\text{P}(X\leq n-1)=1-\text{P}(X=n)=1-0.8^n\geq 0.95$

$$\Rightarrow 0.8^n\leq 0.05\Rightarrow n\ln 0.8\leq \ln 0.05\Rightarrow n\geq \frac{\ln 0.05}{\ln 0.8}=13.43\Rightarrow n\geq 14$$

Exercise 15.3

1. **(a)** $$\int_0^1 f(x)\text{d}x=1\Rightarrow \int_0^1\left(kx^2+\frac{3}{2}\right)\text{d}x=\left(\frac{kx^3}{3}+\frac{3}{2}x\right)\Bigg|_0^1=1\Rightarrow \frac{k}{3}+\frac{3}{2}=1\Rightarrow k=-\frac{3}{2}$$

(b) $$\text{P}(X>0.5)=\int_{0.5}^1\left(-\frac{3}{2}x^2+\frac{3}{2}\right)\text{d}x=\left(-\frac{x^3}{2}+\frac{3}{2}x\right)\Bigg|_{0.5}^1=\frac{5}{16}=0.3125$$

(c) $P(0<X<0.5)=1-P(X>0.5)=1-\frac{5}{16}=\frac{11}{16}=0.6875$

Note: We could have calculated the integral from 0 to 0.5

(d) The **mode** is the most frequent observation, and, as such, it is the maximum of the Probability density function. In this case, it is $x=0$ since the vertex of the parabola lies at $\left(0,\frac{3}{2}\right)$.

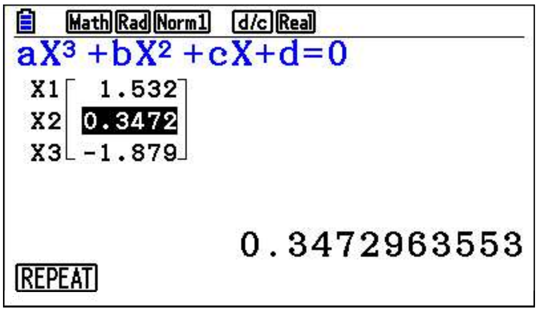
Now, we calculate the **mean** value.

$$\mu=\int_0^1 xf(x)\mathrm{d}x=\int_0^1\left(-\frac{3}{2}x^3+\frac{3}{2}x\right)\mathrm{d}x=\left(-\frac{3x^4}{8}+\frac{3}{4}x^2\right)\Bigg|_0^1=\frac{3}{8}=0.375$$

Letting m be the **median**, we have:

$$\int_0^m\left(-\frac{3}{2}x^2+\frac{3}{2}\right)\mathrm{d}x=0.5\Rightarrow\left(-\frac{x^3}{2}+\frac{3}{2}x\right)\Bigg|_0^m=\frac{3}{2}m-\frac{m^3}{2}=\frac{1}{2}\Rightarrow m^3-3m+1=0$$

This is a cubic equation which does not have an exact zero. Either graphically, or using the solver, the only zero of this function that satisfies the constraints we have is 0.3473



```
aX³ +bX² +cX+d=0
X1   1.532
X2   0.3472
X3  -1.879
                0.3472963553
```

For the **standard deviation**:

$$\sigma^2=\int_0^1 x^2f(x)\mathrm{d}x-\mu^2=\int_0^1\left(-\frac{3}{2}x^4+\frac{3}{2}x^2\right)\mathrm{d}x-\left(\frac{3}{8}\right)^2$$

$$=\left(-\frac{3}{10}x^5+\frac{1}{2}x^3\right)\Bigg|_0^1-\left(\frac{3}{8}\right)^2=\frac{19}{320}\Rightarrow\sigma=\sqrt{\frac{19}{320}}\approx 0.2437$$

2. **(a)** $\int_0^2 f(x)\mathrm{d}x=1\Rightarrow\int_0^2\left(k\left(5-2x\right)\right)\mathrm{d}x=k\left(5x-x^2\right)\Big|_0^2=1\Rightarrow 6k=1\Rightarrow k=\frac{1}{6}$

(b) $P(X>1.5)=\frac{1}{6}\int_{1.5}^2\left(5-2x\right)\mathrm{d}x=\frac{1}{6}\left(5x-x^2\right)\Big|_{1.5}^2=\frac{1}{8}=0.125$

(c) $P(0.5<X<1.5)=\frac{1}{6}\int_{0.5}^{1.5}\left(5-2x\right)\mathrm{d}x=\frac{1}{6}\left(5x-x^2\right)\Big|_{0.5}^{1.5}=\frac{1}{2}=0.5$

(d) Use the formula to calculate the **mean** value.

$$\mu = \int_0^2 xf(x)\mathrm{d}x = \frac{1}{6}\int_0^2 \left(5x-2x^2\right)\mathrm{d}x = \frac{1}{6}\left(\frac{5x^2}{2}-\frac{2x^3}{3}\right)\Bigg|_0^2 = \frac{14}{18} = 0.778$$

Letting m be the **median**, we have:

$$\frac{1}{6}\int_0^m (5-2x)\mathrm{d}x = 0.5 \Rightarrow \frac{1}{6}\left(5x-x^2\right)\Bigg|_0^m = \frac{1}{6}\left(5m-m^2\right) = \frac{1}{2}$$

$$\Rightarrow m^2-5m+3=0 \Rightarrow m = \frac{5\pm\sqrt{13}}{2}$$

Only one of the two solutions satisfies the constraint that it must be in the interval [0, 2]. Thus, the median is $m = \frac{5-\sqrt{13}}{2} \approx 0.6972$

For the **standard deviation**:

$$\sigma^2 = \int_0^2 x^2 f(x)\mathrm{d}x - \mu^2 = \frac{1}{6}\int_0^2 \left(5x^2-2x^3\right)\mathrm{d}x - \left(\frac{7}{9}\right)^2$$

$$= \frac{1}{6}\left(\frac{5x^3}{3}-\frac{x^4}{2}\right)\Bigg|_0^2 - \left(\frac{7}{9}\right)^2 = \frac{23}{81} \Rightarrow \sigma = \sqrt{\frac{23}{81}} \approx 0.5329$$

3. **(a)** $$\int_0^k f(x)\mathrm{d}x = 1 \Rightarrow \int_0^k \left(2x-x^3\right)\mathrm{d}x = \left(x^2-\frac{x^4}{4}\right)\Bigg|_0^k = 1 \Rightarrow k^2-\frac{k^4}{4} = 1$$

$$\Rightarrow k^4-4k^2+4=0 \Rightarrow \left(k^2-2\right)^2 = 0 \Rightarrow k=\sqrt{2}$$

We have only taken the positive value of k, since k cannot be negative.

(b) $$\mathrm{P}(X>0.5) = \int_{0.5}^{\sqrt{2}} \left(2x-x^3\right)\mathrm{d}x = \left(x^2-\frac{x^4}{4}\right)\Bigg|_{0.5}^{\sqrt{2}} = 2-1-\frac{1}{4}+\frac{1}{64} = \frac{49}{64}$$

(c) $$\mathrm{P}(0<X<0.5) = 1-\mathrm{P}(X>0.5) = 1-\frac{49}{64} = \frac{15}{64}$$

Note: We could have calculated the integral from 0 to 0.5

(d) To calculate the **mean** value, we use the formula

$$\mu = \int_0^{\sqrt{2}} xf(x)\mathrm{d}x = \int_0^{\sqrt{2}} \left(2x^2-x^4\right)\mathrm{d}x = \left(\frac{2x^3}{3}-\frac{x^5}{5}\right)\Bigg|_0^{\sqrt{2}} = \frac{8\sqrt{2}}{15} \approx 0.754$$

Letting m be the **median**, we have:

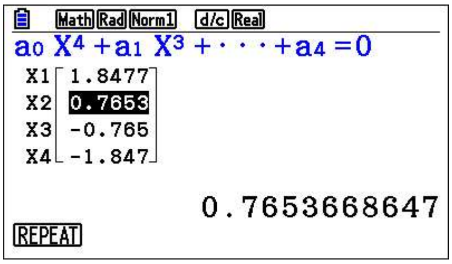
$$\int_0^m \left(2x-x^3\right)\mathrm{d}x = 0.5 \Rightarrow \left(x^2-\frac{x^4}{4}\right)\Bigg|_0^m = m^2-\frac{m^4}{4} = \frac{1}{2} \Rightarrow m^4-4m^2+2=0$$

Again, to find the value of m, we need to solve the biquadratic equation by using a simple substitution.

Let $m^2 = u \Rightarrow u^2-4u+2=0 \Rightarrow u = 2\pm\sqrt{2} \Rightarrow m = \sqrt{2-\sqrt{2}} \approx 0.765$

We have discarded the other possible solution for r, as well as the negative solution for m, since they lie outside the interval $\left[0,\sqrt{2}\right]$.

Alternatively, you can use a GDC to solve the equation and choose the solution that satisfies the constraints.



```
a0 X4 +a1 X3 +· · ·+a4 =0
X1 [ 1.8477 ]
X2 [ 0.7653 ]
X3 [ -0.765 ]
X4 [ -1.847 ]
            0.7653668647
```

For the **standard deviation**:

$$\sigma^2=\int_0^{\sqrt{2}} x^2 f(x)\mathrm{d}x-\mu^2=\int_0^{\sqrt{2}}\left(2x^3-x^5\right)\mathrm{d}x-\left(\frac{8\sqrt{2}}{15}\right)^2$$

$$=\left(\frac{x^4}{2}-\frac{x^6}{6}\right)\Bigg|_0^{\sqrt{2}}-\left(\frac{8\sqrt{2}}{15}\right)^2=\frac{22}{225}\Rightarrow\sigma=\sqrt{\frac{22}{225}}\approx 0.3127$$

4. **(a)** $\int_0^1 f(x)\mathrm{d}x+\int_1^2 f(x)\mathrm{d}x=1\Rightarrow\int_0^1\left(k\left(x+1\right)\right)\mathrm{d}x+\int_1^2 2kx^2\mathrm{d}x=1$

$$\Rightarrow k\left(\frac{x^2}{2}+x\right)\Bigg|_0^1+\frac{2kx^3}{3}\Bigg|_1^2=1\Rightarrow\frac{k}{2}+k+\frac{16k}{3}-\frac{2k}{3}=\frac{37k}{6}=1\Rightarrow k=\frac{6}{37}$$

(b) In this case, since the complementary event is just using one part of the piecewise function, it is simpler to calculate the probability by using the complementary event.

$$\mathrm{P}(X>0.5)=1-\mathrm{P}(X<0.5)=1-\int_0^{0.5}\frac{6}{37}\left(x+1\right)\mathrm{d}x=1-\frac{6}{37}\left(\frac{x^2}{2}+x\right)\Bigg|_0^{0.5}=\frac{133}{148}$$

(c) $\mathrm{P}\left(1<X<1.5\right)=\int_1^{1.5}\frac{12}{37}x^2\mathrm{d}x=\frac{4x^3}{37}\Bigg|_1^{1.5}=\frac{19}{74}$

(d) $\mu=\int_0^2 xf(x)\mathrm{d}x=\int_0^1\frac{6}{37}x\left(x+1\right)\mathrm{d}x+\int_1^2\frac{12}{37}x^3\mathrm{d}x=\left(\frac{2x^3}{37}+\frac{3x^2}{37}\right)\Bigg|_0^1+\frac{3x^4}{37}\Bigg|_1^2$

Thus, $\mu=\frac{5}{37}+\frac{45}{37}=\frac{50}{37}$

In order to find the **median**, let the median be m; then:

$$\int_0^m f(x)\mathrm{d}x=\frac{1}{2}$$

First, we need to find the interval in which the value of m lies.

$$\int_0^1\frac{6}{37}\left(x+1\right)\mathrm{d}x=\frac{6}{37}\left(\frac{x^2}{2}+x\right)\Bigg|_0^1=\frac{9}{37}$$

Since the first part of the area under the probability distribution function is less than 0.5, we conclude that the value of m is in the second interval.

The remaining part of the area is: $\frac{1}{2}-\frac{9}{37}=\frac{19}{74}$, so

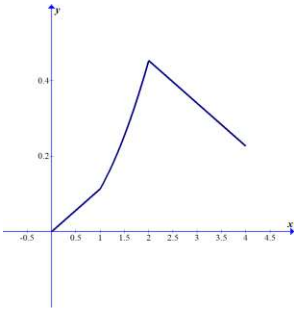
$$\int_1^m \frac{12}{37}x^2\mathrm{d}x=\frac{4x^3}{37}\Bigg|_1^m=\frac{19}{74}\Rightarrow\frac{4\left(m^3-1\right)}{37}=\frac{19}{74}\Rightarrow m=\frac{3}{2}$$

Note: We could have “guessed” this answer from part **(c)**.

$$\sigma^2=\int_{all\ x}x^2f(x)\mathrm{d}x-\mu^2=\int_0^1\frac{6}{37}x^2\left(x+1\right)\mathrm{d}x+\int_1^2\frac{12}{37}x^4\mathrm{d}x-\left(\frac{50}{37}\right)^2$$

$$=\frac{3823}{13690}\approx 0.27925\Rightarrow\sigma=\sqrt{0.27925}\approx 0.5284$$

5. **(a)** To have an accurate sketch, this should be done after part **(b)**.



(b) $\int_0^1 2kx\,\mathrm{d}x+\int_1^2 2kx^2\mathrm{d}x+\int_2^4 k\left(12-2x\right)\mathrm{d}x=1$

$$kx^2\Big|_0^1+\frac{2kx^3}{3}\Bigg|_1^2+k\left(8x-x^2\right)\Big|_2^4=1\Rightarrow\frac{53k}{3}=1\Rightarrow k=\frac{3}{53}$$

(c) For the **mean** we have

$$\mu=\int_0^4 xf(x)\mathrm{d}x=\int_0^1\frac{6x^2}{53}\mathrm{d}x+\int_1^2\frac{6x^3}{53}\mathrm{d}x+\int_2^4\frac{3x}{53}\left(12-2x\right)\mathrm{d}x$$

$$=\frac{2}{53}+\frac{45}{106}+\frac{104}{53}=\frac{257}{106}\approx 2.425$$

To get the **median**, let m denote the median; then:

$$\int_0^m f(x)\mathrm{d}x=\frac{1}{2}$$

We observe that over the first 2 intervals the integral is less than 0.5:

$$\frac{3}{53}+\frac{14}{53}=\frac{17}{53}\approx 0.3208$$

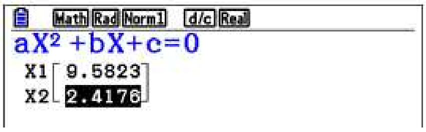
The median is in the third interval. The area still needed is:

$$\frac{1}{2}-\frac{17}{53}=\frac{19}{106}$$

Thus, to find the median we solve the following for m.

$$\int_2^m \frac{3}{53}(12-2x)\,\mathrm{d}x=\frac{19}{106}\Rightarrow\frac{-3\left(m^2-12m+20\right)}{53}=\frac{19}{106}\Rightarrow 6m^2-72m+139=0$$

We use a GDC to find the appropriate estimate, $m\approx 2.4176$



aX²+bX+c=0
X1 9.5823
X2 2.4176

Standard deviation

$$\sigma^2=\int_{all\ x}x^2f(x)\mathrm{d}x-\mu^2=\int_0^1\frac{6}{53}x^3\mathrm{d}x+\int_1^2\frac{6}{53}x^4\mathrm{d}x+\int_2^4\frac{3}{53}\left(12x^2-2x^3\right)\mathrm{d}x-\left(\frac{257}{106}\right)^2$$

$$=\frac{3}{106}+\frac{186}{265}+\frac{312}{53}-\left(\frac{257}{106}\right)^2=\frac{41497}{56180}\approx 0.7386\Rightarrow\sigma=\sqrt{0.7386}\approx 0.8594$$

(d) To find the first quartile, we observe that the first interval has an area of $\frac{3}{53}$ which is less than 0.25. However, the first two intervals add to 0.32. Thus, the first quartile is in the second interval. The area still needed is:

$$\frac{1}{4}-\frac{3}{53}=\frac{41}{212}$$

Let the first quartile be q.

$$\int_1^q\frac{6x^2}{53}\mathrm{d}x=\frac{41}{212}\Rightarrow\frac{2\left(q^3-1\right)}{53}=\frac{41}{212}\Rightarrow q\approx 1.8297$$

The third quartile is in the third interval. The area left to be covered is:

$$\frac{3}{4}-\frac{17}{53}=\frac{91}{212}$$

$$\frac{3}{53}\int_2^q(12-2x)\mathrm{d}x=\frac{91}{212}\Rightarrow-\frac{3\left(q^2-12q+20\right)}{53}=\frac{91}{212}\Rightarrow q\approx 3.0989$$

IQR = 3.0989 – 1.8279 = 1.269

6. **(a)** Remember that X is measured in tens of hours, so our answers must be multiplied by 10.

$$\mu=\int_0^{8\frac{1}{15}}xf(x)\mathrm{d}x=\int_0^1\frac{15}{76}\left(x^5-2x^3+2x\right)\mathrm{d}x-\int_1^{8\frac{1}{15}}\frac{15}{8056}\left(15x^2-121x\right)\mathrm{d}x$$

$$=\frac{15}{76}\left(\frac{x^6}{6}-\frac{2x^4}{4}+x^2\right)\Bigg|_0^1-\frac{15}{8056}\left(15\frac{x^3}{3}-121\frac{x^2}{2}\right)\Bigg|_1^{8\frac{1}{15}}$$

$$=\frac{5}{38}+\frac{8003}{3420}=\frac{8453}{3420}\approx 2.472$$

Since x is measured in tens of hours, the mean life is 24.7 hours.

(b) Since the value of x is measured in tens of hours, $x = 2$.

$$\int_{2}^{8\frac{1}{15}} -\frac{15}{8056}(15x-121)\,\mathrm{d}x \approx 0.514$$

So, the probability that a battery will last at least 20 hours is 0.514

(c) Since both batteries have to work for the unit to work, we need to multiply the individual probabilities that each battery works for more than 20 hours. (The working of one battery is independent of the other battery.)

$$\text{P(unit works)} = 0.514 \times 0.514 \approx 0.264$$

7. **(a)** Remember that X is measured in tens of hours, so our answers must be multiplied by 10.

$$\mu = \int_0^{10} \frac{3}{500} y^2 (10-y)\,\mathrm{d}y = \frac{3}{500}\left(\frac{10y^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^{10} = 5$$

Since y is measured in tens of hours, the mean of a battery life is 50 hours.

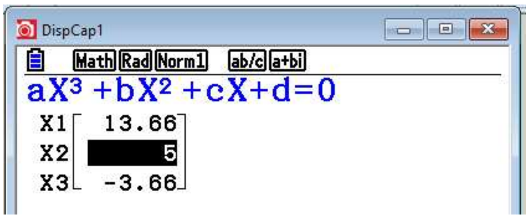
(b) To find the median, let the median be m.

$$\int_0^{m} \frac{3}{500} y(10-y)\,\mathrm{d}y = \frac{1}{2} \Rightarrow \frac{3}{500}\left(5y^2 - \frac{y^3}{3}\right)\Bigg|_0^{m} = \frac{1}{2} \Rightarrow m^3 - 15m^2 + 250 = 0$$

This cubic equation can either be factorised, or use a GDC to find its solutions:

$$(m-5)(m^2 - 10m - 50) = 0 \Rightarrow m = 5$$

Therefore, the median of a battery life is 50 hours.



```
aX³+bX²+cX+d=0
X1  13.66
X2      5
X3  -3.66
```

(c) **Standard deviation**

$$\sigma^2 = \int_{all\ x} x^2 f(x)\mathrm{d}x - \mu^2 = \int_0^{10} \frac{3}{500} y^3 (10-y)\,\mathrm{d}y - 5^2 = \left(\frac{3y^4}{200} - \frac{3y^5}{2500}\right)\Bigg|_0^{10} - 25 = 5$$

$$\sigma = \sqrt{5} \approx 2.24$$

The standard deviation of a battery life is 22.4 hours.

(d) $$\mathrm{P}(Y > 8) = \int_8^{10} \frac{3}{500} y(10-y)\,\mathrm{d}y = \frac{3}{500}\left(5y^2 - \frac{y^3}{3}\right)\Bigg|_8^{10} = \frac{13}{125} \approx 0.104$$

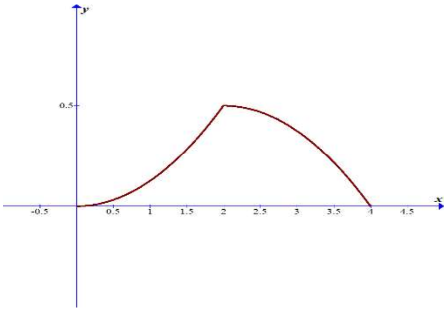
(e) **(i)** Since the bulbs should last for more than 80 hours, the value of the variable y should be greater than 8.

$$\mathrm{P}(\text{both larger than } 80) = \mathrm{P}(Y > 8) \times \mathrm{P}(Y > 8) = \left(\frac{13}{125}\right)^2 \approx 0.010816$$

(ii) ‘At least one has to be replaced’ is the compliment to ‘no battery has to be replaced (both batteries will last for more than 80 hours)’.

$$\mathrm{P}\left(\text{at least one to be replaced}\right) \approx 1 - 0.010816 = 0.989184$$

8. **(a)** PDF of weekly oil production (hundreds of barrels)



(b) $\mu = \int_0^2 \frac{1}{8} y^3 \mathrm{d}y + \int_2^4 \frac{y^2}{8}(4-y)\mathrm{d}y = \frac{7}{3}$. So, the mean weekly production level is approximately $\frac{7}{3} \times 100 \approx 233$ barrels.

(c) Since $\int_0^2 \frac{1}{8} y^2 \mathrm{d}y = \frac{1}{24} y^3 \Big|_0^2 = \frac{1}{3} < 0.25$, the first quartile lies in this interval.

The third quartile is obviously in the second interval.

Let the first quartile be q, and thus

$$\int_0^q \frac{1}{8} y^2 \mathrm{d}y = \frac{1}{24} y^3 \Big|_0^q = \frac{q^3}{24} = 0.25 \Rightarrow q \approx 1.817$$

Let the third quartile be r, and thus

$$\int_r^4 \frac{y}{8}(4-y)\mathrm{d}y = \left(\frac{y^2}{4} - \frac{y^3}{24}\right)\Bigg|_r^4 = 0.25 \Rightarrow r \approx 2.8926$$

IQR = 289.26 – 181.712 ≈ 108 barrels

(d) We need to solve the equation

$\int_0^t \frac{1}{8} y^2 \mathrm{d}y = \frac{1}{24} y^3 \Big|_0^t = \frac{t^3}{24} = 0.10 \Rightarrow t \approx 1.34$, thus the level of production warranting maintenance is 134 barrels.

9. **(a)** The area under the curve must be 1.

$$\int_2^5 \frac{c}{(1-y)(y-6)}\mathrm{d}y = 1 \Rightarrow c\int_2^5\left(\frac{1}{5(y-1)} - \frac{1}{5(y-6)}\right)\mathrm{d}y = 1$$

$$\Rightarrow \frac{c}{5}\left(\ln|y-1| - \ln|y-6|\right)\Big|_2^5 = 1 \Rightarrow \frac{c}{5}\left(\ln\left|\frac{y-1}{y-6}\right|\right)\Big|_2^5 = 1$$

$$\Rightarrow \frac{c}{5}\left(\ln 4 - \ln\frac{1}{4}\right) = 1 \Rightarrow c = \frac{5}{4\ln 2}$$

(b)

$$\mu = \frac{5}{4\ln 2}\int_2^5 \frac{y\mathrm{d}y}{(1-y)(y-6)} = \frac{1}{4\ln 2}\ln\left|\frac{y-1}{(y-6)^6}\right|\Bigg|_2^5$$

$$= \frac{1}{4\ln 2}\left(\ln 4 - \ln\left(\frac{1}{4^6}\right)\right) = \frac{7}{2}$$

Variance

$$\sigma^2 = \int_{all\ y} y^2 f(y)\mathrm{d}y - \mu^2 = \frac{5}{4\ln 2}\int_2^5 \frac{y^2}{(1-y)(y-6)}\mathrm{d}y - \left(\frac{7}{2}\right)^2$$

$$= \frac{5}{4\ln 2}\times\left(\frac{74\ln 2}{5} - 3\right) - \frac{49}{4} \approx 0.83989$$

$$\sigma = \sqrt{0.83989} \approx 0.9165$$

10. **(a)** $a(by - y^2) \geq 0 \Rightarrow y(b-y) \geq 0 \Rightarrow b \geq y$

Since the maximum value of y is 5, $b \geq 5$

$$\int_0^5 a(by - y^2)\mathrm{d}y = 1 \Rightarrow a\left(\frac{by^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^5 = 1$$

$$\Rightarrow a\left(\frac{25b}{2} - \frac{125}{3}\right) = 1 \Rightarrow a = \frac{6}{25(3b-10)}$$

(b) We substitute the value of a found in **(a)** and then solve for b.

$$\mu = \int_0^5 a(by^2 - y^3)\mathrm{d}y = a\left(\frac{by^3}{3} - \frac{y^4}{4}\right)\Bigg|_0^5 = \frac{5}{2}$$

$$\Rightarrow \frac{6}{25(3b-10)}\left(\frac{125b}{3} - \frac{625}{4}\right) = \frac{5}{2} \Rightarrow b = 5; a = \frac{6}{125}$$

(c) **Variance**

$$\sigma^2 = \int_{all\ y} y^2 f(y)\mathrm{d}y - \mu^2 = \int_0^5 a\left(by^3 - y^4\right)\mathrm{d}y - \frac{25}{4}$$

$$= a\left(\frac{by^4}{4} - \frac{y^5}{5}\right)\Bigg|_0^5 - \frac{25}{4} = a\left(\frac{625b}{4} - 625\right) - \frac{25}{4}$$

$$= \frac{6}{125}\left(\frac{3125}{4} - 625\right) - \frac{25}{4} = \frac{5}{4}$$

11. **(a)** $\int_a^b k\,\mathrm{d}x = kx\big|_a^b = k(b-a) = 1 \Rightarrow k = \frac{1}{b-a}$

(b) **(i)** $\mu = \int_a^b \frac{x}{b-a}\mathrm{d}x = \frac{x^2}{2(b-a)}\Bigg|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$

(ii) Since the pdf is constant, the median must be in the middle of the interval, i.e., at $\frac{a+b}{2}$. This can also be done by integration.

(iii) $\sigma^2 = \int_a^b \frac{x^2}{b-a}\mathrm{d}x - \left(\frac{b+a}{2}\right)^2 = \frac{x^3}{3(b-a)}\Bigg|_a^b - \left(\frac{b+a}{2}\right)^2$

$$= \frac{b^3-a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4} = \frac{(b-a)^2}{12}$$

12. **(a)** $\int_{1.2}^{1.7} \frac{5x^4}{31}\mathrm{d}x = \frac{x^5}{31}\Bigg|_{1.2}^{1.7} = \frac{1511}{4000} \approx 0.3778$

(b) $\int_1^m \frac{5x^4}{31}\mathrm{d}x = \frac{1}{2} \Rightarrow \frac{x^5}{31}\Bigg|_1^m = \frac{m^5-1}{31} = \frac{1}{2} \Rightarrow m \approx 1.752$

(c) $\int_k^2 \frac{5x^4}{31}\mathrm{d}x = \frac{1}{4} \Rightarrow \frac{x^5}{31}\Bigg|_k^2 = \frac{32-k^5}{31} = \frac{1}{4} \Rightarrow k \approx 1.892$

(d) This is the complement of 'no observation is larger than 1.5'

$\mathrm{P}(\text{at least one} > 1.5) = 1 - \left(\mathrm{P}(X<1.5)\right)^2$

Now, $\mathrm{P}(X<1.5) = \int_1^{1.5} \frac{5x^4}{31}\mathrm{d}x = \frac{x^5}{31}\Bigg|_1^{1.5} = \frac{211}{992}$, thus:

$\mathrm{P}(\text{at least one} > 1.5) = 1 - \left(\mathrm{P}(X<1.5)\right)^2 = 1 - \left(\frac{211}{992}\right)^2 \approx 0.955$

13. **(a)** $F(x)=k\left(x^3-21x^2+147x-335\right)$ is the cumulative distribution function.

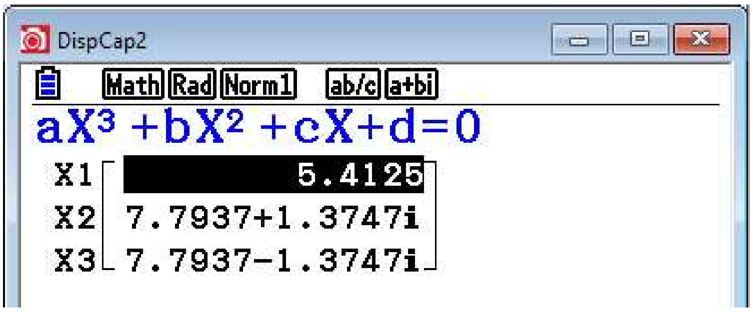
Thus, $F(7)=k\left(7^3-21\times 7^2+147\times 7-335\right)=1\Rightarrow k=\frac{1}{8}$

(b) To find the probability density function, we need to differentiate the cumulative distribution function with respect to the variable *x*.

$$f(x)=F'(x)=\begin{cases}\frac{1}{8}\left(3x^2-42x+147\right) & 5\le x\le 7\\ 0 & \text{otherwise}\end{cases}$$

(c) The median can be found by using the probability density function, or the cumulative distribution function. Since the first method has been demonstrated in the previous question, we are going to use the distribution function.

$$F(m)=\frac{1}{8}\left(m^3-21m^2+147m-335\right)=\frac{1}{2}\Rightarrow m^3-21m^2+147m-339=0$$



aX³ +bX² +cX+d=0

X1 5.4125
X2 7.7937+1.3747i
X3 7.7937−1.3747i

Thus, the median value is 5.41

(d) To find the variance, we need to find the mean value first.

$$\mu=\int_5^7\frac{1}{8}\left(3x^3-42x^2+147x\right)\mathrm{d}x=\frac{1}{8}\left(\frac{3}{4}x^4-14x^3+\frac{147}{2}x^2\right)\Bigg|_5^7=\frac{11}{2}$$

Now for the variance

$$\sigma^2=\int_{all\ x}x^2f(x)\mathrm{d}x-\mu^2=\int_5^7\frac{1}{8}\left(3x^4-42x^3+147x^2\right)\mathrm{d}x-\frac{121}{4}$$

$$=\frac{1}{8}\left(\frac{3}{5}x^5-\frac{21}{2}x^4+49x^3\right)\Bigg|_5^7-\frac{121}{4}=\frac{3}{20}$$

14. **(a)** $\int_0^1 4y^k\mathrm{d}y=1\Rightarrow\frac{4y^{k+1}}{k+1}\Bigg|_0^1=1\Rightarrow k=3$

(b) $\mu=\int_0^1 4y^4\mathrm{d}y=\frac{4y^5}{5}\Bigg|_0^1=\frac{4}{5}$

(c) Obviously, this is the median.

$$\mathrm{P}(Y>a)=0.5\Rightarrow\mathrm{P}(Y<a)=0.5\Rightarrow\int_0^a 4y^3\,\mathrm{d}y=\left.\frac{4y^4}{4}\right|_0^a=0.5\Rightarrow a\approx 0.841$$

15. **(a)** We check two conditions: (1000 hours correspond to 1 unit.)

- $f(Y)=2ye^{-y^2}\geq 0$ since it is the product of two non-negative numbers.
- $\lim\limits_{t\to\infty}\int_0^t 2ye^{-y^2}\,\mathrm{d}y=\lim\limits_{t\to\infty}\left(-e^{-y^2}\right)\Big|_0^t=\lim\limits_{t\to\infty}\left(-e^{-t^2}+e^0\right)=1$

(b) $\mathrm{P}(Y>2)=1-\mathrm{P}(Y<2)=1-\int_0^2 2ye^{-y^2}\,\mathrm{d}y=1-\left(-e^{-y^2}\right)\Big|_0^2=e^{-4}\approx 0.0183$

(c) $\mu=\lim\limits_{t\to\infty}\int_0^t 2y^2e^{-y^2}\,\mathrm{d}y=\lim\limits_{t\to\infty}-ye^{-y^2}\Big|_0^t+\lim\limits_{t\to\infty}\int_0^t e^{-y^2}\,\mathrm{d}y$ (integration by parts)

Now, $\lim\limits_{t\to\infty}-ye^{-y^2}\Big|_0^t=0$ by using l'Hopital's rule, and

$\lim\limits_{t\to\infty}\int_0^t e^{-y^2}\,\mathrm{d}y=\frac{\sqrt{\pi}}{2}$ as given by the hint. Therefore $\mu=\frac{\sqrt{\pi}}{2}$

(d) Let m be the median.

$$\int_0^m 2ye^{-y^2}\,\mathrm{d}y=\left(-e^{-y^2}\right)\Big|_0^m=-e^{-m^2}+1=0.5\Rightarrow e^{-m^2}=0.5$$

$$\Rightarrow -m^2=\ln 0.5\Rightarrow m=\sqrt{-\ln 0.5}=0.8326$$

(e) Lower quartile:

$$\int_0^l 2ye^{-y^2}\,\mathrm{d}y=\left(-e^{-y^2}\right)\Big|_0^l=-e^{-l^2}+1=0.25\Rightarrow e^{-m^2}=0.75$$

$$\Rightarrow -m^2=\ln 0.75\Rightarrow m=\sqrt{-\ln 0.75}=0.5364$$

Upper quartile:

$$\int_0^u 2ye^{-y^2}\,\mathrm{d}y=\left(-e^{-y^2}\right)\Big|_0^u=-e^{-u^2}+1=0.75\Rightarrow e^{-m^2}=0.25$$

$$\Rightarrow -m^2=\ln 0.25\Rightarrow m=\sqrt{-\ln 0.25}=1.1774$$

IQR = 1.1774 – 0.5364 = 0.6410

(f) A valve fails before 200 hours with a probability

$$\mathrm{P}(Y<0.2)=\int_0^{0.2} 2ye^{-y^2}\,\mathrm{d}y=\left(-e^{-y^2}\right)\Big|_0^{0.2}=-e^{-0.2^2}+1=0.03921$$

For an engine to need servicing, at least one valve has to fail, which is the complement of 'neither one fails'. A valve does not fail with probability 1 – 0.03921 = 0.9608

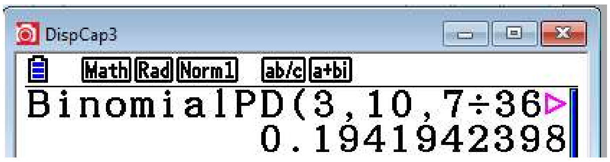
Thus, the probability that the engine (at least one valve) needs servicing before 200 hours of work is $1-0.9608^2\approx 0.07688$

16. **(a)** $\int_0^2\frac{1}{2}\left(cy+\frac{y^2}{3}\right)\mathrm{d}y=1\Rightarrow\frac{1}{2}\left(\frac{cy^2}{2}+\frac{y^3}{9}\right)\Big|_0^2=\frac{1}{2}\left(2c+\frac{8}{9}\right)=1\Rightarrow c=\frac{5}{9}$

(b) $$\int_0^1 \frac{1}{2}\left(\frac{5y}{9}+\frac{y^2}{3}\right)\mathrm{d}y=\frac{1}{2}\left(\frac{5y^2}{18}+\frac{y^3}{9}\right)\Bigg|_0^1=\frac{7}{36}$$

(c) In this part, we have a binomial random variable $X\sim \mathrm{B}\left(10,\frac{7}{36}\right)$

$$\mathrm{P}(X=3)={}_{10}C_3\left(\frac{7}{36}\right)^3\left(\frac{29}{36}\right)^7\approx 0.1942$$



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BinomialPD(3,10,7÷36▷
          0.1941942398
```

(d) Let H be the event that a student needs at least one hour to complete the exam.

$$\mathrm{P}(H)=1-\frac{7}{36}=\frac{29}{36}$$

For Casper,

$$\mathrm{P}(C>1.5)=1-\mathrm{P}(C<1.5)=1-\int_0^{1.5}\frac{1}{2}\left(\frac{5y}{9}+\frac{y^2}{3}\right)\mathrm{d}y$$

$$=1-\frac{1}{2}\left(\frac{5y^2}{18}+\frac{y^3}{9}\right)\Bigg|_0^{1.5}=\frac{1}{2}$$

Therefore, $\mathrm{P}(C>1.5|H)=\frac{\frac{1}{2}}{\frac{29}{36}}\approx 0.6207$

17. **(a)** $$\int_0^5 ky^2(5-y)\mathrm{d}y=1\Rightarrow k\left(\frac{5y^3}{3}-\frac{y^4}{4}\right)\Bigg|_0^5=\frac{625k}{12}=1\Rightarrow k=\frac{12}{625}$$

(b) To find the mode, we find the point at which the pdf has a maximum.

$$f(y)=\frac{12}{625}y^2(5-y)\Rightarrow f'(y)=\frac{12y}{625}(10-3y)$$

$$\Rightarrow f'(y)=0\Rightarrow y=0 \text{ or } y=\frac{10}{3}$$

This implies that the function has a minimum at $y=0$, and a maximum at $y=\frac{10}{3}\approx 3.3$, which is the mode.

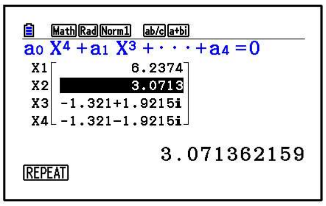
The mean: $\mu=\int_0^5 \frac{12}{625}y^3(5-y)\mathrm{d}y=\frac{12}{625}\left(\frac{5y^4}{4}-\frac{y^5}{5}\right)\Bigg|_0^5=\cdots=3$

For the median, m:

$$\int_0^m \frac{12}{625}y^2(5-y)\mathrm{d}y=\frac{1}{2}\Rightarrow\frac{12}{625}\left(\frac{5y^3}{3}-\frac{y^4}{4}\right)\Bigg|_0^m=\frac{1}{2}$$

$$\Rightarrow\frac{12}{625}\left(\frac{5m^3}{3}-\frac{m^4}{4}\right)=\frac{1}{2}\Rightarrow 6m^4-40m^3+625=0$$

The value of the median given by GDC is 3.1



```
a0 X4 +a1 X3 +···+a4 =0
X1⎡      6.2374⎤
X2⎢      3.0713⎥
X3⎢-1.321+1.9215i⎥
X4⎣-1.321-1.9215i⎦

          3.071362159
```

(c) $\mathrm{P}\left(Y<3\right)=\int_0^3 \frac{12}{625}y^2\left(5-y\right)\mathrm{d}y=\frac{12}{625}\left(\frac{5y^3}{3}-\frac{y^4}{4}\right)\Bigg|_0^3=\frac{297}{625}\approx 0.4752$

(d) Variance:

$$\sigma^2=\int_0^5 \frac{12}{625}y^4\left(5-y\right)\mathrm{d}y-3^2=\frac{12}{625}\left(y^5-\frac{y^6}{6}\right)\Bigg|_0^5-9=\cdots=1$$

$$\Rightarrow \sigma=1$$

(e) $\mu\pm\sigma=\left[2,4\right]$

$$\Rightarrow \mathrm{P}\left(2\le Y\le 4\right)=\int_2^4 \frac{12}{625}y^2\left(5-y\right)\mathrm{d}y$$

$$=\frac{12}{625}\left(\frac{5y^3}{3}-\frac{y^4}{4}\right)\Bigg|_2^4=\cdots\approx 0.64$$

The empirical rule gives this probability as 0.68, which is not very far from the answer we found. So, there is no contradiction.

18. **(a)** To find the mean, we evaluate:

$$\mu=\int_0^5 \frac{4t}{625}\left(5t^3-t^4\right)\mathrm{d}t=\frac{4}{625}\left(t^5-\frac{t^6}{6}\right)\Bigg|_0^5=\cdots=\frac{10}{3}$$

To find the mode, we look for the maximum of the pdf.

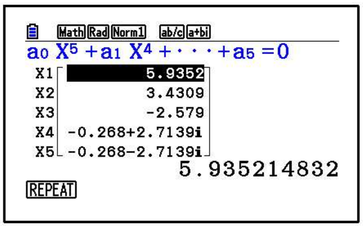
$$f(t)=\frac{4}{625}\left(5t^3-t^4\right)\Rightarrow f'(t)=\frac{4}{625}\left(15t^2-4t^3\right)=\frac{4t^2}{625}\left(15-4t\right)=0$$

This implies that the function has a minimum at $t=0$, and a maximum at $t=\frac{15}{5}=3.75$, which is the mode.

(b) For the median, m

$$\int_0^m \frac{4}{625}\left(5t^3-t^4\right)\mathrm{d}t=\frac{1}{2}\Rightarrow\frac{4}{625}\left(\frac{5t^4}{4}-\frac{t^5}{5}\right)\Bigg|_0^m=\frac{4}{625}\left(\frac{5m^4}{4}-\frac{m^5}{5}\right)=\frac{1}{2}$$

$$\Rightarrow 8m^5-50m^4+3125=0\Rightarrow m=3.43$$



```
a0 X5 +a1 X4 +···+a5 =0
X1  5.9352
X2  3.4309
X3  -2.579
X4  -0.268+2.7139i
X5  -0.268-2.7139i
                5.935214832
```

(c) $$\int_1^2 \frac{4}{625}\left(5t^3 - t^4\right)\mathrm{d}t = \frac{4}{625}\left(\frac{5t^4}{4} - \frac{t^5}{5}\right)\Bigg|_1^2 = \frac{251}{3125} \approx 0.0803$$

(d) Variance:

$$\sigma^2 = \int_0^5 \frac{4}{625}\left(5t^3 - t^4\right)\mathrm{d}t - \left(\frac{10}{3}\right)^2 = \frac{4}{625}\left(\frac{5t^4}{4} - \frac{t^5}{5}\right)\Bigg|_0^5 - \left(\frac{10}{3}\right)^2 = \cdots = \frac{50}{63}$$

$$\sigma = \sqrt{\frac{50}{63}} = 0.8909$$

(e) We need to calculate the probability that one plane has been delayed for more than one hour.

$$\mathrm{P}(t > 1) = 1 - \mathrm{P}(t < 1) = 1 - \frac{4}{625}\left(\frac{5t^4}{4} - \frac{t^5}{5}\right)\Bigg|_0^1 = \frac{3104}{3125} \approx 0.9933$$

(i) Since they are independent, then the probability that both will be delayed is the product of their individual probabilities.

$$\mathrm{P}(t > 1) \times \mathrm{P}(t > 1) = \left(\frac{3104}{3125}\right)^2 \approx 0.987$$

(ii) This is the complement for both being delayed less than 1 hour

$$\mathrm{P}(t < 1) = 1 - \mathrm{P}(T > 1) = 1 - \frac{3104}{3215} = \frac{21}{3125}$$

$$\mathrm{P}(\text{at least one}) = 1 - \left(\mathrm{P}(t < 1)\right)^2 = 1 - \left(\frac{121}{3125}\right)^2 \approx 0.99995$$

(iii) This can happen when the first is delayed but not the second and vice versa

$$\mathrm{P}\left(\text{only one delayed}\right) = \frac{3104}{3215} \times \frac{21}{3125} + \frac{21}{3125} \times \frac{3104}{3215} \approx 0.01335$$

Exercise 15.4

1. Let X be the number of hours it takes to change the batteries: $X \sim \mathrm{N}\left(\mu = 50, \sigma^2 = 7.5^2\right)$

No tables are allowed, so answers depend on your GDC. Almost all GDCs allow you to either find the probabilities of individual events or events in lists. For convenience we chose lists here.

(a) $\mathrm{P}(X \le 50) = \mathrm{P}(X < 50) = 0.5$ This is so, since 50 is the mean value.

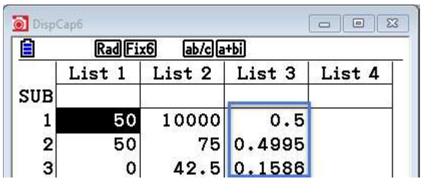
(b) $\mathrm{P}(50 < X < 75) = 0.4996$

(c) $\mathrm{P}(X < 42.5) = 0.15866$

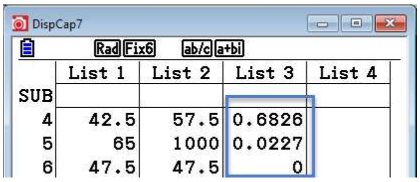
(d) $\mathrm{P}(42.5 < X < 57.5) = 0.68269$

(e) $\mathrm{P}(X > 65) = 0.02275$

(f) $\mathrm{P}(X = 47.5) = 0$, since X is the continuous variable and the probability of obtaining an exact value is 0.



```
     List 1  List 2  List 3  List 4
SUB
  1      50   10000     0.5
  2      50      75  0.4995
  3       0    42.5  0.1586
```



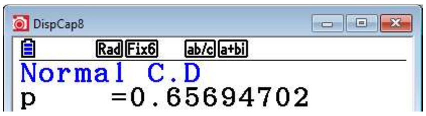
```
     List 1  List 2  List 3  List 4
SUB
  4    42.5    57.5  0.6826
  5      65    1000  0.0227
  6    47.5    47.5       0
```

2. **(a)** $\mathrm{P}(-1.2 < Z < 1.2) \approx 0.7698$

(b) $\mathrm{P}(Z < -1.4) + \mathrm{P}(Z > 1.4) \approx 0.1616$

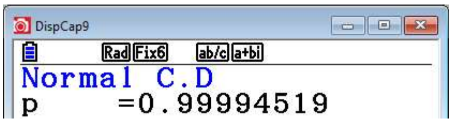
(c) Let X be the normal variable, where $X \sim \mathrm{N}(\mu = 3, \sigma^2 = 3)$.

$\mathrm{P}(X < 3.7) \approx 0.67$



```
Normal C.D
p      =0.65694702
```

(d) $\mathrm{P}(Z > -3.7) \approx 1$ (It is more than 2 standard deviations from the mean.)

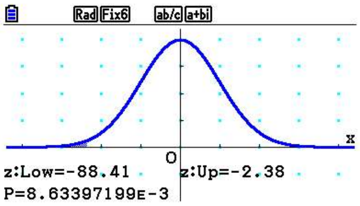


```
Normal C.D
p      =0.99994519
```

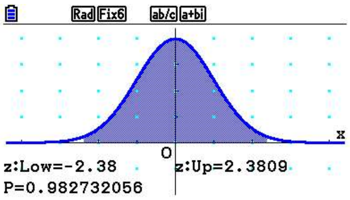
3. Let X be the mileage of a car, where $X \sim \mathrm{N}(11.4, 1.26^2)$

(a) $\mathrm{P}(X < 8.4) \approx 0.008634$

(b) $\mathrm{P}(8.4 < X < 14.4) = 0.98273$

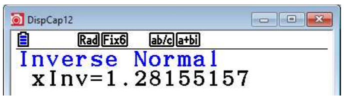


```
z:Low=-88.41    z:Up=-2.38
P=8.63397199E-3
```



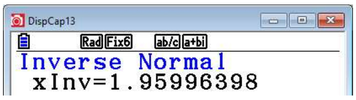
```
z:Low=-2.38    z:Up=2.3809
P=0.982732056
```

4. This is an inverse normal calculation. Due to its nature on a GDC, we need to use its complement, which is 0.90. $z = 1.282$



```
Inverse Normal
 xInv=1.28155157
```

5. If 95% lie within the interval then 5% are outside. These are split 2.5% in each ‘tail’. This is an inverse normal calculation: $z_0 = 1.96$

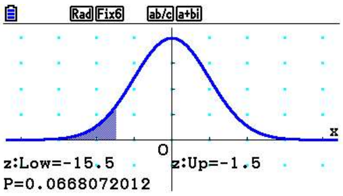


```
Inverse Normal
 xInv=1.95996398
```

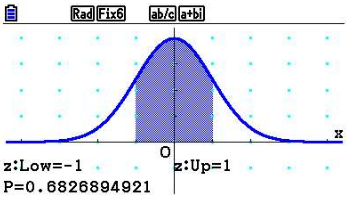
6. Let X be the scores on the examination, where $X \sim \mathrm{N}\left(550, 100^2\right)$

(a) $\mathrm{P}\left(X < 400\right) = 0.066807$

(b) $\mathrm{P}\left(450 < X < 650\right) = 0.68269$

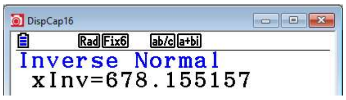


```
z:Low=-15.5   z:Up=-1.5
P=0.0668072012
```



```
z:Low=-1   z:Up=1
P=0.6826894921
```

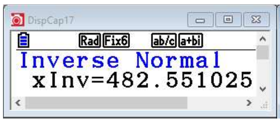
(c) To be in the 90th percentile, 90% of the population should have a score lower than yours. Thus, this is an inverse normal calculation. The score is 678.15, i.e., you will need a score of 679.



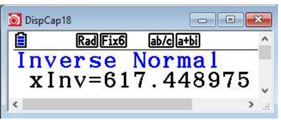
```
Inverse Normal
 xInv=678.155157
```

(d) We need to find the lower quartile, i.e., the number that leaves 25% of the population of scores below it, and the third quartile that leaves 75% of the scores below it.

$$\mathrm{Q}_1 = 482.55, \mathrm{Q}_2 = 617.45, \Rightarrow \mathrm{IQR} = 617.45 - 482.55 = 134.90$$



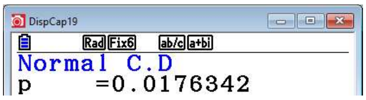
```
Inverse Normal
 xInv=482.551025
```



```
Inverse Normal
 xInv=617.448975
```

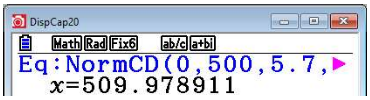
7. **(a)** In this case, $X \sim \mathrm{N}\left(512, 5.7^2\right)$

The question is simply $\mathrm{P}\left(X < 500\right) = 1.76\%$



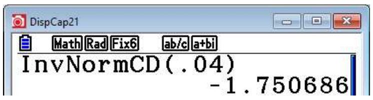
```
Normal C.D
p      =0.0176342
```

(b) **(i)** We can use the equation-solver on a GDC: $\mu \approx 509.98$



```
Eq:NormCD(0,500,5.7,▶
  x=509.978911
```

(ii) Alternatively, we standardise the variables and solve as follows. If the area to the left is to be 4%, then the z value corresponding to this is the inverse normal variable $z = -1.75$



```
InvNormCD(.04)
           -1.750686
```

Then, $-1.75 = \dfrac{500-\mu}{5.7} \Rightarrow \mu \approx 500 + 1.75 \times 5.7 = 509.98$

(c) **(i)** We can use the solver on GDC: $\sigma = 5.712$

(ii) We standardise and solve. As above $z = -1.75$

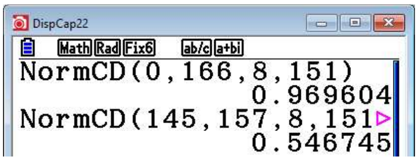
Then, $-1.75 = \dfrac{500-510}{\sigma} \Rightarrow \sigma \approx \dfrac{10}{1.75} = 5.714$

This agrees with the above, correct to 2 decimal places.

8. Let X represent the heights of the students, where $X \sim \mathrm{N}\left(151, 8^2\right)$

(a) $\mathrm{P}(X < 166) = 0.9696$

(b) $\mathrm{P}(145 < X < 157) = 0.5467$



```
NormCD(0,166,8,151)
            0.969604
NormCD(145,157,8,151▷
            0.546745
```

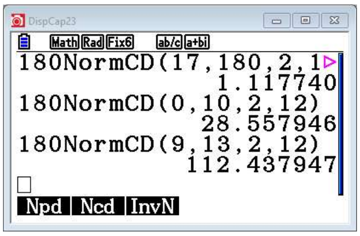
9. Let X represent the number of minutes, where $X \sim \mathrm{N}\left(\mu = 12, \sigma^2 = 4\right)$

Calculated probabilities will be multiplied by 180 school days.

(a) $n = 180\mathrm{P}(X > 17) = 1$

(b) $n = 180\mathrm{P}(X < 10) = 29$

(c) $n = 180\mathrm{P}(9 < X < 13) = 112$



```
180NormCD(17,180,2,1▷
            1.117740
180NormCD(0,10,2,12)
           28.557946
180NormCD(9,13,2,12)
          112.437947
```

10. We will demonstrate two methods here. In the rest of the exercises, we will use GCD solver only.

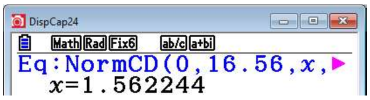
Method 1

We standardise the variables and solve as follows.

If the area to the left is to be 64%, then the z value corresponding to this is the inverse normal variable $z = 0.358$. Then, $0.358 = \dfrac{16.56 - 16}{\sigma} \Rightarrow \sigma \approx \dfrac{0.56}{0.358} = 1.56$

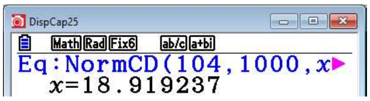
Method 2

Using the GDC solver:



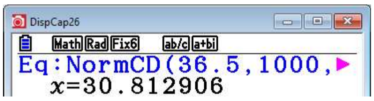
```
Eq:NormCD(0,16.56,x,▶
  x=1.562244
```

11. Using the GDC solver: $\sigma = 18.9192$



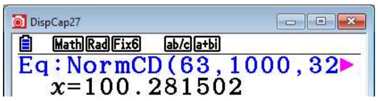
```
Eq:NormCD(104,1000,x▶
  x=18.919237
```

12. Using the GDC solver: $\mu = 30.8129$



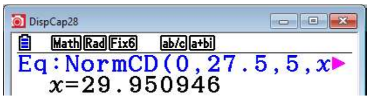
```
Eq:NormCD(36.5,1000,▶
  x=30.812906
```

13. Using the GDC solver: $\mu = 100.2815$



```
Eq:NormCD(63,1000,32▶
  x=100.281502
```

14. Using the GDC solver: $\mu = 29.9509$



```
Eq:NormCD(0,27.5,5,x▶
  x=29.950946
```

15. In cases where the mean and standard deviations are missing, we need to resort to the standard normal distribution and solve a system of equations. By now, you must be familiar enough with your GDC in order to find inverse standard normal values corresponding to given probabilities.

$$\left\{\begin{array}{l} \dfrac{14.6-\mu}{\sigma} = -1.514 \Rightarrow 14.6 - \mu = -1.514\sigma \\ \dfrac{29.6-\mu}{\sigma} = 2.014 \Rightarrow 29.6 - \mu = 2.014\sigma \end{array}\right\}$$

$$\Rightarrow 15 = 3.528\sigma \Rightarrow \sigma \approx 4.2517; \mu \approx 21.03707$$

16. Similar to question **15**. We solve a system of equations.

$$\left\{\begin{array}{l}\dfrac{19.6-\mu}{\sigma}=0.9945\Rightarrow 19.6-\mu=0.9945\sigma\\ \dfrac{17.6-\mu}{\sigma}=-2.2571\Rightarrow 17.6-\mu=-2.2571\sigma\end{array}\right\}$$

$$\Rightarrow 2=3.2516\sigma\Rightarrow\sigma\approx 0.6151;\mu\approx 18.9883$$

17. Again, we solve a system of equations.

$$\left\{\begin{array}{l}\dfrac{162-\mu}{\sigma}=1.165\Rightarrow 162-\mu=1.165\sigma\\ \dfrac{56-\mu}{\sigma}=-1.917\Rightarrow 56-\mu=-1.917\sigma\end{array}\right\}$$

$$\Rightarrow 106=3.082\sigma\Rightarrow\sigma\approx 34.39;\mu\approx 121.932$$

18. **(a)** This part is similar to previous cases. We set up and solve a system of equations.

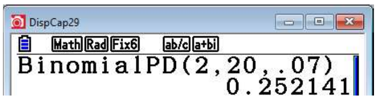
$$\left\{\begin{array}{l}\dfrac{6.3-\mu}{\sigma}=-2.0537\Rightarrow 6.3-\mu=-2.0537\sigma\\ \dfrac{7.5-\mu}{\sigma}=1.6449\Rightarrow 7.5-\mu=1.6449\sigma\end{array}\right\}$$

$$\Rightarrow 1.2=3.6986\sigma\Rightarrow\sigma\approx 0.3244;\mu\approx 6.9663$$

(b) The probability to be rejected is 2% + 5% = 7%

Let X be the number of rejected poles, then $X\sim \mathrm{B}(20,0.07)$

$\mathrm{P}(X=2)=0.252$

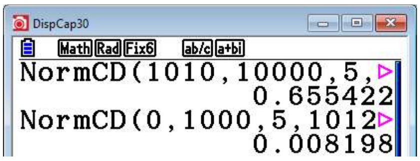


```
BinomialPD(2,20,.07)
             0.252141
```

19. Let X be the number of millilitres of water in a bottle, where $X\sim \mathrm{N}(1012,25)$

(a) $\mathrm{P}(X>1010)=0.6554$

(b) $\mathrm{P}(X<1000)=0.0082$



```
NormCD(1010,10000,5,▷
             0.655422
NormCD(0,1000,5,1012▷
             0.008198
```

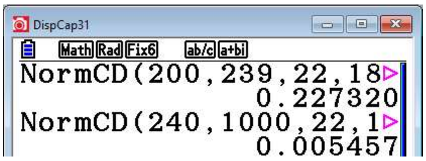
(c) Let Y be the number of under-filled bottles

$\mathrm{E}(Y)=np=10000\times\mathrm{P}(X<1000)=10000\times 0.0082=82$

20. Let X represent the cholesterol level, where $X\sim \mathrm{N}(184,22^2)$

(a) $\mathrm{P}(200 < X < 239) = 22.73\%$

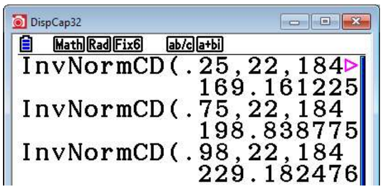
(b) $\mathrm{P}(X > 240) = 0.546\%$



```
NormCD(200,239,22,18▷
              0.227320
NormCD(240,1000,22,1▷
              0.005457
```

(c) $\mathrm{Q}_1 = 169.161; \mathrm{Q}_3 = 198.839 \Rightarrow \mathrm{IQR} = 29.678$

(d) This is an inverse normal probability calculation. $X = 229.183$



```
InvNormCD(.25,22,184▷
          169.161225
InvNormCD(.75,22,184
          198.838775
InvNormCD(.98,22,184
          229.182476
```

21. Let X represent the tread life of the tyres, where $X \sim \mathrm{N}(52000, 4000^2)$

(a) $\mathrm{P}(X > 64\,000) = 0.00135 = 0.135\%$

So, it is not likely (0.135% chance) that a set of tyres will last more than 64 000 km

(b) $\mathrm{P}(X < 48\,000) = 15.87\%$

You would expect 15.87% of the tyres to last less than 48 000 km

(c) $\mathrm{P}(48\,000 < X < 56\,000) = 68.27\%$

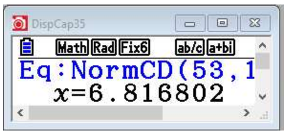
You would expect 68.27% to last between 48 000 km and 56 000 km

(d) $\mathrm{Q}_1 = 49302; \mathrm{Q}_3 = 54\,698 \Rightarrow \mathrm{IQR} = 5396 \text{ km}$

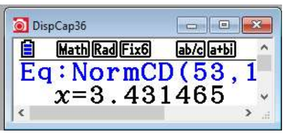
(e) So, the minimum life the company should guarantee is 43 785 km.

22. **(a)** We use GDC solver to find the standard deviation.
$\sigma = 6.817$

(b) Again, use a GDC solver.
$\sigma = 3.431$



```
Eq:NormCD(53,1
  x=6.816802
```



```
Eq:NormCD(53,1
  x=3.431465
```

(c) We need to solve a system of equations.

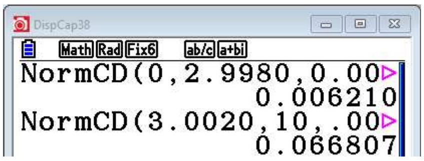
$$\left\{\begin{array}{l} \dfrac{53-\mu}{\sigma} = -1.4758 \Rightarrow 53-\mu = -1.4758\sigma \\ \dfrac{73-\mu}{\sigma} = 1.175 \Rightarrow 73-\mu = 1.175\sigma \end{array}\right\}$$

$$\Rightarrow 20 = 2.6508\sigma \Rightarrow \sigma \approx 7.545; \mu \approx 64.135$$

So, the mean value is 64.135 and the standard deviation is 7.545

23. $X \sim \mathrm{N}\left(3.0005,\ 0.0010^2\right)$

We need to find $\mathrm{P}\left(X < 2.9980\right)$ and $\mathrm{P}\left(X > 3.0020\right)$



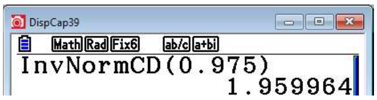
```
NormCD(0,2.9980,0.00▷
              0.006210
NormCD(3.0020,10,.00▷
              0.066807
```

$\mathrm{P}\left(X < 2.9980 \text{ or } X > 3.0020\right) = 0.006210 + 0.066807 = 0.073017 \approx 7.3\%$

24. **(a)** Since both parts will need solving for an unknown, we will use the standard normal distribution.

$$\mathrm{P}\left(X > 237\right) = 0.01 \Rightarrow z = \frac{237-\mu}{9} = 2.3263 \Rightarrow \mu = 216.06$$

(b) We need to find the z value that leaves 2.5% in each tail of the distribution.



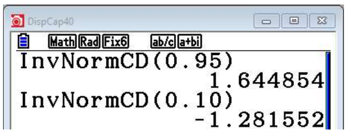
```
InvNormCD(0.975)
            1.959964
```

$$\mathrm{P}\left(\left|x-\mu\right| < 30\right) = 0.95 \Rightarrow \mathrm{P}\left(x-\mu < 30\right) = 0.975$$

$$z = \frac{30}{\sigma} = 1.96 \Rightarrow \sigma = 15.306$$

25. **(a)** As before, we need to solve a system of equations.

$$\begin{cases} \dfrac{140-\mu}{\sigma} = 1.6449 \Rightarrow 140-\mu = 1.6449\sigma \\ \dfrac{90-\mu}{\sigma} = -1.2816 \Rightarrow 90-\mu = -1.2816\sigma \end{cases}$$

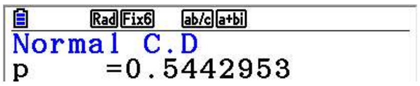


```
InvNormCD(0.95)
             1.644854
InvNormCD(0.10)
            -1.281552
```

$$\Rightarrow 50 = 2.9265\sigma \Rightarrow \sigma \approx 17.085; \mu \approx 111.897$$

(b) We need to find $\mathrm{P}(X > 110)$ given that $X \sim \mathrm{N}(111.897, 17.085^2)$

There about 54% of cars at speeds exceeding 110 km h^{-1}.



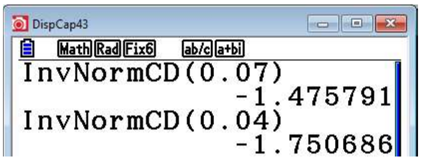
```
Normal C.D
p      =0.5442953
```

26. We solve a system of equations.

$$\begin{cases} \dfrac{10-\mu}{\sigma} = 0.4399 \Rightarrow 10-\mu = 0.4399\sigma \\ \dfrac{12-\mu}{\sigma} = 1.5301 \Rightarrow 12-\mu = 1.5301\sigma \end{cases}$$

$$\Rightarrow 2 = 1.0902\sigma \Rightarrow \sigma \approx 1.8345; \mu \approx 9.1930$$

27. **(a)** **(i)** Even though we can use GDC solver, we will instead use the standard normal distribution.

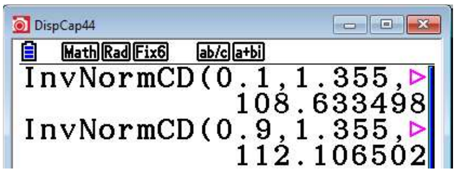


```
InvNormCD(0.07)
             -1.475791
InvNormCD(0.04)
             -1.750686
```

$$\frac{108-110}{\sigma}=-1.475791 \Rightarrow \sigma=\frac{2}{1.475791}=1.355$$

(ii) $$\frac{108-\mu}{1.355}=-1.750686 \Rightarrow 108-\mu=-2.372179 \Rightarrow \mu=110.37$$

(b) This is an inverse normal calculation. *A* leaves 10% below it and *B* leaves 90%.



```
InvNormCD(0.1,1.355,▷
             108.633498
InvNormCD(0.9,1.355,▷
             112.106502
```

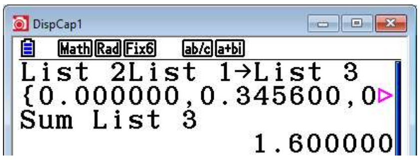
$A = 108.63$; $B = 112.11$

Exercise 15.5

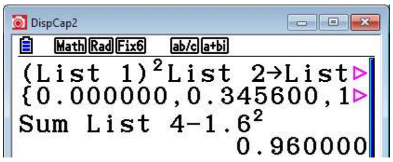
1. **(a)** This is just adding entries from the table:

$\mathrm{P}(X \geq 2)=0.5248$; $\mathrm{P}(1 \leq X \leq 3)=0.8448$

(b) $$\mathrm{E}(X)=\sum x\mathrm{P}(x)=0\times 0.1296+\cdots+4\times 0.0256=1.6$$



```
List 2List 1→List 3
{0.000000,0.345600,0▷
Sum List 3
             1.600000
```

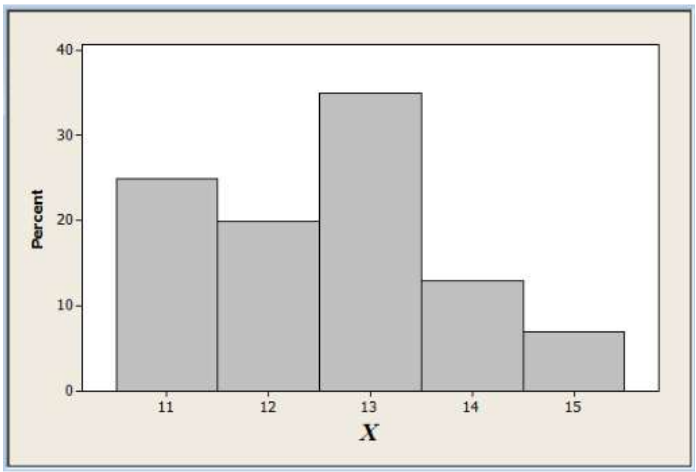


```
(List 1)²List 2→List▷
{0.000000,0.345600,1▷
Sum List 4-1.6²
             0.960000
```

$$\mathrm{V}(X)=\sum x^2\mathrm{P}(x)-(\mathrm{E}(X))^2=0.96$$

(c) $$\mathrm{E}(Y)=9-2\mathrm{E}(X)=5.8;\ \mathrm{V}(Y)=4\mathrm{V}(X)=3.84$$

2. **(a)** $k = 1 - \sum p(x) = 0.13$



(b) Adding entries from the table:

$$\mathrm{P}(12 < X \le 14) = \mathrm{P}(X = 13) + \mathrm{P}(X = 14) = 0.48$$

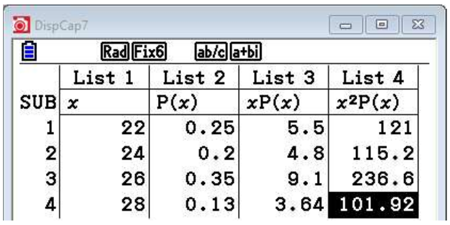
$$\mathrm{P}(X \ge 14) = 0.13 + 0.07 = 0.20$$

(c) $\mathrm{E}(X) = \sum x\mathrm{P}(x) = 11 \times 0.25 + \cdots + 15 \times 0.07 = 12.57$

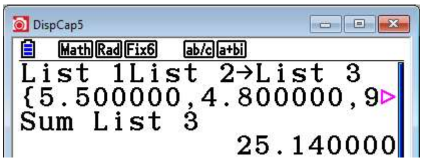
$$\mathrm{V}(X) = \sum x^2\mathrm{P}(x) - (\mathrm{E}(X))^2 = 11^2 \times 0.25 + \cdots + 15^2 \times 0.07 - 12.57^2 = 1.4251$$

(d) **(i)** $\mathrm{E}(Y) = 2\mathrm{E}(X) = 25.14\,;\, \mathrm{V}(Y) = 4\mathrm{V}(X) = 5.7004$

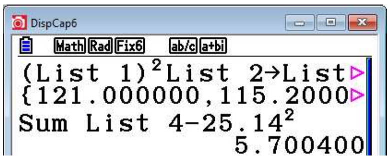
(ii) See GDC output below.



```
      List 1   List 2   List 3   List 4
SUB   x        P(x)     xP(x)    x²P(x)
  1       22     0.25      5.5      121
  2       24      0.2      4.8    115.2
  3       26     0.35      9.1    236.6
  4       28     0.13     3.64   101.92
```



```
List 1List 2→List 3
{5.500000,4.800000,9▷
Sum List 3
                25.140000
```



```
(List 1)²List 2→List▷
{121.000000,115.2000▷
Sum List 4-25.14²
                 5.700400
```

(e) **(i)** $\mathrm{E}(Z)=2\mathrm{E}(X)=25.14;\ \mathrm{V}(Z)=2\mathrm{V}(X)=2.8502$

(ii) This can sensibly be done using a spreadsheet.

x_1	x_2	$\mathrm{p}(x_1)$	$\mathrm{p}(x_2)$	Z	$\mathrm{P}(Z)$	$Z\mathrm{P}(Z)$	$Z^2\mathrm{P}(Z)$
11	11	0.25	0.25	22	0.0625	1.375	30.25
11	12	0.25	0.2	23	0.05	1.15	26.45
11	13	0.25	0.35	24	0.0875	2.1	50.4
11	14	0.25	0.13	25	0.0325	0.8125	20.3125
11	15	0.25	0.07	26	0.0175	0.455	11.83
12	11	0.2	0.25	23	0.05	1.15	26.45
12	12	0.2	0.2	24	0.04	0.96	23.04
12	13	0.2	0.35	25	0.07	1.75	43.75
12	14	0.2	0.13	26	0.026	0.676	17.576
12	15	0.2	0.07	27	0.014	0.378	10.206
13	11	0.35	0.25	24	0.0875	2.1	50.4
13	12	0.35	0.2	25	0.07	1.75	43.75
13	13	0.35	0.35	26	0.1225	3.185	82.81
13	14	0.35	0.13	27	0.0455	1.2285	33.1695
13	15	0.35	0.07	28	0.0245	0.686	19.208
14	11	0.13	0.25	25	0.0325	0.8125	20.3125
14	12	0.13	0.2	26	0.026	0.676	17.576
14	13	0.13	0.35	27	0.0455	1.2285	33.1695
14	14	0.13	0.13	28	0.0169	0.4732	13.2496
14	15	0.13	0.07	29	0.0091	0.2639	7.6531
15	11	0.07	0.25	26	0.0175	0.455	11.83
15	12	0.07	0.2	27	0.014	0.378	10.206
15	13	0.07	0.35	28	0.0245	0.686	19.208
15	14	0.07	0.13	29	0.0091	0.2639	7.6531
15	15	0.07	0.07	30	0.0049	0.147	4.41
						25.14	**2.8502**

The last two entries give the following:

$$\mu_Z=\sum Z\mathrm{P}(Z)=25.14;\sigma_Z^2=\sum Z^2\mathrm{P}(Z)-\mu_Z^2=2.8502$$

You will not be asked to perform such calculations on exams. The exercise is meant to help you demonstrate the theory mentioned in the chapter.

3. **(a)**

X	P(X)	Y	P(Y)
1	0.166667	1	0.25
2	0.166667	2	0.25
3	0.166667	3	0.25
4	0.166667	4	0.25
5	0.166667		
6	0.166667		

(b)

$$\mathrm{E}(X)=\sum x\mathrm{P}(x)=1\times 0.166667+\cdots+6\times 0.166667=3.5$$

$$\mathrm{V}(X)=\sum x^2\mathrm{P}(x)-35^2=1^2\times 0.16667+\cdots+36\times 0.16667-3.5^2=2.917$$

$$\mathrm{E}(Y)=\sum y\mathrm{P}(y)=1\times 0.25+\cdots+4\times 0.25=2.5$$

$$\mathrm{V}(Y)=\sum y^2\mathrm{P}(y)-2.5^2=1^2\times 0.25+\cdots+16\times 0.25-2.5^2=1.25$$

(c) Let $Z = X + Y$. Then the possible values of Z are 2, 3, …, 9. $Z = 2$ happens when both dice land on 1. $\mathrm{P}(Z=2)=\mathrm{P}(X=1)\times\mathrm{P}(Y=1)=0.166667\times 0.25=0.041667$

$Z = 3$ happens when $X = 1$, $Y = 2$, or when $X = 2$, and $Y = 1$.
$\mathrm{P}(Z=3)=0.166667\times 0.25+0.166667\times 0.25=0.083333$
The table shows all the possibilities.

Z	P(Z)
2	0.041667
3	0.083333
4	0.125000
5	0.166667
6	0.166667
7	0.166667
8	0.125000
9	0.083333

(d) **(i)**

$$\mathrm{E}(Z)=\sum z\mathrm{P}(z)=2\times 0.166667+\cdots+9\times 0.083333=6$$

$$\mathrm{V}(Z)=\sum z^2\mathrm{P}(z)=2^2\times 0.16667+\cdots+81\times 0.083333-6^2=4.167$$

(ii)

$$\mathrm{E}(Z)=\mathrm{E}(X)+\mathrm{E}(Y)=3.5+2.5=6$$

$$\mathrm{V}(Z)=\mathrm{V}(X)+\mathrm{V}(Y)=2.917+1.25=4.167$$

4. Using the results found in the section, $\mathrm{E}\left(\bar{X}\right)=\mu$ and $\mathrm{Var}\left(\bar{X}\right)=\frac{\sigma^2}{n}$, we first find

$$\mu=\sum x\mathrm{P}(x)=1\times\frac{1}{6}+\cdots+6\times\frac{1}{6}=3.5,\text{ and}$$

$$\sigma^2=\sum x^2\mathrm{P}(x)=1^2\times\frac{1}{6}+\cdots+6^2\times\frac{1}{6}-3.5^2=2.917$$

$$\mathrm{E}\left(\bar{X}\right)=\mu=3.5\text{ and }\mathrm{Var}\left(\bar{X}\right)=\frac{\sigma^2}{n}=\frac{2.917}{36}\Rightarrow\sqrt{\mathrm{Var}\left(\bar{X}\right)}=\sqrt{\frac{2.917}{36}}=0.2847$$

5. **(a)** **(i)** $\sum\mathrm{P}(m)=1\Rightarrow 10k^2+139k-14=0\Rightarrow k=0.1$

We ignore the negative value of k because it will result in a negative probability.

(ii) The updated pdf is:

m	1	2	3	4	5
P(*m*)	0.01	0.48	0.06	0.20	0.25

$$\mathrm{E}(M)=1\times0.01+2\times0.48+\cdots+5\times0.25=3.2$$

(iii) $\mathrm{V}(M)=1^2\times0.01+2^2\times0.48+\cdots+5^2\times0.25-3.2^2=1.68$

(b) **(i)** $\mathrm{E}(N)=2\mathrm{E}(M)+3\mathrm{E}(M)=5\mathrm{E}(M)=16$

(ii) $\mathrm{Var}(N)=4\mathrm{Var}(m)+9\mathrm{Var}(M)=21.84$

6. **(a)** $\mathrm{E}(X+Y)=\mathrm{E}(X)+\mathrm{E}(Y)=10,\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)=3$

(b) $\mathrm{E}(X-Y)=\mathrm{E}(X)-\mathrm{E}(Y)=-4,\mathrm{Var}(X-Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)=3$

(c) $\mathrm{E}(2X+3Y)=2\mathrm{E}(X)+3\mathrm{E}(Y)=27,\mathrm{Var}(2X+3Y)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=17$

(d) $\mathrm{E}(2X-3Y)=2\mathrm{E}(X)-3\mathrm{E}(Y)=-15,\ \mathrm{Var}(2X-3Y)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=17$

7. We first need to find E(X) and E(Y). Recall the formula for the variance:

$$\mathrm{Var}(X)=\sum x^2\mathrm{P}(x)-\left(\mathrm{E}(X)\right)^2=\mathrm{E}\left(X^2\right)-\left(\mathrm{E}(X)\right)^2$$

$$\Rightarrow\mathrm{E}(X)=\sqrt{\mathrm{E}\left(X^2\right)-\mathrm{Var}(X)}$$

Thus, $\mathrm{E}\left(X\right)=\sqrt{9-2}=\sqrt{7}$, and $\mathrm{E}\left(Y\right)=\sqrt{16-3}=\sqrt{13}$

Note: The chance of error here is mixing up what the statement of the problem says. X and Y are independent. This does not mean that X and X or Y and Y are independent. $\mathrm{E}\left(X^2\right)\neq\mathrm{E}\left(X\right)\mathrm{E}\left(X\right)$

(a) $\mathrm{E}(X+Y)=\mathrm{E}(X)+\mathrm{E}(Y)=\sqrt{7}+\sqrt{13},\ \mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)=5$

(b) $\mathrm{E}(X-Y)=\mathrm{E}(X)-\mathrm{E}(Y)=\sqrt{7}-\sqrt{13},\ \mathrm{Var}(X-Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)=5$

(c) $\mathrm{E}(2X+3Y)=2\mathrm{E}(X)+3\mathrm{E}(Y)=2\sqrt{7}+3\sqrt{13},$
$\mathrm{Var}(2X+3Y)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=35$

(d) $\mathrm{E}(2X-3Y)=2\mathrm{E}(X)-3\mathrm{E}(Y)=2\sqrt{7}-3\sqrt{13},$
$\mathrm{Var}(2X-3Y)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=35$

8. $\mathrm{E}(X)=\sqrt{12-5}=\sqrt{7}\,,\mathrm{E}(Y)=\sqrt{6-2}=2$

(a) $\mathrm{E}(2X+Y)=2\mathrm{E}(X)+\mathrm{E}(Y)=2\sqrt{7}+2, \mathrm{Var}(2X+Y)=4\mathrm{Var}(X)+\mathrm{Var}(Y)=22$

(b) $\mathrm{E}(X-3Y)=\mathrm{E}(X)-3\mathrm{E}(Y)=\sqrt{7}-6, \mathrm{Var}(X-3Y)=\mathrm{Var}(X)+9\mathrm{Var}(Y)=23$

(c) $\mathrm{E}(2X+3Y)=2\mathrm{E}(X)+3\mathrm{E}(Y)=2\sqrt{7}+6,$

$\mathrm{Var}(2X+3Y)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=38$

(d) $\mathrm{E}(2X-3Y)=2\mathrm{E}(X)-3\mathrm{E}(Y)=2\sqrt{7}-6,$

$\mathrm{Var}(2X-3Y)=4\mathrm{Var}(X)+9\mathrm{Var}(Y)=38$

9. **(a)** $\mathrm{E}(X)=\sum x\mathrm{P}(x)=1.05\times0.6+0.95\times0.4=1.01$

$\mathrm{Var}(X)=\sum x^2\mathrm{P}(x)-\left(\mathrm{E}(X)\right)^2=1.05^2\times0.6+0.95^2\times0.4-1.01^2=0.0024$

(b) Possible lengths: 1.05 + 1.05, 1.05 + 0.95, or 0.95 + 0.95

x	2.1	2	1.9
P(*x*)	0.36	0.48	0.16

$\mathrm{E}(X)=\sum x\mathrm{P}(x)=2.1\times0.36+2\times0.48+1.9\times0.16=2.02,$

$\mathrm{Var}(X)=\sum x^2\mathrm{P}(x)-\left(\mathrm{E}(X)\right)^2$

$=2.1^2\times0.36+2^2\times0.48+1.9^2\times0.16-2.02^2=0.0048$

(c) Possible lengths: 1.05 + 1.05 + 1.05 = 3.15, …

l	2.85	2.95	3.05	3.15
***P*(*l*)**	0.064	0.288	0.432	0.216

$\mathrm{E}(X)=\sum x\mathrm{P}(x)=2.85\times0.064+\cdots+3.15\times0.216=3.03,$

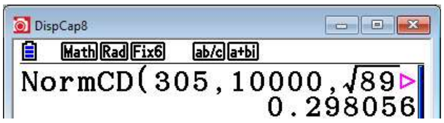
$\mathrm{Var}(X)=\sum x^2\mathrm{P}(x)-\left(\mathrm{E}(X)\right)^2$

$=2.85^2\times0.064+\cdots+3.15^2\times0.216-3.03^2=0.0072$

10. **(a)** The amount of juice is the sum of two normally distributed variables.
Let *J* be the amount of juice dispensed, and so

$J\sim\mathrm{N}\left(\mu=40+260,\sigma^2=5^2+8^2\right)=\mathrm{N}(300,89)$

$\mathrm{P}(J>305)=0.298$

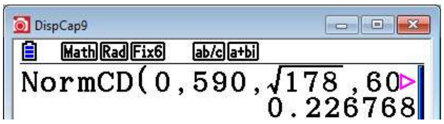


```
NormCD(305,10000,√89▷
              0.298056
```

(b) The amount of a double, D, is the sum of two J variables.

$$D \sim \mathrm{N}(600, 2\times 89) = \mathrm{N}(600, 178)$$

$$\mathrm{P}(D < 590) = 0.227$$

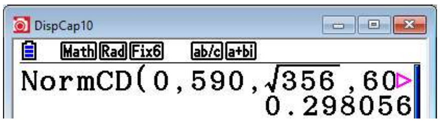


```
NormCD(0,590,√178,60▷
                0.226768
```

(c) The amount of a double, D, is twice the J variable.

$$D \sim \mathrm{N}(600, 4\times 89) = \mathrm{N}(600, 356)$$

$$\mathrm{P}(D < 590) = 0.298$$

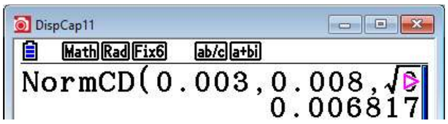


```
NormCD(0,590,√356,60▷
                0.298056
```

11. The difference between the diameters is a difference between two normally distributed variables.

$$D \sim \mathrm{N}\left(0.9-0.8, 0.05^2+0.006^2\right) = \mathrm{N}(0.1, 0.002536)$$

$$\mathrm{P}(0.003 < D < 0.008) = 0.007$$



```
NormCD(0.003,0.008,√▷
                0.006817
```

Chapter 15 practice questions

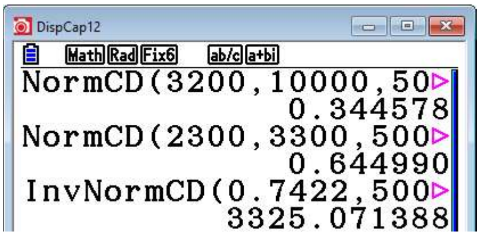
1. Let X be the amount of savings ($), where $X \sim \mathrm{N}\left(3000, 500^2\right)$

(a) $\mathrm{P}(X > 3200) = 0.3446 \Rightarrow 34.5\%$

(b) First, we calculate $\mathrm{P}(2300 < X < 3300) = 0.6449$

For two people $0.6449^2 \approx 0.416$

(c) $\mathrm{P}(X < d) = 0.7422 \Rightarrow d$ is the normal inverse of 0.7422, thus $d = 3325$

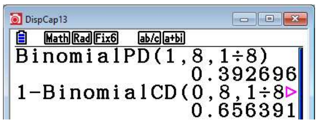


```
NormCD(3200,10000,50▷
                0.344578
NormCD(2300,3300,500▷
                0.644990
InvNormCD(0.7422,500▷
             3325.071388
```

2. Let X represent the number of black discs, where $X \sim \mathrm{B}\left(8, \frac{5}{40} = \frac{1}{8}\right)$

(a) **(i)** $\mathrm{P}(X = 1) = 0.393$

(ii) $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 0.656$



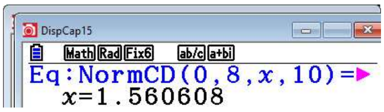
```
BinomialPD(1,8,1÷8)
                0.392696
1-BinomialCD(0,8,1÷8▷
                0.656391
```

(b) Now we change the number of trials to 400. Hence, the expected number of black discs that would be drawn is $np = 400 \times \frac{1}{8} = 50$

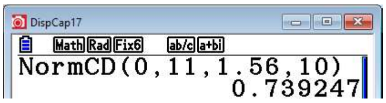
3. **(a)** The area of the shaded region is 0.1

(b) Since the areas are the same, $\mathrm{P}(X < 8) = \mathrm{P}(X > 12) = 0.1$, we can find the mean value as the average of the two numbers: $\mu = \frac{8+12}{2} = 10$. This can also be done along with part **(c)**, by solving a system of equations as you have seen before.

(c) To find the standard deviation we can use GDC solver as seen below.



```
Eq:NormCD(0,8,x,10)=▶
  x=1.560608
```



```
NormCD(0,11,1.56,10)
              0.739247
```

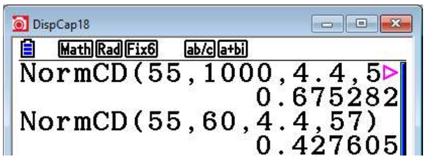
(d) $\mathrm{P}(X < 11) = 0.739$ (see screen above-right)

4. **(a)** a and b are the standard normal values corresponding to 55 and 60, thus,

$$a = \frac{55-57}{4.4} = -0.4545; b = \frac{60-57}{4.4} = 0.682$$

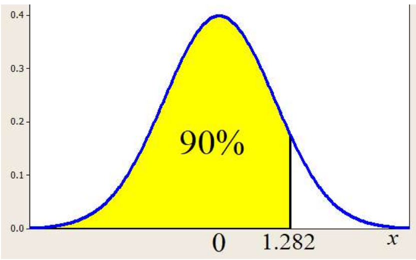
(b) **(i)** $\mathrm{P}(X > 55) = 0.675$

(ii) $\mathrm{P}(55 < X < 60) = 0.428$



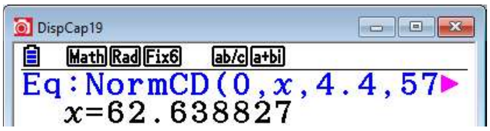
```
NormCD(55,1000,4.4,5▷
              0.675282
NormCD(55,60,4.4,57)
              0.427605
```

(c)



(d) If 90% of the insects die after *t*, then their life span must be less than *t*. We can find *t* by going from the standard value of 1.282 (found with inverse normal calculation) which we have in **(c)** to the raw data, or by using GDC solver.

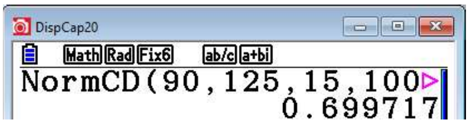
$\frac{t-57}{4.4} = 1.282 \Rightarrow t = 57 + 4.4 \times 1.282 = 62.64$, or



```
Eq:NormCD(0,x,4.4,57▶
  x=62.638827
```

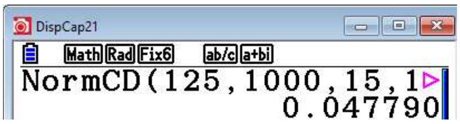
5. **(a)** Normal probability reading from GDC

$\mathrm{P}(90 < X < 125) = 0.6997 \Rightarrow 69.97\%$



```
NormCD(90,125,15,100▷
            0.699717
```

(b) First, we find the probability that one person has an IQ larger than 125.

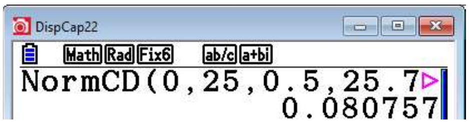


```
NormCD(125,1000,15,1▷
            0.047790
```

$\mathrm{P}(\text{both} > 125) = 0.04779^2 = 0.00228$

6. **(a)** Normal probability reading from GDC

$\mathrm{P}(X < 25) = 0.0808$



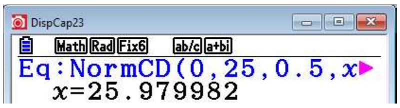
```
NormCD(0,25,0.5,25.7▷
            0.080757
```

(b) This can be done by using a GDC solver, or by using standard normal calculation, which we will demonstrate here for the last time.

The lower tail of the standard normal distribution corresponding to 2.5% is −1.96,

thus, $-1.96 = \frac{25-\mu}{0.5} \Rightarrow \mu = 25 + 0.5 \times 1.96 = 25.98 \approx 26$

Here is what a GDC solver gives:



```
Eq:NormCD(0,25,0.5,x▶
  x=25.979982
```

(c) We need to solve a system of equations as we have done earlier in the chapter. 1.96 and –1.96 are the standard normal values for the upper and lower 2.5%, thus:

$$\left\{\begin{array}{l} \dfrac{25-\mu}{\sigma}=-1.96 \Rightarrow 25-\mu=-1.96\sigma \\ \dfrac{26-\mu}{\sigma}=1.96 \Rightarrow 26-\mu=1.96\sigma \end{array}\right\}$$

$$\Rightarrow 51-2\mu=0 \Rightarrow \mu=25.5; \sigma \approx 0.255$$

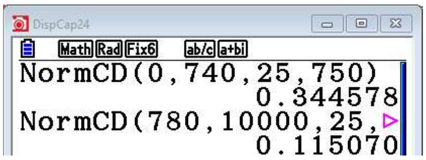
(d) With a mean of 25.5, the new machine will be saving 0.5 kg per bag. The cost saved per bag is then $0.80 \times 0.5 = \$0.40$. Thus, the number of bags needed to recover the \$5000 is $\dfrac{5000}{0.4}=12\,500$

7. Let X represent the mass of the packets, where $X \sim \mathrm{N}\left(750, 25^2\right)$

(a) **(i)** $\mathrm{P}(X<740)=0.345$

(ii) $\mathrm{P}(X>780)=0.115$

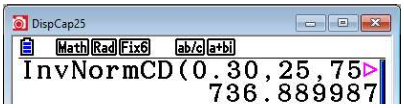
(iii) $\mathrm{P}(740<X<780)=1-0.345-0.115=0.540$



```
NormCD(0,740,25,750)
              0.344578
NormCD(780,10000,25,▷
              0.115070
```

(b) $\mathrm{P}(\text{both}<740)=0.345^2=0.119$

(c) We need the packet where 30% of the masses are less than its weight. This is an inverse normal calculation. $X = 737$

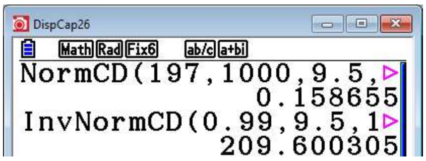


```
InvNormCD(0.30,25,75▷
            736.889987
```

8. Let X represent the height of adults in Tallopia, where $X \sim \mathrm{N}\left(187.5, 9.5^2\right)$

(a) $\mathrm{P}(X>197)=0.159 \Rightarrow 15.9\%$

(b) We first find the 99th percentile for the heights. This is an inverse normal distribution calculation. $X = 209.60 \approx 210$ cm. So the height of a standard doorway is 227 cm.



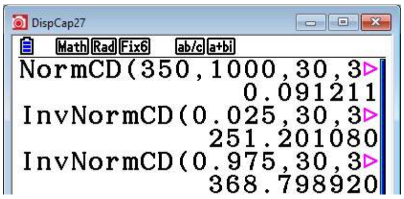
```
NormCD(197,1000,9.5,▷
              0.158655
InvNormCD(0.99,9.5,1▷
            209.600305
```

9. Let X represent the mass of a lion, where $X \sim \mathrm{N}\left(310, 30^2\right)$

(a) $\mathrm{P}(X > 350) = 0.0912$

(b) Since a and b are symmetrical with respect to the mean, we can write: $\mathrm{P}(a < X < b) = 0.95$. This is an inverse normal calculation where a is the 2.5th percentile and b is the 97.5th percentile.

$A = 251.2$, $b = 368.8$



```
NormCD(350,1000,30,3▷
                0.091211
InvNormCD(0.025,30,3▷
              251.201080
InvNormCD(0.975,30,3▷
              368.798920
```

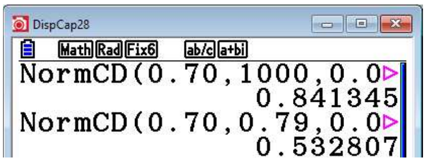
10. Let X represent the reaction time measured in seconds, where $X \sim \mathrm{N}\left(0.76, 0.06^2\right)$

(a) a and b are the standard values for 0.70 and 0.79

$$a = \frac{0.70 - 0.76}{0.06} = -1 \text{, and } b = \frac{0.79 - 0.76}{0.06} = 0.5$$

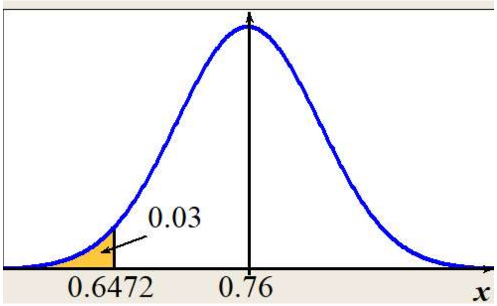
(b) **(i)** $\mathrm{P}(X > 0.70) = 0.841$

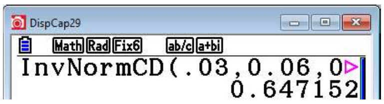
(ii) $\mathrm{P}(0.70 < X < 0.79) = 0.533$



```
NormCD(0.70,1000,0.0▷
                0.841345
NormCD(0.70,0.79,0.0▷
                0.532807
```

(c) **(i) (ii)** This is an inverse normal calculation.



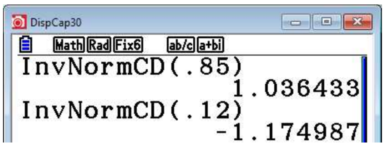


```
InvNormCD(.03,0.06,0▷
                0.647152
```

11. As before, we need to set up a system of two equations with two unknowns.

$$\begin{cases} \dfrac{90-\mu}{\sigma}=1.036 \Rightarrow 90-\mu=1.036\sigma \\ \dfrac{40-\mu}{\sigma}=-1.175 \Rightarrow 40-\mu=-1.175\sigma \end{cases}$$

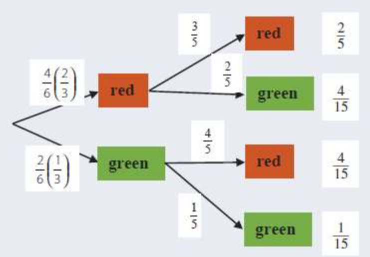
$$\Rightarrow 50=2.211\sigma \Rightarrow \sigma \approx 22.61; \mu=66.58$$



```
InvNormCD(.85)
             1.036433
InvNormCD(.12)
            -1.174987
```

12. **(a)** $\mathrm{E}(X)=\sum x\mathrm{P}(x)=0\times\dfrac{3}{10}+1\times\dfrac{6}{10}+2\times\dfrac{1}{10}=\dfrac{4}{5}$

(b) **(i)**



(ii)

Y	0	1	2
$\mathrm{P}(Y=y)$	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{2}{5}$

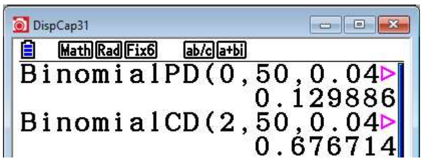
(c) $\mathrm{P}(RR)=\underset{1\text{ or }6}{\dfrac{2}{6}}\times\underbrace{\dfrac{1}{10}}_{Bag\ A}+\underset{2,3,4,or6}{\dfrac{4}{6}}\times\underbrace{\dfrac{1}{10}}_{Bag\ B}=\dfrac{3}{10}$

(d) $\mathrm{P}(A|RR)=\dfrac{\mathrm{P}(A\cap RR)}{\mathrm{P}(RR)}=\dfrac{\dfrac{2}{6}\times\dfrac{1}{10}}{\dfrac{3}{10}}=\dfrac{1}{9}$

13. Let X represent the number of defective ball bearings, where $X\sim\mathrm{B}(50,0.04)$

(a) $\mathrm{P}(X=0)=0.129886$

(b) $\mathrm{P}(X\le 2)=0.676714$



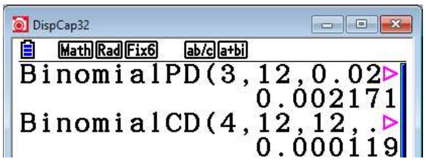
```
BinomialPD(0,50,0.04▷
             0.129886
BinomialCD(2,50,0.04▷
             0.676714
```

(c) $\mathrm{E}(X)=np=50\times0.04=2$

14. **(a)** Let X represent the number of small tomatoes, where $X \sim \mathrm{B}(12, 0.023)$

(i) $\mathrm{P}(X = 3) = 0.002171$

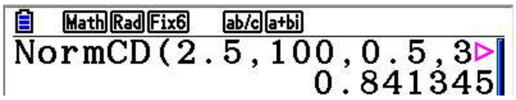
(ii) $\mathrm{P}(X \geq 4) = 0.0.00120$



```
BinomialPD(3,12,0.02▷
             0.002171
BinomialCD(4,12,12,.▷
             0.000119
```

(b) Let Y represent the size of a tomato, where $Y \sim \mathrm{N}(3, 0.5^2)$

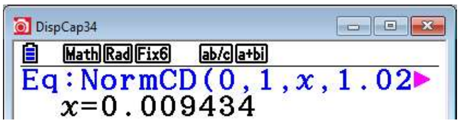
$\mathrm{P}(Y > 2.5) = 0.8413$



```
NormCD(2.5,100,0.5,3▷
             0.841345
```

15. This can be done by using GDC solver.

$\sigma \approx 0.00943\,\text{kg} \approx 9.4\text{ g}$



```
Eq:NormCD(0,1,x,1.02▶
  x=0.009434
```

16. **(a)** $\int_0^1 \left(\mathrm{e} - k\mathrm{e}^{kx}\right)\mathrm{d}x = 1 \Rightarrow \left(\mathrm{e}x - \mathrm{e}^{kx}\right)\Big|_0^1 = \mathrm{e} - \mathrm{e}^k + 1 = 1 \Rightarrow \mathrm{e} = \mathrm{e}^k \Rightarrow k = 1$

(b) $\int_{\frac{1}{4}}^{\frac{1}{2}} \left(\mathrm{e} - \mathrm{e}^x\right)\mathrm{d}x = \left(\mathrm{e}x - \mathrm{e}^x\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}} = \frac{\mathrm{e}}{2} - \sqrt{\mathrm{e}} - \frac{\mathrm{e}}{4} + \sqrt[4]{\mathrm{e}} = \frac{\mathrm{e}}{4} - \sqrt{\mathrm{e}} + \sqrt[4]{\mathrm{e}}$

(c) $\mu = \int_0^1 xf(x)\mathrm{d}x = \int_0^1 \left(\mathrm{e}x - x\mathrm{e}^x\right)\mathrm{d}x$

Use integration by parts to evaluate the second integral

$\int_0^1 x\mathrm{e}^x\mathrm{d}x = \left(x\mathrm{e}^x - \mathrm{e}^x\right)\Big|_0^1 = 1 \Rightarrow \mu = \int_0^1 \left(\mathrm{e}x - x\mathrm{e}^x\right)\mathrm{d}x = \int_0^1 \mathrm{e}x\mathrm{d}x - 1 = \frac{\mathrm{e}}{2} - 1$

$\sigma^2 = \int_{all\ x} x^2 f(x)\mathrm{d}x - \mu^2 = \int_0^1 \left(\mathrm{e}x^2 - x^2\mathrm{e}^x\right)\mathrm{d}x - \left(\frac{\mathrm{e}}{2} - 1\right)^2$

Use integration by parts to evaluate $\int_0^1 x^2\mathrm{e}^x\mathrm{d}x$

$\int_0^1 x^2\mathrm{e}^x\mathrm{d}x = \left(x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x\right)\Big|_0^1 = \mathrm{e} - 2$

$$\Rightarrow \sigma^2 = \int_0^1 \left(\mathrm{e}x^2 - x^2\mathrm{e}^x\right)\mathrm{d}x - \left(\frac{\mathrm{e}}{2} - 1\right)^2 = \frac{\mathrm{e}}{3} - \mathrm{e} + 2 - \left(\frac{\mathrm{e}}{2} - 1\right)^2 = 1 + \frac{\mathrm{e}}{3} - \frac{\mathrm{e}^2}{4}$$

(d) $$\mathrm{P}(X>0.5)=1-\mathrm{P}(X<0.5)=1-\int_0^{\frac{1}{2}}\left(\mathrm{e}-\mathrm{e}^x\right)\mathrm{d}x$$

$$=1-\left(\mathrm{e}x-\mathrm{e}^x\right)\Big|_0^{\frac{1}{2}}=1-\frac{\mathrm{e}}{2}+\sqrt{\mathrm{e}}-1=\sqrt{\mathrm{e}}-\frac{\mathrm{e}}{2}$$

(e) **(i)** $$\mathrm{P}(XXX)=\left(\sqrt{\mathrm{e}}-\frac{\mathrm{e}}{2}\right)^3$$

(ii) This is the event where 2 did not fail and one failed

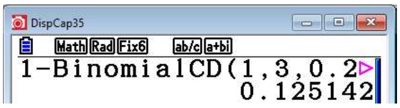
and can happen in ${}_3C_2$ ways. Thus:

$$\mathrm{P}(1\text{ fails})={}_3C_2\left(\sqrt{\mathrm{e}}-\frac{\mathrm{e}}{2}\right)^2\left(1-\sqrt{\mathrm{e}}+\frac{\mathrm{e}}{2}\right)=3\left(\sqrt{\mathrm{e}}-\frac{\mathrm{e}}{2}\right)^2\left(1-\sqrt{\mathrm{e}}+\frac{\mathrm{e}}{2}\right)$$

17. **(a)** $$\mathrm{P}(Y<0.5)=\int_0^{0.5}0.5\mathrm{e}^{-\frac{y}{2}}\mathrm{d}y=0.5\left(-2\mathrm{e}^{-\frac{y}{2}}\right)\Big|_0^{0.5}=1-\mathrm{e}^{-0.25}\approx 0.2212$$

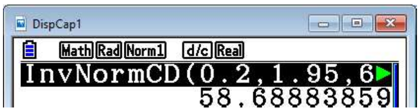
(b) At least 2 components continue to work is the complement of at most one fails. The variable X that 1 fails is a binomial with $n = 3$ and $p = 0.2212$

$\mathrm{P}(\text{at least 2 work})=1-\mathrm{P}(X\le 1)=0.125$



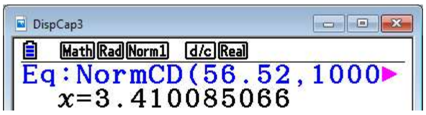
```
1-BinomialCD(1,3,0.2▷
            0.125142
```

18. **(a)** This is an inverse normal calculation. If 80% of the throws were longer than x, then 20% were shorter. $X = 58.69$



```
InvNormCD(0.2,1.95,6▶
          58.68883859
```

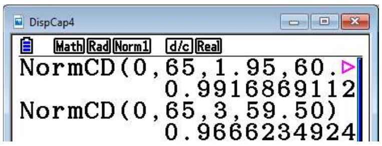
(b) We will use GDC solver for this. $\sigma\approx 3.41$



```
Eq:NormCD(56.52,1000▶
  x=3.410085066
```

(c) **(i)** For Ian to have a throw less than 65: $\mathrm{P}(X<65)\approx 0.9916869$

For Karl to have a throw less than 65: $\mathrm{P}(X<65)\approx 0.9666236$



```
NormCD(0,65,1.95,60.▷
         0.9916869112
NormCD(0,65,3,59.50)
         0.9666234924
```

For Ian to have at least one throw more than 65:

$$1-\left(\mathrm{P}(X,65)\right)^3=1-0.9916869^3=0.0247$$

For Karl to have at least one throw more than 65:

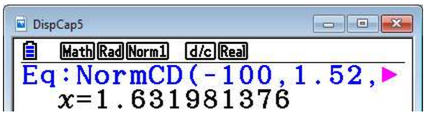
$$1-\left(\mathrm{P}(X,65)\right)^3=1-0.9666236^3=0.0968$$

Thus, Karl has better chances.

(ii) If both qualify, the probability is the product of each qualifying:

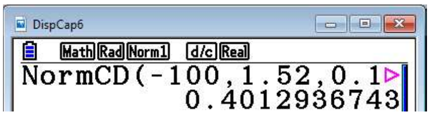
$$\mathrm{P}(\text{both qualify}) = 0.0247 \times 0.0968 = 0.00239$$

19. **(a)** We will use GDC solver for this: $\mu \approx 1.632$



```
Eq:NormCD(-100,1.52,▶
  x=1.631981376
```

(b) $\mathrm{P}(X < 1.52)$ from manufacturer A is 0.4013



```
NormCD(-100,1.52,0.1▷
            0.4012936743
```

$$\mathrm{P}(X < 1.52) = \mathrm{P}((X < 1.52) \cap A) + \mathrm{P}((X < 1.52) \cap B)$$
$$= 0.44 \times 0.4013 + 0.56 \times 0.242 = 0.312$$

(c) $$\mathrm{P}(B \mid X < 1.52) = \frac{\mathrm{P}((X < 1.52) \cap B)}{\mathrm{P}(X < 1.52)} = \frac{0.56 \times 0.242}{0.312} = 0.434$$

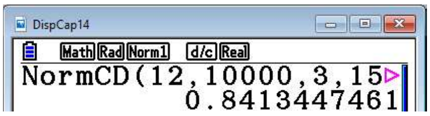
(d) The probabilities and numbers produced from each type are:

Diameter = *d*	$d < 1.52$	$1.52 < d < 1.83$	$d > 1.83$
P(*d*)	0.242	0.650	0.108
Number	1936	5200	834

The expected profit = $-0.85 \times 1936 + 1.50 \times 5200 + 0.50 \times 834 \approx 6600$

(Answers will differ with the number of decimal places used.)

20. **(a)** $\mathrm{P}(T > 12) = 0.841$

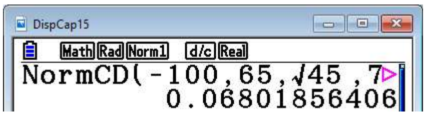


```
NormCD(12,10000,3,15▷
            0.8413447461
```

(b) **(i)** Let *T* be total time for 5 days, i.e., $T = T_1 + T_2 + \cdots$

Then $T \sim \mathrm{N}(5 \times 15, 5 \times 3^2)$

$$\mathrm{P}(T < 65) = 0.0680$$



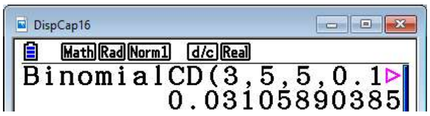
```
NormCD(-100,65,√45 ,7▷
           0.06801856406
```

(ii) We first use the original distribution $T \sim \mathrm{N}\left(15, 3^2\right)$

$\mathrm{P}(T < 12) = 0.1587$

To wait on at least 3 days is a binomial with $n = 5$ and $p = 0.1587$

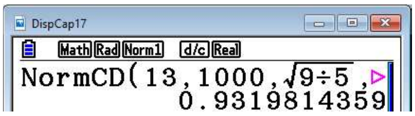
$\mathrm{P}(\text{wait on at least 3 days}) = 0.0311$



```
BinomialCD(3,5,5,0.1▷
           0.03105890385
```

(iii) Average waiting time is Normally distributed with mean of 15 minutes and a variance of $\frac{9}{5}$

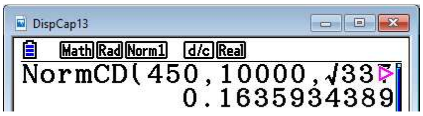
$\mathrm{P}(T > 13) = 0.932$



```
NormCD(13,1000,√9÷5 ,▷
            0.9319814359
```

21. $X \sim \mathrm{N}\left(6 \times 72, 6 \times 7.5^2\right)$

$\mathrm{P}(X > 450) = 0.164$



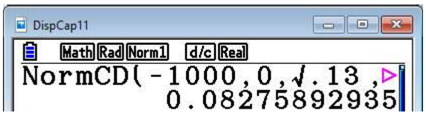
```
NormCD(450,10000,√33▷
            0.1635934389
```

22. **(a)** **(i)** $\mathrm{E}(2Y - X) = 2\mathrm{E}(Y) - \mathrm{E}(X) = 2 \times 2.5 - 4.5 = 0.5$

$\mathrm{Var}(2Y - X) = 4\mathrm{Var}(Y) + \mathrm{Var}X = 4 \times 0.15^2 + 0.2^2 = 0.13$

(ii) $(2Y - X) \sim \mathrm{N}(0.5, 0.13)$

$\mathrm{P}(2Y - X < 0) = 0.0828$

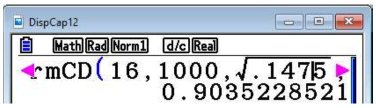


```
NormCD(-1000,0,√.13 ,▷
           0.08275892935
```

(b) Let the total weight be $T = X_1 + X_2 + Y_1 + Y_2 + Y_3$

Then $T \sim \mathrm{N}\left(2\times4.5+3\times2.5, 2\times0.2^2+3\times0.15^2\right)=\mathrm{N}(16.5, 0.1475)$

$\mathrm{P}(T>16)=0.904$



```
rmCD(16,1000,√.1475
                 0.9035228521
```

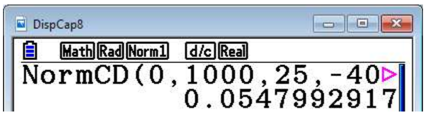
23. **(a)** Let X, Y (grams) denote respectively the weights of a randomly chosen apple, pear.

Then, we are interested in $\mathrm{P}(X>2Y)\Rightarrow\mathrm{P}(X-2Y>0)$

Thus, we are interested in the variable $X-2Y$.

$X-2Y\sim\mathrm{N}\left(200-2\times120, 15^2+2^2\times10^2\right)=\mathrm{N}(-40, 625)=\mathrm{N}\left(-40, 25^2\right)$

$\mathrm{P}(X-2Y>0)=0.0548$

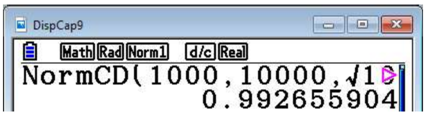


```
NormCD(0,1000,25,-40
              0.0547992917
```

(b) Let $T=X_1+X_2+X_3+Y_1+Y_2+Y_3+Y_4$ be the total weight of the apples and pears.

$T\sim\mathrm{N}\left(3\times200+4\times120, 3\times15^2+4\times10^2\right)=\mathrm{N}(1080, 1075)$

$\mathrm{P}(T>1000)\approx0.993$



```
NormCD(1000,10000,√1
               0.992655904
```

24. **(a)** **(i)** $\mathrm{E}(2Y+3)=2\mathrm{E}(Y)+3=6\Rightarrow\mathrm{E}(Y)=\dfrac{3}{2}$

(ii) $\mathrm{Var}(2-3Y)=9\mathrm{Var}(Y)=11\Rightarrow\mathrm{Var}(Y)=\dfrac{11}{9}$

(iii) $\mathrm{Var}(Y)=\sum y^2\mathrm{d}y-(\mathrm{E}(Y))^2=\mathrm{E}\left(Y^2\right)-(\mathrm{E}(Y))^2$

$$\Rightarrow\mathrm{E}\left(Y^2\right)=\mathrm{Var}(Y)+(\mathrm{E}(Y))^2=\frac{11}{9}+\left(\frac{3}{2}\right)^2=\frac{125}{36}$$

(b) $V\sim\mathrm{N}(3\times8-4\times5, 9\times2+16\times1)=\mathrm{N}(4, 34)$

$\mathrm{P}(V>5)\approx0.432$

Exercise 16.1

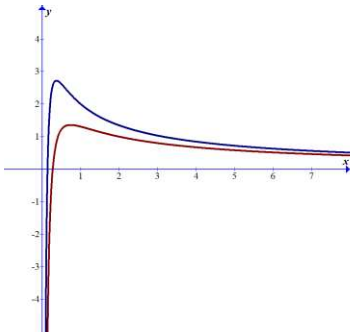
Conventions for this chapter:
DE = Differential equation, LHS = left-hand side, RHS = right-hand side, IF = integrating factor

1. **(a)** We need to show that $f\left(x\right)=\frac{1}{x}\left(\ln x+k\right)$ satisfies the DE $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy-1=0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-k-\ln x}{x^2}$$

$$\Rightarrow x^2\times\frac{1-k-\ln x}{x^2}+x\left(\frac{1}{x}\left(\ln x+k\right)\right)-1=1-k-\ln x+\ln x+k-1=0$$

(b) $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy-1=0\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x}=1-xy\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-xy}{x^2}$



(c) $f\left(1\right)=2\Rightarrow 2=\frac{1}{1}\left(\ln 1+k\right)\Rightarrow k=2\Rightarrow f\left(x\right)=\frac{1}{x}\left(\ln x+2\right)$

$f\left(2\right)=1\Rightarrow 1=\frac{1}{2}\left(\ln 2+k\right)\Rightarrow k=2-\ln 2\Rightarrow f\left(x\right)=\frac{1}{x}\left(\ln x+2-\ln 2\right)$

2. **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{y-x}=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x-y}=\mathrm{e}^{x}\mathrm{e}^{-y}\Rightarrow\mathrm{e}^{y}\mathrm{d}y=\mathrm{e}^{x}\mathrm{d}x$

Integrate both sides: $\mathrm{e}^{y}=\mathrm{e}^{x}+C\Rightarrow y=\ln\left(\mathrm{e}^{x}+C\right)$

(b) Separate variables first

$$\frac{y\mathrm{d}y}{\mathrm{d}x} = y^2x + x = x\left(y^2+1\right) \Rightarrow \frac{y\mathrm{d}y}{y^2+1} = x\mathrm{d}x$$

$$\Rightarrow \frac{1}{2}\ln\left(y^2+1\right) = \frac{x^2}{2} + c \Rightarrow \ln\left(y^2+1\right) = x^2 + 2c$$

$$\Rightarrow y^2 + 1 = \mathrm{e}^{x^2+2c} = \mathrm{e}^{x^2}\mathrm{e}^{2c} = C\mathrm{e}^{x^2} \Rightarrow y^2 = C\mathrm{e}^{x^2} - 1$$

(c) $\mathrm{e}^{x-y}\mathrm{d}y = x\mathrm{d}x \Rightarrow \mathrm{e}^{x}\mathrm{e}^{-y}\mathrm{d}y = x\mathrm{d}x \Rightarrow \mathrm{e}^{-y}\mathrm{d}y = \mathrm{e}^{-x}x\mathrm{d}x$

The LHS is a direct evaluation while the RHS needs integration by parts:

$$-\mathrm{e}^{-y} = -\mathrm{e}^{-x}\left(x+1\right) + c \Rightarrow \mathrm{e}^{-y} = \mathrm{e}^{-x}\left(x+1\right) + C$$

$$\Rightarrow -y = \ln\left(\mathrm{e}^{-x}\left(x+1\right) + C\right) \Rightarrow y = -\ln\left(\mathrm{e}^{-x}\left(x+1\right) + C\right)$$

$$\Rightarrow y = \ln\left(\frac{\mathrm{e}^{x}}{\left(x+1\right) + C\mathrm{e}^{x}}\right) = x - \ln\left(x + 1 + C\mathrm{e}^{x}\right)$$

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^2 - x - y^2 + 1 = x\left(y^2-1\right) - \left(y^2-1\right) = \left(y^2-1\right)\left(x-1\right)$

$\Rightarrow \dfrac{\mathrm{d}y}{y^2-1} = \left(x-1\right)\mathrm{d}x$, and using partial fractions in integrating LHS

$$\frac{1}{2}\left(\ln\left(y-1\right) - \ln\left(y+1\right)\right) = \frac{\left(x-1\right)^2}{2} + c \Rightarrow \ln\left(\frac{y-1}{y+1}\right) = \left(x-1\right)^2 + 2c$$

$$\frac{y-1}{y+1} = \mathrm{e}^{\left(x-1\right)^2+2c} = C\mathrm{e}^{\left(x-1\right)^2}$$

(e) $\left(xy\ln x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left(y+1\right)^2 \Rightarrow \dfrac{y\mathrm{d}y}{\left(y+1\right)^2} = \dfrac{\mathrm{d}x}{x\ln x}$

LHS can be integrated by substituting $u = y + 1$ or by partial fractions.
RHS can be integrated with substituting $u = \ln x$.

$$\ln\left(y+1\right) + \frac{1}{y+1} = \ln\left(\ln x\right) + c \Rightarrow \ln\left(y+1\right) + y\ln\left(y+1\right) + 1 = (y+1)\left(\ln\left(\ln x\right) + c\right)$$

(f) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1+2y^2}{y\sin^2 x}\Rightarrow\frac{y\mathrm{d}y}{1+2y^2}=\frac{\mathrm{d}x}{\sin^2 x}$$

LHS can be integrated with the substitution $u=1+2y^2$, and RHS a standard cosecant^2x integral.

$$\int\frac{y\mathrm{d}y}{1+2y^2}=\int\frac{\mathrm{d}x}{\sin^2 x}\Rightarrow\frac{1}{4}\ln\left(1+2y^2\right)=-\cot x+c\Rightarrow\ln\left(1+2y^2\right)=-4\cot x+4c$$
$$\Rightarrow 1+2y^2=\mathrm{e}^{-4\cot x+4c}\Rightarrow 2y^2=\mathrm{e}^{-4\cot x}\mathrm{e}^{4c}-1\Rightarrow y^2=\frac{\mathrm{e}^{4c}}{2}\mathrm{e}^{-4\cot x}-\frac{1}{2}=C\mathrm{e}^{-4\cot x}-\frac{1}{2}$$

(g) $$\left(1+\tan y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+1\Rightarrow\left(1+\tan y\right)\mathrm{d}y=\left(x^2+1\right)$$

LHS is a direct trigonometric integral and RHS is a polynomial.

$$y+\ln\left|\sec y\right|=\frac{1}{3}x^3+x+c$$

(h) $$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{t\mathrm{e}^t}{y\sqrt{y^2+1}}\Rightarrow y\sqrt{y^2+1}\,\mathrm{d}y=t\mathrm{e}^t\mathrm{d}t$$

LHS integral can be evaluated with the substitution $u=y^2+1$, and the RHS with integration by parts.

$$\int y\sqrt{y^2+1}\,\mathrm{d}y=\int t\mathrm{e}^t\mathrm{d}t\Rightarrow\frac{1}{3}\left(y^2+1\right)^{\frac{3}{2}}=\mathrm{e}^t\left(t-1\right)+c$$
$$\Rightarrow\sqrt{\left(y^2+1\right)^3}=3\mathrm{e}^t\left(t-1\right)+C$$

(i) $$y\sec\theta\,\mathrm{d}y=\mathrm{e}^y\sin^2\theta\,\mathrm{d}\theta\Rightarrow y\mathrm{e}^{-y}\,\mathrm{d}y=\sin^2\theta\cos\theta\mathrm{d}\theta$$

LHS integral can be evaluated with integration by parts and the RHS
with the substitution $u=\sin\theta$.

$$-\mathrm{e}^{-y}\left(y+1\right)=\frac{\sin^3\theta}{3}+c\Rightarrow\mathrm{e}^{-y}\left(y+1\right)=-\frac{\sin^3\theta}{3}+c$$

(j) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x\left(1+y^2\right)\Rightarrow\frac{\mathrm{d}y}{1+y^2}=\mathrm{e}^x\mathrm{d}x\Rightarrow\arctan y=\mathrm{e}^x+c\Rightarrow y=\tan\left(\mathrm{e}^x+c\right)$$

3. **(a)** $x^{-3}\mathrm{d}y = 4y\mathrm{d}x \Rightarrow \frac{\mathrm{d}y}{y} = 4x^3\mathrm{d}x \Rightarrow \ln|y| = x^4 + c$, and with initial condition

$\ln|3| = c \Rightarrow \ln|y| = x^4 + \ln 3 \Rightarrow y = 3\mathrm{e}^{x^4}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy \Rightarrow \frac{\mathrm{d}y}{y} = x\mathrm{d}x \Rightarrow \ln|y| = \frac{x^2}{2} + c$, and with initial condition

$c = \ln|1| = 0 \Rightarrow y = \mathrm{e}^{\frac{x^2}{2}}$

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} - xy^2 = 0 \Rightarrow \frac{\mathrm{d}y}{y^2} = x\mathrm{d}x \Rightarrow -\frac{1}{y} = \frac{x^2}{2} + c$, and with initial condition

$-\frac{1}{2} = \frac{1}{2} + c \Rightarrow c = -1 \Rightarrow -\frac{1}{y} = \frac{x^2}{2} - 1 \Rightarrow y = \frac{2}{2 - x^2}$

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 0 \Rightarrow \frac{\mathrm{d}y}{y^2} = \mathrm{d}x \Rightarrow -\frac{1}{y} = x + c$, and with initial condition

$-\frac{1}{1} = 2 + c \Rightarrow c = -3 \Rightarrow -\frac{1}{y} = x - 3 \Rightarrow y = \frac{1}{3 - x}$

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^y = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{e}^y} = \mathrm{d}x \Rightarrow -\mathrm{e}^{-y} = x + c$, and with initial conditions

$-\mathrm{e}^{-1} = 0 + c \Rightarrow c = \frac{-1}{\mathrm{e}} \Rightarrow \mathrm{e}^{-y} = \frac{1}{\mathrm{e}} - x \Rightarrow -y = \ln\left(\frac{1 - \mathrm{e}x}{\mathrm{e}}\right) \Rightarrow y = \ln\left(\frac{\mathrm{e}}{1 - \mathrm{e}x}\right)$

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = y^{-2}x + y^{-2} \Rightarrow y^2\mathrm{d}y = (x+1)\mathrm{d}x \Rightarrow \frac{y^3}{3} = \frac{(x+1)^2}{2} + c$, and with initial conditions

$\frac{1}{3} = \frac{1^2}{2} + c \Rightarrow c = -\frac{1}{6} \Rightarrow \frac{y^3}{3} = \frac{(x+1)^2}{2} - \frac{1}{6} \Rightarrow y^3 = \frac{3}{2}(x+1)^2 - \frac{1}{2}$

(g) $x\mathrm{d}y - y^2\mathrm{d}x = -\mathrm{d}y \Rightarrow \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{x+1} \Rightarrow -\frac{1}{y} = \ln|x+1| + c$, and with initial conditions

$-\frac{1}{1} = \ln|1| + c \Rightarrow c = -1 \Rightarrow y = \frac{1}{1 - \ln|x+1|}$

(h) $y^2\mathrm{d}y - x\mathrm{d}x = \mathrm{d}x - \mathrm{d}y \Rightarrow \left(y^2+1\right)\mathrm{d}y = \left(x+1\right)\mathrm{d}x \Rightarrow \frac{y^3}{3} + y = \frac{(x+1)^2}{2} + c$

$\frac{3^3}{3} + 3 = \frac{(1)^2}{2} + c \Rightarrow c = \frac{23}{2} \Rightarrow 2y^3 + 6y = 3x^2 + 6x + 72$

(i) $y\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + x \Rightarrow \frac{y\mathrm{d}y}{y^2+1} = x\mathrm{d}x \Rightarrow \frac{1}{2}\ln\left(y^2+1\right) = \frac{1}{2}x^2 + c$, with initial conditions

$\frac{1}{2}\ln(1) = c \Rightarrow c = 0 \Rightarrow \ln\left(y^2+1\right) = x^2 \Rightarrow y^2 = \mathrm{e}^{x^2} - 1$

(j) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy-y}{y+1} \Rightarrow \left(1+\frac{1}{y}\right)\mathrm{d}y = \left(x-1\right)\mathrm{d}x \Rightarrow y + \ln|y| = \frac{x^2}{2} - x + c$,

$1 + \ln|1| = 2 - 2 + c \Rightarrow c = 1 \Rightarrow y + \ln|y| = \frac{x^2}{2} - x + 1$

(k) Separate variables and integrate:

$\frac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{\frac{1-y^2}{1-x^2}} \Rightarrow \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \frac{x\mathrm{d}x}{\sqrt{1-x^2}} \Rightarrow \arcsin y = -\sqrt{1-x^2} + c$, with initial conditions, $\arcsin 0 = -\sqrt{1-0} + c \Rightarrow c = 1 \Rightarrow \arcsin y = 1 - \sqrt{1-x^2}$

(l) $\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\mathrm{e}^x\right) = \mathrm{e}^{x-y} \Rightarrow \mathrm{e}^y\mathrm{d}y = \frac{\mathrm{e}^x\mathrm{d}x}{1+\mathrm{e}^x} \Rightarrow \mathrm{e}^y = \ln\left(1+\mathrm{e}^x\right) + c$, with initial conditions,

$\mathrm{e}^0 = \ln\left(1+\mathrm{e}^1\right) + c \Rightarrow c = 1 - \ln\left(1+\mathrm{e}\right) \Rightarrow \mathrm{e}^y = \ln\left(1+\mathrm{e}^x\right) + 1 - \ln\left(1+\mathrm{e}\right)$

$\mathrm{e}^y = \ln\left(\frac{\mathrm{e}\left(1+\mathrm{e}^x\right)}{1+\mathrm{e}}\right) \Rightarrow y = \ln\left(\ln\left(\frac{\mathrm{e}\left(1+\mathrm{e}^x\right)}{1+\mathrm{e}}\right)\right)$

(m) $\left(y+1\right)\mathrm{d}y = \left(x^2y - y\right)\mathrm{d}x \Rightarrow \left(1+\frac{1}{y}\right)\mathrm{d}y = \left(x^2-1\right)\mathrm{d}x \Rightarrow y + \ln|y| = \frac{x^3}{3} - x + c$,

$1 + \ln|1| = 9 - 3 + c \Rightarrow c = -5 \Rightarrow y + \ln|y| = \frac{x^3}{3} - x - 5$

(n) $$x\frac{\mathrm{d}y}{\mathrm{d}x}-y=2x^2y\Rightarrow x\mathrm{d}y=\left(2x^2+1\right)y\mathrm{d}x\Rightarrow\frac{\mathrm{d}y}{y}=\left(2x+\frac{1}{x}\right)\mathrm{d}x\Rightarrow\ln|y|=x^2+\ln|x|+c$$

$$\ln|1|=1^2+\ln|1|+c\Rightarrow c=-1\Rightarrow\ln|y|=x^2+\ln|x|-1\Rightarrow|y|=\mathrm{e}^{x^2+\ln|x|-1}=|x|\mathrm{e}^{x^2-1}$$

(o) $$xy\mathrm{d}x+\mathrm{e}^{-x^2}\left(y^2-1\right)\mathrm{d}y=0\Rightarrow x\mathrm{e}^{x^2}\mathrm{d}x=\left(\frac{1}{y}-y\right)\mathrm{d}y\Rightarrow\frac{\mathrm{e}^{x^2}}{2}+c=\ln|y|-\frac{y^2}{2}$$

$$\frac{\mathrm{e}^0}{2}+c=\ln|1|-\frac{1}{2}\Rightarrow c=-1\Rightarrow\frac{\mathrm{e}^{x^2}}{2}-1=\ln|y|-\frac{y^2}{2}\Rightarrow 2\ln|y|-y^2=\mathrm{e}^{x^2}-2$$

4. **(a)** $2x\frac{\mathrm{d}y}{\mathrm{d}x}=x+y\Rightarrow 2x\mathrm{d}y=\left(x+y\right)\mathrm{d}x$. This is a homogeneous DE. We use the substitution $y=vx\Rightarrow \mathrm{d}y=v\mathrm{d}x+x\mathrm{d}v$, and then separate variables.

$$2x\mathrm{d}y=\left(x+y\right)\mathrm{d}x\Rightarrow 2x\left(v\mathrm{d}x+x\mathrm{d}v\right)=\left(x+vx\right)\mathrm{d}x\Rightarrow\left(xv-x\right)\mathrm{d}x=-2x^2\mathrm{d}v$$
$$\Rightarrow\frac{\mathrm{d}x}{x}=\frac{2\mathrm{d}v}{1-v}\Rightarrow\ln|x|=-2\ln|1-v|+c\Rightarrow\ln|x|=\ln\frac{1}{\left|1-\frac{y}{x}\right|^2}+c\Rightarrow\ln|x|=\ln\frac{x^2}{|y-x|^2}+c$$
$$\Rightarrow\ln|x|=\ln|x-y|^2+c\Rightarrow|x|=\mathrm{e}^{\ln|x-y|^2+c}=\mathrm{e}^c|x-y|^2=C\left(x-y\right)^2$$

(b) This is also homogeneous. We use the same substitution as before.

$$\left(x+y\right)\mathrm{d}y=\left(x-y\right)\mathrm{d}x\Rightarrow\left(x+vx\right)\left(v\mathrm{d}x+x\mathrm{d}v\right)=\left(x-vx\right)\mathrm{d}x$$
$$\Rightarrow x^2\left(1+v\right)\mathrm{d}v=\left(x-vx-vx-xv^2\right)\mathrm{d}x=x\left(1-2v-v^2\right)\mathrm{d}x$$
$$\Rightarrow\frac{\mathrm{d}x}{x}=\frac{\left(1+v\right)\mathrm{d}v}{1-2v-v^2}\Rightarrow\ln|x|+c=-\frac{1}{2}\ln\left|1-2v-v^2\right|$$

Substitute the value of v back and the result can be simplified to:

$$\left|y^2+2xy-x^2\right|=C$$

(Remember that $\ln x^2=2\ln x$)

(c) $$\left(x^2-y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=xy \Rightarrow \left(x^2-y^2\right)\mathrm{d}y=xy\mathrm{d}x;\ y=vx \Rightarrow \mathrm{d}y=v\mathrm{d}x+x\mathrm{d}v$$

$$\left(x^2-v^2x^2\right)\left(v\mathrm{d}x+x\mathrm{d}v\right)=xvx\mathrm{d}x \Rightarrow \left(x^3-v^2x^3\right)\mathrm{d}v=v^3x^2\mathrm{d}x$$

$$\Rightarrow \left(\frac{1}{v^3}-\frac{1}{v}\right)\mathrm{d}v=\frac{\mathrm{d}x}{x} \Rightarrow \ln|x|=\frac{-1}{2v^2}-\ln|v|+c \Rightarrow \ln|x|=-\frac{x^2}{2y^2}-\ln\left|\frac{y}{x}\right|+c$$

$$\ln|y|=-\frac{x^2}{2y^2}+c \Rightarrow y=\mathrm{e}^{-\frac{x^2}{2y^2}+c}=C\mathrm{e}^{-\frac{x^2}{2y^2}}$$

(d) $$x\mathrm{d}y-\left(2x\mathrm{e}^{-\frac{y}{x}}+y\right)\mathrm{d}x=0 \Rightarrow x\left(v\mathrm{d}x+x\mathrm{d}v\right)-\left(2x\mathrm{e}^{-v}+vx\right)\mathrm{d}x=0$$

$$x\left(v\mathrm{d}x+x\mathrm{d}v\right)-\left(2x\mathrm{e}^{-v}+vx\right)\mathrm{d}x=0 \Rightarrow x^2\mathrm{d}v=2x\mathrm{e}^{-v}\mathrm{d}x$$

$$\Rightarrow \mathrm{e}^{v}\mathrm{d}v=\frac{2}{x}\mathrm{d}x \Rightarrow \mathrm{e}^{\frac{y}{x}}=2\ln|x|+c$$

With $y(1)$=0, $\mathrm{e}^{\frac{y}{x}}=2\ln|x|+1$

(e) $$\left(x\sec\frac{y}{x}+y\right)\mathrm{d}x=x\mathrm{d}y \Rightarrow \left(x\sec v+vx\right)\mathrm{d}x=x\left(v\mathrm{d}x+x\mathrm{d}v\right)$$

$$\Rightarrow x\sec v\mathrm{d}x=x^2\mathrm{d}v \Rightarrow \frac{\mathrm{d}x}{x}=\cos v\mathrm{d}v \Rightarrow \ln|x|=\sin\left(\frac{y}{x}\right)+c$$

$$(1,0) \Rightarrow c=0 \Rightarrow x=\mathrm{e}^{\sin\left(\frac{y}{x}\right)}$$

(f) $$\left(x^2+y^2\right)\mathrm{d}x=\left(xy-x^2\right)\mathrm{d}y \Rightarrow \left(x^2+v^2x^2\right)\mathrm{d}x=\left(x^2v-x^2\right)\left(v\mathrm{d}x+x\mathrm{d}v\right)$$

$$\Rightarrow x^2\left(1+v\right)\mathrm{d}x=x^3\left(v-1\right)\mathrm{d}v \Rightarrow \frac{\mathrm{d}x}{x}=\left(1-\frac{2}{v+1}\right)\mathrm{d}v$$

$$\Rightarrow \ln|x|+c=\frac{y}{x}-2\ln\left|\frac{y}{x}+1\right|$$

This can be simplified to different forms which also partly depends where we place the constant term, c. Possible forms for the answer:

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x} \text{ or } (x+y)^2 = cxe^{\frac{y}{x}}$$

(g) $y\mathrm{d}x = \left(x+\sqrt{xy}\right)\mathrm{d}y \Rightarrow vx\mathrm{d}x = \left(x+x\sqrt{v}\right)\left(v\mathrm{d}x + x\mathrm{d}v\right)$

$$vx\mathrm{d}x = \left(x+x\sqrt{v}\right)\left(v\mathrm{d}x + x\mathrm{d}v\right) \Rightarrow -xv\sqrt{v}\mathrm{d}x = x^2\left(1+\sqrt{v}\right)\mathrm{d}v$$

$$\Rightarrow -\frac{\mathrm{d}x}{x} = \left(v^{-\frac{3}{2}} + \frac{1}{v}\right)\mathrm{d}v \Rightarrow -\ln|x| + c = -2\sqrt{\frac{x}{y}} + \ln\left|\frac{y}{x}\right| \Rightarrow 2\sqrt{\frac{x}{y}} = \ln|y| - c$$

$$\Rightarrow 4x = y\left(\ln|y| - c\right)^2$$

5. **(a)** $\frac{\mathrm{d}T}{\mathrm{d}t} = m(T-21) \Rightarrow \frac{\mathrm{d}T}{T-21} = m\mathrm{d}t \Rightarrow \ln|T-21| = mt + c$

$\Rightarrow |T-21| = \mathrm{e}^{mt+c}$

With T in a kettle which is at least at room temperature, $T - 21 \geq 0$, and thus,

$$|T-21| = T-21 = \mathrm{e}^{mt+c} \Rightarrow T = \mathrm{e}^{mt}\mathrm{e}^{c} + 21 = C\mathrm{e}^{mt} + 21$$

(b) **(i)** $T = C\mathrm{e}^{mt} + 21 \Rightarrow T(0) = 99 \Rightarrow 99 = C + 21 \Rightarrow C = 78$

$$T(15) = 69 \Rightarrow 69 = 78\mathrm{e}^{15m} + 21 \Rightarrow m = \frac{1}{15}\ln\frac{8}{13}$$

6. **(a)** The growth of the tiger population can be modelled by

$\frac{\mathrm{d}p}{\mathrm{d}t} = kp\left(1 - \frac{p}{200}\right)$, $25 \leq p \leq 200$, where t is the number of years.

This logistic DE can be solved using the model given in the text.
(Review pages 848–850)

$$p = \frac{200}{1 + b\mathrm{e}^{-kt}}$$

Since $p(0) = 25$, then $25 = \frac{200}{1+b\mathrm{e}^{-0}} \Rightarrow 1+b=8 \Rightarrow b=7 \Rightarrow p = \frac{200}{1+7\mathrm{e}^{-kt}}$

We also know that $p(2) = 39$, then

$$39 = \frac{200}{1+7\mathrm{e}^{-2k}} \Rightarrow 7\mathrm{e}^{-2k} = 4.12821 \Rightarrow \mathrm{e}^{-2k} = 0.589744$$
$$\Rightarrow k = -\frac{\ln 0.589744}{2} \approx 0.2640 \Rightarrow p = \frac{200}{1+7\mathrm{e}^{-0.2640t}}$$

(b) $p(5) = \frac{200}{1+7\mathrm{e}^{-0.2640\times 5}} \approx 69.69 \approx 70\text{ tigers}$

(c) $100 = \frac{200}{1+7\mathrm{e}^{-0.2640t}} \Rightarrow 1+7\mathrm{e}^{-0.2640t} = 2 \Rightarrow \mathrm{e}^{-0.2640t} = \frac{1}{7} \Rightarrow t = \frac{\ln\left(\frac{1}{7}\right)}{-0.2640} \approx 7.37\text{ years}$

7. **(a)** This situation corresponds to $t = 0$, $P(0) = \frac{2100}{1+29\mathrm{e}^{0}} = 70$

(b) Carrying capacity, as clear from the model is 2100, thus,

$$1050 = \frac{2100}{1+29\mathrm{e}^{-0.75t}} \Rightarrow 1+29\mathrm{e}^{-0.75t} = 2 \Rightarrow \mathrm{e}^{-0.75t} = \frac{1}{29} \Rightarrow t = \frac{\ln\left(\frac{1}{29}\right)}{-0.75} \approx 4.49\text{ years}$$

(c) The logistic model is of the form $P(t) = \frac{L}{1+b\mathrm{e}^{-kt}}$ and thus, $k = 0.75$, and the logistic DE corresponding to this is of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1-\frac{y}{L}\right) \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} = 0.75P\left(1-\frac{P}{2100}\right)$$

8. **(a)** This is a typical growth model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Rightarrow \frac{\mathrm{d}P}{P} = k\mathrm{d}t \Rightarrow \ln P = kt + c \Rightarrow P = \mathrm{e}^{kt+c} = \mathrm{e}^{c}\mathrm{e}^{kt}$$

$$P(0) = 500 = \mathrm{e}^{c} \Rightarrow P = 500\mathrm{e}^{kt}$$

$$P(2) = 2000 = 500\mathrm{e}^{2k} \Rightarrow k = \frac{\ln 4}{2} = \ln 2 \Rightarrow P = 500\mathrm{e}^{t\ln 2}$$

$$P\left(12\right) = 500\mathrm{e}^{12\times\ln 2} = 2\,048\,000$$

(b) This is the time it takes the bacteria to reach 1000.

$$1000 = 500\mathrm{e}^{t\ln 2} \Rightarrow t = 1 \text{ hour}$$

Exercise 16.2

1. A linear DE can be reduced to the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$

(a) $3\frac{\mathrm{d}y}{\mathrm{d}x} + xy = y^2$ is not a linear DE since the right side is not linear in y.

(b) $x^3\frac{\mathrm{d}y}{\mathrm{d}x} - 3y + 2x^2 = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x^3}y = -\frac{2}{x}$ is a linear DE.

(c) $5xy' = 3x - 2y \Rightarrow 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{5x}y = \frac{3}{5}$ is a linear DE.

(d) $2y\frac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2y}\sin 2x = 0$ is not a linear DE since the term in y is not linear.

2. $x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x$ is a linear DE with IF $= \mathrm{e}^{\int -\frac{2\mathrm{d}x}{x}} = \mathrm{e}^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$

We multiply the equation with the IF and then solve:

$$\frac{1}{x^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}\cdot\frac{2}{x}y = \frac{1}{x^2}\cdot x \Rightarrow \frac{1}{x^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \frac{1}{x} \Rightarrow \mathrm{d}\left(\frac{1}{x^2}\cdot y\right) = \frac{\mathrm{d}x}{x}$$

$$\Rightarrow \frac{1}{x^2}\cdot y = \ln|x| + c \Rightarrow y = x^2\ln|x| + cx^2$$

3. $$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=(1-2x)\sqrt{x^2+1}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{x^2+1}\cdot y=\frac{1-2x}{\sqrt{x^2+1}}$$

This is a linear DE with IF = $\mathrm{e}^{\int\frac{x\mathrm{d}x}{x^2+1}}=\mathrm{e}^{\frac{1}{2}\ln\left(x^2+1\right)}=\sqrt{x^2+1}$

We multiply the equation with the IF and then solve:

$$\sqrt{x^2+1}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{\sqrt{x^2+1}}\cdot y=1-2x\Rightarrow\mathrm{d}\left(y\sqrt{x^2+1}\right)=(1-2x)\mathrm{d}x$$

$$\Rightarrow y\sqrt{x^2+1}=\int(1-2x)\mathrm{d}x=x-x^2+c\Rightarrow y=\frac{x-x^2+c}{\sqrt{x^2+1}}$$

$$2=\frac{1-1+c}{\sqrt{1+1}}\Rightarrow c=2\sqrt{2}\Rightarrow y=\frac{x-x^2+2\sqrt{2}}{\sqrt{x^2+1}}$$

4. $$\frac{\mathrm{d}y}{\mathrm{d}x}=x-y\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+y=x\Rightarrow\text{IF}=\mathrm{e}^{\int\mathrm{d}x}=\mathrm{e}^x$$

$$\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^xy=x\mathrm{e}^x\Rightarrow\mathrm{d}\left(\mathrm{e}^xy\right)=x\mathrm{e}^x\mathrm{d}x\Rightarrow\mathrm{e}^xy=\int x\mathrm{e}^x\mathrm{d}x$$

$$\Rightarrow\mathrm{e}^xy=\mathrm{e}^x(x-1)+c\Rightarrow y=x-1+c\mathrm{e}^{-x}$$

5. $$(1+\sin x)\frac{\mathrm{d}y}{\mathrm{d}x}-y\cos x=(1+\sin x)^4\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}-y\cdot\frac{\cos x}{1+\sin x}=\frac{(1+\sin x)^4}{1+\sin x}=(1+\sin x)^3$$

A linear DE with IF $=\mathrm{e}^{\int\frac{-\cos x}{1+\sin x}\mathrm{d}x}=\mathrm{e}^{-\ln(1+\sin x)}=\frac{1}{1+\sin x}$

$$\left(\frac{1}{1+\sin x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}-y\cdot\frac{\cos x}{(1+\sin x)^2}=(1+\sin x)^2\Rightarrow\mathrm{d}\left(\left(\frac{1}{1+\sin x}\right)\cdot y\right)=(1+\sin x)^2\,\mathrm{d}x$$

$$\Rightarrow\frac{y}{1+\sin x}=\int\left(1+2\sin x+\sin^2x\right)\mathrm{d}x\Rightarrow\frac{y}{1+\sin x}=\frac{1}{4}(6x-8\cos x-\sin 2x+c)$$

$$\Rightarrow y=(1+\sin x)\left(\frac{1}{4}(6x-8\cos x-\sin 2x+c)\right)$$

Given $y(0)=1\Rightarrow c=12\Rightarrow y=\frac{1}{4}(1+\sin x)(6x-\sin 2x-8\cos x+12)$

6. **(a)** $\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}+2\mathrm{e}^x y=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+2y=\mathrm{e}^{-x};\mathrm{IF}=\mathrm{e}^{\int 2\mathrm{d}x}=\mathrm{e}^{2x}$

$\mathrm{e}^{2x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+2\mathrm{e}^{2x}\cdot y=\mathrm{e}^x\Rightarrow\mathrm{d}\left(\mathrm{e}^{2x}\cdot y\right)=\mathrm{e}^x\mathrm{d}x\Rightarrow\mathrm{e}^{2x}\cdot y=\mathrm{e}^x+c$

$y=\mathrm{e}^{-x}+c\mathrm{e}^{-2x}$

(b) $x\frac{\mathrm{d}y}{\mathrm{d}x}+3y=\frac{\sin x}{x^2}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3}{x}y=\frac{\sin x}{x^3}\Rightarrow\mathrm{IF}=\mathrm{e}^{\int\frac{3}{x}\mathrm{d}x}=\mathrm{e}^{\ln x^3}=x^3$

$x^3\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2y=\sin x\Rightarrow\mathrm{d}\left(x^3y\right)=\sin x\mathrm{d}x\Rightarrow x^3y=\int\sin x\mathrm{d}x=-\cos x+c$

$y=\frac{-\cos x+c}{x^3}$

(c) $(x-1)^3\frac{\mathrm{d}y}{\mathrm{d}x}+4y(x-1)^2=x+1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{4y}{x-1}=\frac{x+1}{(x-1)^3}$

$\Rightarrow\mathrm{IF}=\mathrm{e}^{\int\frac{4\mathrm{d}x}{x-1}}=\mathrm{e}^{\ln(x-1)^4}=(x-1)^4$

$\Rightarrow(x-1)^4\frac{\mathrm{d}y}{\mathrm{d}x}+4(x-1)^3y=(x+1)(x-1)\Rightarrow\mathrm{d}\left(y(x-1)^4\right)=\left(x^2-1\right)\mathrm{d}x$

$\Rightarrow y(x-1)^4=\frac{x^3}{3}-x+c\Rightarrow y=\frac{x^3-3x+C}{3(x-1)^4}$

(d) $\sin x\frac{\mathrm{d}y}{\mathrm{d}x}+y\cos x=\tan x$ is already in a simplified form.

$\Rightarrow\mathrm{d}(y\sin x)=\tan x\mathrm{d}x\Rightarrow y\sin x=\ln|\sec x|+c\Rightarrow y=\frac{\ln|\sec x|+c}{\sin x}$

(e) $x\frac{\mathrm{d}y}{\mathrm{d}x}-2y=x^2\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{2}{x}y=x\Rightarrow\mathrm{IF}=\mathrm{e}^{\int-\frac{2}{x}\mathrm{d}x}=\mathrm{e}^{\ln\frac{1}{x^2}}=\frac{1}{x^2}$

$\Rightarrow\frac{1}{x^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{2}{x^3}y=\frac{1}{x}\Rightarrow\mathrm{d}\left(\frac{y}{x^2}\right)=\frac{\mathrm{d}x}{x}\Rightarrow\frac{y}{x^2}=\ln|x|+c\Rightarrow y=x^2\left(\ln|x|+c\right)$

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y = x \Rightarrow \text{IF} = \mathrm{e}^{\int \mathrm{d}x} = \mathrm{e}^{x}$

$$\mathrm{e}^{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{x}y = x\mathrm{e}^{x} \Rightarrow \mathrm{d}\left(\mathrm{e}^{x}y\right) = x\mathrm{e}^{x}\mathrm{d}x \Rightarrow \mathrm{e}^{x}y = \int x\mathrm{e}^{x}\mathrm{d}x$$
$$\Rightarrow \mathrm{e}^{x}y = \mathrm{e}^{x}\left(x-1\right) + c \Rightarrow y = x - 1 + c\mathrm{e}^{-x}$$

(g) $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sqrt{x} \Rightarrow x\mathrm{d}y + y\mathrm{d}x = \sqrt{x}\mathrm{d}x$, so, it is already in simplified form.

$$\mathrm{d}\left(xy\right) = \sqrt{x}\mathrm{d}x \Rightarrow xy = \frac{2x\sqrt{x}}{3} + c \Rightarrow y = \frac{2\sqrt{x}}{3} + \frac{c}{x}$$

(h) $\sin x\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \sin\left(x^2\right) \Rightarrow \mathrm{d}\left(y\sin x\right) = \sin\left(x^2\right)\mathrm{d}x$

Unfortunately, we cannot evaluate $\int \sin\left(x^2\right)\mathrm{d}x$ symbolically, thus the solution of this differential equation will be possible only if we have initial conditions. Then, we can evaluate it numerically. The only thing we can do is just to give the form of the answer:

$$y\sin x = \int \sin\left(x^2\right)\mathrm{d}x + C \Rightarrow y = \frac{1}{\sin x}\left(\int \sin\left(x^2\right)\mathrm{d}x + C\right)$$

(i) $\left(1+t\right)\frac{\mathrm{d}u}{\mathrm{d}t} + u = 1 + t$. Notice that the LHS is nothing but $\frac{\mathrm{d}}{\mathrm{d}t}\left(u\left(1+t\right)\right)$, therefore

$$\mathrm{d}\left(u\left(1+t\right)\right) = \left(1+t\right)\mathrm{d}t \Rightarrow u\left(1+t\right) = \int\left(1+t\right)\mathrm{d}t = t + \frac{t^2}{2} + c \text{ or } \frac{\left(1+t\right)^2}{2} + c$$
$$\Rightarrow u = \frac{\left(1+t\right)}{2} + \frac{c}{1+t}$$

This is only one of a few forms of the answer.

(j) $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^{x^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{\mathrm{e}^{x^2}}{x} \Rightarrow \text{IF} = \mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = \mathrm{e}^{\ln x^2} = x^2$

$$\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\mathrm{e}^{x^2} \Rightarrow \mathrm{d}\left(yx^2\right) = x\mathrm{e}^{x^2}\mathrm{d}x \Rightarrow yx^2 = \frac{1}{2}\mathrm{e}^{x^2} + c \Rightarrow y = \frac{\mathrm{e}^{x^2}}{2x^2} + \frac{c}{x^2}$$

(k) $$\cos x\frac{\mathrm{d}y}{\mathrm{d}x}-\sin 2x=y\sin x\Rightarrow\cos x\frac{\mathrm{d}y}{\mathrm{d}x}-y\sin x=\sin 2x\Rightarrow\mathrm{d}\left(y\cos x\right)=\sin 2x\mathrm{d}x$$

$$\Rightarrow y\cos x=-\frac{1}{2}\cos 2x+c\Rightarrow y=-\frac{\cos 2x}{2\cos x}+\frac{c}{\cos x}=\frac{\sec x}{2}-\cos x+C\sec x$$

(l) $$x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\mathrm{e}^{-x}-xy\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+y\frac{1+x}{x}=\frac{\mathrm{e}^{-x}}{x}\Rightarrow\mathrm{IF}=\mathrm{e}^{\int\left(\frac{1}{x}+1\right)\mathrm{d}x}=\mathrm{e}^{(\ln x+x)}=x\mathrm{e}^{x}$$

$$\Rightarrow x\mathrm{e}^{x}\frac{\mathrm{d}y}{\mathrm{d}x}+y\mathrm{e}^{x}\left(1+x\right)=1\Rightarrow\mathrm{d}\left(x\mathrm{e}^{x}y\right)=\mathrm{d}x\Rightarrow x\mathrm{e}^{x}y=x+c$$

$$y=\mathrm{e}^{-x}\left(1+\frac{c}{x}\right)$$

7. **(a)** $$\frac{\mathrm{d}y}{\mathrm{d}x}+2y=x\Rightarrow\mathrm{IF}=\mathrm{e}^{\int 2\mathrm{d}x}=\mathrm{e}^{2x}\Rightarrow\mathrm{e}^{2x}\frac{\mathrm{d}y}{\mathrm{d}x}+2\mathrm{e}^{2x}y=x\mathrm{e}^{2x}$$

$\Rightarrow\mathrm{d}\left(\mathrm{e}^{2x}y\right)=x\mathrm{e}^{2x}\mathrm{d}x$, RHS can be evaluated by parts.

$\Rightarrow\mathrm{e}^{2x}y=\mathrm{e}^{2x}\left(\frac{x}{2}-\frac{1}{4}\right)+c$. Now apply initial conditions.

$$\Rightarrow\mathrm{e}^{0}=\mathrm{e}^{0}\left(0-\frac{1}{4}\right)+c\Rightarrow c=\frac{5}{4}\Rightarrow\Rightarrow\mathrm{e}^{2x}y=\mathrm{e}^{2x}\left(\frac{x}{2}-\frac{1}{4}\right)+\frac{5}{4}$$

$$\Rightarrow y=\frac{x}{2}-\frac{1}{4}+\frac{5}{4}\mathrm{e}^{-2x}$$

(b) $$x\frac{\mathrm{d}y}{\mathrm{d}x}-2y=x^{3}\sec x\tan x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{2}{x}y=x^{2}\sec x\tan x\Rightarrow\mathrm{IF}=\mathrm{e}^{\int-\frac{2}{x}\mathrm{d}x}=\mathrm{e}^{\ln x^{-2}}=x^{-2}$$

$$\Rightarrow x^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{2}{x^{3}}y=\sec x\tan x\Rightarrow\mathrm{d}\left(\frac{y}{x^{2}}\right)=\sec x\tan x\mathrm{d}x\Rightarrow\frac{y}{x^{2}}=\sec x+c,$$

Apply initial conditions: $\dfrac{2}{\frac{\pi^{2}}{9}}=2+c\Rightarrow c=2\left(\frac{9}{\pi^{2}}-1\right)$

Thus, the solution can be written as $y=x^{2}\left(\sec x+2\left(\frac{9}{\pi^{2}}-1\right)\right)$

(c) $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=\ln x\Rightarrow x^2\mathrm{d}y+2xy\mathrm{d}x=\ln x\mathrm{d}x$. LHS is $\mathrm{d}\left(x^2y\right)$ and RHS can be evaluated using integration by parts, thus $x^2y=\int\ln x\mathrm{d}x=x\ln x-x+c$.

Apply initial conditions: $2=0-1+c\Rightarrow c=3$. Therefore the solution can be written as $x^2y=x\ln x-x+3\Rightarrow y=\frac{1}{x}\ln x-\frac{1}{x}+\frac{3}{x^2}$

(d) $\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{3}{t}u=t\Rightarrow \mathrm{IF}=\mathrm{e}^{\int-\frac{3}{t}\mathrm{d}t}=\mathrm{e}^{\ln t^{-3}}=t^{-3}$

$t^{-3}\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{3}{t^4}u=t^{-2}\Rightarrow\mathrm{d}\left(t^{-3}u\right)=t^{-2}\mathrm{d}t\Rightarrow t^{-3}u=-t^{-1}+c$. Apply initial conditions:

$t^{-3}u=-t^{-1}+c\Rightarrow\frac{1}{2}=-\frac{1}{2}+c\Rightarrow c=1\Rightarrow t^{-3}u=-t^{-1}+1\Rightarrow u=t^3-t^2$

(e) $x\frac{\mathrm{d}y}{\mathrm{d}x}=y+x^2\sin x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x}=x\sin x\Rightarrow\mathrm{IF}=\mathrm{e}^{\int-\frac{\mathrm{d}x}{x}}=\mathrm{e}^{\ln x^{-1}}=x^{-1}$

Multiply with the IF and simplify:

$x^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}-y=\sin x\Rightarrow\mathrm{d}\left(yx^{-1}\right)=\sin x\mathrm{d}x\Rightarrow yx^{-1}=-\cos x+c$

With initial conditions: $0=1+c\Rightarrow c=-1\Rightarrow y=-x\cos x-x$

(f) $\frac{y'}{2y}-x=\frac{x\mathrm{e}^{x^2}}{y}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}-2yx=2x\mathrm{e}^{x^2}\Rightarrow\mathrm{IF}=\mathrm{e}^{\int-2x\mathrm{d}x}=\mathrm{e}^{-x^2}$

$\Rightarrow\mathrm{e}^{-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}-2\mathrm{e}^{-x^2}yx=2x\Rightarrow\mathrm{d}\left(y\mathrm{e}^{-x^2}\right)=2x\mathrm{d}x\Rightarrow y\mathrm{e}^{-x^2}=x^2+c$

With initial conditions: $3=c\Rightarrow y=\left(x^2+3\right)\mathrm{e}^{x^2}$

8. The basic Kirchhoff's law for the charge is given as

$$R\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{C}Q=E\left(t\right)\Rightarrow 5\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{0.05}Q=12\Rightarrow\frac{\mathrm{d}Q}{\mathrm{d}t}+4Q=12$$

This is a linear DE which can be solved using an IF of $\mathrm{e}^{\int 4\mathrm{d}t} = \mathrm{e}^{4t}$

$$\mathrm{e}^{4t}\frac{\mathrm{d}Q}{\mathrm{d}t} + 4\mathrm{e}^{4t}Q = 12\mathrm{e}^{4t} \Rightarrow \mathrm{d}\left(\mathrm{e}^{4t}Q\right) = 12\mathrm{e}^{4t}\mathrm{d}t \Rightarrow \mathrm{e}^{4t}Q = 3\mathrm{e}^{4t} + c$$

With $Q(0) = 0,\ 0 = 3 + c \Rightarrow c = -3 \Rightarrow Q = 3 - 3\mathrm{e}^{-4t}$

$I = 12\mathrm{e}^{-4t}$

Exercise 16.3

1. $y' = y + xy,\ y(0) = 1 \Rightarrow F(x, y) = y + xy, x_0 = 0, y_0 = 1, h = 0.1$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.1)(y_n + x_n y_n)$$

Arranging the process in a table is an efficient organisation of the work involved.

n	x_n	y_n	y_{n+1}
0	0	1	$1 + 0.1 \times 1 = 1.1$
1	0.1	1.1	$1.1 + 0.1(1.1 + 0.1 \times 1.1) = 1.221$
2	0.2	1.221	1.36752
3	0.3	1.36752	1.5453
4	0.4	1.5453	**1.7616**

2. **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 3x^2 y,\quad y(0) = 3 \Rightarrow F(x, y) = 6x^2 - 3x^2 y, x_0 = 0, y_0 = 3$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + h\left(6x_n^2 - 3x_n^2 y_n\right)$$

Since there is a large number of iterations required (given the small sizes of the steps), it is best if we set up a spreadsheet to do the calculations.

$h \rightarrow$	0.1		0.01		0.001	
n	x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	3	0	3	0	3
1	0.1	3	0.01	3	0.001	3
↓	↓	↓	↓	↓	↓	↓
10	1	**2.392794**	0.1	2.999145	0.01	2.999999
11			0.11	2.998846	0.011	2.999999
↓			↓	↓	↓	↓
100			1	**2.370111**	0.1	2.999015
101					0.101	2.998985
↓					↓	↓
1000					1	**2.3681**

(b) We substitute the suggested solution into the DE:

$$y = \mathrm{e}^{-x^3} + 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2\mathrm{e}^{-x^3} = -3x^2\left(y-2\right) = 6x^2 - 3x^2 y$$

(c) Exact value: $y = \mathrm{e}^{-x^3} + 2 = \mathrm{e}^{-1} + 2 \approx 2.367879$

Error when $h = 0.1$: $2.367879 - 2.392794 = -0.024915$

Error when $h = 0.01$: $2.367879 - 2.370111 = -0.002232$

Error when $h = 0.001$: $2.367879 - 2.3681 = -0.000221$

It appears as if the error is divided by 10 (approximately).

3. $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x - 2y + 1, \quad y\left(1\right) = 2 \Rightarrow F\left(x, y\right) = 3x - 2y + 1, x_0 = 1, y_0 = 2$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF\left(x_n, y_n\right) = y_n + \left(0.5\right)\left(3x_n - 2y_n + 1\right)$$

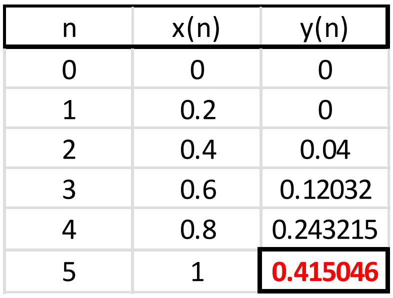
$$y_{n+1} = y_n + (0.5)(3x_n - 2y_n + 1)$$
$$y_{0+1} = y_0 + (0.5)(3x_0 - 2y_0 + 1) = 2 + 0.5(3 - 2\times 2 + 1) = 2$$
$$y_{1+1} = y_1 + (0.5)(3x_1 - 2y_1 + 1) = 2 + 0.5(3\times 1.5 - 2\times 2 + 1) = 2.75$$
$$y_{2+1} = y_2 + (0.5)(3x_2 - 2y_2 + 1) = 2.75 + 0.5(3\times 2 - 2\times 2.75 + 1) = 3.5$$
$$y_{3+1} = y_3 + (0.5)(3x_3 - 2y_3 + 1) = 3.5 + 0.5(3\times 2.5 - 2\times 3.5 + 1) = 4.25$$

4. $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2, \quad y(0) = 0 \Rightarrow F(x,y) = x + y^2, x_0 = 0, y_0 = 0$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.2)(x_n + y_n^2)$$

We will use a spreadsheet for the iterations.



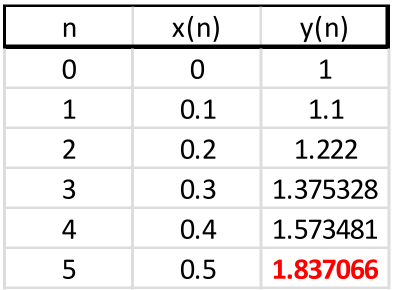
n	x(n)	y(n)
0	0	0
1	0.2	0
2	0.4	0.04
3	0.6	0.12032
4	0.8	0.243215
5	1	0.415046

5. $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2, \quad y(0) = 1 \Rightarrow F(x,y) = x^2 + y^2, x_0 = 0, y_0 = 1$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.1)(x_n^2 + y_n^2)$$

Spreadsheet output is given below



n	x(n)	y(n)
0	0	1
1	0.1	1.1
2	0.2	1.222
3	0.3	1.375328
4	0.4	1.573481
5	0.5	1.837066

6. **(a)** $\frac{dy}{dt} = y(3-ty), \quad y(0)=0.5 \Rightarrow F(t,y) = y(3-ty), t_0 = 0, y_0 = 0.5$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (h)y_n(3 - t_n y_n)$$

Spreadsheet output:

$h \rightarrow$	0.1		0.05		0.01	
n	x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	0.5	0	0.5	0	0.5
1	0.1	0.65	0.05	0.575	0.01	0.515
↓	↓	↓	↓	↓	↓	↓
5	0.5	1.703081	0.25	0.989081	0.05	0.57933
6	0.6	2.06898	0.3	1.125214	0.06	0.596542
↓	↓	↓	↓	↓	↓	↓
10	1	3.06605	0.5	1.795475	0.1	0.670188
↓	↓	↓	↓	↓	↓	↓
15	1.5	2.440297	0.75	2.686308	0.15	0.773706
↓	↓	↓	↓	↓	↓	↓
30	3	1.119248	1.5	2.432919	0.3	1.168512
↓	↓	↓	↓	↓	↓	↓
50			2.5	1.377951	0.5	1.877339
↓	↓	↓	↓	↓	↓	↓
60			3	1.121911	0.6	2.261424
↓	↓	↓	↓	↓	↓	↓
150					1.5	2.426722
↓	↓	↓	↓	↓	↓	↓
300					3	1.124112

(b) $y' = 5 - 3\sqrt{y}, \quad y(0) = 2 \Rightarrow F(x, y) = 5 - 3\sqrt{y}, x_0 = 0, y_0 = 2$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (h)\left(5 - 3\sqrt{y_n}\right)$$

Spreadsheet output:

$h \rightarrow$	0.1		0.05		0.01	
n	x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	2	0	2	0	2
1	0.1	2.075736	0.05	2.037868	0.01	2.007574
↓	↓	↓	↓	↓	↓	↓
5	0.5	2.307998	0.25	2.170531	0.05	2.037075
↓	↓	↓	↓	↓	↓	↓
10	1	2.490062	0.5	2.301666	0.1	2.072241
↓	↓	↓	↓	↓	↓	↓
15	1.5	2.600226	0.75	2.403337	0.15	2.105611
↓	↓	↓	↓	↓	↓	↓
30	3	2.73521	1.5	2.593517	0.3	2.195958
↓	↓	↓	↓	↓	↓	↓
50			2.5	2.705188	0.5	2.296863
↓	↓	↓	↓	↓	↓	↓
60			3	2.73209	0.6	2.340239
↓	↓	↓	↓	↓	↓	↓
150					1.5	2.588297
↓	↓	↓	↓	↓	↓	↓
300					3	2.729585

(c) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{4-ty}{1+y^2}, y(0) = -2 \Rightarrow F(t,y) = \frac{4-ty}{1+y^2}, t_0 = 0, y_0 = -2$

Thus, the recursive formula for y_n is:

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (h)\left(\frac{4-t_n y_n}{1+y_n^2}\right)$$

Spreadsheet output:

$h \rightarrow$	0.1		0.05		0.01	
n	t(n)	y(n)	t(n)	y(n)	t(n)	y(n)
0	0	-2	0	-2	0	-2
1	0.1	-1.92	0.05	-1.96	0.01	-1.992
↓	↓	↓	↓	↓	↓	↓
5	0.5	-1.48849	0.25	-1.775	0.05	-1.95907
↓	↓	↓	↓	↓	↓	↓
10	1	-0.41234	0.5	-1.46909	0.1	-1.91572
↓	↓	↓	↓	↓	↓	↓
15	1.5	1.046866	0.75	-1.02416	0.15	-1.86977
↓	↓	↓	↓	↓	↓	↓
30	3	1.51971	1.5	1.053515	0.3	-1.71402
↓	↓	↓	↓	↓	↓	↓
50			2.5	1.530002	0.5	-1.45212
↓	↓	↓	↓	↓	↓	↓
60			3	1.50549	0.6	-1.28842
↓	↓	↓	↓	↓	↓	↓
150					1.5	1.059414
↓	↓	↓	↓	↓	↓	↓
300					3	1.494898

7. The DE $\frac{\mathrm{d}y}{\mathrm{d}x} - y = \cos x$ is linear. The IF is $\mathrm{e}^{\int -\mathrm{d}x} = \mathrm{e}^{-x}$

Now, multiply with the IF and simplify:

$$e^{-x}\frac{dy}{dx}-e^{-x}y=e^{-x}\cos x \Rightarrow d\left(e^{-x}y\right)=e^{-x}\cos x\,dx \Rightarrow e^{-x}y=\int e^{-x}\cos x\,dx$$

The RHS can be evaluated using integration by parts:

$e^{-x}y=\frac{e^{-x}}{2}\left(\sin x-\cos x\right)+c$, and with initial conditions $0=\frac{1}{2}\left(0-1\right)+c \Rightarrow c=\frac{1}{2}$

The exact particular solution is $e^{-x}y=\frac{e^{-x}}{2}\left(\sin x-\cos x\right)+\frac{1}{2} \Rightarrow y=\frac{1}{2}\left(\sin x-\cos x+e^{x}\right)$

Finding the exact values is done by simple substitution of the given numbers into the solution formula found earlier.

For example, $y(0)=\frac{1}{2}\left(\sin 0-\cos 0+e^{0}\right)=0$. The rest of the row is similarly done.

Using $h = 0.1$ or 0.2 to approximate the solution using Euler's method:

We first rewrite the equation if the form $y'=F\left(x,y\right)$.

$\frac{dy}{dx}-y=\cos x \Rightarrow \frac{dy}{dx}=y+\cos x$, and then do the iterations using

$y_{n+1}=y_n+hF\left(x_n,y_n\right)=y_n+h\left(y_n+\cos x_n\right)$. 15 iterations are needed, thus it is more efficient to use a spreadsheet for the calculations.

Exact		0.2		0.1	
x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	0	0	0	0
0.2	0.220003	0.2	0.2	0.1	0.1
0.4	0.480091	0.4	0.436013	0.2	0.2095
0.6	0.780713	0.6	0.707428	0.3	0.328457
0.8	1.123095	0.8	1.013981	0.4	0.456836
1	1.509725	1	1.356118	0.5	0.594626
				0.6	0.741847
				0.7	0.898565
				0.8	1.064906
				0.9	1.241067
				1	1.427335

Exercise 16.4

1. **(a)** $f(x)=\frac{1}{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)$. Use the Maclaurin expansion for e^{x},

$$\mathrm{e}^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots,\text{ and }\mathrm{e}^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^{n}}{n!}=1-x+\frac{x^{2}}{2!}-\frac{x^{3}}{3!}+\cdots$$

$$f(x)=\frac{1}{2}\left(1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots-\left(1-x+\frac{x^{2}}{2!}-\frac{x^{3}}{3!}+\cdots\right)\right)$$

$$=x+\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

(b) $f(x)=\cos x^{2}$. Use the expansion for $\cos x$ and substitute x^{2} for x.

$$\cos x=\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{(2n)!}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\cdots$$

$$\Rightarrow\cos x^{2}=\sum_{k=0}^{\infty}(-1)^{k}\frac{\left(x^{2}\right)^{2k}}{k!}=1-\frac{x^{4}}{2!}+\frac{x^{8}}{4!}-\frac{x^{12}}{6!}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^{k}x^{4k}}{(2k)!}$$

(c) $f(x)=x^{2}\sin 2x$. Use the expansion for $\sin x$, then substitute $2x$ for x and finally multiply the expression with x^{2}.

$$\sin x=\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots$$

$$\Rightarrow\sin 2x=\sum_{n=0}^{\infty}(-1)^{n}\frac{(2x)^{2n+1}}{(2n+1)!}=2x-\frac{(2x)^{3}}{3!}+\frac{(2x)^{5}}{5!}-\frac{(2x)^{7}}{7!}+\cdots$$

$$\Rightarrow\sin 2x=\sum_{n=0}^{\infty}(-1)^{n}\frac{2^{2n+1}x^{2n+1}}{(2n+1)!}=2x-\frac{2^{3}x^{3}}{3!}+\frac{2^{5}x^{5}}{5!}-\frac{2^{7}x^{7}}{7!}+\cdots$$

$$\Rightarrow x^{2}\sin 2x=\sum_{n=0}^{\infty}(-1)^{n}\frac{2^{2n+1}x^{2n+3}}{(2n+1)!}=2x^{3}-\frac{2^{3}x^{5}}{3!}+\frac{2^{5}x^{7}}{5!}-\frac{2^{7}x^{9}}{7!}+\cdots$$

(d) $f(x)=\sin^{2}x$. Use the provided hint, then expand the result using the expression for $\cos x$, replacing x with $2x$, and simplify.

$$\sin^{2}x=\frac{1}{2}(1-\cos 2x)$$

$$\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$\Rightarrow 1 - \cos 2x = 1 - \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots$$

$$= \frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \cdots$$

$$\Rightarrow \frac{1}{2}(1 - \cos 2x) = \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$$

(e) $f(x) = \ln(1 + x^2)$. Use the expansion for $\ln(1+x)$ and substitute x^2 for x.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^2)^n}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n}$$

(f) $f(x) = x^2 e^{-x^2}$. First use the expansion for e^x and replace x with $-x^2$.

Then multiply the resulting expression with x^2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\Rightarrow x^2 e^{-x^2} = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

(g) $f(x) = \begin{cases} \ln(1+x)^{1/x} & x \neq 0 \\ 1 & x = 0 \end{cases}$. Use logarithmic properties first and then the expansion for $\ln(1+x)$, and finally simplify.

$$\ln(1+x)^{1/x}=\frac{1}{x}\ln(1+x)=\frac{1}{x}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}==\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)$$

$$=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots=1+\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n+1}$$

2. **(a)** $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\Rightarrow e^{-0.02}\approx1-0.02+\frac{0.02^2}{2!}-\frac{0.02^3}{3!}=0.9802$

(b) $\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

$$\Rightarrow\sin0.1\approx0.1-\frac{0.1^3}{3!}+\frac{0.1^5}{5!}-\frac{0.1^7}{7!}=0.0998$$

(c) $\ln(1-x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

$$\Rightarrow\ln0.9=\ln(1-0.1)\approx-0.1-\frac{0.1^2}{2}-\frac{0.1^3}{3}-\frac{0.1^4}{4}=-0.1054$$

(d) We use the expansion found in (**1**, **(b)**):

$$\int_0^{0.5}\cos x^2\mathrm{d}x=\int_0^{0.5}\left(\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{(2k)!}\right)\mathrm{d}x\approx\int_0^{0.5}\left(1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}\right)\mathrm{d}x$$

$$=\left(x-\frac{x^5}{10}+\frac{x^9}{216}-\frac{x^{13}}{9360}\right)_0^{0.5}=0.4969$$

(e) We use the expansion found in (**1**, **(f)**):

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$$

$$\Rightarrow\int_0^1 e^{-x^2}\mathrm{d}x\approx\int_0^1\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}\right)\mathrm{d}x=\left(x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}\right)_0^1=0.7429$$

3. **(a)** $f(x)=x\sin x=x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)=x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\cdots$

(b) $$g\left(x\right)=\tan x=\frac{\sin x}{\cos x}=\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots}.$$

We use long division and stop when we get to the third term.

$$\begin{array}{r|l}
 & x+\frac{x^3}{3}+\frac{2x^5}{15} \\
\hline
1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots & x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \\
 & x-\frac{x^3}{2!}+\frac{x^5}{4!}-\cdots \\
\cline{2-2}
 & +\frac{x^3}{3}-\frac{x^5}{30}+\cdots \\
 & +\frac{x^3}{3}-\frac{x^5}{6}+\cdots \\
\cline{2-2}
 & \frac{2x^5}{15}+\cdots
\end{array}$$

Thus, $\tan x\approx x+\frac{x^3}{3}+\frac{2x^5}{15}$

(c) $$f\left(x\right)=\mathrm{e}^x\ln\left(1-x\right)=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)$$

$$\Rightarrow f\left(x\right)=1\cdot\left(-x-\frac{x^2}{2}-\frac{x^3}{3}+\cdots\right)+x\cdot\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right)+\frac{x^2}{2}\cdot\left(-x-\frac{x^2}{2}-\cdots\right)$$

$$=-x-\frac{x^2}{2}-\frac{x^3}{3}-x^2-\frac{x^3}{2}-\cdots-\frac{x^3}{2}-\cdots=-x-\frac{3x^2}{2}-\frac{4}{3}x^3$$

4. $$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots+x^n+\cdots$$

$$\Rightarrow\left(\frac{1}{1-x}\right)'=\frac{1}{\left(1-x\right)^2}=\sum_{n=0}^{\infty}nx^{n-1}=0+1+2x+3x^2+\cdots+nx^{n-1}+\cdots$$

5. $\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$. Use results from previous exercises to replace the functions involved:

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right)&=\ln(1+x)-\ln(1-x)\\&=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)\\&=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\cdots=\sum_{n=1}^{\infty}\frac{2}{2n-1}x^{2n-1}\end{aligned}$$

6. **(a)** $f(x)=\frac{e^x+e^{-x}}{2}\Rightarrow f'(x)=\frac{e^x-e^{-x}}{2}\Rightarrow f''(x)=\frac{e^x+e^{-x}}{2}\Rightarrow f'''(x)=\frac{e^x-e^{-x}}{2}$

Notice that only the coefficient of the term in e^{-x} is alternating its sign, thus,

$$f^{(n)}(x)=\frac{e^x+(-1)^n e^{-x}}{2}$$

(b) We find the derivatives up to 4th order:

$$\begin{array}{ll} f^{(0)}(x)=f(x)=\frac{e^x+e^{-x}}{2} & f(0)=1\\ f'(x)=\frac{e^x-e^{-x}}{2} & f'(0)=0\\ f''(x)=\frac{e^x+e^{-x}}{2} & f''(0)=1\\ f^{(3)}(x)=\frac{e^x-e^{-x}}{2} & f^{(3)}(0)=0\\ \vdots & \vdots\\ f^{(n)}(x)=\frac{e^x+(-1)^n e^{-x}}{2} & f^{(n)}(0)=0 \text{ or } 1\end{array}$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n=1+\frac{x^2}{2!}+\frac{x^4}{4!}$$

(c) $f(x)\approx 1+\frac{x^2}{2!}+\frac{x^4}{4!}\Rightarrow f\left(\frac{1}{2}\right)\approx 1+\frac{1}{8}+\frac{1}{384}=\frac{433}{384}$

7. There are several ways of approaching this question. One way is to consider the result of (**2**, (**b**)) and the formula $\sec^2 x = 1 + \tan^2 x$.

$$\tan x \approx x + \frac{x^3}{3} + \frac{2x^5}{15} \Rightarrow \tan^2 x \approx \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \right)^2$$

$$\Rightarrow \sec^2 x = 1 + \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \right)^2 = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \frac{4x^{10}}{225} + \cdots$$

8. **(a)** Using series to represent the numerator and denominator functions.

$$\lim_{x\to 0} \frac{1-\cos x}{e^x - 1 - x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x} = \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x}$$

$$= \lim_{x\to 0} \frac{1 - \frac{x^2}{12} + \frac{x^4}{360} - \cdots}{1 + \frac{x}{3} + \cdots} = 1$$

Notice that, in this case, using l'Hopital's rule twice would be more efficient.

(b) We will use series as requested.

$$\lim_{x\to 0} \frac{x - \arctan x}{x^3} = \lim_{x\to 0} \frac{x - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} + \cdots}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} + \cdots}{x^3}$$

$$= \lim_{x\to 0} \frac{x^3\left(\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} + \cdots\right)}{x^3} = \frac{1}{3}$$

9. **(a)** $y' + y = 0$. We assume that $y = \sum_{k=0}^{\infty} a_k x^k$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = y' = \sum_{k=0}^{\infty} k a_k x^{k-1}$

Substituting these values into the differential equation gives

$\sum_{k=0}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} a_k x^k = 0$, which implies

$$\left(a_1+2a_2x+3a_3x^2+\cdots+ka_kx^{k-1}+\cdots\right)+\left(a_0+a_1x+a_2x^2+\cdots+a_{k-1}x^{k-1}+\cdots\right)=0$$
$$\Rightarrow\left(a_1+a_0\right)+\left(2a_2+a_1\right)x+\left(3a_3+a_2\right)x^2+\cdots+\left(ka_k+a_{k-1}\right)x^{k-1}+\cdots=0$$

Since this series is equal to zero for all values of x, the coefficients must be zero:

$$\begin{aligned}a_1&=-a_0\\2a_2&=-a_1\\3a_3&=-a_2\\&\vdots\\ka_k&=-a_{k-1}\end{aligned}$$

Using these equations, we can solve for each coefficient in terms of the one preceding it.

$$\begin{aligned}a_1&=-a_0\\a_2&=-\frac{1}{2}a_1=\frac{1}{2}a_0\\a_3&=-\frac{1}{3}a_2=-\frac{1}{3}\cdot\frac{1}{2}a_0\\&\vdots\\a_k&=-\frac{1}{k}a_{k-1}=\frac{1}{k}\cdot\frac{1}{k-1}a_{k-2}=-\frac{1}{k}\cdot\frac{1}{k-1}\cdot\frac{1}{k-2}a_{k-3}\cdots=\frac{(-1)^k}{k(k-1)(k-2)\cdots}a_0=\frac{(-1)^k}{k!}a_0\end{aligned}$$

Returning to the series representation for y:

$$y=\sum_{k=0}^{\infty}a_kx^k=\sum_{k=0}^{\infty}\frac{(-1)^k a_0}{k!}x^k=a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^k=a_0\mathrm{e}^{-x}$$

Note: This DE is variables separable and easily done without series expansion. The exercise is simply meant to give you practice solving more complex DEs.

(b) $y'-6y=0$. We assume that $y=\sum_{k=0}^{\infty}a_kx^k$, so that $\frac{\mathrm{d}y}{\mathrm{d}x}=y'=\sum_{k=0}^{\infty}ka_kx^{k-1}$

Substituting these values into the differential equation gives

$\sum_{k=0}^{\infty}ka_kx^{k-1}-6\sum_{k=0}^{\infty}a_kx^k=0,$ which implies

$$\left(a_1+2a_2x+3a_3x^2+\cdots+ka_kx^{k-1}+\cdots\right)-6\left(a_0+a_1x+a_2x^2+\cdots+a_{k-1}x^{k-1}+\cdots\right)=0$$
$$\Rightarrow\left(a_1-6a_0\right)+\left(2a_2-6a_1\right)x+\left(3a_3-6a_2\right)x^2+\cdots+\left(ka_k-6a_{k-1}\right)x^{k-1}+\cdots=0$$

Since this series is equal to zero for all values of x, the coefficients must be zero:

$$\left.\begin{aligned} a_1&=6a_0\\ a_2&=\frac{6}{2}a_1\\ a_3&=\frac{6}{3}a_2\\ &\vdots\\ a_k&=\frac{6}{k}a_{k-1}\end{aligned}\right\}\Rightarrow\left\{\begin{aligned} a_1&=6a_0\\ a_2&=\frac{6}{2}a_1=\frac{6}{2}\cdot 6a_0\\ a_3&=\frac{6}{3}a_2=\frac{6}{3}\cdot\frac{6}{2}\cdot\frac{6}{1}a_0\\ &\vdots\\ a_k&=\frac{6}{k}a_{k-1}=\frac{6}{k}\cdot\frac{6}{k-1}\cdots\frac{6}{3}\cdot\frac{6}{2}\cdot\frac{6}{1}a_0=\frac{6^k}{k!}a_0\end{aligned}\right.$$

Returning to the series representation for y:

$$y=\sum_{k=0}^{\infty}a_kx^k=\sum_{k=0}^{\infty}a_0\frac{6^k}{k!}x^k=a_0\sum_{k=0}^{\infty}\frac{(6x)^k}{k!}=a_0\mathrm{e}^{6x}$$

(c) $y'-2xy=0$. We assume that $y=\sum_{k=0}^{\infty}a_kx^k$, so that $\frac{\mathrm{d}y}{\mathrm{d}x}=y'=\sum_{k=0}^{\infty}ka_kx^{k-1}$

Substituting these values into the differential equation gives:

$\sum_{k=0}^{\infty}ka_kx^{k-1}-2x\sum_{k=0}^{\infty}a_kx^k=0$, which implies

$$\left(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots+ka_kx^{k-1}\cdots\right)$$
$$-2x\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_{k-1}x^{k-1}\cdots\right)=0$$
$$\Rightarrow\left(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots+ka_kx^{k-1}+(k+1)a_{k+1}x^k\cdots\right)$$
$$-\left(2xa_0+2a_1x^2+2a_2x^3+2a_3x^4\cdots+2a_{k-2}x^k+\cdots\right)$$
$$\Rightarrow a_1+\left(2a_2-2a_0\right)x+\left(3a_3-2a_1\right)x^2+\left(4a_4-2a_2\right)x^3+\left(5a_5-2a_3\right)x^4\cdots$$
$$+\left((k+1)a_{k+1}-2a_{k-1}\right)x^k+\cdots=0$$

Since this series is equal to zero for all values of x, the coefficients must be zero:

$$\left.\begin{aligned} a_1 &= 0 \\ a_2 &= a_0 \\ a_3 &= \frac{2}{3}a_1 = 0 \\ a_4 &= \frac{2}{4}a_2 \\ &\vdots \\ a_{2k} &= \frac{2}{2n}a_{2(n-1)} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{2}{2}a_0 \\ a_3 &= 0 \\ a_4 &= \frac{2}{4}a_2 = \frac{1}{2}a_0 \\ &\vdots \\ a_6 &= \frac{2}{6}a_4 = \frac{1}{6}a_0 \\ &\vdots \\ a_{2n} &= \frac{1}{(2n)!}a_0 \end{aligned}\right.$$

Notice that $a_1 = a_3 = a_{2n-1} = 0$, i.e., the coefficients of odd powers are all zero.

Thus, $y = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_0 \frac{x^{2k}}{k!} = a_0 \sum_{k=0}^{\infty} \frac{\left(x^2\right)^k}{k!} = a_0 \mathrm{e}^{x^2}$

(d) $y' + 4y = 0, \quad y(0) = 1$. We assume that $y = \sum_{k=0}^{\infty} a_k x^k$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = y' = \sum_{k=0}^{\infty} k a_k x^{k-1}$

Substituting these values into the differential equation gives:

$\sum_{k=0}^{\infty} k a_k x^{k-1} + 4\sum_{k=0}^{\infty} a_k x^k = 0$, which implies

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + k a_k x^{k-1} + \cdots\right) + 4\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} + \cdots\right) = 0$$
$$\Rightarrow \left(a_1 + 4a_0\right) + \left(2a_2 + 4a_1\right)x + \left(3a_3 + 4a_2\right)x^2 + \cdots + \left(k a_k + 4a_{k-1}\right)x^{k-1} + \cdots = 0$$

Since this series is equal to zero for all values of x, the coefficients must be zero:

$$\left.\begin{array}{l} a_1 = -4a_0 \\ a_2 = -2a_1 \\ a_3 = -\frac{4}{3}a_2 \\ \quad\vdots \\ a_k = -\frac{4}{k}a_{k-1}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_1 = -4a_0 \\ a_2 = -2a_1 = -2\cdot -4a_0 \\ a_3 = -\frac{4}{3}a_2 = -\frac{4}{3}\cdot -\frac{4}{2}\cdot -\frac{4}{1}a_0 \\ \quad\vdots \\ a_k = -\frac{4}{k}a_{k-1} = -\frac{4}{k}\cdot -\frac{4}{k-1}\cdots -\frac{4}{3}\cdot -\frac{4}{2}\cdot -\frac{4}{1}a_0 = (-1)^k\frac{4^k}{k!}a_0\end{array}\right.$$

Returning to the series representation for y:

$$y=\sum_{k=0}^{\infty}a_kx^k=\sum_{k=0}^{\infty}(-1)^k\frac{4^k}{k!}a_0x^k=a_0\sum_{k=0}^{\infty}(-1)^k\frac{(4x)^k}{k!}=a_0\mathrm{e}^{-4x}$$

With the initial condition that $y(0)=1$, $y=a_0\mathrm{e}^{-4x}\Rightarrow 1=a_0$, thus, $y=\mathrm{e}^{-4x}$

(e) $(x+1)y'=3y\Rightarrow(x+1)y'-3y=0$. We assume that $y=\sum_{k=0}^{\infty}a_kx^k$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y'=\sum_{k=0}^{\infty}ka_kx^{k-1}$$

Substituting these values into the differential equation gives:

$(x+1)\sum_{k=0}^{\infty}ka_kx^{k-1}-3\sum_{k=0}^{\infty}a_kx^k=0$, which implies

$$\begin{aligned}&(x+1)\left(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots+ka_kx^{k-1}+\cdots\right)\\&\qquad-3\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_{k-1}x^{k-1}+\cdots\right)=0\\&\left(a_1x+2a_2x^2+3a_3x^3+4a_4x^4+5a_5x^5+\cdots+ka_kx^k+\cdots\right)\\&\qquad+\left(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots+ka_kx^{k-1}+\cdots\right)\\&\qquad-3\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+a_{k-1}x^{k-1}+\cdots\right)=0\\&\Rightarrow\left(a_1-3a_0\right)+\left(a_1+2a_2-3a_1\right)x+\left(2a_2+3a_3-3a_2\right)x^2+\left(3a_3+4a_4-3a_3\right)x^3\\&\qquad+\left(4a_4+5a_5-3a_4\right)x^4\cdots+\left(ka_k+(k+1)a_{k+1}-3a_k\right)x^k+\cdots=0\\&\Rightarrow\left(a_1-3a_0\right)+\left(2a_2-2a_1\right)x+\left(3a_3-a_2\right)x^2+\cdots\\&\qquad+\left((k+1)a_{k+1}+\left(k-3\right)a_k\right)x^k+\cdots=0\end{aligned}$$

Since this series is equal to zero for all values of x, the coefficients must be zero:

$$\left.\begin{aligned} a_1 &= 3a_0 \\ 2a_2 &= 2a_1 \\ a_3 &= \frac{1}{3}a_2 \\ 4a_4 &= 0 \\ 5a_5 &= -a_4 \\ &\vdots \\ a_k &= \frac{k-4}{k}a_{k-1} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a_1 &= 3a_0 \\ a_2 &= 3a_0 \\ a_3 &= \frac{1}{3}a_2 = a_0 \\ a_4 &= 0 \\ a_5 &= 0 \\ &\vdots \\ a_k &= 0 \end{aligned}\right.$$

Returning to the series representation for y:

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + 3a_0 x + 3a_0 x^2 + a_0 x^3 + 0 = a_0\left(x+1\right)^3$$

(f) $\left(x^2+1\right)y' = -2xy \Rightarrow \left(x^2+1\right)y' + 2xy = 0$.

We assume that $y = \sum_{k=0}^{\infty} a_k x^k$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = y' = \sum_{k=0}^{\infty} k a_k x^{k-1}$

Substituting these values into the differential equation gives:

$\left(x^2+1\right)\sum_{k=0}^{\infty} k a_k x^{k-1} + 2x\sum_{k=0}^{\infty} a_k x^k = 0$, which implies

$$\begin{aligned} &\left(x^2+1\right)\left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots + k a_k x^{k-1} + \cdots\right) \\ &\quad + 2x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{k-1}x^{k-1} + \cdots\right) = 0 \\ &\left(a_1 x^2 + 2a_2 x^3 + 3a_3 x^4 + 4a_4 x^5 + 5a_5 x^6 + \cdots + k a_k x^{k+1} + \cdots\right) \\ &\quad + \left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots + k a_k x^{k-1} + \cdots\right) \\ &\quad + \left(2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + \cdots + 2a_{k-1}x^k + \cdots\right) = 0 \\ &\Rightarrow a_1 + \left(2a_0 + 2a_2\right)x + \left(3a_1 + 3a_3\right)x^2 + \left(4a_2 + 4a_4\right)x^3 \\ &\quad + \left(5a_3 + 5a_5\right)x^4 \cdots + \left(\left(k+1\right)a_{k-1} + (k+1)a_{k+1}\right)x^k + \cdots = 0 \end{aligned}$$

Since this series is equal to zero for all values of x, the coefficients must be zero:

$$\left.\begin{aligned} a_1 &= 0 \\ 2a_2 &= -2a_0 \\ 3a_3 &= -3a_1 \\ 4a_4 &= -4a_2 \\ 5a_5 &= -5a_3 \\ &\vdots \\ a_k &= -ka_{k-2} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a_1 &= 0 \\ a_2 &= -a_0 \\ a_3 &= -a_1 = 0 \\ a_4 &= a_0 \\ a_5 &= 0 \\ &\vdots \end{aligned}\right.$$

Notice that $a_1 = a_3 = a_{2n-1} = 0$, i.e., the coefficients of odd powers are all zero.

Thus, $y = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (-1)^k a_0 x^{2k} = a_0 \sum_{k=0}^{\infty} (-1)^k x^{2k}$

10. **(a)** $f(x) = \sqrt[4]{1+x} = (1+x)^{\frac{1}{4}}$, the expansion is a binomial series.

Recall that the binomial series is given by

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Thus,

$$\begin{aligned} f(x) &= \sqrt[4]{1+x} = (1+x)^{\frac{1}{4}} \\ &= (1+x)^{\frac{1}{4}} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} x^n = 1 + \frac{1}{4}x + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}x^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{4}x - \frac{3}{4^2 2!}x^2 + \frac{3\times 7}{4^3 3!}x^3 - \frac{3\times 7\times 11}{4^4 4!}x^4 + \cdots \\ &= 1 + \frac{1}{4}x + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}\cdot 3\cdot 7\cdot 11\cdot\cdots\cdot(4k-5)x^k}{4^k k!} \end{aligned}$$

(b) $f(x) = \frac{1}{\sqrt[3]{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{3}}$

This can be done with the binomial series.

$$f(x)=\left(1-x^2\right)^{-\frac{1}{3}}$$

$$=\sum_{n=0}^{\infty}\begin{pmatrix}-\frac{1}{3}\\ n\end{pmatrix}\left(-x^2\right)^n=1+\frac{1}{3}x^2+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2!}\left(-x^2\right)^2+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}\left(-x^2\right)^3+\cdots$$

$$=1+\frac{1}{3}x^2+\frac{4}{3^2 2!}x^4+\frac{4\times 7}{3^3 3!}x^6+\cdots$$

$$=1+\sum_{k=1}^{\infty}\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3k-2)x^{2k}}{3^k k!}$$

(c) $$f(x)=\frac{1}{\sqrt[3]{1+x}}=(1+x)^{-\frac{1}{3}}$$

This can be done with the binomial series.

$$f(x)=(1+x)^{-\frac{1}{3}}$$

$$=\sum_{n=0}^{\infty}\begin{pmatrix}-\frac{1}{3}\\ n\end{pmatrix}x^n=1-\frac{1}{3}x+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2!}x^2+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}x^3+\cdots$$

$$=1-\frac{1}{3}x+\frac{4}{3^2 2!}x^2-\frac{4\times 7}{3^3 3!}x^3+\cdots$$

$$=1+\sum_{k=1}^{\infty}(-1)^k\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3k-2)x^k}{3^k k!}$$

(d) $$f(x)=\sqrt[3]{27+x}=3\left(1+\frac{x}{27}\right)^{\frac{1}{3}}$$

Again, with the binomial series.

$$f(x)=3(1+x)^{\frac{1}{3}}$$

$$=3\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{3}\\ n\end{pmatrix}\left(\frac{x}{27}\right)^n=3\left(1+\frac{1}{3}\cdot\frac{x}{27}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{27}\right)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{27}\right)^3+\cdots\right)$$

$$=3+\frac{x}{27}-\frac{2}{3^7 2!}x^2+\frac{2\times 5}{3^{11}3!}x^3+\cdots$$

$$=3+\frac{x}{27}+\sum_{k=2}^{\infty}\frac{(-1)^{k+1}\cdot 2\cdot 5\cdot 8\cdot 11\cdot\cdots\cdot(3k-4)x^k}{3^{4k-1}k!}$$

11. **(a)** $\sqrt{1.03}=\sqrt{1+0.03}=\sqrt{1+x}=1+\frac{0.03}{2}-\frac{0.03^2}{2^2\cdot 2!}\approx 1.01489$

(b) $\sqrt{99}=\sqrt{100-1}=10\sqrt{1-0.01}=10\sqrt{1-x}\approx 10\left(1-\frac{0.01}{2}-\frac{0.01^2}{8}\right)=9.9499$

(c) $\int_0^{0.4}\sqrt[3]{1+x^4}\,\mathrm{d}x=\int_0^{0.4}\left(1+x^4\right)^{\frac{1}{3}}\mathrm{d}x=\int_0^{0.4}\left(1+\frac{1}{3}x^4-\frac{1}{9}x^8+\cdots\right)\mathrm{d}x\approx 0.4007$

Chapter 16 practice questions

1. Separate the variables and integrate both sides

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{xy}{\sqrt{1+x^2}}\Rightarrow\frac{\mathrm{d}y}{y}=\frac{x\mathrm{d}x}{\sqrt{1+x^2}}\Rightarrow\int\frac{\mathrm{d}y}{y}=\int\frac{x\mathrm{d}x}{\sqrt{1+x^2}}$$
$$\Rightarrow\ln|y|=\sqrt{1+x^2}+c$$

With initial condition

$$\ln|1|=\sqrt{1+0}+c\Rightarrow c=-1\Rightarrow\ln|y|=\sqrt{1+x^2}-1$$
$$\Rightarrow y=\mathrm{e}^{\sqrt{1+x^2}-1}=\frac{\mathrm{e}^{\sqrt{1+x^2}}}{\mathrm{e}}$$

2. $\frac{\mathrm{d}y}{\mathrm{d}x}=\sin x\cos^2 y\Rightarrow\sec^2 y\,\mathrm{d}y=\sin x\,\mathrm{d}x\Rightarrow\tan y=-\cos x+C$

With initial condition

$$\tan\frac{\pi}{4}=-\cos\frac{\pi}{2}+C\Rightarrow C=1\Rightarrow\tan y=1-\cos x\Rightarrow y=\arctan\left(1-\cos x\right)$$

3. This is also a variables separable DE.

$$x\frac{\mathrm{d}y}{\mathrm{d}x}=y\left(3-y\right)\Rightarrow\frac{\mathrm{d}y}{y\left(3-y\right)}=\frac{\mathrm{d}x}{x}$$

The LHS can be simplified using partial fractions and then integrated.

$$\frac{\mathrm{d}y}{y(3-y)}=\frac{\mathrm{d}x}{x}\Rightarrow\frac{\mathrm{d}y}{3y}-\frac{\mathrm{d}y}{3(y-3)}=\frac{\mathrm{d}x}{x}\Rightarrow\frac{1}{3}\left(\ln|y|-\ln|y-3|\right)=\ln|x|+c$$

$$\Rightarrow\ln\left|\frac{y}{y-3}\right|=\ln\left|x^3\right|+3c\Rightarrow\left|\frac{y}{y-3}\right|=K\left|x^3\right|$$

With initial condition

$$\frac{y}{y-3}=Kx^3\Rightarrow\frac{2}{-1}=8K\Rightarrow K=-\frac{1}{4}$$

$$\frac{y}{y-3}=-\frac{1}{4}x^3\Rightarrow -4y=x^3(y-3)\Rightarrow y=\frac{3x^3}{x^3+4}$$

4. If $y=Cx^{\ln\sqrt{x}}$, then

$$\ln y=\ln\left(Cx^{\ln\sqrt{x}}\right)=\ln C+\ln\sqrt{x}\ln x=\ln C+\frac{1}{2}\ln x\cdot\ln x=\ln C+\frac{1}{2}(\ln x)^2$$

$$\Rightarrow\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\times 2\times\ln x\times\frac{1}{x}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=y\times\frac{\ln x}{x}$$

$$\Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x}=x\times y\times\frac{\ln x}{x}=y\ln x$$

5. $$\frac{\mathrm{d}Q}{\mathrm{d}t}=kQ\Rightarrow\frac{\mathrm{d}Q}{Q}=k\mathrm{d}t\Rightarrow\ln Q=kt+c\Rightarrow Q=Ae^{kt}$$

At time $t = 0$, there will be the original amount, Q_0, and thus the model is $Q=Q_0\mathrm{e}^{kt}$.

The half-life is 1620 means that

$$\frac{1}{2}Q_0=Q_0\mathrm{e}^{1620k}\Rightarrow 1620k=-\ln 2\Rightarrow k\approx -1.00042787$$

Remains of 10 grams after 25 years is $Q\approx 10\mathrm{e}^{-0.000427867\times 25}=9.89$ grams

6. **(a)** $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{y}\Rightarrow y\mathrm{d}y=2x\mathrm{d}x\Rightarrow\frac{y^2}{2}=x^2+c\Rightarrow y^2=2x^2+k$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2}{x^2}\Rightarrow\frac{\mathrm{d}y}{y^2}=\frac{\mathrm{d}x}{x^2}\Rightarrow-\frac{1}{y}=-\frac{1}{x}+c\Rightarrow y=\frac{1}{\frac{1}{x}-c}=\frac{x}{1-cx}$$

(c) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}=y^2-y\Rightarrow\dfrac{\mathrm{d}y}{y^2-y}=\dfrac{\mathrm{d}x}{x^2}$. LHS can be evaluated with partial fractions

$$\frac{1}{y(y-1)}=\frac{1}{y-1}-\frac{1}{y}\Rightarrow\int\frac{\mathrm{d}y}{y^2-y}=\ln|y-1|-\ln|y|=\int\frac{\mathrm{d}x}{x^2}=-\frac{1}{x}+c$$

$$\ln\left|\frac{y-1}{y}\right|=-\frac{1}{x}+c\Rightarrow\frac{y-1}{y}=\mathrm{e}^{-\frac{1}{x}+c}=A\mathrm{e}^{-\frac{1}{x}}$$

(d) $x\dfrac{\mathrm{d}y}{\mathrm{d}x}=\tan y\Rightarrow\dfrac{\cos y\mathrm{d}y}{\sin y}=\dfrac{\mathrm{d}x}{x}\Rightarrow\ln|\sin y|=\ln|x|+c\Rightarrow\sin y=\mathrm{e}^{\ln|x|+c}=A|x|$

$\Rightarrow y=\arcsin Ax$

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=xy\Rightarrow\dfrac{\mathrm{d}y}{y}=x\mathrm{d}x\Rightarrow\ln|y|=\dfrac{x^2}{2}+c\Rightarrow y=\mathrm{e}^{\frac{x^2}{2}+c}=A\mathrm{e}^{\frac{x^2}{2}}$

(f) $\sqrt{x^2+1}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x}{y}\Rightarrow y\mathrm{d}y=\dfrac{x\mathrm{d}x}{\sqrt{x^2+1}}\Rightarrow y^2=2\sqrt{x^2+1}+c$

(g) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y^2-1}{\mathrm{e}^x}\Rightarrow\dfrac{\mathrm{d}y}{y^2-1}=\mathrm{e}^{-x}\mathrm{d}x\Rightarrow\dfrac{\mathrm{d}y}{2(y+1)}+\dfrac{\mathrm{d}y}{2(y-1)}=\mathrm{e}^{-x}\mathrm{d}x$

(Using partial fractions for the LHS.)

$$\frac{\mathrm{d}y}{2(y+1)}+\frac{\mathrm{d}y}{2(y-1)}=\mathrm{e}^{-x}\mathrm{d}x\Rightarrow\frac{1}{2}\ln\left(y^2-1\right)=-\mathrm{e}^{-x}+c\Rightarrow\ln\sqrt{y^2-1}=c-\mathrm{e}^{-x}$$

(h) $\ln y\dfrac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow\ln y\,\mathrm{d}y=\mathrm{d}x$

Use integration by parts on the LHS.

$y\ln|y|-y=x+c$

7. Factor numerators and denominator and separate the variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{xy+y}{xy+x}=\frac{y(x+1)}{x(y+1)}\Rightarrow\frac{y+1}{y}\mathrm{d}y=\frac{x+1}{x}\mathrm{d}x$$

$$\int\frac{y+1}{y}\mathrm{d}y=\int\frac{x+1}{x}\mathrm{d}x\Rightarrow y+\ln|y|=x+\ln|x|+c$$

$$\Rightarrow \mathrm{e}^{y}\mathrm{e}^{\ln y}=\mathrm{e}^{x}\mathrm{e}^{\ln x}\mathrm{e}^{c}\Rightarrow y\mathrm{e}^{y}=Ax\mathrm{e}^{x}$$

8. $$y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x\Rightarrow y\mathrm{d}y=\cos x\,\mathrm{d}x\Rightarrow\int y\mathrm{d}y=\int\cos x\,\mathrm{d}x$$

$$\Rightarrow\frac{y^2}{2}=\sin x+c\Rightarrow y^2=2\sin x+C\Rightarrow y=\pm\sqrt{2\sin x+C}$$

The constant C cannot be completely arbitrary because $2\sin x+C\geq 0$. If $C<-2$, then $2\sin x+C$ will be negative for all values of x. If $-2\leq C\leq 2$, then $2\sin x+C$ will be positive for some values of x.

9. In order to be able to find the limiting value of the population, we first solve the DE.

$$\frac{\mathrm{d}p}{\mathrm{d}t}=5p-2p^2\Rightarrow\frac{\mathrm{d}p}{p(5-2p)}=\mathrm{d}t,\text{ LHS evaluated using partial fractions:}$$

$$\int\frac{\mathrm{d}p}{p(5-2p)}=\int\mathrm{d}t\Rightarrow\frac{1}{5}\ln\left|\frac{p}{2p-5}\right|=t+c\Rightarrow\ln\left|\frac{p}{2p-5}\right|=5t+K\quad,$$

$$\Rightarrow\frac{p}{2p-5}=\mathrm{e}^{5t+K}=A\mathrm{e}^{5t}$$

then, with initial condition

$$\frac{p}{2p-5}=A\mathrm{e}^{5t}\Rightarrow\frac{4}{3}=A\Rightarrow\frac{p}{2p-5}=\frac{4}{3}\mathrm{e}^{5t}$$

Solving for p, we have

$$p=\frac{20e^{5t}}{8\mathrm{e}^{5t}-3}$$

(a) $$\lim_{t\to\infty} p = \lim_{t\to\infty}\frac{20e^{5t}}{8e^{5t}-3} = \frac{20}{8} = \frac{5}{2}$$

(b) With the new initial condition:

$$\frac{p}{2p-5} = Ae^{5t} \Rightarrow \frac{0.5}{-4} = A \Rightarrow \frac{p}{2p-5} = -\frac{1}{8}e^{5t}$$

Solving for p, we have

$$p = \frac{5e^{5t}}{8+2e^{5t}}, \text{ and}$$

$$\lim_{t\to\infty} p = \lim_{t\to\infty}\frac{5e^{5t}}{8+2e^{5t}} = \frac{5}{2}$$

(c) Regardless of the initial value of the population, as time increases, the population stabilises at 2500.

10. $$\frac{dy}{dx} = \frac{2x+\sec^2 x}{2y} \Rightarrow 2y\,dy = \left(2x+\sec^2 x\right)dx \Rightarrow y^2 = x^2 + \tan x + c$$

$$y(0) = -5 \Rightarrow 25 = 0+0+c \Rightarrow y^2 = x^2 + \tan x + 25$$

11. **(a)** $$\left(1+x^2\right)\frac{dy}{dx} + 1 + y^2 = 0 \Rightarrow \left(1+x^2\right)dy + \left(1+y^2\right)dx = 0 \Rightarrow \frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$$

$$\Rightarrow \int\frac{dy}{1+y^2} + \int\frac{dx}{1+x^2} = c \Rightarrow \arctan y + \arctan x = c.$$

With initial condition

$$\arctan\left(-1\right) + \arctan 0 = c \Rightarrow c = -\frac{\pi}{4}, \text{ thus, } \arctan y + \arctan x = -\frac{\pi}{4}$$

(b) $$\tan\left(\arctan y + \arctan x\right) = -1 \Rightarrow \frac{y+x}{1-xy} = -1 \Rightarrow y + x = xy - 1$$

$$y - xy = -1 - x \Rightarrow y = \frac{-1-x}{1-x} = \frac{x+1}{x-1}$$

12. $$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y^2 \Rightarrow \frac{\mathrm{d}y}{1+y^2}=\frac{\mathrm{d}x}{1+x^2} \Rightarrow \int\frac{\mathrm{d}y}{1+y^2}=\int\frac{\mathrm{d}x}{1+x^2} \Rightarrow \arctan y=\arctan x+c$$

$$y(2)=3 \Rightarrow \arctan 3=\arctan 2+c \Rightarrow c=\arctan 3-\arctan 2$$

$$\Rightarrow \arctan y-\arctan x=\arctan 3-\arctan 2$$

$$\Rightarrow \tan\left(\arctan y-\arctan x\right)=\tan\left(\arctan 3-\arctan 2\right) \Rightarrow \frac{y-x}{1+xy}=\frac{1}{1+6}$$

$$7y-7x=1+xy \Rightarrow y\left(7-x\right)=1+7x \Rightarrow y=\frac{1+7x}{7-x}$$

13. **(a)** $$\frac{1}{x^2-x-2}=\frac{1}{\left(x-2\right)\left(x+1\right)} \Rightarrow \frac{1}{\left(x-2\right)\left(x+1\right)} \equiv \frac{A}{x-2}+\frac{B}{x+1}$$

$$\Rightarrow 1 \equiv A\left(x+1\right)+B\left(x-2\right) \Rightarrow A=\frac{1}{3}, B=-\frac{1}{3}$$

$$\Rightarrow \frac{1}{x^2-x-2} \equiv \frac{1}{3\left(x-2\right)}-\frac{1}{3\left(x+1\right)}$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2}{x^2-x-2} \Rightarrow \frac{\mathrm{d}y}{y^2}=\frac{\mathrm{d}x}{x^2-x-2} \Rightarrow \int\frac{\mathrm{d}y}{y^2}=\int\frac{\mathrm{d}x}{3\left(x-2\right)}-\int\frac{\mathrm{d}x}{3\left(x+1\right)}$$

$$\Rightarrow -\frac{1}{y}=\frac{1}{3}\ln\left(x-2\right)-\frac{1}{3}\ln\left(x+1\right) \Rightarrow \frac{1}{y}=\frac{1}{3}\ln\left(x+1\right)-\frac{1}{3}\ln\left(x-2\right)=\frac{1}{3}\ln\frac{x+1}{x-2}$$

$$\Rightarrow \frac{3}{y}=\ln\frac{x+1}{x-2}+c$$

With initial values

$$\Rightarrow \frac{3}{1}=\ln\frac{5+1}{5-2}+c \Rightarrow c=3-\ln 2$$

$$\Rightarrow \frac{3}{y}=\ln\frac{x+1}{x-2}+3-\ln 2 \Rightarrow \frac{3-3y}{y}+\ln 2=\ln\frac{x+1}{x-2}$$

$$\Rightarrow \mathrm{e}^{\frac{3-3y}{y}+\ln 2}=2\mathrm{e}^{\frac{3-3y}{y}}=\frac{x+1}{x-2}$$

14. **(a)**

$$\left(1-x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=2x\Rightarrow\left(1-x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2x-2xy=2x\left(1-y\right)$$

$$\Rightarrow\frac{\mathrm{d}y}{1-y}=\frac{2x\mathrm{d}x}{1-x^2}\Rightarrow\ln\left|y-1\right|=\ln\left|x^2-1\right|+c$$

$$\Rightarrow y-1=\mathrm{e}^{\ln\left|x^2-1\right|+c}=\mathrm{e}^c\mathrm{e}^{\ln\left|x^2-1\right|}=C\left(x^2-1\right)\Rightarrow y=C\left(x^2-1\right)+1$$

(b)

$$\left(1-x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=2x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2x}{1-x^2}y=\frac{2x}{1-x^2}$$

$$\Rightarrow\mathrm{IF}=\mathrm{e}^{\int\frac{2x}{1-x^2}\mathrm{d}x}=\mathrm{e}^{-\ln\left(1-x^2\right)}=\frac{1}{1-x^2}$$

Now multiply with the IF and simplify

$$\frac{1}{1-x^2}\left[\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2x}{1-x^2}y=\frac{2x}{1-x^2}\right]\Rightarrow\frac{\mathrm{d}y}{1-x^2}+\frac{2x\mathrm{d}x}{\left(1-x^2\right)^2}y=\frac{2x\mathrm{d}x}{\left(1-x^2\right)^2}$$

$$\Rightarrow\mathrm{d}\left(\frac{y}{1-x^2}\right)=\frac{2x\mathrm{d}x}{\left(1-x^2\right)^2}\Rightarrow\frac{y}{1-x^2}=\int\frac{2x\mathrm{d}x}{\left(1-x^2\right)^2}=\frac{1}{1-x^2}+c$$

$$\Rightarrow y=1+c\left(1-x^2\right)$$

The two answers are, in fact, the same. Just consider $c=-C$.

15. **(a)**

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{2}{x}\right)y=6x^3\Rightarrow\mathrm{IF}=\mathrm{e}^{\int\frac{2}{x}\mathrm{d}x}=x^2$$

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=6x^5\Rightarrow\mathrm{d}\left(x^2y\right)=6x^5\mathrm{d}x\Rightarrow x^2y=x^6+c\Rightarrow y=x^4+\frac{c}{x^2}$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x}-xy=x\Rightarrow\mathrm{IF}=\mathrm{e}^{\int-x\mathrm{d}x}=\mathrm{e}^{-\frac{x^2}{2}}$$

$$\mathrm{e}^{-\frac{x^2}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}-x\mathrm{e}^{-\frac{x^2}{2}}y=x\mathrm{e}^{-\frac{x^2}{2}}\Rightarrow\mathrm{d}\left(\mathrm{e}^{-\frac{x^2}{2}}y\right)=x\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$$

$$\Rightarrow\mathrm{e}^{-\frac{x^2}{2}}y=-\mathrm{e}^{-\frac{x^2}{2}}+c\Rightarrow y=c\mathrm{e}^{\frac{x^2}{2}}-1$$

(c) $$\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x}=x^3 \Rightarrow \text{IF}=\mathrm{e}^{\int -\frac{\mathrm{d}x}{x}}=\frac{1}{x}$$

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x^2}=x^2 \Rightarrow \mathrm{d}\left(\frac{y}{x}\right)=x^2\mathrm{d}x \Rightarrow \frac{y}{x}=\frac{1}{3}x^3+c \Rightarrow y=\frac{x^4}{3}+cx$$

(d) $$\frac{\mathrm{d}y}{\mathrm{d}x}+y\sin x=\mathrm{e}^{\cos x} \Rightarrow \text{IF}=\mathrm{e}^{\int \sin x\mathrm{d}x}=\mathrm{e}^{-\cos x}$$

$$\Rightarrow \mathrm{e}^{-\cos x}\frac{\mathrm{d}y}{\mathrm{d}x}+y\mathrm{e}^{-\cos x}\sin x=1 \Rightarrow \mathrm{d}\left(\mathrm{e}^{-\cos x}y\right)=\mathrm{d}x \Rightarrow \mathrm{e}^{-\cos x}y=x+c$$
$$\Rightarrow y=\mathrm{e}^{\cos x}\left(x+c\right)$$

(e) $$\frac{\mathrm{d}y}{\mathrm{d}x}-3x^2y=\mathrm{e}^{x^3} \Rightarrow \text{IF}=\mathrm{e}^{\int -3x^2\mathrm{d}x}=\mathrm{e}^{-x^3}$$

$$\Rightarrow \mathrm{e}^{-x^3}\frac{\mathrm{d}y}{\mathrm{d}x}-3x^2\mathrm{e}^{-x^3}y=1 \Rightarrow \mathrm{d}\left(\mathrm{e}^{-x^3}y\right)=\mathrm{d}x$$
$$\Rightarrow \mathrm{e}^{-x^3}y=x+c \Rightarrow y=\mathrm{e}^{x^3}\left(x+c\right)$$

(f) $$x\frac{\mathrm{d}y}{\mathrm{d}x}=x+y \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x}y=1 \Rightarrow \text{IF}=\mathrm{e}^{\int \frac{-\mathrm{d}x}{x}}=\mathrm{e}^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$$

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x^2}y=\frac{1}{x} \Rightarrow \mathrm{d}\left(\frac{y}{x}\right)=\frac{\mathrm{d}x}{x} \Rightarrow \frac{y}{x}=\ln|x|+c \Rightarrow y=x\ln|x|+cx$$

16. $$\tan x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\csc x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\cos x}{\sin x}y=\frac{\cos x}{\sin^2 x}$$

$$\Rightarrow \text{IF}=\mathrm{e}^{\int \frac{\cos x}{\sin x}\mathrm{d}x}=\mathrm{e}^{\ln(\sin x)}=\sin x$$

$$\Rightarrow \sin x\frac{\mathrm{d}y}{\mathrm{d}x}+y\cos x=\frac{\cos x}{\sin x} \Rightarrow \mathrm{d}\left(y\sin x\right)=\frac{\cos x}{\sin x}\mathrm{d}x$$
$$\Rightarrow y\sin x=\int\frac{\cos x}{\sin x}\mathrm{d}x=\ln\left(\sin x\right)+C \Rightarrow y=\frac{1}{\sin x}\ln\left(\sin x\right)+C\csc x$$

17. (a) $$\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{xy}{1-x^2}=1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}+y\left(\frac{-x}{1-x^2}\right)=1$$

(b) $\text{IF} = \mathrm{e}^{\int \frac{-x}{1-x^2}\mathrm{d}x} = \mathrm{e}^{\frac{1}{2}\ln\left(1-x^2\right)} = \sqrt{1-x^2}$

(c) Let $x = \sin u \Rightarrow \mathrm{d}x = \cos u\,\mathrm{d}u$

$$\int\sqrt{1-x^2}\mathrm{d}x = \int\sqrt{1-x^2}\mathrm{d}x = \int\cos u \times \cos u\,\mathrm{d}u = \int\cos^2 u\,\mathrm{d}u$$
$$= \frac{1}{2}\int\left(\cos 2u + 1\right)\mathrm{d}u = \frac{\sin 2u}{4} + \frac{u}{2} + c$$
$$= \frac{\sin u \cos u}{2} + \frac{u}{2} + c$$
$$= \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + c$$

(d) Multiply the IF by the equation in **(a)**

$$\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + y\left(\frac{-x}{\sqrt{1-x^2}}\right) = \sqrt{1-x^2} \Rightarrow \mathrm{d}\left(y\sqrt{1-x^2}\right) = \sqrt{1-x^2}\mathrm{d}x$$
$$\Rightarrow y\sqrt{1-x^2} = \int\sqrt{1-x^2}\mathrm{d}x = \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C$$
$$\Rightarrow y = \frac{x}{2} + \frac{\arcsin x}{2\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}$$
$$y(0) = 1 \Rightarrow C = 1 \Rightarrow y = \frac{x}{2} + \frac{\arcsin x}{2\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

18. **(a)** $\int\tan x\,\mathrm{d}x = \int\frac{\sin x}{\cos x}\mathrm{d}x$

Let $\cos x = u \Rightarrow \mathrm{d}u = -\sin x$

$$\int\frac{\sin x}{\cos x}\mathrm{d}x = -\int\frac{\mathrm{d}u}{u} = -\ln|u| = -\ln|\cos x|$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y\tan x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y\left(-\tan x\right) = 1$

(c) $\text{IF} = \mathrm{e}^{\int -\tan x\mathrm{d}x} = \mathrm{e}^{\ln(\cos x)} = \cos x$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x}+\cos x\,y\left(-\tan x\right)=\cos x\Rightarrow\cos x\frac{\mathrm{d}y}{\mathrm{d}x}+y\left(-\sin x\right)=\cos x$$
$$\Rightarrow\mathrm{d}\left(y\cos x\right)=\cos x\,\mathrm{d}x\Rightarrow y\cos x=\sin x+c\Rightarrow y=\tan x+c\sec x$$

19. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2\ln x-y}{x}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{x}y=x\ln x\Rightarrow\text{IF}=\mathrm{e}^{\int\frac{\mathrm{d}x}{x}}=\mathrm{e}^{\ln x}=x$

$x\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2\ln x\Rightarrow\mathrm{d}\left(xy\right)=x^2\ln x$

The RHS can be evaluated using integration by parts.

$\Rightarrow xy=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+c\Rightarrow y=\frac{1}{3}x^2\ln x-\frac{1}{9}x^2+\frac{c}{x}$, and with initial value

$1=0-\frac{1}{9}+\frac{c}{1}\Rightarrow c=\frac{10}{9}\Rightarrow y=\frac{1}{3}x^2\ln x-\frac{1}{9}x^2+\frac{10}{9x}$

20. $x^2\frac{\mathrm{d}y}{\mathrm{d}x}-x^3+xy=0\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy=x^3\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{x}y=x$ which is now a linear DE in standard form.

$\text{IF}=\mathrm{e}^{\int\frac{\mathrm{d}x}{x}}=\mathrm{e}^{\ln x}=x$

$x\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2\Rightarrow\mathrm{d}\left(xy\right)=x^2\mathrm{d}x\Rightarrow xy=\frac{x^3}{3}+c\Rightarrow y=\frac{x^2}{3}+\frac{c}{x}$

21. Use the substitution $y=vx\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$

For $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3y-x}{3x-y}$,replace the LHS with its value from above and divide numerator and denominator by x and simplify.

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{3v-1}{3-v}\Rightarrow 3v-1=3v-v^2+3x\frac{\mathrm{d}v}{\mathrm{d}x}-vx\frac{\mathrm{d}v}{\mathrm{d}x}$$
$$\Rightarrow v^2-1=x\left(3-v\right)\frac{\mathrm{d}v}{\mathrm{d}x}\Rightarrow\frac{3-v}{v^2-1}\mathrm{d}v=\frac{\mathrm{d}x}{x}$$

Use partial fractions to simplify the LHS.

$$\frac{\mathrm{d}v}{v-1}-\frac{2\mathrm{d}v}{v+1}=\frac{\mathrm{d}x}{x}\Rightarrow\int\frac{\mathrm{d}v}{v-1}-\int\frac{2\mathrm{d}v}{v+1}=\int\frac{\mathrm{d}x}{x}$$

$$\Rightarrow\ln(v-1)-2\ln(v+1)=\ln x+c\Rightarrow\ln\left(\frac{v-1}{(v+1)^2}\right)=\ln x+c$$

$$\Rightarrow\frac{v-1}{(v+1)^2}=\mathrm{e}^{\ln x+c}=Cx\Rightarrow\frac{\frac{y}{x}-1}{\left(\frac{y}{x}+1\right)^2}=Cx$$

$$\Rightarrow\frac{y-x}{x}\cdot\frac{x^2}{(y+x)^2}=Cx\Rightarrow C=\frac{y-x}{(y+x)^2}$$

22. In parts of this exercise, we will use the substitution $y=vx\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$ when it is needed.

(a) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x+1}\Rightarrow\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}x}{x+1}\Rightarrow\ln|y|=\ln|x+1|+c\Rightarrow y=A(x+1)$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x+2y}{x}\Rightarrow v+x\frac{\mathrm{d}v}{\mathrm{d}x}=1+2v\Rightarrow\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}v}{v+1}$$

$$\ln|v+1|=\ln|x|+c\Rightarrow v+1=Ax\Rightarrow\frac{y}{x}=Ax-1\Rightarrow y=Ax^2-x$$

Notice that $A=\mathrm{e}^c$

(c)

$$x\frac{\mathrm{d}y}{\mathrm{d}x}=2x+3y\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=2+3\frac{y}{x}\Rightarrow y=vx\Rightarrow v+x\frac{\mathrm{d}v}{\mathrm{d}x}=2+3v$$

$$\Rightarrow\frac{\mathrm{d}v}{2+2v}=\frac{\mathrm{d}x}{x}\Rightarrow\frac{1}{2}\ln|1+v|=\ln|x|+c\Rightarrow\ln\left|1+\frac{y}{x}\right|=\ln x^2+2c$$

$$\Rightarrow1+\frac{y}{x}=Ax^2\Rightarrow\frac{y}{x}=Ax^2-1\Rightarrow y=Ax^3-x$$

(d) $$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x^2+y^2}{2xy+3y^2}\Rightarrow v+x\frac{\mathrm{d}v}{dx}=-\frac{2+v^2}{2v+3v^2}$$

$$-2-v^2=2v^2+3v^3+2xv\frac{\mathrm{d}v}{\mathrm{d}x}+3v^2x\frac{\mathrm{d}v}{dx}$$

$$\Rightarrow -\frac{3v^2+2v}{3v^3+3v^2+2}\mathrm{d}v=\frac{\mathrm{d}x}{x},$$

Let $u=3v^3+3v^2+2\Rightarrow \mathrm{d}u=3\left(3v^2+2v\right)\mathrm{d}v$

$$\Rightarrow -\frac{3v^2+2v}{3v^3+3v^2+2}\mathrm{d}v=-\frac{1}{3}\frac{\mathrm{d}u}{u}=\frac{\mathrm{d}x}{x}\Rightarrow -\frac{1}{3}\int\frac{\mathrm{d}u}{u}=\int\frac{\mathrm{d}x}{x}$$

$$\Rightarrow -\frac{1}{3}\ln|u|=\ln|x|+c\Rightarrow \ln|u|=-3\ln|x|+C\Rightarrow u=\frac{A}{x^3}$$

$$\Rightarrow 3v^3+3v^2+2=\frac{A}{x^3}$$

$$\Rightarrow 3\left(\frac{y}{x}\right)^3+3\left(\frac{y}{x}\right)^2+2=\frac{A}{x^3}\Rightarrow 2x^3+3xy^2+3y^3=A$$

(e) Divide by xy first, then use the above substitution

$$xy\frac{\mathrm{d}y}{\mathrm{d}x}=x^2-y^2\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{v}-v\Rightarrow v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{v}-v$$

$$\Rightarrow\frac{v\mathrm{d}v}{1-2v^2}=\frac{\mathrm{d}x}{x}\Rightarrow\int\frac{v\mathrm{d}v}{1-2v^2}=\int\frac{\mathrm{d}x}{x}\Rightarrow -\frac{1}{4}\ln\left|1-2v^2\right|=\ln|x|+c$$

$$\Rightarrow\ln\left|1-2v^2\right|=-4\ln|x|+C\Rightarrow 1-2\left(\frac{y}{x}\right)^2=\frac{1}{x^4}+C$$

(f) $$x\left(y-x\right)\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(x+y\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\left(x+y\right)}{x\left(y-x\right)}$$

Dividing numerator and denominator of the RHS by x and using the previously mentioned substitution:

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v\cdot\frac{v+1}{v-1} \Rightarrow 2v = x\left(v-1\right)\frac{\mathrm{d}v}{\mathrm{d}x} \Rightarrow \frac{v-1}{2v}\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

$$\int\left(\frac{1}{2}-\frac{1}{2v}\right)\mathrm{d}v = \int\frac{\mathrm{d}x}{x} \Rightarrow \frac{1}{2}\left(v-\ln v\right) = \ln\left|x\right| + c \Rightarrow v - \ln v = 2\ln\left|x\right| + C$$

$$\ln v = v - \ln x^2 - C \Rightarrow v = \mathrm{e}^{v-\ln x^2 - C} \Rightarrow \frac{y}{x} = \mathrm{e}^{-C}\mathrm{e}^{v}\frac{1}{x^2} \Rightarrow y = A\frac{\mathrm{e}^{\frac{y}{x}}}{x}$$

$$\mathrm{e}^{\frac{y}{x}} = Bxy \Rightarrow \frac{y}{x} = \ln\left(Bxy\right) \Rightarrow y = x\ln\left(Bxy\right)$$

23. **(a)** $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{3y-2x} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+2\left(\frac{y}{x}\right)}{3\left(\frac{y}{x}\right)-2} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+2v}{3v-2}$$

(b) Multiply and collect like terms.

$$3v^2 - 2v + x\left(3v-2\right)\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v \Rightarrow -\frac{3v-2}{3v^2-4v-1}\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

$$\Rightarrow -\frac{1}{2}\int\frac{6v-4}{3v^2-4v-1}\,\mathrm{d}v = \int\frac{\mathrm{d}x}{x} \Rightarrow \ln\left|3v^2-4v-1\right| = -2\ln\left|x\right| + c$$

$$\Rightarrow 3v^2 - 4v - 1 = -\frac{1}{x^2} + c \Rightarrow 3\times\frac{y^2}{x^2} - 4\times\frac{y}{x} - 1 = -\frac{1}{x^2} + c$$

$$\Rightarrow 3y^2 - 4xy - x^2 = -1 + cx^2$$

With initial value

$$0 - 0 - 1 = -1 + c \Rightarrow c = 0 \Rightarrow 3y^2 - 4xy - x^2 + 1 = 0$$

24. Substitute $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

$$y^2 - x^2 + xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow v^2 - 1 + v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = 0$$

$$\Rightarrow v^2 - 1 + v^2 + vx\frac{\mathrm{d}v}{\mathrm{d}x} = 0 \Rightarrow \int\frac{v\mathrm{d}v}{1-2v^2} = \int\frac{\mathrm{d}x}{x}$$

$$\Rightarrow -\frac{1}{4}\ln\left|2v^2 - 1\right| = \ln|x| + c \Rightarrow \ln\left|\frac{1}{2v^2-1}\right| = \ln x^4 + 4c$$

$$\Rightarrow \frac{1}{2\frac{y^2}{x^2} - 1} = \mathrm{e}^{4c}x^4 \Rightarrow \frac{x^2}{2y^2 - x^2} = Ax^4 \Rightarrow 1 = A\left(2x^2y^2 - x^4\right)$$

$C = 2x^2y^2 - x^4$, where $C = \frac{1}{A}$

25. **(a)** Substitute $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{x^2 - y^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sqrt{1-v^2}$$

(b) The equation above can be simplified to $x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{1-v^2}$, then,

$$\int\frac{\mathrm{d}v}{\sqrt{1-v^2}} = \int\frac{\mathrm{d}x}{x} \Rightarrow \arcsin v = \ln|x| + c \Rightarrow \arcsin\frac{y}{x} = \ln|x| + c$$

With the initial value

$$\arcsin\frac{1}{1} = \ln|1| + c \Rightarrow c = \frac{\pi}{2} \Rightarrow \arcsin\frac{y}{x} = \ln|x| + \frac{\pi}{2}$$

26. **(a)** $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + y}{x} \Rightarrow x\mathrm{d}y = \left(y^2 + y\right)\mathrm{d}x \Rightarrow \int\frac{\mathrm{d}y}{y^2 + y} = \int\frac{\mathrm{d}x}{x}$$

The LHS can be evaluated using partial fractions.

$$\ln|y| - \ln|y+1| = \ln|x| + c \Rightarrow \left|\frac{y}{y+1}\right| = \mathrm{e}^c|x| = C|x|,$$

(b) $\left|\dfrac{y}{y+1}\right| = C|x| \Rightarrow \dfrac{1}{2} = C \Rightarrow \left|\dfrac{y}{y+1}\right| = \dfrac{1}{2}|x|$

(c)(d) Use a spreadsheet for Euler's method calculations.

$y_{n+1} = y_n + hF(x_n, y_n)$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = F(x, y) = \dfrac{y^2 + y}{x}$, we have

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + h\left(\frac{y_n^2 + y_n}{x_n}\right)$$

For example, $y_{0+1} = y_0 + hF(x_0, y_0) = 1 + 0.2\left(\dfrac{1+1}{1}\right) = 1.4$

x_n	approx. y_n	exact y_n	% error
1.2	1.400	1.5	$6.\overline{6}$
1.4	1.960	$2.\overline{3}$	16
1.6	2.789	4	30.3
1.8	4.110	9	54.3

27. $y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + h\left(x_n y_n^2\right)$

If 5 steps are needed, then $h = 0.2$

For example, $y_{0+1} = y_0 + hF(x_0, y_0) = 1 + 0.2(0) = 1$, and

$y_{1+1} = y_1 + hF(x_1, y_1) = 1 + 0.2(0.2 \times 1) = 1.04$,

x(n)	y(n)
0	1
0.2	1
0.4	1.04
0.6	1.126528
0.8	1.278816
1	1.540475

28. $y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + 0.1\left(\mathrm{e}^{x_n y_n}\right)$

For example, $y_{0+1} = y_0 + 0.1F(x_0, y_0) = 1 + 0.1\left(\mathrm{e}^{0\times1}\right) = 1.1$

x(n)	y(n)
0	1
0.1	1.1
0.2	1.211628
↓	↓
0.9	3.539766
1	5.958404

29. Substitute $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+3y^2}{2xy} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+3v^2}{2v} \Rightarrow \frac{2v\mathrm{d}v}{1+v^2} = \frac{\mathrm{d}x}{x}$$
$$\Rightarrow \ln\left(1+v^2\right) = \ln|x| + c \Rightarrow 1+v^2 = A|x|$$
$$\Rightarrow 1 + \frac{y^2}{x^2} = Ax \Rightarrow y^2 = Ax^3 - x^2$$

30. $y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + 0.1\left(x_n\sqrt{y_n}\right)$

For example, $y_{0+1} = y_0 + 0.1F(x_0, y_0) = 4 + 0.1\left(1\times\sqrt{4}\right) = 4.2$

x(n)	y(n)
1	4
1.1	4.2
1.2	4.425433
1.3	4.677873
1.4	4.959043
1.5	5.270807

31. **(a)** This is a linear DE: $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + y = x \Rightarrow \mathrm{IF} = \mathrm{e}^{\int \mathrm{d}x} = \mathrm{e}^x$

$$\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^x y = x\mathrm{e}^x \Rightarrow \mathrm{d}\left(\mathrm{e}^x y\right) = x\mathrm{e}^x\mathrm{d}x \Rightarrow \mathrm{e}^x y = \int x\mathrm{e}^x\mathrm{d}x$$

RHS can be evaluated with by parts $\mathrm{e}^x y = \mathrm{e}^x\left(x-1\right)+c \Rightarrow y = x-1+c\mathrm{e}^{-x}$

With initial value: $0 = -1+c\mathrm{e}^0 \Rightarrow c = 1 \Rightarrow y = x-1+\mathrm{e}^{-x}$

(b)(c) With 5 steps, we will need $h = 0.2$; with 10 steps, we will need $h = 0.1$

$$y_{n+1} = y_n + hF\left(x_n, y_n\right) \Rightarrow y_{n+1} = y_n + h\left(x_n - y_n\right)$$

0.2		0.1	
x(n)	y(n)	x(n)	y(n)
0	0	0	0
0.2	0	0.1	0
0.4	0.04	0.2	0.01
0.6	0.112	0.3	0.029
0.8	0.2096	0.4	0.0561
1	0.32768	0.5	0.09049
		0.6	0.131441
		0.7	0.178297
		0.8	0.230467
		0.9	0.28742
		1	0.348678

(d) Actual value to 10 s.f. is $y(1) \approx 0.3678794412$. Notice that with 10 steps the discrepancy is less than with 5 steps. Thus, using more steps (and a smaller step size) gives a better approximation.

32. $\dfrac{\mathrm{d}\alpha}{\mathrm{d}t} = -k\left(\alpha-20\right) \Rightarrow \dfrac{\mathrm{d}\alpha}{\alpha-20} = -k\mathrm{d}t \Rightarrow \ln\left(\alpha-20\right) = -kt+c$

$\Rightarrow \alpha - 20 = A\mathrm{e}^{-kt} \Rightarrow \alpha = A\mathrm{e}^{-kt}+20$

With the 2 initial values (0, 70) and (10, 50), we can find the values of A and k.

$\left(0,70\right)$: $\alpha = A\mathrm{e}^{-kt}+20 \Rightarrow 70 = A+20 \Rightarrow A = 50 \Rightarrow \alpha = 50\mathrm{e}^{-kt}+20$

$\left(10,50\right)$: $\alpha = 50\mathrm{e}^{-kt}+20 \Rightarrow 50 = 50\mathrm{e}^{-10k}+20 \Rightarrow \mathrm{e}^{-10k} = \dfrac{3}{5} \Rightarrow k = -\dfrac{1}{10}\ln\dfrac{3}{5}$

33. **(a)** $x\frac{\mathrm{d}y}{\mathrm{d}x}-3y=x^4\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{3}{x}y=x^3$. This is a linear DE with $\mathrm{IF}=\mathrm{e}^{\int-\frac{3\mathrm{d}x}{x}}=\mathrm{e}^{\ln\left(\frac{1}{x^3}\right)}=\frac{1}{x^3}$

$$\frac{1}{x^3}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{3}{x^4}y=1\Rightarrow\mathrm{d}\left(\frac{y}{x^3}\right)=\mathrm{d}x\Rightarrow\frac{y}{x^3}=x+c\Rightarrow y=x^4+cx^3$$

(b) $2=1+c\Rightarrow c=1\Rightarrow y=x^4+x^3$

34. **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x^2+x}\Rightarrow m_{\text{tangent}}=\frac{6}{9+3}=\frac{1}{2}\Rightarrow m_{\text{normal}}=-2$

Equation of normal: $y-y_0=m\left(x-x_0\right)\Rightarrow y-6=-2\left(x-3\right)\Rightarrow y=-2x+12$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x^2+x}\Rightarrow\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}x}{x\left(x+1\right)}\Rightarrow\ln y=\ln x-\ln\left(x+1\right)+c$$

(b) $\ln y=\ln\frac{x}{x+1}+c\Rightarrow y=\frac{Ax}{x+1}$

$\left(3,6\right)$: $6=\frac{3A}{4}\Rightarrow A=8\Rightarrow y=\frac{8x}{x+1}$

35. **(a)** Use the substitution $y=vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3y^2+x^2}{2xy}\Rightarrow v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{3v^2+1}{2v}$$

(b) When $x=1,y=2$

$$\frac{2v\mathrm{d}v}{1+v^2}=\frac{\mathrm{d}x}{x}\Rightarrow\ln\left(1+v^2\right)=\ln\left|x\right|+c\Rightarrow1+v^2=A\left|x\right|$$

$$\Rightarrow1+\frac{y^2}{x^2}=Ax\Rightarrow y^2=Ax^3-x^2$$

$$4=A-1\Rightarrow A=5\Rightarrow y^2=5x^3-x^2$$

36. **(a)** $f(x)=\ln(1+\sin x)\Rightarrow f'(x)=\dfrac{\cos x}{1+\sin x}$

Using the quotient rule for derivatives

$$f''(x)=\frac{-\sin x(1+\sin x)-\cos x\cdot\cos x}{(1+\sin x)^2}=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}=\frac{-1}{1+\sin x}$$

(b) To determine the Maclaurin's series, we will need derivatives until the 4th order.

$$f(0)=0$$
$$f'(0)=1$$
$$f''(0)=-1$$

$$f'''(x)=(f''(x))'=\left(\frac{-1}{1+\sin x}\right)'=\frac{\cos x}{(1+\sin x)^2}\Rightarrow f'''(0)=1$$

$$f^{(iv)}(x)=-\frac{\sin x\cos^2 x+2(1+\sin x)}{(1+\sin x)^4}\Rightarrow f^{(iv)}(0)=-2$$

Therefore,

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}x^k=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(k)}(0)}{k!}x^k+\cdots$$

$$=0+x-\frac{x^2}{2}+\frac{x^3}{6}-2\cdot\frac{x^4}{24}+\cdots=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\cdots$$

(c) Replace x by $-x$ in the expansion in **(b)**

$$f(x)=\ln(1-\sin x)=\ln(1+(-\sin x))$$

$$=(-x)-\frac{(-x)^2}{2}+\frac{(-x)^3}{6}-\frac{(-x)^4}{12}+\cdots=-x-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{12}+\cdots$$

(d) Since $\sec x = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1-\sin^2 x}} = \dfrac{1}{\sqrt{1+\sin x}} \cdot \dfrac{1}{\sqrt{1-\sin x}}$, then

$$\begin{aligned}
\ln\left(\sec x\right) &= \ln\left(\frac{1}{\sqrt{1+\sin x}} \cdot \frac{1}{\sqrt{1-\sin x}}\right) = -\frac{1}{2}\ln\left(1+\sin x\right) - \frac{1}{2}\ln\left(1-\sin x\right) \\
&= -\frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots - x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \cdots\right) \\
&= -\frac{1}{2}\left(-x^2 - \frac{x^4}{6} - \cdots\right) = \frac{x^2}{2} + \frac{x^4}{12} + \cdots .
\end{aligned}$$

(e) $\displaystyle \lim_{x\to 0} \frac{\ln\left(\sec x\right)}{x\sqrt{x}} = \lim_{x\to 0} \frac{\frac{x^2}{2} + \frac{x^4}{12} + \cdots}{x\sqrt{x}} = \lim_{x\to 0}\left(\frac{\sqrt{x}}{2} + \frac{x^2\sqrt{x}}{12} + \cdots\right) = 0$