HAESE MATHEMATICS

Mathematics

Applications and Interpretation HL 2

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for use with

IB Diploma Programme

WORKED SOLUTIONS

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MATHEMATICS:
APPLICATIONS AND INTERPRETATION HL WORKED SOLUTIONS

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Published by:
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National Library of Australia Card Number & ISBN 978-1-925489-87-3

First Edition 2020

Artwork by Brian Houston and Bronson Mathews.

Typeset in Australia by Charlotte Frost and Deanne Gallasch. Typeset in Times Roman 10.

This book is available on Snowflake only.

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FOREWORD

This book gives you fully worked solutions for every question in Exercises, Review Sets, Activities, and Investigations (which do not involve student experimentation) in each chapter of our textbook *Mathematics: Applications and Interpretation HL*.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modelled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

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Chapter 1

EXPONENTIALS

EXERCISE 1A

1 **a** $\sqrt[5]{2} = 2^{\frac{1}{5}}$

b $\frac{1}{\sqrt[5]{2}} = \frac{1}{2^{\frac{1}{5}}}$
$= 2^{-\frac{1}{5}}$

c $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}}$
$= 2^{1+\frac{1}{2}}$
$= 2^{\frac{3}{2}}$

d $4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}}$
$= 2^{2+\frac{1}{2}}$
$= 2^{\frac{5}{2}}$

e $\frac{1}{\sqrt[3]{2}} = \frac{1}{2^{\frac{1}{3}}}$
$= 2^{-\frac{1}{3}}$

f $2 \times \sqrt[3]{2} = 2^1 \times 2^{\frac{1}{3}}$
$= 2^{1+\frac{1}{3}}$
$= 2^{\frac{4}{3}}$

g $\frac{4}{\sqrt{2}} = \frac{2^2}{2^{\frac{1}{2}}}$
$= 2^{2-\frac{1}{2}}$
$= 2^{\frac{3}{2}}$

h $(\sqrt{2})^3 = (2^{\frac{1}{2}})^3$
$= 2^{3\times\frac{1}{2}}$
$= 2^{\frac{3}{2}}$

i $\frac{1}{\sqrt[3]{16}} = \frac{1}{\sqrt[3]{2^4}}$
$= \frac{1}{2^{\frac{4}{3}}}$
$= 2^{-\frac{4}{3}}$

j $\frac{1}{\sqrt{8}} = \frac{1}{\sqrt{2^3}}$
$= \frac{1}{2^{\frac{3}{2}}}$
$= 2^{-\frac{3}{2}}$

2 **a** $\sqrt[3]{3} = 3^{\frac{1}{3}}$

b $\frac{1}{\sqrt[3]{3}} = \frac{1}{3^{\frac{1}{3}}}$
$= 3^{-\frac{1}{3}}$

c $\sqrt[4]{3} = 3^{\frac{1}{4}}$

d $3\sqrt{3} = 3^1 \times 3^{\frac{1}{2}}$
$= 3^{\frac{3}{2}}$

e $\frac{1}{9\sqrt{3}} = \frac{1}{3^2 \times 3^{\frac{1}{2}}}$
$= \frac{1}{3^{\frac{5}{2}}}$
$= 3^{-\frac{5}{2}}$

3 **a** $\sqrt[3]{7} = 7^{\frac{1}{3}}$

b $\sqrt[4]{27} = \sqrt[4]{3^3}$
$= \left(3^3\right)^{\frac{1}{4}}$
$= 3^{\frac{3}{4}}$

c $\sqrt[5]{16} = \sqrt[5]{2^4}$
$= \left(2^4\right)^{\frac{1}{5}}$
$= 2^{\frac{4}{5}}$

d $\sqrt[3]{32} = \sqrt[3]{2^5}$
$= \left(2^5\right)^{\frac{1}{3}}$
$= 2^{\frac{5}{3}}$

e $\sqrt[7]{49} = \sqrt[7]{7^2}$
$= \left(7^2\right)^{\frac{1}{7}}$
$= 7^{\frac{2}{7}}$

f $\frac{1}{\sqrt[3]{7}} = \frac{1}{7^{\frac{1}{3}}}$
$= 7^{-\frac{1}{3}}$

g $\frac{1}{\sqrt[4]{27}} = \frac{1}{3^{\frac{3}{4}}}$
$= 3^{-\frac{3}{4}}$

h $\frac{1}{\sqrt[5]{16}} = \frac{1}{2^{\frac{4}{5}}}$
$= 2^{-\frac{4}{5}}$

i $\dfrac{1}{\sqrt[3]{32}} = \dfrac{1}{2^{\frac{5}{3}}} = 2^{-\frac{5}{3}}$

j $\dfrac{1}{\sqrt[7]{49}} = \dfrac{1}{7^{\frac{2}{7}}} = 7^{-\frac{2}{7}}$

4 **a** $\sqrt{x} = x^{\frac{1}{2}}$

b $x\sqrt{x} = x^1 \times x^{\frac{1}{2}} = x^{\frac{3}{2}}$

c $\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$

d $x^2\sqrt{x} = x^2 \times x^{\frac{1}{2}} = x^{\frac{5}{2}}$

e $\dfrac{1}{x\sqrt{x}} = \dfrac{1}{x^{\frac{3}{2}}} = x^{-\frac{3}{2}}$

5 **a** $3^{\frac{3}{4}} \approx 2.28$

b $4^{-\frac{3}{5}} \approx 0.435$

c $\sqrt[4]{8} \approx 1.68$

d $\sqrt[5]{27} \approx 1.93$

e $\dfrac{1}{\sqrt[3]{7}} \approx 0.523$

6 **a** $5^{\frac{1}{3}} = \sqrt[3]{5}$

b $3^{-\frac{1}{2}} = \dfrac{1}{3^{\frac{1}{2}}} = \dfrac{1}{\sqrt{3}}$

c $3^{\frac{5}{2}} = 3^2 \times 3^{\frac{1}{2}} = 9\sqrt{3}$

d $m^{\frac{3}{2}} = m \times m^{\frac{1}{2}} = m\sqrt{m}$

e $x^{\frac{7}{2}} = x^3 \times x^{\frac{1}{2}} = x^3\sqrt{x}$

7 **a** $4^{\frac{3}{2}} = \left(2^2\right)^{\frac{3}{2}} = 2^3 = 8$

b $8^{\frac{5}{3}} = \left(2^3\right)^{\frac{5}{3}} = 2^5 = 32$

c $16^{\frac{3}{4}} = \left(2^4\right)^{\frac{3}{4}} = 2^3 = 8$

d $25^{\frac{3}{2}} = \left(5^2\right)^{\frac{3}{2}} = 5^3 = 125$

e $32^{\frac{2}{5}} = \left(2^5\right)^{\frac{2}{5}} = 2^2 = 4$

f $4^{-\frac{1}{2}} = \left(2^2\right)^{-\frac{1}{2}} = 2^{-1} = \frac{1}{2}$

g $9^{-\frac{3}{2}} = \left(3^2\right)^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$

h $8^{-\frac{4}{3}} = \left(2^3\right)^{-\frac{4}{3}} = 2^{-4} = \frac{1}{16}$

i $27^{-\frac{4}{3}} = \left(3^3\right)^{-\frac{4}{3}} = 3^{-4} = \frac{1}{81}$

j $125^{-\frac{2}{3}} = \left(5^3\right)^{-\frac{2}{3}} = 5^{-2} = \frac{1}{25}$

EXERCISE 1B

1 **a** $x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$
$= x^0$
$= 1$

b $x^{\frac{3}{2}} \times x^{-\frac{1}{2}}$
$= x^1$
$= x$

c $x^2 \times x^{-\frac{3}{2}}$
$= x^{\frac{1}{2}}$ or $\sqrt{x}$

2 **a** $x^2(x^3 + 2x^2 + 1)$
$= x^2 \times x^3 + x^2 \times 2x^2 + x^2 \times 1$
$= x^5 + 2x^4 + x^2$

b $2^x(2^x + 1)$
$= 2^x \times 2^x + 2^x \times 1$
$= 2^{2x} + 2^x$

c $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$
$= x^{\frac{1}{2}} \times x^{\frac{1}{2}} + x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$
$= x^1 + x^0$
$= x + 1$

d $7^x(7^x + 2)$
$= 7^x \times 7^x + 7^x \times 2$
$= 7^{2x} + 2(7^x)$

e $3^x(2 - 3^{-x})$
$= 3^x \times 2 - 3^x \times 3^{-x}$
$= 2(3^x) - 3^0$
$= 2(3^x) - 1$

f $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$
$= x^{\frac{1}{2}} \times x^{\frac{3}{2}} + x^{\frac{1}{2}} \times 2x^{\frac{1}{2}} + x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}}$
$= x^2 + 2x^1 + 3x^0$
$= x^2 + 2x + 3$

g $2^{-x}(2^x + 5)$
$= 2^{-x} \times 2^x + 2^{-x} \times 5$
$= 2^0 + 5(2^{-x})$
$= 1 + 5(2^{-x})$

h $5^{-x}(5^{2x} + 5^x)$
$= 5^{-x} \times 5^{2x} + 5^{-x} \times 5^x$
$= 5^x + 5^0$
$= 5^x + 1$

i $x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}})$
$= x^{-\frac{1}{2}} \times x^2 + x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times x^{\frac{1}{2}}$
$= x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^0$
$= x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$

j $3^x(3^x + 5 + 3^{-x})$
$= 3^x \times 3^x + 3^x \times 5 + 3^x \times 3^{-x}$
$= 3^{2x} + 5(3^x) + 3^0$
$= 3^{2x} + 5(3^x) + 1$

k $x^{-\frac{1}{2}}(2x^2 - x + 5x^{\frac{1}{2}})$
$= x^{-\frac{1}{2}} \times 2x^2 - x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times 5x^{\frac{1}{2}}$
$= 2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5x^0$
$= 2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5$

l $2^{2x}(2^x - 3 - 2^{-2x})$
$= 2^{2x} \times 2^x - 2^{2x} \times 3 - 2^{2x} \times 2^{-2x}$
$= 2^{3x} - 3(2^{2x}) - 2^0$
$= 2^{3x} - 3(2^{2x}) - 1$

3 **a** $(2^x - 1)(2^x + 3)$
$= 2^x \times 2^x + 2^x \times 3 - 1 \times 2^x - 3$
$= 2^{2x} + 2(2^x) - 3$
$= 2^{2x} + 2^{x+1} - 3$

b $(3^x + 2)(3^x + 5)$
$= 3^x \times 3^x + 3^x \times 5 + 2 \times 3^x + 10$
$= 3^{2x} + 7(3^x) + 10$

c
$$\begin{aligned}&(5^x-2)(5^x-4)\\&=5^x\times5^x-5^x\times4-2\times5^x+8\\&=5^{2x}-6(5^x)+8\end{aligned}$$

d
$$\begin{aligned}&(2^x+3)^2\\&=(2^x)^2+2\times2^x\times3+3^2\\&=2^{2x}+6(2^x)+9\end{aligned}$$

e
$$\begin{aligned}&(3^x-1)^2\\&=(3^x)^2-2\times3^x\times1+1^2\\&=3^{2x}-2(3^x)+1\end{aligned}$$

f
$$\begin{aligned}&(4^x+7)^2\\&=(4^x)^2+2\times4^x\times7+7^2\\&=4^{2x}+14(4^x)+49\end{aligned}$$

g
$$\begin{aligned}&(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)\\&=(x^{\frac{1}{2}})^2-2^2\\&=x-4\end{aligned}$$

h
$$\begin{aligned}&(2^x+3)(2^x-3)\\&=(2^x)^2-3^2\\&=2^{2x}-9\\&=4^x-9\end{aligned}$$

i
$$\begin{aligned}&(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})\\&=(x^{\frac{1}{2}})^2-(x^{-\frac{1}{2}})^2\\&=x^1-x^{-1}\\&=x-\frac{1}{x}\end{aligned}$$

j
$$\begin{aligned}&\left(x+\frac{2}{x}\right)^2\\&=x^2+2\times x\times\left(\frac{2}{x}\right)+\left(\frac{2}{x}\right)^2\\&=x^2+4+\frac{4}{x^2}\end{aligned}$$

k
$$\begin{aligned}&(7^x-7^{-x})^2\\&=(7^x)^2-2\times7^x\times7^{-x}+(7^{-x})^2\\&=7^{2x}-2\times7^0+7^{-2x}\\&=7^{2x}-2+7^{-2x}\end{aligned}$$

l
$$\begin{aligned}&(5-2^{-x})^2\\&=5^2-2\times5\times2^{-x}+(2^{-x})^2\\&=25-10(2^{-x})+2^{-2x}\end{aligned}$$

4 **a**
$$\begin{aligned}&5^{2x}+5^x\\&=5^x\times5^x+5^x\\&=5^x(5^x+1)\end{aligned}$$

b
$$\begin{aligned}&3^{n+2}+3^n\\&=3^n\times3^2+3^n\\&=3^n(3^2+1)\\&=10(3^n)\end{aligned}$$

c
$$\begin{aligned}&7^n+7^{3n}\\&=7^n+7^n\times7^{2n}\\&=7^n(1+7^{2n})\end{aligned}$$

d
$$\begin{aligned}&5^{n+1}-5\\&=5\times5^n-5\\&=5(5^n-1)\end{aligned}$$

e
$$\begin{aligned}&6^{n+2}-6\\&=6\times6^{n+1}-6\\&=6(6^{n+1}-1)\end{aligned}$$

f
$$\begin{aligned}&4^{n+2}-16\\&=4^2\times4^n-16\\&=16\times4^n-16\\&=16(4^n-1)\end{aligned}$$

g
$$\begin{aligned}&2^{2n}-2^{n+3}\\&=2^n\times2^n-2^n\times2^3\\&=2^n\times2^n-2^n\times8\\&=2^n(2^n-8)\end{aligned}$$

h
$$\begin{aligned}&2^{n+1}+2^{n-1}\\&=2^{n-1}\times2^2+2^{n-1}\\&=2^{n-1}\times4+2^{n-1}\\&=5(2^{n-1})\end{aligned}$$

i
$$\begin{aligned}&4^{n+1}+2^{2n-1}\\&=(2^2)^{n+1}+2^{2n-1}\\&=2^{2n+2}+2^{2n-1}\\&=2^{2n-1}\times2^3+2^{2n-1}\\&=2^{2n-1}\times8+2^{2n-1}\\&=9(2^{2n-1})\end{aligned}$$

5 **a**
$$\begin{aligned}&9^x-4\\&=(3^x)^2-2^2\\&=(3^x+2)(3^x-2)\end{aligned}$$

b
$$\begin{aligned}&4^x-25\\&=(2^x)^2-5^2\\&=(2^x+5)(2^x-5)\end{aligned}$$

c
$$\begin{aligned}&16-9^x\\&=4^2-(3^x)^2\\&=(4+3^x)(4-3^x)\end{aligned}$$

d $25-4^x$
$=5^2-(2^x)^2$
$=(5+2^x)(5-2^x)$

e 9^x-4^x
$=(3^x)^2-(2^x)^2$
$=(3^x+2^x)(3^x-2^x)$

f $4^x+6(2^x)+9$
$=(2^x)^2+6(2^x)+9$
$=(2^x+3)^2$
$\{a^2+6a+9=(a+3)^2\}$

g $9^x+10(3^x)+25$
$=(3^x)^2+10(3^x)+25$
$=(3^x+5)^2$
$\{a^2+10a+25=(a+5)^2\}$

h $4^x-14(2^x)+49$
$=(2^x)^2-14(2^x)+49$
$=(2^x-7)^2$
$\{a^2-14a+49=(a-7)^2\}$

i $25^x-4(5^x)+4$
$=(5^x)^2-4(5^x)+4$
$=(5^x-2)^2$
$\{a^2-4a+4=(a-2)^2\}$

6 **a** $(2^x)^2-2^x-2$
$=(2^x+1)(2^x-2)$
$\{a^2-a-2=(a+1)(a-2)\}$

b $(3^x)^2+3^x-6$
$=(3^x+3)(3^x-2)$
$\{a^2+a-6=(a+3)(a-2)\}$

c $4^x-7(2^x)+12$
$=(2^x)^2-7(2^x)+12$
$=(2^x-3)(2^x-4)$
$\{a^2-7a+12=(a-3)(a-4)\}$

d $4^x+9(2^x)+18$
$=(2^x)^2+9(2^x)+18$
$=(2^x+3)(2^x+6)$
$\{a^2+9a+18=(a+3)(a+6)\}$

e 4^x-2^x-20
$=(2^x)^2-2^x-20$
$=(2^x+4)(2^x-5)$
$\{a^2-a-20=(a+4)(a-5)\}$

f $9^x+9(3^x)+14$
$=(3^x)^2+9(3^x)+14$
$=(3^x+2)(3^x+7)$
$\{a^2+9a+14=(a+2)(a+7)\}$

g $9^x+4(3^x)-5$
$=(3^x)^2+4(3^x)-5$
$=(3^x+5)(3^x-1)$
$\{a^2+4a-5=(a+5)(a-1)\}$

h 25^x+5^x-2
$=(5^x)^2+5^x-2$
$=(5^x+2)(5^x-1)$
$\{a^2+a-2=(a+2)(a-1)\}$

i $49^x-7^{x+1}+12$
$=(7^x)^2-7(7^x)+12$
$=(7^x-4)(7^x-3)$
$\{a^2-7a+12=(a-4)(a-3)\}$

7 **a** $\dfrac{12^n}{6^n}$ **or** $\dfrac{12^n}{6^n}$
$=\dfrac{2^n6^n}{6^n}$ $\quad=\left(\frac{12}{6}\right)^n$
$=2^n$ $\quad=2^n$

b $\dfrac{20^a}{2^a}$ **or** $\dfrac{20^a}{2^a}$
$=\dfrac{2^a10^a}{2^a}$ $\quad=\left(\frac{20}{2}\right)^a$
$=10^a$ $\quad=10^a$

c $\dfrac{6^b}{2^b}$ or $\dfrac{6^b}{2^b}$

$= \dfrac{2^b 3^b}{2^b}$ $= \left(\frac{6}{2}\right)^b$

$= 3^b$ $= 3^b$

d $\dfrac{4^n}{20^n}$ or $\dfrac{4^n}{20^n}$

$= \dfrac{4^n}{4^n 5^n}$ $= \left(\frac{4}{20}\right)^n$

$= \dfrac{1}{5^n}$ $= \left(\frac{1}{5}\right)^n$

e $\dfrac{35^x}{7^x}$ or $\dfrac{35^x}{7^x}$

$= \dfrac{5^x 7^x}{7^x}$ $= \left(\frac{35}{7}\right)^x$

$= 5^x$ $= 5^x$

f $\dfrac{6^a}{8^a}$ or $\dfrac{6^a}{8^a}$

$= \dfrac{2^a 3^a}{2^a 4^a}$ $= \left(\frac{6}{8}\right)^a$

$= \dfrac{3^a}{4^a}$ $= \left(\frac{3}{4}\right)^a$

g $\dfrac{24^k}{9^k}$ or $\dfrac{24^k}{9^k}$

$= \dfrac{3^k 8^k}{3^k 3^k}$ $= \left(\frac{24}{9}\right)^k$

$= \dfrac{8^k}{3^k}$ $= \left(\frac{8}{3}\right)^k$

h $\dfrac{5^{n+1}}{5^n}$

$= \dfrac{5^n 5^1}{5^n}$

$= 5$

i $\dfrac{5^{n+1}}{5}$

$= \dfrac{5^n 5^1}{5}$

$= 5^n$

8 **a** $\dfrac{6^m + 2^m}{2^m}$

$= \dfrac{2^m 3^m + 2^m}{2^m}$

$= \dfrac{2^m(3^m + 1)}{2^m}$

$= 3^m + 1$

b $\dfrac{2^n + 12^n}{2^n}$

$= \dfrac{2^n + 2^n 6^n}{2^n}$

$= \dfrac{2^n(1 + 6^n)}{2^n}$

$= 1 + 6^n$

c $\dfrac{8^n + 4^n}{2^n}$

$= \dfrac{2^n 4^n + 2^n 2^n}{2^n}$

$= \dfrac{2^n(4^n + 2^n)}{2^n}$

$= 4^n + 2^n$

d $\dfrac{12^x - 3^x}{3^x}$

$= \dfrac{3^x 4^x - 3^x}{3^x}$

$= \dfrac{3^x(4^x - 1)}{3^x}$

$= 4^x - 1$

e $\dfrac{6^n + 12^n}{1 + 2^n}$

$= \dfrac{6^n + 6^n 2^n}{1 + 2^n}$

$= \dfrac{6^n(1 + 2^n)}{1 + 2^n}$

$= 6^n$

f $\dfrac{5^{n+1} - 5^n}{4}$

$= \dfrac{5^n \times 5 - 5^n}{4}$

$= \dfrac{5^n(5 - 1)}{4}$

$= 5^n$

g $\dfrac{5^{n+1} - 5^n}{5^n}$

$= \dfrac{5^n \times 5 - 5^n}{5^n}$

$= \dfrac{5^n(5 - 1)}{5^n}$

$= 4$

h $\dfrac{4^n - 2^n}{2^n}$

$= \dfrac{2^n 2^n - 2^n}{2^n}$

$= \dfrac{2^n(2^n - 1)}{2^n}$

$= 2^n - 1$

i $\dfrac{2^n - 2^{n-1}}{2^n}$

$= \dfrac{2^{n-1} \times 2 - 2^{n-1}}{2^{n-1} \times 2}$

$= \dfrac{2^{n-1}(2 - 1)}{2^{n-1} \times 2}$

$= \frac{1}{2}$

9 **a** $2^n(n+1)+2^n(n-1)$
$= 2^n(n+1+n-1)$
$= 2^n(2n)$
$= n2^{n+1}$

b $3^n\left(\frac{n-1}{6}\right) - 3^n\left(\frac{n+1}{6}\right)$
$= 3^n\left(\frac{n-1}{6} - \frac{n+1}{6}\right)$
$= 3^n(-\frac{1}{3})$
$= 3^n \times -3^{-1}$
$= -3^{n-1}$

EXERCISE 1C

1 **a** $y = x^3 - 2$ is not an exponential function as the variable x does not appear in the exponent.

b $f(x) = 7 - 2^{-x}$ is an exponential function as the variable x appears in the exponent.

c $g(x) = \sqrt{x}+5$ is not an exponential function as the variable x does not appear in the exponent.

d $f(x) = -3 \times 5^{\frac{x}{2}}$ is an exponential function as the variable x appears in the exponent.

2 $f(x) = 2^x - 3$

a $f(2) = 2^2 - 3$
$= 4 - 3$
$= 1$

b $f(1) = 2^1 - 3$
$= 2 - 3$
$= -1$

c $f(0) = 2^0 - 3$
$= 1 - 3$
$= -2$

d $f(-1) = 2^{-1} - 3$
$= \frac{1}{2} - 3$
$= -\frac{5}{2}$ or $-2\frac{1}{2}$

e $f(-2) = 2^{-2} - 3$
$= \frac{1}{2^2} - 3$
$= -\frac{11}{4}$ or $-2\frac{3}{4}$

3 $f(x) = 5 \times 3^x$

a $f(1) = 5 \times 3^1$
$= 5 \times 3$
$= 15$

b $f(3) = 5 \times 3^3$
$= 5 \times 27$
$= 135$

c $f(0) = 5 \times 3^0$
$= 5 \times 1$
$= 5$

d $f(-4) = 5 \times 3^{-4}$
$= 5 \times \frac{1}{3^4}$
$= \frac{5}{81}$

e $f(-1) = 5 \times 3^{-1}$
$= 5 \times \frac{1}{3}$
$= \frac{5}{3}$ or $1\frac{2}{3}$

4 $f(x) = 2 \times 2^x$

a $f(4) = 2 \times 2^4$
$= 2^5$
$= 32$

b $f(0) = 2 \times 2^0$
$= 2 \times 1$
$= 2$

c $f(1) = 2 \times 2^1$
$= 2^2$
$= 4$

d $f(-1) = 2 \times 2^{-1}$
$= 2^0$
$= 1$

e $f(-5) = 2 \times 2^{-5}$
$= 2^{-4}$
$= \frac{1}{16}$

5 $g(x) = 5^{-x}$

a $g(1) = 5^{-1} = \frac{1}{5}$

b $g(3) = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

c $g(0) = 5^0 = 1$

d $g(-2) = 5^{-(-2)} = 5^2 = 25$

e $g(-3) = 5^{-(-3)} = 5^3 = 125$

6 $h(x) = 3 \times (1.1)^x$

a $h(0) = 3 \times (1.1)^0 = 3$

b $h(1) = 3 \times (1.1)^1 = 3.3$

c $h(5) = 3 \times (1.1)^5 \approx 4.83$

d $h(-2) = 3 \times (1.1)^{-2} \approx 2.48$

e $h(3.8) = 3 \times (1.1)^{3.8} \approx 4.31$

7 **a** $y = 2^x + 1$

When $x = 3$, $y = 2^3 + 1 = 8 + 1 = 9$

$\therefore$ the point $(3, 9)$ satisfies $y = 2^x + 1$.

b $f(x) = 5^{2x}$

$\therefore f(1) = 5^{2\times 1} = 25$

$\therefore$ the point $(1, 5)$ does not satisfy $f(x) = 5^{2x}$.

c $f(x) = 3^{-x} - 2$

$\therefore f(0) = 3^0 - 2 = 1 - 2 = -1$

$\therefore$ the point $(0, -1)$ satisfies $f(x) = 3^{-x} - 2$.

d $y = 6 \times 2^x$

When $x = -1$, $y = 6 \times 2^{-1} = 6 \times \frac{1}{2} = 3$

$\therefore$ the point $(-1, 3)$ satisfies $y = 6 \times 2^x$.

e $f(x) = -4 \times 3^x + 1$

$\therefore f(2) = -4 \times 3^2 + 1 = -4 \times 9 + 1 = -35$

$\therefore$ the point $(2, -13)$ does not satisfy $f(x) = -4 \times 3^x + 1$.

f $y = 4^{-3x} + 2$

When $x = -1$, $y = 4^{-3(-1)} + 2 = 4^3 + 2 = 64 + 2 = 66$

$\therefore$ the point $(-1, 66)$ satisfies $y = 4^{-3x} + 2$.

8 $f(x) = 3^x - 1$

$\therefore f(0) = 3^0 - 1 = 1 - 1 = 0$

The function passes through the origin $(0, 0)$, so the axes intercepts are both zero.

9 $f(x) = 2^{-x} - 8$

a $f(0) = 2^0 - 8$
$= 1 - 8$
$= -7$
$\therefore$ the y-intercept is -7.

b $f(-3) = 2^{-(-3)} - 8$
$= 2^3 - 8$
$= 8 - 8$
$= 0$
$\therefore$ the x-intercept is -3.

EXERCISE 1D

1 $y = 4^x$

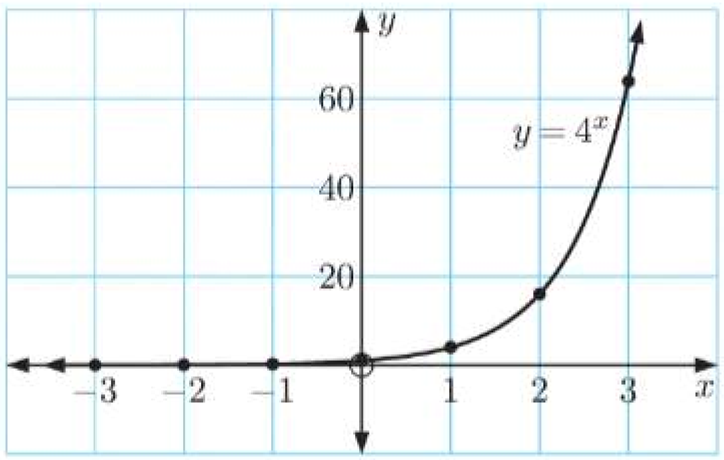
a

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

b **i** If x is increased by 1, the value of y is quadrupled.

ii If x is decreased by 1, the value of y is divided by 4.

c



d **i** As $x \to \infty$, $y \to \infty$. **ii** As $x \to -\infty$, $y \to 0^+$.

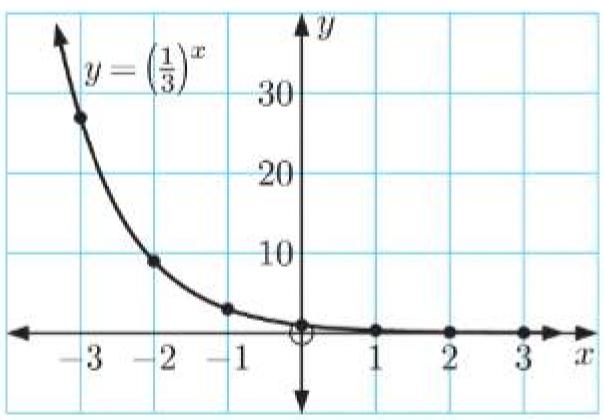
e From **d ii**, as $x \to -\infty$, $y \to 0^+$.
$\therefore$ the horizontal asymptote is $y = 0$.

2 $y = \left(\frac{1}{3}\right)^x$

a

x	-3	-2	-1	0	1	2	3
y	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

b



c The graph of $y = \left(\frac{1}{3}\right)^x$ is decreasing.

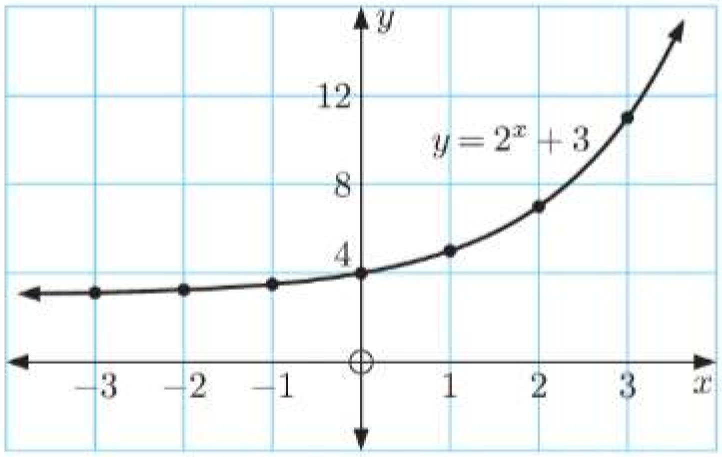
d **i** As $x \to \infty$, $y \to 0^+$. **ii** As $x \to -\infty$, $y \to \infty$.

e From **d i**, as $x \to -\infty$, $y \to 0^+$.

$\therefore$ the horizontal asymptote is $y = 0$.

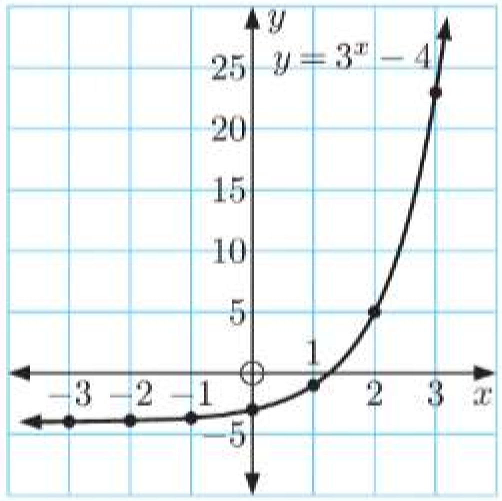
3 **a** $y = 2^x + 3$

x	-3	-2	-1	0	1	2	3
y	$3\frac{1}{8}$	$3\frac{1}{4}$	$3\frac{1}{2}$	4	5	7	11



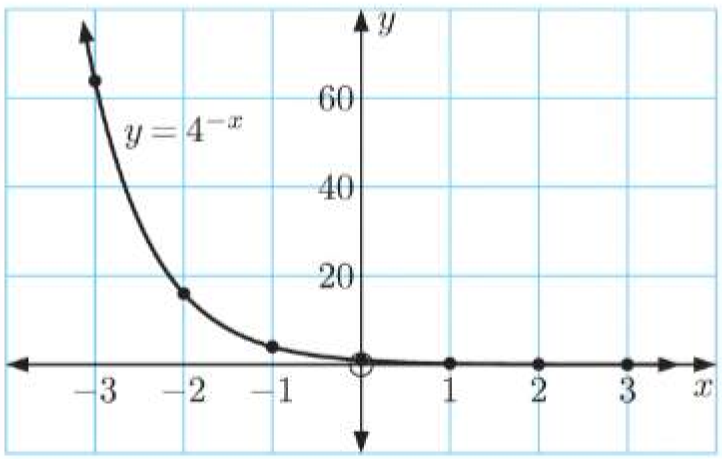
b $y = 3^x - 4$

x	-3	-2	-1	0	1	2	3
y	$-3\frac{26}{27}$	$-3\frac{8}{9}$	$-3\frac{2}{3}$	-3	-1	5	23



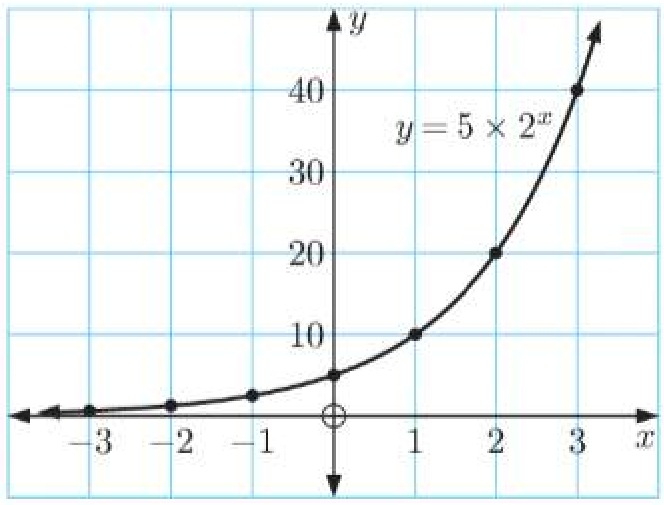
c $y = 4^{-x}$

x	-3	-2	-1	0	1	2	3
y	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$



d $y = 5 \times 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{5}{8}$	$\frac{5}{4}$	$\frac{5}{2}$	5	10	20	40



4 **a** **i** $y = 2^{-x}$

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

ii $y = \left(\frac{1}{2}\right)^x$

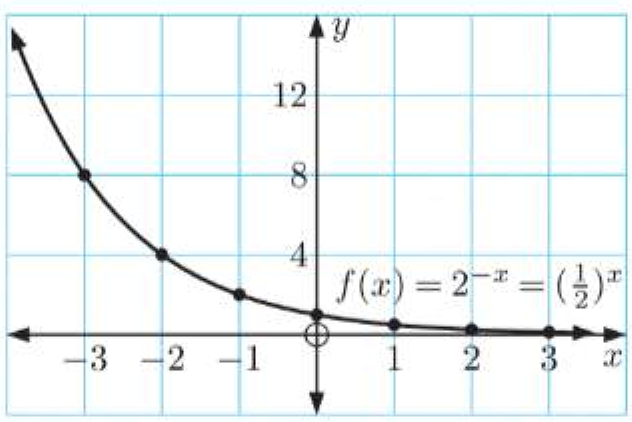
x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The corresponding y-values are the same for each function.

b $2^{-x} = (2^{-1})^x$

$= \left(\frac{1}{2}\right)^x$

c



INVESTIGATION 1 GRAPHS OF EXPONENTIAL FUNCTIONS

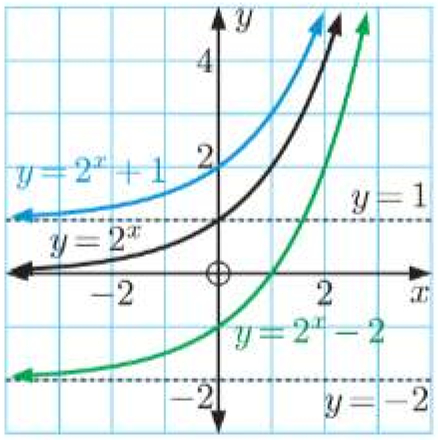
1 **a** A vertical translation through c units maps $y = a^x$ to $y = a^x + c$.

b **i** The shape of the graph will remain the same.

ii The graph is translated vertically through c units.

iii The horizontal asymptote $y = 0$ is transformed to $y = c$.

c



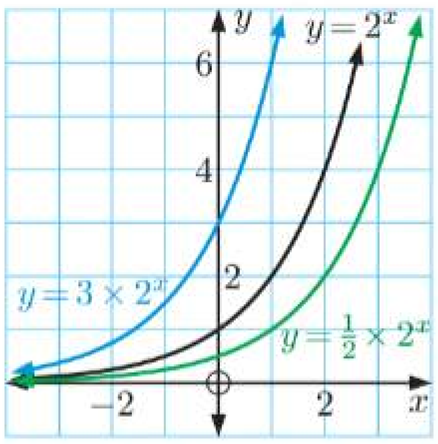
2 **a** A vertical stretch with scale factor $k > 0$ will map $y = a^x$ to $y = k \times a^x$.

b **i** Each point will become k times its previous distance from the x-axis.

ii The graph is stretched vertically with invariant x-axis and scale factor k.

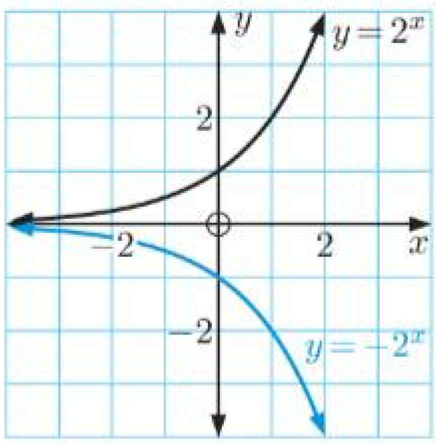
iii The horizontal asymptote $y = 0$ will remain the same.

c



3 **a** A reflection in the x-axis maps $y = a^x$ to $y = -a^x$.

b $y = -2^x$ is a reflection of $y = 2^x$ in the x-axis.



c For the family $y = k \times a^x$, the sign of k determines whether the graph lies above or below the asymptote.

If $k > 0$, the graph lies above the asymptote.

If $k < 0$, the graph lies below the asymptote.

4 **a** A horizontal stretch with scale factor $\frac{1}{q}$ maps $y = a^x$ to $y = a^{qx}$, $q > 0$.

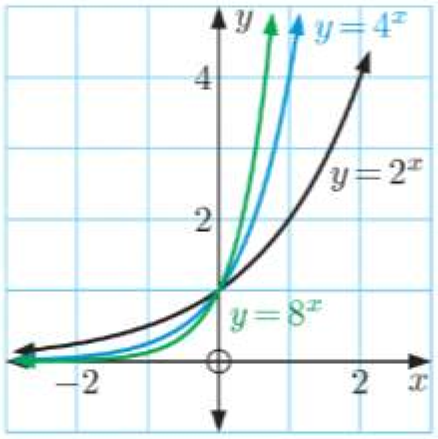
b **i** Each point will become $\frac{1}{q}$ times its previous distance from the y-axis.

If $0 < q < 1$, the graph becomes flatter, if $q > 1$, the graph becomes steeper, and if $q = 1$, the graph remains unchanged.

ii The graph is stretched horizontally with invariant y-axis and scale factor $\frac{1}{q}$.

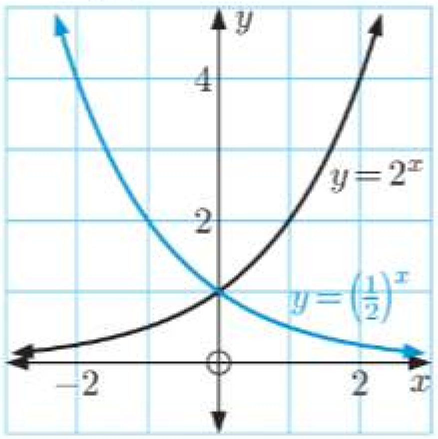
iii The horizontal asymptote $y = 0$ will remain the same.

c



5 **a** A reflection in the y-axis will map $y = a^x$ to $y = a^{-x}$.

b $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ is a reflection of $y = 2^x$ in the y-axis.



c $a^{-x} = (a^{-1})^x$

$= \left(\frac{1}{a}\right)^x, \quad a > 0, \ a \neq 1$

Since $a^{-x} = \left(\frac{1}{a}\right)^x$ for all $a > 0$, $a \neq 1$, the graphs of $y = a^{-x}$ and $y = \left(\frac{1}{a}\right)^x$ are identical.

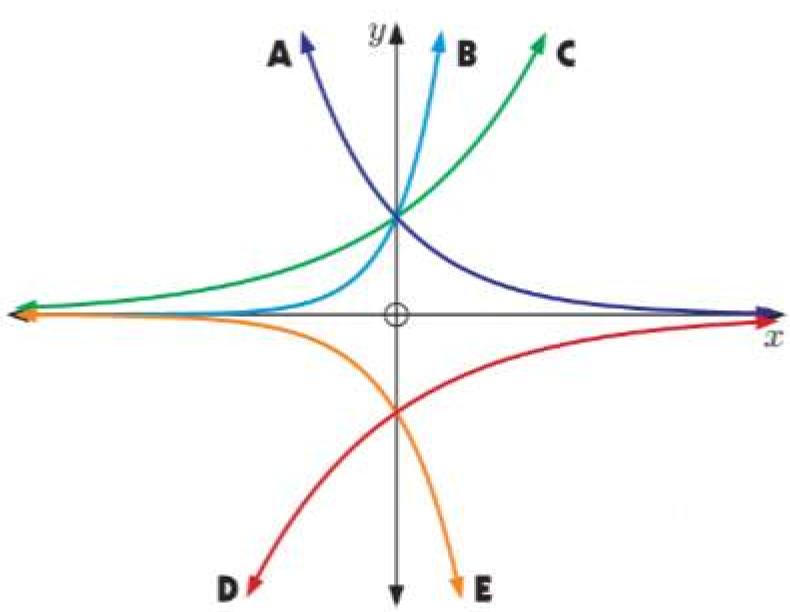
EXERCISE 1E

1 **a, b** Both $y = 2^x$ and $y = 10^x$ have $k > 0$ and $a > 1$.

$\therefore$ both graphs lie above the horizontal asymptote $y = 0$ and are increasing.

$y = 10^x$ is steeper than $y = 2^x$ as $10 > 2$.

$\therefore$ $y = 2^x$ corresponds to **C**, and $y = 10^x$ corresponds to **B**.



c $y = -5^x$ has $k < 0$ and $a > 1$.

$\therefore$ the graph lies below the horizontal asymptote $y = 0$ and is decreasing.

$\therefore$ $y = -5^x$ corresponds to **E**.

d $y = \left(\frac{1}{3}\right)^x$ has $k > 0$ and $0 < a < 1$.

$\therefore$ the graph lies above the horizontal asymptote $y = 0$ and is decreasing.

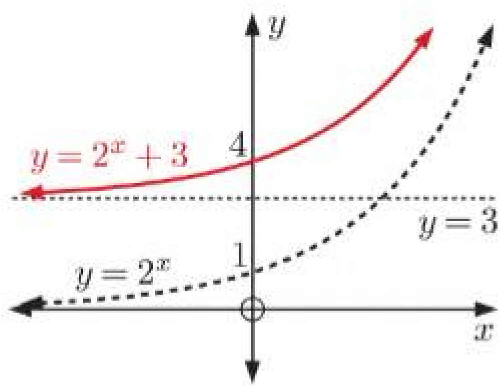
$\therefore$ $y = \left(\frac{1}{3}\right)^x$ corresponds to **A**.

e $y = -\left(\frac{1}{2}\right)^x$ has $k < 0$ and $0 < a < 1$.

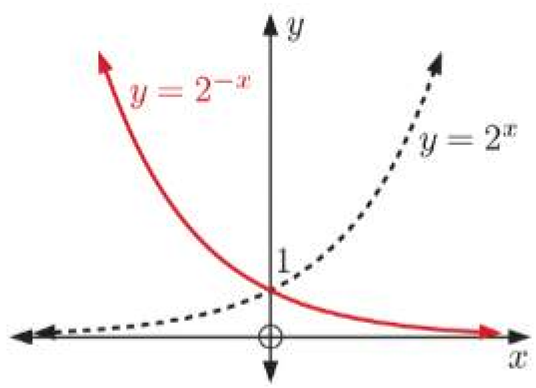
$\therefore$ the graph lies below the horizontal asymptote $y = 0$ and is increasing.

$\therefore$ $y = -\left(\frac{1}{2}\right)^x$ corresponds to **D**.

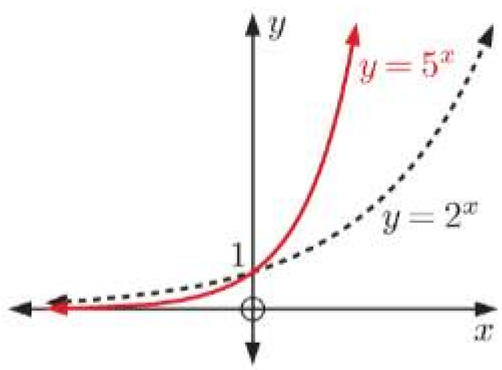
2 **a**



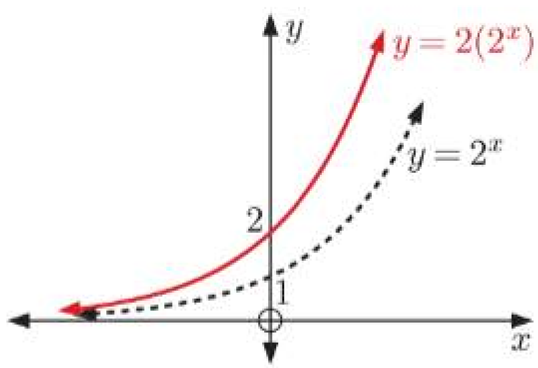
b



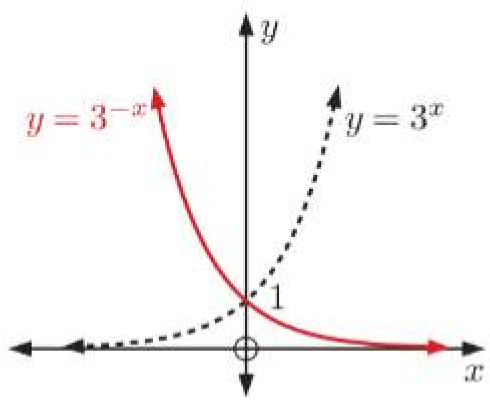
c



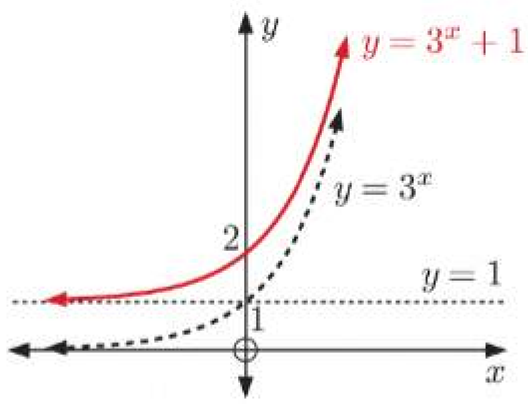
d



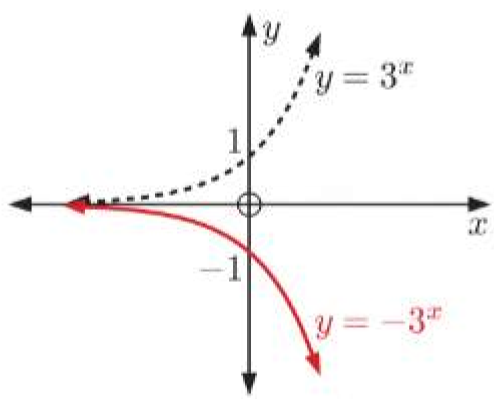
3 **a**



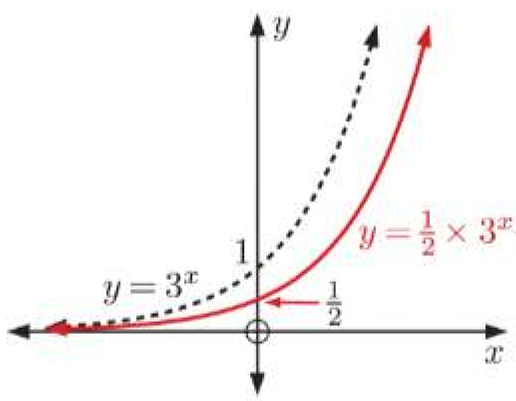
b



c



d



4 **a** The graph of $y = 5^x - 1$ has horizontal asymptote $y = -1$.

b The graph of $y = 2^{-x} + 4$ has horizontal asymptote $y = 4$.

c The graph of $y = 3 \times 4^x + 1$ has horizontal asymptote $y = 1$.

d The graph of $y = -\left(\frac{1}{2}\right)^x - 5$ has horizontal asymptote $y = -5$.

5 **a**
$$\begin{aligned} & y = 3^x + 4 \\ \text{When } x = 0, \quad & y = 3^0 + 4 \\ & \quad = 1 + 4 \\ & \quad = 5 \end{aligned}$$
$\therefore$ the y-intercept is 5.

b
$$\begin{aligned} & y = 6^{-x} - 2 \\ \text{When } x = 0, \quad & y = 6^0 - 2 \\ & \quad = 1 - 2 \\ & \quad = -1 \end{aligned}$$
$\therefore$ the y-intercept is -1.

c
$$\begin{aligned} & y = 3 \times 2^x + 7 \\ \text{When } x = 0, \quad & y = 3 \times 2^0 + 7 \\ & \quad = 3 \times 1 + 7 \\ & \quad = 10 \end{aligned}$$
$\therefore$ the y-intercept is 10.

d
$$\begin{aligned} & y = -\tfrac{1}{2} \times 5^x + 6 \\ \text{When } x = 0, \quad & y = -\tfrac{1}{2} \times 5^0 + 6 \\ & \quad = -\tfrac{1}{2} \times 1 + 6 \\ & \quad = 5\tfrac{1}{2} \end{aligned}$$
$\therefore$ the y-intercept is $5\frac{1}{2}$.

6 $f(x) = 3^x - 2$

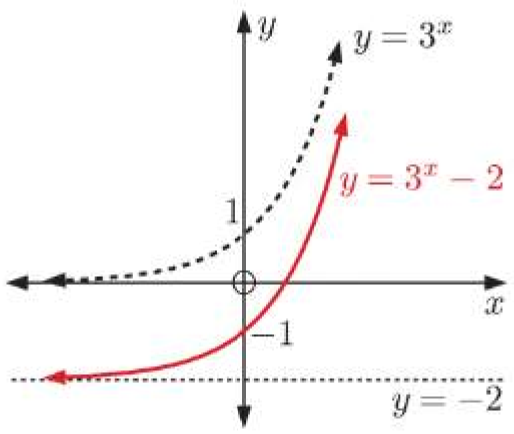
a **i** $f(0) = 3^0 - 2 = 1 - 2 = -1$

ii $f(2) = 3^2 - 2 = 9 - 2 = 7$

iii $f(-2) = 3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9} = -1\frac{8}{9}$

b The graph of $y = 3^x - 2$ has horizontal asymptote $y = -2$.

c



d The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > -2\}$.

7 $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$

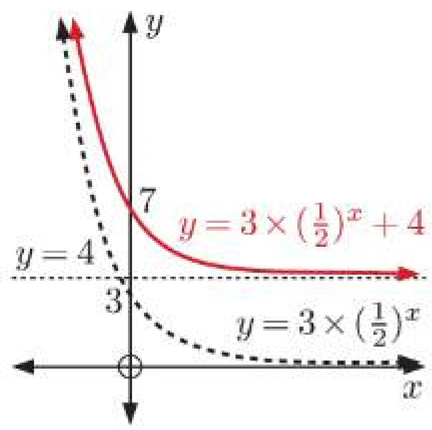
a **i** $g(0) = 3 \times \left(\frac{1}{2}\right)^0 + 4 = 3 \times 1 + 4 = 7$

ii $g(2) = 3 \times \left(\frac{1}{2}\right)^2 + 4 = 3 \times \frac{1}{4} + 4 = \frac{19}{4} = 4\frac{3}{4}$

iii $g(-2) = 3 \times \left(\frac{1}{2}\right)^{-2} + 4 = 3 \times 2^2 + 4 = 3 \times 4 + 4 = 16$

b The graph of $y = 3 \times \left(\frac{1}{2}\right)^x + 4$ has horizontal asymptote $y = 4$.

c



d The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > 4\}$.

8 **a** $y = 2^x + 1$

i When $x = 0$, $y = 2^0 + 1$
$= 1 + 1$
$= 2$

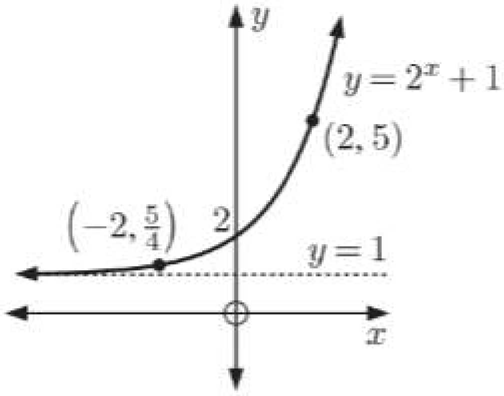
$\therefore$ the y-intercept is 2.

ii The horizontal asymptote is $y = 1$.

iii When $x = 2$, $y = 2^2 + 1$
$= 4 + 1$
$= 5$

When $x = -2$, $y = 2^{-2} + 1$
$= \frac{1}{4} + \frac{4}{4}$
$= \frac{5}{4}$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 1\}$.

b $y = 3^{-x} + 4$

i When $x = 0$, $y = 3^0 + 4$
$= 1 + 4$
$= 5$

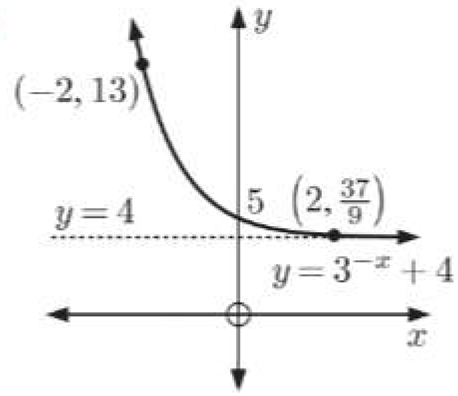
$\therefore$ the y-intercept is 5.

ii The horizontal asymptote is $y = 4$.

iii When $x = 2$, $y = 3^{-2} + 4$
$= \frac{1}{3^2} + 4$
$= \frac{1}{9} + \frac{36}{9}$
$= \frac{37}{9}$

When $x = -2$, $y = 3^{-(-2)} + 4$
$= 3^2 + 4$
$= 9 + 4$
$= 13$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 4\}$.

c $y = \left(\frac{2}{5}\right)^x$

i When $x = 0$, $y = \left(\frac{2}{5}\right)^0 = 1$

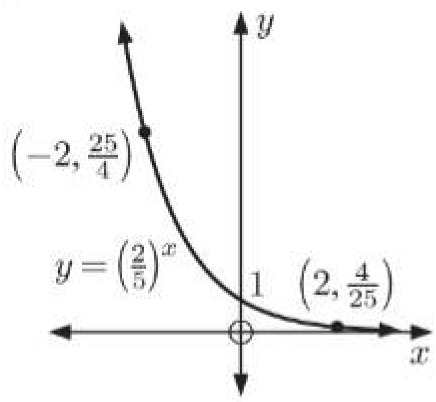
$\therefore$ the y-intercept is 1.

ii The horizontal asymptote is $y = 0$.

iii When $x = 2$,
$$y = \left(\tfrac{2}{5}\right)^2 = \frac{2^2}{5^2} = \tfrac{4}{25}$$

When $x = -2$,
$$y = \left(\tfrac{2}{5}\right)^{-2} = \frac{2^{-2}}{5^{-2}} = \frac{5^2}{2^2} = \tfrac{25}{4}$$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 0\}$.

d $y = \left(\frac{1}{2}\right)^x - 3$

i When $x = 0$,
$$y = \left(\tfrac{1}{2}\right)^0 - 3 = 1 - 3 = -2$$

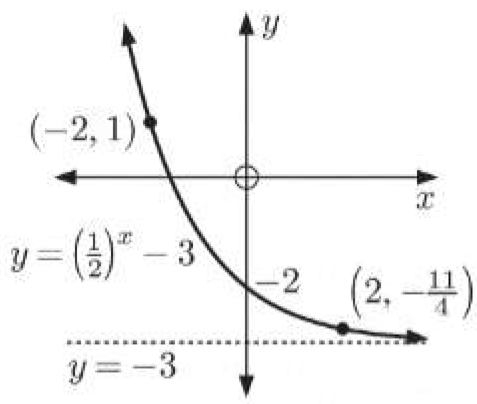
$\therefore$ the y-intercept is -2.

ii The horizontal asymptote is $y = -3$.

iii When $x = 2$,
$$y = \left(\tfrac{1}{2}\right)^2 - 3 = \frac{1^2}{2^2} - 3 = \tfrac{1}{4} - \tfrac{12}{4} = -\tfrac{11}{4}$$

When $x = -2$,
$$y = \left(\tfrac{1}{2}\right)^{-2} - 3 = \frac{1^{-2}}{2^{-2}} - 3 = \frac{2^2}{1^2} - 3 = 4 - 3 = 1$$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > -3\}$.

e $y = 2 - 2^x$

i When $x = 0$, $y = 2 - 2^0$
$= 2 - 1$
$= 1$

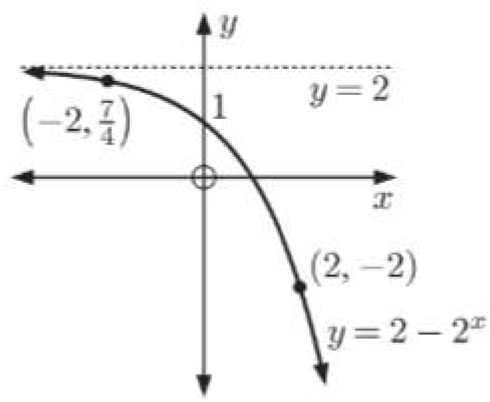
$\therefore$ the y-intercept is 1.

ii The horizontal asymptote is $y = 2$.

iii When $x = 2$, $y = 2 - 2^2$
$= 2 - 4$
$= -2$

When $x = -2$, $y = 2 - 2^{-2}$
$= 2 - \frac{1}{2^2}$
$= \frac{8}{4} - \frac{1}{4}$
$= \frac{7}{4}$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 2\}$.

f $y = 4^{-x} + 3$

i When $x = 0$, $y = 4^0 + 3$
$= 1 + 3$
$= 4$

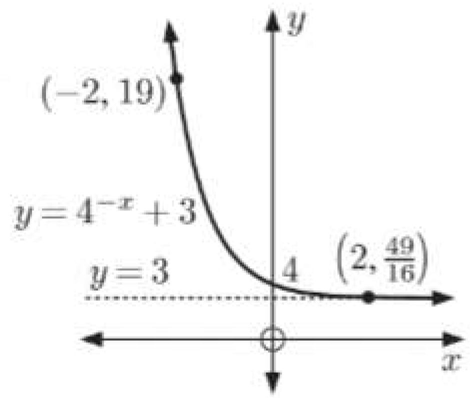
$\therefore$ the y-intercept is 4.

ii The horizontal asymptote is $y = 3$.

iii When $x = 2$, $y = 4^{-2} + 3$
$= \frac{1}{4^2} + 3$
$= \frac{1}{16} + \frac{48}{16}$
$= \frac{49}{16}$

When $x = -2$, $y = 4^{-(-2)} + 3$
$= 4^2 + 3$
$= 16 + 3$
$= 19$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 3\}$.

g $y = 3 - 2^{-x}$

i When $x = 0$, $y = 3 - 2^0$
$= 3 - 1$
$= 2$

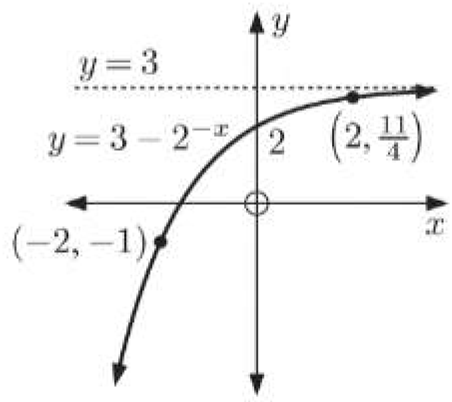
$\therefore$ the y-intercept is 2.

ii The horizontal asymptote is $y = 3$.

iii When $x = 2$, $y = 3 - 2^{-2}$
$= 3 - \dfrac{1}{2^2}$
$= \frac{12}{4} - \frac{1}{4}$
$= \frac{11}{4}$

When $x = -2$, $y = 3 - 2^{-(-2)}$
$= 3 - 2^2$
$= 3 - 4$
$= -1$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 3\}$.

h $y = -\frac{1}{2} \times 3^{-x} + 1$

i When $x = 0$, $y = -\frac{1}{2} \times 3^0 + 1$
$= -\frac{1}{2} \times 1 + 1$
$= \frac{1}{2}$

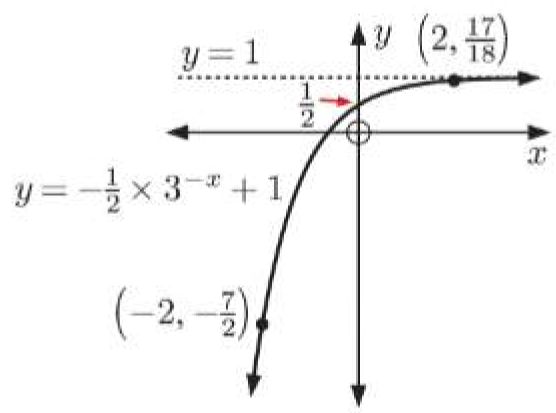
$\therefore$ the y-intercept is $\frac{1}{2}$.

ii The horizontal asymptote is $y = 1$.

iii When $x = 2$, $y = -\frac{1}{2} \times 3^{-2} + 1$
$= -\frac{1}{2} \times \dfrac{1}{3^2} + 1$
$= -\frac{1}{2} \times \frac{1}{9} + 1$
$= -\frac{1}{18} + \frac{18}{18}$
$= \frac{17}{18}$

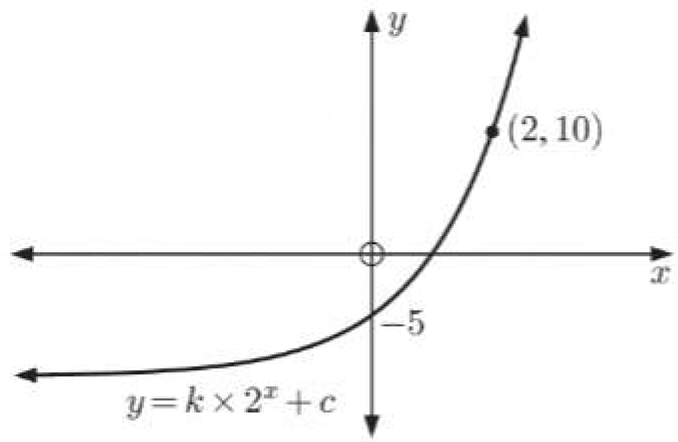
When $x = -2$, $y = -\frac{1}{2} \times 3^{-(-2)} + 1$
$= -\frac{1}{2} \times 3^2 + 1$
$= -\frac{1}{2} \times 9 + 1$
$= -\frac{9}{2} + \frac{2}{2}$
$= -\frac{7}{2}$

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 1\}$.

9 **a** $y = k \times 2^x + c$



Substituting $(0, -5)$ into the equation gives

$$-5 = k \times 2^0 + c$$

$\therefore\ k + c = -5$ .... (1)

Substituting $(2, 10)$ into the equation gives

$$10 = k \times 2^2 + c$$

$\therefore\ 4k + c = 10$ .... (2)

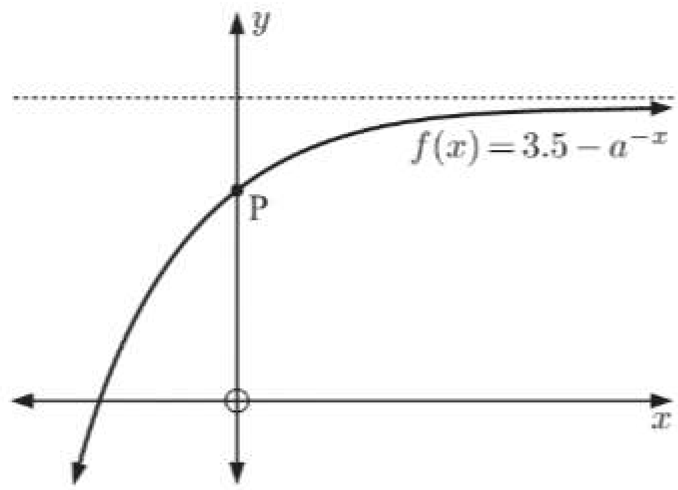
Now $(2) - (1)$ gives $3k = 15$

$\therefore\ k = 5$ and so $c = -10$.

b $y = 5 \times 2^x - 10$

When $x = 6$, $y = 5 \times 2^6 - 10$
$= 5 \times 64 - 10$
$= 310$

10 **a** $f(0) = 3.5 - a^0$
$= 3.5 - 1$
$= 2.5$



$\therefore$ the y-intercept is 2.5.

$\therefore$ P is $(0, 2.5)$.

b The point $(-1, 2)$ lies on the graph.

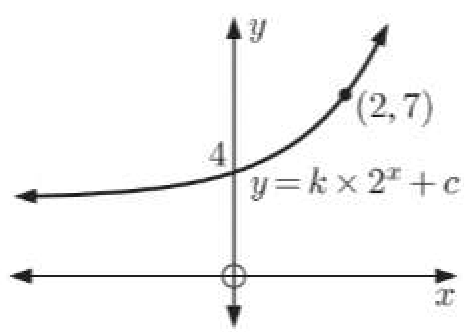
$\therefore\ f(-1) = 2$

$\therefore\ 3.5 - a^1 = 2$

$\therefore\ a = 1.5$

c $f(x) = 3.5 - 1.5^{-x}$ has horizontal asymptote $y = 3.5$.

11 **a** $y = k \times 2^x + c$



Substituting $(0, 4)$ into the equation gives

$$4 = k \times 2^0 + c$$

$\therefore\ k + c = 4$ .... (1)

Substituting $(2, 7)$ into the equation gives

$$7 = k \times 2^2 + c$$

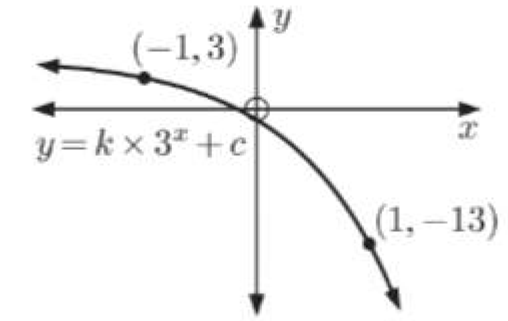
$\therefore\ 4k + c = 7$ .... (2)

Now $(2) - (1)$ gives $3k = 3$

$\therefore\ k = 1$ and so $c = 3$.

$\therefore$ the exponential model is $y = 2^x + 3$.

b $y = k \times 3^x + c$



Substituting $(-1, 3)$ into the equation gives

$$3 = k \times 3^{-1} + c$$

$\therefore \ \frac{1}{3}k + c = 3$ .... (1)

Substituting $(1, -13)$ into the equation gives

$$-13 = k \times 3^1 + c$$

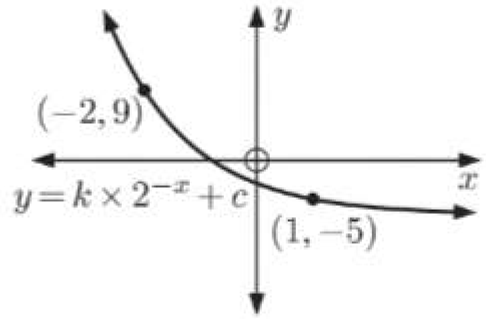
$\therefore \ 3k + c = -13$ .... (2)

Now (2) − (1) gives $\frac{8}{3}k = -16$

$\therefore \ k = -6$ and so $c = 5$.

$\therefore$ the exponential model is $y = -6 \times 3^x + 5$.

c $y = k \times 2^{-x} + c$



Substituting $(-2, 9)$ into the equation gives

$$9 = k \times 2^{-(-2)} + c$$
$$= k \times 2^2 + c$$

$\therefore \ 4k + c = 9$ .... (1)

Substituting $(1, -5)$ into the equation gives

$$-5 = k \times 2^{-1} + c$$

$\therefore \ \frac{1}{2}k + c = -5$ .... (2)

Now (1) − (2) gives $\frac{7}{2}k = 14$

$\therefore \ k = 4$ and so $c = -7$.

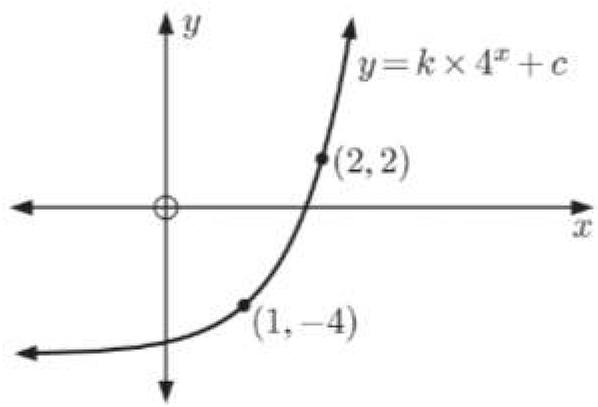
$\therefore$ the exponential model is $y = 4 \times 2^{-x} - 7$.

12 **a** **i** The graph lies above the horizontal asymptote.

$\therefore$ k must be positive.

ii The horizontal asymptote lies below the x-axis.

$\therefore$ c must be negative.



b $y = k \times 4^x + c$

Substituting $(1, -4)$ into the equation gives $-4 = k \times 4^1 + c$

$\therefore \ 4k + c = -4$ .... (1)

Substituting $(2, 2)$ into the equation gives $2 = k \times 4^2 + c$

$\therefore \ 16k + c = 2$ .... (2)

Now (2) − (1) gives $12k = 6$

$\therefore \ k = \frac{1}{2}$ and so $c = -6$.

$\therefore$ the exponential function is $y = \frac{1}{2} \times 4^x - 6$.

c When $x = 0$, $y = \frac{1}{2} \times 4^0 - 6$
$= \frac{1}{2} \times 1 - 6$
$= -\frac{11}{2}$

$\therefore$ the y-intercept is $-\frac{11}{2}$.

d The horizontal asymptote of $y = \frac{1}{2} \times 4^x - 6$ is $y = -6$.

13 $f(x) = k \times a^x + c$

The graph has horizontal asymptote $y = 2$, so $c = 2$.

The y-intercept is 10, so $f(0) = 10$

$\therefore\ k \times a^0 + 2 = 10$

$\therefore\ k = 8$

The graph passes through $(5, 258)$, so $f(5) = 258$

$\therefore\ 8 \times a^5 + 2 = 258$

$\therefore\ 8a^5 = 256$

$\therefore\ a^5 = 32$

$\therefore\ a = 2$

So, the exponential function is $f(x) = 8 \times 2^x + 2$.

14 $f(x) = k \times a^{-x} + c$

The graph has horizontal asymptote $y = 4$, so $c = 4$.

The y-intercept is 1, so $f(0) = 1$

$\therefore\ k \times a^0 + 4 = 1$

$\therefore\ k = -3$

The graph passes through $(1, 2)$, so $f(1) = 2$

$\therefore\ -3 \times a^{-1} + 4 = 2$

$\therefore\ -\dfrac{3}{a} = -2$

$\therefore\ a = \frac{3}{2}$

So, the exponential function is $f(x) = -3 \times \left(\frac{3}{2}\right)^{-x} + 4$.

15 $f(x) = ka^x + c$

$f(1) = 11$ $\therefore\ ka + c = 11$ .... (1)

$f(2) = 17$ $\therefore\ ka^2 + c = 17$ .... (2)

$f(3) = 29$ $\therefore\ ka^3 + c = 29$ .... (3)

Now (2) – (1) gives $ka^2 - ka = 6$

$\therefore\ ka(a-1) = 6$ .... (4)

and (3) – (2) gives $ka^3 - ka^2 = 12$

$\therefore\ ka^2(a-1) = 12$

$\therefore\ a[ka(a-1)] = 12$

$\therefore\ 6a = 12$ {using (4)}

$\therefore\ a = 2$

Substituting $a = 2$ into (4) gives $k(2)(1) = 6$

$\therefore \; k = 3$

Substituting $a = 2$ and $k = 3$ into (1) gives $3(2) + c = 11$

$\therefore \; c = 5$

$\therefore$ the exponential function is $f(x) = 3 \times 2^x + 5$.

$f(0) = 3 \times 2^0 + 5 = 8$

$\therefore$ the y-intercept is 8.

16 $f(x) = ka^{-x} + c$

$f(-2) = 21 \qquad \therefore \; ka^2 + c = 21 \quad .... \; (1)$

$f(1) = 0 \qquad \therefore \; ka^{-1} + c = 0 \quad .... \; (2)$

$f(2) = -\frac{3}{2} \qquad \therefore \; ka^{-2} + c = -\frac{3}{2} \quad .... \; (3)$

Now $(1) - (2)$ gives $ka^2 - ka^{-1} = 21$

$$\therefore \; ka^3 - k = 21a$$

$$\therefore \; k(a^3 - 1) = 21a$$

$$\therefore \; k = \frac{21a}{a^3 - 1} \quad .... \; (4)$$

and $(2) - (3)$ gives $ka^{-1} - ka^{-2} = \frac{3}{2}$

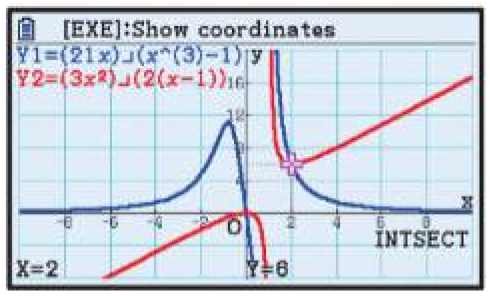
$$\therefore \; ka - k = \tfrac{3}{2}a^2$$

$$\therefore \; k(a - 1) = \tfrac{3}{2}a^2$$

$$\therefore \; k = \frac{3a^2}{2(a-1)} \quad .... \; (5)$$

Equating (4) and (5) gives $\dfrac{21a}{a^3 - 1} = \dfrac{3a^2}{2(a-1)}$

Using technology, $a = 2 \quad \{a > 0\}$



Substituting $a = 2$ into (4) gives $k = \dfrac{21(2)}{2^3 - 1}$

$$\therefore \; k = \frac{42}{7}$$

$$\therefore \; k = 6$$

Substituting $a = 2$ and $k = 6$ into (1) gives $6(2)^2 + c = 21$

$\therefore \; 24 + c = 21$

$\therefore \; c = -3$

So, the exponential function is $f(x) = 6 \times 2^{-x} - 3$.

a $f(0) = 6 \times 2^0 - 3$

$= 3$

$\therefore$ the y-intercept is 3.

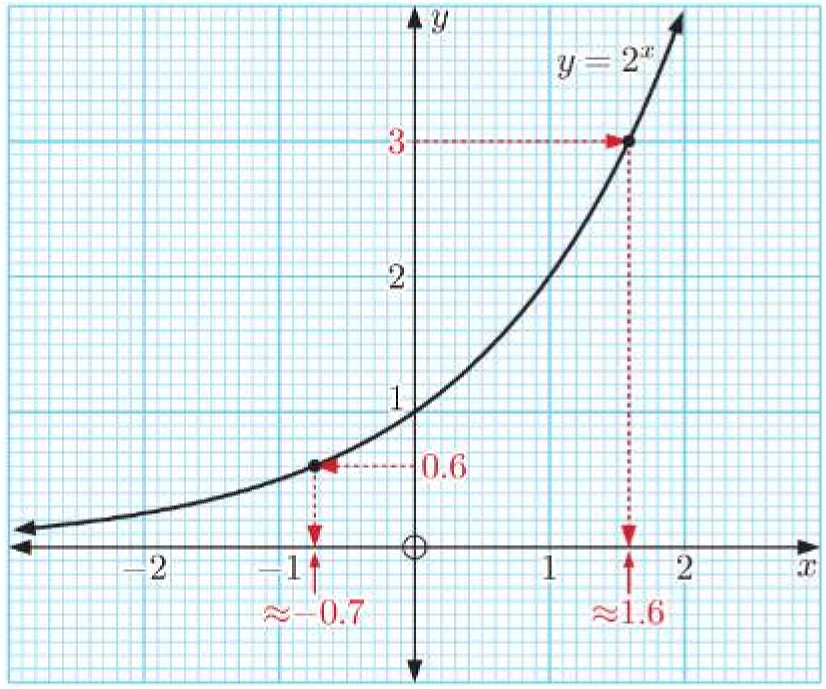
b The horizontal asymptote is $y = -3$.

EXERCISE 1F

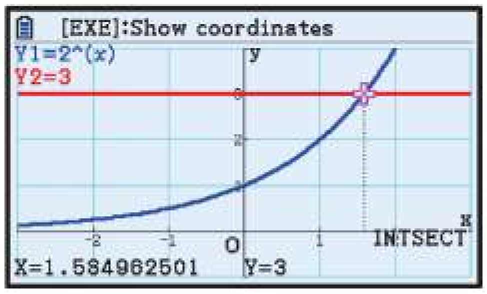
1 **a** From the graph:

i $2^x = 3$ when $x \approx 1.6$.

ii $2^x = 0.6$ when $x \approx -0.7$.

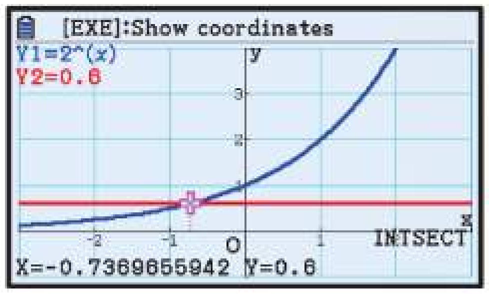


b **i** We graph $Y_1 = 2^X$ and $Y_2 = 3$ on the same set of axes, and find their point of intersection.



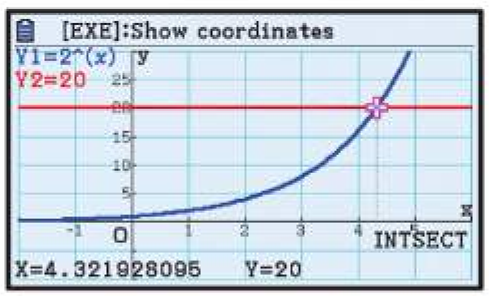
The solution is $x \approx 1.58$. ✓

ii We graph $Y_1 = 2^X$ and $Y_2 = 0.6$ on the same set of axes, and find their point of intersection.



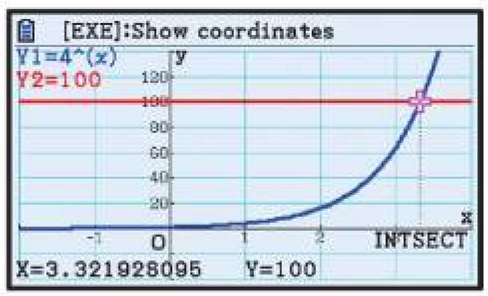
The solution is $x \approx -0.737$. ✓

2 **a** We graph $Y_1 = 2^X$ and $Y_2 = 20$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 4.32$.

b We graph $Y_1 = 4^X$ and $Y_2 = 100$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 3.32$.

c We graph $Y_1 = 3^X$ and $Y_2 = 30$ on the same set of axes, and find their point of intersection.

The solution is $x \approx 3.10$.

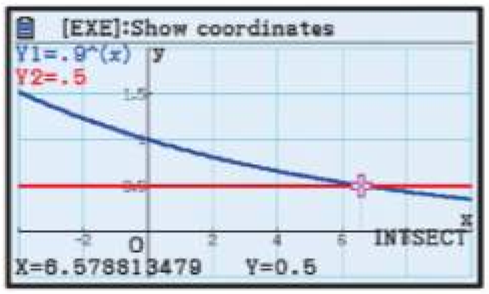
d We graph $Y_1 = (1.2)^X$ and $Y_2 = 3$ on the same set of axes, and find their point of intersection.

The solution is $x \approx 6.03$.

e We graph $Y_1 = (1.04)^X$ and $Y_2 = 4.238$ on the same set of axes, and find their point of intersection.

The solution is $x \approx 36.8$.

f We graph $Y_1 = (0.9)^X$ and $Y_2 = 0.5$ on the same set of axes, and find their point of intersection.

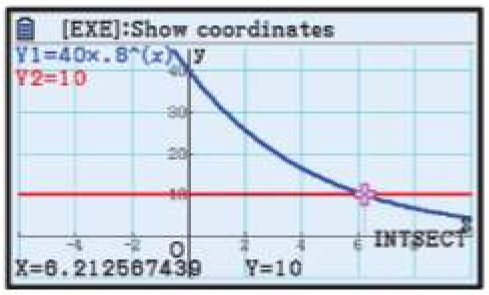


The solution is $x \approx 6.58$.

3 **a** We graph $Y_1 = 3 \times 2^X$ and $Y_2 = 93$ on the same set of axes, and find their point of intersection.

The solution is $x \approx 4.95$.

b We graph $Y_1 = 40 \times (0.8)^X$ and $Y_2 = 10$ on the same set of axes, and find their point of intersection.

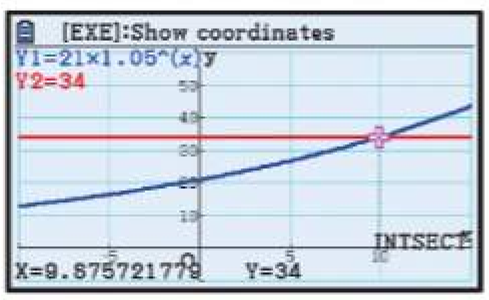


The solution is $x \approx 6.21$.

c We graph $Y_1 = 8 \times 3^X$ and $Y_2 = 120$ on the same set of axes, and find their point of intersection.

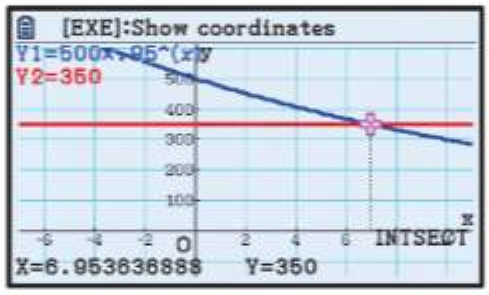
The solution is $x \approx 2.46$.

d We graph $Y_1 = 21 \times (1.05)^X$ and $Y_2 = 34$ on the same set of axes, and find their point of intersection.



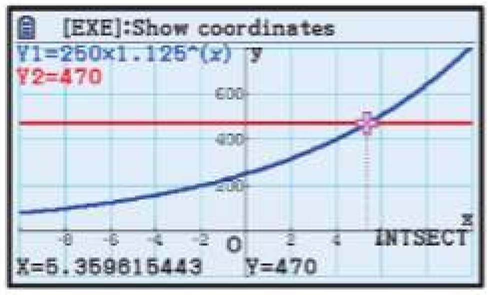
The solution is $x \approx 9.88$.

e We graph $Y_1 = 500 \times (0.95)^X$ and $Y_2 = 350$ on the same set of axes, and find their point of intersection.



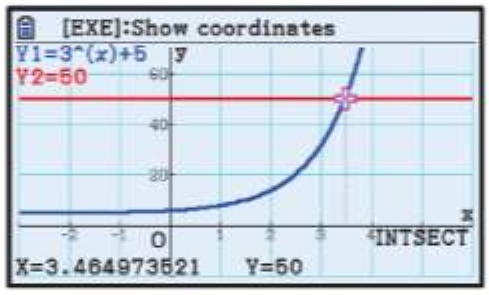
The solution is $x \approx 6.95$.

f We graph $Y_1 = 250 \times (1.125)^X$ and $Y_2 = 470$ on the same set of axes, and find their point of intersection.



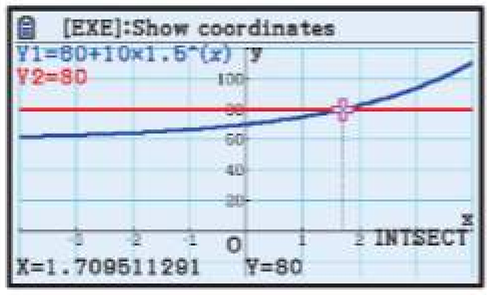
The solution is $x \approx 5.36$.

g We graph $Y_1 = 3^X + 5$ and $Y_2 = 50$ on the same set of axes, and find their point of intersection.



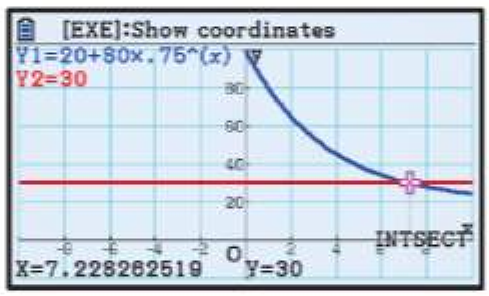
The solution is $x \approx 3.46$.

h We graph $Y_1 = 60 + 10 \times (1.5)^X$ and $Y_2 = 80$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 1.71$.

i We graph $Y_1 = 20 + 80 \times (0.75)^X$ and $Y_2 = 30$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 7.23$.

4 **a** $y = 4 \times a^x - 7, \quad a > 0$

When $x = -2, \quad y = 5$

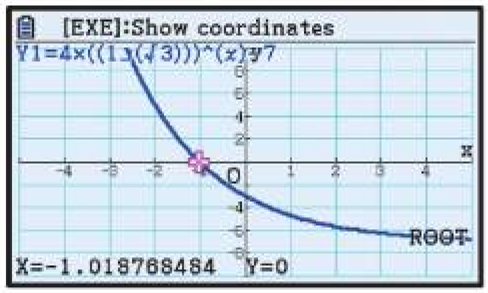
$$\therefore\ 5 = 4 \times a^{-2} - 7$$
$$\therefore\ 12 = 4a^{-2}$$
$$\therefore\ a^2 = \tfrac{1}{3}$$
$$\therefore\ a = \tfrac{1}{\sqrt{3}} \quad \{a > 0\}$$

b $y = 4 \times \left(\frac{1}{\sqrt{3}}\right)^x - 7$

The x-intercept occurs where $y = 0$

$$\therefore \quad 4 \times \left(\frac{1}{\sqrt{3}}\right)^x - 7 = 0$$

We graph $\mathrm{Y}_1 = 4 \times \left(\frac{1}{\sqrt{3}}\right)^{\mathrm{X}} - 7$ using technology.



The x-intercept is ≈ -1.02.

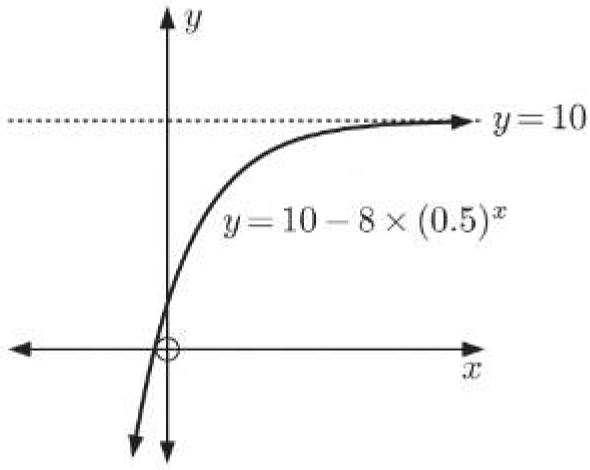
5 The graph of $y = 10 - 8 \times (0.5)^x$ lies below its horizontal asymptote $y = 10$.

$\therefore \quad y < 10$ for all x.

So, the equation $10 - 8 \times (0.5)^x = k$ has:

a 1 solution for $k < 10$

b no solutions for $k \geqslant 10$.



6 $y = k \times 2^x + c$

When $x = -1$, $y = -\frac{7}{8}$

$$\therefore \quad k \times 2^{-1} + c = -\frac{7}{8}$$

$$\therefore \quad \frac{k}{2} + c = -\frac{7}{8} \quad .... (1)$$

When $x = 5$, $y = 7$

$$\therefore \quad k \times 2^5 + c = 7$$

$$\therefore \quad 32k + c = 7 \quad .... (2)$$

Now (2) − (1) gives $32k - \frac{k}{2} = 7 - \left(-\frac{7}{8}\right)$

$$\therefore \quad \frac{63}{2}k = \frac{63}{8}$$

$$\therefore \quad k = \frac{1}{4}$$

Substituting $k = \frac{1}{4}$ into (2) gives $32\left(\frac{1}{4}\right) + c = 7$

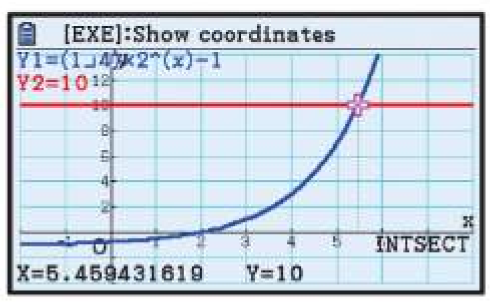
$$\therefore \quad 8 + c = 7$$

$$\therefore \quad c = -1$$

$\therefore$ the exponential function is $y = \frac{1}{4} \times 2^x - 1$.

When $y = 10$, $\frac{1}{4} \times 2^x - 1 = 10$

We graph $\mathrm{Y}_1 = \frac{1}{4} \times 2^{\mathrm{X}} - 1$ and $\mathrm{Y}_2 = 10$ on the same set of axes, and find their point of intersection.
The solution is $x \approx 5.46$.



ACTIVITY 1 EQUATING INDICES

1 **a** $2^x = 32$
$\therefore\ 2^x = 2^5$
$\therefore\ x = 5$

b $5^x = 25$
$\therefore\ 5^x = 5^2$
$\therefore\ x = 2$

c $3^x = 81$
$\therefore\ 3^x = 3^4$
$\therefore\ x = 4$

d $7^x = 1$
$\therefore\ 7^x = 7^0$
$\therefore\ x = 0$

e $3^x = \frac{1}{3}$
$\therefore\ 3^x = 3^{-1}$
$\therefore\ x = -1$

f $2^x = \sqrt{2}$
$\therefore\ 2^x = 2^{\frac{1}{2}}$
$\therefore\ x = \frac{1}{2}$

g $5^x = \frac{1}{125}$
$\therefore\ 5^x = 5^{-3}$
$\therefore\ x = -3$

h $4^x = 64$
$\therefore\ 4^x = 4^3$
$\therefore\ x = 3$

i $2^{x-2} = \frac{1}{32}$
$\therefore\ 2^{x-2} = 2^{-5}$
$\therefore\ x - 2 = -5$
$\therefore\ x = -3$

j $3^{x+1} = \frac{1}{27}$
$\therefore\ 3^{x+1} = 3^{-3}$
$\therefore\ x + 1 = -3$
$\therefore\ x = -4$

k $7^{x+1} = \frac{1}{\sqrt{7}}$
$\therefore\ 7^{x+1} = 7^{-\frac{1}{2}}$
$\therefore\ x + 1 = -\frac{1}{2}$
$\therefore\ x = -\frac{3}{2}$

l $5^{1-2x} = \frac{1}{5}$
$\therefore\ 5^{1-2x} = 5^{-1}$
$\therefore\ 1 - 2x = -1$
$\therefore\ -2x = -2$
$\therefore\ x = 1$

2 Equating indices allows us to solve exponential equations without drawing graphs or using technology.

However, it is not always practical to express both sides of an exponential equation as powers of the same base. For example, $2^x = 3$.

EXERCISE 1G.1

1 $A(t) = 3 \times (1.08)^t$ square metres

a $A(0) = 3 \times (1.08)^0$
$= 3$

The initial area covered by the weed was $3\ \text{m}^2$.

b The multiplier is 1.08, so the area increases by 8% each day.

c **i** $A(2) = 3 \times (1.08)^2$
≈ 3.50

The area covered after 2 days is about $3.50\ \text{m}^2$.

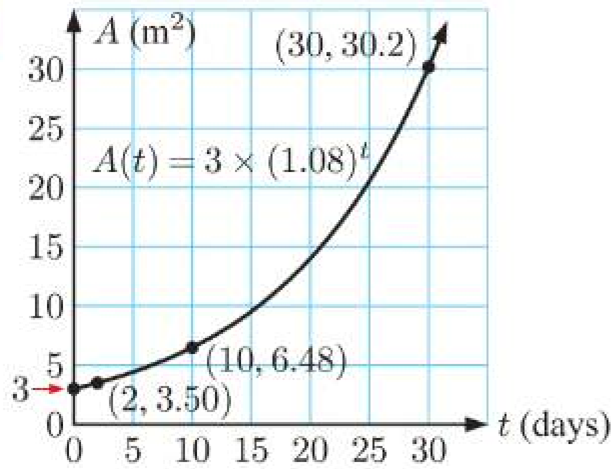
ii $A(10) = 3 \times (1.08)^{10}$
≈ 6.48

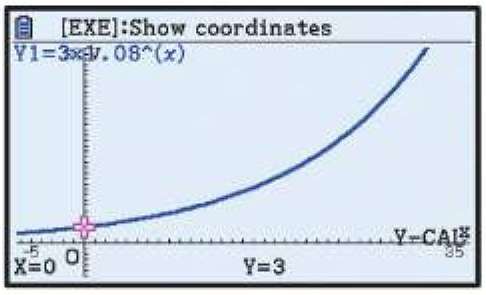
The area covered after 10 days is about $6.48\ \text{m}^2$.

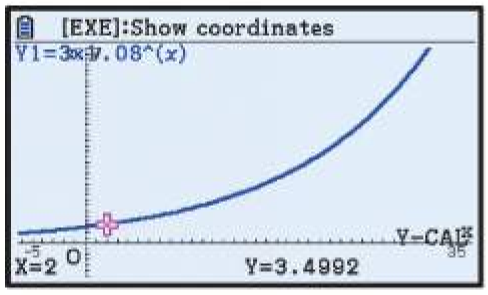
iii $A(30) = 3 \times (1.08)^{30}$
≈ 30.2

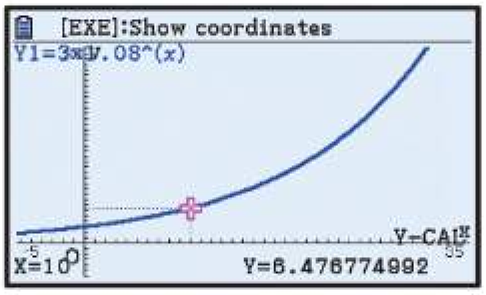
The area covered after 30 days is about $30.2\ \text{m}^2$.

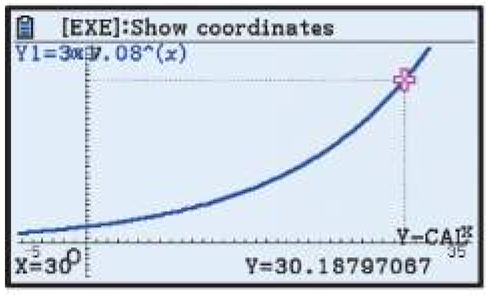
d











2 $W(t) = 100 \times (1.07)^t$ grams

a $W(0) = 100 \times (1.07)^0$
$= 100 \times 1$
$= 100$

$\therefore$ the initial weight was 100 grams.

b The value 1.07 means that the weight is increasing by 7% every hour.

c **i** $W(4) = 100 \times (1.07)^4$
≈ 131

The weight after 4 hours is about 131 grams.

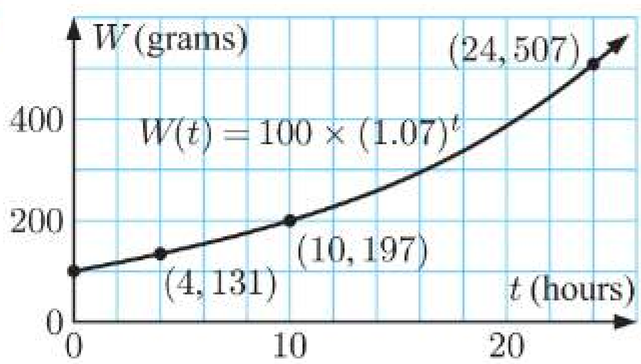
ii $W(10) = 100 \times (1.07)^{10}$
≈ 197

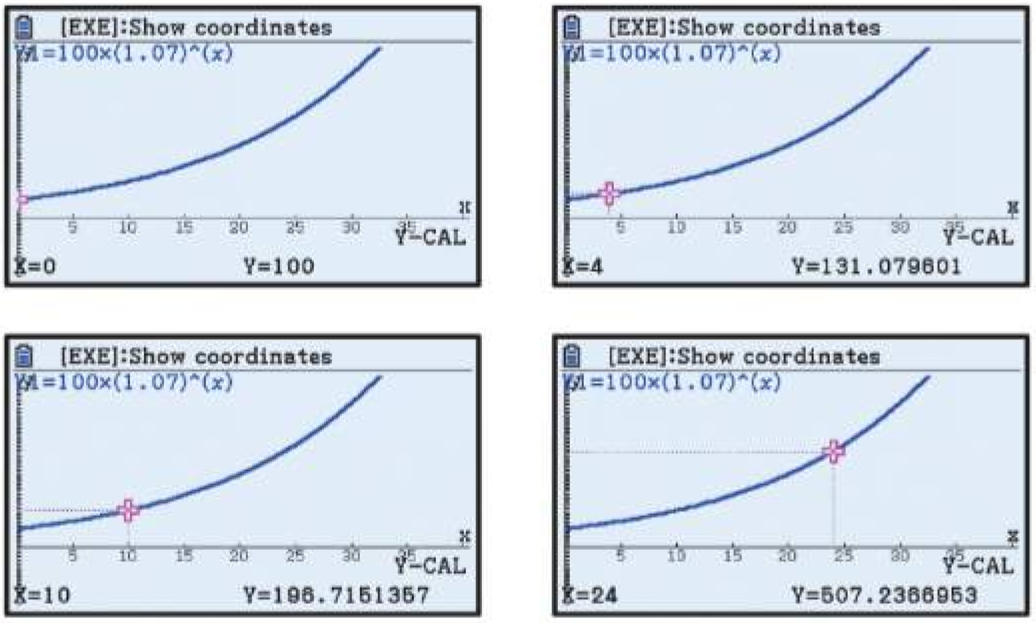
The weight after 10 hours is about 197 grams.

iii $W(24) = 100 \times (1.07)^{24}$
≈ 507

The weight after 24 hours is about 507 grams.

d





3 $P(n) = P_0 \times (1.23)^n$ possums

a The initial population is 50 possums, so $P(0) = 50$

$$\therefore \; P_0 \times (1.23)^0 = 50$$
$$\therefore \; P_0 = 50$$

b $P(n) = 50 \times (1.23)^n$

i $P(2) = 50 \times (1.23)^2$
≈ 75.6

After 2 years, the expected population is about 76 possums.

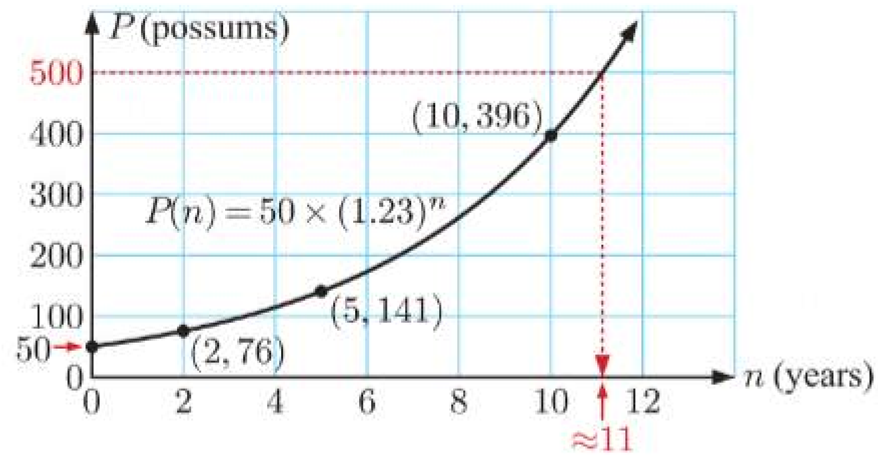
ii $P(5) = 50 \times (1.23)^5$
≈ 141

After 5 years, the expected population is about 141 possums.

iii $P(10) = 50 \times (1.23)^{10}$
≈ 396

After 10 years, the expected population is about 396 possums.

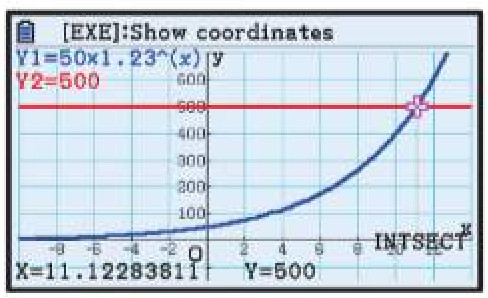
c



d From the graph in **c**, it appears that it would take about 11 years for the population to reach 500.

When $P = 500$, $500 = 50 \times (1.23)^n$.

Using technology, $n \approx 11.1$.



It will take about 11.1 years for the population to reach 500.

4 $N = 4 \times 1.332^t, \quad t \geqslant 0$

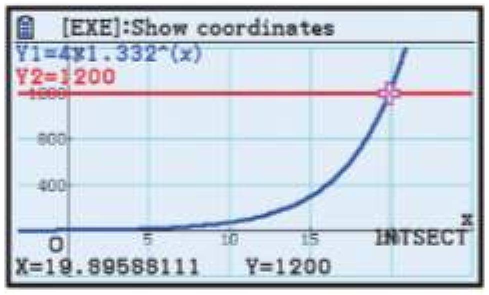
a When $t = 0$, $N = 4 \times 1.332^0$
$= 4 \times 1$
$= 4$

$\therefore$ the number of people initially infected was 4.

b When $t = 16$, $N = 4 \times 1.332^{16}$
≈ 393

$\therefore$ the number of people infected after 16 days was about 393.

c When $N = 1200$, $1200 = 4 \times 1.332^t$.
Using technology, $t \approx 19.9$.



It will take about 19.9 days for everybody in the school to catch the flu.

d Since there are 1200 people in the school and it takes about 19.9 days for all of them to catch the flu (from **c**), it is only reasonable to use this model for $0 \leqslant t \leqslant 19.9$.

5 $B(t) = B_0 \times a^t$ bears

a There were initially 200 bears introduced to the island, so $B(0) = 200$

$\therefore B_0 \times a^0 = 200$

$\therefore B_0 = 200$

b 2000 is 2 years after 1998, so $t = 2$.

$B(t) = 200 \times a^t$

$B(2) = 200 \times a^2$

$\therefore 242 = 200 \times a^2$

$\therefore \frac{242}{200} = a^2$

$\therefore a = \sqrt{\frac{242}{200}} = 1.1 \quad \{a > 0\}$

The bear population is increasing by 10% every year.

c 2018 is 20 years after 1998, so $t = 20$.

$B(20) = 200 \times (1.1)^{20}$
≈ 1350

The expected bear population in 2018 is about 1350 bears.

d 2008 is 10 years after 1998, so $t = 10$.

$B(10) = 200 \times (1.1)^{10}$

Percentage increase from 2008 to 2018 $= \left(\dfrac{B(20) - B(10)}{B(10)}\right) \times 100\%$

$= \left(\dfrac{200 \times (1.1)^{20} - 200 \times (1.1)^{10}}{200 \times (1.1)^{10}}\right) \times 100\%$

$\approx 159\%$

e When $B(t) = 2000, \quad 2000 = 200 \times (1.1)^t$.

Using technology, $t \approx 24.2$.

It will take about 24.2 years for the population to reach 2000.

6 **a** **i** $V(0) = V_0 \times 2^{0.05(0)}$

$= V_0 \times 2^0$

$= V_0 \times 1$

$= V_0$

So, the reaction speed at $0°C$ is V_0.

ii $V(20) = V_0 \times 2^{0.05(20)}$

$= V_0 \times 2^1$

$= 2V_0$

So, the reaction speed at $20°C$ is $2V_0$.

b Percentage increase at $20°C$ compared with $0°C$

$$= \left(\frac{V(20) - V(0)}{V(0)}\right) \times 100\%$$

$$= \left(\frac{2V_0 - V_0}{V_0}\right) \times 100\%$$

$$= \left(\frac{V_0}{V_0}\right) \times 100\%$$

$$= 100\%$$

So, there is a 100% increase in reaction speed at $20°C$ compared with $0°C$.

c $$\left(\frac{V(50) - V(20)}{V(20)}\right) \times 100\% = \left(\frac{V_0 \times 2^{0.05(50)} - 2V_0}{2V_0}\right) \times 100\%$$

$$= \left(\frac{V_0 \times 2^{2.5} - 2V_0}{2V_0}\right) \times 100\%$$

$$= \left(\frac{V_0(2^{2.5} - 2)}{V_0(2)}\right) \times 100\%$$

$$= \left(\frac{2^{2.5} - 2}{2}\right) \times 100\%$$

$$\approx 183\%$$

This means that there is about a 183% increase in reaction speed at $50°C$ compared with $20°C$.

7 **a** $A(t) = 5000 \times (1.1)^t$, where t is the number of years since Kayla deposited £5000.

b **i** $A(2) = 5000 \times (1.1)^2$

$= 6050$

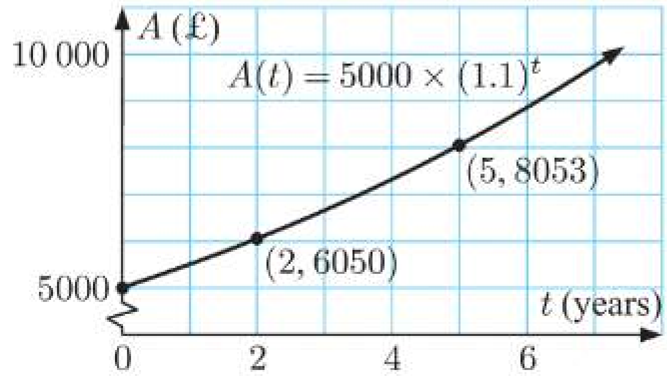
So there was £6050 in the account after 2 years.

ii $A(5) = 5000 \times (1.1)^5$

$= 8052.55$

So there was £8052.55 in the account after 5 years.

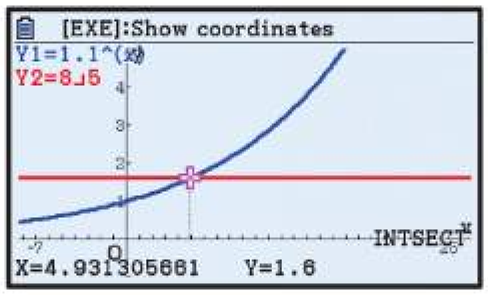
c



d

$$A(t) = 8000$$
$$\therefore \; 5000 \times (1.1)^t = 8000$$
$$\therefore \; (1.1)^t = \tfrac{8}{5}$$

We graph $Y_1 = (1.1)^X$ and $Y_2 = \frac{8}{5}$ on the same set of axes and find their point of intersection.

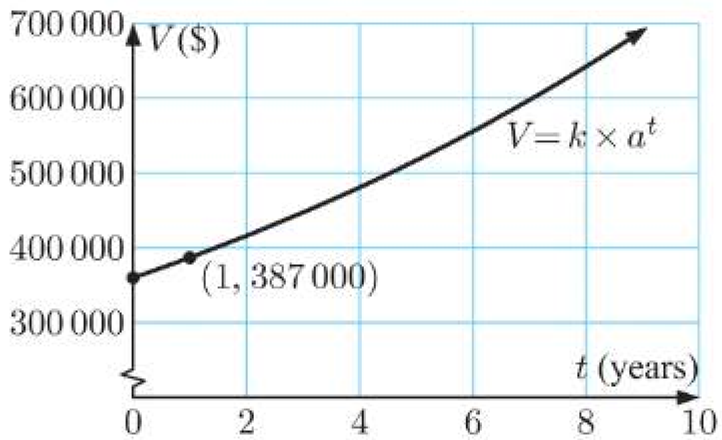


The solution is $t \approx 4.93$.

It will take about 4.93 years for the amount in the account to reach £8000.

8 **a** $V = k \times a^t$ dollars, $t \geqslant 0$

The value increases by 7.5% each year, so the multiplier $a = 1.075$.



b

$$V = k \times (1.075)^t$$

When $t = 1$, $V = 387\,000$

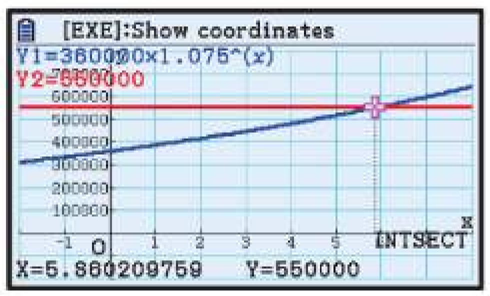
$$\therefore \; k \times (1.075)^1 = 387\,000$$
$$\therefore \; k = \frac{387\,000}{1.075}$$
$$\therefore \; k = 360\,000$$

The original value of the house was \$360 000.

c $V = 360\,000 \times (1.075)^t$

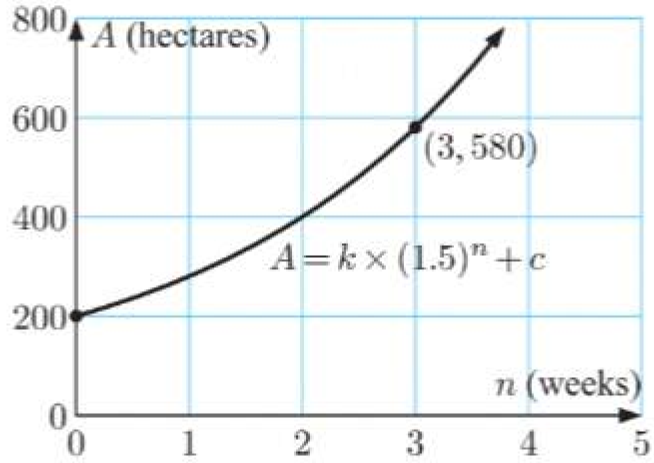
When $V = 550\,000$, $550\,000 = 360\,000 \times (1.075)^t$.

Using technology, $t \approx 5.86$.



It will take about 5.86 years for the house's value to reach \$550 000.

9 $A(n) = k \times (1.5)^n + c$ hectares



a From the graph, when $n = 0$, $A = 200$

$\therefore\ k \times (1.5)^0 + c = 200$

$\therefore\ k + c = 200$

Also, when $n = 3$, $A = 580$

$\therefore\ k \times (1.5)^3 + c = 580$

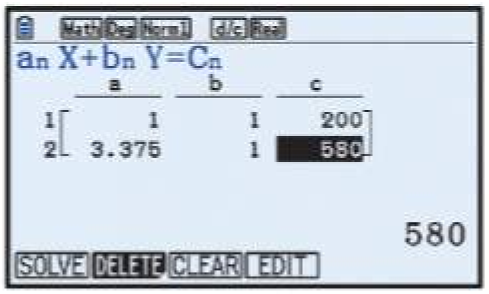
$\therefore\ 3.375k + c = 580$

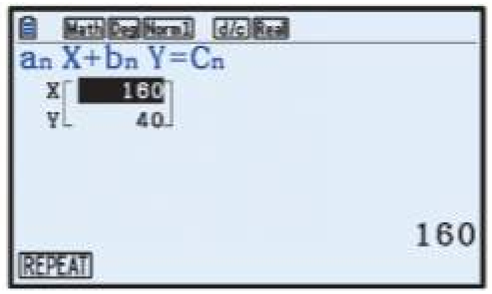
Solving the system of equations

$$\begin{cases} k + c = 200 \\ 3.375k + c = 580 \end{cases}$$

simultaneously gives

$k = 160$, $c = 40$.





b $A(n) = 160 \times (1.5)^n + 40$

When $A(n) = 1000$, $1000 = 160 \times (1.5)^n + 40$.

Using technology, $n \approx 4.42$.

It will take about 4.42 weeks for the infested area to reach 1000 hectares.

10 $V = c - k \times (0.8)^t$ m s^{-1}

a When $t = 0$, $V = c - k \times (0.8)^0$

$= c - k$

Now, $V = 0$ at time $t = 0$

$\therefore\ 0 = c - k$

$\therefore\ c = k$

b $V = k - k \times (0.8)^t$

When $t = 2$, $V = k - k \times (0.8)^2$

$\therefore\ 21.6 = k - 0.64k$

$\therefore\ 21.6 = 0.36k$

$\therefore\ k = \dfrac{21.6}{0.36}$

$\therefore\ k = 60$

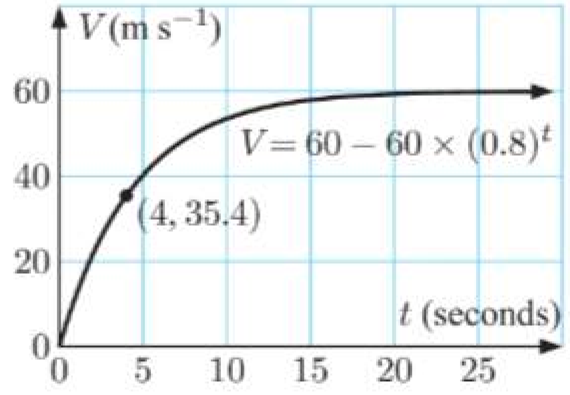
So, the exponential model is $V = 60 - 60 \times (0.8)^t$ m s^{-1}.

c When $t = 4$, $V = 60 - 60 \times (0.8)^4$

$= 60 - 60 \times 0.4096$

$= 35.424$

After 4 seconds, the speed of the parachutist is about 35.4 m s^{-1}.

d 

e From the graph, the parachutist accelerates rapidly until he approaches his terminal velocity of 60 m s^{-1}.

EXERCISE 1G.2

1 $W(t) = 250 \times (0.998)^t$ grams

a $W(0) = 250 \times (0.998)^0$
$= 250 \times 1$
$= 250$

$\therefore$ there was initially 250 grams of radioactive substance set aside.

b **i** $W(400) = 250 \times (0.998)^{400}$
≈ 112

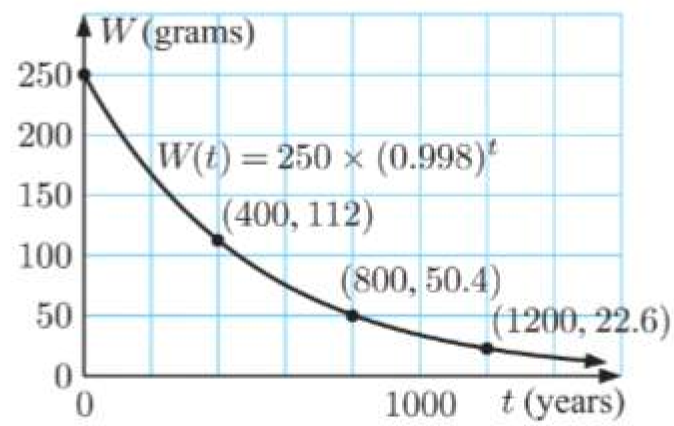
The weight was about 112 grams after 400 years.

ii $W(800) = 250 \times (0.998)^{800}$
≈ 50.4

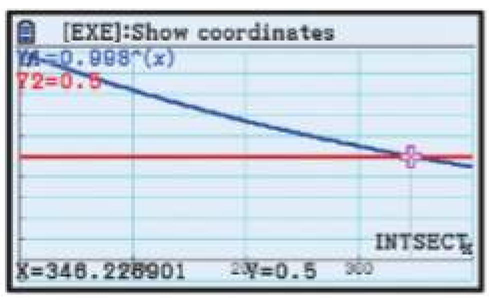
The weight was about 50.4 grams after 800 years.

iii $W(1200) = 250 \times (0.998)^{1200}$
≈ 22.6

The weight was about 22.6 grams after 1200 years.

c 

d $W(t) = 125$
$\therefore\ 250 \times (0.998)^t = 125$
$\therefore\ (0.998)^t = 0.5$
$\therefore\ t \approx 346.2$ {using technology}



It takes approximately 346 years for the substance to decay to 125 grams.

2 $T(t) = 100 \times (1.02)^{-t}$ °C

a $T(0) = 100 \times (1.02)^0$
$= 100$

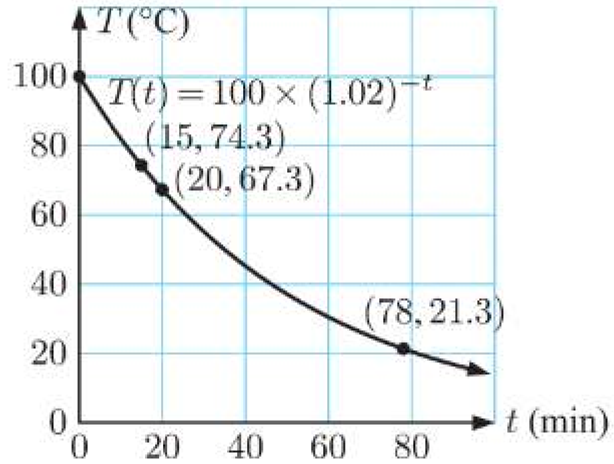
The initial temperature of the liquid was 100°C.

b **i** $T(15) = 100 \times (1.02)^{-15}$
≈ 74.3
The temperature was about 74.3°C after 15 minutes.

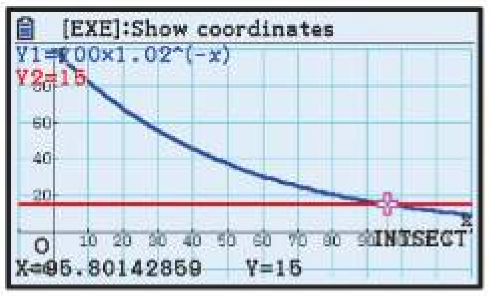
ii $T(20) = 100 \times (1.02)^{-20}$
≈ 67.3
The temperature was about 67.3°C after 20 minutes.

iii $T(78) = 100 \times (1.02)^{-78}$
≈ 21.3
The temperature was about 21.3°C after 78 minutes.

c



d When $T(t) = 15$, $15 = 100 \times (1.02)^{-t}$.
Using technology, $t \approx 95.8$.



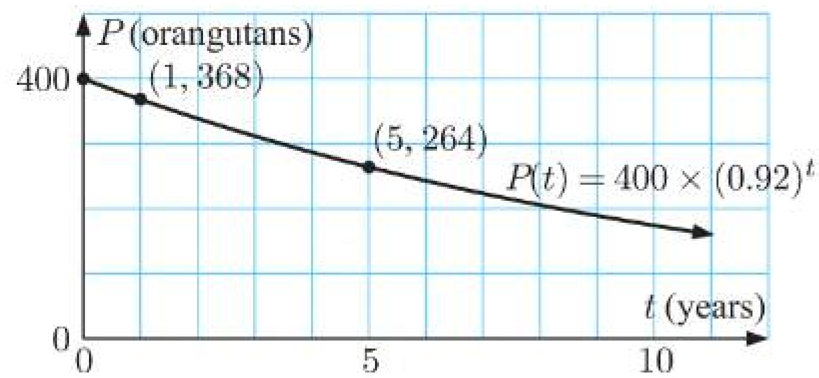
It will take about 95.8 minutes for the temperature of the liquid to fall to 15°C.

3 **a** $P(t) = 400 \times (0.92)^t$ orangutans

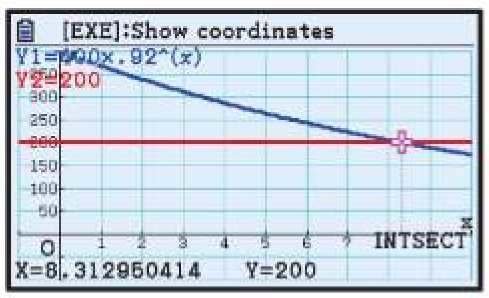
b **i** $P(1) = 400 \times (0.92)^1$
$= 368$
There were 368 orangutans after 1 year.

ii $P(5) = 400 \times (0.92)^5$
≈ 264
There were about 264 orangutans after 5 years.

c



d When $P(t) = 200$, $200 = 400 \times (0.92)^t$.
Using technology, $t \approx 8.31$.



The population will fall to 200 after about 8.31 years, or about 8 years and 114 days.

4 $L = 10 \times a^d$ units, $a > 0$, $d \geqslant 0$

a As the intensity of light diminishes as the depth increases, we would expect that $0 < a < 1$.

b When $d = 1$, $L = 9.5$

$$\therefore\ 10 \times a^1 = 9.5$$
$$\therefore\ 10a = 9.5$$
$$\therefore\ a = 0.95$$

c $L = 10 \times (0.95)^d$

When $d = 25$, $L = 10 \times (0.95)^{25}$

≈ 2.77

$\therefore$ the intensity of light 25 m below the surface is about 2.77 units.

d When $L = 4$, $4 = 10 \times (0.95)^d$.

Using technology, $d \approx 17.9$.

A depth of about 17.9 m has a light intensity of 4 units.

e When $L = 3$, $3 = 10 \times (0.95)^d$.

Using technology, $d \approx 23.5$.

A depth of about 23.5 m has a light intensity of 3 units.

When $L = 1$, $1 = 10 \times (0.95)^d$.

Using technology, $d \approx 44.9$.

A depth of about 44.9 m has a light intensity of 1 unit.

A depth between approximately 23.5 m and 44.9 m will have a light intensity between 1 and 3 units.

5 $P = k \times a^t$ turtles

a The population decreased by 5% each year, so the multiplier $a = 0.95$.

b $P = k \times (0.95)^t$

When $t = 1$, $P = 323$

$$\therefore\ k \times (0.95)^1 = 323$$
$$\therefore\ k = \frac{323}{0.95}$$
$$\therefore\ k = 340$$

The initial population of turtles was 340.

c 2015 is 10 years after 2005, so $t = 10$.

$P = 340 \times (0.95)^t$

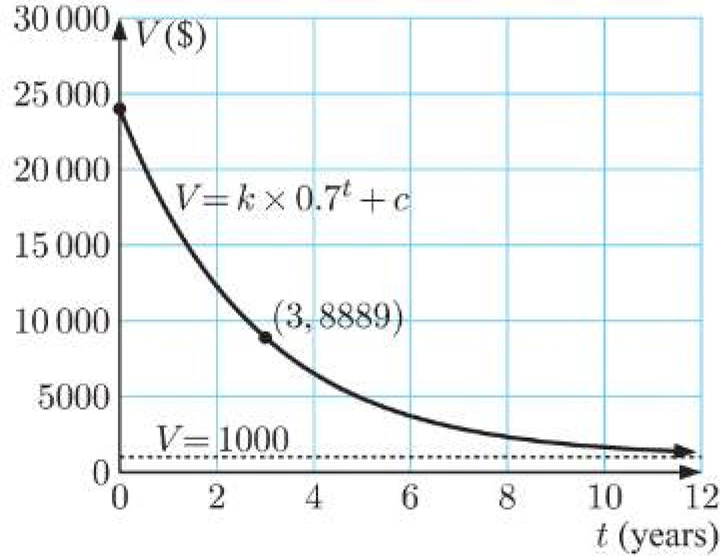
When $t = 10$, $P = 340 \times (0.95)^{10}$

≈ 204

The population of turtles in 2015 was about 204.

d No, it would not be reasonable to apply this model for negative values of t. The model is based on data collected since 2005. Before 2005 would be an extrapolation and therefore unreliable.

6 $V = k \times 0.7^t + c$ dollars, $t \geqslant 0$



a The horizontal asymptote is $V = 1000$, so $c = 1000$.

The value of the car approaches a minimum "salvage value" of \$1000.

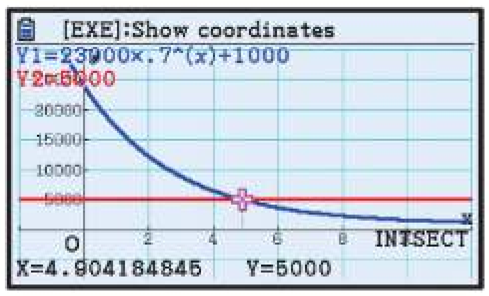
b
$$V = k \times 0.7^t + 1000$$
When $t = 3$, $V = 8889$
$$\therefore \quad k \times 0.7^3 + 1000 = 8889$$
$$\therefore \quad 0.343k = 7889$$
$$\therefore \quad k = \frac{7889}{0.343}$$
$$\therefore \quad k = 23\,000$$

c
$$V = 23\,000 \times 0.7^t + 1000$$
When $t = 0$,
$$V = 23\,000 \times 0.7^0 + 1000 = 23\,000 + 1000 = 24\,000$$

The initial value of the car was \$24 000.

d When $V = 5000$, $5000 = 23\,000 \times 0.7^t$.

Using technology, $t \approx 4.90$.



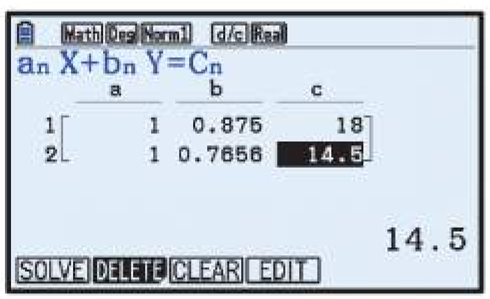
It will take about 5 years for the value of the car to reduce to \$5000.

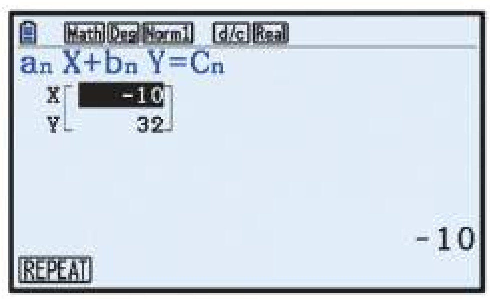
e No, the car depreciates by the same percentage each year *above* the salvage value of \$1000.

7 $T(t) = c + k \times 0.875^t$ °C

a
$$T(1) = 18 \qquad\qquad T(2) = 14.5$$
$$\therefore \quad c + k \times 0.875^1 = 18 \qquad\qquad \therefore \quad c + k \times 0.875^2 = 14.5$$
$$\therefore \quad c + 0.875k = 18 \qquad\qquad \therefore \quad c + 0.765\,625k = 14.5$$

Solving the system of equations
$$\begin{cases} c + 0.875k = 18 \\ c + 0.765\,625k = 14.5 \end{cases}$$
simultaneously gives $c = -10$, $k = 32$.





b $T(t) = -10 + 32 \times 0.875^t$

The horizontal asymptote is $T = -10$ which suggests that the temperature inside the freezer is -10°C.

c **i** $T(0) = -10 + 32 \times 0.875^0$
$= -10 + 32$
$= 22$

The temperature of the peas was $22°C$ when first placed in the freezer.

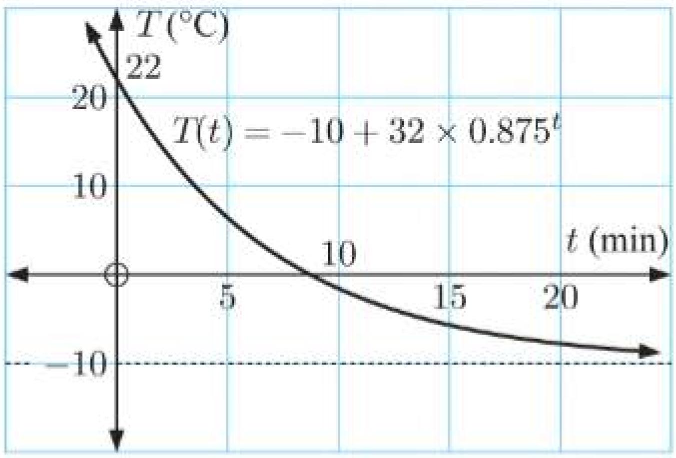
ii $T(5) = -10 + 32 \times 0.875^5$
≈ 6.41

The temperature of the peas was about $6.41°C$ after 5 minutes.

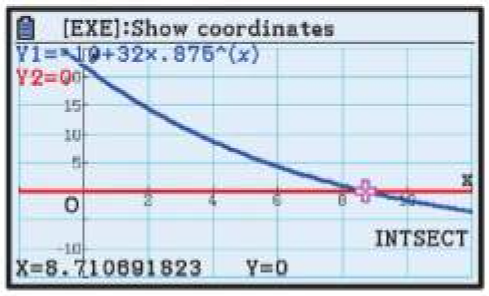
iii $T(10) = -10 + 32 \times 0.875^{10}$
≈ -1.58

The temperature of the peas was about $-1.58°C$ after 10 minutes.

d



e When $T(t) = 0$, $0 = -10 + 32 \times 0.875^t$.
Using technology, $t \approx 8.71$.



It takes about 8.71 minutes for the temperature of the peas to fall to $0°C$.

8 $D(t) = 120 \times (0.9)^t$ mg

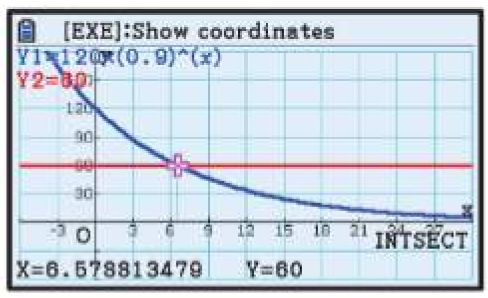
a $D(0) = 120 \times (0.9)^0$
$= 120$

The initial drug dose was 120 mg.

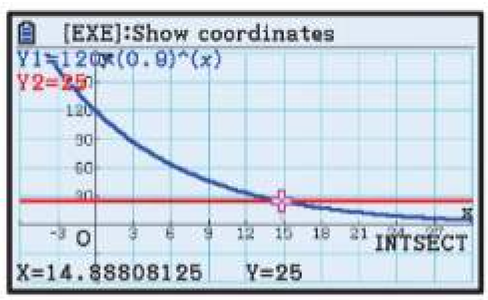
b Half of the initial drug dose is 60 mg.
$\therefore \quad 120 \times (0.9)^t = 60$

Using technology, it takes about 6.58 hours for the amount of drug remaining to fall to 60 mg.

So, the half-life of the drug is about 6.58 hours.



c When $D(t) = 25$, $25 = 120 \times (0.9)^t$
Using technology, $t \approx 14.9$.



There is only 25 mg of drug left in the body after about 14.9 hours.

9 $W(t) = 800 \times (0.96)^t$

a $W(0) = 800 \times (0.96)^0$
$= 800$

The initial weight was 800 g.

b Half of the initial weight is 400 g.

$\therefore \ 800 \times (0.96)^t = 400$

Using technology, it takes about 17.0 years for the substance's weight to fall to 400 g.

So, the half-life of the substance is about 17.0 years.

c It takes about 17.0 years for the weight to fall from 800 g to 400 g.

$\therefore$ it takes about 17.0 years for the weight to fall from 400 g to 200 g.

$\therefore$ it takes about 17.0 years for the weight to fall from 200 g to 100 g.

$\therefore$ it takes about 17.0 years for the weight to fall from 100 g to 50 g.

$\therefore$ it takes about $17.0 \times 4 \approx 68$ years for the weight of the substance to reach 50 g.

10 $W(t) = 10 \times a^t$ mg

a The value 10 indicates that the initial weight of the isotope is 10 mg.

b Fermium-253 has a half-life of 3 days.

So, after 3 days, its weight is $\frac{1}{2} \times 10 = 5$ mg.

Now, $W(3) = 10 \times a^3$

$\therefore \ 5 = 10 \times a^3$

$\therefore \ \frac{5}{10} = a^3$

$\therefore \ a = \sqrt[3]{\frac{1}{2}}$

$\therefore \ a \approx 0.7937$

Each day the isotope's weight decreases by about $(1 - 0.7937) \times 100\% \approx 20.63\%$.

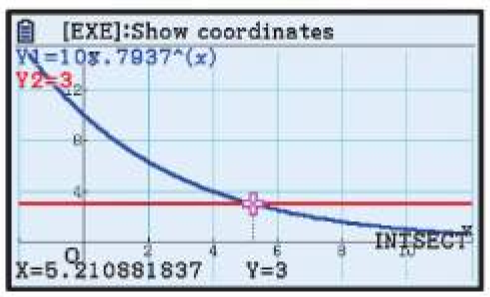
c $W(t) \approx 10 \times (0.7937)^t$

$\therefore \ W(2) \approx 10 \times (0.7937)^2$
≈ 6.30

The weight of fermium-253 after 2 days is about 6.30 mg.

d **i** When $W(t) = 3$, $3 = 10 \times (0.7937)^t$.

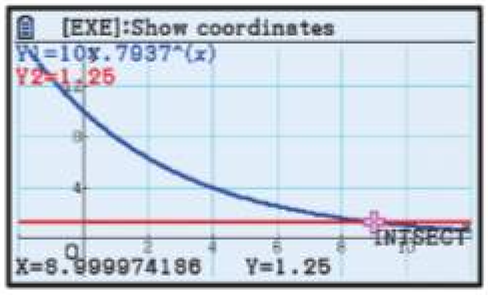
Using technology, $t \approx 5.21$.



It will take about 5.21 days for the weight of fermium-253 to fall to 3 mg.

ii When $W(t) = 1.25, \quad 1.25 = 10 \times (0.7937)^t$.

Using technology, $t \approx 9.00$.



It will take about 9 days for the weight of fermium-253 to fall to 1.25 mg.

11 $W = k \times a^t$ grams, $k > 0, \ 0 < a < 1$

When $t = 2, \quad W = 300$

$\therefore \ k \times a^2 = 300 \quad .... \ (*)$

When $t = 6, \quad W = 100$

$$\begin{aligned} \therefore \ k \times a^6 &= 100 \\ \therefore \ (k \times a^2) \times a^4 &= 100 \quad \{\text{using } (*)\} \\ \therefore \ 300 \times a^4 &= 100 \\ \therefore \ a^4 &= \tfrac{1}{3} \\ \therefore \ a &= \tfrac{1}{\sqrt[4]{3}} \quad \{0 < a < 1\} \end{aligned}$$

Substituting $a = \frac{1}{\sqrt[4]{3}}$ into $(*)$ gives $k \times \left(\frac{1}{\sqrt[4]{3}}\right)^2 = 300$

$$\begin{aligned} \therefore \ k \times \tfrac{1}{\sqrt{3}} &= 300 \\ \therefore \ k &= 300\sqrt{3} \end{aligned}$$

$\therefore \ W = 300\sqrt{3} \times \left(\frac{1}{\sqrt[4]{3}}\right)^t$

When $t = 0, \quad W = 300\sqrt{3} \times \left(\frac{1}{\sqrt[4]{3}}\right)^0$

$\qquad\qquad = 300\sqrt{3}$

$\therefore$ the initial weight is $300\sqrt{3}$ g.

Half of the initial weight is $150\sqrt{3}$ g.

$\therefore \ 300\sqrt{3} \times \left(\frac{1}{\sqrt[4]{3}}\right)^t = 150\sqrt{3}$

Using technology, it takes about 2.52 hours for the substance's weight to fall to $150\sqrt{3}$ g.

So, the half-life of the substance is about 2.52 hours.

12 The amount of carbon-14 in the piece of wood t years after being cut is given by the exponential model $C = k \times a^t, \ k > 0, \ 0 < a < 1$.

When $t = 0, \quad C = k \times a^0$

$\qquad\qquad = k$

$\therefore$ the initial amount of carbon-14 is k atoms.

Half the initial amount is $\frac{k}{2}$ atoms, and the half-life of carbon-14 is 5700 years.

$$\therefore \quad k \times a^{5700} = \frac{k}{2}$$

$$\therefore \quad a^{5700} = \tfrac{1}{2} \qquad \{k > 0\}$$

$$\therefore \quad a = \left(\tfrac{1}{2}\right)^{\frac{1}{5700}} \qquad \{0 < a < 1\}$$

So, $C = k \times \left(\tfrac{1}{2}\right)^{\frac{t}{5700}}$.

Let D atoms be the amount of carbon-12 in the piece of wood after being cut, which is constant. The ratio of carbon-12 atoms to carbon-14 atoms is initially $10^{12} : 1$.

So, when $t = 0$, $\dfrac{D}{C} = \dfrac{10^{12}}{1}$

$$\therefore \quad \frac{D}{k} = 10^{12}$$

$$\therefore \quad D = k \times 10^{12}$$

The ratio of carbon-12 atoms to carbon-14 atoms is now $10^{13} : 1$.

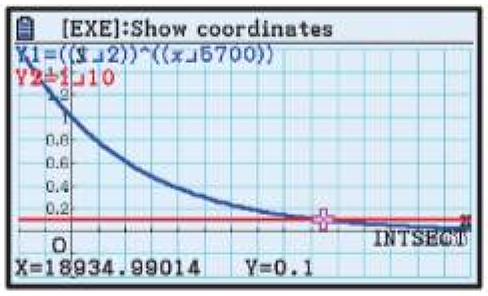
So, $\dfrac{D}{C} = \dfrac{10^{13}}{1}$

$$\therefore \quad \frac{k \times 10^{12}}{k \times \left(\frac{1}{2}\right)^{\frac{t}{5700}}} = 10^{13}$$

$$\therefore \quad \left(\tfrac{1}{2}\right)^{\frac{t}{5700}} = \tfrac{1}{10}$$

Using technology, $t \approx 18\,900$.

So, the piece of wood is about 18 900 years old.



INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

1 $u_n = u_0(1+i)^n, \quad u_0 = 1000$

a Interest paid annually:

$n = 1, \; i = 6\% = 0.06$

$$\therefore \quad u_1 = 1000(1+0.06)^1 = 1060$$

The final amount is \$1060.

b Interest paid quarterly:

$n = 4, \; i = \dfrac{6\%}{4} = 0.015$

$$\therefore \quad u_4 = 1000(1+0.015)^4 \approx 1061.36$$

The final amount is \$1061.36.

c Interest paid monthly:

$n = 12, \; i = \dfrac{6\%}{12} = 0.005$

$$\therefore \quad u_{12} = 1000(1+0.005)^{12} \approx 1061.68$$

The final amount is \$1061.68.

d Interest paid daily:

$n = 365.25, \; i = \dfrac{6\%}{365.25}$

$$\therefore \quad u_{365.25} = 1000\left(1 + \frac{0.06}{365.25}\right)^{365.25} \approx 1061.83$$

The final amount is \$1061.83.

e Interest paid by the second:

$n = 365.25 \times 24 \times 60 \times 60 = 31\,557\,600, \quad i = \dfrac{6\%}{31\,557\,600}$

$$\therefore \quad u_{31\,557\,600} = 1000\left(1 + \frac{0.06}{31\,557\,600}\right)^{31\,557\,600}$$
$$\approx 1061.84$$

The final amount is \$1061.84.

f Interest paid by the millisecond:

$n = 31\,557\,600 \times 1000 = 31\,557\,600\,000, \quad i = \dfrac{6\%}{31\,557\,600\,000}$

$$\therefore \quad u_{31\,557\,600\,000} = 1000\left(1 + \frac{0.06}{31\,557\,600\,000}\right)^{31\,557\,600\,000}$$
$$\approx 1061.84$$

The final amount is \$1061.84.

Given a fixed interest rate per annum, paying out the interest more frequently results in a higher final amount, but seems to approach a particular value.

2
$$\begin{aligned} u_n &= u_0(1+i)^n \\ &= u_0\left(1+\frac{r}{N}\right)^{Nt} \qquad \{i = \frac{r}{N},\ n = Nt\} \\ &= u_0\left(1+\frac{1}{a}\right)^{art} \qquad \{a = \frac{N}{r},\ N = ar\} \\ &= u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt} \end{aligned}$$

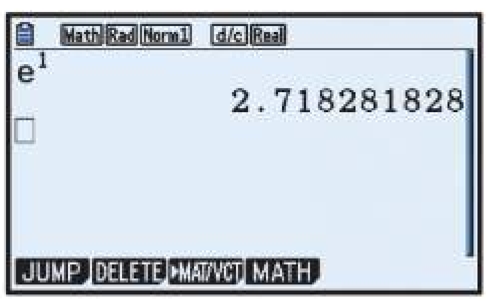
3 **a** $a = \dfrac{N}{r}$, so as $N \to \infty$, $a \to \infty$

b

a	$\left(1+\frac{1}{a}\right)^a$
10	2.593 724 46
100	2.704 813 829
1000	2.716 923 932
10 000	2.718 145 927
100 000	2.718 268 237
1 000 000	2.718 280 469
10 000 000	2.718 281 693

4 $e^1 \approx 2.718\,281\,828$

This appears to be the value of $\left(1+\dfrac{1}{a}\right)^a$ as $a \to \infty$.



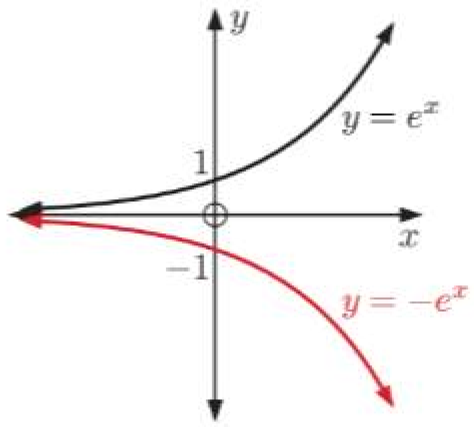
5 $u_n = u_0 e^{rt}$, $u_0 = 1000$, $r = 0.06$, $t = 1$

$\therefore \quad u_n = 1000 \times e^{0.06 \times 1} \approx 1061.84$

The final amount is \$1061.84.

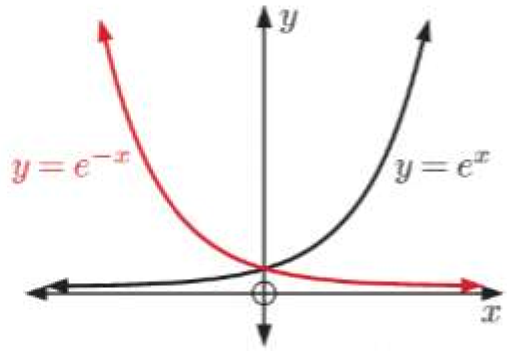
EXERCISE 1H

1



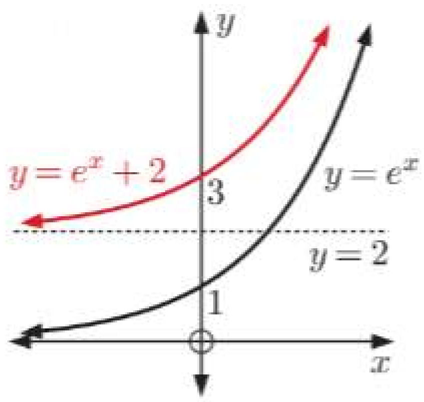
$y = -e^x$ is the reflection of $y = e^x$ in the x-axis.

2



$y = e^{-x}$ is the reflection of $y = e^x$ in the y-axis.

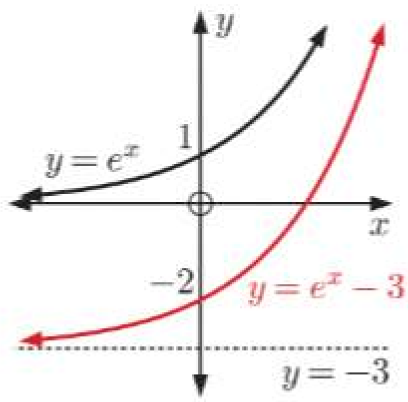
3 **a**



The y-intercept is 3.

The horizontal asymptote is $y = 2$.

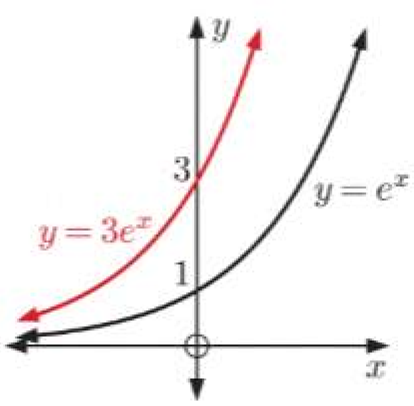
b



The y-intercept is -2.

The horizontal asymptote is $y = -3$.

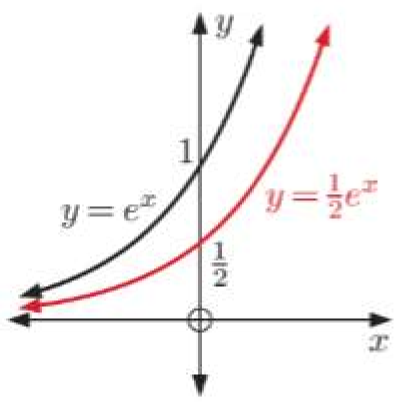
c



The y-intercept is 3.

The horizontal asymptote is $y = 0$.

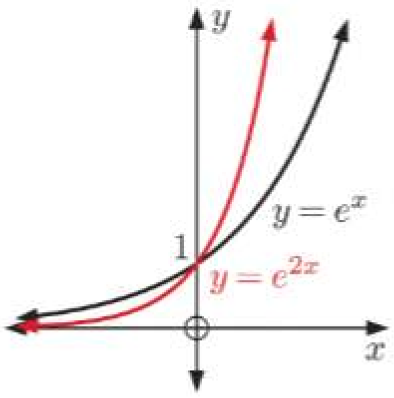
d



The y-intercept is $\frac{1}{2}$.

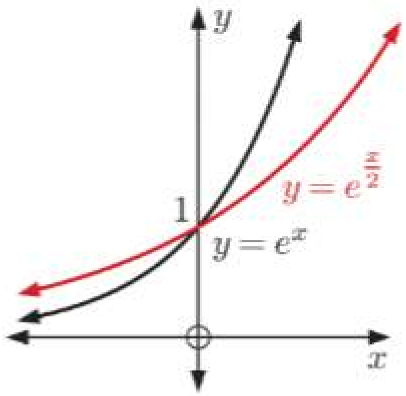
The horizontal asymptote is $y = 0$.

e



The y-intercept is 1.
The horizontal asymptote is $y = 0$.

f



The y-intercept is 1.
The horizontal asymptote is $y = 0$.

4 $y = ke^{rx} + c$

When $x = 0$, $\begin{aligned} y &= ke^0 + c \\ &= k + c \end{aligned}$

$\therefore$ the y-intercept is $k + c$.

5 **a** $e^2 \approx 7.39$ **b** $e^3 \approx 20.1$ **c** $e^{0.7} \approx 2.01$ **d** $\sqrt{e} \approx 1.65$
e $e^{-1} \approx 0.368$

6 **a** $\sqrt{e} = e^{\frac{1}{2}}$ **b** $\begin{aligned} \frac{1}{\sqrt{e}} &= \frac{1}{e^{\frac{1}{2}}} \\ &= e^{-\frac{1}{2}} \end{aligned}$ **c** $\frac{1}{e^2} = e^{-2}$ **d** $\begin{aligned} e\sqrt{e} &= e^1 \times e^{\frac{1}{2}} \\ &= e^{\frac{3}{2}} \end{aligned}$

7 **a** $e^{2.31} \approx 10.074$ **b** $e^{-2.31} \approx 0.099\,261$ **c** $e^{4.829} \approx 125.09$
d $e^{-4.829} \approx 0.007\,994\,5$ **e** $50e^{-0.1764} \approx 41.914$ **f** $80e^{-0.6342} \approx 42.429$
g $1000e^{1.2642} \approx 3540.3$ **h** $0.25e^{-3.6742} \approx 0.006\,342\,4$

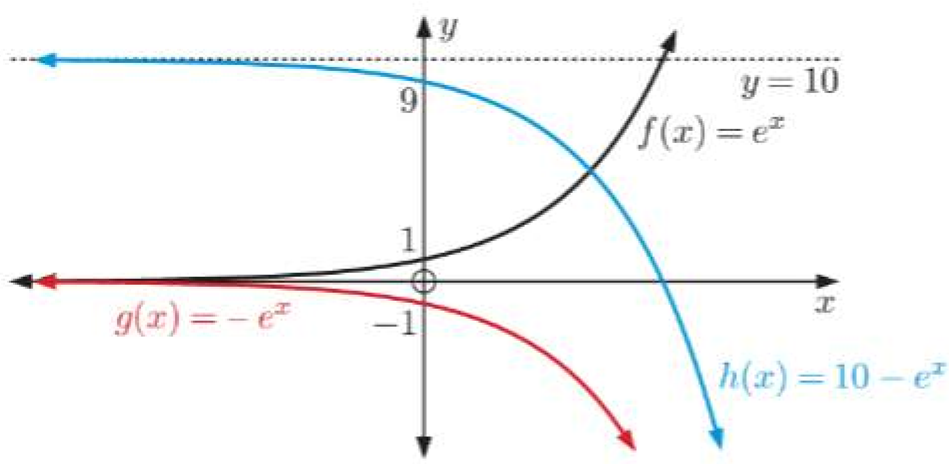
8 **a** $\begin{aligned} (e^x+1)^2 &= (e^x)^2 + 2 \times e^x \times 1 + 1^2 \\ &= e^{2x} + 2e^x + 1 \end{aligned}$ **b** $\begin{aligned} (1+e^x)(1-e^x) &= 1^2 - (e^x)^2 \\ &= 1 - e^{2x} \end{aligned}$

c $\begin{aligned} e^x(e^{-x} - 3) &= e^x \times e^{-x} - e^x \times 3 \\ &= e^0 - 3e^x \\ &= 1 - 3e^x \end{aligned}$

9 **a** $\begin{aligned} e^{2x} + e^x &= e^x \times e^x + e^x \\ &= e^x(e^x + 1) \end{aligned}$ **b** $\begin{aligned} e^{2x} - 16 &= (e^x)^2 - 4^2 \\ &= (e^x + 4)(e^x - 4) \end{aligned}$

c $e^{2x} - 8e^x + 12 = (e^x - 2)(e^x - 6)$ $\{a^2 - 8a + 12 = (a-2)(a-6)\}$

10 **a**



b The domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$.

The range of f is $\{y \mid y > 0\}$. The range of g is $\{y \mid y < 0\}$.

The range of h is $\{y \mid y < 10\}$.

c For f: as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to 0^+$

For g: as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 0^-$

For h: as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 10^-$

11 $W(t) = 2e^{\frac{t}{2}}$ grams

a **i** $W(0) = 2e^0$
$= 2 \times 1$
$= 2$

The weight of the culture is 2 grams initially.

ii $t = 30$ min $= \frac{1}{2}$ hour

$W(\frac{1}{2}) = 2e^{\frac{1}{4}}$
≈ 2.57

The weight of the culture is about 2.57 grams after 30 minutes.

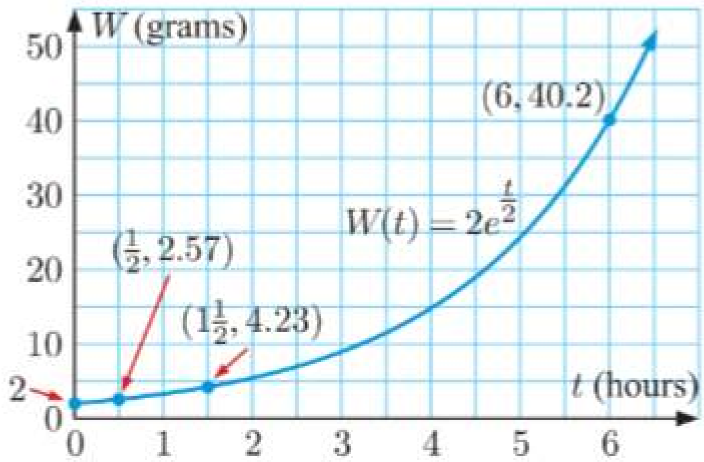
iii $W(1\frac{1}{2}) = 2e^{\frac{3}{4}}$
≈ 4.23

The weight of the culture is about 4.23 grams after $1\frac{1}{2}$ hours.

iv $W(6) = 2e^3$
≈ 40.2

The weight of the culture is about 40.2 grams after 6 hours.

b



12 $I(t) = 75e^{-0.15t}$ amps

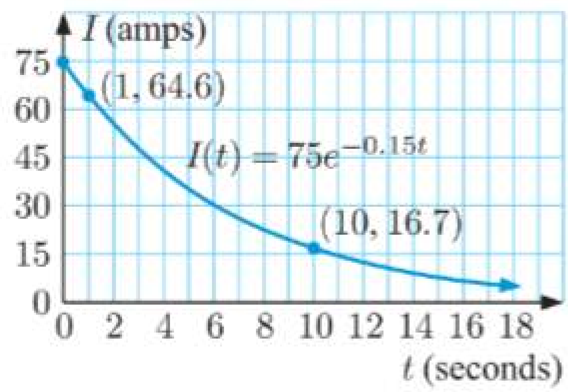
a **i** $I(1) = 75e^{-0.15}$
≈ 64.6

About 64.6 amps of current is still flowing after 1 second.

ii $I(10) = 75e^{-1.5}$
≈ 16.7

About 16.7 amps of current is still flowing after 10 seconds.

b

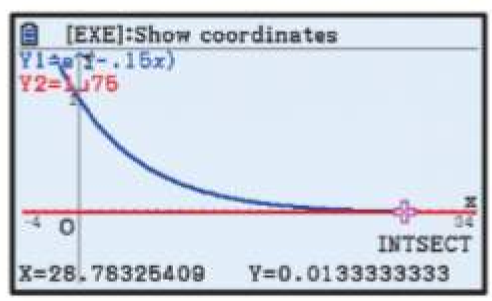


c We need to solve $75e^{-0.15t} = 1$

$\therefore\ e^{-0.15t} = \frac{1}{75}$

$\therefore\ t \approx 28.8$ {using technology}

It will take about 28.8 seconds for the current to fall to 1 amp.



13 $A = k \times e^{-0.5t} + c$ kL

a The horizontal asymptote is $A = 3$, so $c = 3$.

$$A = k \times e^{-0.5t} + 3$$

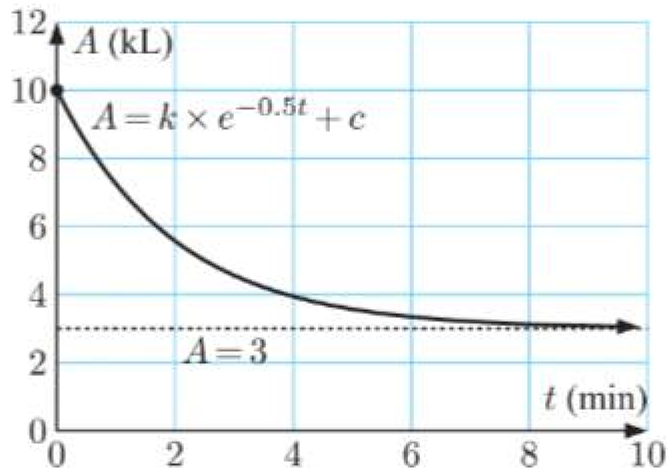
When $t = 0$, $A = 10$

$\therefore\ k \times e^{-0.5(0)} + 3 = 10$

$\therefore\ k + 3 = 10$

$\therefore\ k = 7$

So, the exponential model is $A = 7e^{-0.5t} + 3$.



b The hole is in the side of the tank as the tank never completely empties.

c When $t = 2$, $A = 7e^{-0.5(2)} + 3$

$= 7e^{-1} + 3$

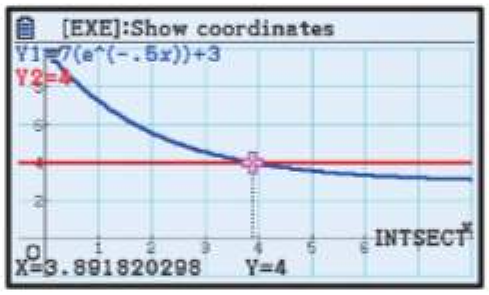
≈ 5.58

There is about 5.58 kL of water in the tank after 2 minutes.

d There is initially 10 kL of water in the tank, so the amount of water in the tank after losing 6 kL is $10 - 6 = 4$ kL.

When $A = 4$, $4 = 7e^{-0.5t} + 3$.

Using technology, $t \approx 3.89$.



It will take about 3.89 minutes for the tank to lose 6 kL of water.

14 $V(t) = 650(4 + 2 \times e^{-0.1t})$

a As $t \to \infty$, $e^{-0.1t} \to 0^+$

$\therefore$ the speed of the meteor is decreasing.

b **i** $V(0) = 650(4 + 2 \times e^{-0.1 \times 0})$

$= 650(4 + 2 \times 1)$

$= 650(6)$

$= 3900$

The speed of the meteor when it was first sighted was 3900 m s^{-1}.

ii $V(120) = 650(4 + 2 \times e^{-0.1 \times 120})$

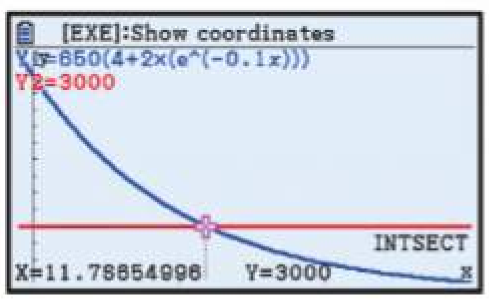
$= 650(4 + 2 \times e^{-12})$

$\approx 2600\ \text{m s}^{-1}$

The speed of the meteor after 2 minutes was about 2600 m s^{-1}.

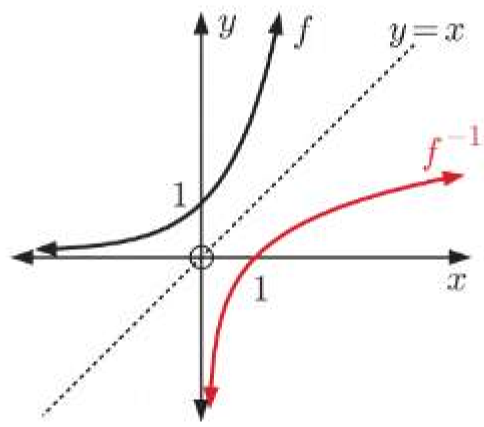
c When $V(t) = 3000$, $3000 = 650(4 + 2 \times e^{-0.1t})$.

Using technology, $t \approx 11.8$.



It will take about 11.8 seconds for the meteor's speed to reach 3000 m s^{-1}.

15 **a** f^{-1} is a reflection of f in the line $y = x$



b The domain of f^{-1} is $\{x \mid x > 0\}$.
The range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$.

16 $e^1 \approx \sum_{i=0}^{19} \frac{1}{i!} 1^i \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... + \frac{1}{19!} \approx 2.718\,281\,828$

EXERCISE 1I

1 $P(t) = \dfrac{5000}{1 + 24e^{-0.1t}}$

a $P(0) = \dfrac{5000}{1+24}$
$= 200$

Initially, 200 people inhabited the island.

b **i** $P(20) = \dfrac{5000}{1 + 24e^{-0.1(20)}}$
≈ 1177.0

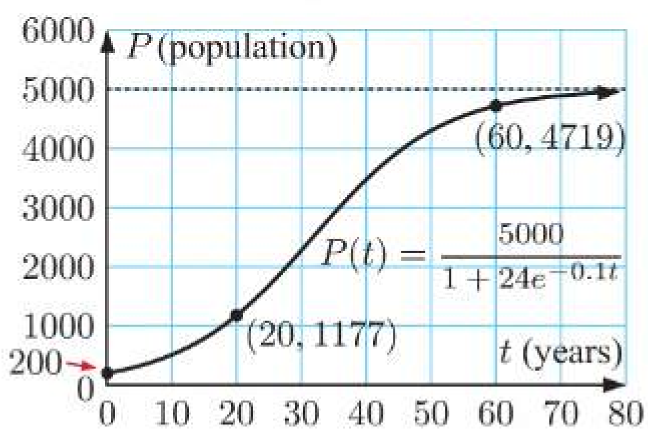
After 20 years, the population was about 1177.

ii $P(60) = \dfrac{5000}{1 + 24e^{-0.1(60)}}$
≈ 4719.3

After 60 years, the population was about 4719.

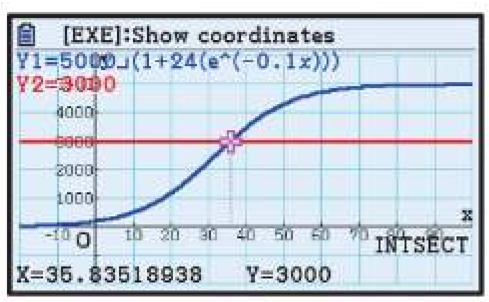
c The limiting population of the island is 5000.

d



e When $P(t) = 3000$, $3000 = \dfrac{5000}{1 + 24e^{-0.1t}}$

Using technology, $t \approx 35.8$



So, it took about 35.8 years for the population to reach 3000.

2 **a** $N = \dfrac{80}{1+Ce^{-t}}$

When $t = 0$, $N = 1$

$$\therefore \quad 1 = \frac{80}{1+C}$$

$$\therefore \quad 1 + C = 80$$

$$\therefore \quad C = 79$$

b $N = \dfrac{80}{1+79e^{-t}}$

As $t \to \infty$, $e^{-t} \to 0^+$

$$\therefore \quad N \to \frac{80}{1+0^+} = 80^-$$

c **i** When $t = 5$, $N = \dfrac{80}{1+79e^{-5}}$

≈ 52.2

After 5 days, there were about 52 people infected.

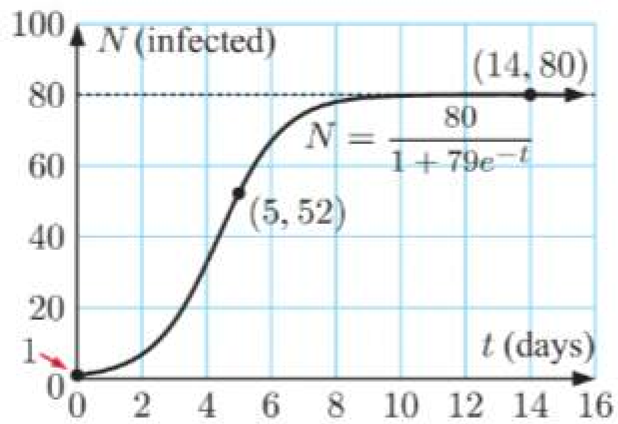
ii 2 weeks = 14 days

When $t = 14$, $N = \dfrac{80}{1+79e^{-14}}$

≈ 80.0

After 2 weeks, there were about 80 people infected.

d



3 **a** $H(t) = \dfrac{L}{1+2e^{-\frac{t}{4}}}$ cm

Now $H(0) = 54$

$$\therefore \quad 54 = \frac{L}{1+2}$$

$$\therefore \quad L = 162$$

b $H(t) = \dfrac{162}{1+2e^{-\frac{t}{4}}}$ cm

i $H(3) = \dfrac{162}{1+2e^{-\frac{3}{4}}}$

≈ 83.3

Paige was about 83.3 cm tall when she was 3 years old.

ii $H(10) = \dfrac{162}{1+2e^{-\frac{10}{4}}}$

≈ 139

Paige was about 139 cm tall when she was 10 years old.

c The limiting height is 162 cm.

$\therefore$ Paige is 162 cm tall now that she is fully grown.

d Piper's height at age t years is given by $h(t) = \dfrac{L}{1+Ce^{-kt}}$ cm.

Piper has the same height as Paige as adults $\therefore \; L = 162$.

So, $h(t) = \dfrac{162}{1+Ce^{-kt}}$.

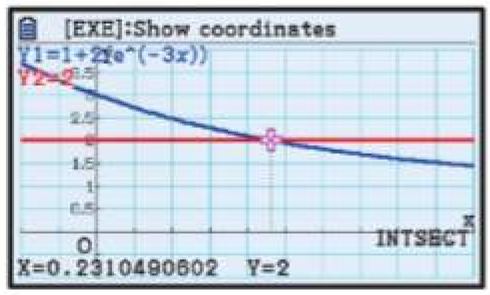
Piper had the same birth height as Paige.

$$\therefore \; h(0) = 54$$
$$\therefore \; 54 = \frac{162}{1+C}$$
$$\therefore \; 1+C = \frac{162}{54} = 3$$
$$\therefore \; C = 2$$

So, $h(t) = \dfrac{162}{1+2e^{-kt}}$.

Now $h(3) = 81$

$$\therefore \; 81 = \frac{162}{1+2e^{-3k}}$$
$$\therefore \; 1+2e^{-3k} = \frac{162}{81} = 2$$
$$\therefore \; k \approx 0.231 \quad \{\text{using technology}\}$$



So, $h(t) \approx \dfrac{162}{1+2e^{-0.231t}}$

$$\therefore \; h(10) \approx \frac{162}{1+2e^{-0.231(10)}}$$
$$\approx 135$$

Piper was about 135 cm tall when she was 10 years old.

4 **a** $N = \dfrac{L}{1+Ce^{-kt}}$

When $t = 0$, $N = 2$

$$\therefore \; 2 = \frac{L}{1+C}$$
$$\therefore \; L = 2(1+C) \quad .... \; (1)$$

10 am is 2 hours after 8 am.

When $t = 2$, $N = 10$

$$\therefore \ 10 = \frac{L}{1 + Ce^{-k(2)}}$$

$$\therefore \ 10 = \frac{2(1+C)}{1 + Ce^{-2k}} \qquad \{\text{using (1)}\}$$

$$\therefore \ 10(1 + Ce^{-2k}) = 2(1+C)$$

$$\therefore \ 5(1 + Ce^{-2k}) = 1 + C$$

$$\therefore \ 5 + 5Ce^{-2k} = 1 + C$$

$$\therefore \ 4 = C - 5Ce^{-2k}$$

$$\therefore \ 4 = C(1 + 5e^{-2k})$$

$$\therefore \ C = \frac{4}{1 - 5e^{-2k}} \qquad .... (2)$$

Noon is 4 hours after 8 am.

When $t = 4$, $N = 45$

$$\therefore \ 45 = \frac{L}{1 + Ce^{-k(4)}}$$

$$\therefore \ 45 = \frac{2(1+C)}{1 + Ce^{-4k}} \qquad \{\text{using (1)}\}$$

$$\therefore \ 45(1 + Ce^{-4k}) = 2(1+C)$$

$$\therefore \ 45 + 45Ce^{-4k} = 2 + 2C$$

$$\therefore \ 43 = 2C - 45Ce^{-4k}$$

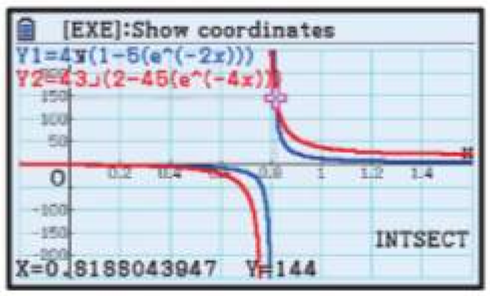
$$\therefore \ 43 = C(2 - 45e^{-4k})$$

$$\therefore \ C = \frac{43}{2 - 45e^{-4k}} \qquad .... (3)$$

Equating (2) and (3) gives $\dfrac{4}{1 - 5e^{-2k}} = \dfrac{43}{2 - 45e^{-4k}}$

Using technology, $k \approx 0.818\,80 \quad \{k > 0\}$

$$\therefore \ k \approx 0.819$$



Substituting $k \approx 0.818\,80$ into (2) gives $C \approx \dfrac{4}{1 - 5e^{-2(0.818\,80)}}$

$$\therefore \ C \approx 144$$

Substituting $C \approx 144$ into (1) gives $L \approx 2(1 + 144)$

$$\therefore \ L \approx 290$$

So, $N \approx \dfrac{290}{1 + 144e^{-0.819t}}$.

b 3 pm is 7 hours after 8 am.

When $t=7$, $N \approx \dfrac{290}{1+144e^{-0.819(7)}}$
≈ 197.8

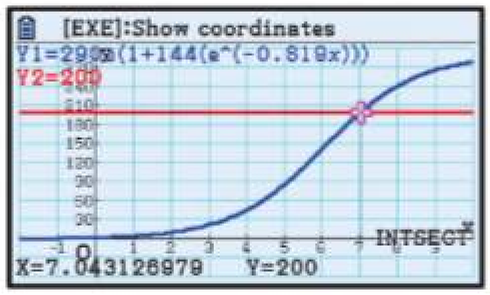
About 198 people have heard the rumour by 3 pm.

c The limiting value of N is about 290.

$\therefore$ there are about 290 people living in the town.

d When $N=200$, $200 \approx \dfrac{290}{1+144e^{-0.819t}}$

Using technology, $t \approx 7.04$.



So, 200 people have heard the rumour after about 7.04 hours or by about 3:03 pm.

REVIEW SET 1A

1 **a** $\sqrt[4]{2} = 2^{\frac{1}{4}}$

b $8\sqrt{2} = 2^3 \times 2^{\frac{1}{2}}$
$= 2^{3+\frac{1}{2}}$
$= 2^{\frac{7}{2}}$

c $\dfrac{1}{\sqrt[3]{4}} = \dfrac{1}{\sqrt[3]{2^2}}$
$= \dfrac{1}{2^{\frac{2}{3}}}$
$= 2^{-\frac{2}{3}}$

2 **a** $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$
$= 2^2$
$= 4$

b $27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}}$
$= 3^{-2}$
$= \dfrac{1}{3^2}$
$= \frac{1}{9}$

c $81^{-\frac{1}{4}} = (3^4)^{-\frac{1}{4}}$
$= 3^{-1}$
$= \frac{1}{3}$

3 **a** $e^x(e^{-x}+e^x)$
$= e^x \times e^{-x} + e^x \times e^x$
$= e^0 + e^{2x}$
$= 1 + e^{2x}$

b $(2^x+5)^2$
$= (2^x)^2 + 2 \times 2^x \times 5 + 5^2$
$= 2^{2x} + 10(2^x) + 25$

c $(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+7)$
$= (x^{\frac{1}{2}})^2 - 7^2$
$= x^1 - 49$
$= x - 49$

4 **a** $y = 4^x - 1$

When $x=2$, $y = 4^2 - 1$
$= 16 - 1$
$= 15$

$\therefore$ the point $(2, 15)$ satisfies $y = 4^x - 1$.

b $f(x) = 5 \times 3^{-x}$
$\therefore f(-1) = 5 \times 3^{-(-1)}$
$= 5 \times 3$
$= 15$

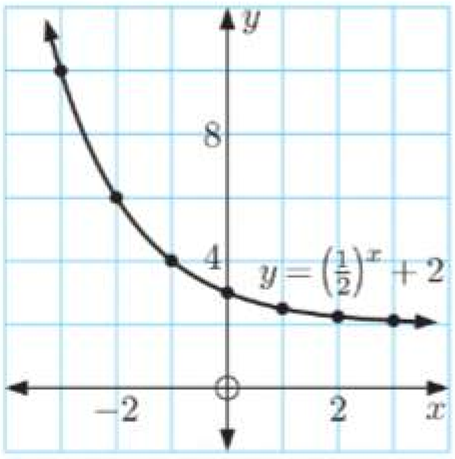
$\therefore$ the point $(-1, \frac{5}{3})$ does not satisfy $f(x) = 5 \times 3^{-x}$.

5 $y = \left(\frac{1}{2}\right)^x + 2$

a

x	-3	-2	-1	0	1	2	3
y	10	6	4	3	$2\frac{1}{2}$	$2\frac{1}{4}$	$2\frac{1}{8}$

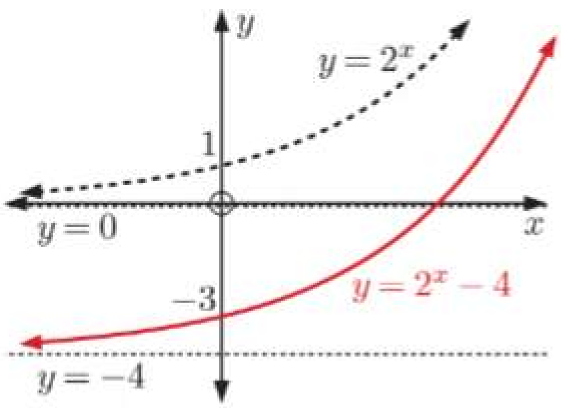
b



c **i** As $x \to \infty$, $y \to 2^+$. **ii** As $x \to -\infty$, $y \to \infty$.

d The horizontal asymptote is $y = 2$.

6



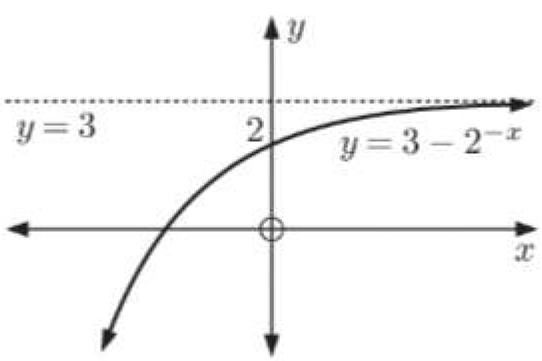
$y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$.
$y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$.

7 $y = 3 - 2^{-x}$

a When $x = 0$, $y = 3 - 2^0 = 3 - 1 = 2$
When $x = 1$, $y = 3 - 2^{-1} = 3 - \frac{1}{2} = 2\frac{1}{2}$
When $x = 2$, $y = 3 - 2^{-2} = 3 - \frac{1}{4} = 2\frac{3}{4}$
When $x = -1$, $y = 3 - 2^1 = 3 - 2 = 1$
When $x = -2$, $y = 3 - 2^2 = 3 - 4 = -1$

b As $x \to \infty$, $2^{-x} \to 0^+$
and so $y \to 3^-$
As $x \to -\infty$, $2^{-x} \to \infty$
and so $y \to -\infty$

c



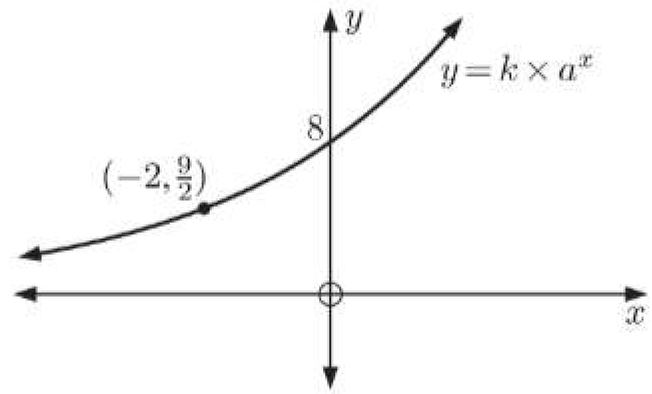
d $y = 3$ is the horizontal asymptote.

8 **a** $y = k \times a^x$

Substituting $(0, 8)$ into the equation gives

$8 = k \times a^0$

$\therefore\ k = 8$



Substituting $(-2, \frac{9}{2})$ into the equation gives

$\frac{9}{2} = k \times a^{-2}$

$\therefore\ \frac{9}{2} = \frac{8}{a^2}$ {since $k = 8$}

$\therefore\ 9a^2 = 16$

$\therefore\ a^2 = \frac{16}{9}$

$\therefore\ a = \pm\sqrt{\frac{16}{9}}$

$\therefore\ a = \frac{4}{3}$ {since $a > 0$}

$\therefore\ k = 8,\ a = \frac{4}{3}$

b As $x \to -\infty$, $y \to 0^+$, so the horizontal asymptote is $y = 0$.

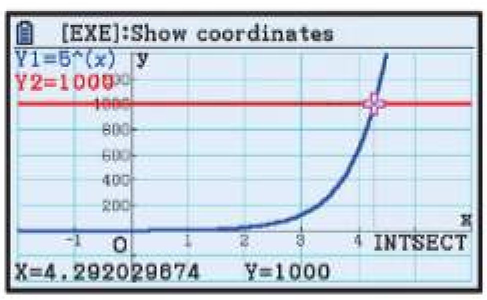
c $y = 8 \times \left(\frac{4}{3}\right)^x$

When $x = 2$, $y = 8 \times \left(\frac{4}{3}\right)^2$

$\therefore\ y = 8 \times \frac{16}{9}$

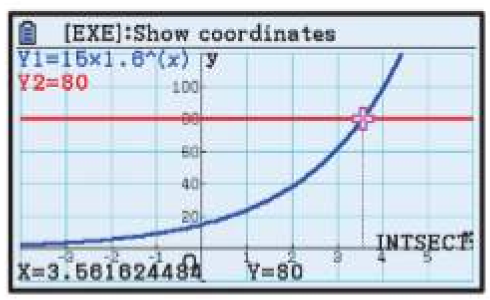
$\therefore\ y = \frac{128}{9}$

9 **a** We graph $Y_1 = 5^X$ and $Y_2 = 1000$ on the same set of axes, and find their point of intersection.



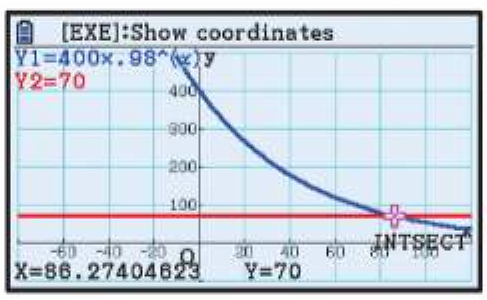
The solution is $x \approx 4.29$.

b We graph $Y_1 = 15 \times (1.6)^X$ and $Y_2 = 80$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 3.56$.

c We graph $Y_1 = 400 \times (0.98)^X$ and $Y_2 = 70$ on the same set of axes, and find their point of intersection.



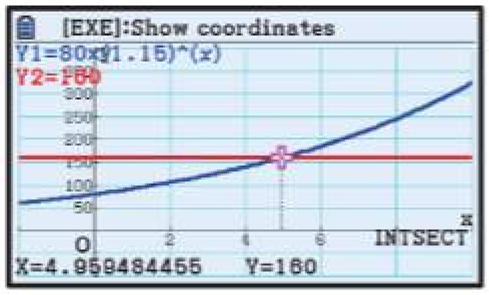
The solution is $x \approx 86.3$.

10 $P(t) = 80 \times (1.15)^t$ seals

a $P(0) = 80 \times (1.15)^0$
$= 80 \times 1$
$= 80$

So the initial population was 80 seals.

b When $P(t) = 2 \times 80 = 160$, $160 = 80 \times (1.15)^t$.
Using technology, $t \approx 4.96$.



$\therefore$ it took about 4.96 years for the population to double in size.

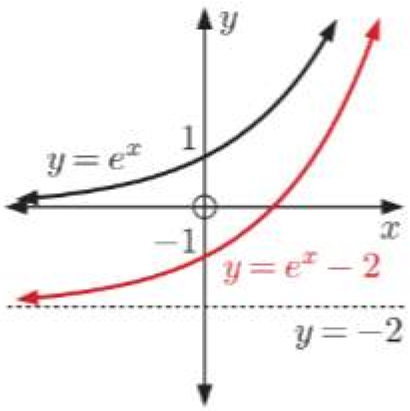
c Percentage increase in first 4 years $= \left(\dfrac{P(4) - P(0)}{P(0)}\right) \times 100\%$

$$= \left(\frac{80 \times (1.15)^4 - 80}{80}\right) \times 100\%$$

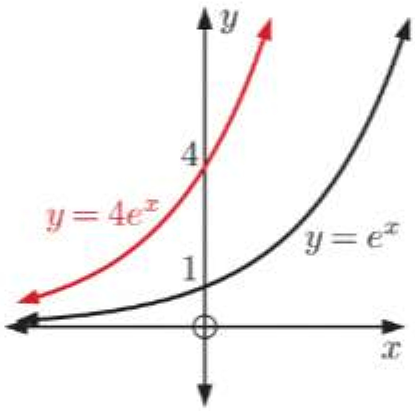
$$\approx 74.9\%$$

11 **a** $e^4 \approx 54.6$ **b** $e^{-2} \approx 0.135$ **c** $10e^{3.5} \approx 331$ **d** $40e^{-2.53} \approx 3.19$

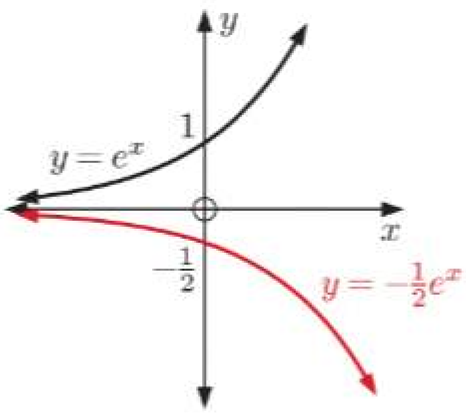
12 **a**



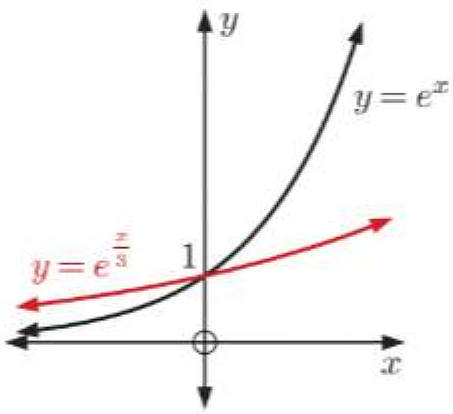
b



c



d



13 **a** $V = k \times 0.6^t + c$ pounds

Substituting $(0, 800)$ into the equation gives

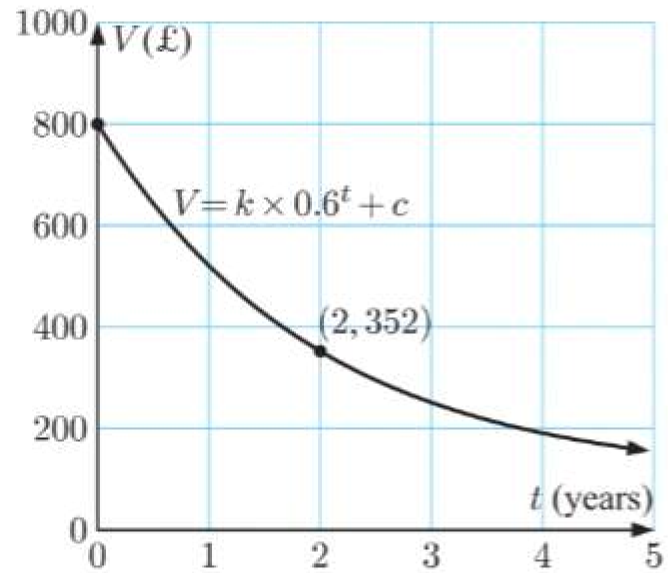
$$800 = k \times 0.6^0 + c$$

$\therefore\ k + c = 800$

Substituting $(2, 352)$ into the equation gives

$$352 = k \times 0.6^2 + c$$

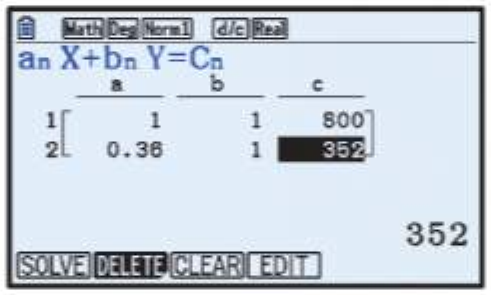
$\therefore\ 0.36k + c = 352$

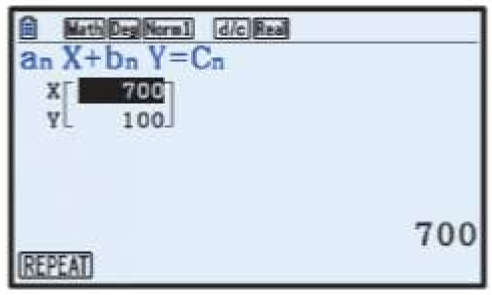


Solving the system of equations

$$\begin{cases} k + c = 800 \\ 0.36k + c = 352 \end{cases}$$

simultaneously gives $k = 700$, $c = 100$.





b No, the value of the computer does not decrease by 40% each year. The computer depreciates by 40% each year above the computer's *minimum* price, but not overall.

c $V = 700 \times 0.6^t + 100$

When $t = 3$, $V = 700 \times 0.6^3 + 100$

$= 251.2$

The value of the computer after 3 years is £251.20.

d The horizontal asymptote is $V = 100$. This means the computer will never be worth less than £100.

14 $W = 1500 \times (0.993)^t$ grams

a When $t = 0$, $W = 1500 \times (0.993)^0$

$= 1500$

The original amount was 1500 g.

b Half the original amount is 750 g.

$\therefore\ 1500 \times (0.993)^t = 750$

Using technology, it takes about 98.7 years for the substance's weight to fall to 750 g.

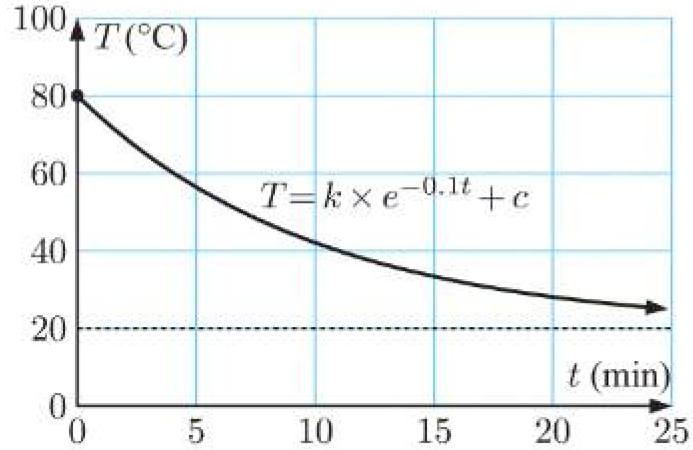
So, the half-life of the substance is about 98.7 years.

c It takes about 98.7 years for the weight to fall from 1500 g to 750 g.

$\therefore$ it takes about 98.7 years for the weight to fall from 750 g to 375 g.

$\therefore$ it takes about $98.7 \times 2 \approx 197$ years for the weight to reduce to 375 g.

15 **a** $T = k \times e^{-0.1t} + c$



The horizontal asymptote is $T = 20$,
so $c = 20$

$\therefore \quad T = k \times e^{-0.1t} + 20$

Substituting $(0, 80)$ into the equation gives

$$80 = k \times e^{-0.1(0)} + 20$$

$\therefore \quad k + 20 = 80$

$\therefore \quad k = 60$

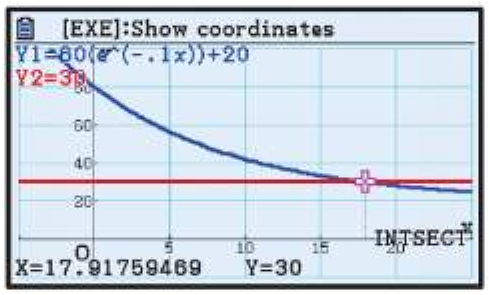
$\therefore$ the exponential model is $T = 60e^{-0.1t} + 20$.

b When $t = 10$, $T = 60e^{-0.1(10)} + 20$
$= 60e^{-1} + 20$
≈ 42.1

The temperature of the water after 10 minutes is about 42.1°C.

c When $T = 30$, $30 = 60e^{-0.1t} + 20$.

Using technology, $t \approx 17.9$.



It will take about 17.9 minutes for the temperature of the water to fall to 30°C.

16 **a** $H(t) = \dfrac{L}{1 + 5e^{-\frac{t}{2}}}$ cm

Now $H(0) = 10$

$\therefore \quad 10 = \dfrac{L}{1+5}$

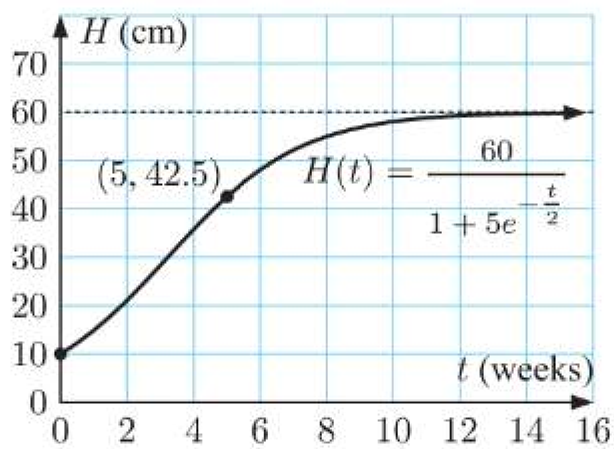
$\therefore \quad L = 60$

b $H(t) = \dfrac{60}{1 + 5e^{-\frac{t}{2}}}$

$\therefore \quad H(5) = \dfrac{60}{1 + 5e^{-\frac{5}{2}}}$
≈ 42.5

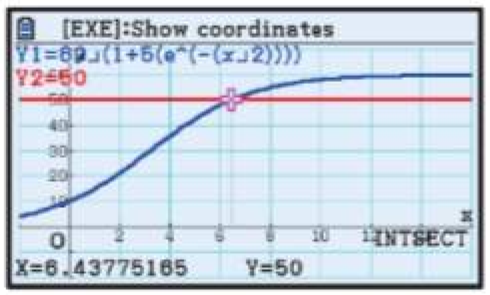
After 5 weeks, the seedling is about 42.5 cm tall.

c



d When $H(t) = 50$, $50 = \dfrac{60}{1 + 5e^{-\frac{t}{2}}}$

Using technology, $t \approx 6.44$.



So, it took about 6.44 weeks after planting for the seedling's height to reach 50 cm.

REVIEW SET 1B

1 **a** $\sqrt[3]{x} = x^{\frac{1}{3}}$

b $x^3\sqrt{x} = x^3 \times x^{\frac{1}{2}}$
$= x^{3+\frac{1}{2}}$
$= x^{\frac{7}{2}}$

c $\dfrac{1}{x^2\sqrt{x}} = \dfrac{1}{x^2 \times x^{\frac{1}{2}}}$
$= \dfrac{1}{x^{2+\frac{1}{2}}}$
$= \dfrac{1}{x^{\frac{5}{2}}}$
$= x^{-\frac{5}{2}}$

2 **a** $3^{\frac{5}{4}} \approx 3.95$ **b** $27^{-\frac{1}{5}} \approx 0.517$ **c** $\sqrt[4]{100} \approx 3.16$

3 **a** $(3 - e^x)^2 = 3^2 - 2 \times 3 \times e^x + (e^x)^2$
$= 9 - 6e^x + e^{2x}$

b $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$
$= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} - x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times x^{-\frac{1}{2}}$
$= x^1 - 2x^0 - x^{-1}$
$= x - 2 - x^{-1}$

c $2^{-x}(2^{2x} + 2^x) = 2^{-x} \times 2^{2x} + 2^{-x} \times 2^x$
$= 2^x + 2^0$
$= 2^x + 1$

4 **a** $3^{x+2} - 3^x$
$= 3^x(3^2 - 1)$
$= 3^x(9 - 1)$
$= 8(3^x)$

b $4^x - 2^x - 12$
$= (2^x)^2 - 2^x - 12$
$= (2^x + 3)(2^x - 4)$ $\{a^2 - a - 12 = (a + 3)(a - 4)\}$

c $e^{2x} + 2e^x - 15$
$= (e^x)^2 + 2e^x - 15$
$= (e^x + 5)(e^x - 3)$ $\{a^2 + 2a - 15 = (a + 5)(a - 3)\}$

5 $y = 2e^{-x} + 1$

a When $x = 0$, $y = 2e^0 + 1 = 3$

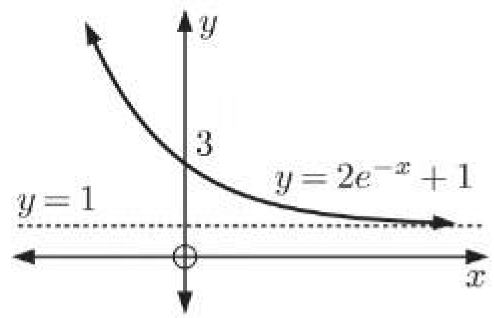
When $x = 1$, $y = 2e^{-1} + 1 \approx 1.74$

When $x = 2$, $y = 2e^{-2} + 1 \approx 1.27$

When $x = -1$, $y = 2e^1 + 1 \approx 6.44$

When $x = -2$, $y = 2e^2 + 1 \approx 15.8$

c



b As $x \to \infty$, $y \to 1^+$

As $x \to -\infty$, $y \to \infty$

d $y = 1$ is a horizontal asymptote.

6 **a** The clock cost £500 and increases in value by 5% each year.

$\therefore$ the value of the clock 1 year after purchase $= £500 \times 1.05$
$= £525$

The vase cost £400 and increases in value by 7% each year.

$\therefore$ the value of the vase 1 year after purchase $= £400 \times 1.07$
$= £428$

b The clock will have value $V(t) = 500 \times (1.05)^t$ pounds, t years after purchase.

The vase will have value $V(t) = 400 \times (1.07)^t$ pounds, t years after purchase.

c For the clock, $V(15) = 500 \times (1.05)^{15}$
$\approx £1039.46$

For the vase, $V(15) = 400 \times (1.07)^{15}$
$\approx £1103.61$

$\therefore$ the vase is more valuable than the clock, 15 years after purchase.

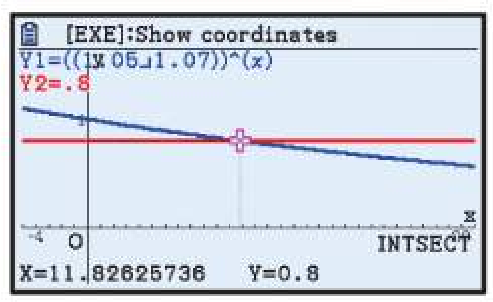
d To find when the items are equal in value,

we set $500 \times (1.05)^t = 400 \times (1.07)^t$ and solve for t.

$$\therefore \frac{(1.05)^t}{(1.07)^t} = \frac{400}{500}$$

$$\therefore \left(\frac{1.05}{1.07}\right)^t = 0.8$$

$$\therefore t \approx 11.8 \quad \{\text{using technology}\}$$



So, the items are equal in value after about 11.8 years.

7 $f(x) = 3^x$

a **i** $f(4) = 3^4$
$= 81$

ii $f(-1) = 3^{-1}$
$= \frac{1}{3}$

b $f(x+2) = kf(x),\ k \in \mathbb{Z}$

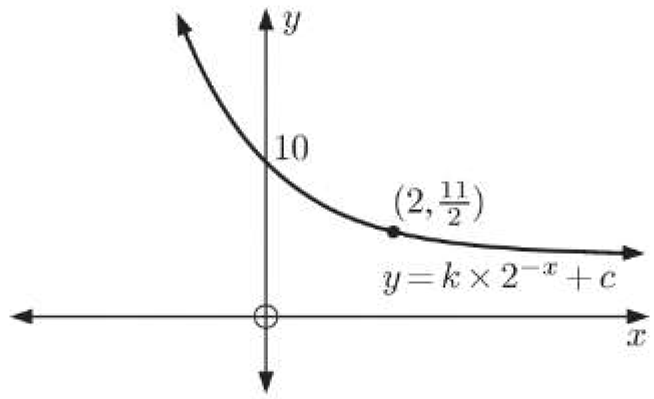
$\therefore 3^{x+2} = k \times 3^x$

$\therefore 3^2 \times 3^x = k \times 3^x$

$\therefore k = 3^2 \quad \{\text{as } 3^x \neq 0\}$

$\therefore k = 9$

8 **a** $y = k \times 2^{-x} + c$



Substituting $(0, 10)$ into the equation gives

$$10 = k \times 2^0 + c$$

$\therefore \ k + c = 10 \quad .... (1)$

Substituting $(2, \frac{11}{2})$ into the equation gives

$$\tfrac{11}{2} = k \times 2^{-2} + c$$

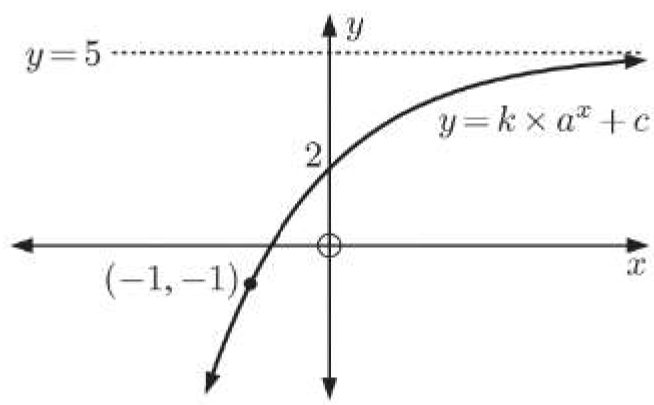
$\therefore \ \frac{1}{4}k + c = \frac{11}{2} \quad .... (2)$

Now $(1) - (2)$ gives $\frac{3}{4}k = \frac{9}{2}$

$\therefore \ k = 6$ and so $c = 4$.

So, the exponential model is $y = 6 \times 2^{-x} + 4$.

b $y = k \times a^x + c$



The horizontal asymptote is $y = 5$, so $c = 5$.

$\therefore \ y = k \times a^x + 5$

Substituting $(0, 2)$ into the equation gives

$$2 = k \times a^0 + 5$$

$\therefore \ k + 5 = 2$

$\therefore \ k = -3$

$\therefore \ y = -3 \times a^x + 5$

Substituting $(-1, -1)$ into the equation gives

$$-1 = -3 \times a^{-1} + 5$$

$\therefore \ -6 = \dfrac{-3}{a}$

$\therefore \ -6a = -3$

$\therefore \ a = \frac{1}{2}$

So, the exponential model is $y = -3 \times \left(\frac{1}{2}\right)^x + 5$.

9 $y = k \times 5^{-x} + c$

a The x-intercept is 1.

When $x = 1$, $y = 0$

$\therefore \ k \times 5^{-1} + c = 0$

$\therefore \ \dfrac{k}{5} + c = 0 \quad .... (1)$

The y-intercept is 8.

When $x = 0$, $y = 8$

$\therefore \ k \times 5^0 + c = 8$

$\therefore \ k + c = 8 \quad .... (2)$

Now $(2) - (1)$ gives $\frac{4}{5}k = 8$

$\therefore \ k = 10$ and so $c = -2$.

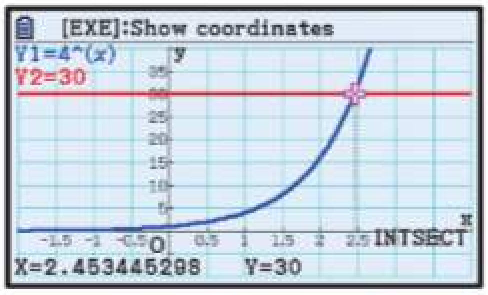
So, the exponential model is $y = 10 \times 5^{-x} - 2$.

b When $x = -1$, $y = 10 \times 5 - 2$

$= 50 - 2$

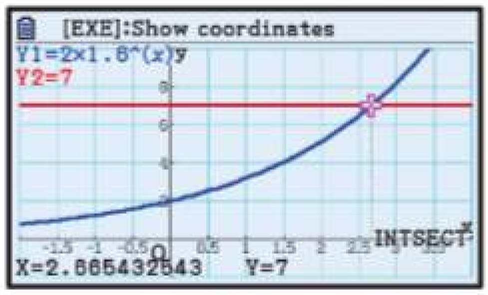
$= 48$

10 **a** We graph $Y_1 = 4^X$ and $Y_2 = 30$ on the same set of axes, and find their point of intersection.



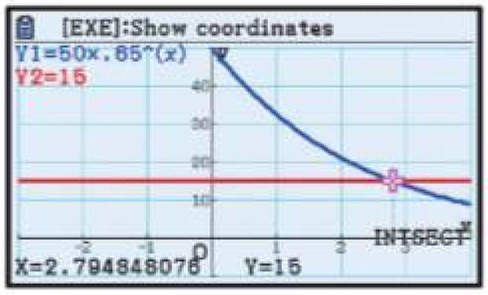
The solution is $x \approx 2.45$.

b We graph $Y_1 = 2 \times (1.6)^X$ and $Y_2 = 7$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 2.67$.

c We graph $Y_1 = 50 \times (0.65)^X$ and $Y_2 = 15$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 2.79$.

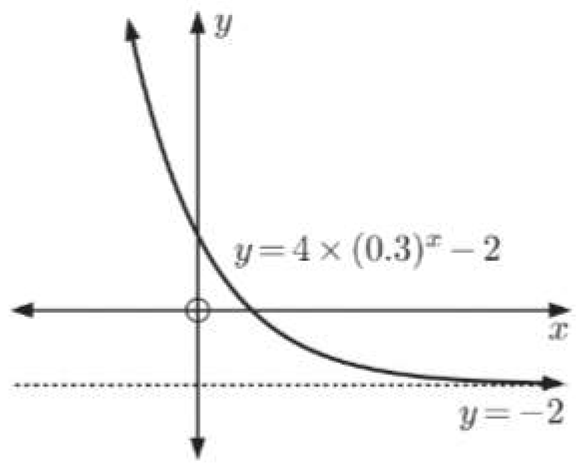
11 The graph of $y = 4 \times (0.3)^x - 2$ lies above its horizontal asymptote $y = -2$.

$\therefore$ $y > -2$ for all x.

So, the equation $4 \times (0.3)^x - 2 = k$ has:

a 1 solution for $k > -2$

b no solutions for $k \leqslant -2$.



12 $P = k \times a^t$ birds, $t \geqslant 0$

a The population increases by 15% each year, so the multiplier $a = 1.15$.

b $$P = k \times (1.15)^t$$

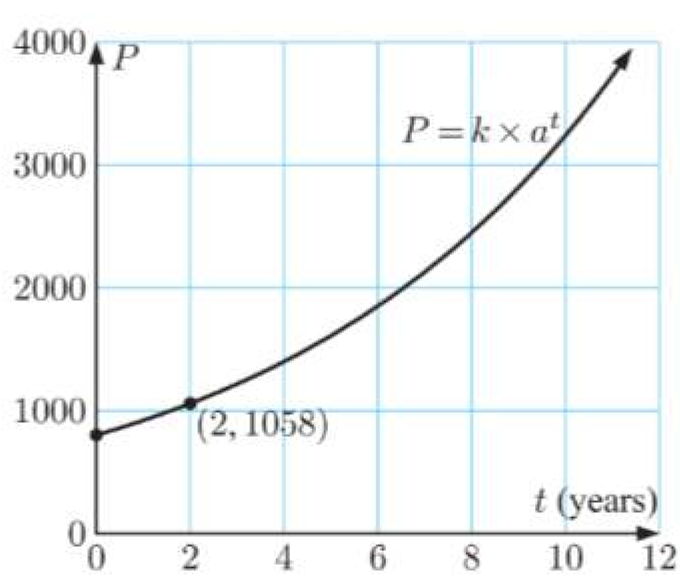
When $t = 2$, $P = 1058$

$$\therefore\ k \times (1.15)^2 = 1058$$

$$\therefore\ k = \frac{1058}{(1.15)^2}$$

$$\therefore\ k = 800$$

The initial population was 800 birds.



c When $t = 5$, $P = 800 \times (1.15)^5$
≈ 1610

The population after 5 years was about 1610 birds.

d No, this model is unlikely to be accurate for very large values of t as the population of birds would be unrealistically large. The birds would eventually reach a limiting population.

13 $W = k \times a^t$ grams

When $t = 0$, $W = k \times a^0$
$= k$

$\therefore$ the initial weight is k grams.

Half the initial weight is $\frac{k}{2}$ grams, and the half-life of the substance is 5 days.

$$\therefore\ k \times a^5 = \frac{k}{2}$$
$$\therefore\ a^5 = \tfrac{1}{2}$$
$$\therefore\ a = \left(\tfrac{1}{2}\right)^{\frac{1}{5}}$$

So, $W = k \times \left(\frac{1}{2}\right)^{\frac{t}{5}}$.

When $t = 2$, $W = 400$

$$\therefore\ 400 = k \times \left(\tfrac{1}{2}\right)^{\frac{2}{5}}$$
$$\therefore\ k = \frac{400}{\left(\frac{1}{2}\right)^{\frac{2}{5}}} \approx 528$$

So, $W \approx 528 \times \left(\frac{1}{2}\right)^{\frac{t}{5}}$

When $t = 9$, $W \approx 528 \times \left(\frac{1}{2}\right)^{\frac{9}{5}}$
≈ 152

After 9 days, the weight of the substance was about 152 g.

14 Use the exponential function $y = ka^x + c$.

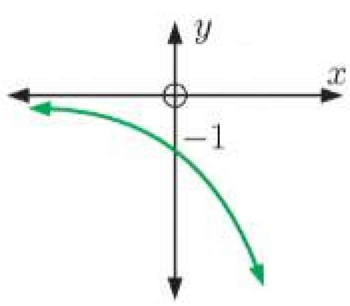
a $y = -e^x$

$$\left.\begin{array}{ll} k = -1 & \therefore\ k < 0 \\ a = e & \therefore\ a > 1 \end{array}\right\} \text{function is below horizontal asymptote and is decreasing}$$

When $x = 0$, $y = -e^0 = -1$

$\therefore$ the y-intercept is -1.

$\therefore$ the graph is **C**.



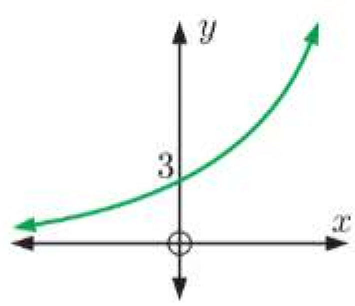
b $y = 3 \times 2^x$

$$\left.\begin{array}{ll} k = 3 & \therefore\ k > 0 \\ a = 2 & \therefore\ a > 1 \end{array}\right\} \text{function is above horizontal asymptote and is increasing}$$

When $x = 0$, $y = 3 \times 2^0 = 3$

$\therefore$ the y-intercept is 3.

$\therefore$ the graph is **E**.



c $y = e^x + 1$

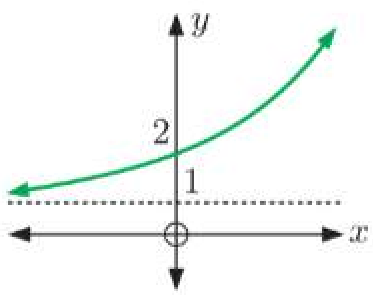
$\left.\begin{array}{l} k = 1 \quad \therefore \ k > 0 \\ a = e \quad \therefore \ a > 1 \end{array}\right\}$ function is above horizontal asymptote and is increasing

When $x = 0$, $y = e^0 + 1 = 2$

$\therefore$ the y-intercept is 2.

$c = 1$, so $y = 1$ is a horizontal asymptote.

$\therefore$ the graph is **A**.



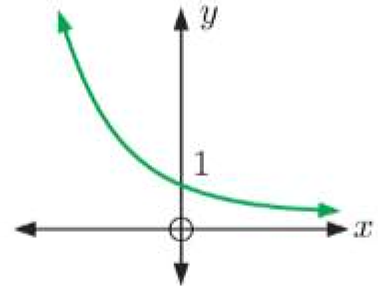
d $y = 3^{-x} = \dfrac{1}{3^x} = \left(\frac{1}{3}\right)^x$

$\left.\begin{array}{l} k = 1 \quad \therefore \ k > 0 \\ a = \frac{1}{3} \quad \therefore \ 0 < a < 1 \end{array}\right\}$ function is above horizontal asymptote and is decreasing

When $x = 0$, $y = 3^0 = 1$

$\therefore$ the y-intercept is 1.

$\therefore$ the graph is **B**.



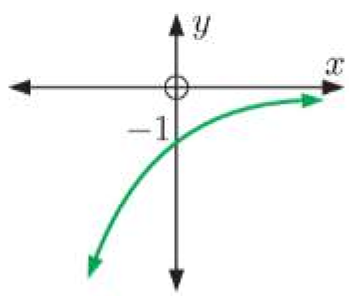
e $y = -e^{-x} = -\dfrac{1}{e^x} = -\left(\dfrac{1}{e}\right)^x$

$\left.\begin{array}{l} k = -1 \quad \therefore \ k < 0 \\ a = \frac{1}{e} \quad \therefore \ 0 < a < 1 \end{array}\right\}$ function is below horizontal asymptote and is increasing

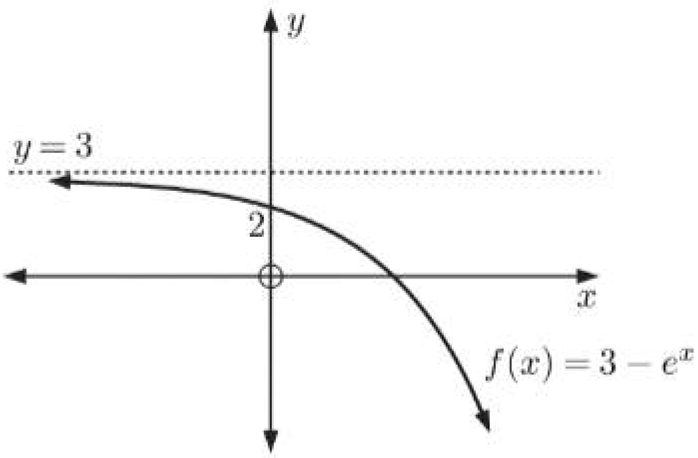
When $x = 0$, $y = -e^0 = -1$

$\therefore$ the y-intercept is -1.

$\therefore$ the graph is **D**.



15 **a**



b The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y < 3\}$.

c As $x \to \infty$, $f(x) \to -\infty$.

As $x \to -\infty$, $f(x) \to 3^-$.

16 **a** $N = \dfrac{L}{1 + Ce^{-\frac{t}{10}}}$

When $t = 0$, $N = 10$

$$\therefore \ 10 = \frac{L}{1 + C}$$

$$\therefore \ L = 10(1 + C) \quad .... \ (*)$$

When $t = 10$, $N = 20$

$$\therefore \quad 20 = \frac{L}{1 + Ce^{-\frac{10}{10}}}$$

$$\therefore \quad 20 = \frac{10(1 + C)}{1 + Ce^{-1}} \qquad \{\text{using } (*)\}$$

$$\therefore \quad 20(1 + Ce^{-1}) = 10(1 + C)$$

$$\therefore \quad 2(1 + Ce^{-1}) = 1 + C$$

$$\therefore \quad 2 + 2Ce^{-1} = 1 + C$$

$$\therefore \quad 1 = C - 2Ce^{-1}$$

$$\therefore \quad e = Ce - 2C$$

$$\therefore \quad e = C(e - 2)$$

$$\therefore \quad C = \frac{e}{e - 2} \approx 3.78$$

Substituting $C = \dfrac{e}{e-2}$ into $(*)$ gives
$$\begin{aligned} L &= 10\left(1 + \frac{e}{e-2}\right) \\ &= 10\left(\frac{e - 2 + e}{e - 2}\right) \\ &= 10\left(\frac{2e - 2}{e - 2}\right) \\ &= \frac{20e - 20}{e - 2} \approx 47.8 \end{aligned}$$

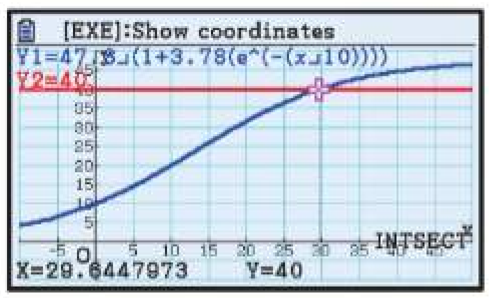
b $N \approx \dfrac{47.8}{1 + 3.78e^{-\frac{t}{10}}}$

When $t = 15$, $N \approx \dfrac{47.8}{1 + 3.78e^{-\frac{15}{10}}}$
≈ 25.9

After 15 months, there were about 26 adult females on the island.

c When $N = 40$, $40 \approx \dfrac{47.8}{1 + 3.78e^{-\frac{t}{10}}}$

Using technology, $t \approx 29.6$.



So, it took about 29.6 months for the number of adult females to reach 40.

Chapter 2

LOGARITHMS

EXERCISE 2A

1 **a** $\log 10\,000 = \log(10^4)$
$= 4$

b $\log(0.001) = \log(10^{-3})$
$= -3$

c $\log 10 = \log(10^1)$
$= 1$

d $\log 1 = \log(10^0)$
$= 0$

e $\log \sqrt{10} = \log(10^{\frac{1}{2}})$
$= \frac{1}{2}$

f $\log \sqrt[3]{10} = \log(10^{\frac{1}{3}})$
$= \frac{1}{3}$

g $\log\left(\frac{1}{\sqrt[4]{10}}\right) = \log(10^{-\frac{1}{4}})$
$= -\frac{1}{4}$

h $\log(10\sqrt{10}) = \log(10^{\frac{3}{2}})$
$= \frac{3}{2}$ or $1\frac{1}{2}$

i $\log \sqrt[3]{100} = \log((10^2)^{\frac{1}{3}})$
$= \log(10^{\frac{2}{3}})$
$= \frac{2}{3}$

j $\log\left(\frac{100}{\sqrt{10}}\right) = \log\left(\frac{10^2}{10^{\frac{1}{2}}}\right)$
$= \log(10^{\frac{3}{2}})$
$= \frac{3}{2}$ or $1\frac{1}{2}$

k $\log(10 \times \sqrt[3]{10}) = \log(10^1 \times 10^{\frac{1}{3}})$
$= \log(10^{\frac{4}{3}})$
$= \frac{4}{3}$ or $1\frac{1}{3}$

l $\log(1000\sqrt{10}) = \log(10^3 \times 10^{\frac{1}{2}})$
$= \log(10^{\frac{7}{2}})$
$= \frac{7}{2}$ or $3\frac{1}{2}$

2 **a** $\log(10^n) = n$

b $\log(10^a \times 100) = \log(10^a \times 10^2)$
$= \log(10^{a+2})$
$= a + 2$

c $\log\left(\frac{10}{10^m}\right) = \log(10^{1-m})$
$= 1 - m$

d $\log\left(\frac{10^a}{10^b}\right) = \log(10^{a-b})$
$= a - b$

3 **a** $100 < 237 < 1000$
$\therefore \log 100 < \log 237 < \log 1000$
$\therefore \log(10^2) < \log 237 < \log(10^3)$
$\therefore 2 < \log 237 < 3$

b $\log 237 \approx 2.37$

4 **a** We know that $\log 1 = \log(10^0) = 0$ and $\log(0.1) = \log(10^{-1}) = -1$.
Also, $0.1 < 0.6 < 1 \quad \therefore \log(0.1) < \log(0.6) < \log 1$
$\therefore -1 < \log(0.6) < 0$

b $\log(0.6) \approx -0.22$ which is between -1 and 0. ✓

5 **a** $\log 76 \approx 1.88$ **b** $\log 114 \approx 2.06$ **c** $\log 3 \approx 0.48$
d $\log 831 \approx 2.92$ **e** $\log(0.4) \approx -0.40$ **f** $\log 3247 \approx 3.51$
g $\log(0.008) \approx -2.10$ **h** $\log(-7)$ does not exist

6 **a** $\log x > 0$
$\therefore\ x > 10^0$
$\therefore\ x > 1$

b $\log x = 0$
$\therefore\ x = 10^0$
$\therefore\ x = 1$

c $\log x < 0$
$\therefore\ x < 10^0$
$\therefore\ x < 1$
but $\log x$ is only defined for $x > 0$
$\therefore\ \log x$ is negative when $0 < x < 1$.

d $\log x$ is undefined when $x \leqslant 0$.

7 **a** $6 = 10^{\log 6} \approx 10^{0.7782}$
b $60 = 10^{\log 60} \approx 10^{1.7782}$
c $6000 = 10^{\log 6000} \approx 10^{3.7782}$
d $0.6 = 10^{\log(0.6)} \approx 10^{-0.2218}$
e $0.006 = 10^{\log(0.006)} \approx 10^{-2.2218}$
f $15 = 10^{\log 15} \approx 10^{1.1761}$
g $1500 = 10^{\log 1500} \approx 10^{3.1761}$
h $1.5 = 10^{\log(1.5)} \approx 10^{0.1761}$
i $0.15 = 10^{\log(0.15)} \approx 10^{-0.8239}$
j $0.000\,15 = 10^{\log(0.000\,15)} \approx 10^{-3.8239}$

8 **a** **i** $\log 3 \approx 0.477$ **ii** $\log 300 \approx 2.477$

b
$$\begin{aligned}\log 300 &= \log(3 \times 100)\\ &= \log(10^{\log 3} \times 10^2)\\ &= \log(10^{\log 3 + 2})\\ &= \log 3 + 2\end{aligned}$$

9 **a** **i** $\log 5 \approx 0.699$ **ii** $\log(0.05) \approx -1.301$

b
$$\begin{aligned}\log(0.05) &= \log\left(5 \times \tfrac{1}{100}\right)\\ &= \log(10^{\log 5} \times 10^{-2})\\ &= \log(10^{\log 5 - 2})\\ &= \log 5 - 2\end{aligned}$$

10 **a** $\log x = 2$
$\therefore\ 10^{\log x} = 10^2$
$\therefore\ x = 10^2$
$\therefore\ x = 100$

b $\log x = 1$
$\therefore\ 10^{\log x} = 10^1$
$\therefore\ x = 10^1$
$\therefore\ x = 10$

c $\log x = 0$
$\therefore\ 10^{\log x} = 10^0$
$\therefore\ x = 10^0$
$\therefore\ x = 1$

d $\log x = -1$
$\therefore\ 10^{\log x} = 10^{-1}$
$\therefore\ x = 10^{-1}$
$\therefore\ x = \frac{1}{10}$

e $\log x = \frac{1}{2}$
$\therefore\ 10^{\log x} = 10^{\frac{1}{2}}$
$\therefore\ x = 10^{\frac{1}{2}}$
$\therefore\ x = \sqrt{10}$

f $\log x = -\frac{1}{2}$
$\therefore\ 10^{\log x} = 10^{-\frac{1}{2}}$
$\therefore\ x = 10^{-\frac{1}{2}}$
$\therefore\ x = \dfrac{1}{10^{\frac{1}{2}}}$
$\therefore\ x = \frac{1}{\sqrt{10}}$

g $\log x = 4$
$\therefore\ 10^{\log x} = 10^{4}$
$\therefore\ x = 10^{4}$
$\therefore\ x = 10\,000$

h $\log x = -5$
$\therefore\ 10^{\log x} = 10^{-5}$
$\therefore\ x = 10^{-5}$
$\therefore\ x = 0.000\,01$

i $\log x \approx 0.8351$
$\therefore\ 10^{\log x} \approx 10^{0.8351}$
$\therefore\ x \approx 10^{0.8351}$
$\therefore\ x \approx 6.84$

j $\log x \approx 2.1457$
$\therefore\ 10^{\log x} \approx 10^{2.1457}$
$\therefore\ x \approx 10^{2.1457}$
$\therefore\ x \approx 140$

k $\log x \approx -1.378$
$\therefore\ 10^{\log x} \approx 10^{-1.378}$
$\therefore\ x \approx 10^{-1.378}$
$\therefore\ x \approx 0.0419$

l $\log x \approx -3.1997$
$\therefore\ 10^{\log x} \approx 10^{-3.1997}$
$\therefore\ x \approx 10^{-3.1997}$
$\therefore\ x \approx 0.000\,631$

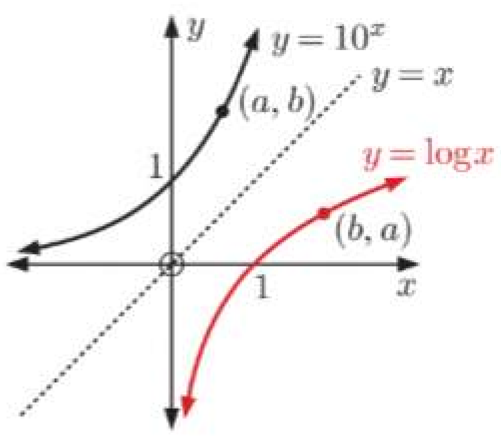
11 **a** (a, b) lies on $y = 10^x$
$\therefore$ at (a, b), $b = 10^a$

b $b = 10^a$ {from **a**}
$\therefore\ \log b = \log(10^a)$
$\therefore\ \log b = a$

c When $x = b$, $y = \log b = a$ {from **b**}
$\therefore$ (b, a) lies on the graph of $y = \log x$.

d $y = 10^x$ and $y = \log x$ are inverse functions.

e



f The y-intercept of $y = 10^x$ is 1.
$\therefore$ the x-intercept of $y = \log x$ is 1.

g The domain is $\{x \mid x > 0\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

INVESTIGATION 1 DISCOVERING THE LAWS OF LOGARITHMS

1 **a** **i** $\log 2 + \log 3 \approx 0.778$ **ii** $\log 3 + \log 7 \approx 1.32$ **iii** $\log 4 + \log 20 \approx 1.90$
iv $\log 6 \approx 0.778$ **v** $\log 21 \approx 1.32$ **vi** $\log 80 \approx 1.90$

b $\log m + \log n = \log(mn)$

2 **a** **i** $\log 6 - \log 2 \approx 0.477$ **ii** $\log 12 - \log 3 \approx 0.602$ **iii** $\log 3 - \log 5 \approx -0.222$

iv $\log 3 \approx 0.477$ **v** $\log 4 \approx 0.602$ **vi** $\log(0.6) \approx -0.222$

b $\log m - \log n = \log\left(\frac{m}{n}\right)$

3 **a** **i** $3\log 2 \approx 0.903$ **ii** $2\log 5 \approx 1.40$ **iii** $-4\log 3 \approx -1.91$

iv $\log(2^3) \approx 0.903$ **v** $\log(5^2) \approx 1.40$ **vi** $\log(3^{-4}) \approx -1.91$

b $m\log b = \log(b^m)$

EXERCISE 2B

1 **a** $\log 8 + \log 2$
$= \log(8 \times 2)$
$= \log 16$

b $\log 4 + \log 5$
$= \log(4 \times 5)$
$= \log 20$

c $\log 40 - \log 5$
$= \log\left(\frac{40}{5}\right)$
$= \log 8$

d $\log p - \log m$
$= \log\left(\frac{p}{m}\right)$

e $\log 8 - \log 2$
$= \log\left(\frac{8}{2}\right)$
$= \log 4$

f $\log 5 + \log(0.4)$
$= \log(5 \times 0.4)$
$= \log 2$

g $\log 250 + \log 4$
$= \log(250 \times 4)$
$= \log 1000$
$= \log(10^3)$
$= 3$

h $\log 100 - \log 4$
$= \log\left(\frac{100}{4}\right)$
$= \log 25$

i $\log 2 + \log 3 + \log 4$
$= \log(2 \times 3 \times 4)$
$= \log 24$

j $\log 5 + \log 4 - \log 2$
$= \log\left(\frac{5\times 4}{2}\right)$
$= \log 10$
$= 1$

k $\log 6 - \log 2 - \log 3$
$= \log(6 \div 2 \div 3)$
$= \log 1$
$= \log(10^0)$
$= 0$

l $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$
$= \log\left(\frac{4}{3} \times 3 \times 7\right)$
$= \log 28$

2 **a** $\log 7 + 2$
$= \log 7 + \log(10^2)$
$= \log(7 \times 100)$
$= \log 700$

b $\log 4 - 1$
$= \log 4 - \log(10^1)$
$= \log\left(\frac{4}{10}\right)$
$= \log\left(\frac{2}{5}\right)$

c $1 + \log 3$
$= \log(10^1) + \log 3$
$= \log(10 \times 3)$
$= \log 30$

d $\log 5 - 2$
$= \log 5 - \log(10^2)$
$= \log 5 - \log 100$
$= \log\left(\frac{5}{100}\right)$
$= \log\left(\frac{1}{20}\right)$

e $2 + \log 2$
$= \log(10^2) + \log 2$
$= \log(100 \times 2)$
$= \log 200$

f $\log 50 - 4$
$= \log 50 - \log(10^4)$
$= \log 50 - \log 10\,000$
$= \log\left(\frac{50}{10\,000}\right)$
$= \log(0.005)$

g
$$\begin{aligned} & t + \log w \\ &= \log(10^t) + \log w \\ &= \log(10^t \times w) \end{aligned}$$

h
$$\begin{aligned} & \log 40 - 2 \\ &= \log 40 - \log(10^2) \\ &= \log 40 - \log 100 \\ &= \log\left(\tfrac{40}{100}\right) \\ &= \log\left(\tfrac{2}{5}\right) \end{aligned}$$

i
$$\begin{aligned} & 3 - \log 50 \\ &= \log(10^3) - \log 50 \\ &= \log 1000 - \log 50 \\ &= \log\left(\tfrac{1000}{50}\right) \\ &= \log 20 \end{aligned}$$

3 **a**
$$\begin{aligned} & 5\log 2 + \log 3 \\ &= \log(2^5) + \log 3 \\ &= \log 32 + \log 3 \\ &= \log(32 \times 3) \\ &= \log 96 \end{aligned}$$

b
$$\begin{aligned} & 2\log 3 + 3\log 2 \\ &= \log(3^2) + \log(2^3) \\ &= \log 9 + \log 8 \\ &= \log(9 \times 8) \\ &= \log 72 \end{aligned}$$

c
$$\begin{aligned} & 3\log 4 - \log 8 \\ &= \log(4^3) - \log 8 \\ &= \log 64 - \log 8 \\ &= \log\left(\tfrac{64}{8}\right) \\ &= \log 8 \end{aligned}$$

d
$$\begin{aligned} & 2\log 5 - 3\log 2 \\ &= \log(5^2) - \log(2^3) \\ &= \log 25 - \log 8 \\ &= \log\left(\tfrac{25}{8}\right) \end{aligned}$$

e
$$\begin{aligned} & \tfrac{1}{2}\log 4 + \log 3 \\ &= \log(4^{\frac{1}{2}}) + \log 3 \\ &= \log 2 + \log 3 \\ &= \log(2 \times 3) \\ &= \log 6 \end{aligned}$$

f
$$\begin{aligned} & \tfrac{1}{3}\log\left(\tfrac{1}{8}\right) \\ &= \log((\tfrac{1}{8})^{\frac{1}{3}}) \\ &= \log((2^{-3})^{\frac{1}{3}}) \\ &= \log(2^{-1}) \\ &= \log\left(\tfrac{1}{2}\right) \end{aligned}$$

g
$$\begin{aligned} & 3 - \log 4 - 2\log 5 \\ &= \log(10^3) - \log 4 - \log(5^2) \\ &= \log 1000 - \log 4 - \log 25 \\ &= \log(1000 \div 4 \div 25) \\ &= \log 10 \\ &= 1 \end{aligned}$$

h
$$\begin{aligned} & 1 - 3\log 2 + \log 20 \\ &= \log(10^1) - \log(2^3) + \log 20 \\ &= \log 10 - \log 8 + \log 20 \\ &= \log(10 \div 8 \times 20) \\ &= \log 25 \end{aligned}$$

i
$$\begin{aligned} & 2 - \tfrac{1}{2}\log 4 - \log 5 \\ &= \log(10^2) - \log(4^{\frac{1}{2}}) - \log 5 \\ &= \log 100 - \log 2 - \log 5 \\ &= \log(100 \div 2 \div 5) \\ &= \log 10 \\ &= 1 \end{aligned}$$

4 **a**
$$\begin{aligned} & \frac{\log 4}{\log 2} \\ &= \frac{\log(2^2)}{\log 2} \\ &= \frac{2\log 2}{\log 2} \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} & \frac{\log 27}{\log 9} \\ &= \frac{\log(3^3)}{\log(3^2)} \\ &= \frac{3\log 3}{2\log 3} \\ &= \tfrac{3}{2} \end{aligned}$$

c
$$\begin{aligned} & \frac{\log 8}{\log 2} \\ &= \frac{\log(2^3)}{\log 2} \\ &= \frac{3\log 2}{\log 2} \\ &= 3 \end{aligned}$$

d
$$\begin{aligned} & \frac{\log 3}{\log 9} \\ &= \frac{\log 3}{\log(3^2)} \\ &= \frac{\log 3}{2\log 3} \\ &= \tfrac{1}{2} \end{aligned}$$

e $\frac{\log 25}{\log(0.2)}$
$= \frac{\log(5^2)}{\log(5^{-1})}$
$= \frac{2\log 5}{-1\log 5}$
$= -2$

f $\frac{\log 8}{\log(0.25)}$
$= \frac{\log(2^3)}{\log(2^{-2})}$ $\{0.25 = \frac{1}{4} = \frac{1}{2^2}\}$
$= \frac{3\log 2}{-2\log 2}$
$= -\frac{3}{2}$

5 **a** $\log 9 = \log(3^2)$
$= 2\log 3$ ✓

b $\log\sqrt{2} = \log(2^{\frac{1}{2}})$
$= \frac{1}{2}\log 2$ ✓

c $\log(\frac{1}{8}) = \log\left(\frac{1}{2^3}\right)$
$= \log(2^{-3})$
$= -3\log 2$ ✓

d $\log(\frac{1}{5}) = \log(5^{-1})$
$= -1 \times \log 5$
$= -\log 5$ ✓

e $\log 5 = \log(\frac{10}{2})$
$= \log(10^1) - \log 2$
$= 1 - \log 2$ ✓

f $\log 5000$
$= \log(\frac{10\,000}{2})$
$= \log(10^4) - \log 2$
$= 4 - \log 2$ ✓

6 $\log(a \times 10^k)$
$= \log a + \log 10^k$
$= \log a + k$

7 **a** $\log 6$
$= \log(2 \times 3)$
$= \log 2 + \log 3$
$= p + q$

b $\log 45$
$= \log(9 \times 5)$
$= \log(3^2 \times 5)$
$= \log(3^2) + \log 5$
$= 2\log 3 + \log 5$
$= 2q + r$

c $\log 108$
$= \log(4 \times 27)$
$= \log(2^2 \times 3^3)$
$= \log(2^2) + \log(3^3)$
$= 2\log 2 + 3\log 3$
$= 2p + 3q$

d $\log\left(\frac{5\sqrt{3}}{2}\right)$
$= \log(5 \times 3^{\frac{1}{2}}) - \log 2$
$= \log 5 + \log(3^{\frac{1}{2}}) - \log 2$
$= \log 5 + \frac{1}{2}\log 3 - \log 2$
$= r + \frac{1}{2}q - p$

e $\log(\frac{5}{32})$
$= \log 5 - \log 32$
$= \log 5 - \log(2^5)$
$= \log 5 - 5\log 2$
$= r - 5p$

f $\log(\frac{2}{9})$
$= \log 2 - \log 9$
$= \log 2 - \log(3^2)$
$= \log 2 - 2\log 3$
$= p - 2q$

8 **a** $\log(PR)$
$= \log P + \log R$
$= x + z$

b $\log(RQ^2)$
$= \log R + \log(Q^2)$
$= \log R + 2\log Q$
$= z + 2y$

c $\log\left(\frac{PR}{Q}\right)$
$= \log(PR) - \log Q$
$= \log P + \log R - \log Q$
$= x + z - y$

d $\log\left(P^2\sqrt{Q}\right)$
$= \log(P^2) + \log(Q^{\frac{1}{2}})$
$= 2\log P + \frac{1}{2}\log Q$
$= 2x + \frac{1}{2}y$

e

$$\log\left(\frac{Q^3}{\sqrt{R}}\right)$$
$$= \log(Q^3) - \log(R^{\frac{1}{2}})$$
$$= 3\log Q - \tfrac{1}{2}\log R$$
$$= 3y - \tfrac{1}{2}z$$

f

$$\log\left(\frac{R^2\sqrt{Q}}{P^3}\right)$$
$$= \log(R^2) + \log(Q^{\frac{1}{2}}) - \log(P^3)$$
$$= 2\log R + \tfrac{1}{2}\log Q - 3\log P$$
$$= 2z + \tfrac{1}{2}y - 3x$$

9 **a**

$$\log(N^2) = 1.72$$
$$\therefore\ 2\log N = 1.72$$
$$\therefore\ \log N = 0.86$$

b

$$\log(MN)$$
$$= \log M + \log N$$
$$= 1.29 + 0.86$$
$$= 2.15$$

c

$$\log\left(\frac{N^2}{\sqrt{M}}\right)$$
$$= \log(N^2) - \log(M^{\frac{1}{2}})$$
$$= 1.72 - \tfrac{1}{2}\log M$$
$$= 1.72 - \tfrac{1}{2}(1.29)$$
$$= 1.075$$

10

$$\log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!)$$
$$= \log\left(\frac{8!}{7!}\right) + \log\left(\frac{6!}{5!}\right) + \log\left(\frac{4!}{3!}\right) + \log\left(\frac{2!}{1!}\right)$$
$$= \log\left(\frac{8\times 7!}{7!}\right) + \log\left(\frac{6\times 5!}{5!}\right) + \log\left(\frac{4\times 3!}{3!}\right) + \log\left(\frac{2\times 1!}{1!}\right)$$
$$= \log 8 + \log 6 + \log 4 + \log 2$$
$$= \log(8\times 6\times 4\times 2)$$
$$= \log 384$$

11

$$\log(5!) = \log(1\times 2\times 3\times 4\times 5)$$
$$= \log 120$$
$$= \log(10\times 12)$$
$$= \log 10 + \log 12$$
$$= 1 + \log 12$$

EXERCISE 2C

1 **a**

$$\ln(e^2)$$
$$= 2$$

b

$$\ln(e^4)$$
$$= 4$$

c

$$\ln((\sqrt{e})^3)$$
$$= \ln((e^{\frac{1}{2}})^3)$$
$$= \ln(e^{\frac{3}{2}})$$
$$= \tfrac{3}{2}$$

d

$$\ln 1$$
$$= \ln(e^0)$$
$$= 0$$

e

$$\ln\left(\frac{1}{e}\right)$$
$$= \ln(e^{-1})$$
$$= -1$$

f

$$\ln\sqrt[3]{e}$$
$$= \ln(e^{\frac{1}{3}})$$
$$= \tfrac{1}{3}$$

g

$$\ln\left(\frac{1}{e^2}\right)$$
$$= \ln(e^{-2})$$
$$= -2$$

h

$$\ln\left(\frac{1}{\sqrt{e}}\right)$$
$$= \ln(e^{-\frac{1}{2}})$$
$$= -\tfrac{1}{2}$$

2 **a** $e^{\ln 3}$
$= 3$

b $e^{2\ln 3}$
$= \left(e^{\ln 3}\right)^2$
$= 3^2$
$= 9$

c $e^{-\ln 5}$
$= \left(e^{\ln 5}\right)^{-1}$
$= 5^{-1}$
$= \frac{1}{5}$

d $e^{-2\ln 2}$
$= \left(e^{\ln 2}\right)^{-2}$
$= 2^{-2}$
$= \dfrac{1}{2^2}$
$= \frac{1}{4}$

e $\ln(e^a)$
$= a$

f $\ln(e \times e^a)$
$= \ln(e^{1+a})$
$= 1 + a$

g $\ln(e^a \times e^b)$
$= \ln(e^{a+b})$
$= a + b$

h $\ln\left((e^a)^b\right)$
$= \ln(e^{ab})$
$= ab$

3 **a** $e^3 \approx 20.0855, \quad e^4 \approx 54.5982$

b $\therefore\ e^3 < 40 < e^4$
$\therefore\ \ln(e^3) < \ln 40 < \ln(e^4)$
$\therefore\ 3 < \ln 40 < 4$
So, $\ln 40$ lies between 3 and 4.

c $\ln 40 \approx 3.689$

4 **a** $\ln 12 \approx 2.485$ **b** $\ln 68 \approx 4.220$ **c** $\ln(1.4) \approx 0.336$

d $\ln(0.7) \approx -0.357$ **e** $\ln 500 \approx 6.215$

5 x does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.
$\therefore\ \ln(-2)$ and $\ln 0$ do not exist.

6 **a** $6 = e^{\ln 6}$
$\approx e^{1.7918}$

b $60 = e^{\ln 60}$
$\approx e^{4.0943}$

c $6000 = e^{\ln 6000}$
$\approx e^{8.6995}$

d $0.6 = e^{\ln(0.6)}$
$\approx e^{-0.5108}$

e $0.006 = e^{\ln(0.006)}$
$\approx e^{-5.1160}$

f $15 = e^{\ln 15}$
$\approx e^{2.7081}$

g $1500 = e^{\ln 1500}$
$\approx e^{7.3132}$

h $1.5 = e^{\ln(1.5)}$
$\approx e^{0.4055}$

i $0.15 = e^{\ln(0.15)}$
$\approx e^{-1.8971}$

j $0.000\,15 = e^{\ln(0.000\,15)}$
$\approx e^{-8.8049}$

7 **a** $\ln x = 3$
$\therefore\ x = e^3$
$\therefore\ x \approx 20.1$

b $\ln x = 1$
$\therefore\ x = e^1$
$\therefore\ x = e$

c $\ln x = 0$
$\therefore\ x = e^0$
$\therefore\ x = 1$

d $\ln x = -1$
$\therefore\ x = e^{-1}$
$\therefore\ x \approx 0.368$

e $\ln x = -5$
$\therefore\ x = e^{-5}$
$\therefore\ x \approx 0.006\,74$

f $\ln x \approx 0.835$
$\therefore\ x \approx e^{0.835}$
$\therefore\ x \approx 2.30$

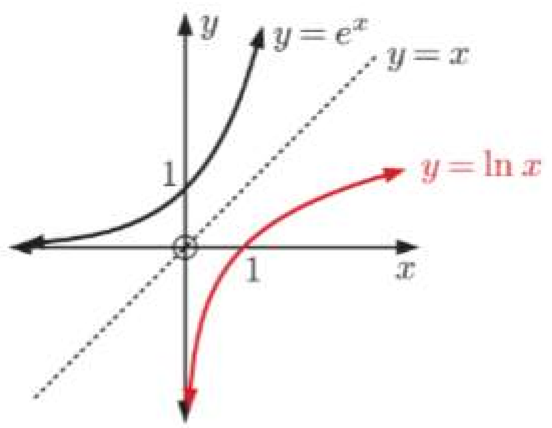
g $\ln x \approx 2.145$
$\therefore\ x \approx e^{2.145}$
$\therefore\ x \approx 8.54$

h $\ln x \approx -3.2971$
$\therefore\ x \approx e^{-3.2971}$
$\therefore\ x \approx 0.0370$

8 **a** **i** $\ln(e^x) = x$ **ii** $e^{\ln x} = x$

b $y = e^x$ and $y = \ln x$ are inverses of each other.

c



d The domain is $\{x \mid x > 0\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

9 **a** $\ln 15 + \ln 3 = \ln(15 \times 3) = \ln 45$

b $\ln 15 - \ln 3 = \ln\left(\frac{15}{3}\right) = \ln 5$

c $\ln 20 - \ln 5 = \ln\left(\frac{20}{5}\right) = \ln 4$

d $\ln 4 + \ln 6 = \ln(4 \times 6) = \ln 24$

e $\ln 5 + \ln(0.2) = \ln(5 \times 0.2) = \ln 1 = 0$

f $\ln 2 + \ln 3 + \ln 5 = \ln(2 \times 3 \times 5) = \ln 30$

g $1 + \ln 4 = \ln(e^1) + \ln 4 = \ln(e \times 4) = \ln(4e)$

h $\ln 6 - 1 = \ln 6 - \ln(e^1) = \ln\left(\frac{6}{e}\right)$

i $\ln 5 + \ln 8 - \ln 2 = \ln(5 \times 8 \div 2) = \ln 20$

j $2 + \ln 4 = \ln(e^2) + \ln 4 = \ln(e^2 \times 4) = \ln(4e^2)$

k $\ln 20 - 2 = \ln 20 - \ln(e^2) = \ln\left(\frac{20}{e^2}\right)$

l $\ln 12 - \ln 4 - \ln 3 = \ln(12 \div 4 \div 3) = \ln 1 = 0$

10 **a** $5\ln 3 + \ln 4 = \ln(3^5) + \ln 4 = \ln(243 \times 4) = \ln 972$

b $3\ln 2 + 2\ln 5 = \ln(2^3) + \ln(5^2) = \ln(8 \times 25) = \ln 200$

c $3\ln 2 - \ln 8 = \ln(2^3) - \ln 8 = \ln\left(\frac{8}{8}\right) = \ln 1 = 0$

d $3\ln 4 - 2\ln 2 = \ln(4^3) - \ln(2^2) = \ln\left(\frac{64}{4}\right) = \ln 16$

e $\frac{1}{3}\ln 8 + \ln 3 = \ln(8^{\frac{1}{3}}) + \ln 3 = \ln(2 \times 3) = \ln 6$

f $\frac{1}{3}\ln\left(\frac{1}{27}\right) = \ln\left(\left(\frac{1}{27}\right)^{\frac{1}{3}}\right) = \ln\left(\frac{1}{27^{\frac{1}{3}}}\right) = \ln\left(\frac{1}{3}\right)$

g $-\ln 2 = \ln(2^{-1}) = \ln\left(\frac{1}{2}\right)$

h $-\ln\left(\frac{1}{2}\right) = \ln\left(\left(\frac{1}{2}\right)^{-1}\right) = \ln 2$

i $-2\ln\left(\frac{1}{4}\right) = \ln\left(\left(\frac{1}{4}\right)^{-2}\right) = \ln(4^2) = \ln 16$

j $4\ln 2+2$
$=\ln(2^4)+\ln(e^2)$
$=\ln 16+\ln(e^2)$
$=\ln(16e^2)$

k $\frac{1}{2}\ln 9-1$
$=\ln(9^{\frac{1}{2}})-\ln(e^1)$
$=\ln 3-\ln e$
$=\ln\left(\frac{3}{e}\right)$

l $-3\ln 2+\frac{1}{2}$
$=\ln(2^{-3})+\ln(e^{\frac{1}{2}})$
$=\ln(\frac{1}{8})+\ln\sqrt{e}$
$=\ln\left(\frac{\sqrt{e}}{8}\right)$

11 **a** $\ln 27$
$=\ln(3^3)$
$=3\ln 3$

b $\ln\sqrt{3}$
$=\ln(3^{\frac{1}{2}})$
$=\frac{1}{2}\ln 3$

c $\ln(\frac{1}{16})$
$=\ln\left(\frac{1}{2^4}\right)$
$=\ln(2^{-4})$
$=-4\ln 2$

d $\ln(\frac{1}{6})$
$=\ln(6^{-1})$
$=-1\times\ln 6$
$=-\ln 6$

e $\ln\left(\frac{1}{\sqrt{2}}\right)$
$=\ln(2^{-\frac{1}{2}})$
$=-\frac{1}{2}\ln 2$

f $\ln\left(\frac{e}{5}\right)$
$=\ln(e^1)-\ln 5$
$=1-\ln 5$

g $\ln(6e)$
$=\ln 6+\ln e$
$=\ln 6+1$

h $\ln\sqrt[3]{5}$
$=\ln(5^{\frac{1}{3}})$
$=\frac{1}{3}\ln 5$

i $\ln\left(\frac{1}{\sqrt[5]{2}}\right)$
$=\ln(2^{-\frac{1}{5}})$
$=-\frac{1}{5}\ln 2$

j $\ln\left(\frac{e^2}{8}\right)$
$=\ln(e^2)-\ln 8$
$=\ln(e^2)-\ln(2^3)$
$=2-3\ln 2$

k $\ln\left(\frac{\sqrt{3}}{e^4}\right)$
$=\ln(3^{\frac{1}{2}})-\ln(e^4)$
$=\frac{1}{2}\ln 3-4$

l $\ln\left(\frac{1}{16\times\sqrt[3]{e}}\right)$
$=\ln 1-\ln(16\times\sqrt[3]{e})$
$=0-(\ln 16+\ln(e^{\frac{1}{3}}))$
$=-(\ln(2^4)+\frac{1}{3})$
$=-(4\ln 2+\frac{1}{3})$
$=-4\ln 2-\frac{1}{3}$

EXERCISE 2D

1 **a** $f(x)=\ln x-4$

i $f(x)$ is a translation of $y=\ln x$ through $\begin{pmatrix}0\\-4\end{pmatrix}$.

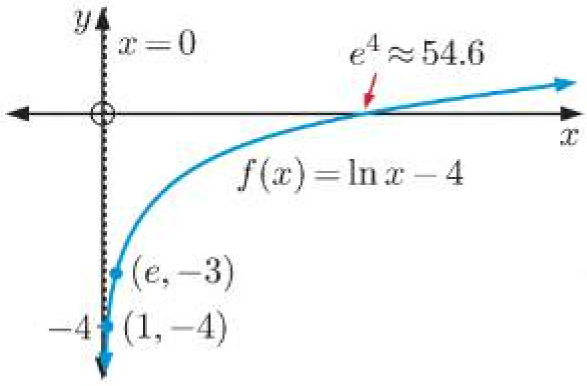
ii Domain is $\{x \mid x>0\}$, Range is $\{y \mid y\in\mathbb{R}\}$

iii $f(1)=\ln 1-4$ and $f(e)=\ln e-4$
$=-4$ $\qquad$ $=-3$

iv When $f(x)=0$, $\ln x=4$
$\therefore\ x=e^4\approx 54.6$

So, the x-intercept is e^4.

v



vi f is defined by $y = \ln x - 4$

$\therefore$ f^{-1} is defined by $x = \ln y - 4$

$\therefore$ $x + 4 = \ln y$

$\therefore$ $y = e^{x+4}$

$\therefore$ $f^{-1}(x) = e^{x+4}$

b $f(x) = \ln(x-1) + 2$

i $f(x)$ is a translation of $y = \ln x$ through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

ii Domain is $\{x \mid x > 1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii $f(1) = \ln(1-1) + 2$ and $f(e) = \ln(e-1) + 2$

$= \ln 0 + 2$ ≈ 2.54

$\therefore$ $f(1)$ is undefined.

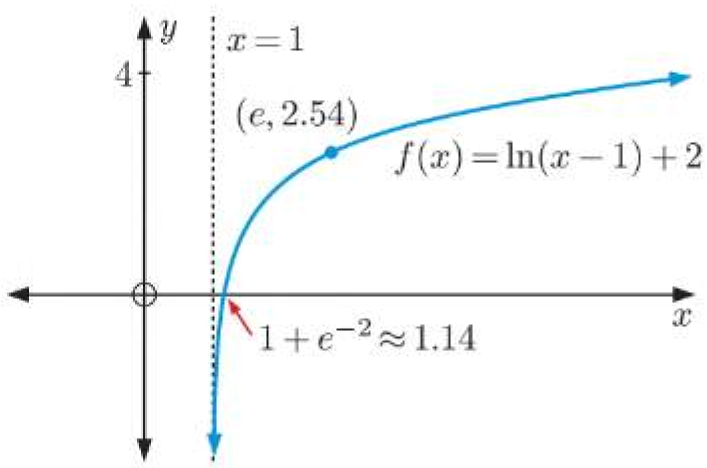
iv When $f(x) = 0$, $\ln(x-1) = -2$

$\therefore$ $x - 1 = e^{-2}$

$\therefore$ $x = 1 + e^{-2} \approx 1.14$

So, the x-intercept is $1 + e^{-2}$.

v



vi f is defined by $y = \ln(x-1) + 2$

$\therefore$ f^{-1} is defined by $x = \ln(y-1) + 2$

$\therefore$ $x - 2 = \ln(y-1)$

$\therefore$ $y - 1 = e^{x-2}$

$\therefore$ $y = e^{x-2} + 1$

$\therefore$ $f^{-1}(x) = e^{x-2} + 1$

c $f(x) = 3\ln x - 1$

i $f(x)$ is a vertical stretch of $y = \ln x$ with scale factor 3, then a translation through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii $f(1) = 3\ln 1 - 1$ and $f(e) = 3\ln e - 1$

$= -1$ $= 2$

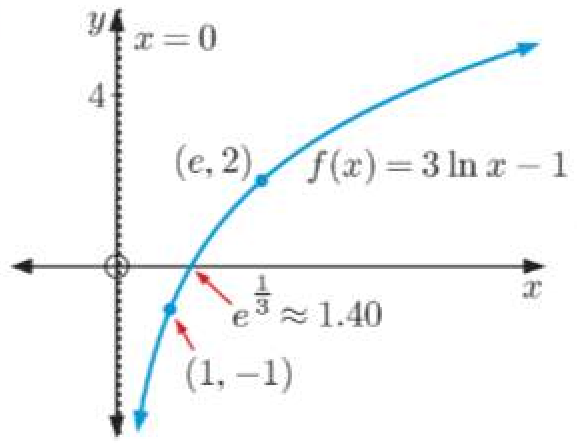
iv When $f(x) = 0$, $3\ln x = 1$

$\therefore$ $\ln x = \frac{1}{3}$

$\therefore$ $x = e^{\frac{1}{3}} \approx 1.40$

So, the x-intercept is $e^{\frac{1}{3}}$.

v



vi f is defined by $y = 3\ln x - 1$

$\therefore$ f^{-1} is defined by $x = 3\ln y - 1$

$$\therefore\ x + 1 = 3\ln y$$
$$\therefore\ \ln y = \frac{x+1}{3}$$
$$\therefore\ y = e^{\frac{x+1}{3}}$$
$$\therefore\ f^{-1}(x) = e^{\frac{x+1}{3}}$$

d $f(x) = 3 - \ln x$

i $f(x)$ is a reflection of $y = \ln x$ in the x-axis, then a translation through $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

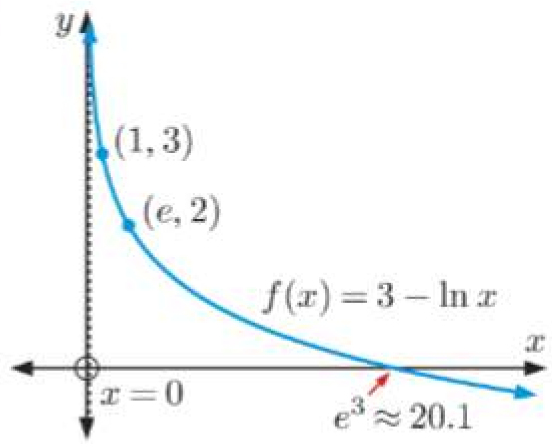
ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii $f(1) = 3 - \ln 1 = 3$ and $f(e) = 3 - \ln e = 2$

iv When $f(x) = 0$, $3 - \ln x = 0$
$$\therefore\ \ln x = 3$$
$$\therefore\ x = e^3 \approx 20.1$$

So, the x-intercept is e^3.

v



vi f is defined by $y = 3 - \ln x$

$\therefore$ f^{-1} is defined by $x = 3 - \ln y$

$$\therefore\ x - 3 = -\ln y$$
$$\therefore\ \ln y = 3 - x$$
$$\therefore\ y = e^{3-x}$$
$$\therefore\ f^{-1}(x) = e^{3-x}$$

2 **a** $f(x) = \log x - 2$

i $\log x$ is defined when $x > 0$

So, the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to 0^+$, $f(x) \to -\infty$, so the vertical asymptote is $x = 0$.

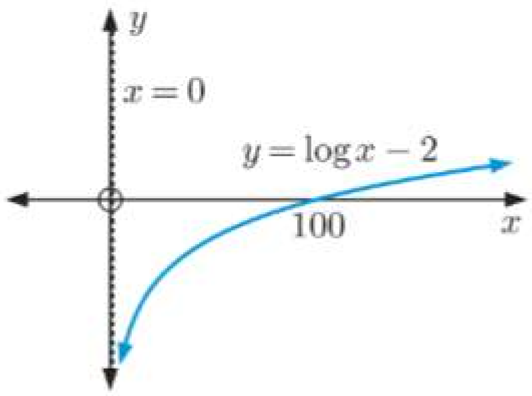
As $x \to \infty$, $f(x) \to \infty$, so there is no horizontal asymptote.

$f(0)$ is undefined, so there is no y-intercept.

When $f(x) = 0$, $\log x = 2$
$$\therefore\ x = 10^2$$
$$\therefore\ x = 100$$

So, the x-intercept is 100.

iii



b $f(x) = \log(x+1)$

i $\log(x+1)$ is defined when $x + 1 > 0$, that is, when $x > -1$

So, the domain is $\{x \mid x > -1\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to -1^+$, $f(x) \to -\infty$, so the vertical asymptote is $x = -1$.

As $x \to \infty$, $f(x) \to \infty$, so there is no horizontal asymptote.

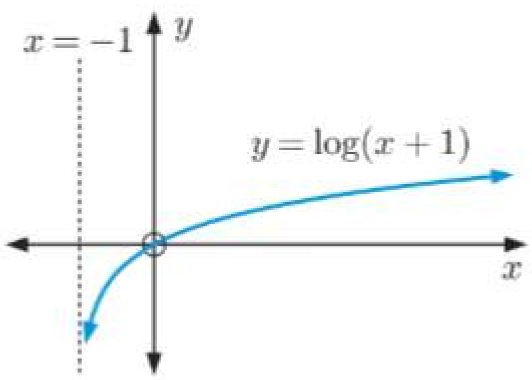
$f(0) = \log 1 = 0$, so the y-intercept is 0.

When $f(x) = 0$, $\log(x+1) = 0$

$$\therefore\ x + 1 = 10^0$$
$$\therefore\ x = 0$$

So, the x-intercept is 0.

iii



c $f(x) = 2\log x + 1$

i $\log x$ is defined when $x > 0$

So, the domain is $\{x \mid x > 0\}$ and the range is $\{y \mid y \in \mathbb{R}\}$.

ii As $x \to 0^+$, $f(x) \to -\infty$, so the vertical asymptote is $x = 0$.

As $x \to \infty$, $f(x) \to \infty$, so there is no horizontal asymptote.

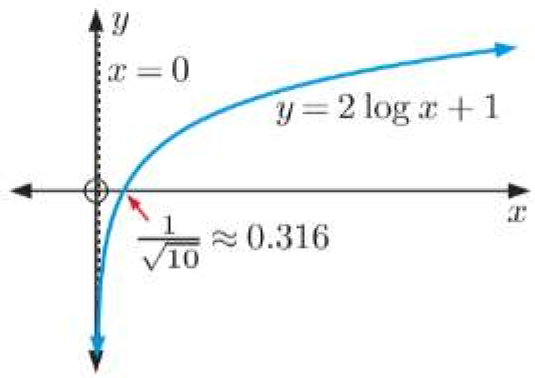
$f(0)$ is undefined, so there is no y-intercept.

When $f(x) = 0$, $2\log x + 1 = 0$

$$\therefore\ 2\log x = -1$$
$$\therefore\ \log x = -\tfrac{1}{2}$$
$$\therefore\ x = 10^{-\frac{1}{2}} = \tfrac{1}{\sqrt{10}}$$

So, the x-intercept is $\frac{1}{\sqrt{10}}$.

iii



d $f(x)=1-\log x$

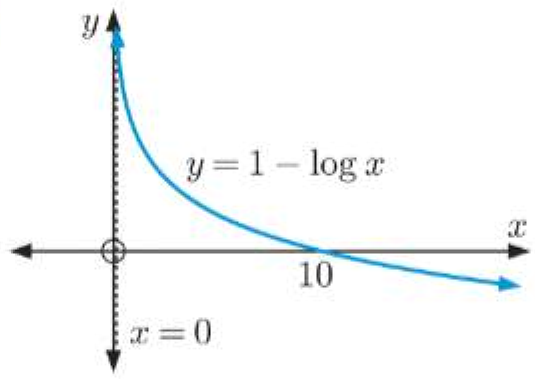
i $\log x$ is defined when $x>0$
So, the domain is $\{x \mid x>0\}$ and the range is $\{y \mid y\in\mathbb{R}\}$.

ii As $x\to 0^+$, $f(x)\to\infty$, so the vertical asymptote is $x=0$.
As $x\to\infty$, $f(x)\to-\infty$, so there is no horizontal asymptote.
$f(0)$ is undefined, so there is no y-intercept.
When $f(x)=0$, $1-\log x=0$

$$\therefore \log x = 1$$
$$\therefore x = 10$$

So, the x-intercept is 10.

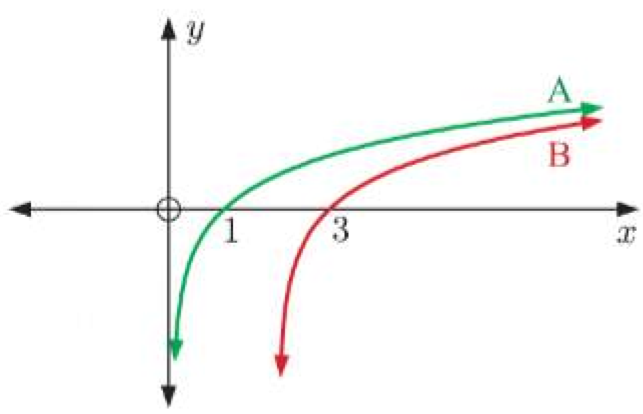
iii



3 **a** For $y=\ln x$, when $y=0$, $\ln x=0$

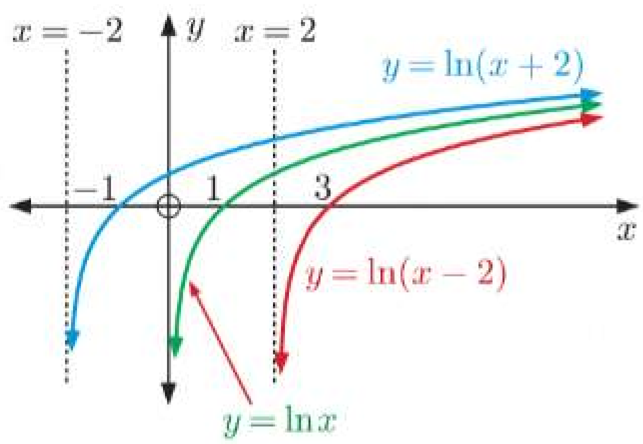
$$\therefore x=e^0$$
$$\therefore x=1$$

$\therefore$ A is $y=\ln x$ as its x-intercept is 1,
so B must be $y=\ln(x-2)$.



b $y=\ln(x-2)$ is a horizontal translation of $y=\ln x$, 2 units to the right.

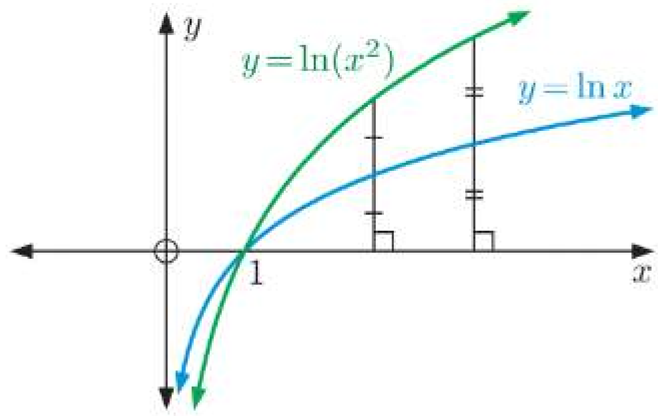
$y=\ln(x+2)$ is a horizontal translation of $y=\ln x$, 2 units to the left.



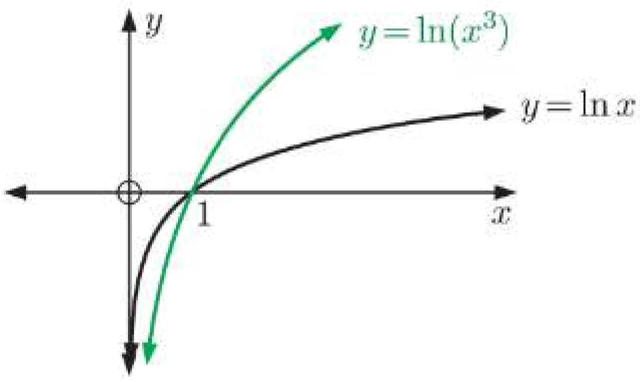
c $y = \ln x$ has domain $\{x \mid x > 0\}$.
As $x \to 0^+$, $y \to -\infty$, so $y = \ln x$ has vertical asymptote $x = 0$.
$y = \ln(x-2)$ has domain $\{x \mid x > 2\}$.
As $x \to 2^+$, $y \to -\infty$, so $y = \ln(x-2)$ has vertical asymptote $x = 2$.
$y = \ln(x+2)$ has domain $\{x \mid x > -2\}$.
As $x \to -2^+$, $y \to -\infty$, so $y = \ln(x+2)$ has vertical asymptote $x = -2$.

4 $y = \ln(x^2)$, $x > 0$
$= 2\ln x$ $\{m \ln b = \ln(b^m)\}$

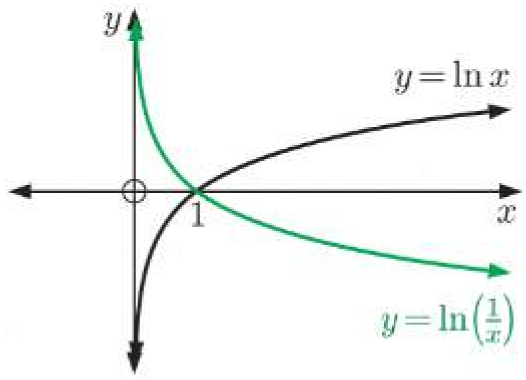
So, the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$. Therefore, yes, Kelly is correct.



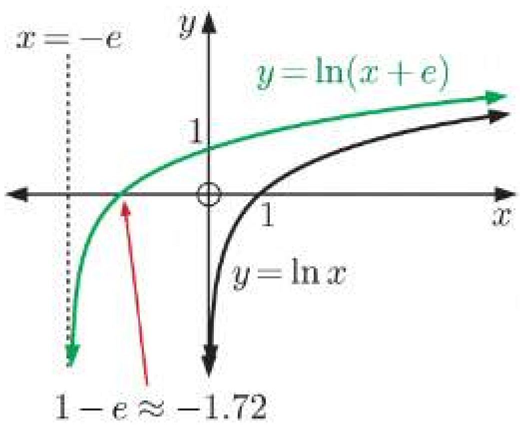
5 **a** $y = \ln(x^3) = 3\ln x$ is a vertical stretch of $y = \ln x$ with scale factor 3.



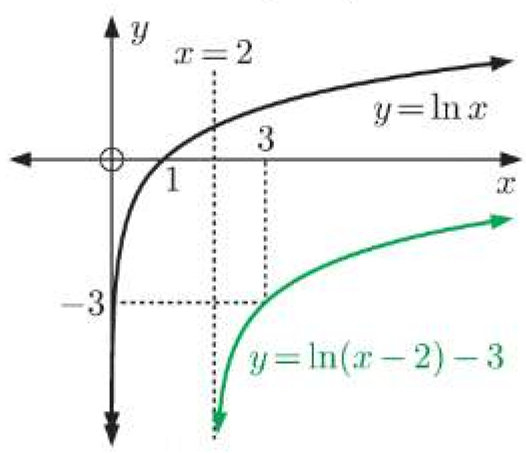
b $y = \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x$ is a reflection of $y = \ln x$ in the x-axis.



c $y = \ln(x+e)$ is a horizontal translation of $y = \ln x$, e units to the left.

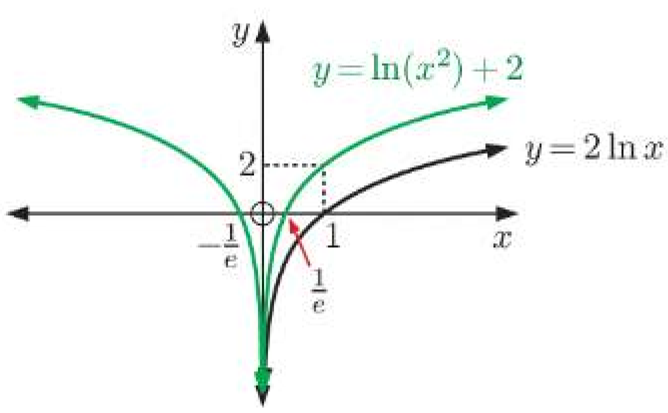


d $y = \ln(x-2) - 3$ is a translation of $y = \ln x$ through $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

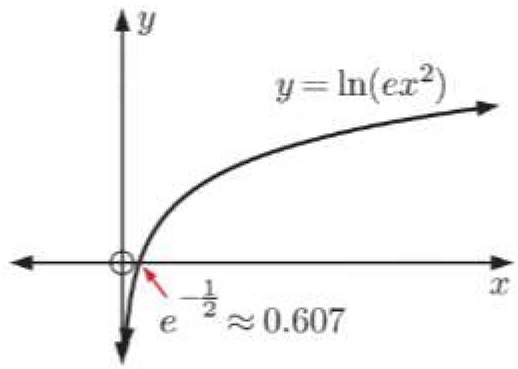


e For $x > 0$, $y = \ln(x^2) + 2 = 2\ln x + 2$ is a vertical translation of $y = 2\ln x$, 2 units upwards.

For $x < 0$, $y = \ln(x^2) + 2 = \ln((-x)^2) + 2$ is a reflection of $y = \ln(x^2) + 2$, $x > 0$, in the y-axis.

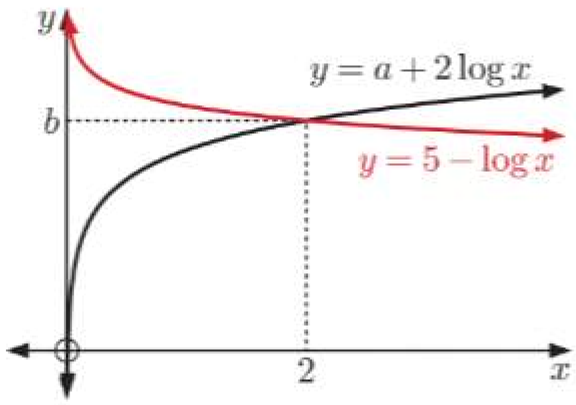


6 **a** $\ln(ex^2) = \ln e + \ln(x^2)$
$= 1 + 2\ln x \quad \{x > 0\}$

b 

7 **a** When $x = 2$, $y = b$

$\therefore\ b = 5 - \log 2$
$= \log(10^5) - \log 2$
$= \log\left(\frac{100\,000}{2}\right)$
$= \log 50\,000$



b When $x = 2$, $y = b = \log 50\,000$ {from **a**}

$\therefore\ a + 2\log 2 = \log 50\,000$
$\therefore\ a = \log 50\,000 - 2\log 2$
$= \log 50\,000 - \log(2^2)$
$= \log\left(\frac{50\,000}{4}\right)$
$= \log 12\,500$

8 $f : x \mapsto e^{2x}$, $g : x \mapsto 2x - 1$

a f is defined by $y = e^{2x}$
$\therefore\ f^{-1}$ is defined by $x = e^{2y}$
$\therefore\ 2y = \ln x$
$\therefore\ y = \frac{1}{2}\ln x$
$\therefore\ f^{-1}(x) = \frac{1}{2}\ln x$

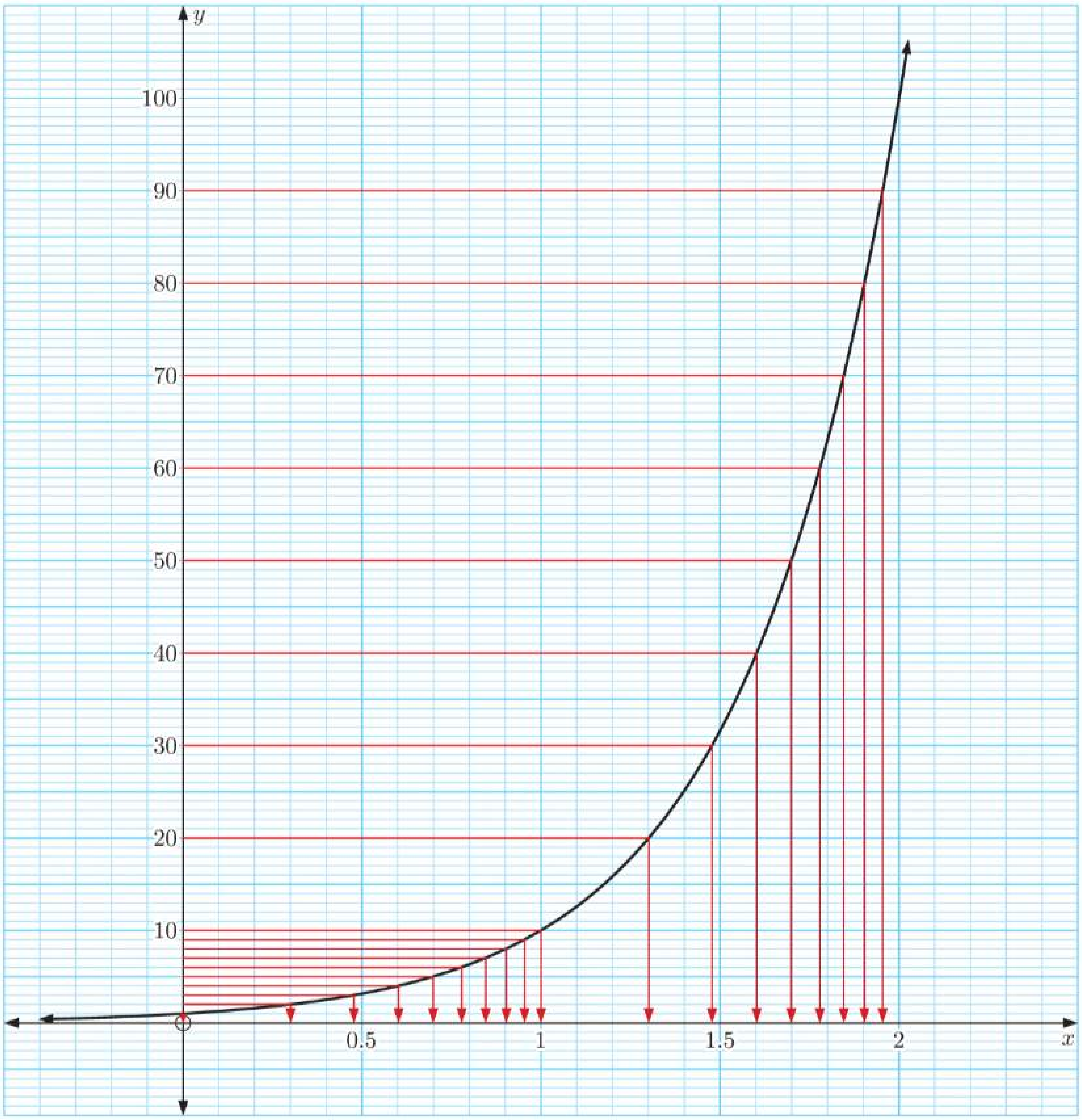
$(f^{-1} \circ g)(x) = f^{-1}(g(x))$
$= f^{-1}(2x - 1)$
$= \frac{1}{2}\ln(2x - 1)$

b $(g \circ f)(x) = g(f(x))$
$= g(e^{2x})$
$= 2e^{2x} - 1$

$(g \circ f)(x)$ is defined by $y = 2e^{2x} - 1$
$\therefore\ (g \circ f)^{-1}(x)$ is defined by $x = 2e^{2y} - 1$
$\therefore\ x + 1 = 2e^{2y}$
$\therefore\ e^{2y} = \dfrac{x+1}{2}$
$\therefore\ 2y = \ln\left(\dfrac{x+1}{2}\right)$
$\therefore\ y = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$
$\therefore\ (g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$

INVESTIGATION 2 LOGARITHMIC SCALES

1



2 As the y-values approach a power of 10, the minor tick marks become closer together. The spacing of the y-values $1, 2, ..., 9,$ and the y-values $10, 20, ..., 90,$ matches the spacing of the numbers on the slide rules in the **Opening Problem**.

3 From the graph, each distance measures ≈ 4.7 cm.

$$\begin{aligned}&\log 10 - \log 1\\ &= 1 - 0\\ &= 1\end{aligned}$$

$$\begin{aligned}&\log 20 - \log 2\\ &= \log\left(\tfrac{20}{2}\right)\\ &= \log 10\\ &= 1\end{aligned}$$

$$\begin{aligned}&\log 50 - \log 5\\ &= \log\left(\tfrac{50}{5}\right)\\ &= \log 10\\ &= 1\end{aligned}$$

4 $\log(5\times10^2) = \log 5 + \log(10^2)$ and $\log(10^{2.5}) = 2.5$

$= \log 5 + 2$

≈ 2.70

Since $\log(5\times10^2) > \log(10^{2.5})$, 5×10^2 is further to the right on the logarithmic scale.

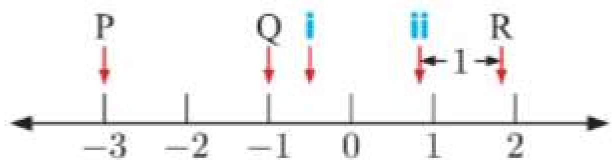
EXERCISE 2E

1 **a** P corresponds to the value $10^{-3} = \frac{1}{1000}$

Q corresponds to the value $10^{-1} = \frac{1}{10}$

b **i** $\log\left(\frac{1}{\sqrt{10}}\right) = \log\left(10^{-\frac{1}{2}}\right)$

$= -\frac{1}{2}$



ii $\log\left(\frac{R}{10}\right) = \log R - \log 10$

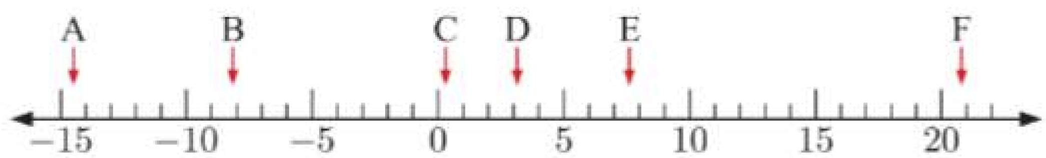
$= \log R - 1$

c No, it is not possible to represent 0 on the number line since $\log 0$ is undefined.

2 **a**

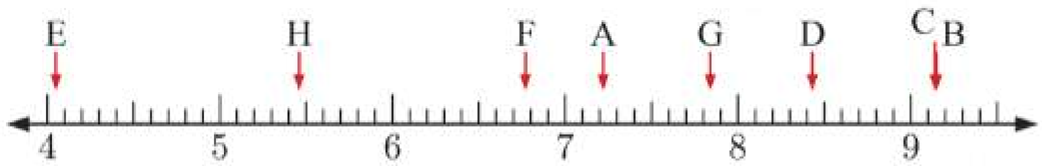
	Process	*Energy* (J)	*Logarithm*
A	average kinetic energy of a human red blood cell	3×10^{-15}	≈ -14.5
B	energy needed to raise a grain of sand by 1 mm	7×10^{-9}	≈ -8.15
C	kinetic energy when a 200 g apple falls 1 m	2×10^{0}	≈ 0.301
D	solar energy received by 1 m^2 of the Earth's surface directly under the Sun in 1 second	1.4×10^{3}	≈ 3.15
E	energy from the combustion of 1 m^3 of natural gas	4×10^{7}	≈ 7.60
F	total world energy consumption in 2016	6×10^{20}	≈ 20.8

b



3 **a**

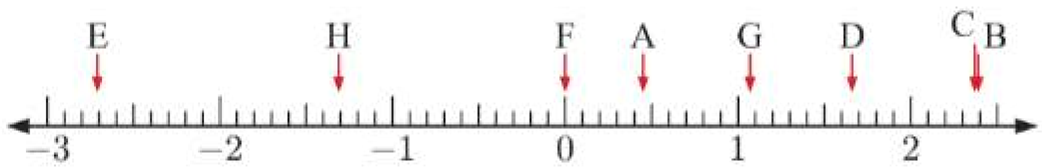
	Country	Population	Logarithm
A	Cambodia	16 482 646	≈ 7.22
B	China	1 420 062 022	≈ 9.15
C	India	1 368 737 513	≈ 9.14
D	Indonesia	269 536 482	≈ 8.43
E	Nauru	11 260	≈ 4.05
F	Singapore	5 836 496	≈ 6.77
G	Thailand	69 306 160	≈ 7.84
H	Vanuatu	288 017	≈ 5.46



b

	Country	Population	Population relative to Singapore	Logarithm
A	Cambodia	16 482 646	≈ 2.82	≈ 0.451
B	China	1 420 062 022	≈ 243	≈ 2.39
C	India	1 368 737 513	≈ 235	≈ 2.37
D	Indonesia	269 536 482	≈ 46.2	≈ 1.66
E	Nauru	11 260	$\approx 0.001\,93$	≈ -2.71
F	Singapore	5 836 496	1	0
G	Thailand	69 306 160	≈ 11.9	≈ 1.07
H	Vanuatu	288 017	≈ 0.0493	≈ -1.31

c



d Our scale in **c** is obtained by translating the scale in **a** ≈ 6.77 units left. This occurs since for any population P, $\log\left(\frac{P}{P_S}\right) = \log P - \log P_S$, where P_S is the population of Singapore.

4 **a** **i** If $M = 0$, then $\frac{I}{I_0} = 1$, and so $I = I_0$. This means the earthquake intensity is equal to the reference intensity.

ii If $M = 1$, then $\frac{I}{I_0} = 10$, and so $I = 10I_0$. This means the earthquake intensity is 10 times greater than the reference intensity.

b The magnitude of an earthquake follows a logarithmic scale. A magnitude 6 earthquake is $10^{6-3} = 1000$ times more intense than a magnitude 3 earthquake.

c **i** The intensity I_5 of a magnitude 5 earthquake obeys $5 = \log\left(\frac{I_5}{I_0}\right)$

$$\therefore \frac{I_5}{I_0} = 10^5$$

$$\therefore I_5 = 10^5 I_0$$

The magnitude of an earthquake with 10% of this intensity is

$$\begin{aligned} M &= \log\left(\frac{0.1 \times 10^5 I_0}{I_0}\right) \\ &= \log(10^4) \\ &= 4 \end{aligned}$$

ii The intensity I_6 of an earthquake with magnitude 6 obeys

$$\begin{aligned} 6 &= \log\left(\frac{I_6}{I_0}\right) \\ \therefore \quad \frac{I_6}{I_0} &= 10^6 \\ \therefore \quad I_6 &= 10^6 I_0 \end{aligned}$$

The magnitude of an earthquake with half this intensity is

$$\begin{aligned} M &= \log\left(\frac{\frac{1}{2} \times 10^6 I_0}{I_0}\right) \\ &= \log\tfrac{1}{2} + \log(10^6) \\ &= 6 - \log 2 \\ &\approx 5.70 \end{aligned}$$

d Let I_{H} be the intensity of the Honduras earthquake and let I_{F} be the intensity of the Fiji earthquake.

$\therefore \quad 7.5 = \log\left(\frac{I_{\mathrm{H}}}{I_0}\right)$ and $8.2 = \log\left(\frac{I_{\mathrm{F}}}{I_0}\right)$

$\therefore \quad \frac{I_{\mathrm{H}}}{I_0} = 10^{7.5}$ and $\frac{I_{\mathrm{F}}}{I_0} = 10^{8.2}$

$\therefore \quad I_{\mathrm{H}} = 10^{7.5} I_0$ and $I_{\mathrm{F}} = 10^{8.2} I_0$

$\therefore$ the Fiji earthquake was $10^{8.2-7.5} = 10^{0.7} \approx 5.01$ times more intense than the Honduras earthquake.

5 **a** A logarithmic scale would be useful in describing the acidity of a solution as the possible concentrations of H_3O^+ take extremely small values that would otherwise be impossible to compare.

b $\mathrm{pH} = -\log C$

i If $C = 0.000\,234$ mol L^{-1},

$$\begin{aligned} \mathrm{pH} &= -\log(0.000\,234) \\ &\approx 3.63 \end{aligned}$$

ii If $\mathrm{pH} = 7$,

$$\begin{aligned} 7 &= -\log C \\ \therefore \quad \log C &= -7 \\ \therefore \quad C &= 10^{-7} \text{ mol L}^{-1} \end{aligned}$$

c **i** Let C_{M} be the concentration of H_3O^+ in milk and let C_{V} be the concentration of H_3O^+ in vinegar.

$$\begin{aligned} -\log C_{\mathrm{M}} &= 6.6 \\ \therefore \quad \log C_{\mathrm{M}} &= -6.6 \\ \therefore \quad C_{\mathrm{M}} &= 10^{-6.6} \\ &\approx 2.51 \times 10^{-7} \end{aligned}$$

$$\begin{aligned} -\log C_{\mathrm{V}} &= 2.4 \\ \therefore \quad \log C_{\mathrm{V}} &= -2.4 \\ \therefore \quad C_{\mathrm{V}} &= 10^{-2.4} \\ &\approx 0.003\,98 \end{aligned}$$

Since $C_{\mathrm{V}} > C_{\mathrm{M}}$, vinegar is more acidic.

ii If $C = 200C_M$, $\text{pH} = -\log(200C_M)$
$$\begin{aligned} &= -\log 200 - \log C_M \\ &= -\log 200 + 6.6 \\ &\approx 4.30 \end{aligned}$$

iii The H_3O^+ concentration in vinegar is about $10^{-2.4-(-4.30)} \approx 10^{1.90} \approx 79.2$ times greater than in the substance in **ii**.

d If $C > 1$, then $\log C > 0$
$\therefore\ \text{pH} = -\log C < 0$

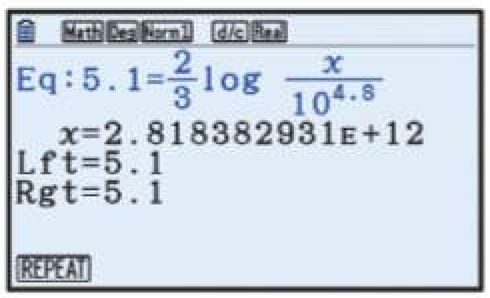
So, it is possible for a solution to have negative pH. In this case, the concentration of H_3O^+ is greater than 1 mol L^{-1}.

6 $M = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right)$

a When $E = 6.2 \times 10^{13}$, $M = \frac{2}{3}\log\left(\frac{6.2 \times 10^{13}}{10^{4.8}}\right)$
≈ 5.99

b When $M = 5.1$, $5.1 = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right)$

Using technology, $E \approx 2.82 \times 10^{12}$ joules.



7 $n \approx 3.322\log\left(\frac{f}{261.6}\right)$

a "Middle C" is 0 octaves above "Middle C", so $n = 0$
$$\begin{aligned} \therefore\ 3.322\log\left(\frac{f}{261.6}\right) &\approx 0 \\ \therefore\ \log\left(\frac{f}{261.6}\right) &\approx 0 \\ \therefore\ \frac{f}{261.6} &\approx 1 \quad \{\text{since } \log 1 = 0\} \\ \therefore\ f &\approx 261.6 \end{aligned}$$

So, the frequency of "Middle C" is about 261.6 Hz.

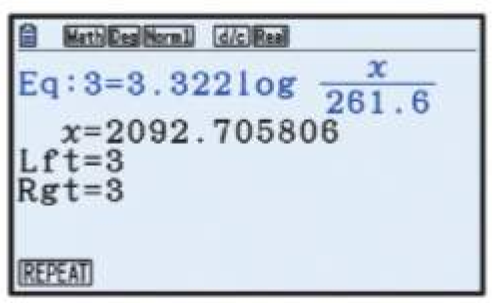
b When $f = 784$, $n \approx 3.322\log\left(\frac{784}{261.6}\right)$
≈ 1.58

So, a note with frequency 784 Hz is about 1.58 octaves above "Middle C".

c **i** When $n = 3$, $3 \approx 3.322\log\left(\frac{f}{261.6}\right)$

Using technology, $f \approx 2090$

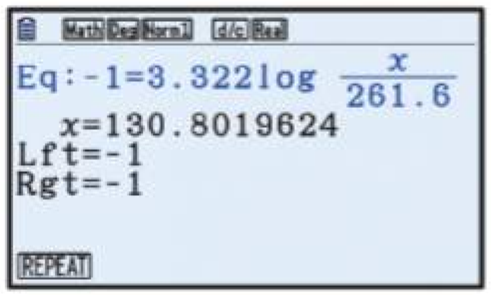
The note which is 3 octaves above "Middle C" has frequency of about 2090 Hz.



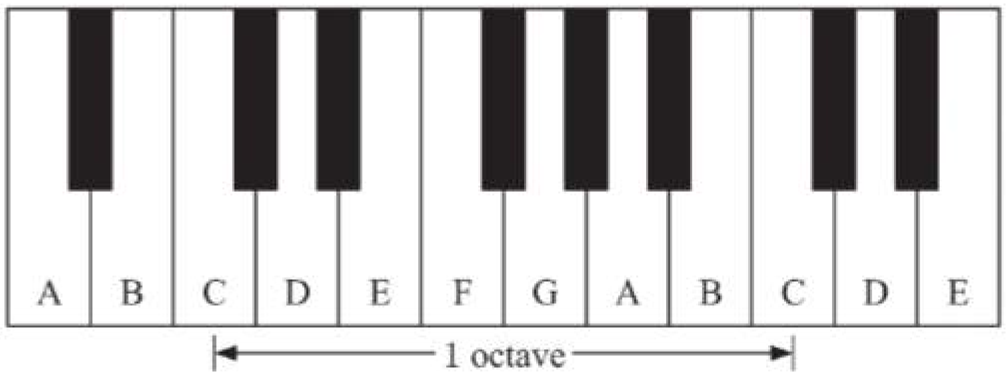
ii When $n = -1$, $\quad -1 \approx 3.322 \log\left(\dfrac{f}{261.6}\right)$

Using technology, $\quad f \approx 131$

The note which is 1 octave below "Middle C" has frequency of about 131 Hz.



d



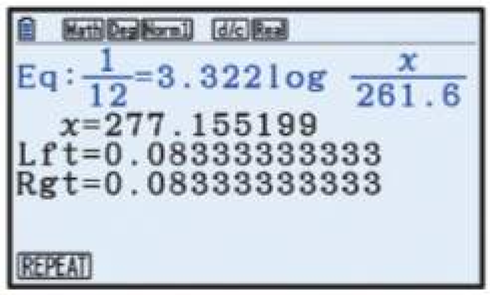
There are 12 notes in an octave, so each note is $\frac{1}{12}$ of an octave above the note before it.

The note which is adjacent to "Middle C" is $\frac{1}{12}$ of an octave above "Middle C", so $n = \frac{1}{12}$.

When $n = \frac{1}{12}$, $\quad \frac{1}{12} \approx 3.322 \log\left(\dfrac{f}{261.6}\right)$

Using technology, $\quad f \approx 277.2$

So the note adjacent to "Middle C" has frequency of about 277.2 Hz.



Since the frequency of "Middle C" is 261.6 Hz, the ratio of frequencies between two adjacent notes is $\quad 277.2 : 261.6 \approx 1.06 : 1$.

8 $\phi = -\log_2\left(\dfrac{D}{D_0}\right), \quad D_0 = 1$ mm

a If D is measured in mm, then $\dfrac{D}{D_0} = \dfrac{D}{1} = D$

$$\therefore \quad \phi = \log_2 D$$
$$\therefore \quad 2^{\phi} = D$$
$$\therefore \quad \log(2^{\phi}) = \log D$$
$$\therefore \quad \phi \log 2 = \log D$$
$$\therefore \quad \phi = \frac{\log D}{\log 2}$$

b If $\phi = -3.4$, then $\quad -\dfrac{\log D}{\log 2} = -3.4$

$$\therefore \quad \log D = 3.4 \log 2$$
$$\therefore \quad D = 10^{3.4 \log 2}$$
$$\approx 10.6 \text{ mm}$$

ACTIVITY THE BRIGHTNESS OF THE STARS

1 Let B_n be the brightness of a star with magnitude n.

$\therefore \ \dfrac{B_1}{B_6} = 100$

$\therefore \ B_6 = \frac{1}{100} B_1$

Let r be the ratio of B_{n+1} to B_n.

$$\therefore \ \frac{B_{n+1}}{B_n} = r$$

$$\begin{aligned} \therefore \ B_{n+1} &= B_n r \\ &= B_{n-1}\, r^2 \\ &= B_{n-2}\, r^3 \\ &\ \ \vdots \\ &= B_1 r^n \end{aligned}$$

Now $B_6 = \frac{1}{100} B_1$

$\therefore \ B_1 r^5 = \frac{1}{100} B_1$

$\therefore \ r^5 = \frac{1}{100}$

$\therefore \ r = \frac{1}{\sqrt[5]{100}}$

$\therefore$ a magnitude increase of 1 corresponds to a decrease in brightness by the factor $\sqrt[5]{100}$.

2 $M = -2.5 \log\left(\dfrac{F}{F_0}\right)$

a A higher apparent magnitude indicates a fainter object.

b When $F = F_1$, $M = 1$

$$\therefore \ -2.5 \log\left(\frac{F_1}{F_0}\right) = 1$$

$$\therefore \ \log\left(\frac{F_1}{F_0}\right) = -\frac{1}{2.5} = -\tfrac{2}{5}$$

$$\therefore \ \frac{F_1}{F_0} = 10^{-\frac{2}{5}}$$

$$\therefore \ F_1 = 10^{-\frac{2}{5}} F_0$$

When $F = F_6$, $M = 6$

$$\therefore \ -2.5 \log\left(\frac{F_6}{F_0}\right) = 6$$

$$\therefore \ \log\left(\frac{F_6}{F_0}\right) = -\frac{6}{2.5} = -\tfrac{12}{5}$$

$$\therefore \ \frac{F_6}{F_0} = 10^{-\frac{12}{5}}$$

$$\therefore \ F_6 = 10^{-\frac{12}{5}} F_0$$

$$\therefore \quad \frac{F_1}{F_6} = \frac{10^{-\frac{2}{5}} F_0}{10^{-\frac{12}{5}} F_0}$$
$$= 10^{-\frac{2}{5}-(-\frac{12}{5})}$$
$$= 10^{\frac{10}{5}}$$
$$= 10^2$$
$$= 100$$

c If a star has a negative apparent magnitude, then $\log\left(\dfrac{F}{F_0}\right) > 0$

$$\therefore \quad \frac{F}{F_0} > 1$$
$$\therefore \quad F > F_0$$

$\therefore$ the observed flux density of the star is greater than the reference flux density.

d **i** Let F_S be the observed flux density of the Sun and
let F_M be the observed flux density of the full Moon.

$$\therefore \quad -2.5\log\left(\frac{F_S}{F_0}\right) = -26.74 \quad \text{and} \quad -2.5\log\left(\frac{F_M}{F_0}\right) = -12.74$$
$$\therefore \quad \log\left(\frac{F_S}{F_0}\right) = \frac{26.74}{2.5} = 10.696 \quad \text{and} \quad \log\left(\frac{F_M}{F_0}\right) = \frac{12.74}{2.5} = 5.096$$
$$\therefore \quad \frac{F_S}{F_0} = 10^{10.696} \quad \text{and} \quad \frac{F_M}{F_0} = 10^{5.096}$$
$$\therefore \quad F_S = 10^{10.696} F_0 \quad \text{and} \quad F_M = 10^{5.096} F_0$$

$\therefore$ the Sun is $10^{10.696-5.096} = 10^{5.6} \approx 3.98 \times 10^5$ times brighter than the full Moon when viewed from Earth.

ii The apparent magnitude of a star 1 000 000 times fainter than the full Moon is

$$M = -2.5\log\left(\frac{10^{5.096} F_0 \div 1\,000\,000}{F_0}\right)$$
$$= -2.5\log\left(\frac{10^{5.096}}{10^6}\right)$$
$$= -2.5\log(10^{-0.904})$$
$$= -2.5(-0.904)$$
$$= 2.26$$

3 **a** **i** If star B is further away than star A, but has the same apparent magnitude, then star B must be brighter.
A lower absolute magnitude corresponds to a brighter object, so star B must have the lower absolute magnitude.

ii Let F_A be the observed flux density of star A if it was 10 parsecs from Earth, and let F_B be the observed flux density of star B if it was 10 parsecs from Earth.

$$\therefore \quad -2.5\log\left(\frac{F_A}{F_0}\right) = -6$$
$$\therefore \quad \log\left(\frac{F_A}{F_0}\right) = \frac{6}{2.5} = 2.4$$
$$\therefore \quad \frac{F_A}{F_0} = 10^{2.4}$$
$$\therefore \quad F_A = 10^{2.4}F_0$$

Now, the brightness of an object varies inversely with the square of its distance from the observer, so if star B is 2 times further away than star A but has the same apparent magnitude, then star B must be $2^2 = 4$ times brighter than star A.

$\therefore \quad F_B = 4F_A$

$\therefore$ the absolute magnitude of star B is
$$\begin{aligned} M &= -2.5\log\left(\frac{4\times 10^{2.4}F_0}{F_0}\right) \\ &= -2.5\log(4\times 10^{2.4}) \\ &= -2.5(\log 4 + \log(10^{2.4})) \\ &= -2.5(\log 4 + 2.4) \\ &\approx -7.51 \end{aligned}$$

b From **2 d i**, the observed flux density of the Sun is $10^{10.696}F_0$.
If the Sun was 10 parsecs away from the Earth, then the observed flux density would be

$$10^{10.696}F_0 \times \left(\frac{4.848\times 10^{-6}}{10}\right)^2 \approx 2.35\times 10^{-2.304}F_0$$

$\therefore$ the Sun has absolute magnitude
$$\begin{aligned} &\approx -2.5\log\left(\frac{2.35\times 10^{-2.304}F_0}{F_0}\right) \\ &\approx -2.5\log(2.35\times 10^{-2.304}) \\ &\approx -2.5(\log 2.35 - 2.304) \\ &\approx 4.83 \end{aligned}$$

REVIEW SET 2A

1 **a** $\log\sqrt{10}$
$= \log(10^{\frac{1}{2}})$
$= \frac{1}{2}$

b $\log\left(\frac{1}{\sqrt[3]{10}}\right)$
$= \log(10^{-\frac{1}{3}})$
$= -\frac{1}{3}$

c $\log(10^a \times 10^{b+1})$
$= \log(10^{a+b+1})$
$= a+b+1$

2 **a** $\log 27$
≈ 1.431

b $\log(0.58)$
≈ -0.237

c $\log 400$
≈ 2.602

d $\ln 40$
≈ 3.689

3 **a** $32 = 10^{\log 32}$
$\approx 10^{1.5051}$

b 0.0013
$= 10^{\log(0.0013)}$
$\approx 10^{-2.8861}$

c 8.963×10^{-5}
$= 10^{\log(8.963)}\times 10^{-5}$
$= 10^{\log(8.963)-5}$
$\approx 10^{-4.0475}$

4 **a** $4\ln 2+2\ln 3$
$=\ln(2^4)+\ln(3^2)$
$=\ln(16\times 9)$
$=\ln 144$

b $\frac{1}{2}\ln 9-\ln 2$
$=\ln(9^{\frac{1}{2}})-\ln 2$
$=\ln 3-\ln 2$
$=\ln(\frac{3}{2})$

c $2\ln 5-1$
$=\ln(5^2)-\ln(e^1)$
$=\ln\left(\frac{25}{e}\right)$

d $\frac{1}{4}\ln 81$
$=\ln(3^4)^{\frac{1}{4}}$
$=\ln(3^1)$
$=\ln 3$

5 **a** $\log 16+2\log 3$
$=\log 16+\log(3^2)$
$=\log(16\times 9)$
$=\log 144$

b $\log 16-2\log 3$
$=\log 16-\log(3^2)$
$=\log(\frac{16}{9})$

c $2+\log 5$
$=\log(10^2)+\log 5$
$=\log(100\times 5)$
$=\log 500$

6 **a** $\log 36$
$=\log(4\times 9)$
$=\log(2^2\times 3^2)$
$=\log(2^2)+\log(3^2)$
$=2\log 2+2\log 3$
$=2A+2B$

b $\log 54$
$=\log(2\times 27)$
$=\log(2\times 3^3)$
$=\log 2+\log(3^3)$
$=\log 2+3\log 3$
$=A+3B$

c $\log(8\sqrt{3})$
$=\log(2^3\times 3^{\frac{1}{2}})$
$=\log(2^3)+\log(3^{\frac{1}{2}})$
$=3\log 2+\frac{1}{2}\log 3$
$=3A+\frac{1}{2}B$

d $\log(\sqrt{6})$
$=\log(6^{\frac{1}{2}})$
$=\frac{1}{2}\log 6$
$=\frac{1}{2}\log(2\times 3)$
$=\frac{1}{2}(\log 2+\log 3)$
$=\frac{1}{2}(A+B)$

e $\log(20.25)$
$=\log(\frac{81}{4})$
$=\log\left(\frac{3^4}{2^2}\right)$
$=\log(3^4)-\log(2^2)$
$=4\log 3-2\log 2$
$=4B-2A$

f $\log(\frac{8}{9})$
$=\log\left(\frac{2^3}{3^2}\right)$
$=\log(2^3)-\log(3^2)$
$=3\log 2-2\log 3$
$=3A-2B$

7 **a** $\log x=-3$
$\therefore\ x=10^{-3}$
$\therefore\ x=\frac{1}{1000}$

b $\ln x=5$
$\therefore\ x=e^5$

8 $f(x)=3\ln x+3$

a $f(x)$ is a vertical stretch of $y=\ln x$ with scale factor 3, then a translation through $\begin{pmatrix}0\\3\end{pmatrix}$.

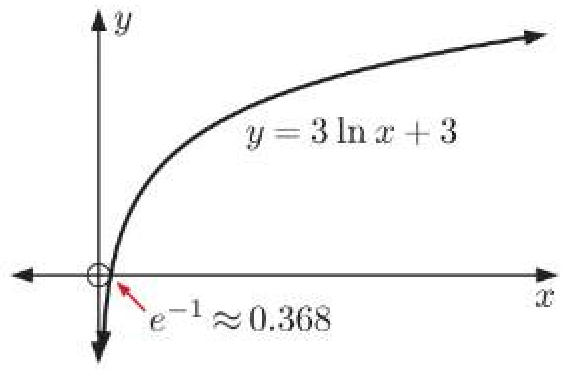
b $f(1)=3\ln 1+3$
$=3$

c $f(0)$ is undefined, so there is no y-intercept.

When $f(x)=0$, $3\ln x+3=0$
$\therefore\ 3\ln x=-3$
$\therefore\ \ln x=-1$
$\therefore\ x=e^{-1}$

So, the x-intercept is e^{-1}.

d



e f is defined by $y = 3\ln x + 3$

$\therefore$ f^{-1} is defined by $x = 3\ln y + 3$

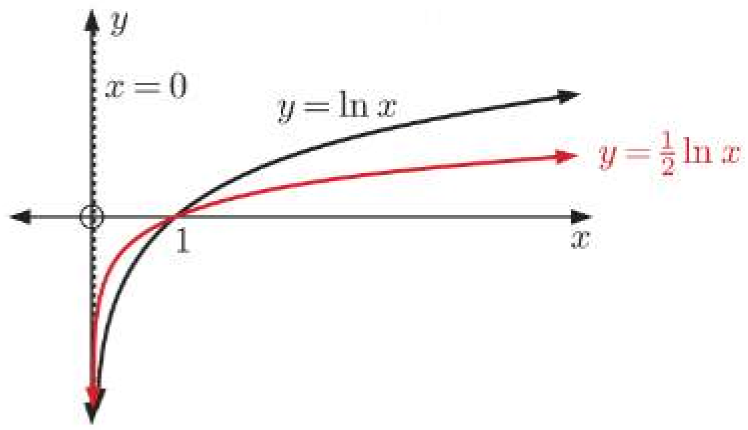
$\therefore\ x - 3 = 3\ln y$

$\therefore\ \dfrac{x-3}{3} = \ln y$

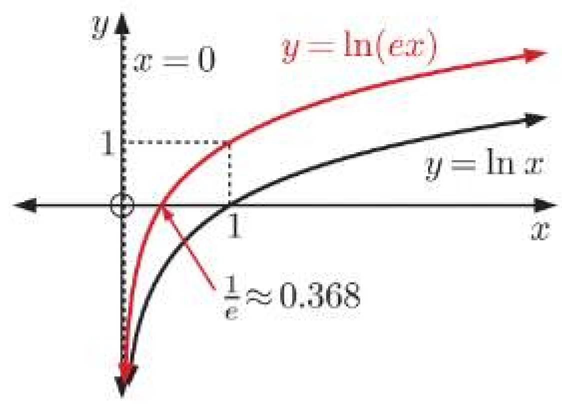
$\therefore\ e^{\frac{x-3}{3}} = y$

$\therefore\ f^{-1}(x) = e^{\frac{x-3}{3}}$

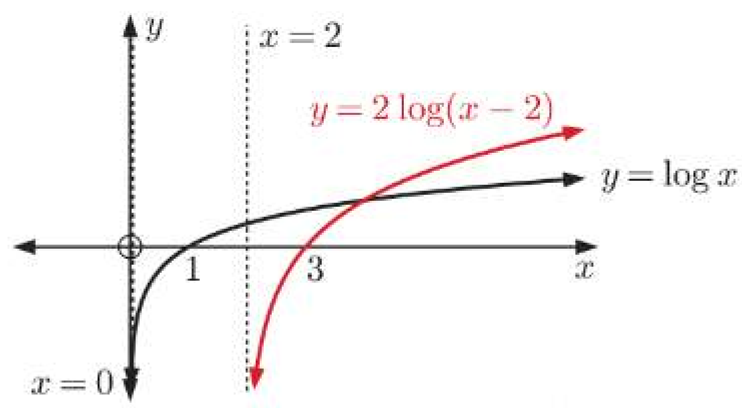
9 **a** $y = \frac{1}{2}\ln x$ is a vertical stretch of $y = \ln x$ with scale factor $\frac{1}{2}$.



b $y = \ln(ex)$ is a horizontal stretch of $y = \ln x$ with scale factor $\dfrac{1}{e}$.



c $y = 2\log(x-2)$ is a horizontal translation of $y = \log x$ 2 units to the right followed by a vertical stretch with scale factor 2.



10 $\ln k > 1$

$\therefore\ e^{\ln k} > e^1$

$\therefore\ k > e$

Also, $\log k < 1$

$\therefore\ 10^{\log k} < 10^1$

$\therefore\ k < 10$

$\therefore$ k lies between e and 10.

11 **a** $f(x) = 10^x$

$\therefore\ f(2) = 10^2$

$= 100$

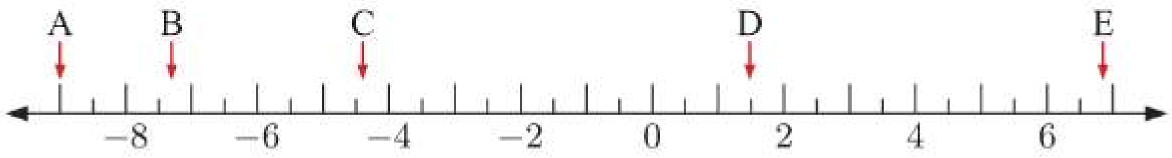
$\therefore$ the y-coordinate of P is 100.

b $f(x) = 10^x$ and $f^{-1}(x) = \log x$ are inverse functions.

$\therefore$ the point corresponding to P(2, 100) on the inverse function is (100, 2).

12

	Electromagnetic radiation type	*Wavelength* (m)	*Logarithm*
A	X-rays	0.000 000 001	-9
B	Ultraviolet	0.000 000 05	≈ -7.30
C	Infrared	0.000 04	≈ -4.40
D	High frequency microwaves	30	≈ 1.48
E	Super low frequency radio waves	7 000 000	≈ 6.85



REVIEW SET 2B

1 **a** $\log\sqrt{1000}$
$= \log((10^3)^{\frac{1}{2}})$
$= \log(10^{\frac{3}{2}})$
$= \frac{3}{2}$

b $\log\left(\dfrac{10}{\sqrt[3]{10}}\right)$
$= \log\left(\dfrac{10^1}{10^{\frac{1}{3}}}\right)$
$= \log(10^{\frac{2}{3}})$
$= \frac{2}{3}$

c $\log\left(\dfrac{10^a}{10^{-b}}\right)$
$= \log\left(10^{a-(-b)}\right)$
$= \log(10^{a+b})$
$= a+b$

2 **a** $\ln\left(e\sqrt{e}\right)$
$= \ln(e^1 e^{\frac{1}{2}})$
$= \ln(e^{\frac{3}{2}})$
$= \frac{3}{2}$

b $\ln\left(\dfrac{1}{e^3}\right)$
$= \ln(e^{-3})$
$= -3$

c $\ln(e^{2x}) = 2x$

d $\ln\left(\dfrac{e}{e^x}\right)$
$= \ln(e^{1-x})$
$= 1-x$

3 **a** $\dfrac{\log 25}{\log 125} = \dfrac{\log(5^2)}{\log(5^3)}$
$= \dfrac{2\log 5}{3\log 5}$
$= \frac{2}{3}$

b $\dfrac{\log 64}{\log 32} = \dfrac{\log(2^6)}{\log(2^5)}$
$= \dfrac{6\log 2}{5\log 2}$
$= \frac{6}{5}$

c $\dfrac{\log 81}{\log\sqrt{3}} = \dfrac{\log(3^4)}{\log(3^{\frac{1}{2}})}$
$= \dfrac{4\log 3}{\frac{1}{2}\log 3}$
$= \dfrac{4}{\frac{1}{2}}$
$= 8$

4 **a** $e^{4\ln x}$
$= \left(e^{\ln x}\right)^4$
$= x^4$

b $\ln(e^5) = 5$

c $\ln\left(\sqrt{e}\right) = \ln(e^{\frac{1}{2}})$
$= \frac{1}{2}$

d $$\begin{aligned}&10^{\log x+\log 3}\\&=10^{\log x}\times 10^{\log 3}\\&=x\times 3\\&=3x\end{aligned}$$

e $$\begin{aligned}\ln\left(\frac{1}{e^x}\right)&=\ln(e^{-x})\\&=-x\end{aligned}$$

f $$\begin{aligned}&\frac{\log(x^2)}{\log(0.01)}\\&=\frac{\log(x^2)}{\log(10^{-2})}\\&=\frac{\log(x^2)}{-2}\\&=-\tfrac{1}{2}\log(x^2)\\&=-\log x\end{aligned}$$

5 **a** $$\begin{aligned}20&=e^{\ln 20}\\&\approx e^{2.9957}\end{aligned}$$

b $$\begin{aligned}3000&=e^{\ln 3000}\\&\approx e^{8.0064}\end{aligned}$$

c $$\begin{aligned}0.075&=e^{\ln(0.075)}\\&\approx e^{-2.5903}\end{aligned}$$

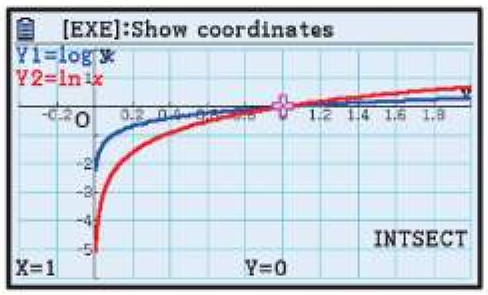
6 **a** $$\begin{aligned}&\ln 60-\ln 20\\&=\ln(\tfrac{60}{20})\\&=\ln 3\end{aligned}$$

b $$\begin{aligned}&\ln 4+\ln 1\\&=\ln 4+0\\&=\ln 4\end{aligned}$$

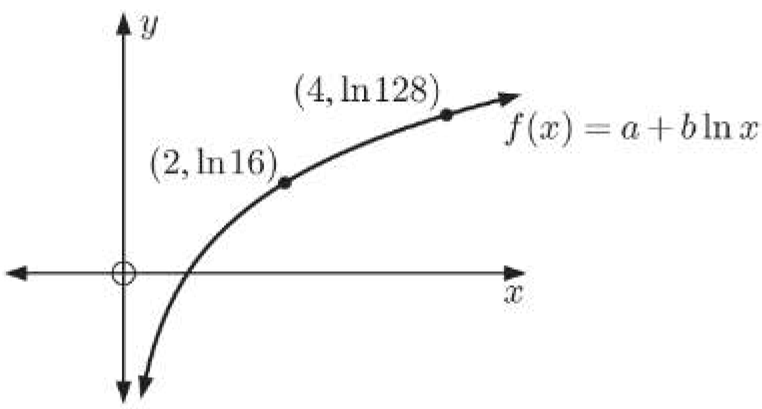
c $$\begin{aligned}&\ln 200-\ln 8+\ln 5\\&=\ln(\tfrac{200}{8})+\ln 5\\&=\ln(\tfrac{200}{8}\times 5)\\&=\ln 125\end{aligned}$$

7 Using technology, the graphs of $y=\log x$ and $y=\ln x$ intersect where $x=1$.

$\therefore$ $\log x=\ln x$ when $x=1$.



8 $$\begin{aligned}f(2)&=\ln 16\\ \therefore\ a+b\ln 2&=\ln 16\\ \therefore\ a+b\ln 2&=\ln(2^4)\\ \therefore\ a+b\ln 2&=4\ln 2\\ \therefore\ a&=4\ln 2-b\ln 2\\ \therefore\ a&=(4-b)\ln 2 \quad .... (*)\end{aligned}$$



$$\begin{aligned}f(4)&=\ln 128\\ \therefore\ a+b\ln 4&=\ln 128\\ \therefore\ a+b\ln(2^2)&=\ln(2^7)\\ \therefore\ a+2b\ln 2&=7\ln 2\\ \therefore\ (4-b)\ln 2+2b\ln 2&=7\ln 2 \quad \{\text{using } (*)\}\\ \therefore\ 4+b&=7\\ \therefore\ b&=3\end{aligned}$$

Substituting into $(*)$ gives $a=(4-3)\ln 2=\ln 2$.

9 **a** f is defined by $y = e^{5x-1}$

$\therefore$ f^{-1} is defined by $x = e^{5y-1}$

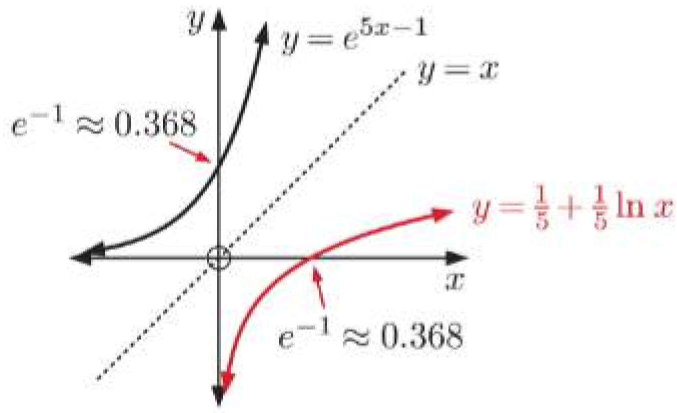
$\therefore$ $\ln x = 5y - 1$

$\therefore$ $\ln x + 1 = 5y$

$\therefore$ $\frac{1}{5}\ln x + \frac{1}{5} = y$

$\therefore$ $f^{-1}(x) = \frac{1}{5} + \frac{1}{5}\ln x$

b



10 $L = 10\log\left(\dfrac{I}{I_0}\right)$

a Let I_{50} be the intensity of a 50 dB sound and let I_{30} be the intensity of a 30 dB sound.

$\therefore$ $50 = 10\log\left(\dfrac{I_{50}}{I_0}\right)$ and $30 = 10\log\left(\dfrac{I_{30}}{I_0}\right)$

$\therefore$ $5 = \log\left(\dfrac{I_{50}}{I_0}\right)$ and $3 = \log\left(\dfrac{I_{30}}{I_0}\right)$

$\therefore$ $\dfrac{I_{50}}{I_0} = 10^5$ and $\dfrac{I_{30}}{I_0} = 10^3$

$\therefore$ $I_{50} = 10^5 I_0$ and $I_{30} = 10^3 I_0$

$\therefore$ a 50 dB sound is $10^{5-3} = 10^2 = 100$ times more intense than a 30 dB sound.

b **i** The sound intensity I_J of a jackhammer obeys $90 = 10\log\left(\dfrac{I_J}{I_0}\right)$

$\therefore$ $9 = \log\left(\dfrac{I_J}{I_0}\right)$

$\therefore$ $\dfrac{I_J}{I_0} = 10^9$

$\therefore$ $I_J = 10^9 I_0$

The SIL of the passing car $= 10\log\left(\dfrac{0.02 \times 10^9 I_0}{I_0}\right)$

$= 10\log(0.02 \times 10^9)$

$= 10(\log 0.02 + \log(10^9))$

$= 10(\log 0.02 + 9)$

≈ 73.0 dB

ii The jackhammer will be perceived to be about $2^{\frac{90-73.0}{10}} \approx 3.25$ times louder than the car.

11 $H = \dfrac{\ln 0.5}{\ln(1 - \frac{k}{100})}$ years

a **i** When $k = 5$, $H = \dfrac{\ln 0.5}{\ln(1 - \frac{5}{100})}$

≈ 13.5

The half-life of the substance is about 13.5 years.

ii When $k = 2.2$, $H = \dfrac{\ln 0.5}{\ln(1 - \frac{2.2}{100})}$

≈ 31.2

The half-life of the substance is about 31.2 years.

b When $k = 1.5$, $H = \dfrac{\ln 0.5}{\ln(1 - \frac{1.5}{100})}$
≈ 45.9

Substance A's half-life is about 45.9 years.

When $k = 0.7$, $H = \dfrac{\ln 0.5}{\ln(1 - \frac{0.7}{100})}$
≈ 98.7

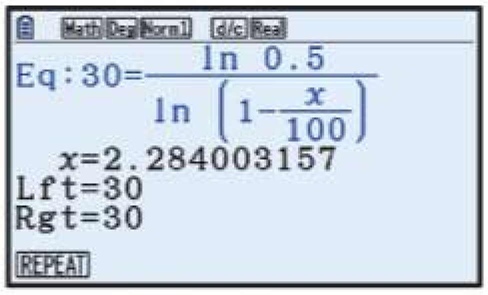
Substance B's half-life is about 98.7 years.

So, substance B's half-life is $98.7 - 45.9 \approx 52.8$ years longer than substance A's half-life.

c When $H = 30$, $30 = \dfrac{\ln 0.5}{\ln(1 - \frac{k}{100})}$.

Using technology, $k \approx 2.28$.

The substance decays by about 2.28% each year.



Chapter 3

APPROXIMATIONS AND ERROR

EXERCISE 3A

1 **a** The tape measure is accurate to $\pm\frac{1}{2}$ cm.

b The measuring cylinder is accurate to $\pm\frac{1}{2}$ mL.

c The beaker is accurate to $\pm\frac{1}{2} \times 100$ mL $= \pm 50$ mL.

d The set of scales is accurate to $\pm\frac{1}{2} \times 500$ g $= \pm 250$ g.

e The thermometer is accurate to $\pm\frac{1}{2} \times 0.1\,°\text{C} = \pm 0.05°\text{C}$.

2 **a** The range of values is $27 \pm \frac{1}{2}$ mm.
So, the actual value is in the range
26.5 mm to 27.5 mm.

b 38.3 cm $=$ 383 mm
The range of values is $383 \pm \frac{1}{2}$ mm.
So, the actual value is in the range
382.5 mm to 383.5 mm
or 38.25 cm to 38.35 cm.

c 4.8 m $=$ 480 cm
The range of values is $480 \pm \frac{1}{2} \times 10$ cm
$= 480 \pm 5$ cm.
So, the actual value is in the range
475 cm to 485 cm
or 4.75 m to 4.85 m.

d 1.5 kg $=$ 1500 g
The range of values is $1500 \pm \frac{1}{2} \times 100$ g
$= 1500 \pm 50$ g.
So, the actual value is in the range
1450 g to 1550 g
or 1.45 kg to 1.55 kg.

e The range of values is $25 \pm \frac{1}{2}$ g.
So, the actual value is in the range
24.5 g to 25.5 g.

f 3.75 kg $=$ 3750 g
The range of values is $3750 \pm \frac{1}{2} \times 10$ g
$= 3750 \pm 5$ g.
So, the actual value is in the range
3745 g to 3755 g
or 3.745 kg to 3.755 kg.

3 The thermometer reads 36.4°C, so it must be accurate to $\pm\frac{1}{2} \times 0.1°\text{C} = \pm 0.05\,°\text{C}$.
$\therefore$ the range of values is $36.4 \pm 0.05\,°\text{C}$.
Tom's actual temperature lies between 36.35°C and 36.45°C.
$\therefore$ $36.35°\text{C} < T < 36.45°\text{C}$

4 For distances less than 10 km, the exercise watch is accurate to $\pm\dfrac{0.01}{2}$ km $= \pm 0.005$ km.

For distances between 10 km and 100 km, the exercise watch is accurate to $\pm\dfrac{0.1}{2}$ km $= \pm 0.05$ km.

a The range of values is 1.06 ± 0.005 km, or 1.055 km to 1.065 km.
$\therefore$ if the watch displays 1.06 km, the least distance Joanne could have run is 1.055 km.

b The range of values is 9.72 ± 0.005 km, or 9.715 km to 9.725 km.
$\therefore$ if the watch displays 9.72 km, the least distance Joanne could have run is 9.715 km.

c The range of values is 10.1 ± 0.05 km, or 10.05 km to 10.15 km.

$\therefore$ if the watch displays 10.1 km, the least distance Joanne could have run is 10.05 km.

5 **a** The tape measure is accurate to ± 0.05 m

$\therefore$ for a measurement of 6.1 m, the range is 6.05 m to 6.15 m
for a measurement of 6.4 m, the range is 6.35 m to 6.45 m
for a measurement of 6.0 m, the range is 5.95 m to 6.05 m
for a measurement of 6.1 m, the range is 6.05 m to 6.15 m

$\therefore$ 6.4 m is likely to be the incorrect measurement, as its range of values is furthest from any of the other measurements.

b 6.05 m, as it is in the range of values for a measurement of 6.0 m or 6.1 m.

c 10 cm graduations, as all of the measurements were given to the nearest 0.1 m, or 10 cm.

6 **a** $2.4 \text{ m} = 240 \text{ cm}$

The range of values is $240 \pm \frac{1}{2} \times 10$ cm
$= 240 \pm 5$ cm.

So, the actual length of a rope is in the range 235 cm to 245 cm.

$\therefore \; 2.35 \text{ m} < l < 2.45 \text{ m}$

b If each rope is 2.35 m, then n of these ropes will have total length $2.35n$ m.
If each rope is 2.45 m, then n of these ropes will have total length $2.45n$ m.

$\therefore \; 2.35n \text{ m} < L < 2.45n \text{ m}$

7 The times are accurate to $\pm \frac{1}{2}$ s.

The range of Jiao's times is 2 min $8 \pm \frac{1}{2}$ s.

So, Jiao's actual time is in the range 2 min 7.5 s to 2 min 8.5 s.

The range of Liang's times is 2 min $13 \pm \frac{1}{2}$ s.

So, Liang's actual time is in the range 2 min 12.5 s to 2 min 13.5 s.

$\therefore$ the least time t by which Jiao beat Liang is $12.5 - 8.5 = 4$ s
and the greatest time t by which Jiao beat Liang is $13.5 - 7.5 = 6$ s

$\therefore \; 4 \text{ s} < t < 6 \text{ s}$

8 The length of the bath mat could be from $85\frac{1}{2}$ cm to $86\frac{1}{2}$ cm.

The width of the bath mat could be from $37\frac{1}{2}$ cm to $38\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the perimeter is $2 \times 85\frac{1}{2} + 2 \times 37\frac{1}{2} = 246$ cm
and the upper boundary of the perimeter is $2 \times 86\frac{1}{2} + 2 \times 38\frac{1}{2} = 250$ cm

The perimeter is between 246 cm and 250 cm, which is 248 ± 2 cm.

9 The length of the garden bed could be from $251\frac{1}{2}$ cm to $252\frac{1}{2}$ cm.

The width of the garden bed could be from $142\frac{1}{2}$ cm to $143\frac{1}{2}$ cm.

$\therefore$ the lower boundary for the length of edging l is $2 \times 251\frac{1}{2} + 2 \times 142\frac{1}{2} = 788$ cm
and the upper boundary for the length of edging l is $2 \times 252\frac{1}{2} + 2 \times 143\frac{1}{2} = 792$ cm

$\therefore \; 788 \text{ cm} < l < 792 \text{ cm}$

10 The length of the rectangle could be from 5.5 cm to 6.5 cm.
The width of the rectangle could be from 7.5 cm to 8.5 cm.

a Largest area $= 6.5 \times 8.5 \text{ cm}^2$
$= 55.25 \text{ cm}^2$

b Smallest area $= 5.5 \times 7.5 \text{ cm}^2$
$= 41.25 \text{ cm}^2$

11 The length of the glass window could be from $41\frac{1}{2}$ cm to $42\frac{1}{2}$ cm.

The width of the glass window could be from $25\frac{1}{2}$ cm to $26\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $41\frac{1}{2} \times 25\frac{1}{2} = 1058.25 \text{ cm}^2$

and the upper boundary of the area is $42\frac{1}{2} \times 26\frac{1}{2} = 1126.25 \text{ cm}^2$

The area is between 1058.25 cm^2 and 1126.25 cm^2, which is $1092.25 \pm 34 \text{ cm}^2$.

12 The base of the triangle could be from 8.5 cm to 9.5 cm.

The height of the triangle could be from 7.5 cm to 8.5 cm.

$\therefore$ the lower boundary of the area A is $\frac{1}{2} \times 8.5 \times 7.5 = 31.875 \text{ cm}^2$

and the upper boundary of the area A is $\frac{1}{2} \times 9.5 \times 8.5 = 40.375 \text{ cm}^2$

$\therefore \ 31.875 \text{ cm}^2 < A < 40.375 \text{ cm}^2$

13 The length of the box could be from 3.5 cm to 4.5 cm.
The width of the box could be from 7.5 cm to 8.5 cm.
The depth of the box could be from 5.5 cm to 6.5 cm.

$\therefore$ the lower boundary of the volume is $3.5 \times 7.5 \times 5.5 = 144.375 \text{ cm}^3$
and the upper boundary of the volume is $4.5 \times 8.5 \times 6.5 = 248.625 \text{ cm}^3$

The volume is between 144.375 cm^3 and 248.625 cm^3, which is $196.5 \pm 52.125 \text{ cm}^3$.

14 The length of the house brick could be from 21.25 cm to 21.35 cm.
The width of the house brick could be from 9.75 cm to 9.85 cm.
The depth of the house brick could be from 7.25 cm to 7.35 cm.

$\therefore$ the lower boundary of the volume is $21.25 \times 9.75 \times 7.25 \approx 1502.11 \text{ cm}^3$
and the upper boundary of the volume is $21.35 \times 9.85 \times 7.35 \approx 1545.69 \text{ cm}^3$

$\therefore \ 1502.11 \text{ cm}^3 < V < 1545.69 \text{ cm}^3$

15 The radius of the cylinder could be from 4.5 cm to 5.5 cm.
The height of the cylinder could be from 14.5 cm to 15.5 cm.

$\therefore$ the lower boundary of the volume is $\pi \times 4.5^2 \times 14.5 \approx 922.45 \text{ cm}^3$
and the upper boundary of the volume is $\pi \times 5.5^2 \times 15.5 \approx 1473.01 \text{ cm}^3$

The volume is between about 922.45 cm^3 and 1473.01 cm^3, which is about $1197.73 \pm 275.28 \text{ cm}^3$.

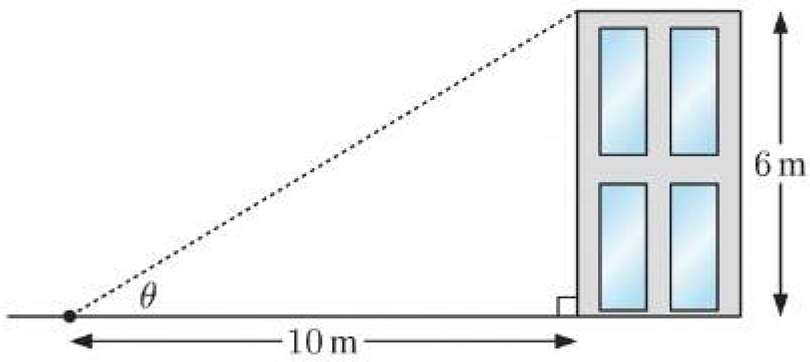
16 The radius of the cone could be from 8.35 cm to 8.45 cm.
The height of the cone could be from 4.55 cm to 4.65 cm.

$\therefore$ the lower boundary of the volume is $\frac{1}{3} \times \pi \times 8.35^2 \times 4.55 \approx 332.21 \text{ cm}^3$

and the upper boundary of the volume is $\frac{1}{3} \times \pi \times 8.45^2 \times 4.65 \approx 347.69 \text{ cm}^3$

The volume is between about 332.21 cm^3 and 347.69 cm^3, which is about $339.95 \pm 7.74 \text{ cm}^3$.

17 The height of the building could be from 5.5 m to 6.5 m.
The distance to the foot of the building could be from 9.5 m to 10.5 m.



$\therefore$ the lower boundary for θ is given by $\tan\theta = \dfrac{5.5}{10.5}$

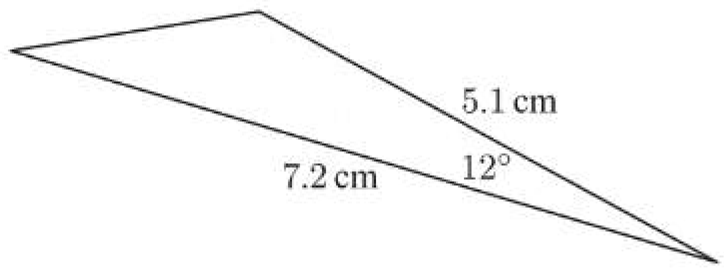
$$\therefore\ \theta = \tan^{-1}\left(\frac{5.5}{10.5}\right) \approx 27.6^\circ$$

and the upper boundary for θ is given by $\tan\theta = \dfrac{6.5}{9.5}$

$$\therefore\ \theta = \tan^{-1}\left(\frac{6.5}{9.5}\right) \approx 34.4^\circ$$

$\therefore\ 27.6^\circ < \theta < 34.4^\circ$

18 The shorter side could be from 5.05 cm to 5.15 cm.
The longer side could be from 7.15 cm to 7.25 cm.
The included angle could be from 11.5° to 12.5°.



a The lower boundary of the area is $\frac{1}{2} \times 5.05 \times 7.15 \times \sin 11.5^\circ \approx 3.60\ \text{cm}^2$
and the upper boundary of the area is $\frac{1}{2} \times 5.15 \times 7.25 \times \sin 12.5^\circ \approx 4.04\ \text{cm}^2$
$\therefore\ 3.60\ \text{cm}^2 < A < 4.04\ \text{cm}^2$

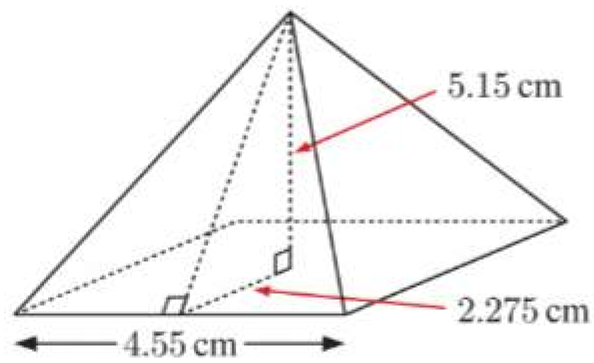
b The lower boundary of the perimeter is
$5.05 + 7.15 + \sqrt{5.05^2 + 7.15^2 - 2(5.05)(7.15)\cos 11.5^\circ} \approx 14.62$ cm
The upper boundary of the perimeter is
$5.15 + 7.25 + \sqrt{5.15^2 + 7.25^2 - 2(5.15)(7.25)\cos 12.5^\circ} \approx 14.89$ cm
$\therefore\ 14.62\ \text{cm} < P < 14.89\ \text{cm}$

19 Volume of a sphere $= \frac{4}{3}\pi r^3$, surface area of a sphere $= 4\pi r^2$
The rounding will have more effect on the volume, as the error is multiplied through 3 times rather than twice.

20 The base side length of the square-based pyramid could be from 4.55 cm to 4.65 cm.
The height of the square-based pyramid could be from 5.15 cm to 5.25 cm.

a The lower boundary of the volume is $\frac{1}{3} \times 4.55^2 \times 5.15 \approx 35.54\ \text{cm}^3$
and the upper boundary of the volume is $\frac{1}{3} \times 4.65^2 \times 5.25 \approx 37.84\ \text{cm}^3$
The volume is between about $35.54\ \text{cm}^3$ and $37.84\ \text{cm}^3$, which is about $36.69 \pm 1.15\ \text{cm}^3$.

b The lower boundary of the surface area is found when the base side length is 4.55 cm and the height is 5.15 cm.



The height of a triangular face

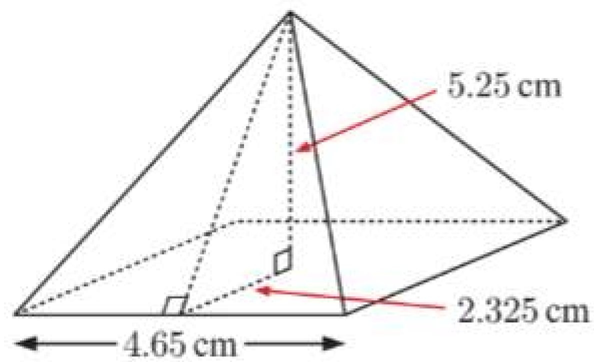
$= \sqrt{2.275^2 + 5.15^2}$ cm {Pythagoras}

The lower boundary of the surface area

$= 4 \times (\frac{1}{2} \times 4.55 \times \sqrt{2.275^2 + 5.15^2}) + 4.55^2$

$\approx 71.94 \text{ cm}^2$

The upper boundary of the surface area is found when the base side length is 4.65 cm and the height is 5.25 cm.



The height of a triangular face

$= \sqrt{2.325^2 + 5.25^2}$ cm {Pythagoras}

The upper boundary of the surface area

$= 4 \times (\frac{1}{2} \times 4.65 \times \sqrt{2.325^2 + 5.25^2}) + 4.65^2$

$\approx 75.02 \text{ cm}^2$

The surface area is between about 71.94 cm^2 and 75.02 cm^2, which is about $73.48 \pm 1.54 \text{ cm}^2$.

EXERCISE 3B

1 **a** Absolute error $= |V_A - V_E|$
$= €|1\,370\,000 - 1\,367\,540|$
$= €2460$

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{2460}{1\,367\,540} \times 100\%$
$\approx 0.180\%$

b Absolute error $= |V_A - V_E|$
$= |31\,000 - 31\,467|$ people
$= |-467|$ people
$= 467$ people

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{467}{31\,467} \times 100\%$
$\approx 1.48\%$

c Absolute error $= |V_A - V_E|$
$= \$|460\,000 - 458\,110|$
$= \$1890$

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{1890}{458\,110} \times 100\%$
$\approx 0.413\%$

d Absolute error $= |V_A - V_E|$
$= |3000 - 2811|$ cars
$= 189$ cars

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{189}{2811} \times 100\%$
$\approx 6.72\%$

2 **a** Absolute error $= |V_A - V_E|$
$= |5 - 6.238|$ kg
$= |-1.238|$ kg
$= 1.238$ kg

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{1.238}{6.238} \times 100\%$
$\approx 19.8\%$

b Absolute error $= |V_A - V_E|$
$= |100 - 97.6|$ m
$= 2.4$ m

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{2.4}{97.6} \times 100\%$
$\approx 2.46\%$

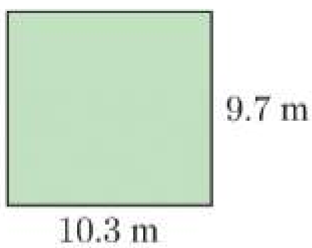
c Absolute error $= |V_A - V_E|$
$= |20 - 23.8|$ L
$= |-3.8|$ L
$= 3.8$ L

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{3.8}{23.8} \times 100\%$
$\approx 16.0\%$

d Absolute error $= |V_A - V_E|$
$= |50 - 72|$ hours
$= |-22|$ hours
$= 22$ hours

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{22}{72} \times 100\%$
$\approx 30.6\%$

3 **a** Actual area $= 10.3 \text{ m} \times 9.7 \text{ m}$
$= 99.91 \text{ m}^2$



b Area $\approx 10 \text{ m} \times 10 \text{ m}$
$\approx 100 \text{ m}^2$

c Absolute error $= |V_A - V_E|$
$= |100 - 99.91|$ m^2
$= 0.09$ m^2

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{0.09}{99.91} \times 100\%$
$\approx 0.0901\%$

4 **a** Actual volume $= 23.9 \text{ cm} \times 14.8 \text{ cm} \times 9.2 \text{ cm}$
$= 3254.224 \text{ cm}^3$

b Volume $\approx 24 \text{ cm} \times 15 \text{ cm} \times 9 \text{ cm}$
$\approx 3240 \text{ cm}^3$

c Absolute error $= |V_A - V_E|$
$= |3240 - 3254.224|$ cm^3
$= |-14.224|$ cm^3
$= 14.224$ cm^3

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{14.224}{3254.224} \times 100\%$
$\approx 0.437\%$

5 **a** **i** Abigail measured her height to be 173 cm, but it could be from 172.5 cm to 173.5 cm.

$\therefore$ maximum absolute error $= |V_A - V_E| = |173 - 172.5|$ cm $= 0.5$ cm

If her exact height V_E was 172.5 cm,

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|173 - 172.5|}{172.5} \times 100\% \approx 0.290\%$$

If her exact height V_E was 173.5 cm,

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|173 - 173.5|}{173.5} \times 100\% \approx 0.288\%$$

$\therefore$ the maximum percentage error in the estimate $\approx 0.290\%$.

ii Abigail measured the length of her finger to be 6 cm, but it could be from 5.5 cm to 6.5 cm.

$\therefore$ maximum absolute error $= |V_A - V_E| = |6 - 5.5|$ cm $= 0.5$ cm

If the exact length V_E was 5.5 cm,

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|6 - 5.5|}{5.5} \times 100\% \approx 9.09\%$$

If the exact length V_E was 6.5 cm,

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|6 - 6.5|}{6.5} \times 100\% \approx 7.69\%$$

b The estimate of Abigail's height is more accurate since the maximum percentage error is much lower.

6 **a** Area $\approx 8 \text{ m} \times 9 \text{ m} \approx 72 \text{ m}^2$

b Cost $= \$85 \times 72 = \$6120$

c Actual area $= 8.2 \text{ m} \times 9.4 \text{ m} = 77.08 \text{ m}^2$

d
$$\text{Percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\% = \frac{|72 - 77.08|}{77.08} \times 100\% = \frac{|-5.08|}{77.08} \times 100\% \approx 6.59\%$$

e Exact area $77.08 \text{ m}^2 >$ approximate area 72 m^2

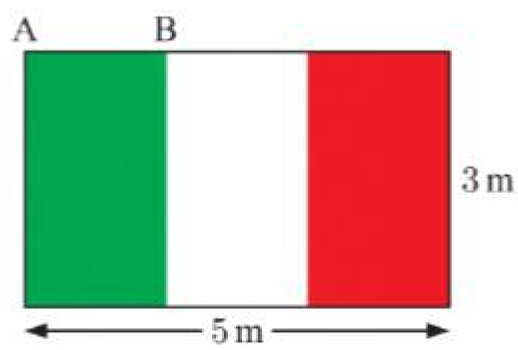
$\therefore$ there will not be enough grass to cover the courtyard.

f Area $\approx 9 \text{ m} \times 10 \text{ m} \approx 90 \text{ m}^2$

Cost $= \$85 \times 90 = \$7650$

7 **a** Area of flag $= 5 \text{ m} \times 3 \text{ m} = 15 \text{ m}^2$

$\therefore$ area of green section $= \frac{1}{3} \times 15 \text{ m}^2 = 5 \text{ m}^2$



b $AB = \frac{5}{3} \approx 1.666....$ cm ≈ 1.7 cm

c Area of green section $\approx 1.7 \text{ m} \times 3 \text{ m} \approx 5.1 \text{ m}^2$

d Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$$= \frac{|5.1 - 5|}{5} \times 100\%$$

$$= \frac{0.1}{5} \times 100\%$$

$$= 2\%$$

8 **a** Average speed $= \dfrac{\text{distance}}{\text{time}}$

$$= \frac{87 \text{ km}}{\frac{4}{3} \text{ hours}}$$

$$= 65.25 \text{ km h}^{-1}$$

b Absolute error $= |V_A - V_E|$

$$= |70 - 65.25| \text{ km h}^{-1}$$

$$= 4.75 \text{ km h}^{-1}$$

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$$= \frac{4.75}{65.25} \times 100\%$$

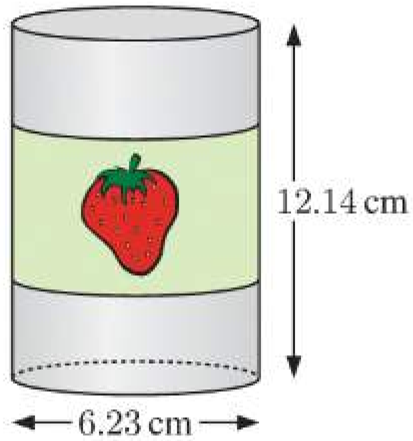
$$\approx 7.28\%$$

9 **a** Actual surface area

$$= 2\pi \times \frac{6.23}{2} \times 12.14 + 2 \times \pi \times \left(\frac{6.23}{2}\right)^2$$

$$= 95.038\,65\pi \text{ cm}^2$$

$$\approx 298.6 \text{ cm}^2$$



Estimated surface area

$$= 2\pi \times \frac{6.2}{2} \times 12.1 + 2 \times \pi \times \left(\frac{6.2}{2}\right)^2$$

$$= 94.24\pi \text{ cm}^2$$

$$\approx 296.1 \text{ cm}^2$$

$\therefore$ percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$$= \frac{|94.24\pi - 95.038\,65\pi|}{95.038\,65\pi} \times 100\%$$

$$\approx 0.840\%$$

b Actual volume $= \pi \times \left(\dfrac{6.23}{2}\right)^2 \times 12.14$

$$= 117.797\,151\,5\pi \text{ cm}^3$$

$$\approx 370.1 \text{ cm}^3$$

$\therefore$ actual capacity ≈ 370.1 mL

Estimated volume $= \pi \times \left(\dfrac{6.2}{2}\right)^2 \times 12.1$

$$= 116.281\pi \text{ cm}^3$$

$$\approx 365.3 \text{ cm}^3$$

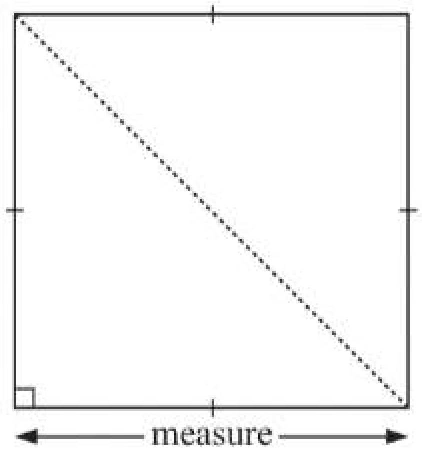
$\therefore$ estimated capacity ≈ 365.3 mL

$\therefore$ percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$$= \frac{|116.281\pi - 117.797\,151\,5\pi|}{117.797\,151\,5\pi} \times 100\%$$

$$\approx 1.29\%$$

10 **a** Let the square have side length x.



$$\begin{aligned}\text{Length of diagonal} &= \sqrt{x^2+x^2} \quad \{\text{Pythagoras}\}\\ &= \sqrt{2x^2}\\ &= \sqrt{2}\times\sqrt{x^2}\\ &= \sqrt{2}x \qquad \{\text{as } x>0\}\end{aligned}$$

$\therefore$ the diagonal of a square is $\sqrt{2}$ times the measure of its side.

b We let the measure be 1, so that the length of the diagonal is $\sqrt{2}$.

$$\therefore\ \sqrt{2} \approx 1+\underbrace{\frac{1}{3}}_{\substack{\text{measure}\\ \text{increased}\\ \text{by its third}}} + \underbrace{\frac{1}{3\times 4}}_{\substack{\text{increased}\\ \text{again by a}\\ \text{fourth of}\\ \text{this third}}} - \underbrace{\frac{1}{3\times 4\times 34}}_{\substack{\text{less the}\\ \text{thirtyfourth}\\ \text{part of that}\\ \text{fourth}}}$$

$$\begin{aligned}&\approx 1+\tfrac{1}{3}+\tfrac{1}{12}-\tfrac{1}{408}\\ &\approx \tfrac{408}{408}+\tfrac{136}{408}+\tfrac{34}{408}-\tfrac{1}{408}\\ &\approx \tfrac{577}{408}\end{aligned}$$

c
$$\begin{aligned}\text{Percentage error} &= \frac{|V_A-V_E|}{V_E}\times 100\%\\ &\approx \frac{\left|\frac{577}{408}-\sqrt{2}\right|}{\sqrt{2}}\times 100\%\\ &\approx 1.50\times 10^{-4}\,\%\end{aligned}$$

11 **a**
$$\begin{aligned}\text{Area} &\approx 2.3\text{ m}\times 1.4\text{ m}\\ &\approx 3.22\text{ m}^2\end{aligned}$$

b The length of the rectangle could be from 2.25 m to 2.35 m.
The width of the rectangle could be from 1.35 m to 1.45 m.

$\therefore$ the lower boundary of the area is $2.25\times 1.35 = 3.0375\text{ m}^2$
and the upper boundary of the area is $2.35\times 1.45 = 3.4075\text{ m}^2$

c If the exact area V_E was 3.0375 m^2, the

$$\begin{aligned}\text{percentage error} &= \frac{|V_A-V_E|}{V_E}\times 100\%\\ &= \frac{|3.22-3.0375|}{3.0375}\times 100\%\\ &\approx 6.01\%\end{aligned}$$

If the exact area V_E was 3.4075 m^2, the

$$\begin{aligned}\text{percentage error} &= \frac{|V_A-V_E|}{V_E}\times 100\%\\ &= \frac{|3.22-3.4075|}{3.4075}\times 100\%\\ &\approx 5.50\%\end{aligned}$$

$\therefore$ the maximum percentage error in the estimate is about 6.01%.

12 **a**
$$\begin{aligned}\text{Volume} &\approx (\pi\times 4^2)\text{ cm}^2\times 5\text{ cm}\\ &\approx 80\pi\text{ cm}^3\\ &\approx 251\text{ cm}^3\end{aligned}$$

b The radius of the can could be from 3.5 cm to 4.5 cm.
The height of the can could be from 4.5 cm to 5.5 cm.

$\therefore$ the lower boundary of the volume of the can is $\pi \times 3.5^2 \times 4.5 = 55.125\pi \text{ cm}^3 \approx 173 \text{ cm}^3$

and the upper boundary of the volume of the can is $\pi \times 4.5^2 \times 5.5 = 111.375\pi \text{ cm}^3 \approx 350 \text{ cm}^3$

$\therefore$ $173 \text{ cm}^3 < V < 350 \text{ cm}^3$

c If the exact volume V_E was $55.125\pi \text{ cm}^3$, the percentage error

$$= \frac{|V_A - V_E|}{V_E} \times 100\%$$
$$= \frac{|80\pi - 55.125\pi|}{55.125\pi} \times 100\%$$
$$\approx 45.1\%$$

If the exact volume V_E was $111.375\pi \text{ cm}^3$, the percentage error

$$= \frac{|V_A - V_E|}{V_E} \times 100\%$$
$$= \frac{|80\pi - 111.375\pi|}{111.375\pi} \times 100\%$$
$$\approx 28.2\%$$

$\therefore$ the maximum percentage error in the estimate is about 45.1%.

13 **a** Time taken $= \dfrac{\text{distance}}{\text{speed}}$

$$\approx \frac{250 \text{ km}}{56.8 \text{ km h}^{-1}}$$
$$\approx \frac{625}{142} \text{ hours}$$
$$\approx 4.40 \text{ hours} \quad (\approx 4 \text{ h } 24 \text{ min } 5 \text{ s})$$

b The length of the trip could be from 249.5 km to 250.5 km.
The average speed for the trip could be from 56.75 km h^{-1} to 56.85 km h^{-1}.

$\therefore$ the lower boundary of the time taken is $\dfrac{249.5}{56.85} = \dfrac{4990}{1137} \approx 4.39$ hours

and the upper boundary of the time taken is $\dfrac{250.5}{56.75} = \dfrac{1002}{227} \approx 4.41$ hours

If the exact time V_E was $\dfrac{4990}{1137}$ hours, the

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$
$$= \frac{\left|\frac{625}{142} - \frac{4990}{1137}\right|}{\frac{4990}{1137}} \times 100\%$$
$$\approx 0.289\%$$

If the exact time V_E was $\dfrac{1002}{227}$ hours, the

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$
$$= \frac{\left|\frac{625}{142} - \frac{1002}{227}\right|}{\frac{1002}{227}} \times 100\%$$
$$\approx 0.287\%$$

$\therefore$ the maximum percentage error in the estimate is about 0.289%.

REVIEW SET 3A

1 **a** The tape measure is accurate to $\pm\frac{1}{2}$ cm.

b The range of values is $36 \pm \frac{1}{2}$ cm.

So, the possible values are in the range 35.5 cm to 36.5 cm.

c If the side length is 35.5 cm, then the area A is $35.5 \times 35.5 = 1260.25 \text{ cm}^2$

If the side length is 36.5 cm, then the area A is $36.5 \times 36.5 = 1332.25 \text{ cm}^2$

$\therefore\ 1260.25 \text{ cm}^2 < A < 1332.25 \text{ cm}^2$

2 The length of 15 cm could be from 14.5 cm to 15.5 cm.

The width of 10 cm could be from 9.5 cm to 10.5 cm.

$\therefore$ the lower boundary of the perimeter is $2 \times 14.5 + 2 \times 9.5 = 48$ cm

and the upper boundary of the perimeter is $2 \times 15.5 + 2 \times 10.5 = 52$ cm

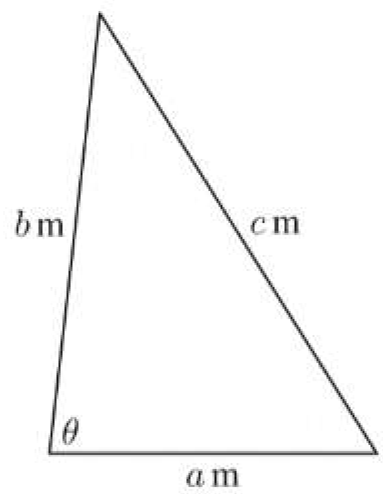
The perimeter is between 48 cm and 52 cm.

3 The first side length a m could be from 8.5 m to 9.5 m.

The second side length b m could be from 11.5 m to 12.5 m.

The third side length could be from 13.5 m to 14.5 m.

The smallest area occurs when $a = 8.5$, $b = 11.5$, and $c = 13.5$.



$$\therefore\ \cos\theta = \frac{8.5^2 + 11.5^2 - 13.5^2}{2(8.5)(11.5)} \approx 0.113\,81$$

$$\therefore\ \sin\theta = \sqrt{1 - \cos^2\theta} \approx 0.994$$

$$\therefore\ A = \tfrac{1}{2}ab\sin\theta \approx \tfrac{1}{2} \times 8.5 \times 11.5 \times 0.994 \approx 48.6 \text{ m}^2$$

The largest area occurs when $a = 9.5$, $b = 12.5$, and $c = 14.5$.

$$\therefore\ \cos\theta = \frac{9.5^2 + 12.5^2 - 14.5^2}{2(9.5)(12.5)} \approx 0.153$$

$$\therefore\ \sin\theta = \sqrt{1 - \cos^2\theta} \approx 0.988$$

$$\therefore\ A = \tfrac{1}{2}ab\sin\theta \approx \tfrac{1}{2} \times 9.5 \times 12.5 \times 0.988 \approx 58.7 \text{ m}^2$$

$\therefore\ 48.6 \text{ m}^2 < A < 58.7 \text{ m}^2$

4 **a** Absolute error $= |\,V_A - V_E\,|$
$= \$\,|\,2000 - 2590\,|$
$= \$\,|-590\,|$
$= \$590$

percentage error $= \dfrac{|\,V_A - V_E\,|}{V_E} \times 100\%$
$= \dfrac{590}{2590} \times 100\%$
$\approx 22.8\%$

b Absolute error $= |\,V_A - V_E\,|$
$= |\,26 - 26.109\,|$ cm
$= |-0.109\,|$ cm
$= 0.109$ cm

percentage error $= \dfrac{|\,V_A - V_E\,|}{V_E} \times 100\%$
$= \dfrac{0.109}{26.109} \times 100\%$
$\approx 0.417\%$

c Absolute error $= |\,V_A - V_E\,|$
$= |\,4000 - 4386\,|$ people
$= |-386\,|$ people
$= 386$ people

percentage error $= \dfrac{|\,V_A - V_E\,|}{V_E} \times 100\%$
$= \dfrac{386}{4386} \times 100\%$
$\approx 8.80\%$

5 $V_A = 5.29$ runs, $V_E = \frac{37}{7}$ runs

Percentage error $= \dfrac{|\,V_A - V_E\,|}{V_E} \times 100\%$

$= \dfrac{\left|5.29 - \frac{37}{7}\right|}{\frac{37}{7}} \times 100\%$

$\approx 0.0811\%$

6 **a** The radius of the gazebo is $\dfrac{2.8}{2} = 1.4$ m
$\therefore$ the exact area of the gazebo is $\pi \times 1.4^2 = 1.96\pi \approx 6.16\ \text{m}^2$

b If the diameter 2.8 m is rounded up to 3 m, then the radius is estimated to be $\dfrac{3}{2} = 1.5$ m.
$\therefore$ the approximate area of the gazebo is $\pi \times 1.5^2 = 2.25\pi \approx 7.07\ \text{m}^2$

c Absolute error $= |\,V_A - V_E\,|$
$= |\,2.25\pi - 1.96\pi\,|\ \text{m}^2$
$= 0.29\pi\ \text{m}^2$
$\approx 0.911\ \text{m}^2$

percentage error $= \dfrac{|\,V_A - V_E\,|}{V_E} \times 100\%$
$= \dfrac{0.29\pi}{1.96\pi} \times 100\%$
$\approx 14.8\%$

REVIEW SET 3B

1 **a** Dafne's time could be from 14.85 s to 14.95 s.
$\therefore$ 14.85 s $< t <$ 14.95 s

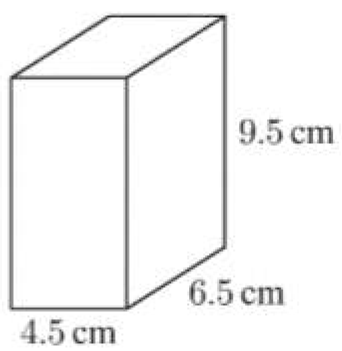
b Dafne's time could be from 14.85 s to 14.95 s.

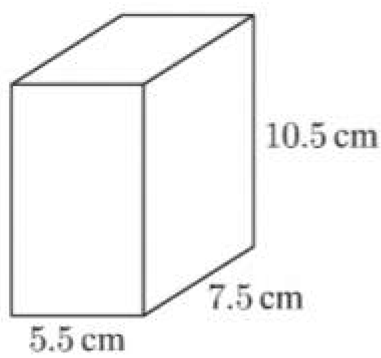
$\therefore$ the lower boundary of Dafne's speed s is $\dfrac{100}{14.95} \approx 6.69 \text{ m s}^{-1}$

and the upper boundary of Dafne's speed s is $\dfrac{100}{14.85} \approx 6.73 \text{ m s}^{-1}$

$\therefore$ $6.69 \text{ m s}^{-1} < s < 6.73 \text{ m s}^{-1}$

2 The length of the box could be from 4.5 cm to 5.5 cm.
The width of the box could be from 6.5 cm to 7.5 cm.
The height of the box could be from 9.5 cm to 10.5 cm.





lower boundary for the surface area
$= 2 \times (4.5 \times 6.5) + 2 \times (6.5 \times 9.5)$
$\quad + 2 \times (4.5 \times 9.5)$
$= 267.5 \text{ cm}^2$

upper boundary for the surface area
$= 2 \times (5.5 \times 7.5) + 2 \times (7.5 \times 10.5)$
$\quad + 2 \times (5.5 \times 10.5)$
$= 355.5 \text{ cm}^2$

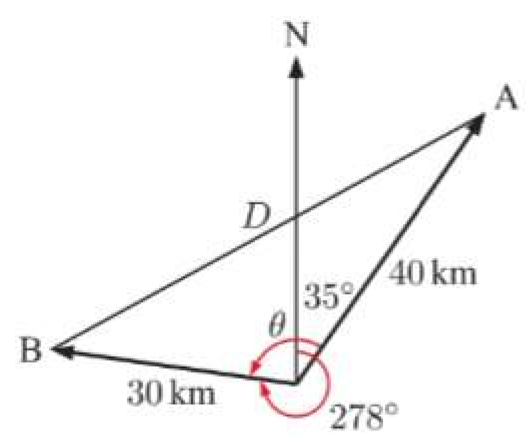
$\therefore$ $267.5 \text{ cm}^2 < A < 355.5 \text{ cm}^2$

3 After 2 hours:

- Boat A has travelled $20 \times 2 = 40$ km
- Boat B has travelled $15 \times 2 = 30$ km.

The bearing that Boat A travels in could be from $034.5°$ to $035.5°$.
The bearing that Boat B travels in could be from $277.5°$ to $278.5°$.

$\therefore$ the angle θ between A and B could be from
$34.5° + 360° - 278.5° = 116°$ to
$35.5° + 360° - 277.5° = 118°$.



$\therefore$ the lower boundary of D is $\sqrt{40^2 + 30^2 - 2(40)(30)\cos 116°} \approx 59.6$ km

and the upper boundary of D is $\sqrt{40^2 + 30^2 - 2(40)(30)\cos 118°} \approx 60.2$ km.

$\therefore$ $59.6 \text{ km} < D < 60.2 \text{ km}$

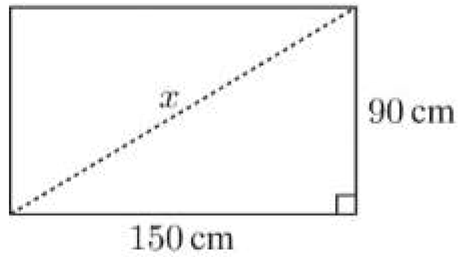
4 **a** $x^2 = 150^2 + 90^2$ {Pythagoras}

$\therefore\ x = \sqrt{150^2 + 90^2}$ {as $x > 0$}

$= \sqrt{30\,600}$

$= 30\sqrt{34}$

≈ 175



$\therefore$ the length of the diagonal of the screen is approximately 175 cm.

b Absolute error $= |V_A - V_E|$

$= \left|30\sqrt{34} - 177.8\right|$ cm

≈ 2.87 cm

percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{\left|30\sqrt{34} - 177.8\right|}{177.8} \times 100\%$

$\approx 1.61\%$

5 **a** **i** $\sqrt{5}$ m ≈ 2 m

ii $\sqrt{5}$ m ≈ 2.24 m ≈ 224 cm

iii $\sqrt{5}$ m ≈ 2.236 m ≈ 2236 mm

b Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

percentage error for nearest metre $= \dfrac{\left|2 - \sqrt{5}\right|}{\sqrt{5}} \times 100\%$

$\approx 10.6\%$

percentage error for nearest centimetre $= \dfrac{\left|2.24 - \sqrt{5}\right|}{\sqrt{5}} \times 100\%$

$\approx 0.176\%$

percentage error for nearest millimetre $= \dfrac{\left|2.236 - \sqrt{5}\right|}{\sqrt{5}} \times 100\%$

$\approx 0.003\,04\%$

The nearest centimetre length, 2.24 m, and the nearest millimetre length, 2.236 m, satisfy the architect's requirements.

6 **a** Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{|3 - \pi|}{\pi} \times 100\%$

$\approx 4.51\%$

b Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{|3.1 - \pi|}{\pi} \times 100\%$

$\approx 1.32\%$

c Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{|3.14 - \pi|}{\pi} \times 100\%$

$\approx 0.0507\%$

d Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{\left|\frac{22}{7} - \pi\right|}{\pi} \times 100\%$

$\approx 0.0402\%$

e Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{\left|\frac{355}{113} - \pi\right|}{\pi} \times 100\%$

$\approx 8.49 \times 10^{-6}\ \%$

7 **a** Area $\approx \pi \times 3.5^2 \approx 12.25\pi \approx 38.5\ \text{cm}^2$

b The radius of the circle could be from 3.45 cm to 3.55 cm.

$\therefore$ the lower bound of the area A is $\pi \times 3.45^2 = 11.9025\pi \approx 37.4\ \text{cm}^2$
and the upper bound of the area A is $\pi \times 3.55^2 = 12.6025\pi \approx 39.6\ \text{cm}^2$

$\therefore$ $37.4\ \text{cm}^2 < A < 39.6\ \text{cm}^2$

c If the exact area V_E was $11.9025\pi\ \text{cm}^2$, the percentage error

$= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{|12.25\pi - 11.9025\pi|}{11.9025\pi} \times 100\%$

$\approx 2.92\%$

If the exact time V_E was $12.6025\pi\ \text{cm}^2$, the percentage error

$= \dfrac{|V_A - V_E|}{V_E} \times 100\%$

$= \dfrac{|12.25\pi - 12.6025\pi|}{12.6025\pi} \times 100\%$

$\approx 2.80\%$

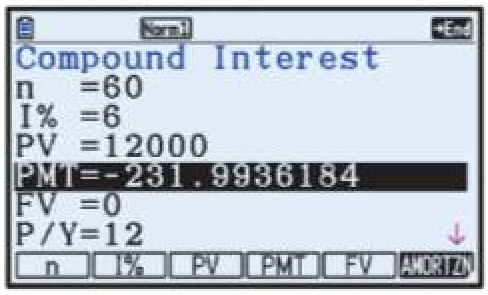
$\therefore$ the maximum percentage error in the estimate is about 2.92%.

Chapter 4

LOANS AND ANNUITIES

EXERCISE 4A

1 **a** $N = 5 \times 12 = 60,\quad I\% = 6,\quad PV = 12\,000,\quad FV = 0,\quad P/Y = 12,\quad C/Y = 12$



$\therefore\quad PMT \approx -232.00$

The monthly repayment is \$232.

b Total repayment = monthly repayment × number of months

$= \$232 \times 60$

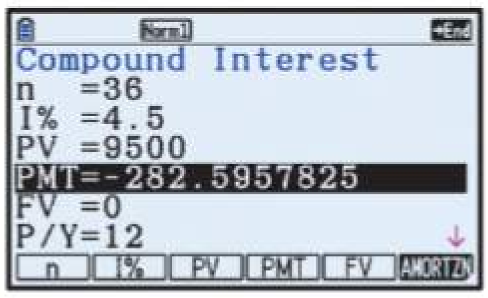
$= \$13\,920$

c Interest = total repayment − amount borrowed

$= \$13\,920 - \$12\,000$

$= \$1920$

2 **a** $N = 3 \times 12 = 36,\quad I\% = 4.5,\quad PV = 9500,\quad FV = 0,\quad P/Y = 12,\quad C/Y = 12$



$\therefore\quad PMT \approx -282.60$

The monthly repayment is £282.60 .

b Total repayment = monthly repayment × number of months

$= £282.60 \times 36$

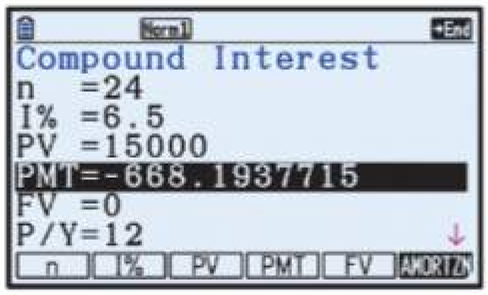
$= £10\,173.60$

c Interest = total repayment − amount borrowed

$= £10\,173.60 - £9500$

$= £673.60$

3 **a** $N = 2 \times 12 = 24, \quad I\% = 6.5, \quad PV = 15\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -668.20$

The monthly repayment is \$668.20.

Total repayment = monthly repayment × number of months

$= \$668.20 \times 24$

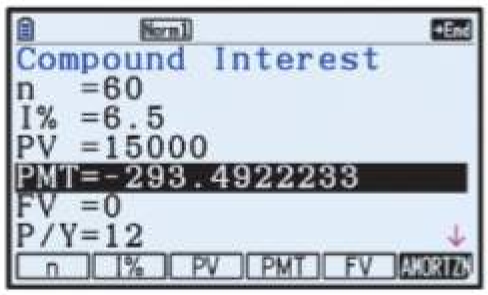
$= \$16\,036.80$

Interest = total repayment − amount borrowed

$= \$16\,036.80 - \$15\,000$

$= \$1036.80$

b $N = 5 \times 12 = 60, \quad I\% = 6.5, \quad PV = 15\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -293.50$

The monthly repayment is \$293.50.

Total repayment = monthly repayment × number of months

$= \$293.50 \times 60$

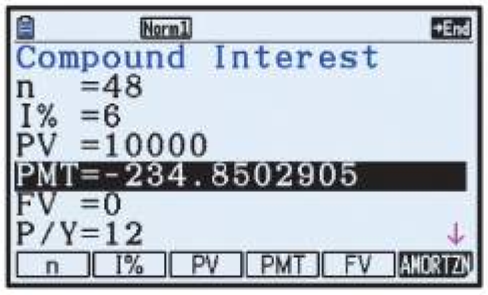
$= \$17\,610$

Interest = total repayment − amount borrowed

$= \$17\,610 - \$15\,000$

$= \$2610$

4 $N = 4 \times 12 = 48, \quad I\% = 6, \quad PV = 10\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -234.86$

The monthly repayment is €234.86 .

Total repayment = monthly repayment × number of months

$= €234.86 \times 48$

$= €11\,273.28$

Interest = total repayment − amount borrowed

$= €11\,273.28 - €10\,000$

$= €1273.28$

Total cost of taking out the loan

= interest + application fee + monthly service fee × number of months

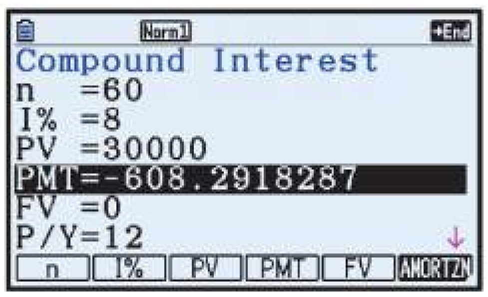
$= €1273.28 + €150 + €10 \times 48$

$= €1903.28$

5

	A	B	C	D	E
1	LOAN SPREADSHEET				
2					
3	Loan amount	$30,000.00			
4	Number of years	5			
5	Rate p.a.	8.00%			
6	Periods p.a.	12			
7	Rate per period	0.667%			
8	Repayment	$608.30			
9					
10	Month	Amount	Interest	Repayment	Balance
11	1	$30,000.00	$200.00	$608.30	$29,591.70
12	2	$29,591.70	$197.28	$608.30	$29,180.68
13	3	$29,180.68	$194.54	$608.30	$28,766.92
14	4	$28,766.92	$191.78	$608.30	$28,350.40
15	5	$28,350.40	$189.00	$608.30	$27,931.10
16	6	$27,931.10	$186.21	$608.30	$27,509.01

a $N = 5 \times 12 = 60, \quad I\% = 8, \quad PV = 30\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -608.30$

The monthly repayment is $608.30 which is the same as that given in the spreadsheet.

b From the spreadsheet, the account balance after six months is $27 509.01 .

c **i** From the spreadsheet, \$200 is paid in interest in month 1.

ii From the spreadsheet, \$186.21 is paid in interest in month 6.

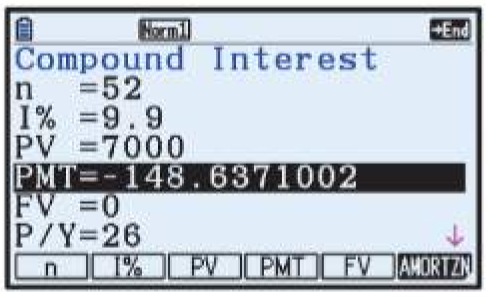
The balance of the loan is less in month 6 which means the interest paid will also be less.

d **i** Using the spreadsheet given, \$4.02 is paid in interest in the 60th month.

ii Using the spreadsheet given, \$6497.40 is paid in interest in total (or \$6498 using technology).

e The monthly repayment was rounded up so every month the payments have reduced the balance by a little extra which means that the final payment is slightly less.

6 **a** $N = 2 \times 26 = 52, \quad I\% = 9.9, \quad PV = 7000, \quad FV = 0, \quad P/Y = 26, \quad C/Y = 26$



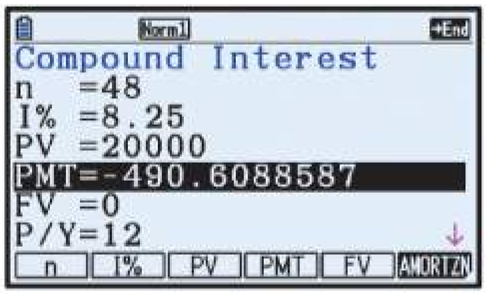
$\therefore \quad PMT \approx -148.64$

The fortnightly repayment is \$148.64 .

b Total interest = total repayment − starting principal

$$\begin{aligned} &= \$148.64 \times 52 - \$7000 \\ &= \$7729.28 - \$7000 \\ &= \$729.28 \end{aligned}$$

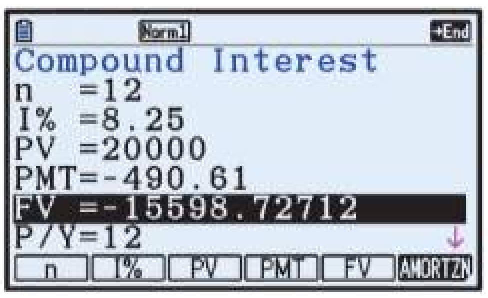
7 **a** $N = 4 \times 12 = 48, \quad I\% = 8.25, \quad PV = 20\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -490.61$

The monthly repayment is £490.61 .

b $N = 1 \times 12 = 12, \quad I\% = 8.25, \quad PV = 20\,000, \quad PMT = -490.61, \quad P/Y = 12, \quad C/Y = 12$

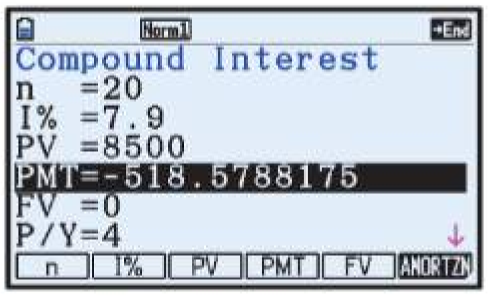


$\therefore \quad FV \approx -15\,598.73$

The outstanding balance on the loan after 1 year is £15 598.73 .

8 **a** Jacob will borrow $\$12\,000 - \$3500 = \$8500$.

b **i** $N = 5 \times 4 = 20, \quad I\% = 7.9, \quad PV = 8500, \quad FV = 0, \quad P/Y = 4, \quad C/Y = 4$



$\therefore \quad PMT \approx -518.58$

The quarterly repayment is $\$518.58$.

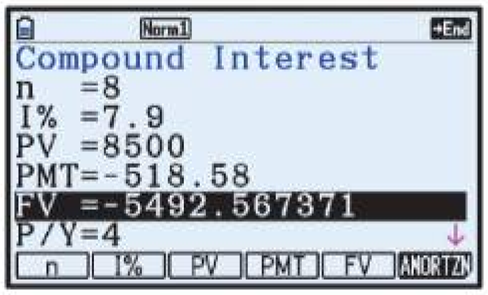
ii Total interest = total repayment − starting principal

$= \$518.58 \times 20 - \$8500$

$= \$10\,371.60 - \$8500$

$= \$1871.60$

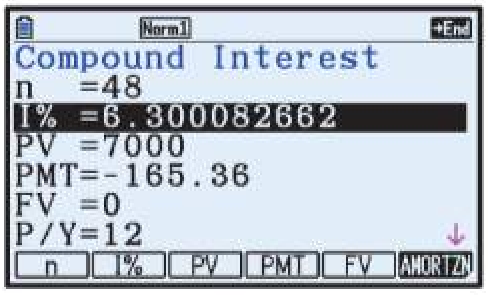
iii $N = 2 \times 4 = 8, \quad I\% = 7.9, \quad PV = 8500, \quad PMT = -518.58, \quad P/Y = 4, \quad C/Y = 4$



$\therefore \quad FV \approx -5492.57$

The outstanding balance on the loan after 2 years is $\$5492.57$.

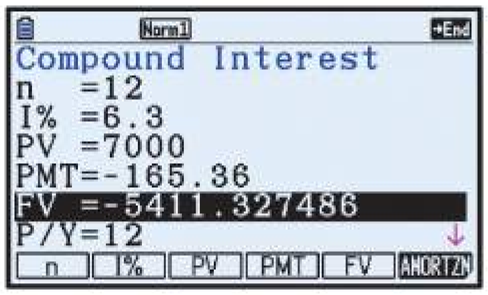
9 **a** $N = 4 \times 12 = 48, \quad PV = 7000, \quad FV = 0, \quad PMT = -165.36, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad I\% \approx 6.30$

So, the annual interest rate is 6.30% p.a.

b $N = 1 \times 12 = 12,\ \ I\% = 6.30,\ \ PV = 7000,\ \ PMT = -165.36,\ \ P/Y = 12,\ \ C/Y = 12$



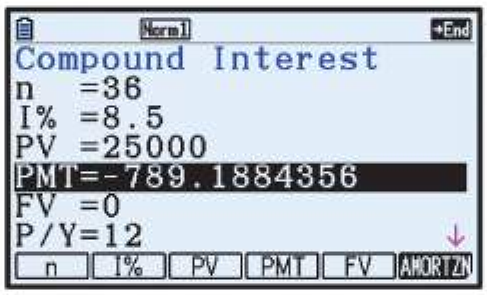
$\therefore\ \ FV \approx -5411.33$

So, the outstanding balance on the loan after 1 year is \$5411.33 .

$\therefore$ Simon has paid off \$7000 − \$5411.33 = \$1588.67 of the loan after 1 year.

Total interest paid in first year = total repayment − amount paid off in first year
= \$165.36 × 12 − \$1588.67
= \$1984.32 − \$1588.67
= \$395.65

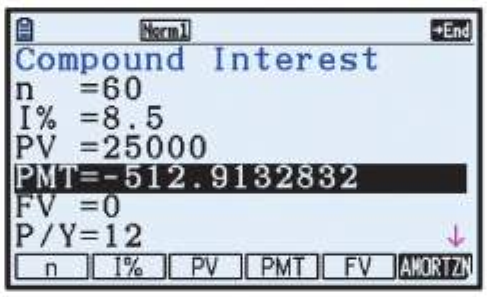
10 **a** **i** $N = 3 \times 12 = 36,\ \ I\% = 8.5,\ \ PV = 25\,000,\ \ FV = 0,\ \ P/Y = 12,\ \ C/Y = 12$



$\therefore\ \ PMT \approx -789.19$

So, the monthly repayment is \$789.19 .

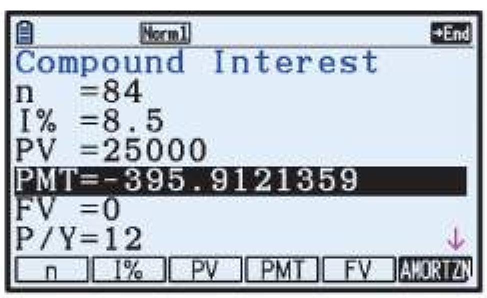
ii $N = 5 \times 12 = 60,\ \ I\% = 8.5,\ \ PV = 25\,000,\ \ FV = 0,\ \ P/Y = 12,\ \ C/Y = 12$



$\therefore\ \ PMT \approx -512.92$

So, the monthly repayment is \$512.92 .

iii $N = 7 \times 12 = 84,\ \ I\% = 8.5,\ \ PV = 25\,000,\ \ FV = 0,\ \ P/Y = 12,\ \ C/Y = 12$



$\therefore\ \ PMT \approx -395.92$

So, the monthly repayment is \$395.92 .

b Total interest charged for 3 year loan = total repayment − starting principal

$$\begin{aligned} &= \$789.19 \times 36 - \$25\,000 \\ &= \$28\,410.84 - \$25\,000 \\ &= \$3410.84 \end{aligned}$$

Total interest charged for 5 year loan = total repayment − starting principal

$$\begin{aligned} &= \$512.92 \times 60 - \$25\,000 \\ &= \$30\,775.20 - \$25\,000 \\ &= \$5775.20 \end{aligned}$$

Total interest charged for 7 year loan = total repayment − starting principal

$$\begin{aligned} &= \$395.92 \times 84 - \$25\,000 \\ &= \$33\,257.28 - \$25\,000 \\ &= \$8257.28 \end{aligned}$$

The 3 year loan charges the least interest of \$3410.84 as more is paid off the balance each month and therefore less interest is charged overall.

11 **a** $N = 5 \times 12 = 60$, $I\% = 10.5$, $PV = 18\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

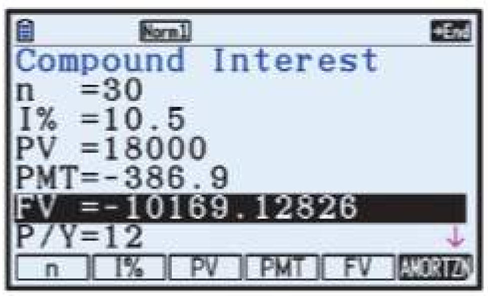
$\therefore$ $PMT \approx -386.90$

So, the monthly repayment is €386.90.

b Total interest = total repayment − starting principal

$$\begin{aligned} &= €386.90 \times 60 - €18\,000 \\ &= €23\,214 - €18\,000 \\ &= €5214 \end{aligned}$$

c $N = 2.5 \times 12 = 30$, $I\% = 10.5$, $PV = 18\,000$, $PMT = -386.90$, $P/Y = 12$, $C/Y = 12$

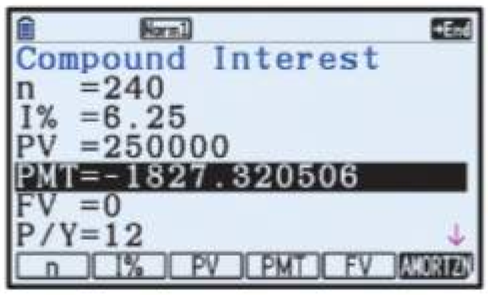


$\therefore$ $FV \approx -10\,169.13$

So, the outstanding balance on the loan after $2\frac{1}{2}$ years is €10 169.13.

d Ally pays more interest in the first $2\frac{1}{2}$ years than in the second $2\frac{1}{2}$ years, so she still has more than half the loan to pay off when half the loan period has passed.

12 **a** $N = 20 \times 12 = 240, \quad I\% = 6.25, \quad PV = 250\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



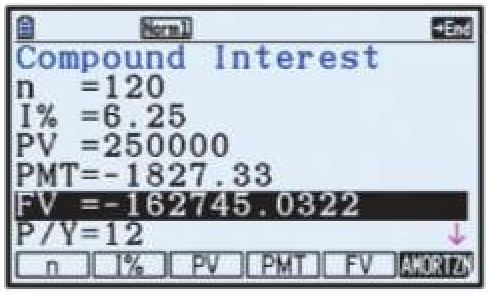
$\therefore \quad PMT \approx -1827.33$

So, the monthly repayment is \$1827.33.

b Total interest = total repayment − principal

$$\begin{aligned} &= \$1827.33 \times 240 - \$250\,000 \\ &= \$438\,559.20 - \$250\,000 \\ &= \$188\,559.20 \end{aligned}$$

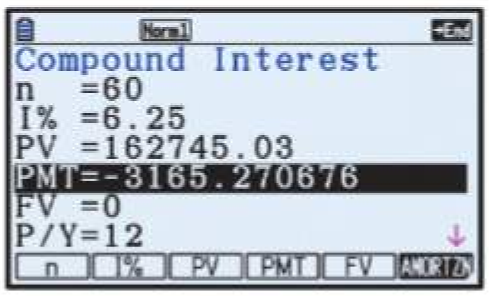
c $N = 10 \times 12 = 120, \quad I\% = 6.25, \quad PV = 250\,000, \quad PMT = -1827.33, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad FV \approx -162\,745.03$

The outstanding balance on the loan after 10 years is \$162 745.03.

d **i** $N = 5 \times 12 = 60, \quad I\% = 6.25, \quad PV = 162\,745.03, \quad FV = 0, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad PMT \approx -3165.28$

The monthly repayment for the next 5 years is \$3165.28.

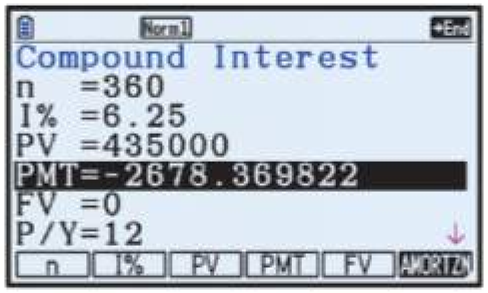
ii Total interest

= (total repayment in first 10 years − principal paid in first 10 years)
+ (total repayment in last 5 years − principal paid in last 5 years)

$$\begin{aligned} &= (\$1827.33 \times 120 - (\$250\,000 - \$162\,745.03)) + (\$3165.28 \times 60 - \$162\,745.03) \\ &= \$132\,024.63 + \$27\,171.77 \\ &= \$159\,196.40 \end{aligned}$$

iii Total interest saved $= \$188\,559.20 - \$159\,196.40$
$= \$29\,362.80$

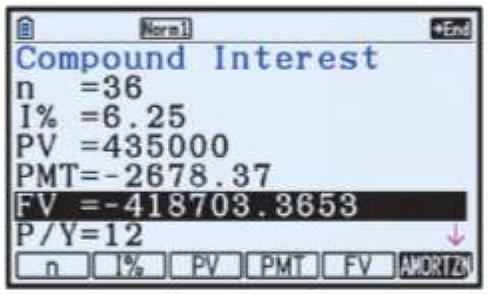
13 **a** $N = 30 \times 12 = 360, \quad I\% = 6.25, \quad PV = 435\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -2678.37$

So, the monthly repayment is $2678.37.

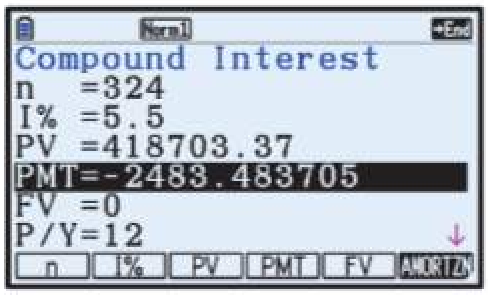
b $N = 3 \times 12 = 36, \quad I\% = 6.25, \quad PV = 435\,000, \quad PMT = -2678.37, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad FV \approx -418\,703.37$

So, the outstanding debt after 3 years is $418 703.37.

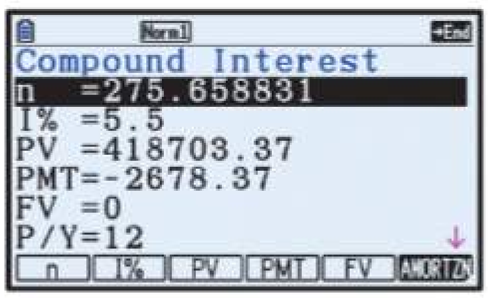
c **i** $N = 27 \times 12 = 324, \quad I\% = 5.5, \quad PV = 418\,703.37, \quad FV = 0, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad PMT \approx -2483.49$

So, the new monthly repayment is $2483.49.

ii $I\% = 5.5, \quad PV = 418\,703.37, \quad FV = 0, \quad PMT = -2678.37, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad N \approx 275.7$

So, it will now take 276 more months for the loan to be repaid.
The loan would be paid off $324 - 276 = 48$ months, or 4 years, earlier.
We have assumed that the new interest rate is fixed for the remainder of the loan period, and that the couple can continue to pay the higher repayments.

ACTIVITY 1

At the time of writing, the interest rates $\approx 2.5\%$.

1

Monthly repayments

	4%	6%	8%	10%	12%	2.5%
5 years	\$7366.61	\$7733.13	\$8110.56	\$8498.82	\$8897.78	\$7098.95
10 years	\$4049.81	\$4440.83	\$4853.11	\$5286.03	\$5738.84	\$3770.80
15 years	\$2958.76	\$3375.43	\$3822.61	\$4298.43	\$4800.68	\$2667.16
20 years	\$2423.93	\$2865.73	\$3345.77	\$3860.09	\$4404.35	\$2119.62

Total interest paid

	4%	6%	8%	10%	12%	2.5%
5 years	\$41 966.60	\$63 987.80	\$86 633.60	\$109 929.20	\$133 866.80	\$25 937.00
10 years	\$85 977.20	\$132 899.60	\$182 373.20	\$234 323.60	\$288 660.80	\$52 496.00
15 years	\$132 576.80	\$207 577.40	\$288 069.80	\$373 717.40	\$464 122.40	\$80 088.80
20 years	\$181 743.20	\$287 775.20	\$402 984.80	\$526 421.60	\$657 044.00	\$108 708.80

2 A higher interest rate will increase the total amount of interest paid over the course of a loan. Paying off a loan over a shorter time period will decrease the total interest paid, sometimes even when the interest rate is higher. For example, in the table above we can see that less interest is paid at 10% over 5 years than at 6% over 10 years. However, we have to consider whether or not a family can meet their regular payments since higher interest rates and shorter loan periods will increase the necessary regular repayment.

ACTIVITY 2 THE LOAN REPAYMENTS FORMULA

1 **a** The balance after 2 repayment periods $= PV(1+i)^2 - p(1+i) - p$

The balance after 3 repayment periods $= [PV(1+i)^2 - p(1+i) - p](1+i) - p$

$$= PV(1+i)^3 - p(1+i)^2 - p(1+i) - p$$

b The balance after 4 repayment periods $= [PV(1+i)^3 - p(1+i)^2 - p(1+i) - p](1+i) - p$

$$= PV(1+i)^4 - p(1+i)^3 - p(1+i)^2 - p(1+i) - p$$

2 Following the pattern found in **1**, the balance of the loan after n payment periods

$$= PV(1+i)^n - p(1+i)^{n-1} - p(1+i)^{n-2} - \ldots. - p(1+i) - p$$

$$= PV(1+i)^n - [p + p(1+i) + \ldots. + p(1+i)^{n-2} + p(1+i)^{n-1}]$$

Now $p + p(1+i) + \ldots. + p(1+i)^{n-2} + p(1+i)^{n-1}$ is a finite geometric series with $u_1 = p$, $r = 1 + i$.

$$\therefore \text{ the balance of the loan after } n \text{ payment periods} = PV(1+i)^n - \sum_{k=0}^{n-1} p(1+i)^k$$
$$= PV(1+i)^n - \frac{p[(1+i)^n - 1]}{1+i-1}$$
$$= PV(1+i)^n - p\,\frac{(1+i)^n - 1}{i}$$

3 If the loan is completely repaid after n payments, then

the balance of the loan after n payment periods $= 0$

$$\therefore\ PV(1+i)^n - p\,\frac{(1+i)^n - 1}{i} = 0$$
$$\therefore\ p\,\frac{(1+i)^n - 1}{i} = PV(1+i)^n$$
$$\therefore\ p = \frac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$$

4 $PV = 16\,500, \quad i = \dfrac{0.055}{12}, \quad n = 4 \times 12 = 48$

$$\therefore\ p = \frac{16\,500 \times \frac{0.055}{12} \times \left(1 + \frac{0.055}{12}\right)^{48}}{\left(1 + \frac{0.055}{12}\right)^{48} - 1}$$
$$\approx 383.74 \quad \{\text{rounded up}\}$$

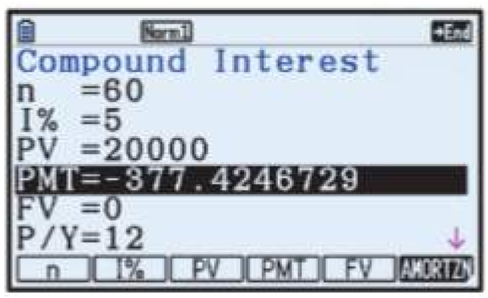
This agrees with our answer to **Example 1**.

6 $PV = 20\,000, \quad i = \dfrac{0.05}{12}, \quad n = 5 \times 12 = 60$

$$\therefore\ p = \frac{20\,000 \times \frac{0.05}{12} \times \left(1 + \frac{0.05}{12}\right)^{60}}{\left(1 + \frac{0.05}{12}\right)^{60} - 1}$$
$$\approx 377.43 \quad \{\text{rounded up}\}$$

Using technology,

$N = 5 \times 12 = 60, \quad I\% = 5, \quad PV = 20\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore\ PMT \approx -377.43$

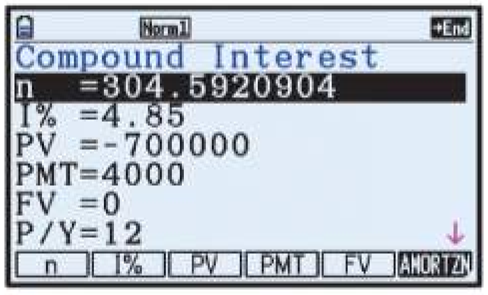
The monthly repayment is \$377.43.

8 When we know the formula and can implement it in a spreadsheet, we have greater flexibility in our calculations, and we can perform calculations very quickly.

EXERCISE 4B

1 **a** We need to find how long it will take for the future value to fall to \$0.

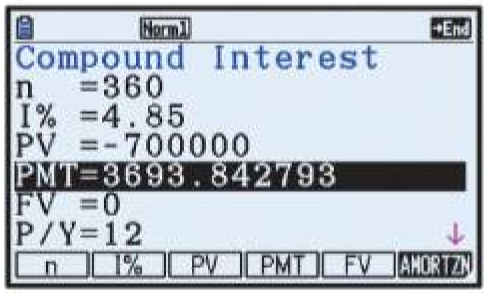
$I\% = 4.85, \quad PV = -700\,000, \quad PMT = 4000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad N \approx 305$

Sue will be able to withdraw \$4000 per month for 304 months, and then less in the 305th month (after 25 years 5 months).

b $N = 30 \times 12 = 360, \quad I\% = 4.85, \quad PV = -700\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$

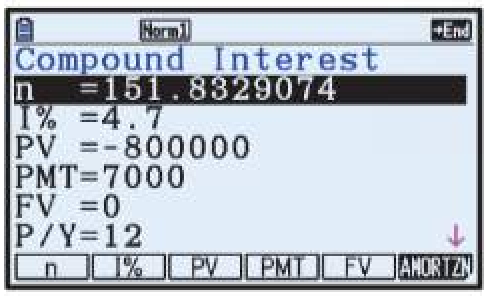


$\therefore \quad PMT \approx 3693.84$

Sue can afford to withdraw \$3693.84 each month.

2 **a** We need to find how long it will take for the future value to fall to \$0.

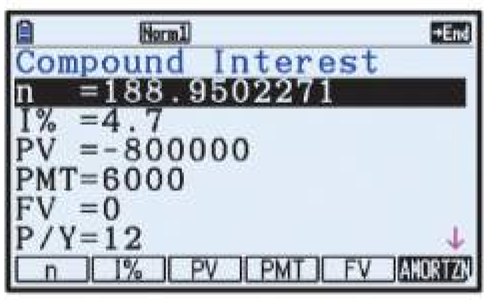
$I\% = 4.7, \quad PV = -800\,000, \quad PMT = 7000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad N \approx 152$

Terence will be able to withdraw \$7000 per month for 151 months, and then less in the 152nd month (after 12 years 8 months).

b $I\% = 4.7, \quad PV = -800\,000, \quad PMT = 6000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$

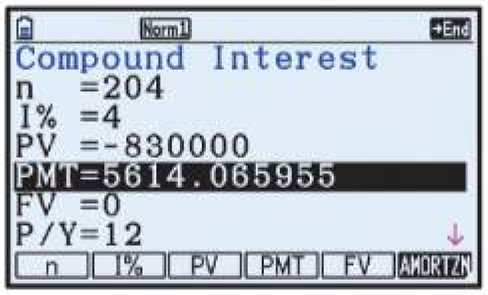


$\therefore \quad N \approx 189 \quad$ {we round up since the money will last until the 189th month}

Terence's money will last $189 - 152 = 37$ months (or 3 years 1 month) longer if he only withdraws \$6000 each month.

3 **a** Henry wants the money to last $85 - 68 = 17$ years.

$N = 17 \times 12 = 204$, $I\% = 4$, $PV = -830\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



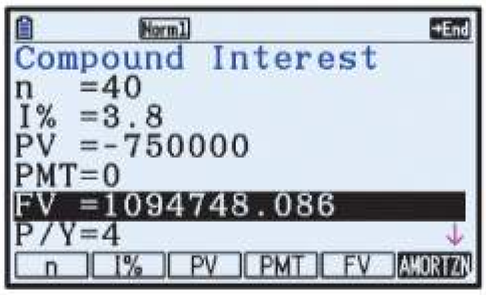
$\therefore \quad PMT \approx 5614.06$

Henry can afford to withdraw £5614.06 each month.

b No, he can only afford to spend £5614.06 per month if he wants the money to last until he is 85, which is not enough to maintain his current lifestyle.

If he spends £6000 per month then his money will run out before he turns 84.

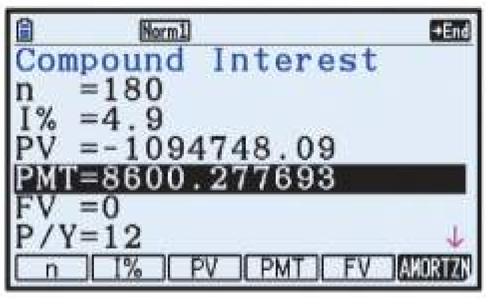
4 **a** $N = 10 \times 4 = 40$, $I\% = 3.8$, $PV = -750\,000$, $PMT = 0$, $P/Y = 4$, $C/Y = 4$



$\therefore \quad FV \approx 1\,094\,748.09$

There will be \$1 094 748.09 in the account after 10 years.

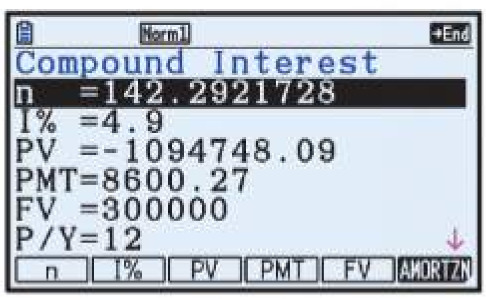
b **i** $N = 15 \times 12 = 180$, $I\% = 4.9$, $PV = -1\,094\,748.09$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx 8600.27$

Tamsyn can afford to withdraw \$8600.27 each month.

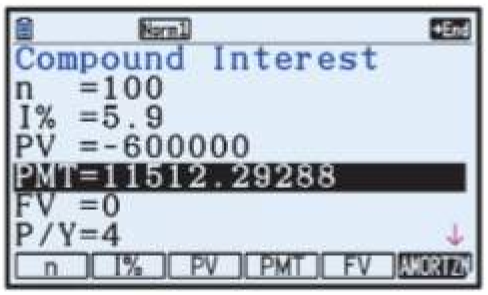
ii $I\% = 4.9$, $PV = -1\,094\,748.09$, $PMT = 8600.27$, $FV = 300\,000$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad N \approx 143$ {we round up since after 142 months there will be more than \$300 000 in the account}

It will take 143 months (or 11 years 11 months) for the balance of the fund to fall below \$300 000.

5 **a** $N = 25 \times 4 = 100,\quad I\% = 5.9,\quad PV = -600\,000,\quad FV = 0,\quad P/Y = 4,\quad C/Y = 4$

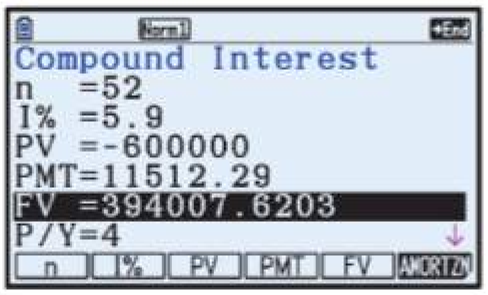


$\therefore\quad PMT \approx 11\,512.29$

Danny can afford to withdraw £11 512.29 each quarter.

b Danny is 55, so it will be 13 years until he turns 68.

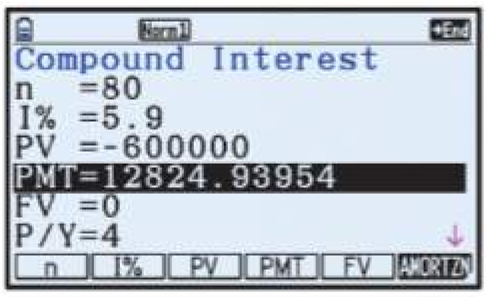
$N = 13 \times 4 = 52,\quad I\% = 5.9,\quad PV = -600\,000,\quad PMT = 11\,512.29,\quad P/Y = 4,$
$C/Y = 4$



$\therefore\quad FV \approx 394\,007.62$

There will be £394 007.62 in the account when Danny is 68.

c $N = 20 \times 4 = 80,\quad I\% = 5.9,\quad PV = -600\,000,\quad FV = 0,\quad P/Y = 4,\quad C/Y = 4$



$\therefore\quad PMT \approx 12\,824.93$

If Danny's money only needed to last 20 years, he could withdraw
£12 824.93 − £11 512.29 = £1312.64 more each quarter.

6 **a** Maggie needs the money to last $90 - 70 = 20$ years.
20 years $= 20 \times 12 = 240$ months.
If Maggie withdraws \$4500 each month, she will withdraw a total of
$\$4500 \times 240 = \$1\,080\,000$ from her annuity account.

b Maggie will earn interest on the money in the annuity account as she makes her regular withdrawals.

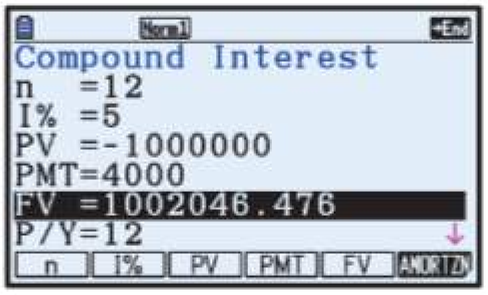
c $N = 20 \times 12 = 240, \quad I\% = 6.2, \quad PMT = 4500, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$

$\therefore \quad PV \approx -618\,117.53$

Maggie will need \$618 117.53 in savings when she retires.

7 We find the balance of the fund after 1 year.

$N = 1 \times 12 = 12, \quad I\% = 5, \quad PV = -1\,000\,000, \quad PMT = 4000, \quad P/Y = 12, \quad C/Y = 12$



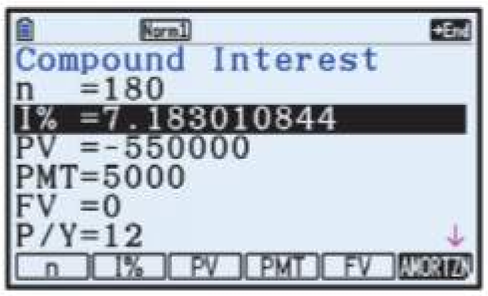
$\therefore \quad FV \approx 1\,002\,046.48$

So, after 1 year, the amount in the fund has increased.

$\therefore$ the amount in the fund will continue to increase.

$\therefore$ the money will last forever.

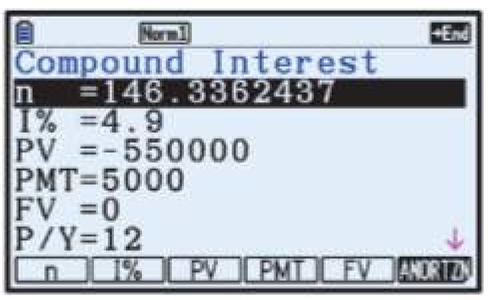
8 **a** $N = 15 \times 12 = 180, \quad PV = -550\,000, \quad PMT = 5000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad I\% \approx 7.19$

Igor will require an interest rate of 7.19% p.a. compounded monthly.

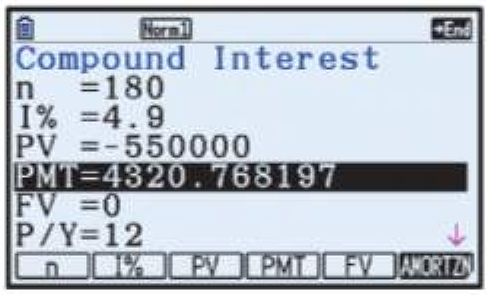
b **i** $I\% = 4.9, \quad PV = -550\,000, \quad PMT = 5000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad N \approx 147 \quad$ {we round up since the money will last until the 147th month}

The money will last $180 - 147 = 33$ months (or 2 years 9 months) less time.

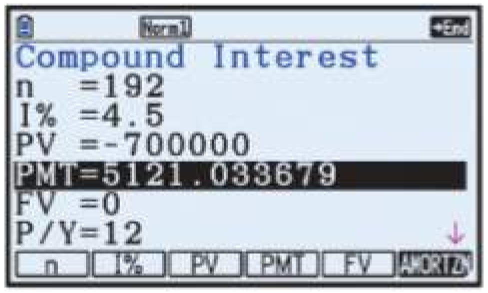
ii $N = 15 \times 12 = 180$, $I\% = 4.9$, $PV = -550\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx 4320.76$

If Igor wants the money to last 15 years, he needs to withdraw €5000 − €4320.76 = €679.24 less each month.

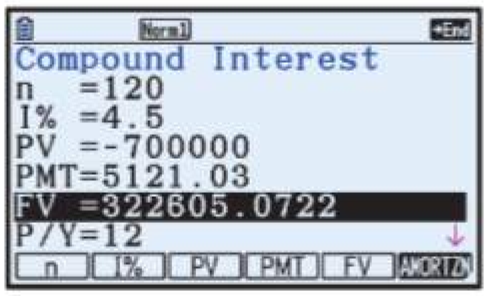
9 **a** $N = 16 \times 12 = 192$, $I\% = 4.5$, $PV = -700\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx 5121.03$

Luke can withdraw \$5121.03 each month.

b $N = 10 \times 12 = 120$, $I\% = 4.5$, $PV = -700\,000$, $PMT = 5121.03$, $P/Y = 12$, $C/Y = 12$



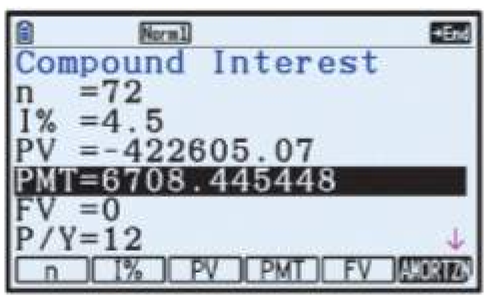
$\therefore \quad FV \approx 322\,605.07$

The balance of the fund after 10 years is \$322 605.07.

c Luke wants the money to last another $16 - 10 = 6$ years.

After receiving the inheritance, there is \$322 605.07 + \$100 000 = \$422 605.07 in the fund.

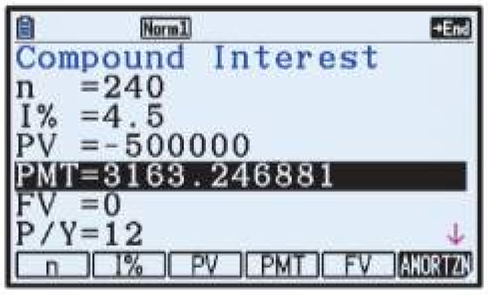
$N = 6 \times 12 = 72$, $I\% = 4.5$, $PV = -422\,605.07$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx 6708.44$

Luke is able to withdraw \$6708.44 each month for the remaining 6 years.

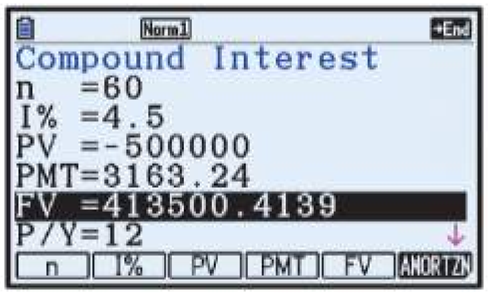
10 **a** $N = 20 \times 12 = 240, \quad I\% = 4.5, \quad PV = -500\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx 3163.24$

Célia can afford to withdraw €3163.24 each month.

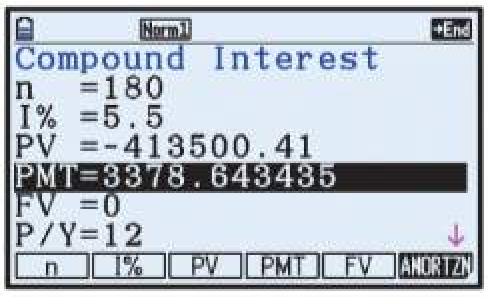
b $N = 5 \times 12 = 60, \quad I\% = 4.5, \quad PV = -500\,000, \quad PMT = 3163.24, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad FV \approx 413\,500.41$

After 5 years, the outstanding balance of Célia's fund will be €413 500.41 .

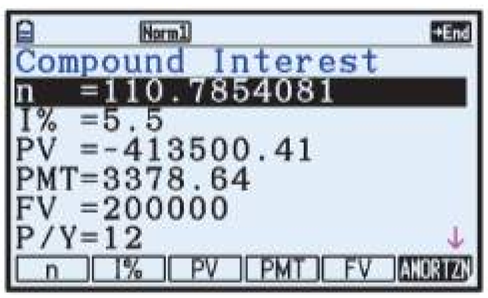
c **i** $N = 15 \times 12 = 180, \quad I\% = 5.5, \quad PV = -413\,500.41, \quad FV = 0, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad PMT \approx 3378.64$

Célia can afford to withdraw €3378.64 each month for the remaining 15 years.

ii $I\% = 5.5, \quad PV = -413\,500.41, \quad PMT = 3378.64, \quad FV = 200\,000, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad N \approx 110.8$

The balance of the fund will fall below €200 000 after another 111 months, or 9 years 3 months.

ACTIVITY 3 PERPETUITIES

1 **a** We expect to pay $PV = \dfrac{PMT}{r} = \dfrac{\$300}{0.05} = \$6000.$

b We expect to pay $PV = \dfrac{PMT}{r} = \dfrac{\$4000}{0.032} = \$125\,000.$

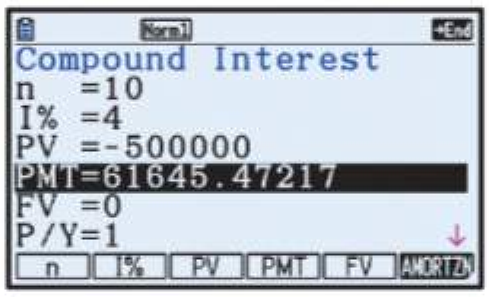
c We expect to pay $PV = \dfrac{PMT}{r} = \dfrac{\$18\,000}{0.045} = \$400\,000.$

2 **a** We expect to receive $PMT = PV \times r = €1000 \times 0.035 = €35$ per year.

b We expect to receive $PMT = PV \times r = €25\,000 \times 0.048 = €1200$ per year.

3 **a** We expect to receive $PMT = PV \times r = \$500\,000 \times 0.04 = \$20\,000$ per year.

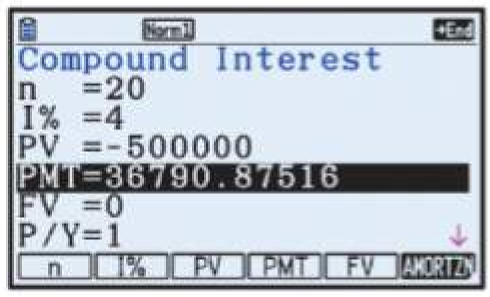
b **i** $N = 10,\ I\% = 4,\ PV = -500\,000,$
$FV = 0,\ P/Y = 1,\ C/Y = 1$



$\therefore\ PMT \approx 61\,645.47$

The money will last for 10 years if \$61 645.47 is withdrawn each year.

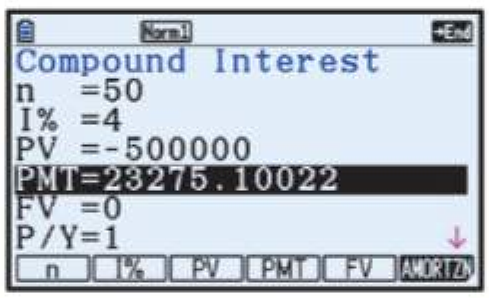
ii $N = 20,\ I\% = 4,\ PV = -500\,000,$
$FV = 0,\ P/Y = 1,\ C/Y = 1$



$\therefore\ PMT \approx 36\,790.87$

The money will last for 20 years if \$36 790.87 is withdrawn each year.

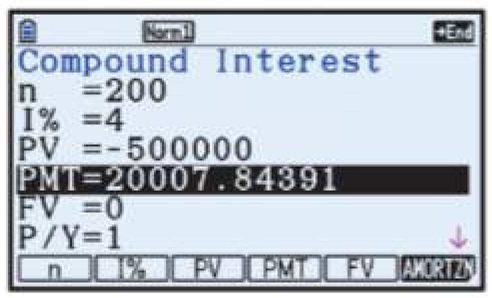
iii $N = 50,\ I\% = 4,\ PV = -500\,000,$
$FV = 0,\ P/Y = 1,\ C/Y = 1$



$\therefore\ PMT \approx 23\,275.10$

The money will last for 50 years if \$23 275.10 is withdrawn each year.

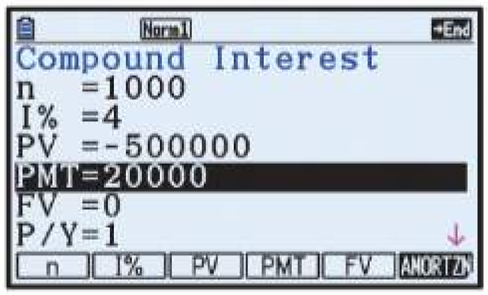
iv $N = 200,\ I\% = 4,\ PV = -500\,000,$
$FV = 0,\ P/Y = 1,\ C/Y = 1$



$\therefore\ PMT \approx 20\,007.84$

The money will last for 200 years if \$20 007.84 is withdrawn each year.

v $N = 1000,\ I\% = 4,\ PV = -500\,000,$
$FV = 0,\ P/Y = 1,\ C/Y = 1$



$\therefore \quad PMT \approx 20\,000$

The money will last for 1000 years if \$20 000 is withdrawn each year.

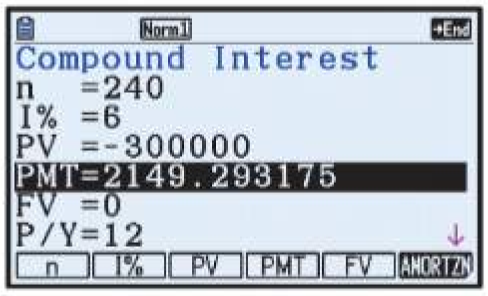
c As the number of years required increases, the amount that can be withdrawn approaches \$20 000. This is the same value as the payment received from the perpetuity, which is meant to continue indefinitely.

ACTIVITY 4 GROWING ANNUITIES

1 **b** The withdrawals are increasing over time.

c \$2709.58 is withdrawn in the final time period.

d $N = 20 \times 12 = 240,\quad I\% = 6,\quad PV = -300\,000,\quad FV = 0,\quad P/Y = 12,\quad C/Y = 12$



$\therefore \quad PMT \approx 2149.29$

The regular withdrawal for this investment would be \$2149.29, which lies between the initial and final withdrawals in the growing annuity.

e We would set the growth rate to 0%.

2 **a** **i** Using the spreadsheet, the initial monthly withdrawal is \$1888.51 .

ii We enter an expression such as "= SUM(D14 : D313)" to add up all of the values in the withdrawal column.

The total amount withdrawn over 25 years is \$842 289.06 .

b Total interest earned = total withdrawn − starting principal
= \$842 289.06 − \$500 000
= \$342 289.06

This agrees with the value obtained by adding up all of the values in the interest column.

3 **a** Increasing the amount originally deposited will *increase* the initial withdrawal.

b Increasing the duration of the annuity will *decrease* the initial withdrawal.

c Increasing the interest rate will *increase* the initial withdrawal.

d Increasing the growth rate will *decrease* the initial withdrawal.

4 **a** $w = \dfrac{PV \times (i-g) \times (1+i)^n}{(1+i)^n - (1+g)^n}$

In question **1** we have $PV = 300\,000$, $i = \dfrac{0.06}{12}$, $g = \dfrac{0.02}{12}$, and $n = 20 \times 12 = 240$.

$$\therefore \quad w = \frac{300\,000 \times \left(\frac{0.06}{12} - \frac{0.02}{12}\right) \times \left(1 + \frac{0.06}{12}\right)^{240}}{\left(1 + \frac{0.06}{12}\right)^{240} - \left(1 + \frac{0.02}{12}\right)^{240}}$$

≈ 1819.92 ✓

In question **2** we have $PV = 500\,000$, $i = \dfrac{0.04}{12}$, $g = \dfrac{0.03}{12}$, and $n = 25 \times 12 = 300$.

$$\therefore \quad w = \frac{500\,000 \times \left(\frac{0.04}{12} - \frac{0.03}{12}\right) \times \left(1 + \frac{0.04}{12}\right)^{300}}{\left(1 + \frac{0.04}{12}\right)^{300} - \left(1 + \frac{0.03}{12}\right)^{300}}$$

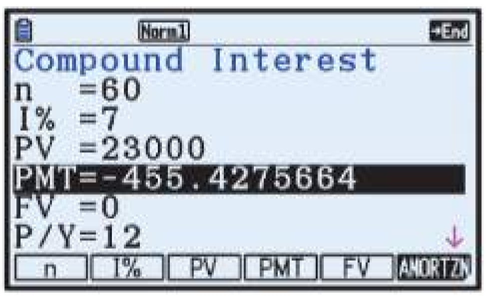
≈ 1888.51 ✓

b If $g = 0$, then $w = \dfrac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$

which is the formula of the regular withdrawal amount from **Activity 2**.

REVIEW SET 4A

1 **a** $N = 5 \times 12 = 60$, $I\% = 7$, $PV = 23\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



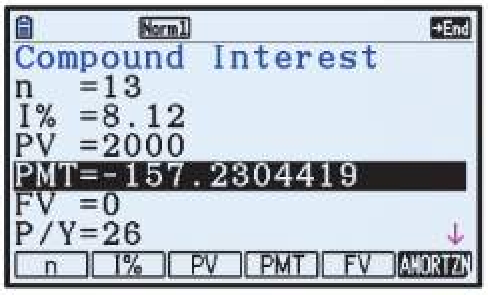
$\therefore \quad PMT \approx -455.43$

The monthly repayment is \$455.43.

b Total repayment = monthly repayment × number of months
= \$455.43 × 60
= \$27 325.80

c Interest = total repayment − amount borrowed
= \$27 325.80 − \$23 000
= \$4325.80

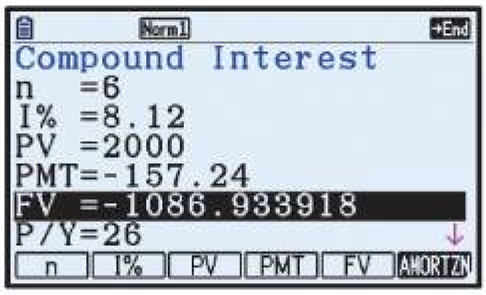
2 **a** $N = \frac{1}{2} \times 26 = 13, \quad I\% = 8.12, \quad PV = 2000, \quad FV = 0, \quad P/Y = 26, \quad C/Y = 26$



$\therefore \quad PMT \approx -157.24$

The fortnightly repayment is €157.24.

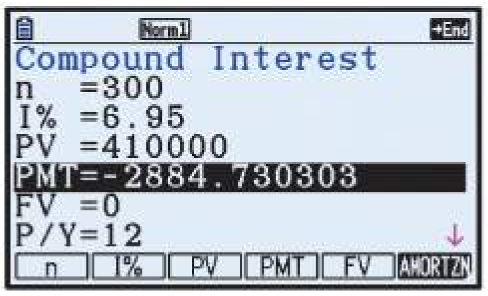
b $N = 6, \quad I\% = 8.12, \quad PV = 2000, \quad PMT = -157.24, \quad P/Y = 26, \quad C/Y = 26$



$\therefore \quad FV \approx -1086.93$

The outstanding balance on the loan after 6 fortnights is €1086.93.

3 **a** $N = 25 \times 12 = 300, \quad I\% = 6.95, \quad PV = 410\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PMT \approx -2884.74$

The minimum monthly repayment is \$2884.74.

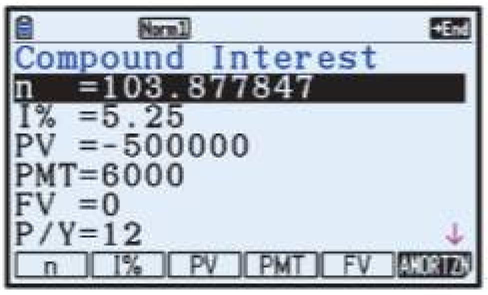
b Total interest = total repayment − principal

$$\begin{aligned} &= \$2884.74 \times 300 - \$410\,000 \\ &= \$865\,422 - \$410\,000 \\ &= \$455\,422 \end{aligned}$$

Simone will therefore pay more in interest than the amount she originally borrowed.

4 **a** We need to find how long it will take for the future value to fall to \$0.

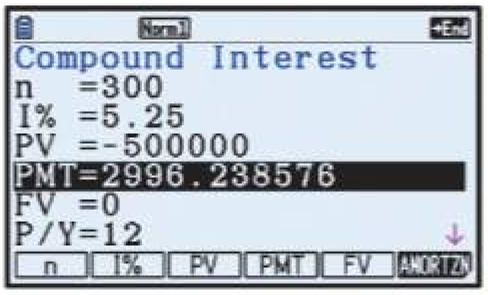
$I\% = 5.25, \quad PV = -500\,000, \quad PMT = 6000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad N \approx 104$

Yasmin will be able to withdraw \$6000 per month for 103 months, and then less in the 104th month (after 8 years 8 months).

b $N = 25 \times 12 = 300, \quad I\% = 5.25, \quad PV = -500\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$

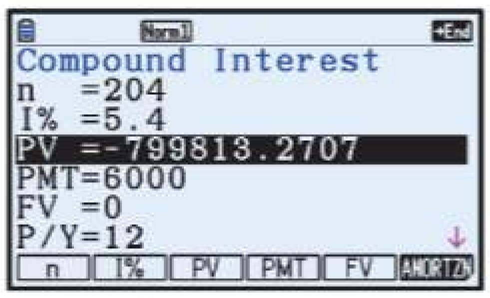


$\therefore \quad PMT \approx 2996.23$

Yasmin can afford to withdraw \$2996.23 per month.

5 **a** Scott's money needs to last for $85 - 68 = 17$ years.

$N = 17 \times 12 = 204, \quad I\% = 5.4, \quad PMT = 6000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$

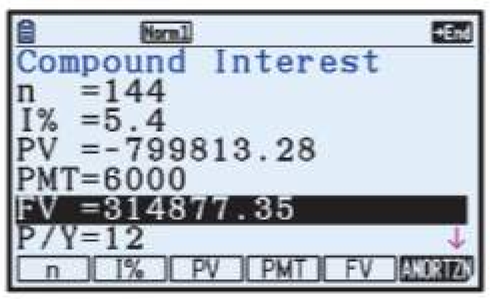


$\therefore \quad PV \approx -799\,813.28$

Scott will need to have \$799 813.28 in savings when he retires.

b Scott is 68 now, so he will be 80 in 12 years' time.

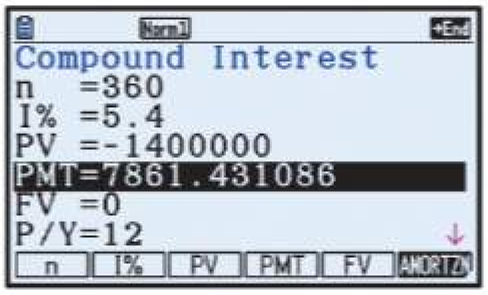
$N = 12 \times 12 = 144, \quad I\% = 5.4, \quad PV = -799\,813.28, \quad PMT = 6000, \quad P/Y = 12,$
$C/Y = 12$



$\therefore \quad FV \approx 314\,877.35$

When Scott is 80 there will be \$314 877.35 left in his account.

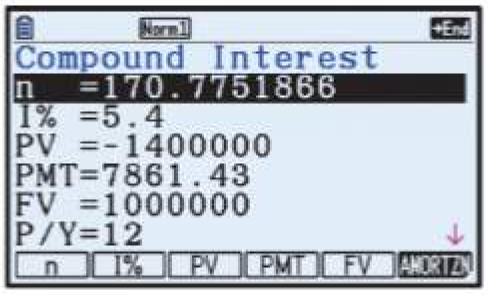
6 **a** $N = 30 \times 12 = 360$, $I\% = 5.4$, $PV = -1\,400\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx 7861.43$

Vasili can withdraw €7861.43 each month.

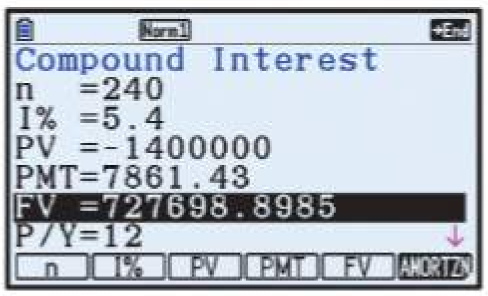
b $I\% = 5.4$, $PV = -1\,400\,000$, $PMT = 7861.43$, $FV = 1\,000\,000$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad N \approx 171$ {we round up since after 170 months the balance is still greater than €1 000 000}

It will take 171 months (or 14 years 3 months) for the balance of the fund to fall below €1 000 000.

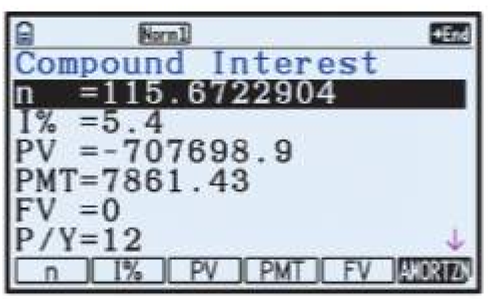
c $N = 20 \times 12 = 240$, $I\% = 5.4$, $PV = -1\,400\,000$, $PMT = 7861.43$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad FV \approx 727\,698.90$

After 20 years, there is €727 698.90 left in the fund.

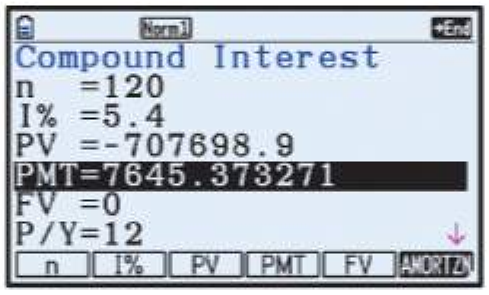
d **i** $I\% = 5.4$, $PV = -707\,698.90$, $PMT = 7861.43$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad N \approx 115.7$

Vasili will be able to withdraw the same amount each month for an additional 115 months, and then less in the 116th month (after 9 years 8 months).

ii $N = 10 \times 12 = 120$, $I\% = 5.4$, $PV = -707\,698.90$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

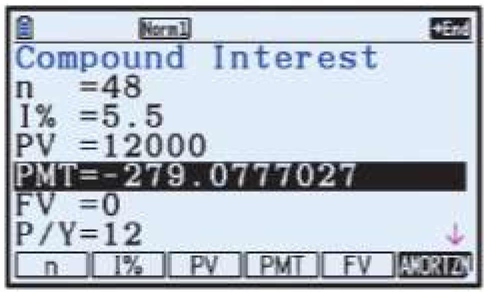


$\therefore \quad PMT \approx 7645.37$

Vasili can withdraw €7645.37 each month for the remaining 10 years.

REVIEW SET 4B

1 **a** $N = 4 \times 12 = 48$, $I\% = 5.5$, $PV = 12\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx -279.08$

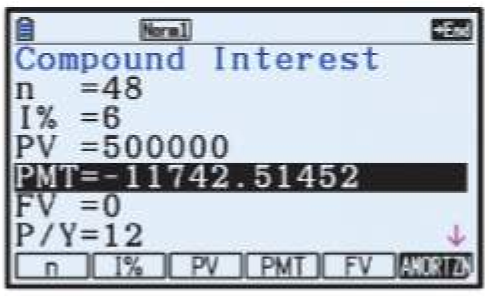
The monthly repayment is \$279.08 .

b Total interest = total repayment − amount borrowed

$= \$279.08 \times 48 - \$12\,000$

$= \$1395.84$

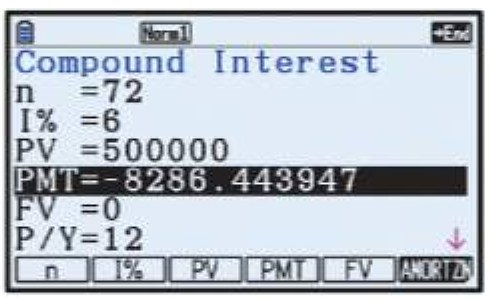
2 **a** **i** $N = 4 \times 12 = 48$, $I\% = 6$, $PV = 500\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx -11\,742.52$

The monthly loan repayments over 4 years would be 11 742.52 pesos.

ii $N = 6 \times 12 = 72$, $I\% = 6$, $PV = 500\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad PMT \approx -8286.45$

The monthly loan repayments over 6 years would be 8286.45 pesos.

b 4 year loan total interest = total repayment − amount borrowed

$$= 11\,742.52 \times 48 - 500\,000$$
$$= 563\,640.96 - 500\,000$$
$$= 63\,640.96 \text{ pesos}$$

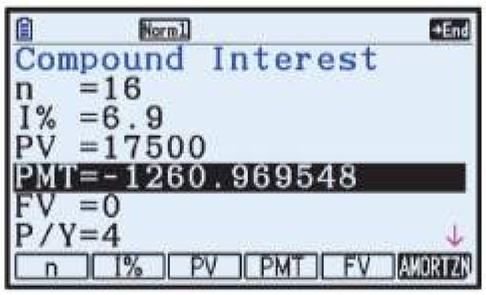
6 year loan total interest = total repayment − amount borrowed

$$= 8286.45 \times 72 - 500\,000$$
$$= 596\,624.40 - 500\,000$$
$$= 96\,624.40$$

The 4 year loan charges the least interest of 63 640.96 pesos as more is paid off each month and therefore less interest is charged.

3 **a** Peter will borrow $22 000 − $4500 = $17 500.

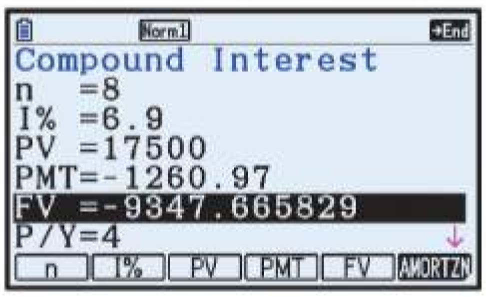
b $N = 4 \times 4 = 16$, $I\% = 6.9$, $PV = 17\,500$, $FV = 0$, $P/Y = 4$, $C/Y = 4$



$\therefore \quad PMT \approx -1260.97$

The quarterly repayment is $1260.97.

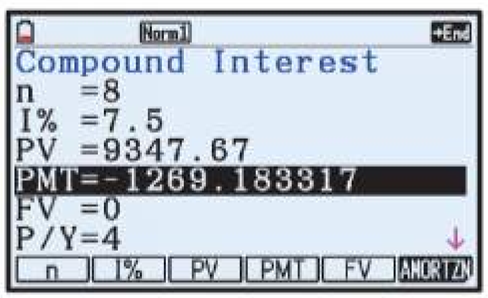
c $N = 2 \times 4 = 8$, $I\% = 6.9$, $PV = 17\,500$, $PMT = -1260.97$, $P/Y = 4$, $C/Y = 4$



$\therefore \quad FV \approx -9347.67$

The outstanding balance on the loan after 2 years is $9347.67.

d **i** $N = 2 \times 4 = 8$, $I\% = 7.5$, $PV = 9347.67$, $FV = 0$, $P/Y = 4$, $C/Y = 4$



$\therefore \quad PMT \approx -1269.19$

Peter must pay $1269.19 each quarter for the remaining 2 years.

ii Without the interest rate rise, Peter would have paid $\$1260.97 \times 8 = \$10\,087.76$ in the last 2 years. {using **b**}

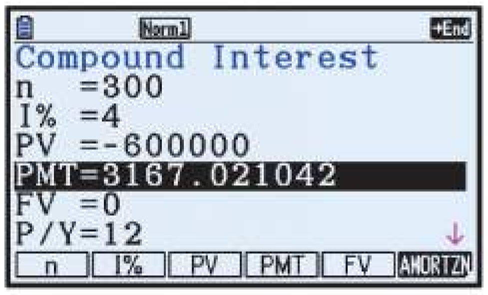
With the interest rate rise, Peter must pay $\$1269.19 \times 8 = \$10\,153.52$ in the last 2 years. {using **d i**}

So, Peter must pay an additional $\$10\,153.52 - \$10\,087.76 = \$65.76$ in interest as a result of the interest rate rise.

4 **a** An annuity fund is an investment where an individual makes a lump-sum deposit, and then makes regular *withdrawals* from the account. We have previously considered compound interest investments that make regular *deposits* into an account.

b Diane is technically correct, but she will be able to withdraw more than £2000 per month since the money in the fund will earn interest.

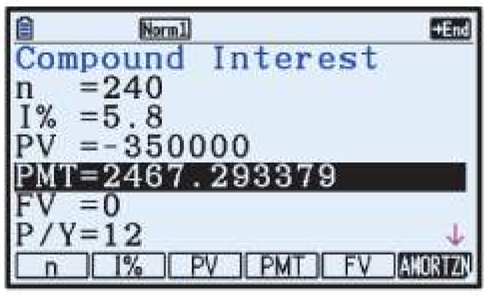
c $N = 25 \times 12 = 300,\quad I\% = 4,\quad PV = -600\,000,\quad FV = 0,\quad P/Y = 12,\quad C/Y = 12$



$\therefore\quad PMT \approx 3167.02$

Diane will be able to withdraw £3167.02 each month.

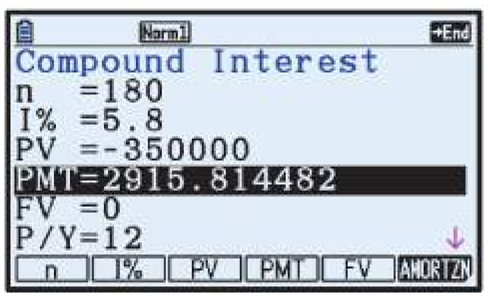
5 **a** $N = 20 \times 12 = 240,\quad I\% = 5.8,\quad PV = -350\,000,\quad FV = 0,\quad P/Y = 12,\quad C/Y = 12$



$\therefore\quad PMT \approx 2467.29$

Pia will be able to withdraw €2467.29 each month.

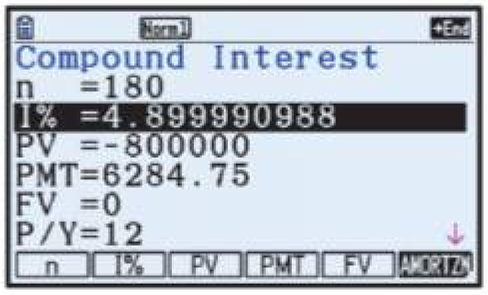
b $N = 15 \times 12 = 180,\quad I\% = 5.8,\quad PV = -350\,000,\quad FV = 0,\quad P/Y = 12,\quad C/Y = 12$



$\therefore\quad PMT \approx 2915.81$

Pia would be able to withdraw €2915.81 − €2467.29 = €448.52 more each month if the money only needed to last 15 years.

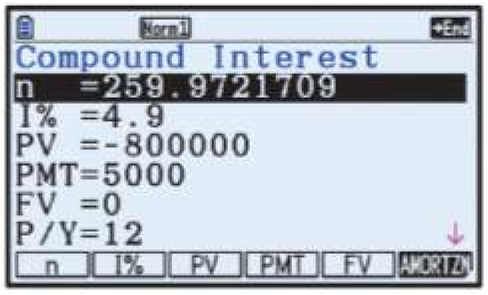
6 **a** $N = 15 \times 12 = 180$, $PV = -800\,000$, $PMT = 6284.75$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore$ $I\% \approx 4.90$

The account pays 4.9% p.a. compounded monthly.

b $I\% = 4.90$, $PV = -800\,000$, $PMT = 5000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore$ $N \approx 260$ {we round up since the money will last until the 260th month}

Harold's money will last $260 - 180 = 80$ months (or 6 years 8 months) longer if he withdraws £5000 each month.

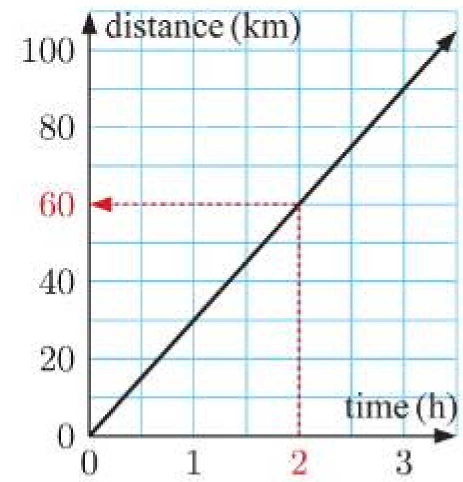
Chapter 5

MODELLING

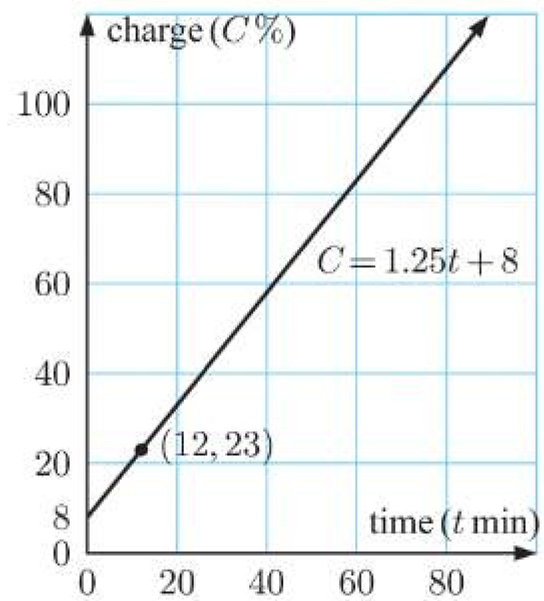
EXERCISE 5A

1 **a** We have assumed that the cyclist travelled at a constant speed of 30 km h^{-1} the entire time. This is not realistic, as the cyclist will travel at different speeds uphill, downhill, and on flat ground.

b From the diagram, the model predicts that the cyclist will travel 60 km in 2 hours.



2 **a** Briony constructed her model by finding the equation of the line through $(0, 8)$ and $(12, 23)$. Briony has assumed that the laptop will charge at a constant rate, and indefinitely. These assumptions are not realistic, but they may be satisfactory for this problem.



b $0 \leqslant C \leqslant 100, \quad t \geqslant 0$

Briony's model suggests that it is possible to have charge greater than 100%, which is not possible.

c The gradient of the line $= \dfrac{23-8}{12-0}$

$= \dfrac{15}{12}$

$= \frac{5}{4}$

The C-intercept is 8.

$\therefore \quad C = \frac{5}{4}t + 8$

When $C = 100$, $\frac{5}{4}t + 8 = 100$

$\therefore \quad \frac{5}{4}t = 92$

$\therefore \quad t = 73.6$

$\therefore$ Briony's model predicts that it will take 73.6 minutes for the laptop to be fully charged.

d **i** The laptop likely charges at a faster rate earlier on, then at a slower rate as it approaches a full charge.

ii Yes, 73.6 minutes was a useful estimate.

3 **a** Rick takes 15 seconds to run 100 metres.

We assume that Rick runs at a constant speed of $\frac{100}{15} = \frac{20}{3}\ \text{m s}^{-1}$.

Let t be the time in seconds it takes Rick to run d metres.

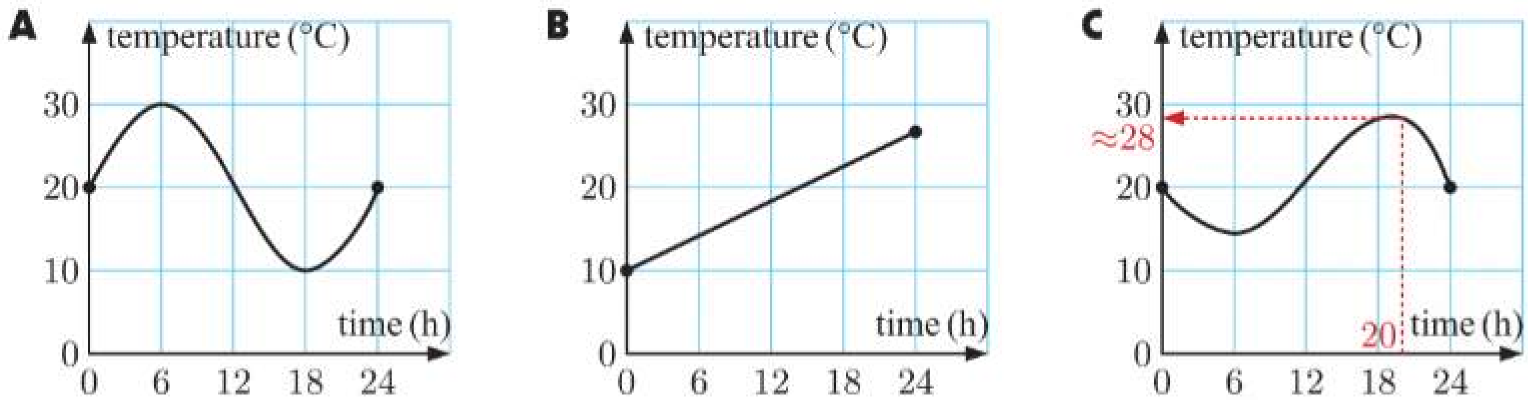
$$\text{Now} \quad \text{speed} = \frac{\text{distance}}{\text{time}}$$
$$\therefore \quad \text{time} = \frac{\text{distance}}{\text{speed}}$$
$$\therefore \quad t = \frac{d}{\frac{20}{3}}$$
$$\therefore \quad t = \tfrac{3}{20}d \ \text{seconds}$$

b When $d = 500$, $\quad t = \frac{3}{20}(500)$
$$= 75$$

$\therefore$ we predict that Rick will take 75 seconds to run 500 metres.

c Rick will take a longer time to run 500 metres than our prediction. He will not be able to run 500 metres at the same pace that he runs 100 metres.

4 **a**

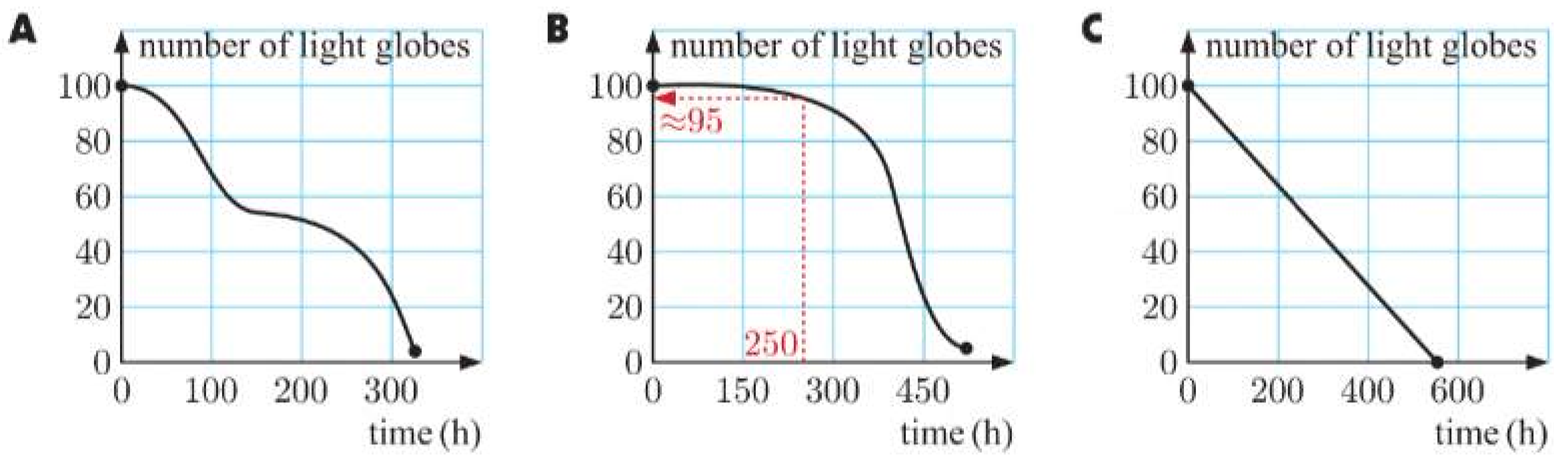


The most appropriate model for the temperature of a city on a particular day is **C**. It is usually coldest at dawn, and warmest in the afternoon. Daily temperature should be roughly periodic.

b 8 pm is 20 hours after midnight.

Using model **C**, we predict that it will be about 28°C at 8 pm.

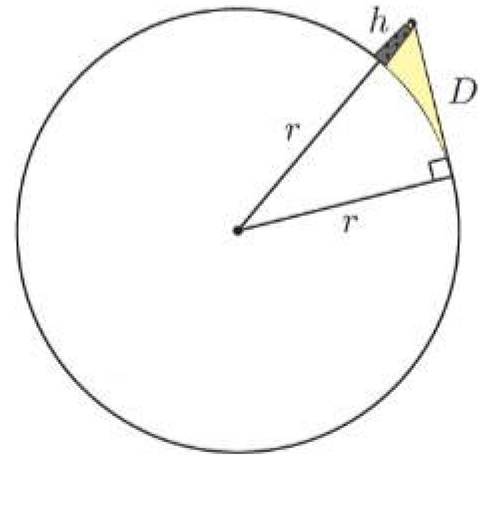
5 **a**



The most appropriate model for the number of light globes still working at any given time is **B**. Most light globes last for 200 hours, then the number of working globes quickly decreases.

b Using model **B**, we predict that about 95 globes will still be working after 250 hours.

6 **a** Darren has assumed that the Earth is perfectly spherical, and that the lighthouse is perpendicular to the Earth's surface. These are reasonable assumptions.



b $r \approx 6370$ km, $h = 40$ m
$= 0.04$ km

$\therefore \; r^2 + D^2 = (r+h)^2$ {Pythagoras}

$\therefore \; 6370^2 + D^2 \approx (6370 + 0.04)^2$

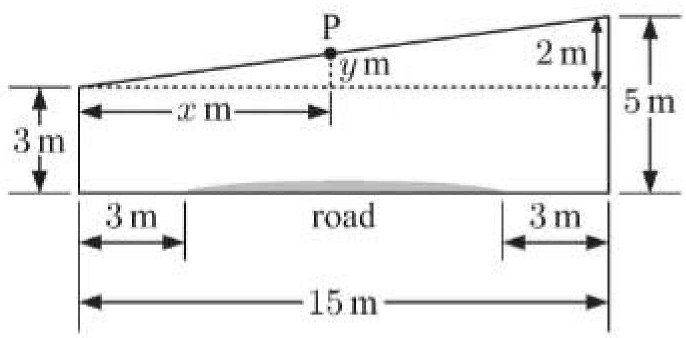
$\therefore \; 6370^2 + D^2 \approx 6370.04^2$

$\therefore \; D \approx \sqrt{6370.04^2 - 6370^2}$ $\{D > 0\}$

$\therefore \; D \approx 22.6$ km

7 **a** Using similar triangles, $\frac{y}{2} = \frac{x}{15}$

$\therefore \; y = \frac{2}{15}x$



$\therefore$ the height of P above ground level is

$h = y + 3$ m

$= (\frac{2}{15}x + 3)$ m

We have assumed that the power line is hanging perfectly straight and that the poles are directly opposite.

b It is reasonable to apply this model for x such that $0 \leqslant x \leqslant 15$.

c The middle of the road corresponds to $x = \frac{15}{2}$.

When $x = \frac{15}{2}$, $h = \frac{2}{15}(\frac{15}{2}) + 3$ m

$= 1 + 3$ m

$= 4$ m

$\therefore$ we predict that the height of the power line above ground level as it passes over the middle of the road is 4 m.

d Yes, the answer in **c** seems reasonable. If the poles are 3 m and 5 m high, we expect the height of the power line above the middle of the road to be 4 m.

e Yes, the model is useful to Hashni. Provided he knows the width and height of the truck, he should be able to reasonably decide whether or not the truck can fit.

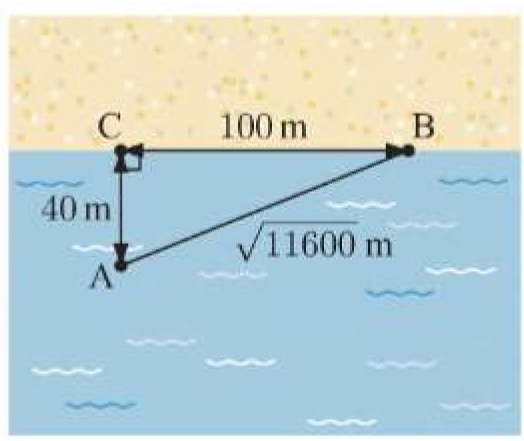
8 **a** **i** $\text{AB} = \sqrt{40^2 + 100^2}$ {Pythagoras}

$= \sqrt{11\,600}$ m

Time taken to swim directly to B

$= \frac{\sqrt{11\,600}\text{ m}}{1.5\text{ m s}^{-1}}$

≈ 71.8 s



ii Time taken to swim to C $= \dfrac{40 \text{ m}}{1.5 \text{ m s}^{-1}}$ ≈ 26.7 s

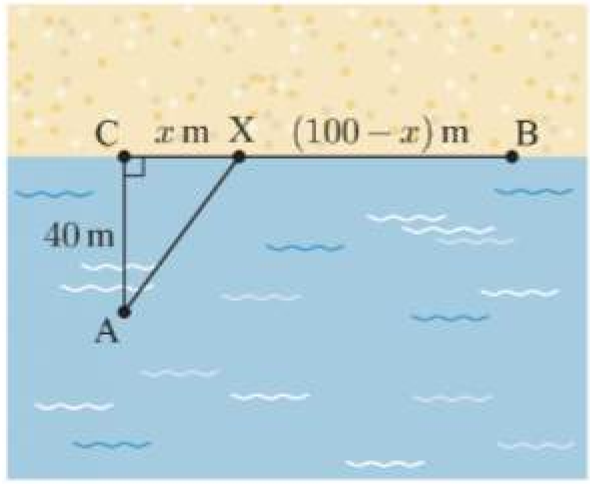
Time taken to jog to B $= \dfrac{100 \text{ m}}{4 \text{ m s}^{-1}}$ $= 25$ s

$\therefore$ total time taken $\approx 26.7 \text{ s} + 25 \text{ s}$
≈ 51.7 s

We have assumed that there is no current, and that Antonio can swim/jog at a constant speed, in all situations.

b Swimming to C then jogging to B appears to be quicker.

c Suppose Antonio swims to a point X, which is x m from C, then jogs to B.



$\text{AX} = \sqrt{40^2 + x^2}$ {Pythagoras}
$= \sqrt{1600 + x^2}$ m

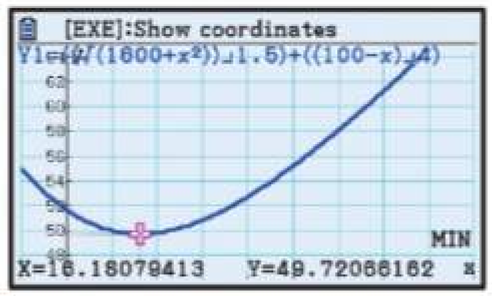
Time taken to swim to X $= \dfrac{\sqrt{1600 + x^2}}{1.5}$ s.

Time taken to jog to B $= \dfrac{100 - x}{4}$ s.

$\therefore$ total time taken $= \left(\dfrac{\sqrt{1600 + x^2}}{1.5} + \dfrac{100 - x}{4} \right)$ s

Using technology, $\dfrac{\sqrt{1600 + x^2}}{1.5} + \dfrac{100 - x}{4}$ has a minimum value of ≈ 49.7, when $x \approx 16.2$.

So, Antonio could swim to the point about 16.2 m from C and jog about 83.8 m to B in about 49.7 seconds.



9 Kate can dig $\frac{1}{7}$ of a hole each hour, and Lenny can dig $\frac{1}{12}$ of a hole each hour.

We assume Kate and Lenny can work together without getting in each other's way. So, working together they will dig $\frac{1}{7} + \frac{1}{12} = \frac{19}{84}$ of a hole each hour.

$\therefore$ it would take them $\frac{84}{19} = 4\frac{8}{19}$ hours $\approx$ 4 hours 25 minutes to dig a hole working together.

10 Pulling the plug would drain 1200 litres in 15 minutes, or $\dfrac{1200}{15} = 80$ litres in 1 minute.

A hole in the tank would drain 1200 litres in 25 minutes, or $\dfrac{1200}{25} = 48$ litres in 1 minute.

$\therefore$ pulling the plug, and having a hole in the tank would drain $80 + 48 = 128$ litres each minute.

It would take $\dfrac{1200}{128} = 9\frac{3}{8}$ minutes $\approx$ 9 minutes 23 seconds for the tank to empty.

11 Independent of one another, Aaron can make 4 widgets per hour, Bonnie can make 3 widgets per hour, and Calum can make 2 widgets per hour.

In one hour they will make $4 + 3 + 2 = 9$ widgets.

$\therefore$ it will take them $\dfrac{135}{9} = 15$ hours to make 135 widgets working together.

12 Suppose Beitidh can paint the room on her own in x hours.

$\therefore$ Beitidh can paint $\dfrac{1}{x}$ of the room each hour.

Angus can paint $\frac{1}{3}$ of the room each hour.

Working together they can paint $\frac{1}{3} + \dfrac{1}{x}$ of the room each hour.

Now, $\dfrac{1}{\frac{1}{3}+\frac{1}{x}} = 2$ (it would take them 2 hours to paint the room working together)

$\therefore \frac{1}{3} + \dfrac{1}{x} = \frac{1}{2}$

$\therefore \dfrac{1}{x} = \frac{1}{6}$

$\therefore x = 6$

$\therefore$ it would take Beitidh 6 hours to paint the room on her own.

13 Suppose Huiliang can complete a jigsaw puzzle in H hours, Lifen can complete a jigsaw puzzle in L hours, and Suyin can complete a jigsaw puzzle in S hours.

So, Huiliang can complete $\dfrac{1}{H}$ of a puzzle each hour, Lifen can complete $\dfrac{1}{L}$ of a puzzle each hour, and Suyin can complete $\dfrac{1}{S}$ of a puzzle each hour.

Working together, Huiliang and Lifen can complete $\dfrac{1}{H} + \dfrac{1}{L}$ of a puzzle each hour.

Now, $\dfrac{1}{\frac{1}{H}+\frac{1}{L}} = 2,$ as it would take 2 hours for Huiliang and Lifen to complete a puzzle.

$\therefore \dfrac{1}{H} + \dfrac{1}{L} = \dfrac{1}{2}$ .... (1)

Similarly, $\dfrac{1}{L} + \dfrac{1}{S} = \dfrac{1}{3}$ .... (2)

and $\dfrac{1}{H} + \dfrac{1}{S} = \dfrac{1}{4}$ .... (3)

Adding (1), (2), and (3) together gives $\left(\dfrac{1}{H}+\dfrac{1}{L}\right) + \left(\dfrac{1}{L}+\dfrac{1}{S}\right) + \left(\dfrac{1}{H}+\dfrac{1}{S}\right) = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}$

$$\therefore \frac{2}{H} + \frac{2}{L} + \frac{2}{S} = \frac{13}{12}$$

$$\therefore 2\left(\frac{1}{H} + \frac{1}{L} + \frac{1}{S}\right) = \frac{13}{12}$$

$$\therefore \frac{1}{H} + \frac{1}{L} + \frac{1}{S} = \frac{13}{24}$$

So, Huiliang, Lifen, and Suyin can complete $\frac{13}{24}$ of a puzzle each hour if they work together.

$\therefore$ it would take them $\frac{24}{13} = 1\frac{11}{13}$ hours ≈ 1 hour 51 minutes to complete a jigsaw puzzle working together.

INVESTIGATION CALENDARS

1 **a** Difference between a tropical year and 365 days $\approx 365.242\,199 - 365$
$\approx 0.242\,199$ days

b $\dfrac{1}{0.242\,199} \approx 4.13$

$\therefore$ it would take about 4.13 years for the difference between 365 days and a tropical year to accumulate to one whole day.

c Assume that six months $= 183$ days.

$\dfrac{183}{0.242\,199} \approx 756$

$\therefore$ it would take about 756 years for the difference between 365 days and a tropical year to accumulate to six months.

2 **a** $365.25 - 365.242\,199 = 0.007\,801$

$\therefore$ the Julian calendar is inaccurate by about $0.007\,801$ days per year.

b **i** $\dfrac{1}{0.007\,801} \approx 128$

$\therefore$ it would take about 128 years for the inaccuracy to accumulate to one whole day.

ii Assume that six months $= 183$ days.

$\dfrac{183}{0.007\,801} \approx 23\,459$

$\therefore$ it would take about $23\,459$ years for the inaccuracy to accumulate to six months.

3 **a** In a period of 400 years, the Gregorian calendar adds $100 - 3 = 97$ leap years (the 100th, 200th, and 300th years are not leap years).

b 303 years with 365 days and 97 years with 366 days gives a total of
$303 \times 365 + 97 \times 366 = 146\,097$ days in a period of 400 years.

$\therefore$ the average year in the Gregorian calendar has $\dfrac{146\,097}{400} = 365.2425$ days.

c $365.2425 - 365.242\,199 = 0.000\,301$

$\therefore$ the Gregorian calendar is inaccurate by about $0.000\,301$ days per year.

$\dfrac{1}{0.000\,301} \approx 3322$

$\therefore$ it would take about 3322 years for the inaccuracy to accumulate to one whole day.

4 **a** **i** $4 = 2^2$

$\therefore$ if the year number is divisible by 4, then it ends in "00".

ii $128 = 2^7$

$\therefore$ if the year number is divisible by 128, then it ends in "0000000".

b **i** There is a leap year if the year number (in base 2) ends in "00", except if it ends in "0000000".

So, using **a**, there is a leap year if the year number (in base 10) is divisible by 4, but not divisible by 128.

The Julian calendar has a leap year if the year number is divisible by 4, and an approximate error of one day per 128 years.

The binary calendar improves the Julian calendar by removing one day per 128 years to reduce this error.

ii The rules for determining whether a given year is a leap year are much simpler in the binary calendar compared to the Gregorian calendar.

iii In a period of 128 years, there are $\frac{128}{4} = 32$ years in which the year number is divisible by 4.

Only one of these 32 years have a year number divisible by 128.

$\therefore$ the binary calendar adds $32 - 1 = 31$ leap years in a period of 128 years.

iv 97 years with 365 days and 31 years with 366 days gives a total of
$97 \times 365 + 31 \times 366 = 46\,751$ days in a period of 128 years.

$\therefore$ the average year in the binary calendar has $\frac{46\,751}{128} = 365.242\,187\,5$ days.

v $365.242\,199 - 365.242\,187\,5 = 0.000\,011\,5$

$\therefore$ the binary calendar is inaccurate by about $0.000\,011\,5$ days per year.

$\frac{1}{0.000\,011\,5} \approx 86\,957$

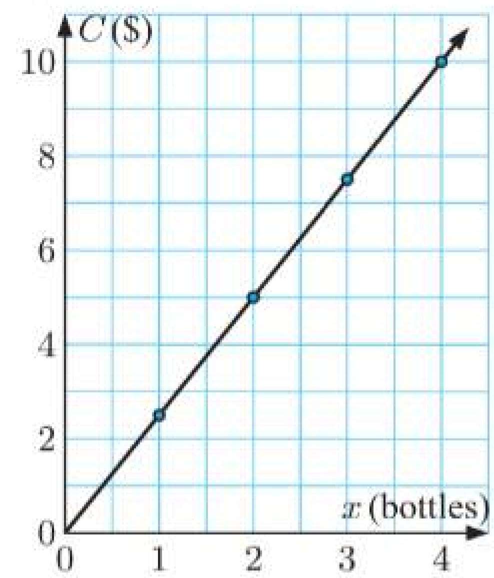
$\therefore$ it would take about $86\,957$ years for the inaccuracy to accumulate to one whole day.

EXERCISE 5B

1 **a** The gradient of the graph is $\frac{5-0}{2-0} = 2.5$

The C-intercept is 0.

$\therefore$ $C = 2.5x$



b The model fits the data points exactly.

$\therefore$ the model is exact.

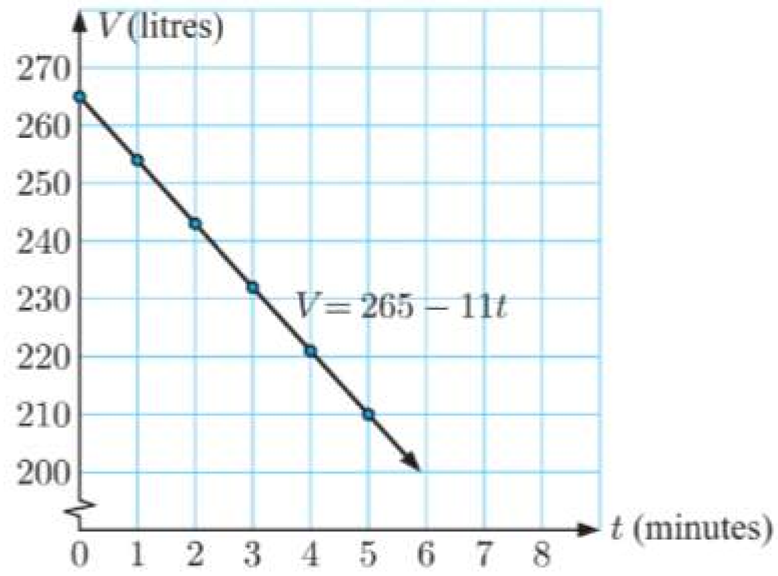
c Yes, if $x = 12$, $C = 2.5(12)$
$= 30$

$\therefore$ the cost of 12 bottles of juice is \$30.

2 **a**

Time (t minutes)	0	1	2	3	4	5
Volume (V litres)	265	254	243	232	221	210

-11 -11 -11 -11 -11

b 

c The gradient of the graph is -11.
The V-intercept is 265.
$\therefore \quad V = 265 - 11t$

d **i** When $t = 15$, $V = 265 - 11(15)$
$= 265 - 165$
$= 100$

$\therefore$ there are 100 litres of water left in the tank after 15 minutes.

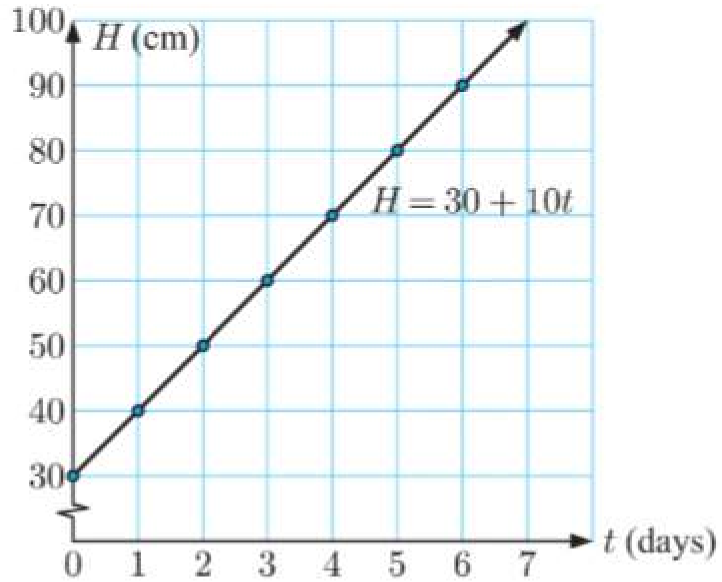
ii When $V = 0$, $265 - 11t = 0$
$\therefore \quad 11t = 265$
$\therefore \quad t = \frac{265}{11}$
$\therefore \quad t \approx 24.1$

$\therefore$ it will take about 24.1 minutes for the tank to empty.

3 **a**

t (days)	0	1	2	3	4	5	6
H (cm)	30	40	50	60	70	80	90

+10 +10 +10 +10 +10 +10

b 

c It does not make sense to extend the model for $t < 0$ since the number of days after planting cannot be negative. It does not make sense to extend the model indefinitely for $t > 6$ since the plant will not continue to grow at this rate forever. However, we will assume that it does for the purposes of this model. $H(t)$ has domain $\{t \mid t \geqslant 0\}$.

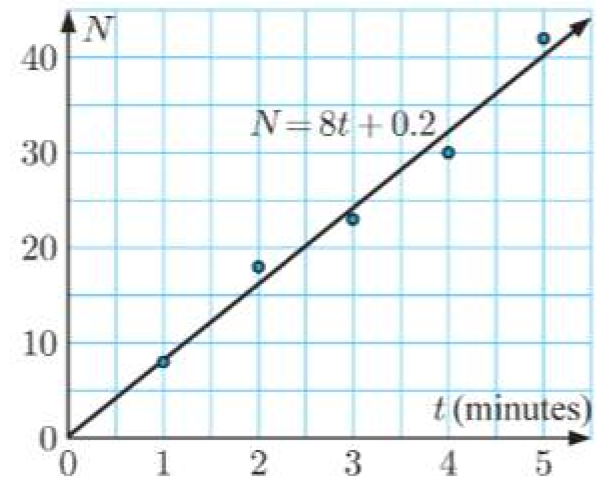
d The gradient of the graph is 10.
The H-intercept is 30.
$\therefore \quad H = 10t + 30$

e When $H = 100$, $10t + 30 = 100$
$\therefore \quad 10t = 70$
$\therefore \quad t = 7$

$\therefore$ it will take 7 days for the bamboo to be 1 m high.

4 **a** The points do not lie exactly on the line.

$\therefore$ the model is approximate.



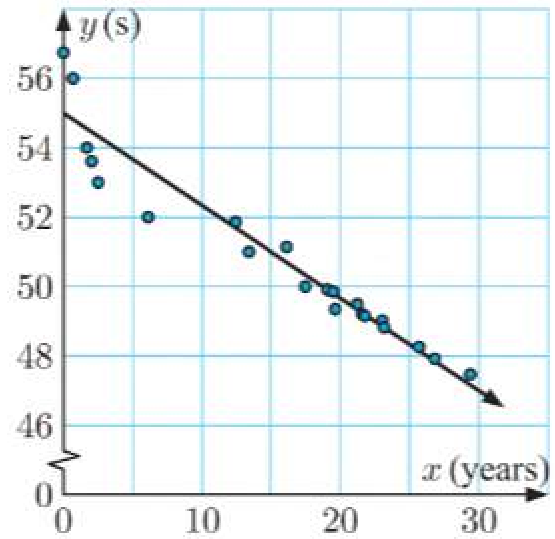
b $N = 8t + 0.2$

When $t = 20$, $N = 8(20) + 0.2$
$= 160.2$

$\therefore$ the model predicts that Jack will pick up about 160 pieces of litter in 20 minutes. This estimate is likely to be inaccurate as it is an extrapolation. There would probably not be this much litter for Jack to pick up.

5 **a** $y = 54.85 - 0.2660x$

When $x = 0$, $y = 54.85 - 0.2660(0)$
$= 54.85$



The model predicts that in 1957 $(x = 0)$ the world record time will be 54.85 s, while the actual time was 57.0 s.

The values differ since the model is not exact. Some times will be higher or lower than what is predicted.

b The model would not be accurate for years much earlier than 1957, as the model would begin to predict unrealistically large times.

c When $y = 41.96$, $54.85 - 0.2660x = 41.96$

$\therefore 0.2660x = 12.89$

$\therefore x \approx 48.5$

The model predicts that the women's 400 m will be completed in 41.96 s in the 48th year after 1957, which is 2005.

d It is not suitable to extrapolate future times, since improvements in time will eventually stop following a linear trend.

EXERCISE 5C.1

1 **a** A is (0, 50) and B is (25, 100).

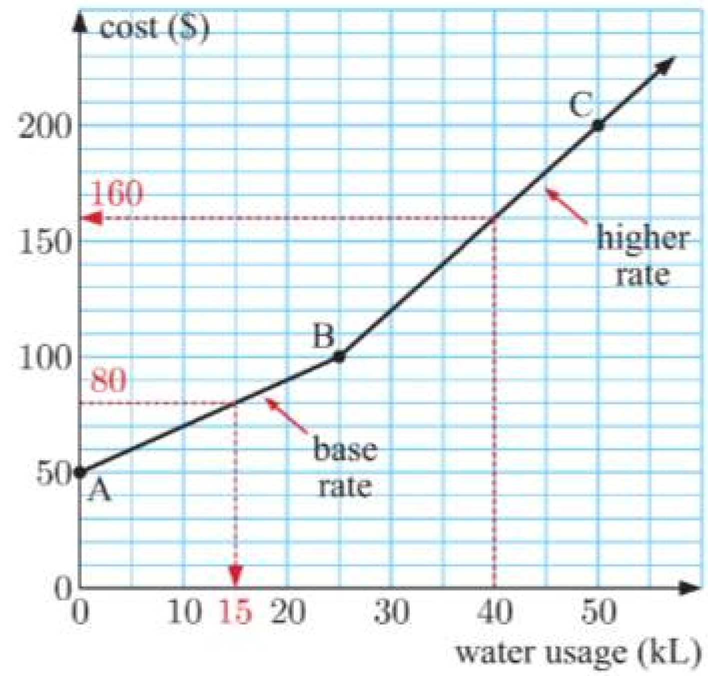
$\therefore$ gradient of [AB] $= \dfrac{100-50}{25-0} = 2$

The first 25 kL of water used costs \$2 per kL. This is the base rate.

B is (25, 100) and C is (50, 200).

$\therefore$ gradient of [BC] $= \dfrac{200-100}{50-25} = 4$

Each kL of water used above 25 kL costs \$4 per kL. This is the higher rate.

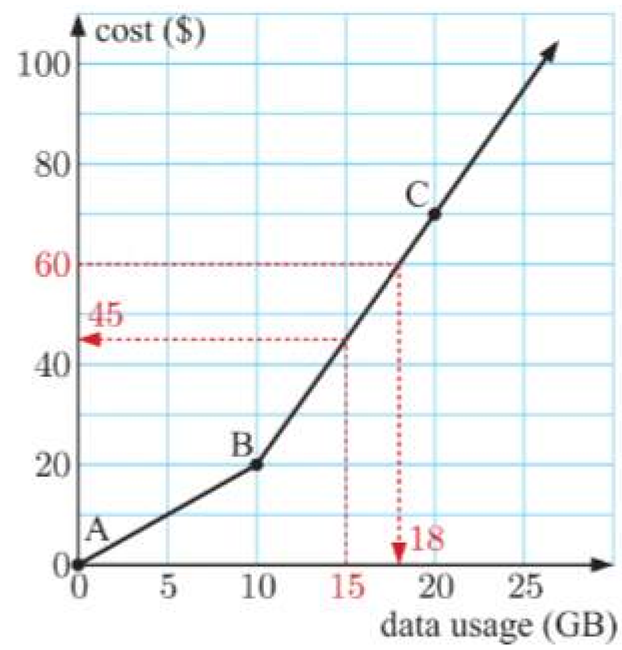


b The cost of using 40 kL of water is \$160.

c Kelly's last water bill of \$80 was for 15 kL of water used.

2 **a** Colin pays one rate up to his allowance and then a higher rate after that. The gradient of the graph changes at the point (10, 20), which is when Colin's allowance must end.

So, Colin has 10 GB of data allowance.



b B is (10, 20) and C is (20, 70).

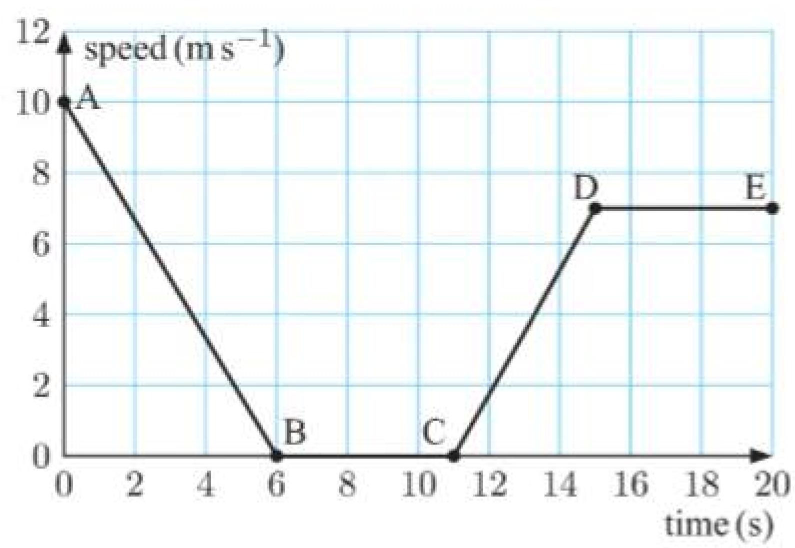
$\therefore$ gradient of [BC] $= \dfrac{70-20}{20-10} = 5$

Colin is charged \$5 per GB for excess data.

c Colin needs to pay \$45 for using 15 GB of data in a month.

d A bill of \$60 would result by using 18 GB of data in a month.

3 **a** The car was initially travelling at 10 $\text{m}\,\text{s}^{-1}$, then braked for 6 seconds to a complete stop. The car stopped for 5 seconds before accelerating for 4 seconds to a speed of 7 $\text{m}\,\text{s}^{-1}$ which it maintained for the remaining 5 seconds.



b We have assumed that the car accelerates and decelerates at a constant rate. These assumptions are reasonable.

c **i** The car decelerates at a constant rate from A to B.

$\therefore$ average speed between A and B $= \dfrac{10+0}{2} = 5\ \text{m}\,\text{s}^{-1}$.

ii The car travels at an average speed of 5 $\text{m}\,\text{s}^{-1}$ for 6 s.

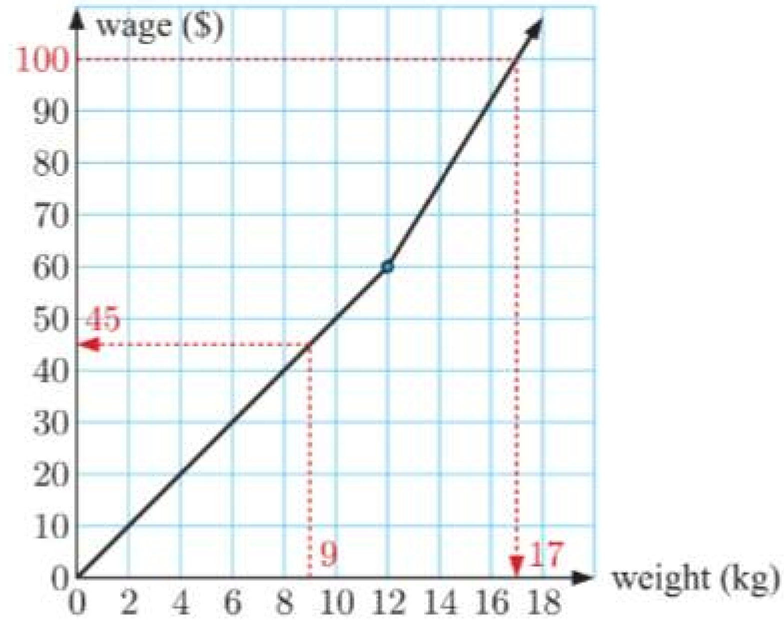
$\therefore$ distance travelled between A and B $= 5 \times 6 = 30$ m.

d

Segment	*Average speed* ($m\,s^{-1}$)	*Time* (s)	*Distance travelled* (m)
AB	5	6	$5 \times 6 = 30$
BC	0	5	$0 \times 5 = 0$
CD	$\frac{7+0}{2} = 3.5$	4	$3.5 \times 4 = 14$
DE	7	5	$7 \times 5 = 35$

$\therefore$ the car travelled a total distance of $30 + 0 + 14 + 35 = 79$ m.

4 **a** For the first 12 kg of berries picked, the gradient of the graph is 5.
For any berries picked over 12 kg, the gradient of the graph is 8.



b Jasper earns \$45 for picking 9 kg of berries.

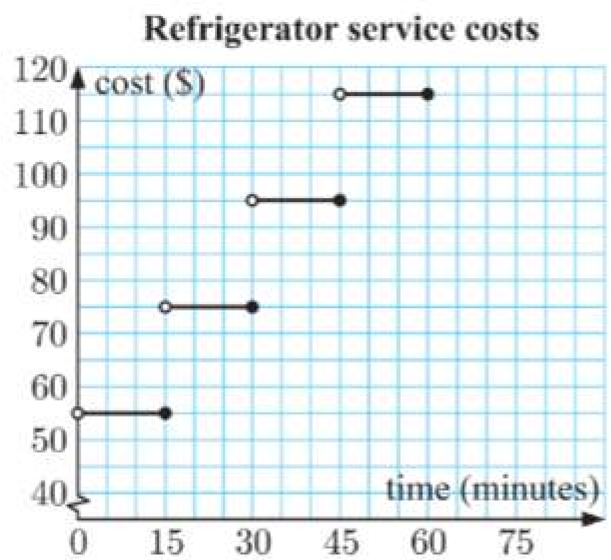
c Jasper must pick 17 kg of berries to earn \$100 in a day.

5 **a** **i** From the graph, a service which takes 20 minutes costs \$75.

ii From the graph, a service which takes 45 minutes costs \$95.

b **i** From the graph, the maximum time for a service costing \$55 is 15 minutes.

ii From the graph, the maximum time for a service costing \$115 is 60 minutes.



c Looking at the graph, a service costs \$55 for the first 15 minutes, and an extra \$20 for each additional 15 minutes or part thereof.
We assume that this trend continues indefinitely.

80 minutes $= 1 \times 15$ minutes $+ 4\frac{1}{3} \times 15$ minutes

So a service which takes 80 minutes will be charged

$\$55$ for the first 15 minutes $+\ 5 \times \$20$ for the next 5 lots of 15 minutes

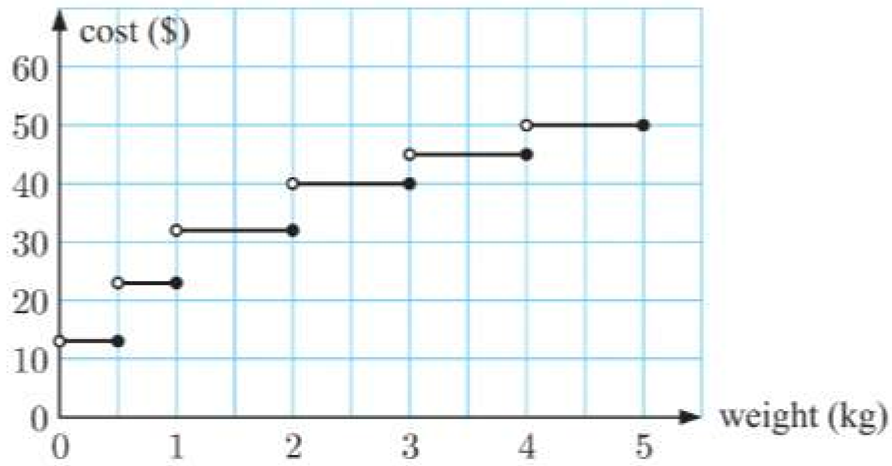
$= \$55 + 5 \times \$20$

$= \$155$

6

Weight	*Cost*
Up to 500 g	\$13
Over 500 g up to 1 kg	\$23
Over 1 kg up to 2 kg	\$32
Over 2 kg up to 3 kg	\$40
Extra kg or part thereof	\$5

a



b **i** It costs \$23 to send a parcel weighing 750 g.

ii It costs \$32 to send a parcel weighing 1.6 kg.

iii It costs \$45 to send a parcel weighing 3.4 kg.

c It costs \$32 to send a parcel weighing 1.7 kg and \$40 to send a parcel weighing 2.8 kg.

$\therefore$ it costs a total of $\$32 + \$40 = \$72$ to send the parcels separately.

If Heather puts the two parcels together as a single package, it will weigh $1.7 \text{ kg} + 2.8 \text{ kg} = 4.5 \text{ kg}$.

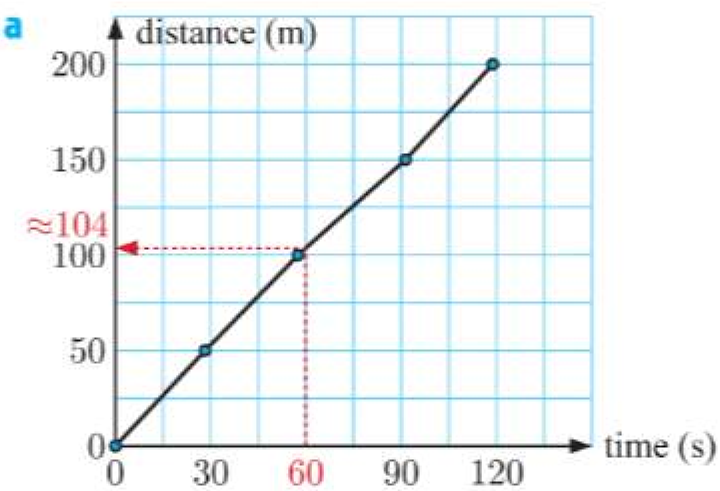
It costs \$50 to send a parcel weighing 4.5 kg.

Heather will save $\$72 - \$50 = \$22$ if she sends the parcels together as a single package.

7

Stroke	*Time* (s)
Butterfly	28.20
Backstroke	29.28
Breaststroke	33.91
Freestyle	27.43

a



b We assume Laura swims at a constant rate in each 50 m leg.

c Using our graph from **a**, we estimate that Laura had swum approximately 104 m after 1 minute.

8

Taxable income (per year)	*Tax paid*
Up to £11 850	Nil
£11 851 - £46 350	£0.20 for each £1 above £11 850
£46 351 - £150 000	£6900 + £0.40 for each £1 above £46 350
Over £150 000	£48 360 + £0.45 for each £1 above £150 000

a Taxable income $= £7.83$ per hour $\times$ 36 hours per week $\times$ 50 weeks per year
$= £14\,094$ per year

Now $£14\,094 = £11\,850 + £2244$
$\therefore$ tax paid $= £0.20 \times 2244$
$= £448.80$

b **i** $£53\,172 = £46\,350 + £6822$
$\therefore$ tax paid $= £6900 + £0.40 \times 6822$
$= £9628.80$

ii After-tax income $= £53\,172 - £9628.80$
$= £43\,543.20$

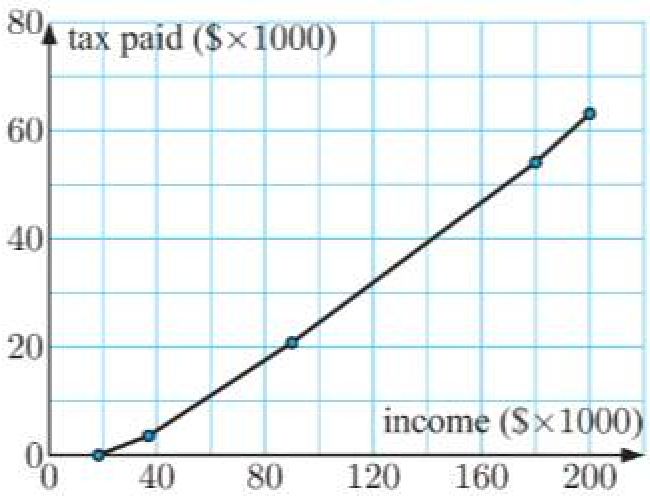
c **i** $£153\,907 = £150\,000 + £3907$
$\therefore$ tax paid $= £48\,360 + £0.45 \times 3907$
$= £50\,118.15$

ii Percentage of income paid as income tax $= \dfrac{£50\,118.15}{£153\,907} \times 100\%$
$\approx 32.6\%$

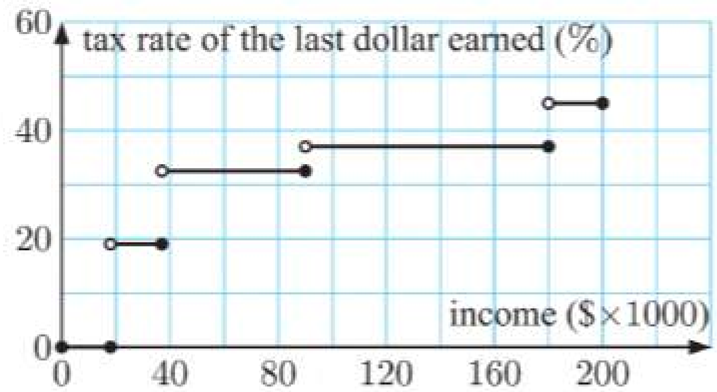
9

Taxable income (per year)	*Tax paid*
Up to \$18 200	Nil
\$18 201 - \$37 000	\$0.19 for each \$1 over \$18 200
\$37 001 - \$90 000	\$3572 plus \$0.325 for each \$1 over \$37 000
\$90 001 - \$180 000	\$20 797 plus \$0.37 for each \$1 over \$90 000
\$180 001 and over	\$54 097 plus \$0.45 for each \$1 over \$180 000

a



b



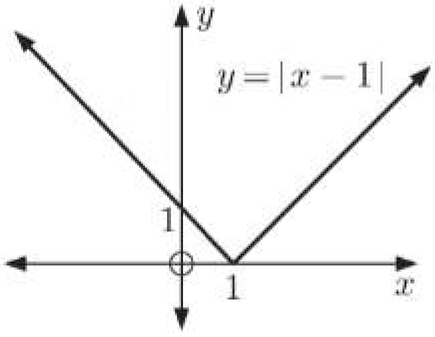
c **i** $\$28\,000 = \$18\,200 + \$9800$
$\therefore$ income tax $= \$0.19 \times 9800$
$= \$1862$

ii $\$48\,300 = \$37\,000 + \$11\,300$
$\therefore$ income tax $= \$3572 + \$0.325 \times 11\,300$
$= \$7244.50$

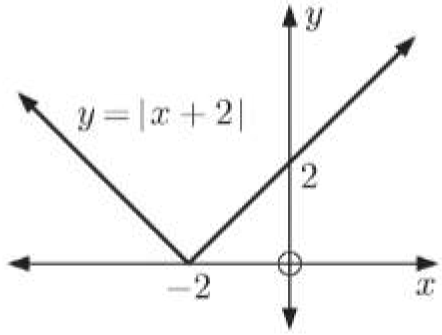
iii $\$96\,150 = \$90\,000 + \$6150$
$\therefore$ income tax $= \$20\,797 + \$0.37 \times 6150$
$= \$23\,072.50$

EXERCISE 5C.2

1 **a** $|x-1| = \begin{cases} x-1 & \text{if } x \geqslant 1 \\ 1-x & \text{if } x < 1 \end{cases}$

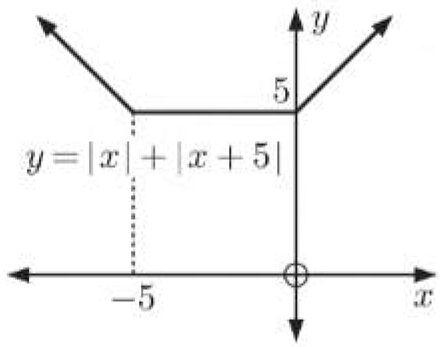


b $|x+2| = \begin{cases} x+2 & \text{if } x \geqslant -2 \\ -x-2 & \text{if } x < -2 \end{cases}$



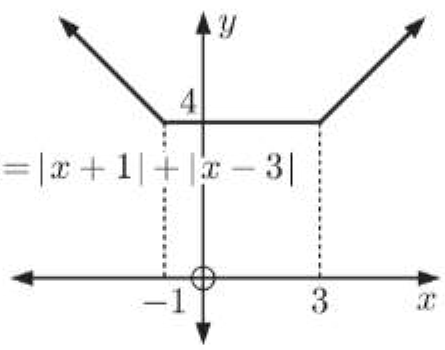
c $|x| + |x+5| = \begin{cases} x+(x+5) & \text{if } x \geqslant 0 \\ -x+(x+5) & \text{if } -5 \leqslant x < 0 \\ -x+(-x-5) & \text{if } x < -5 \end{cases}$

$= \begin{cases} 2x+5 & \text{if } x \geqslant 0 \\ 5 & \text{if } -5 \leqslant x < 0 \\ -2x-5 & \text{if } x < -5 \end{cases}$



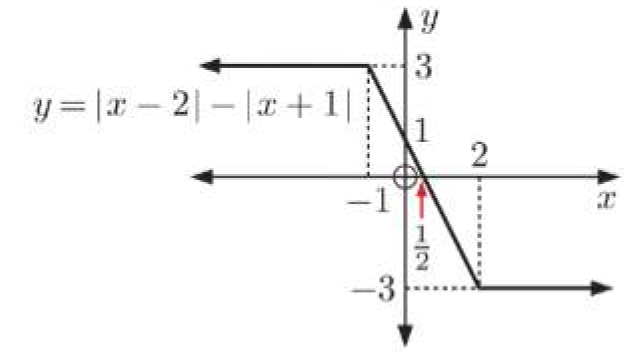
d $|x+1| + |x-3| = \begin{cases} (x+1)+(x-3) & \text{if } x \geqslant 3 \\ (x+1)+(3-x) & \text{if } -1 \leqslant x < 3 \\ (-x-1)+(3-x) & \text{if } x < -1 \end{cases}$

$= \begin{cases} 2x-2 & \text{if } x \geqslant 3 \\ 4 & \text{if } -1 \leqslant x < 3 \\ 2-2x & \text{if } x < -1 \end{cases}$



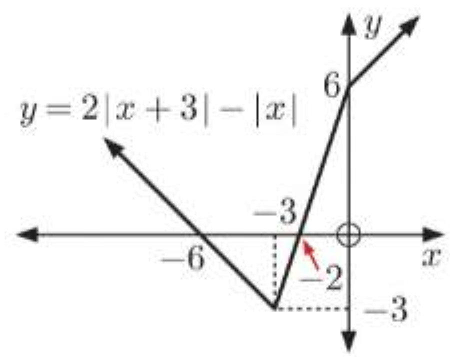
e $|x-2|-|x+1| = \begin{cases} (x-2)-(x+1) & \text{if } x \geqslant 2 \\ (2-x)-(x+1) & \text{if } -1 \leqslant x < 2 \\ (2-x)-(-x-1) & \text{if } x < -1 \end{cases}$

$= \begin{cases} -3 & \text{if } x \geqslant 2 \\ 1-2x & \text{if } -1 \leqslant x < 2 \\ 3 & \text{if } x < -1 \end{cases}$



f $2|x+3|-|x| = \begin{cases} 2(x+3)-x & \text{if } x \geqslant 0 \\ 2(x+3)-(-x) & \text{if } -3 \leqslant x < 0 \\ 2(-x-3)-(-x) & \text{if } x < -3 \end{cases}$

$= \begin{cases} x+6 & \text{if } x \geqslant 0 \\ 3x+6 & \text{if } -3 \leqslant x < 0 \\ -x-6 & \text{if } x < -3 \end{cases}$



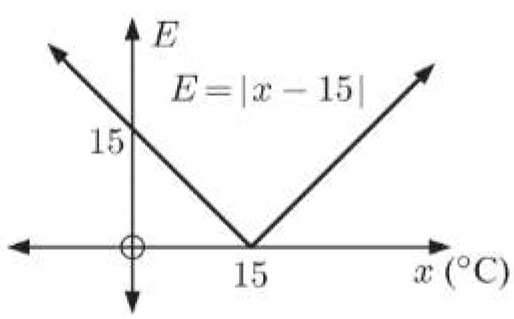
2 **a** The absolute error of the forecast is $E = |x-15|$ °C.

b **i** When $x = 18$,
$E = |18-15|$
$= |3|$
$= 3$°C

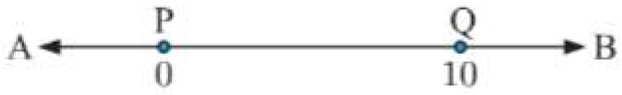
ii When $x = 15$,
$E = |15-15|$
$= |0|$
$= 0$°C

iii When $x = 13$,
$E = |13-15|$
$= |-2|$
$= 2$°C

c $|x-15| = \begin{cases} x-15 & \text{if } x \geqslant 15 \\ 15-x & \text{if } x < 15 \end{cases}$



3



a Susie attends school P.

$\therefore$ Susie must cycle $|x - 0| = |x|$ km.

Sarah attends school Q.

$\therefore$ Sarah must cycle $|x - 10|$ km.

So, the total distance $D = |x| + |x - 10|$ km.

b **i** When $x = 7$,

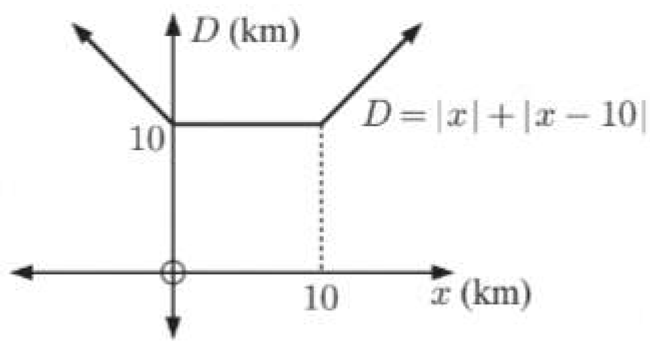
$$\begin{aligned} D &= |7| + |7 - 10| \\ &= 7 + |-3| \\ &= 7 + 3 \\ &= 10 \text{ km} \end{aligned}$$

ii When $x = -3$,

$$\begin{aligned} D &= |-3| + |-3 - 10| \\ &= 3 + |-13| \\ &= 3 + 13 \\ &= 16 \text{ km} \end{aligned}$$

c $$|x| + |x - 10| = \begin{cases} x + (x - 10) & \text{if } x \geqslant 10 \\ x + (10 - x) & \text{if } 0 \leqslant x < 10 \\ -x + (10 - x) & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 2x - 10 & \text{if } x \geqslant 10 \\ 10 & \text{if } 0 \leqslant x < 10 \\ 10 - 2x & \text{if } x < 0 \end{cases}$$



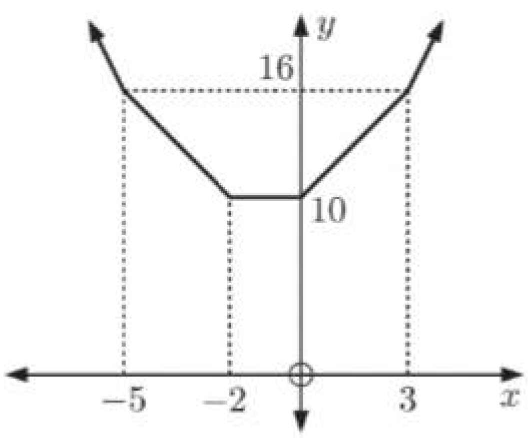
d From the graph in **c**, D is minimised when $0 \leqslant x \leqslant 10$.

The minimum value of D is 10 km.

4 **a** $|x + 5| + |x + 2| + |x| + |x - 3|$

$$= \begin{cases} (x + 5) + (x + 2) + x + (x - 3) & \text{if } x \geqslant 3 \\ (x + 5) + (x + 2) + x + (3 - x) & \text{if } 0 \leqslant x < 3 \\ (x + 5) + (x + 2) + (-x) + (3 - x) & \text{if } -2 \leqslant x < 0 \\ (x + 5) + (-x - 2) + (-x) + (3 - x) & \text{if } -5 \leqslant x < -2 \\ (-x - 5) + (-x - 2) + (-x) + (3 - x) & \text{if } x < -5 \end{cases}$$

$$= \begin{cases} 4x + 4 & \text{if } x \geqslant 3 \\ 2x + 10 & \text{if } 0 \leqslant x < 3 \\ 10 & \text{if } -2 \leqslant x < 0 \\ 6 - 2x & \text{if } -5 \leqslant x < -2 \\ -4x - 4 & \text{if } x < -5 \end{cases}$$



b 

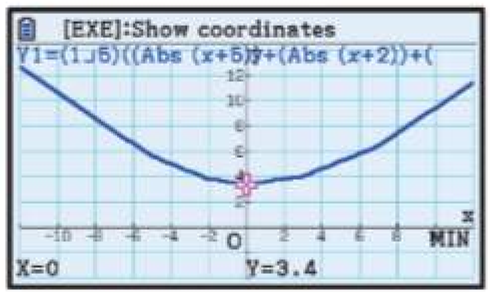
i The total distance from x to the four beaches O, P, Q, and R is
$|x+5|+|x+2|+|x|+|x-3|$ km.
$\therefore$ the average distance to any beach is $\frac{1}{4}(|x+5|+|x+2|+|x|+|x-3|)$ km.

ii From the graph in **a**, $y = |x+5|+|x+2|+|x|+|x-3|$ is minimised when $-2 \leqslant x \leqslant 0$.
So, to minimise the average distance to any of the four beaches, the ambulance should locate its premises anywhere between O and Q inclusive.

iii The total distance from x to the five beaches O, P, Q, R, and S is
$|x+5|+|x+2|+|x|+|x-3|+|x-7|$ km.
$\therefore$ the average distance to any beach is

$$\tfrac{1}{5}(|x+5|+|x+2|+|x|+|x-3|+|x-7|) \text{ km.}$$

Using technology,
$y = \frac{1}{5}(|x+5|+|x+2|+|x|+|x-3|+|x-7|)$ is minimised when $x = 0$.



So, the ambulance service should now locate its premises at O.

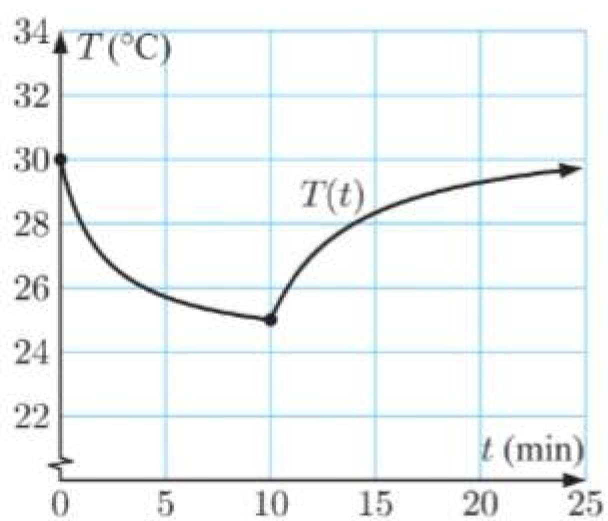
EXERCISE 5C.3

1 $$T(t) = \begin{cases} 24 + \dfrac{12}{t+2}, & 0 \leqslant t < 10 \\[2ex] 31 - \dfrac{24}{t-6}, & t \geqslant 10 \end{cases}$$

a $$\begin{aligned} T(0) &= 24 + \frac{12}{2} \quad \{\text{as } 0 \leqslant t < 10\} \\ &= 30 \end{aligned}$$

The initial temperature of the room was 30°C.

b From the graph, the temperature is decreasing for $0 \leqslant t \leqslant 10$ minutes, and is increasing for $t \geqslant 10$ minutes.
So, the air conditioner was turned on for 10 minutes.



c The air conditioner was turned off after 10 minutes.

$$T(10) = 31 - \frac{24}{10-6} \quad \{\text{as } t \geqslant 10\}$$
$$= 25$$

The temperature of the room was 25°C when the air conditioner was turned off.

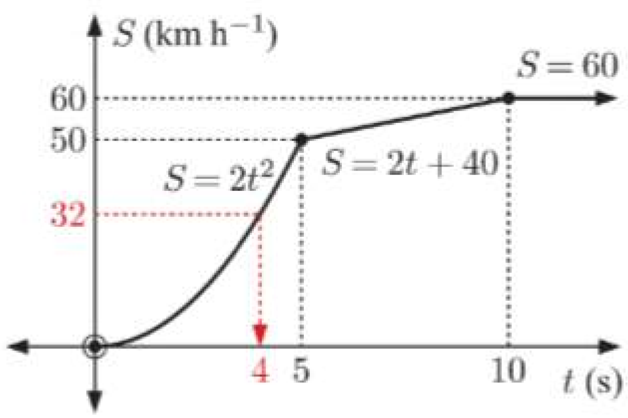
d The room returned to its initial temperature when $31 - \dfrac{24}{t-6} = 30$ {using **a**}

$$\therefore\ 1 = \frac{24}{t-6}$$
$$\therefore\ t - 6 = 24$$
$$\therefore\ t = 30$$

So, it took 30 minutes for the room to return to its original temperature.

2 $S(t) = \begin{cases} 2t^2, & 0 \leqslant t < 5 \\ 2t + 40, & 5 \leqslant t < 10 \\ 60, & t \geqslant 10 \end{cases}$

a



b From the graph in **a**, $S(t)$ has a maximum value of 60.

$\therefore$ the speed limit where the car is driving is 60 km h^{-1}.

c **i** $S(3) = 2(3)^2 \quad \{\text{as } 0 \leqslant t < 5\}$
$= 18$

After 3 seconds, the speed of the car was 18 km h^{-1}.

ii $S(8) = 2(8) + 40 \quad \{\text{as } 5 \leqslant t < 10\}$
$= 56$

After 8 seconds, the speed of the car was 56 km h^{-1}.

iii $S(12) = 60 \quad \{\text{as } t \geqslant 10\}$

After 12 seconds, the speed of the car was 60 km h^{-1}.

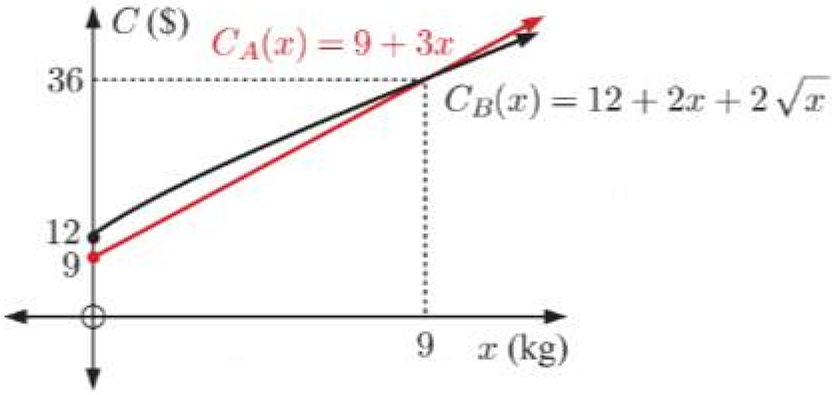
d From the graph in **a**, $S(t) = 32$ when $0 \leqslant t < 5$, so $2t^2 = 32$

$$\therefore\ t^2 = 16$$
$$\therefore\ t = 4 \quad \{t \geqslant 0\}$$

So, the speed of the car was 32 km h^{-1} after 4 seconds.

3 $C_A(x) = 9 + 3x$ and $C_B(x) = 12 + 2x + 2\sqrt{x}$ dollars

a



b From the graph in **a**, $C_A(x) \leqslant C_B(x)$ for $0 \leqslant x < 9$,
and $C_B(x) \leqslant C_A(x)$ for $x \geqslant 9$.

$$\therefore \quad C = \begin{cases} 9 + 3x, & 0 \leqslant x < 9 \\ 12 + 2x + 2\sqrt{x}, & x \geqslant 9 \end{cases}$$

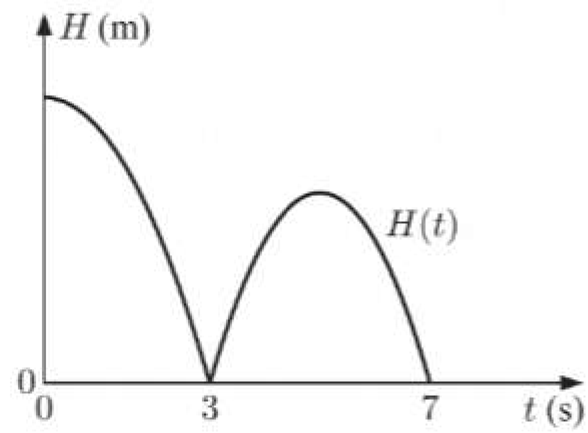
c **i** When $x = 6$, $C = 9 + 3(6)$ {as $0 \leqslant x < 9$}
$= 27$

The cost of 6 kg of tomatoes is $27.

ii When $x = 9$, $C = 12 + 2(9) + 2\sqrt{9}$ {as $x \geqslant 9$}
$= 36$

The cost of 9 kg of tomatoes is $36.

4 **a** $H(t) = \begin{cases} -5t^2 + c, & 0 \leqslant t < 3 \\ -5t^2 + dt + e, & 3 \leqslant t \leqslant 7 \end{cases}$



The ball hits the ground after 3 seconds.

$\therefore \; H(3) = 0$

$\therefore \; -5(3)^2 + 3d + e = 0$ {as $3 \leqslant t \leqslant 7$}

$\therefore \; 3d + e = 45$ .... (1)

The ball hits the ground again after 7 seconds.

$\therefore \; H(7) = 0$

$\therefore \; -5(7)^2 + 7d + e = 0$ {as $3 \leqslant t \leqslant 7$}

$\therefore \; 7d + e = 245$ .... (2)

Now (2) − (1) gives $4d = 200$

$\therefore \; d = 50$

Substituting $d = 50$ into (1) gives $3(50) + e = 45$

$\therefore \; e = -105$

So, $H(t) = \begin{cases} -5t^2 + c, & 0 \leqslant t < 3 \\ -5t^2 + 50t - 105, & 3 \leqslant t \leqslant 7 \end{cases}$

From the graph, when $t = 3$, $-5t^2 + c = -5t^2 + 50t - 105$

$\therefore \; c = 50(3) - 105$

$\therefore \; c = 45$

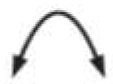
b $H(t) = \begin{cases} -5t^2 + 45, & 0 \leqslant t < 3 \\ -5t^2 + 50t - 105, & 3 \leqslant t \leqslant 7 \end{cases}$

Now $H(0) = -5(0)^2 + 45$ $\{$as $0 \leqslant t < 3\}$
$= 45$

$\therefore$ the tower is 45 m high.

c After the first bounce is when $3 \leqslant t \leqslant 7$.

For $3 \leqslant t \leqslant 7$, $H(t) = -5t^2 + 50t - 105$ which is a quadratic with $a = -5$, $b = 50$, and $c = -105$.

Since $a < 0$, the shape is  .

The maximum height occurs when $x = -\dfrac{b}{2a} = -\dfrac{50}{2(-5)} = 5$.

$H(5) = -5(5)^2 + 50(5) - 105$
$= 20$

$\therefore$ the maximum height reached by the ball after the first bounce was 20 m above ground level.

d $H(t) = 15$ when $-5t^2 + 45 = 15$, $0 \leqslant t < 3$

$\therefore -5t^2 = -30$

$\therefore t^2 = 6$

$\therefore t = \sqrt{6} \approx 2.45$ $\{0 \leqslant t < 3\}$

or $-5t^2 + 50t - 105 = 15$, $3 \leqslant t \leqslant 7$

$\therefore -5t^2 + 50t - 120 = 0$

$\therefore t^2 - 10t + 24 = 0$

$\therefore (t-4)(t-6) = 0$

$\therefore t = 4$ or 6

So, the ball was 15 m above ground level after about 2.45 seconds, 4 seconds, and 6 seconds.

EXERCISE 5D

1 **a** Substituting $(4, 3)$ into the model gives

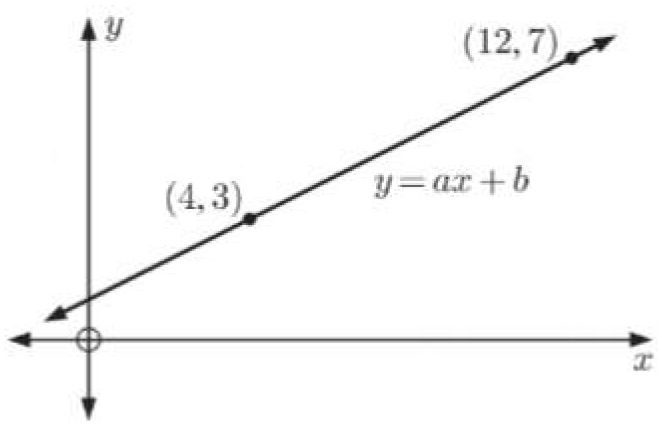
$3 = a(4) + b$

$\therefore 4a + b = 3$

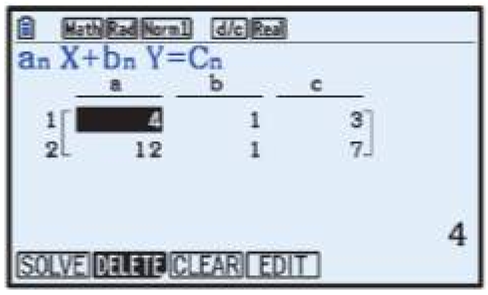
Substituting $(12, 7)$ into the model gives

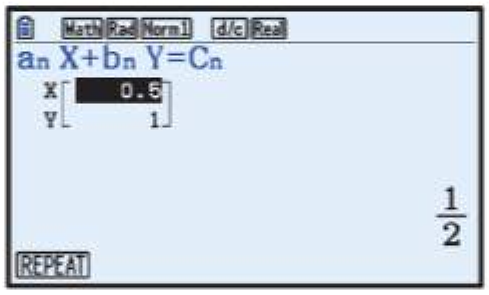
$7 = a(12) + b$

$\therefore 12a + b = 7$



So, we have the system of equations $\begin{cases} 4a + b = 3 \\ 12a + b = 7 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = \frac{1}{2}$ and $b = 1$.

So, the model is $y = \frac{1}{2}x + 1$.

b When $x = 4$,
$$\begin{aligned} y &= \tfrac{1}{2}(4) + 1 \\ &= 2 + 1 \\ &= 3 \ \checkmark \end{aligned}$$

When $x = 12$,
$$\begin{aligned} y &= \tfrac{1}{2}(12) + 1 \\ &= 6 + 1 \\ &= 7 \ \checkmark \end{aligned}$$

2 **a** Substituting $(1, 7)$ into the model gives $7 = a(1)^2 + b(1) + c$

$\therefore\ a + b + c = 7$

Substituting $(2, 10)$ into the model gives $10 = a(2)^2 + b(2) + c$

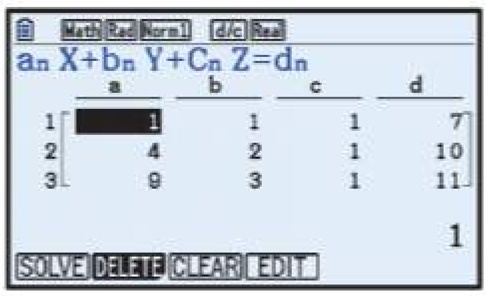
$\therefore\ 4a + 2b + c = 10$

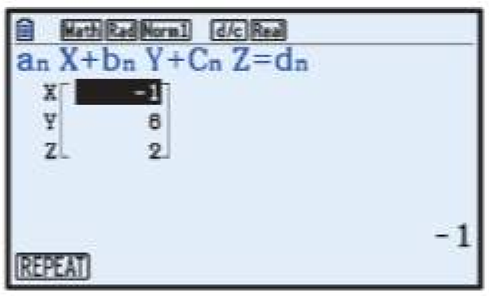
Substituting $(3, 11)$ into the model gives $11 = a(3)^2 + b(3) + c$

$\therefore\ 9a + 3b + c = 11$

So, we have the system of equations $\begin{cases} a + b + c = 7 \\ 4a + 2b + c = 10 \\ 9a + 3b + c = 11 \end{cases}$.

b





Solving the system of equations in **a** using technology, we find that $a = -1$, $b = 6$, and $c = 2$.

c The model is $y = -x^2 + 6x + 2$.

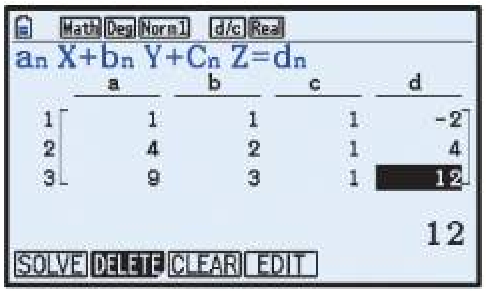
3 **a** Let the quadratic function be $y = ax^2 + bx + c$.

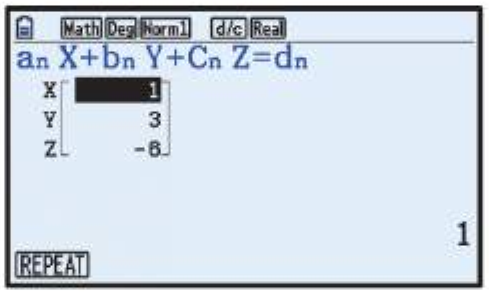
When $x = 1$, $y = -2$ $\therefore\ -2 = a(1)^2 + b(1) + c$ or $a + b + c = -2$

When $x = 2$, $y = 4$ $\therefore\ 4 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 4$

When $x = 3$, $y = 12$ $\therefore\ 12 = a(3)^2 + b(3) + c$ or $9a + 3b + c = 12$

We solve the system of equations $\begin{cases} a + b + c = -2 \\ 4a + 2b + c = 4 \\ 9a + 3b + c = 12 \end{cases}$ simultaneously using technology.





We find that $a = 1$, $b = 3$, and $c = -6$.

So, the function is $y = x^2 + 3x - 6$.

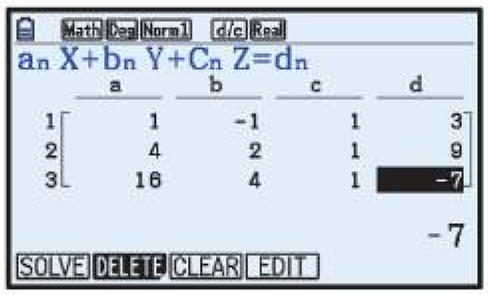
b Let the quadratic function be $y = ax^2 + bx + c$.

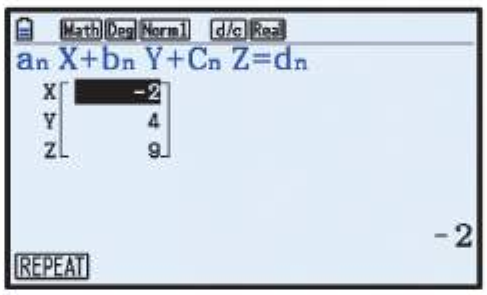
When $x = -1$, $y = 3$ $\therefore$ $3 = a(-1)^2 + b(-1) + c$ or $a - b + c = 3$

When $x = 2$, $y = 9$ $\therefore$ $9 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 9$

When $x = 4$, $y = -7$ $\therefore$ $-7 = a(4)^2 + b(4) + c$ or $16a + 4b + c = -7$

We solve the system of equations $\begin{cases} a - b + c = 3 \\ 4a + 2b + c = 9 \\ 16a + 4b + c = -7 \end{cases}$ simultaneously using technology.





We find that $a = -2$, $b = 4$, and $c = 9$.

So, the function is $y = -2x^2 + 4x + 9$.

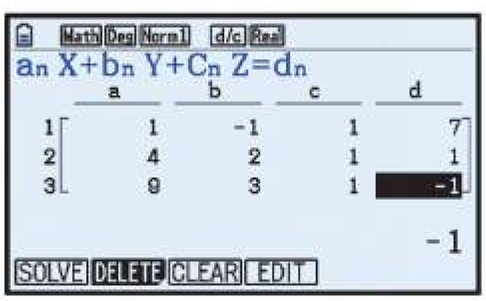
4 Let the quadratic function be $y = ax^2 + bx + c$.

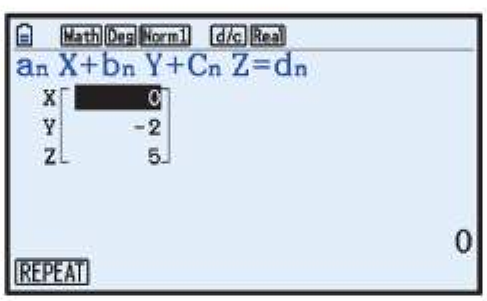
When $x = -1$, $y = 7$ $\therefore$ $7 = a(-1)^2 + b(-1) + c$ or $a - b + c = 7$

When $x = 2$, $y = 1$ $\therefore$ $1 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 1$

When $x = 3$, $y = -1$ $\therefore$ $-1 = a(3)^2 + b(3) + c$ or $9a + 3b + c = -1$

We solve the system of equations $\begin{cases} a - b + c = 7 \\ 4a + 2b + c = 1 \\ 9a + 3b + c = -1 \end{cases}$ simultaneously using technology.



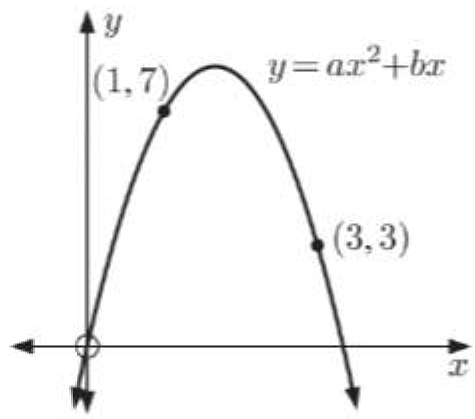


We find that $a = 0$, $b = -2$, and $c = 5$.

So, the three points lie on the line $y = -2x + 5$, but not on a quadratic.

5 **a** Substituting $(1, 7)$ into the model gives $7 = a(1)^2 + b(1)$

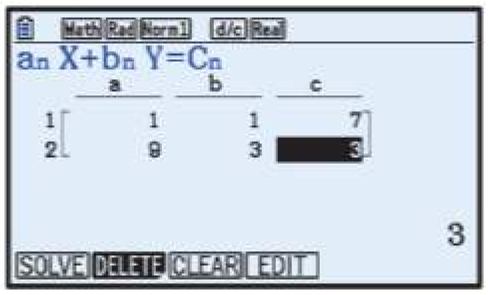
$\therefore\ a + b = 7$

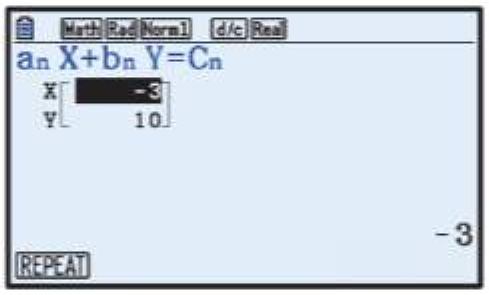


Substituting $(3, 3)$ into the model gives $3 = a(3)^2 + b(3)$

$\therefore\ 9a + 3b = 3$

So, we have the system of equations $\begin{cases} a + b = 7 \\ 9a + 3b = 3 \end{cases}$.





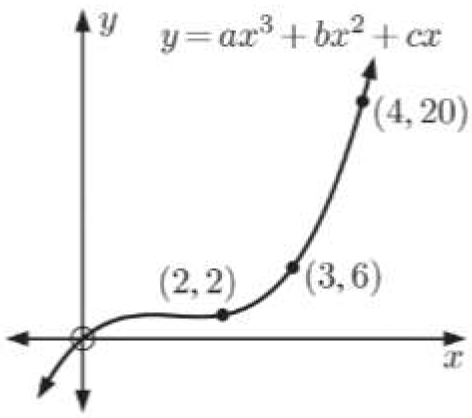
Solving these equations simultaneously using technology, we find that $a = -3$ and $b = 10$.

The model is $y = -3x^2 + 10x$.

b Substituting $(2, 2)$ into the model gives

$2 = a(2)^3 + b(2)^2 + c(2)$

$\therefore\ 8a + 4b + 2c = 2$



Substituting $(3, 6)$ into the model gives

$6 = a(3)^3 + b(3)^2 + c(3)$

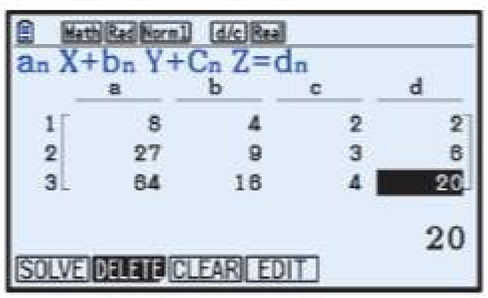
$\therefore\ 27a + 9b + 3c = 6$

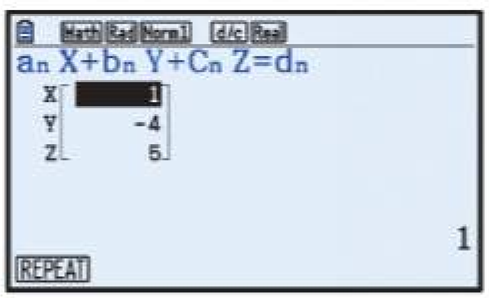
Substituting $(4, 20)$ into the model gives

$20 = a(4)^3 + b(4)^2 + c(4)$

$\therefore\ 64a + 16b + 4c = 20$

So, we have the system of equations $\begin{cases} 8a + 4b + 2c = 2 \\ 27a + 9b + 3c = 6 \\ 64a + 16b + 4c = 20 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 1$, $b = -4$, and $c = 5$.

The model is $y = x^3 - 4x^2 + 5x$.

c Substituting $(1, 5)$ into the model gives

$$5 = a(1) + \frac{b}{1} + c$$

$\therefore \ a + b + c = 5$

Substituting $(2, 4)$ into the model gives

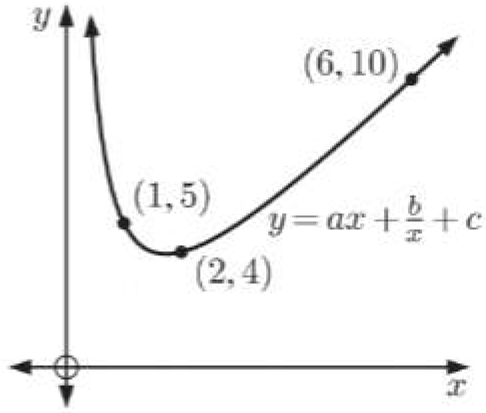
$$4 = a(2) + \frac{b}{2} + c$$

$\therefore \ 2a + \frac{1}{2}b + c = 4$

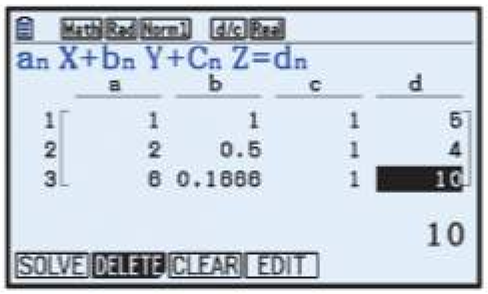
Substituting $(6, 10)$ into the model gives

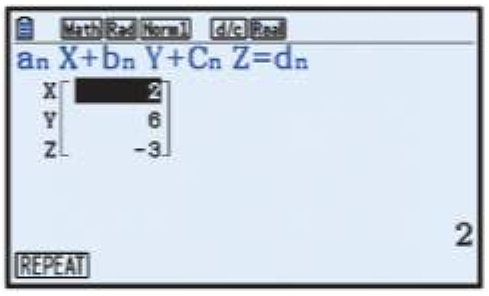
$$10 = a(6) + \frac{b}{6} + c$$

$\therefore \ 6a + \frac{1}{6}b + c = 10$



So, we have the system of equations $\begin{cases} a + b + c = 5 \\ 2a + \frac{1}{2}b + c = 4 \\ 6a + \frac{1}{6}b + c = 10 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 2$, $b = 6$, and $c = -3$.

The model is $y = 2x + \dfrac{6}{x} - 3$.

6 When $t = 1$, $D = 1.7$

$\therefore \ 1.7 = a(1)^2 + b(1) + c\sqrt{1}$

$\therefore \ a + b + c = 1.7$

t (seconds)	1	4	9
D (metres)	1.7	7.2	23.7

When $t = 4$, $D = 7.2$

$\therefore \ 7.2 = a(4)^2 + b(4) + c\sqrt{4}$

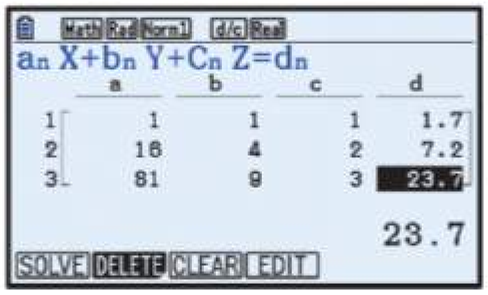
$\therefore \ 16a + 4b + 2c = 7.2$

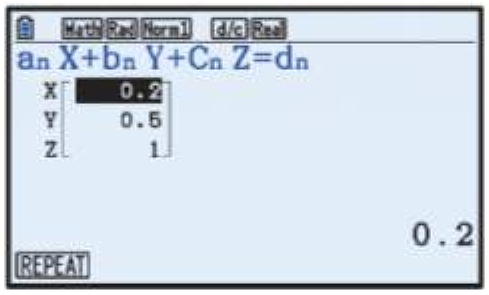
When $t = 9$, $D = 23.7$

$\therefore \ 23.7 = a(9)^2 + b(9) + c\sqrt{9}$

$\therefore \ 81a + 9b + 3c = 23.7$

So, we have the system of equations $\begin{cases} a + b + c = 1.7 \\ 16a + 4b + 2c = 7.2 \\ 81a + 9b + 3c = 23.7 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 0.2$, $b = 0.5$, and $c = 1$.

7 **a** Substituting $(2, 16)$ into the model gives

$$16 = \frac{a}{2} + b(2)$$

$\therefore\ \frac{1}{2}a + 2b = 16$

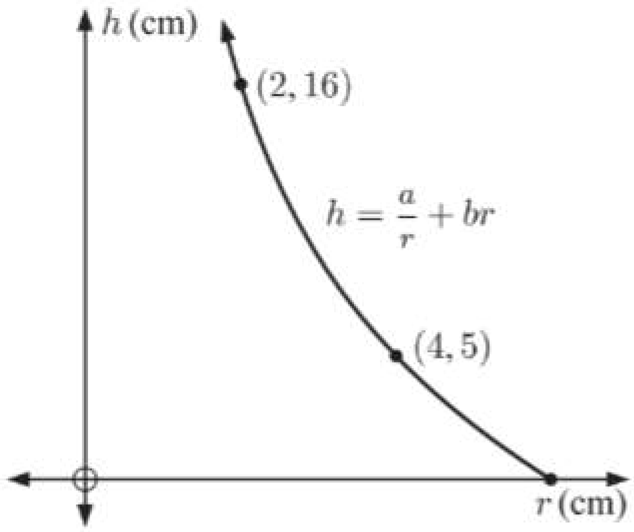
Substituting $(4, 5)$ into the model gives

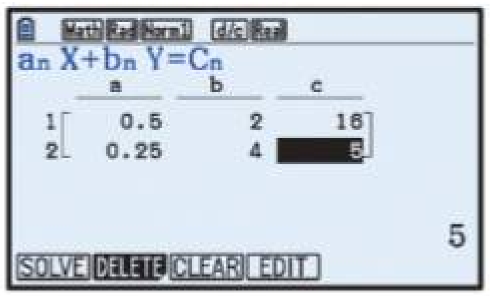
$$5 = \frac{a}{4} + b(4)$$

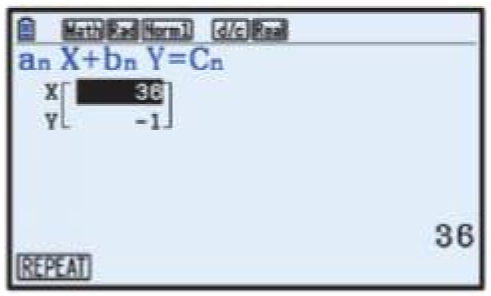
$\therefore\ \frac{1}{4}a + 4b = 5$

So, we have the system of equations

$$\begin{cases} \frac{1}{2}a + 2b = 16 \\ \frac{1}{4}a + 4b = 5 \end{cases}.$$







Solving these equations simultaneously using technology, we find that $a = 36$, $b = -1$.

The model is $h = \dfrac{36}{r} - r$.

b The surface area of a cylinder is $A = 2\pi rh + 2\pi r^2$

$$\therefore\ 2\pi rh = A - 2\pi r^2$$

$$\therefore\ h = \frac{A}{2\pi r} - r$$

$$\therefore\ h = \frac{\frac{A}{2\pi}}{r} - r$$

which has the same form as the model in **a**. The model in **a** therefore seems reasonable.

c $A = 2\pi rh + 2\pi r^2$

$= 2\pi r\left(\dfrac{36}{r} - r\right) + 2\pi r^2$ {using **a**}

$= 72\pi - 2\pi r^2 + 2\pi r^2$

$= 72\pi$

$\therefore$ the constant surface area of these particular cylinders is 72π cm^2.

d When $r = 9$, $h = \dfrac{36}{9} - 9$

$$= 4 - 9$$
$$= -5 \text{ cm}$$

This implies that the cylinder has negative height, which is not reasonable.

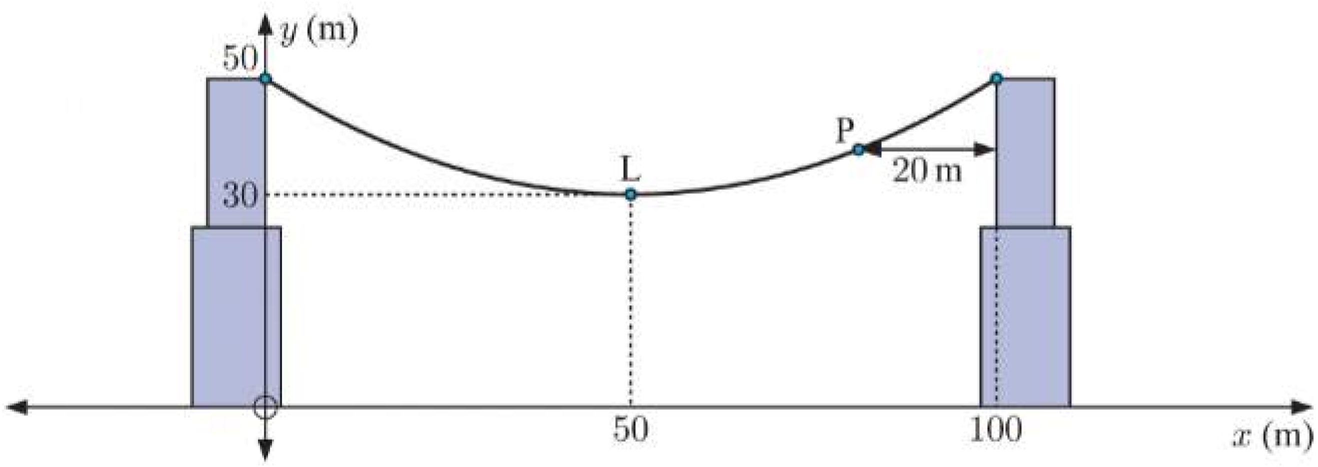
e We require the radius and height to be positive.

$\therefore\ r > 0$ and $h > 0$

$$\frac{36}{r} - r > 0$$
$$\therefore\ \frac{36}{r} > r$$
$$\therefore\ 36 > r^2 \quad \{r > 0\}$$
$$\therefore\ r < 6 \quad \{r > 0\}$$

$\therefore$ it is reasonable to apply this model for $0 < r < 6$.

8



a Let the quadratic model be $y = ax^2 + bx + c$.

When $x = 0$, $y = 50$

$$\therefore\ 50 = a(0)^2 + b(0) + c$$
$$\therefore\ c = 50$$

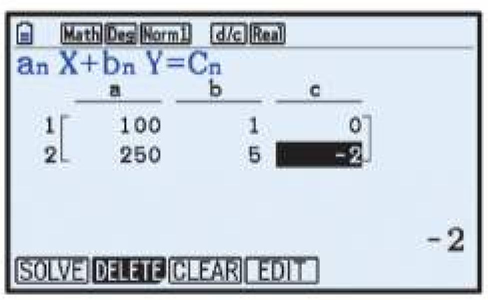
The vertex L has x-coordinate 50 $\quad \therefore\ -\dfrac{b}{2a} = 50$

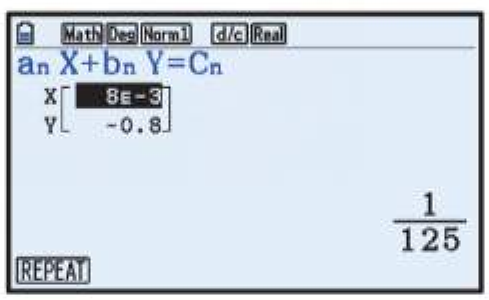
$$\therefore\ -b = 100a$$
$$\therefore\ 100a + b = 0$$

When $x = 50$, $y = 30$ $\quad \therefore\ 30 = a(50)^2 + b(50) + 50$ or $2500a + 50b = -20$

$$\therefore\ 250a + 5b = -2$$

We solve the system of equations $\begin{cases} 100a + b = 0 \\ 250a + 5b = -2 \end{cases}$ simultaneously using technology.





We find that $a = \frac{1}{125}$ and $b = -\frac{4}{5}$.

So, the quadratic model is $y = \frac{1}{125}x^2 - \frac{4}{5}x + 50$.

b Point P has x-coordinate $100 - 20 = 80$.

When $x = 80$, $y = \frac{1}{125}(80)^2 - \frac{4}{5}(80) + 50$

$= \frac{256}{5} - 64 + 50$

$= 37.2$

$\therefore$ the tightrope is 37.2 m above ground level at point P.

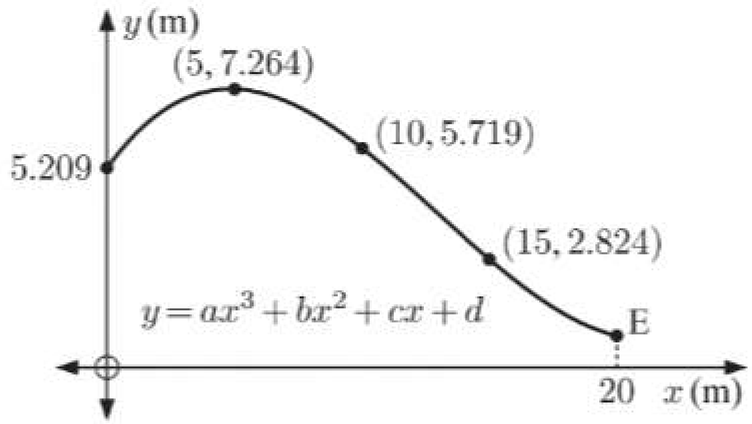
c The platforms are 100 apart.

$\therefore$ the quadratic model is valid for $0 \leqslant x \leqslant 100$.

9 **a** When $x = 0$, $y = 5.209$.

$\therefore$ $5.209 = a(0)^3 + b(0)^2 + c(0) + d$

$\therefore$ $d = 5.209$



b Substituting $(5, 7.264)$ into the model gives $7.264 = a(5)^3 + b(5)^2 + c(5) + 5.209$

$\therefore$ $125a + 25b + 5c = 2.055$

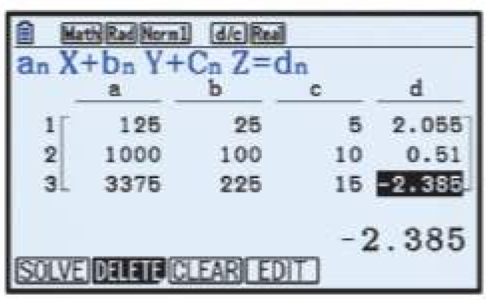
Substituting $(10, 5.719)$ into the model gives $5.719 = a(10)^3 + b(10)^2 + c(10) + 5.209$

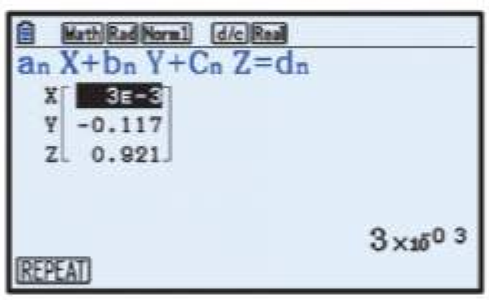
$\therefore$ $1000a + 100b + 10c = 0.51$

Substituting $(15, 2.824)$ into the model gives $2.824 = a(15)^3 + b(15)^2 + c(15) + 5.209$

$\therefore$ $3375a + 225b + 15c = -2.385$

So, we have the system of equations $\begin{cases} 125a + 25b + 5c = 2.055 \\ 1000a + 100b + 10c = 0.51 \\ 3375a + 225b + 15c = -2.385 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 0.003$, $b = -0.117$, and $c = 0.921$.

c The model is $y = 0.003x^3 - 0.117x^2 + 0.921x + 5.209$

When $x = 20$, $y = 0.003(20)^3 - 0.117(20)^2 + 0.921(20) + 5.209$

$= 0.829$

$\therefore$ the height of the mound at point E is 0.829 m.

REVIEW SET 5A

1 **a** Ben can paddle 100 metres in 40 seconds.

We assume that Ben paddles at a constant speed of $\frac{100}{40} = \frac{5}{2}\ \text{m s}^{-1}$.

Let t be the time in seconds it takes Ben to kayak D metres.

Now $\text{speed} = \frac{\text{distance}}{\text{time}}$

$\therefore$ distance = speed × time

$\therefore$ $D = \frac{5}{2}t$ m

b 10 minutes $= 60 \times 10 = 600$ seconds

When $t = 600$, $D = \frac{5}{2}(600)$
$= 1500$

$\therefore$ we predict that Ben can kayak 1500 m in 10 minutes.

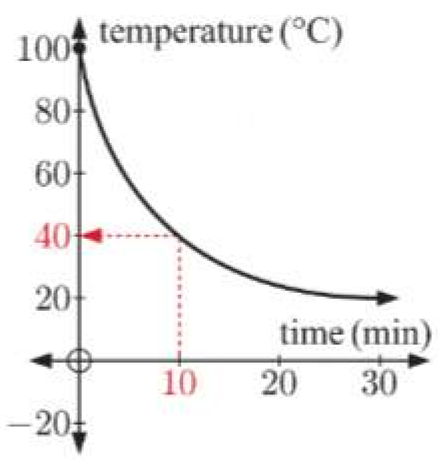
c The actual distance will be less than our prediction. Ben will not be able to kayak at the same speed for 10 minutes as he can for 40 seconds.

2 **a** Amy can wire $\frac{1}{3}$ of the house in one day, and Bernard can wire $\frac{1}{3}$ of the house in one day. Assuming they can work together without getting in each other's way, Amy and Bernard could wire $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ of the house in one day.

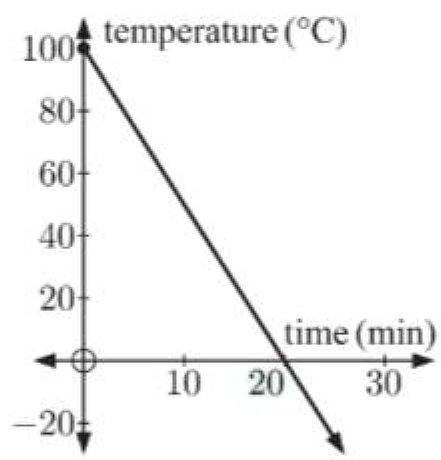
$\therefore$ it would take them $\frac{3}{2} = 1\frac{1}{2}$ days to wire the house if they work together.

b The assumption is not reasonable. The work that Amy or Bernard does is likely to affect the other person, given the complexity of the task. The job will take more time than our prediction in **a**.

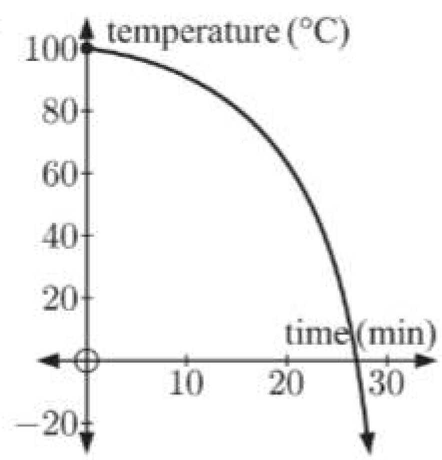
3 **a** **A**



B



C

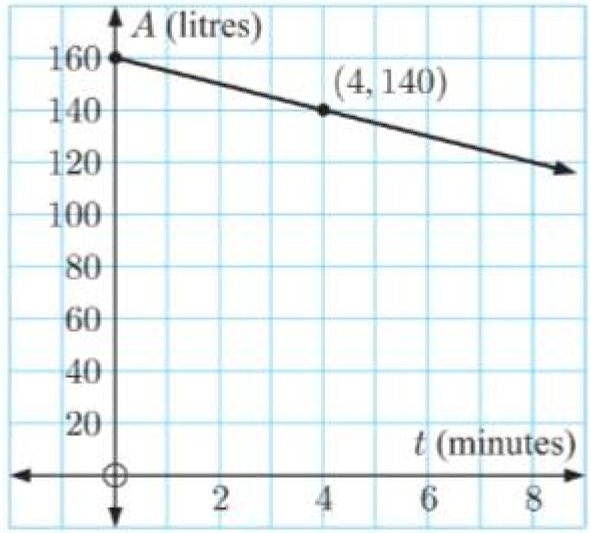


The most appropriate model for the temperature of the water is **A**. Initially, the temperature will decrease quickly, then the rate of temperature decrease will slow as it approaches room temperature.

b Using model **A**, we predict that the temperature of the water after 10 minutes will be about 40°C.

4 **a** The line passes through $(0, 160)$ and $(4, 140)$, so the gradient is $\frac{140-160}{4-0} = -5$. This means that oil is leaking out of the barrel at a rate of 5 L per minute.

The A-intercept is 160. This means that the barrel initially contained 160 litres of oil.



b Using **a**, the equation connecting A and t is $A = 160 - 5t$.

c When $t = 15$,
$$\begin{aligned} A &= 160 - 5(15) \\ &= 160 - 75 \\ &= 85 \end{aligned}$$

There are 85 litres of oil left after 15 minutes.

d We require $t \geqslant 0$ and $A \geqslant 0$
$$\begin{aligned} \therefore\ 160 - 5t &\geqslant 0 \\ \therefore\ 5t &\leqslant 160 \\ \therefore\ t &\leqslant 32 \end{aligned}$$

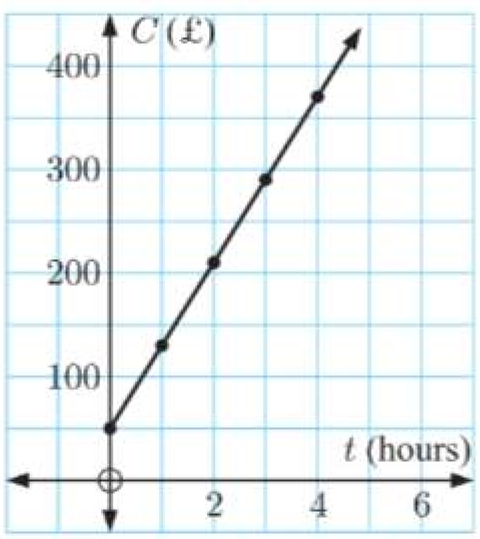
$\therefore$ it is reasonable to apply this model for $0 \leqslant t \leqslant 32$.

5 **a**

Time (t hours)	0	1	2	3	4
Cost (£C)	50	130	210	290	370

+80 +80 +80 +80

b



c The line passes through $(0, 50)$ and $(1, 130)$, so the gradient is $\frac{130-50}{1-0} = 80$.

The C-intercept is 50.

$\therefore\ C = 80t + 50$

d When $t = 6$,
$$\begin{aligned} C &= 80(6) + 50 \\ &= 480 + 50 \\ &= 530 \end{aligned}$$

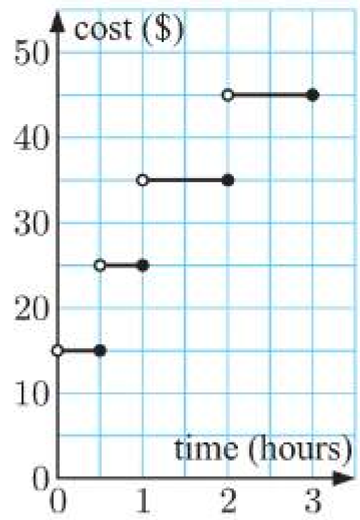
$\therefore$ the cost of a job which takes 6 hours is £530.

6 **a** **i** From the graph, the cost of hiring the court for 45 minutes is \$25.

ii From the graph, the cost of hiring the court for 2 hours is \$35.

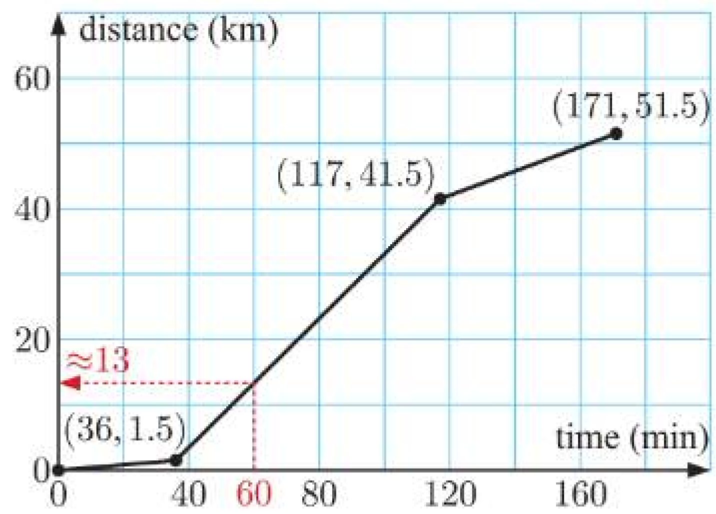
b Kate and Peggy can spend no more than \$15 each, or \$30 in total.

The longest time Kate and Peggy can hire the court for is 1 hour.



7 **a**

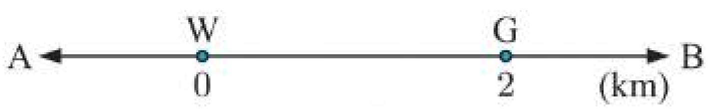
Leg	*Time* (min)
Swim	36
Bicycle ride	81
Run	54



b We have assumed that Alana travels at a constant speed during each leg of the race.

c Using the graph in **a**, we estimate that Alana had travelled about 13 km after 1 hour.

8



a The length of water pipe required is $|x-0| = |x|$ km.

The length of gas pipe required is $|x-2|$ km.

$\therefore$ the total length of pipe required is $L = |x| + |x-2|$ km.

b **i** When $x = 1.5$,

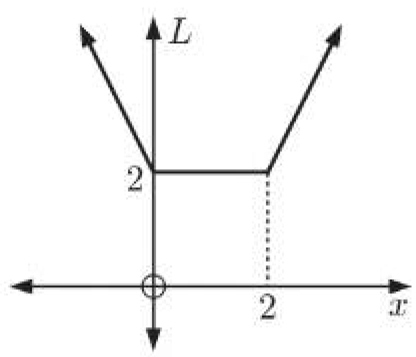
$$\begin{aligned} L &= |1.5| + |1.5-2| \\ &= 1.5 + |-0.5| \\ &= 1.5 + 0.5 \\ &= 2 \text{ km} \end{aligned}$$

ii When $x = -1$,

$$\begin{aligned} L &= |-1| + |-1-2| \\ &= 1 + |-3| \\ &= 1 + 3 \\ &= 4 \text{ km} \end{aligned}$$

c $$|x| + |x-2| = \begin{cases} x + (x-2) & \text{if } x \geqslant 2 \\ x + (2-x) & \text{if } 0 \leqslant x < 2 \\ -x + (2-x) & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 2x-2 & \text{if } x \geqslant 2 \\ 2 & \text{if } 0 \leqslant x < 2 \\ 2-2x & \text{if } x < 0 \end{cases}$$



From the graph, L is minimised when $0 \leqslant x \leqslant 2$.

The minimum value of L is 2 km.

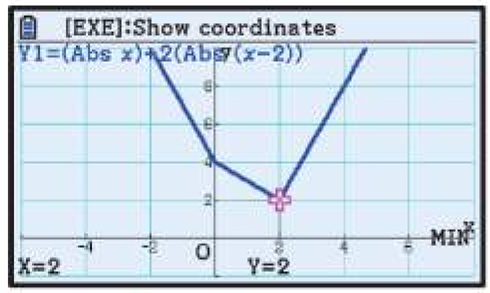
d Let C be the cost of one metre of water pipe.

$\therefore$ $2C$ is the cost of one metre of gas pipe.

So, the total cost of the pipes $= 1000C\,|\,x\,| + 2000C\,|\,x-2\,|$

$= 1000C(|\,x\,| + 2\,|\,x-2\,|)$

Using technology, $y = |\,x\,| + 2\,|\,x-2\,|$ is minimised when $x = 2$.



So, the house should be built at G.

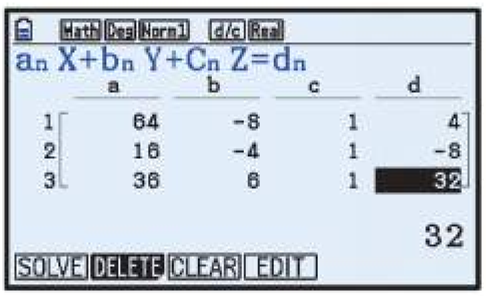
9 Let the quadratic function be $y = ax^2 + bx + c$.

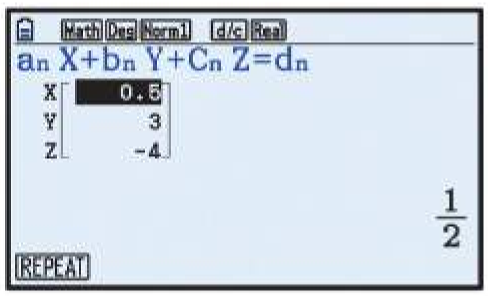
When $x = -8$, $y = 4$ $\quad\therefore\ 4 = a(-8)^2 + b(-8) + c$ or $64a - 8b + c = 4$

When $x = -4$, $y = -8$ $\quad\therefore\ -8 = a(-4)^2 + b(-4) + c$ or $16a - 4b + c = -8$

When $x = 6$, $y = 32$ $\quad\therefore\ 32 = a(6)^2 + b(6) + c$ or $36a + 6b + c = 32$

We solve the system of equations $\begin{cases} 64a - 8b + c = 4 \\ 16a - 4b + c = -8 \\ 36a + 6b + c = 32 \end{cases}$ simultaneously using technology.





We find that $a = \frac{1}{2}$, $b = 3$, and $c = -4$.

So, the function is $y = \frac{1}{2}x^2 + 3x - 4$.

10 **a** Substituting $(1, 7)$ into the model gives

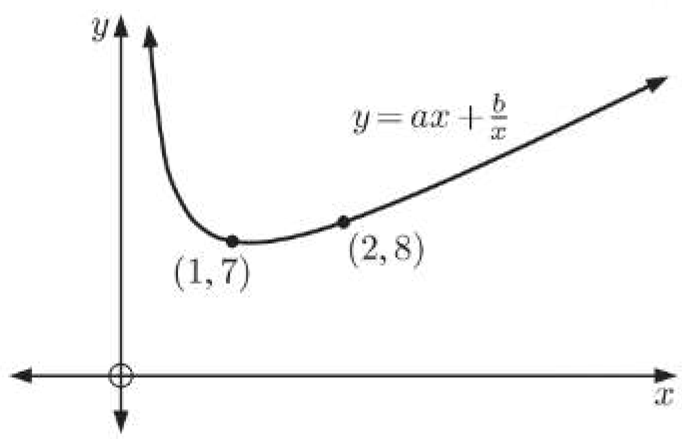
$$7 = a(1) + \frac{b}{1}$$

$\therefore$ $a + b = 7$

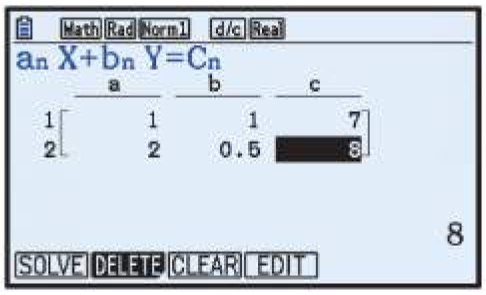
Substituting $(2, 8)$ into the model gives

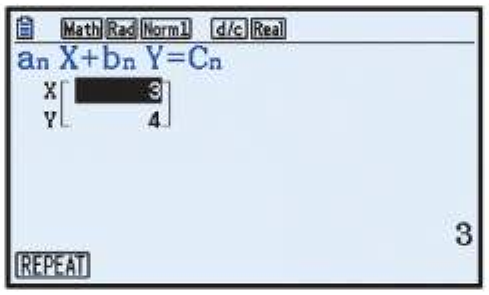
$$8 = a(2) + \frac{b}{2}$$

$\therefore$ $2a + \frac{1}{2}b = 8$



So, we have the system of equations $\begin{cases} a + b = 7 \\ 2a + \frac{1}{2}b = 8 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 3$ and $b = 4$.

The model is $y = 3x + \dfrac{4}{x}$.

b Substituting $(1, 1)$ into the model gives

$$1 = a(1)^3 + b(1)^2 + c(1)$$

$\therefore\ a + b + c = 1$

Substituting $(2, 6)$ into the model gives

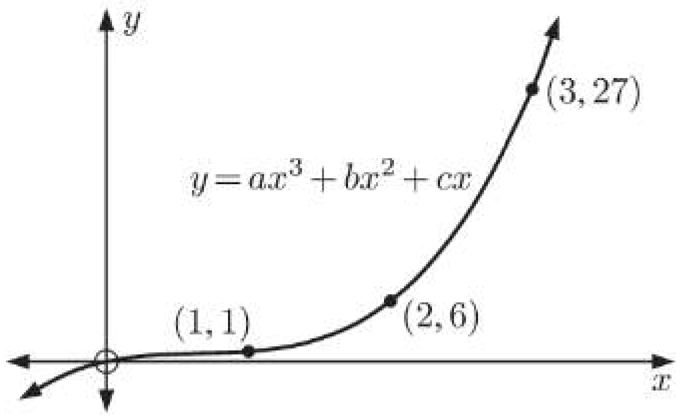
$$6 = a(2)^3 + b(2)^2 + c(2)$$

$\therefore\ 8a + 4b + 2c = 6$

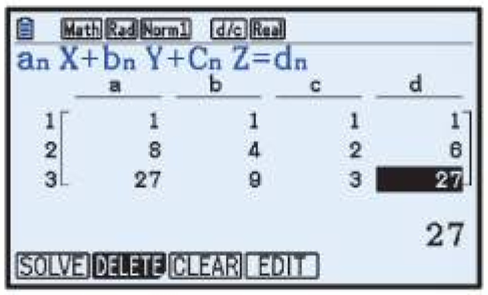
Substituting $(3, 27)$ into the model gives

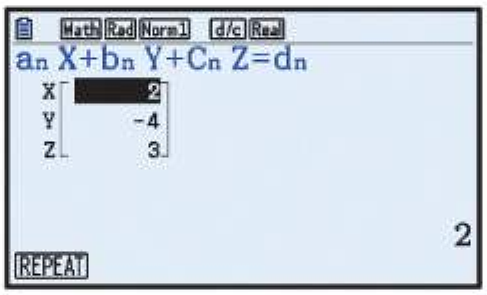
$$27 = a(3)^3 + b(3)^2 + c(3)$$

$\therefore\ 27a + 9b + 3c = 27$



So, we have the system of equations $\begin{cases} a + b + c = 1 \\ 8a + 4b + 2c = 6 \\ 27a + 9b + 3c = 27 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 2$, $b = -4$, and $c = 3$.

The model is $y = 2x^3 - 4x^2 + 3x$.

11 **a** Substituting $(10, 19)$ into the model gives

$$19 = a(10)^2 + b(10)$$

$\therefore\ 100a + 10b = 19$

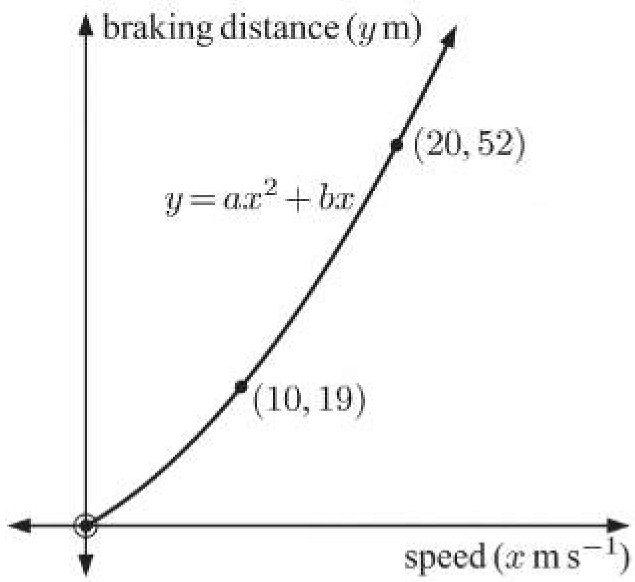
Substituting $(20, 52)$ into the model gives

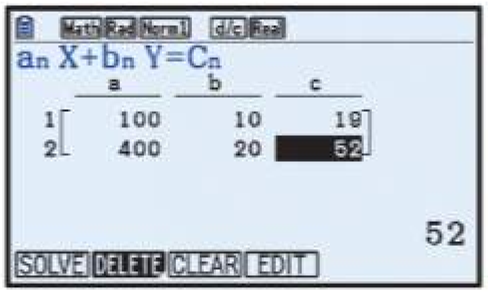
$$52 = a(20)^2 + b(20)$$

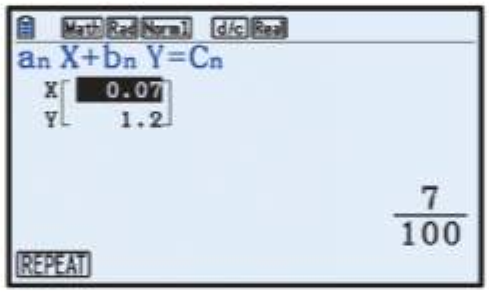
$\therefore\ 400a + 20b = 52$

So, we have the system of equations

$\begin{cases} 100a + 10b = 19 \\ 400a + 20b = 52 \end{cases}$.







Solving these equations simultaneously using technology, we find that $a = \frac{7}{100}$ and $b = \frac{6}{5}$.

b The model is $y = \frac{7}{100}x^2 + \frac{6}{5}x$

When $x = 30$, $y = \frac{7}{100}(30)^2 + \frac{6}{5}(30)$

$= 99$

$\therefore$ we predict that Daniel's braking distance when he is travelling at $30\ \text{m s}^{-1}$ is 99 metres.

c No, we cannot use this model to predict the braking distance for a different person. Braking distance is dependent on the car and the reaction time of the driver.

REVIEW SET 5B

1 Todd can weed $\frac{1}{3}$ of the garden each hour, and Sophie can weed $\frac{1}{3.5} = \frac{2}{7}$ of the garden each hour. We assume Todd and Sophie can work together without getting in each other's way. So, working together they will weed $\frac{1}{3} + \frac{2}{7} = \frac{13}{21}$ of the garden each hour.

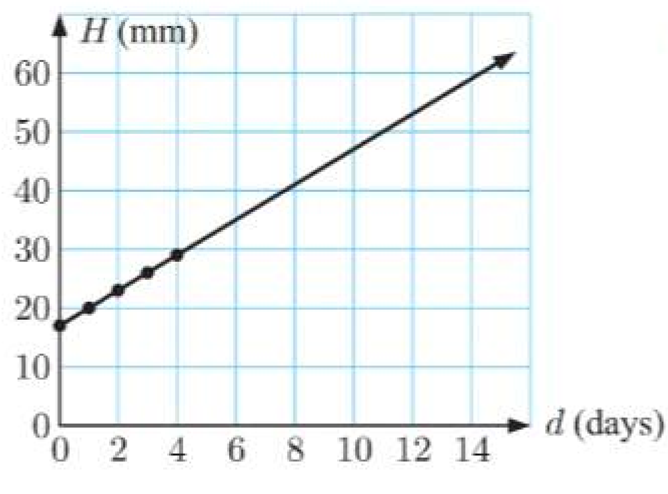
$\therefore$ it would take them $\frac{21}{13} = 1\frac{8}{13}$ hours $\approx$ 1 hour 37 minutes to weed the garden together.

2 **a**

d (days)	0	1	2	3	4
H (mm)	17	20	23	26	29

+3 +3 +3 +3

b



c The line passes through $(0, 17)$ and $(1, 20)$, so the gradient is $\dfrac{20-17}{1-0} = 3$.

The H-intercept is 17.

$\therefore$ $H = 3d + 17$

d When $d = 12$, $H = 3(12) + 17$

$= 36 + 17$

$= 53$

$\therefore$ after 12 days, the height of the lawn is 53 mm.

e When the lawn is 8 cm high, $H = 80$

$$\therefore\ 3d + 17 = 80$$
$$\therefore\ 3d = 63$$
$$\therefore\ d = 21$$

$\therefore$ Rohan mows the lawn every 21 days (or 3 weeks).

3 **a** Yes, these are reasonable assumptions. Not all of the wood in a tree is usable, and assuming that the trees are cylindrical makes it easier to perform calculations.

b $V = 80\% \times \pi r^2 h$

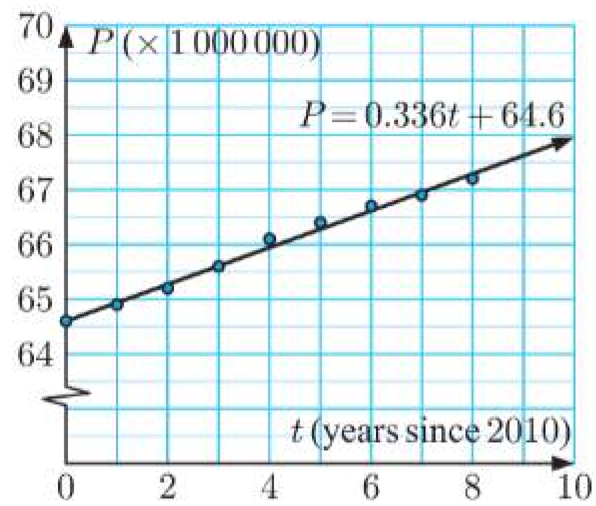
$$= 0.8 \times \pi \left(\frac{0.45}{2}\right)^2 \times h \text{ m}^3$$
$$= 0.8 \times \pi (0.225)^2 h \text{ m}^3$$
$$= \frac{81\pi h}{2000} \text{ m}^3$$

c When $h = 15$, $V = \dfrac{81\pi(15)}{2000}$

$$\approx 1.91 \text{ m}^3$$

$\therefore$ we predict that there is about 1.91 m^3 of usable wood in a 15 m high tree.

4 **a** The points do not lie exactly on the line.
$\therefore$ the model is approximate.



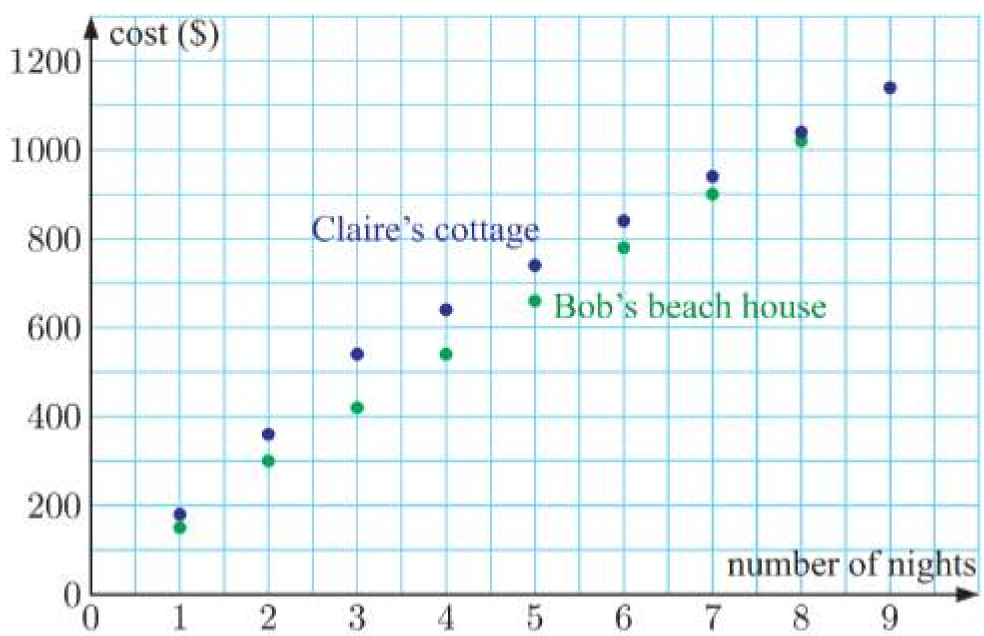
b When $P = 75$, $75 = 0.336t + 64.6$

$$\therefore\ 0.336t = 10.4$$
$$\therefore\ t \approx 30.96$$

$\therefore$ the model predicts that France's population will reach 75 million in the 31st year after the start of 2010, which corresponds to the year 2040.

This is an extrapolation, so this prediction is not likely to be reasonable.

5 **a**



b **i** Cost of staying for 4 nights at Bob's Beach House $= 2 \times \$150 + 2 \times \$120$

$$= \$540$$

ii Cost of staying for 4 nights at Claire's Cottage $= 3 \times \$180 + 1 \times \$100$
$= \$640$

c From the graph, the cost is the same at each place when staying for 9 nights.

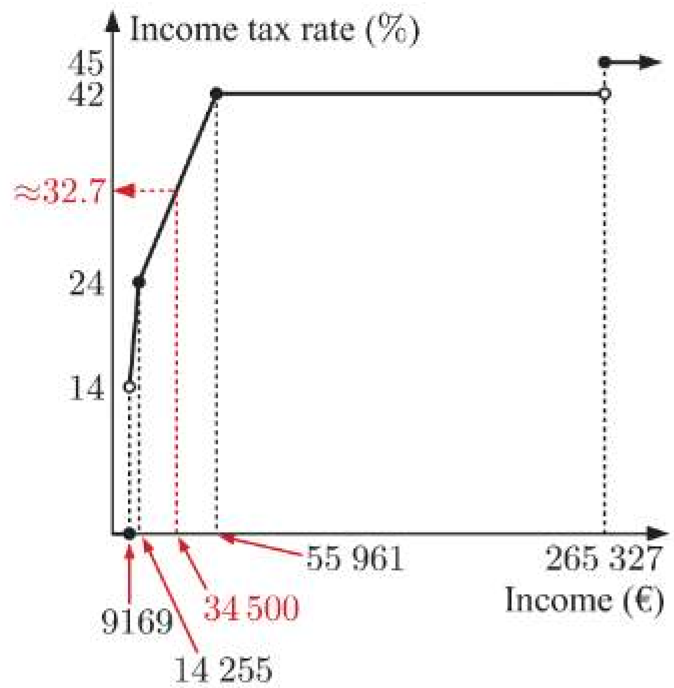
d Cost of staying for 8 nights at Bob's Beach House $= 2 \times \$150 + 6 \times \$120$
$= \$1020$

Cost of staying for 8 nights at Claire's Cottage $= 3 \times \$180 + 5 \times \$100$
$= \$1040$

It is $\$1040 - \$1020 = \$20$ cheaper to stay 8 nights at Bob's Beach House.

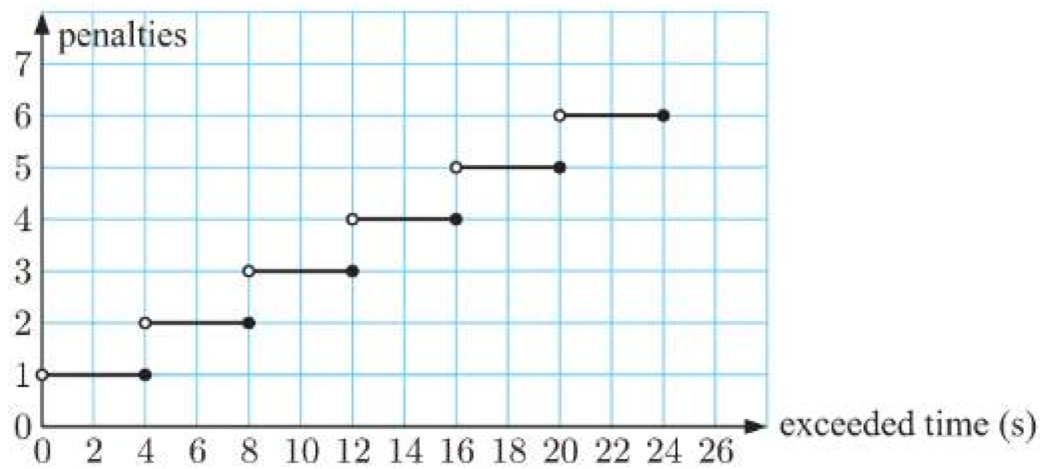
6 **a** This is a piecewise linear model.

b Using the graph, we find that the rate of income tax payable for Gert-Jan, who has an annual income of €34 500, is about 32.7%.



c €108 609 = €55 961 + €52 648
$\therefore$ tax payable = €14 729.32 + 0.42 × €52 648
= €14 729.32 + €22 112.16
= €36 841.48

7 **a**



b **i** 83.1 seconds is $83.1 - 82 = 1.1$ seconds over the time allowed.
The rider would be given 1 time penalty.

ii 87.9 seconds is $87.9 - 82 = 5.9$ seconds over the time allowed.
The rider would be given 2 time penalties.

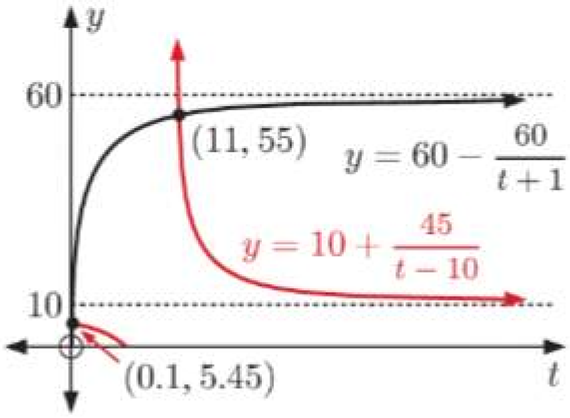
iii 81.5 seconds is less than the time allowed, so the rider would not be given any time penalties.

iv 96.3 seconds is $96.3 - 82 = 14.3$ seconds over the time allowed.
The rider would be given 4 time penalties.

8 **a** $S(t) = \begin{cases} 60 - \dfrac{k}{t+1}, & 0 \leqslant t < T \\ 10 + \dfrac{45}{t-10}, & t \geqslant T \end{cases}$

Now $S(0) = 0 \quad \therefore \quad 60 - \dfrac{k}{0+1} = 0 \quad \{\text{as } 0 \leqslant t < T\}$

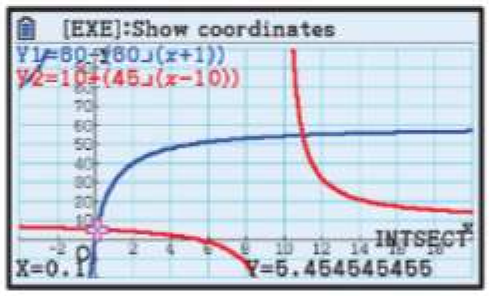
$\therefore \quad k = 60$

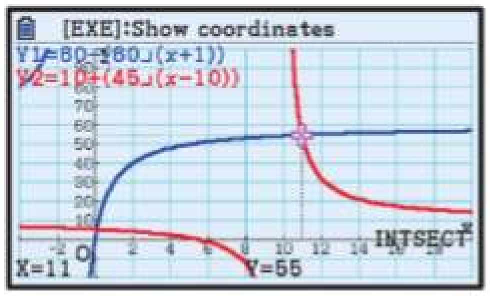
b 

c The skydiver opened her parachute when

$60 - \dfrac{60}{t+1} = 10 + \dfrac{45}{t-10}$

Using technology,
$t = 0.1$ or 11





From the graph in **b**, it is unreasonable that the parachute was opened after 0.1 seconds.
So, the skydiver opened her parachute after $T = 11$ seconds.

d If the skydiver had not opened her parachute, then her downward speed would be

$S(t) = 60 - \dfrac{60}{t+1}\ \text{m}\,\text{s}^{-1}, \quad t \geqslant 0.$

From the graph in **b**, as $t \to \infty$, $S(t) \to 60$.

So, the terminal velocity would be $60\ \text{m}\,\text{s}^{-1}$.

e $S(t) = \begin{cases} 60 - \dfrac{60}{t+1}, & 0 \leqslant t < 11 \\ 10 + \dfrac{45}{t-10}, & t \geqslant 11 \end{cases}$

i $S(5) = 60 - \dfrac{60}{5+1} \quad \{\text{as } 0 \leqslant t < 11\}$

$= 50$

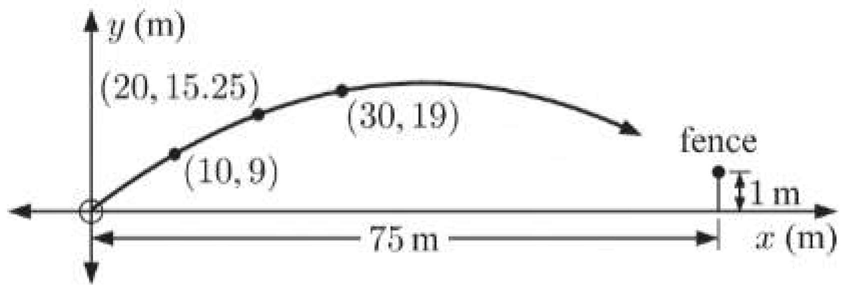
After 5 seconds, the speed of the skydiver was $50\ \text{m}\,\text{s}^{-1}$.

ii $S(15) = 10 + \dfrac{45}{15-10} \quad \{\text{as } t \geqslant 11\}$

$= 19$

After 15 seconds, the speed of the skydiver was $19\ \text{m}\,\text{s}^{-1}$.

9



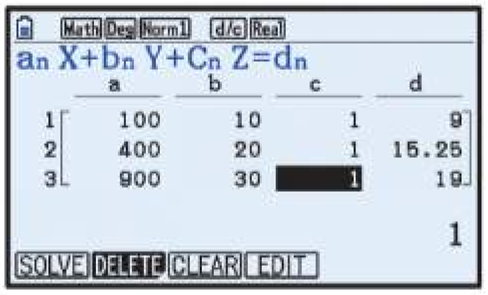
a Let the quadratic model be $y = ax^2 + bx + c$.

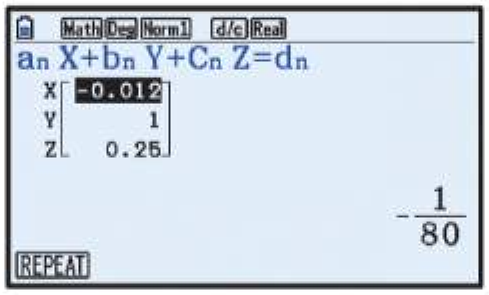
When $x = 10$, $y = 9$ $\quad\therefore\ 9 = a(10)^2 + b(10) + c$ or $100a + 10b + c = 9$

When $x = 20$, $y = 15.25$ $\quad\therefore\ 15.25 = a(20)^2 + b(20) + c$ or $400a + 20b + c = 15.25$

When $x = 30$, $y = 19$ $\quad\therefore\ 19 = a(30)^2 + b(30) + c$ or $900a + 30b + c = 19$

We solve the system of equations $\begin{cases} 100a + 10b + c = 9 \\ 400a + 20b + c = 15.25 \\ 900a + 30b + c = 19 \end{cases}$ simultaneously using technology.





We find that $a = -\frac{1}{80}$, $b = 1$, and $c = \frac{1}{4}$.

So, the quadratic model is $y = -\frac{1}{80}x^2 + x + \frac{1}{4}$.

b For the quadratic model y, $a = -\frac{1}{80}$, $b = 1$, and $c = \frac{1}{4}$.

Since $a < 0$, the shape is  .

The maximum height occurs when $x = -\dfrac{b}{2a} = -\dfrac{1}{2(-\frac{1}{80})} = 40$

Now, when $x = 40$, $\quad y = -\frac{1}{80}(40)^2 + 40 + \frac{1}{4}$

$= -20 + 40 + \frac{1}{4}$

$= 20.25$

$\therefore$ the maximum height reached by the ball is 20.25 m.

c When $x = 75$, $\quad y = -\frac{1}{80}(75)^2 + 75 + \frac{1}{4}$

$= -\frac{1125}{16} + 75 + \frac{1}{4}$

$= 4.9375$

So, the ball is 4.9375 m above the ground after travelling 75 m horizontally.

$\therefore$ the ball will clear the boundary fence.

10 **a** Substituting $(1,\ 3)$ into the model gives

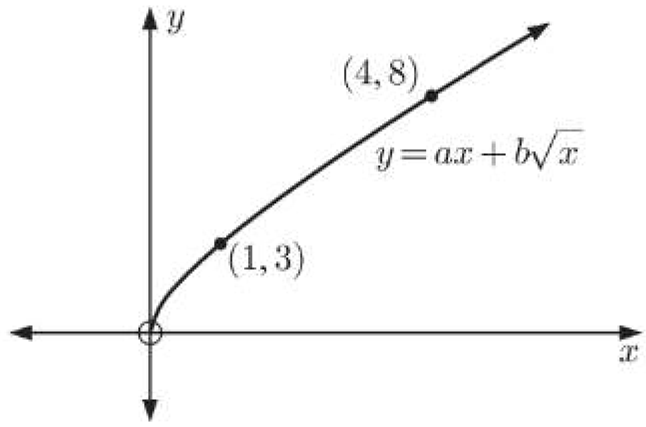
$$3 = a(1) + b\sqrt{1}$$

$\therefore\ a + b = 3$

Substituting $(4,\ 8)$ into the model gives

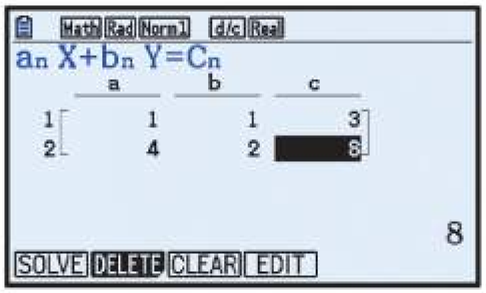
$$8 = a(4) + b\sqrt{4}$$

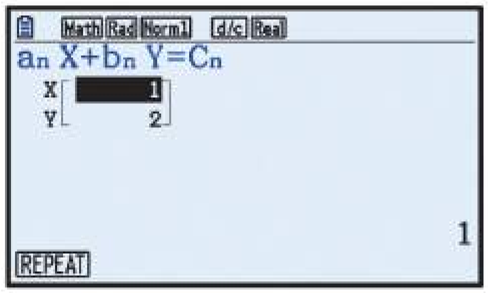
$\therefore\ 4a + 2b = 8$



So, we have the system of equations

$$\begin{cases} a + b = 3 \\ 4a + 2b = 8 \end{cases}.$$





Solving these equations simultaneously using technology, we find that $a = 1$ and $b = 2$.

The model is $y = x + 2\sqrt{x}$.

b Substituting $(1,\ 12)$ into the model gives

$$12 = a(1)^2 + b(1) + \frac{c}{1}$$

$\therefore\ a + b + c = 12$

Substituting $(2,\ 10)$ into the model gives

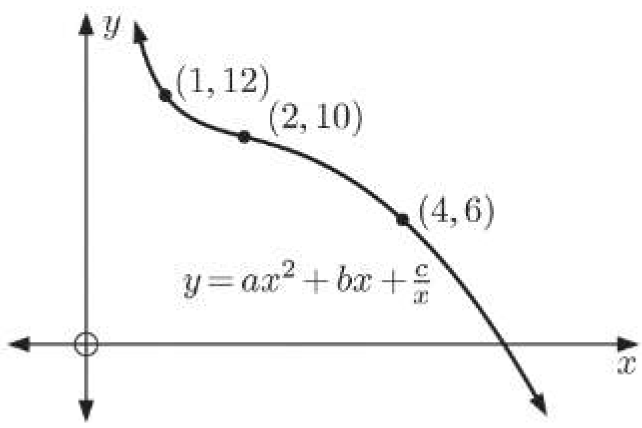
$$10 = a(2)^2 + b(2) + \frac{c}{2}$$

$\therefore\ 4a + 2b + \frac{1}{2}c = 10$

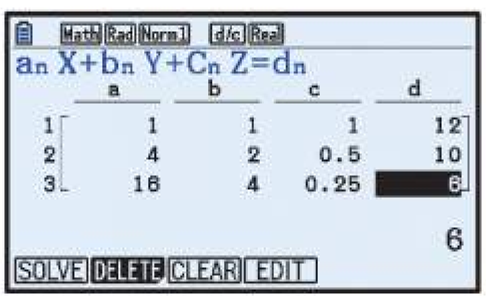
Substituting $(4,\ 6)$ into the model gives

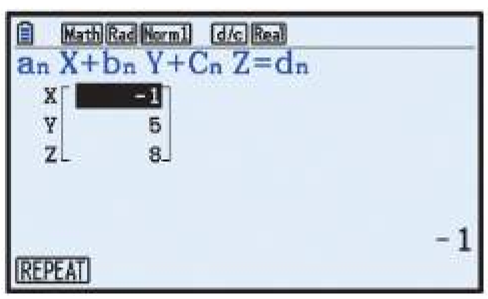
$$6 = a(4)^2 + b(4) + \frac{c}{4}$$

$\therefore\ 16a + 4b + \frac{1}{4}c = 6$



So, we have the system of equations $\begin{cases} a + b + c = 12 \\ 4a + 2b + \frac{1}{2}c = 10 \\ 16a + 4b + \frac{1}{4}c = 6 \end{cases}.$





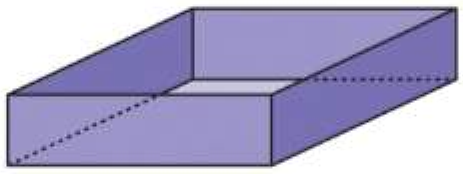
Solving these equations simultaneously using technology, we find that $a = -1$, $b = 5$, and $c = 8$.

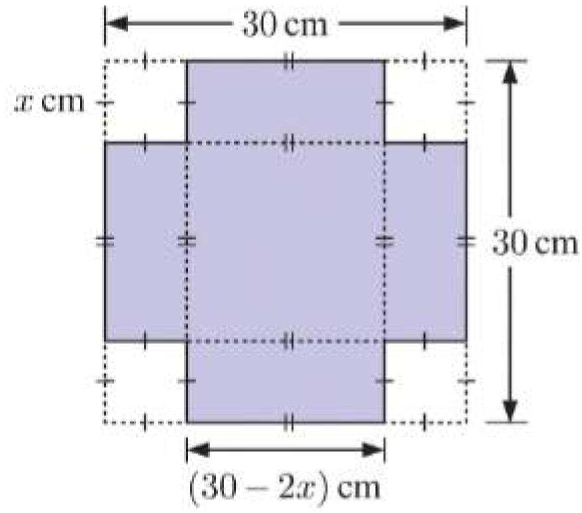
The model is $y = -x^2 + 5x + \dfrac{8}{x}$.

11 **a** If $x = 0$, then we are not cutting out squares from each corner, and the sheet of metal cannot be folded into a tray.

The volume $V = a(0)^3 + b(0)^2 + c(0) + d = 0$

$\therefore \; d = 0$





b When $x = 2$, $V = 1352$

$\therefore \; 1352 = a(2)^3 + b(2)^2 + c(2)$

$\therefore \; 8a + 4b + 2c = 1352$

x (cm)	2	5	10
V (cm^3)	1352	2000	1000

When $x = 5$, $V = 2000$

$\therefore \; 2000 = a(5)^3 + b(5)^2 + c(5)$

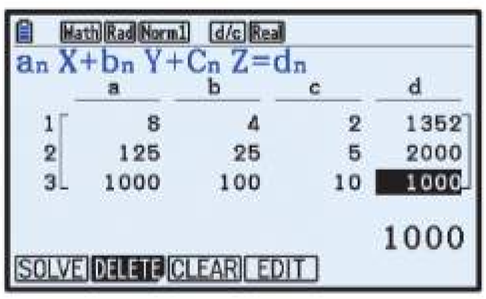
$\therefore \; 125a + 25b + 5c = 2000$

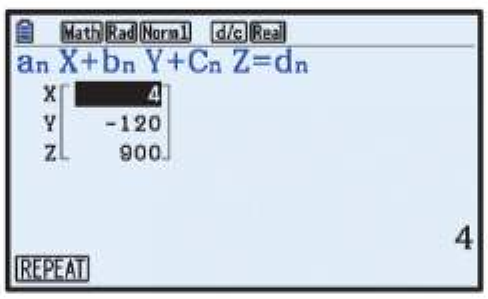
When $x = 10$, $V = 1000$

$\therefore \; 1000 = a(10)^3 + b(10)^2 + c(10)$

$\therefore \; 1000a + 100b + 10c = 1000$

So, we have the system of equations $\begin{cases} 8a + 4b + 2c = 1352 \\ 125a + 25b + 5c = 2000 \\ 1000a + 100b + 10c = 1000 \end{cases}$.





Solving these equations simultaneously using technology, we find that $a = 4$, $b = -120$, and $c = 900$.

c The volume of the rectangular prism $=$ length $\times$ width $\times$ height

$= (30 - 2x)$ cm $\times$ $(30 - 2x)$ cm $\times$ x cm

$= (900 - 120x + 4x^2) \times x$ cm^3

$= 4x^3 - 120x^2 + 900x$ cm^3

which is in the form $V = ax^3 + bx^2 + cx + d$ cm^3, where $d = 0$.

So a model of this form is reasonable.

d We must make a cut greater than 0 cm otherwise the sheet cannot be folded.

We also require $30 - 2x > 0$ as the side lengths must be positive

$\therefore \; 2x < 30$

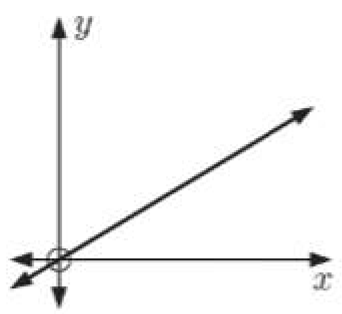
$\therefore \; x < 15$

$\therefore$ it is reasonable to apply this model for $0 < x < 15$.

Chapter 6

DIRECT AND INVERSE VARIATION

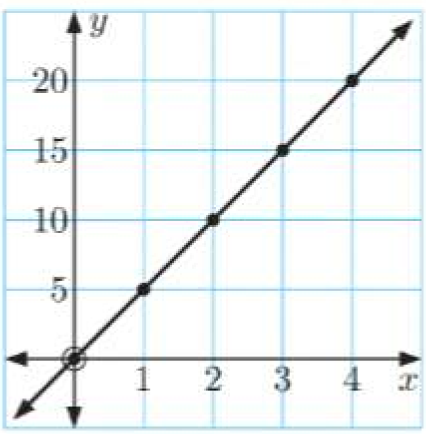
EXERCISE 6A

1 

D indicates that y is directly proportional to x, as it is a straight line which passes through the origin.

2 **a**

x	0	1	2	3	4
y	0	5	10	15	20

i 

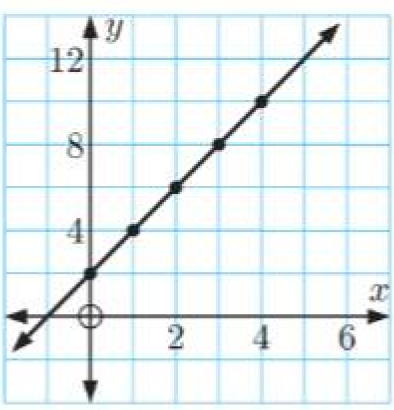
ii The points lie on a straight line which passes through the origin, so $y \propto x$.

The gradient of the line $= \frac{5-0}{1-0} = 5$

$\therefore$ the proportionality constant $k = 5$.

b

x	0	1	2	3	4
y	2	4	6	8	10

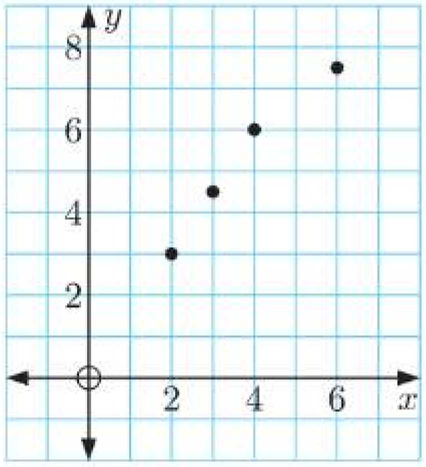
i 

ii The points lie on a straight line which does not pass through the origin, so y is not directly proportional to x.

c

x	2	3	4	6
y	3	4.5	6	7.5

i

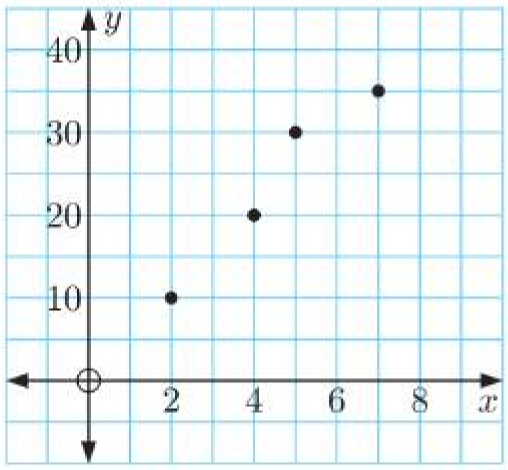


ii The points do not lie on a straight line, so y is not directly proportional to x.

d

x	2	4	5	7
y	10	20	30	35

i



ii The points do not lie on a straight line, so y is not directly proportional to x.

3

Time (t hours)	*Cost* (€C)
$0 < t \leqslant 1$	6
$1 < t \leqslant 2$	12
$2 < t \leqslant 3$	18
$3 < t \leqslant 4$	24
$4 < t \leqslant 5$	30

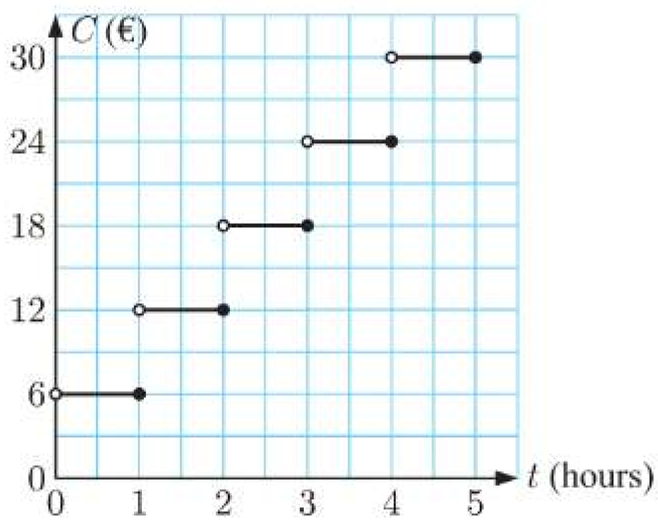
a **i** When $t = 1$, the cost is €6.

ii When $t = 2$, the cost is €12.

iii When $t = 2.5$, the cost is €18.

iv When $t = 4$, the cost is €24.

b



The graph of C against t is not a straight line through the origin.

$\therefore$ C and t are not directly proportional.

4 $y \propto x$

a If x is doubled, then
x is multiplied by 2
$\therefore$ y is multiplied by 2
$\therefore$ y is doubled.

b If y is multiplied by 6, then
x is multiplied by 6.

c If x is divided by 3, then
x is multiplied by $\frac{1}{3}$
$\therefore$ y is multiplied by $\frac{1}{3}$
$\therefore$ y is divided by 3.

d If y is increased by 20%, then
y is multiplied by 1.2
$\therefore$ x is multiplied by 1.2
$\therefore$ x is increased by 20%.

e If x is decreased by 70%, then
x is multiplied by $1 - 0.7 = 0.3$
$\therefore$ y is multiplied by 0.3
$\therefore$ y is decreased by 70%.

f Now $y = kx$ or $x = \dfrac{y}{k}$ where k is the proportionality constant.
If y is increased by 10, then
y becomes $y + 10$

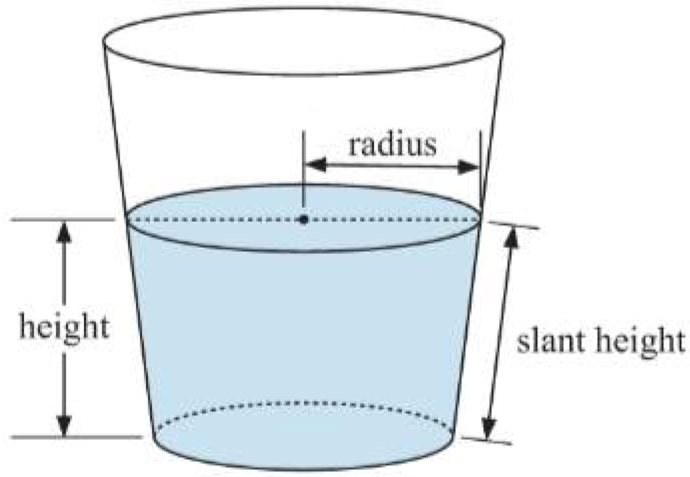
$$\therefore \; x \text{ becomes } \frac{y+10}{k} = \frac{y}{k} + \frac{10}{k} = x + \frac{10}{k}$$

$\therefore$ x is increased by $\dfrac{10}{k}$.

5 When the temperature in °C is 0, the temperature in °F is 32, so the graph of these variables does not pass through the origin.
$\therefore$ these variables are not directly proportional.

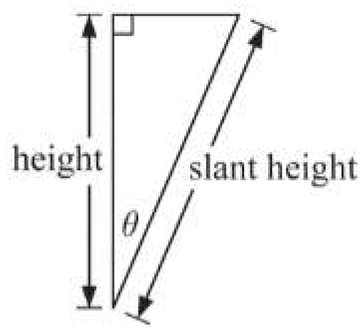
6 We assume the glass is a truncated cone, which is initially empty.

a The *volume of water* is increasing at a constant rate, and there is initially no water in the glass. So, *volume of water* and *time* are **directly proportional**.



b The *volume of water* and *height* are both initially 0, but adding a unit of volume will add less and less to the height as more water is added, as the radius is getting larger. So, *volume of water* and *height* are **not directly proportional**.

c *Height* and *slant height* are both initially 0, and form two sides of a right angled triangle.



Now $\cos\theta = \dfrac{\text{height}}{\text{slant height}}$

$\therefore$ height $=$ slant height $\times \cos\theta$
where $\cos\theta$ is a constant.

So, *height* and *slant height* are **directly proportional**.

d The *height* is initially 0, but the *radius* is not. So, *height* and *radius* are **not directly proportional**.

e *Slant height* is directly proportional to *height* (from **c**), and *weight of water* is directly proportional to *volume of water*. However *height* and *volume of water* are not directly proportional (from **b**). So, *slant height* and *weight of water* are **not directly proportional**.

7 $y \propto x$

a

$\times 3$

x	7	21
y	24	?

To change x from 7 to 21, we multiply by 3.

Since $y \propto x$, we must also multiply y by 3.

$\therefore \quad y = 24 \times 3 = 72$

b

x	7	?
y	24	30

$\times \frac{30}{24}$

To change y from 24 to 30, we multiply by $\dfrac{30}{24} = \dfrac{5}{4}$.

Since $y \propto x$, we must also multiply x by $\frac{5}{4}$.

$$\therefore \quad x = 7 \times \tfrac{5}{4}$$
$$= \tfrac{35}{4}$$

8 **a** Energy usage E and time t are directly proportional, so $E = kt$ where k is the proportionality constant.

When $t = 2$ hours, $E = 3.6$ kWh, so $3.6 = 2k$

$\therefore \quad k = 1.8$

This means the heater uses 1.8 kWh of energy each hour.

b When $t = 1.5$ hours, $E = 1.8 \times 1.5 = 2.7$ kWh.

If the heater is on for 1.5 hours, it uses 2.7 kWh of energy.

c When $E = 6$ kWh, $6 = 1.8t$

$$\therefore \quad t = \frac{6}{1.8} = 3\tfrac{1}{3} \text{ hours}$$

The heater uses 6 kWh of energy when it is on for $3\frac{1}{3}$ hours or 3 hours 20 minutes.

9 Let D m be represented as d cm on Isabella's diagram.

$\therefore \quad D \propto d$ so $D = kd$ where k is the proportionality constant.

a When $D = 60$, $d = 3$, so $60 = k(3)$

$\therefore \quad k = 20$

So, 1 cm on the diagram represents an actual length of 20 m.

b When $D = 110$, $110 = 20d$

$\therefore \quad d = 5.5$

$\therefore$ the football pitch is 5.5 cm long on Isabella's diagram.

c When $d = 15$, $D = 20 \times 15$

$= 300$

$\therefore$ the actual length of the fence is 300 m.

10 The player's momentum p is directly proportional to his velocity v.

$\therefore\ p = kv$ where k is a constant.

When $v = 5\ \text{m s}^{-1}$, $p = 610\ \text{kg m s}^{-1}$, so $610 = k(5)$

$\therefore\ k = 122$

So, $p = 122v$.

a When $v = 3\ \text{m s}^{-1}$, $p = 122 \times 3$
$= 366\ \text{kg m s}^{-1}$

b When $p = 420\ \text{kg m s}^{-1}$, $420 = 122v$

$\therefore\ v = \frac{420}{122}$

$\approx 3.44\ \text{m s}^{-1}$

11 **a** **i** If the plumber is called out but spends no time doing labour $(t = 0)$, then the charge $C = 30$.

$\therefore$ the graph of C against t has C-intercept 30, so does not pass through the origin.

$\therefore$ C and t are not directly proportional.

ii When $t = 0$, $C = 30$

$\therefore$ when $t = 0$, $C - 30 = 30 - 30 = 0$

The plumber charges a constant rate per hour, so the graph of $(C - 30)$ against t passes through the origin and is a straight line.

$\therefore$ $(C - 30)$ and t are directly proportional.

b $(C - 30) \propto t$, so $C - 30 = kt$

When $t = 1.5$, $C = 120$

$\therefore\ 120 - 30 = k \times 1.5$

$\therefore\ k = \frac{90}{1.5} = 60$

So, $C = 60t + 30$

When $t = 5$, $C = 60 \times 5 + 30$
$= 330$

$\therefore$ the charge for a 5 hour job is £330.

EXERCISE 6B

1 **a** $A = \pi r^2$, so $A \propto r^2$ and $k = \pi$.

b $V = \frac{4}{3}\pi r^3$, so $V \propto r^3$ and $k = \frac{4}{3}\pi$.

c $T = \frac{3n^4}{4} = \frac{3}{4}n^4$, so $T \propto n^4$ and $k = \frac{3}{4}$.

2 $y \propto x^3$

a If x is doubled, then

x is multiplied by 2

$\therefore\ x^3$ is multiplied by $2^3 = 8$

$\therefore\ y$ is multiplied by 8.

b If x is divided by 10, then

x is multiplied by $\frac{1}{10}$

$\therefore\ x^3$ is multiplied by $\left(\frac{1}{10}\right)^3 = \frac{1}{1000}$

$\therefore\ y$ is multiplied by $\frac{1}{1000}$

$\therefore\ y$ is divided by 1000.

c If x is increased by 20%, then
x is multiplied by 1.2
$\therefore$ x^3 is multiplied by $(1.2)^3 = 1.728$
$\therefore$ y is multiplied by 1.728
$\therefore$ y is increased by 72.8%.

d If y is multiplied by 2.5, then
x^3 is multiplied by 2.5
$\therefore$ x is multiplied by $\sqrt[3]{2.5}$.

e If y is decreased by 30%, then
y is multiplied by 0.7
$\therefore$ x^3 is multiplied by 0.7
$\therefore$ x is multiplied by $\sqrt[3]{0.7} \approx 0.888$
$\therefore$ x is decreased by $\approx 11.2\%$.

f If y is increased by 100%, then
y is multiplied by 2
$\therefore$ x^3 is multiplied by 2
$\therefore$ x is multiplied by $\sqrt[3]{2} \approx 1.260$
$\therefore$ x is increased by $\approx 26.0\%$.

3 $M \propto t^2$, $t > 0$

a

$\times\frac{9}{6}$

t	6	9
M	40	

t is multiplied by $\frac{9}{6} = \frac{3}{2}$
$\therefore$ t^2 is multiplied by $(\frac{3}{2})^2 = \frac{9}{4}$
$\therefore$ M is multiplied by $\frac{9}{4}$ $\{$as $M \propto t^2\}$
$\therefore$ $M = 40 \times \frac{9}{4} = 90$

b

t	6	
M	40	120

$\times 3$

M is multiplied by 3
$\therefore$ t^2 is multiplied by 3 $\{$as $M \propto t^2\}$
$\therefore$ t is multiplied by $\sqrt{3}$ $\{$as $t > 0\}$
$\therefore$ $t = 6\sqrt{3} \approx 10.4$

4 $V \propto y^3$

a

$\times 4$

y	3	12
V	30	

y is multiplied by 4
$\therefore$ y^3 is multiplied by $4^3 = 64$
$\therefore$ V is multiplied by 64 $\{$as $V \propto y^3\}$
$\therefore$ $V = 30 \times 64 = 1920$

b

y	3	
V	30	180

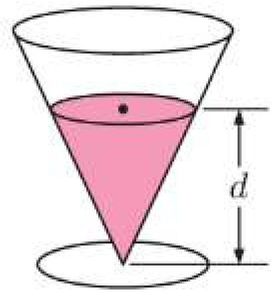
$\times 6$

V is multiplied by 6
$\therefore$ y^3 is multiplied by 6 $\{$as $V \propto y^3\}$
$\therefore$ y is multiplied by $\sqrt[3]{6}$
$\therefore$ $y = 3\sqrt[3]{6} \approx 5.45$

5 The mass of glass m g is directly proportional to the square of the length l cm.
$\therefore$ $m = kl^2$ where k is a constant.
When $m = 900$, $l = 30$, so $900 = k \times 30^2$
$\therefore$ $900 = k \times 900$
$\therefore$ $k = 1$
So, $m = l^2$.
When $l = 50$, $m = 50^2 = 2500$
$\therefore$ a 50 cm square sheet of glass has mass 2500 g.

6 The amount of medicine $V \propto d^3$

$\therefore \; V = kd^3$ where k is a constant.



When $d = 6$ cm, $V = 40$ mL, so $40 = k \times 6^3$

$\therefore \; 40 = k \times 216$

$\therefore \; k = \frac{40}{216} = \frac{5}{27}$

So $V = \frac{5}{27}d^3$.

a When $d = 4$ cm, $V = \frac{5}{27} \times 4^3$

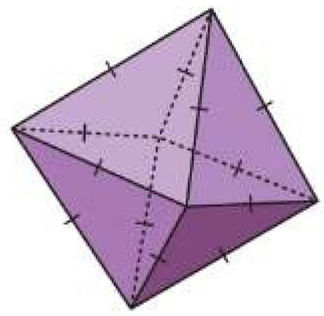
$= \frac{320}{27} \approx 11.9$ mL

b When $V = 30$ mL, $30 = \frac{5}{27}d^3$

$\therefore \; d^3 = 27 \times 6$

$\therefore \; d = 3\sqrt[3]{6} \approx 5.45$ cm

7 The volume V is directly proportional to the cube of its side lengths l, so $V \propto l^3$.



a If l is increased by 5%, then

l is multiplied by 1.05

$\therefore \; l^3$ is multiplied by $(1.05)^3 = 1.157\,625$

$\therefore \; V$ is multiplied by $1.157\,625$ {as $V \propto l^3$}

So, the volume is increased by about 15.8%.

b V is multiplied by 2

$\therefore \; l^3$ is multiplied by 2 {as $V \propto l^3$}

$\therefore \; l$ is multiplied by $\sqrt[3]{2} \approx 1.260$

So, the side length is increased by about 26.0%.

8 **a** $E \propto m$ with proportionality constant $k = \frac{1}{2}v^2$.

b $E \propto v^2$ with proportionality constant $k = \frac{1}{2}m$.

c If v is decreased by 10%, then

v is multiplied by $1 - 0.1 = 0.9$

$\therefore \; v^2$ is multiplied by $(0.9)^2$

$\therefore \; E$ is multiplied by $(0.9)^2 = 0.81$ {as $E \propto v^2$ from **b**}

So, the kinetic energy decreases by 19%.

d We assume that the amount of heat generated is directly proportional to the stopping distance d.

So, $d \propto E$ and $E \propto v^2$ {from **b**}

$\therefore \; d \propto v^2$

9 $$z \propto x^2$$

$\therefore \quad z = kx^2$ where k is a constant

$\therefore \quad z^{\frac{1}{4}} = \pm k^{\frac{1}{4}} x^{\frac{2}{4}}$

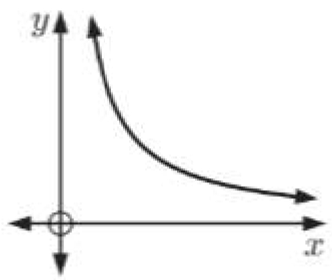
$= \pm k^{\frac{1}{4}} \sqrt{x}$

$\therefore \quad z^{\frac{1}{4}} \propto \sqrt{x}$

But $y \propto \sqrt{x}$, so $y \propto z^{\frac{1}{4}}$.

EXERCISE 6C

1



B indicates that y is inversely proportional to x, as the graph is a hyperbola.

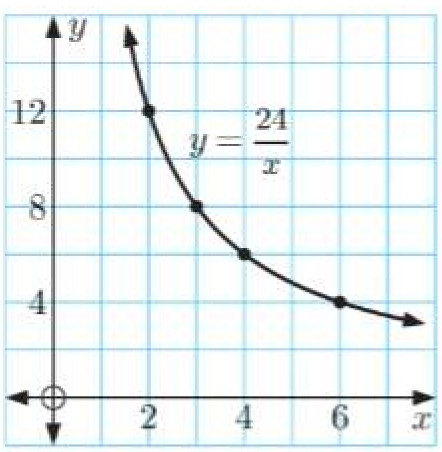
2 **a**

x	2	3	4	6
y	12	8	6	4
xy	24	24	24	24

$xy = 24$ for each point.

$\therefore$ x and y are inversely proportional,

and $y = \dfrac{24}{x}$.



b

x	1	2	4	5
y	20	10	6	4
xy	20	20	24	20

xy is not the same value for each point.

$\therefore$ x and y are not inversely proportional.

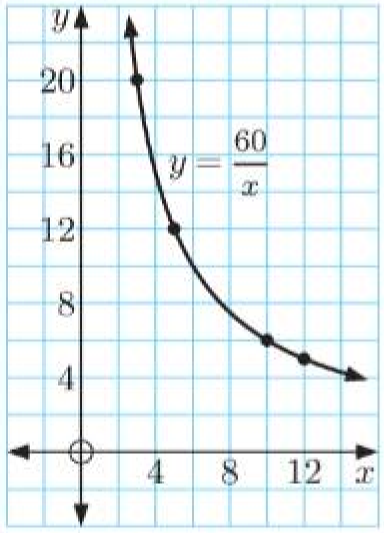
c

x	3	5	10	12
y	20	12	6	5
xy	60	60	60	60

$xy = 60$ for each point.

$\therefore$ x and y are inversely proportional,

and $y = \dfrac{60}{x}$.



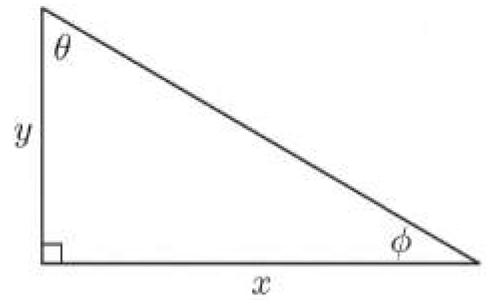
3 **a** No, $\theta = 90 - \phi$. There is no constant k such that $\theta \times \phi = k$.

$\therefore$ θ and ϕ are not inversely proportional.

b $\tan\theta = \dfrac{x}{y}$ and $\tan\phi = \dfrac{y}{x}$

$\therefore$ $\tan\theta = \dfrac{1}{\tan\phi}$

$\therefore$ $\tan\theta$ and $\tan\phi$ are inversely proportional.



4 $y \propto \dfrac{1}{x}$

a If x is doubled, then

x is multiplied by 2

$\therefore$ y is multiplied by $\frac{1}{2}$

$\therefore$ y is halved.

b If x is divided by 7, then

x is multiplied by $\frac{1}{7}$

$\therefore$ y is multiplied by 7.

c If x is multiplied by $\frac{9}{5}$, then

y is multiplied by $\frac{5}{9}$.

d If x is increased by 30%, then

x is multiplied by 1.3

$\therefore$ y is multiplied by $\dfrac{1}{1.3} \approx 0.769$

$\therefore$ y is decreased by about 23.1%.

5 $C \propto \dfrac{1}{t}$

a

$\times 3$

t	6	18
C	15	

t is multiplied by 3

$\therefore$ C is multiplied by $\frac{1}{3}$ $\{$as $C \propto \dfrac{1}{t}\}$

$\therefore$ $C = \frac{15}{3} = 5$

b

t	6	
C	15	20

$\times\frac{20}{15}$

C is multiplied by $\dfrac{20}{15} = \dfrac{4}{3}$

$\therefore$ t is multiplied by $\frac{3}{4}$ $\{$as $C \propto \dfrac{1}{t}\}$

$\therefore$ $t = 6 \times \frac{3}{4} = \frac{9}{2} = 4.5$

6 The time taken t is inversely proportional to the number of gardeners n, so $t = \frac{k}{n}$.

$t = 6$ hours when $n = 5$, so $6 = \frac{k}{5}$

$\therefore\ k = 30$

So when $n = 3$, $t = \frac{30}{3} = 10$ hours

$\therefore$ it would take 3 gardeners 10 hours to do the task.

7 The object's acceleration a is inversely proportional to its mass m, so $a = \frac{k}{m}$.

$a = 1.5\ \text{m s}^{-2}$ when $m = 5$ kg, so $1.5 = \frac{k}{5}$

$\therefore\ k = 7.5$

So, $a = \frac{7.5}{m}$.

a When $m = 2$ kg, $a = \frac{7.5}{2} = 3.75\ \text{m s}^{-2}$.

b When $a = 10\ \text{m s}^{-2}$, $\frac{7.5}{m} = 10$

$\therefore\ m = 0.75$ kg

8 **a** The *amount* in Wendy's savings account is constant, and the cost of the total number of shares is the *share price* multiplied by the *number of shares*.

This is a relationship of the form $xy = k$, so *number of shares* is inversely proportional to *share price*.

b **i** If the share price drops by £0.15 from £6.25, then it has been multiplied by $\frac{6.25 - 0.15}{6.25} = \frac{122}{125}$.

This means the number of shares Wendy can buy is multiplied by $\frac{125}{122}$, so she can now buy $\frac{125}{122}n$ shares (rounded down).

ii For Wendy to be able to buy 1.5 times as many shares, each share must be $\frac{1}{1.5} = \frac{2}{3}$ times the price.

This means the price must decrease by $33\frac{1}{3}\%$.

EXERCISE 6D

1 $y \propto \frac{1}{x^3}$

a If x is doubled, then

x is multiplied by 2

$\therefore\ x^3$ is multiplied by $2^3 = 8$

$\therefore\ \frac{1}{x^3}$ is multiplied by $\frac{1}{8}$

$\therefore\ y$ is multiplied by $\frac{1}{8}$

$\therefore\ y$ is divided by 8.

b If x is multiplied by $\frac{3}{5}$, then

x^3 is multiplied by $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$

$\therefore\ \frac{1}{x^3}$ is multiplied by $\frac{125}{27}$

$\therefore\ y$ is multiplied by $\frac{125}{27}$.

c If y is multiplied by 64, then

$\frac{1}{x^3}$ is multiplied by 64

$\therefore$ x^3 is multiplied by $\frac{1}{64}$

$\therefore$ x is multiplied by $\sqrt[3]{\frac{1}{64}} = \frac{1}{4}$

$\therefore$ x is divided by 4.

2 $y \propto \frac{1}{x^2}$

a

$\times 3$

x	8	24
y	27	

x is multiplied by 3

$\therefore$ x^2 is multiplied by $3^2 = 9$

$\therefore$ $\frac{1}{x^2}$ is multiplied by $\frac{1}{9}$

$\therefore$ y is multiplied by $\frac{1}{9}$ $\{$as $y \propto \frac{1}{x^2}\}$

$\therefore$ $y = 27 \times \frac{1}{9} = 3$

b

x	8	
y	27	75

$\times \frac{75}{27}$

y is multiplied by $\frac{75}{27} = \frac{25}{9}$

$\therefore$ $\frac{1}{x^2}$ is multiplied by $\frac{25}{9}$ $\{$as $y \propto \frac{1}{x^2}\}$

$\therefore$ x^2 is multiplied by $\frac{9}{25}$

$\therefore$ x is multiplied by $\sqrt{\frac{9}{25}} = \frac{3}{5}$ $\{$as $x > 0\}$

$\therefore$ $x = 8 \times \frac{3}{5} = \frac{24}{5}$

3 $M \propto \frac{1}{c^3}$

a

$\times \frac{8}{12}$

c	12	8
M	64	

c is multiplied by $\frac{8}{12} = \frac{2}{3}$

$\therefore$ c^3 is multiplied by $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

$\therefore$ $\frac{1}{c^3}$ is multiplied by $\frac{27}{8}$

$\therefore$ M is multiplied by $\frac{27}{8}$ $\{$as $M \propto \frac{1}{c^3}\}$

$\therefore$ $M = 6.4 \times \frac{27}{8} = 216$

b

c	12	
M	64	1

$\times \frac{1}{64}$

M is multiplied by $\frac{1}{64}$

$\therefore$ $\frac{1}{c^3}$ is multiplied by $\frac{1}{64}$ $\{$as $M \propto \frac{1}{c^3}\}$

$\therefore$ c^3 is multiplied by 64

$\therefore$ c is multiplied by $\sqrt[3]{64} = 4$

$\therefore$ $c = 12 \times 4 = 48$

4 **a** The volume of a cylinder $V = \pi r^2 h$, where h is the height and r is the radius

$$\therefore \; h = \frac{V}{\pi} \times \frac{1}{r^2}$$

$$\therefore \; h \propto \frac{1}{r^2} \quad \{\text{since } V \text{ and } \pi \text{ are constants}\}$$

So, the height of the can is inversely proportional to the square of its radius.

b If r is multiplied by $\frac{3.39}{3.04}$, then

h is multiplied by $\left(\frac{3.04}{3.39}\right)^2$ $\{\text{as } h \propto \frac{1}{r^2}\}$

$\therefore$ the height is $12.9 \times \left(\frac{3.04}{3.39}\right)^2 \approx 10.4$ cm.

c If h is multiplied by $\frac{15.3}{12.9}$, then

$\frac{1}{r^2}$ is multiplied by $\frac{15.3}{12.9}$ $\{\text{as } h \propto \frac{1}{r^2}\}$

$\therefore$ r is multiplied by $\sqrt{\frac{12.9}{15.3}}$ $\{\text{as } r > 0\}$

$\therefore$ the radius is $3.04 \times \sqrt{\frac{12.9}{15.3}} \approx 2.79$ cm.

d The cans would otherwise be too narrow or too wide for practical use.

5 The tidal acceleration a is inversely proportional to the cube of the distance d, so $a \propto \frac{1}{d^3}$.

a If d is increased by 10%, then

d is multiplied by 1.1

$\therefore$ a is multiplied by $\frac{1}{(1.1)^3} \approx 0.751$ $\{\text{as } a \propto \frac{1}{d^3}\}$

So, the tidal acceleration is decreased by about 24.9%.

b If a is tripled, then

a is multiplied by 3

$\therefore$ $\frac{1}{d^3}$ is multiplied by 3 $\{\text{as } a \propto \frac{1}{d^3}\}$

$\therefore$ d^3 is multiplied by $\frac{1}{3}$

$\therefore$ d is multiplied by $\frac{1}{\sqrt[3]{3}} \approx 0.693$

So, the distance is decreased by about 30.7%.

6 $y \propto \dfrac{1}{x}$

$\therefore\ y = \dfrac{k}{x}$ where k is a constant

$\therefore\ y^2 = \dfrac{k^2}{x^2}$

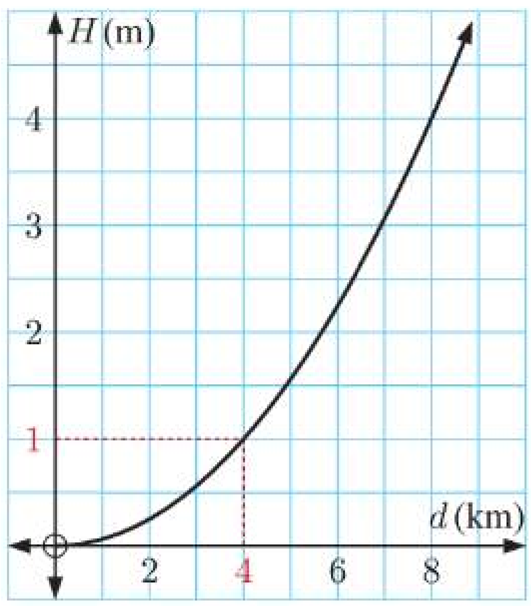
$\therefore\ \dfrac{1}{y^2} = \dfrac{1}{k^2}x^2$

$\therefore\ \dfrac{1}{y^2} \propto x^2$

But $z \propto \dfrac{1}{y^2}$, so $z \propto x^2$.

EXERCISE 6E

1 $H \propto d^2$, so $H = kd^2$ where k is a constant.



a From the graph, we see that $H = 1$ when $d = 4$.

$\therefore\ 1 = k \times 4^2$

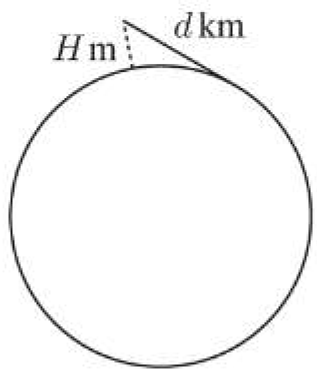
$\therefore\ k = \frac{1}{16}$

So, the equation of the variation model is

$H = \dfrac{d^2}{16}$.

b When $d = 16$, $H = \frac{1}{16} \times 16^2 = 16$.

So, Teresa is 16 m above sea level.



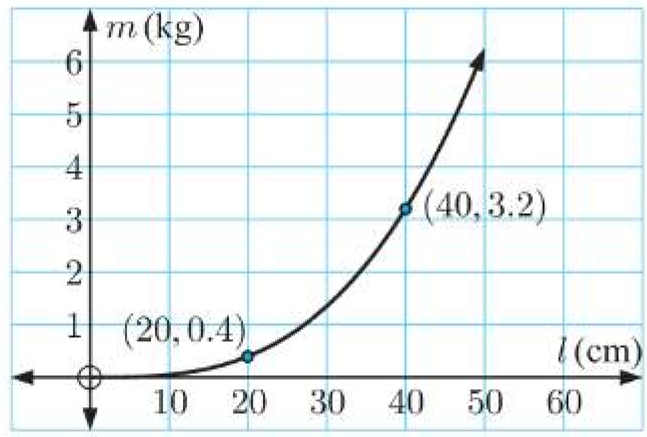
2 **a** $m \propto l^3$, so $m = kl^3$ where k is a constant.

$m = 0.4$ when $l = 20$, so $0.4 = k \times 20^3$

$\therefore\ 0.4 = 8000k$

$\therefore\ k = \frac{1}{20\,000}$

So, the variation model is $m = \dfrac{l^3}{20\,000}$.



b When $l = 50$, $m = \dfrac{50^3}{20\,000}$

$= \dfrac{125\,000}{20\,000} = 6.25$

So, the mass of a 50 cm long model car is 6.25 kg.

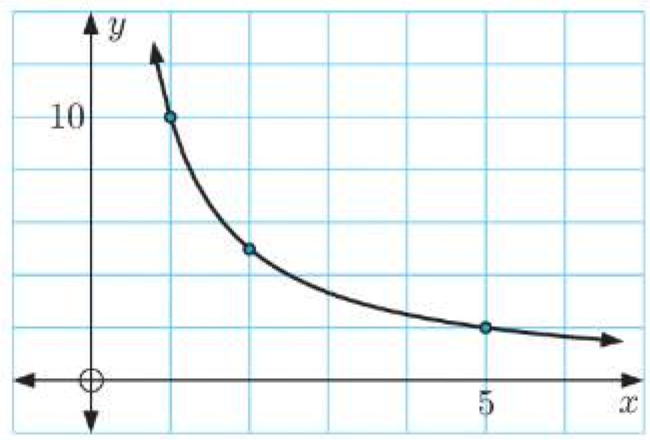
c When $m = l$, $\dfrac{l^3}{20\,000} = 1$

$\therefore\ l^3 = 20\,000$

$\therefore\ l = \sqrt[3]{20\,000} \approx 27.1$

So, the length of a model car with mass 1 kg is about 27.1 cm.

3 **a** The points are (1, 10), (2, 5), and (5, 2).
$xy = 10$ for each point, so it is reasonable to assume that y varies inversely with x.



b $xy = 10$

$\therefore\ y = \dfrac{10}{x}$

c When $x = 8$, $y = \dfrac{10}{8}$

$= \frac{5}{4}$

4 **a**

x	0.25	0.5	1	2
y	80	20	5	1.25
x^2y	5	5	5	5

$x^2y = 5$ for every data point

$\therefore\ y = \dfrac{5}{x^2}$

$\therefore\ k = 5$

b When $y = 0.5$, $\dfrac{5}{x^2} = 0.5$

$\therefore\ x^2 = 10$

$\therefore\ x = \sqrt{10}$ $\{$as $x > 0\}$

5

v	10	20	30	40
R	0.5	4	13.5	32

a **i** $R \propto v^2$, so $R = kv^2$ where k is a constant

Using the first point, $0.5 = k \times 10^2$

$$\therefore \quad k = \frac{1}{200}$$

$$\therefore \quad R = \frac{v^2}{200}$$

ii When $v = 20$, $R = \dfrac{20^2}{200} = 2 \neq 4$

When $v = 30$, $R = \dfrac{30^2}{200} = 4.5 \neq 13.5$

When $v = 40$, $R = \dfrac{40^2}{200} = 8 \neq 32$

$\therefore$ this model is incorrect.

b

v	10	20	30	40
R	0.5	4	13.5	32
$\dfrac{R}{v^3}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$

$\dfrac{R}{v^3} = \frac{1}{2000}$ for every data point, so $R = \frac{1}{2000}v^3$.

c When $v = 50$, $R = \frac{1}{2000} \times 50^3$

$$= \frac{125\,000}{2000} = 62.5$$

So, when the car's velocity is 50 km h^{-1}, its air resistance is 62.5 units.

d If R is to be reduced by 20%, then

R is multiplied by 0.8

$\therefore$ v^3 is multiplied by 0.8

$\therefore$ v is multiplied by $\sqrt[3]{0.8} \approx 0.9283$

$\therefore$ v must be reduced by $\approx 7.17\%$.

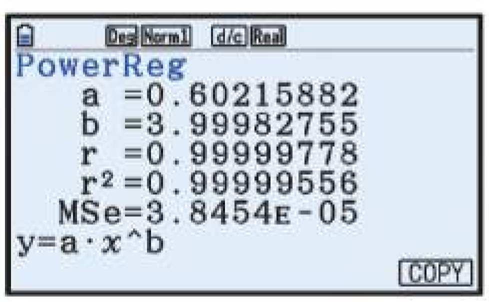
EXERCISE 6F

1 **a**

x	1	2	3	4
y	0.6	9.7	48.8	153.5

The power is very close to 4, so it is reasonable to conclude that y is directly proportional to x^4.

The model is $y \approx 0.602x^4$.

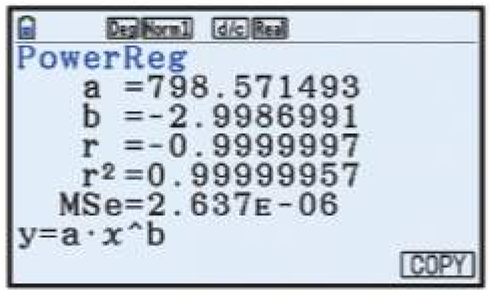


b

x	2	3	6	9
y	100	29.6	3.7	1.1

The power is very close to -3, so it is reasonable to conclude that y is inversely proportional to x^3.

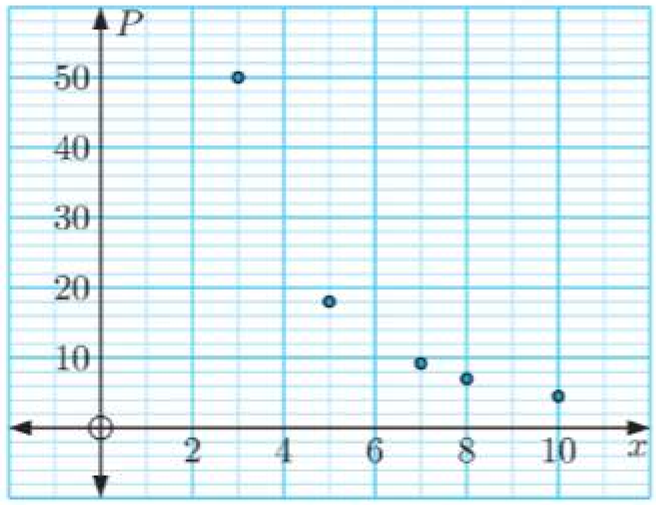
The model is $y \approx \dfrac{799}{x^3}$.



2 **a**

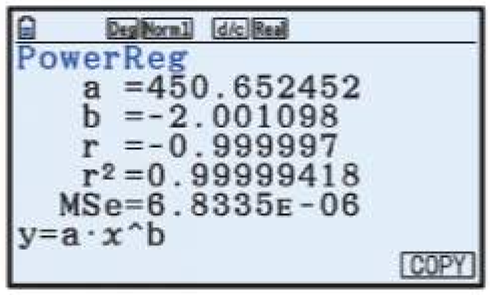
x	3	5	7	8	10
P	50	18	9.2	7	4.5

We expect inverse variation between the variables, as the points appear to lie on a curve which is asymptotic to both axes.



b The power is very close to -2, so it is reasonable to conclude that P is inversely proportional to x^2.

The model is $P \approx \dfrac{451}{x^2}$.



c When $x = 4$, $P \approx \dfrac{451}{4^2}$

≈ 28.2

3

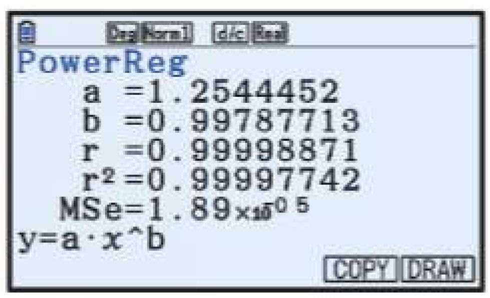
Time (t minutes)	8	25	32	45
Percentage charge ($C\%$)	10	31	40	56

a We would expect direct variation between C and t because:

- Zach only charges his phone when the battery has completely run out, so when $t = 0$, $C = 0$. Hence the graph of C against t should pass through the origin.
- We expect the charge to increase as time increases.

b The power is very close to 1, so it is reasonable to conclude that C is directly proportional to t.

The model is $C \approx 1.25t$.



c When $t = 56$, $C \approx 1.25 \times 56$

≈ 70

So, after 56 minutes, the phone will receive about 70% charge.

d The percentage charge $C\%$ is always between 0% and 100%, and $C \approx 1.25t$,

so $0 \leqslant 1.25t \leqslant 100$

$\therefore\ 0 \leqslant t \leqslant 80$

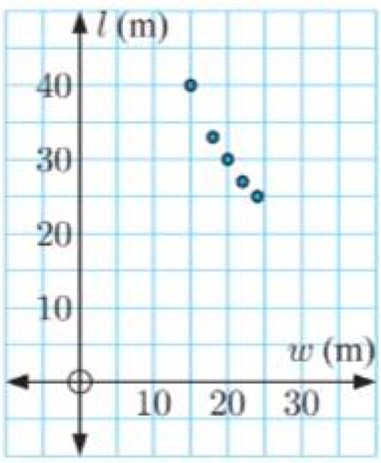
4

Width (w metres)	15	18	20	22	24
Length (l metres)	40	33	30	27	25
lw	600	594	600	594	600

a $lw \approx 600$ for all l and w

$\therefore$ we expect inverse variation between l and w.

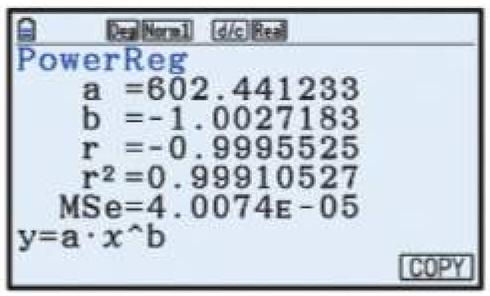
b



Yes, this diagram is consistent with the answer to **a**. The points appear to lie on a curve which is asymptotic to both axes, as expected for inverse variation.

c The power is very close to -1, so it is reasonable to conclude that l and w are inversely proportional.

The model is $l \approx \dfrac{602}{w}$.



d When $w = 23$, $l \approx \dfrac{602}{23}$

≈ 26.2

So a 23 m wide block will be about 26.2 m long.

e Most houses would require a block at least 15 m wide, and most blocks would be longer than they are wide.

If $l = w$ then $w \approx \sqrt{602} \approx 24.5$, so $15 \leqslant w \leqslant 24.5$.

5

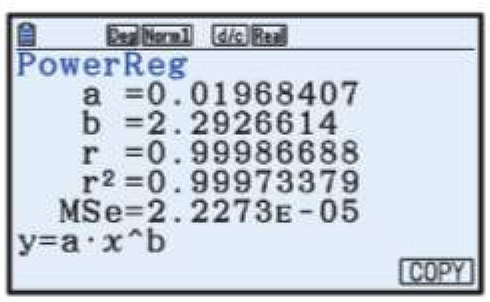
Speed (s m s^{-1})	5.56	6.00	6.41	6.80	7.17	7.52
Turning radius (R m)	1.0	1.2	1.4	1.6	1.8	2.0

a The model is $R \approx 0.0197s^{2.29}$.

b When $s = 10$, $R \approx 0.0197 \times 10^{2.29}$

≈ 3.84

So the turning radius of a car travelling at 10 m s^{-1} is about 3.84 m.

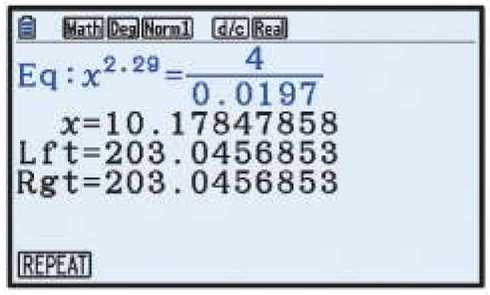


c When $R = 4$, $0.0197s^{2.29} \approx 4$

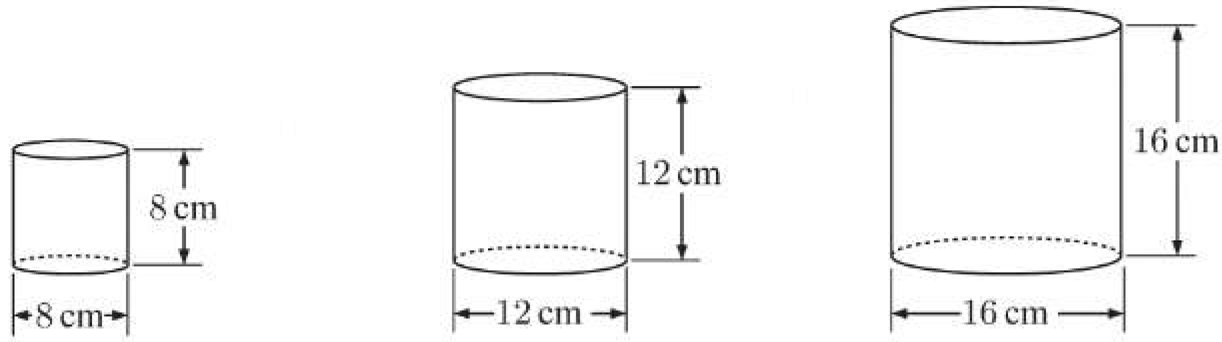
$$\therefore \quad s^{2.29} \approx \frac{4}{0.0197}$$

Using technology, $s \approx 10.2$

So a car with a turning radius of 4 m is travelling at about 10.2 m s^{-1}.

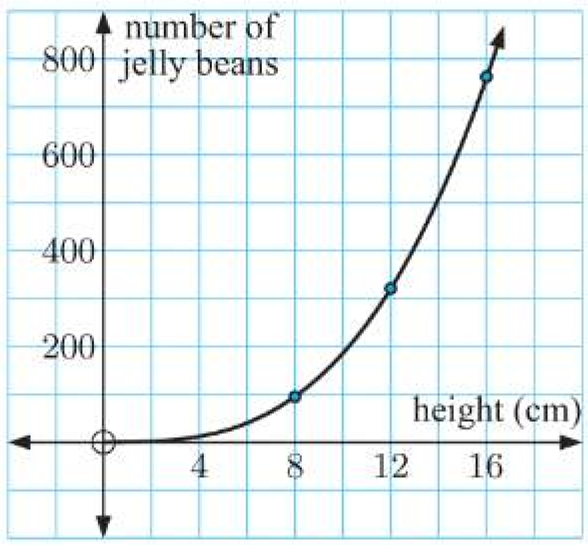


6



95 jelly beans 321 jelly beans 763 jelly beans

a



b No, the *number of jelly beans* does not increase *in proportion* with the *jar height* as the graph between these variables is not a straight line.

c The volume of the jar $V = \pi r^2 h$, and its radius $r = \dfrac{h}{2}$ where h is the height of the jar.

$$\therefore \quad V = \tfrac{\pi}{4}h^3$$

$$\therefore \quad V \propto h^3 \quad \{\tfrac{\pi}{4} \text{ is a constant}\}$$

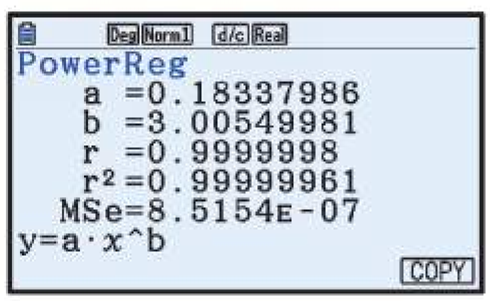
The number of jelly beans N which a jar can hold is directly proportional to its volume V, and its volume is proportional to the cube of its height.

So, we expect the *number of jelly beans* to increase in proportion to the cube of the *jar height*.

d We find the power model which best fits the points $(8, 95)$, $(12, 321)$, and $(16, 763)$.

The power is very close to 3, so it is reasonable to conclude that N is directly proportional to h^3.

The model is $N \approx 0.183h^3$.



e When $h = 20$, $N \approx 0.183 \times 20^3$

$$\approx 1464$$

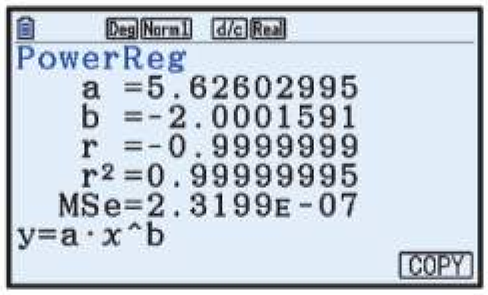
Jill should guess 1464 jelly beans.

7

Distance (d m)	0.1	0.25	0.5	0.75	1	1.5
Force (F N)	563	90.0	22.5	10.0	5.63	2.50

a The power is very close to -2, so it is reasonable to conclude that F is inversely proportional to d^2.

The model is $F \approx \dfrac{5.63}{d^2}$.



b When $d = 0.4$, $F \approx \dfrac{5.63}{0.4^2}$

≈ 35.2

So when the spheres are 0.4 m apart, the force will be about 35.2 N.

c When $F = 650$, $\dfrac{5.63}{d^2} \approx 650$

$\therefore\ d \approx \sqrt{\dfrac{5.63}{650}}$ $\{\text{as } d > 0\}$

≈ 0.0931

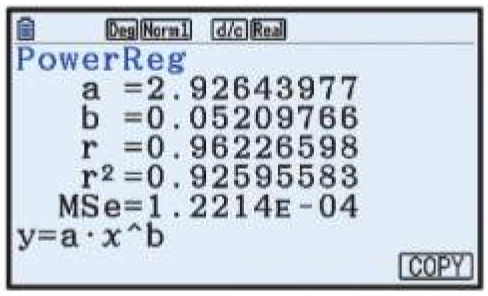
So the spheres are about 0.0931 m apart when the force is 650 N.

8

Temperature (T °C)	5	10	15	20	25	30	35
Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39	3.45	3.51	3.57

a The model is $P \approx 2.93 \times T^{0.0521}$.

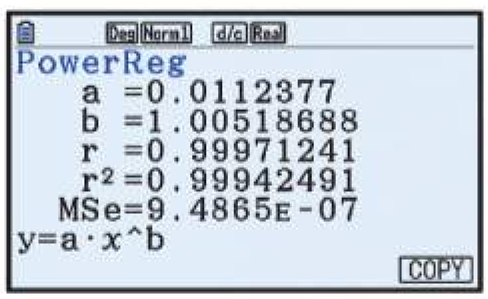
The power of T is not close to 1, so P is not directly proportional to T.



b **i**

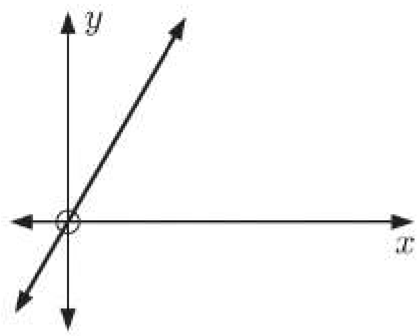
Temperature (T K)	278.15	283.15	288.15	293.15	298.15	303.15	308.15
Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39	3.45	3.51	3.57

ii The model is $P \approx 0.0112T$ where T is measured in kelvin.



iii Yes, it is reasonable to assume that pressure and temperature are directly proportional when temperature is measured in kelvin, as the power is now very close to 1, which suggests that $P \propto T$.

REVIEW SET 6A

1 

C indicates that y is directly proportional to x, as the graph is a straight line which passes through the origin.

2 $A \propto t$

a If t is multiplied by 4, then A is multiplied by 4.

b If A is increased by 5%, then A is multiplied by 1.05
$\therefore$ t is multiplied by 1.05
$\therefore$ t is increased by 5%.

3 A person's weight on the Moon M is directly proportional to their weight on Earth E, so $M = kE$ where k is a constant.

When $M = 124$ N, $E = 750$ N, so $124 = k(750)$

$$\therefore \ k = \frac{124}{750} = \frac{62}{375}$$

So, $M = \dfrac{62}{375}E$.

When $E = 640$ N, $M = \dfrac{62}{375} \times 640 \approx 106$ N.

So, if John weighs 640 N on Earth, then he weighs about 106 N on the Moon.

4 **a** $y = 5x^2$, so $y \propto x^2$ and $k = 5$

b $P = \dfrac{2n^4}{3} = \frac{2}{3}n^4$, so $P \propto n^4$ and $k = \frac{2}{3}$

c $V = \frac{\sqrt{5}}{4}a^3$, so $V \propto a^3$ and $k = \frac{\sqrt{5}}{4}$

5 **a** $y \propto x^2$, so $y = kx^2$ where k is a constant.

When $x = 8$ and $y = 30$, $30 = k \times 8^2$

$$\therefore \ k = \frac{30}{64} = \frac{15}{32}$$

So the model is $y = \frac{15}{32}x^2$.

b **i** When $x = 4$,
$$\begin{aligned} y &= \tfrac{15}{32} \times 4^2 \\ &= \tfrac{15}{2} \\ &= 7.5 \end{aligned}$$

ii When $y = 150$,
$$\begin{aligned} \tfrac{15}{32}x^2 &= 150 \\ \therefore \ x^2 &= 320 \\ \therefore \ x &= 8\sqrt{5} \quad \{\text{as } x > 0\} \\ &\approx 17.9 \end{aligned}$$

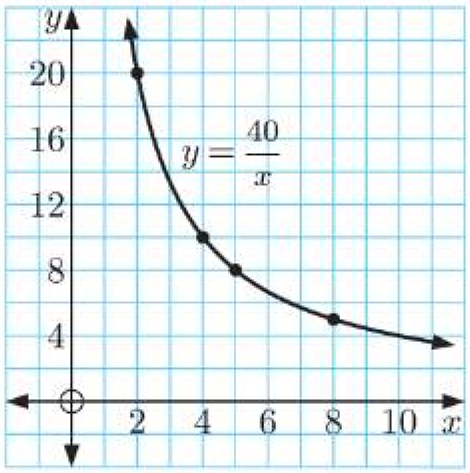
6 **a**

x	2	4	5	8
y	20	10	8	5
xy	40	40	40	40

$xy = 40$ for each point.

$\therefore$ x and y are inversely proportional,

and $y = \dfrac{40}{x}$.



b

x	3	5	8	10
y	20	12	8	6
xy	60	60	64	60

xy is not the same value for each point.

$\therefore$ x and y are not inversely proportional.

7 The frequency f is inversely proportional to wavelength λ, so $f = \dfrac{k}{\lambda}$ where k is a constant.

When $f = 500$ THz, $\lambda = 600$ nm, so $500 = \dfrac{k}{600}$

$$\therefore\ k = 300\,000$$

So, $f = \dfrac{300\,000}{\lambda}$.

When $\lambda = 480$ nm, $f = \dfrac{300\,000}{480}$

$= 625$ THz

$\therefore$ the frequency of the blue light wave is 625 THz.

8 The resistance r is inversely proportional to the square of the diameter d, so $r = \dfrac{k}{d^2}$ where k is a constant.

$r = 0.24$ ohms when $d = 0.44$ cm, so $0.24 = \dfrac{k}{(0.44)^2}$

$$\therefore\ k = 0.24(0.44)^2$$
$$= 0.046\,464$$

a When $d = 0.3$ cm, $r = \dfrac{0.046\,464}{(0.3)^2}$

≈ 0.516 ohms

b When $r = 0.45$ ohms, $\dfrac{0.046\,464}{d^2} = 0.45$

$$\therefore\ d^2 = \frac{0.046\,464}{0.45}$$

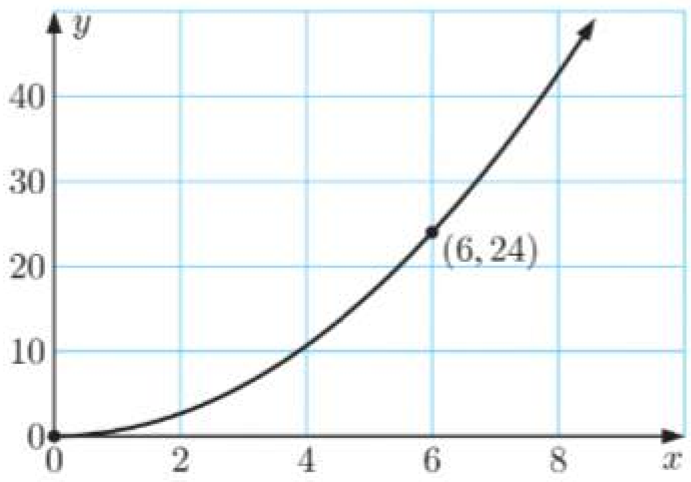
$$\therefore\ d \approx 0.321 \text{ cm}$$

9 **a** $y \propto x^2$, so $y = kx^2$ where k is a constant.

$y = 24$ when $x = 6$, so $24 = k \times 6^2$

$\therefore \; k = \frac{24}{36} = \frac{2}{3}$

The model is $y = \frac{2}{3}x^2$.



b When $x = 11$, $y = \frac{2}{3} \times 11^2$

$= \frac{242}{3} = 80\frac{2}{3}$

10

h	2	4	6
V	3.2	25.6	86.4

a If the height of a regular pyramid increases, then each side length also increases.
This means the pyramid gets larger in all 3 dimensions as the height increases, so we should expect that V is directly proportional to h^3.

b $V \propto h^3$, so $V = kh^3$ where k is a constant.

$V = 3.2$ when $h = 2$, so $3.2 = k \times 2^3$

$\therefore \; k = \frac{3.2}{8} = 0.4$

$\therefore \; V = \frac{2}{5}h^3$

c When $h = 4$, $V = \frac{2}{5} \times 4^3$

$= \frac{2}{5} \times 64$

$= \frac{128}{5} = 25.6$ ✓

When $h = 6$, $V = \frac{2}{5} \times 6^3$

$= \frac{2}{5} \times 216$

$= \frac{432}{5} = 86.4$ ✓

So the model $V = \frac{2}{5}h^3$ satisfies every data point.

d **i** When $h = 8$, $V = \frac{2}{5} \times 8^3$

$= \frac{2}{5} \times 512$

$= \frac{1024}{5} = 204.8$

ii When $V = 50$, $\frac{2}{5}h^3 = 50$

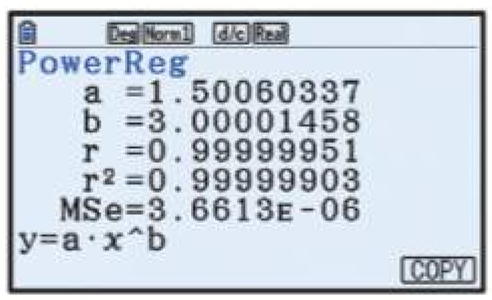
$\therefore \; h^3 = 125$

$\therefore \; h = 5$

11 **a**

x	2	4	5	7
y	12	96	188	514

The power is very close to 3, so it is reasonable to assume that y is directly proportional to x^3.
The model is $y \approx 1.50x^3$.

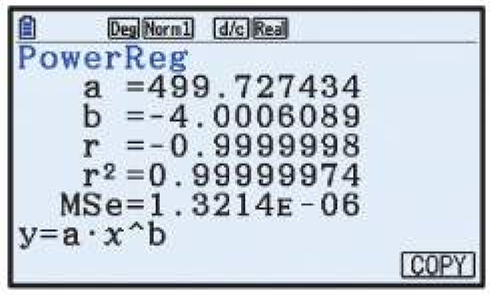


b

x	0.9	1.4	2.2	2.5
y	762	130	21.3	12.8

The power is very close to -4, so it is reasonable to assume that y is inversely proportional to x^4.

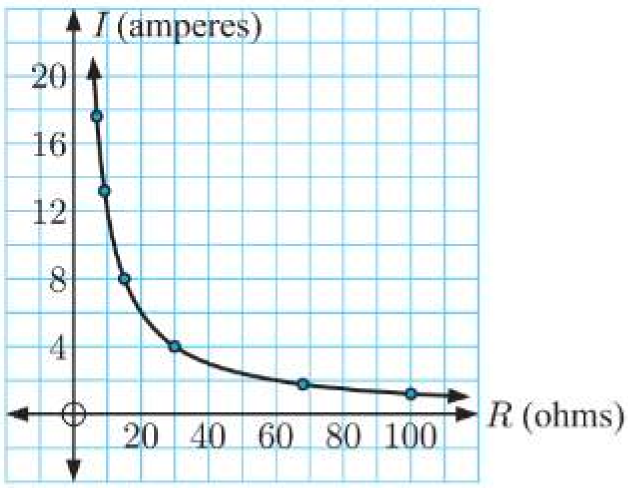
The model is $y \approx \frac{500}{x^4}$.



12

Resistance (R ohms)	6.8	9.1	15	30	68	100
Current (I amperes)	17.6	13.2	8.00	4.00	1.76	1.20
RI	119.68	120.12	120	120	119.68	120

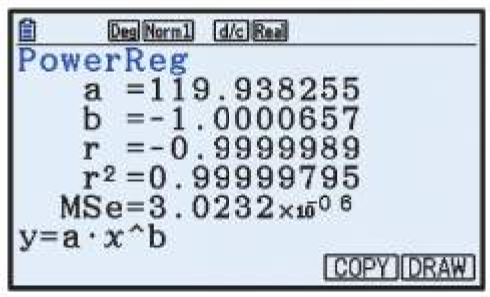
a



b We expect inverse variation between the variables, as $RI \approx 120$ for each point and the points appear to lie on a line which is asymptotic to both axes.

c The power is very close to -1, so it is reasonable to assume that I and R are inversely proportional.

The model is $I \approx \frac{120}{R}$.



d When $R = 250$, $I \approx \frac{120}{250}$

≈ 0.48

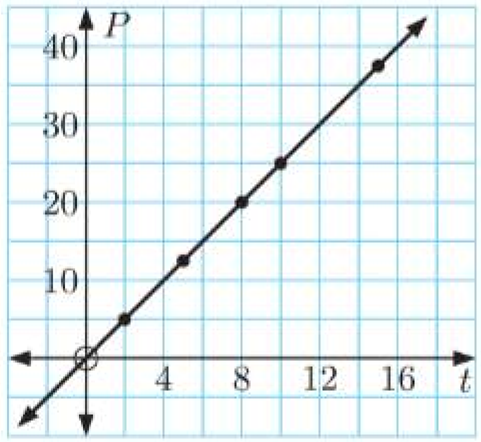
So the current passing through a 250 ohm resistor will be about 0.48 amperes.

REVIEW SET 6B

1

t	2	5	8	10	15
P	5	12.5	20	25	37.5

a



b The graph of P against t is a straight line which passes through the origin.

c The gradient of the line is $\dfrac{12.5-5}{5-2} = 2.5$

$\therefore \quad P = 2.5t$

2 The current C and the force F are directly proportional, so $F = kC$ where k is a constant.

When $C = 1.09$ A, $F = 2.18$ N, so $2.18 = k \times 1.09$

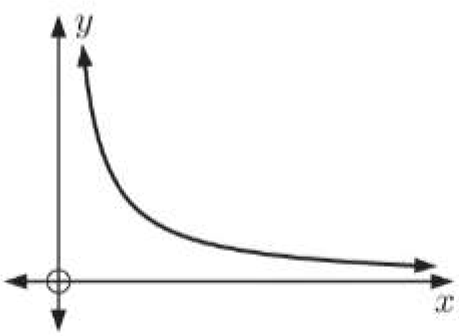
$\therefore \quad k = 2$

$\therefore \quad F = 2t$

a When $C = 1.45$ A, $F = 2 \times 1.45$
$= 2.9$ N

b When $F = 3.6$ N, $2C = 3.6$
$\therefore \quad C = 1.8$ A

3



B indicates that y is inversely proportional to x, as the graph is asymptotic to both axes.

4 **a** The area of the garden bed is

$A = \pi r^2$, where r is the radius

$\therefore \quad A \propto r^2$

The amount of compost is directly proportional to A.

$\therefore$ the amount of compost is directly proportional to the square of the radius.

b Let C be the amount of compost used.

If r is increased by 15%, then

r is multiplied by 1.15

$\therefore$ C is multiplied by $(1.15)^2 = 1.3225$ {as $C \propto r^2$ from **a**}

$\therefore$ C is increased by 32.25%.

So, Tamzin needs $250 \times 0.3225 = 80.625$ kg extra compost.

c If C is increased by 40, then

C is multiplied by $\dfrac{250+40}{250} = \dfrac{29}{25}$

$\therefore\ r^2$ is multiplied by $\frac{29}{25}$ {as $C \propto r^2$ from **a**}

$\therefore\ r$ is multiplied by $\sqrt{\frac{29}{25}} = \frac{\sqrt{29}}{5} \approx 1.0770$ {as $r > 0$}

$\therefore\ r$ is increased by about 7.70%.

So, Tamzin can extend the radius by about $3 \times 0.0770 \approx 0.231$ m which is about 23.1 cm.

5 **a** $y \propto \dfrac{1}{x^3}$, so $y = \dfrac{k}{x^3}$ where k is a constant.

$y = 16$ when $x = 6$, so $16 = \dfrac{k}{6^3}$

$\therefore\ k = 16 \times 216 = 3456$

So the model is $y = \dfrac{3456}{x^3}$.

b **i** When $x = 4$, $y = \dfrac{3456}{4^3}$

$= 54$

ii When $y = -2$, $\dfrac{3456}{x^3} = -2$

$\therefore\ x^3 = -1728$

$\therefore\ x = -12$

6 The number of workers n is inversely proportional to the number of days t to paint the silo, so $n = \dfrac{k}{t}$ where k is a constant.

When $n = 3$, $t = 18$, so $3 = \dfrac{k}{18}$

$\therefore\ k = 54$

So $n = \dfrac{54}{t}$.

When $n = 8$, $8 = \dfrac{54}{t}$

$\therefore\ t = \frac{54}{8} = 6.75$

$\therefore$ it will take 6.75 days for 8 people to paint the silo.

7 The radius r is inversely proportional to the square of the orbital speed s, so $r \propto \dfrac{1}{s^2}$.

If r is increased by 20%, then

r is multiplied by 1.2

$\therefore\ \dfrac{1}{s^2}$ is multiplied by 1.2 {as $r \propto \dfrac{1}{s^2}$}

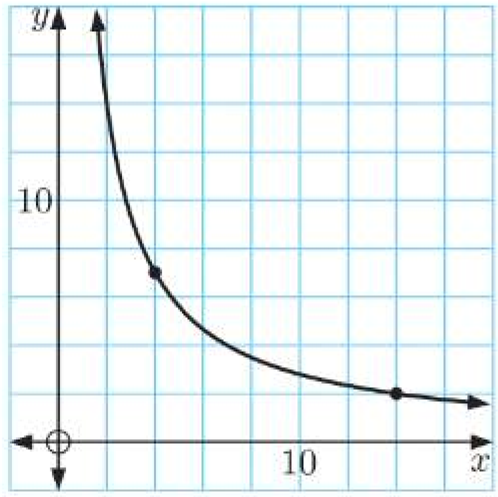
$\therefore\ s^2$ is multiplied by $\dfrac{1}{1.2} = \dfrac{5}{6}$

$\therefore\ s$ is multiplied by $\sqrt{\frac{5}{6}} \approx 0.9129$ {as $s > 0$}

So, the orbital speed decreases by about 8.71%.

8 **a** $y \propto \frac{1}{x}$, so $y = \frac{k}{x}$ where k is a constant.

The points (4, 7) and (14, 2) are marked on the graph.



Using (4, 7), $y = 7$ when $x = 4$, so $7 = \frac{k}{4}$

$\therefore \; k = 28$

$\therefore \; y = \frac{28}{x}$

Check: When $x = 14$, $y = \frac{28}{14} = 2$. ✓

b When $x = 0.1$, $y = \frac{28}{0.1} = 280$.

9 **a** $P \propto v^3$, so $P = kv^3$ where k is a constant

When $v = 35\ \text{km h}^{-1}$, $P = 262$ W, so $262 = k(35^3)$

$\therefore \; k = \frac{262}{42\,875}$

So, $P = \frac{262}{42\,875}v^3$.

b **i** When $v = 45\ \text{km h}^{-1}$, $P = \frac{262}{42\,875} \times 45^3$

≈ 557 W

To maintain a speed of $45\ \text{km h}^{-1}$, about 557 W of power is needed.

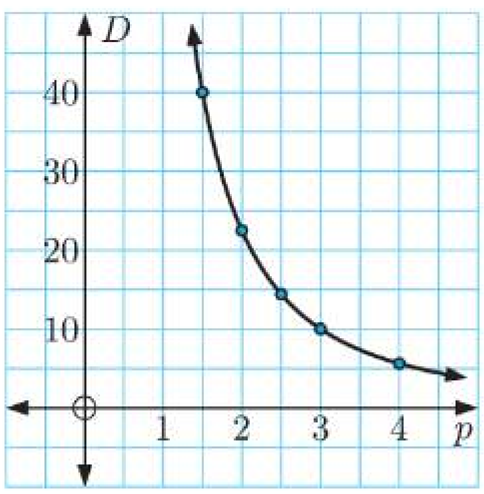
ii When $P = 205$ W, $\frac{262}{42\,875}v^3 = 205$

$\therefore \; v = \sqrt[3]{205 \times \frac{42\,875}{262}}$

$\approx 32.3\ \text{km h}^{-1}$

When exerting 205 W of power on flat ground, the cyclist's speed will be about $32.3\ \text{km h}^{-1}$.

10 **a**



b The points appear to lie on a curve which is asymptotic to both axes, which is what we would expect to see if $D \propto \frac{1}{p^2}$.

c

p	1.5	2	2.5	3	4
D	40	22.5	14.4	10	5.625
p^2D	90	90	90	90	90

$p^2D = 90$ at every point

$\therefore\ k = 90$

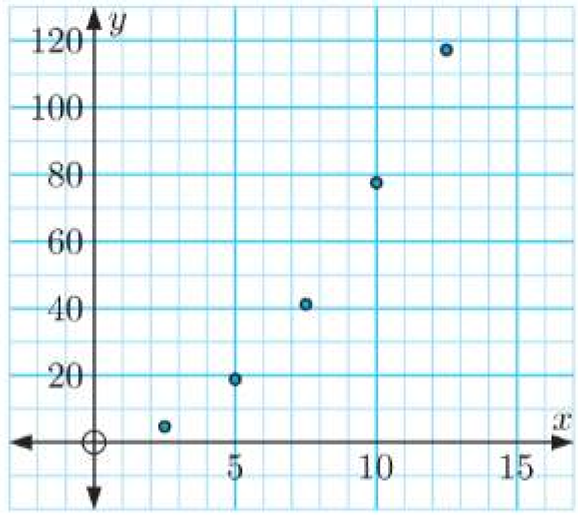
d When $p = 5$, $D = \frac{90}{5^2}$

$= \frac{18}{5} = 3.6$

11 **a**

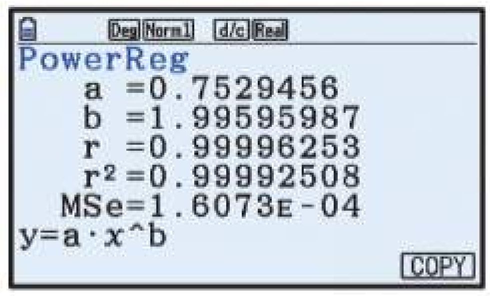
x	2.5	5	7.5	10	12.5
y	4.7	18.8	41.2	75	117.2

We expect direct variation between the variables, as the graph of y against x apppears to be a curve which passes through the origin.



b The power is very close to 2, so it is reasonable to assume that y is directly proportional to x^2.

The model is $y \approx 0.753x^2$.



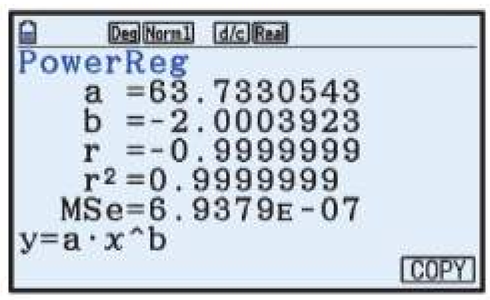
12 **a**

Distance (d m)	1	5	10	15	20
Sound intensity (I W m^{-2})	63.7	2.55	0.637	0.283	0.159
$I \times d$	63.7	12.75	6.37	4.245	3.18

The values of $I \times d$ are not constant, so I and d are *not* inversely proportional, and Abbas is incorrect.

b The power is very close to -2, so it is reasonable to assume that I is inversely proportional to d^2.

The model is $I \approx \frac{63.7}{d^2}$.



c If d is increased by 40%, then

d is multiplied by 1.4

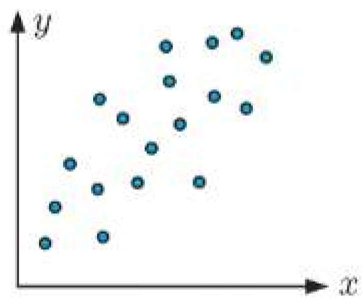
$\therefore$ I is multiplied by $\frac{1}{(1.4)^2} \approx 0.510$ $\{$as $I \propto \frac{1}{d^2}$ from **b**$\}$

$\therefore$ I is decreased by about 49.0%.

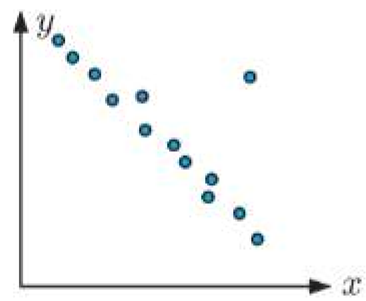
Chapter 7

BIVARIATE STATISTICS

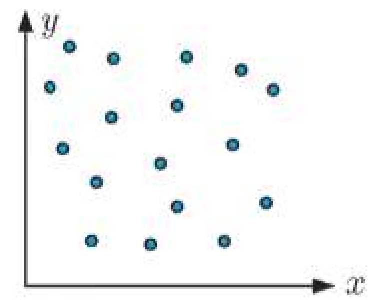
EXERCISE 7A

1 **a** 

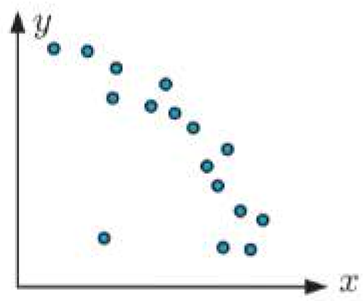
There is a weak, positive, linear correlation with no outliers.

b 

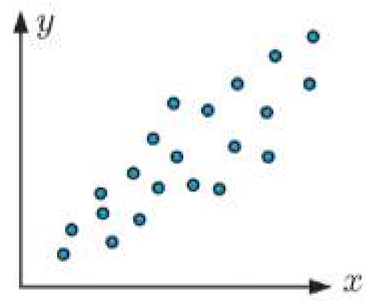
There is a strong, negative, linear correlation with one outlier.

c 

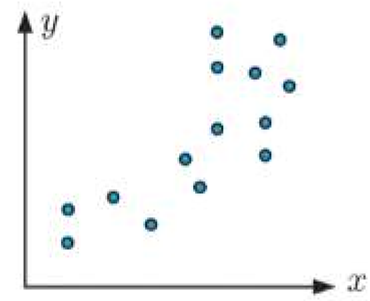
There is no correlation.

d 

There is a strong, negative, non-linear correlation with one outlier.

e 

There is a moderate, positive, linear correlation with no outliers.

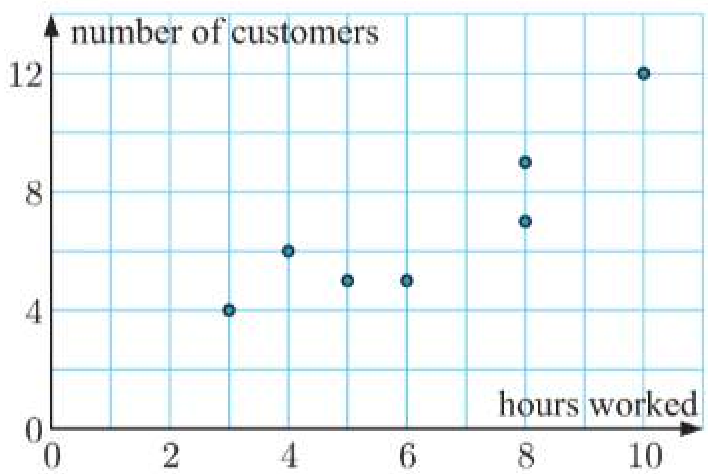
f 

There is a weak, positive, non-linear correlation with no outliers.

2

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

a *Hours worked* is the explanatory variable.
Number of customers is the response variable.

b 

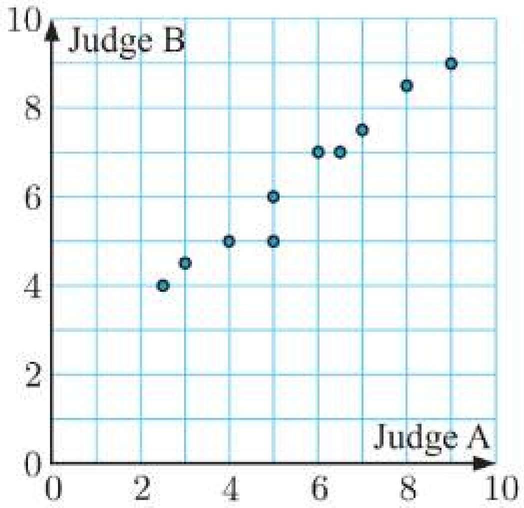
c **i** Tiffany worked the same number of hours (8 hours) on Monday and Friday.
ii Tiffany had the same number of customers (5 customers) on Wednesday and Sunday.

d The more hours that Tiffany works, the more customers she is likely to have, so we would expect a positive correlation between the variables.

3

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

a



b There appears to be **strong**, **positive**, **linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.

c No, an increase in Judge A's scores are not likely to cause an increase in Judge B's scores. It is much more likely that both scores are related to the quality of the ice skaters' performances.

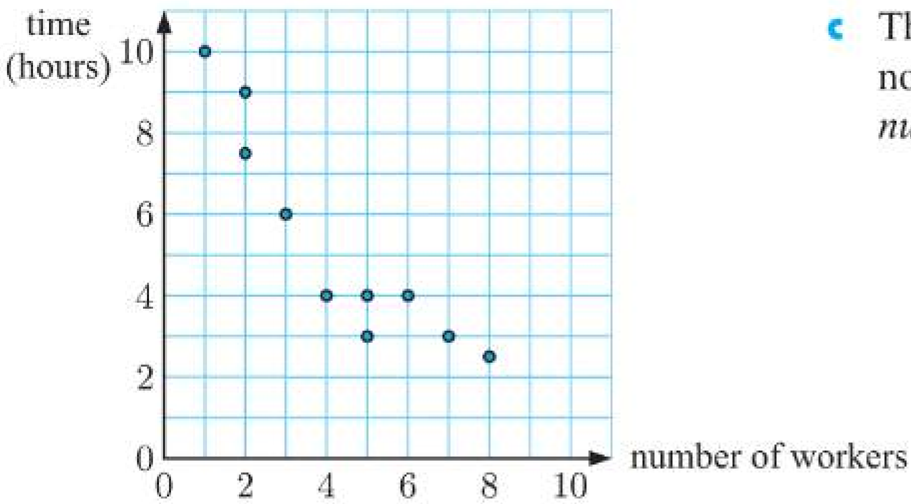
4

Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

a **i** Job G took the longest to complete (10 hours).

ii Job C involved the most workers (8 workers).

b



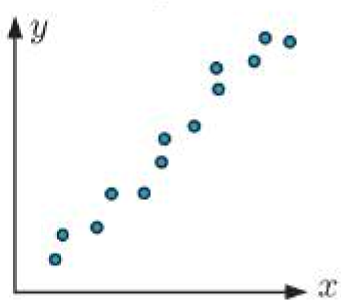
c There is a strong, negative, non-linear correlation between the *number of workers* and *time*.

5 **a** x = the number of apples bought by customers

y = the total cost of apples bought

We expect strong, positive, linear correlation. This corresponds to **D**.

D

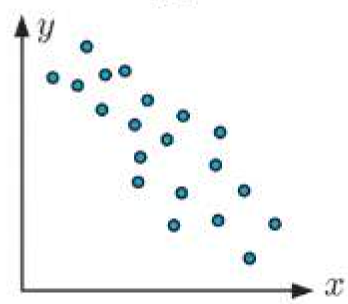


b x = the number of pushups a student can perform in one minute

y = the time taken for a student to run 100 metres

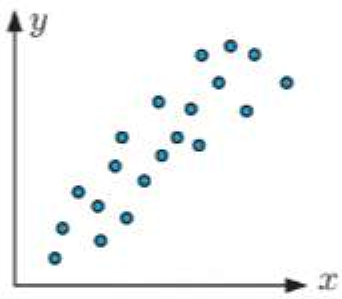
We expect moderate, negative, linear correlation. This corresponds to **A**.

A



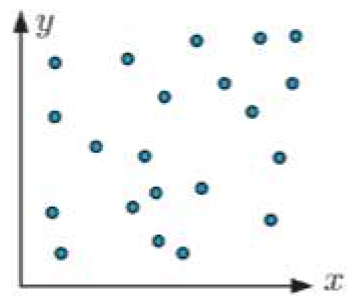
c $x =$ the height of a person
$y =$ the weight of the person
We expect moderate, positive, linear correlation. This corresponds to **B**.

B



d $x =$ the distance a student travels to school
$y =$ the height of the student's uncle
We expect no correlation. This corresponds to **C**.

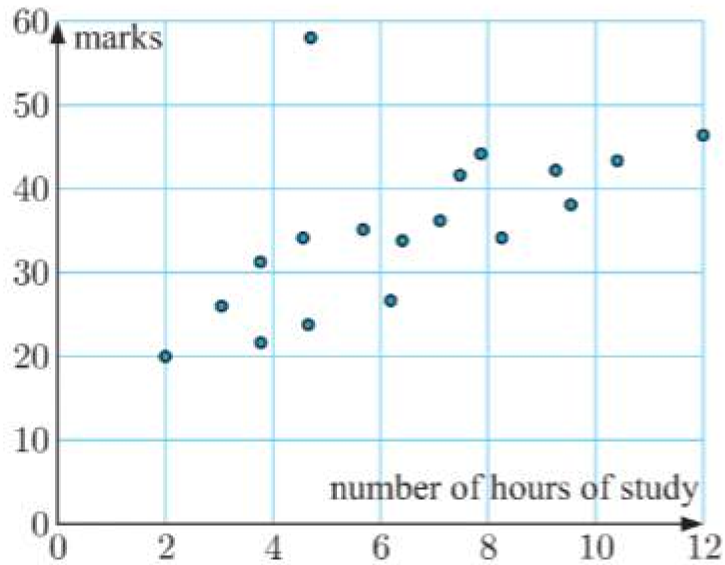
C



6 **a** There is a moderate, positive, linear correlation between the *number of hours of study* and the *marks obtained.*

b As the test is out of 50 marks and the outlier is greater than 50, we can assume it is an error and discard it.

c Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.



7 **a** Not causal, dependent on genetics and/or age.

b Not causal, dependent on the size/location of the fire.

c Causal, an increase in advertising is likely to cause an increase in sales.

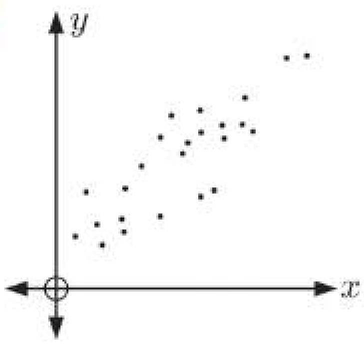
d Causal, the childrens' adult height is determined by the genetics inherited from their parents to a great extent.

e Not causal, dependent on the population of the towns.

EXERCISE 7B

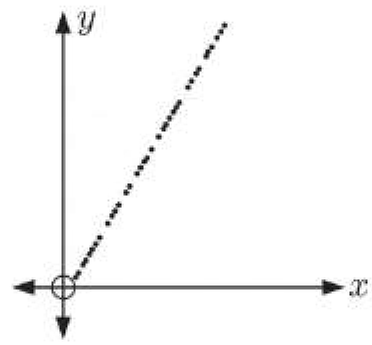
1 $r = 0.556$
There is a weak, positive correlation between the *number of employees of a company* and its *export earnings.*

2 **a**



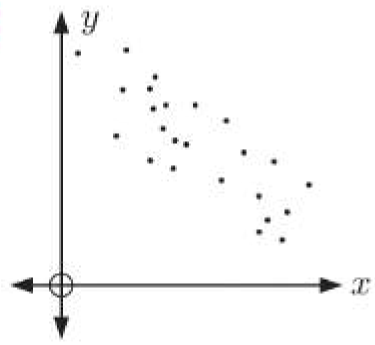
B $r = 0.6$

b



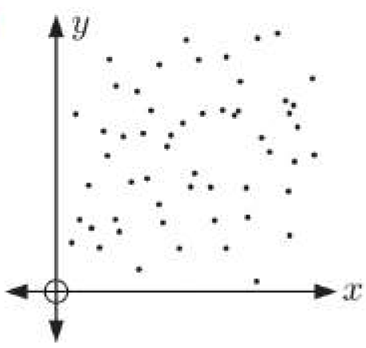
A $r = 1$

c



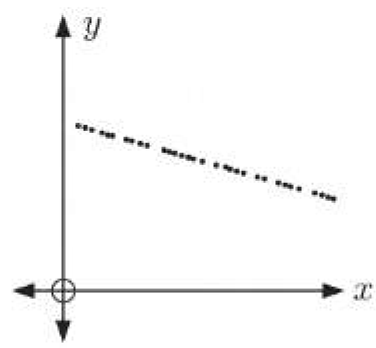
D $r = -0.7$

d



C $r = 0$

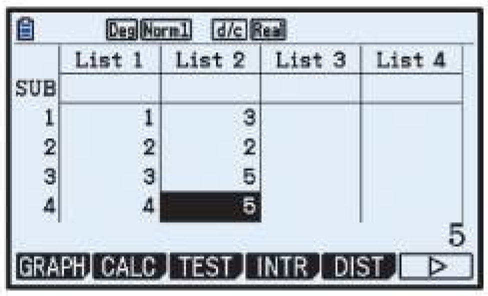
e

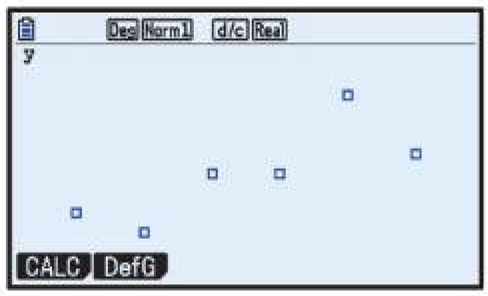


E $r = -1$

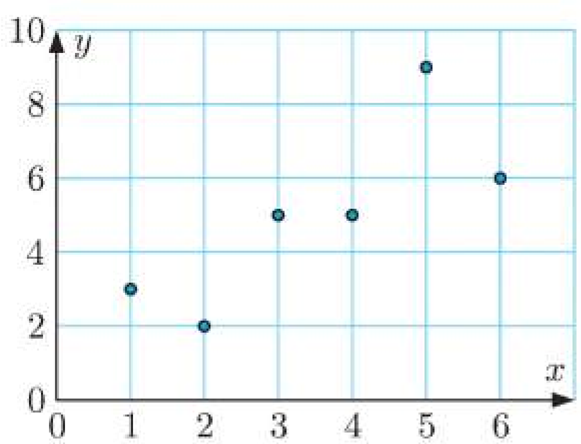
3 **a**

x	1	2	3	4	5	6
y	3	2	5	5	9	6

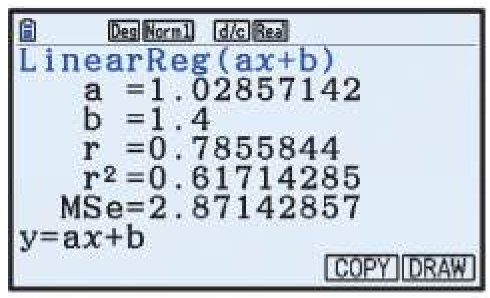




i



ii

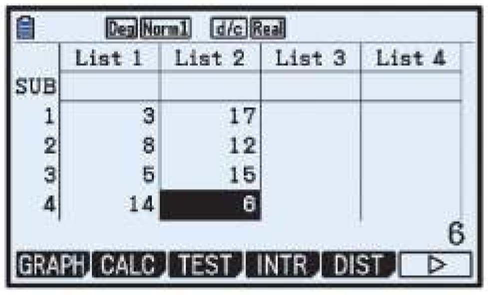


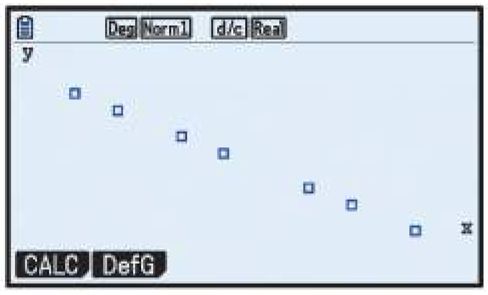
So, $r \approx 0.786$.

iii There is a moderate, positive correlation between x and y.

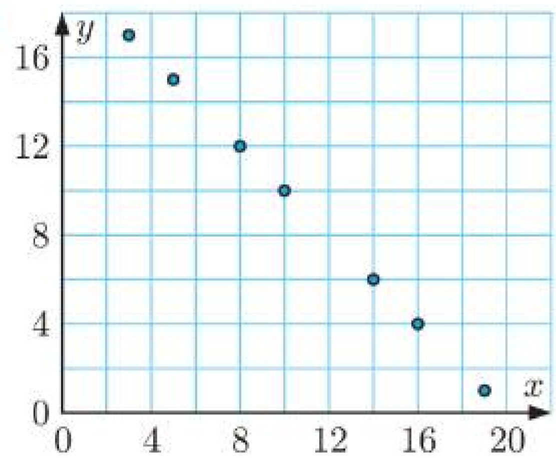
b

x	3	8	5	14	19	10	16
y	17	12	15	6	1	10	4

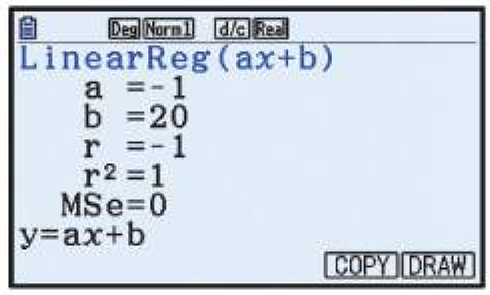




i



ii

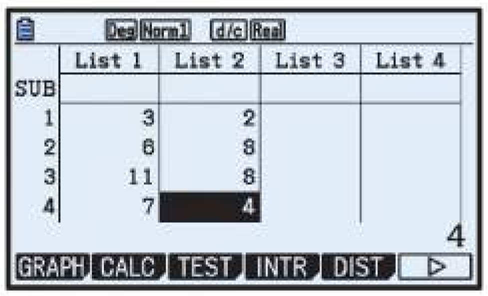


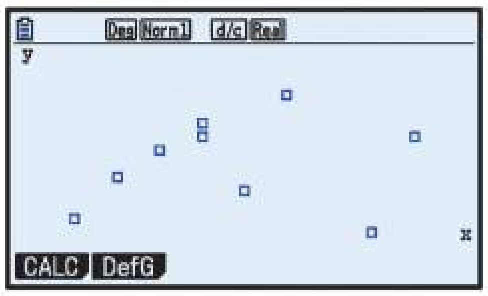
So, $r = -1$.

iii There is a perfect, negative correlation between x and y.

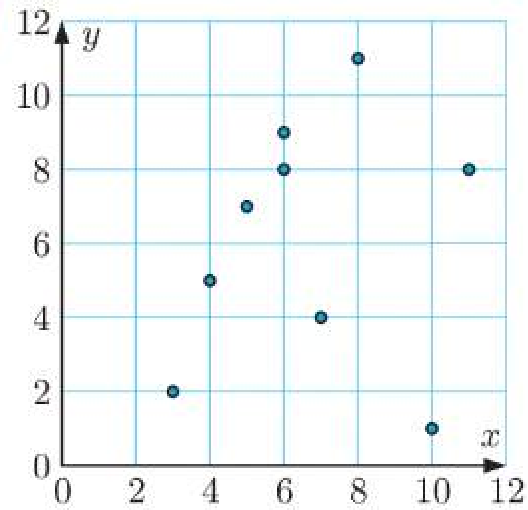
c

x	3	6	11	7	5	6	8	10	4
y	2	8	8	4	7	9	11	1	5

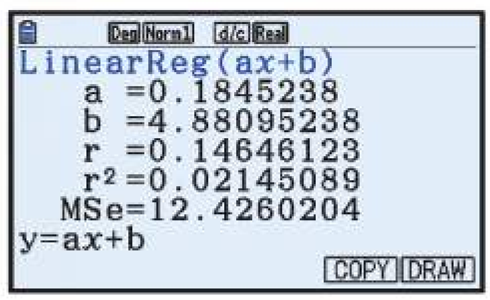




i



ii

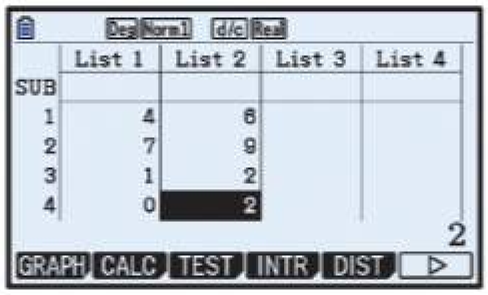


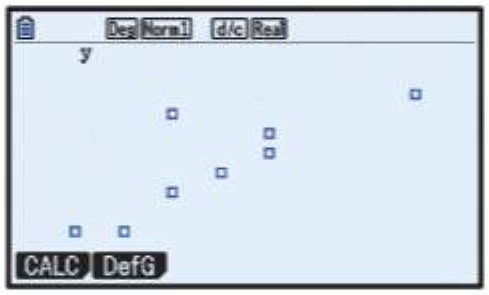
So, $r \approx 0.146$.

iii There is a very weak, positive correlation between x and y.

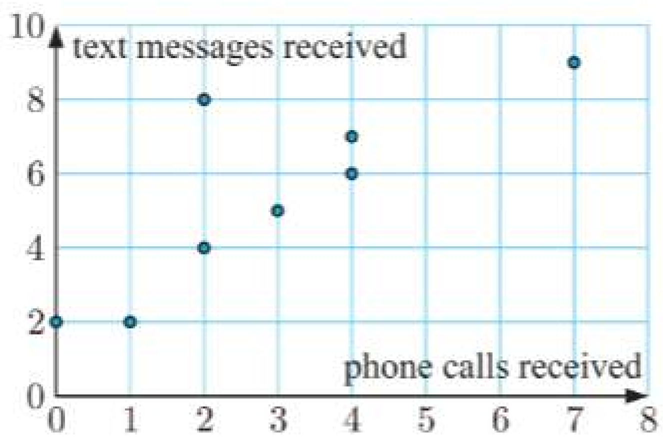
4

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

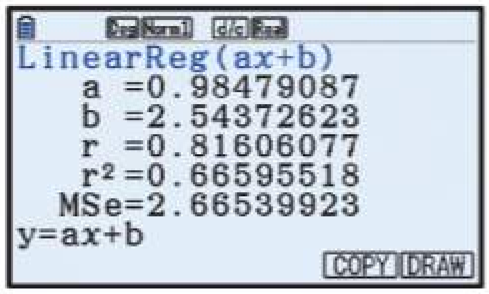




a



b



So, $r \approx 0.816$.

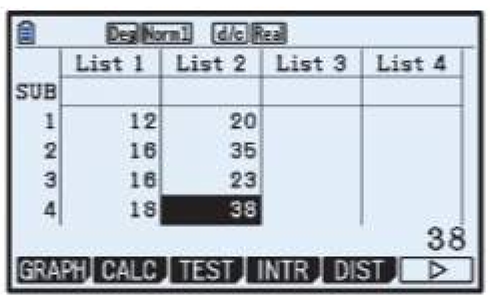
c There is a moderate, positive correlation between *phone calls received* and *text messages received*.

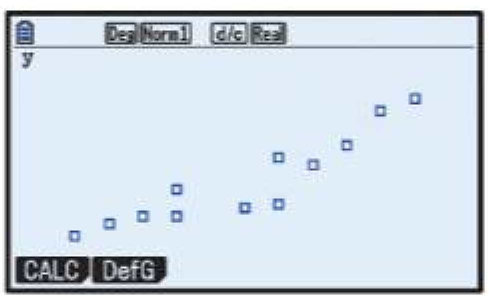
d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

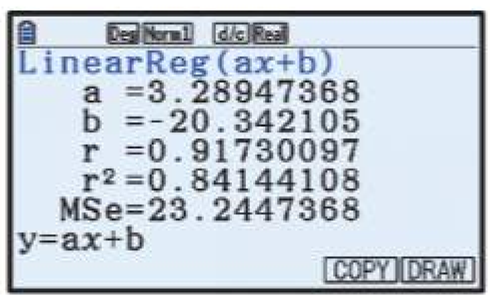
5

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

a





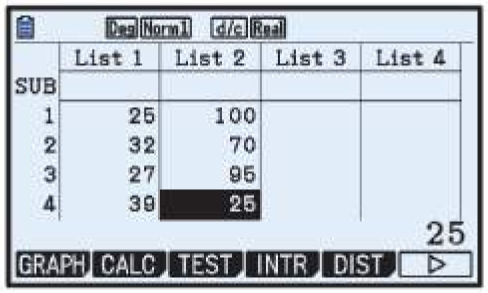


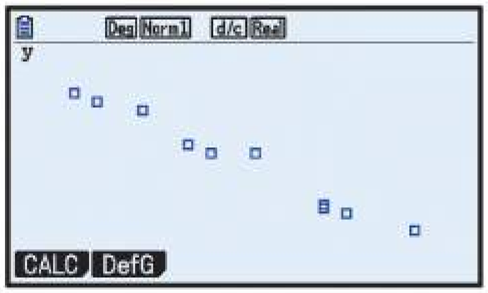
So, $r \approx 0.917$.

b There is a strong, positive correlation between the *age* of the young athlete and the *distance thrown*. In general, the higher the young athlete's age, the further they can throw a discus.

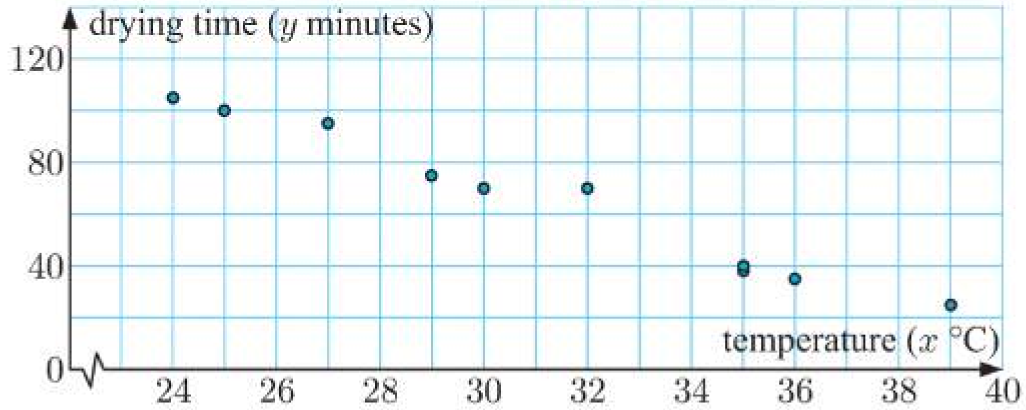
6

Temperature (x °C)	25	32	27	39	35	24	30	36	29	35
Drying time (y minutes)	100	70	95	25	38	105	70	35	75	40

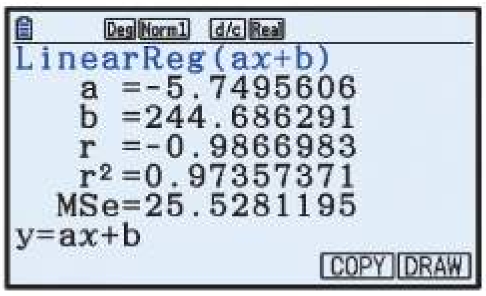




a



b

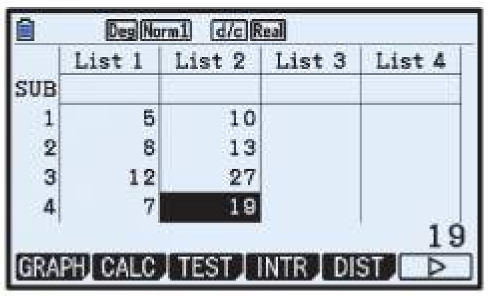


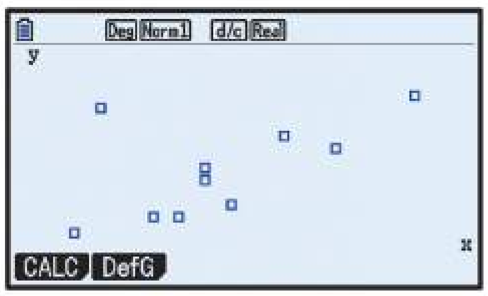
So, $r \approx -0.987$.

c There is a very strong, negative correlation between *temperature* and *drying time*. In general, the higher the temperature, the lower the drying time.

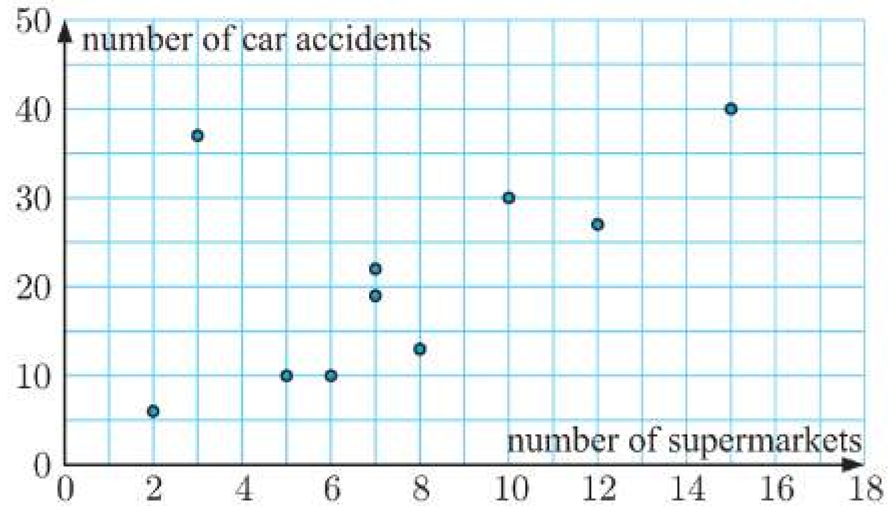
7

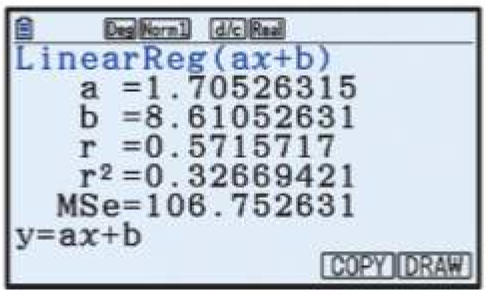
Number of supermarkets	5	8	12	7	6	2	15	10	7	3
Number of car accidents	10	13	27	19	10	6	40	30	22	37





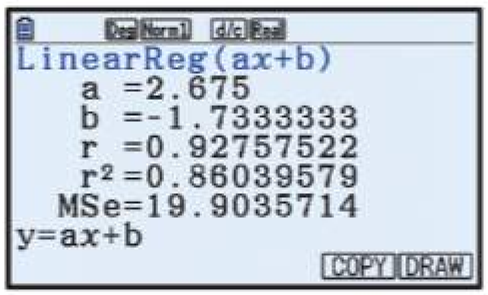
a



b 

So, $r \approx 0.572$.

c The point $(3, 37)$, which represents 37 car accidents in a town with 3 supermarkets, is an outlier.

d **i** 

So, $r \approx 0.928$.

ii There is a strong, positive correlation between the *number of supermarkets* and the *number of car accidents.*

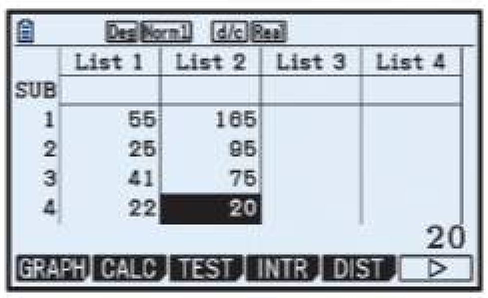
iii By removing the outlier, the value of r increased significantly.

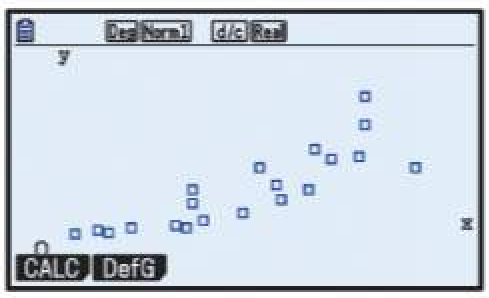
e No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

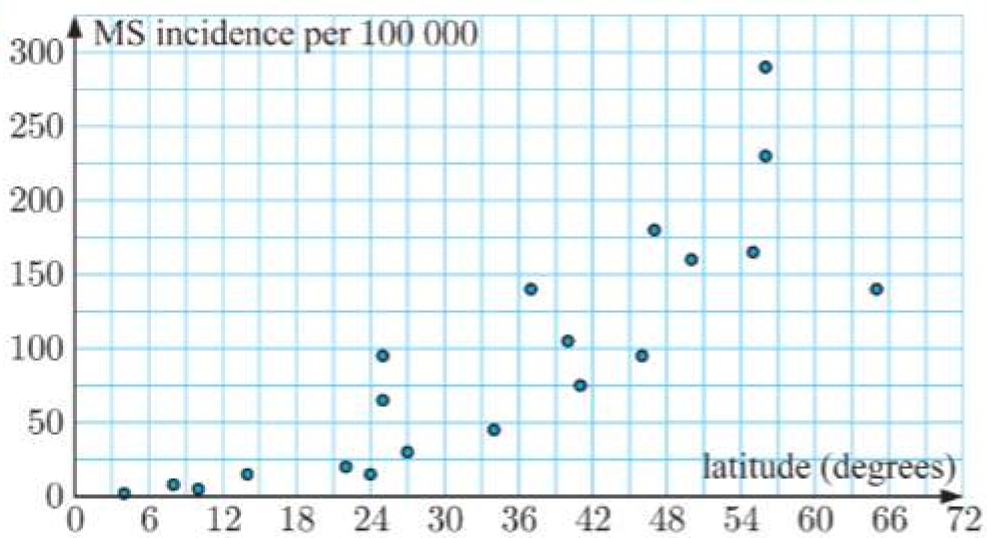
8

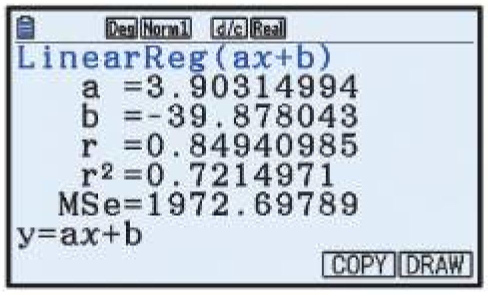
Latitude (degrees)	55	25	41	22	47	37	56	14	34	25
MS incidence per 100 000	165	95	75	20	180	140	230	15	45	65

Latitude (degrees)	27	65	10	24	4	56	46	8	50	40
MS incidence per 100 000	30	140	5	15	2	290	95	8	160	105





a 

b 

So, $r \approx 0.849$.

c There is a moderate, positive correlation between *latitude* and *MS incidence*.

d The incidence of MS is higher near the poles.

EXERCISE 7C

1 57.8% of the variation in the *number of visitors* can be explained by the variation in *maximum temperature*.

2
$$r = 0.7732$$
$$\therefore\ r^2 = (0.7732)^2 \approx 0.598$$

About 59.8% of the variation in the amount of *money lost* can be explained by the variation in *time spent gambling*.

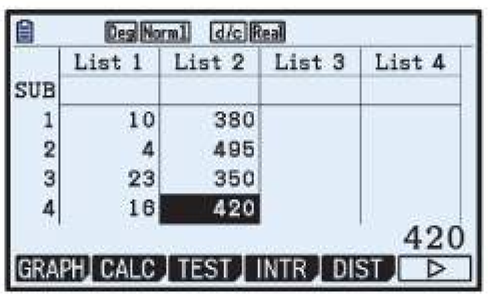
3
$$r = -0.365$$
$$\therefore\ r^2 = (-0.365)^2 \approx 0.133$$

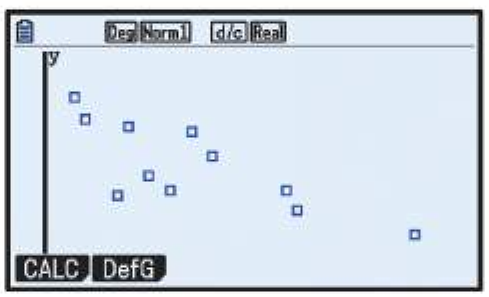
About 13.3% of the variation in *heart rate* can be explained by the variation in *age*.

4
$$r^2 = 0.821$$
$$\therefore\ r = -\sqrt{0.821} \quad \{r < 0 \text{ as } \textit{weight} \text{ and } \textit{running speed} \text{ are negatively correlated}\}$$
$$\approx -0.906$$

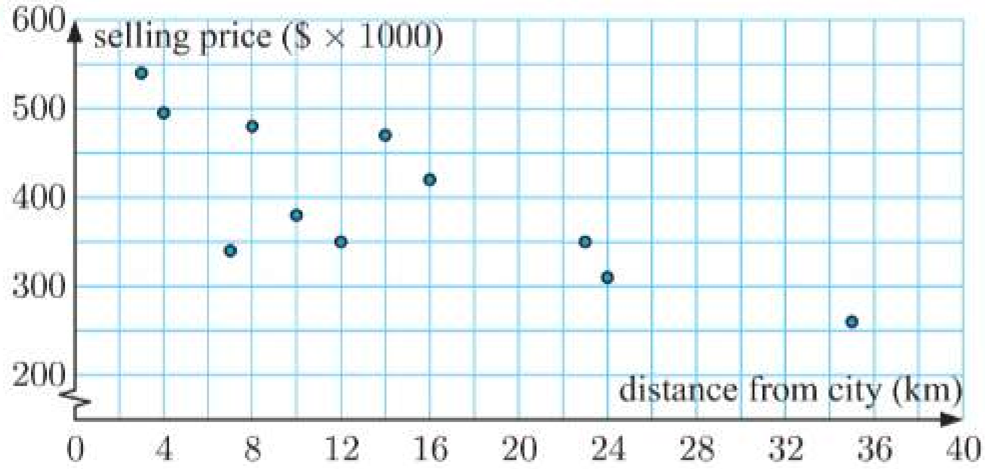
5

Distance from city (km)	10	4	23	16	3	35	8	7	12	24	14	12
Selling price (× \$1000)	380	495	350	420	540	260	480	340	350	310	470	350

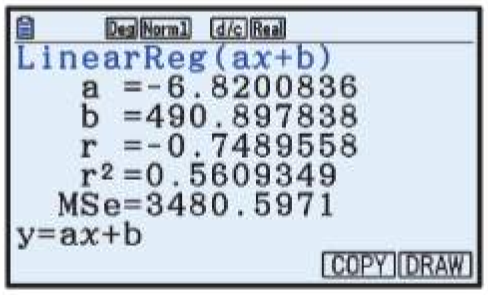




a



b



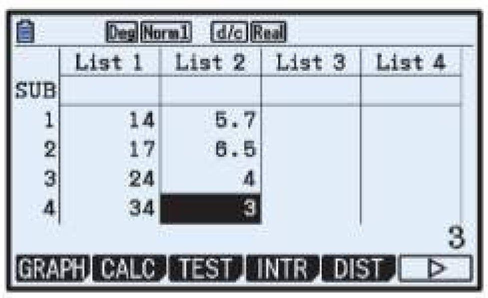
So, $r^2 \approx 0.561$.

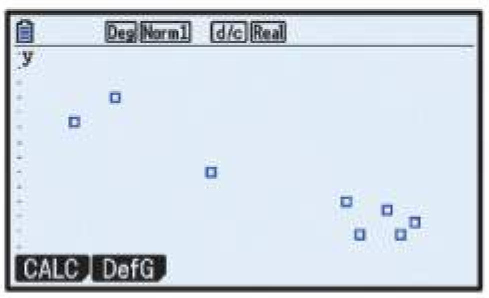
c About 56.1% of the variation in *selling price* can be explained by the variation in the *distance from the city*.

d The age of the house, or the size of the land could explain the variation in *selling price*.

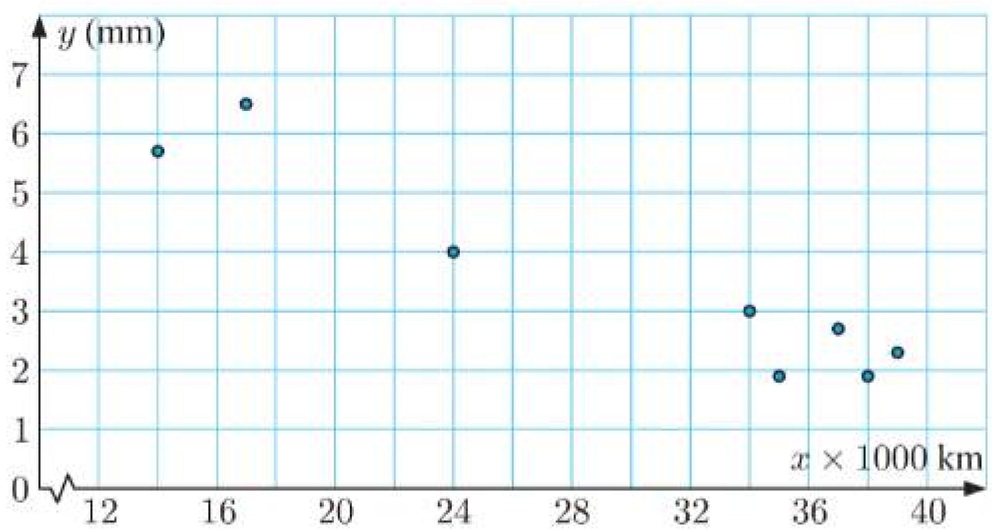
6

Distance (x km × 1000)	14	17	24	34	35	37	38	39
Tread depth (y mm)	5.7	6.5	4.0	3.0	1.9	2.7	1.9	2.3





a



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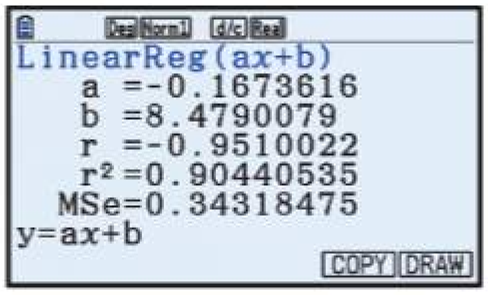


So, $r^2 \approx 0.904$.

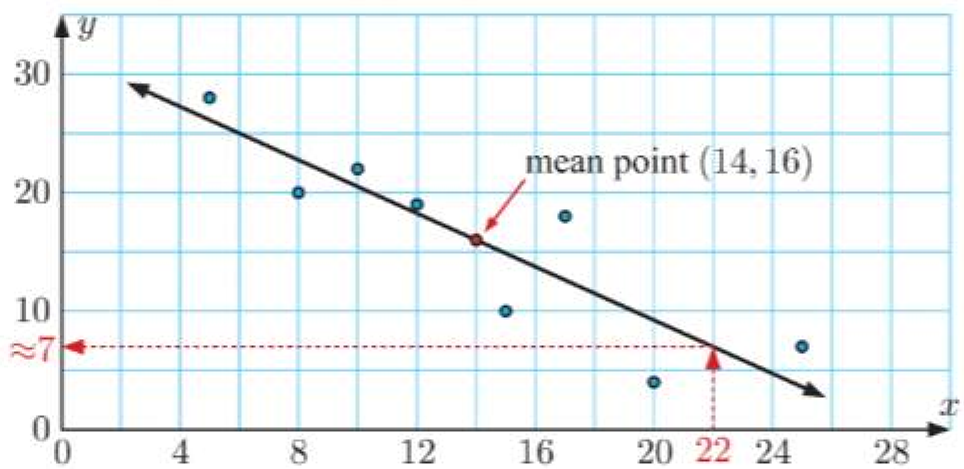
About 90.4% of the variation in *tread depth* can be explained by the variation in the *distance travelled.*

EXERCISE 7D

1

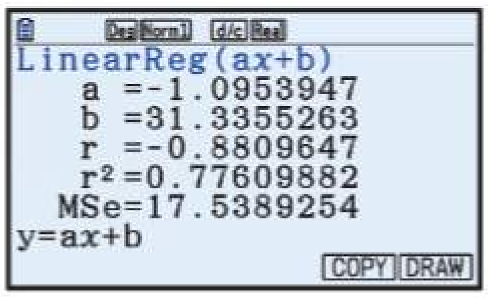
x	5	12	20	17	10	8	25	15
y	28	19	4	18	22	20	7	10

a, f



b The data appears to be negatively correlated.

c



So, $r \approx -0.881$.

d There is a strong, negative correlation between x and y.

e $\overline{x} = \dfrac{5+12+20+17+10+8+25+15}{8}$, $\quad \overline{y} = \dfrac{28+19+4+18+22+20+7+10}{8}$

$= 14 \qquad\qquad = 16$

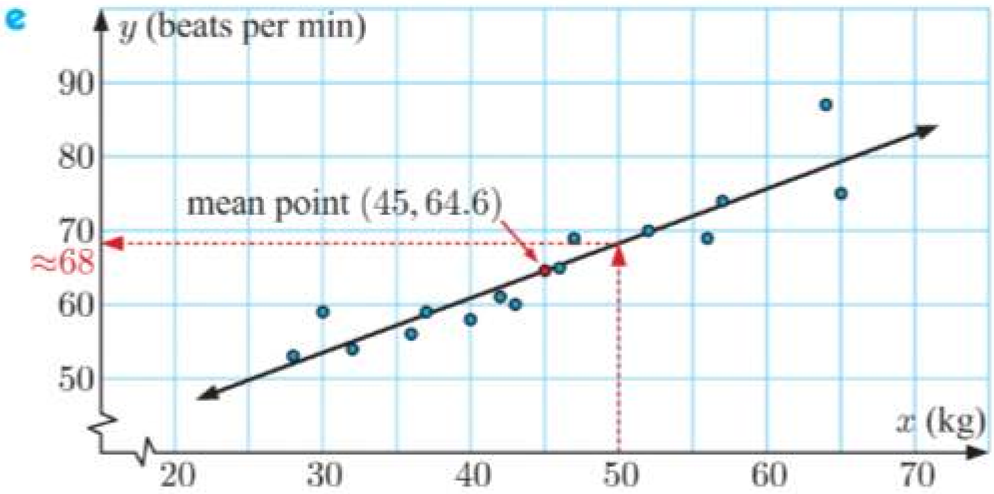
So the mean point is $(14, 16)$.

g When $x = 22$, $y \approx 7$.

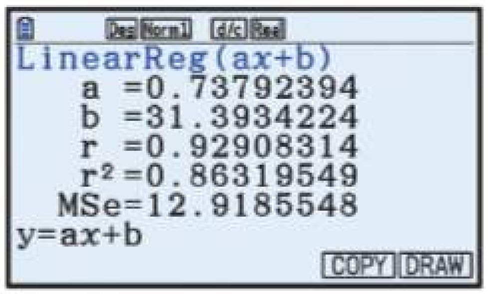
2

Weight (x kg)	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
Pulse rate (y beats per min)	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58

a, e



b



So, $r \approx 0.929$.

c There is a strong, positive correlation between the *weight* of a student and their *pulse rate.*

d $\overline{x} = \dfrac{46 + 37 + ... + 28 + 40}{15}$, $\quad \overline{y} = \dfrac{65 + 59 + ... + 53 + 58}{15}$

$= 45 \qquad\qquad = 64.6$

So the mean point is $(45, 64.6)$.

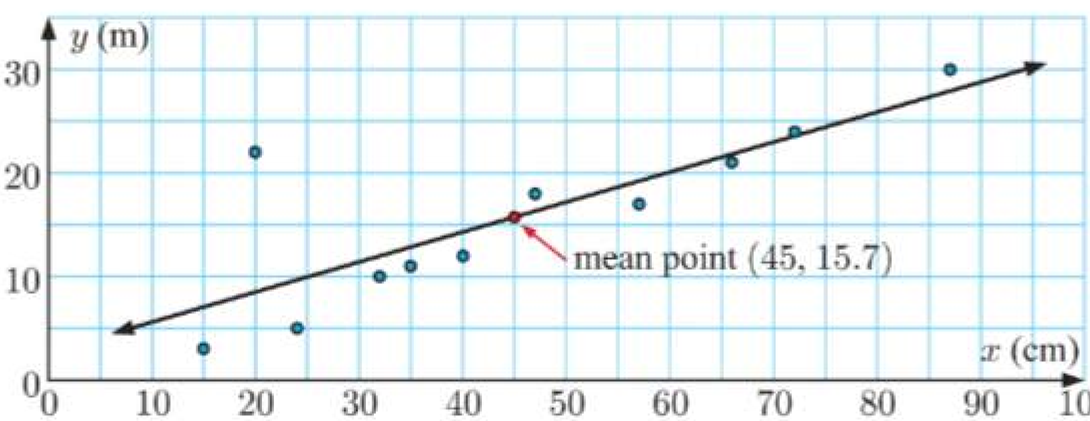
f When $x = 50$, $y \approx 68$.

A student who weighs 50 kg will have a pulse rate of approximately 68 beats per minute. This is an interpolation, so the estimate is reliable.

3

Trunk width (x cm)	35	47	72	40	15	87	20	66	57	24	32
Height (y m)	11	18	24	12	3	30	22	21	17	5	10

a, e



b $(20, 22)$ is an outlier as it appears separated from the rest of the data.

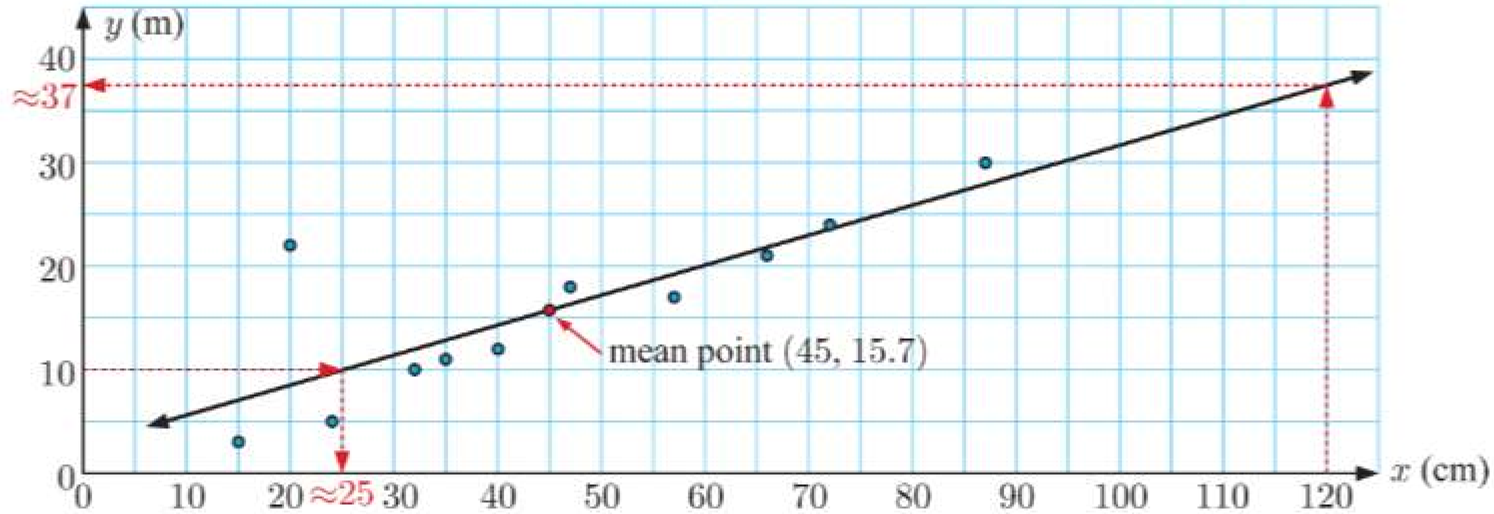
c The tree represented by the outlier would be very tall and thin.

d $\overline{x} = \dfrac{35 + 47 + ... + 24 + 32}{11}$, $\quad \overline{y} = \dfrac{11 + 18 + ... + 5 + 10}{11}$

$= 45 \qquad\qquad \approx 15.7$

So the mean point is $(45, 15.7)$.

f We extend the scatter diagram from **a** to include the value $x = 120$:



When $x = 120$, $y \approx 37$.

A tree with trunk width 120 cm will have a height of approximately 37 m. This is an extrapolation, so the prediction may not be reliable.

g When $y = 10$, $x \approx 25$.

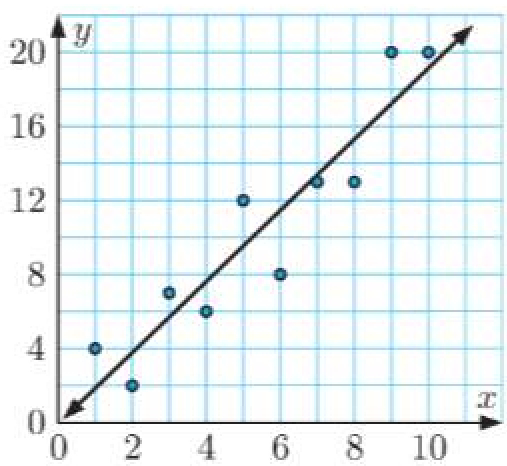
A tree with height 10 m will have a trunk width of approximately 25 cm. This is an interpolation, so the estimate is reliable.

EXERCISE 7E

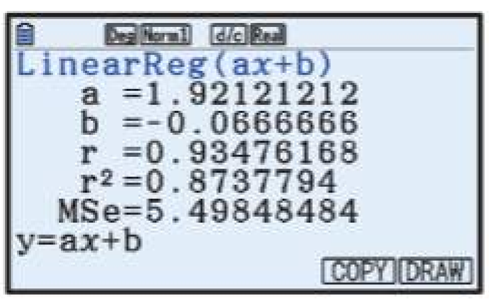
1

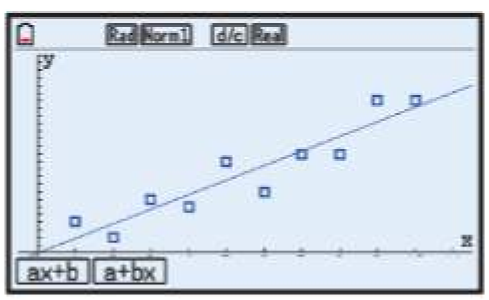
x	10	4	6	8	9	5	7	1	2	3
y	20	6	8	13	20	12	13	4	2	7

a, c



b



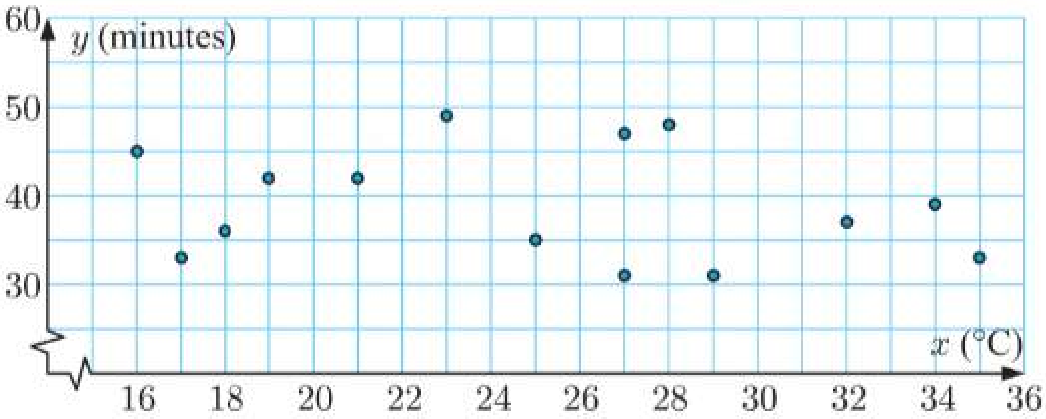


Using technology, the least squares regression line is $y \approx 1.92x - 0.0667$.

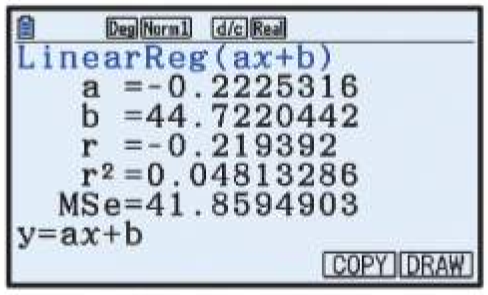
2

Temperature (x °C)	25	19	23	27	32	35	29	27	21	18	16	17	28	34
Time (y minutes)	35	42	49	31	37	33	31	47	42	36	45	33	48	39

a



b



So, $r \approx -0.219$ and $r^2 \approx 0.0481$.

c There is a very weak, negative correlation between *temperature* and *time*.

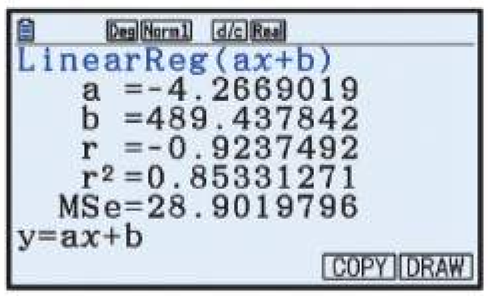
d No, it is not reasonable to find a line of best fit for this data as there is almost no correlation, and only about 4.81% of the variation in y is explained by the variation in x.

3

Petrol price (x cents per litre)	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
Number of customers (y)	45	42	25	48	43	15	19	10

Petrol price (x cents per litre)	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
Number of customers (y)	30	23	42	50	12	24	32	17

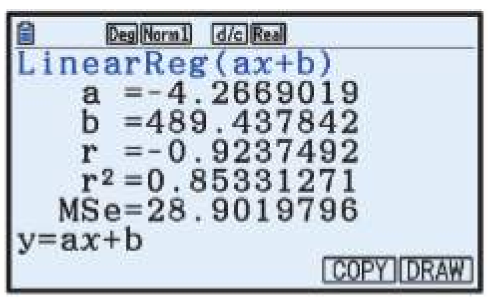
a



So, $r \approx -0.924$ and $r^2 \approx 0.853$.

b There is a strong, negative correlation between *petrol price* and the *number of customers*.

c



Using technology, the least squares regression line is $y \approx -4.27x + 489$.

d Approximately 85.3% of the variation in the *number of customers* can be explained by the variation in *petrol price*.

So, the linear model fits the data quite well.

e The gradient of the least squares regression line ≈ -4.27. This means that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

f When $x = 115.9$, $y \approx -4.27(115.9) + 489$

$$\approx -5.10$$

So, when petrol is 115.9 cents per litre, we would expect about -5.10 customers per hour.

g When $y = 40$, $40 \approx -4.27x + 489$

$$\therefore \quad -449 \approx -4.27x$$

$$\therefore \quad x \approx 105.3$$

So, a petrol station which has 40 customers per hour would sell petrol at approximately 105.3 cents per litre.

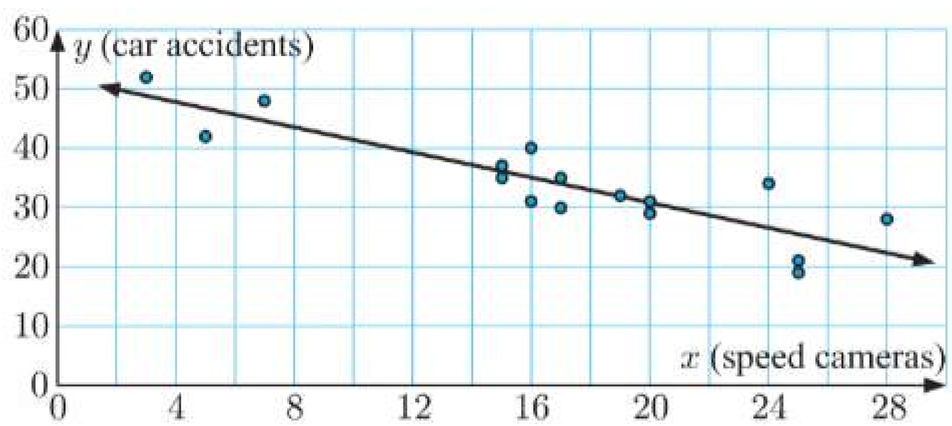
h In **f**, it is impossible to have a negative number of customers. This extrapolation is not valid.

In **g**, this is an interpolation, so this estimate is likely to be reliable.

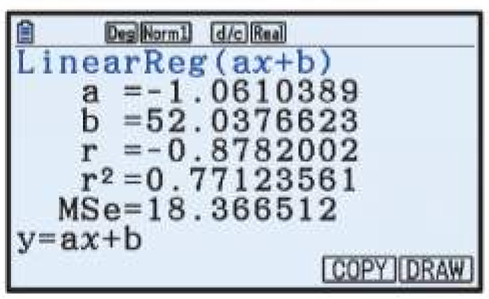
4

Number of speed cameras (x)	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
Number of car accidents (y)	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32

a



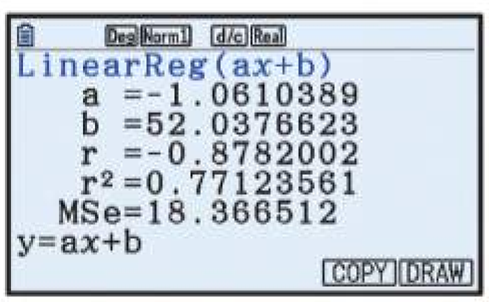
b



So, $r \approx -0.878$ and $r^2 \approx 0.771$.

c There is a strong, negative correlation between the *number of speed cameras* and the *number of car accidents*.

d



Using technology, the least squares regression line is $y \approx -1.06x + 52.0$.

e Approximately 77.1% of the variation in the *number of car accidents* can be explained by the variation in the *number of speed cameras*.

So, the linear model fits the data moderately well.

f The gradient of the least squares regression line ≈ -1.06. This indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.

The y-intercept of the least squares regression line ≈ 52.0. This indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

g When $x = 10$, $y \approx -1.06(10) + 52.0$

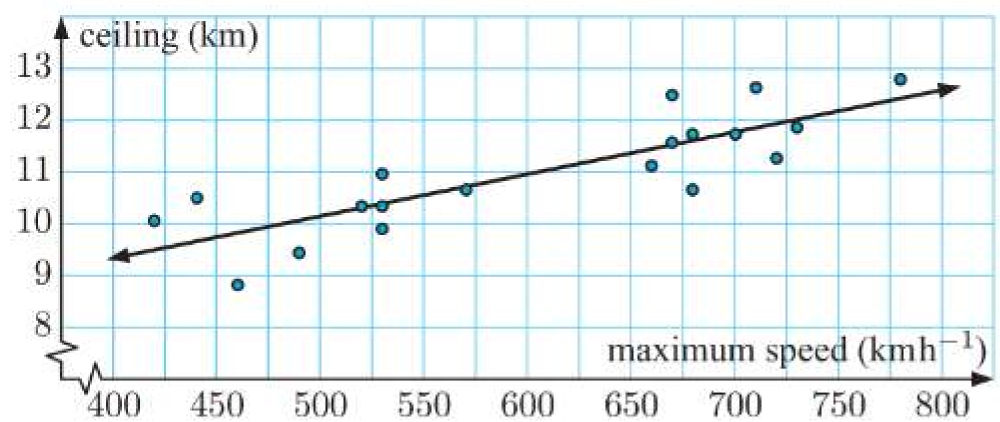
≈ 41.4

So, there will be approximately 41.4 car accidents per week in a city with 10 speed cameras.

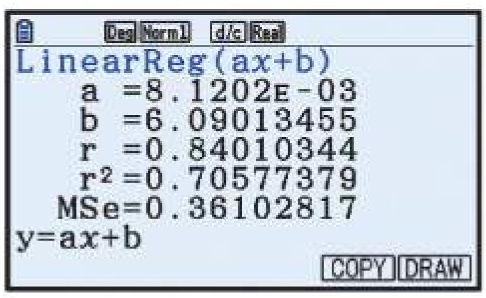
5

Maximum speed	*Ceiling*
460	8.84
420	10.06
530	10.97
530	9.906
490	9.448
530	10.36
680	11.73
680	10.66
720	11.27
710	12.64
660	11.12
780	12.80
730	11.88
670	12.49
570	10.66
440	10.51
670	11.58
700	11.73
520	10.36

a, d



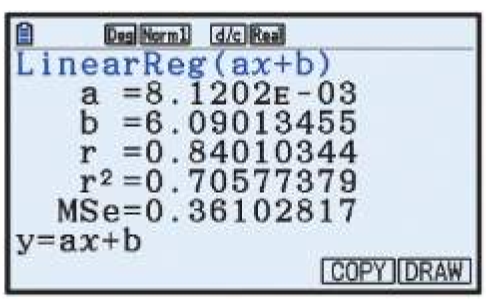
b



So, $r \approx 0.840$ and $r^2 \approx 0.706$.

c There is a moderate, positive, linear correlation between *maximum speed* and *ceiling*.

d



Using technology, the least squares regression line is $y \approx 0.008\,12x + 6.09$.

e Approximately 70.6% of the variation in the *ceiling* can be explained by the variation in the *maximum speed*.

So, the linear model fits the data moderately well.

f The gradient of the least squares regression line $\approx 0.008\,12$. This indicates that for each additional $\text{km}\,\text{h}^{-1}$, the ceiling increases by an average of approximately 0.008 12 km or 8.12 m.

g When $x = 600$, $y \approx 0.008\,12(600) + 6.09$
≈ 11.0

So, a fighter plane with maximum speed 600 $\text{km}\,\text{h}^{-1}$ would have a ceiling of approximately 11.0 km.

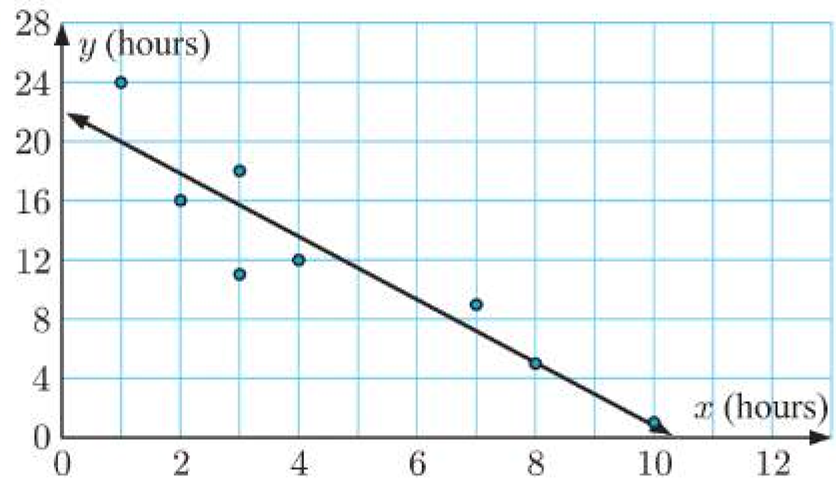
h When $y = 11$, $11 \approx 0.008\,12x + 6.09$
$\therefore\ 4.91 \approx 0.008\,12x$
$\therefore\ x \approx 605$

So, a fighter plane with a ceiling of 11 km would have maximum speed of approximately 605 $\text{km}\,\text{h}^{-1}$.

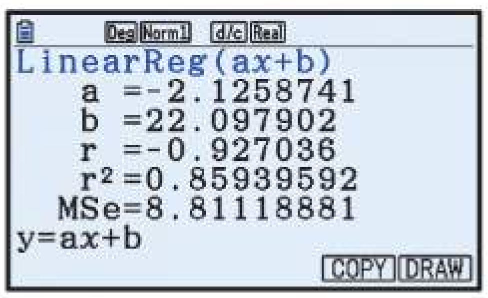
6

Exercise (x hours per week)	4	1	8	7	10	3	3	2
Television (y hours per week)	12	24	5	9	1	18	11	16

a, d



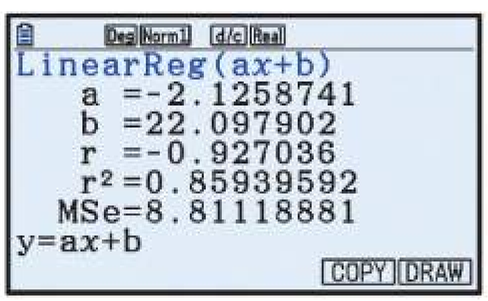
b



So, $r \approx -0.927$ and $r^2 \approx 0.859$.

c There is a strong, negative, linear correlation between *time exercising* and *time watching television*.

d



Using technology, the least squares regression line is $y \approx -2.13x + 22.1$.

e Approximately 85.9% of the variation in the *time watching television* can be explained by the variation in the *time exercising*.

So, the linear model fits the data quite well.

f The gradient of the least squares regression line ≈ -2.13. This indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.

The y-intercept of the least squares regression line ≈ 22.1. This indicates that for children who do not spend time exercising, they would watch television for an average of about 22.1 hours per week.

g **i** From the table, the student who exercised for 7 hours each week watched 9 hours of television each week.

ii When $x = 7$, $y \approx -2.126(7) + 22.1$

≈ 7.22

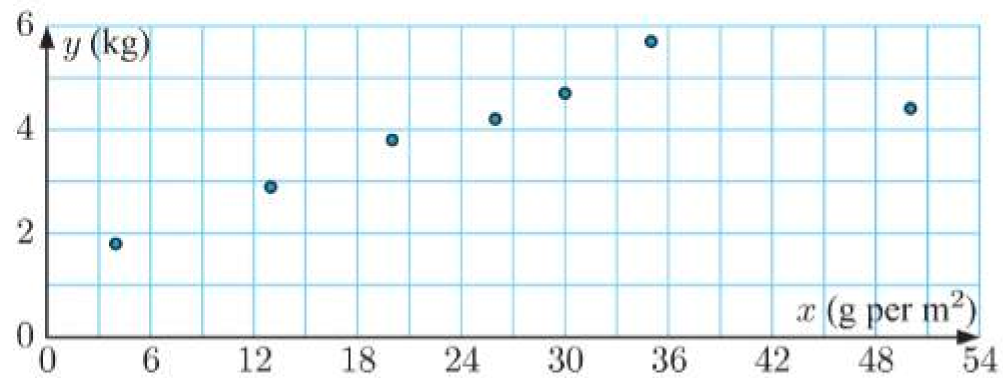
Using the least squares regression line, a child who exercises for 7 hours each week watches approximately 7.22 hours of television each week.

iii This particular child spent more time watching television than predicted.

7

Fertiliser (x g per m^2)	4	13	20	26	30	35	50
Yield (y kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4

a

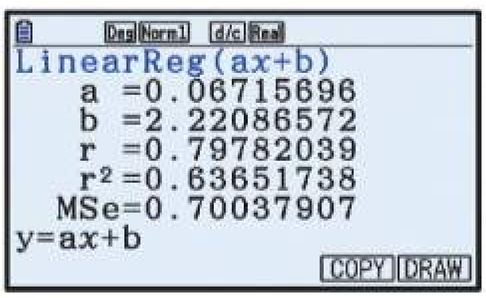


$(50,\ 4.4)$ is the outlier.

b **i** The outlier reduces the strength of correlation of the data.

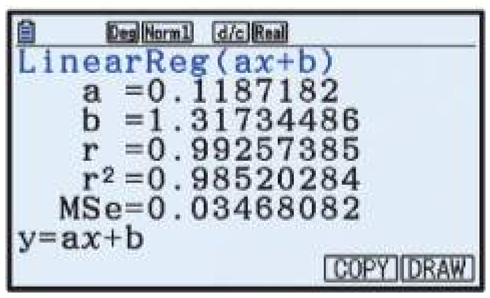
ii The outlier decreases the gradient of the least squares regression line.

c **i**



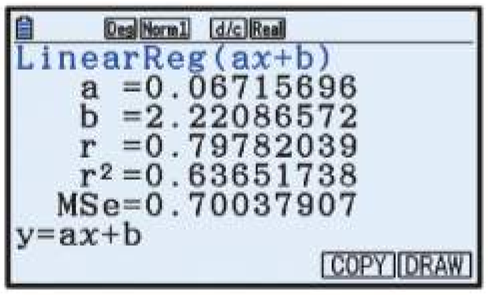
So, $r \approx 0.798$ and $r^2 \approx 0.637$.

ii



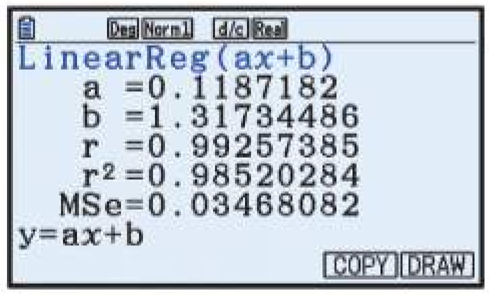
So, $r \approx 0.993$ and $r^2 \approx 0.985$.

d **i**



Using technology, the least squares regression line is $y \approx 0.0672x + 2.22$.

ii



Using technology, the least squares regression line is $y \approx 0.119x + 1.32$.

e The regression line which excludes the outlier should be used to estimate the yield when 15 g per m^2 of fertiliser is used. This will be more accurate for an interpolation.

f Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

ACTIVITY 2 FITTING A LINE THROUGH THE ORIGIN

$y = \alpha x$

1 $$\begin{aligned} SS_{\text{res}} &= \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2 \\ &= \sum_{i=1}^{n} (y_i - \alpha x_i)^2 \\ &= \sum_{i=1}^{n} ({y_i}^2 - 2\alpha x_i y_i + \alpha^2 {x_i}^2) \\ &= \alpha^2 \sum_{i=1}^{n} ({x_i}^2) - 2\alpha \sum_{i=1}^{n} (x_i y_i) + \sum_{i=1}^{n} ({y_i}^2) \end{aligned}$$

2 SS_{res} is a quadratic in α with $a = \sum_{i=1}^{n} ({x_i}^2)$, $b = -2\sum_{i=1}^{n} (x_i y_i)$, and $c = \sum_{i=1}^{n} ({y_i}^2)$.

Since $a > 0$, the shape is  .

So, SS_{res} is minimised when $$\begin{aligned} \alpha &= -\frac{b}{2a} \\ &= -\frac{-2\sum_{i=1}^{n} (x_i y_i)}{2\sum_{i=1}^{n} ({x_i}^2)} \\ &= \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} {x_i}^2} \end{aligned}$$

3

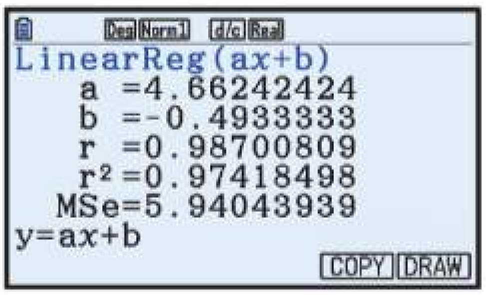
x	1	2	3	4	5	6	7	8	9	10
y	2.6	9.6	11.0	22.8	23.5	24.5	33.0	38.2	42.2	44.1

a The zero-intercept model $y = \alpha x$ is obtained by minimising SS_{res}.

From **2**,
$$\alpha = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$
$$= \frac{1 \times 2.6 + 2 \times 9.6 + ... + 9 \times 42.2 + 10 \times 44.1}{1^2 + 2^2 + ... + 9^2 + 10^2}$$
$$= \frac{1767.9}{385}$$
$$\approx 4.59$$

So, the zero-intercept model is $y \approx 4.59x$.

b



So, the regular linear regression model is $y \approx 4.66x - 0.493$.

c The regular linear regression model has a slightly steeper gradient than the zero-intercept model. This is because the regular regression model has a lower y-intercept than the zero-intercept model which has y-intercept 0.

4 **a** To fit the model $y = ax + c$, where c is a known constant, to a data set, first set $z_i = y_i - c$ for each data value. Then fit the zero intercept model $z = ax$ to the transformed data.

So, $z = ax$
$\therefore$ $y - c = ax$
$\therefore$ $y = ax + c$

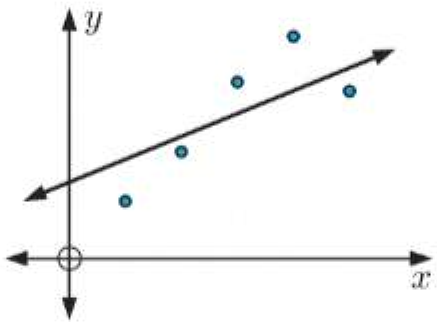
b To fit the model $y = ax + b + c$, where c is a known constant, to a data set, first set $z_i = y_i - c$ for each data value. Then fit the regular regression model $z = ax + b$ to the transformed data.

So, $z = ax + b$
$\therefore$ $y - c = ax + b$
$\therefore$ $y = ax + b + c$

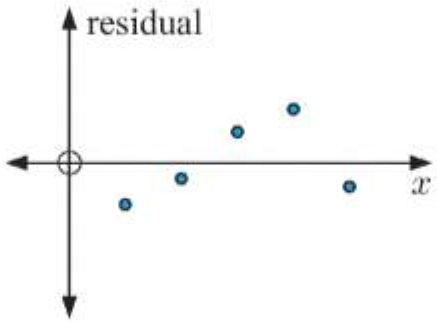
ACTIVITY 3 RESIDUAL PLOTS

PART 1: CONSTRUCTING A RESIDUAL PLOT

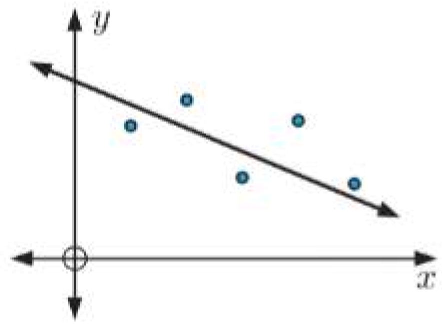
1 **a**



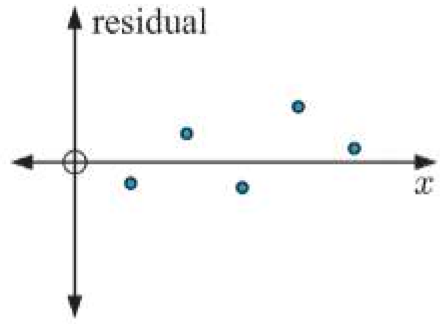
This scatter diagram has 2 data points above the least squares regression line and 3 points below the regression line.
This pattern is shown in the residual plot in **B**.



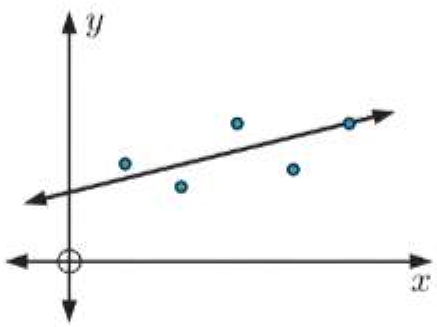
b



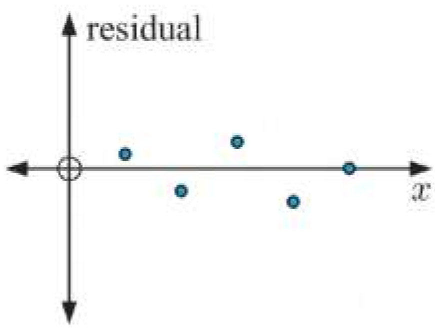
This scatter diagram has 3 data points above the least squares regression line and 2 points below the regression line.
This pattern is shown in the residual plot in **C**.



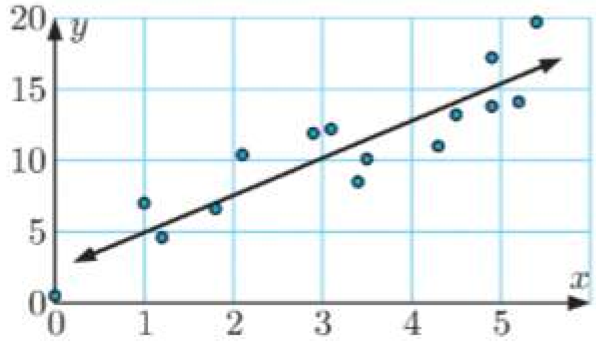
c



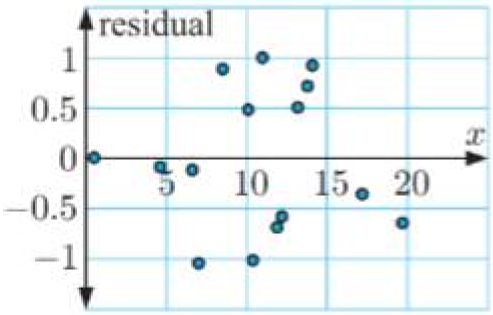
This scatter diagram has 2 data points above the least squares regression line, 2 points below the regression line, and 1 point on the regression line.
This pattern is shown in the residual plot in **A**.



2



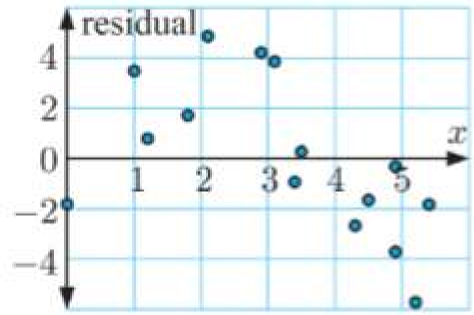
A



The values on the x-axis do not correspond to those on the scatter diagram.

So **A** is not the correct residual plot.

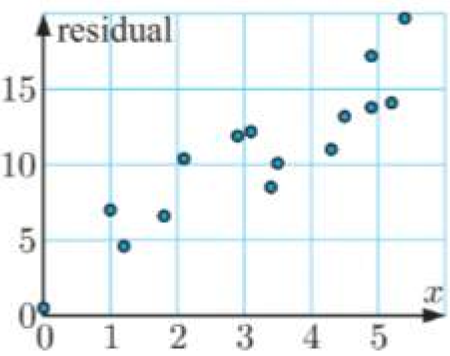
B



The residuals in this plot indicate that there are values which are more than 4 units from the regression line. The scatter diagram however shows that the values are within about ± 3 of the regression line.

So **B** is not the correct residual plot.

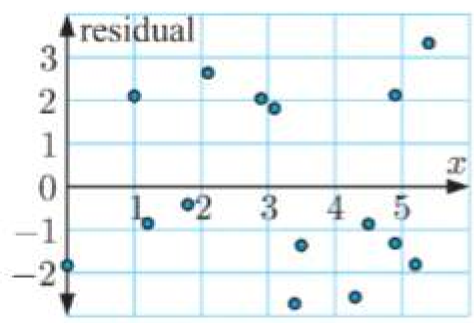
C



The residuals in this plot indicate that all values are above the regression line. The scatter diagram however shows that there are values below the regression line.

So **C** is not the correct residual plot.

D



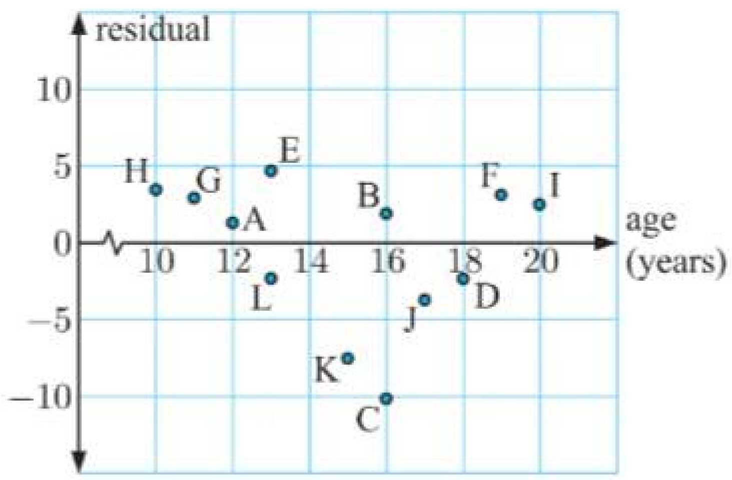
The residuals in this plot indicate that all values are within about ± 3 of the regression line. The scatter diagram shows that the values are within about ± 3 of the regression line.

So **D** is the correct residual plot.

3 **a** The athletes corresponding to the points above the x-axis threw the discus further than expected. These were athletes H, G, A, E, B, F, and I.

b The athlete closest to the x-axis is A. So athlete A performed closest to what the linear model predicted.

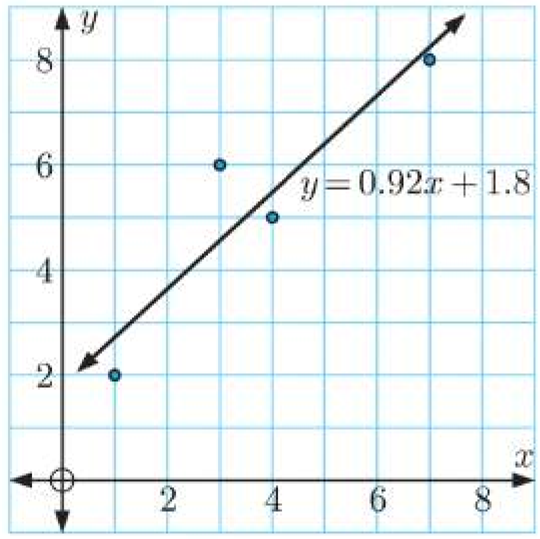
c No, it is not possible to determine which athlete threw the discus furthest. The residual plot only shows the difference between the actual distance and the predicted distance.



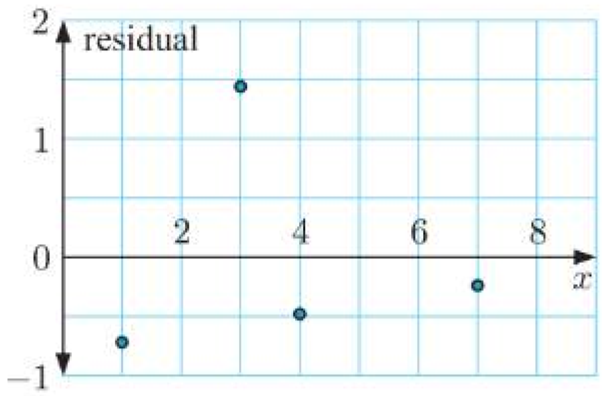
4 **a** When $x = 3$, $y = 0.92(3) + 1.8 = 4.56$
When $x = 4$, $y = 0.92(4) + 1.8 = 5.48$
When $x = 7$, $y = 0.92(7) + 1.8 = 8.24$

So, the table is:

x	y_{obs}	y_{pred}	residual $= y_{\text{obs}} - y_{\text{pred}}$
1	2	2.72	-0.72
3	6	4.56	1.44
4	5	5.48	-0.48
7	8	8.24	-0.24



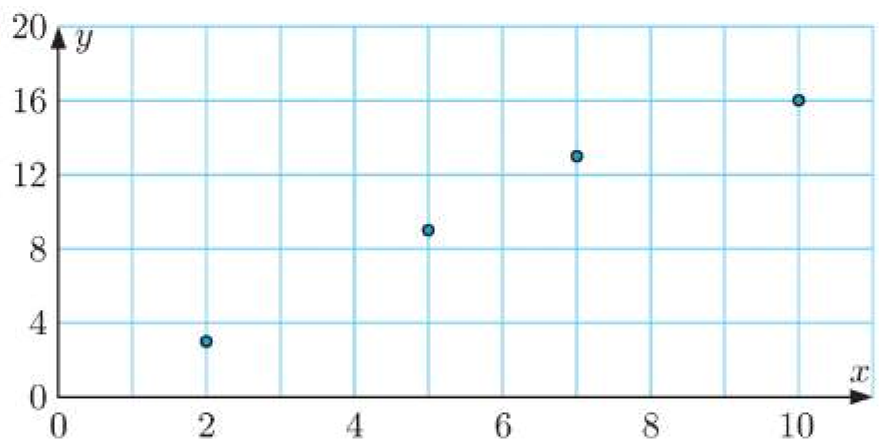
b



5 **a**

x	2	5	7	10
y	3	9	13	16

i



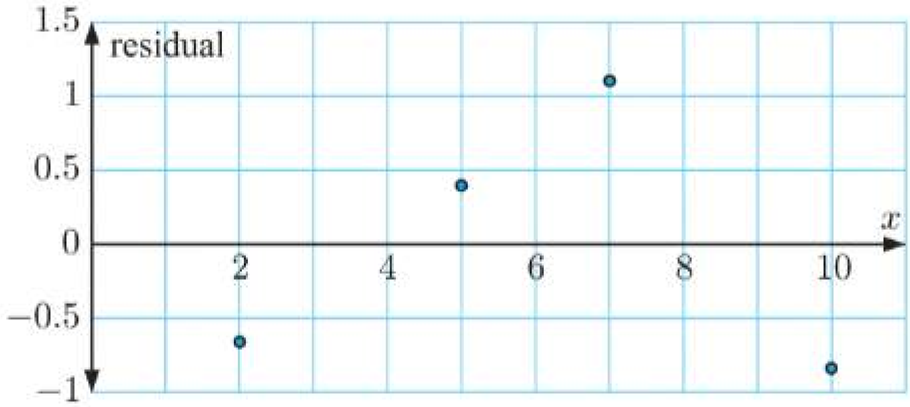
ii

```
LinearReg(ax+b)
   a =1.64705882
   b =0.36764705
   r =0.98664054
   r² =0.97345956
  MSe=1.25735294
y=ax+b
                COPY DRAW
```

Using technology, the least squares regression line is $y \approx 1.65x + 0.368$.

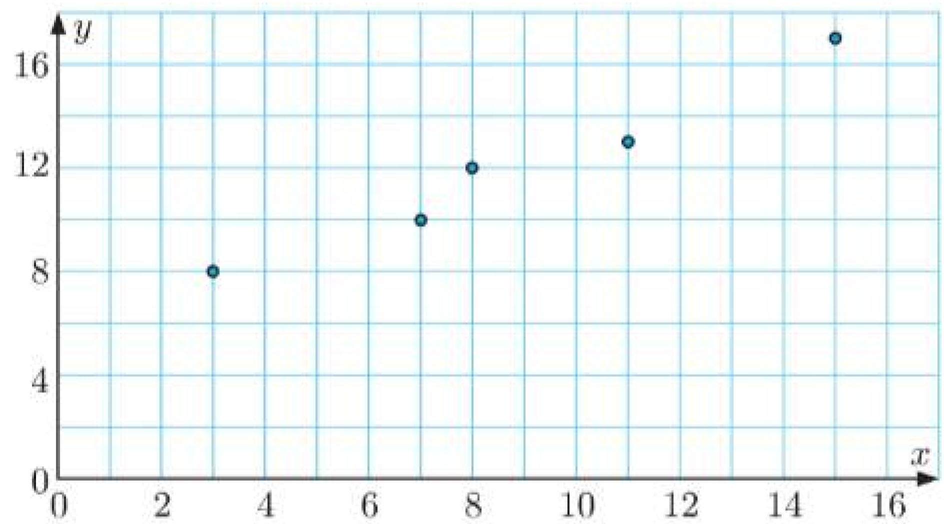
iii We find y_{pred} for each data point by evaluating $y \approx 1.65x + 0.368$ for each of the x-values.

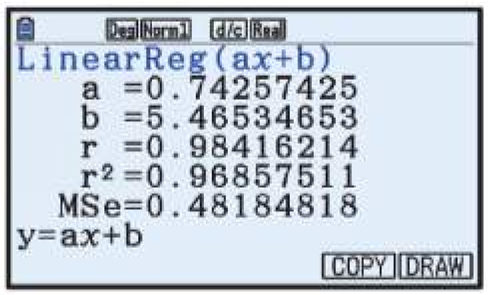
x	y_{obs}	y_{pred}	residual $= y_{\text{obs}} - y_{\text{pred}}$
2	3	3.66	-0.66
5	9	8.60	0.40
7	13	11.90	1.10
10	16	16.84	-0.84

iv 

b

x	3	7	8	11	15
y	8	10	12	13	17

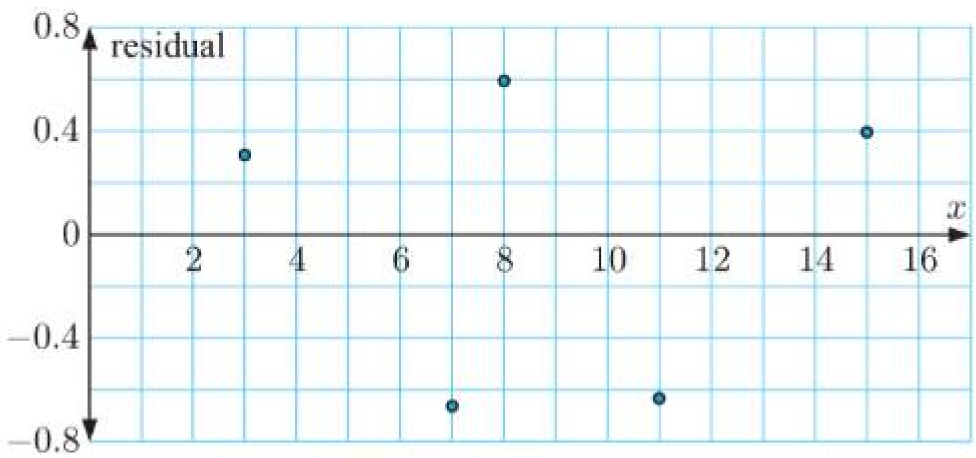
i 

ii 

Using technology, the least squares regression line is $y \approx 0.743x + 5.47$.

iii We find y_{pred} for each data point by evaluating $y \approx 0.743x + 5.47$ for each of the x-values.

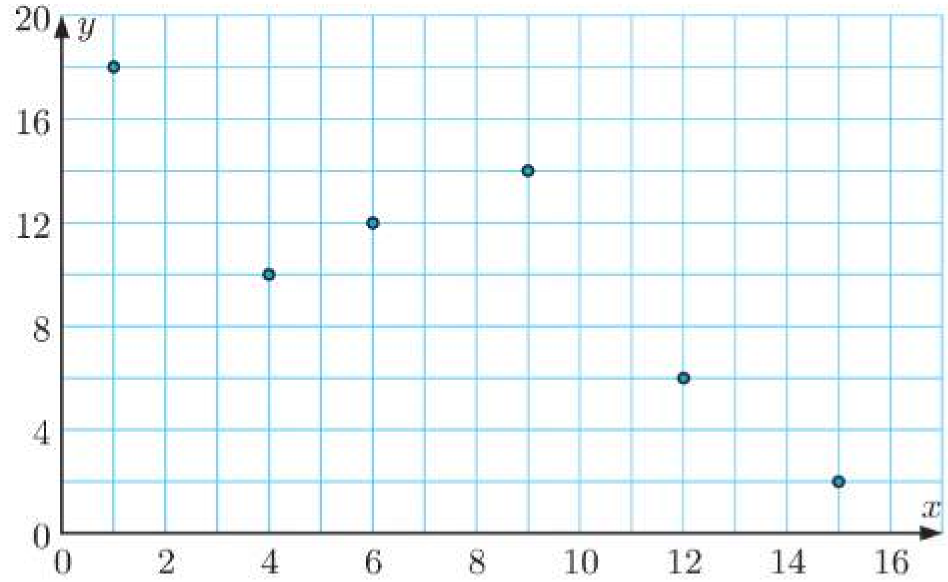
x	y_{obs}	y_{pred}	residual $= y_{\text{obs}} - y_{\text{pred}}$
3	8	7.69	0.31
7	10	10.66	−0.66
8	12	11.41	0.59
11	13	13.63	−0.63
15	17	16.60	0.40

iv 

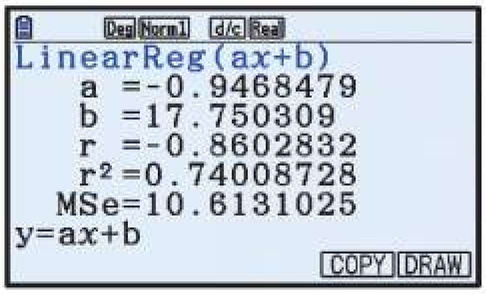
c

x	1	9	6	15	4	12
y	18	14	12	2	10	6

i



ii

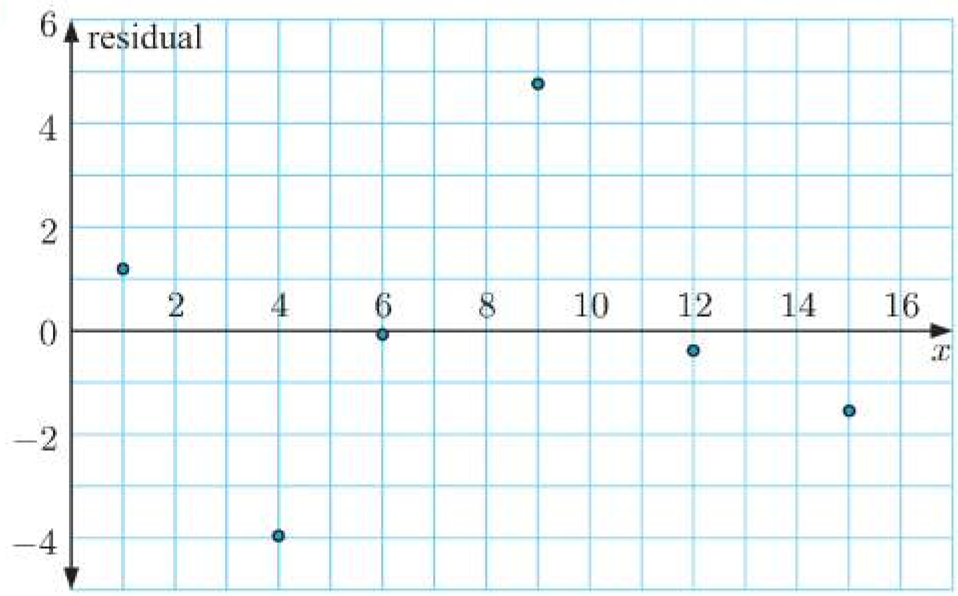


Using technology, the least squares regression line is $y \approx -0.947x + 17.8$.

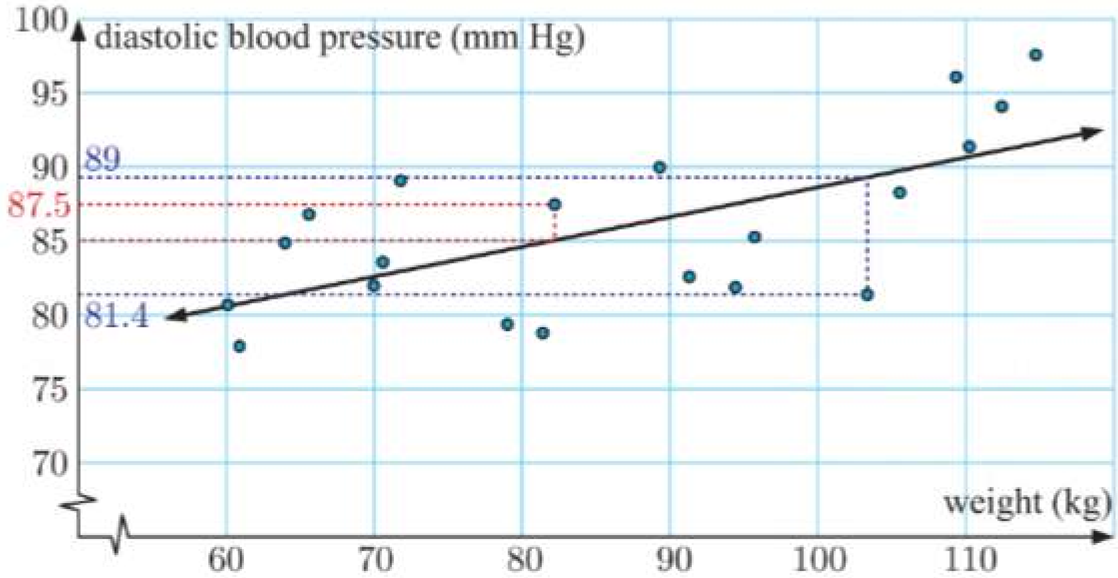
iii We find y_{pred} for each data point by evaluating $y \approx -0.947x + 17.8$ for each of the x-values.

x	y_{obs}	y_{pred}	residual $= y_{\text{obs}} - y_{\text{pred}}$
1	18	16.80	1.20
9	14	9.23	4.77
6	12	12.07	−0.07
15	2	3.55	−1.55
4	10	13.96	−3.96
12	6	6.39	−0.39

iv



6 a



b i From the graph, the residual for the point (82, 87.5) is about 2.5.

From the equation, when $x = 82$, $y_{\text{pred}} = 68.5 + 0.2 \times 82$

$$= 84.9$$

$$\therefore \text{ the residual} = y_{\text{obs}} - y_{\text{pred}}$$
$$= 87.5 - 84.9$$
$$= 2.6$$

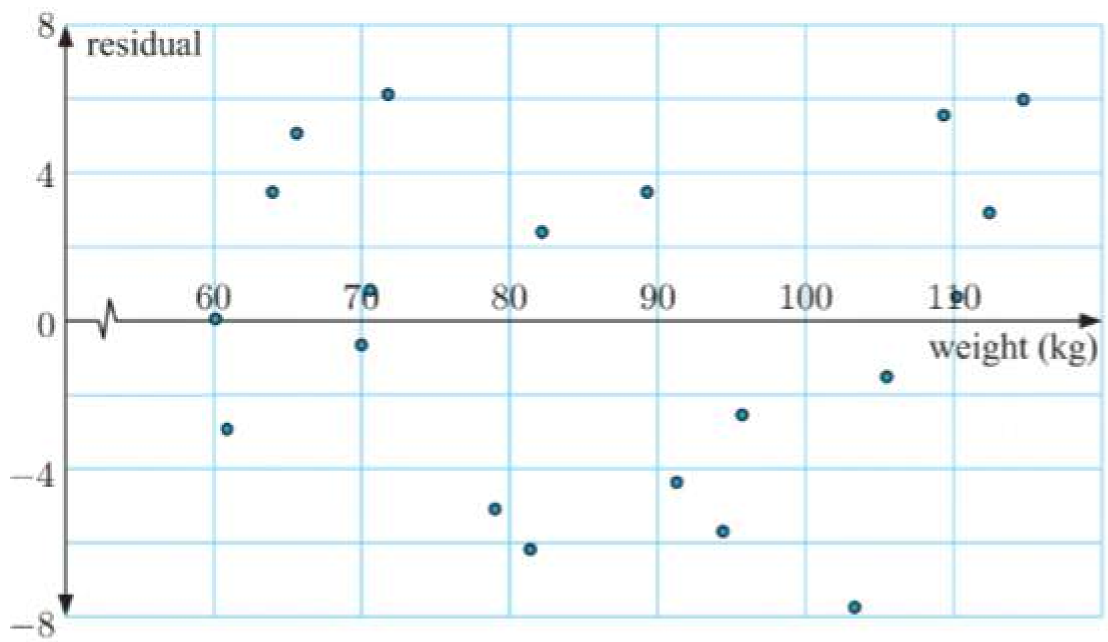
ii From the graph, the residual for the point (103.3, 81.4) is about -8.

From the equation, when $x = 103.3$, $y_{\text{pred}} = 68.5 + 0.2 \times 103.3$

$$= 89.16$$

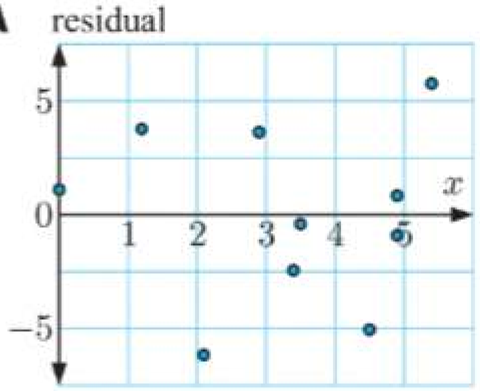
$$\therefore \text{ the residual} = y_{\text{obs}} - y_{\text{pred}}$$
$$= 81.4 - 89.16$$
$$= -7.76$$

c

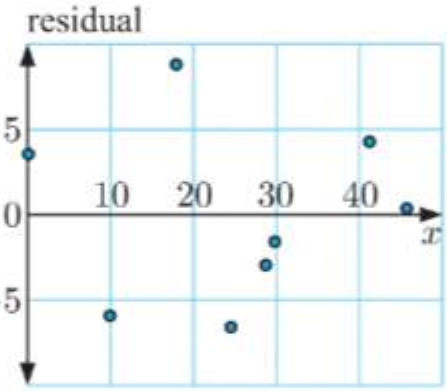


PART 2: ANALYSING RESIDUAL PLOTS

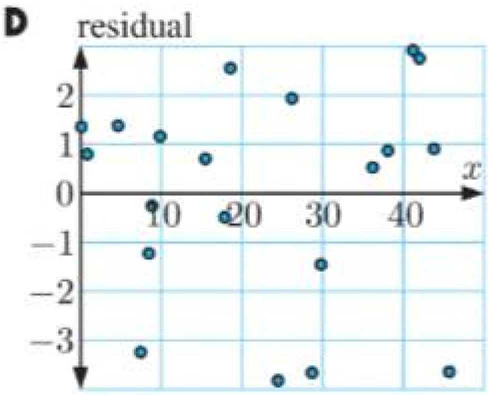
1 **A**



B

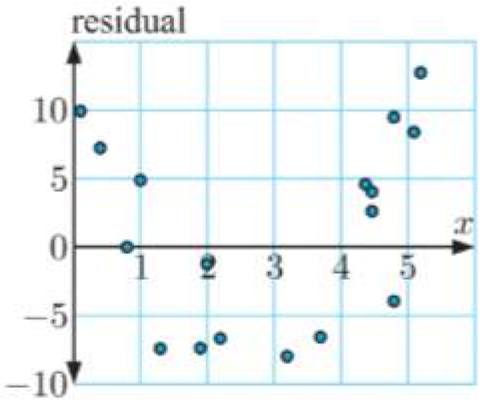


D



The residual plots for **A**, **B**, and **D** show points randomly scattered about the x-axis, with no obvious pattern.

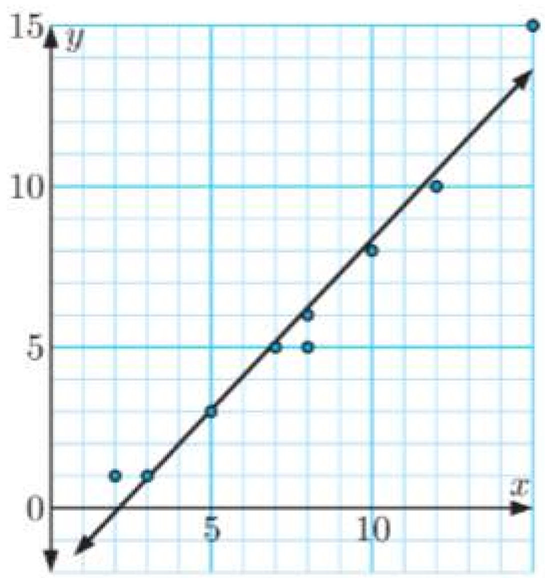
C



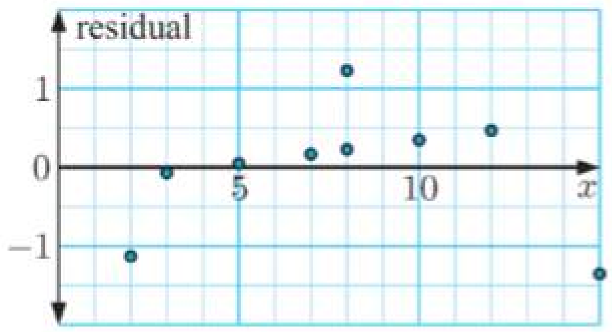
The residual plot for **C** however shows a clear, non-random pattern.

So the residual plot for **C** shows a regression line which is not a good fit for the data.

2



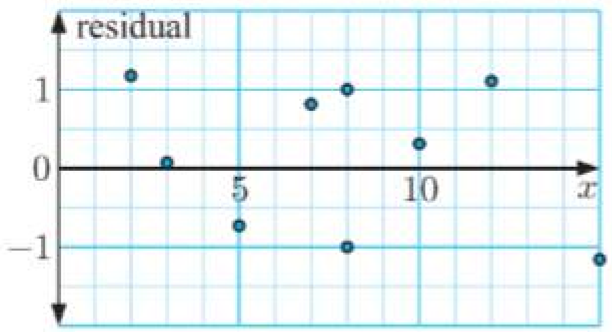
a **A**



Most of the residuals in this plot are above the x-axis. The scatter diagram however shows that only three data values are above the regression line.

So **A** is not the correct residual plot.

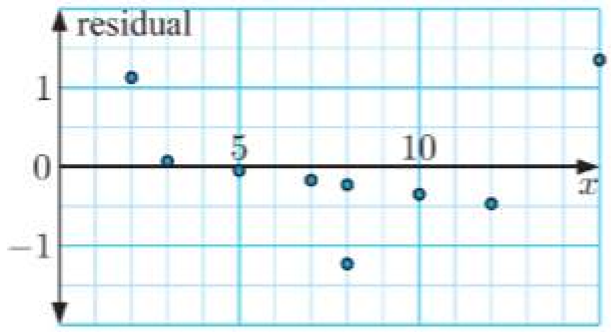
B



Most of the residuals in this plot are above the x-axis. The scatter diagram however shows that only three data values are above the regression line.

So **B** is not the correct residual plot.

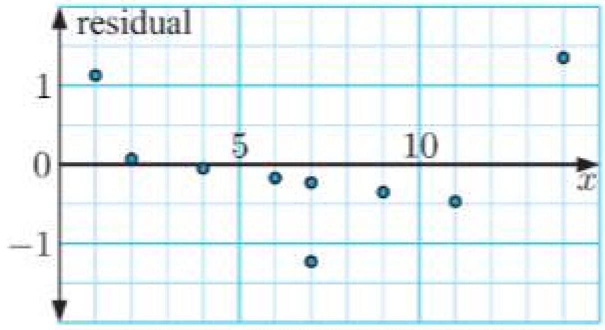
C



Most of the residuals in this plot are below the x-axis, with only three residuals above it. The scatter diagram shows that only three data values are above the regression line, and the values on the x-axis correspond to those on the scatter diagram.

So **C** is the correct residual plot.

D

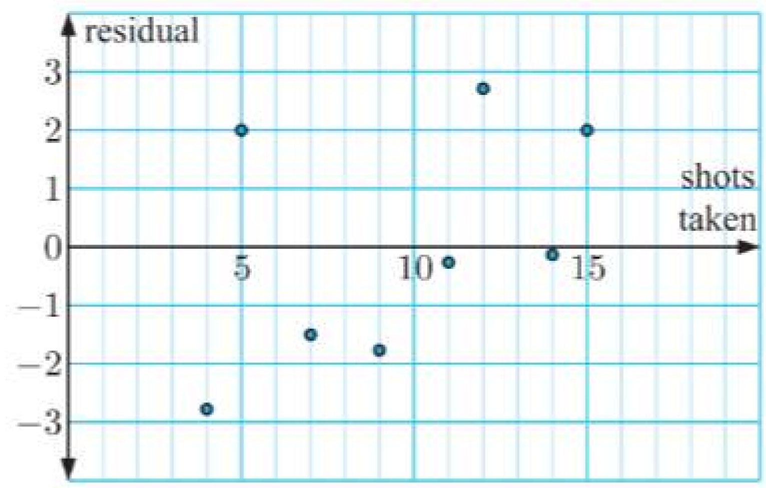


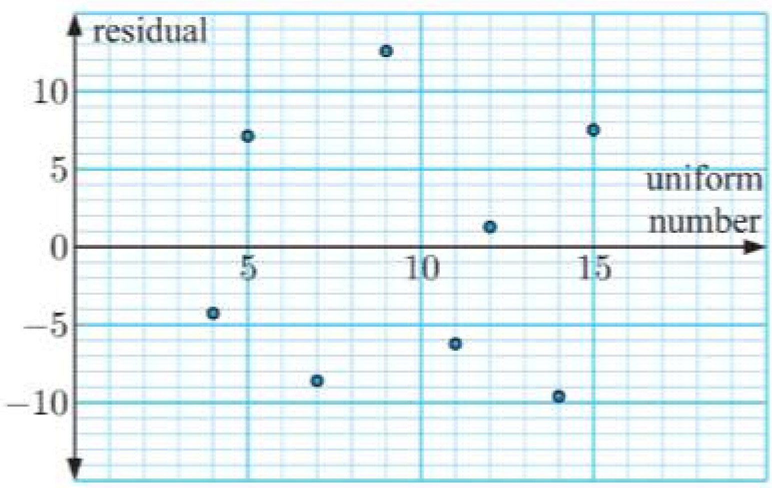
This is very similar to residual plot **C**, except the values on the x-axis do not correspond to those on the scatter diagram.

So **D** is not the correct residual plot.

b The residual plot does not appear to be random, so a linear model may not be appropriate for the data.

3





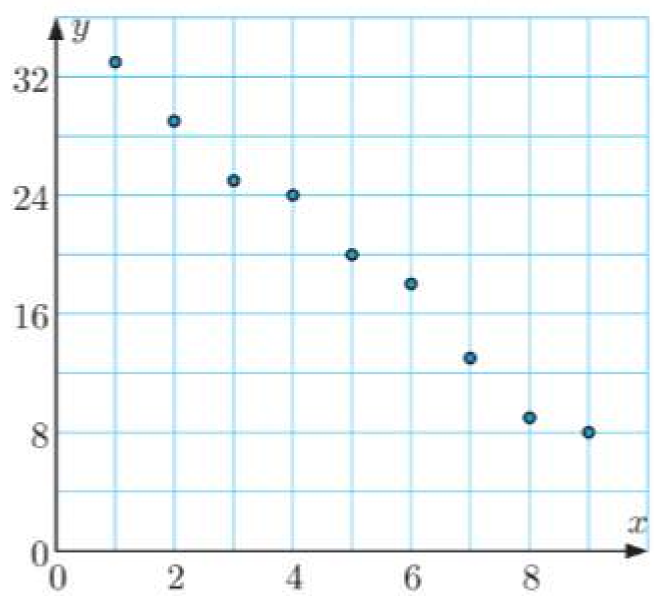
a Yes, the points in both plots appear to be randomly scattered.

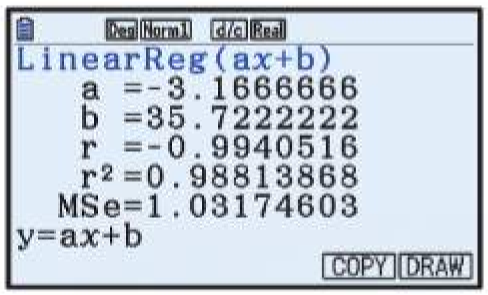
b A linear model is most appropriate for the *points scored* vs *shots taken* data set. The residuals in this plot are generally smaller, which means that the points are generally closer to the least squares regression line.

4 **a**

x	1	2	3	4	5	6	7	8	9
y	33	29	25	24	20	18	13	9	8

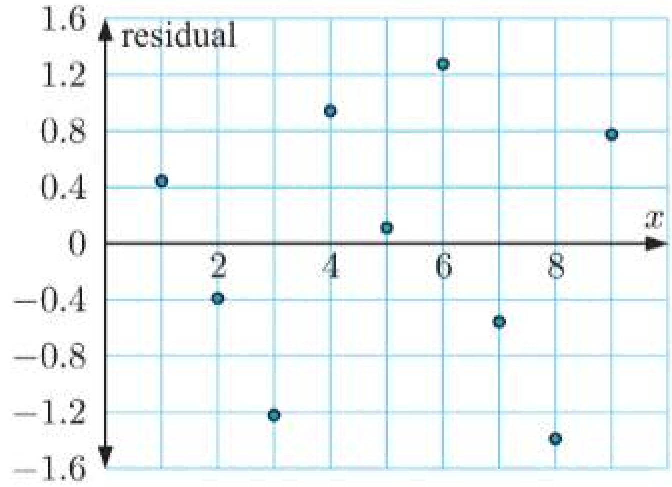
i



ii 

Using technology, the least squares regression line is $y \approx -3.17x + 35.7$, and $r^2 \approx 0.988$.

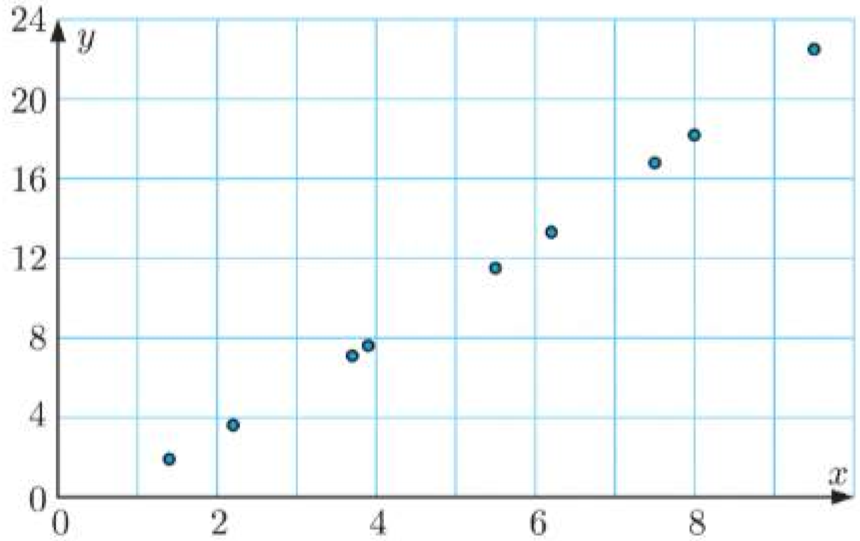
iii Using technology, the residual plot is:

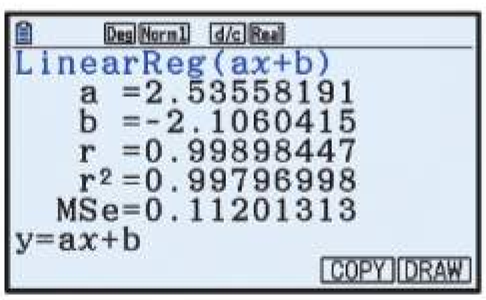


iv The value of r^2 is very high and the residuals are randomly scattered, so the least squares regression line is appropriate to model the data.

b

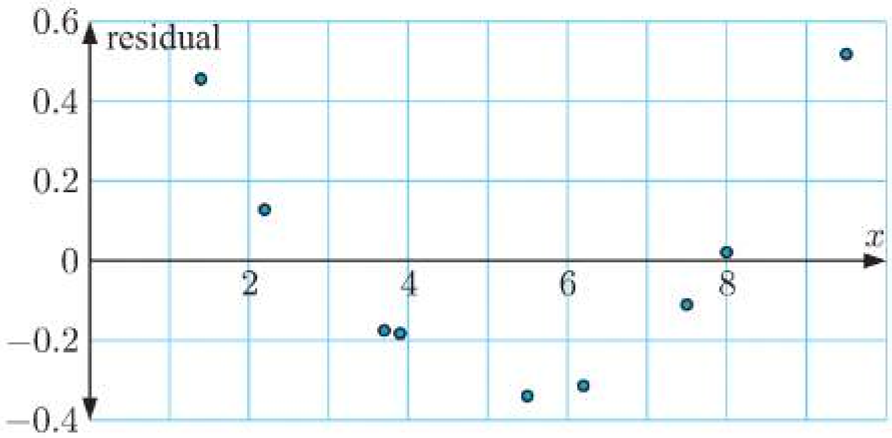
x	2.2	3.7	9.5	6.2	1.4	3.9	7.5	8	5.5
y	3.6	7.1	22.5	13.3	1.9	7.6	16.8	18.2	11.5

i 

ii 

Using technology, the least squares regression line is $y \approx 2.54x - 2.11$, and $r^2 \approx 0.998$.

iii Using technology, the residual plot is:

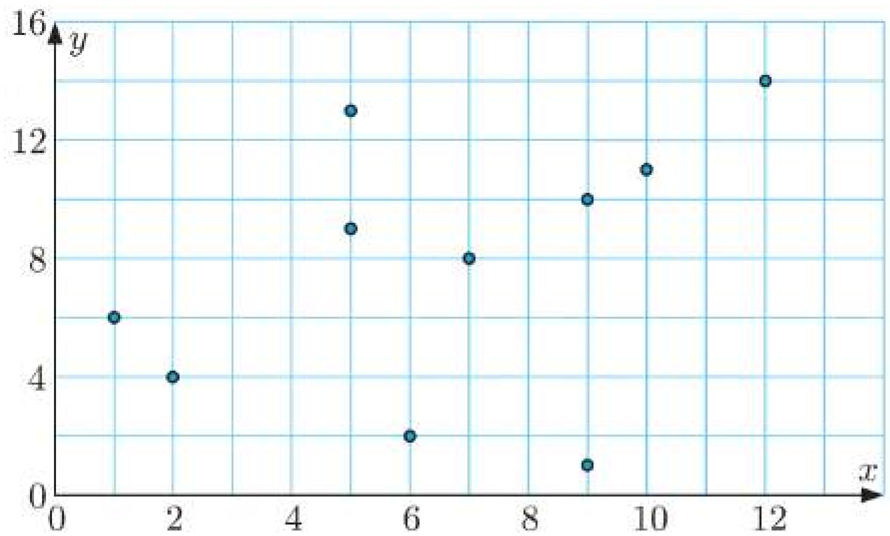


iv The residual plot shows a clear pattern and does not appear random. So the least squares regression line is not appropriate to model the data.

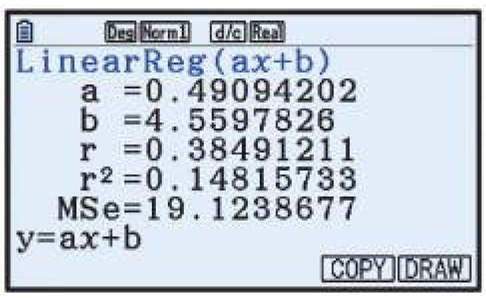
c

x	5	9	1	12	6	5	9	7	2	10
y	13	1	6	14	2	9	10	8	4	11

i

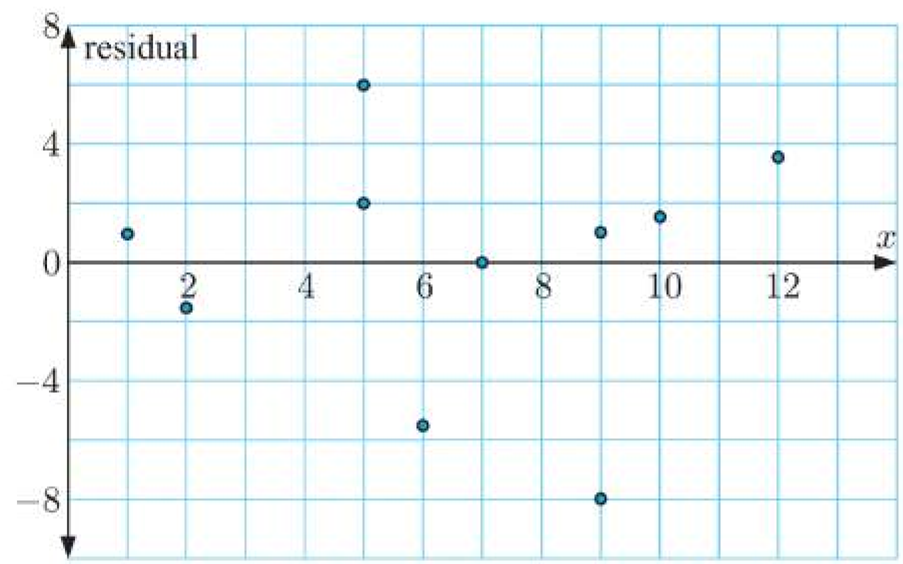


ii



Using technology, the least squares regression line is $y \approx 0.491x + 4.56$, and $r^2 \approx 0.148$.

iii Using technology, the residual plot is:

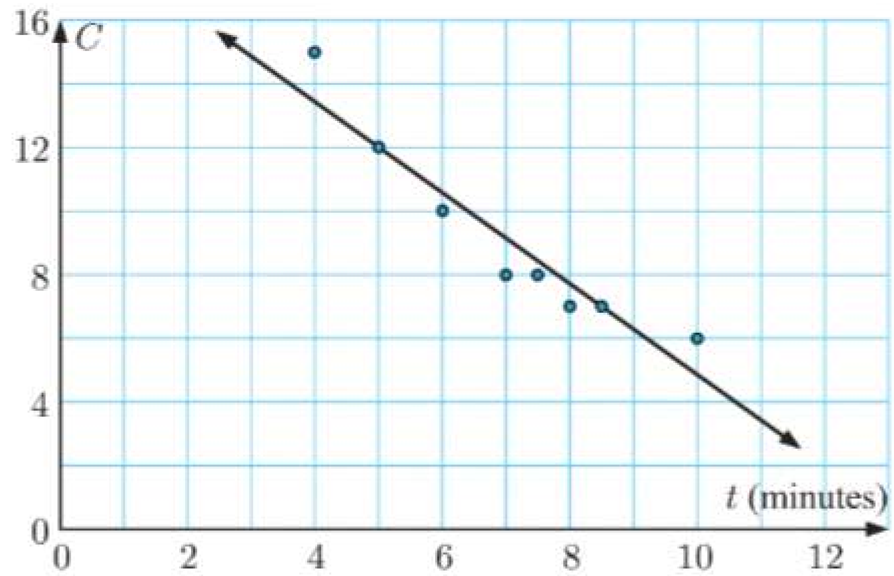


iv The value of r^2 is very small which indicates that the linear model does not explain a significant proportion of the variation in y. So the least squares regression line is not appropriate to model the data.

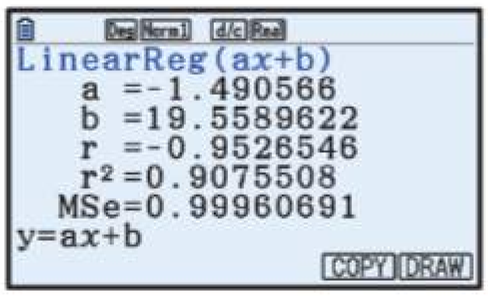
5

Time taken (t minutes)	6	8.5	4	5	8	7.5	10	7
Cranes made (C)	10	7	15	12	7	8	6	8

a, c

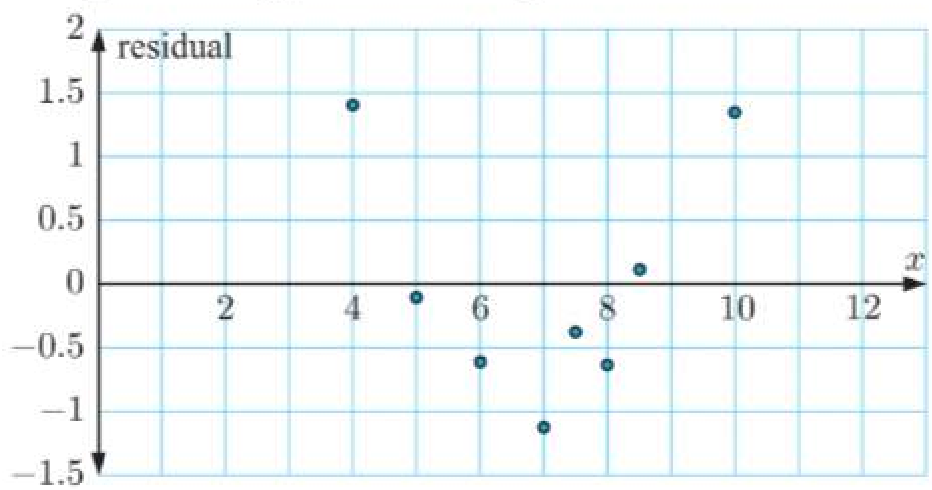


b



Using technology, the least squares regression line is $y \approx -1.49x + 19.6$, and $r^2 \approx 0.908$.

d Using technology, the residual plot is:

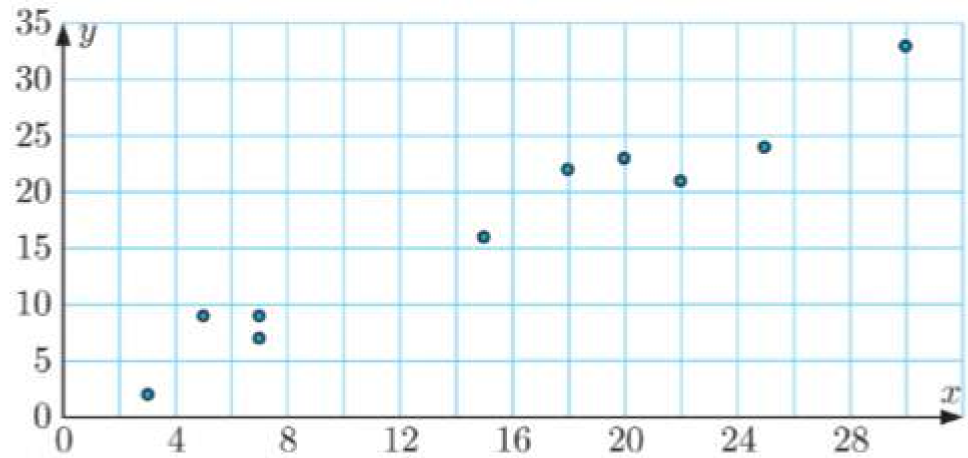


e The residual plot shows a clear pattern, and does not appear random. So the least squares regression line is not appropriate to model the data.

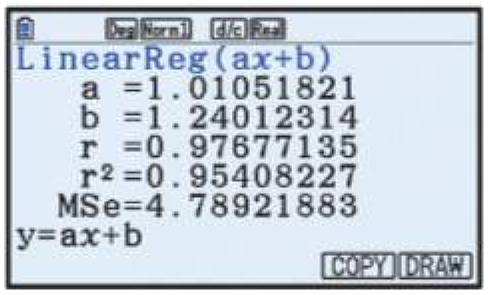
6

Text messages sent (x)	18	3	7	22	15	5	20	30	7	25
Text messages received (y)	22	2	9	21	16	9	23	33	7	24

a



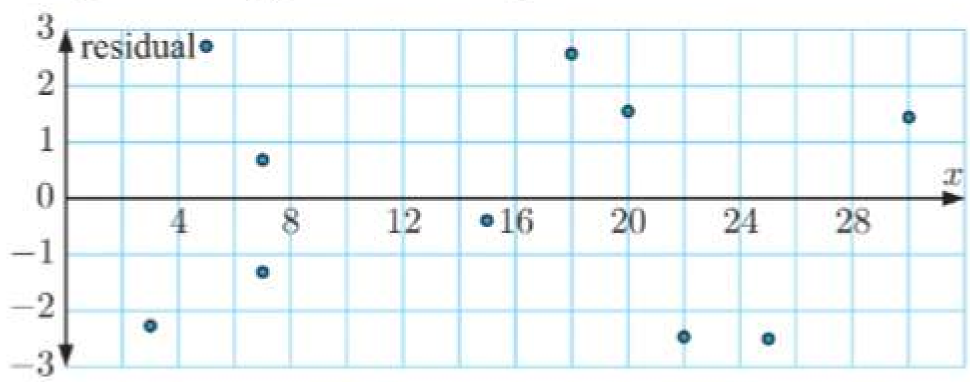
b



Using technology, the least squares regression line is $y \approx 1.01x + 1.24$, and $r^2 \approx 0.954$.

c From the screenshot in **b**, $r \approx 0.977$. There is very strong, positive correlation between *text messages sent* and *text messages received.*

d Using technology, the residual plot is:



e There is very strong, positive correlation, the value of r^2 is very high, and the residuals are randomly scattered. So the least squares regression line is appropriate to model the data.

f **i** When $y = 10$, $10 \approx 1.01x + 1.24$

$\therefore\ 8.76 \approx 1.01x$

$\therefore\ x \approx 8.67$

≈ 9 {rounded to nearest integer}

So, we estimate Ted sent about 9 text messages.

ii As the estimate is an interpolation with strongly correlated data, it is fairly reliable.

EXERCISE 7F

1 **a** Each test has the same number of sentences to translate, and is done under the same conditions by Frederik every evening. However, the tests are not identical as the 10 sentences are randomly selected each evening.
Parallel forms reliability is being considered here.

b Each test is identical and is done under the same conditions by Lucien every morning. Test-retest reliability is being considered here.

c Each test is identical and is done under the same conditions by each javelin thrower at the end of each training session. Test-retest reliability is being considered here.

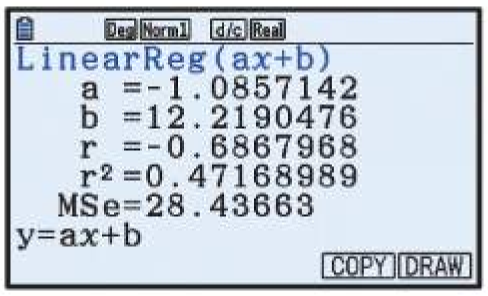
2 **a** To ensure Jo's measurements are reliable:
- the same treadmill should be used
- each measurement should be taken at approximately the same time of day (if they are taken on different days)
- each volunteer should wear the same footwear for each measurement
- Jo should ask each volunteer to keep their eating, sleeping, and exercise routines the same throughout the duration of the experiment.

b Test-retest reliability is more relevant as Jo is interested in consistent measurements over time, and familiarity with the test will not affect the performance of the volunteers.

3

ATP rank	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of grand slams	14	17	20	0	1	0	1	0	0	0	0	0	0	0	0

a



So, $r \approx -0.687$.

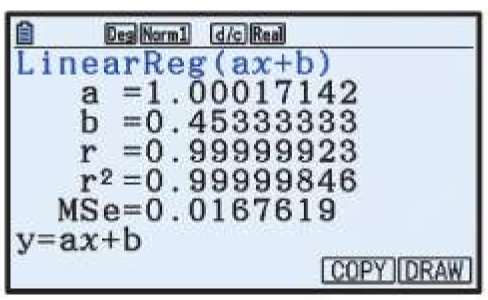
b Criterion validity is being considered here as the *ATP rank* of each player is being used to predict the *number of grand slams* won, the latter of which is the criterion variable.

c There is only a weak, negative correlation between the variables. So, the predictor variable *ATP rank* has little criterion validity.

4

Standard weight (grams)	50	100	150	200	250	300
Measured weight (grams)	50.4	100.5	150.4	200.7	250.5	300.4

a



So, $r \approx 0.999\,999\,2$.

b The post office worker is concerned about the validity of the scales' measurements as they want to know if the set of scales is measuring accurately.

c This is an example of criterion validity. The post office worker wants to know if the *measured weight* accurately predicts the *standard weight*.

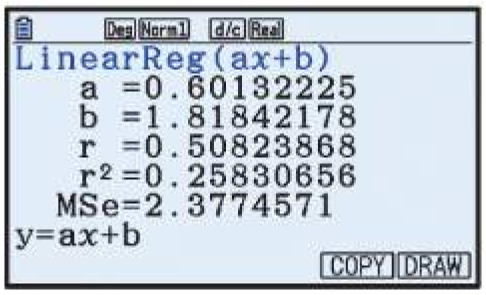
d By comparing each measured weight with the corresponding standard weight, we can see there is a systematic error in the measured weight (between 400 mg to 700 mg over the expected standard weight). This is not detectable if only Pearson's product-moment correlation is considered.

5

Child	1	2	3	4	5	6	7	8	9	10
1st game (minutes)	4.4	8.3	5.8	4.0	2.9	5.2	5.2	5.1	3.8	4.3
2nd game (minutes)	3.1	4.7	8.3	6.3	2.8	5.8	5.4	8.4	3.7	3.7

Child	11	12	13	14	15	16	17	18	19	20
1st game (minutes)	4.9	5.3	4.3	4.5	5.7	7.6	6.8	5.4	4.1	1.9
2nd game (minutes)	4.2	4.9	4.6	2.0	4.5	6.1	6.1	4.8	4.8	2.0

a



So, $r \approx 0.508$.

b Parallel forms reliability is being considered here.

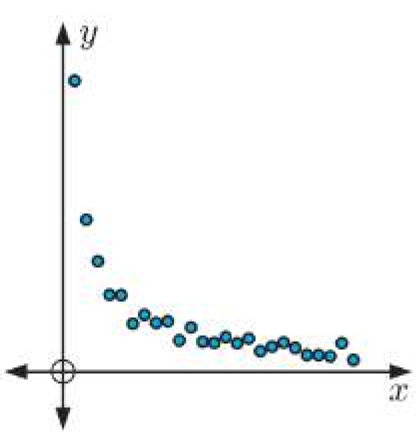
The same set of cards is used, but they are placed randomly each time. So, each repetition of the test is different.

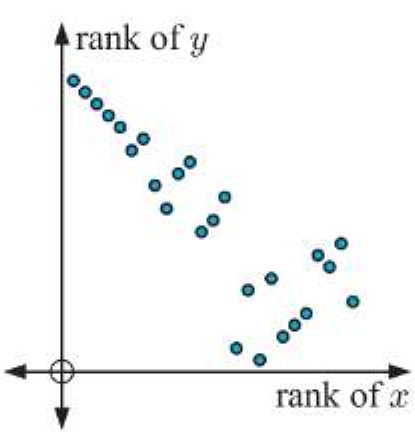
c The correlation between the *1st game* and *2nd game* times is weak. So, the test is not very reliable.

d The cards are likely to be in different positions in the 2nd game so remembering where the cards are in the first game will not give the child an advantage.

EXERCISE 7G

1 **a**

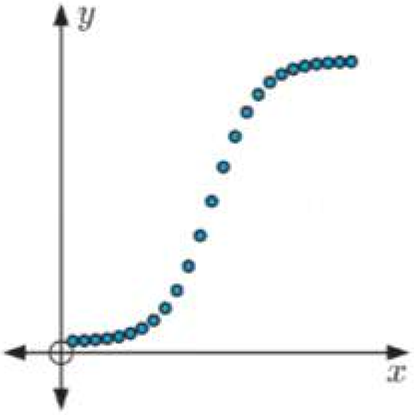
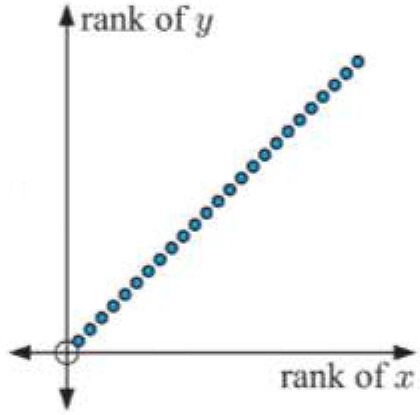




In general, as x increases, y decreases.

So a higher rank of x generally corresponds to a lower rank of y.

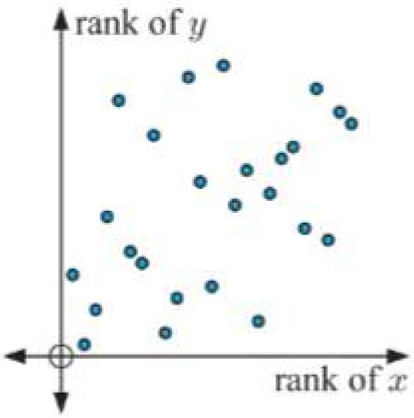
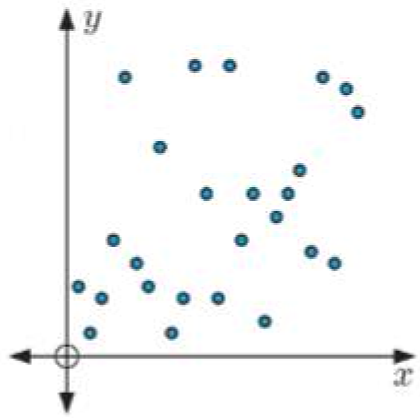
The correct scatter rank diagram is **B**.

b  

As x increases, y always increases.

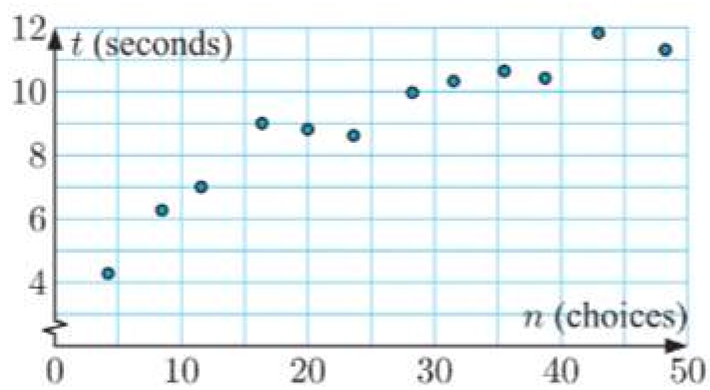
So for each data point (x, y), rank of x = rank of y.

The correct scatter rank diagram is **C**.

c  

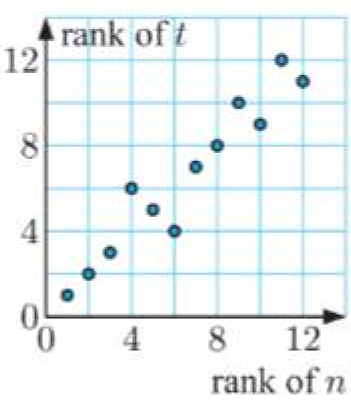
There appears to be no correlation in the scatter diagram. So we expect to see no correlation in the corresponding rank scatter diagram.

The correct scatter rank diagram is **A**.

2 **a** 

As the number of choices n increases, the time taken t generally increases.

So as the rank of n increases, the rank of t generally increases.



Looking at the first few points, we see that **A** is the correct scatter rank diagram.

b The scatter rank diagram has a strong, positive linear correlation, so the correct value of Spearman's rank correlation coefficient is $r_s \approx 0.958$ (**A**).

3 **a** Since $r_s \approx -0.7$, the trend in the ranks is negative.

$\therefore$ as x increases, y decreases.

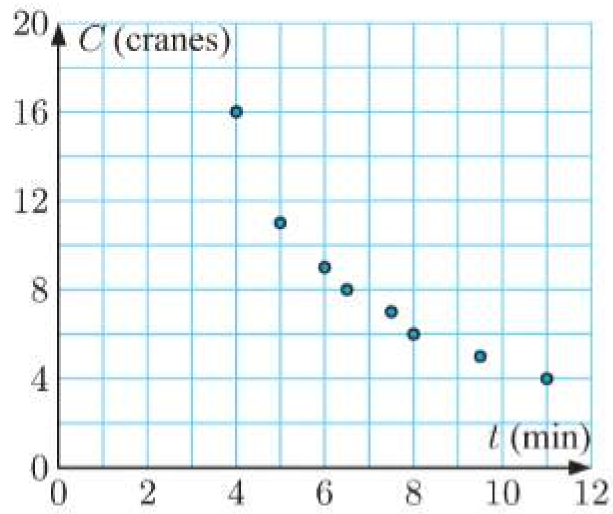
$\therefore$ the trend in the data is negative.

b No, Spearman's rank correlation coefficient indicates a positive or negative trend in the data but does not tell us about the linearity of the data set.

4

Time taken (t min)	6	9.5	4	5	8	7.5	11	6.5
Cranes made (C)	9	5	16	11	6	7	4	8

a

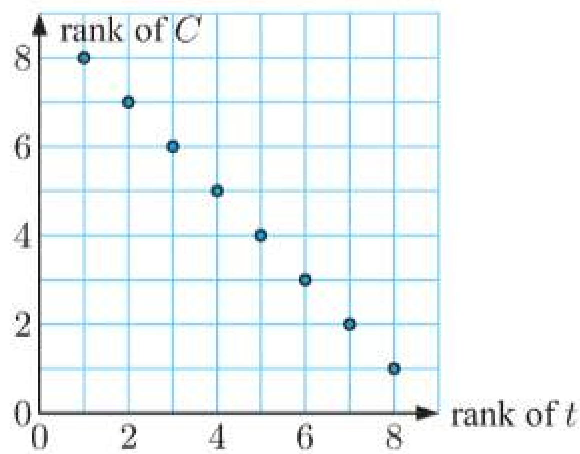


b

Using technology, $r_p \approx -0.921$.

c

t	4	5	6	6.5	7.5	8	9.5	11
rank of t	1	2	3	4	5	6	7	8
C	16	11	9	8	7	6	5	4
rank of C	8	7	6	5	4	3	2	1



d We can see that the scatter rank diagram shows a perfect negative linear correlation.

$\therefore \quad r_s = -1$

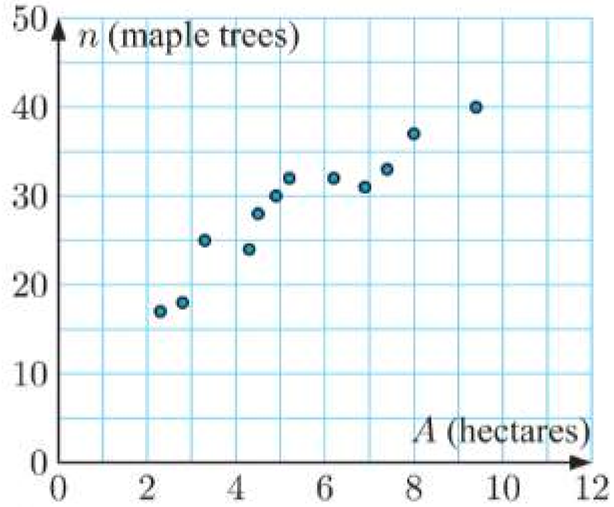
e The scatter diagram of the raw data shows a non-linear trend. Spearman's rank correlation coefficient is therefore more appropriate.

f Using **d** and **e**, there is a very strong, negative, non-linear correlation between *time taken* and *number of cranes made.*

5

Area (A hectares)	2.8	6.9	7.4	4.3	2.3	9.4	5.2	8.0	4.9	6.2	3.3	4.5
Number of maple trees (n)	18	31	33	24	17	40	32	37	30	32	25	28

a

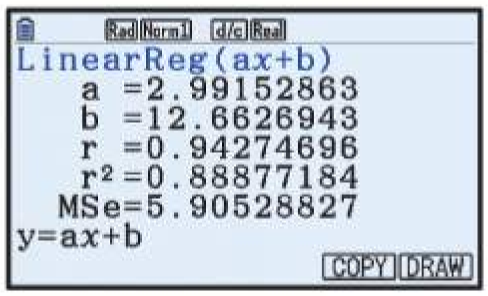


b There is a positive, linear correlation in the scatter diagram of the raw data. We expect to see the same in the rank scatter diagram. r_p and r_s will be very similar.

c

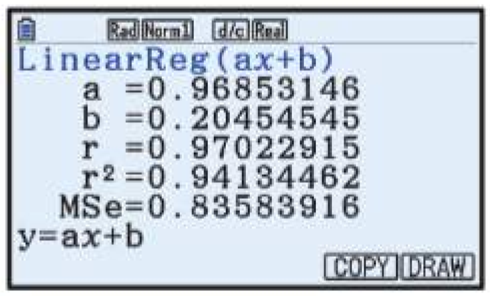
Area (A hectares)	2.8	6.9	7.4	4.3	2.3	9.4	5.2	8.0	4.9	6.2	3.3	4.5
rank of A	2	9	10	4	1	12	7	11	6	8	3	5
Number of maple trees (n)	18	31	33	24	17	40	32	37	30	32	25	28
rank of n	2	7	10	3	1	12	8.5	11	6	8.5	4	5

Original data:



So, $r_p \approx 0.943$

Ranked data:

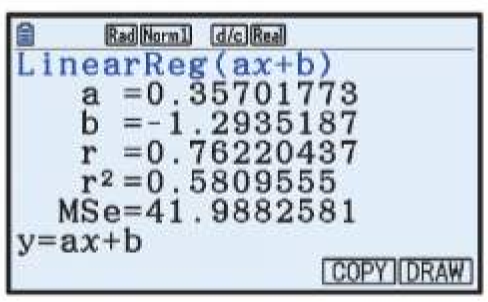


So, $r_s \approx 0.970$

6 **a**

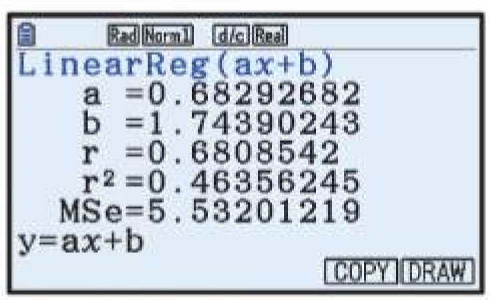
Number of words (x)	40	53	20	65	35	60	85	49	35	76
rank of x	4	6	1	8	2.5	7	10	5	2.5	9
Number of errors (y)	11	15	2	20	4	22	30	16	27	25
rank of y	3	4	1	6	2	7	10	5	9	8

Original data:



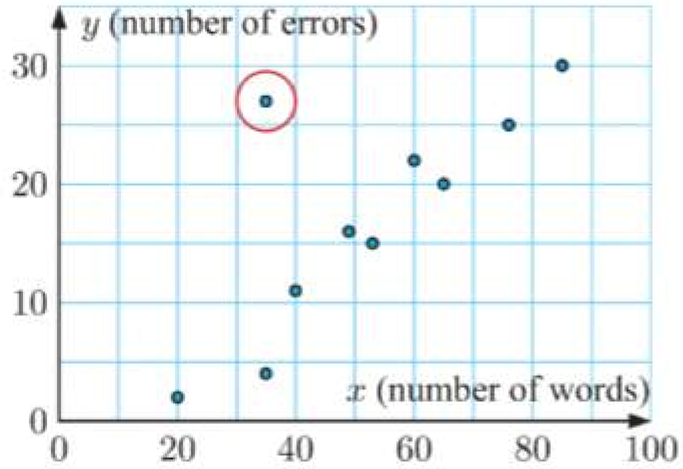
So, $r_p \approx 0.762$

Ranked data:



So, $r_s \approx 0.681$

b

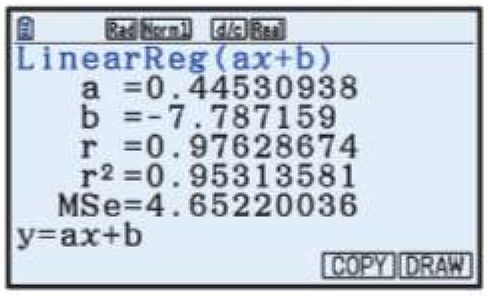


The outlier (35, 27) is circled.

c

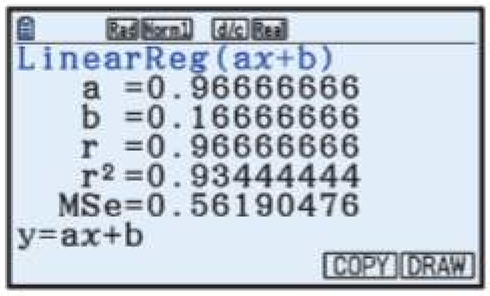
Number of words (x)	40	53	20	65	35	60	85	49	76
rank of x	3	5	1	7	2	6	9	4	8
Number of errors (y)	11	15	2	20	4	22	30	16	25
rank of y	3	4	1	6	2	7	9	5	8

Original data (without outlier):



So, $r_p \approx 0.976$

Ranked data (without outlier):



So, $r_s \approx 0.967$

d r_s was more affected by the presence of the outlier.

7 a

Longest jump (y m)	5.29	5.22	4.64	4.62	4.58
Placing	1	2	3	4	5

Longest jump (y m)	4.38	4.31	4.28	3.94	3.89
Placing	6	7	8	9	10

b *Placing* is the **dependent** variable since it depends on the length of the longest jump.
$\therefore$ *longest jump* is the independent variable.

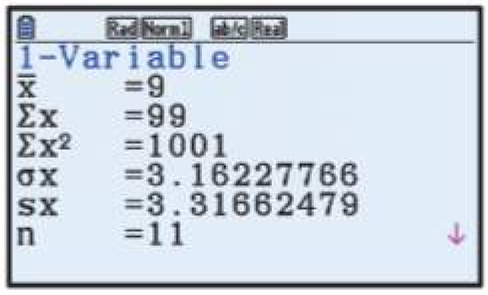
c **i** The variable *placing* has the values 1 to 10 which act as a ranking.

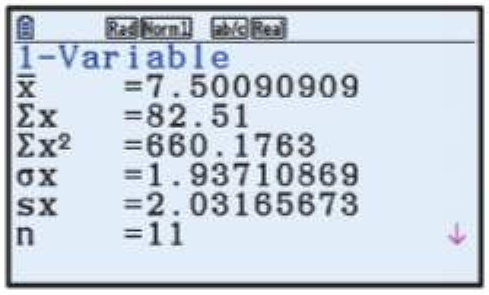
ii Spearman's correlation coefficient for the variables *longest jump* and *placing* must be exactly -1 since the longest jump always decreases as the placing increases.

ACTIVITY 4 ANSCOMBE'S QUARTET

1 *Data set A*:

x	10	8	13	9	11	14	6	4	12	7	5
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

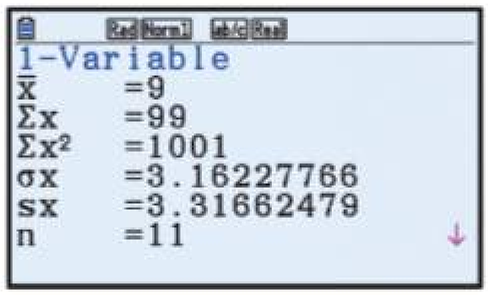


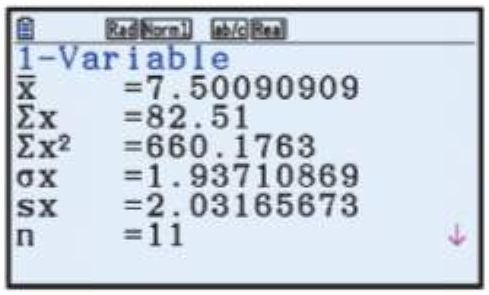


$\mu_x = 9,\ \ \sigma_x \approx 3.16$ $\mu_y \approx 7.50,\ \ \sigma_y \approx 1.94$

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

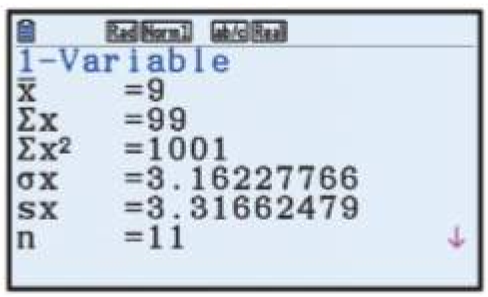


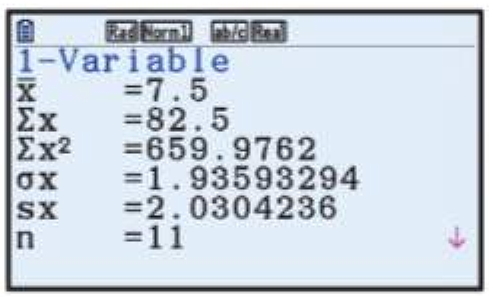


$\mu_x = 9,\ \ \sigma_x \approx 3.16$ $\mu_y \approx 7.50,\ \ \sigma_y \approx 1.94$

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

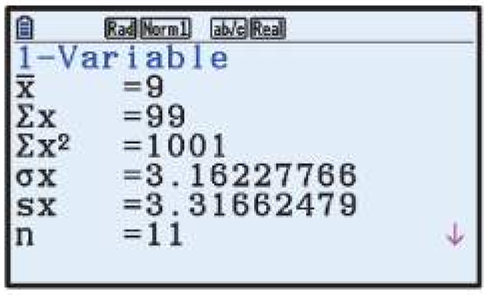




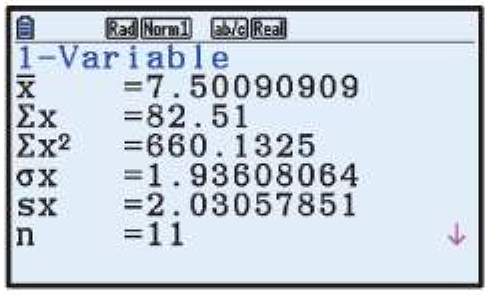
$\mu_x = 9,\ \ \sigma_x \approx 3.16$ $\mu_y = 7.5,\ \ \sigma_y \approx 1.94$

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89



$\mu_x = 9, \ \ \sigma_x \approx 3.16$

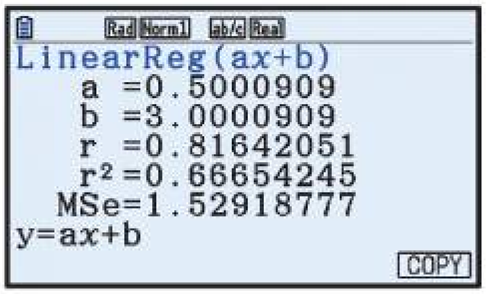


$\mu_y \approx 7.50, \ \ \sigma_y \approx 1.94$

a In each data set: The mean of x is 9.
The mean of y is 7.5 (or very close to 7.5).

b In each data set: The population standard deviation of $x \approx 3.16$.
The population standard deviation of $y \approx 1.94$.

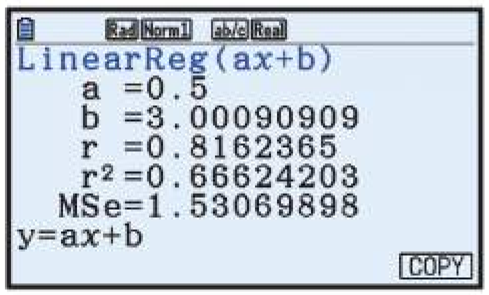
2 *Data set A*:



The regression line is $y \approx 0.500x + 3.00$.

$r_\mathrm{p} \approx 0.816$

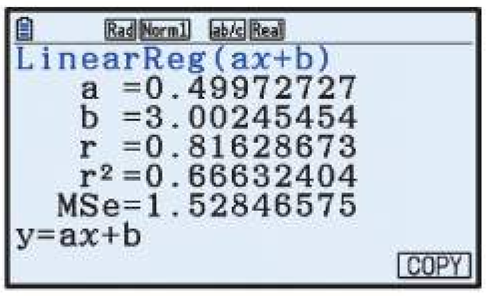
Data set B:



The regression line is $y \approx 0.5x + 3.00$.

$r_\mathrm{p} \approx 0.816$

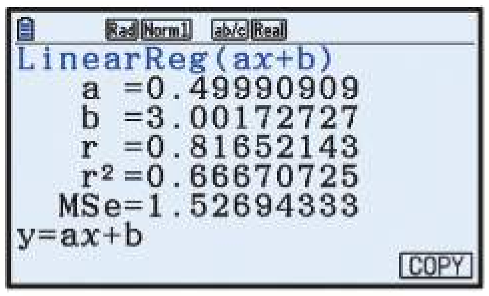
Data set C:



The regression line is $y \approx 0.500x + 3.00$.

$r_\mathrm{p} \approx 0.816$

Data set D:

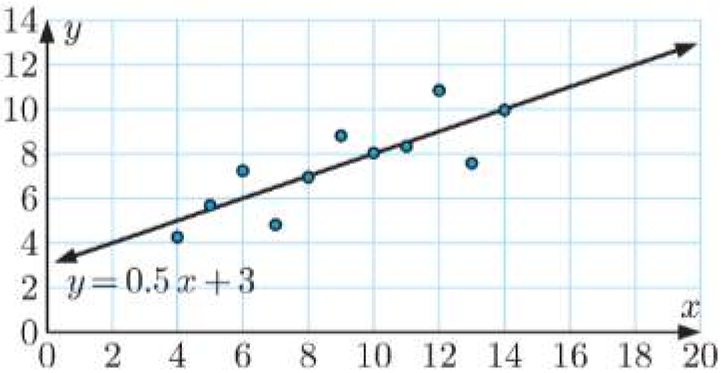


The regression line is $y \approx 0.500x + 3.00$.

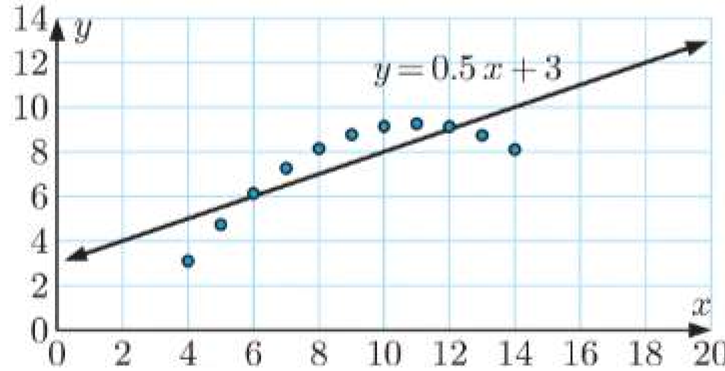
$r_\mathrm{p} \approx 0.817$

The regression lines and the values of Pearson's product-moment correlation coefficient are almost identical for each data set.

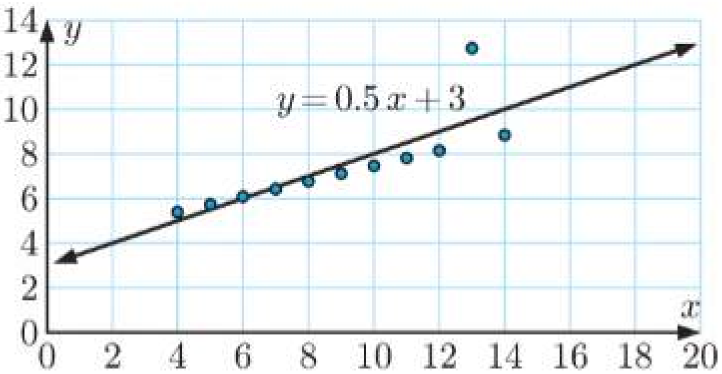
3 *Data set A*:



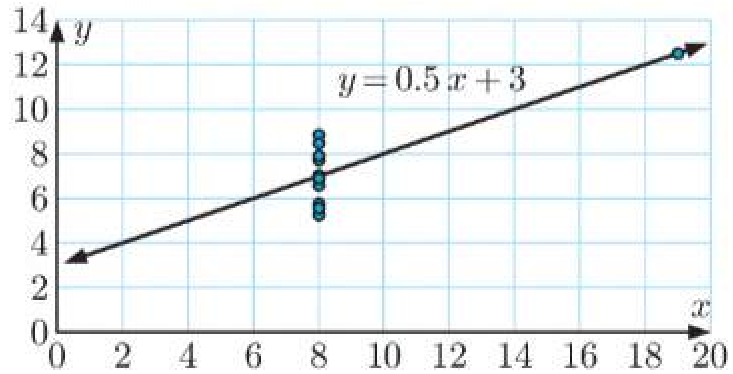
Data set B:



Data set C:



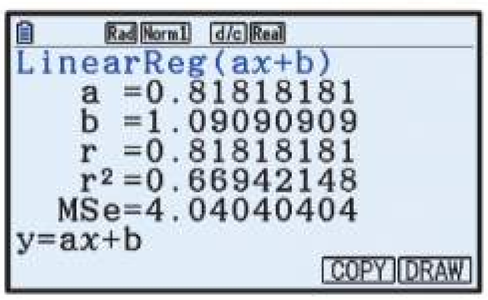
Data set D:



4 Each data set has the same mean and standard deviation for both variables, and the same regression line. However, we see that the scatter diagrams for each data set are wildly different from each other. A linear model is not necessarily appropriate for each data set.

5 *Data set A*:

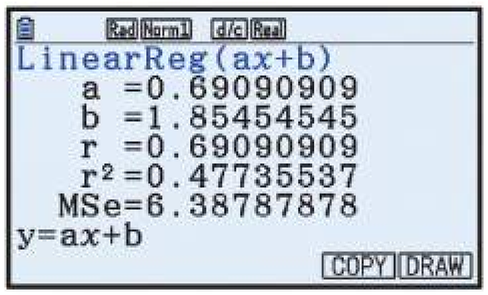
x	10	8	13	9	11	14	6	4	12	7	5
rank of x	7	5	10	6	8	11	3	1	9	4	2
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68
rank of y	7	4	6	9	8	10	5	1	11	2	3



$r_s \approx 0.818$

Data set B:

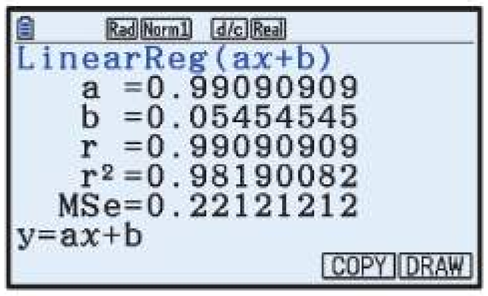
x	10	8	13	9	11	14	6	4	12	7	5
rank of x	7	5	10	6	8	11	3	1	9	4	2
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74
rank of y	10	6	7	8	11	5	3	1	9	4	2



$r_s \approx 0.691$

Data set C:

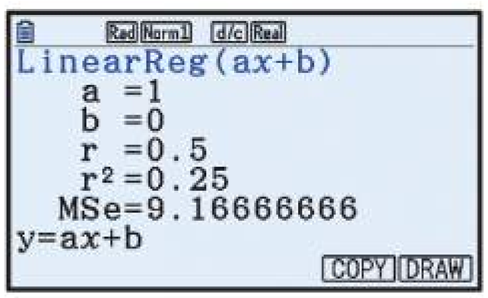
x	10	8	13	9	11	14	6	4	12	7	5
rank of x	7	5	10	6	8	11	3	1	9	4	2
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73
rank of y	7	5	11	6	8	10	3	1	9	4	2



$r_s \approx 0.991$

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
rank of x	5.5	5.5	5.5	5.5	5.5	5.5	5.5	11	5.5	5.5	5.5
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89
rank of y	4	3	7	10	9	6	1	11	2	8	5

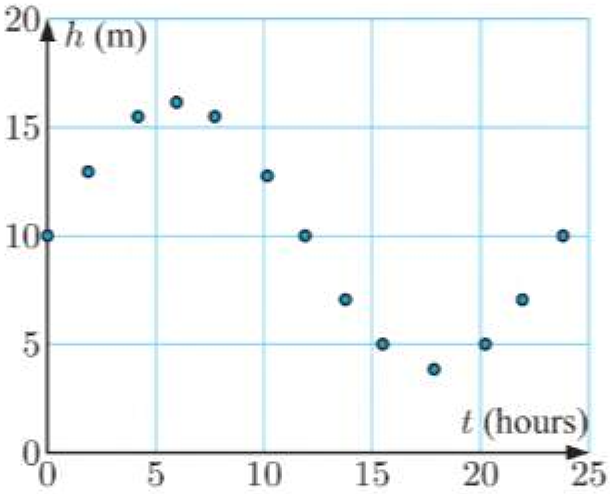


$r_s = 0.5$

The values of Spearman's rank correlation coefficient vary greatly between each data set. The values alone cannot predict the trend in the original data sets.

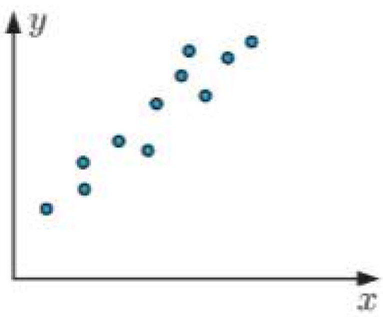
6 A scatter diagram allows us to see patterns in data that cannot be conveyed with descriptive statistics alone.

In the scatter diagram alongside, neither Pearson's product-moment correlation coefficient nor Spearman's rank correlation coefficient will be indicative of the periodic trend in the data.



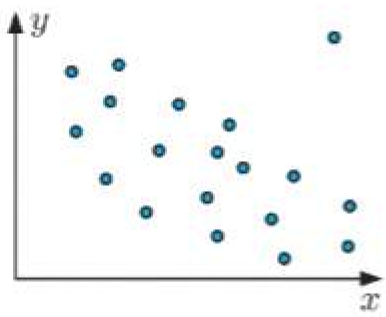
REVIEW SET 7A

1 **a**



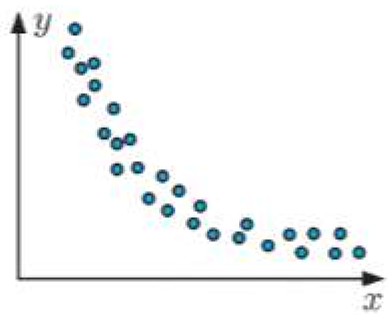
There is a strong, positive, linear correlation, with no outliers.

b



There is a weak, negative, linear correlation, with one outlier.

c



There is a strong, negative, non-linear correlation, with no outliers.

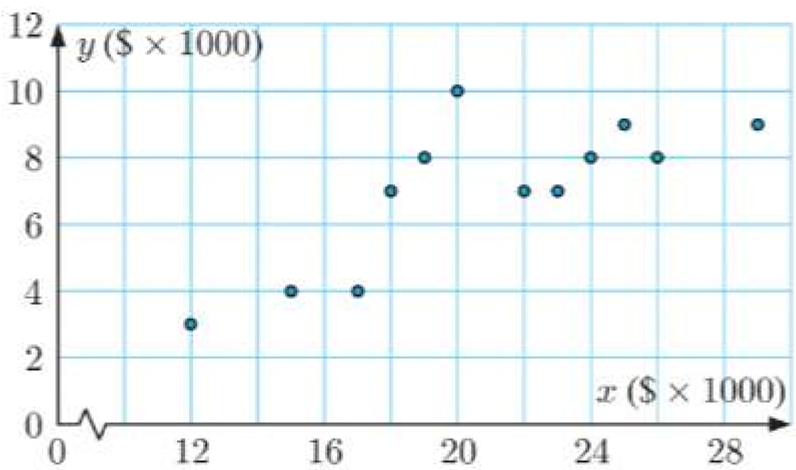
2 **a** The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.

b No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

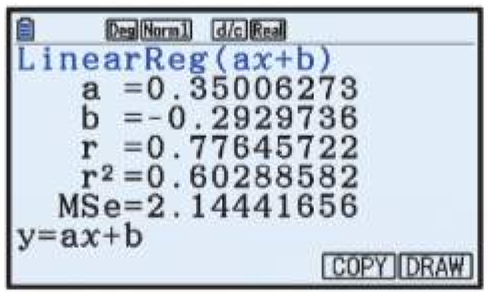
3

Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8

a



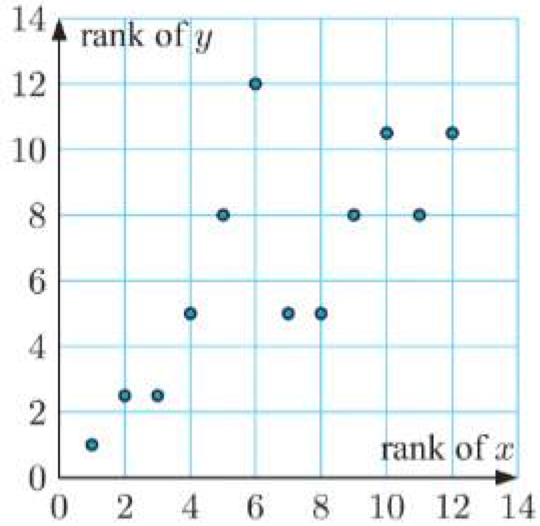
b



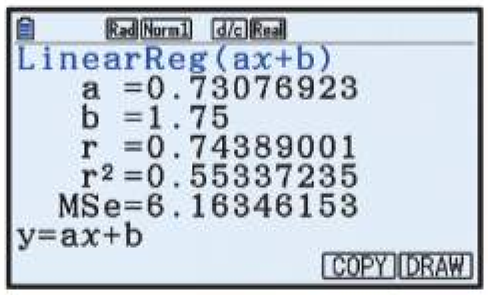
So, $r_p \approx 0.776$.

c

Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
rank of x	10	7	2	5	1	3	9	6	4	8	12	11
Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8
rank of y	10.5	5	2.5	8	1	2.5	8	12	5	5	10.5	8



d



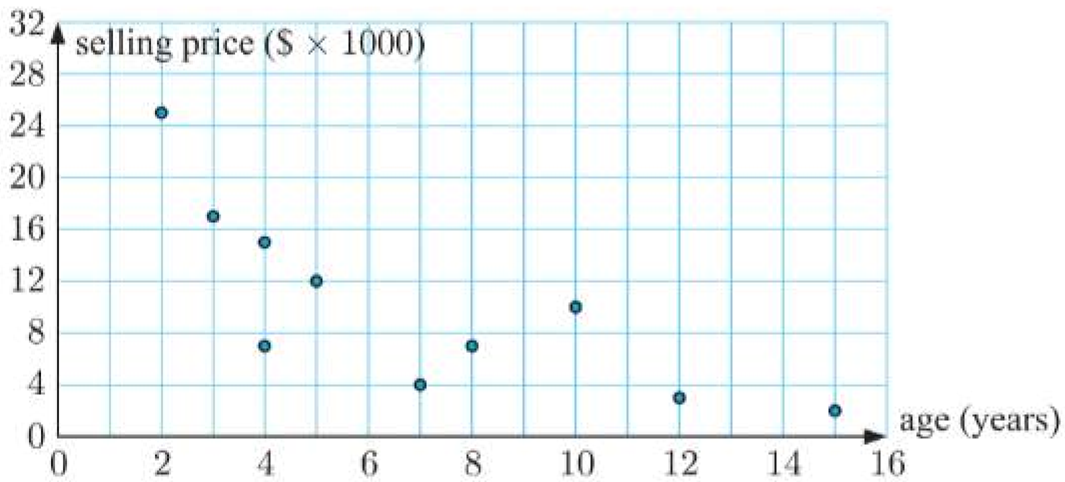
So, $r_s \approx 0.744$.

e There is a moderate, positive correlation between *ticket sales* and *beverage sales*.

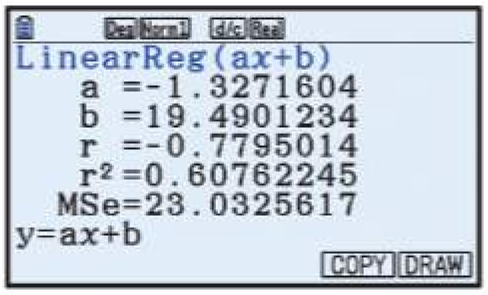
4

Age (years)	2	10	5	4	3	4	7	12	15	8
Selling price ($\$ \times 1000$)	25	10	12	7	17	15	4	3	2	7

a



b There appears to be a moderate, negative correlation between *age* and *selling price*.

c 

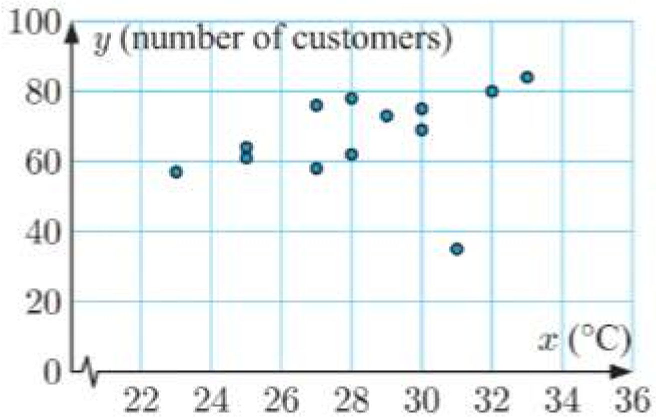
So, $r^2 \approx 0.608$.

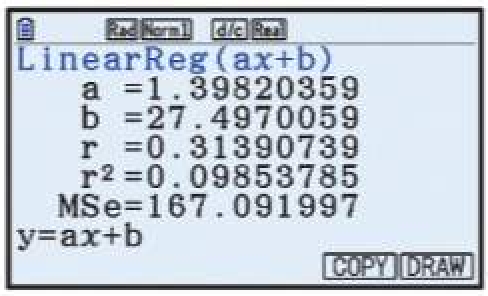
d About 60.8% of the variation in *selling price* can be explained by the variation in *age*.

e The mileage, condition, and features of the car could explain the variation in *selling price*.

5

Temperature (x°C)	23	25	28	30	30	27	25	28	32	31	33	29	27
Number of customers (y)	57	64	62	75	69	58	61	78	80	35	84	73	76

a 

b 

So, $r_p \approx 0.314$.

c

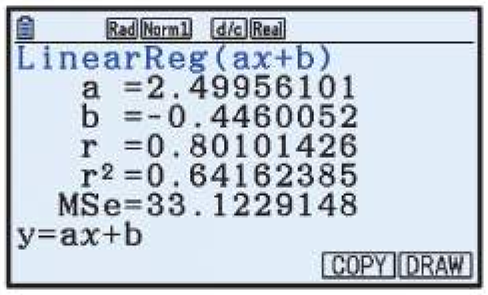
Temperature (x°C)	23	25	28	30	30	27	25	28	32	31	33	29	27
rank of x	1	2.5	6.5	9.5	9.5	4.5	2.5	6.5	12	11	13	8	4.5
Number of customers (y)	57	64	62	75	69	58	61	78	80	35	84	73	76
rank of y	2	6	5	9	7	3	4	11	12	1	13	8	10

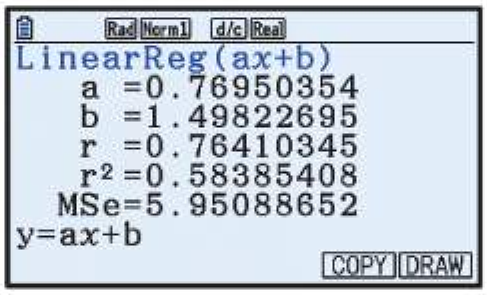
So, $r_s \approx 0.511$.

d The outlier is $(31, 35)$ as it is greatly separated from the rest of the data.

e

Temperature (x°C)	23	25	28	30	30	27	25	28	32	33	29	27
rank of x	1	2.5	6.5	9.5	9.5	4.5	2.5	6.5	11	12	8	4.5
Number of customers (y)	57	64	62	75	69	58	61	78	80	84	73	76
rank of y	1	5	4	8	6	2	3	10	11	12	7	9





$r_p \approx 0.801$ $\qquad$ $r_s \approx 0.764$

f r_p was more affected by the presence of the outlier.

g There is a moderate, positive, linear correlation between the *number of customers* and the *noon temperature* at the garden centre.

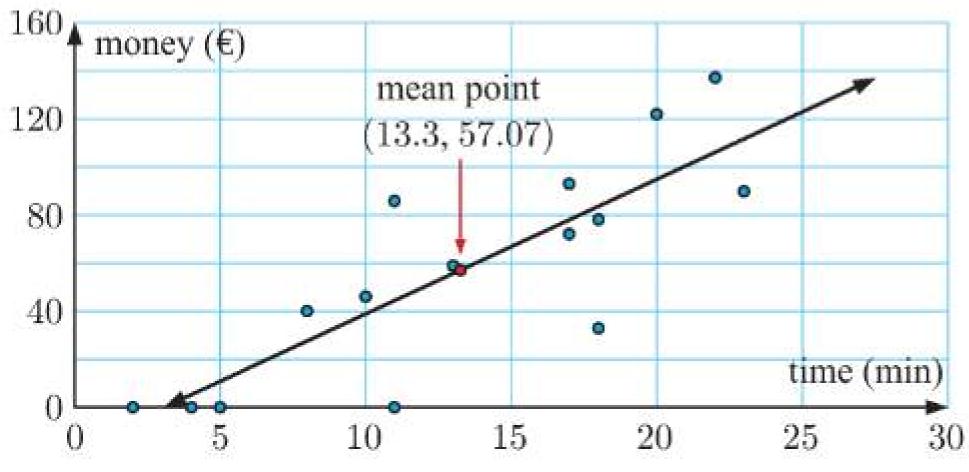
6

Time (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
Money (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a $\overline{x} = \dfrac{8+18+....+22+17}{15} = \dfrac{199}{15} \approx 13.3$ $\qquad$ $\overline{y} = \dfrac{40+78+....+137+93}{15} = \dfrac{856}{15} \approx 57.07$

So the mean time is about 13.3 minutes, and the mean spending is about €57.07 .

b



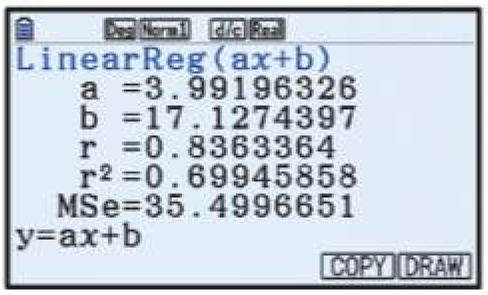
c There is a moderate, positive, linear correlation between *time in the store* and *money spent.*

7 Thien should consider test-retest reliability, as each titration should be identical.

8

Student	A	B	C	D	E	F	G
Short response (marks)	13.0	17.0	14.5	19.0	16.5	15.5	16.0
Mid-year exam (%)	70	81	71	94	90	72	83

Student	H	I	J	K	L	M	N
Short response (marks)	17.0	16.5	17.0	18.5	17.0	13.5	11.0
Mid-year exam (%)	86	86	72	96	86	80	59

a 

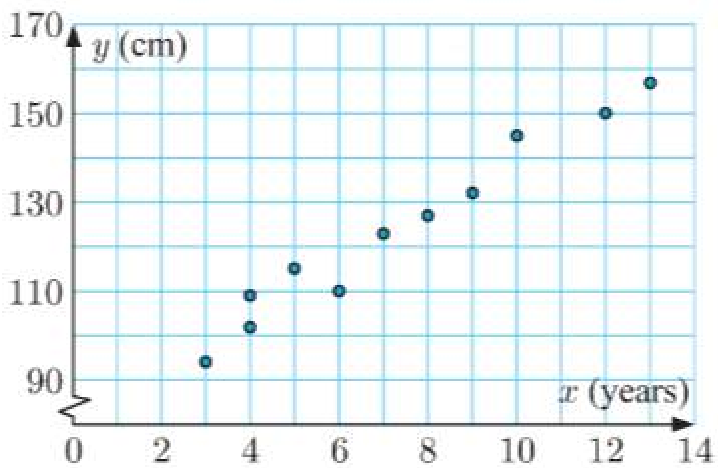
So, $r \approx 0.836$.

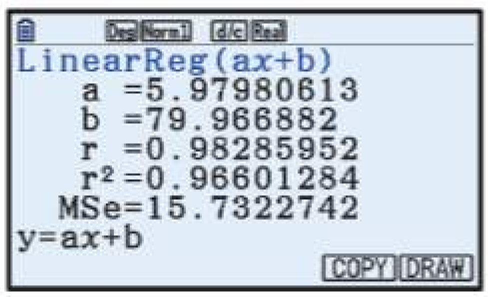
b Criterion validity is being considered here as the *short response* marks are being used to predict the *mid-year exam* results, the latter of which is the criterion variable.

c There is a moderate, positive correlation between the variables. So, the predictor variable *short response* has moderate criterion validity.

9

Age (x years)	3	9	7	4	4	12	8	6	5	10	13
Height (y cm)	94	132	123	102	109	150	127	110	115	145	157

a 

b 

Using technology, the least squares regression line is $y \approx 5.98x + 80.0$.

c The gradient of the least squares regression line ≈ 5.98. This indicates that each year, a child grows taller by an average of 5.98 cm.

d When $x = 5$, $y \approx 5.98(5) + 80.0$

≈ 110

So, a 5 year old child would be approximately 110 cm tall.

e When $y = 140$, $140 \approx 5.98x + 80$

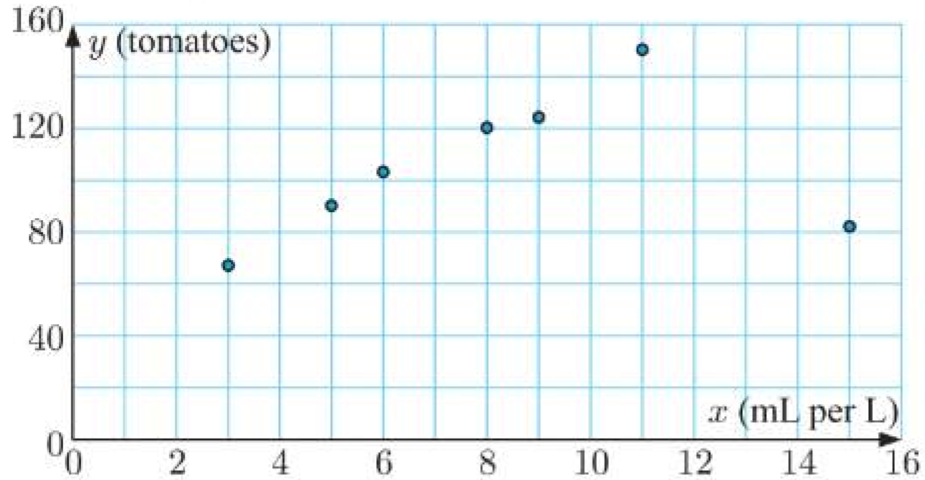
$$5.98x \approx 60$$
$$x \approx 10.0$$

A child would be expected to reach 140 cm in height at age 10 years.

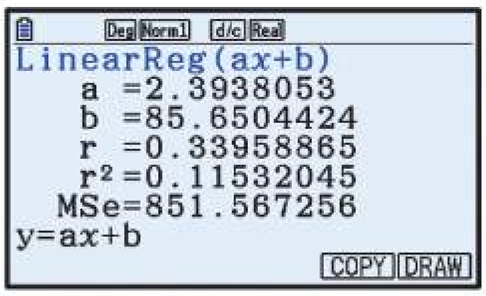
10

Spray concentration (x mL per L)	3	5	6	8	9	11	15
Yield of tomatoes (y per bush)	67	90	103	120	124	150	82

a



b

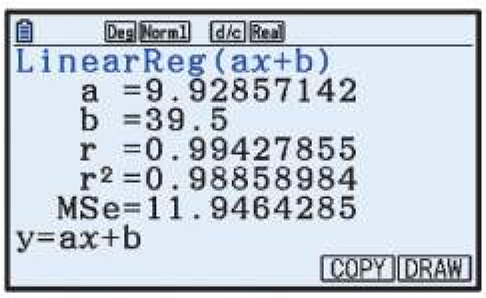


So, $r^2 \approx 0.115$.

About 11.5% of the variation in *yield of tomatoes* can be explained by the variation in *spray concentration.*

c Yes, (15, 82) is an outlier which is affecting the correlation.

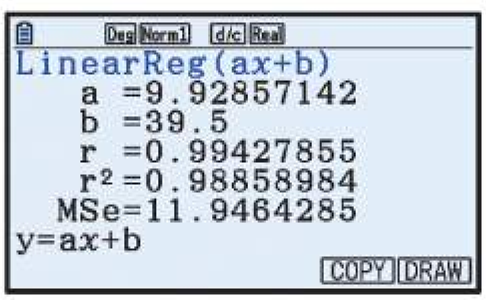
d



So, $r^2 \approx 0.989$.

Yes it is now reasonable to draw a least squares regression line.

e



Using technology, the least squares regression line is $y \approx 9.93x + 39.5$.

f The gradient of the least squares regression line ≈ 9.93. This indicates that for every additional mL per L the spray concentration increases, the yield of tomatoes per bush increases on average by 9.93 tomatoes.

The y-intercept of the least squares regression line ≈ 39.5. This indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g **i** When $x = 7$, $y \approx 9.93(7) + 39.5$
≈ 109

If the spray concentration is 7 mL per L, the yield will be approximately 109 tomatoes per bush.

ii When $y = 200$, $200 \approx 9.93x + 39.5$
$9.93x \approx 160.5$
$x \approx 16.2$

If the yield is 200 tomatoes per bush, the spray concentration would be approximately 16.2 mL per L.

h In **g i**, this is an interpolation, so this estimate is likely to be reliable.

In **g ii**, this is an extrapolation, so this estimate may not be reliable.

REVIEW SET 7B

1 **a** The variables are likely to be negatively correlated, as prices increase, the number of tickets sold is likely to decrease.

This is a causal relationship as less people will be able to afford tickets as the prices increase.

b The variables are likely to be positively correlated, as ice cream sales increase, the number of shark attacks is likely to increase.

This is not a causal relationship as both of these variables are dependent on the time of year.

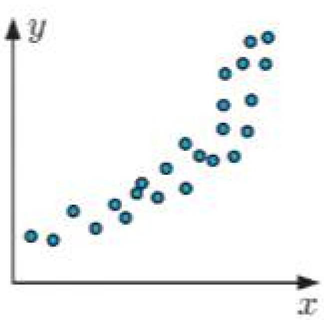
2 **a** There is a moderate, positive correlation between *fat content* and *energy*.

b $r = 0.787$

$\therefore\ r^2 = (0.787)^2$
≈ 0.619

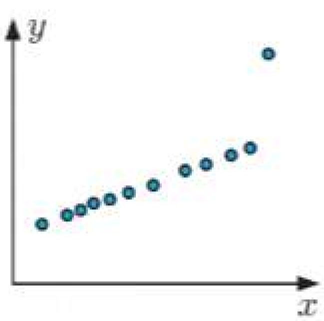
About 61.9% of the variation in *energy* can be explained by the variation in *fat content*.

3 **a**

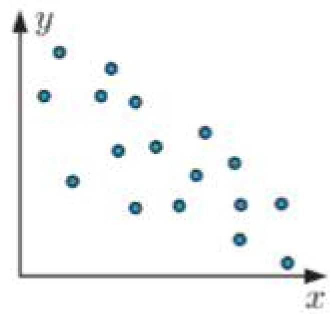


In general, as x increases, y increases.
So in general, as the rank of x increases, the rank of y increases. The ranks are positively correlated.
The rank correlation coefficient is $r_s = 0.7$ (**C**).

b



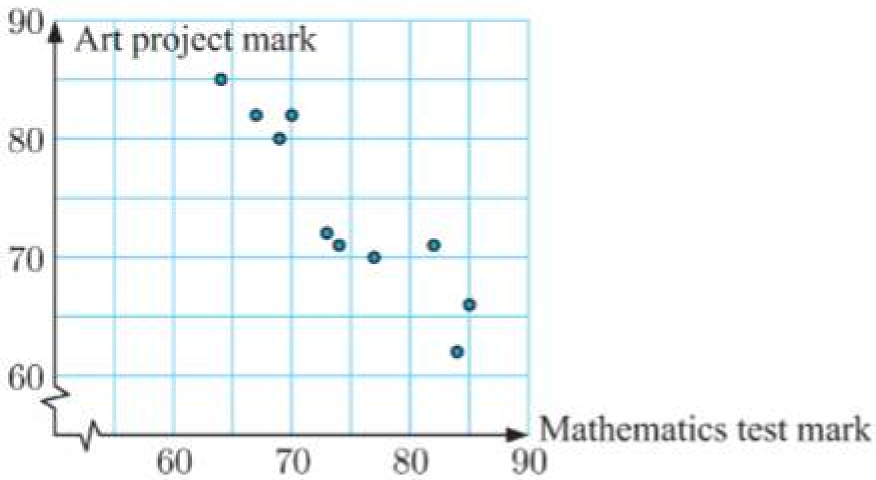
As x increases, y always increases.
So the ranks are perfectly positively correlated.
The rank correlation coefficient is $r_s = 1$ (**A**).

c 

In general, as x increases, y decreases.
So in general, as the rank of x increases, the rank of y decreases. The ranks are negatively correlated.
The rank correlation coefficient is $r_s = -0.4$ (**B**).

4

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66

a 

b There is a strong, negative, linear correlation between the Mathematics and Art marks.

c $r^2 = 0.864$

$\therefore\ r = -\sqrt{0.864}$ $\{r < 0$ as the Mathematics and Art marks are negatively correlated$\}$

≈ -0.930

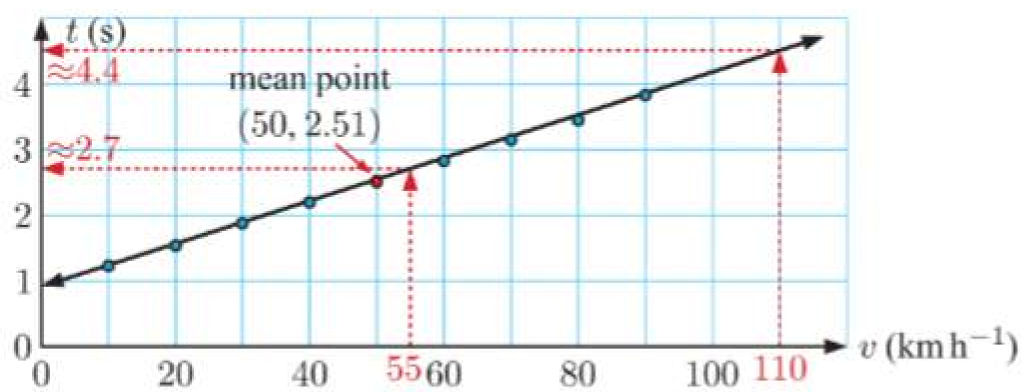
5

Speed (v km h^{-1})	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a $\overline{v} = \dfrac{10 + 20 + 30 + ... + 80 + 90}{9}$ $\qquad$ $\overline{t} = \dfrac{1.23 + 1.54 + 1.88 + ... + 3.45 + 3.83}{9}$

$= 50$ $\qquad$ ≈ 2.51

$\therefore$ the mean point $(\overline{v}, \overline{t})$ is $(50, 2.51)$.

b 

c **i** We estimate that the stopping time for a speed of 55 km h^{-1} is about 2.7 seconds.

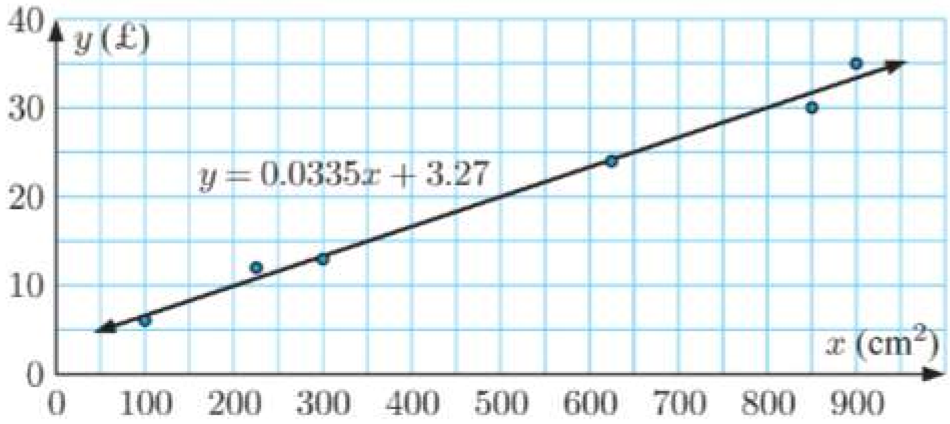
ii We estimate that the stopping time for a speed of 110 km h^{-1} is about 4.4 seconds.

d The estimate in **c i** is more likely to be reliable, since it is an interpolation.

6

Area (x cm^2)	100	225	300	625	850	900
Price (£y)	6	12	13	24	30	35

a, d



b

So, $r \approx 0.994$ and $r^2 \approx 0.988$.

c There is a very strong, positive correlation between the *area* of a canvas and its *price*.

d The regression line is $y \approx 0.0335x + 3.27$.

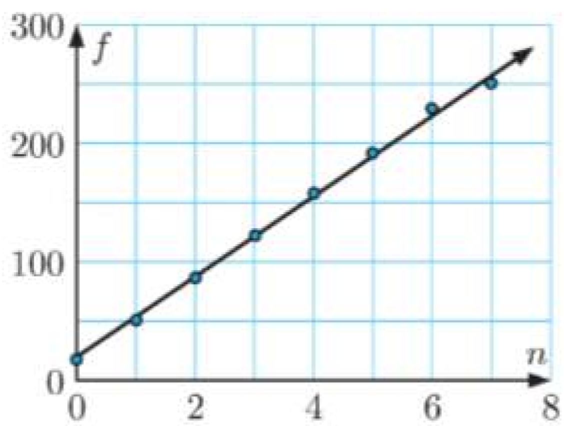
e When $x = 1200$, $y \approx 0.0335(1200) + 3.27$
≈ 43.42

We estimate that a canvas with area 1200 cm^2 will cost about £43.42. This is an extrapolation however, so it may be unreliable.

7

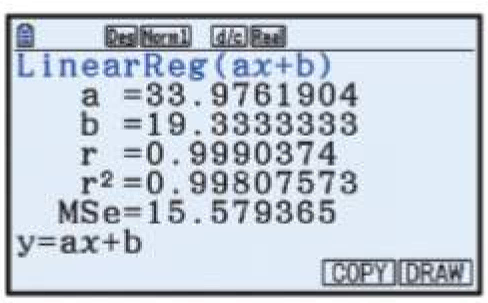
Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250

a, e



There is a very strong, positive correlation between number of waterings and flowers produced.

b



The regression line is $f \approx 34.0n + 19.3$.

c From the screenshot in **b**, $r^2 \approx 0.998$.

About 99.8% of the variation in the number of *flowers produced* can be explained by the variation in the *number of waterings*.

So, the linear model fits the data extremely well.

d Yes, plants need water to grow. So it is expected that an increase in watering will result in an increase in flower production.

f **i** 5 times a fortnight $\equiv$ 2.5 times a week

When $n = 2.5$, $f \approx 34.0(2.5) + 19.3$
≈ 104

Violet can expect about 104 flowers from this bed.

When $n = 10$, $f \approx 34.0(10) + 19.3$
≈ 359

Violet can expect about 359 flowers from this bed.

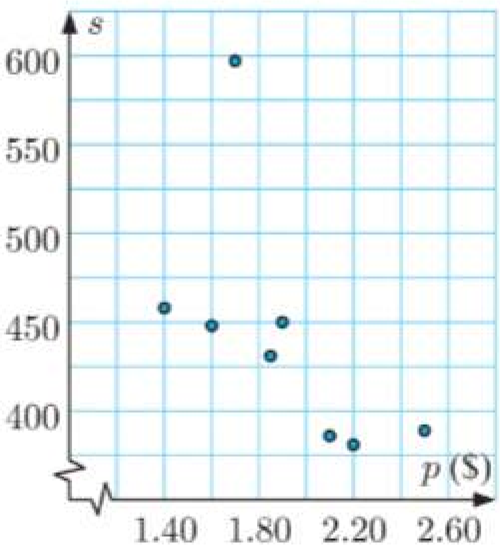
ii The estimate for $n = 2.5$ is reliable as it is an interpolation.

The estimate for $n = 10$ is unreliable as it is an extrapolation and over-watering could be a problem.

8

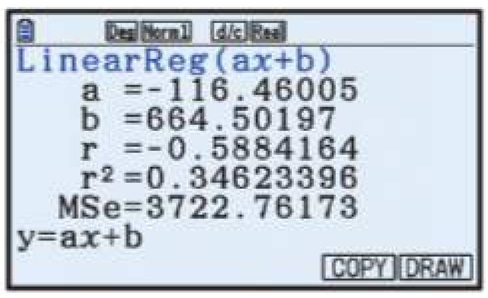
Price (\$$p$)	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
Sales (s)	389	450	448	386	381	458	597	431

a



b Yes, the point $(1.70, 597)$ is an outlier. It should not be deleted as there is no evidence that it is a recording error.

c



The regression line is $s \approx -116p + 665$.

d The gradient of the least squares regression line ≈ -116. This indicates that for every additional dollar the price increases by, the number of sales decreases by 116.

e From the screenshot in **c**, $r^2 \approx 0.346$.

About 34.6% of the variation in *sales* can be explained by the variation in *price*.

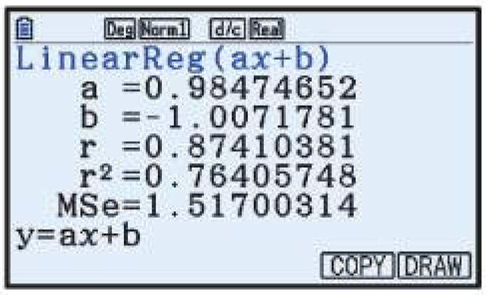
So, the linear model does not fit the data very well.

f No, the prediction of sales of Supa-fizz if it was priced at 50 cents would not be accurate, as it is an extrapolation well beyond the range of data values given.

9

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
Week 1	18	14	17	20	19	16	18	15	20	20	17	19	12	16	17	20	18	15	16	19	20	17
Week 2	16	13	15	20	17	16	17	12	18	20	18	16	11	15	16	20	14	15	14	17	19	16

a



So, $r \approx 0.874$

b Each test has the same number of characters and is done under the same conditions by the same number of students. However, the tests are not identical as the 20 characters are randomly selected each week.

Parallel forms reliability is being considered here.

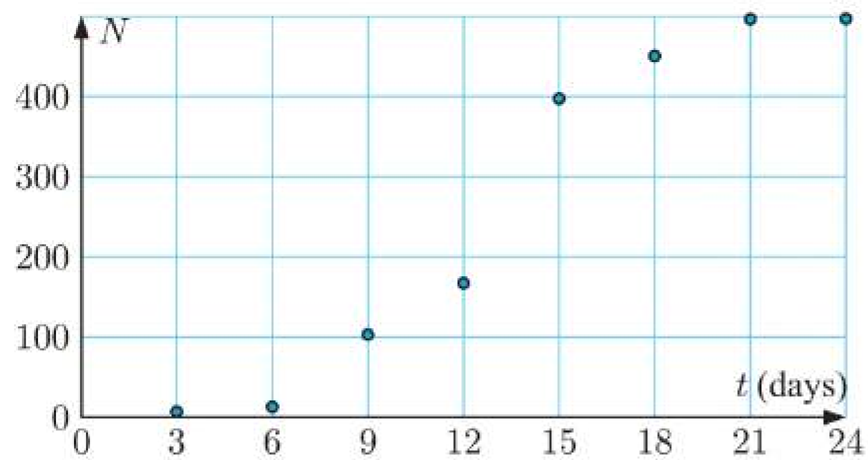
c There is a strong, positive correlation between the *Week 1* and *Week 2* results. So, Mrs Wang's tests are very reliable.

d No, this test does not have content validity for assessing the students' language skills. It does not consider the students' oral skills, grammar skills, sentence structure, and so on. Also, it is only relevant to Chinese. No other languages are considered.

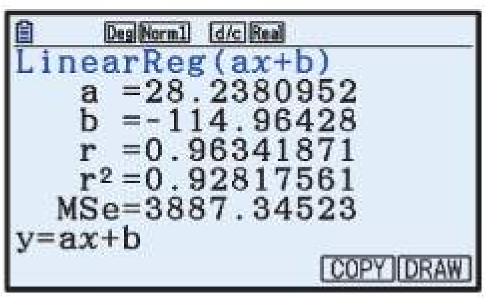
10

Time (t days)	3	6	9	12	15	18	21	24
Number of people (N)	7	13	103	167	397	450	496	497

a



b



So, $r_p \approx 0.963$.

c From the data and the scatter diagram, as t increases, N always increases.

$\therefore \quad r_s = 1$

d The scatter diagram shows a clear non-linear trend in the data. So, Spearman's rank correlation coefficient is more appropriate for this data.

Chapter 8

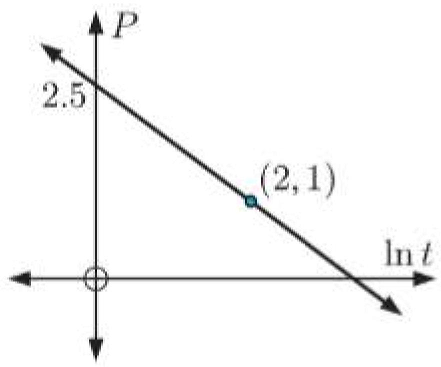
NON-LINEAR MODELLING

EXERCISE 8A

1 **a** The graph of P against $\ln t$ is linear, with gradient

$m = \dfrac{1-2.5}{2-0} = -0.75$ and P-intercept $c = 2.5$.

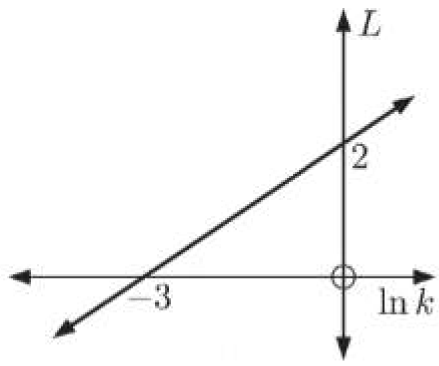
$\therefore \quad P = 2.5 - 0.75 \ln t$



b The graph of L against $\ln k$ is linear, with gradient

$m = \dfrac{2-0}{0-(-3)} = \dfrac{2}{3}$ and L-intercept $c = 2$.

$\therefore \quad L = 2 + \frac{2}{3} \ln k$



c The graph of A against $\ln r$ is linear, with gradient

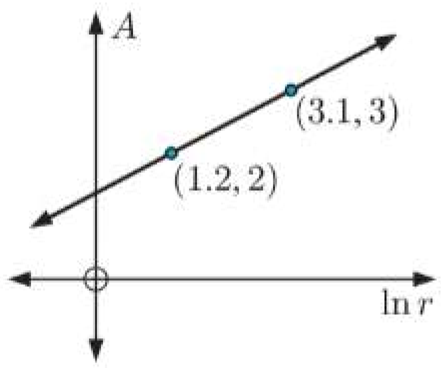
$m = \dfrac{3-2}{3.1-1.2} = \dfrac{10}{19}$ and passes through $(1.2, 2)$.

$\therefore \quad A = c + \frac{10}{19} \ln r$

Now $\quad 2 = c + \frac{10}{19}(1.2)$

$\therefore \quad c = \frac{26}{19}$

$\therefore \quad A = \frac{26}{19} + \frac{10}{19} \ln r$



2 **a** $y = p + q\ln(rx)$

$= p + q(\ln r + \ln x)$

$= (p + q\ln r) + q\ln x$

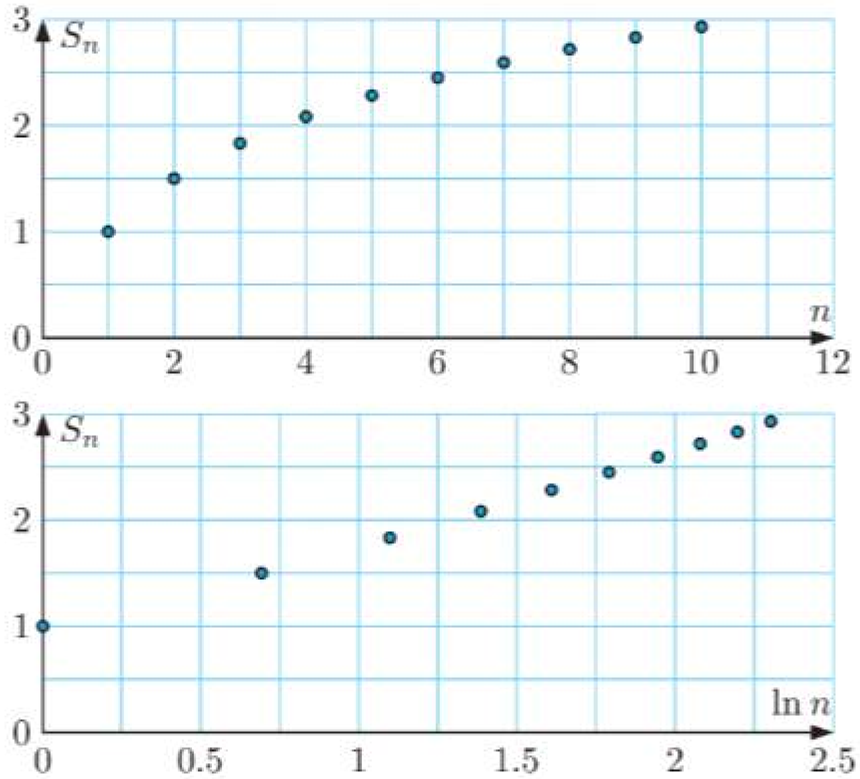
Letting $a = p + q\ln r$ and $b = q$, we see that $y = p + q\ln(rx)$ is equivalent to the form $y = a + b\ln x$, assuming a, b, p, q, and r are constants.

b The laws of logarithms cannot be used to write $\ln(x-\gamma)$ in the form $\ln x +$ constant, so $y = \alpha + \beta\ln(x-\gamma)$ is *not* equivalent to the form $y = a + b\ln x$.

3

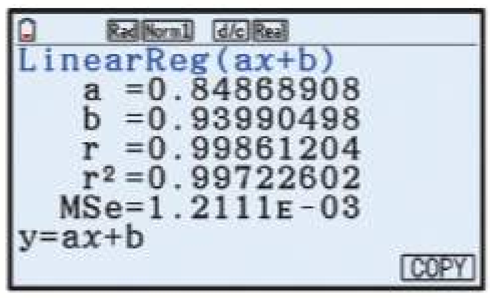
n	1	2	3	4	5	6	7	8	9	10
S_n	1	1.5	≈ 1.833	≈ 2.083	≈ 2.283	2.45	≈ 2.593	≈ 2.718	≈ 2.829	≈ 2.929

a



b The scatter diagram of S_n against $\ln n$ appears to be linear, and so a logarithmic model is appropriate.

c Using technology, the linear model connecting S_n and $\ln n$ is $S_n \approx 0.940 + 0.849 \ln n$. This is the logarithmic model connecting S_n and n.



d As $n \to \infty$, $\ln n \to \infty$ and so $S_n \to \infty$.

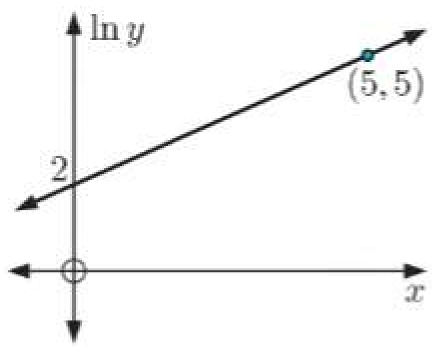
Also, as $n \to \infty$, $S_n \to S$.

$\therefore$ the infinite series $S = \sum\limits_{i=1}^{\infty} \frac{1}{i}$ is not convergent.

EXERCISE 8B

1 **a** The graph of $\ln y$ against x is linear, with gradient

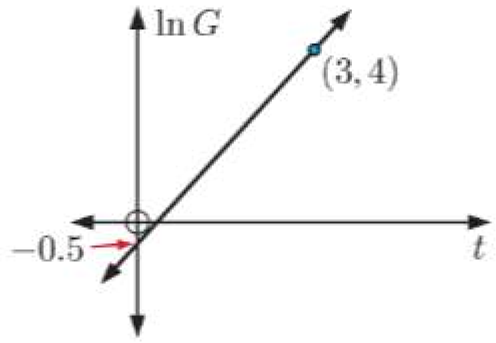
$m = \frac{5-2}{5-0} = \frac{3}{5}$ and vertical axis intercept $c = 2$.



$\therefore \ln y = \frac{3}{5}x + 2$

$$\therefore y = e^{\frac{3}{5}x+2}$$
$$= e^{\frac{3}{5}x} \times e^2$$
$$= e^2 \times \left(e^{\frac{3}{5}}\right)^x$$
$$\approx 7.39 \times 1.82^x$$

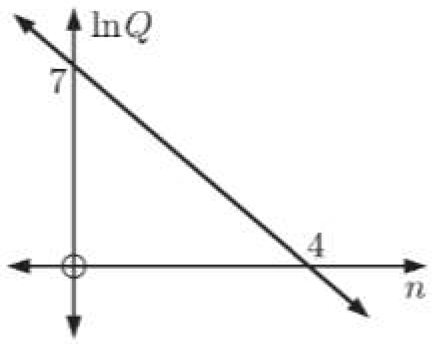
b The graph of $\ln G$ against t is linear, with gradient $m = \frac{4-(-0.5)}{3-0} = 1.5$ and vertical axis intercept $c = -0.5$.



$\therefore \ \ln G = 1.5t - 0.5$

$$\begin{aligned}\therefore \ G &= e^{1.5t-0.5} \\ &= e^{1.5t} \times e^{-0.5} \\ &= e^{-0.5} \times (e^{1.5})^t \\ &\approx 0.607 \times 4.48^t\end{aligned}$$

c The graph of $\ln Q$ against n is linear, with gradient $m = \frac{0-7}{4-0} = -\frac{7}{4}$ and vertical axis intercept $c = 7$.

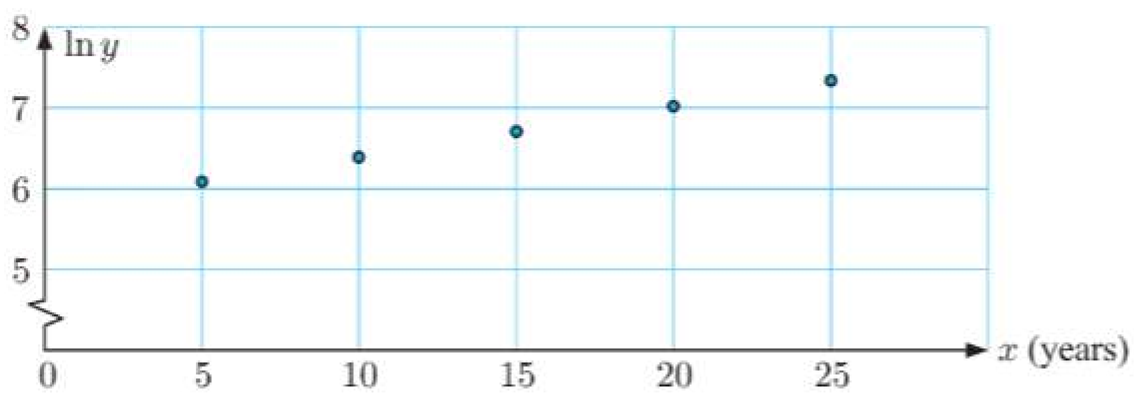


$\therefore \ \ln Q = -\frac{7}{4}n + 7$

$$\begin{aligned}\therefore \ Q &= e^{-\frac{7}{4}n+7} \\ &= e^{-\frac{7}{4}n} \times e^{7} \\ &= e^{7} \times \left(e^{-\frac{7}{4}}\right)^n \\ &\approx 1100 \times 0.174^n\end{aligned}$$

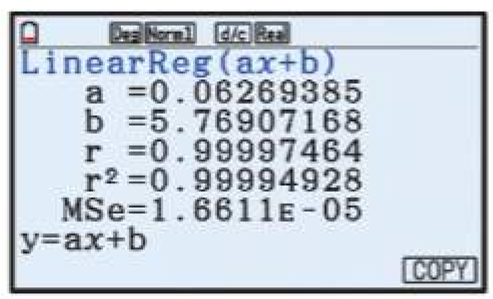
2 **a**

x (years)	5	10	15	20	25
y (individuals)	440	597	819	1120	1540
$\ln y$	≈ 6.09	≈ 6.39	≈ 6.71	≈ 7.02	≈ 7.34



b The graph of $\ln y$ against x appears to be linear, so an exponential model is appropriate.

c Using technology, the linear model connecting $\ln y$ and x is $\ln y \approx 0.0627x + 5.77$.



d Using **c**, the exponential model connecting x and y is $y \approx e^{0.0627x+5.77}$

$\therefore \ y \approx e^{0.0627x} \times e^{5.77}$

$\therefore \ y \approx e^{5.77} \times (e^{0.0627})^x$

$\therefore \ y \approx 320 \times 1.06^x$

e **i** In 1998, when $x = 8$, $y \approx 320 \times 1.06^8 \approx 529$.

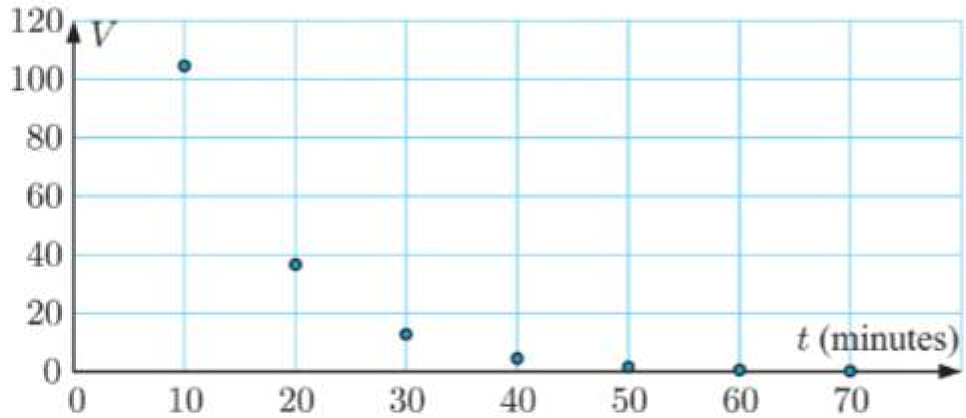
So, the population of the herd in 1998 was about 529.

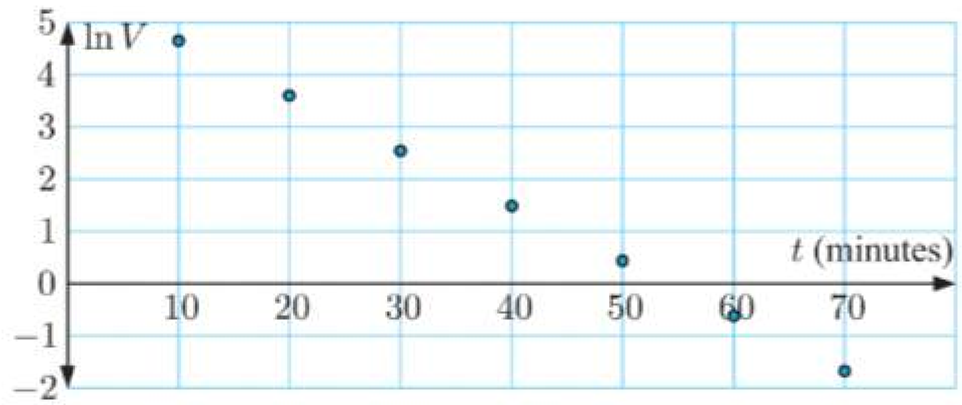
ii In 2025, when $x = 35$, $y \approx 320 \times 1.06^{35} \approx 2874$.

So, the population of the herd in 2025 will be about 2874.

3 **a**

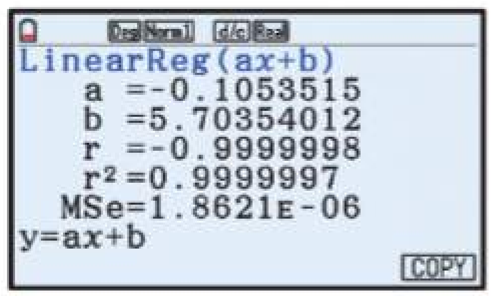
t (minutes)	10	20	30	40	50	60	70
V (units)	104.6	36.5	12.7	4.43	1.55	0.539	0.188
$\ln V$	≈ 4.65	≈ 3.60	≈ 2.54	≈ 1.49	≈ 0.438	≈ -0.618	≈ -1.67





b The graph of $\ln V$ against t appears to be linear, so an exponential model is appropriate.

c Using technology, the linear model connecting $\ln V$ and t is $\ln V \approx -0.105t + 5.70$.



d Using **c**, the exponential model connecting V and t is $V \approx e^{-0.105t+5.70}$

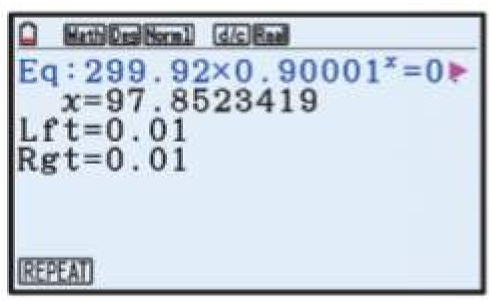
$\therefore \quad V \approx e^{-0.105t} \times e^{5.70}$

$\therefore \quad V \approx e^{5.70} \times (e^{-0.105})^t$

$\therefore \quad V \approx 300 \times 0.900^t$

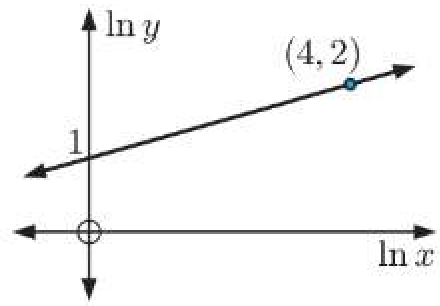
e Using technology, $V = 0.01$ when $t \approx 97.9$.

So, the victim will be considered "safe" after about 97.9 minutes.



EXERCISE 8C

1 **a** The graph of $\ln y$ against $\ln x$ is linear, with gradient $m = \dfrac{2-1}{4-0} = \dfrac{1}{4}$ and vertical axis intercept $c = 1$.



$\therefore \ln y = \frac{1}{4}\ln x + 1$

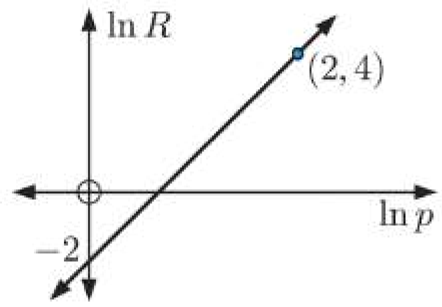
$\therefore \ln y = \ln(x^{\frac{1}{4}}) + \ln(e^1)$

$\therefore \ln y = \ln(x^{\frac{1}{4}} \times e)$

$\therefore y = x^{\frac{1}{4}} \times e$

$\therefore y \approx 2.72 \times x^{0.25}$

b The graph of $\ln R$ against $\ln p$ is linear, with gradient $m = \dfrac{4-(-2)}{2-0} = 3$ and vertical axis intercept $c = -2$.



$\therefore \ln R = 3\ln p - 2$

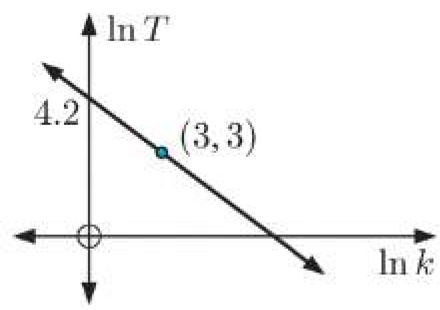
$\therefore \ln R = \ln(p^3) + \ln(e^{-2})$

$\therefore \ln R = \ln(p^3 \times e^{-2})$

$\therefore R = p^3 \times e^{-2}$

$\therefore R \approx 0.135 \times p^3$

c The graph of $\ln T$ against $\ln k$ is linear, with gradient $m = \dfrac{3-4.2}{3-0} = -0.4$ and vertical axis intercept $c = 4.2$.



$\therefore \ln T = -0.4\ln k + 4.2$

$\therefore \ln T = \ln(k^{-0.4}) + \ln(e^{4.2})$

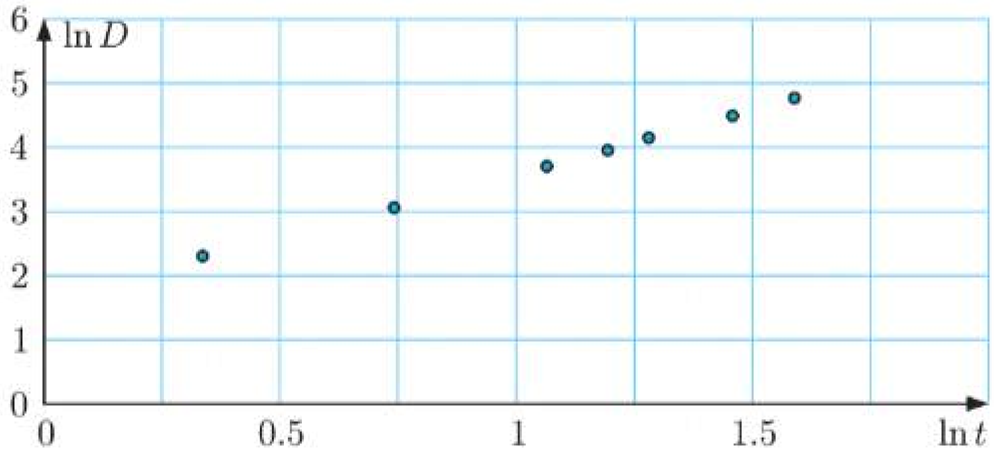
$\therefore \ln T = \ln(k^{-0.4} \times e^{4.2})$

$\therefore T = k^{-0.4} \times e^{4.2}$

$\therefore T \approx 66.7 \times k^{-0.4}$

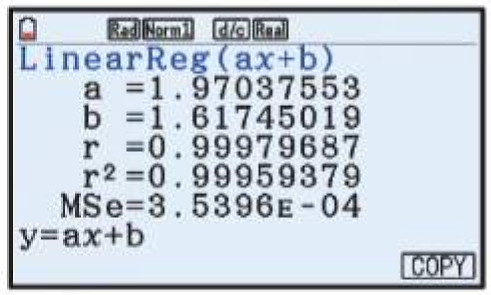
2 **a**

t (s)	1.4	2.1	2.9	3.3	3.6	4.3	4.9
D (m)	10.0	21.3	40.6	52.1	63.4	89.2	117.7
$\ln t$	≈ 0.336	≈ 0.742	≈ 1.06	≈ 1.19	≈ 1.28	≈ 1.46	≈ 1.59
$\ln D$	≈ 2.30	≈ 3.06	≈ 3.70	≈ 3.95	≈ 4.15	≈ 4.49	≈ 4.77



b The graph of $\ln D$ against $\ln t$ appears to be linear, so a power model is appropriate.

c Using technology, the linear model connecting $\ln D$ and $\ln t$ is $\ln D \approx 1.97 \ln t + 1.62$.



d Using **c**, $\ln D \approx \ln(t^{1.97}) + \ln(e^{1.62})$

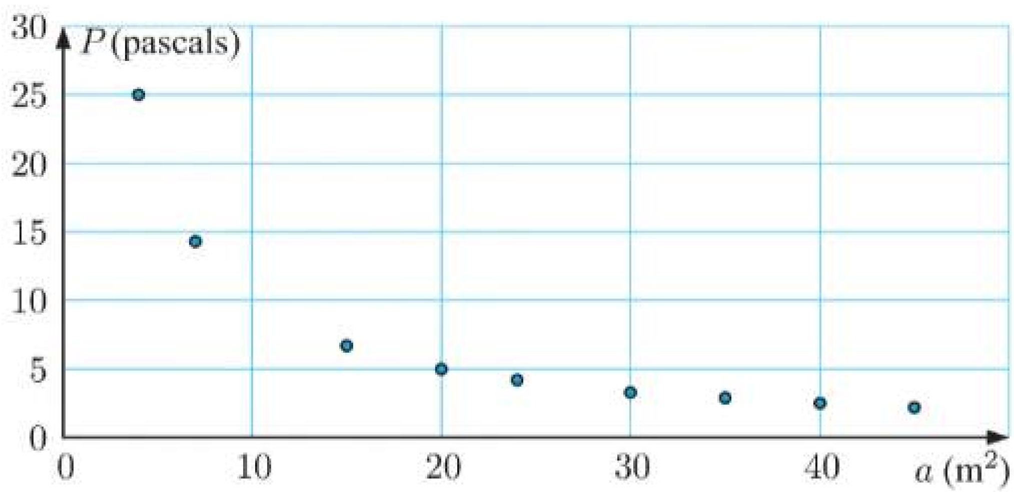
$\therefore \ \ln D \approx \ln(t^{1.97} \times e^{1.62})$

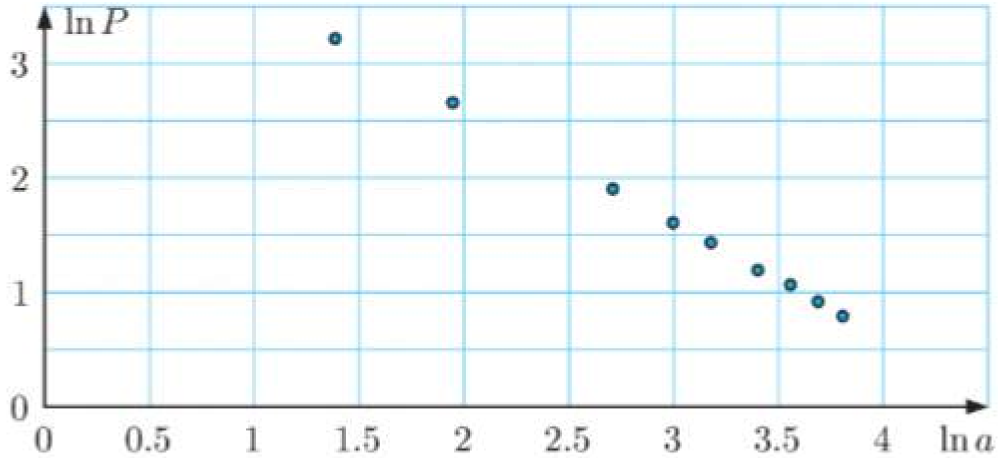
$\therefore \ D \approx e^{1.62} \times t^{1.97}$

$\therefore \ D \approx 5.04 \times t^{1.97}$ is the power model connecting D and t.

3 **a**

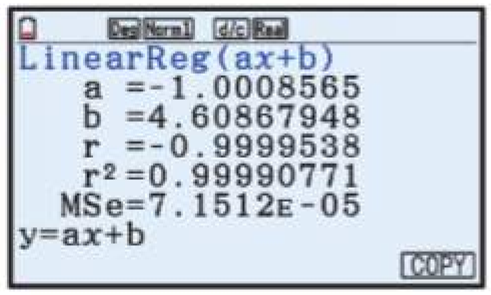
a (m^2)	4	7	15	20	24	30	35	40	45
P (pascals)	25	14.3	6.7	5	4.2	3.3	2.9	2.5	2.2
$\ln a$	≈ 1.39	≈ 1.95	≈ 2.71	≈ 3.00	≈ 3.18	≈ 3.40	≈ 3.56	≈ 3.69	≈ 3.81
$\ln P$	≈ 3.22	≈ 2.66	≈ 1.90	≈ 1.61	≈ 1.44	≈ 1.19	≈ 1.06	≈ 0.916	≈ 0.788





b The scatter diagram of $\ln P$ against $\ln a$ appears to be linear, so a power model is appropriate.

c Using technology, the linear model connecting $\ln P$ and $\ln a$ is $\ln P \approx -1.00 \ln a + 4.61$.



d Using **c**, $\ln P \approx \ln(a^{-1.00}) + \ln(e^{4.61})$

$\therefore \ \ln P \approx \ln(a^{-1.00} \times e^{4.61})$

$\therefore \ P \approx e^{4.61} \times a^{-1.00}$

$\therefore \ P \approx 100 \times a^{-1.00}$ is the power model connecting P and a.

e When $a = 10$, $P \approx 100 \times 10^{-1.00}$

≈ 10.0

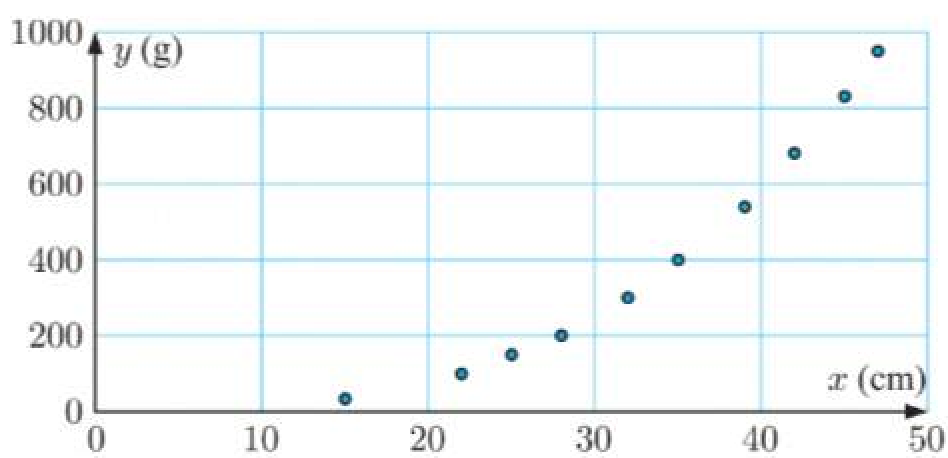
So, when the 100 N force is applied over 10 m^2, the pressure is about 10.0 pascals.
The estimate is an interpolation, so it is likely to be reliable.

EXERCISE 8D

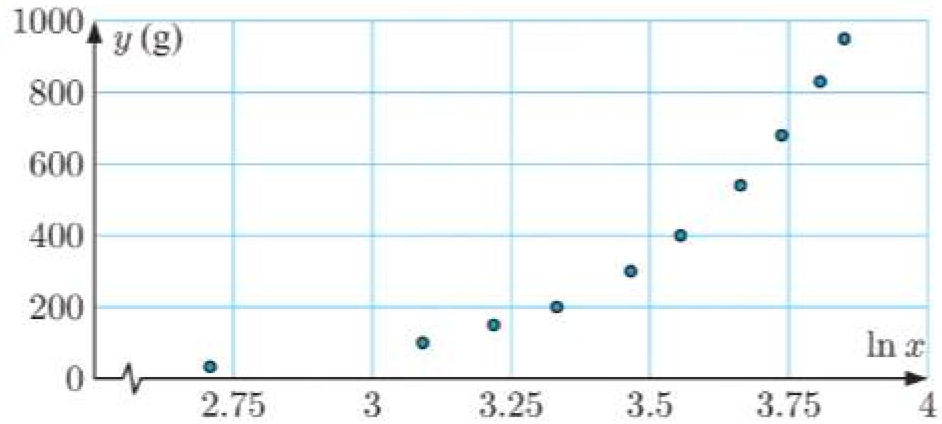
1 **a**

x (cm)	32	28	15	42	39	22	25	45	35	47
y (g)	300	200	33	680	540	100	150	830	400	950
$\ln x$	≈ 3.47	≈ 3.33	≈ 2.71	≈ 3.74	≈ 3.66	≈ 3.09	≈ 3.22	≈ 3.81	≈ 3.56	≈ 3.85
$\ln y$	≈ 5.70	≈ 5.30	≈ 3.50	≈ 6.52	≈ 6.29	≈ 4.61	≈ 5.01	≈ 6.72	≈ 5.99	≈ 6.86

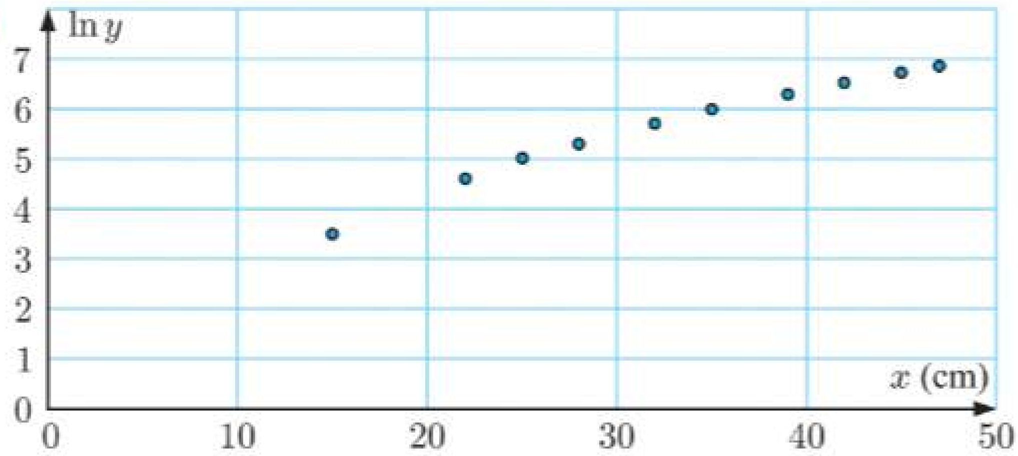
i



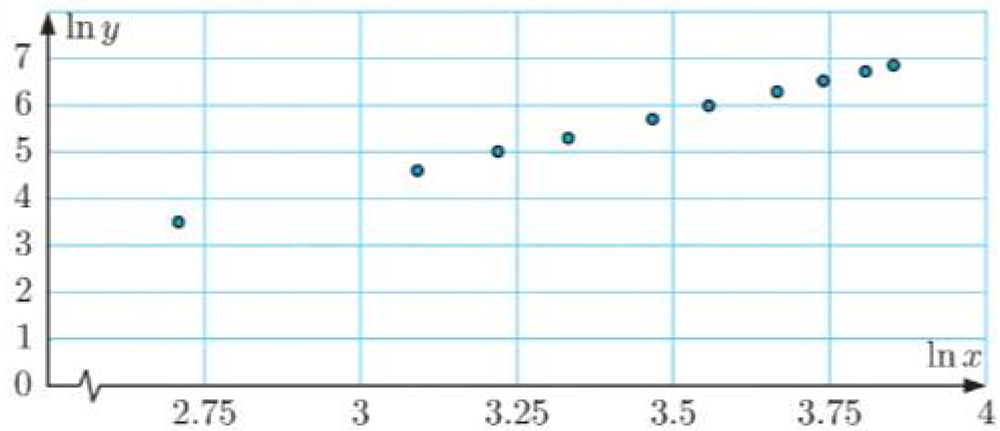
ii



iii

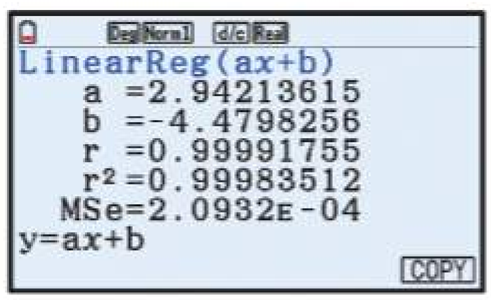


iv



b The only graph in **a** that appears to be linear is the graph of $\ln y$ against $\ln x$, so a power model is most appropriate.

c Using technology, the linear model connecting $\ln y$ and $\ln x$ is $\ln y \approx 2.94 \ln x - 4.48$, and $r^2 \approx 0.9998$.



d Using **c**, $\ln y \approx \ln(x^{2.94}) + \ln(e^{-4.48})$

$\therefore \ln y \approx \ln(x^{2.94} \times e^{-4.48})$

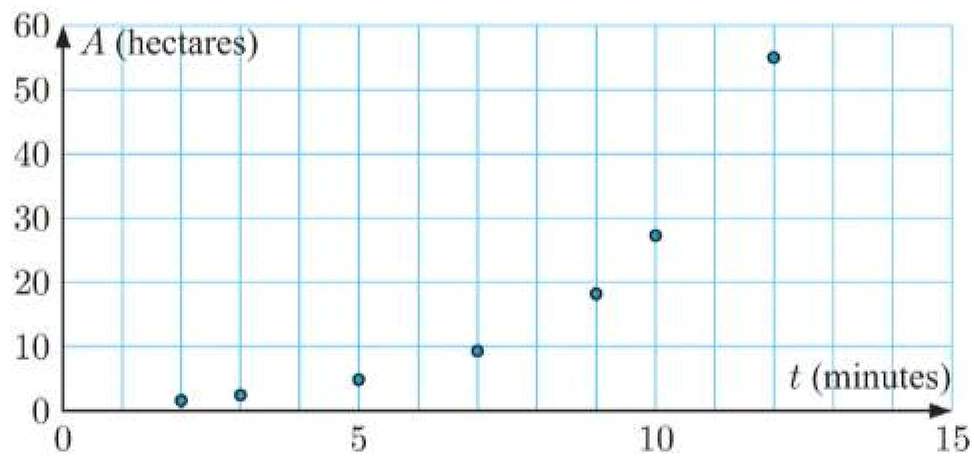
$\therefore y \approx e^{-4.48} \times x^{2.94}$

$\therefore y \approx 0.0113 \times x^{2.94}$ is the power model connecting x and y.

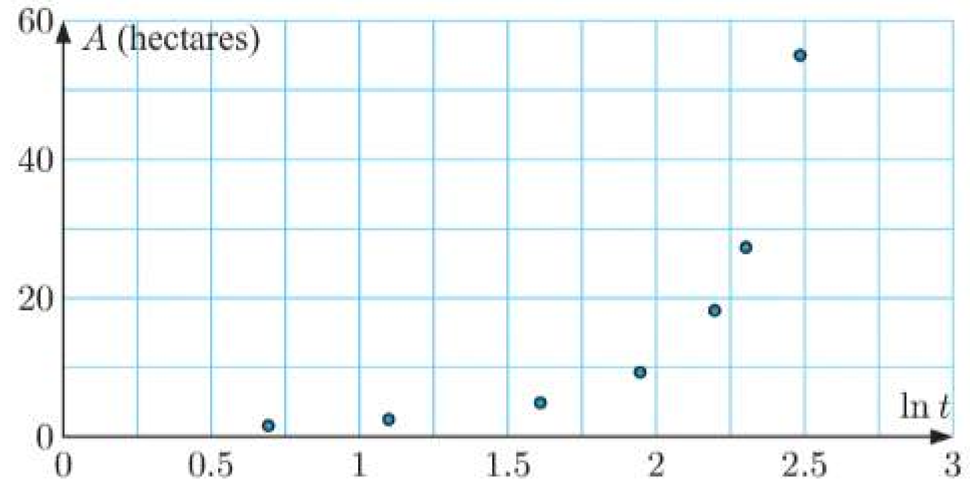
2 **a**

t (minutes)	2	3	5	7	9	10	12
A (hectares)	1.6	2.5	4.9	9.3	18.2	27.3	55
$\ln t$	≈ 0.693	≈ 1.10	≈ 1.61	≈ 1.95	≈ 2.20	≈ 2.30	≈ 2.48
$\ln A$	≈ 0.470	≈ 0.916	≈ 1.59	≈ 2.23	≈ 2.90	≈ 3.31	≈ 4.01

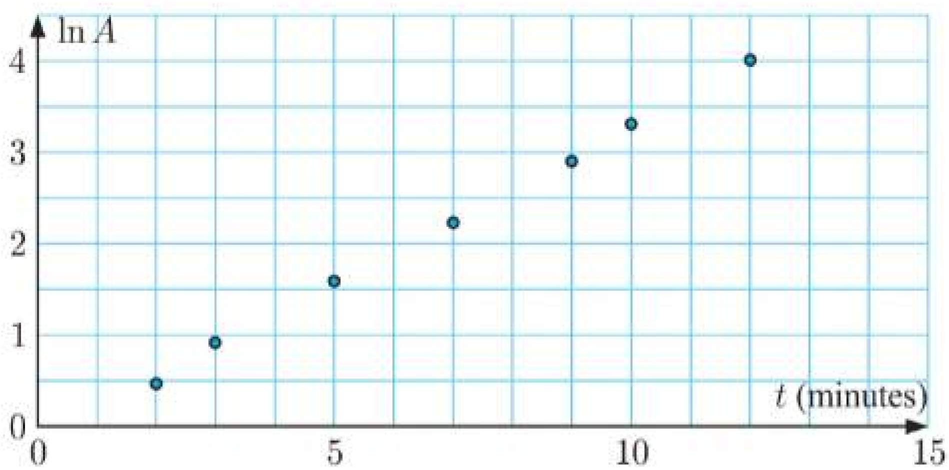
i



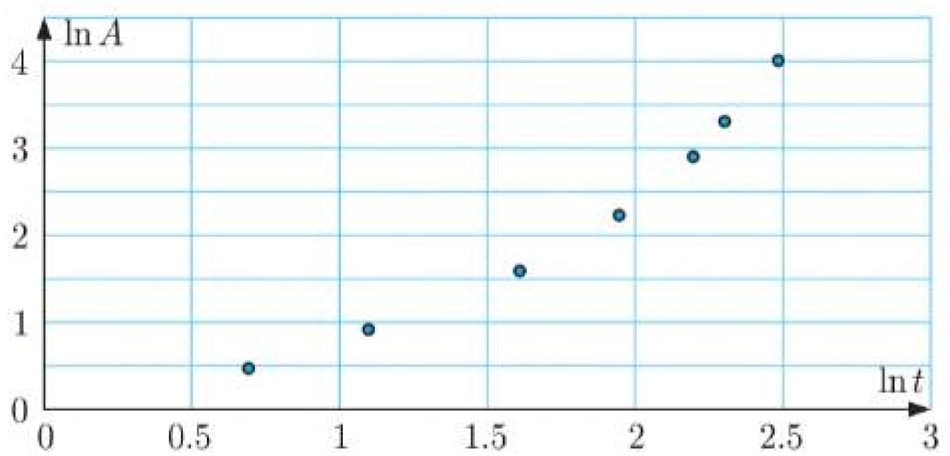
ii



iii

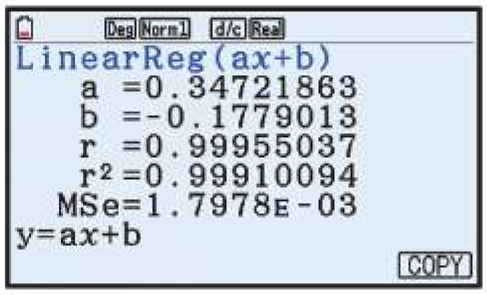


iv



b The only graph in **a** which appears to be linear is the graph of $\ln A$ against t, so an exponential model is most appropriate.

c Using technology, the linear model connecting $\ln A$ and t is $\ln A \approx 0.347t - 0.178$, and $r^2 \approx 0.9991$.



d Using **c**, the exponential model connecting t and A is $A \approx e^{0.347t-0.178}$

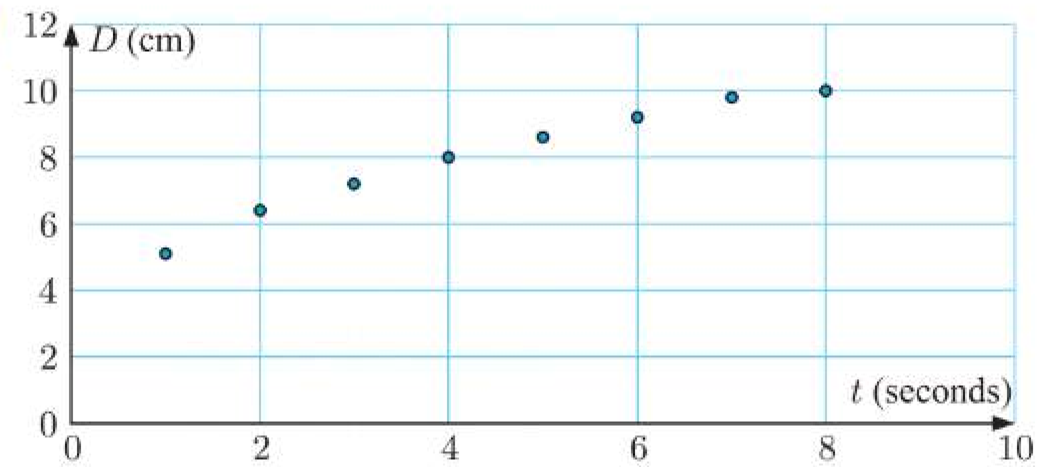
$$\therefore \quad A \approx e^{-0.178} \times (e^{0.347})^t$$

$$\therefore \quad A \approx 0.837 \times 1.42^t$$

3 **a**

t (seconds)	1	2	3	4	5	6	7	8
D (cm)	5.1	6.4	7.2	8.0	8.6	9.2	9.8	10
$\ln t$	0	≈ 0.693	≈ 1.10	≈ 1.39	≈ 1.61	≈ 1.79	≈ 1.95	≈ 2.08
$\ln D$	≈ 1.63	≈ 1.86	≈ 1.97	≈ 2.08	≈ 2.15	≈ 2.22	≈ 2.28	≈ 2.30

i



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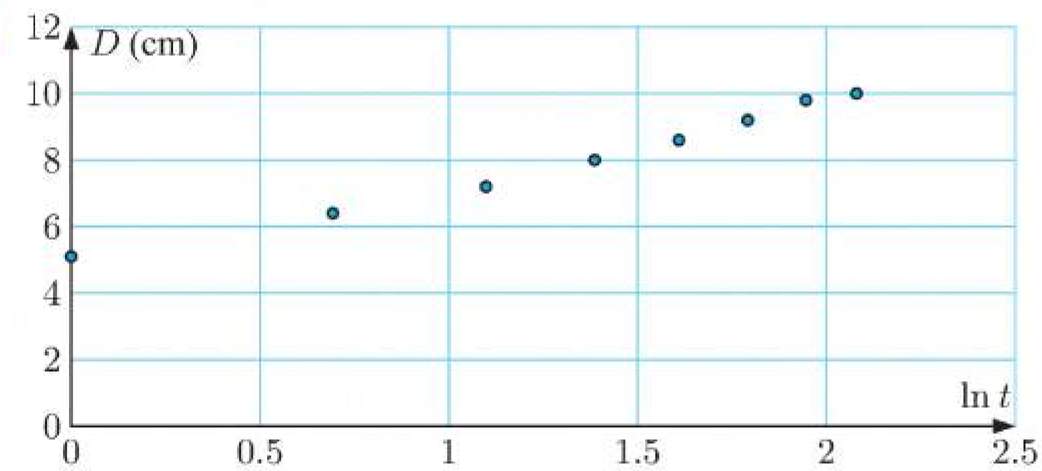
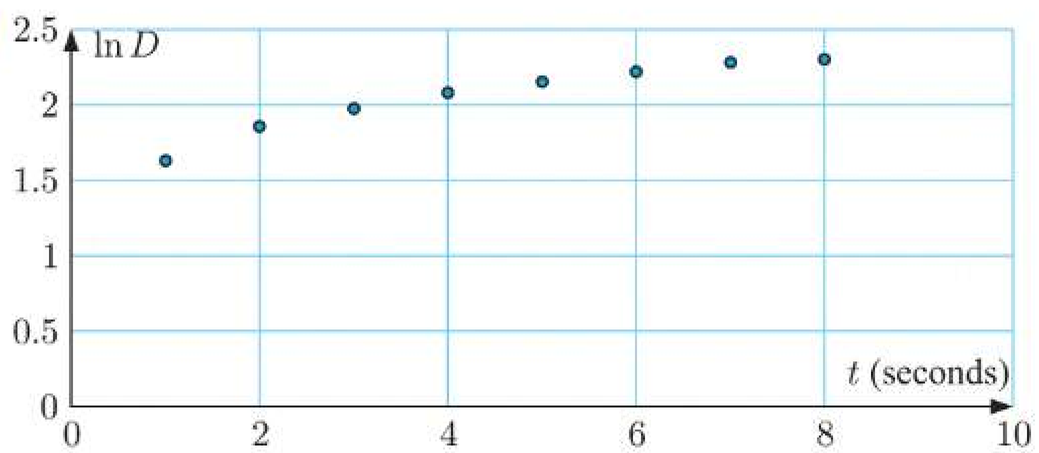
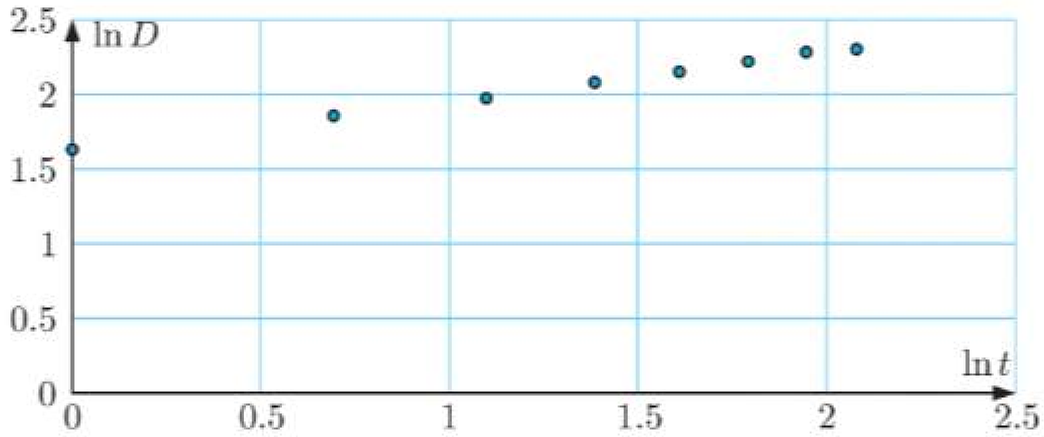


iii

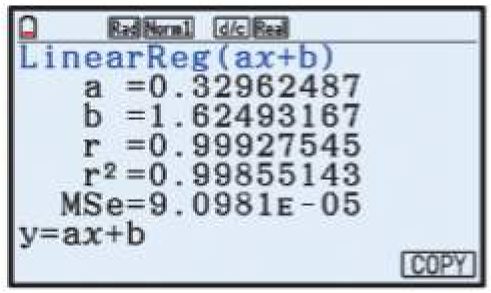


iv



b The only graph in **a** which appears to be linear is the graph of $\ln D$ against $\ln t$, so a power model is most appropriate.

c Using technology, the linear model connecting $\ln D$ and $\ln t$



is $\ln D \approx 0.330 \ln t + 1.62$

$\therefore \ln D \approx \ln(t^{0.330}) + \ln(e^{1.62})$

$\therefore \ln D \approx \ln(t^{0.330} \times e^{1.62})$

$\therefore D \approx e^{1.62} \times t^{0.330}$

$\therefore D \approx 5.08 \times t^{0.330}$ is the model connecting D and t.

d When $t = 3.5$, $D \approx 5.08 \times (3.5)^{0.330}$

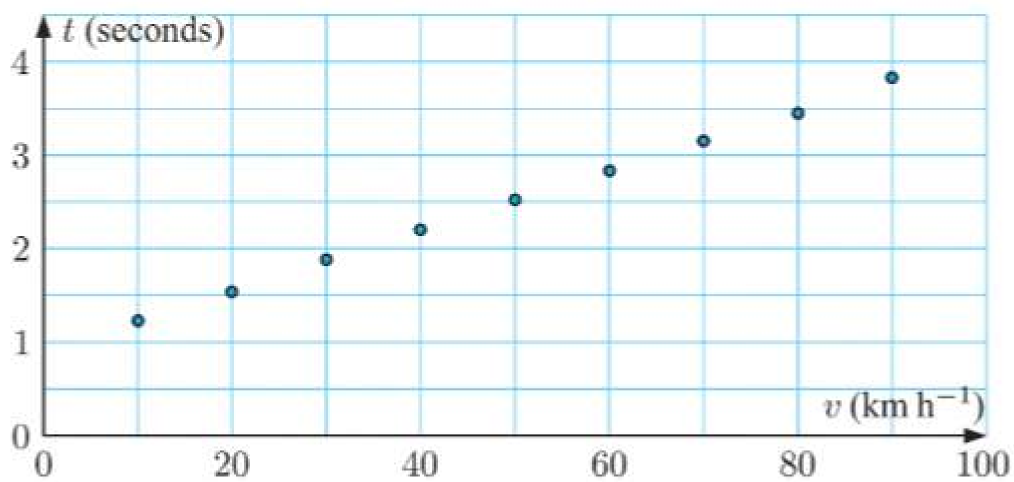
≈ 7.67

So, after 3.5 seconds the diameter of the water balloon is about 7.67 cm.
This estimate is an interpolation, so it is likely to be reliable.

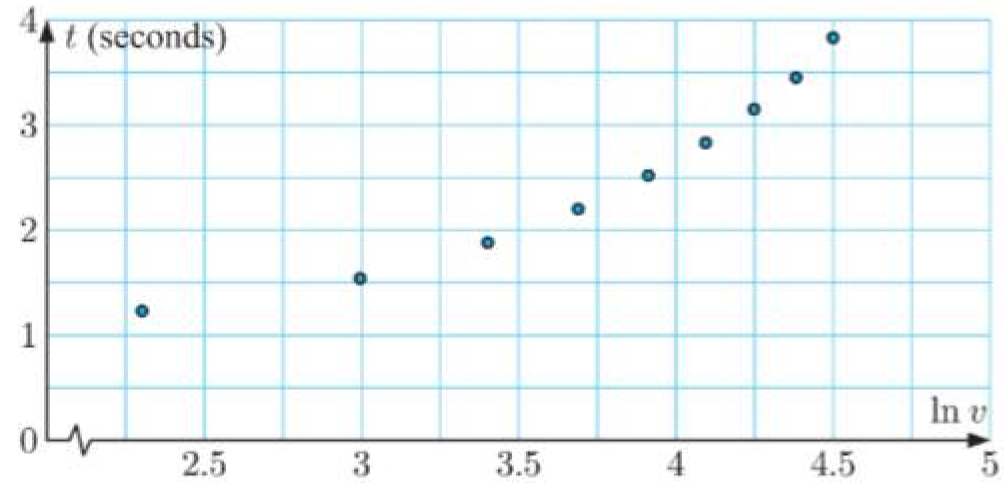
4 **a**

v (km h^{-1})	10	20	30	40	50	60	70	80	90
t (seconds)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83
$\ln v$	≈ 2.30	≈ 3.00	≈ 3.40	≈ 3.69	≈ 3.91	≈ 4.09	≈ 4.25	≈ 4.38	≈ 4.50
$\ln t$	≈ 0.207	≈ 0.432	≈ 0.631	≈ 0.788	≈ 0.924	≈ 1.04	≈ 1.15	≈ 1.24	≈ 1.34

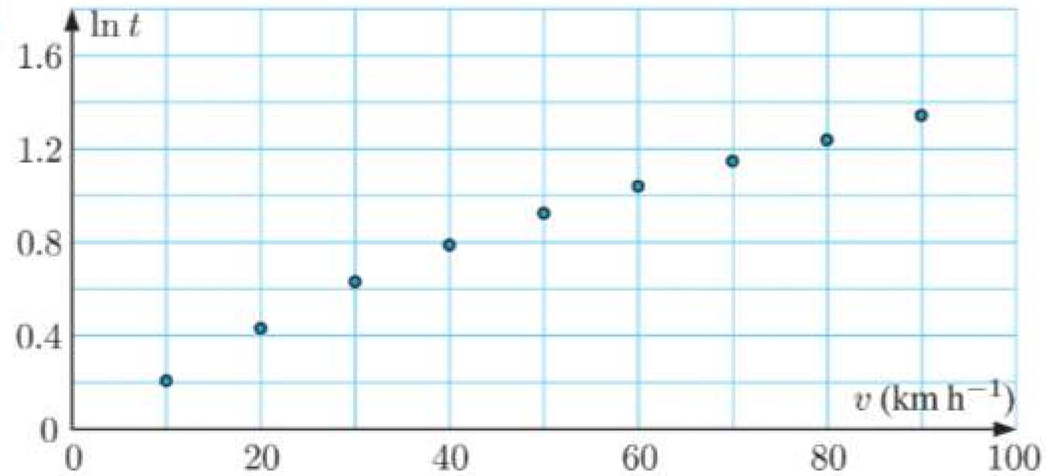
i



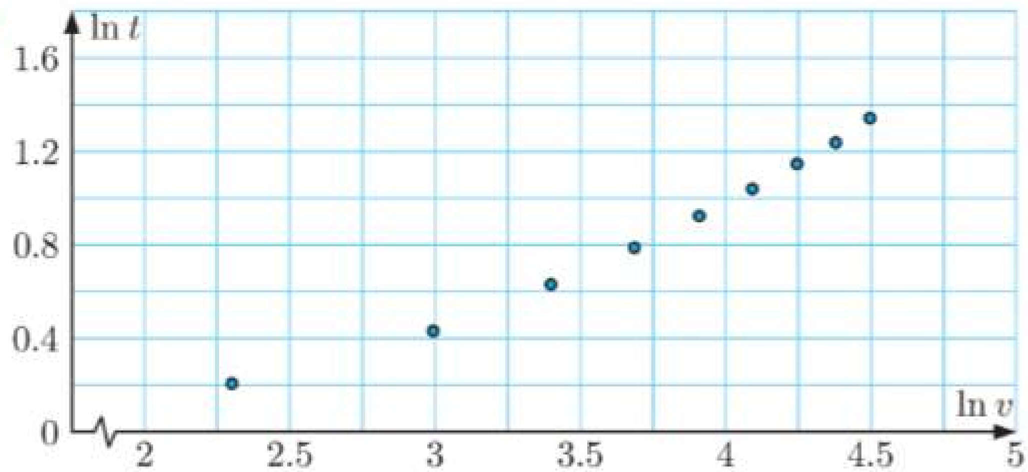
ii



iii

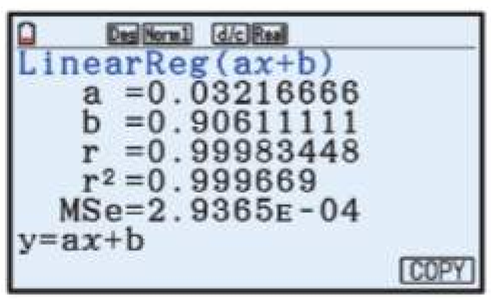


iv



b The only graph in **a** which appears to be linear is the graph of t against v, so a linear model is most appropriate.

c Using technology, the linear model connecting t and v is $t \approx 0.0322v + 0.906$.



d When $v = 110$, $t \approx 0.0322 \times 110 + 0.906$

≈ 4.44

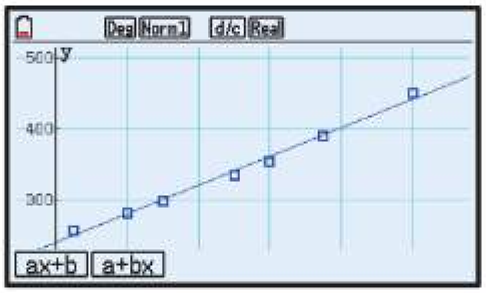
So, a car travelling at 110 km h^{-1} will stop in about 4.44 seconds.

5

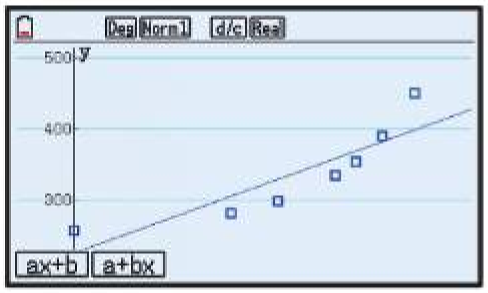
Time (t years)	1	4	6	10	12	15	20
Value (£$V \times 1000$)	257	281	299	335	354	390	450

a Using technology, we draw the scatter plot and the line of best fit of:

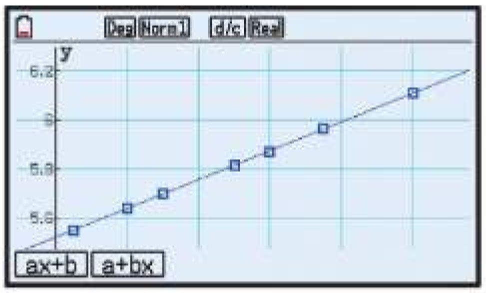
V against t



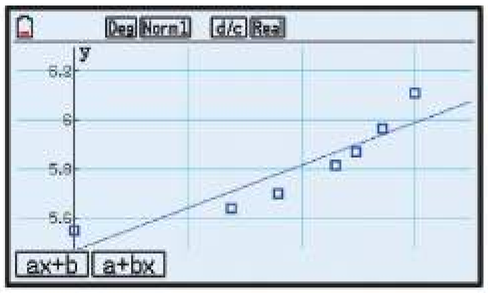
V against $\ln t$



$\ln V$ against t



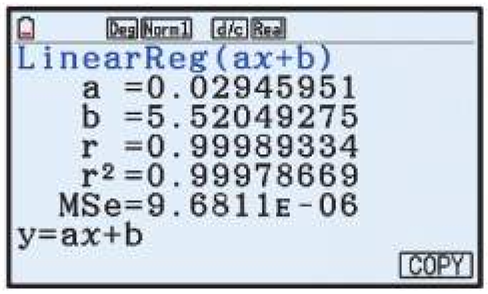
$\ln V$ against $\ln t$



The only graph which appears to be linear is $\ln V$ against t, so the most appropriate linear model is

$$\ln V \approx 0.0295t + 5.52$$
$$\therefore \quad V \approx e^{0.0295t+5.52}$$
$$\therefore \quad V \approx e^{5.52} \times (e^{0.0295})^t$$
$$\therefore \quad V \approx 250 \times 1.03^t$$

is the most appropriate model connecting V and t.



b When $t = 25$, $V \approx 250 \times 1.03^{25}$
≈ 522

So, 25 years after it is built, the house will be worth about £522 000.

6

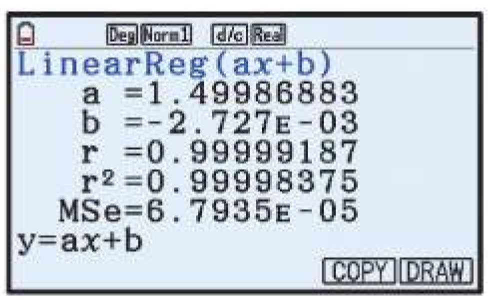
Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Orbital radius (r)	0.387	0.723	1.000	1.542	5.202	9.539
Orbital period (T)	0.241	0.615	1.000	1.881	11.862	29.457

a $T = ar^k$ is a power model, so we find a and k by calculating a linear model which connects $\ln T$ and $\ln r$.

Using technology, $\ln T \approx 1.50 \ln r - 0.002\,73$

$$\therefore \quad \ln T \approx \ln(r^{1.50} \times e^{-0.002\,73})$$
$$\therefore \quad T \approx e^{-0.002\,73} \times r^{1.50}$$
$$\therefore \quad T \approx 0.997 \times r^{1.50}$$

So equating coefficients, $a \approx 0.997$ and $k \approx 1.50$.



b From the screenshot in **a**, we see that $r^2 \approx 0.999\,98$. This value is very close to 1, so the model is appropriate for the data.

c A planet's *orbital radius* is its distance from the sun, and its *orbital period* is the length of its year.

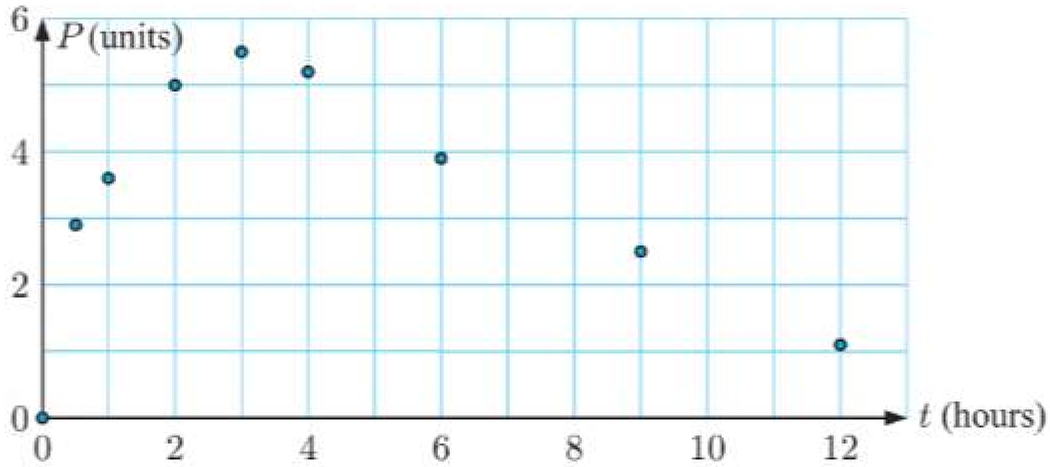
From **a**, $T = ar^k$ with $k \approx \frac{3}{2}$, so $T \propto r^{\frac{3}{2}}$

$\therefore\ T^2 \propto r^3$, which agrees with Kepler's conclusion.

ACTIVITY 2 SURGE, TERMINAL VELOCITY, AND LOGISTIC MODELS

PART 1: THE SURGE MODEL

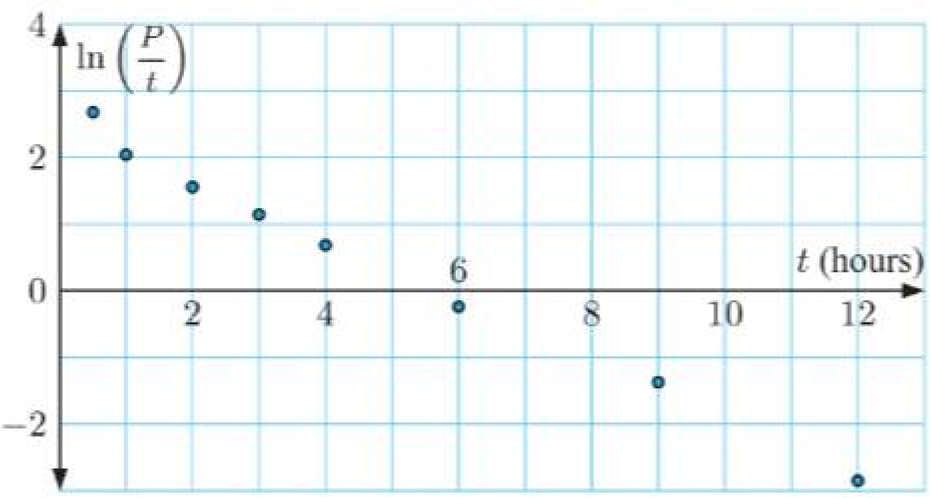
1 **a**

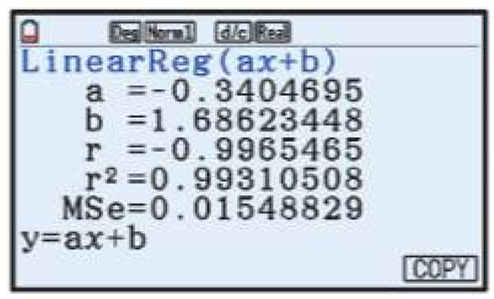


The graph rapidly rises to its maximum, and then decays more slowly back towards 0, which is consistent with a surge model.

b

t (hours)	0	0.5	1	2	3	4	6	9	12
P (units)	0	2.9	3.6	5.0	5.5	5.2	3.9	2.5	1.1
$\ln\left(\frac{P}{t}\right)$	undefined	≈ 1.76	≈ 1.28	≈ 0.916	≈ 0.606	≈ 0.262	≈ -0.431	≈ -1.28	≈ -2.39





Using technology, the linear model connecting $\ln\left(\frac{P}{t}\right)$ and t is $\ln\left(\frac{P}{t}\right) \approx -0.340t + 1.69$.

c Using **b**, $\frac{P}{t} \approx e^{-0.340t+1.69}$

$\therefore\ P \approx e^{1.69} \times t \times e^{-0.340t}$

$\therefore\ P \approx 5.40te^{-0.340t}$

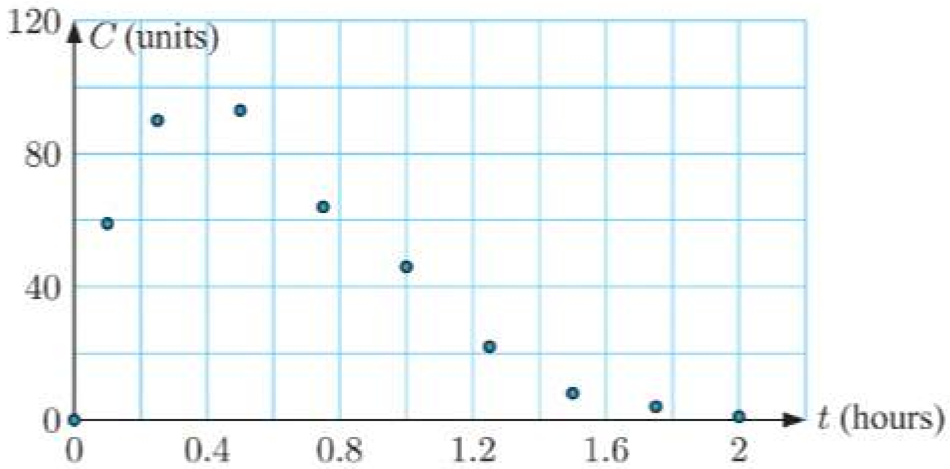
d **i** When $t = 5$, $P \approx 5.40 \times 5 \times e^{-0.340\times5} \approx 4.92$

So, after 5 hours the APRE will be about 4.92 units.

ii When $t = 10$, $P \approx 5.40 \times 10 \times e^{-0.340\times10} \approx 1.79$

So, after 10 hours the APRE will be about 1.79 units.

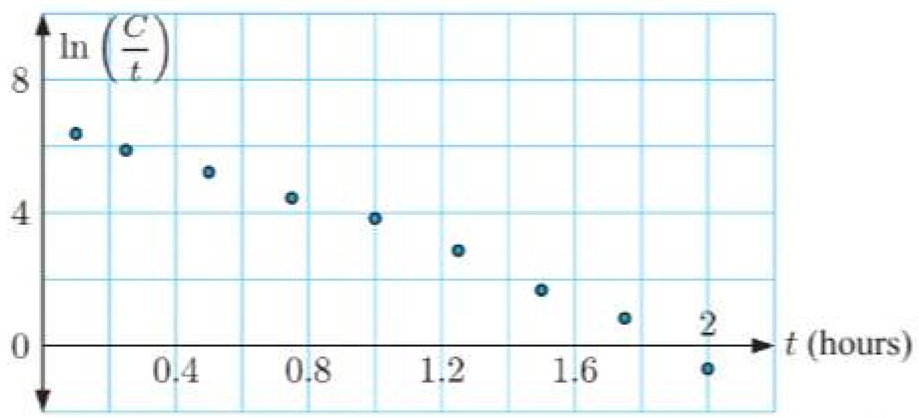
2 **a**

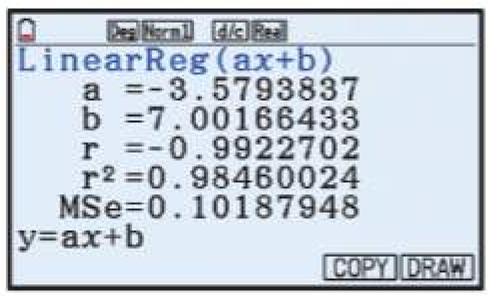


The graph rapidly rises to its maximum, and then decays more slowly back towards 0, which is consistent with a surge model.

b

t (hours)	0.00	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
C (units)	0	59	90	93	64	46	22	8	4	1
$\ln\left(\frac{C}{t}\right)$	undefined	≈ 6.38	≈ 5.89	≈ 5.23	≈ 4.45	≈ 3.83	≈ 2.87	≈ 1.67	≈ 0.827	≈ -0.693





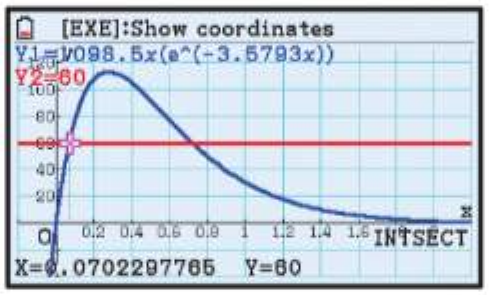
Using technology, the linear model connecting $\ln\left(\frac{C}{t}\right)$ and t is $\ln\left(\frac{C}{t}\right) \approx -3.58t + 7.00$.

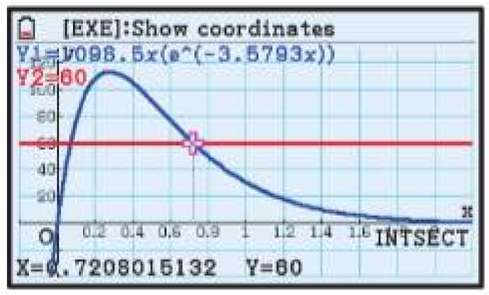
c Using **b**, $\frac{C}{t} \approx e^{-3.58t+7.00}$

$\therefore \; C \approx e^{7.00} \times t \times e^{-3.58t}$

$\therefore \; C \approx 1100te^{-3.58t}$

d Using technology, $C = 60$ when $t \approx 0.0702$ or $t \approx 0.721$.





i The operation can commence after about 0.0702 hours, which is about 4 minutes 13 seconds.

ii The surgeon has about $0.721 - 0.0702 \approx 0.651$ hours to carry out the operation, which is about 39 minutes 2 seconds.

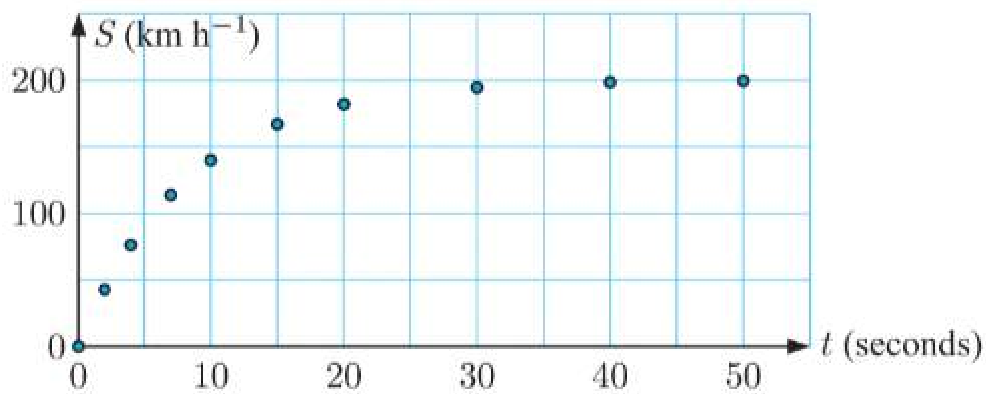
3 $y = At^n e^{-bt}$

$\therefore \dfrac{y}{t^n} = Ae^{-bt}$ where A, n, and b are constants

$\therefore$ there is an exponential relationship between $\dfrac{y}{t^n}$ and t.

PART 2: TERMINAL VELOCITY

1 **a**

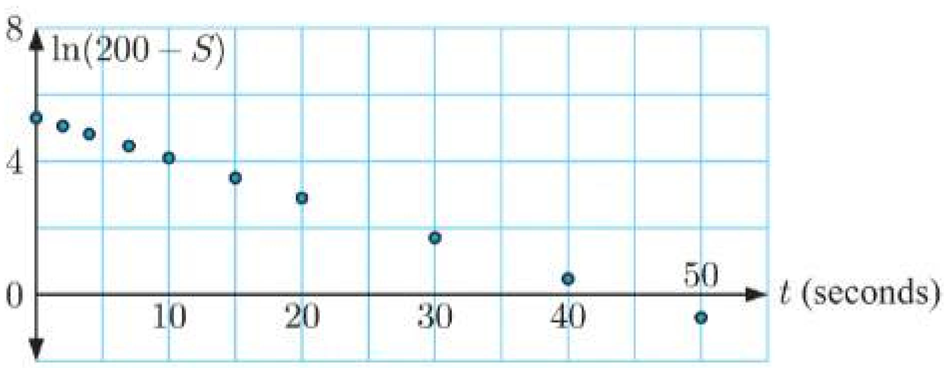


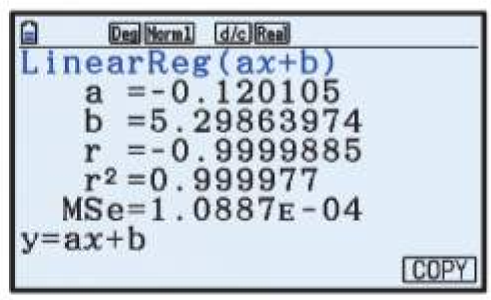
The speed starts at 0 and increases towards a limiting value, which suggests a terminal velocity model is appropriate.

b As t increases, S appears to approach $A = 200$.

c

t (seconds)	0	2	4	7	10	15	20	30	40	50
S (km h^{-1})	0	42.7	76.2	113.7	139.8	166.9	181.9	194.5	198.4	199.5
$\ln(200 - S)$	≈ 5.30	≈ 5.06	≈ 4.82	≈ 4.46	≈ 4.10	≈ 3.50	≈ 2.90	≈ 1.70	≈ 0.470	≈ -0.693





Using technology, the linear model connecting $\ln(200 - S)$ and t is $\ln(200 - S) \approx -0.120t + 5.30$.

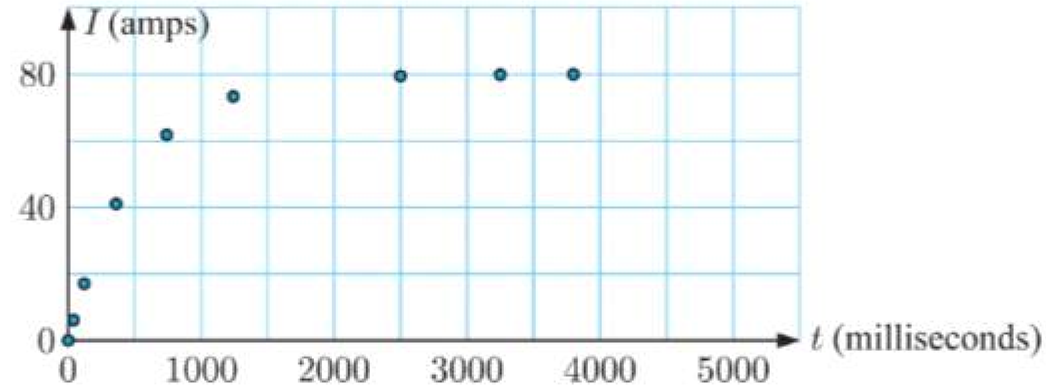
d Using **c**, $200 - S \approx e^{-0.120t+5.30}$

$$\therefore\ S \approx 200 - e^{-0.120t} \times e^{5.30}$$
$$\therefore\ S \approx 200 - e^{-0.120t} \times 200$$
$$\therefore\ S \approx 200(1 - e^{-0.120t})$$

e When $t = 6$, $S \approx 200(1 - e^{-0.120\times 6}) \approx 103$

So, after 6 seconds the skydiver's speed will be about 103 km h^{-1}.

2 **a**

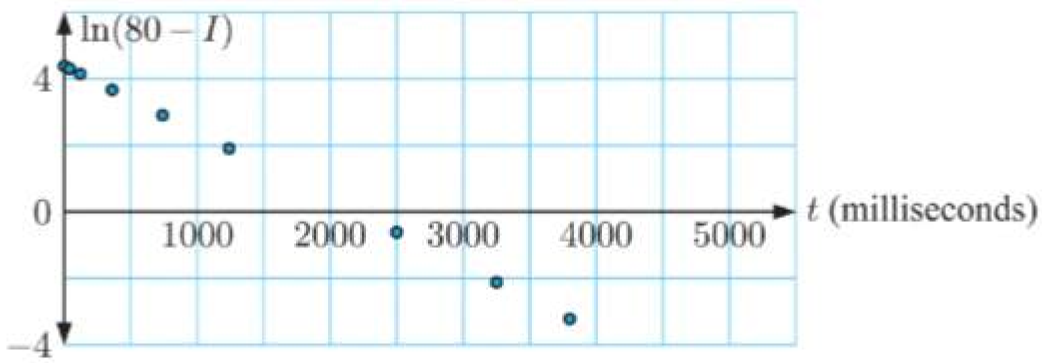


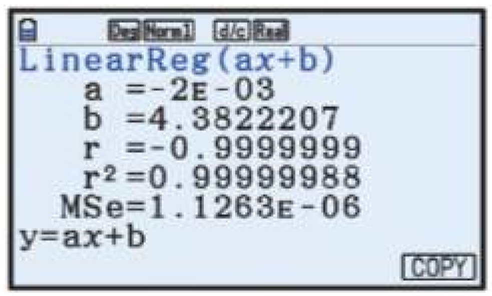
The current starts at 0 and increases towards a limiting value, which suggests a terminal velocity-type model is appropriate.

b As t increases, I appears to approach $A = 80$.

c

t (milliseconds)	0	40	120	360	740	1240	2500	3250	3800
I (amps)	0	6.15	17.07	41.06	61.79	73.30	79.46	79.88	79.96
$\ln(80 - I)$	≈ 4.38	≈ 4.30	≈ 4.14	≈ 3.66	≈ 2.90	≈ 1.90	≈ -0.616	≈ -2.12	≈ -3.22





Using technology, the linear model connecting $\ln(80 - I)$ and t is $\ln(80 - I) \approx -0.002\,00t + 4.38$.

d Using **c**, $80 - I \approx e^{-0.002\,00t+4.38}$

$$\therefore\ I \approx 80 - e^{-0.002\,00t} \times e^{4.38}$$
$$\therefore\ I \approx 80 - e^{-0.002\,00t} \times 80.0$$
$$\therefore\ I \approx 80.0(1 - e^{-0.002\,00t})$$

e **i** When $t = 500$, $I \approx 80.0(1 - e^{-0.002\,00\times 500}) \approx 50.6$

So, after 500 milliseconds the current will be about 50.6 amps.

ii When $t = 8000$, $I \approx 80.0(1 - e^{-0.002\,00\times 8000}) \approx 80.0$

So, after 8000 milliseconds the current will be about 80.0 amps.

PART 3: THE LOGISTIC MODEL

1 In a logistic model, $y = \dfrac{C}{1 + Ae^{-bt}}$

$$\therefore \ 1 + Ae^{-bt} = \frac{C}{y}$$

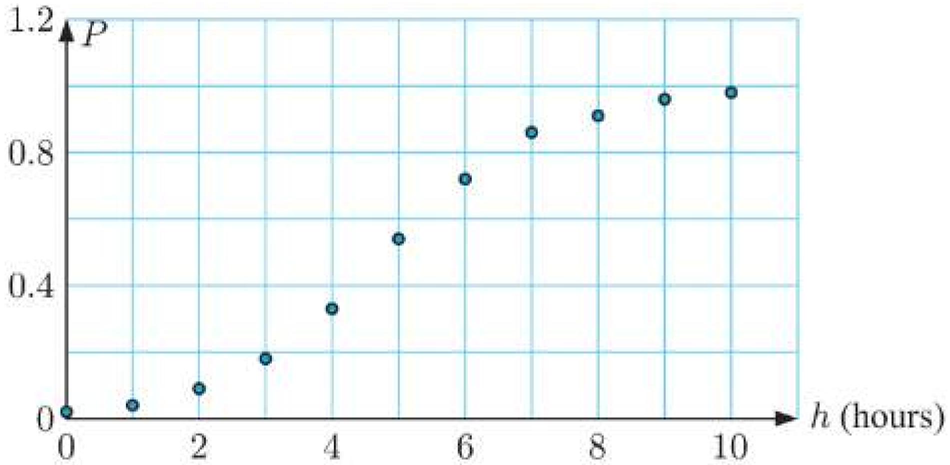
$$\therefore \ \frac{C}{y} - 1 = Ae^{-bt}$$

So, we expect to find an approximately linear relationship between $\ln\left(\dfrac{C}{y} - 1\right)$ and t.

2

Hours (h)	0	1	2	3	4	5	6	7	8	9	10
Proportion (P)	0.02	0.04	0.09	0.18	0.33	0.54	0.72	0.86	0.91	0.96	0.98

a

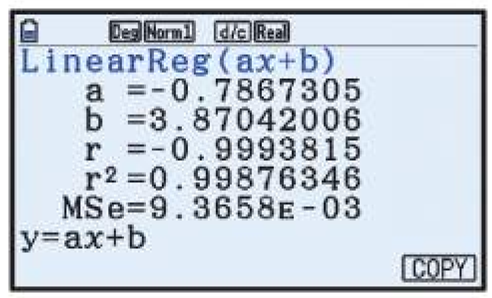


There is initially exponential growth, but this slows down and approaches a limiting value, so a logistic model is appropriate.

b As h increases, P appears to approach 1.
So, the limiting value of the proportion P is $C = 1$.

c To find the model of best fit, we perform a linear regression of $\ln\left(\dfrac{1}{P} - 1\right)$ against h.

Using technology,



$$\ln\left(\frac{1}{P} - 1\right) \approx -0.787h + 3.87$$

$$\therefore \ \frac{1}{P} - 1 \approx e^{-0.787h + 3.87}$$

$$\therefore \ \frac{1}{P} \approx e^{-0.787h} \times e^{3.87} + 1$$

$$\therefore \ P \approx \frac{1}{1 + 48.0e^{-0.787h}}$$ is the model of best fit.

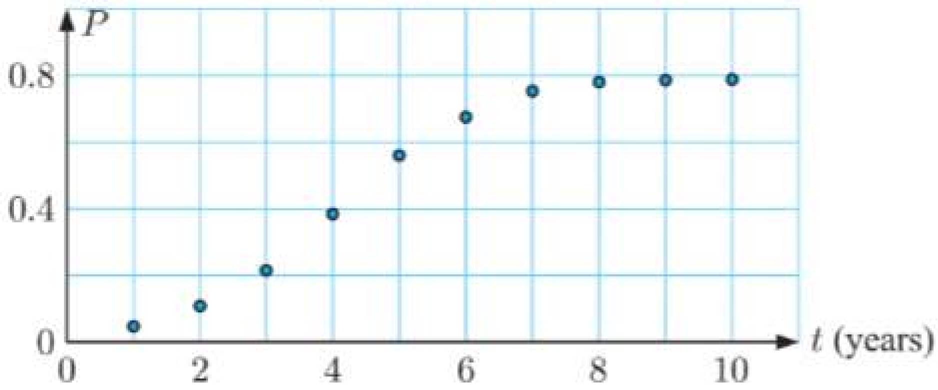
d When $h = 5.5$, $P \approx \dfrac{1}{1 + 48.0 \times e^{-0.787 \times 5.5}} \approx 0.612$

So, after 5.5 hours about 61.2% of the community have heard the rumour.

3

Time (t years)	1	2	3	4	5	6	7	8	9	10
Lizard population (P)	47	108	215	384	560	675	753	780	786	788

a



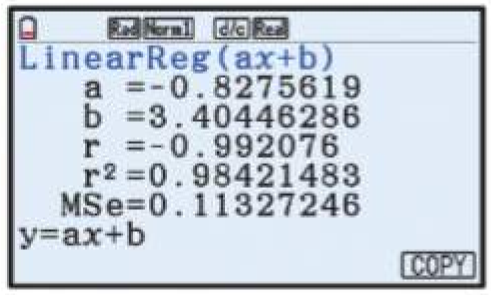
There is initially exponential growth, but this slows down and approaches a limiting value, so a logistic model is appropriate.

b As t increases, P appears to approach 800.

So, the limiting value of the population P is $C = 800$.

To find the model of best fit, we perform a linear regression of $\ln\left(\frac{800}{P} - 1\right)$ against t.

Using technology,



$$\ln\left(\frac{800}{P} - 1\right) \approx -0.828t + 3.40$$

$$\therefore \frac{800}{P} - 1 \approx e^{-0.828t+3.40}$$

$$\therefore \frac{800}{P} \approx e^{3.40} \times e^{-0.828t} + 1$$

$$\therefore P \approx \frac{800}{1 + 30.1e^{-0.828t}}$$ is the model of best fit.

c When $t = 4.5$, $P \approx \dfrac{800}{1 + 30.1 \times e^{-0.828 \times 4.5}} \approx 463$

So, after 4.5 years, the lizard population will be about 463.

ACTIVITY 3 USING TECHNOLOGY TO PERFORM NON-LINEAR REGRESSION

1

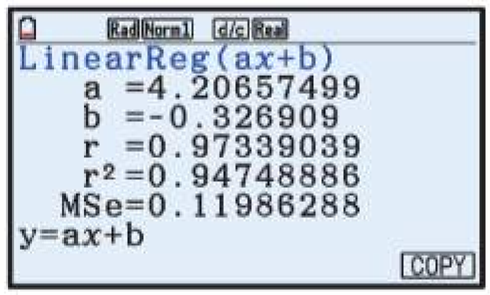
x	-0.3	0.2	0	-0.4	0.1	0.3	-0.5	-0.2	0.4	0.5	-0.1
y	0.20	1.47	1.42	0.13	0.65	3.12	0.07	0.43	4.58	4.52	0.40

a The values in the column on the far left represent a range of different possible values of λ, which we are testing to find the particular value that best fits the data.

b The spreadsheet estimates λ by finding the value of λ which minimises SS_{res}. This is represented by λ^*.

c The *precision* of the estimate can be improved by testing values of λ which are closer together. For example, we could test $\lambda = -5, -4.99, -4.98, ...$ and so on.

d Using technology, the linear model connecting $\ln y$ and x is



$\ln y \approx 4.21x - 0.327$

$\therefore \quad y \approx e^{4.21x-0.327}$

$\therefore \quad y \approx e^{-0.327} \times e^{4.21x}$

$\therefore \quad y \approx 0.721e^{4.21x}$

Calculating SS_{res} for this model using technology, we see that $SS_{\text{res}} \approx 3.50$.
This is higher than the SS_{res} calculated in the spreadsheet, so the model from the non-linear regression is a better fit for the data.

2

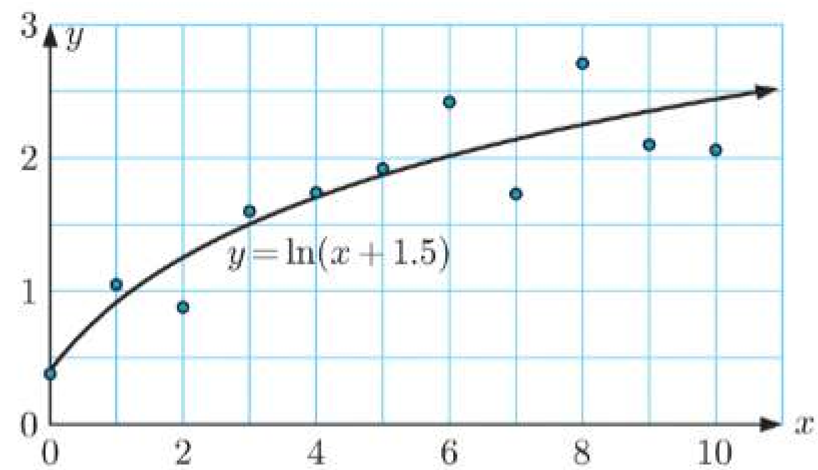
x	0	1	2	3	4	5	6	7	8	9	10
y	0.38	1.05	0.88	1.60	1.74	1.92	2.42	1.73	2.71	2.10	2.06

a $0 < y < 3$, so $1 \leqslant x + c \leqslant e^3 \approx 20$
Also $0 \leqslant x \leqslant 10$, so $1 \leqslant c \leqslant 10$.

b SPREADSHEET

From the spreadsheet, $c \approx 1.5$.
So, the model is $y \approx \ln(x + 1.5)$.

c



d The model appears to fit the data well.
$SS_{\text{res}} \approx 0.922 < 1$, which confirms that the fit is good.

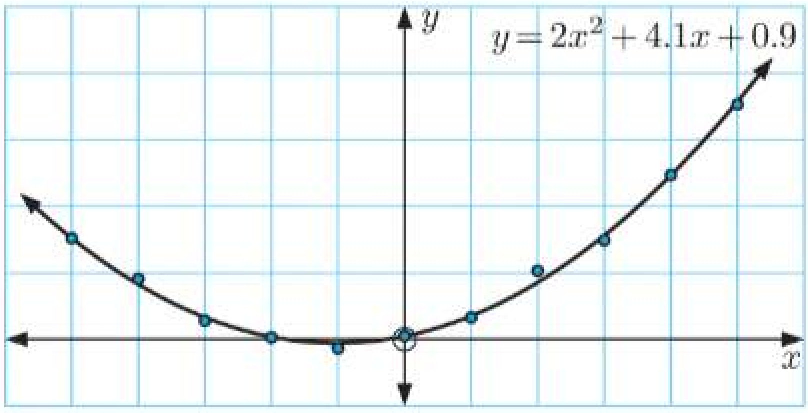
3

x	-3	-2	0	2	-4	3	-1	5	1	-5	4
y	5.61	0.56	0.99	20.61	18.15	29.8	-2.72	70.57	6.55	30.38	49.38

a By using the arrows to sort the values of a, b, and c, we see that $1.5 \leqslant a \leqslant 2.5$, $3.5 \leqslant b \leqslant 4.5$, and $0.5 \leqslant c \leqslant 1.5$.

b By using a^*, b^*, and c^* we see that SS_{res} is minimised when $a \approx 2$, $b \approx 4.1$, and $c \approx 0.9$.

c



d In this example, a, b, and c each take 11 values, and every combination of these values is tested.

So $11^3 = 1331$ rows are needed to calculate a, b, and c to 1 decimal place.

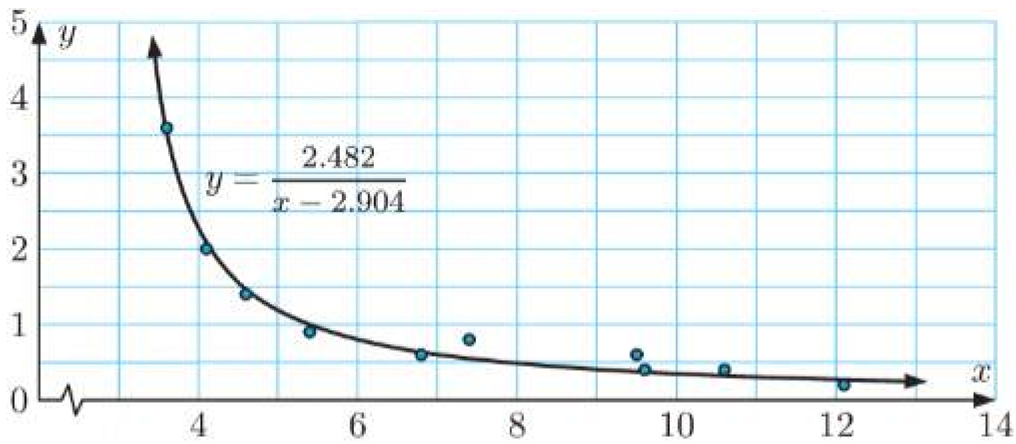
If we increased the accuracy to 2 decimal places, we would instead need $101^3 = 1\,030\,301$ rows.

So, the number of rows needed to calculate more accurate estimates quickly grows too large.

The same principle applies if we want to test a greater range for each parameter, or if the model has a greater number of parameters.

EXERCISE 8E

1 **a**



b

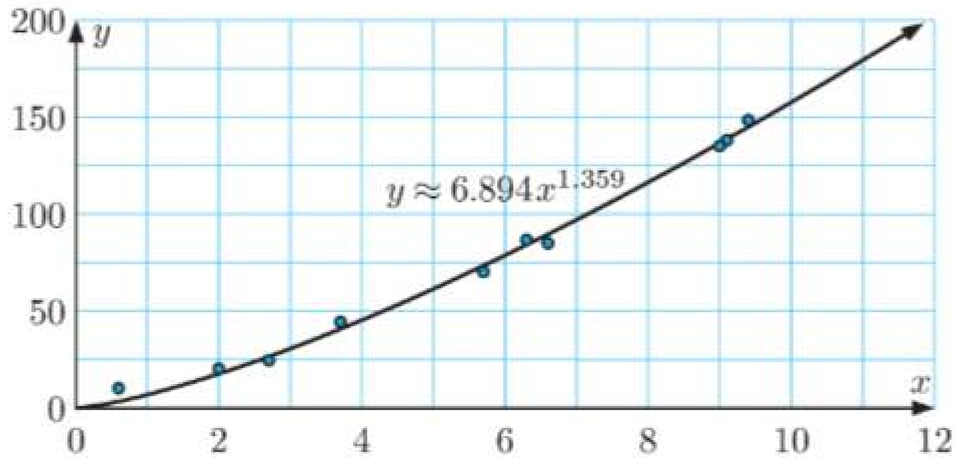
x	y	$\frac{2.482}{x - 2.904}$	*Residual*
9.6	0.4	≈ 0.3707	≈ 0.0293
7.4	0.8	≈ 0.5520	≈ 0.2480
3.6	3.6	≈ 3.5661	≈ 0.0339
4.6	1.4	≈ 1.4634	≈ -0.0634
5.4	0.9	≈ 0.9944	≈ -0.0944
10.6	0.4	≈ 0.3225	≈ 0.0775
12.1	0.2	≈ 0.2699	≈ -0.0699
4.1	2.0	≈ 2.0753	≈ -0.0753
6.8	0.6	≈ 0.6371	≈ -0.0371
9.5	0.6	≈ 0.3763	≈ 0.2237

c $SS_{\text{res}} \approx (0.0293)^2 + (0.2480)^2 + ... + (0.2237)^2$

≈ 0.144

d From **a** we see that the model fits the data well, which is confirmed by the small value of SS_{res} calculated in **c**.

2 **a**



b

x	y	$6.894x^{1.359}$	*Residual*
2.7	24.6	≈ 26.589	≈ -1.99
3.7	44.4	≈ 40.800	≈ 3.60
5.7	70.2	≈ 73.402	≈ -3.20
9.1	138.1	≈ 138.614	≈ -0.514
2.0	20.2	≈ 17.684	≈ 2.52
9.0	135.1	≈ 136.548	≈ -1.45
9.4	148.4	≈ 144.861	≈ 3.54
6.6	85.0	≈ 89.584	≈ -4.58
6.3	86.7	≈ 84.096	≈ 2.60
0.6	10.2	≈ 3.443	≈ 6.76

c $SS_{\text{res}} \approx (-1.99)^2 + (3.60)^2 + ... + (6.76)^2$
≈ 122

d Using technology, the linear model connecting $\ln x$ and $\ln y$ is

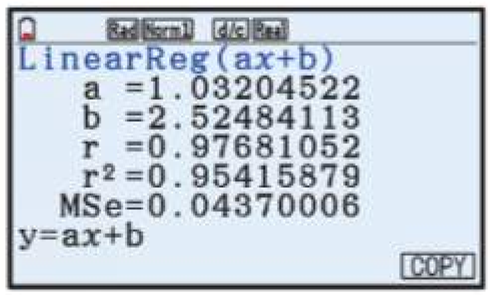
$\ln y \approx 1.03 \ln x + 2.52$

$\therefore \ \ln y \approx \ln(x^{1.03}) + \ln(e^{2.52})$

$\therefore \ \ln y \approx \ln(x^{1.03} \times e^{2.52})$

$\therefore \ y \approx e^{2.52} \times x^{1.03}$

$\therefore \ y \approx 12.5x^{1.03}$



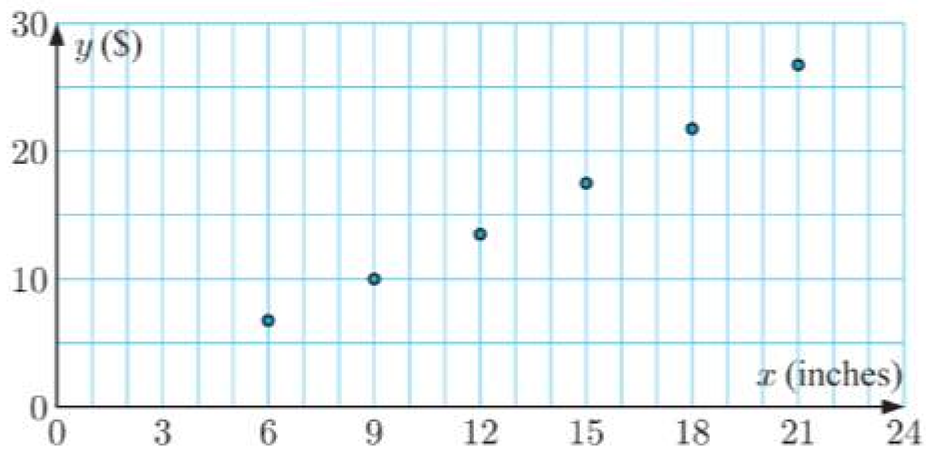
Calculating SS_{res} for this model using technology, we see that $SS_{\text{res}} \approx 1160$, which is much greater than the value found in **c**.

This model does not fit the data as well as the model obtained using non-linear regression.

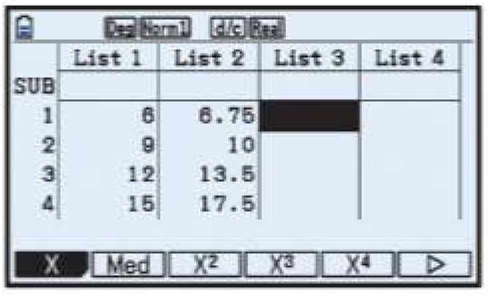
3

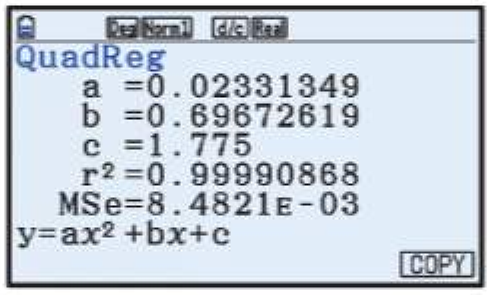
Diameter (x inches)	6	9	12	15	18	21
Price ($\$y$)	6.75	10.00	13.50	17.50	21.75	26.75

a



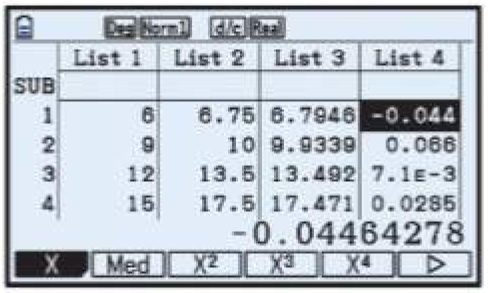
b

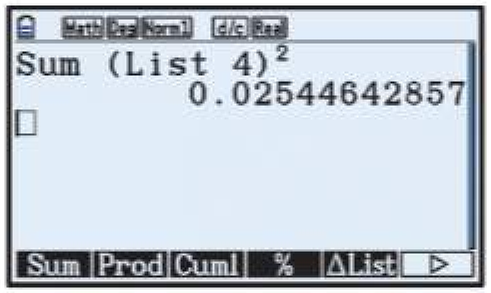




Using technology, the quadratic model that best fits the data is $y \approx 0.0233x^2 + 0.697x + 1.78$.

c





Using technology, $SS_{\text{res}} \approx 0.0254$.

d SS_{res} is very small which suggests that the model is a good fit for the data.

Also, we can assume that price $\propto$ area of each pizza

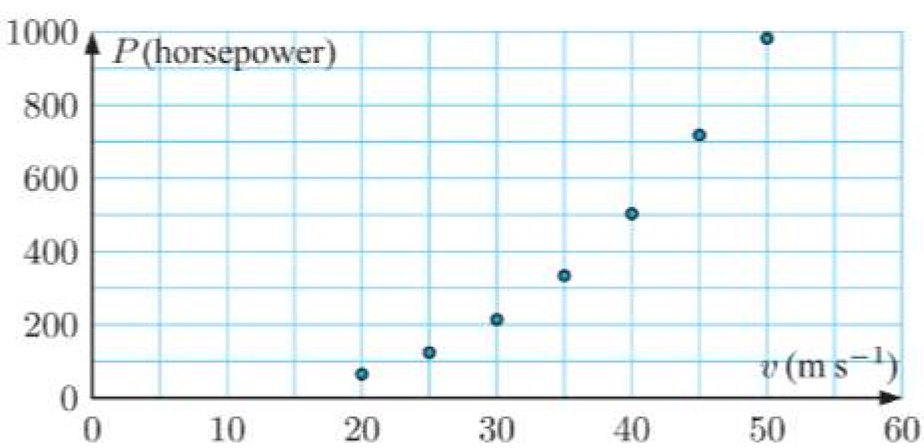
$$\propto (\text{radius})^2$$

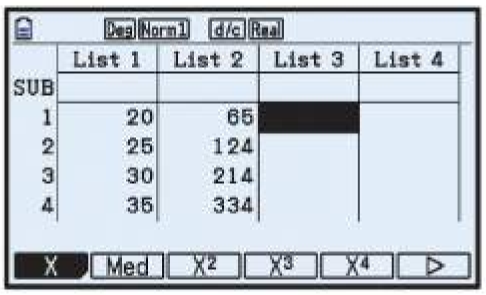
$$\propto (\text{diameter})^2$$

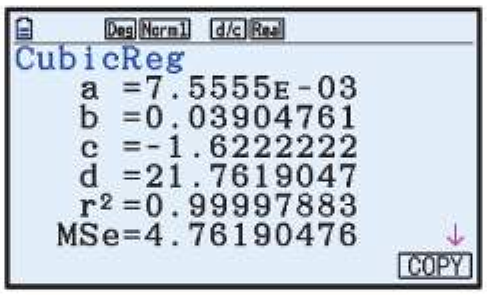
4

Speed (v m s^{-1})	20	25	30	35	40	45	50
Power (P horsepower)	65	124	214	334	503	718	982

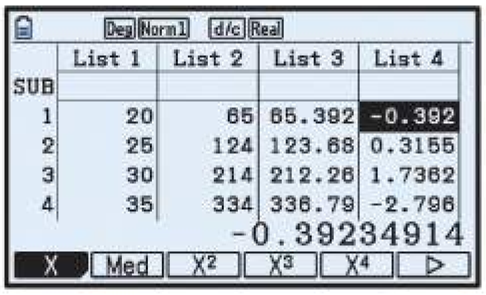
a

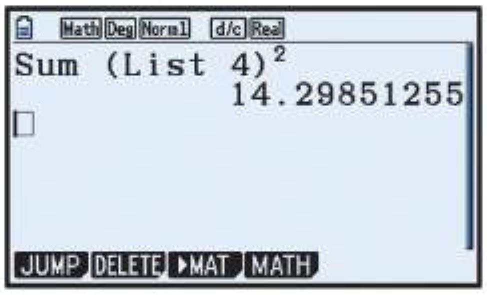


b 



Using technology, the cubic model that best fits the data is
$P \approx 0.007\,56v^3 + 0.0390v^2 - 1.62v + 21.8$.

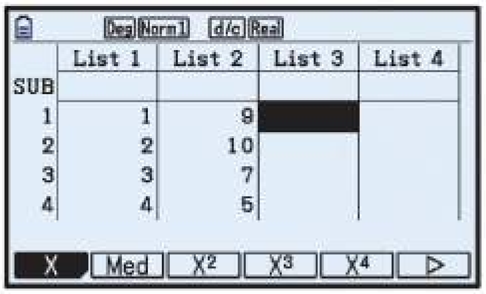
c 

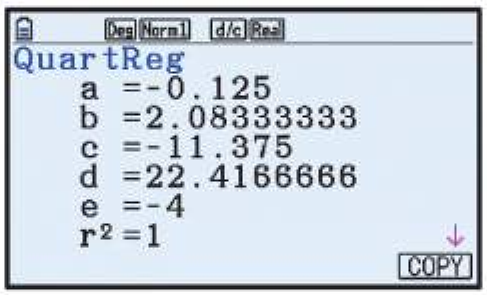


Using technology, $SS_{\text{res}} \approx 14.3$.

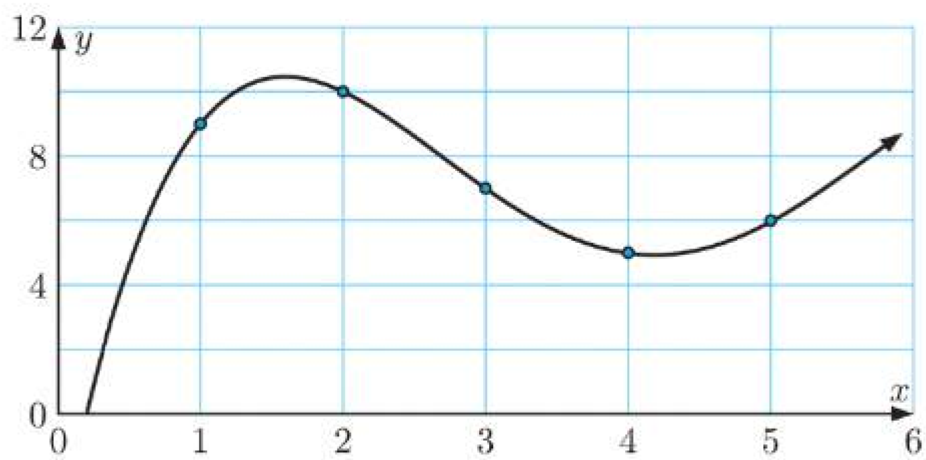
5

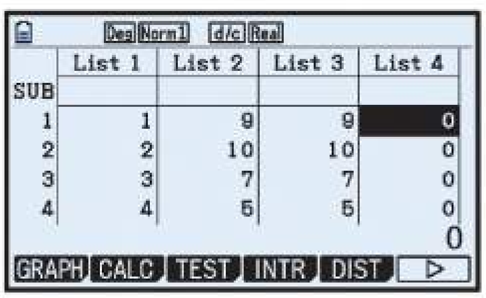
x	1	2	3	4	5
y	9	10	7	5	6

a 



Using technology, the quartic model that best fits the data is
$y \approx -0.125x^4 + 2.08x^3 - 11.4x^2 + 22.4x - 4$.
Or, expressing the coefficients as fractions, $y = -\frac{1}{8}x^4 + \frac{25}{12}x^3 - \frac{91}{8}x^2 + \frac{269}{12}x - 4$.

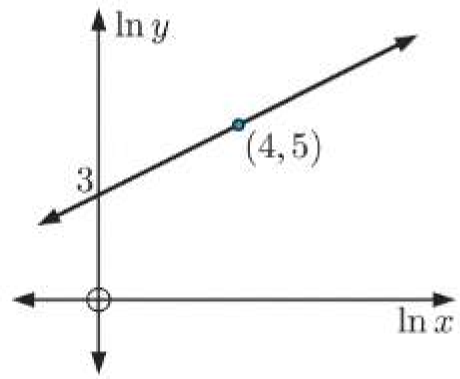
b 

c 

$SS_{\text{res}} = 0$
The quartic model fitted passes through each data point.

REVIEW SET 8A

1 **a** The graph of $\ln y$ against $\ln x$ is linear with gradient $m = \dfrac{5-3}{4-0} = \dfrac{1}{2}$ and vertical axis intercept $c = 3$.



$\therefore \ \ln y = \frac{1}{2}\ln x + 3$

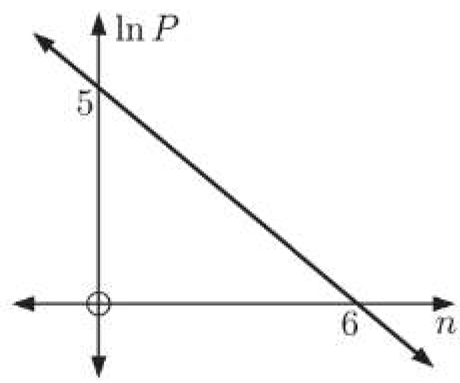
$\therefore \ \ln y = \ln(x^{\frac{1}{2}}) + \ln(e^3)$

$\therefore \ \ln y = \ln(x^{\frac{1}{2}} \times e^3)$

$\therefore \ y = e^3 \times x^{\frac{1}{2}}$

$\therefore \ y \approx 20.1x^{0.5}$

b The graph of $\ln P$ against n is linear with gradient $m = \dfrac{0-5}{6-0} = -\dfrac{5}{6}$ and vertical axis intercept $c = 5$.



$\therefore \ \ln P = -\frac{5}{6}n + 5$

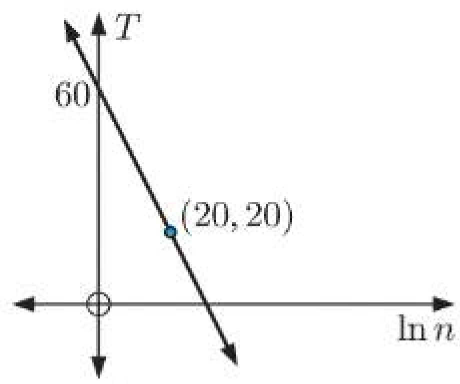
$\therefore \ P = e^{-\frac{5}{6}n+5}$

$\therefore \ P = e^{-\frac{5}{6}n} \times e^5$

$\therefore \ P = e^5 \times (e^{-\frac{5}{6}})^n$

$\therefore \ P \approx 148 \times 0.435^n$

c The graph of T against $\ln n$ is linear with gradient $m = \dfrac{20-60}{20-0} = -2$ and vertical axis intercept $c = 60$.

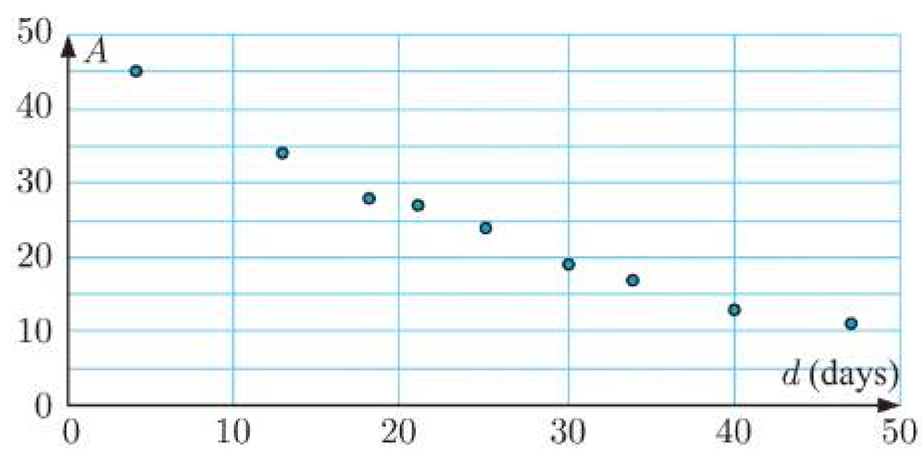


$\therefore \ T = 60 - 2\ln n$

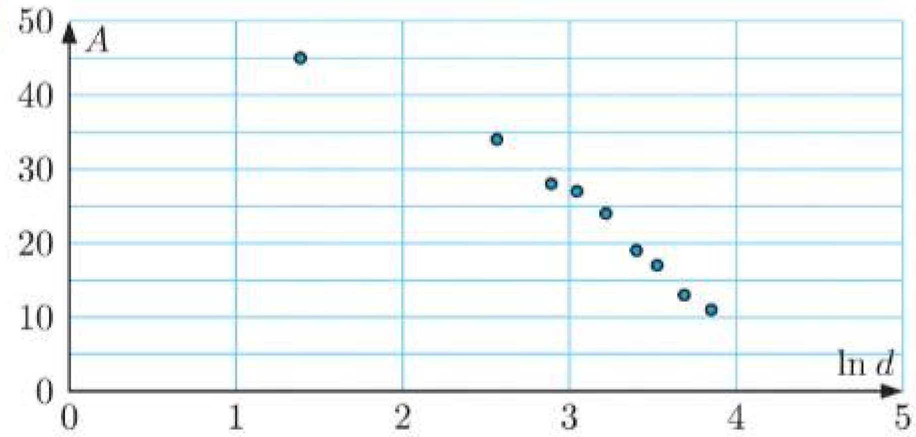
2

Days after release (d)	4	13	18	21	25	30	34	40	47
Audience members (A)	45	34	28	27	24	19	17	13	11

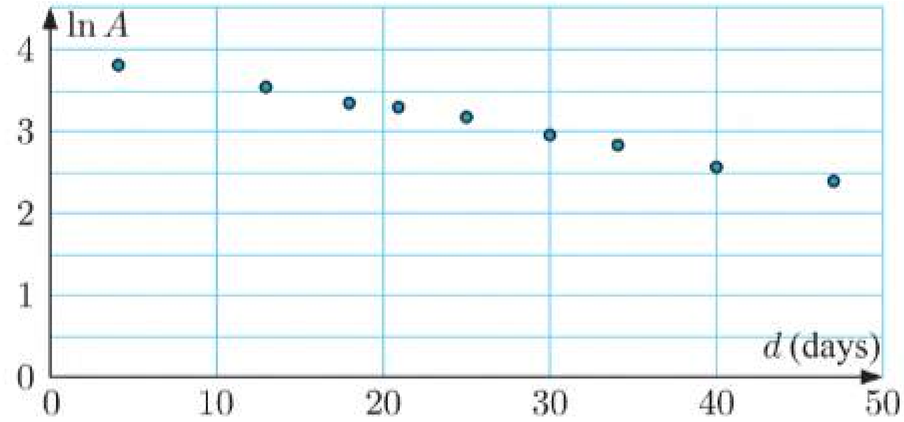
a **i**



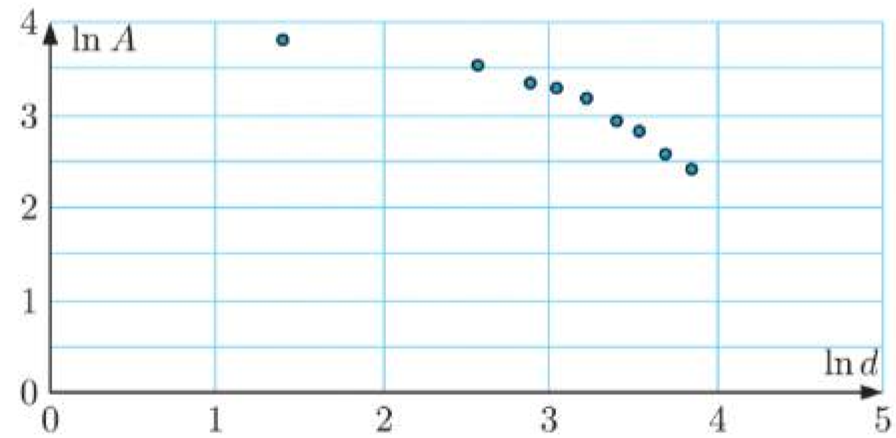
ii



iii

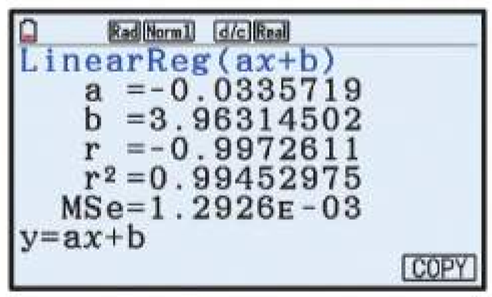


iv



b The only graph in **a** that appears to be linear is the graph of $\ln A$ against d, so an exponential model is most appropriate.

c Using technology, the linear model connecting $\ln A$ and d is $\ln A \approx -0.0336d + 3.96$.



d Using **c**, $A \approx e^{-0.0336d+3.96}$

$\therefore \; A \approx e^{-0.0336d} \times e^{3.96}$

$\therefore \; A \approx e^{3.96} \times (e^{-0.0336})^d$

$\therefore \; A \approx 52.6 \times 0.967^d$ is the exponential model connecting A and d.

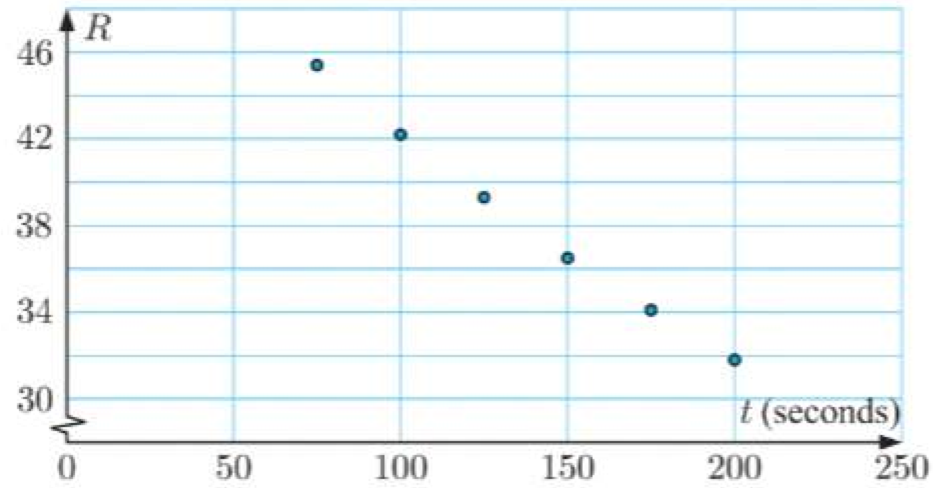
e When $d = 0$, $A \approx 52.6 \times 0.967^0 \approx 52.6$

The average number of audience members when the movie was first released was about 53.

3

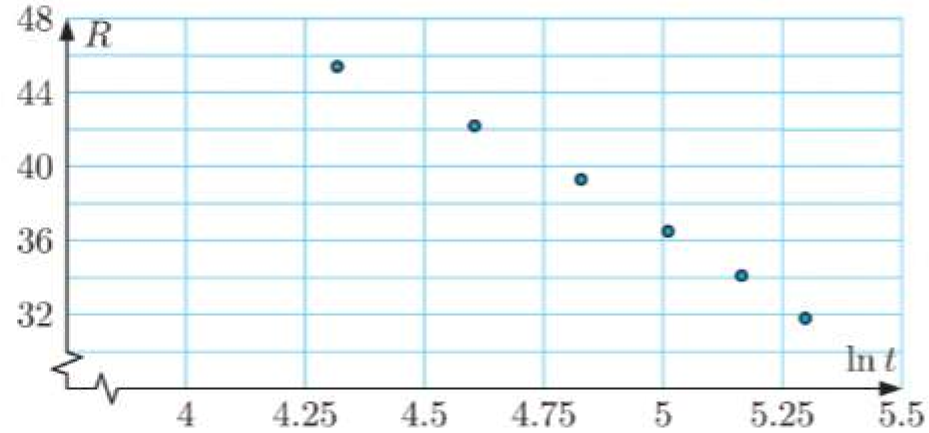
Time (t seconds)	75	100	125	150	175	200
Rate of reaction (R)	45.4	42.2	39.3	36.5	34.1	31.8

a **i**



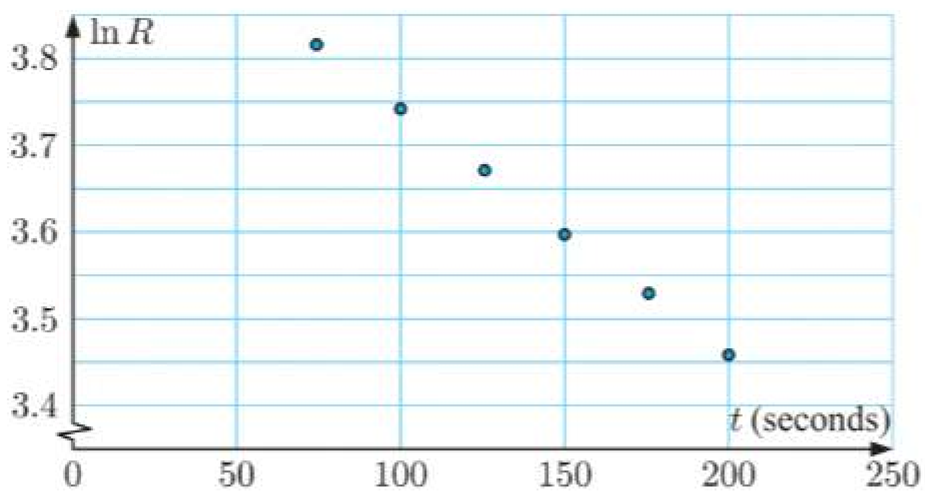
$r^2 \approx 0.996$

ii



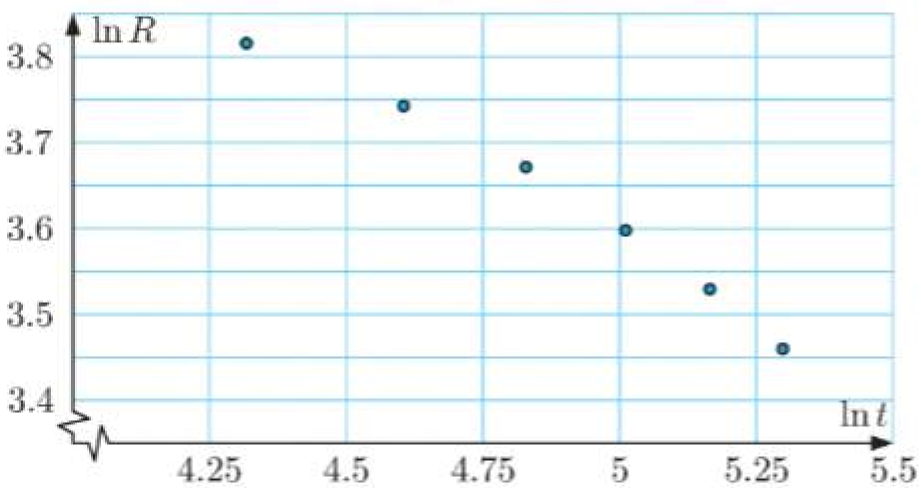
$r^2 \approx 0.993$

iii



$r^2 \approx 0.999$

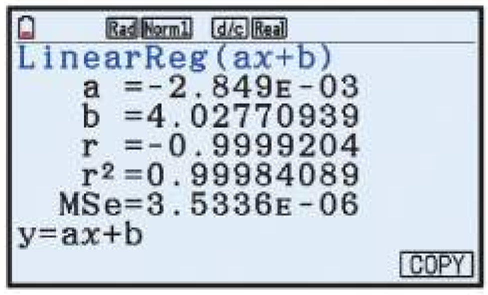
iv



$r^2 \approx 0.983$

b All the graphs in **a** appear to be approximately linear. The graph of $\ln R$ against t has the highest value of r^2, so an exponential model is the most appropriate.

c Using technology, the linear model connecting $\ln R$ and t is $\ln R \approx -0.002\,85t + 4.03$



$\therefore\ R \approx e^{-0.002\,85t+4.03}$

$\therefore\ R \approx e^{-0.002\,85t} \times e^{4.03}$

$\therefore\ R \approx e^{4.03} \times (e^{-0.002\,85})^t$

$\therefore\ R \approx 56.1 \times 0.997^t$ is the exponential model connecting R and t.

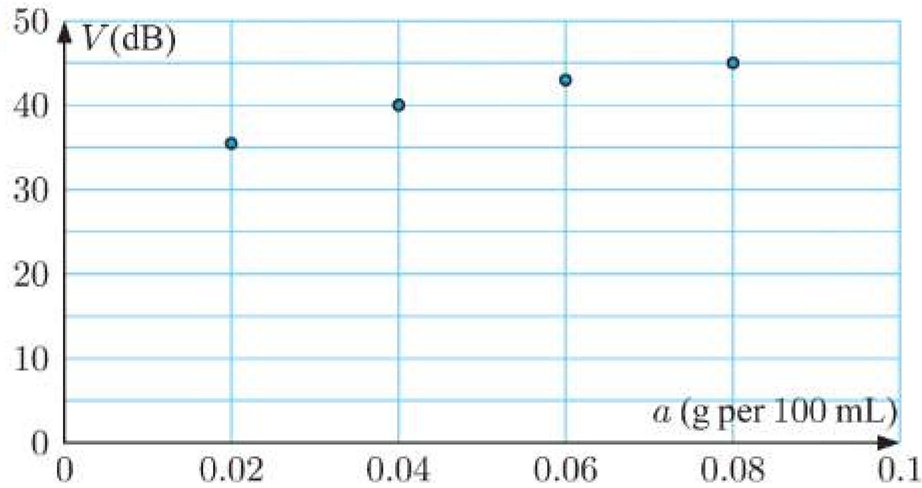
d When $t = 160$, $R \approx 56.1 \times 0.997^{160} \approx 35.6$

So after 160 seconds the rate of reaction will be about 35.6 units. This estimate is an interpolation, and the value of r^2 is very close to 1, so it is likely to be reliable.

4

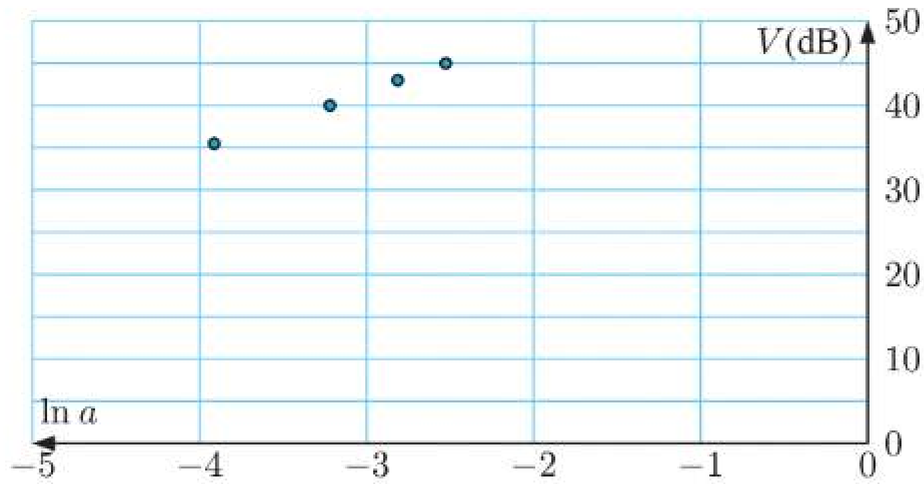
Alcohol reading (a g per 100 mL)	0.02	0.04	0.06	0.08
Volume (V dB)	35.5	40	43	45

a **i**



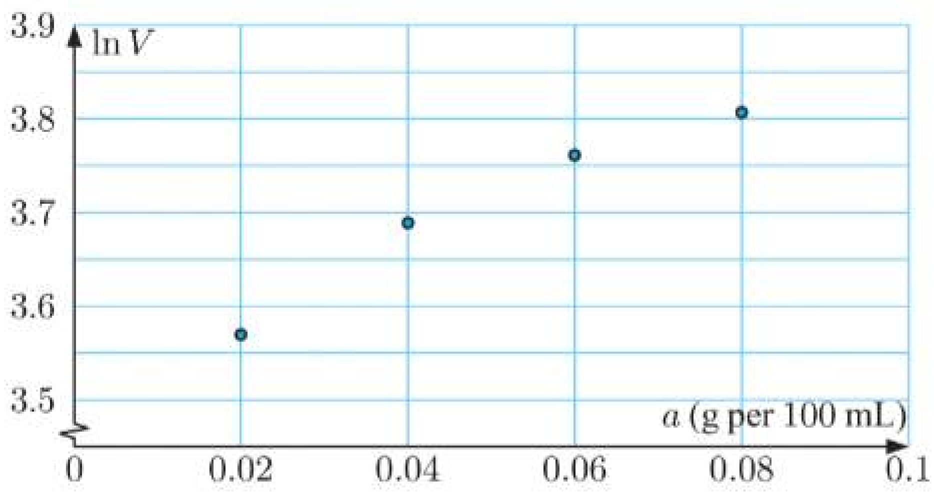
$r^2 \approx 0.969$

ii



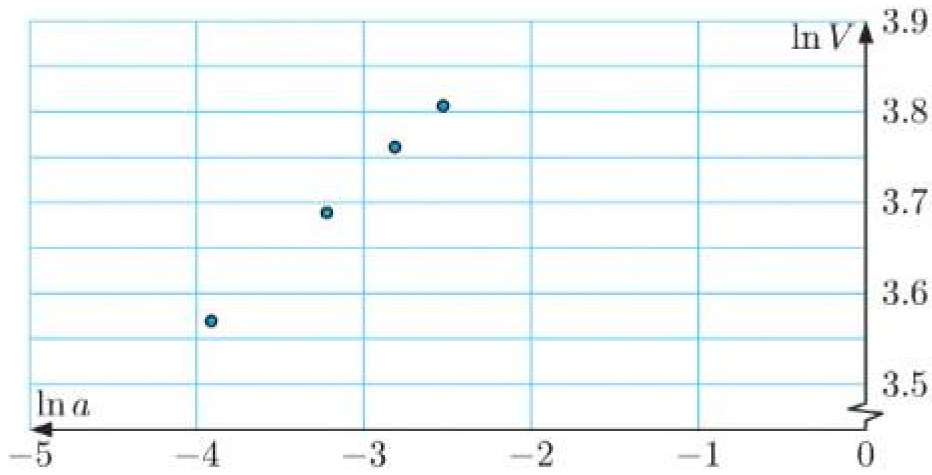
$r^2 \approx 0.9991$

iii



$r^2 \approx 0.957$

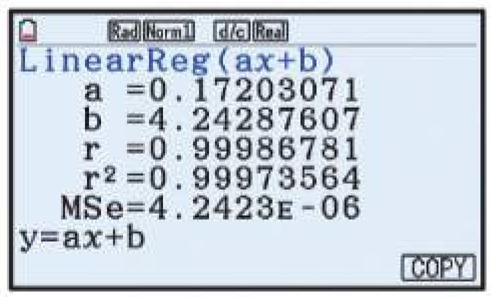
iv



$r^2 \approx 0.9997$

b The graphs of V against $\ln a$ and $\ln V$ against $\ln a$ both appear to be linear, but the graph of $\ln V$ against $\ln a$ has the highest r^2 value, so a power model is the most appropriate.

c Using technology, the linear model connecting $\ln V$ and $\ln a$ is $\ln V \approx 0.172 \ln a + 4.24$



$\therefore \ln V \approx \ln(a^{0.172}) + \ln(e^{4.24})$

$\therefore \ln V \approx \ln(a^{0.172} \times e^{4.24})$

$\therefore V \approx e^{4.24} \times a^{0.172}$

$\therefore V \approx 69.6 \times a^{0.172}$ is the power model connecting V and a.

d **i** When $a = 0.15$, $V \approx 69.6 \times 0.15^{0.172} \approx 50.2$

An alarm volume of about 50.2 dB is required to wake a person with an alcohol reading of 0.15 g per 100 mL.

ii When $a = 0.05$, $V \approx 69.6 \times 0.05^{0.172} \approx 41.6$

An alarm volume of about 41.6 dB is required to wake a person with an alcohol reading of 0.05 g per 100 mL.

5

Mass (m kg)	10	20	30	40	50	60	70	80
Acceleration (a m s^{-2})	12.4	6.3	4.2	3.1	2.5	2.1	1.8	1.5

a Using technology, we draw the scatter plot and line of best fit of:

a against m

a against $\ln m$

$\ln a$ against m

$\ln a$ against $\ln m$

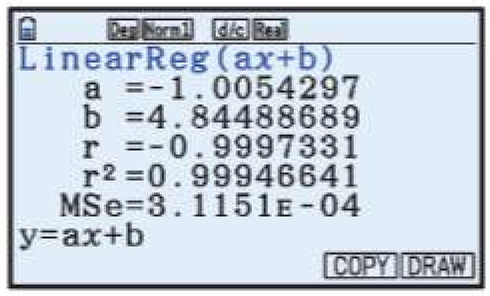
The only graph which appears to be linear is $\ln a$ against $\ln m$, so a power model is most appropriate.

Using technology, the linear model connecting $\ln a$ and $\ln m$ is $\ln a \approx -1.01 \ln m + 4.84$

$\therefore \ln a \approx \ln(m^{-1.01}) + \ln(e^{4.84})$

$\therefore \ln a \approx \ln(e^{4.84} \times m^{-1.01})$

$\therefore a \approx 127m^{-1.01}$



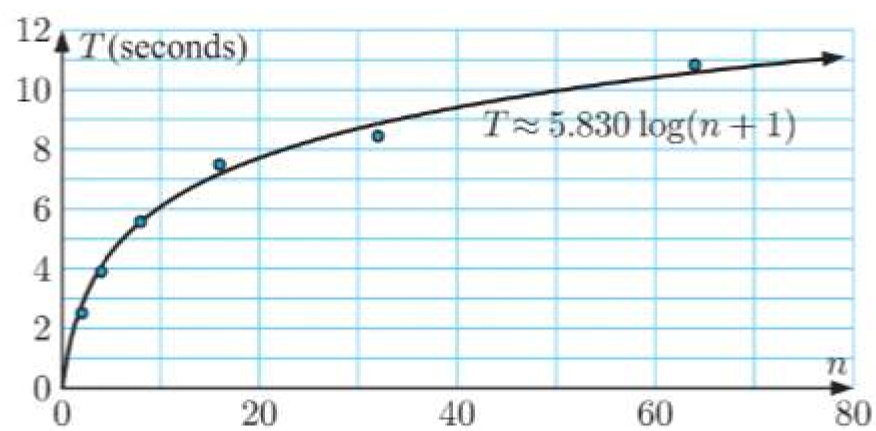
b From Newton's second law: $F = ma$

$$\therefore 125 = ma$$

$$\therefore a = \frac{125}{m} = 125m^{-1}$$

The coefficients for the power model in **a** are quite close to the theoretical model. So the model found in **a** fits the theoretical model reasonably well.

6 **a**



b

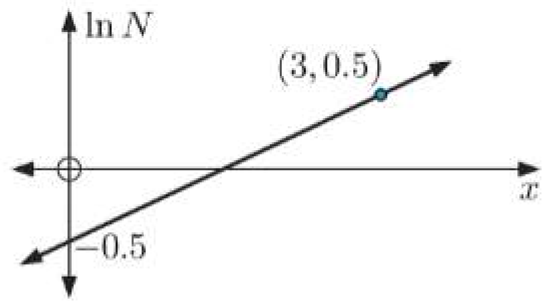
n (choices)	T (seconds)	$5.830\log(n+1)$	*Residual*
2	2.51	$\approx 2.781\,62$	≈ -0.272
4	3.90	$\approx 4.075\,00$	≈ -0.175
8	5.57	$\approx 5.563\,23$	$\approx 0.006\,77$
16	7.49	$\approx 7.173\,52$	≈ 0.316
32	8.44	$\approx 8.852\,94$	≈ -0.413
64	10.83	$\approx 10.569\,28$	≈ 0.261

c $SS_{\text{res}} \approx (-0.272)^2 + (-0.175)^2 + (0.006\,77)^2 + (0.316)^2 + (-0.413)^2 + (0.261)^2$
≈ 0.443

d From **a** we see that the model fits the data well, which is confirmed by the small value of SS_{res} calculated in **c**.

REVIEW SET 8B

1 **a** The graph of $\ln N$ against x is linear with gradient $m = \dfrac{0.5-(-0.5)}{3-0} = \dfrac{1}{3}$ and vertical axis intercept $c = -0.5$.



$\therefore\ \ln N = \frac{1}{3}x - \frac{1}{2}$

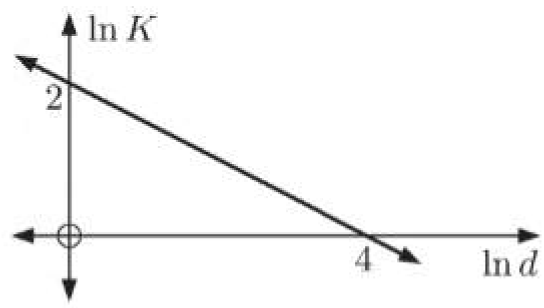
$\therefore\ N = e^{\frac{1}{3}x - \frac{1}{2}}$

$\therefore\ N = e^{\frac{1}{3}x} \times e^{-\frac{1}{2}}$

$\therefore\ N = e^{-\frac{1}{2}} \times (e^{\frac{1}{3}})^x$

$\therefore\ N \approx 0.607 \times 1.40^x$

b The graph of $\ln K$ against $\ln d$ is linear with gradient $m = \dfrac{0-2}{4-0} = -\dfrac{1}{2}$ and vertical axis intercept $c = 2$.



$\therefore\ \ln K = -\frac{1}{2}\ln d + 2$

$\therefore\ \ln K = \ln(d^{-\frac{1}{2}}) + \ln(e^2)$

$\therefore\ \ln K = \ln(d^{-\frac{1}{2}} \times e^2)$

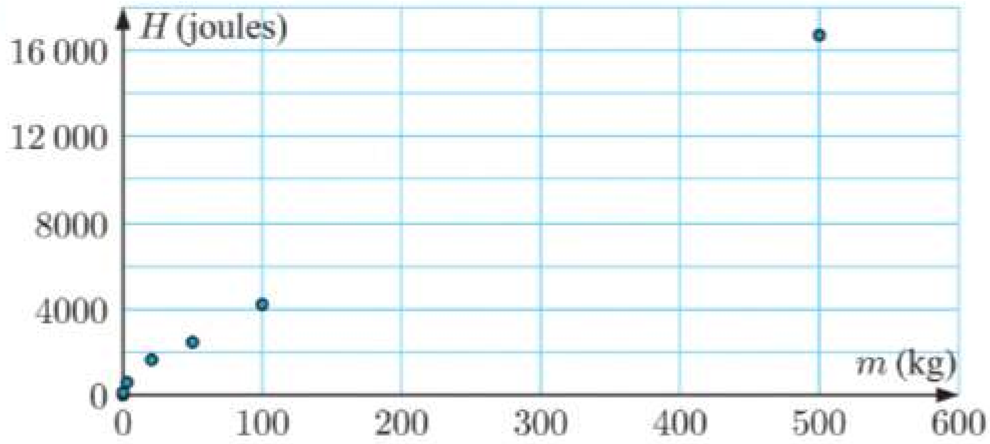
$\therefore\ K = \dfrac{e^2}{\sqrt{d}}$

$\therefore\ K \approx 7.39 \times d^{-0.5}$

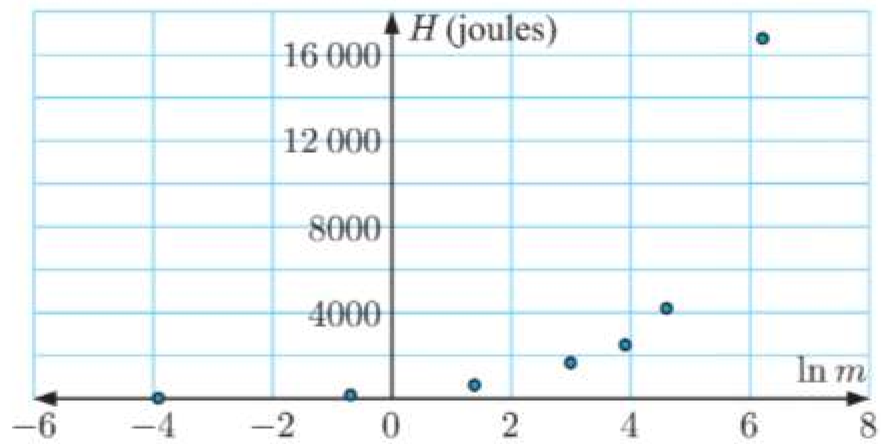
2

Mass (m kg)	0.02	0.5	4	20	50	100	500
Heat produced (H joules)	25	170	630	1670	2500	4200	16 750

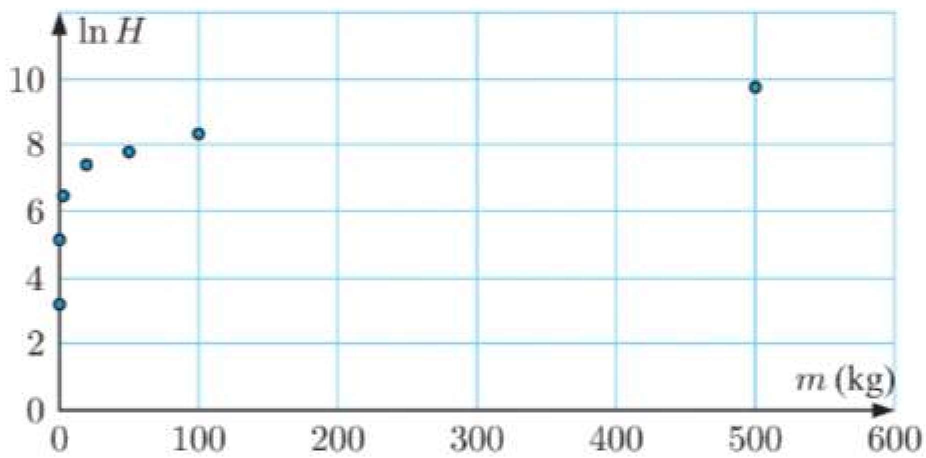
a **i**



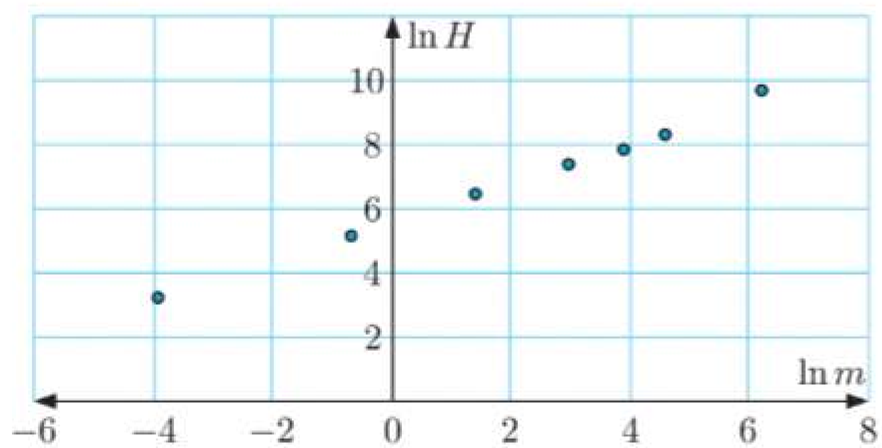
ii



iii

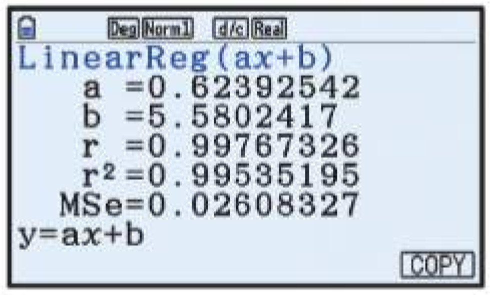


iv



b The only graph in **a** which appears to be linear is the graph of $\ln H$ against $\ln m$, so a power model is most suitable.

c Using technology, the linear model connecting $\ln H$ and $\ln m$ is $\ln H \approx 0.624 \ln m + 5.58$, and $r^2 \approx 0.995$.



d Using **c**, $\ln H \approx \ln(m^{0.624}) + \ln(e^{5.58})$

$\therefore \ln H \approx \ln(m^{0.624} \times e^{5.58})$

$\therefore H \approx e^{5.58} \times m^{0.624}$

$\therefore H \approx 265 \times m^{0.624}$

e 5 tonnes $\equiv$ 5000 kg

When $m = 5000$, $H \approx 265 \times 5000^{0.624} \approx 53\,900$

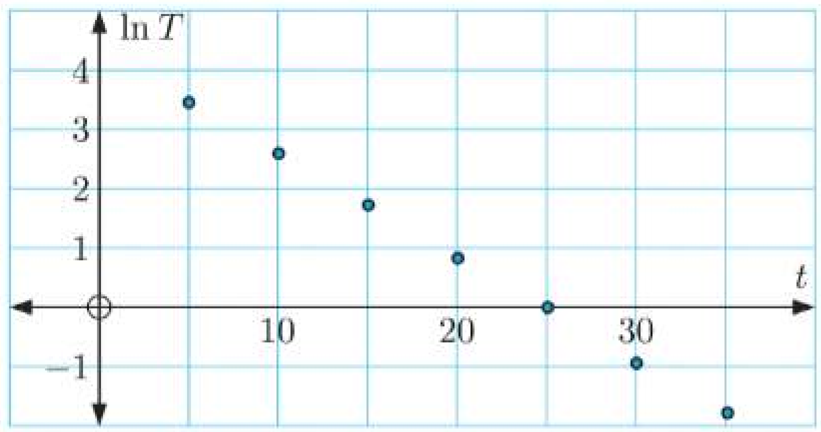
So, an elephant weighing 5 tonnes produces about 53 900 joules of heat.

This estimate is an extrapolation, so it may not be reliable.

3

Time of cooling (t minutes)	5	10	15	20	25	30	35
Temperature (T °C above room temperature)	31.5	13.3	5.6	2.3	1.0	0.41	0.17

a



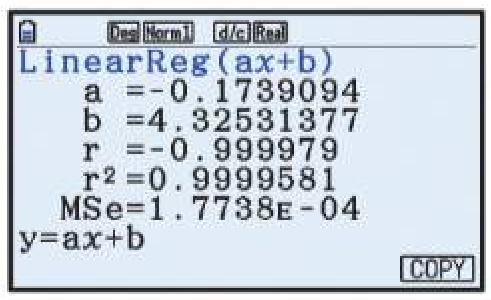
b Yes an exponential model seems appropriate as the scatter diagram of $\ln T$ against t appears to be linear.

c Using technology, the linear model connecting $\ln T$ and t is $\ln T \approx -0.174t + 4.33$

$\therefore T \approx e^{-0.174t+4.33}$

$\therefore T \approx e^{4.33} \times (e^{-0.174})^t$

$\therefore T \approx 75.6 \times 0.840^t$



When $t = 0$, $T \approx 75.6 \times 0.840^0 \approx 75.6$

The coffee was about 75.6°C above room temperature when the experiment started, so it was about $75.6°\text{C} + 25°\text{C} \approx 101°\text{C}$.

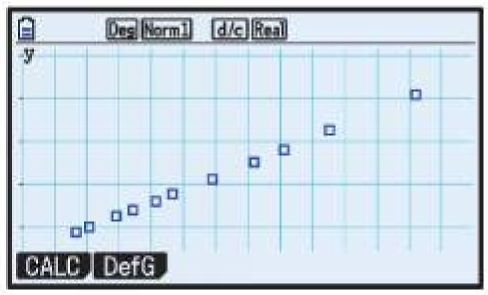
d When $t = 2$, $T \approx 75.6 \times 0.840^2 \approx 53.4$

After 2 minutes, the coffee was about $53.4°\text{C} + 25°\text{C} \approx 78.4°\text{C}$.

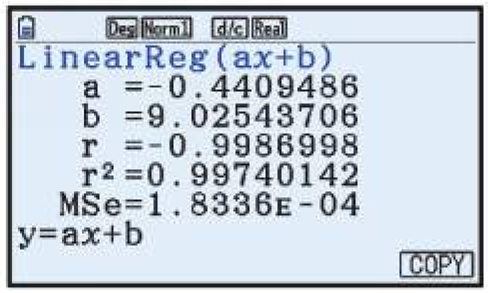
4

Partial pressure of CO_2 (P torr)	10	15	20	25	30	35	40	45	50	55	60
Blood pH (b)	8.02	7.81	7.70	7.63	7.53	7.44	7.40	7.35	7.31	7.25	7.22

a Using technology, the graph of b against $\ln P$ appears to be linear, so a logarithmic model is suitable for this data.



b Using technology, the linear model connecting b and $\ln P$ is $b \approx 9.03 - 0.441 \ln P$. This is the logarithmic model connecting b and P.



c When $b = 7.5$, $7.5 \approx 9.03 - 0.441 \ln P$

$$\therefore \frac{-1.53}{-0.441} \approx \ln P$$

$$\therefore P \approx e^{3.46}$$

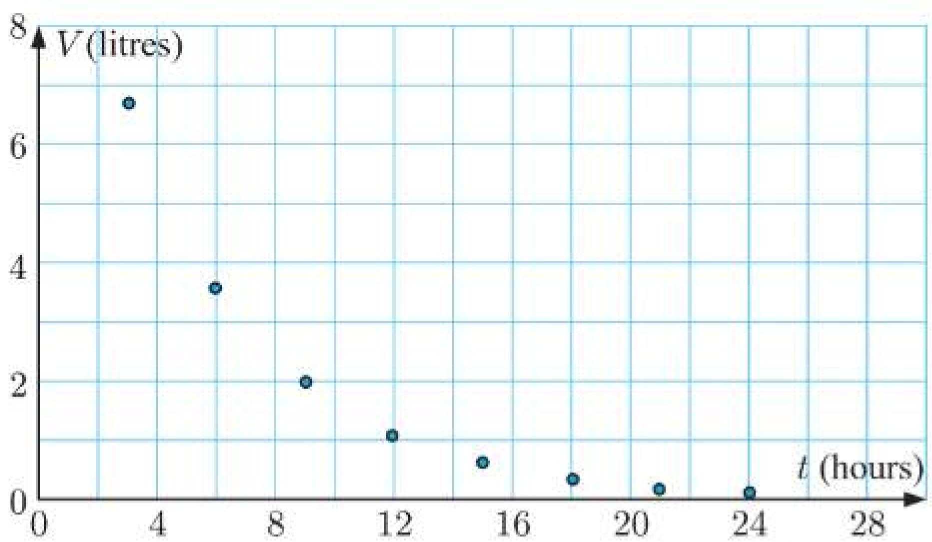
$$\therefore P \approx 31.8$$

If a person's blood has a pH of 7.5, then we estimate that their partial pressure of CO_2 is about 31.8 torr.

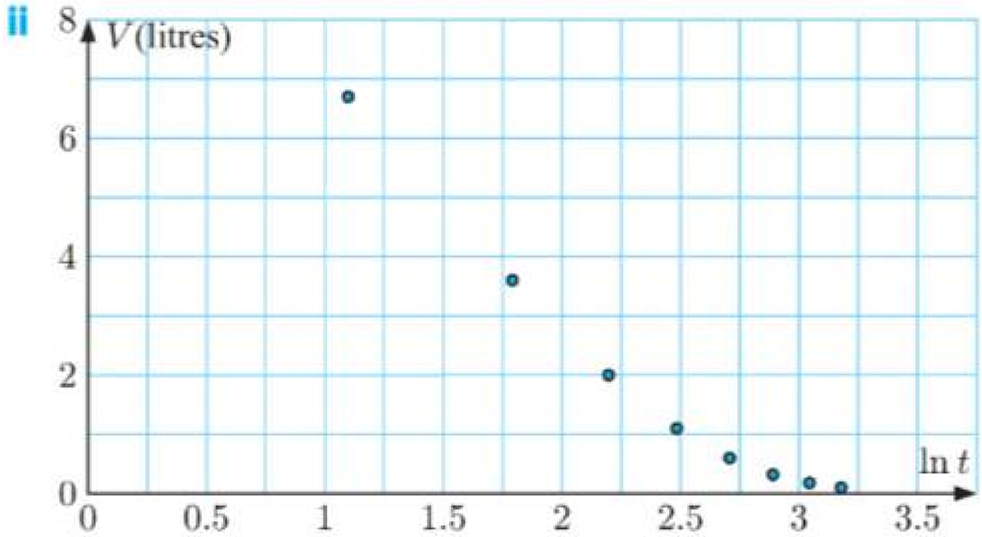
5

Time (t hours)	3	6	9	12	15	18	21	24
Water remaining (V litres)	6.7	3.6	2	1.1	0.6	0.32	0.18	0.10

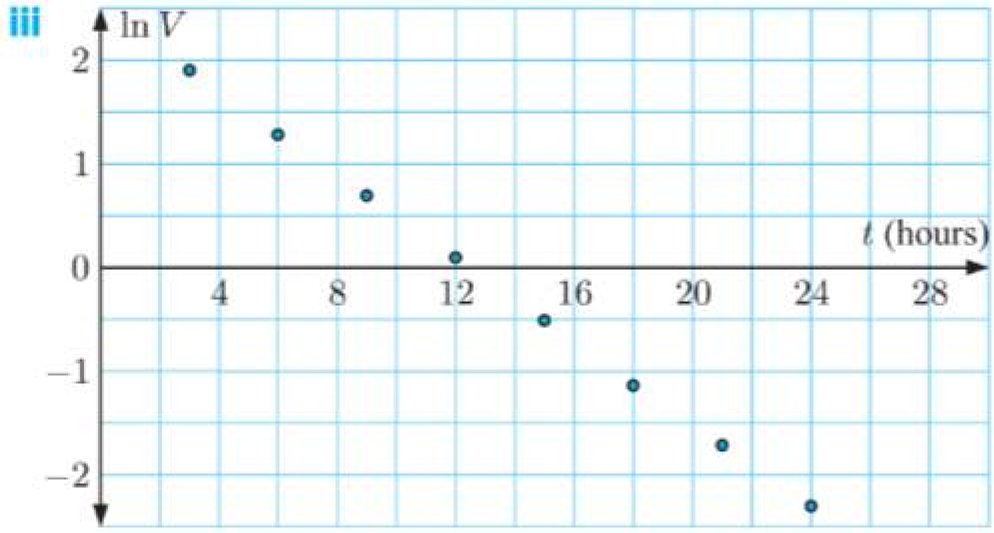
a **i**



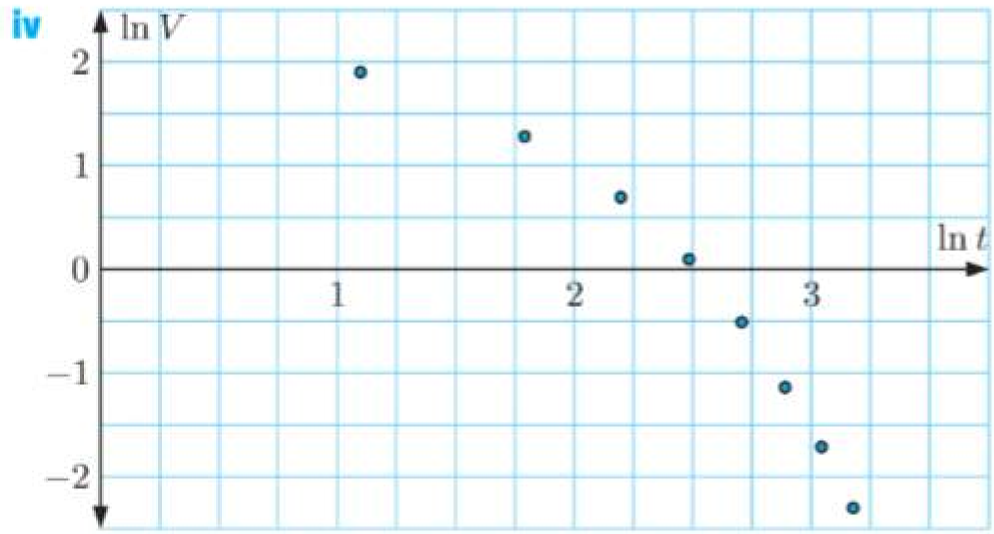
ii



iii



iv



b The only graph in **a** which appears to be linear is $\ln V$ against t, so an exponential model is most appropriate.

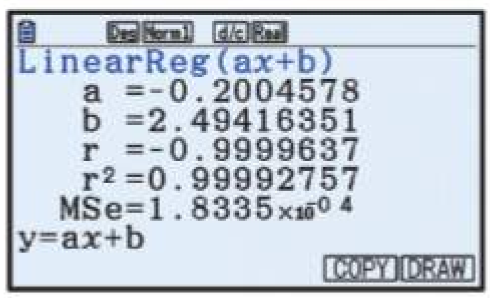
Using technology, the linear model connecting $\ln V$ and t

is $\ln V \approx -0.200t + 2.49$

$\therefore \; V \approx e^{-0.200t+2.49}$

$\therefore \; V \approx e^{2.49} \times (e^{-0.200})^t$

$\therefore \; V \approx 12.1 \times 0.818^t$ is the model connecting V and t.



c When $t = 5$, $V \approx 12.1 \times 0.818^5 \approx 4.45$

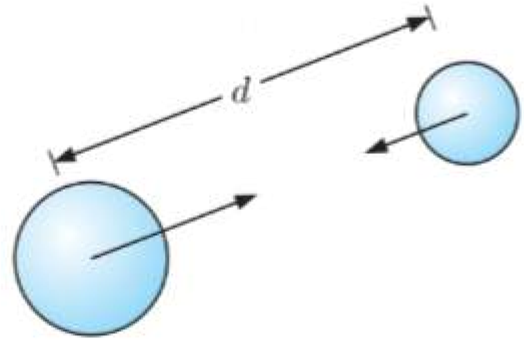
After 5 hours, there was about 4.45 L of water remaining in the bird bath.

d When $t = 0$, $V \approx 12.1 \times 0.818^0 \approx 12.1$ and when $t = 10$, $V \approx 12.1 \times 0.818^{10} \approx 1.63$

So, after 10 hours, about $12.1 - 1.63 \approx 10.5$ L of water has evaporated.

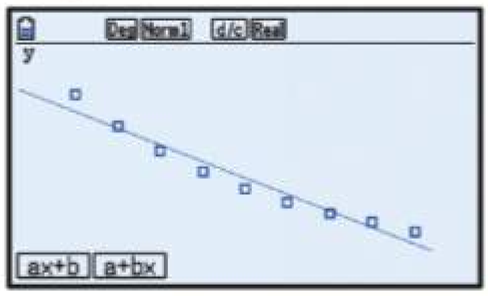
6

d (cm)	24	26	28	30	32	34	36	38	40
F (newtons)	0.237	0.202	0.174	0.152	0.133	0.118	0.105	0.095	0.085

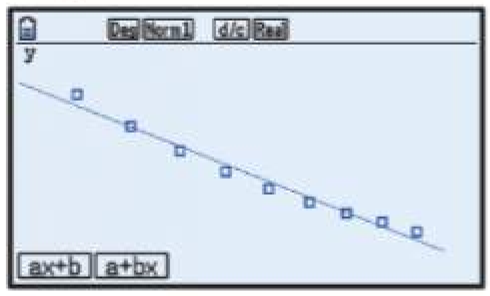


a Using technology, we draw the scatter plot and line of best fit of:

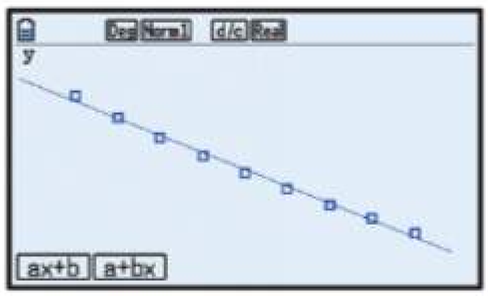
F against d



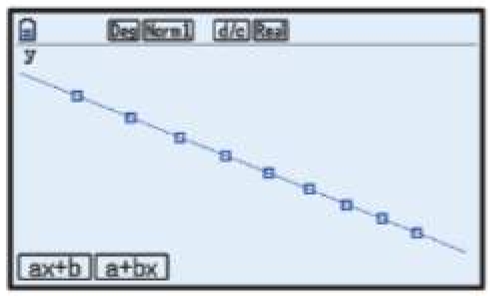
F against $\ln d$



$\ln F$ against d

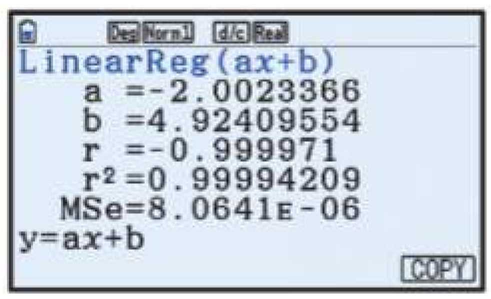


$\ln F$ against $\ln d$



The only graph which appears to be linear is the graph of $\ln F$ against $\ln d$, so a power model is most appropriate.

Using technology, the linear model connecting $\ln F$ and $\ln d$ is $\ln F \approx -2.00 \ln d + 4.92$



$\therefore \quad \ln F \approx \ln(d^{-2.00}) + \ln(e^{4.92})$

$\therefore \quad \ln F \approx \ln(d^{-2.00} \times e^{4.92})$

$\therefore \quad F \approx e^{4.92} \times d^{-2.00}$

$\therefore \quad F \approx 138 \times d^{-2.00}$ is the power model which best fits the data.

b **i** When $d = 29$, $F \approx 138 \times 29^{-2.00} \approx 0.162$

When the two spheres are 29 cm apart, the gravitational force between them is about 0.162 newtons.

ii When $d = 20$, $F \approx 138 \times 20^{-2.00} \approx 0.342$

When the two spheres are 20 cm apart, the gravitational force between them is about 0.342 newtons.

iii $50 \text{ m} \equiv 5000 \text{ cm}$

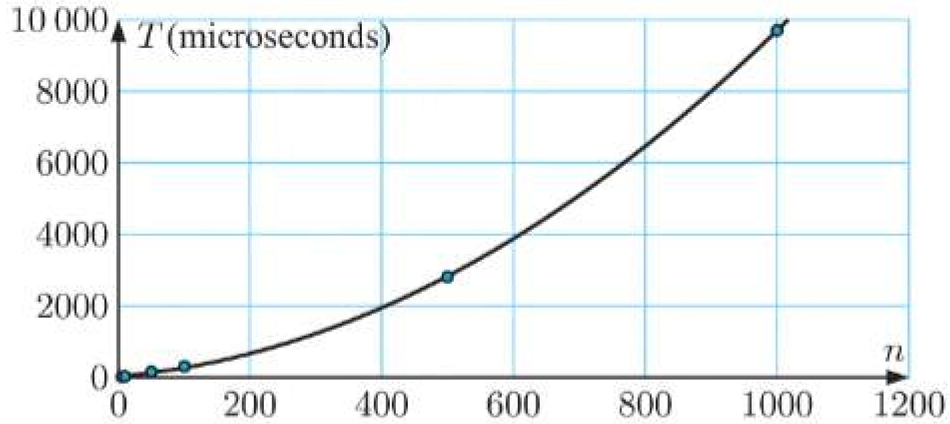
When $d = 5000$, $F \approx 138 \times 5000^{-2.00} \approx 5.39 \times 10^{-6}$

When the two spheres are 50 m apart, the gravitational force between them is about 5.39×10^{-6} newtons.

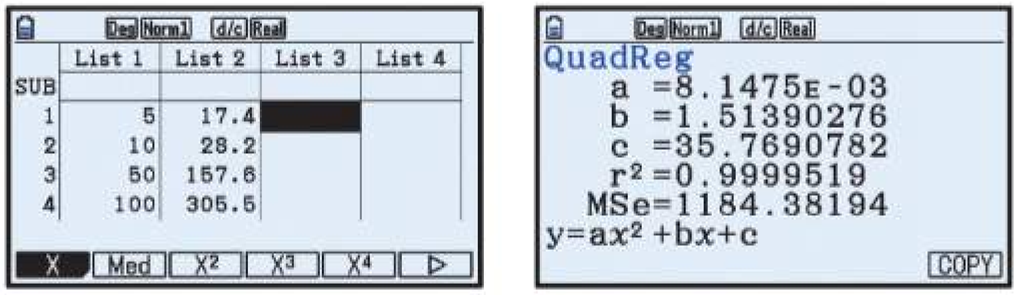
7

n (items)	5	10	50	100	500	1000
T (microseconds)	17.4	28.2	157.6	305.5	2812.9	9701.0

a

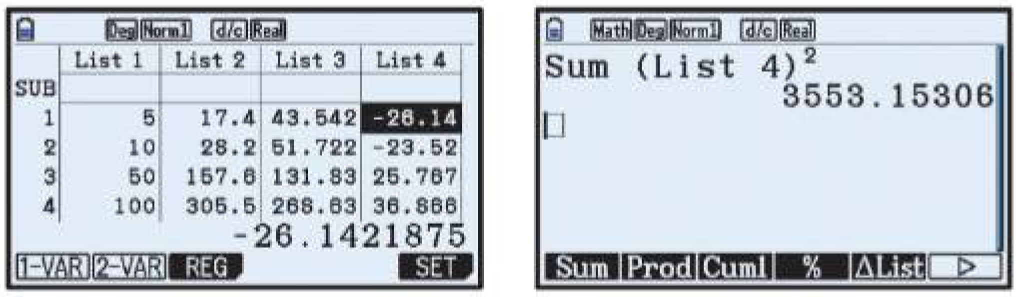


b



Using technology, the quadratic model that best fits the data is $T \approx 0.008\,15n^2 + 1.51n + 35.8$.

c



Using technology, $SS_{\text{res}} \approx 3550$

d When $n = 10\,000$, $T \approx 0.008\,15(10\,000)^2 + 1.5(10\,000) + 35.8$
$\approx 830\,000$

So, it would take the selection sort algorithm about 830 000 microseconds or about 0.830 seconds to sort 10 000 items.

e The prediction in **d** is not very reliable because:

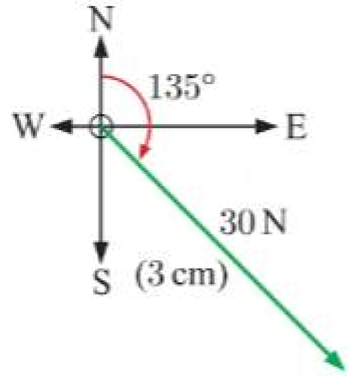
- it is an extrapolation far beyond the upper pole
- the value of SS_{res} is very large, suggesting a poor fit.

Chapter 9

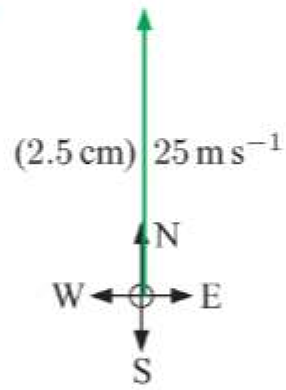
VECTORS

EXERCISE 9A.1

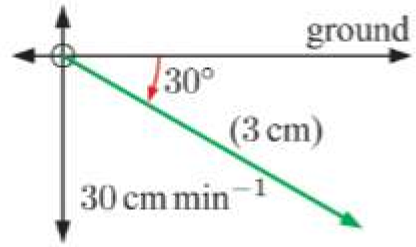
1 **a**



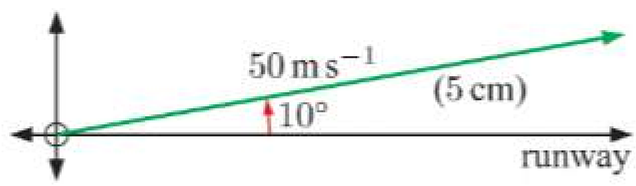
b



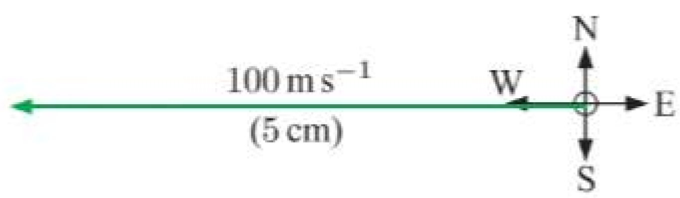
c



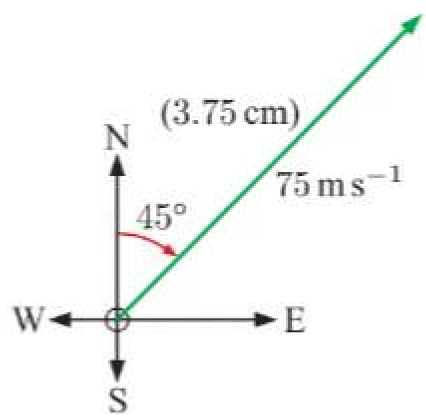
d



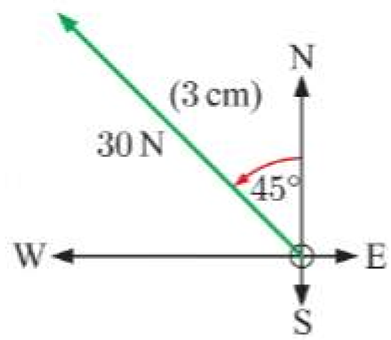
2 **a**



b

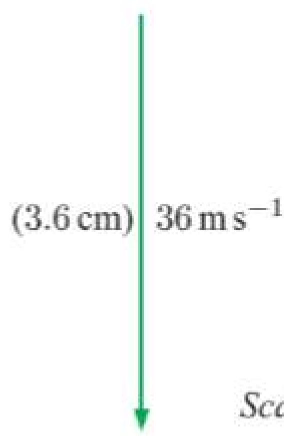


3 **a**



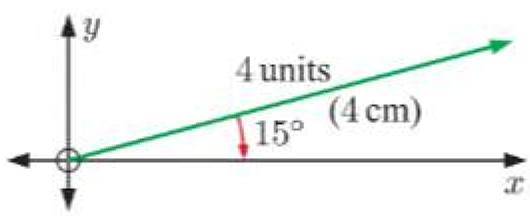
Scale: 1 cm $\equiv$ 10 N

b



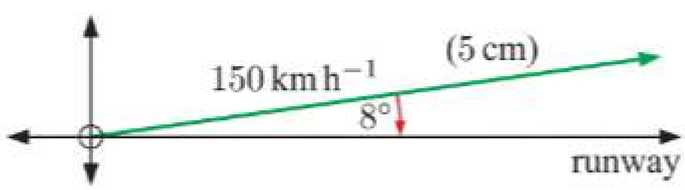
Scale: 1 cm $\equiv$ 10 m s^{-1}

c



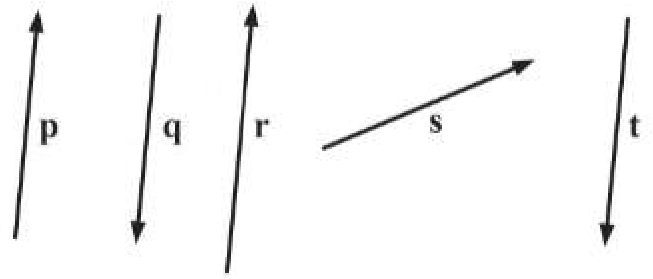
Scale: 1 cm $\equiv$ 1 unit

d



Scale: 1 cm $\equiv$ 30 km h^{-1}

EXERCISE 9A.2

1 

a If they are equal in magnitude, they have the same length. These are **p**, **q**, **s**, and **t**.

b Those parallel are **p**, **q**, **r**, and **t**.

c Those in the same direction are: **p** and **r**, **q** and **t**.

d To be equal they must have the same direction and be equal in length $\therefore$ $\mathbf{q} = \mathbf{t}$.

e **p** and **q** are negatives (equal length, but opposite direction). Likewise, **p** and **t** are negatives. We write $\mathbf{p} = -\mathbf{q}$ and $\mathbf{p} = -\mathbf{t}$.

2 a True, as they have the same length and direction.

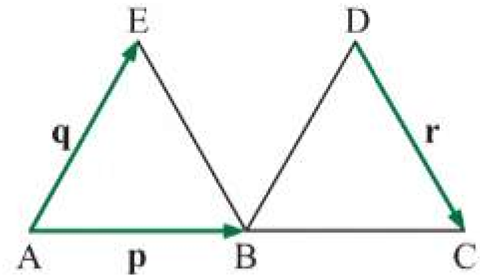
b True, as they are sides of an equilateral triangle.

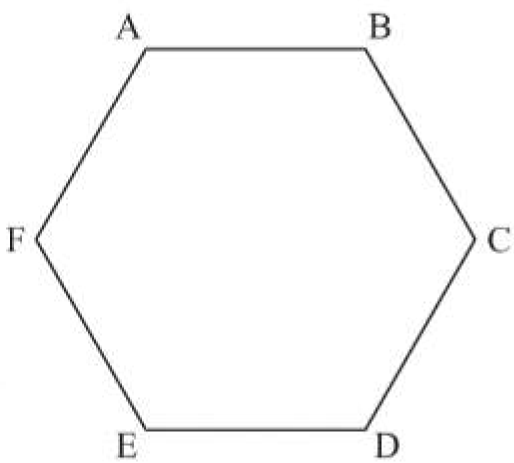
c False, as they do not have the same direction.

d False, as they have opposite directions.

e True, as they have the same length and direction.

f False, as they do not have the same direction.



3 

a i $\overrightarrow{BC}$ is the vector which originates at B and terminates at C.

ii $\overrightarrow{ED} = \overrightarrow{AB}$, as they have the same length and direction.

b i $\overrightarrow{FE}$ and $\overrightarrow{BC}$ are negatives of $\overrightarrow{EF}$, as they both have the same length but opposite direction.

ii All sides of the hexagon are equal in length

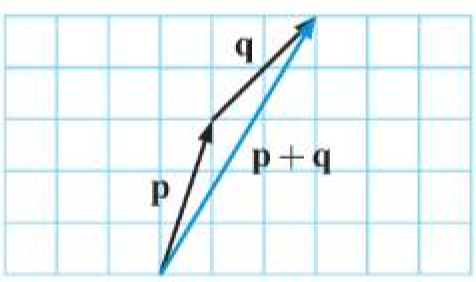
$\therefore$ the vectors with the same length as $\overrightarrow{ED}$ are $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FE}$, $\overrightarrow{FA}$, $\overrightarrow{AF}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, and $\overrightarrow{DC}$.

c The vector $\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and twice its length.

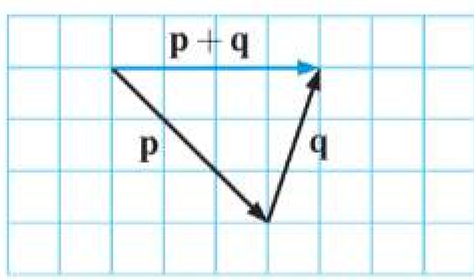
$\overrightarrow{CF}$ is also parallel to $\overrightarrow{AB}$ and twice its length (but in the opposite direction).

EXERCISE 9B.1

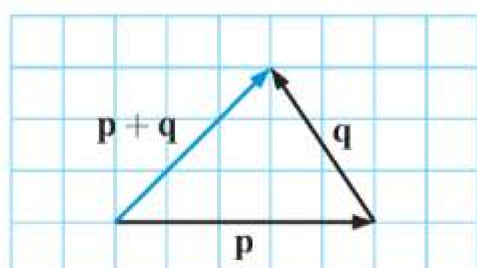
1 a



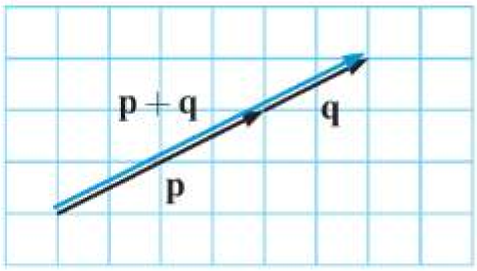
b



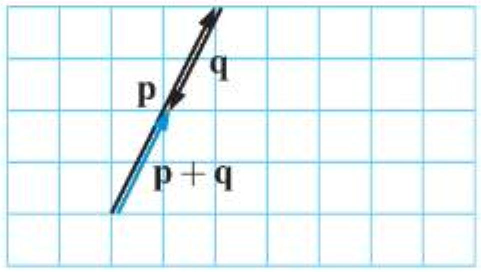
c



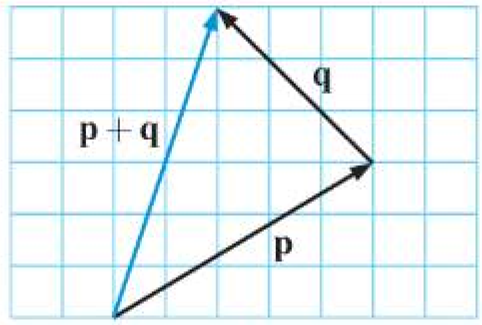
d



e



f



2 **a** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

b $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

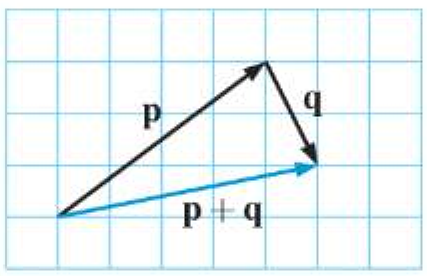
c $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$
$= \mathbf{0}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

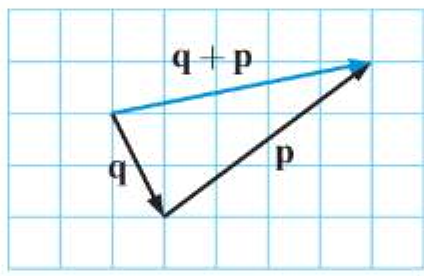
e $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$
$= \overrightarrow{AB} + \overrightarrow{BD}$
$= \overrightarrow{AD}$

f $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$
$= \overrightarrow{BA} + \overrightarrow{AB}$
$= \overrightarrow{BB}$
$= \mathbf{0}$

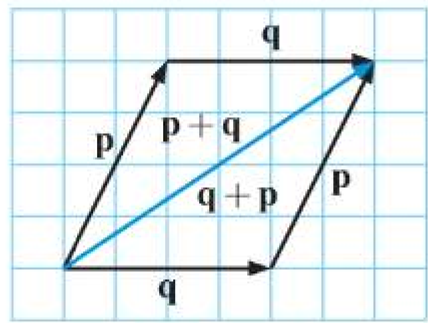
3 **a** **i**



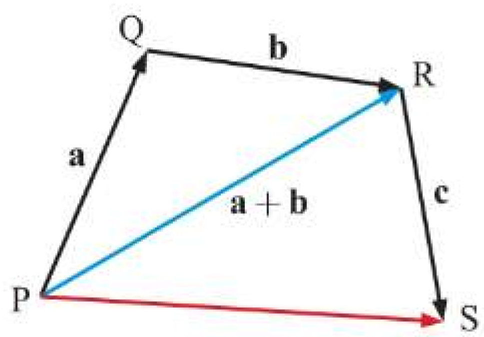
ii

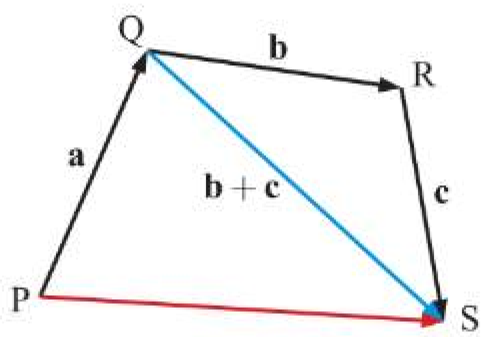


b yes



4



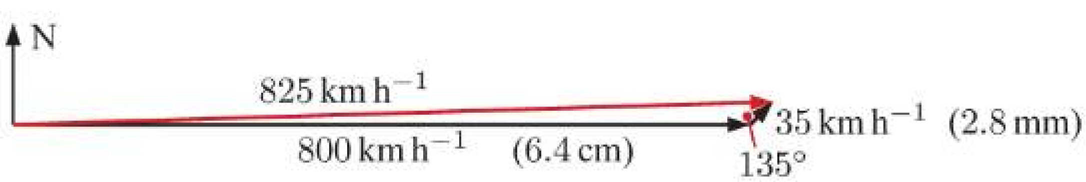


$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$
$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

But $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$
$= \mathbf{a} + (\mathbf{b} + \mathbf{c})$

$\therefore \quad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

5 **a**



Scale: 1 cm ≡ 125 km h^{-1}

b We need to perform vector addition to find the effect of the wind on the aeroplane.

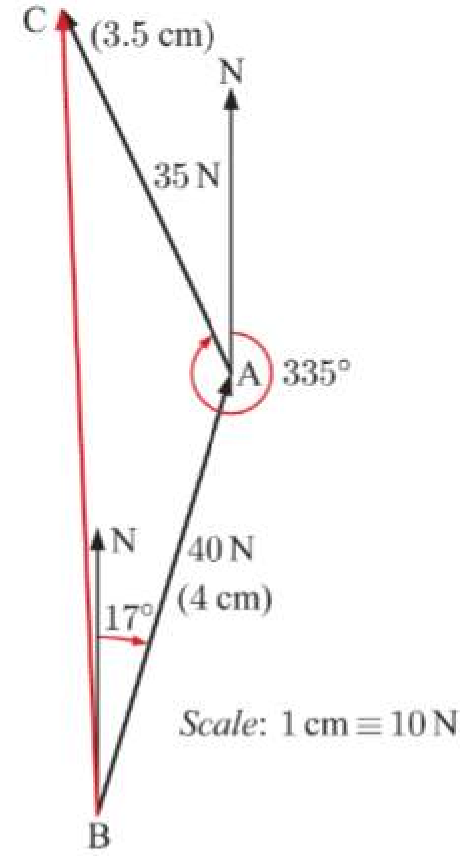
c Measuring the length of the resulting vector, we get 66 mm, or 6.6 cm.

$\therefore$ the resulting speed of the plane is $6.6 \times 125 = 825$ km h^{-1}.

Using a protractor to measure the angle between 'true north' and the resulting vector, we get 88°.

$\therefore$ the direction of the aeroplane is 88° east of north.

6 **a**



Measuring the resulting vector, we get ≈ 7 cm.

$\therefore$ the resultant force is $\approx 7 \times 10$ N
≈ 70 N

Using a protractor to measure the angle between 'true north' and the resulting vector, we get $\approx 357°$.

$\therefore$ the direction of the force is $\approx 357°$ or $3°$ west of north.

b Using the cosine rule,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 35^2 + 40^2 - 2 \times 35 \times 40 \times \cos 138° \\ &\approx 2825 - (-2080) \\ a &\approx \sqrt{4905} \\ &\approx 70.0 \text{ N} \end{aligned}$$

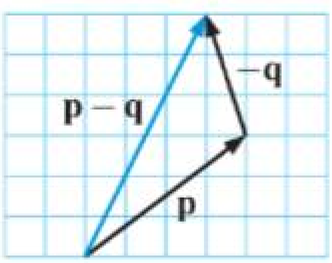
$$\begin{aligned} \cos B &= \frac{c^2 + a^2 - b^2}{2ac} \\ &\approx \frac{40^2 + 70^2 - 35^2}{2 \times 70 \times 40} \\ &\approx 0.942 \\ B &\approx 19.6° \end{aligned}$$

and $19.6° - 17° = 2.6°$

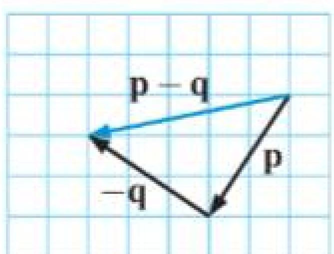
$\therefore$ the resultant force is ≈ 70.0 N in the direction $\approx 357°$ or $\approx 3°$ west of north.

EXERCISE 9B.2

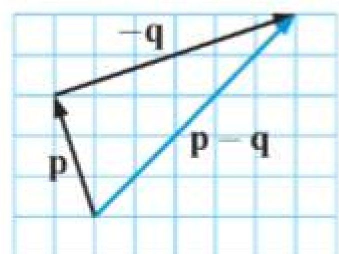
1 **a**



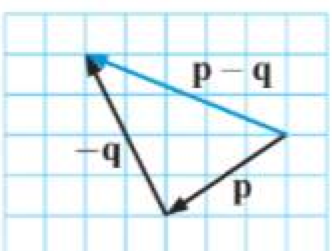
b



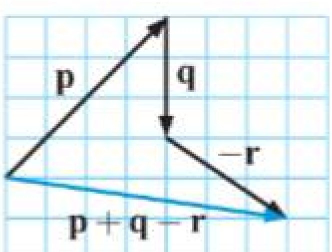
c



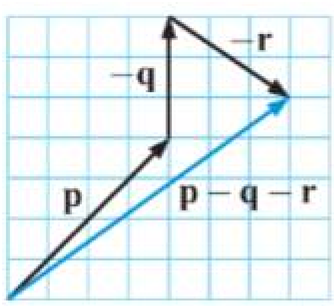
d



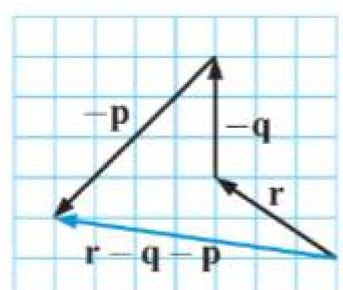
2 **a**



b



c



3 **a** $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

b $\overrightarrow{AD} - \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{DB}$
$= \overrightarrow{AB}$

c $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA}$
$= \mathbf{0}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

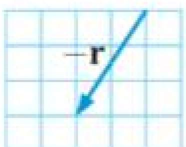
e $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$
$= \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CB}$
$= \overrightarrow{BC} + \overrightarrow{CB}$
$= \overrightarrow{BB}$
$= \mathbf{0}$

f $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$
$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

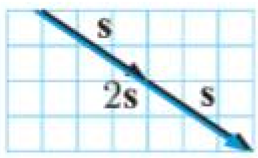
EXERCISE 9B.3

1

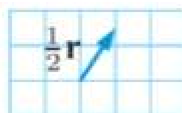
a



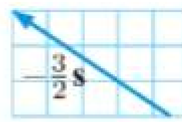
b



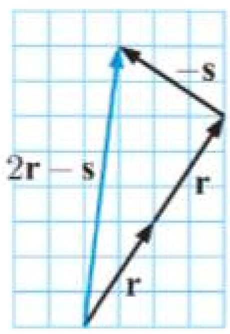
c



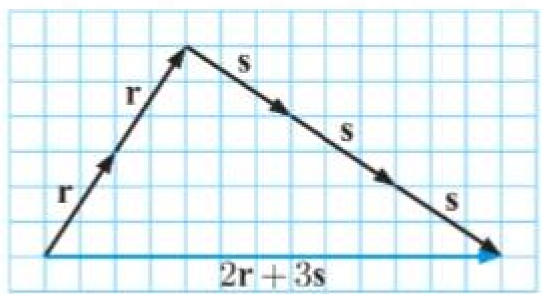
d



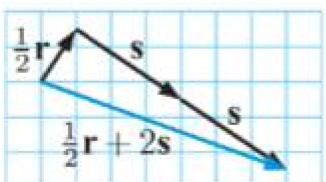
e



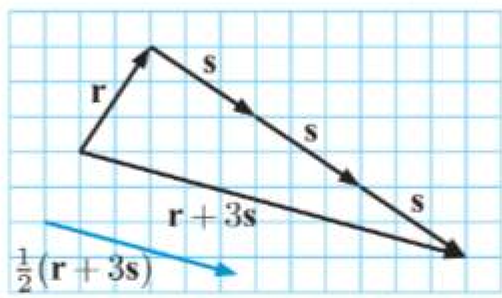
f



g



h



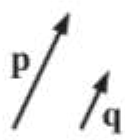
2 **a**

$\mathbf{p} = \mathbf{q}$

b

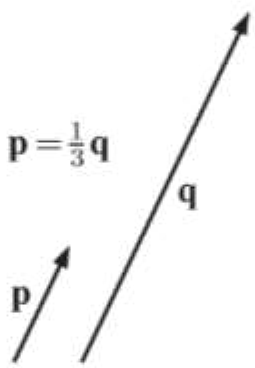
$\mathbf{p} = -\mathbf{q}$

c

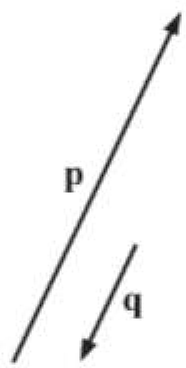


$\mathbf{p} = 2\mathbf{q}$

d $\mathbf{p} = \frac{1}{3}\mathbf{q}$

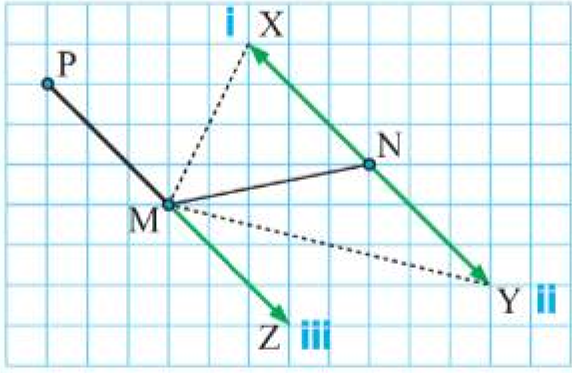


e



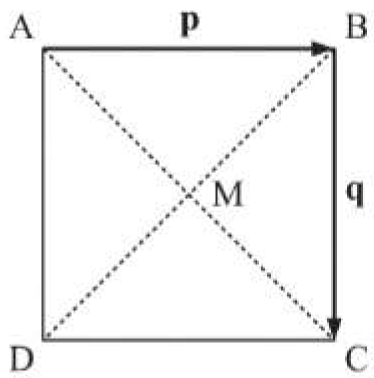
$\mathbf{p} = -3\mathbf{q}$

3 **a**



b MNYZ is a parallelogram.

4



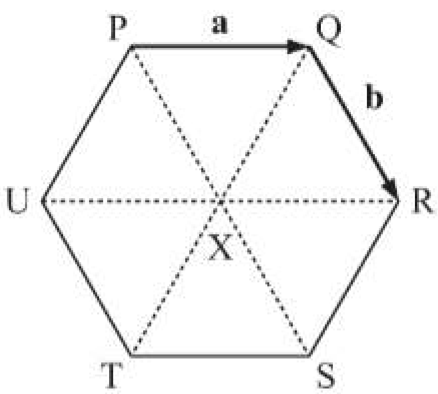
a $\overrightarrow{CD} = -\overrightarrow{AB}$
$= -\mathbf{p}$

b $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= \mathbf{p} + \mathbf{q}$

c $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$
$= \frac{1}{2}(\mathbf{p} + \mathbf{q})$
{using **b**}

d $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
$= \mathbf{q} + (-\mathbf{p})$
{using **a**}
$= \mathbf{q} - \mathbf{p}$
and $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BD}$
$= \frac{1}{2}(\mathbf{q} - \mathbf{p})$

5



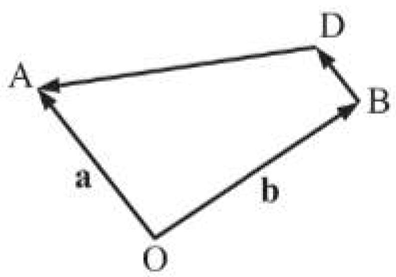
a $\overrightarrow{PX} = \overrightarrow{QR}$
$= \mathbf{b}$

b $\overrightarrow{PS} = 2\,\overrightarrow{PX}$
$= 2\mathbf{b}$
{using **a**}

c $\overrightarrow{QX} = \overrightarrow{QR} + \overrightarrow{RX}$
$= \mathbf{b} + (-\mathbf{a})$
$= \mathbf{b} - \mathbf{a}$

d $\overrightarrow{RS} = \overrightarrow{QX}$
$= \mathbf{b} - \mathbf{a}$
{using **c**}

6



a $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{OA}$
$= \frac{1}{2}\mathbf{a}$

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= (-\mathbf{a}) + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

c $\overrightarrow{BA} = -\overrightarrow{AB}$
$= -(\mathbf{b} - \mathbf{a})$
{using **b**}
$= -\mathbf{b} + \mathbf{a}$

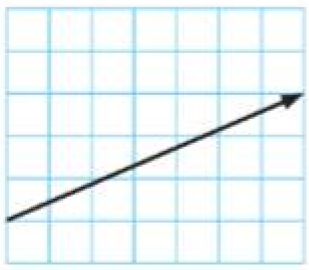
d $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$
$= \mathbf{b} + \frac{1}{2}\mathbf{a}$
{using **a**}

e $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BD}$
$= (-\mathbf{a}) + \mathbf{b} + \frac{1}{2}\mathbf{a}$
{using **a**}
$= \mathbf{b} - \frac{1}{2}\mathbf{a}$

f $\overrightarrow{DA} = -\overrightarrow{AD}$
$= -(\mathbf{b} - \frac{1}{2}\mathbf{a})$
{using **e**}
$= \frac{1}{2}\mathbf{a} - \mathbf{b}$

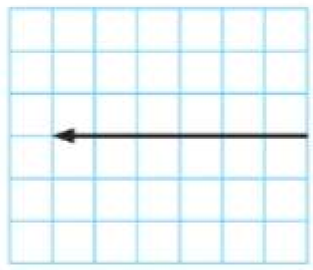
EXERCISE 9C

1 **a**



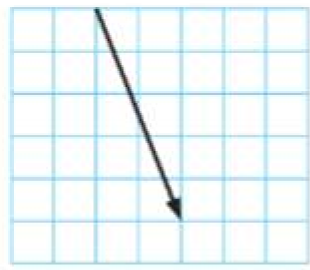
$\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 7\mathbf{i} + 3\mathbf{j}$

b



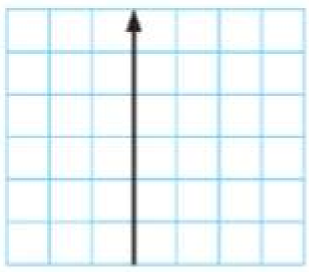
$\begin{pmatrix} -6 \\ 0 \end{pmatrix} = -6\mathbf{i}$

c



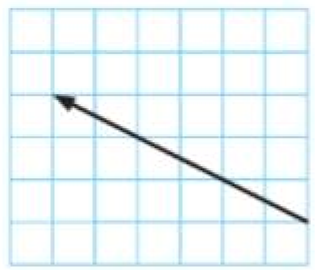
$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j}$

d



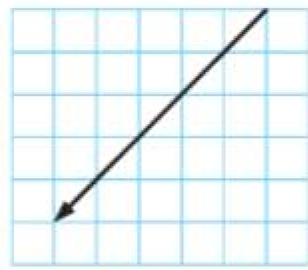
$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = 6\mathbf{j}$

e



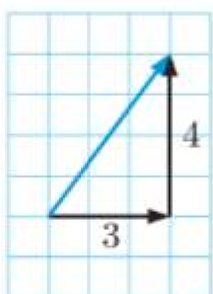
$\begin{pmatrix} -6 \\ 3 \end{pmatrix} = -6\mathbf{i} + 3\mathbf{j}$

f

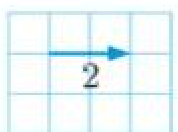


$\begin{pmatrix} -5 \\ -5 \end{pmatrix} = -5\mathbf{i} - 5\mathbf{j}$

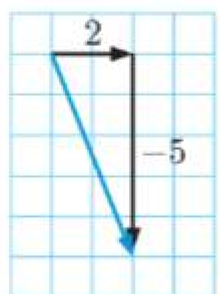
2 **a** $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j}$



b $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$

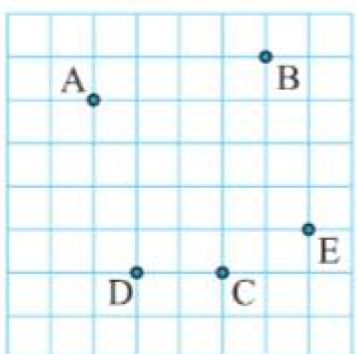


c $\begin{pmatrix} 2 \\ -5 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j}$



d $\begin{pmatrix} -1 \\ -3 \end{pmatrix} = -\mathbf{i} - 3\mathbf{j}$

3



a **i** $\overrightarrow{BA} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}$ **ii** $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -5 \end{pmatrix} = -\mathbf{i} - 5\mathbf{j}$

iii $\overrightarrow{DC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$ **iv** $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 3\mathbf{i} - 4\mathbf{j}$

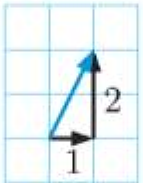
v $\overrightarrow{CA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -3\mathbf{i} + 4\mathbf{j}$ **vi** $\overrightarrow{DB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\mathbf{i} + 5\mathbf{j}$

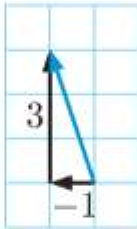
vii $\overrightarrow{AE} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} = 5\mathbf{i} - 3\mathbf{j}$ **viii** $\overrightarrow{CE} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{i} + \mathbf{j}$

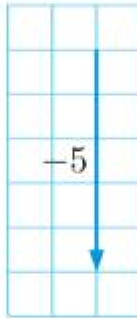
ix $\overrightarrow{ED} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}$

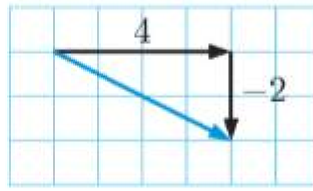
b $\overrightarrow{BA} = \overrightarrow{ED}$, $\overrightarrow{AB} = \overrightarrow{DE}$, $\overrightarrow{AD} = \overrightarrow{BE}$, $\overrightarrow{DA} = \overrightarrow{EB}$

4 **a** $\mathbf{i}+2\mathbf{j}=\begin{pmatrix}1\\2\end{pmatrix}$ **b** $-\mathbf{i}+3\mathbf{j}=\begin{pmatrix}-1\\3\end{pmatrix}$ **c** $-5\mathbf{j}=\begin{pmatrix}0\\-5\end{pmatrix}$ **d** $4\mathbf{i}-2\mathbf{j}=\begin{pmatrix}4\\-2\end{pmatrix}$









5 The zero vector $\mathbf{0}$ in component form is $\begin{pmatrix}0\\0\end{pmatrix}$.

6 **a** $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$

$\therefore\ a=3,\ b=5$

b $\begin{pmatrix}a+1\\2b-8\end{pmatrix}=\begin{pmatrix}9-a\\a\end{pmatrix}$

$\therefore\ a+1=9-a$ and $2b-8=a$

$\therefore\ 2a=8$ $\therefore\ 2b-8=4$

$\therefore\ a=4$ $\therefore\ 2b=12$

$\therefore\ b=6$

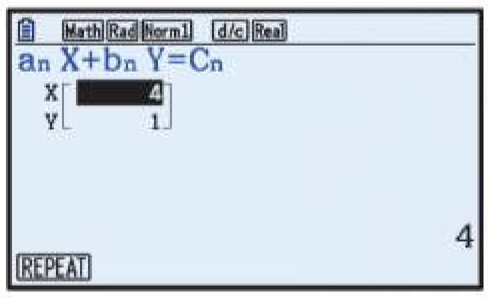
c $\begin{pmatrix}2x+3y\\x-2\end{pmatrix}=\begin{pmatrix}11\\2y\end{pmatrix}$

$\therefore\ 2x+3y=11$ .... (1) and $x-2=2y$

$\therefore\ x-2y=2$ .... (2)

Solving (1) and (2) simultaneously,

$x=4,\ y=1$



EXERCISE 9D

1 **a** $\left|\begin{pmatrix}3\\4\end{pmatrix}\right|=\sqrt{3^2+4^2}$
$=\sqrt{9+16}$
$=\sqrt{25}=5$ units

b $\left|\begin{pmatrix}-4\\3\end{pmatrix}\right|=\sqrt{(-4)^2+3^2}$
$=\sqrt{16+9}$
$=\sqrt{25}=5$ units

c $\left|\begin{pmatrix}2\\0\end{pmatrix}\right|=\sqrt{2^2+0^2}$
$=\sqrt{4}$
$=2$ units

d $\left|\begin{pmatrix}-2\\2\end{pmatrix}\right|=\sqrt{(-2)^2+2^2}$
$=\sqrt{4+4}$
$=\sqrt{8}$ units

e $\left|\begin{pmatrix}0\\-3\end{pmatrix}\right|=\sqrt{0^2+(-3)^2}$
$=\sqrt{9}$
$=3$ units

2 **a** As $\mathbf{i}+\mathbf{j}=\begin{pmatrix}1\\1\end{pmatrix}$,

$$\begin{aligned}|\mathbf{i}+\mathbf{j}| &= \sqrt{1^2+1^2}\\ &= \sqrt{2}\text{ units}\end{aligned}$$

b As $5\mathbf{i}-12\mathbf{j}=\begin{pmatrix}5\\-12\end{pmatrix}$,

$$\begin{aligned}|5\mathbf{i}-12\mathbf{j}| &= \sqrt{5^2+(-12)^2}\\ &= \sqrt{25+144}\\ &= \sqrt{169}\\ &= 13\text{ units}\end{aligned}$$

c As $-\mathbf{i}+4\mathbf{j}=\begin{pmatrix}-1\\4\end{pmatrix}$,

$$\begin{aligned}|-\mathbf{i}+4\mathbf{j}| &= \sqrt{(-1)^2+4^2}\\ &= \sqrt{1+16}\\ &= \sqrt{17}\text{ units}\end{aligned}$$

d As $3\mathbf{i}=\begin{pmatrix}3\\0\end{pmatrix}$,

$$\begin{aligned}|3\mathbf{i}| &= \sqrt{3^2+0^2}\\ &= \sqrt{9}\\ &= 3\text{ units}\end{aligned}$$

e As $k\mathbf{j}=\begin{pmatrix}0\\k\end{pmatrix}$,

$$\begin{aligned}|k\mathbf{j}| &= \sqrt{0^2+k^2}\\ &= \sqrt{k^2}\\ &= |\,k\,|\text{ units}\end{aligned}$$

3 **a** $$\begin{aligned}\left|\begin{pmatrix}0\\-1\end{pmatrix}\right| &= \sqrt{0^2+(-1)^2}\\ &= 1\end{aligned}$$

$\therefore\ \begin{pmatrix}0\\-1\end{pmatrix}$ is a unit vector.

b $$\begin{aligned}\left|\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\right| &= \sqrt{(-\tfrac{1}{\sqrt{2}})^2+(\tfrac{1}{\sqrt{2}})^2}\\ &= \sqrt{\tfrac{1}{2}+\tfrac{1}{2}}\\ &= 1\end{aligned}$$

$\therefore\ \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ is a unit vector.

c $$\begin{aligned}\left|\begin{pmatrix}\frac{2}{3}\\ \frac{1}{3}\end{pmatrix}\right| &= \sqrt{(\tfrac{2}{3})^2+(\tfrac{1}{3})^2}\\ &= \sqrt{\tfrac{4}{9}+\tfrac{1}{9}}\\ &= \tfrac{\sqrt{5}}{3}\end{aligned}$$

$\therefore\ \begin{pmatrix}\frac{2}{3}\\ \frac{1}{3}\end{pmatrix}$ is not a unit vector.

d $$\begin{aligned}\left|\begin{pmatrix}-\frac{3}{5}\\ -\frac{4}{5}\end{pmatrix}\right| &= \sqrt{(-\tfrac{3}{5})^2+(-\tfrac{4}{5})^2}\\ &= \sqrt{\tfrac{9}{25}+\tfrac{16}{25}}\\ &= 1\end{aligned}$$

$\therefore\ \begin{pmatrix}-\frac{3}{5}\\ -\frac{4}{5}\end{pmatrix}$ is a unit vector.

e $\left|\begin{pmatrix}\frac{2}{7}\\-\frac{5}{7}\end{pmatrix}\right| = \sqrt{(\frac{2}{7})^2+(-\frac{5}{7})^2}$

$= \sqrt{\frac{4}{49}+\frac{25}{49}}$

$= \frac{\sqrt{29}}{7}$

$\therefore$ $\begin{pmatrix}\frac{2}{7}\\-\frac{5}{7}\end{pmatrix}$ is not a unit vector.

4 **a** Since $\begin{pmatrix}0\\k\end{pmatrix}$ is a unit vector,

$$\left|\begin{pmatrix}0\\k\end{pmatrix}\right| = \sqrt{0^2+k^2} = 1$$
$$\therefore\ k^2 = 1$$
$$\therefore\ k = \pm 1$$

b Since $\begin{pmatrix}k\\0\end{pmatrix}$ is a unit vector,

$$\left|\begin{pmatrix}k\\0\end{pmatrix}\right| = \sqrt{k^2+0} = 1$$
$$\therefore\ k^2 = 1$$
$$\therefore\ k = \pm 1$$

c Since $\begin{pmatrix}k\\1\end{pmatrix}$ is a unit vector,

$$\left|\begin{pmatrix}k\\1\end{pmatrix}\right| = \sqrt{k^2+1} = 1$$
$$\therefore\ k^2+1 = 1$$
$$\therefore\ k^2 = 0$$
$$\therefore\ k = 0$$

d Since $\begin{pmatrix}k\\k\end{pmatrix}$ is a unit vector,

$$\left|\begin{pmatrix}k\\k\end{pmatrix}\right| = \sqrt{k^2+k^2} = 1$$
$$\therefore\ 2k^2 = 1$$
$$\therefore\ k^2 = \tfrac{1}{2}$$
$$\therefore\ k = \pm\sqrt{\tfrac{1}{2}} = \pm\tfrac{1}{\sqrt{2}}$$

e Since $\begin{pmatrix}\frac{1}{2}\\k\end{pmatrix}$ is a unit vector,

$$\left|\begin{pmatrix}\frac{1}{2}\\k\end{pmatrix}\right| = \sqrt{\left(\tfrac{1}{2}\right)^2+k^2} = 1$$
$$\therefore\ \tfrac{1}{4}+k^2 = 1$$
$$\therefore\ k^2 = \tfrac{3}{4}$$
$$\therefore\ k = \pm\sqrt{\tfrac{3}{4}} = \pm\tfrac{\sqrt{3}}{2}$$

5 If $|\mathbf{v}| = \sqrt{73}$ units, then $\sqrt{8^2+p^2} = \sqrt{73}$ $\left\{\mathbf{v} = \begin{pmatrix}8\\p\end{pmatrix}\right\}$

$$\therefore\ 64+p^2 = 73$$
$$\therefore\ p^2 = 9$$
$$\therefore\ p = \pm 3$$

EXERCISE 9E

1 **a** $\mathbf{a}+\mathbf{b} = \begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}$

$= \begin{pmatrix}-2\\6\end{pmatrix}$

b $\mathbf{b}+\mathbf{a} = \begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}$

$= \begin{pmatrix}-2\\6\end{pmatrix}$

c $\mathbf{b}+\mathbf{c}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}$
$=\begin{pmatrix}-1\\-1\end{pmatrix}$

d $\mathbf{c}+\mathbf{b}=\begin{pmatrix}-2\\-5\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}$
$=\begin{pmatrix}-1\\-1\end{pmatrix}$

e $\mathbf{a}+\mathbf{c}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}$
$=\begin{pmatrix}-5\\-3\end{pmatrix}$

f $\mathbf{c}+\mathbf{a}=\begin{pmatrix}-2\\-5\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}$
$=\begin{pmatrix}-5\\-3\end{pmatrix}$

g $\mathbf{a}+\mathbf{a}=\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}$
$=\begin{pmatrix}-6\\4\end{pmatrix}$

h $\mathbf{b}+\mathbf{a}+\mathbf{c}=\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-3\\2\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}$
$=\begin{pmatrix}-2\\6\end{pmatrix}+\begin{pmatrix}-2\\-5\end{pmatrix}$
$=\begin{pmatrix}-4\\1\end{pmatrix}$

2 **a** $\mathbf{p}-\mathbf{q}=\begin{pmatrix}-4\\2\end{pmatrix}-\begin{pmatrix}-1\\-5\end{pmatrix}$
$=\begin{pmatrix}-3\\7\end{pmatrix}$

b $\mathbf{q}-\mathbf{r}=\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}$
$=\begin{pmatrix}-4\\-3\end{pmatrix}$

c $\mathbf{p}+\mathbf{q}-\mathbf{r}=\begin{pmatrix}-4\\2\end{pmatrix}+\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}$
$=\begin{pmatrix}-8\\-1\end{pmatrix}$

d $\mathbf{p}-\mathbf{q}-\mathbf{r}=\begin{pmatrix}-4\\2\end{pmatrix}-\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}$
$=\begin{pmatrix}-6\\9\end{pmatrix}$

e $\mathbf{q}-\mathbf{r}-\mathbf{p}=\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix}$
$=\begin{pmatrix}0\\-5\end{pmatrix}$

f $\mathbf{r}+\mathbf{q}-\mathbf{p}=\begin{pmatrix}3\\-2\end{pmatrix}+\begin{pmatrix}-1\\-5\end{pmatrix}-\begin{pmatrix}-4\\2\end{pmatrix}$
$=\begin{pmatrix}6\\-9\end{pmatrix}$

3 **a** $\mathbf{a}+\mathbf{0}=\begin{pmatrix}a_1\\a_2\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$
$=\begin{pmatrix}a_1+0\\a_2+0\end{pmatrix}$
$=\begin{pmatrix}a_1\\a_2\end{pmatrix}$
$=\mathbf{a}$

b $\mathbf{a}-\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}-\begin{pmatrix}a_1\\a_2\end{pmatrix}$
$=\begin{pmatrix}a_1-a_1\\a_2-a_2\end{pmatrix}$
$=\begin{pmatrix}0\\0\end{pmatrix}$
$=\mathbf{0}$

4 **a** $-3\mathbf{p} = -3\begin{pmatrix}1\\5\end{pmatrix}$
$= \begin{pmatrix}-3\\-15\end{pmatrix}$

b $\frac{1}{2}\mathbf{q} = \frac{1}{2}\begin{pmatrix}-2\\4\end{pmatrix}$
$= \begin{pmatrix}-1\\2\end{pmatrix}$

c $2\mathbf{p} + \mathbf{q} = 2\begin{pmatrix}1\\5\end{pmatrix} + \begin{pmatrix}-2\\4\end{pmatrix}$
$= \begin{pmatrix}2\\10\end{pmatrix} + \begin{pmatrix}-2\\4\end{pmatrix}$
$= \begin{pmatrix}0\\14\end{pmatrix}$

d $\mathbf{p} - 2\mathbf{q} = \begin{pmatrix}1\\5\end{pmatrix} - 2\begin{pmatrix}-2\\4\end{pmatrix}$
$= \begin{pmatrix}1\\5\end{pmatrix} - \begin{pmatrix}-4\\8\end{pmatrix}$
$= \begin{pmatrix}5\\-3\end{pmatrix}$

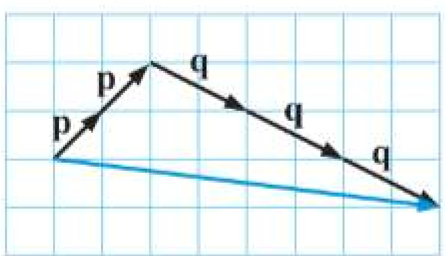
e $\mathbf{p} - \frac{1}{2}\mathbf{r} = \begin{pmatrix}1\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-3\\-1\end{pmatrix}$
$= \begin{pmatrix}1\\5\end{pmatrix} - \begin{pmatrix}-\frac{3}{2}\\-\frac{1}{2}\end{pmatrix}$
$= \begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\end{pmatrix}$

f $2\mathbf{p} + 3\mathbf{r} = 2\begin{pmatrix}1\\5\end{pmatrix} + 3\begin{pmatrix}-3\\-1\end{pmatrix}$
$= \begin{pmatrix}2\\10\end{pmatrix} + \begin{pmatrix}-9\\-3\end{pmatrix}$
$= \begin{pmatrix}-7\\7\end{pmatrix}$

g $2\mathbf{q} - 3\mathbf{r} = 2\begin{pmatrix}-2\\4\end{pmatrix} - 3\begin{pmatrix}-3\\-1\end{pmatrix}$
$= \begin{pmatrix}-4\\8\end{pmatrix} - \begin{pmatrix}-9\\-3\end{pmatrix}$
$= \begin{pmatrix}5\\11\end{pmatrix}$

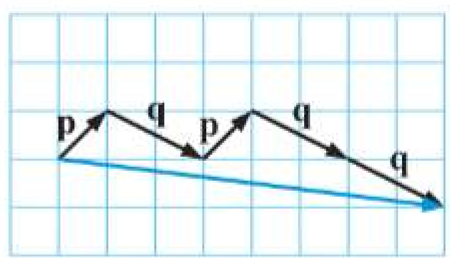
h $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$
$= 2\begin{pmatrix}1\\5\end{pmatrix} - \begin{pmatrix}-2\\4\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-3\\-1\end{pmatrix}$
$= \begin{pmatrix}2\\10\end{pmatrix} - \begin{pmatrix}-2\\4\end{pmatrix} + \begin{pmatrix}-1\\-\frac{1}{3}\end{pmatrix}$
$= \begin{pmatrix}3\\\frac{17}{3}\end{pmatrix}$

5 **a** **i**



$\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q} = \begin{pmatrix}8\\-1\end{pmatrix}$

ii



$\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q} = \begin{pmatrix}8\\-1\end{pmatrix}$

iii

$$\mathbf{q}+\mathbf{p}+\mathbf{q}+\mathbf{p}+\mathbf{q}=\begin{pmatrix}8\\-1\end{pmatrix}$$

b The vector expressions are equal, as each consists of the sum of 2 **p**s and 3 **q**s. Each expression is equal to $2\mathbf{p}+3\mathbf{q}$.

6 **a**
$$\begin{pmatrix}2\\-5\end{pmatrix}+k\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}20\\-11\end{pmatrix}$$
$$\therefore \begin{pmatrix}2+3k\\-5-k\end{pmatrix}=\begin{pmatrix}20\\-11\end{pmatrix}$$
$$\therefore\ 2+3k=20$$
$$\therefore\ 3k=18$$
$$\therefore\ k=6$$
Check: $-5-6=-11$ ✓

b
$$\tfrac{1}{3}\begin{pmatrix}-1\\k\end{pmatrix}+\tfrac{4}{3}\begin{pmatrix}7\\k+1\end{pmatrix}=\begin{pmatrix}9\\-1\end{pmatrix}$$
$$\therefore \begin{pmatrix}-\frac{1}{3}\\ \frac{1}{3}k\end{pmatrix}+\begin{pmatrix}\frac{28}{3}\\ \frac{4}{3}k+\frac{4}{3}\end{pmatrix}=\begin{pmatrix}9\\-1\end{pmatrix}$$
$$\therefore \begin{pmatrix}9\\ \frac{5}{3}k+\frac{4}{3}\end{pmatrix}=\begin{pmatrix}9\\-1\end{pmatrix}$$
$$\therefore\ \tfrac{5}{3}k+\tfrac{4}{3}=-1$$
$$\therefore\ \tfrac{5}{3}k=-\tfrac{7}{3}$$
$$\therefore\ k=-\tfrac{7}{5}$$

7 **a** $|\mathbf{r}|=\sqrt{2^2+3^2}$
$=\sqrt{13}$ units

b $|\mathbf{s}|=\sqrt{(-1)^2+4^2}$
$=\sqrt{17}$ units

c $\mathbf{r}+\mathbf{s}$
$=\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}$
$=\begin{pmatrix}1\\7\end{pmatrix}$
$\therefore\ |\mathbf{r}+\mathbf{s}|$
$=\sqrt{1^2+7^2}$
$=\sqrt{1+49}$
$=\sqrt{50}$
$=5\sqrt{2}$ units

d $\mathbf{r}-\mathbf{s}$
$=\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}-1\\4\end{pmatrix}$
$=\begin{pmatrix}3\\-1\end{pmatrix}$
$\therefore\ |\mathbf{r}-\mathbf{s}|$
$=\sqrt{3^2+(-1)^2}$
$=\sqrt{9+1}$
$=\sqrt{10}$ units

e $\mathbf{s}-2\mathbf{r}$
$=\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}4\\6\end{pmatrix}$
$=\begin{pmatrix}-5\\-2\end{pmatrix}$
$\therefore\ |\mathbf{s}-2\mathbf{r}|$
$=\sqrt{(-5)^2+(-2)^2}$
$=\sqrt{25+4}$
$=\sqrt{29}$ units

f $2\mathbf{r} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$\therefore\ |2\mathbf{r}| = \sqrt{4^2 + 6^2}$
$= \sqrt{16 + 36}$
$= \sqrt{52}$
$= 2\sqrt{13}$ units

g $\mathbf{r} + 2\mathbf{s} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 8 \end{pmatrix}$
$= \begin{pmatrix} 0 \\ 11 \end{pmatrix}$

$\therefore\ |\mathbf{r} + 2\mathbf{s}| = 11$ units

h $2\mathbf{r} - \mathbf{s} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
$= \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\therefore\ |2\mathbf{r} - \mathbf{s}| = \sqrt{5^2 + 2^2}$
$= \sqrt{25 + 4}$
$= \sqrt{29}$ units

8 a i $|\mathbf{p}| = \sqrt{1^2 + 3^2}$
$= \sqrt{1 + 9}$
$= \sqrt{10}$ units

ii $2\mathbf{p} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$\therefore\ |2\mathbf{p}| = \sqrt{2^2 + 6^2}$
$= \sqrt{4 + 36}$
$= \sqrt{40}$
$= 2\sqrt{10}$ units

iii $-2\mathbf{p} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

$\therefore\ |-2\mathbf{p}| = \sqrt{(-2)^2 + (-6)^2}$
$= \sqrt{4 + 36}$
$= \sqrt{40}$
$= 2\sqrt{10}$ units

iv $3\mathbf{p} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$

$\therefore\ |3\mathbf{p}| = \sqrt{3^2 + 9^2}$
$= \sqrt{9 + 81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

v $-3\mathbf{p} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$

$\therefore\ |-3\mathbf{p}| = \sqrt{(-3)^2 + (-9)^2}$
$= \sqrt{9 + 81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

b $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$

$\therefore\ |k\mathbf{v}| = \sqrt{(kv_1)^2 + (kv_2)^2}$
$= \sqrt{k^2{v_1}^2 + k^2{v_2}^2}$
$= \sqrt{k^2({v_1}^2 + {v_2}^2)}$
$= \sqrt{k^2}\sqrt{{v_1}^2 + {v_2}^2}$
$= |k|\sqrt{{v_1}^2 + {v_2}^2}$
$= |k|\,|\mathbf{v}|$

9 $$k\mathbf{x} = \mathbf{a}$$

$$\therefore\ k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\therefore\ kx_1 = a_1 \quad \text{and} \quad kx_2 = a_2$$

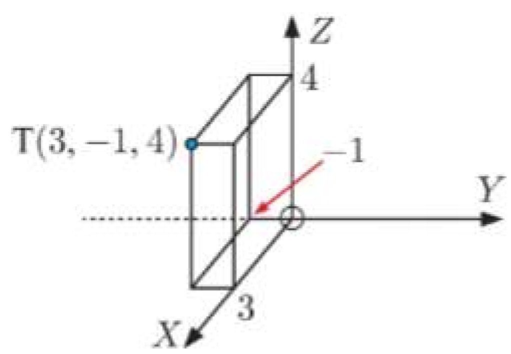
$$\therefore\ x_1 = \tfrac{1}{k}a_1 \quad \text{and} \quad x_2 = \tfrac{1}{k}a_2, \quad k \neq 0$$

$$\therefore\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

and so $\mathbf{x} = \frac{1}{k}\mathbf{a}, \quad k \neq 0$

EXERCISE 9F

1 **a**



b $\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

c $\text{OT} = \sqrt{3^2 + (-1)^2 + 4^2}$
$= \sqrt{26}$ units

2 **a** $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2\mathbf{i} + 4\mathbf{k}$

b $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

c $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = -2\mathbf{j}$

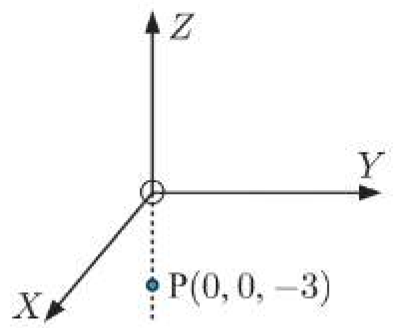
3 **a** $\left|\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\right| = 3$ units

b $\left|\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}\right| = \sqrt{(-4)^2 + 0^2 + 3^2}$
$= \sqrt{25}$
$= 5$ units

c $\left|\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}\right| = \sqrt{4^2 + 1^2 + (-2)^2}$
$= \sqrt{21}$ units

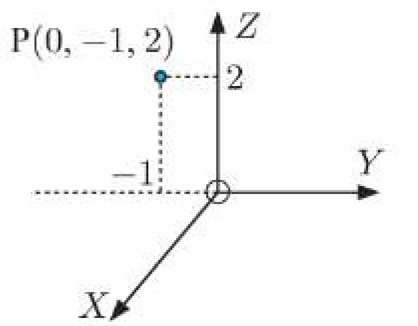
d $\left|\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}\right| = \sqrt{(-2)^2 + 2^2 + (-1)^2}$
$= \sqrt{9}$
$= 3$ units

4 **a**



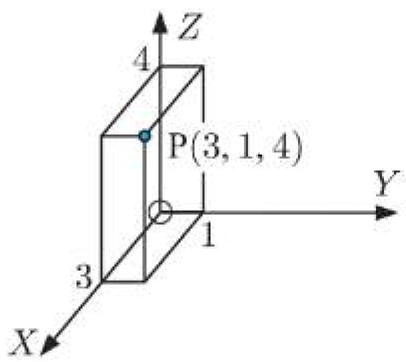
$$\begin{aligned}\text{OP} &= \sqrt{0^2+0^2+(-3)^2}\\ &= \sqrt{9}\\ &= 3 \text{ units}\end{aligned}$$

b



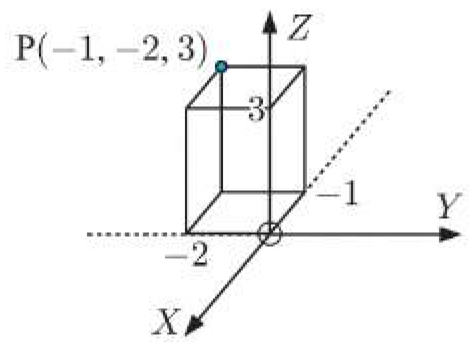
$$\begin{aligned}\text{OP} &= \sqrt{0^2+(-1)^2+2^2}\\ &= \sqrt{5} \text{ units}\end{aligned}$$

c



$$\begin{aligned}\text{OP} &= \sqrt{3^2+1^2+4^2}\\ &= \sqrt{26} \text{ units}\end{aligned}$$

d



$$\begin{aligned}\text{OP} &= \sqrt{(-1)^2+(-2)^2+3^2}\\ &= \sqrt{14} \text{ units}\end{aligned}$$

5 **a** $\mathbf{a} = \begin{pmatrix}2\\1\\4\end{pmatrix}$, $\quad |\,\mathbf{a}\,| = \sqrt{2^2+1^2+4^2} = \sqrt{21}$ units

b $\mathbf{b} = \begin{pmatrix}-3\\0\\4\end{pmatrix}$, $\quad |\,\mathbf{b}\,| = \sqrt{(-3)^2+0^2+4^2} = \sqrt{25} = 5$ units

c $\mathbf{c} = \begin{pmatrix}0\\2\\-3\end{pmatrix}$, $\quad |\,\mathbf{c}\,| = \sqrt{0^2+2^2+(-3)^2} = \sqrt{13}$ units

6 **a** $\begin{pmatrix}a-4\\b-3\\c+2\end{pmatrix} = \begin{pmatrix}1\\3\\-4\end{pmatrix}$

$\therefore \quad a-4=1, \quad b-3=3, \quad c+2=-4$

$\therefore \quad a=5, \ b=6, \ c=-6$

b $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$

$\therefore\ a-5=3-a,\quad b-2=2-b,\quad \text{and}\quad c+3=5-c$

$\therefore\ 2a=8,\quad 2b=4,\quad \text{and}\quad 2c=2$

$\therefore\ a=4,\quad b=2,\quad \text{and}\quad c=1$

$\therefore\ a=4,\ b=2,\ c=1$

c $\begin{pmatrix} 3a+4 \\ b-c \\ 4-5b \end{pmatrix} = \begin{pmatrix} 12-a \\ a-4 \\ 5a-22 \end{pmatrix}$

$\therefore\ 3a+4=12-a$

$\therefore\ 4a=8$

$\therefore\ a=2$

Now $4-5b=5a-22$

$\therefore\ 4-5b=5(2)-22$

$\therefore\ -5b=-16$

$\therefore\ b=\frac{16}{5}$

and $b-c=a-4$

$\therefore\ \frac{16}{5}-c=2-4$

$\therefore\ c=\frac{26}{5}$

$\therefore\ a=2,\ b=\frac{16}{5},\ c=\frac{26}{5}$

7 **a** $\left|\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right| = 1$ unit

$\therefore\ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ is a unit vector.

b $\left|\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right| = \sqrt{0^2+(\frac{1}{2})^2+(\frac{1}{2})^2}$

$= \sqrt{\frac{1}{4}+\frac{1}{4}}$

$= \sqrt{\frac{1}{2}}$

$= \frac{1}{\sqrt{2}}$ units

$\therefore\ \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ is not a unit vector.

c $\left|\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}\right| = \sqrt{(-\frac{2}{3})^2 + (\frac{1}{3})^2 + (\frac{2}{3})^2}$

$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}}$

$= \sqrt{1}$

$= 1$ unit

$\therefore$ $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ is a unit vector.

d $\left|\begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{5}{3\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \end{pmatrix}\right|$

$= \sqrt{(\frac{1}{3\sqrt{3}})^2 + (\frac{5}{3\sqrt{3}})^2 + (-\frac{1}{3\sqrt{3}})^2}$

$= \sqrt{\frac{1}{27} + \frac{25}{27} + \frac{1}{27}}$

$= \sqrt{1}$

$= 1$ unit

$\therefore$ $\begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{5}{3\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \end{pmatrix}$ is a unit vector.

8 **a** $\sqrt{(-\frac{1}{2})^2 + k^2 + (\frac{1}{4})^2} = 1$

$\therefore \sqrt{\frac{1}{4} + k^2 + \frac{1}{16}} = 1$

$\therefore \sqrt{k^2 + \frac{5}{16}} = 1$

$\therefore k^2 + \frac{5}{16} = 1$

$\therefore k^2 = \frac{11}{16}$

$\therefore k = \pm\frac{\sqrt{11}}{4}$

b $\sqrt{k^2 + (\frac{2}{3})^2 + (-\frac{1}{3})^2} = 1$

$\therefore \sqrt{k^2 + \frac{4}{9} + \frac{1}{9}} = 1$

$\therefore \sqrt{k^2 + \frac{5}{9}} = 1$

$\therefore k^2 + \frac{5}{9} = 1$

$\therefore k^2 = \frac{4}{9}$

$\therefore k = \pm\frac{2}{3}$

c $\sqrt{k^2 + k^2 + k^2} = 1$

$\therefore \sqrt{3k^2} = 1$

$\therefore 3k^2 = 1$

$\therefore k^2 = \frac{1}{3}$

$\therefore k = \pm\frac{1}{\sqrt{3}}$

d $\sqrt{\left(\frac{1}{k}\right)^2 + \left(\frac{2}{k}\right)^2 + \left(\frac{3}{k}\right)^2} = 1$

$\therefore \sqrt{\frac{1}{k^2} + \frac{4}{k^2} + \frac{9}{k^2}} = 1$

$\therefore \sqrt{\frac{14}{k^2}} = 1$

$\therefore \frac{14}{k^2} = 1$

$\therefore k^2 = 14$

$\therefore k = \pm\sqrt{14}$

9 **a** $\left|\begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix}\right| = \sqrt{14}$ units

$\therefore \sqrt{(-2)^2 + m^2 + 1^2} = \sqrt{14}$

$\therefore 4 + m^2 + 1 = 14$

$\therefore m^2 = 9$

$\therefore m = \pm 3$

b $\left|\begin{pmatrix} 4 \\ -3 \\ m \end{pmatrix}\right| = 6$ units

$\therefore \sqrt{4^2 + (-3)^2 + m^2} = 6$

$\therefore 16 + 9 + m^2 = 36$

$\therefore m^2 = 11$

$\therefore m = \pm\sqrt{11}$

10 $\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = 5$ units

$\therefore\ |a|,\ |b| \leqslant 5$ and $\sqrt{a^2+b^2} = 5$

$\therefore\ a^2+b^2 = 25$

$\left|\begin{pmatrix} b \\ c \end{pmatrix}\right| = 3$ units

$\therefore\ |b|,\ |c| \leqslant 3$ and $\sqrt{b^2+c^2} = 3$

$\therefore\ b^2+c^2 = 9$

$\therefore\ a^2+2b^2+c^2 = 34$

Now $$\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{a^2+b^2+c^2}$$
$$= \sqrt{a^2+2b^2+c^2-b^2}$$
$$= \sqrt{34-b^2}$$

which is maximised when $b = 0$, and minimised when $b = \pm 3$ $\{-3 \leqslant b \leqslant 3\}$.

When $b = 0$, $\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{34}$ units

When $b = \pm 3$, $$\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \sqrt{34-(\pm 3)^2}$$
$$= \sqrt{34-9}$$
$$= \sqrt{25}$$
$$= 5 \text{ units}$$

$$\therefore\ 5 \leqslant \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| \leqslant \sqrt{34}$$

EXERCISE 9G

1 **a** $$\mathbf{a}+\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 2+1 \\ -1+2 \\ 1+(-3) \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

b $$\mathbf{a}-\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 2-1 \\ -1-2 \\ 1-(-3) \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

c $\mathbf{b}+2\mathbf{c}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+2\begin{pmatrix}0\\1\\-3\end{pmatrix}$

$=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\begin{pmatrix}0\\2\\-6\end{pmatrix}$

$=\begin{pmatrix}1+0\\2+2\\-3+(-6)\end{pmatrix}$

$=\begin{pmatrix}1\\4\\-9\end{pmatrix}$

d $\mathbf{c}-\frac{1}{2}\mathbf{a}=\begin{pmatrix}0\\1\\-3\end{pmatrix}-\frac{1}{2}\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$=\begin{pmatrix}0\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$

$=\begin{pmatrix}0-1\\1-(-\frac{1}{2})\\-3-\frac{1}{2}\end{pmatrix}$

$=\begin{pmatrix}-1\\\frac{3}{2}\\-\frac{7}{2}\end{pmatrix}$

e $\mathbf{a}-\mathbf{b}-\mathbf{c}$

$=\begin{pmatrix}2\\-1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}$

$=\begin{pmatrix}2-1-0\\-1-2-1\\1-(-3)-(-3)\end{pmatrix}$

$=\begin{pmatrix}1\\-4\\7\end{pmatrix}$

f $2\mathbf{b}-\mathbf{c}+\mathbf{a}$

$=2\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$=\begin{pmatrix}2\\4\\-6\end{pmatrix}-\begin{pmatrix}0\\1\\-3\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$=\begin{pmatrix}2-0+2\\4-1+(-1)\\-6-(-3)+1\end{pmatrix}$

$=\begin{pmatrix}4\\2\\-2\end{pmatrix}$

2 **a** $|\,\mathbf{a}\,|=\sqrt{1^2+0^2+3^2}$

$=\sqrt{10}$ units

b $|\,\mathbf{b}\,|=\sqrt{(-2)^2+1^2+1^2}$

$=\sqrt{6}$ units

c $2\,|\,\mathbf{a}\,|=2\sqrt{10}$ units {using **a**}

d $2\mathbf{a}=2\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}2\\0\\6\end{pmatrix}$

$\therefore\ |\,2\mathbf{a}\,|=\sqrt{2^2+0^2+6^2}$

$=\sqrt{40}$

$=2\sqrt{10}$ units

e $-3\,|\,\mathbf{b}\,| = -3\sqrt{6}$ units {using **b**}

f $-3\mathbf{b} = -3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$

$$\begin{aligned} \therefore\ |-3\mathbf{b}\,| &= \sqrt{6^2 + (-3)^2 + (-3)^2} \\ &= \sqrt{54} \\ &= 3\sqrt{6} \text{ units} \end{aligned}$$

g $$\begin{aligned} \mathbf{a} + \mathbf{b} &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + (-2) \\ 0 + 1 \\ 3 + 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore\ |\,\mathbf{a} + \mathbf{b}\,| &= \sqrt{(-1)^2 + 1^2 + 4^2} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \text{ units} \end{aligned}$$

h $$\begin{aligned} \mathbf{a} - \mathbf{b} &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - (-2) \\ 0 - 1 \\ 3 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore\ |\,\mathbf{a} - \mathbf{b}\,| &= \sqrt{3^2 + (-1)^2 + 2^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

3 **a** $$\begin{aligned} 2\mathbf{a} - \mathbf{c} &= 2\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 - (-2) \\ 2 - 2 \\ 6 - 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

b $$\begin{aligned} 3\mathbf{b} + \tfrac{1}{2}\mathbf{c} &= 3\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -9 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -8 \\ 8 \end{pmatrix} \end{aligned}$$

c $-\mathbf{a}+3\mathbf{b} = -\begin{pmatrix}-1\\1\\3\end{pmatrix} + 3\begin{pmatrix}1\\-3\\2\end{pmatrix}$

$= \begin{pmatrix}1\\-1\\-3\end{pmatrix} + \begin{pmatrix}3\\-9\\6\end{pmatrix}$

$= \begin{pmatrix}1+3\\-1+(-9)\\-3+6\end{pmatrix}$

$= \begin{pmatrix}4\\-10\\3\end{pmatrix}$

d $\mathbf{a}-5\mathbf{b}+4\mathbf{c}$

$= \begin{pmatrix}-1\\1\\3\end{pmatrix} - 5\begin{pmatrix}1\\-3\\2\end{pmatrix} + 4\begin{pmatrix}-2\\2\\4\end{pmatrix}$

$= \begin{pmatrix}-1\\1\\3\end{pmatrix} - \begin{pmatrix}5\\-15\\10\end{pmatrix} + \begin{pmatrix}-8\\8\\16\end{pmatrix}$

$= \begin{pmatrix}-14\\24\\9\end{pmatrix}$

e $\mathbf{b}+\mathbf{c} = \begin{pmatrix}1\\-3\\2\end{pmatrix} + \begin{pmatrix}-2\\2\\4\end{pmatrix}$

$= \begin{pmatrix}-1\\-1\\6\end{pmatrix}$

$\therefore\ |\mathbf{b}+\mathbf{c}| = \sqrt{(-1)^2+(-1)^2+6^2}$

$= \sqrt{38}$ units

f $\mathbf{a}-\mathbf{c} = \begin{pmatrix}-1\\1\\3\end{pmatrix} - \begin{pmatrix}-2\\2\\4\end{pmatrix}$

$= \begin{pmatrix}1\\-1\\-1\end{pmatrix}$

$\therefore\ |\mathbf{a}-\mathbf{c}| = \sqrt{1^2+(-1)^2+(-1)^2}$

$= \sqrt{3}$ units

g $|\mathbf{a}| = \left|\begin{pmatrix}-1\\1\\3\end{pmatrix}\right|$

$= \sqrt{(-1)^2+1^2+3^2}$

$= \sqrt{11}$ units

$\therefore\ |\mathbf{a}|\,\mathbf{b} = \sqrt{11}\begin{pmatrix}1\\-3\\2\end{pmatrix}$

$= \begin{pmatrix}\sqrt{11}\\-3\sqrt{11}\\2\sqrt{11}\end{pmatrix}$

h $\frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{\sqrt{11}}\begin{pmatrix}-1\\1\\3\end{pmatrix}$ {using **g**}

$= \begin{pmatrix}-\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{3}{\sqrt{11}}\end{pmatrix}$

4 **a** $2\begin{pmatrix}1\\0\\3a\end{pmatrix} = \begin{pmatrix}b\\c-1\\2\end{pmatrix}$

$\therefore\ \begin{pmatrix}2\\0\\6a\end{pmatrix} = \begin{pmatrix}b\\c-1\\2\end{pmatrix}$

$\therefore\ 2=b,\ 0=c-1,$ and $6a=2$

$\therefore\ a=\frac{1}{3},\ b=2,$ and $c=1$

b $$a\begin{pmatrix}1\\1\\0\end{pmatrix}+b\begin{pmatrix}2\\0\\-1\end{pmatrix}+c\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}a\\a\\0\end{pmatrix}+\begin{pmatrix}2b\\0\\-b\end{pmatrix}+\begin{pmatrix}0\\c\\c\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}a+2b\\a+c\\-b+c\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$$

So, $a+2b=-1$ .... (1)

$a+c=3$

$\therefore\ c=3-a$ .... (2)

$-b+c=3$

$\therefore\ c=b+3$ .... (3)

Equating (2) and (3), we get

$3-a=b+3$

$\therefore\ -a=b$

Substituting into (1), we get

$a+2(-a)=-1$

$\therefore\ -a=-1$

$\therefore\ a=1$

$\therefore\ b=-1$

and $c=-1+3=2$ {using (3)}

$\therefore\ a=1,\ b=-1,$ and $c=2$

c $$a\begin{pmatrix}2\\-3\\1\end{pmatrix}+b\begin{pmatrix}1\\7\\2\end{pmatrix}=\begin{pmatrix}7\\c\\2\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}2a\\-3a\\a\end{pmatrix}+\begin{pmatrix}b\\7b\\2b\end{pmatrix}=\begin{pmatrix}7\\c\\2\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}2a+b\\-3a+7b\\a+2b\end{pmatrix}=\begin{pmatrix}7\\c\\2\end{pmatrix}$$

So, $2a+b=7$

$\therefore\ b=7-2a$ .... (1)

$-3a+7b=c$ .... (2)

$a+2b=2$ .... (3)

Substituting (1) into (3), we get

$a+2(7-2a)=2$

$\therefore\ a+14-4a=2$

$\therefore\ -3a=-12$

$\therefore\ a=4$

and so $b=7-2(4)=-1$

Substituting $a=4$ and $b=-1$ into (2),

we get $-3(4)+7(-1)=c$

$\therefore\ -12-7=c$

$\therefore\ c=-19$

$\therefore\ a=4,\ b=-1,\ c=-19$

d $$2\begin{pmatrix} a \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} b \\ c \\ 4 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2a \\ 2 \\ -4 \end{pmatrix} + \begin{pmatrix} b \\ c \\ 4 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2a+b \\ 2+c \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix}$$

So, $2a+b=c$ .... (1)

$2+c=-c$

$\therefore\ 2c=-2$

$\therefore\ c=-1$ .... (2)

$ab=0$

$\therefore\ a=0$ or $b=0$

If $a=0$, then

$b=c=-1$ {using (1) and (2)}

If $b=0$, then

$2a=c=-1$ {using (1) and (2)}

$\therefore\ a=-\frac{1}{2}$

$\therefore\ a=0,\ b=-1,\ c=-1$ *or* $a=-\frac{1}{2},\ b=0,\ c=-1$

EXERCISE 9H

1 **a** $\overrightarrow{AB} = \begin{pmatrix} 4-2 \\ 7-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

b $\overrightarrow{AB} = \begin{pmatrix} 1-3 \\ 4-(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

c $\overrightarrow{AB} = \begin{pmatrix} 1-(-2) \\ 4-7 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

2 **a** $\overrightarrow{AB} = \begin{pmatrix} 6-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

b $\overrightarrow{BA} = -\overrightarrow{AB}$

$= -\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ {using **a**}

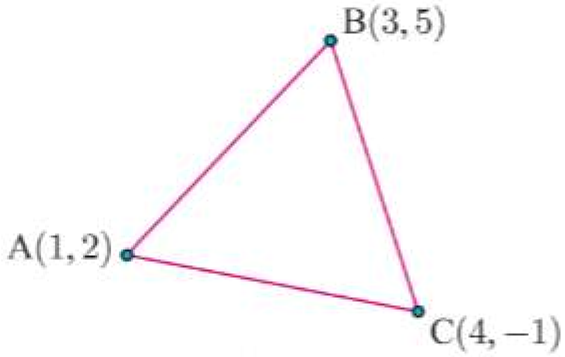
$= \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

c $|\overrightarrow{AB}| = \left|\begin{pmatrix} -4 \\ 1 \end{pmatrix}\right|$ {using **a**}

$= \sqrt{4^2+(-1)^2}$

$= \sqrt{17}$ units

3



a $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ 5-2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 4-1 \\ -1-2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b $\overrightarrow{BC}$

$= \overrightarrow{BA} + \overrightarrow{AC}$

$= -\overrightarrow{AB} + \overrightarrow{AC}$

c $\overrightarrow{BC}$

$= -\overrightarrow{AB} + \overrightarrow{AC}$

$= -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

d $\overrightarrow{BC} = \begin{pmatrix} 4-3 \\ -1-5 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ ✓

4 **a** $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= -\overrightarrow{BA} + \overrightarrow{BC}$

$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

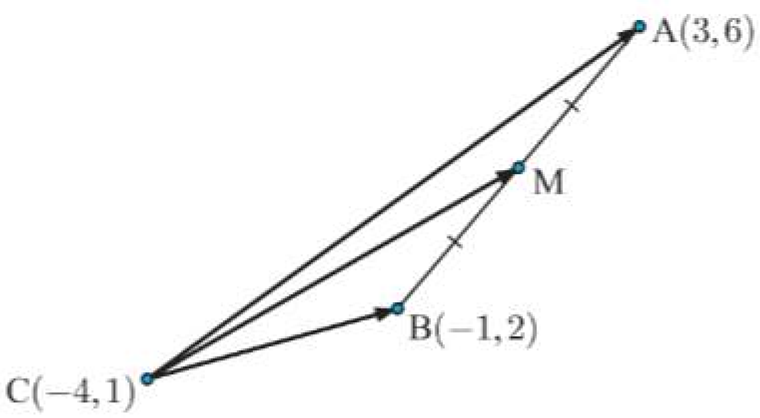
b $\overrightarrow{SP} = \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$

$= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$

$= -\begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

5



a M is $\left(\frac{3+(-1)}{2}, \frac{6+2}{2}\right)$

$\therefore$ M is $(1, 4)$

b $\overrightarrow{CA} = \begin{pmatrix} 3-(-4) \\ 6-1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

$\overrightarrow{CM} = \begin{pmatrix} 1-(-4) \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix} -1-(-4) \\ 2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c $\frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$

$= \frac{1}{2}\begin{pmatrix} 7 \\ 5 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ {from **b**}

$= \begin{pmatrix} \frac{7}{2} \\ \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$= \overrightarrow{CM}$ {from **b**}

6 **a** $\overrightarrow{PQ} = \begin{pmatrix} 3-1 \\ 5-0 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} 6-5 \\ 2-2 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$

c $\overrightarrow{PQ} = \begin{pmatrix} 1-(-2) \\ 4-3 \\ -3-0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$

d $\overrightarrow{PQ} = \begin{pmatrix} -1-4 \\ -5-(-1) \\ 3-5 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ -2 \end{pmatrix}$

7 $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$$= -\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$\therefore\ AB = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$ units

8 **a** $\overrightarrow{AB} = \begin{pmatrix} 1-(-3) \\ 0-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \quad \overrightarrow{BA} = \begin{pmatrix} -3-1 \\ 1-0 \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26}$ units,

$|\overrightarrow{BA}| = \sqrt{(-4)^2 + 1^2 + 3^2} = \sqrt{26}$ units

9 **a** The displacement vector of M relative to N

$= \overrightarrow{NM}$

$= \begin{pmatrix} 4-(-1) \\ -2-2 \\ -1-0 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$

b The displacement vector of N relative to M

$= \overrightarrow{MN}$

$= \begin{pmatrix} -1-4 \\ 2-(-2) \\ 0-(-1) \end{pmatrix}$

$= \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$

c $MN = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{42}$ units

10 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ -2-3 \\ 1-(-2) \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$

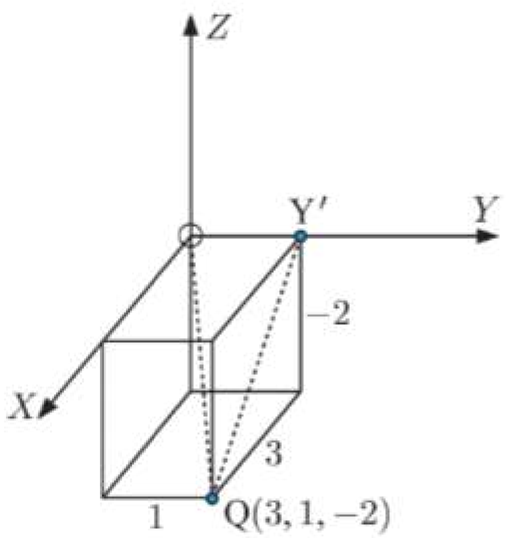
$= 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

b $|\overrightarrow{AB}| = \sqrt{4^2 + (-5)^2 + 3^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$ units

11



a The distance from Q to the Y-axis is the distance from Q to Y′(0, 1, 0).

$\therefore\ QY' = \sqrt{(3-0)^2 + (1-1)^2 + (-2-0)^2}$

$= \sqrt{9+0+4}$

$= \sqrt{13}$ units

b The distance from Q to the origin is

$QO = \sqrt{(3-0)^2 + (1-0)^2 + (-2-0)^2}$

$= \sqrt{9+1+4}$

$= \sqrt{14}$ units

c The distance from Q to the YZ-plane is the distance from Q to (0, 1, −2), which is 3 units.

12 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= (\mathbf{i} - \mathbf{j} + \mathbf{k}) + (-2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$

$= -\mathbf{i} - 2\mathbf{k}$

13 **a** $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$= \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$

b $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\overrightarrow{AC} + \overrightarrow{AB}$

$= -\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$

c $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD}$

$= \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$ {using **b**}

$= \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$

14 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ k-3 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ k-3 \end{pmatrix}$

$\therefore |\overrightarrow{AB}| = \sqrt{4^2+(k-3)^2}$

$= \sqrt{16+(k-3)^2}$ units

b $|\overrightarrow{AB}| = 5$

$\therefore \sqrt{16+(k-3)^2} = 5$ {using **a**}

$\therefore 16+k^2-6k+9 = 25$

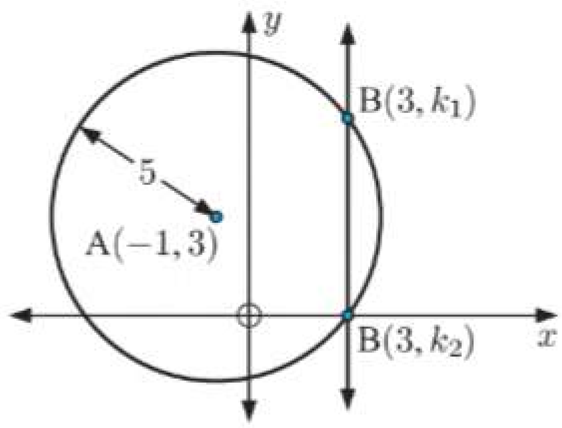
$\therefore k^2-6k = 0$

$\therefore k(k-6) = 0$

$\therefore k = 0$ or $k-6 = 0$

$\therefore k = 0$ or 6

c



15 **a** Let B have coordinates (b_1, b_2).

$\therefore \overrightarrow{AB} = \begin{pmatrix} b_1-1 \\ b_2-4 \end{pmatrix}$

$\therefore \begin{pmatrix} b_1-1 \\ b_2-4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\therefore b_1-1 = 3$ and $b_2-4 = -2$

$\therefore b_1 = 4$ and $b_2 = 2$

$\therefore$ B has coordinates $(4, 2)$.

b Let C have coordinates (c_1, c_2).

$\therefore \overrightarrow{CA} = \begin{pmatrix} 1-c_1 \\ 4-c_2 \end{pmatrix}$

$\therefore \begin{pmatrix} 1-c_1 \\ 4-c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\therefore 1-c_1 = -1$ and $4-c_2 = 2$

$\therefore c_1 = 2$ and $c_2 = 2$

$\therefore$ C has coordinates $(2, 2)$.

16 **a** $\overrightarrow{PC} = \begin{pmatrix} 1-(-1) \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b Let Q have coordinates (q_1, q_2).

$\therefore \overrightarrow{CQ} = \begin{pmatrix} q_1-1 \\ q_2-2 \end{pmatrix}$

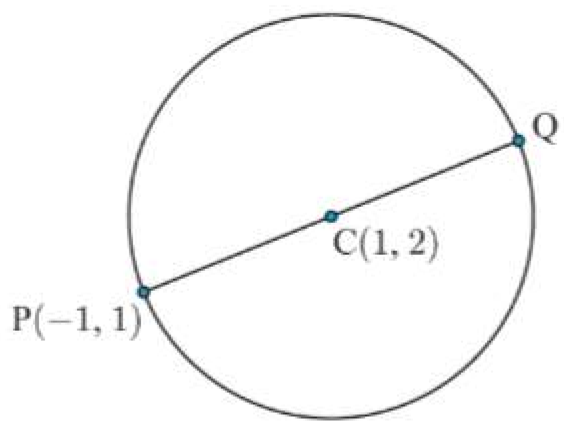
But $\overrightarrow{CQ} = \overrightarrow{PC}$

$\therefore \begin{pmatrix} q_1-1 \\ q_2-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore q_1-1 = 2$ and $q_2-2 = 1$

$\therefore q_1 = 3$ and $q_2 = 3$

$\therefore$ Q has coordinates $(3, 3)$.



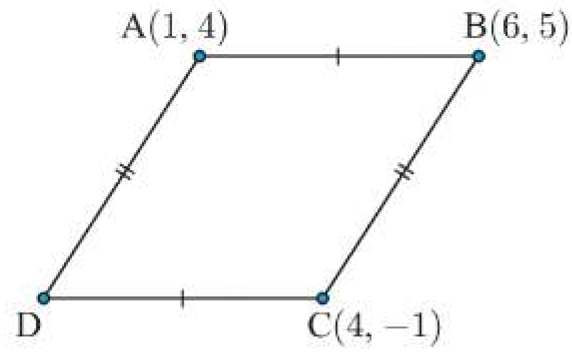
17 **a** $\overrightarrow{AB} = \begin{pmatrix} 6-1 \\ 5-4 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

b $\overrightarrow{CD} = -\overrightarrow{AB}$

$= -\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -5 \\ -1 \end{pmatrix}$



c Let D have coordinates (d_1, d_2).

$\therefore \overrightarrow{CD} = \begin{pmatrix} d_1 - 4 \\ d_2 - (-1) \end{pmatrix} = \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix}$

But $\overrightarrow{CD} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

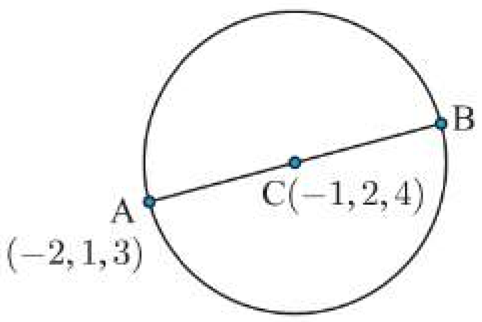
$\therefore \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

$\therefore d_1 - 4 = -5$ and $d_2 + 1 = -1$

$\therefore d_1 = -1$ and $d_2 = -2$

$\therefore$ D has coordinates $(-1, -2)$.

18



If B is (a, b, c) then $\frac{a-2}{2} = -1$, $\frac{b+1}{2} = 2$, $\frac{c+3}{2} = 4$

$\therefore a = 0, \ b = 3, \ c = 5$

$\therefore$ B is $(0, 3, 5)$

$r = AC = \sqrt{(-1-(-2))^2 + (2-1)^2 + (4-3)^2}$

$= \sqrt{1+1+1}$

$= \sqrt{3}$ units

19 **a** $\overrightarrow{AB} = \begin{pmatrix} 2-(-1) \\ 0-2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$

$\therefore AB = \sqrt{3^2 + (-2)^2 + (-2)^2}$

$= \sqrt{17}$ units

b $\overrightarrow{AC} = \begin{pmatrix} -3-(-1) \\ 1-2 \\ 0-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}$

$\therefore AC = \sqrt{(-2)^2 + (-1)^2 + (-5)^2}$

$= \sqrt{30}$ units

c $\overrightarrow{CB} = \begin{pmatrix} 2-(-3) \\ 0-1 \\ 3-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$\therefore CB = \sqrt{5^2 + (-1)^2 + 3^2}$

$= \sqrt{35}$ units

d Triangle ABC has $AC = \sqrt{30}$ units, $CB = \sqrt{35}$ units, and $AB = \sqrt{17}$ units.
All the side lengths are different, and
$(\sqrt{17})^2 + (\sqrt{30})^2 \neq (\sqrt{35})^2$.
So, triangle ABC is not right angled.
$\therefore$ triangle ABC is scalene.

20 P(0, 4, 4), Q(2, 6, 5), R(1, 4, 3)

$$\begin{aligned} PQ &= \sqrt{(2-0)^2+(6-4)^2+(5-4)^2} \\ &= \sqrt{4+4+1} \\ &= 3 \end{aligned}$$

$$\begin{aligned} PR &= \sqrt{(1-0)^2+(4-4)^2+(3-4)^2} \\ &= \sqrt{1+0+1} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(1-2)^2+(4-6)^2+(3-5)^2} \\ &= \sqrt{1+4+4} \\ &= 3 \end{aligned}$$

$\therefore$ PQ = QR and so triangle PQR is isosceles.

21 **a** $\overrightarrow{AB} = \begin{pmatrix} 6-5 \\ 12-6 \\ 9-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}$ $\qquad$ $\overrightarrow{BC} = \begin{pmatrix} 2-6 \\ 4-12 \\ 2-9 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \\ -7 \end{pmatrix}$

$$\begin{aligned} \therefore\ AB &= \sqrt{1^2+6^2+11^2} \\ &= \sqrt{158} \text{ units} \end{aligned}$$

$$\begin{aligned} \therefore\ BC &= \sqrt{(-4)^2+(-8)^2+(-7)^2} \\ &= \sqrt{129} \text{ units} \end{aligned}$$

$\overrightarrow{AC} = \begin{pmatrix} 2-5 \\ 4-6 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$

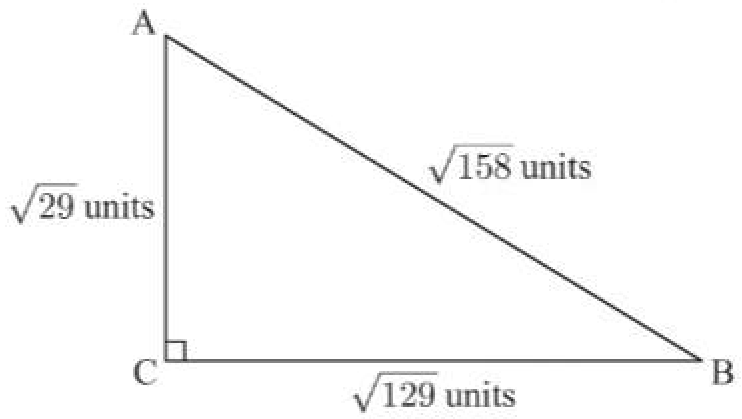
$$\begin{aligned} \therefore\ AC &= \sqrt{(-3)^2+(-2)^2+4^2} \\ &= \sqrt{29} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Now, } (\sqrt{29})^2 + (\sqrt{129})^2 &= 29+129 \\ &= 158 \\ &= (\sqrt{158})^2 \end{aligned}$$

So, $AC^2 + BC^2 = AB^2$

$\therefore$ triangle ABC is right angled with the right angle at C.

b
$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times \sqrt{129} \times \sqrt{29} \\ &\approx 30.6 \text{ units}^2 \end{aligned}$$



22 A(−1, 3, 4), B(2, 5, −1), C(−1, 2, −2), D(r, s, t)

a If $\overrightarrow{AC} = \overrightarrow{BD}$ then $\begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t-(-1) \end{pmatrix}$

$\therefore$ $r-2=0$, $s-5=-1$, and $t+1=-6$

$\therefore$ $r=2$, $s=4$, and $t=-7$

b If $\overrightarrow{AB} = \overrightarrow{DC}$ then $\begin{pmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1-r \\ 2-s \\ -2-t \end{pmatrix}$

$\therefore$ $-1-r=3$, $2-s=2$, and $-2-t=-5$

$\therefore$ $r=-4$, $s=0$, and $t=3$

23 A(1, 2, 3), B(3, −3, 2), C(7, −4, 5), D(5, 1, 6)

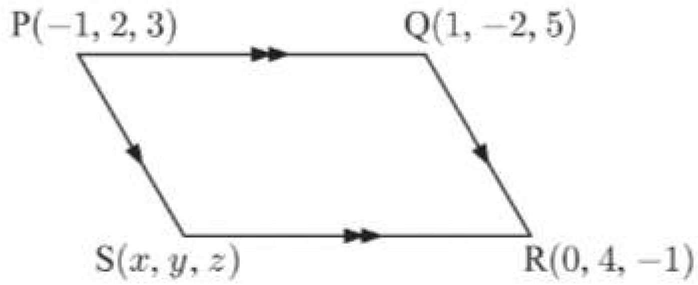
a $\overrightarrow{AB} = \begin{pmatrix} 3-1 \\ -3-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 7-5 \\ -4-1 \\ 5-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$.

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

24 **a** Suppose S is (x, y, z).

$\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}

$\therefore \begin{pmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0-x \\ 4-y \\ -1-z \end{pmatrix}$



$\therefore \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ 4-y \\ -1-z \end{pmatrix}$

$\therefore\ -x = 2, \quad 4-y = -4, \quad \text{and} \quad -1-z = 2$

$\therefore\ x = -2, \quad y = 8, \quad \text{and} \quad z = -3$

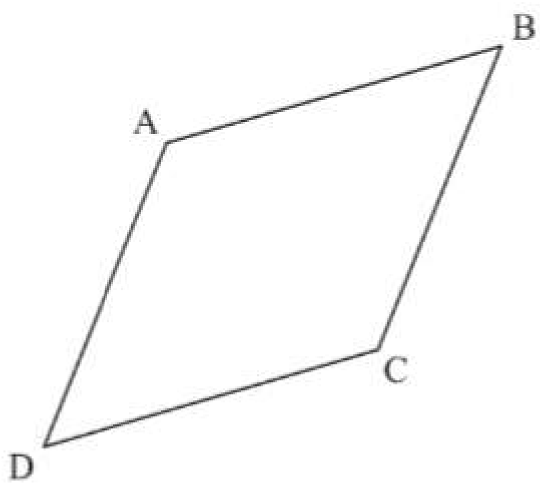
$\therefore$ S is $(-2, 8, -3)$.

b The midpoint of [PR] is $\left(\frac{-1+0}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

The midpoint of [QS] is $\left(\frac{1+(-2)}{2}, \frac{-2+8}{2}, \frac{5+(-3)}{2}\right)$ which is $(-\frac{1}{2}, 3, 1)$.

So, [PR] and [QS] have the same midpoint. ✓

25



a $\overrightarrow{AB} = \begin{pmatrix} -1-5 \\ 2-0 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$ and

$\overrightarrow{DC} = \begin{pmatrix} 4-10 \\ -3-(-5) \\ 6-5 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$

So, $\overrightarrow{AB} = \overrightarrow{DC}$

Sides [AB] and [DC] are equal in length and parallel.

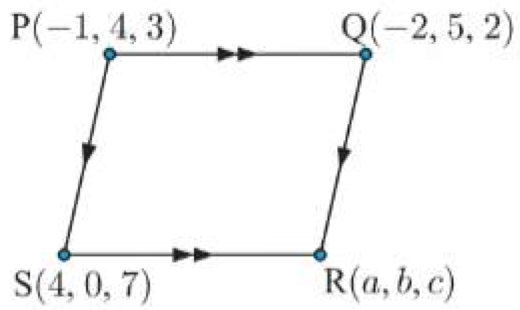
$\therefore$ ABCD is a parallelogram.

b $\overrightarrow{AB} = \begin{pmatrix} 1-2 \\ 4-(-3) \\ -1-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} -2-(-1) \\ 6-(-1) \\ -2-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -4 \end{pmatrix}$

So, $\overrightarrow{AB} \neq \overrightarrow{DC}$

$\therefore$ ABCD is not a parallelogram.

26 **a**



Let R be (a, b, c).

Now $\overrightarrow{SR} = \overrightarrow{PQ}$

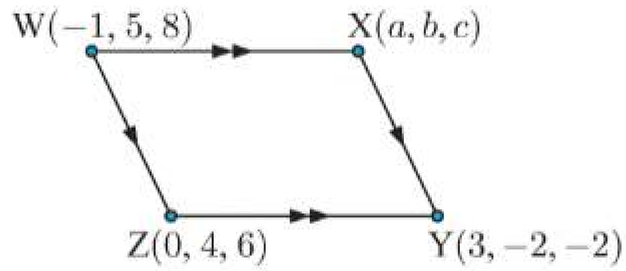
$$\therefore \begin{pmatrix} a-4 \\ b-0 \\ c-7 \end{pmatrix} = \begin{pmatrix} -2-(-1) \\ 5-4 \\ 2-3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$\therefore \ a = 3, \ b = 1, \ c = 6$

So, R is $(3, 1, 6)$.

b



Let X be (a, b, c).

Now $\overrightarrow{WX} = \overrightarrow{ZY}$

$$\therefore \begin{pmatrix} a-(-1) \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3-0 \\ -2-4 \\ -2-6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$$

$\therefore \ a = 2, \ b = -1, \ c = 0$

So, X is $(2, -1, 0)$.

EXERCISE 9I

1 **a** $\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ 15 \end{pmatrix}$

$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 15 \end{pmatrix}$ are parallel.

b There is no scalar k such that

$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = k\begin{pmatrix} 2 \\ -10 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -10 \end{pmatrix}$ are not parallel.

c $\begin{pmatrix} 6 \\ -12 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} -4 \\ 8 \end{pmatrix}$

$\therefore \begin{pmatrix} 6 \\ -12 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$ are parallel.

d $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix}$ are parallel.

e $\begin{pmatrix} 8 \\ 0 \\ -20 \end{pmatrix} = -\frac{4}{3}\begin{pmatrix} -6 \\ 0 \\ 15 \end{pmatrix}$

$\therefore \begin{pmatrix} 8 \\ 0 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 0 \\ 15 \end{pmatrix}$ are parallel.

f There is no scalar k such that

$\begin{pmatrix} 6 \\ 10 \\ -2 \end{pmatrix} = k\begin{pmatrix} 24 \\ 40 \\ 8 \end{pmatrix}$

$\therefore \begin{pmatrix} 6 \\ 10 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 24 \\ 40 \\ 8 \end{pmatrix}$ are not parallel.

2 **a** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ a \end{pmatrix}$ are parallel.

$\therefore \begin{pmatrix} 2 \\ -1 \end{pmatrix} = k \begin{pmatrix} -6 \\ a \end{pmatrix}$ for some scalar k.

$\therefore\ 2 = -6k$ and $-1 = ka$

$\therefore\ k = -\frac{1}{3}$ and $-1 = -\frac{1}{3}a$

$\therefore\ a = 3$

b $\begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ are parallel.

$\therefore \begin{pmatrix} a \\ 2 \end{pmatrix} = k \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ for some scalar k.

$\therefore\ a = 3k$ and $2 = -k$

$\therefore\ a = -6$

3 **a** Since **a** and **b** are parallel, $\mathbf{b} = k\mathbf{a}$.

$$\therefore \begin{pmatrix} -6 \\ r \\ s \end{pmatrix} = k \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2k \\ -k \\ 3k \end{pmatrix}$$

$\therefore\ 2k = -6$

$\therefore\ k = -3$

$\therefore\ r = 3,\ s = -9$

b Since **a** and **b** are parallel, $\mathbf{b} = k\mathbf{a}$.

$$\therefore \begin{pmatrix} r \\ 2 \\ s \end{pmatrix} = k \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3k \\ -k \\ 2k \end{pmatrix}$$

$\therefore\ 2 = -k$

$\therefore\ k = -2$

$\therefore\ r = -6,\ s = -4$

4 **a** $\overrightarrow{AB} = 3\overrightarrow{CD}$ means that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and 3 times its length.

b $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ means that $\overrightarrow{RS}$ is parallel to $\overrightarrow{KL}$, half its length, and in the opposite direction.

c

$\overrightarrow{AB} = 2\overrightarrow{BC}$ means that A, B, and C are collinear and the length of $\overrightarrow{AB}$ is twice the length of $\overrightarrow{BC}$.

5 $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$, $\overrightarrow{OR} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\overrightarrow{OS} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{QS} &= \overrightarrow{QO} + \overrightarrow{OS} \\ &= -\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix} \\ &= 2\begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\overrightarrow{PR} \end{aligned}$$

Thus [PR] and [QS] are parallel and PR : QS = 1 : 2.

6 **a** The vector in the same direction as **a** and twice its length is $2\mathbf{a}$.

$$2\mathbf{a} = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

b The vector in the opposite direction to **a** and half its length is $-\frac{1}{2}\mathbf{a}$.

$$-\tfrac{1}{2}\mathbf{a} = -\tfrac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} \\ -\frac{4}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

7 **a** $\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{1^2 + 2^2} = \sqrt{5}$ units.

$\therefore$ the unit vector in the same direction as $\mathbf{i} + 2\mathbf{j}$ is $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$.

b $2\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{2^2 + (-3)^2} = \sqrt{13}$ units.

$\therefore$ the unit vector in the same direction as $2\mathbf{i} - 3\mathbf{j}$ is $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j}) = \frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$.

c $-2\mathbf{i} + 2\mathbf{j}$ has length $\sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ units.

$\therefore$ the unit vector in the same direction as $-2\mathbf{i} + 2\mathbf{j}$ is
$\frac{1}{2\sqrt{2}}(-2\mathbf{i} + 2\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$.

8 **a** $2\mathbf{i} + 3\mathbf{k}$ has length $\sqrt{2^2 + 3^2} = \sqrt{13}$ units

$\therefore$ the unit vector $= \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{k}) = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{k}$

b $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ has length $\sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$ units

$\therefore$ the unit vector $= \frac{1}{\sqrt{6}}(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

c $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ has length $\sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$ units

$\therefore$ the unit vector is $\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

9 **a** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ has length $\sqrt{2^2 + (-1)^2} = \sqrt{5}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\therefore$ the vector of length 3 units in the same direction is $\frac{3}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$

b $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ has length $\sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{\sqrt{17}}\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\therefore$ the vector of length 2 units in the opposite direction is $\frac{2}{\sqrt{17}}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$

c $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ has length $\sqrt{(-1)^2+4^2+1^2} = \sqrt{18} = 3\sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$\therefore$ the vector of length 6 units in the same direction is

$$\frac{6}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

d $\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$ has length $\sqrt{(-1)^2+(-2)^2+(-2)^2} = \sqrt{9} = 3$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\therefore$ the vector of length 5 units in the opposite direction is $\frac{5}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix}$

10 **a** $|\mathbf{a}| = \sqrt{2^2+(-1)^2+(-2)^2}$
$= \sqrt{4+1+4}$
$= 3$ units

$\therefore$ the vectors of length 1 unit parallel to **a** are $\pm\frac{1}{3}\mathbf{a}$.

$\therefore$ the vectors are

$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

b $|\mathbf{b}| = \sqrt{(-2)^2+(-1)^2+2^2}$
$= \sqrt{4+1+4}$
$= 3$ units

$\therefore$ the vectors of length 2 units parallel to **b** are $\pm\frac{2}{3}\mathbf{b}$.

$\therefore$ the vectors are

$\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$ and $\begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix}$.

11 **a** $\overrightarrow{AB}$ is a vector in the same direction as $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with length 4 units.

Now, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has length $\sqrt{1^2+(-1)^2} = \sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ the vector of length 4 units in the same direction is $\frac{4}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$\therefore$ $\overrightarrow{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

b $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

Now $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$$\therefore \overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 3+2\sqrt{2} \\ 2-2\sqrt{2} \end{pmatrix}$$

c If $\overrightarrow{OB} = \begin{pmatrix} 3+2\sqrt{2} \\ 2-2\sqrt{2} \end{pmatrix}$, then the coordinates of B are $(3+2\sqrt{2},\ 2-2\sqrt{2})$.

12 **a** $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ has length $\sqrt{3^2+4^2} = 5$ units.

$\therefore$ the vector of length 5 units in the same direction is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Now $\begin{pmatrix} 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$\therefore$ the point 5 units from $(1, -5)$ in the direction $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $X(4, -1)$.

b $\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ has length $\sqrt{(-3)^2+0^2+4^2} = \sqrt{25} = 5$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{5}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$.

$\therefore$ the vector of length 6 units in the same direction is $\frac{6}{5}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{18}{5} \\ 0 \\ \frac{24}{5} \end{pmatrix}$.

Now $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -\frac{18}{5} \\ 0 \\ \frac{24}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{5} \\ 3 \\ \frac{14}{5} \end{pmatrix}$

$\therefore$ the point 6 units from $(1, 3, -2)$ in the direction $\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ is $X(-\frac{13}{5}, 3, \frac{14}{5})$.

c $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ has length $\sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

$\therefore$ the vector of length 4 units in the same direction is $\frac{4}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{6}} \\ -\frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} \end{pmatrix}$.

Now $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{4}{\sqrt{6}} \\ -\frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2+\frac{4}{\sqrt{6}} \\ -1-\frac{4}{\sqrt{6}} \\ 4+\frac{8}{\sqrt{6}} \end{pmatrix}$

$\therefore$ the point 4 units from $(2, -1, 4)$ in the direction $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ is

$X\left(2+\frac{4}{\sqrt{6}},\ -1-\frac{4}{\sqrt{6}},\ 4+\frac{8}{\sqrt{6}}\right)$.

13 **a** $\overrightarrow{AB} = \begin{pmatrix} 4-(-1) \\ 6-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 1-4 \\ 3-6 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$

So, $\overrightarrow{AB} = -\frac{5}{3}\overrightarrow{BC}$

$\therefore$ A, B, and C are collinear.

b $\overrightarrow{PQ} = \begin{pmatrix} 6-3 \\ -3-3 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} 1-6 \\ 7-(-3) \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$

So, $\overrightarrow{PQ} = -\frac{3}{5}\overrightarrow{QR}$

$\therefore$ P, Q, and R are collinear.

c $\overrightarrow{AB} = \begin{pmatrix} 4-(-2) \\ 3-1 \\ 0-4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 19-4 \\ 8-3 \\ -10-0 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix}$

So, $\overrightarrow{AB} = \frac{2}{5}\overrightarrow{BC}$

$\therefore$ A, B, and C are collinear.

d $\overrightarrow{PQ} = \begin{pmatrix} 5-2 \\ -5-1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} -1-5 \\ 7-(-5) \\ 4-(-2) \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ 6 \end{pmatrix}$

So, $\overrightarrow{PQ} = -\frac{1}{2}\overrightarrow{QR}$

$\therefore$ P, Q, and R are collinear.

14 **a** $\overrightarrow{AB} = \begin{pmatrix} 11-2 \\ -9-(-3) \\ 7-4 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -13-11 \\ a-(-9) \\ b-7 \end{pmatrix} = \begin{pmatrix} -24 \\ a+9 \\ b-7 \end{pmatrix}$

A, B, and C are collinear.

$\therefore\ \overrightarrow{AB} = k\,\overrightarrow{BC}$

$\therefore\ 9 = k \times -24$

$\therefore\ k = -\frac{3}{8}$

$\therefore\ \overrightarrow{AB} = -\frac{3}{8}\overrightarrow{BC}$

$\therefore\ -\frac{8}{3}\overrightarrow{AB} = \overrightarrow{BC}$

So, $-\frac{8}{3} \times -6 = a+9$

$\therefore\ 16 = a+9$

$\therefore\ a = 7$

and $-\frac{8}{3} \times 3 = b-7$

$\therefore\ -8 = b-7$

$\therefore\ b = -1$

b $\overrightarrow{KL} = \begin{pmatrix} 4-1 \\ -3-(-1) \\ 7-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$

$\overrightarrow{LM} = \begin{pmatrix} a-4 \\ 2-(-3) \\ b-7 \end{pmatrix} = \begin{pmatrix} a-4 \\ 5 \\ b-7 \end{pmatrix}$

K, L, and M are collinear.

$\therefore\ \overrightarrow{KL} = k\overrightarrow{LM}$

$\therefore\ -2 = k \times 5$

$\therefore\ k = -\frac{2}{5}$

$\therefore\ \overrightarrow{KL} = -\frac{2}{5}\,\overrightarrow{LM}$

$\therefore\ -\frac{5}{2}\,\overrightarrow{KL} = \overrightarrow{LM}$

So, $-\frac{5}{2} \times 3 = a - 4$

$\therefore\ -\frac{15}{2} = a - 4$

$\therefore\ a = -\frac{7}{2}$

and $-\frac{5}{2} \times 7 = b - 7$

$\therefore\ -\frac{35}{2} = b - 7$

$\therefore\ b = -\frac{21}{2}$

15 **a** $\overrightarrow{CP} = \begin{pmatrix} x-X \\ y-Y \\ z-Z \end{pmatrix}$

b $|\,\overrightarrow{CP}\,| = \left|\begin{pmatrix} x-X \\ y-Y \\ z-Z \end{pmatrix}\right|$

$\therefore\ r = \sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}$

$\therefore\ (x-X)^2 + (y-Y)^2 + (z-Z)^2 = r^2$

This is a Cartesian equation for the sphere with radius r units and centre $C(X, Y, Z)$.

INVESTIGATION 1 LINEAR COMBINATIONS

1 **a** **i** $r\begin{pmatrix} 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -22 \end{pmatrix}$

$\therefore\ \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} -4 \\ -22 \end{pmatrix}$

$\therefore\ r = -4,\ s = -22$

ii $r\begin{pmatrix} 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -19 \end{pmatrix}$

$\therefore\ \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 7 \\ -19 \end{pmatrix}$

$\therefore\ r = 7,\ s = -19$

b Yes, $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2 **a** **i** $r\begin{pmatrix}1\\0\end{pmatrix}+s\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-4\\-22\end{pmatrix}$

$\therefore \begin{pmatrix}r\\0\end{pmatrix}+\begin{pmatrix}s\\s\end{pmatrix}=\begin{pmatrix}-4\\-22\end{pmatrix}$

$\therefore \begin{pmatrix}r+s\\s\end{pmatrix}=\begin{pmatrix}-4\\-22\end{pmatrix}$

$\therefore\ s=-22$ and $r-22=-4$

$\therefore\ r=18$

ii $r\begin{pmatrix}1\\0\end{pmatrix}+s\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}7\\-19\end{pmatrix}$

$\therefore \begin{pmatrix}r\\0\end{pmatrix}+\begin{pmatrix}s\\s\end{pmatrix}=\begin{pmatrix}7\\-19\end{pmatrix}$

$\therefore \begin{pmatrix}r+s\\s\end{pmatrix}=\begin{pmatrix}7\\-19\end{pmatrix}$

$\therefore\ s=-19$ and $r-19=7$

$\therefore\ r=26$

b Yes, $\begin{pmatrix}x\\y\end{pmatrix}=(x-y)\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}1\\1\end{pmatrix}$.

3 **a** **i** $r\begin{pmatrix}2\\1\end{pmatrix}+s\begin{pmatrix}-4\\-2\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$

$\therefore \begin{pmatrix}2r\\r\end{pmatrix}+\begin{pmatrix}-4s\\-2s\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$

$\therefore \begin{pmatrix}2r-4s\\r-2s\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$

So we have $2r-4s=8$ and $r-2s=4$, but these are equivalent since they are multiples of each other. There are infinitely many values of r and s which satisfy the equation.

ii $r\begin{pmatrix}2\\1\end{pmatrix}+s\begin{pmatrix}-4\\-2\end{pmatrix}=\begin{pmatrix}7\\-19\end{pmatrix}$

$\therefore \begin{pmatrix}2r\\r\end{pmatrix}+\begin{pmatrix}-4s\\-2s\end{pmatrix}=\begin{pmatrix}7\\-19\end{pmatrix}$

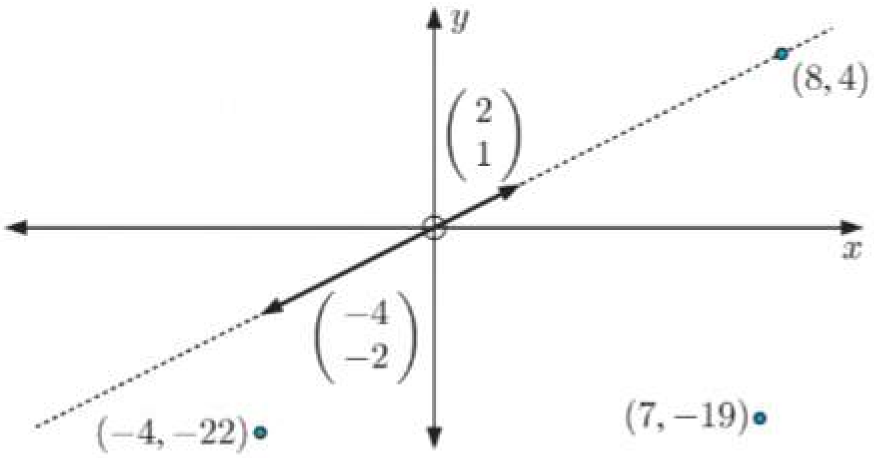
$\therefore \begin{pmatrix}2r-4s\\r-2s\end{pmatrix}=\begin{pmatrix}7\\-19\end{pmatrix}$

$\therefore\ 2r-4s=7$ and $r-2s=-19$

$\therefore\ 2r-4s=7$ and $2r-4s=-38$

But $2r-4s$ cannot be 7 and -38 simultaneously, so there are no solutions.

b $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$ are multiples of each other, so every linear combination of these vectors is a vector of the form $k\begin{pmatrix}2\\1\end{pmatrix}$. These vectors are represented by the line on the diagram below.



4 **a** $r\begin{pmatrix}4\\1\end{pmatrix} + s\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-6\\7\end{pmatrix}$

$\therefore \begin{pmatrix}4r\\r\end{pmatrix} + \begin{pmatrix}3s\\5s\end{pmatrix} = \begin{pmatrix}-6\\7\end{pmatrix}$

$\therefore \begin{pmatrix}4r+3s\\r+5s\end{pmatrix} = \begin{pmatrix}-6\\7\end{pmatrix}$

$\therefore 4r + 3s = -6$ .... (1)

$r + 5s = 7$ .... (2)

Solving simultaneously, $4r + 3s = -6$ $\{(1)\}$

$4r + 20s = 28$ $\{4 \times (2)\}$

Subtracting, $-17s = -34$

$\therefore s = 2$

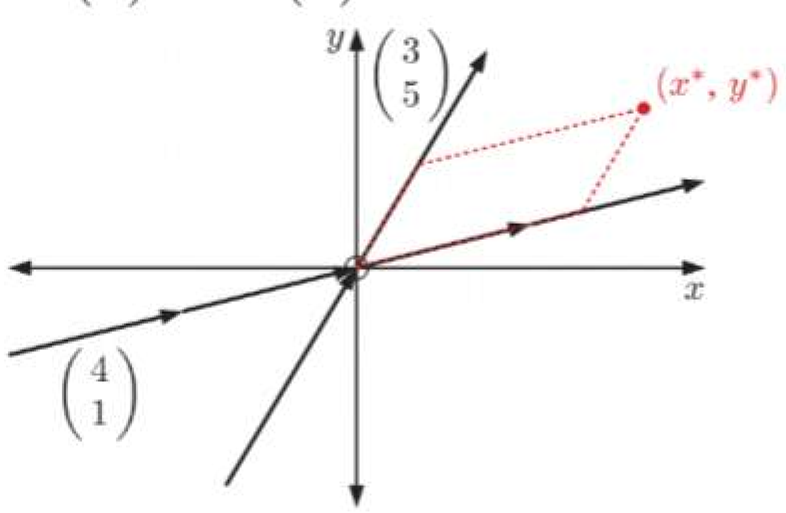
$\therefore 4r + 3(2) = -6$

$\therefore 4r = -12$

$\therefore r = -3$

$\therefore r = -3, \ s = 2$

b Yes, any vector in the plane can be written as a linear combination of $\begin{pmatrix}4\\1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$, since $\begin{pmatrix}4\\1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ are **not** multiples of each other.



We can arrive at any (x^*, y^*) by moving along multiples of $\begin{pmatrix}4\\1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$.

EXERCISE 9J

1 **a** $\begin{pmatrix} 2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2 \times 3 + 5 \times 1$
$= 6 + 5$
$= 11$

b $\begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 5 \times 2 + (-1) \times 4$
$= 10 - 4$
$= 6$

c $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = 3 \times 4 + 0 \times (-2) + (-2) \times 5$
$= 12 + 0 - 10$
$= 2$

d $\begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 9 \\ -6 \\ -1 \end{pmatrix} = (-4) \times 9 + (-5) \times (-6) + 3 \times (-1)$
$= -36 + 30 - 3$
$= -9$

2 **a** $\mathbf{q} \bullet \mathbf{p} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$= (-1) \times 3 + 5 \times 2$
$= -3 + 10$
$= 7$

b $\mathbf{q} \bullet \mathbf{r} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix}$
$= (-1) \times (-2) + 5 \times 4$
$= 2 + 20$
$= 22$

c $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right]$
$= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 6 \end{pmatrix}$
$= (-1) \times 1 + 5 \times 6$
$= -1 + 30$
$= 29$

d $3\mathbf{r} \bullet \mathbf{q} = 3 \begin{pmatrix} -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
$= \begin{pmatrix} -6 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$
$= (-6) \times (-1) + 12 \times 5$
$= 6 + 60$
$= 66$

e $2\mathbf{p} \bullet 2\mathbf{p} = 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \bullet 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \end{pmatrix}$
$= 6 \times 6 + 4 \times 4$
$= 36 + 16$
$= 52$

f $\mathbf{i} \bullet \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$= 1 \times 3 + 0 \times 2$
$= 3 + 0$
$= 3$

g $\mathbf{q} \bullet \mathbf{j} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= (-1) \times 0 + 5 \times 1$
$= 0 + 5$
$= 5$

h $\mathbf{i} \bullet \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$= 1 \times 1 + 0 \times 0$
$= 1 + 0$
$= 1$

3 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$
$= 2(-1) + 1(1) + 3(1)$
$= -2 + 1 + 3$
$= 2$

b $\mathbf{b} \bullet \mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$= -1(2) + 1(1) + 1(3)$
$= -2 + 1 + 3$
$= 2$

c $|\mathbf{a}|^2 = \left(\sqrt{2^2 + 1^2 + 3^2}\right)^2$
$= 14$

d $\mathbf{a} \bullet \mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$= 2(2) + 1(1) + 3(3)$
$= 14$

e $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$
$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \left[\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right]$
$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$
$= 2(-1) + 1(0) + 3(2)$
$= 4$

f $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$
$= 2 + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ {using **a**}
$= 2 + 2(0) + 1(-1) + 3(1)$
$= 4$

4 **a** $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k})$
$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$
$= 1(0) + 1(2) + (-1)(1)$
$= 1$

b $\mathbf{i} \bullet \mathbf{i}$
$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$= 1(1) + 0(0) + 0(0)$
$= 1$

c $(3\mathbf{i} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j})$
$= \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
$= 3(1) + 0(1) + (-2)(0)$
$= 3$

d $(2\mathbf{i} + \mathbf{k}) \bullet (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$
$= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$
$= 2(-1) + 0(3) + 1(2)$
$= 0$

e $\mathbf{i} \bullet \mathbf{j}$
$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$= 1(0) + 0(1) + 0(0)$
$= 0$

f $(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \bullet (3\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$
$= \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}$
$= 1(3) + (-4)(-3) + 2(-5)$
$= 5$

5 $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$

a **i** $\mathbf{a} \bullet \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 1(3) + 0(-1) + 4(5) = 23$

ii $\mathbf{a} \bullet \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = 1(0) + 0(-2) + 4(3) = 12$

iii $\mathbf{b} \bullet \mathbf{c} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = -2(3) + 1(-1) + 0(5) = -7$

iv $\mathbf{b} \bullet \mathbf{d} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = -2(0) + 1(-2) + 0(3) = -2$

b
$$\begin{aligned}(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{c}+\mathbf{d}) &= \left[\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right] \bullet \left[\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}\right] \\ &= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 8 \end{pmatrix} \\ &= -1(3) + 1(-3) + 4(8) \\ &= 26\end{aligned}$$

and $\mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} = 23 + 12 + (-7) + (-2) = 26$

$\therefore$ $(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{c}+\mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$

6 $\mathbf{a} \bullet \mathbf{b}$ is a scalar and so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is the scalar product of a scalar and a vector, which is meaningless.

7 $|\mathbf{a}| = 5$, $|\mathbf{b}| = 3$

$$\begin{aligned}(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b}) &= \mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} \\ &= \mathbf{a} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2 \\ &= 5^2 - 3^2 \\ &= 16\end{aligned}$$

EXERCISE 9K

1 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \end{pmatrix}$
$= 3(2) + 1(5)$
$= 6 + 5$
$= 11$

b $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\,\mathbf{a}\,|\,|\,\mathbf{b}\,|}$
$= \dfrac{11}{\sqrt{9+1}\sqrt{4+25}}$
$= \dfrac{11}{\sqrt{290}}$
$\therefore\ \theta = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.8°$

2 **a** $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
$= 3(-1) + 3(2)$
$= -3 + 6$
$= 3$

Now $\cos\theta = \dfrac{\mathbf{r} \bullet \mathbf{s}}{|\,\mathbf{r}\,|\,|\,\mathbf{s}\,|}$
$= \dfrac{3}{\sqrt{3^2+3^2}\sqrt{(-1)^2+2^2}}$
$= \dfrac{3}{\sqrt{90}}$
$\therefore\ \theta = \cos^{-1}\left(\frac{3}{\sqrt{90}}\right) \approx 71.6°$

b $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \end{pmatrix}$
$= (-1)(2) + (-3)(5)$
$= -2 - 15$
$= -17$

Now $\cos\theta = \dfrac{\mathbf{r} \bullet \mathbf{s}}{|\,\mathbf{r}\,|\,|\,\mathbf{s}\,|}$
$= \dfrac{-17}{\sqrt{(-1)^2+(-3)^2}\sqrt{2^2+5^2}}$
$= \dfrac{-17}{\sqrt{290}}$
$\therefore\ \theta = \cos^{-1}\left(\frac{-17}{\sqrt{290}}\right) \approx 177°$

c $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
$= (1)(2) + (-1)(-1)$
$= 2 + 1$
$= 3$

Now $\cos\theta = \dfrac{\mathbf{r} \bullet \mathbf{s}}{|\,\mathbf{r}\,|\,|\,\mathbf{s}\,|}$
$= \dfrac{3}{\sqrt{1^2+(-1)^2}\sqrt{2^2+(-1)^2}}$
$= \dfrac{3}{\sqrt{10}}$
$\therefore\ \theta = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right) \approx 18.4°$

d $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$= 1(1) + 0(1)$
$= 1 + 0$
$= 1$

Now $\cos\theta = \dfrac{\mathbf{r} \bullet \mathbf{s}}{|\,\mathbf{r}\,|\,|\,\mathbf{s}\,|}$
$= \dfrac{1}{\sqrt{1^2+0^2}\sqrt{1^2+1^2}}$
$= \dfrac{1}{\sqrt{2}}$
$\therefore\ \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

3 $\begin{pmatrix}1\\1\\5\end{pmatrix} \bullet \begin{pmatrix}2\\3\\-1\end{pmatrix} = 1(2) + 1(3) + 5(-1) = 0$

$\therefore$ $\begin{pmatrix}1\\1\\5\end{pmatrix}$ and $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ are perpendicular.

4 $\mathbf{a} \bullet \mathbf{b}$

$= \begin{pmatrix}3\\1\\2\end{pmatrix} \bullet \begin{pmatrix}-1\\1\\1\end{pmatrix}$

$= 3(-1) + 1(1) + 2(1)$

$= 0$

$\mathbf{b} \bullet \mathbf{c}$

$= \begin{pmatrix}-1\\1\\1\end{pmatrix} \bullet \begin{pmatrix}1\\5\\-4\end{pmatrix}$

$= -1(1) + 1(5) + 1(-4)$

$= 0$

$\mathbf{a} \bullet \mathbf{c}$

$= \begin{pmatrix}3\\1\\2\end{pmatrix} \bullet \begin{pmatrix}1\\5\\-4\end{pmatrix}$

$= 3(1) + 1(5) + 2(-4)$

$= 0$

$\therefore$ **a**, **b**, and **c** are mutually perpendicular.

5 **a** $$\cos\theta = \frac{\begin{pmatrix}4\\0\\-2\end{pmatrix} \bullet \begin{pmatrix}2\\-3\\1\end{pmatrix}}{\left|\begin{pmatrix}4\\0\\-2\end{pmatrix}\right|\left|\begin{pmatrix}2\\-3\\1\end{pmatrix}\right|}$$

$$= \frac{4(2) + 0(-3) + (-2)(1)}{\sqrt{4^2 + 0^2 + (-2)^2}\sqrt{2^2 + (-3)^2 + 1^2}}$$

$$= \frac{8 + 0 - 2}{\sqrt{20}\sqrt{14}}$$

$$= \frac{6}{\sqrt{280}}$$

$\therefore$ $\theta = \cos^{-1}\left(\frac{6}{\sqrt{280}}\right) \approx 69.0°$

b $$\cos\theta = \frac{\begin{pmatrix}3\\-1\\2\end{pmatrix} \bullet \begin{pmatrix}-2\\1\\3\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\2\end{pmatrix}\right|\left|\begin{pmatrix}-2\\1\\3\end{pmatrix}\right|}$$

$$= \frac{3(-2) + (-1)(1) + 2(3)}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{(-2)^2 + 1^2 + 3^2}}$$

$$= \frac{-6 - 1 + 6}{\sqrt{14}\sqrt{14}}$$

$$= -\frac{1}{14}$$

$\therefore$ $\theta = \cos^{-1}(-\frac{1}{14}) \approx 94.1°$

c $$\cos\theta = \frac{\begin{pmatrix}0\\2\\-1\end{pmatrix} \bullet \begin{pmatrix}1\\0\\2\end{pmatrix}}{\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|\left|\begin{pmatrix}1\\0\\2\end{pmatrix}\right|}$$

$$= \frac{0(1) + 2(0) + (-1)(2)}{\sqrt{0^2 + 2^2 + (-1)^2}\sqrt{1^2 + 0^2 + 2^2}}$$

$$= \frac{-2}{\sqrt{5}\sqrt{5}}$$

$$= -\frac{2}{5}$$

$\therefore$ $\theta = \cos^{-1}(-\frac{2}{5}) \approx 114°$

6 **a** Since **p** and **q** are perpendicular,

$$\mathbf{p} \bullet \mathbf{q} = 0$$
$$\therefore \begin{pmatrix} 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0$$
$$\therefore 3(-2) + t(1) = 0$$
$$\therefore -6 + t = 0$$
$$\therefore t = 6$$

b Since **r** and **s** are perpendicular,

$$\mathbf{r} \bullet \mathbf{s} = 0$$
$$\therefore \begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 0$$
$$\therefore t(3) + (t+2)(-4) = 0$$
$$\therefore 3t - 4t - 8 = 0$$
$$\therefore t = -8$$

c Since **a** and **b** are perpendicular,

$$\mathbf{a} \bullet \mathbf{b} = 0$$
$$\therefore \begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 2-3t \\ t \end{pmatrix} = 0$$
$$\therefore t(2-3t) + (t+2)(t) = 0$$
$$\therefore 2t - 3t^2 + t^2 + 2t = 0$$
$$\therefore -2t^2 + 4t = 0$$
$$\therefore t^2 - 2t = 0$$
$$\therefore t(t-2) = 0$$
$$\therefore t = 0 \text{ or } 2$$

d Since **a** and **b** are perpendicular,

$$\mathbf{a} \bullet \mathbf{b} = 0$$
$$\therefore \begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix} \bullet \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix} = 0$$
$$\therefore 3(2t) + (-1)(-3) + t(-4) = 0$$
$$\therefore 6t + 3 - 4t = 0$$
$$\therefore 2t = -3$$
$$\therefore t = -\tfrac{3}{2}$$

e Since **p** and **q** are perpendicular,

$$\mathbf{p} \bullet \mathbf{q} = 0$$
$$\therefore \begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix} = 0$$
$$\therefore 3(1-t) + t(-3) + (-2)(4) = 0$$
$$\therefore 3 - 3t - 3t - 8 = 0$$
$$\therefore -6t = 5$$
$$\therefore t = -\tfrac{5}{6}$$

f Since **a** and **b** are perpendicular,

$$\mathbf{a} \bullet \mathbf{b} = 0$$
$$\therefore \begin{pmatrix} 2 \\ t \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} t \\ 3 \\ t \end{pmatrix} = 0$$
$$\therefore 2t + 3t + t(t-2) = 0$$
$$\therefore 2t + 3t + t^2 - 2t = 0$$
$$\therefore t(t+3) = 0$$
$$\therefore t = 0 \text{ or } -3$$

7 Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular,

$\mathbf{a} \bullet \mathbf{b} = 0$, $\mathbf{b} \bullet \mathbf{c} = 0$, and $\mathbf{a} \bullet \mathbf{c} = 0$

$$\therefore \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix} = 0 \qquad \therefore 2 + 4 + 3r = 0$$
$$\therefore 3r = -6$$
$$\therefore r = -2$$

and $\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \qquad \therefore 2s + 2t - 2 = 0$

$$\therefore s + t = 1 \quad .... (1)$$

and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \qquad \therefore s + 2t + 3 = 0$

$$\therefore s + 2t = -3 \quad .... (2)$$

(2) − (1) gives $t = -4$ and so $s = 5$

$\therefore r = -2$, $s = 5$, and $t = -4$

8 **a** $\begin{pmatrix} 3 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \end{pmatrix} = (3)(4) + (-4)(3) = 0$

$\therefore$ vectors of the form $k\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $k \neq 0$ are perpendicular to $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

Now $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ has length $\sqrt{16+9} = 5$ units

$\therefore$ $k = \frac{1}{5}$ for a unit vector.

$\therefore$ a vector of length 10 units which is perpendicular to $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is $\frac{10}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, which is $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

b $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1)(1) + (-1)(1) = 0$

$\therefore$ vectors of the form $k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $k \neq 0$ are perpendicular to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Now $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has length $\sqrt{1+1} = \sqrt{2}$ units

$\therefore$ $k = \frac{1}{\sqrt{2}}$ for a unit vector.

$\therefore$ a vector of length $3\sqrt{2}$ units which is perpendicular to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is $\frac{3\sqrt{2}}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

c $\begin{pmatrix} -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \end{pmatrix} = (-2)(-1) + (-1)(2) = 0$

$\therefore$ vectors of the form $k\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $k \neq 0$ are perpendicular to $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$.

Now $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ has length $\sqrt{1+4} = \sqrt{5}$ units

$\therefore$ $k = \frac{1}{\sqrt{5}}$ for a unit vector.

$\therefore$ a vector of length $\sqrt{20}$ units which is perpendicular to $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ is $\frac{\sqrt{20}}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, which is $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

9 $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \end{pmatrix} = (2)(-3) + (3)(2) = 0$

$\therefore$ vectors of the form $k\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $k \neq 0$ are perpendicular to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Now $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ has length $\sqrt{9+4} = \sqrt{13}$ units

$\therefore$ $k = \frac{1}{\sqrt{13}}$ or $-\frac{1}{\sqrt{13}}$ for a unit vector.

$\therefore$ the vectors of length 5 units which are perpendicular to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are $\pm\frac{5}{\sqrt{13}}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, which are $\begin{pmatrix} -\frac{15}{\sqrt{13}} \\ \frac{10}{\sqrt{13}} \end{pmatrix}$ and $\begin{pmatrix} \frac{15}{\sqrt{13}} \\ -\frac{10}{\sqrt{13}} \end{pmatrix}$.

10 $$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta \\ &= \sqrt{10} \times 3 \times \cos 71^\circ \\ &\approx 3.089 \end{aligned}$$

11 **a** $$\begin{aligned} \mathbf{p} \bullet \mathbf{q} &= |\,\mathbf{p}\,|\,|\,\mathbf{q}\,| \cos\theta \\ &= 2 \times 5 \times \cos 60^\circ \\ &= 5 \end{aligned}$$

b $$\begin{aligned} \mathbf{p} \bullet \mathbf{q} &= |\,\mathbf{p}\,|\,|\,\mathbf{q}\,| \cos\theta \\ &= 6 \times 3 \times \cos \tfrac{2\pi}{3} \\ &= -9 \end{aligned}$$

12 **a** $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular as $\mathbf{a} \bullet \mathbf{b} \neq 0$.

b $\mathbf{a} \bullet \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta$

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\theta = 0^\circ$ or 180°

$\therefore \cos\theta = \pm 1$

$\therefore$ $-12 = |\,\mathbf{a}\,| \times 1 \times (\pm 1)$

$\therefore$ $|\,\mathbf{a}\,| = \pm 12$

$\therefore$ $|\,\mathbf{a}\,| = 12$ units $\quad \{\,|\,\mathbf{a}\,| > 0\}$

Note: This means that $\cos\theta$ must be -1, so the angle between $\mathbf{a}$ and $\mathbf{b}$ is 180°

$\therefore$ $\mathbf{a}$ and $\mathbf{b}$ are in opposite directions.

13 **a** $$\begin{aligned} \mathbf{c} \bullet \mathbf{d} &= |\,\mathbf{c}\,|\,|\,\mathbf{d}\,| \cos\theta \\ \therefore\ 5 &= \sqrt{5}\sqrt{5}\cos\theta \\ \therefore\ 5 &= 5\cos\theta \\ \therefore\ \cos\theta &= 1 \\ \therefore\ \theta &= 0^\circ \end{aligned}$$

So, $\mathbf{c} = \mathbf{d}$

b $$\begin{aligned} \mathbf{c} \bullet \mathbf{d} &= |\,\mathbf{c}\,|\,|\,\mathbf{d}\,| \cos\theta \\ \therefore\ -5 &= \sqrt{5}\sqrt{5}\cos\theta \\ \therefore\ -5 &= 5\cos\theta \\ \therefore\ \cos\theta &= -1 \\ \therefore\ \theta &= 180^\circ \end{aligned}$$

So, $\mathbf{c} = -\mathbf{d}$

14 For example, let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, and $\mathbf{c} = \mathbf{k}$

$\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = 0$ and $\mathbf{j} \neq \mathbf{k}$

15 If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$.

So, to find a vector perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, we pick two non-zero integer values for a and b, then solve for c.

For example, if $a = 1$, $b = 2$ then $\begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$

$$\therefore\ 1 + 4 - c = 0$$
$$\therefore\ 5 - c = 0$$
$$\therefore\ c = 5$$

So, $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

Repeating this process with a different value of a (or b) will give another vector which is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

If $a = 2$, $b = 2$ then $\begin{pmatrix} 2 \\ 2 \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$

$$\therefore\ 2 + 4 - c = 0$$
$$\therefore\ 6 - c = 0$$
$$\therefore\ c = 6$$

So, $\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$ is also perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, and is not parallel to $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

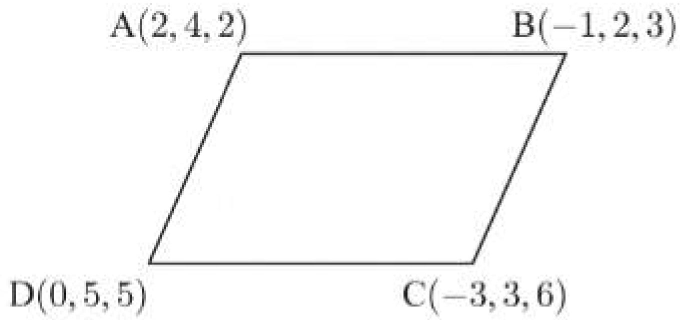
16 A(5, 1, 2), B(6, −1, 0), C(3, 2, 0)

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$, and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$

Now $\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = (-2) + (-2) + 4 = 0$

$\therefore$ $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so triangle ABC is right angled at A.

17 **a**



$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ and $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$.

$\therefore$ ABCD is a parallelogram.

b $|\overrightarrow{AB}| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$ units

and $|\overrightarrow{BC}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}$ units

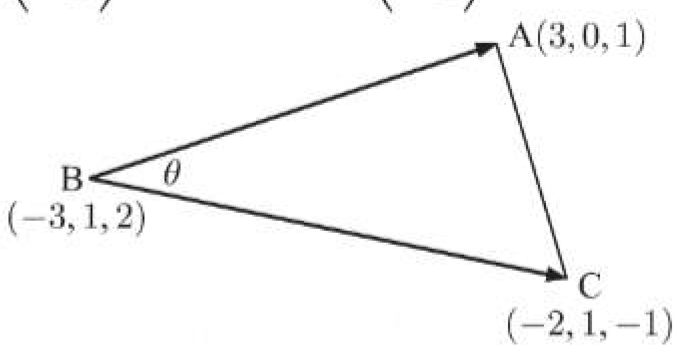
$\therefore$ ABCD is a rhombus.

c $\overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-5)(1) + (-1)(3) + 4(2) = 0$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$ which illustrates that the diagonals of a rhombus are perpendicular.

18 A(3, 0, 1), B(−3, 1, 2), C(−2, 1, −1)

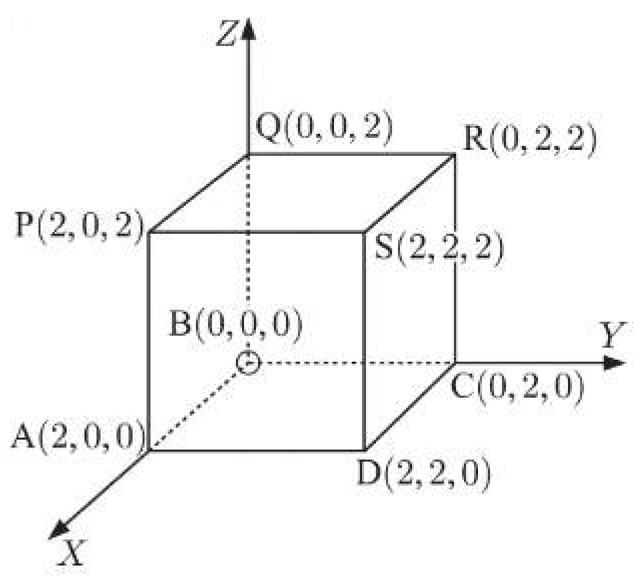
$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$



$$\begin{aligned}\therefore \cos\theta &= \frac{\overrightarrow{BC} \bullet \overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|} \\ &= \frac{(1)(6) + (0)(-1) + (-3)(-1)}{\sqrt{1+0+9}\sqrt{36+1+1}} \\ &= \frac{9}{\sqrt{380}} \\ \therefore \theta &= \cos^{-1}\left(\frac{9}{\sqrt{380}}\right) \\ &\approx 62.5^\circ\end{aligned}$$

If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, which is $\approx 117.5^\circ$.

19 Suppose the origin is at B.



a $\overrightarrow{BA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$$\therefore \quad \overrightarrow{BA} \bullet \overrightarrow{BS} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 0 = 4$$

$$\therefore \quad \cos(A\widehat{B}S) = \frac{4}{\sqrt{4+0+0}\sqrt{4+4+4}} = \frac{4}{2 \times 2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad A\widehat{B}S = \cos^{-1}\left(\tfrac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$

b $\overrightarrow{BR} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ and $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$\therefore \quad \overrightarrow{BR} \bullet \overrightarrow{BP} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 0 + 0 + 4 = 4$$

$$\therefore \quad \cos(R\widehat{B}P) = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}} = \frac{4}{\sqrt{8} \times \sqrt{8}} = \tfrac{1}{2}$$

$$\therefore \quad R\widehat{B}P = 60^\circ$$

c $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

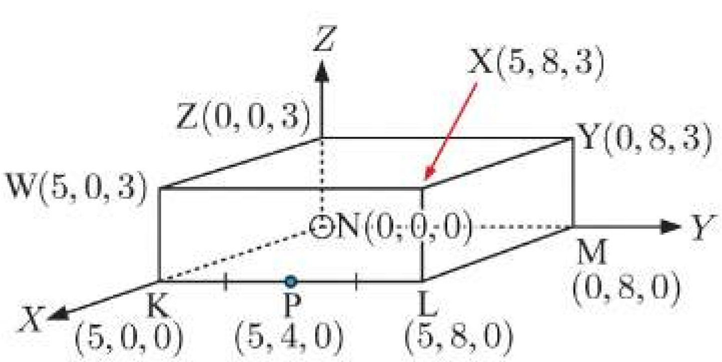
$$\therefore \quad \overrightarrow{BP} \bullet \overrightarrow{BS} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 4 = 8$$

$$\therefore \quad \cos(P\widehat{B}S) = \frac{8}{\sqrt{4+0+4}\sqrt{4+4+4}} = \tfrac{8}{\sqrt{96}}$$

$$\therefore \quad P\widehat{B}S = \cos^{-1}\left(\tfrac{8}{\sqrt{96}}\right) \approx 35.3^\circ$$

20 Suppose the origin is at N.

a



$\overrightarrow{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}$ and $\overrightarrow{NX} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$

$$\overrightarrow{NY} \bullet \overrightarrow{NX} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} = 0 + 64 + 9 = 73$$

$$\therefore \quad \cos(Y\widehat{N}X) = \frac{\overrightarrow{NY} \bullet \overrightarrow{NX}}{|\overrightarrow{NY}|\,|\overrightarrow{NX}|} = \frac{73}{\sqrt{0+64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\tfrac{73}{98}}$$

$$\therefore \quad Y\widehat{N}X = \cos^{-1}\left(\sqrt{\tfrac{73}{98}}\right) \approx 30.3^\circ$$

b $\overrightarrow{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}$ and $\overrightarrow{NP} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

$$\begin{aligned} \overrightarrow{NY} \bullet \overrightarrow{NP} &= \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \\ &= 0 + 32 + 0 \\ &= 32 \end{aligned}$$

$$\begin{aligned} \therefore \cos(\mathrm{Y}\widehat{\mathrm{N}}\mathrm{P}) &= \frac{\overrightarrow{NY} \bullet \overrightarrow{NP}}{|\overrightarrow{NY}||\overrightarrow{NP}|} \\ &= \frac{32}{\sqrt{0+64+9}\sqrt{25+16}} \\ &= \frac{32}{\sqrt{73}\sqrt{41}} \\ \therefore \mathrm{Y}\widehat{\mathrm{N}}\mathrm{P} &= \cos^{-1}\left(\frac{32}{\sqrt{73}\sqrt{41}}\right) \\ &\approx 54.2^\circ \end{aligned}$$

INVESTIGATION 2 THE VECTOR CROSS PRODUCT FORMULA

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= (a_2b_3 - a_3b_2)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (a_1b_3 - a_3b_1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (a_1b_2 - a_2b_1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_3b_1 - a_1b_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\ &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\ &= \mathbf{a} \times \mathbf{b} \quad \checkmark \end{aligned}$$

EXERCISE 9L.1

1 **a** $$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} -3 & 1 \\ 4 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix}\mathbf{k}$$
$$= (6-4)\mathbf{i} - (-4-1)\mathbf{j} + (8+3)\mathbf{k}$$
$$= 2\mathbf{i} - (-5)\mathbf{j} + 11\mathbf{k}$$
$$= \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}$$

b $$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 3 & -1 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix}\mathbf{k}$$
$$= (0+2)\mathbf{i} - (2-6)\mathbf{j} + (1-0)\mathbf{k}$$
$$= 2\mathbf{i} - (-4)\mathbf{j} + \mathbf{k}$$
$$= \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

c $$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 2 \\ -3 & 1 & -5 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 2 \\ -3 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & -1 \\ -3 & 1 \end{vmatrix}\mathbf{k}$$
$$= (5-2)\mathbf{i} - (-20+6)\mathbf{j} + (4-3)\mathbf{k}$$
$$= 3\mathbf{i} - (-14)\mathbf{j} + \mathbf{k}$$
$$= \begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix}$$

d $$(\mathbf{i}+\mathbf{j}-2\mathbf{k}) \times (\mathbf{i}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$
$$= (-1-0)\mathbf{i} - (-1+2)\mathbf{j} + (0-1)\mathbf{k}$$
$$= -\mathbf{i} - \mathbf{j} - \mathbf{k}$$

e $(2\mathbf{i}-\mathbf{k})\times(\mathbf{j}+3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$
$$= (0+1)\mathbf{i} - (6-0)\mathbf{j} + (2-0)\mathbf{k}$$
$$= \mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$$

f $(\mathbf{i}-\mathbf{j}+3\mathbf{k})\times(-2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -2 & 3 & -1 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 3 \\ 3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix}\mathbf{k}$$
$$= (1-9)\mathbf{i} - (-1+6)\mathbf{j} + (3-2)\mathbf{k}$$
$$= -8\mathbf{i} - 5\mathbf{j} + \mathbf{k}$$

2 $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

a $\mathbf{x}\times\mathbf{x} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$
$$= (-1+1)\mathbf{i} - (2-2)\mathbf{j} + (-2+2)\mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

b $\mathbf{y}\times\mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 2 \\ -1 & 3 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix}\mathbf{k}$$
$$= (6-6)\mathbf{i} - (-2+2)\mathbf{j} + (-3+3)\mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

c $\mathbf{x}\times\mathbf{y}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 1\\ -1 & 3 & 2\end{vmatrix}$

$$=\begin{vmatrix}-1 & 1\\ 3 & 2\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & 1\\ -1 & 2\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & -1\\ -1 & 3\end{vmatrix}\mathbf{k}$$
$$=(-2-3)\mathbf{i}-(4+1)\mathbf{j}+(6-1)\mathbf{k}$$
$$=-5\mathbf{i}-5\mathbf{j}+5\mathbf{k}$$
$$=\begin{pmatrix}-5\\-5\\5\end{pmatrix}$$

$$-\mathbf{y}\times\mathbf{x}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -3 & -2\\ 2 & -1 & 1\end{vmatrix}$$
$$=\begin{vmatrix}-3 & -2\\ -1 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & -3\\ 2 & -1\end{vmatrix}\mathbf{k}$$
$$=(-3-2)\mathbf{i}-(1+4)\mathbf{j}+(-1+6)\mathbf{k}$$
$$=-5\mathbf{i}-5\mathbf{j}+5\mathbf{k}$$
$$=\begin{pmatrix}-5\\-5\\5\end{pmatrix}$$
$$=\mathbf{x}\times\mathbf{y}$$

3 $\mathbf{a}=\begin{pmatrix}1\\2\\3\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}-1\\3\\-1\end{pmatrix}$

a $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ -1 & 3 & -1\end{vmatrix}$

$$=\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 3\\ -1 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}\mathbf{k}$$
$$=(-2-9)\mathbf{i}-(-1+3)\mathbf{j}+(3+2)\mathbf{k}$$
$$=-11\mathbf{i}-2\mathbf{j}+5\mathbf{k}$$
$$=\begin{pmatrix}-11\\-2\\5\end{pmatrix}$$

b $\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}-11\\-2\\5\end{pmatrix}$

$$=-11-4+15$$
$$=0$$

$\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})=\begin{pmatrix}-1\\3\\-1\end{pmatrix}\bullet\begin{pmatrix}-11\\-2\\5\end{pmatrix}$

$$=11-6-5$$
$$=0$$

$\therefore\ \mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=0=\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})$

c $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both **a** and **b**.

4 **a** $$\mathbf{i}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$

$$= (0\times 0 - 0\times 0)\mathbf{i} - (1\times 0 - 0\times 1)\mathbf{j} + (1\times 0 - 0\times 1)\mathbf{k}$$

$$= \mathbf{0}$$

$$\mathbf{j}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$

$$= (1\times 0 - 0\times 1)\mathbf{i} - (0\times 0 - 0\times 0)\mathbf{j} + (0\times 1 - 1\times 0)\mathbf{k}$$

$$= \mathbf{0}$$

$$\mathbf{k}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{k}$$

$$= (0\times 1 - 1\times 0)\mathbf{i} - (0\times 1 - 1\times 0)\mathbf{j} + (0\times 0 - 0\times 0)\mathbf{k}$$

$$= \mathbf{0}$$

In each case, we observe that $\mathbf{a}\times\mathbf{a} = \mathbf{0}$.

b $$\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$

$$= (0\times 0 - 0\times 1)\mathbf{i} - (1\times 0 - 0\times 0)\mathbf{j} + (1\times 1 - 0\times 0)\mathbf{k}$$

$$= \mathbf{k}$$

$$\mathbf{j}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{k}$$

$$= (1\times 0 - 0\times 0)\mathbf{i} - (0\times 0 - 0\times 1)\mathbf{j} + (0\times 0 - 1\times 1)\mathbf{k}$$

$$= -\mathbf{k}$$

In each case, we observe that $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

c **i** $\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{k}$

$= (1 \times 1 - 0 \times 0)\mathbf{i} - (0 \times 1 - 0 \times 0)\mathbf{j} + (0 \times 0 - 1 \times 0)\mathbf{k}$

$= \mathbf{i}$

$\mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$

$= (0 \times 0 - 1 \times 1)\mathbf{i} - (0 \times 0 - 1 \times 0)\mathbf{j} + (0 \times 1 - 0 \times 0)\mathbf{k}$

$= -\mathbf{i}$

ii $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{k}$

$= (0 \times 1 - 0 \times 0)\mathbf{i} - (1 \times 1 - 0 \times 0)\mathbf{j} + (1 \times 0 - 0 \times 0)\mathbf{k}$

$= -\mathbf{j}$

$\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k}$

$= (0 \times 0 - 1 \times 0)\mathbf{i} - (0 \times 0 - 1 \times 1)\mathbf{j} + (0 \times 0 - 0 \times 1)\mathbf{k}$

$= \mathbf{j}$

5 **a** $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{k}$

$= (2 - 1)\mathbf{i} - (-4)\mathbf{j} + 2\mathbf{k}$

$= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$= \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

b $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

$= 1 + 12 + 4$

$= 17$

6 **a** $\mathbf{a}\times\mathbf{b}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}\mathbf{k}$$

$$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

b $\mathbf{a}\times\mathbf{c}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}\mathbf{k}$$

$$= -(-1-4)\mathbf{j}$$

$$= 5\mathbf{j}$$

c $$\begin{aligned}(\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c}) &= (2\mathbf{i} - \mathbf{j} - \mathbf{k}) + 5\mathbf{j} \\ &= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}\end{aligned}$$

d $$\begin{aligned}(\mathbf{b}+\mathbf{c}) &= (-\mathbf{j}+\mathbf{k}) + (2\mathbf{i} - \mathbf{k}) \\ &= 2\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\begin{aligned}\therefore\ \mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}\mathbf{k} \\ &= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}\end{aligned}$$

7 **a** $$\begin{aligned}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k} \\ &= (-1-3)\mathbf{i} - (2-3)\mathbf{j} + (2+1)\mathbf{k} \\ &= -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}\end{aligned}$$

The vectors have the form $k\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$.

b $$\begin{aligned}\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 5 & 0 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix}\mathbf{k} \\ &= 6\mathbf{i} - (-2-20)\mathbf{j} - 15\mathbf{k} \\ &= 6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k}\end{aligned}$$

The vectors have the form $k\begin{pmatrix} 6 \\ 22 \\ -15 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$.

c $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i}+\mathbf{j}-2\mathbf{k}$$

The vectors have the form $(-\mathbf{i}+\mathbf{j}-2\mathbf{k})n, \;\; n\in\mathbb{R}, \;\; n\neq 0.$

d $(\mathbf{i}-\mathbf{j}-\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}-3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix}$

$$= \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{k}$$

$$= (3+2)\mathbf{i} - (-3+2)\mathbf{j} + (2+2)\mathbf{k}$$

$$= 5\mathbf{i}+\mathbf{j}+4\mathbf{k}$$

The vectors have the form $(5\mathbf{i}+\mathbf{j}+4\mathbf{k})n, \;\; n\in\mathbb{R}, \;\; n\neq 0.$

8 $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= (6-2)\mathbf{i} - (4+1)\mathbf{j} + (-4-3)\mathbf{k}$$

$$= 4\mathbf{i}-5\mathbf{j}-7\mathbf{k}$$

$\therefore$ vectors perpendicular to **a** and **b** have the form $k\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}, \;\; k\in\mathbb{R}, \;\; k\neq 0.$

Now $|\,\mathbf{a}\times\mathbf{b}\,| = \sqrt{4^2+(-5)^2+(-7)^2}$

$$= \sqrt{90}$$

$$= 3\sqrt{10}$$

$\therefore$ the two vectors of length 5 units which are perpendicular to both **a** and **b** are

$$\pm\tfrac{5}{3\sqrt{10}}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} = \pm\tfrac{5\sqrt{10}}{30}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

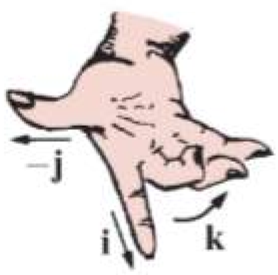
$$= \pm\tfrac{\sqrt{10}}{6}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

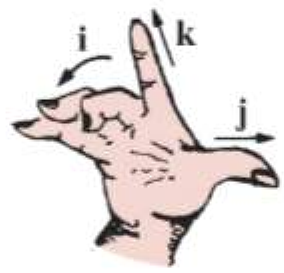
EXERCISE 9L.2

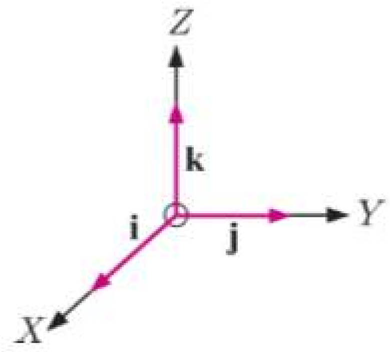
1 **a** $\mathbf{i}\times\mathbf{k}=\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\times1-0\times0\\0\times0-1\times1\\1\times0-0\times0\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}=-\mathbf{j}$

$\mathbf{k}\times\mathbf{i}=\begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\times0-1\times0\\1\times1-0\times0\\0\times0-0\times1\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}=\mathbf{j}$

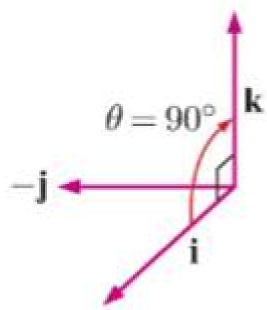
Yes, the **right hand rule** does accurately give the direction.





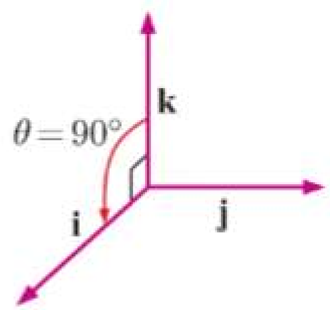


b If **u** is the unit vector in the direction $\mathbf{i}\times\mathbf{k}$, then by the right hand rule, $\mathbf{u}=-\mathbf{j}$.



$$\begin{aligned}|\mathbf{i}|\,|\mathbf{k}|\sin\theta\,\mathbf{u}&=1\times1\times\sin90^\circ\times(-\mathbf{j})\\&=-\mathbf{j}\\&=\mathbf{i}\times\mathbf{k}\end{aligned}$$

If **u** is the unit vector in the direction $\mathbf{k}\times\mathbf{i}$, then by the right hand rule, $\mathbf{u}=\mathbf{j}$.



$$\begin{aligned}|\mathbf{k}|\,|\mathbf{i}|\sin\theta\,\mathbf{u}&=1\times1\times\sin90^\circ\times\mathbf{j}\\&=\mathbf{j}\\&=\mathbf{k}\times\mathbf{i}\end{aligned}$$

2 **a** $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$= 2 + 0 - 3$
$= -1$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{k} \\ &= \mathbf{i} - (-2-3)\mathbf{j} + \mathbf{k} \\ &= \mathbf{i} + 5\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}\end{aligned}$$

b $|\,\mathbf{a}\,| = \sqrt{2^2 + (-1)^2 + 3^2}$
$= \sqrt{4+1+9}$
$= \sqrt{14}$

$|\,\mathbf{b}\,| = \sqrt{1^2 + 0^2 + (-1)^2}$
$= \sqrt{1+0+1}$
$= \sqrt{2}$

$\mathbf{a} \bullet \mathbf{b} = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \cos\theta$

$$\begin{aligned}\therefore\ \cos\theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\,\mathbf{a}\,|\,|\,\mathbf{b}\,|} \\ &= \frac{-1}{\sqrt{14}\sqrt{2}} \qquad \{\mathbf{a} \bullet \mathbf{b} = -1 \text{ from } \mathbf{a}\} \\ &= -\tfrac{1}{\sqrt{28}}\end{aligned}$$

c $\sin^2\theta + \cos^2\theta = 1$

$$\begin{aligned}\therefore\ \sin\theta &= \pm\sqrt{1 - \cos^2\theta} \\ &= \pm\sqrt{1 - \left(\tfrac{1}{\sqrt{28}}\right)^2} \\ &= \pm\sqrt{\tfrac{27}{28}}\end{aligned}$$

But since θ is the angle between two vectors, $0° \leqslant \theta \leqslant 180°$.

$\therefore\ \sin\theta \geqslant 0$

$\therefore\ \sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$

d $|\,\mathbf{a} \times \mathbf{b}\,| = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \sin\theta$

$$\begin{aligned}\sin\theta &= \frac{|\,\mathbf{a} \times \mathbf{b}\,|}{|\,\mathbf{a}\,|\,|\,\mathbf{b}\,|} \\ &= \frac{\left|\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}\right|}{\sqrt{14}\sqrt{2}} \qquad \{\text{from } \mathbf{a} \text{ and } \mathbf{b}\} \\ &= \frac{\sqrt{1+25+1}}{\sqrt{28}} \\ &= \frac{\sqrt{27}}{\sqrt{28}}\end{aligned}$$

3 $(\Rightarrow)$ If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, then $|\mathbf{a}\times\mathbf{b}|=0$

$$\therefore\ |\mathbf{a}|\,|\mathbf{b}|\sin\theta=0$$

$$\therefore\ \sin\theta=0 \qquad \{|\mathbf{a}|,\ |\mathbf{b}|\neq 0 \text{ as } \mathbf{a},\mathbf{b}\neq\mathbf{0}\}$$

$$\therefore\ \theta=0^\circ \text{ or } 180^\circ$$

So, the angle between $\mathbf{a}$ and $\mathbf{b}$ is 0° or 180°.
In either case, $\mathbf{a}$ is parallel to $\mathbf{b}$.

$\therefore$ if $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}\times\mathbf{b}=\mathbf{0} \Rightarrow \mathbf{a}$ is parallel to $\mathbf{b}$.

$(\Leftarrow)$ If $\mathbf{a}$ is parallel to $\mathbf{b}$ then $\mathbf{b}=k\mathbf{a}$, $k\in\mathbb{R}$, $k\neq 0$

$$\begin{aligned}\therefore\ \mathbf{a}\times\mathbf{b}&=\mathbf{a}\times(k\mathbf{a})\\&=k(\mathbf{a}\times\mathbf{a})\\&=k\mathbf{0}\\&=\mathbf{0}\end{aligned}$$

$\therefore$ $\mathbf{a}$ is parallel to $\mathbf{b}$ $\Rightarrow$ $\mathbf{a}\times\mathbf{b}=\mathbf{0}$

$\therefore$ if $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}\times\mathbf{b}=\mathbf{0} \Leftrightarrow \mathbf{a}$ is parallel to $\mathbf{b}$.

4 **a**

$$\begin{aligned}\mathbf{p}\bullet\mathbf{q}&=\begin{pmatrix}2\\1\\-5\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\1\end{pmatrix}\\&=6-1-5\\&=0\end{aligned}$$

$\therefore$ $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

b

$$\begin{aligned}\mathbf{p}\times\mathbf{q}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&-5\\3&-1&1\end{vmatrix}\\&=\begin{vmatrix}1&-5\\-1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&-5\\3&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&1\\3&-1\end{vmatrix}\mathbf{k}\\&=(1-5)\mathbf{i}-(2+15)\mathbf{j}+(-2-3)\mathbf{k}\\&=-4\mathbf{i}-17\mathbf{j}-5\mathbf{k}\\&=\begin{pmatrix}-4\\-17\\-5\end{pmatrix}\end{aligned}$$

c

$$\begin{aligned}|\mathbf{p}|\,|\mathbf{q}|&=\sqrt{2^2+1^2+(-5)^2}\sqrt{3^2+(-1)^2+1^2}\\&=\sqrt{30}\sqrt{11}\\&=\sqrt{330}\end{aligned}$$

$$\begin{aligned}|\mathbf{p}\times\mathbf{q}|&=\sqrt{(-4)^2+(-17)^2+(-5)^2}\\&=\sqrt{16+289+25}\\&=\sqrt{330}\\&=|\mathbf{p}|\,|\mathbf{q}| \quad \checkmark\end{aligned}$$

5 **a** $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$

$\therefore \ \mathbf{0} = \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c}$

$\therefore \ \mathbf{0} = (\mathbf{b} - \mathbf{a}) \times \mathbf{c}$

$\therefore \ \overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$.

b $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\therefore \ \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$\therefore \ \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$

$\therefore \ -\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$

{since $\mathbf{b} \times \mathbf{b} = \mathbf{0}$, $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$}

$\therefore \ \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

c $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}, \quad \mathbf{c} \neq \mathbf{0}$

$\therefore \ \mathbf{b} \times \mathbf{c} - \mathbf{c} \times \mathbf{a} = \mathbf{0}$

$\therefore \ \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{c} = \mathbf{0} \qquad \{-\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}\}$

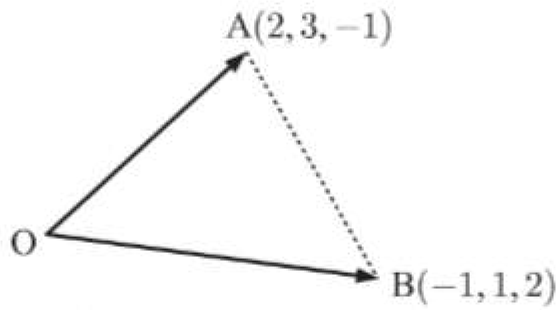
$\therefore \ (\mathbf{b} + \mathbf{a}) \times \mathbf{c} = \mathbf{0}$

$\therefore$ since $\mathbf{c} \neq \mathbf{0}$, $\mathbf{b} + \mathbf{a}$ and $\mathbf{c}$ must be parallel, or $\mathbf{a} + \mathbf{b} = \mathbf{0}$

$\therefore \ \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = k\mathbf{c}, \quad k \in \mathbb{R}$

EXERCISE 9L.3

1 **a** $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$



b $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{k}$$

$$= 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

$$= \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$$

c Area of triangle OAB $= \frac{1}{2} \mid \overrightarrow{OA} \times \overrightarrow{OB} \mid$

$= \frac{1}{2}\sqrt{7^2 + (-3)^2 + 5^2}$

$= \frac{\sqrt{83}}{2}$ units^2

2 **a** A(2, 1, 1), B(4, 3, 0), C(1, 3, −2)

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{aligned} \therefore\ \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\ &= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore\ \text{area} &= \tfrac{1}{2} \,|\, \overrightarrow{AB} \times \overrightarrow{AC} \,| \\ &= \tfrac{1}{2} \,|\, -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} \,| \\ &= \tfrac{1}{2} \sqrt{(-4)^2 + 7^2 + 6^2} \\ &= \tfrac{\sqrt{101}}{2} \text{ units}^2 \end{aligned}$$

b A(0, 0, 0), B(−1, 2, 3), C(1, 2, 6)

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{aligned} \therefore\ \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore\ \text{area} &= \tfrac{1}{2} \,|\, \overrightarrow{AB} \times \overrightarrow{AC} \,| \\ &= \tfrac{1}{2} \,|\, 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k} \,| \\ &= \tfrac{1}{2} \sqrt{6^2 + 9^2 + (-4)^2} \\ &= \tfrac{\sqrt{133}}{2} \text{ units}^2 \end{aligned}$$

c A(1, 3, 2), B(2, −1, 0), C(1, 10, 6)

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 0 \\ 7 \\ 4 \end{pmatrix}$$

$$\therefore \ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 0 & 7 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & -2 \\ 7 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -4 \\ 0 & 7 \end{vmatrix} \mathbf{k}$$

$$= -2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$$

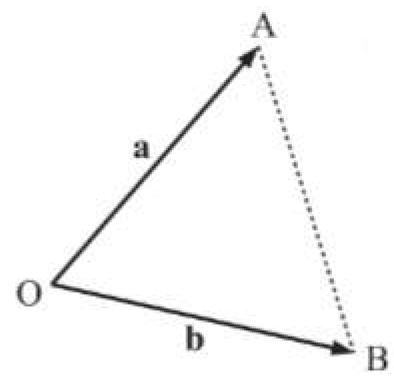
$$\therefore \ \text{area} = \tfrac{1}{2} \,|\, \overrightarrow{AB} \times \overrightarrow{AC} \,|$$

$$= \tfrac{1}{2} \,|-2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}|$$

$$= \tfrac{1}{2}\sqrt{(-2)^2 + (-4)^2 + 7^2}$$

$$= \tfrac{\sqrt{69}}{2} \text{ units}^2$$

3 **a** Area $= \frac{1}{2} \,|\, \mathbf{a} \times \mathbf{b} \,|$ units^2



b **i** $\overrightarrow{BO} = -\overrightarrow{OB}$

$= -\mathbf{b}$

$\overrightarrow{BA} = (-\mathbf{b}) + \mathbf{a}$

$= \mathbf{a} - \mathbf{b}$

ii Area $= \frac{1}{2} \,|\, \overrightarrow{BO} \times \overrightarrow{BA} \,|$

$= \frac{1}{2} \,|-\mathbf{b} \times (\mathbf{a} - \mathbf{b})\,|$ units^2

c

$$-\mathbf{b} \times (\mathbf{a} - \mathbf{b}) = -\mathbf{b} \times (\mathbf{a} + (-\mathbf{b}))$$

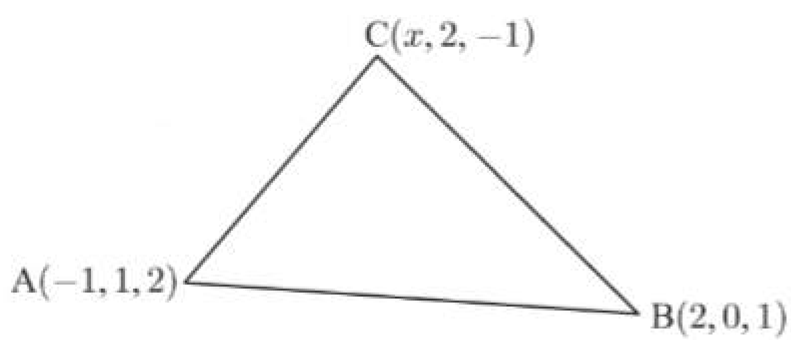
$$= (-\mathbf{b} \times \mathbf{a}) + (-\mathbf{b} \times -\mathbf{b})$$

$$= \mathbf{a} \times \mathbf{b}$$

$$\therefore \ \tfrac{1}{2} \,|-\mathbf{b} \times (\mathbf{a} - \mathbf{b})\,| = \tfrac{1}{2} \,|\, \mathbf{a} \times \mathbf{b} \,|$$

$\therefore$ the areas in **a** and **b ii** are equal.

4 $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} x+1 \\ 1 \\ -3 \end{pmatrix}$



$$\therefore \ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -1 \\ x+1 & 1 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ x+1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ x+1 & 1 \end{vmatrix} \mathbf{k}$$

$$= (3+1)\mathbf{i} - (-9 + (x+1))\mathbf{j} + (3 + (x+1))\mathbf{k}$$

$$= 4\mathbf{i} + (8-x)\mathbf{j} + (x+4)\mathbf{k}$$

$$\text{Area of } \triangle\text{ABC} = \tfrac{1}{2}\,|\,\overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}}\,|$$

$$\therefore\ \sqrt{88} = \tfrac{1}{2}\,|\,4\mathbf{i} + (8-x)\mathbf{j} + (x+4)\mathbf{k}\,|$$

$$\therefore\ \sqrt{352} = \sqrt{16 + (8-x)^2 + (x+4)^2}$$

$$\therefore\ 352 = 16 + 64 - 16x + x^2 + x^2 + 8x + 16$$

$$\therefore\ 2x^2 - 8x - 256 = 0$$

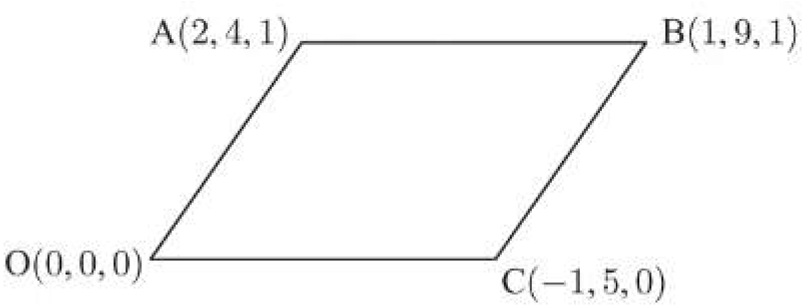
$$\therefore\ x^2 - 4x - 128 = 0$$

$$\therefore\ x = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2}$$

$$= 2 \pm \sqrt{132}$$

$$= 2 \pm 2\sqrt{33}$$

5 **a** $\overrightarrow{\text{OA}} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$



$$\overrightarrow{\text{CB}} = \begin{pmatrix} 1-(-1) \\ 9-5 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{\text{OC}} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{\text{AB}} = \begin{pmatrix} 1-2 \\ 9-4 \\ 1-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

Opposite pairs of sides are parallel and equal in length.

$\therefore$ OABC is a parallelogram.

b Area of OABC $= |\,\overrightarrow{\text{OA}} \times \overrightarrow{\text{OC}}\,|$

$$= \left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 1 \\ -1 & 5 & 0 \end{vmatrix}\right|$$

$$= \left|\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}\mathbf{k}\right|$$

$$= |\,-5\mathbf{i} - \mathbf{j} + 14\mathbf{k}\,|$$

$$= \sqrt{(-5)^2 + (-1)^2 + 14^2}$$

$$= \sqrt{222}\ \text{units}^2$$

6 A$(-1, 2, 2)$, B$(2, -1, 4)$, C$(0, 1, 0)$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$\therefore \quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 2 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 2 \\ -1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{k}$$

$$= 8\mathbf{i} + 8\mathbf{j}$$

$$\therefore \quad \text{area} = |\, \overrightarrow{AB} \times \overrightarrow{AC} \,|$$

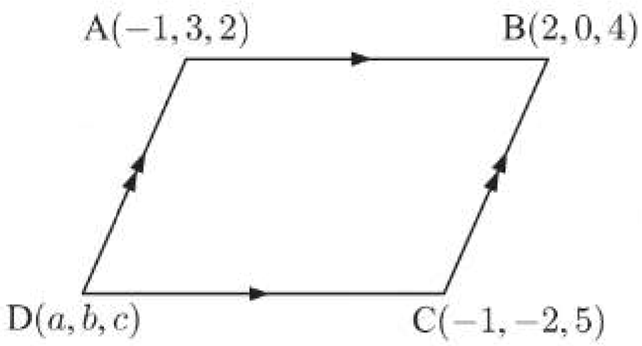
$$= |\, 8\mathbf{i} + 8\mathbf{j} \,|$$

$$= \sqrt{8^2 + 8^2}$$

$$= 8\sqrt{2} \text{ units}^2$$

7 A$(-1, 3, 2)$, B$(2, 0, 4)$, C$(-1, -2, 5)$

a



Suppose D is (a, b, c).

Since $\overrightarrow{AB} = \overrightarrow{DC}$,

$$\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 5-c \end{pmatrix}$$

$\therefore \quad -1 - a = 3, \quad -2 - b = -3, \quad \text{and} \quad 5 - c = 2$

$\therefore \quad a = -4, \quad b = 1, \quad \text{and} \quad c = 3$

$\therefore$ D is $(-4, 1, 3)$.

b $$\overrightarrow{BC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{BA} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$$

$$\therefore \quad \overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -3 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 1 \\ -3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & -2 \\ -3 & 3 \end{vmatrix} \mathbf{k}$$

$$= \mathbf{i} - 9\mathbf{j} - 15\mathbf{k}$$

$$\therefore \quad \text{area} = |\, \overrightarrow{BC} \times \overrightarrow{BA} \,|$$

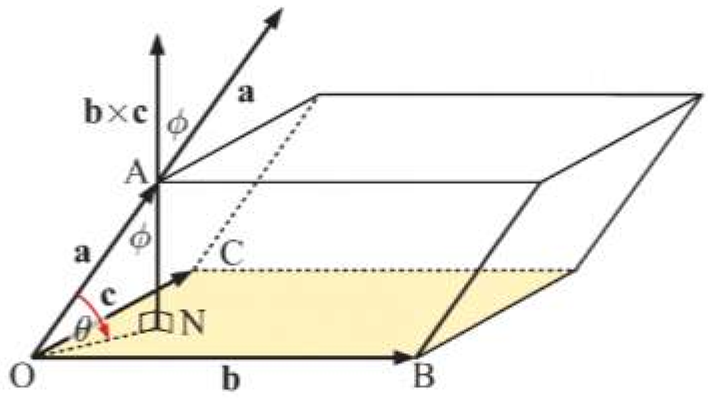
$$= |\, \mathbf{i} - 9\mathbf{j} - 15\mathbf{k} \,|$$

$$= \sqrt{1^2 + (-9)^2 + (-15)^2}$$

$$= \sqrt{307} \text{ units}^2$$

8 **a** **i** Area of the base plane $= |\mathbf{b} \times \mathbf{c}|$ units^2

ii Perpendicular height $= |\mathbf{a}| \sin\theta$ units



b Let ϕ be the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

$\therefore$ perpendicular height $= |\mathbf{a}| \cos\phi$

Volume = area of base × perpendicular height

$$= |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \cos\phi$$

$$= |\mathbf{a}||\mathbf{b} \times \mathbf{c}| \cos\phi$$

$$= |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| \text{ units}^3 \quad \{\phi \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b} \times \mathbf{c},\ \cos\phi \geqslant 0\}$$

c $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\overrightarrow{OB} \times \overrightarrow{OC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k}$$

$$= 3\mathbf{i} - \mathbf{k}$$

$$= \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

Volume of the parallelepiped $= |\overrightarrow{OA} \bullet (\overrightarrow{OB} \times \overrightarrow{OC})|$

$$= \left| \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right|$$

$$= 9 \text{ units}^3$$

EXERCISE 9M

1 **a** $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

i component of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$

$$= \frac{1(-2) + 4(2) + 0(1)}{\sqrt{(-2)^2 + 2^2 + 1^2}}$$

$$= \frac{6}{\sqrt{9}}$$

$$= 2$$

ii $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ -2 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix}\mathbf{k}$

$= 4\mathbf{i} - \mathbf{j} + 10\mathbf{k}$

$\therefore$ component of $\mathbf{a}$ perpendicular to $\mathbf{b} = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$

$= \dfrac{\sqrt{4^2 + (-1)^2 + 10^2}}{3}$ {from **i**}

$= \frac{\sqrt{117}}{3} = \sqrt{13}$

b $\mathbf{a} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

i component of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$

$= \dfrac{-2(-1) + 0(3) + 5(-2)}{\sqrt{(-1)^2 + 3^2 + (-2)^2}}$

$= -\frac{8}{\sqrt{14}}$

ii $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 5 \\ -1 & 3 & -2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 5 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 5 \\ -1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{k}$

$= -15\mathbf{i} - 9\mathbf{j} - 6\mathbf{k}$

$\therefore$ component of $\mathbf{a}$ perpendicular to $\mathbf{b} = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$

$= \dfrac{\sqrt{(-15)^2 + (-9)^2 + (-6)^2}}{\sqrt{14}}$ {from **i**}

$= \frac{\sqrt{342}}{\sqrt{14}} = \frac{\sqrt{171}}{\sqrt{7}}$

c $\mathbf{a} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 3\mathbf{k}$

i component of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$

$= \dfrac{1(-1) + 4(0) + 2(3)}{\sqrt{(-1)^2 + 0^2 + 3^2}}$

$= \frac{5}{\sqrt{10}}$

ii $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ -1 & 0 & 3 \end{vmatrix}$

$= \begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix}\mathbf{k}$

$= 12\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$

$\therefore$ component of $\mathbf{a}$ perpendicular to $\mathbf{b} = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$

$= \dfrac{\sqrt{12^2+(-5)^2+4^2}}{\sqrt{10}}$ {from **i**}

$= \dfrac{\sqrt{185}}{\sqrt{10}} = \dfrac{\sqrt{37}}{\sqrt{2}}$

d $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

i component of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|}$

$= \dfrac{0(1)+2(-1)-4(1)}{\sqrt{1^2+(-1)^2+1^2}}$

$= -\frac{6}{\sqrt{3}}$

ii $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -4 \\ 1 & -1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -4 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{k}$

$= -2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

$\therefore$ component of $\mathbf{a}$ perpendicular to $\mathbf{b} = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$

$= \dfrac{\sqrt{(-2)^2+(-4)^2+(-2)^2}}{\sqrt{3}}$ {from **i**}

$= \dfrac{\sqrt{24}}{\sqrt{3}} = 2\sqrt{2}$

2 $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

a component of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|}$

$= \dfrac{1(5)-3(1)+2(-1)}{\sqrt{5^2+1^2+(-1)^2}}$

$= 0$

So there is no component of $\mathbf{a}$ in the direction of $\mathbf{b}$, which means that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

b $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 5 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix}\mathbf{k}$

$= \mathbf{i} + 11\mathbf{j} + 16\mathbf{k}$

$= \begin{pmatrix} 1 \\ 11 \\ 16 \end{pmatrix}$

Now $|\mathbf{a}\times\mathbf{b}|$ $= \sqrt{1^2+11^2+16^2}$ $= \sqrt{378}$, $|\mathbf{a}|$ $= \sqrt{1^2+(-3)^2+2^2}$ $= \sqrt{14}$ and $|\mathbf{b}|$ $= \sqrt{5^2+1^2+(-1)^2}$ $= \sqrt{27}$

$\therefore$ component of **a** perpendicular to **b** $= \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$

$= \dfrac{\sqrt{378}}{\sqrt{27}}$

$= \sqrt{14} = |\mathbf{a}|$

So all of **a** is perpendicular to **b**, which confirms the result found in **a**.

3 $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \quad \mathbf{q} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

a **i** component of **p** in the direction of $\mathbf{q} = \dfrac{\mathbf{p}\bullet\mathbf{q}}{|\mathbf{q}|}$

$= \dfrac{2(-3)+1(4)-3(1)}{\sqrt{(-3)^2+4^2+1^2}}$

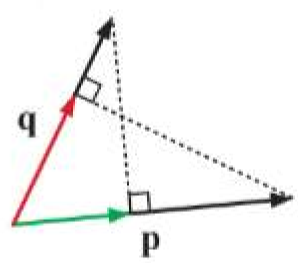
$= -\frac{5}{\sqrt{26}}$

ii component of **q** in the direction of $\mathbf{p} = \dfrac{\mathbf{q}\bullet\mathbf{p}}{|\mathbf{p}|}$

$= \dfrac{-3(2)+4(1)+1(-3)}{\sqrt{2^2+1^2+(-3)^2}}$

$= -\frac{5}{\sqrt{14}}$

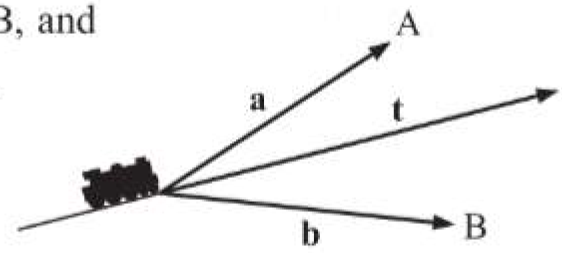
b



The red line is the component of **p** in the direction of **q**, and the green line is the component of **q** in the direction of **p**.

The answers in **a** are not the same because the lines are different lengths.

4 Let $\mathbf{a} = 220\mathbf{i} + 200\mathbf{j} + 40\mathbf{k}$ be the force vector of strongman A,
let $\mathbf{b} = 180\mathbf{i} + 240\mathbf{j} + 60\mathbf{k}$ be the force vector of strongman B, and
let $\mathbf{t} = 7\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ be the direction vector of the train tracks.



a $|\mathbf{a}| = \sqrt{220^2 + 200^2 + 40^2} = \sqrt{90\,000} = 300$

Strongman A exerts 300 N.

$|\mathbf{b}| = \sqrt{180^2 + 240^2 + 60^2} = \sqrt{93\,600} = 60\sqrt{26} \approx 306$

Strongman B exerts about 306 N.

b component of $\mathbf{a}$ in the direction of $\mathbf{t} = \dfrac{\mathbf{a} \bullet \mathbf{t}}{|\mathbf{t}|}$

$$= \frac{220(7) + 200(6) + 40(1)}{\sqrt{7^2 + 6^2 + 1^2}}$$

$$= \frac{2780}{\sqrt{86}} \approx 299.8$$

Strongman A exerts about 299.8 N in the direction of the train's motion.

component of $\mathbf{b}$ in the direction of $\mathbf{t} = \dfrac{\mathbf{b} \bullet \mathbf{t}}{|\mathbf{t}|}$

$$= \frac{180(7) + 240(6) + 60(1)}{\sqrt{7^2 + 6^2 + 1^2}}$$

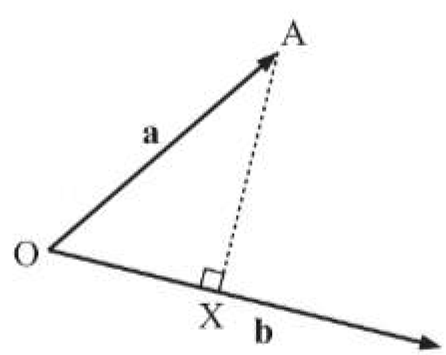
$$= \frac{2760}{\sqrt{86}} \approx 297.6$$

Strongman B exerts about 297.6 N in the direction of the train's motion.

c No, strongman B is exerting the most force in total (from **a**), but strongman A is exerting more force in the direction of the train's motion (from **b**), and therefore is contributing the most to the train's movement.

5 **a** The component of $\mathbf{a}$ perpendicular to $\mathbf{b}$ is in the plane containing $\mathbf{a}$ and $\mathbf{b}$. So, it is perpendicular to $\mathbf{a} \times \mathbf{b}$.
This component is perpendicular to both $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, so it has direction $\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$.

b $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$



i OX = component of $\mathbf{a}$ in the direction of $\mathbf{b}$

$$= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$$

$$= \frac{2(3) + 1(0) + 2(1)}{\sqrt{3^2 + 0^2 + 1^2}}$$

$$= \frac{8}{\sqrt{10}}$$

$\therefore$ $\overrightarrow{OX}$ is the vector in the direction of $\mathbf{b}$ with magnitude $\frac{8}{\sqrt{10}}$.

Now $|\mathbf{b}| = \sqrt{10}$, so the unit vector in the direction of $\mathbf{b}$ is $\frac{1}{\sqrt{10}}\begin{pmatrix}3\\0\\1\end{pmatrix}$.

$$\therefore \ \overrightarrow{\text{OX}} = \tfrac{8}{\sqrt{10}} \times \tfrac{1}{\sqrt{10}}\begin{pmatrix}3\\0\\1\end{pmatrix}$$
$$= \begin{pmatrix}\frac{12}{5}\\0\\\frac{4}{5}\end{pmatrix}$$

ii $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 2\\ 3 & 0 & 1\end{vmatrix}$$
$$= \begin{vmatrix}1 & 2\\0 & 1\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 2\\3 & 1\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & 1\\3 & 0\end{vmatrix}\mathbf{k}$$
$$= \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$
$$= \begin{pmatrix}1\\4\\-3\end{pmatrix}$$

$$\therefore \ \text{XA} = \text{component of } \mathbf{a} \text{ perpendicular to } \mathbf{b}$$
$$= \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$$
$$= \frac{\sqrt{1^2+4^2+(-3)^2}}{\sqrt{10}}$$
$$= \tfrac{\sqrt{26}}{\sqrt{10}}$$

From **a**, $\overrightarrow{\text{XA}}$ is the vector in the direction of $\mathbf{b}\times(\mathbf{a}\times\mathbf{b})$ with magnitude $\frac{\sqrt{26}}{\sqrt{10}}$.

Now $$\mathbf{b}\times(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 0 & 1\\ 1 & 4 & -3\end{vmatrix}$$
$$= \begin{vmatrix}0 & 1\\4 & -3\end{vmatrix}\mathbf{i} - \begin{vmatrix}3 & 1\\1 & -3\end{vmatrix}\mathbf{j} + \begin{vmatrix}3 & 0\\1 & 4\end{vmatrix}\mathbf{k}$$
$$= -4\mathbf{i} + 10\mathbf{j} + 12\mathbf{k}$$
$$= \begin{pmatrix}-4\\10\\12\end{pmatrix}$$

$$\therefore \ |\mathbf{b}\times(\mathbf{a}\times\mathbf{b})| = \sqrt{(-4)^2+10^2+12^2} = \sqrt{260}$$

So the unit vector in the direction of $\mathbf{b}\times(\mathbf{a}\times\mathbf{b})$ is $\frac{1}{\sqrt{260}}\begin{pmatrix}-4\\10\\12\end{pmatrix}$.

$$\therefore \quad \overrightarrow{XA} = \frac{\sqrt{26}}{\sqrt{10}} \times \frac{1}{\sqrt{260}} \begin{pmatrix} -4 \\ 10 \\ 12 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{5} \\ 1 \\ \frac{6}{5} \end{pmatrix}$$

iii (1) $\overrightarrow{OX} \bullet \overrightarrow{XA} = \begin{pmatrix} \frac{12}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} \bullet \begin{pmatrix} -\frac{2}{5} \\ 1 \\ \frac{6}{5} \end{pmatrix} = \frac{12}{5}\left(-\frac{2}{5}\right) + 0(1) + \frac{4}{5}\left(\frac{6}{5}\right) = 0$

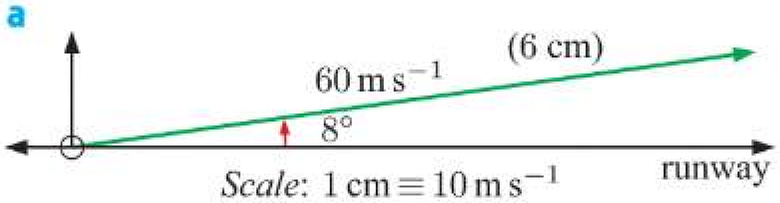
$\therefore$ $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{XA}$.

(2) $\overrightarrow{OX} + \overrightarrow{XA} = \begin{pmatrix} \frac{12}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} + \begin{pmatrix} -\frac{2}{5} \\ 1 \\ \frac{6}{5} \end{pmatrix}$

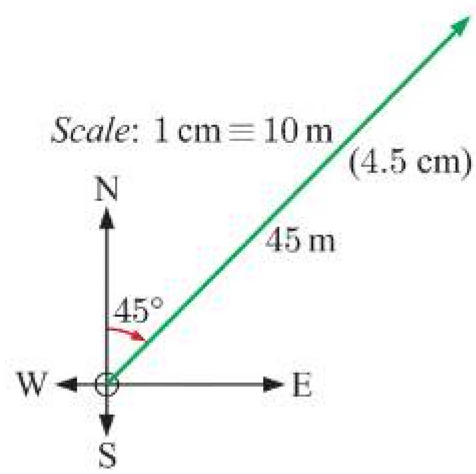
$$= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \mathbf{a}$$

REVIEW SET 9A

1 **a**



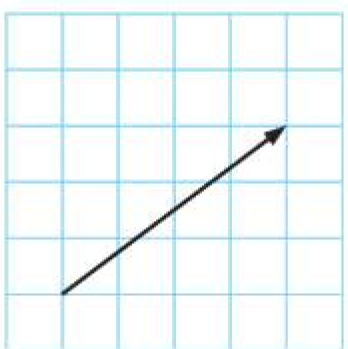
b



2 **a** $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

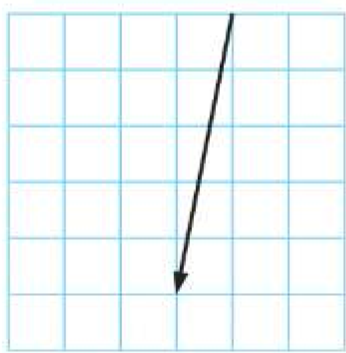
b $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

3 **a**



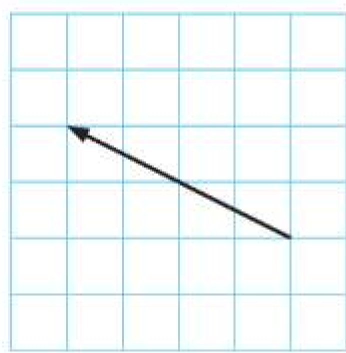
$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\mathbf{i} + 3\mathbf{j}$

b



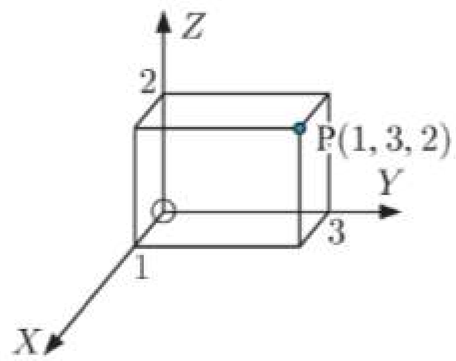
$\begin{pmatrix} -1 \\ -5 \end{pmatrix} = -\mathbf{i} - 5\mathbf{j}$

c



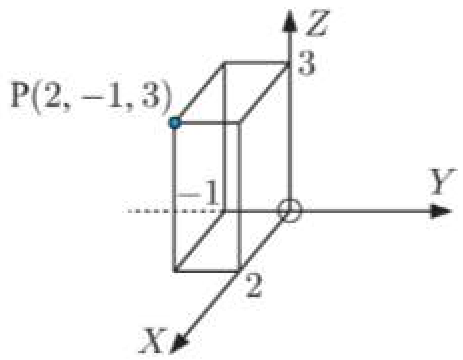
$\begin{pmatrix} -4 \\ 2 \end{pmatrix} = -4\mathbf{i} + 2\mathbf{j}$

4 **a**



$$\begin{aligned} OP &= \sqrt{(1-0)^2+(3-0)^2+(2-0)^2} \\ &= \sqrt{1+9+4} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

b



$$\begin{aligned} OP &= \sqrt{(2-0)^2+(-1-0)^2+(3-0)^2} \\ &= \sqrt{4+1+9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

5 $$\begin{aligned} \overrightarrow{SP} &= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP} \\ &= -\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix} \end{aligned}$$

6 $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}$

a $$\begin{aligned} \mathbf{a}-\mathbf{b} &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 3-6 \\ -1-(-1) \\ 2-(-4) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} \end{aligned}$$

b $$\begin{aligned} |\,\mathbf{a}\,| &= \sqrt{3^2+(-1)^2+2^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

c $2\mathbf{a}+3\mathbf{b}=2\begin{pmatrix}3\\-1\\2\end{pmatrix}+3\begin{pmatrix}6\\-1\\-4\end{pmatrix}$

$=\begin{pmatrix}6\\-2\\4\end{pmatrix}+\begin{pmatrix}18\\-3\\-12\end{pmatrix}$

$=\begin{pmatrix}24\\-5\\-8\end{pmatrix}$

d $-\frac{1}{3}\mathbf{a}=-\frac{1}{3}\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$=\begin{pmatrix}-1\\\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$

e $|\mathbf{b}|\,\mathbf{a}=\sqrt{6^2+(-1)^2+(-4)^2}\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$=\sqrt{53}\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$=\begin{pmatrix}3\sqrt{53}\\-\sqrt{53}\\2\sqrt{53}\end{pmatrix}$

f $\mathbf{b}-2\mathbf{a}=\begin{pmatrix}6\\-1\\-4\end{pmatrix}-2\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$=\begin{pmatrix}6\\-1\\-4\end{pmatrix}-\begin{pmatrix}6\\-2\\4\end{pmatrix}$

$=\begin{pmatrix}0\\1\\-8\end{pmatrix}$

$\therefore\ |\mathbf{b}-2\mathbf{a}|=\sqrt{0^2+1^2+(-8)^2}$

$=\sqrt{65}$ units

7 $3\begin{pmatrix}1\\-2\\-1\end{pmatrix}+a\begin{pmatrix}2\\0\\6\end{pmatrix}-\begin{pmatrix}b\\-1\\-4\end{pmatrix}=\begin{pmatrix}4\\c\\7\end{pmatrix}$

$\therefore\ \begin{pmatrix}3\\-6\\-3\end{pmatrix}+\begin{pmatrix}2a\\0\\6a\end{pmatrix}-\begin{pmatrix}b\\-1\\-4\end{pmatrix}=\begin{pmatrix}4\\c\\7\end{pmatrix}$

$\therefore\ \begin{pmatrix}3+2a-b\\-6+0+1\\-3+6a+4\end{pmatrix}=\begin{pmatrix}4\\c\\7\end{pmatrix}$

$\therefore\ \begin{pmatrix}3+2a-b\\-5\\6a+1\end{pmatrix}=\begin{pmatrix}4\\c\\7\end{pmatrix}$

So, $3+2a-b=4$, $-5=c$, and $6a+1=7$

$\therefore\ b=2a-1$, $c=-5$, and $a=1$

$\therefore\ a=1,\ b=1,\ c=-5$

8 **a** $\overrightarrow{BC} = \begin{pmatrix} -1-5 \\ -4-0 \\ 1-(-1) \end{pmatrix}$ $\overrightarrow{AB} = \begin{pmatrix} 5-2 \\ 0-1 \\ -1-(-4) \end{pmatrix}$

$= \begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$

$= 2\,\overrightarrow{BM}$

$$\begin{aligned}\text{Now}\quad \overrightarrow{AM} &= \overrightarrow{AB} + \overrightarrow{BM} \\ &= \overrightarrow{AB} + \tfrac{1}{2}\,\overrightarrow{BC} \\ &= \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore\ |\,\overrightarrow{AM}\,| &= \sqrt{0^2 + (-3)^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \text{ units}\end{aligned}$$

b Let D have position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Now $\overrightarrow{AC} = \overrightarrow{BD}$

$$\therefore\ \begin{pmatrix} -1-2 \\ -4-1 \\ 1-(-4) \end{pmatrix} = \begin{pmatrix} x-5 \\ y-0 \\ z-(-1) \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} x-5 \\ y \\ z+1 \end{pmatrix}$$

So, $x - 5 = -3$, $y = -5$, and $z + 1 = 5$

$\therefore\ x = 2$ $\therefore\ z = 4$

$\therefore$ D has position vector $2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$.

9 The vectors are parallel, so $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} = k\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$

So, $3k = -12$, $km = -20$, $kn = 2$

$\therefore\ k = -4$

$\therefore\ m = 5$, $n = -\frac{1}{2}$

10 P(−6, 8, 2), Q(4, 6, 8), R(19, 3, 17)

$$\overrightarrow{PQ} = \begin{pmatrix} 4-(-6) \\ 6-8 \\ 8-2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 19-4 \\ 3-6 \\ 17-8 \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

So, $\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{QR}$

$\therefore$ P, Q, and R are collinear.

11 $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ has length $\sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$ units

$\therefore$ a unit vector in the opposite direction is $-\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ a vector of length 5 units in the opposite direction is $-\frac{5}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

12 **a** $\mathbf{p} \bullet \mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$= 3 \times (-1) + (-2) \times 5$

$= -3 - 10$

$= -13$

b $\mathbf{p} - \mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

$\therefore$ $\mathbf{q} \bullet (\mathbf{p} - \mathbf{r}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

$= (-1) \times 6 + 5 \times (-6)$

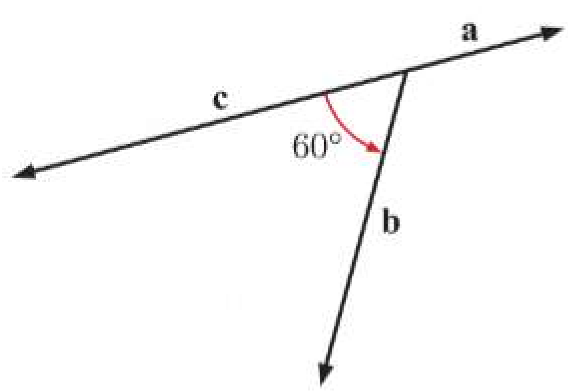
$= -6 - 30$

$= -36$

13 **a** $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$= 2 \times 4 \times \cos 120°$

$= -4$



b $\mathbf{b} \bullet \mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos\theta$

$= 4 \times 5 \times \cos 60°$

$= 10$

c $\mathbf{a} \bullet \mathbf{c} = |\mathbf{a}||\mathbf{c}|\cos\theta$

$= 2 \times 5 \times \cos 180°$

$= -10$

14 **a** $\mathbf{u} \bullet \mathbf{v} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

$= -4(-1) + 2(3) + 1(-2)$

$= 4 + 6 - 2$

$= 8$

b If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then
$$\begin{aligned}\cos\theta &= \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\\ &= \frac{4+6-2}{\sqrt{(-4)^2+2^2+1^2}\sqrt{(-1)^2+3^2+(-2)^2}}\\ &= \frac{8}{\sqrt{21}\sqrt{14}}\\ \therefore\ \theta &= \cos^{-1}\left(\frac{8}{\sqrt{21}\sqrt{14}}\right)\\ &\approx 62.2^\circ\end{aligned}$$

15 If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then
$$\begin{aligned}\cos\theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\\ &= \frac{3(2)+1(5)+(-2)(1)}{\sqrt{9+1+4}\sqrt{4+25+1}}\\ &= \frac{9}{\sqrt{14}\sqrt{30}}\\ \therefore\ \theta &= \cos^{-1}\left(\frac{9}{\sqrt{14}\sqrt{30}}\right)\\ &\approx 64.0^\circ\end{aligned}$$

16
$$\begin{aligned}\begin{pmatrix}2-t\\3\\t\end{pmatrix} \bullet \begin{pmatrix}t\\4\\t+1\end{pmatrix} &= 0\\ \therefore\ (2-t)t+12+t(t+1) &= 0\\ \therefore\ 2t-t^2+12+t^2+t &= 0\\ \therefore\ 3t+12 &= 0\\ \therefore\ t &= -4\end{aligned}$$

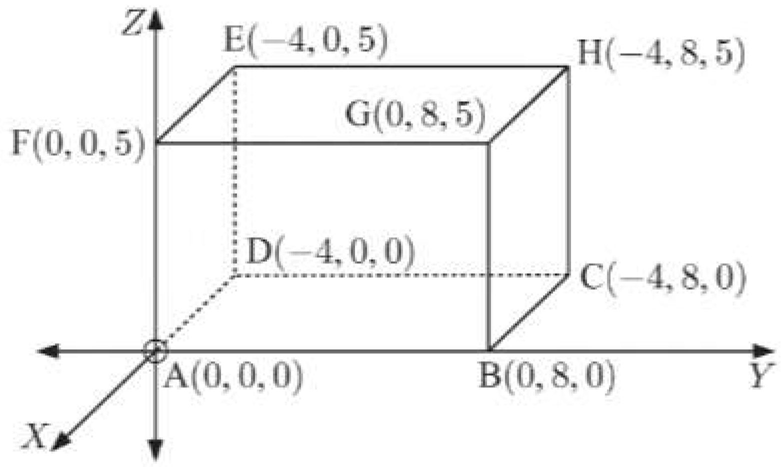
17 Vectors parallel to $\mathbf{i}+r\mathbf{j}+2\mathbf{k}$ have form $k\begin{pmatrix}1\\r\\2\end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$.

If these vectors are perpendicular to $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ then
$$\begin{aligned}k\begin{pmatrix}1\\r\\2\end{pmatrix} \bullet \begin{pmatrix}2\\2\\-1\end{pmatrix} &= 0\\ k(2+2r-2) &= 0\\ 2kr &= 0\\ \text{but } k \neq 0 \quad \therefore\ r &= 0\end{aligned}$$

One of these vectors is $\mathbf{i}+2\mathbf{k}$, which has length $\sqrt{1^2+0^2+2^2} = \sqrt{5}$ units.

$\therefore$ $\frac{1}{\sqrt{5}}(\mathbf{i}+2\mathbf{k}) = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$ is a unit vector parallel to $\mathbf{i}+r\mathbf{j}+2\mathbf{k}$, and perpendicular to $2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

18 Let A be the origin.



$$\overrightarrow{AG} = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$$

$$\begin{aligned}\overrightarrow{AG} \bullet \overrightarrow{AC} &= \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} \\ &= 0 \times (-4) + 8 \times 8 + 5 \times 0 \\ &= 64\end{aligned}$$

$$\begin{aligned}|\overrightarrow{AG}| &= \sqrt{0^2 + 8^2 + 5^2} \\ &= \sqrt{64 + 25} \\ &= \sqrt{89}\end{aligned} \qquad \begin{aligned}|\overrightarrow{AC}| &= \sqrt{(-4)^2 + 8^2 + 0^2} \\ &= \sqrt{16 + 64} \\ &= \sqrt{80}\end{aligned}$$

$$\cos(G\widehat{A}C) = \frac{\overrightarrow{AG} \bullet \overrightarrow{AC}}{|\overrightarrow{AG}||\overrightarrow{AC}|}$$

$$\begin{aligned}\therefore \quad G\widehat{A}C &= \cos^{-1}\left(\frac{\overrightarrow{AG} \bullet \overrightarrow{AC}}{|\overrightarrow{AG}||\overrightarrow{AC}|}\right) \\ &= \cos^{-1}\left(\frac{64}{\sqrt{89}\sqrt{80}}\right) \\ &\approx 40.7^\circ\end{aligned}$$

19 **a**
$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}\mathbf{k} \\ &= -3\mathbf{i} - 7\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} -3 \\ -7 \\ 1 \end{pmatrix}\end{aligned}$$

b
$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} &= \begin{pmatrix} -3 \\ -7 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \\ &= -3(1) + (-7)(0) + 1(3) \\ &= 0\end{aligned}$$

c
$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{k} \\ &= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}\end{aligned}$$

20 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix}\mathbf{k}$

$= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$

$\therefore$ vectors perpendicular to $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ have the form $k\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$.

Now $\left|\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}\right| = \sqrt{3^2 + 7^2 + 1^2}$

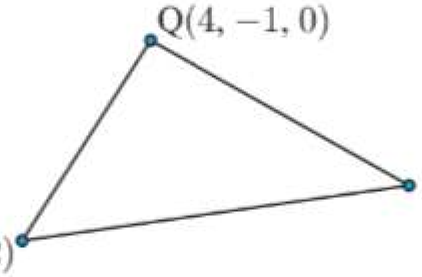
$= \sqrt{9 + 49 + 1}$

$= \sqrt{59}$

$\therefore$ the two vectors of length 3 units which are perpendicular to both $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ are $\pm\frac{3}{\sqrt{59}}\begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$.

21 $\overrightarrow{PQ} = \begin{pmatrix} 4-1 \\ -1-(-3) \\ 0-2 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 5-1 \\ 2-(-3) \\ -1-2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ $\qquad$ $= \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$



$\therefore$ $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -2 \\ 4 & 5 & -3 \end{vmatrix}$

$= \begin{vmatrix} 2 & -2 \\ 5 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}\mathbf{k}$

$= 4\mathbf{i} + \mathbf{j} + 7\mathbf{k}$

$= \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}$

$\therefore$ area of triangle PQR $= \frac{1}{2} \mid \overrightarrow{PQ} \times \overrightarrow{PR} \mid$

$= \frac{1}{2}\sqrt{4^2 + 1^2 + 7^2}$

$= \frac{\sqrt{66}}{2}$ units^2

22 $\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

a component of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$

$$= \frac{4(-1) + 0(2) - 1(3)}{\sqrt{(-1)^2 + 2^2 + 3^2}}$$

$$= -\tfrac{7}{\sqrt{14}} = -\tfrac{\sqrt{7}}{\sqrt{2}}$$

b $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -1 \\ -1 & 2 & 3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -1 \\ -1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 0 \\ -1 & 2 \end{vmatrix} \mathbf{k}$$

$$= 2\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$$

$$= \begin{pmatrix} 2 \\ -11 \\ 8 \end{pmatrix}$$

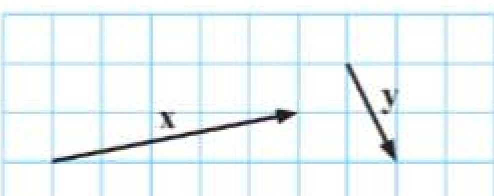
$\therefore$ component of $\mathbf{a}$ perpendicular to $\mathbf{b} = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$

$$= \frac{\sqrt{2^2 + (-11)^2 + 8^2}}{\sqrt{14}} \quad \{\text{from } \mathbf{a}\}$$

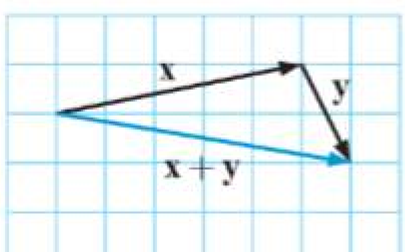
$$= \tfrac{\sqrt{189}}{\sqrt{14}} = \tfrac{3\sqrt{3}}{\sqrt{2}}$$

REVIEW SET 9B

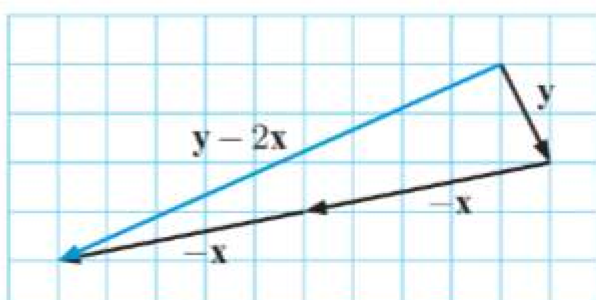
1



a



b



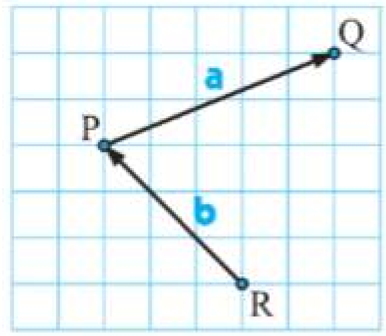
2 **a** $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$

b $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{QR}$

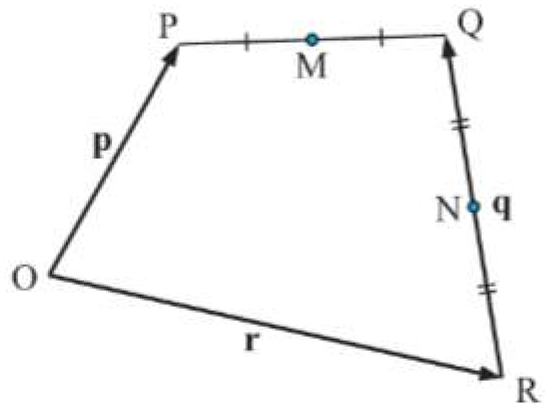
$$= \overrightarrow{PR}$$

3 **a** $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
$= 5\mathbf{i} + 2\mathbf{j}$

b $\overrightarrow{RP} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$
$= -3\mathbf{i} + 3\mathbf{j}$



4



a $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q}$

b $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$

c $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q}$

d $\overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN}$
$= \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}$
$= \frac{1}{2}(-\mathbf{p} + \mathbf{r} + \mathbf{q}) + \frac{1}{2}(-\mathbf{q})$ {from **b**}
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q}$
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r}$

5 **a** $\mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix}$

b $2\mathbf{n} - 3\mathbf{p} = 2\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$
$= \begin{pmatrix} 4 \\ 6 \\ -8 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \\ 18 \end{pmatrix}$
$= \begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix}$

c $\mathbf{m} + \mathbf{p} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$
$= \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}$
$\therefore \ |\mathbf{m} + \mathbf{p}| = \sqrt{25 + 0 + 49}$
$= \sqrt{74}$ units

6 $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

$= -\overrightarrow{AC} + \overrightarrow{AB}$

$= -\begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}$

7 **a** $\overrightarrow{PQ} = \begin{pmatrix} -1-2 \\ 7-(-5) \\ 9-6 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$

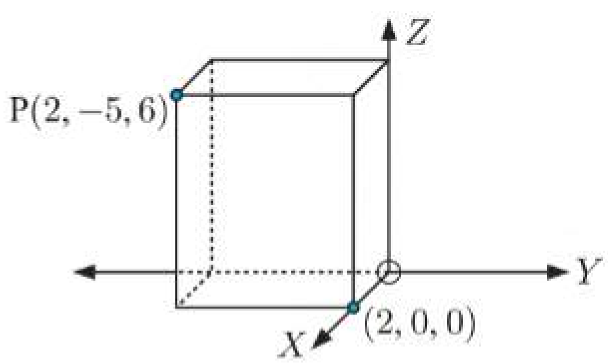
b $PQ = \sqrt{(-3)^2 + 12^2 + 3^2}$

$= \sqrt{9+144+9}$

$= \sqrt{162}$

$= 9\sqrt{2}$ units

c



$\therefore$ the shortest distance from P to the X-axis

$= \sqrt{(2-2)^2 + (0-(-5))^2 + (0-6)^2}$

$= \sqrt{0+25+36}$

$= \sqrt{61}$ units

8 $\sqrt{k^2 + (\frac{1}{\sqrt{2}})^2 + (-k)^2} = 1$

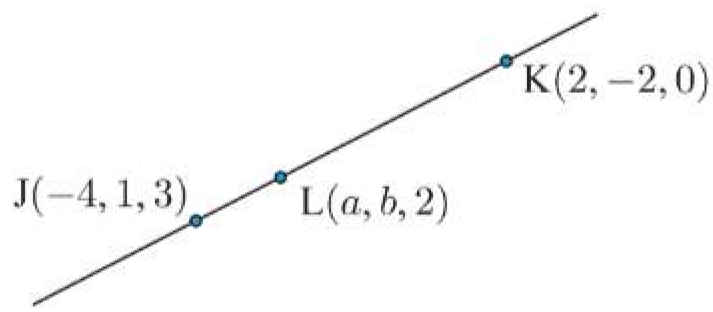
$\therefore\ k^2 + \frac{1}{2} + k^2 = 1$

$\therefore\ 2k^2 = \frac{1}{2}$

$\therefore\ k^2 = \frac{1}{4}$

$\therefore\ k = \pm\frac{1}{2}$

9 J(−4, 1, 3), K(2, −2, 0), L(a, b, 2)



$\overrightarrow{JK} = \begin{pmatrix} 2-(-4) \\ -2-1 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$

$\overrightarrow{JL} = \begin{pmatrix} a-(-4) \\ b-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} a+4 \\ b-1 \\ -1 \end{pmatrix}$

If J, K, and L are collinear then $\overrightarrow{JK}$ is parallel to $\overrightarrow{JL}$.

$\therefore \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} = k \begin{pmatrix} a+4 \\ b-1 \\ -1 \end{pmatrix}$ for some $k \in \mathbb{R},\ k \neq 0$

$\therefore -3 = k(-1)$

$\therefore k = 3$

$\therefore 6 = 3(a+4)$ and $-3 = 3(b-1)$

$\therefore 2 = a+4$ and $-1 = b-1$

$\therefore a = -2$ and $b = 0$

10 $|\,3\mathbf{i} - 2\mathbf{j} + \mathbf{k}\,| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore$ a unit vector in the direction of $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is $\frac{1}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$\therefore$ the two vectors of length 4 units parallel to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ are $\pm\frac{4}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

11 $$\left|\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right| = \sqrt{2^2 + 1^2 + (-2)^2}$$
$$= \sqrt{9}$$
$$= 3 \text{ units}$$

$\therefore$ the vector in the direction of $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ with length 12 units is $\frac{12}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} 8 \\ 4 \\ -8 \end{pmatrix}$.

$\therefore$ if X is 12 units from $(-2, 1, -5)$ in this direction, then X is $(-2+8,\ 1+4,\ -5-8)$
or $(6, 5, -13)$

12 Since $\mathbf{v}$ is parallel to $\mathbf{w}$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is either $0°$ or $180°$.

Now, $$\begin{aligned} \mathbf{v} \bullet \mathbf{w} &= |\,\mathbf{v}\,|\,|\,\mathbf{w}\,| \cos\theta \\ &= 3 \times 2 \times \cos 0° \text{ or } 3 \times 2 \times \cos 180° \\ &= 6(1) \text{ or } 6(-1) \\ &= \pm 6 \end{aligned}$$

13 $\mathbf{p} \bullet (\mathbf{q} - \mathbf{r})$

$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \left[\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix}\right]$

$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 8 \end{pmatrix}$

$= 3 \times (-3) + (-2) \times 8$

$= -9 - 16$

$= -25$

$\mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$

$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$= (3 \times (-2) + (-2) \times 5) - (3 \times 1 + (-2) \times (-3))$

$= (-6 - 10) - (3 + 6)$

$= -16 - 9$

$= -25$

$\therefore \mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) = \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$

14 **a** $\overrightarrow{PQ} = \begin{pmatrix} 4-(-1) \\ 0-2 \\ -1-3 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$

b The vector $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ represents the X-axis, so the angle θ between $\overrightarrow{OX}$ and $\overrightarrow{PQ}$ is the angle between θ and the X-axis.

$$\therefore \cos\theta = \frac{\overrightarrow{PQ} \bullet \overrightarrow{OX}}{|\overrightarrow{PQ}||\overrightarrow{OX}|}$$
$$= \frac{5+0+0}{\sqrt{25+4+16}\sqrt{1+0+0}}$$
$$= \frac{5}{\sqrt{45}}$$
$$\therefore \theta = \cos^{-1}\left(\frac{5}{\sqrt{45}}\right)$$
$$\approx 41.8^\circ$$

15 $\mathbf{p} \bullet \mathbf{q} = (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \bullet (-\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$
$= 2(-1) + (-1)(-4) + 4(2)$
$= -2 + 4 + 8$
$= 10$

$|\mathbf{p}| = \sqrt{2^2 + (-1)^2 + 4^2}$
$= \sqrt{4+1+16}$
$= \sqrt{21}$

$|\mathbf{q}| = \sqrt{(-1)^2 + (-4)^2 + 2^2}$
$= \sqrt{1+16+4}$
$= \sqrt{21}$

If θ is the angle between $\mathbf{p}$ and $\mathbf{q}$, then $\cos\theta = \dfrac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$

$$= \frac{10}{\sqrt{21}\sqrt{21}}$$
$$\therefore \theta = \cos^{-1}\left(\frac{10}{\sqrt{21}\sqrt{21}}\right)$$
$$\approx 61.6^\circ$$

16 As the vectors are perpendicular,

$$\begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix} = 0$$

$$\therefore \quad -4t + (t+2)(1+t) - 3t = 0$$

$$\therefore \quad -4t + t + t^2 + 2 + 2t - 3t = 0$$

$$\therefore \quad t^2 - 4t + 2 = 0$$

$$\therefore \quad t = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2}$$

$$\therefore \quad t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

17 Let $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$
and $\mathbf{v} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

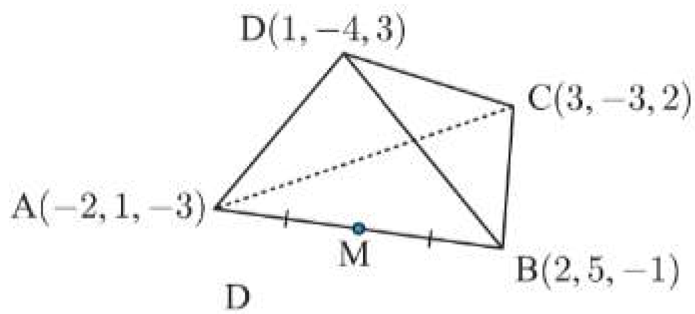
$$\begin{aligned} \mathbf{u} \bullet \mathbf{v} &= 2(-1) - 4(1) + 3(3) \\ &= -2 - 4 + 9 \\ &= 3 \end{aligned}$$

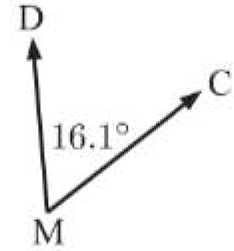
Now
$$\begin{aligned} \cos\theta &= \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \\ &= \frac{3}{\sqrt{4+16+9}\sqrt{1+1+9}} \\ &= \frac{3}{\sqrt{319}} \\ \therefore \quad \theta &= \cos^{-1}\left(\frac{3}{\sqrt{319}}\right) \\ &\approx 80.3^\circ \end{aligned}$$

18 M is $\left(\frac{-2+2}{2}, \frac{1+5}{2}, \frac{-3-1}{2}\right)$ or $(0, 3, -2)$.

$$\therefore \quad \overrightarrow{MD} = \begin{pmatrix} 1-0 \\ -4-3 \\ 3-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix}, \qquad \overrightarrow{MC} = \begin{pmatrix} 3-0 \\ -3-3 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad \cos(D\widehat{M}C) &= \frac{\overrightarrow{MD} \bullet \overrightarrow{MC}}{|\overrightarrow{MD}|\,|\overrightarrow{MC}|} \\ &= \frac{3 + 42 + 20}{\sqrt{1+49+25}\sqrt{9+36+16}} \\ &= \frac{65}{\sqrt{75}\sqrt{61}} \\ \therefore \quad D\widehat{M}C &= \cos^{-1}\left(\frac{65}{\sqrt{75}\sqrt{61}}\right) \\ &\approx 16.1^\circ \end{aligned}$$





19 $$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k}$$
$$= (-2+2)\mathbf{i} - (4+2)\mathbf{j} + (2+1)\mathbf{k}$$
$$= -6\mathbf{j} + 3\mathbf{k}$$
$$= 3(-2\mathbf{j} + \mathbf{k})$$

$\therefore$ perpendicular vectors have the form $k\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$.

20 $$|\mathbf{u} \times \mathbf{v}| = \sqrt{1^2 + (-3)^2 + (-4)^2}$$
$$= \sqrt{1+9+16}$$
$$= \sqrt{26}$$

$$\sin\theta = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$$
$$= \frac{\sqrt{26}}{3 \times 5}$$
$$= \tfrac{\sqrt{26}}{15}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$
$$= \pm\sqrt{1 - \tfrac{26}{225}}$$
$$= \pm\sqrt{\tfrac{199}{225}}$$
$$= \pm\tfrac{\sqrt{199}}{15}$$

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$
$$= 3 \times 5 \times \left(\pm\tfrac{\sqrt{199}}{15}\right)$$
$$= \pm\sqrt{199}$$

So, if θ is acute, $\mathbf{u} \bullet \mathbf{v} = \sqrt{199}$
and if θ is obtuse, $\mathbf{u} \bullet \mathbf{v} = -\sqrt{199}$

21 **a** $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

$$\therefore \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 1 & 1 & -3 \end{vmatrix}$$
$$= \begin{vmatrix} -2 & 2 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{k}$$
$$= 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$$
$$= \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

b $|\mathbf{a}\times\mathbf{b}| = \sqrt{4^2+5^2+3^2}$ {from **a**}

$= \sqrt{50}$

$= 5\sqrt{2}$

$\therefore$ the unit vector in the direction opposite to $\mathbf{a}\times\mathbf{b}$ is $-\frac{1}{5\sqrt{2}}\begin{pmatrix}4\\5\\3\end{pmatrix} = \begin{pmatrix}-\frac{4}{5\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\-\frac{3}{5\sqrt{2}}\end{pmatrix}$

$\therefore$ C is $6\left(-\frac{4}{5\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{3}{5\sqrt{2}}\right)$ which is $\left(-\frac{24}{5\sqrt{2}}, -\frac{6}{\sqrt{2}}, -\frac{18}{5\sqrt{2}}\right)$.

c Area of triangle OAB $= \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|$

$= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

$= \frac{5\sqrt{2}}{2}$ {from **b**}

$= \frac{5}{\sqrt{2}}$ units^2

d Triangle OAB is the base, which has area $\frac{5}{\sqrt{2}}$ units^2.

The apex of the tetrahedron is C, which has a perpendicular height of 6 units above the base.

Volume of the tetrahedron $= \frac{1}{3}$(base area $\times$ perpendicular height)

$= \frac{1}{3}\times\frac{5}{\sqrt{2}}\times 6$

$= \frac{10}{\sqrt{2}}$ units^3

Chapter 10

VECTOR APPLICATIONS

EXERCISE 10A

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ **ii** $x = 1 - \lambda, \quad y = 4 + 2\lambda, \quad \lambda \in \mathbb{R}$

b **i** If the line has direction vector $\mathbf{b}$ perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, then $\mathbf{b} \bullet \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0$

$\therefore \quad \mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ is a reasonable choice

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = 5 - 2\lambda, \quad y = 2 + 5\lambda, \quad \lambda \in \mathbb{R}$

c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ **ii** $x = -6 + 3\lambda, \quad y = 7\lambda, \quad \lambda \in \mathbb{R}$

d **i** If the line has direction vector $\mathbf{b}$ perpendicular to $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, then $\mathbf{b} \bullet \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$

$\therefore \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is a reasonable choice

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = \lambda, \quad y = 2 + 3\lambda, \quad \lambda \in \mathbb{R}$

e **i** Take $(3, 0)$ as our fixed point, so $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

The direction vector $\mathbf{b} = \begin{pmatrix} 7-3 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = 3 + 4\lambda, \quad y = 2\lambda, \quad \lambda \in \mathbb{R}$

f **i** Take $(-2, 5)$ as our fixed point, so $\mathbf{a} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

The direction vector $\mathbf{b} = \begin{pmatrix} 4-(-2) \\ -6-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -11 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = -2 + 6\lambda, \quad y = 5 - 11\lambda, \quad \lambda \in \mathbb{R}$

2 **a** $x = 4 - \lambda, \quad y = -3 + 2\lambda, \quad \lambda \in \mathbb{R}$

b When $\lambda = 0$, $x = 4 - 0 = 4$ and $y = -3 + 2(0) = -3$ $\quad\therefore$ the point is $(4, -3)$.
When $\lambda = 1$, $x = 4 - 1 = 3$ and $y = -3 + 2(1) = -1$ $\quad\therefore$ the point is $(3, -1)$.
When $\lambda = 2$, $x = 4 - 2 = 2$ and $y = -3 + 2(2) = 1$ $\quad\therefore$ the point is $(2, 1)$.
When $\lambda = -1$, $x = 4 - (-1) = 5$ and $y = -3 + 2(-1) = -5$
$\therefore$ the point is $(5, -5)$.
When $\lambda = -3$, $x = 4 - (-3) = 7$ and $y = -3 + 2(-3) = -9$
$\therefore$ the point is $(7, -9)$.

3 **a** When $\lambda = 1$, $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 5+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

$\therefore$ the point is $(0, 8)$.

b $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. It is parallel and in the opposite direction, so it could also be used to describe the direction of the line.

c The line passes through $(0, 8)$ and has direction vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

$\therefore \quad \mathbf{r} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \mu \in \mathbb{R}$ is an alternative vector equation for line L.

4 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = 1 + 2\lambda, \quad y = 3 + \lambda, \quad z = -7 + 3\lambda, \quad \lambda \in \mathbb{R}$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = \lambda, \quad y = 1 + \lambda, \quad z = 2 - 2\lambda, \quad \lambda \in \mathbb{R}$

c **i** Since the line is parallel to the X-axis, it has direction vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = -2 + \lambda, \quad y = 2, \quad z = 1, \quad \lambda \in \mathbb{R}$

d **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = 2\lambda, \quad y = 2 - \lambda, \quad z = -1 + 3\lambda, \quad \lambda \in \mathbb{R}$

e **i** Since the line is perpendicular to the XY-plane, it has direction vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

$$\therefore \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \lambda \in \mathbb{R}$$

ii $x = 3, \quad y = 2, \quad z = -1 + \lambda, \quad \lambda \in \mathbb{R}$

5 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix}-1-1\\3-2\\2-1\end{pmatrix} = \begin{pmatrix}-2\\1\\1\end{pmatrix} \quad \therefore \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\1\end{pmatrix}, \quad \lambda \in \mathbb{R}$

b $\overrightarrow{\text{CD}} = \begin{pmatrix}3-0\\1-1\\-1-3\end{pmatrix} = \begin{pmatrix}3\\0\\-4\end{pmatrix} \quad \therefore \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\-4\end{pmatrix}, \quad \lambda \in \mathbb{R}$

c $\overrightarrow{\text{EF}} = \begin{pmatrix}1-1\\-1-2\\5-5\end{pmatrix} = \begin{pmatrix}0\\-3\\0\end{pmatrix} \quad \therefore \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}0\\-3\\0\end{pmatrix}, \quad \lambda \in \mathbb{R}$

d $\overrightarrow{\text{GH}} = \begin{pmatrix}5-0\\-1-1\\3-(-1)\end{pmatrix} = \begin{pmatrix}5\\-2\\4\end{pmatrix} \quad \therefore \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} + \lambda\begin{pmatrix}5\\-2\\4\end{pmatrix}, \quad \lambda \in \mathbb{R}$

6 **a** $\begin{pmatrix}-2\\0\\3\end{pmatrix}$ **b** $\begin{pmatrix}-1\\1\\-3\end{pmatrix}$

7 $x = 1 - \lambda, \quad y = 3 + \lambda, \quad z = 3 - 2\lambda, \quad \lambda \in \mathbb{R}$

a The line meets the YZ-plane when $x = 0$ $\quad \therefore \; 1 - \lambda = 0$

$\therefore \; \lambda = 1$

When $\lambda = 1$, $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$

$\therefore$ the point is $(0, 4, 1)$.

b The line meets the XZ-plane when $y = 0$ $\quad \therefore \; 3 + \lambda = 0$

$\therefore \; \lambda = -3$

When $\lambda = -3$, $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$

$\therefore$ the point is $(4, 0, 9)$.

c The line meets the XY-plane when $z = 0$ $\quad \therefore \; 3 - 2\lambda = 0$

$\therefore \; \lambda = \frac{3}{2}$

When $\lambda = \frac{3}{2}$, $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$

$\therefore$ the point is $(-\frac{1}{2}, \frac{9}{2}, 0)$.

8 $x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n, \quad \lambda \in \mathbb{R}$

a When $\lambda = 0$, $x = x_0$, $y = y_0$, and $z = z_0$

$\therefore$ the point is (x_0, y_0, z_0).

b $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

9 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$

a A(3, 3, 4) lies on the line if there is a value of t such that $\begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$

$$\therefore\ t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$\therefore\ t = 2$$

So A(3, 3, 4) does lie on the line.

b A point $P(-1+2t,\ 5-t,\ 2t)$ on the line is 9 units from A if $AP = 9$

$$\therefore\ \sqrt{(-1+2t-3)^2 + (5-t-3)^2 + (0+2t-4)^2} = 9$$
$$\therefore\ (2t-4)^2 + (2-t)^2 + (2t-4)^2 = 81$$
$$\therefore\ 4t^2 - 16t + 16 + 4 - 4t + t^2 + 4t^2 - 16t + 16 = 81$$
$$\therefore\ 9t^2 - 36t - 45 = 0$$
$$\therefore\ 9(t^2 - 4t - 5) = 0$$
$$\therefore\ 9(t-5)(t+1) = 0$$
$$\therefore\ t = 5 \text{ or } -1$$

When $t = 5$, P is $(-1+2(5),\ 5-5,\ 2(5))$ or $(9, 0, 10)$

and when $t = -1$, P is $(-1+2(-1),\ 5-(-1),\ 2(-1))$ or $(-3, 6, -2)$

10 $x = 2 - \lambda, \quad y = 3 + 2\lambda, \quad z = 1 + \lambda, \quad \lambda \in \mathbb{R}$

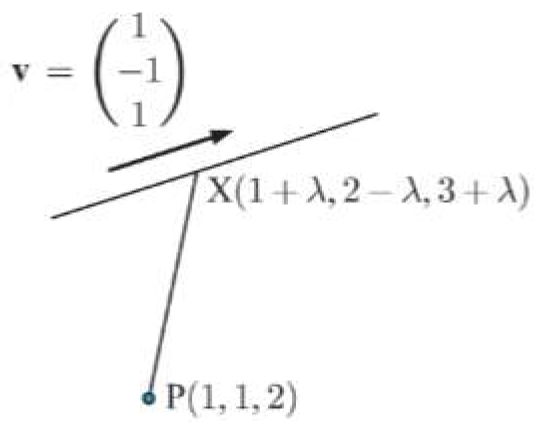
A point $P(2-\lambda,\ 3+2\lambda,\ 1+\lambda)$ on the line is $5\sqrt{3}$ units from $A(1, 0, -2)$ if $AP = 5\sqrt{3}$

$$\therefore\ \sqrt{(2-\lambda-1)^2 + (3+2\lambda-0)^2 + (1+\lambda+2)^2} = 5\sqrt{3}$$
$$\therefore\ (1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2 = 75$$
$$\therefore\ 1 - 2\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 + \lambda^2 + 6\lambda + 9 = 75$$
$$\therefore\ 6\lambda^2 + 16\lambda - 56 = 0$$
$$\therefore\ 3\lambda^2 + 8\lambda - 28 = 0$$
$$\therefore\ (3\lambda + 14)(\lambda - 2) = 0$$
$$\therefore\ \lambda = -\tfrac{14}{3} \text{ or } \lambda = 2$$

When $\lambda = 2$, P is $(2-2,\ 3+2(2),\ 1+2)$ or $(0, 7, 3)$,

and when $\lambda = -\frac{14}{3}$, P is $\left(2-\left(-\frac{14}{3}\right),\ 3+2\left(-\frac{14}{3}\right),\ 1+\left(-\frac{14}{3}\right)\right)$ or $\left(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3}\right)$.

11 **a** Let $X(1+\lambda,\ 2-\lambda,\ 3+\lambda)$ be any point on the line.



Then $\overrightarrow{PX}=\begin{pmatrix}1+\lambda-1\\2-\lambda-1\\3+\lambda-2\end{pmatrix}=\begin{pmatrix}\lambda\\1-\lambda\\1+\lambda\end{pmatrix}$

$$\begin{aligned}\therefore\quad PX&=\sqrt{\lambda^2+(1-\lambda)^2+(1+\lambda)^2}\\&=\sqrt{\lambda^2+(1-2\lambda+\lambda^2)+(1+2\lambda+\lambda^2)}\\&=\sqrt{3\lambda^2+2}\ \text{units}\end{aligned}$$

b PX is minimised when $PX^2=3\lambda^2+2$ is minimised.

This occurs when $\lambda=-\frac{b}{2a}=-\frac{0}{6}=0$.

c When PX is minimised, $\lambda=0$.

When $\lambda=0$, $\overrightarrow{PX}=\begin{pmatrix}0\\1\\1\end{pmatrix}$.

The line has direction vector $\mathbf{v}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

$$\begin{aligned}\text{So,}\quad \overrightarrow{PX}\bullet\mathbf{v}&=\begin{pmatrix}0\\1\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}\\&=0-1+1\\&=0\end{aligned}$$

$\therefore$ $\overrightarrow{PX}$ is perpendicular to the line.

12 **a** A direction vector for the line is $\begin{pmatrix}-2-2\\1-0\\3-3\end{pmatrix}=\begin{pmatrix}-4\\1\\0\end{pmatrix}$.

Using $(2, 0, 3)$ as our fixed point, L_2 has equation $\mathbf{r}=\begin{pmatrix}2\\0\\3\end{pmatrix}+s\begin{pmatrix}-4\\1\\0\end{pmatrix}$, $s\in\mathbb{R}$.

b L_3 has direction vector $\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 4 \\ -4 & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 4 \\ -4 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{k}$$

$$= -4\mathbf{i} - 16\mathbf{j} + 3\mathbf{k}$$

L_3 passes through $(0, 0, -2)$, so its vector equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -16 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

EXERCISE 10B

1 **a** L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

Now $\cos\theta = \dfrac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|}$

$= \dfrac{|-12+2|}{\sqrt{10}\sqrt{20}}$

$= \frac{1}{\sqrt{2}}$

$\therefore\ \theta = 45^\circ$

$\therefore$ the angle between L_1 and L_2 is 45°.

b L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 4 \\ -10 \end{pmatrix}$. L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

Since $\mathbf{b}_1 = -2\,\mathbf{b}_2$, the lines are parallel. The angle between them is 0°.

c L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$.

Now $\cos\theta = \dfrac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|}$

$= \dfrac{|3-2+5|}{\sqrt{30}\sqrt{11}}$

≈ 0.3303

$\therefore\ \theta \approx 70.7^\circ$

$\therefore$ the angle between L_1 and L_2 is about 70.7°.

d L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$.

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$.

Now
$$\begin{aligned} \cos\theta &= \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} \\ &= \frac{|\,0+6-15\,|}{\sqrt{13}\sqrt{35}} \\ &\approx 0.4219 \end{aligned}$$
$\therefore\ \theta \approx 65.0°$

$\therefore$ the angle between L_1 and L_2 is about $65.0°$.

2 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$.

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

Now
$$\begin{aligned} \cos\theta &= \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} \\ &= \frac{\left|\begin{pmatrix} 12 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix}\right|}{\sqrt{144+25}\sqrt{9+16}} \\ &= \frac{|\,36+(-20)\,|}{13 \times 5} \\ &= \tfrac{16}{65} \end{aligned}$$
$\therefore\ \theta = \cos^{-1}\left(\tfrac{16}{65}\right)$

$\therefore\ \theta \approx 75.7°$

$\therefore$ the angle between L_1 and L_2 is about $75.7°$.

3 L_1 has direction vector $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$.

L_2 has direction vector $\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$.

Now $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix} = 20 - 30 + 10 = 0$

$\therefore$ L_1 and L_2 are perpendicular.

4 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Now
$$\begin{aligned}\cos\theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|} \\ &= \frac{\left|\begin{pmatrix} 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \end{pmatrix}\right|}{\sqrt{16+9}\sqrt{25+16}} \\ &= \frac{|20+(-12)|}{\sqrt{25}\sqrt{41}} \\ &= \frac{8}{\sqrt{25}\sqrt{41}}\end{aligned}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{\sqrt{25}\sqrt{41}}\right)$$

$$\therefore \quad \theta \approx 75.5^\circ$$

$\therefore$ the angle between L_1 and L_2 is about 75.5°.

5 **a** L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ and L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$.

Now
$$\begin{aligned}\cos\theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|} \\ &= \frac{\left|\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}\right|}{\sqrt{9+256+49}\sqrt{9+64+25}} \\ &= \frac{|9-128-35|}{\sqrt{314}\sqrt{98}} \\ &= \frac{154}{\sqrt{314}\sqrt{98}}\end{aligned}$$

$$\begin{aligned}\therefore \quad \theta &= \cos^{-1}\left(\frac{154}{\sqrt{314}\sqrt{98}}\right) \\ &\approx 28.6^\circ\end{aligned}$$

$\therefore$ the angle between L_1 and L_2 is about 28.6°.

b **i** Since L_3 is perpendicular to L_1, $\begin{pmatrix} 0 \\ -3 \\ a \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} = 0$

$$\therefore \quad 48 + 7a = 0$$

$$\therefore \quad a = -\tfrac{48}{7}$$

ii L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$ and L_3 has direction vector $\mathbf{b}_3 = \begin{pmatrix} 0 \\ -3 \\ -\frac{48}{7} \end{pmatrix}$.

$$\text{Now} \quad \cos\theta = \frac{|\mathbf{b}_2 \bullet \mathbf{b}_3|}{|\mathbf{b}_2|\,|\mathbf{b}_3|}$$

$$= \frac{\left|\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -3 \\ -\frac{48}{7} \end{pmatrix}\right|}{\sqrt{9+64+25}\sqrt{0+9+\frac{2304}{49}}}$$

$$= \frac{\left|0 - 24 + \frac{240}{7}\right|}{\sqrt{98}\sqrt{\frac{2745}{49}}}$$

$$= \frac{\frac{72}{7}}{7\sqrt{2} \times \frac{\sqrt{2745}}{7}}$$

$$= \frac{72}{7\sqrt{5490}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{72}{7\sqrt{5490}}\right)$$

$$\approx 82.0^\circ$$

$\therefore$ the angle between L_2 and L_3 is about 82.0°.

6 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 0-(-1) \\ 2-5 \\ -2-2 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$$

L_2 is parallel to $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and hence has direction vector $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$.

$$\text{Now} \quad \cos\theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}$$

$$= \frac{\left|\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}\right|}{\sqrt{1+9+16}\sqrt{1+9+16}}$$

$$= \frac{|1+9-16|}{\sqrt{26}\sqrt{26}}$$

$$= \tfrac{6}{26}$$

$$= \tfrac{3}{13}$$

$$\therefore \quad \theta = \cos^{-1}(\tfrac{3}{13})$$

$$\approx 76.7^\circ$$

$\therefore$ the angle between the lines is about 76.7°.

7 L_1 meets the X-axis when $-3+3t = 1-t = 0$

$\therefore\ t = 1$

When $t=1$, $\mathbf{r}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

$\therefore$ L_1 and L_2 meet on the X-axis at $(4, 0, 0)$.

Now, L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

and L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 5-4 \\ -3-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$.

$$\begin{aligned}
\text{So,}\quad \cos\theta &= \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} \\
&= \frac{\left|\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right|}{\sqrt{4+9+1}\sqrt{1+9+1}} \\
&= \frac{|\,2-9-1\,|}{\sqrt{14}\sqrt{11}} \\
&= \frac{8}{\sqrt{14}\sqrt{11}} \\
\therefore\ \theta &= \cos^{-1}\left(\frac{8}{\sqrt{14}\sqrt{11}}\right) \\
&\approx 49.9^\circ
\end{aligned}$$

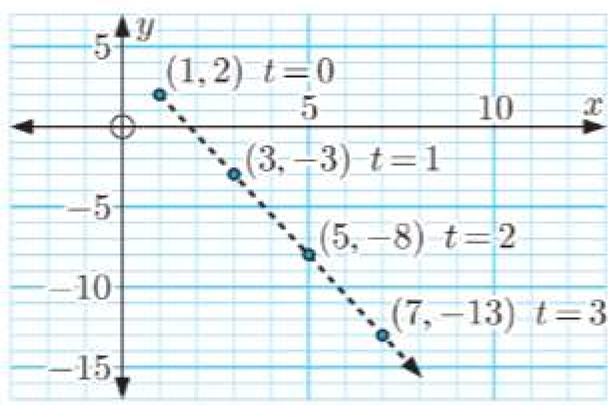
$\therefore$ the angle between the lines is about 49.9°.

EXERCISE 10C

1 **a** $x(0) = 1$ and $y(0) = 2$,

$\therefore$ the initial position is $(1, 2)$.

b



c The velocity vector is $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

d The speed is $\left|\begin{pmatrix} 2 \\ -5 \end{pmatrix}\right| = \sqrt{2^2 + (-5)^2}$

$= \sqrt{29}\ \text{cm s}^{-1}$

e The unit vector with direction $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ is $\frac{1}{\sqrt{29}}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

$\therefore$ the particle travelling with speed 8 cm s^{-1} in the opposite direction has velocity vector $-\frac{8}{\sqrt{29}}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

2 **a** **i** When $t = 0$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$

$\therefore$ the initial position of the object is $(-4, 3, 0)$.

ii The velocity vector of the object is $\begin{pmatrix} 12 \\ 5 \\ 6 \end{pmatrix}$.

iii The speed of the object $= \sqrt{12^2 + 5^2 + 6^2} = \sqrt{205} \approx 14.3\ \text{m s}^{-1}$

b **i** When $t = 0$, $x = 3$, $y = 0$, $z = 4$

$\therefore$ the initial position of the object is $(3, 0, 4)$.

ii The velocity vector of the object is $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

iii The speed of the object $= \sqrt{2^2 + (-1)^2 + (-2)^2} = 3\ \text{m s}^{-1}$

3 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $t \geqslant 0$

b 90 minutes $= 1.5$ hours

When $t = 1.5$, $x = 2 + 4(1.5) = 8$ and $y = 3 - 5(1.5) = -4.5$

$\therefore$ after 90 minutes the boat is at $(8, -4.5)$.

c When the boat is at $(5, -0.75)$

$2 + 4t = 5$ and $3 - 5t = -0.75$

$\therefore$ $4t = 3$ $\quad -5t = -3.75$

$\therefore$ $t = 0.75$ $\quad t = 0.75$ ✓

It will take 0.75 hours $= 45$ minutes for the boat to reach the point $(5, -0.75)$.

4 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $t \geqslant 0$

b When $t = 2.5$, $-3 + 2t = -3 + 5 = 2$
and $-2 + 4t = -2 + 10 = 8$

So, the position vector is $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

c **i** The car is due north when $x = 0$

$\therefore$ $-3 + 2t = 0$

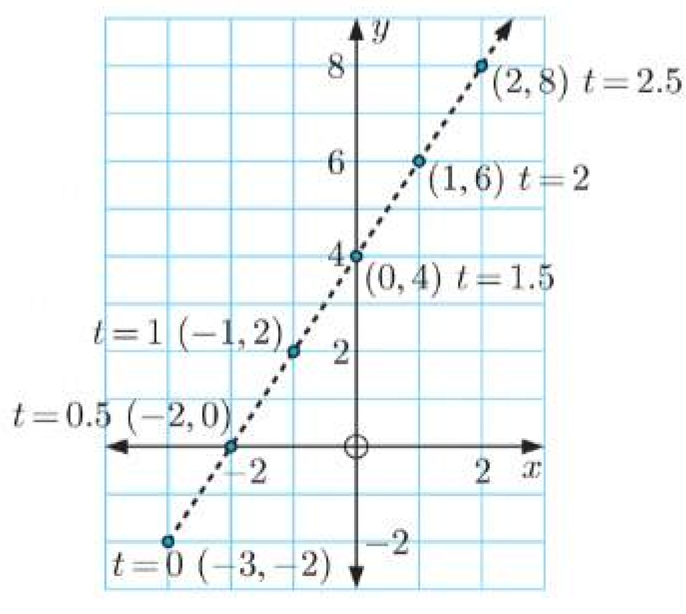
$\therefore$ $t = 1.5$ seconds

ii The car is due west when $y = 0$

$\therefore$ $-2 + 4t = 0$

$\therefore$ $t = 0.5$ seconds

d



5 **a** $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ has length $\sqrt{4^2+(-3)^2}=5$

$\therefore$ $30\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ has length 150

$\therefore$ the velocity vector of the speed boat is $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$.

b $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ has length $\sqrt{(-5)^2+12^2}=13$

$\therefore$ $\frac{3}{5}\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ has length 7.8

$\therefore$ the velocity vector of the jogger is $\begin{pmatrix} -3 \\ 7.2 \end{pmatrix}$.

c $6\mathbf{i}+7\mathbf{j}=\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ has length $\sqrt{6^2+7^2}=\sqrt{85}$

$\therefore$ $\frac{25}{\sqrt{85}}\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ has length 25

$\therefore$ the velocity vector of the ferry is $\begin{pmatrix} \frac{150}{\sqrt{85}} \\ \frac{175}{\sqrt{85}} \end{pmatrix}=\begin{pmatrix} \frac{150\sqrt{85}}{85} \\ \frac{175\sqrt{85}}{85} \end{pmatrix}$

$$=\begin{pmatrix} \frac{30\sqrt{85}}{17} \\ \frac{35\sqrt{85}}{17} \end{pmatrix}$$

d $4\mathbf{i}+7\mathbf{j}+\mathbf{k}=\begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}$ has length $\sqrt{4^2+7^2+1^2}=\sqrt{66}$

$\therefore$ $\frac{33}{\sqrt{66}}\begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}=\frac{\sqrt{66}}{2}\begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}$ has length 33

$\therefore$ the velocity vector of the hot air balloon is $\begin{pmatrix} 2\sqrt{66} \\ \frac{7\sqrt{66}}{2} \\ \frac{\sqrt{66}}{2} \end{pmatrix}$.

e $-2\mathbf{i}+5\mathbf{j}-14\mathbf{k}$ has length $\sqrt{(-2)^2+5^2+(-14)^2}=\sqrt{225}=15$

$\therefore$ $6\begin{pmatrix} -2 \\ 5 \\ -14 \end{pmatrix}$ has length 90

$\therefore$ the velocity vector of the swooping eagle is $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$.

6 Yacht A: $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ Yacht B: $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t \geqslant 0$

a When $t = 0$, $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ $\therefore$ A is initially at $(4, 5)$

and $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix}$ $\therefore$ B is initially at $(1, -8)$.

b The velocity vector of A is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The velocity vector of B is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

c Speed of A $= \sqrt{1^2 + (-2)^2} = \sqrt{5}$ km h^{-1}. Speed of B $= \sqrt{2^2 + 1^2} = \sqrt{5}$ km h^{-1}.
The speed of each yacht does not depend on t and is therefore constant.

d A has direction vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and B has direction vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Since $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 - 2 = 0$, the paths of the yachts are at right angles to each other.

7 **a** The direction vector $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ has length $\sqrt{2^2 + (-2)^2 + 1^2} = 3$.
The cable car moves with speed 4.5 m s^{-1}.

$\therefore$ the velocity vector of the cable car is $1.5\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}$.

b The cable car has position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}$, $t \geqslant 0$.

When $t = 30$,
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + 30\begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 90 \\ -90 \\ 45 \end{pmatrix} = \begin{pmatrix} 100 \\ -87 \\ 45 \end{pmatrix}$$

$\therefore$ after 30 seconds the cable car is at $(100, -87, 45)$.

c After t seconds, the x-coordinate of the cable car is $x = 10 + 3t$.

$550 = 10 + 3t$

$\therefore$ $3t = 540$

$\therefore$ $t = 180$

$\therefore$ the cable car ride lasts 180 seconds.

When $t = 180$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + 180\begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}$

$= \begin{pmatrix} 550 \\ -537 \\ 270 \end{pmatrix}$

$\therefore$ the cable car ride ends at $\begin{pmatrix} 550 \\ -537 \\ 270 \end{pmatrix}$ and begins at $\begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix}$.

$\therefore$ the length of the cable car ride $= \sqrt{(550-10)^2 + (-537-3)^2 + (270-0)^2}$

$= \sqrt{540^2 + (-540)^2 + 270^2}$

$= 810$ m

d The cable car has direction vector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Consider the direction vector $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$ which has the same direction in the horizontal XY-plane, but no vertical component.

$$\cos\theta = \frac{\left|\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ -0 \end{pmatrix}\right|}{\left|\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right|\left|\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}\right|}$$

$$= \frac{|4+4+0|}{\sqrt{2^2+(-2)^2+1^2}\sqrt{2^2+(-2)^2+0^2}}$$

$$= \frac{8}{3\sqrt{8}}$$

$\therefore$ $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{8}}\right) \approx 19.5°$

$\therefore$ the cable car travels at an angle of about $19.5°$ to the horizontal.

8 **a** $\overrightarrow{AB} = \begin{pmatrix} 3-6 \\ 10-9 \\ 2.5-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{(-3)^2 + 1^2 + (-0.5)^2}$

$= \sqrt{10.25}$ km

The helicopter travels $\sqrt{10.25}$ km in 10 minutes.

$\therefore$ the helicopter's speed is $6 \times \sqrt{10.25} \approx 19.2$ km h^{-1}.

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$, $t \in \mathbb{R}$, where t represents $10t$ minutes

d If $z = 0$, $3 + (-0.5)t = 0$

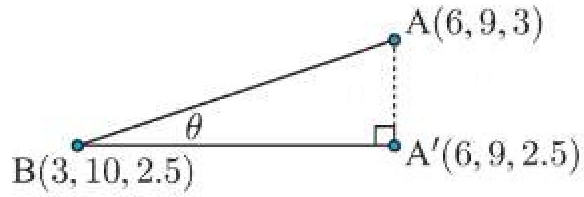
$\therefore\ t = 6$

$t = 1$ represents 10 minutes, so $t = 6$ represents 60 minutes.

$\therefore$ the helicopter lands on the helipad after 1 hour.

e Let A$'$ have coordinates $(6,\ 9,\ 2.5)$, so A$'$ is directly underneath A and has the same z-coordinate as B.

$\therefore$ $\mathrm{A}\widehat{\mathrm{B}}\mathrm{A}' = \theta$ is the angle the helicopter is flying at to the horizontal in the diagram alongside.



Now $\overrightarrow{\mathrm{A'B}} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

$$\therefore\ \cos\theta = \frac{|\,\overrightarrow{\mathrm{AB}} \bullet \overrightarrow{\mathrm{A'B}}\,|}{|\,\overrightarrow{\mathrm{AB}}\,|\,|\,\overrightarrow{\mathrm{A'B}}\,|}$$

$$= \frac{\left|(-3)^2 + (1)^2 + 0\right|}{\sqrt{(-3)^2 + 1^2 + (0.5)^2}\sqrt{(-3)^2 + 1^2}}$$

$$= \frac{2\sqrt{10}}{\sqrt{41}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{2\sqrt{10}}{\sqrt{41}}\right)$$

$$\approx 8.98^\circ$$

EXERCISE 10D

1 **a** Let N be the point on the line closest to P.

N has coordinates $(2 + t,\ 3 + 2t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{\mathrm{PN}}$ is $\begin{pmatrix} 2 + t - 3 \\ 3 + 2t - 2 \end{pmatrix} = \begin{pmatrix} t - 1 \\ 2t + 1 \end{pmatrix}$.

The distance between P and the line is minimised when $\overrightarrow{\mathrm{PN}} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$.

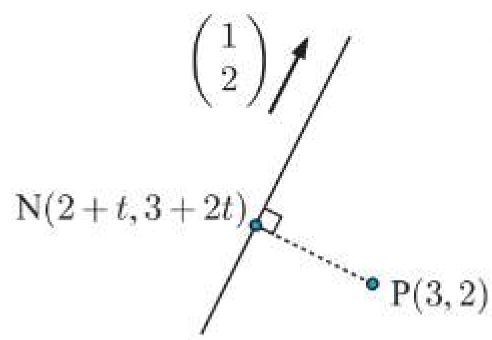
$\therefore\ \begin{pmatrix} t - 1 \\ 2t + 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$

$\therefore\ (t - 1) + 2\,(2t + 1) = 0$

$\therefore\ t - 1 + 4t + 2 = 0$

$\therefore\ 5t = -1$

$\therefore\ t = -\frac{1}{5}$



Thus $\overrightarrow{PN} = \begin{pmatrix} -\frac{1}{5} - 1 \\ -\frac{2}{5} + 1 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$

$= \frac{3}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

and $|\overrightarrow{PN}| = \frac{3}{5}\left|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right| = \frac{3}{5}\sqrt{(-2)^2 + 1^2}$

$= \frac{3\sqrt{5}}{5}$ units

$\therefore$ the shortest distance from P to the line is $\frac{3\sqrt{5}}{5}$ units.

b Let N be the point on the line closest to Q.
N has coordinates $(t, 1-t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{QN}$ is $\begin{pmatrix} t - (-1) \\ 1 - t - 1 \end{pmatrix} = \begin{pmatrix} t+1 \\ -t \end{pmatrix}$.

The distance between Q and the line is minimised when $\overrightarrow{QN} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$.

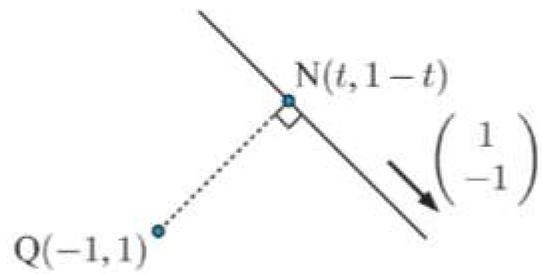
$\therefore \begin{pmatrix} t+1 \\ -t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$\therefore (t+1) + (-1)(-t) = 0$

$\therefore t + 1 + t = 0$

$\therefore 2t = -1$

$\therefore t = -\frac{1}{2}$



Thus $\overrightarrow{QN} = \begin{pmatrix} -\frac{1}{2} + 1 \\ -(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $|\overrightarrow{QN}| = \frac{1}{2}\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right| = \frac{1}{2}\sqrt{1^2 + 1^2}$

$= \frac{\sqrt{2}}{2}$ units

$\therefore$ the shortest distance from Q to the line is $\frac{\sqrt{2}}{2}$ units.

c Let N be the point on the line closest to R.

N has coordinates $(2+s,\ 3-s)$ for some $s \in \mathbb{R}$.

$\overrightarrow{RN}$ is $\begin{pmatrix} 2+s-(-3) \\ 3-s-(-1) \end{pmatrix} = \begin{pmatrix} s+5 \\ 4-s \end{pmatrix}$.

The distance from R to the line is minimised when $\overrightarrow{RN} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$.

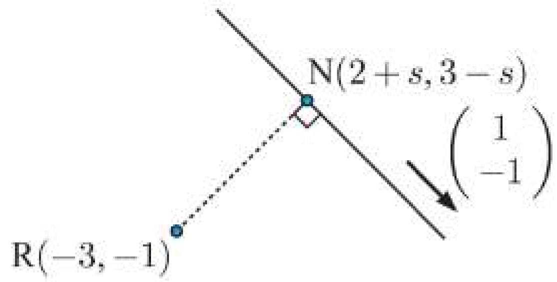
$$\therefore \begin{pmatrix} s+5 \\ 4-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore (s+5)+(-1)(4-s) = 0$$

$$\therefore s+5-4+s = 0$$

$$\therefore 2s = -1$$

$$\therefore s = -\tfrac{1}{2}$$



Thus $\overrightarrow{RN} = \begin{pmatrix} -\frac{1}{2}+5 \\ 4-(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{9}{2} \end{pmatrix}$

$$= \tfrac{9}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $|\overrightarrow{RN}| = \frac{9}{2}\left|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right| = \frac{9}{2}\sqrt{1^2+1^2}$

$$= \tfrac{9\sqrt{2}}{2} \text{ units}$$

$\therefore$ the shortest distance from R to the line is $\frac{9\sqrt{2}}{2}$ units.

d Let N be the point on the line closest to S.

N has coordinates $(2+3t,\ 5-7t)$ for some $t \in \mathbb{R}$.

$\overrightarrow{SN}$ is $\begin{pmatrix} 2+3t-5 \\ 5-7t-(-2) \end{pmatrix} = \begin{pmatrix} 3t-3 \\ 7-7t \end{pmatrix}$.

The distance from S to the line is minimised when $\overrightarrow{SN} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$.

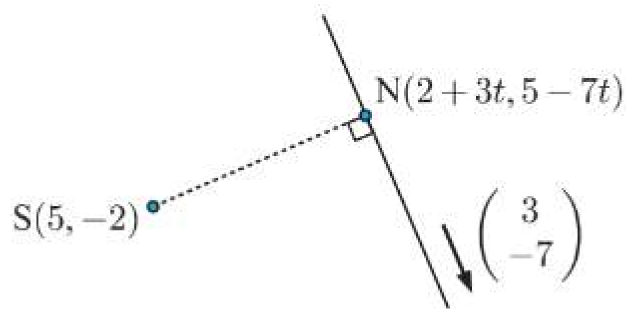
$$\therefore \begin{pmatrix} 3t-3 \\ 7-7t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$$

$$\therefore 3(3t-3)-7(7-7t) = 0$$

$$\therefore 9t-9-49+49t = 0$$

$$\therefore 58t = -58$$

$$\therefore t = 1$$



Thus $\overrightarrow{SN} = \begin{pmatrix} 3(1)-3 \\ 7-7(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore |\overrightarrow{SN}| = 0$

$\therefore$ S actually lies on the line, and the shortest distance is 0 units.

2 **a** $6\mathbf{i} - 6\mathbf{j}$

b $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ has length $\sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$ units

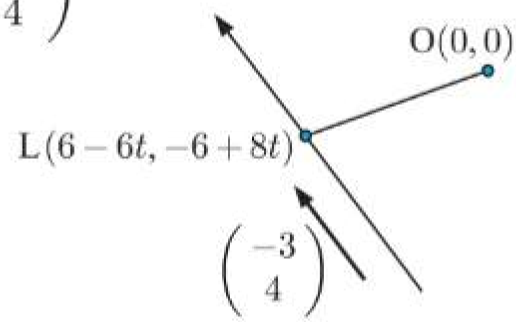
As the speed is 10 km h^{-1}, the liner has velocity vector $2\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$.

$\therefore$ the liner has position vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix}$, $t \geqslant 0$.

c The liner is due east when $y = 0$

$$\therefore\ -6 + 8t = 0$$
$$\therefore\ \text{at } t = \tfrac{3}{4} \text{ hours}$$

d The liner L is nearest the fishing boat O when $\overrightarrow{OL} \perp \begin{pmatrix} -3 \\ 4 \end{pmatrix}$



$$\therefore\ \overrightarrow{OL} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$
$$\therefore\ \begin{pmatrix} 6-6t \\ -6+8t \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$
$$\therefore\ (-18 + 18t) + (-24 + 32t) = 0$$
$$\therefore\ 50t = 42$$
$$\therefore\ t = 0.84 \text{ hours} = 50.4 \text{ minutes}$$

When $t = 0.84$, $\overrightarrow{OL} = \begin{pmatrix} 6 - 6(0.84) \\ -6 + 8(0.84) \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.72 \end{pmatrix}$

$\therefore$ the liner is closest to the fishing boat after 0.84 hours or 50.4 minutes, when it is at (0.96, 0.72).

3 **a** $|\mathbf{b}| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

As the speed is $40\sqrt{10}$ km h^{-1}, the velocity vector is $40\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -120 \\ -40 \end{pmatrix}$.

b $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200 \\ 100 \end{pmatrix} + t\begin{pmatrix} -120 \\ -40 \end{pmatrix}$, $t \geqslant 0$ $\{t = 0$ at 12:00 noon$\}$

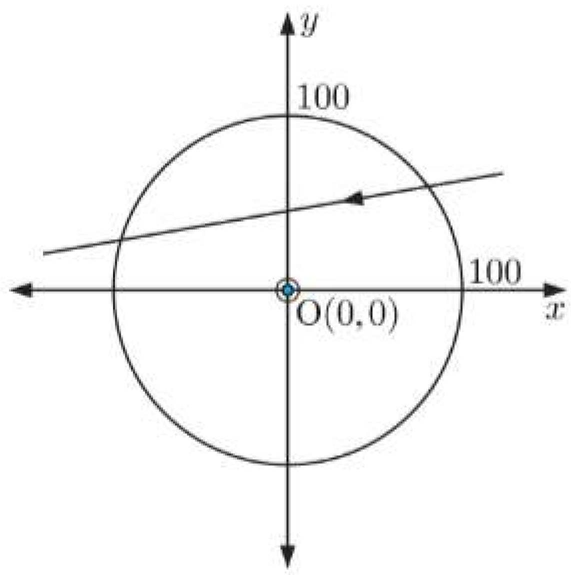
c At 1:00 pm, $t = 1$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200 - 120 \\ 100 - 40 \end{pmatrix} = \begin{pmatrix} 80 \\ 60 \end{pmatrix}$

$\therefore$ the aircraft is at (80, 60).

d The distance from O(0, 0) to (80, 60) is $\left|\begin{pmatrix} 80 \\ 60 \end{pmatrix}\right| = \sqrt{80^2 + 60^2} = 100$ km,

which is when it becomes visible to radar at 1:00 pm. {within 100 km of O(0, 0)}

e



A general point on the path is P$(200-120t,\ 100-40t)$.

Now $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 200-120t \\ 100-40t \end{pmatrix}$,

and for the closest point $\overrightarrow{\mathrm{OP}} \bullet \begin{pmatrix} -3 \\ -1 \end{pmatrix} = 0$

$$\therefore\ -3(200-120t) - 1(100-40t) = 0$$
$$\therefore\ -700 + 400t = 0$$
$$\therefore\ t = \tfrac{7}{4} = 1\tfrac{3}{4} \text{ hours}$$

$\therefore$ the time when the aircraft is closest is 1:45 pm, and at this time

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 200-120(\frac{7}{4}) \\ 100-40(\frac{7}{4}) \end{pmatrix} = \begin{pmatrix} -10 \\ 30 \end{pmatrix}$$

$$\therefore\ d_{\min} = \sqrt{(-10)^2 + 30^2} = \sqrt{1000} = 10\sqrt{10} \text{ km}$$

f The aircraft disappears from radar when $|\overrightarrow{\mathrm{OP}}| = 100$ and $t > 1\frac{3}{4}$

$$\therefore\ \sqrt{(200-120t)^2 + (100-40t)^2} = 100$$
$$\therefore\ 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1600t^2 = 10\,000$$
$$\therefore\ 16\,000t^2 - 56\,000t + 40\,000 = 0$$
$$\therefore\ 2t^2 - 7t + 5 = 0$$
$$\therefore\ (2t-5)(t-1) = 0$$
$$\therefore\ t = \tfrac{5}{2} \qquad \{\text{as } t > 1\tfrac{3}{4}\}$$

So, the aircraft disappears from the radar screen $2\frac{1}{2}$ hours after noon, or at 2:30 pm.

4 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $t \in \mathbb{R}$, P$(5, -1, 0)$

a When $t = 0$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, so the point A$(1, 2, 0)$ lies on the line.

b The direction vector of the line $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$.

c $\overrightarrow{AP} = \begin{pmatrix} 5-1 \\ -1-2 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

$$\therefore \quad \overrightarrow{AP} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 0 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 0 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 0 \\ 3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -3 \\ 3 & -1 \end{vmatrix} \mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$$

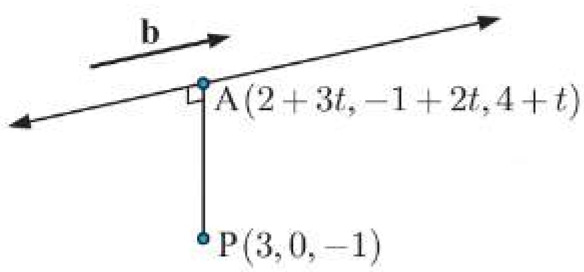
$\therefore \ |\overrightarrow{AP} \times \mathbf{b}| = \sqrt{0^2 + 0^2 + 5^2} = 5$ and $|\mathbf{b}| = \sqrt{3^2 + (-1)^2 + 0^2} = \sqrt{10}$

$\therefore$ the shortest distance between P and the line $= \dfrac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|} = \dfrac{5}{\sqrt{10}} = \dfrac{\sqrt{10}}{2}$ units.

5 **a** The direction vector of the line is $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Let the point $(3, 0, -1)$ be P, and $A(2+3t, \ -1+2t, \ 4+t)$ be any point on the line.

$$\therefore \quad \overrightarrow{PA} = \begin{pmatrix} 2+3t-3 \\ -1+2t-0 \\ 4+t-(-1) \end{pmatrix} = \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix}$$



Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$

$$\therefore \quad \begin{pmatrix} -1+3t \\ -1+2t \\ 5+t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad -3 + 9t - 2 + 4t + 5 + t = 0$$

$$\therefore \quad 14t = 0$$

$$\therefore \quad t = 0$$

Substituting $t = 0$ into the parametric equations, the foot of the perpendicular is $(2, -1, 4)$.

b When $t = 0$, $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$

$$\therefore \ |\overrightarrow{PA}| = \sqrt{1 + 1 + 25}$$

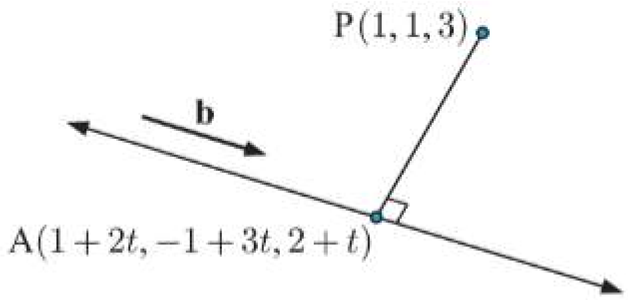
$$= \sqrt{27} \text{ units}$$

So, the shortest distance from the point to the line is $\sqrt{27}$ units.

6 **a** The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let the point $(1, 1, 3)$ be P and $A(1+2t, -1+3t, 2+t)$ be any point on the line.

$$\therefore \overrightarrow{PA} = \begin{pmatrix} 1+2t-1 \\ -1+3t-1 \\ 2+t-3 \end{pmatrix} = \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix}$$



Now $\overrightarrow{PA}$ and $\mathbf{b}$ are perpendicular, so $\overrightarrow{PA} \bullet \mathbf{b} = 0$

$$\therefore \begin{pmatrix} 2t \\ -2+3t \\ -1+t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\therefore 4t - 6 + 9t - 1 + t = 0$$

$$\therefore 14t = 7$$

$$\therefore t = \tfrac{1}{2}$$

Substituting $t = \frac{1}{2}$ into the parametric equations, the foot of the perpendicular is $(2, \frac{1}{2}, \frac{5}{2})$.

b When $t = \frac{1}{2}$, $\overrightarrow{PA} = \begin{pmatrix} 1 \\ -2+\frac{3}{2} \\ -1+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$$\therefore |\overrightarrow{PA}| = \sqrt{1 + \tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{3}{2}} \text{ units}$$

So, the shortest distance from the point to the line is $\sqrt{\frac{3}{2}}$ units.

c $A(1, -1, 2)$, $P(1, 1, 3)$, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$$\overrightarrow{AP} = \begin{pmatrix} 1-1 \\ 1-(-1) \\ 3-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore \overrightarrow{AP} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} \mathbf{k}$$

$$= -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

$$\therefore |\overrightarrow{AP} \times \mathbf{b}| = \sqrt{(-1)^2 + 2^2 + (-4)^2} = \sqrt{21} \quad \text{and} \quad |\mathbf{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$\therefore \text{ the shortest distance from P to the line} = \frac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|} = \frac{\sqrt{21}}{\sqrt{14}} = \sqrt{\tfrac{3}{2}} \text{ units.} \quad \checkmark$$

7 **a** The diver swims with direction vector $\mathbf{b} = \begin{pmatrix} 7-12 \\ -15-25 \\ 0-(-20) \end{pmatrix} = \begin{pmatrix} -5 \\ -40 \\ 20 \end{pmatrix}$.

$|\,\mathbf{b}\,| = \sqrt{(-5)^2 + (-40)^2 + 20^2} = 45$

As the speed is $0.9\ \text{m s}^{-1}$, the velocity vector is $\frac{0.9}{45}\begin{pmatrix} -5 \\ -40 \\ 20 \end{pmatrix} = \begin{pmatrix} -0.1 \\ -0.8 \\ 0.4 \end{pmatrix}$.

b **i** Let $D(12-0.1t,\ 25-0.8t,\ -20+0.4t)$ be any point on the path of the diver.

If the octopus is at $P(12, -8, -5)$, $\overrightarrow{PD} = \begin{pmatrix} 12-0.1t-12 \\ 25-0.8t-(-8) \\ -20+0.4t-(-5) \end{pmatrix}$

$$= \begin{pmatrix} -0.1t \\ 33-0.8t \\ -15+0.4t \end{pmatrix}$$

If D is the closest point on the line to P, then

$$\begin{pmatrix} -0.1t \\ 33-0.8t \\ -15+0.4t \end{pmatrix} \bullet \begin{pmatrix} -0.1 \\ -0.8 \\ 0.4 \end{pmatrix} = 0$$

$\therefore\ -0.1(-0.1t) - 0.8(33-0.8t) + 0.4(-15+0.4t) = 0$

$\therefore\ 0.01t - 26.4 + 0.64t - 6 + 0.16t = 0$

$\therefore\ 0.81t = 32.4$

$\therefore\ t = 40$

$\therefore$ the diver is closest to the octopus after 40 seconds.

ii When $t = 40$, the diver is at $(12-0.1(40),\ 25-0.8(40),\ -20+0.4(40))$
$= (8, -7, -4)$

The distance between $(8, -7, -4)$ and $(12, -8, -5)$

$= \sqrt{(12-8)^2 + (-8-(-7))^2 + (-5-(-4))^2}$

$= \sqrt{4^2 + (-1)^2 + (-1)^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$ m

$\therefore$ the shortest distance from the diver to the octopus is $3\sqrt{2}$ m, which occurs at $t = 40$ seconds.

EXERCISE 10E

1 At time t, the runner is at $(-10+3t,\ 7-2t)$, and the walker is at $(-4+2t,\ -3-t)$.

a **i** When $t = 3$, the runner is at $(-1, 1)$, and the walker is at $(2, -6)$.

$\therefore$ the distance between them is $\sqrt{(2-(-1))^2 + (-6-1)^2} = \sqrt{58} \approx 7.62$ m.

ii When $t = 10$, the runner is at $(20, -13)$, and the walker is at $(16, -13)$.

$\therefore$ the distance between them is $\sqrt{(16-20)^2 + (-13-(-13))^2} = 4$ m.

b At time t, the distance between the runner and the walker is

$$\begin{aligned} D &= \sqrt{[(-4+2t)-(-10+3t)]^2 + [(-3-t)-(7-2t)]^2} \\ &= \sqrt{(6-t)^2 + (-10+t)^2} \\ &= \sqrt{36 - 12t + t^2 + 100 - 20t + t^2} \\ &= \sqrt{2t^2 - 32t + 136} \text{ m} \end{aligned}$$

c D is minimised when $D^2 = 2t^2 - 32t + 136$ is minimised.

This occurs when $t = -\dfrac{-32}{2 \times 2} = 8$.

When $t = 8$, $D = \sqrt{2(8)^2 - 32(8) + 136} = \sqrt{8}$

The shortest distance between the runner and the walker is $\sqrt{8} \approx 2.83$ m, and this occurs after 8 seconds.

2 At time t, the tour bus is at $(-50 + 4t,\ 30 - 3t)$, and the giraffe is at $(-30 + 2t,\ -15 + t)$.

a The bus enters the enclosure when $t = 0$, so the bus is at $(-50, 30)$, and the giraffe is at $(-30, -15)$.

$\therefore$ the distance between them is $\sqrt{(-30-(-50))^2 + (-15-30)^2} = \sqrt{2425} \approx 49.2$ m.

b **i** At time t, the distance between the bus and the giraffe is

$$\begin{aligned} D &= \sqrt{[(-30+2t)-(-50+4t)]^2 + [(-15+t)-(30-3t)]^2} \\ &= \sqrt{(20-2t)^2 + (-45+4t)^2} \\ &= \sqrt{400 - 80t + 4t^2 + 2025 - 360t + 16t^2} \\ &= \sqrt{20t^2 - 440t + 2425} \text{ m} \end{aligned}$$

Now D is minimised when $D^2 = 20t^2 - 440t + 2425$ is minimised.

This occurs when $t = -\dfrac{-440}{2 \times 20} = 11$.

So, the tourist should take the photo 11 seconds after the bus enters the enclosure.

ii When $t = 11$, $D = \sqrt{20(11)^2 - 440(11) + 2425} = \sqrt{5}$

After 11 seconds, the bus is $\sqrt{5} \approx 2.24$ m from the giraffe.

iii When $t = 11$, the giraffe is at $(-30 + 2(11),\ -15 + 11)$, which is $(-8, -4)$.

3 **a** The direction vector of truck A is $\mathbf{b}_1 = \begin{pmatrix} -40 - 68 \\ 60 - (-21) \end{pmatrix} = \begin{pmatrix} -108 \\ 81 \end{pmatrix}$.

A unit vector in this direction is $\dfrac{1}{|\mathbf{b}_1|}\mathbf{b}_1 = \frac{1}{135}\begin{pmatrix} -108 \\ 81 \end{pmatrix}$.

Truck A is travelling at 90 km h^{-1}, so its velocity vector $\mathbf{v}_1 = \frac{90}{135}\begin{pmatrix} -108 \\ 81 \end{pmatrix} = \begin{pmatrix} -72 \\ 54 \end{pmatrix}$.

The direction vector of truck B is $\mathbf{b}_2 = \begin{pmatrix} -40 - 110 \\ 60 - (-20) \end{pmatrix} = \begin{pmatrix} -150 \\ 80 \end{pmatrix}$.

A unit vector in this direction is $\frac{1}{|\mathbf{b}_2|}\mathbf{b}_2 = \frac{1}{170}\begin{pmatrix} -150 \\ 80 \end{pmatrix}$.

Truck B is travelling at 85 km h^{-1}, so its velocity vector $\mathbf{v}_2 = \frac{85}{170}\begin{pmatrix} -150 \\ 80 \end{pmatrix} = \begin{pmatrix} -75 \\ 40 \end{pmatrix}$.

b At time t hours after noon, truck A is at $(68 - 72t,\ -21 + 54t)$.

$\therefore$ truck A is at $X(-40, 60)$ when $68 - 72t = -40$ and $-21 + 54t = 60$

$\therefore\ t = 1.5$ $\qquad\qquad$ $\therefore\ t = 1.5$ ✓

So, truck A passes through X 1.5 hours after noon which is 1:30 pm.

At time t hours after noon, truck B is at $(110 - 75t,\ -20 + 40t)$.

$\therefore$ truck B is at X when $110 - 75t = -40$ and $-20 + 40t = 60$

$\therefore\ t = 2$ $\qquad\qquad$ $\therefore\ t = 2$ ✓

So, truck B passes through X 2 hours after noon which is 2:00 pm.

c **i** Using **b**, the distance between the trucks at time t is

$$\begin{aligned} D &= \sqrt{[(110 - 75t) - (68 - 72t)]^2 + [(-20 + 40t) - (-21 + 54t)]^2} \\ &= \sqrt{(42 - 3t)^2 + (1 - 14t)^2} \\ &= \sqrt{1764 - 252t + 9t^2 + 1 - 28t + 196t^2} \\ &= \sqrt{205t^2 - 280t + 1765} \text{ km} \end{aligned}$$

Now D is minimised when $D^2 = 205t^2 - 280t + 1765$ is minimised.

This occurs when $t = -\dfrac{-280}{2 \times 205} = \dfrac{28}{41}$.

$\frac{28}{41}$ hours $\equiv \frac{28}{41} \times 60 \approx 41$ minutes.

The trucks will be closest to one another at about 41 minutes after noon which is 12:41 pm.

ii When $t = \frac{28}{41}$, $D = \sqrt{205\left(\frac{28}{41}\right)^2 - 280\left(\frac{28}{41}\right) + 1765} \approx 40.9$

The closest the trucks come to one another is about 40.9 km.

4 At time t, P is at $(5 - t,\ -1 - 3t,\ 2 + t)$, and Q is at $(-4 + t,\ -3 - 2t,\ 4 + 2t)$.

a When $t = 0$, P is at $(5, -1, 2)$, and Q is at $(-4, -3, 4)$, so the distance between them is

$\sqrt{(-4 - 5)^2 + (-3 - (-1))^2 + (4 - 2)^2} = \sqrt{89}$ m.

The initial distance between the objects is $\sqrt{89} \approx 9.43$ m.

b At time t, the distance between the objects is

$$\begin{aligned} D &= \sqrt{[(-4 + t) - (5 - t)]^2 + [(-3 - 2t) - (-1 - 3t)]^2 + [(4 + 2t) - (2 + t)]^2} \\ &= \sqrt{(-9 + 2t)^2 + (-2 + t)^2 + (2 + t)^2} \\ &= \sqrt{81 - 36t + 4t^2 + 4 - 4t + t^2 + 4 + 4t + t^2} \\ &= \sqrt{6t^2 - 36t + 89} \text{ m} \end{aligned}$$

Now D is minimised when $D^2 = 6t^2 - 36t + 89$ is minimised.

This occurs when $t = -\dfrac{-36}{2 \times 6} = 3.$

When $t = 3$, $D = \sqrt{6(3)^2 - 36(3) + 89} = \sqrt{35}$

The shortest distance between the objects is $\sqrt{35} \approx 5.92$ m, and this occurs after 3 seconds.

5 **a** Alex's velocity vector is perpendicular to the balloon's velocity vector,

so $\begin{pmatrix} a \\ -3 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ b \\ -1 \end{pmatrix} = 0$

$\therefore\ a(4) - 3(b) + 0(-1) = 0$

$\therefore\ 4a + 3b = 0$

$\therefore\ a = \frac{3}{4}b$

Also, the water balloon is travelling at 9 m s^{-1}, so $|4\mathbf{i} + b\mathbf{j} - \mathbf{k}| = 9$

$\therefore\ \sqrt{4^2 + b^2 + (-1)^2} = 9$

$\therefore\ b^2 + 17 = 81$

$\therefore\ b^2 = 64$

$\therefore\ b = \pm 8$

But Yvonne throws the water balloon *towards* Alex, so $b > 0$.

$\therefore\ b = 8$ and $a = \frac{3}{4}(8) = 6.$

b At time t seconds after the water balloon is thrown, Alex is at $(4 + 6t,\ 2 - 3t,\ 0)$, and the water balloon is at $(8 + 4t,\ -20 + 8t,\ 2 - t)$.

$\therefore$ at time t, the distance between them is

$$\begin{aligned} D &= \sqrt{[(8+4t) - (4+6t)]^2 + [(-20+8t) - (2-3t)]^2 + [(2-t) - 0]^2} \\ &= \sqrt{(4-2t)^2 + (-22+11t)^2 + (2-t)^2} \\ &= \sqrt{16 - 16t + 4t^2 + 484 - 484t + 121t^2 + 4 - 4t + t^2} \\ &= \sqrt{126t^2 - 504t + 504} \end{aligned}$$

Now D is minimised when $D^2 = 126t^2 - 504t + 504$ is minimised.

This occurs when $t = -\dfrac{-504}{2 \times 126} = 2.$

The water balloon is closest to Alex 2 seconds after it is thrown.

c When $t = 2$, $D = \sqrt{126(2)^2 - 504(2) + 504} = 0$

This means that the water balloon hits Alex 2 seconds after it is thrown.

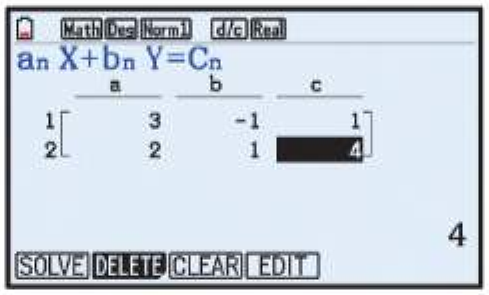
EXERCISE 10F.1

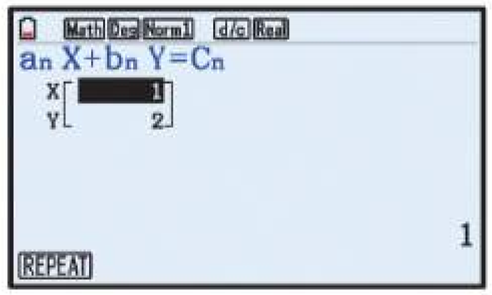
1 The lines meet where $\begin{pmatrix}-1\\6\end{pmatrix}+r\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+s\begin{pmatrix}1\\1\end{pmatrix}$.

$\therefore\ -1+3r=s$ and $6-2r=2+s$

$\therefore\ 3r-s=1$ .... (1) and $2r+s=4$ .... (2)

Solving (1) and (2) simultaneously using technology gives $r=1$, $s=2$.





Using line 1, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\6\end{pmatrix}+1\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$

Checking in line 2, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$ ✓

$\therefore$ the lines meet at $(2, 4)$.

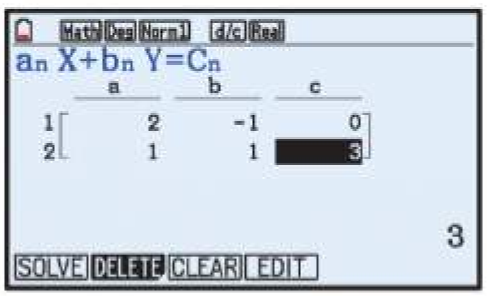
2 **a** (AB) and (AC) meet at A.

$$\therefore\ \begin{pmatrix}0\\2\end{pmatrix}+r\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\5\end{pmatrix}+t\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}2r-t\\r+t\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$$

$\therefore\ 2r-t=0$ .... (1)

and $r+t=3$ .... (2)



Solving (1) and (2) simultaneously using technology gives $r=1$, $t=2$.

$$\therefore\ \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$$

$\therefore$ A is $(2, 3)$.

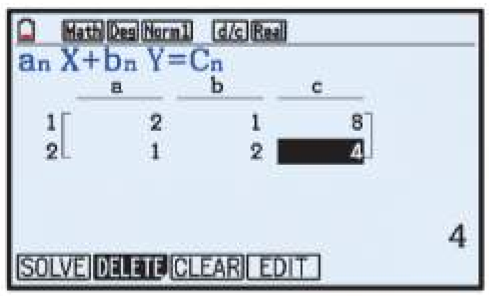
(AB) and (BC) meet at B.

$$\therefore\ \begin{pmatrix}0\\2\end{pmatrix}+r\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}+s\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}2r+s\\r+2s\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$$

$\therefore\ 2r+s=8$ .... (1)

and $r+2s=4$ .... (2)



Solving (1) and (2) simultaneously using technology gives $r=4$, $s=0$.

$$\therefore\ \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+4\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}$$

$\therefore$ B is $(8, 6)$.

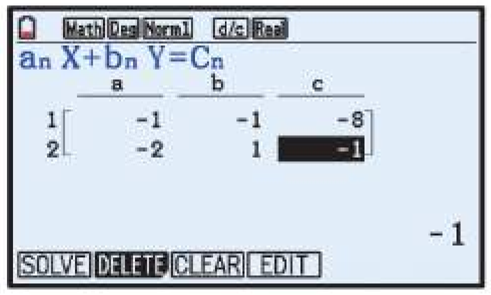
(BC) and (AC) meet at C.

$$\therefore \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -s-t \\ -2s+t \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$$

$$\therefore \ -s-t=-8 \quad .... \ (1) \qquad \text{and} \qquad -2s+t=-1 \quad .... \ (2)$$

Solving (1) and (2) simultaneously using technology gives $s=3, \ t=5.$



$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\therefore$ C is $(5, 0)$.

b A(2, 3), B(8, 6), C(5, 0)

$$\begin{aligned} AB &= \sqrt{(8-2)^2+(6-3)^2} \\ &= \sqrt{36+9} \\ &= \sqrt{45} \text{ units} \end{aligned} \qquad \begin{aligned} BC &= \sqrt{(5-8)^2+(0-6)^2} \\ &= \sqrt{9+36} \\ &= \sqrt{45} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(5-2)^2+(0-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \text{ units} \end{aligned}$$

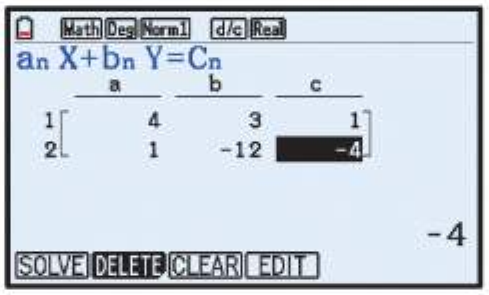
The two equal sides are [AB] and [BC] and they have length $\sqrt{45}$ units. [AC] has length $\sqrt{18}$ units.

3 **a** (AB) and (AD) meet at A.

$$\therefore \begin{pmatrix}2\\5\end{pmatrix} + r\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} + u\begin{pmatrix}-3\\12\end{pmatrix}$$

$$\therefore \begin{pmatrix}4r+3u\\r-12u\end{pmatrix} = \begin{pmatrix}1\\-4\end{pmatrix}$$

$$\therefore\ 4r + 3u = 1 \quad .... (1)$$
$$\text{and}\quad r - 12u = -4 \quad .... (2)$$



Solving (1) and (2) simultaneously using technology gives $r = 0$, $u = \frac{1}{3}$.

$$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$$

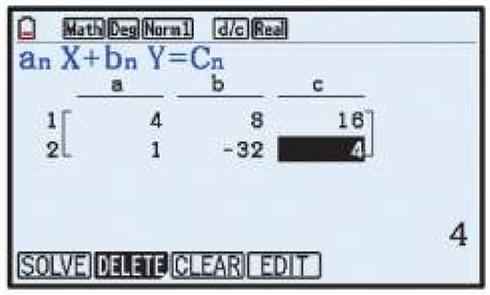
$\therefore$ A is $(2, 5)$.

(AB) and (BC) meet at B.

$$\therefore \begin{pmatrix}2\\5\end{pmatrix} + r\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}18\\9\end{pmatrix} + s\begin{pmatrix}-8\\32\end{pmatrix}$$

$$\therefore \begin{pmatrix}4r+8s\\r-32s\end{pmatrix} = \begin{pmatrix}16\\4\end{pmatrix}$$

$$\therefore\ 4r + 8s = 16 \quad .... (1)$$
$$\text{and}\quad r - 32s = 4 \quad .... (2)$$



Solving (1) and (2) simultaneously using technology gives $r = 4$, $s = 0$.

$$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}18\\9\end{pmatrix}$$

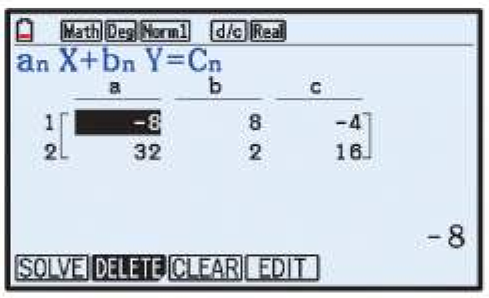
$\therefore$ B is $(18, 9)$.

(BC) and (CD) meet at C.

$$\therefore \begin{pmatrix}18\\9\end{pmatrix} + s\begin{pmatrix}-8\\32\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix}$$

$$\therefore \begin{pmatrix}-8s+8t\\32s+2t\end{pmatrix} = \begin{pmatrix}-4\\16\end{pmatrix}$$

$$\therefore\ -8s + 8t = -4 \quad .... (1)$$
$$\text{and}\quad 32s + 2t = 16 \quad .... (2)$$



Solving (1) and (2) simultaneously using technology gives $s = \frac{1}{2}$, $t = 0$.

$$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix}$$

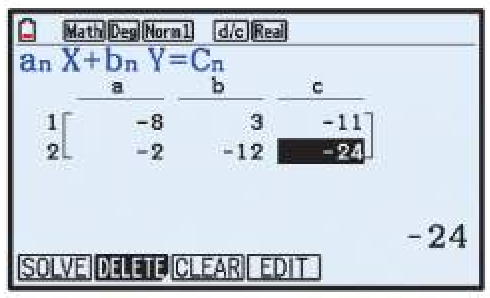
$\therefore$ C is $(14, 25)$.

(CD) and (AD) meet at D.

$$\therefore \begin{pmatrix}14\\25\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} + u\begin{pmatrix}-3\\12\end{pmatrix}$$

$$\therefore \begin{pmatrix}-8t+3u\\-2t-12u\end{pmatrix} = \begin{pmatrix}-11\\-24\end{pmatrix}$$

$$\therefore\ -8t + 3u = -11 \quad .... (1)$$
$$\text{and}\quad -2t - 12u = -24 \quad .... (2)$$



Solving (1) and (2) simultaneously using technology gives $t = 2$, $u = \frac{5}{3}$.

$$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}14\\25\end{pmatrix} + 2\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}-2\\21\end{pmatrix}$$

$\therefore$ D is $(-2, 21)$.

b **i** $\overrightarrow{AC} = \begin{pmatrix} 14-2 \\ 25-5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$

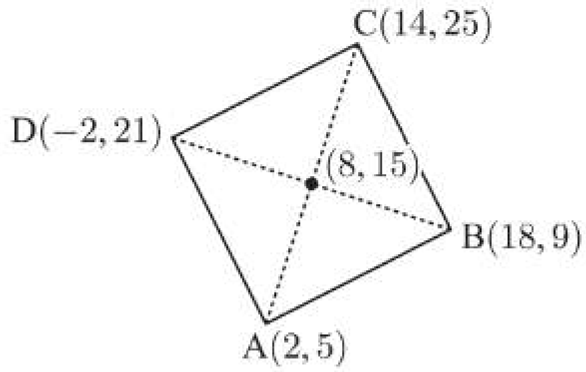
$\overrightarrow{DB} = \begin{pmatrix} 18-(-2) \\ 9-21 \end{pmatrix} = \begin{pmatrix} 20 \\ -12 \end{pmatrix}$

ii $|\overrightarrow{AC}| = \sqrt{12^2 + 20^2} = \sqrt{544}$ units

$|\overrightarrow{DB}| = \sqrt{20^2 + (-12)^2} = \sqrt{544}$ units

iii $\overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$

c The diagonals are perpendicular and equal in length, and as their midpoints are both $(8, 15)$, ABCD is a square.

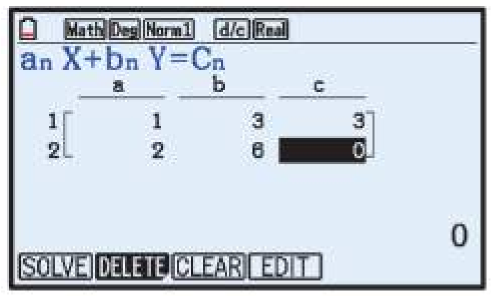


4 The lines intersect where $\begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 3 \\ -6 \end{pmatrix}$.

$\therefore\ 2 - t = -1 + 3s$ and $5 + 2t = 5 - 6s$

$\therefore\ t + 3s = 3$ .... (1) and $2t + 6s = 0$ .... (2)

Solving (1) and (2) simultaneously using technology tells us that there are no solutions.



So, L_1 and L_2 do not intersect. That is what we would expect to see since their direction vectors are multiples of one another, and therefore the lines are parallel.

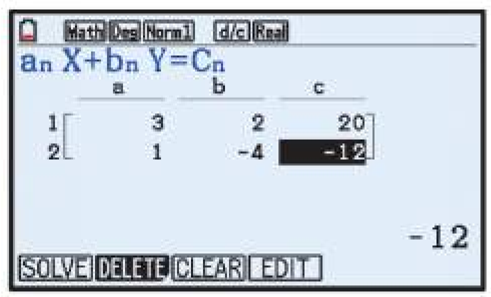
5 **a** At time t, particle A is at $(-13 + 3t,\ -1 + t)$, and particle B is at $(7 - 2t,\ -13 + 4t)$. Their paths are not parallel, so they must intersect.

Suppose particle A reaches the intersection point at time t_A,
and particle B reaches the intersection point at time t_B.

$\therefore\ -13 + 3t_A = 7 - 2t_B$ and $-1 + t_A = -13 + 4t_B$

$\therefore\ 3t_A + 2t_B = 20$ .... (1) and $t_A - 4t_B = -12$ .... (2)

Solving (1) and (2) simultaneously using technology gives $t_A = t_B = 4$.



Using particle A, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -13 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\therefore$ the paths intersect at $(-1, 3)$.

Checking with particle B, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -13 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ ✓

b The particles will collide at $(-1, 3)$, since they both pass through this point after 4 seconds.

6 **a** Jove's speed is 2.5 m s^{-1}.

Fiona's velocity vector is $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, so her speed is $\sqrt{(-2)^2 + (-1)^2} = \sqrt{5} \approx 2.24 \text{ m s}^{-1}$.

So, Jove is moving faster.

b Jove's direction vector is $3\mathbf{i} - 4\mathbf{j}$ and his speed is 2.5 m s^{-1}.

$\therefore$ his velocity vector is $\dfrac{2.5}{|3\mathbf{i} - 4\mathbf{j}|}(3\mathbf{i} - 4\mathbf{j}) = \dfrac{2.5}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix}$.

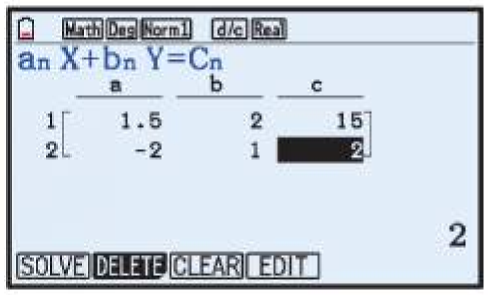
$\therefore$ the vector equation for Jove's position after t seconds is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} + t\begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix}$.

c Their paths will intersect where $\begin{pmatrix} -4 \\ -1 \end{pmatrix} + t_J\begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} + t_F\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, where t_J and t_F are the times at which Jove and Fiona respectively reach the intersection point.

$\therefore \quad -4 + \frac{3}{2}t_J = 11 - 2t_F \quad$ and $\quad -1 - 2t_J = 1 - t_F$

$\therefore \quad \frac{3}{2}t_J + 2t_F = 15 \quad .... (1) \quad$ and $\quad -2t_J + t_F = 2 \quad .... (2)$

Solving (1) and (2) simultaneously using technology gives $t_J = 2$, $t_F = 6$.



Using Jove's equation, $\begin{pmatrix} -4 \\ -1 \end{pmatrix} + 2\begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

$\therefore$ the paths intersect at $(-1, -5)$.

Checking with Fiona's equation, $\begin{pmatrix} 11 \\ 1 \end{pmatrix} + 6\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ ✓

d Fiona and Jove will not collide at this point, as Jove passes through the intersection point after $t_J = 2$ seconds, and Fiona passes through it after $t_F = 6$ seconds.

7 **a** P's torpedo is initially at $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$, and has velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

$\therefore$ the position of P's torpedo can be written as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

b Speed of P's torpedo $= \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ km min}^{-1}$.

c Q's torpedo is initially at $\begin{pmatrix} 15 \\ 7 \end{pmatrix}$, and has velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

Q fires its torpedo after k minutes. At time t, the torpedo has travelled for $(t - k)$ minutes.

$\therefore$ the position of Q's torpedo can be written as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t - k)\begin{pmatrix} -4 \\ -3 \end{pmatrix}$, $t \geqslant k$.

d Let θ be the angle between the paths of the torpedoes.

$$\cos\theta = \frac{\left|\begin{pmatrix}3\\-1\end{pmatrix} \bullet \begin{pmatrix}-4\\-3\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\-1\end{pmatrix}\right|\left|\begin{pmatrix}-4\\-3\end{pmatrix}\right|}$$

$$= \frac{|-12+3|}{\sqrt{3^2+(-1)^2}\sqrt{(-4)^2+(-3)^2}}$$

$$= \frac{9}{5\sqrt{10}}$$

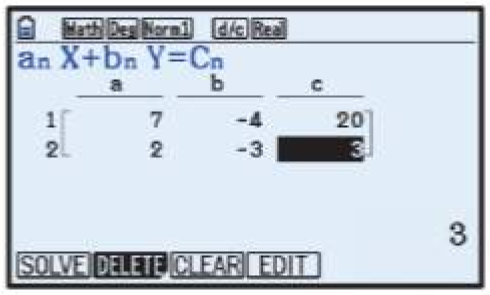
$$\therefore\ \theta = \cos^{-1}\left(\frac{9}{5\sqrt{10}}\right) \approx 55.3°$$

e The torpedoes' paths intersect where $\begin{pmatrix}-5\\4\end{pmatrix} + t\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}15\\7\end{pmatrix} + (t-k)\begin{pmatrix}-4\\-3\end{pmatrix}$.

$\therefore\ -5+3t = 15-4(t-k)$ and $4-t = 7-3(t-k)$

$\therefore\ 7t-4k = 20$ .... (1) and $2t-3k = 3$ .... (2)

Solving (1) and (2) simultaneously using technology gives $t = \frac{48}{13}$, $k = \frac{19}{13}$.



i Q fired its torpedo $\frac{19}{13}$ minutes after 1:34 pm, which is 1:35:28 pm.

ii The collision occurred $\frac{48}{13}$ minutes after 1:34 pm, which is 1:37:42 pm.

iii When $t = \frac{48}{13}$, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-5\\4\end{pmatrix} + \frac{48}{13}\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}\frac{79}{13}\\\frac{4}{13}\end{pmatrix}$.

The torpedoes collide at $\left(\frac{79}{13}, \frac{4}{13}\right)$, which is about $(6.08, 0.308)$.

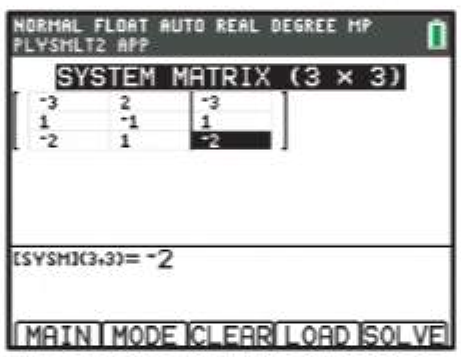
EXERCISE 10F.2

1 **a** Equating x, y, and z values in L_1 and L_2 gives

$1+2t = -2+3s$, $2-t = 3-s$, and $3+t = 1+2s$

$\therefore\ -3s+2t = -3$ .... (1), $s-t = 1$ .... (2), and $-2s+t = -2$ .... (3)

Solving (1), (2), and (3) simultaneously using technology gives $s = 1$, $t = 0$.

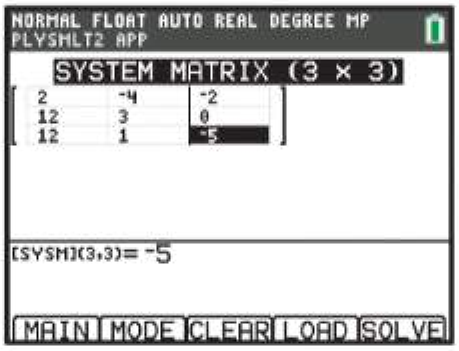


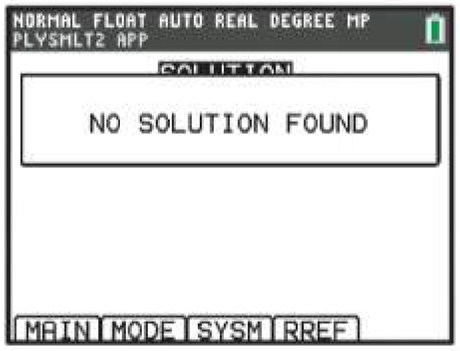
Substituting $t = 0$ into L_1, we find that L_1 and L_2 intersect at $(1, 2, 3)$.

b Equating x, y, and z values in L_1 and L_2 gives

$$-1+2\lambda=-3+4\mu, \qquad 2-12\lambda=2+3\mu, \qquad \text{and} \qquad 4+12\lambda=-1-\mu$$
$$\therefore\ 2\lambda-4\mu=-2 \ \ .... \ (1), \qquad 12\lambda+3\mu=0 \ \ .... \ (2), \qquad \text{and} \qquad 12\lambda+\mu=-5 \ \ .... \ (3)$$

Solving (1), (2), and (3) simultaneously using technology gives no solution. So, the lines do not intersect.



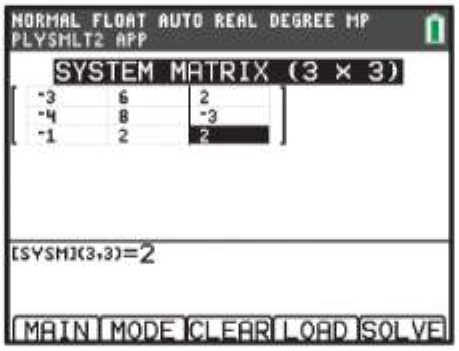


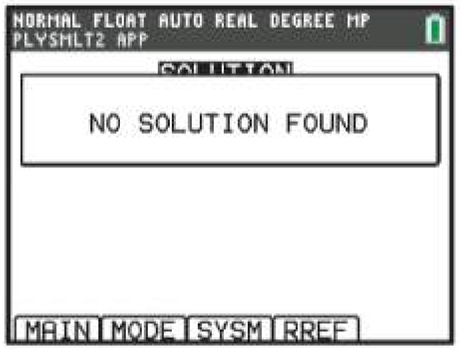
Since $\begin{pmatrix}4\\3\\-1\end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix}2\\-12\\12\end{pmatrix}$ we conclude that the lines are skew.

c Equating x, y, and z values in L_1 and L_2 gives

$$6t=2+3s, \qquad 3+8t=4s, \qquad \text{and} \qquad -1+2t=1+s$$
$$\therefore\ -3s+6t=2 \ \ .... \ (1), \qquad -4s+8t=-3 \ \ .... \ (2), \qquad \text{and} \qquad -s+2t=2 \ \ .... \ (3)$$

Solving (1), (2), and (3) simultaneously using technology gives no solution. So, the lines do not intersect.



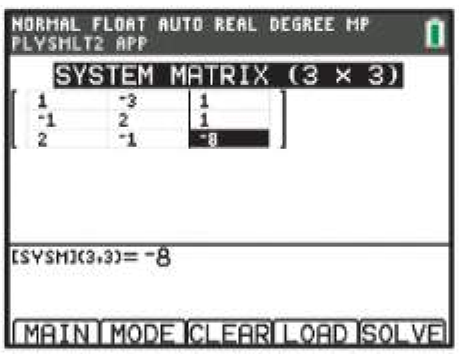


Since $\begin{pmatrix}3\\4\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}6\\8\\2\end{pmatrix}$, we conclude that the lines are parallel.

d Equating x, y, and z values in L_1 and L_2 gives

$$1+\lambda=2+3\mu, \qquad 2-\lambda=3-2\mu, \qquad \text{and} \qquad 3+2\lambda=-5+\mu$$
$$\therefore\ \lambda-3\mu=1 \ \ .... \ (1), \qquad -\lambda+2\mu=1 \ \ .... \ (2), \qquad \text{and} \qquad 2\lambda-\mu=-8 \ \ .... \ (3)$$

Solving (1), (2), and (3) simultaneously using technology gives $\lambda=-5$, $\mu=-2$.



Substituting $\mu=-2$ into L_2, we find that L_1 and L_2 intersect at $(2+3(-2),\ 3-2(-2),\ -5+(-2))$ which is $(-4, 7, -7)$.

2 **a** **i** The bird has velocity vector $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$.

So its speed is $|\,\mathbf{v}_1\,| = \sqrt{4^2+3^2+1^2} = \sqrt{26} \approx 5.10\ \text{m s}^{-1}$.

ii The bat has velocity vector $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$.

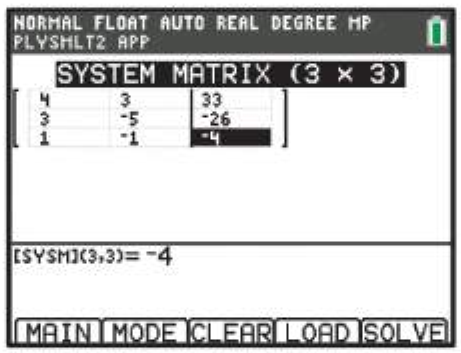
So its speed is $|\,\mathbf{v}_2\,| = \sqrt{(-3)^2+5^2+1^2} = \sqrt{35} \approx 5.92\ \text{m s}^{-1}$.

b The two paths intersect where $\begin{pmatrix} -2 \\ 6 \\ 9 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 31 \\ -20 \\ 5 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$, where t_1 and t_2 are the times at which the bird and the bat respectively reach the point of intersection.

$\therefore\ -2+4t_1 = 31-3t_2, \qquad 6+3t_1 = -20+5t_2, \qquad \text{and} \quad 9+t_1 = 5+t_2$

$\therefore\ 4t_1+3t_2 = 33 \ \ \text{.... (1)}, \qquad 3t_1-5t_2 = -26 \ \ \text{.... (2)}, \qquad \text{and} \quad t_1-t_2 = -4 \ \ \text{.... (3)}$

Solving (1), (2), and (3) simultaneously using technology gives $t_1 = 3,\ \ t_2 = 7$.



Substituting $t_1 = 3$ into $\mathbf{r}_1$, we find that the paths of the bird and the bat intersect at $(-2+4(3),\ 6+3(3),\ 9+3)$ which is $(10,\ 15,\ 12)$.

c The bird and the bat will not collide at the intersection point, as the bird passes through it after 3 seconds, and the bat passes through it after 7 seconds.

3 **a** The direction vector of the shark is $6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, and its speed is $14\ \text{m s}^{-1}$.

So its velocity vector is $\dfrac{14}{|\,6\mathbf{i}-3\mathbf{j}-2\mathbf{k}\,|}(6\mathbf{i}-3\mathbf{j}-2\mathbf{k}) = \frac{14}{7}\begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ -4 \end{pmatrix}$.

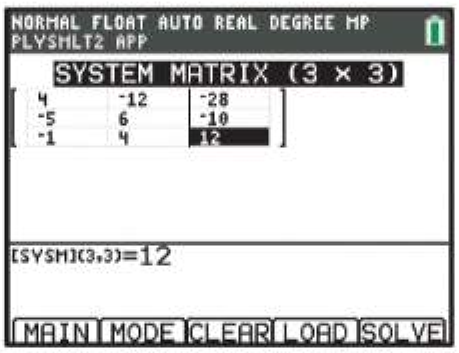
b The shark swims according to $\mathbf{r}_2 = \begin{pmatrix} -35 \\ -6 \\ -10 \end{pmatrix} + t\begin{pmatrix} 12 \\ -6 \\ -4 \end{pmatrix}$.

So the paths intersect when $\begin{pmatrix} -7 \\ 4 \\ -22 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} -35 \\ -6 \\ -10 \end{pmatrix} + t_2 \begin{pmatrix} 12 \\ -6 \\ -4 \end{pmatrix}$, where t_1 and t_2 are the times at which the seal and shark respectively reach the point of intersection.

$\therefore\ -7+4t_1 = -35+12t_2, \qquad 4-5t_1 = -6-6t_2, \qquad \text{and} \quad -22-t_1 = -10-4t_2$

$\therefore\ 4t_1-12t_2 = -28 \ \ \text{.... (1)}, \qquad -5t_1+6t_2 = -10 \ \ \text{.... (2)}, \qquad \text{and} \quad -t_1+4t_2 = 12 \ \ \text{.... (3)}$

Solving (1), (2), and (3) simultaneously using technology gives $t_1 = 8,\ t_2 = 5$.



Substituting $t = 5$ into $\mathbf{r}_2$, we find that the paths intersect at $(-35 + 12(5),\ -6 - 6(5),\ -10 - 4(5))$ which is $(25, -36, -30)$.

c The seal passes through $(25, -36, -30)$ when $t = 8$.

When $t = 8$, $\mathbf{r}_2 = \begin{pmatrix} -35 \\ -6 \\ -10 \end{pmatrix} + 8\begin{pmatrix} 12 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 61 \\ -54 \\ -42 \end{pmatrix}$.

So, the shark is at $(61, -54, -42)$.

$\therefore$ the distance between them is

$$\sqrt{(61-25)^2 + (-54-(-36))^2 + (-42-(-30))^2} = \sqrt{1764} = 42 \text{ m}.$$

4 **a** The bullet moves in the direction $4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ with speed 360 m s^{-1}.

So its velocity vector is $\dfrac{360}{|4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}|}(4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) = \frac{360}{9}\begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 160 \\ 280 \\ 160 \end{pmatrix}$.

b The target's velocity vector is $14\mathbf{i} + 9\mathbf{j} + 8\mathbf{k}$.

So its speed is $\sqrt{14^2 + 9^2 + 8^2} = \sqrt{341} \approx 18.5 \text{ m s}^{-1}$.

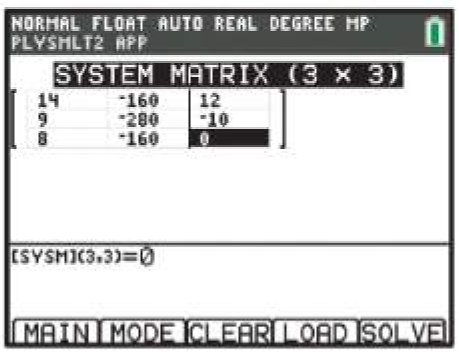
c The paths of the target and bullet intersect when

$\begin{pmatrix} 8 \\ 16 \\ 1 \end{pmatrix} + t_1\begin{pmatrix} 14 \\ 9 \\ 8 \end{pmatrix} = \begin{pmatrix} 20 \\ 6 \\ 1 \end{pmatrix} + t_2\begin{pmatrix} 160 \\ 280 \\ 160 \end{pmatrix}$, where t_1 and t_2 are the times at which the target and bullet respectively reach the intersection point.

$\therefore\ 8 + 14t_1 = 20 + 160t_2, \quad 16 + 9t_1 = 6 + 280t_2, \quad$ and $\quad 1 + 8t_1 = 1 + 160t_2$

$\therefore\ 14t_1 - 160t_2 = 12 \quad .... (1), \quad 9t_1 - 280t_2 = -10 \quad .... (2), \quad$ and $\quad 8t_1 - 160t_2 = 0 \quad .... (3)$

Solving (1), (2), and (3) simultaneously using technology gives $t_1 = 2,\ t_2 = 0.1$.



$\therefore$ the paths intersect.

d **i** The target takes 2 seconds to reach the intersection, and the bullet takes 0.1 seconds, so the bullet must be fired $2 - 0.1 = 1.9$ seconds after the target is released.

ii The bullet is initially at $(20, 6, 1)$.

The bullet is at the point of intersection after 0.1 seconds.

$\therefore$ the point of intersection is $(20 + 160(0.1),\ 6 + 280(0.1),\ 1 + 160(0.1))$ which is $(36, 34, 17)$.

$\therefore$ the bullet travelled $\sqrt{(36-20)^2 + (34-6)^2 + (17-1)^2} = \sqrt{1296} = 36$ m.

REVIEW SET 10A

1 **a** The vector equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix},\quad t \in \mathbb{R}$$

b The parametric equations are

$$x = -6 + 4t,\quad y = 3 - 3t,\quad t \in \mathbb{R}$$

2 $(-3, m)$ lies on the line, so $\begin{pmatrix} -3 \\ m \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$

$$\therefore\ -3 = 18 - 7t \quad\text{and}\quad m = -2 + 4t$$
$$\therefore\ 7t = 21$$
$$\therefore\ t = 3 \quad\text{and so}\quad m = -2 + 4(3) = 10$$

3 **a** $\overrightarrow{AB} = \begin{pmatrix} 0-2 \\ 1-(-1) \\ -1-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$

$\therefore$ since A lies on the line, the line has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix},\quad \lambda \in \mathbb{R}$

b If C lies on (AB) and is 2 units from A, then C has coordinates $(2 - 2\lambda,\ -1 + 2\lambda,\ 3 - 4\lambda)$

and $\overrightarrow{AC}$ has length $\sqrt{(-2\lambda)^2 + (2\lambda)^2 + (-4\lambda)^2} = 2$

$$\therefore\ \sqrt{24\lambda^2} = 2$$
$$\therefore\ 24\lambda^2 = 4$$
$$\therefore\ \lambda^2 = \tfrac{1}{6}$$
$$\therefore\ \lambda = \pm\tfrac{1}{\sqrt{6}}$$

$\therefore$ C has coordinates $\left(2 - \frac{2}{\sqrt{6}},\ -1 + \frac{2}{\sqrt{6}},\ 3 - \frac{4}{\sqrt{6}}\right)$ or $\left(2 + \frac{2}{\sqrt{6}},\ -1 - \frac{2}{\sqrt{6}},\ 3 + \frac{4}{\sqrt{6}}\right)$.

4 L_1 is perpendicular to $2\mathbf{i} - 7\mathbf{j}$, so it has direction vector $\mathbf{b}_1 = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$.

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 3-(-1) \\ -1-4 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

If θ is the angle between the lines, then $\cos\theta = \dfrac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} = \dfrac{|\,28 - 10\,|}{\sqrt{53}\sqrt{41}}$

$$\therefore\ \theta = \cos^{-1}\left(\frac{18}{\sqrt{53}\sqrt{41}}\right)$$
$$\approx 67.3^\circ$$

5 $x(t) = -4 + 8t, \quad y(t) = 3 + 6t, \quad z(t) = 1 - 2t$

a $x(0) = -4, \; y(0) = 3,$ and $z(0) = 1$

The initial position of the particle is $(-4, 3, 1)$.

b $x(4) = -4 + 8(4) = 28, \; y(4) = 3 + 6(4) = 27,$ and $z(4) = 1 - 2(4) = -7$

After 4 seconds, the position of the particle is $(28, 27, -7)$.

c The particle's velocity vector is $\begin{pmatrix} 8 \\ 6 \\ -2 \end{pmatrix}$.

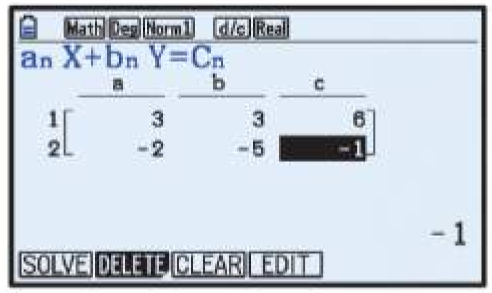
d The speed of the particle is $\sqrt{8^2 + 6^2 + (-2)^2} = \sqrt{104} \approx 10.2 \text{ m s}^{-1}$.

6 The lines meet where $\begin{pmatrix} -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

$\therefore \; -2 + 3\lambda = 4 - 3\mu$ and $-1 - 2\lambda = -2 + 5\mu$

$\therefore \; 3\lambda + 3\mu = 6$ .... (1) and $-2\lambda - 5\mu = -1$ .... (2)

Solving (1) and (2) simultaneously using technology gives $\lambda = 3$, $\mu = -1$.



Using line 1, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$

Checking in line 2, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$ ✓

$\therefore$ the lines meet at $(7, -7)$.

7 **a** Let N be the point on the line closest to P.

N has coordinates $(5 + 2t, \; 1 + t)$ for some t, and $\overrightarrow{PN} = \begin{pmatrix} 5 + 2t - 1 \\ 1 + t - 4 \end{pmatrix} = \begin{pmatrix} 4 + 2t \\ -3 + t \end{pmatrix}$.

The distance between P and the line is minimised when

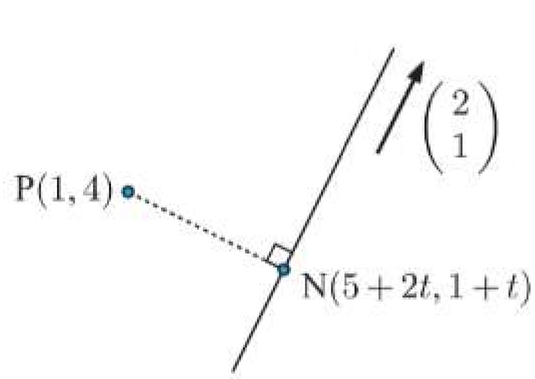
$$\overrightarrow{PN} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore \begin{pmatrix} 4 + 2t \\ -3 + t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore 8 + 4t - 3 + t = 0$$

$$\therefore 5t = -5$$

$$\therefore t = -1$$



Thus $\overrightarrow{PN} = \begin{pmatrix} 4 - 2 \\ -3 - 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

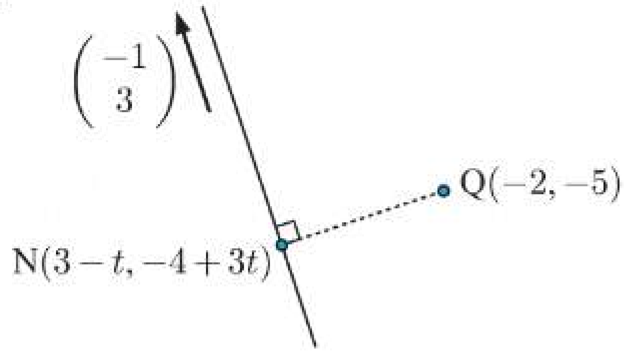
$\therefore |\overrightarrow{PN}| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

$\therefore$ the shortest distance from the point P to the line is $2\sqrt{5} \approx 4.47$ units.

b Let N be the point on the line closest to Q.

N has coordinates $(3-t,\ -4+3t)$ for some t, and $\overrightarrow{QN} = \begin{pmatrix} 3-t-(-2) \\ -4+3t-(-5) \end{pmatrix} = \begin{pmatrix} 5-t \\ 1+3t \end{pmatrix}$.

The distance between Q and the line is minimised when



$$\overrightarrow{QN} \bullet \begin{pmatrix} -1 \\ 3 \end{pmatrix} = 0$$

$$\therefore \begin{pmatrix} 5-t \\ 1+3t \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \end{pmatrix} = 0$$

$$\therefore -5+t+3+9t = 0$$

$$\therefore 10t = 2$$

$$\therefore t = \tfrac{1}{5}$$

Thus $\overrightarrow{QN} = \begin{pmatrix} 5-\frac{1}{5} \\ 1+\frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{24}{5} \\ \frac{8}{5} \end{pmatrix}$

$$\therefore |\overrightarrow{QN}| = \sqrt{\left(\tfrac{24}{5}\right)^2 + \left(\tfrac{8}{5}\right)^2} = \sqrt{\tfrac{640}{25}} = \tfrac{8\sqrt{10}}{5}$$

$\therefore$ the shortest distance from the point Q to the line is $\frac{8\sqrt{10}}{5} \approx 5.06$ units.

8 **a** **i** The yacht is initially at $(-6, 10)$, so its initial position vector is $-6\mathbf{i} + 10\mathbf{j}$.

ii $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\therefore$ $5(-\mathbf{i} - 3\mathbf{j})$ has length $5\sqrt{10}$

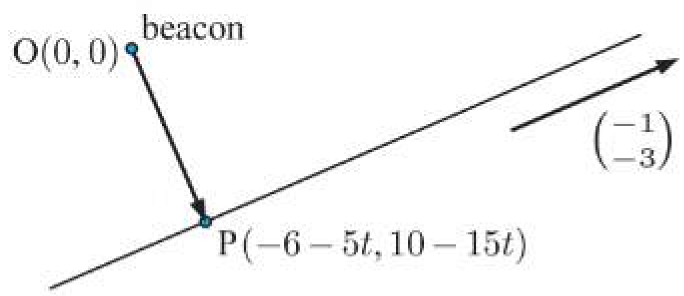
$\therefore$ the velocity vector of the yacht is $-5\mathbf{i} - 15\mathbf{j}$

iii $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \end{pmatrix} + t\begin{pmatrix} -5 \\ -15 \end{pmatrix}$

$\therefore$ the position vector of the yacht after t hours is

$$-6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j})$$
$$= (-6-5t)\mathbf{i} + (10-15t)\mathbf{j}, \quad t \geqslant 0$$

b Let P be the point on the yacht's path when it is closest to the beacon.



Then $\overrightarrow{OP} = \begin{pmatrix} -6-5t \\ 10-15t \end{pmatrix}$ and

$$\overrightarrow{OP} \bullet \begin{pmatrix} -5 \\ -15 \end{pmatrix} = 0$$

$$\therefore -5(-6-5t) - 15(10-15t) = 0$$

$$\therefore 30 + 25t - 150 + 225t = 0$$

$$\therefore 250t = 120$$

$$\therefore t = 0.48 \text{ hours} \quad \text{(or 28.8 minutes)}$$

c When $t = 0.48$, $\overrightarrow{OP} = \begin{pmatrix} -6-5(0.48) \\ 10-15(0.48) \end{pmatrix} = \begin{pmatrix} -8.4 \\ 2.8 \end{pmatrix}$

and $|\overrightarrow{OP}| = \sqrt{(-8.4)^2 + (2.8)^2} \approx 8.85$ km

The closest that the yacht gets to the beacon is about $8.85 \text{ km} > 8 \text{ km}$.

$\therefore$ the yacht will not hit the reef.

9 **a** $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ where $t \geqslant 0$. When $t = 0$, the time is 2:17 pm.

$\therefore$ $x_1(t) = 2 + t$, $y_1(t) = 4 - 3t$, $t \geqslant 0$

b After t minutes have passed, submarine Y18's torpedo has been moving for $(t-2)$ minutes.

$\therefore$ $x_2(t) = 11 - (t-2)$, $\quad y_2(t) = 3 + a(t-2)$

$\therefore$ $x_2(t) = 13 - t$ $\quad y_2(t) = (3-2a) + at$, $t \geqslant 2$

c The interception occurs when $2 + t = 13 - t$ and $4 - 3t = (3-2a) + at$

$\therefore$ $2t = 11$

$\therefore$ $t = \frac{11}{2}$

$\therefore$ the interception occurs $5\frac{1}{2}$ minutes after 2:17 pm which is 2:22:30 pm.

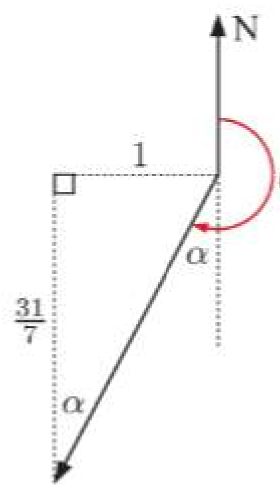
d When $t = \frac{11}{2}$, $4 - 3(\frac{11}{2}) = (3-2a) + a(\frac{11}{2})$

$\therefore$ $-\frac{25}{2} = 3 + \frac{7a}{2}$

$\therefore$ $-25 = 6 + 7a$

$\therefore$ $7a = -31$

$\therefore$ $a = -\frac{31}{7}$



Y18's torpedo has velocity vector $\begin{pmatrix} -1 \\ -\frac{31}{7} \end{pmatrix}$

with speed $= \sqrt{(-1)^2 + \left(-\frac{31}{7}\right)^2}$

≈ 4.54 km min^{-1}

$\tan\alpha = \dfrac{1}{\frac{31}{7}} = \frac{7}{31}$

$\therefore$ $\alpha = \tan^{-1}\left(\frac{7}{31}\right) \approx 12.7°$

So, the torpedo has a speed of about 4.54 km min^{-1}, travelling on the bearing $\approx 192.7°$.

e If θ is the acute angle between the paths of the torpedoes, then
$$\cos\theta = \frac{\left|\begin{pmatrix}1\\-3\end{pmatrix}\bullet\begin{pmatrix}-1\\-\frac{31}{7}\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-3\end{pmatrix}\right|\left|\begin{pmatrix}-1\\-\frac{31}{7}\end{pmatrix}\right|}$$
$$= \frac{\left|-1+\frac{93}{7}\right|}{\sqrt{10}\,\frac{\sqrt{1010}}{7}}$$
$$= \frac{86}{\sqrt{10}\sqrt{1010}}$$
$$\therefore\ \theta = \cos^{-1}\left(\frac{86}{\sqrt{10}\sqrt{1010}}\right)$$
$$\approx 31.2^\circ$$

10 **a** $\overrightarrow{AB} = \begin{pmatrix}1-(-1)\\0-2\\-1-3\end{pmatrix} = \begin{pmatrix}2\\-2\\-4\end{pmatrix}$

$\therefore$ a vector equation for (AB) is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-2\\-4\end{pmatrix}$, $\lambda \in \mathbb{R}$.

b (AB) has direction vector $\mathbf{b} = \begin{pmatrix}2\\-2\\-4\end{pmatrix}$.

Let $X(-1+2\lambda,\ 2-2\lambda,\ 3-4\lambda)$ be any point on the given line.

$\therefore\ \overrightarrow{CX} = \begin{pmatrix}-1+2\lambda-1\\2-2\lambda-3\\3-4\lambda-0\end{pmatrix} = \begin{pmatrix}-2+2\lambda\\-1-2\lambda\\3-4\lambda\end{pmatrix}$

If X is the closest point on (AB) to C, then $\overrightarrow{CX} \perp \mathbf{b}$.

$$\therefore\ \overrightarrow{CX}\bullet\mathbf{b} = 0$$
$$\therefore\ \begin{pmatrix}-2+2\lambda\\-1-2\lambda\\3-4\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\-2\\-4\end{pmatrix} = 0$$
$$\therefore\ -4+4\lambda+2+4\lambda-12+16\lambda = 0$$
$$\therefore\ 24\lambda = 14$$
$$\therefore\ \lambda = \tfrac{7}{12}$$

Substituting $\lambda = \frac{7}{12}$ into the vector equation, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\2\\3\end{pmatrix} + \frac{7}{12}\begin{pmatrix}2\\-2\\-4\end{pmatrix} = \begin{pmatrix}\frac{1}{6}\\\frac{5}{6}\\\frac{2}{3}\end{pmatrix}$.

The foot of the perpendicular from C to (AB) is $\left(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}\right)$.

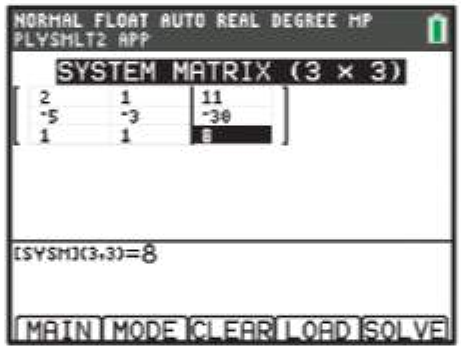
11 **a** After 2 seconds, object A is at $(-2+2(2),\ 14-5(2),\ 2)$, which is $(2, 4, 2)$,
and object B is at $(9-2,\ -16+3(2),\ 8-2)$, which is $(7, -10, 6)$.

b The objects intersect when $\begin{pmatrix}-2\\14\\0\end{pmatrix}+t_A\begin{pmatrix}2\\-5\\1\end{pmatrix}=\begin{pmatrix}9\\-16\\8\end{pmatrix}+t_B\begin{pmatrix}-1\\3\\-1\end{pmatrix}$, where t_A and t_B are the times at which objects A and B respectively pass through the intersection point.

$\therefore\ -2+2t_A=9-t_B,\qquad 14-5t_A=-16+3t_B,\qquad$ and $\qquad t_A=8-t_B$

$\therefore\ 2t_A+t_B=11$ .... (1), $\quad -5t_A-3t_B=-30$ .... (2), $\quad$ and $\quad t_A+t_B=8$ .... (3)

Solving (1), (2), and (3) simultaneously using technology gives $t_A=3,\ t_B=5$.



Substituting $t=3$ into object A's equation, we find that the paths of the objects intersect at $(-2+2(3),\ 14-5(3),\ 3)$ which is $(4, -1, 3)$.

c The objects will not collide at this point, because object A passes the point after 3 seconds, and object B passes the point after 5 seconds.

12 **a** Vivian's puck has direction vector $-3\mathbf{i}+4\mathbf{j}$ and speed $10\ \text{m s}^{-1}$.

So its velocity vector is $\dfrac{10}{|-3\mathbf{i}+4\mathbf{j}|}(-3\mathbf{i}+4\mathbf{j})=\frac{10}{5}\begin{pmatrix}-3\\4\end{pmatrix}=\begin{pmatrix}-6\\8\end{pmatrix}$.

b After 1 second, Chelsea's puck is at $(1+3(1),\ 5+6(1))$, which is $(4, 11)$,
and Vivian's puck is at $(21-6(1),\ 10+8(1))$, which is $(15, 18)$.

$\therefore$ the distance between their pucks after 1 second is $\sqrt{(15-4)^2+(18-11)^2}$
$=\sqrt{170}\approx 13.0$ m.

c After t seconds, Chelsea's puck is at $(1+3t,\ 5+6t)$,
and Vivian's puck is at $(21-6t,\ 10+8t)$.

$\therefore$ at time t, the distance between their pucks is

$$\begin{aligned}D&=\sqrt{[(1+3t)-(21-6t)]^2+[(5+6t)-(10+8t)]^2}\\&=\sqrt{(-20+9t)^2+(-5-2t)^2}\\&=\sqrt{400-360t+81t^2+25+20t+4t^2}\\&=\sqrt{425-340t+85t^2}\ \text{m}\end{aligned}$$

Now D is minimised when $D^2=85t^2-340t+425$ is minimised.

This occurs when $t=-\dfrac{-340}{2\times 85}=2$.

When $t=2$, $\quad D=\sqrt{85(2)^2-340(2)+425}=\sqrt{85}$

The shortest distance between the pucks is $\sqrt{85}\approx 9.22$ m, when Chelsea's puck is at $(1+3(2),\ 5+6(2))$, which is $(7, 17)$.

REVIEW SET 10B

1 **a** When $t = 1$, $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3+2 \\ -3+5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\therefore$ the point is $(5, 2)$.

b $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, so it could also be used to describe the direction of the line.

c The line passes through point $(5, 2)$ and has direction vector $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

$\therefore$ $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, $s \in \mathbb{R}$ is an alternative vector equation for the line.

2 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

ii $x = 2 + 4\lambda$, $y = -3 + 2\lambda$, $z = 1 - \lambda$, $\lambda \in \mathbb{R}$

b **i** The line has direction vector $\begin{pmatrix} 5-(-1) \\ -2-6 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

ii $x = -1 + 6\lambda$, $y = 6 - 8\lambda$, $z = 3 - 3\lambda$, $\lambda \in \mathbb{R}$

3 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} 5-0 \\ -2-3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} -6-(-2) \\ 7-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

If θ is the angle between L_1 and L_1, then
$$\cos\theta = \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|}$$
$$= \frac{|-20-15|}{\sqrt{25+25}\sqrt{16+9}}$$
$$= \frac{35}{5\sqrt{50}} = \frac{7}{\sqrt{50}}$$
$$\therefore \theta = \cos^{-1}\left(\frac{7}{\sqrt{50}}\right) \approx 8.13°$$

4 L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$

If θ is the angle between L_1 and L_2, then
$$\begin{aligned} \cos\theta &= \frac{|\,\mathbf{b}_1 \bullet \mathbf{b}_2\,|}{|\,\mathbf{b}_1\,|\,|\,\mathbf{b}_2\,|} \\ &= \frac{|\,-20-36\,|}{\sqrt{16+9}\sqrt{25+144}} \\ &= \tfrac{56}{65} \\ \therefore\ \theta &= \cos^{-1}\left(\tfrac{56}{65}\right) \\ &\approx 30.5^\circ \end{aligned}$$

5 **a** $$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0-3 \\ 2-(-1) \\ -2-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$$

$\therefore\ |\,\overrightarrow{\mathrm{AB}}\,| = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$ units

b $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \text{where } \lambda = 3t$$

$\therefore\ \mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$, where $\lambda \in \mathbb{R}$

A lies on the line $\mathbf{r}$ when $\lambda = -3$ and B lies on $\mathbf{r}$ when $\lambda = 0$.

$\therefore$ the line between A and B can be described by $\mathbf{r}$.

c The line with equation $t(\mathbf{i} + \mathbf{j} + \mathbf{k})$ has direction vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

If θ is the angle between $\overrightarrow{\mathrm{AB}}$ and the line, then
$$\begin{aligned} \cos\theta &= \frac{\left|\,\overrightarrow{\mathrm{AB}} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|}{|\,\overrightarrow{\mathrm{AB}}\,|\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|} \\ &= \frac{|-3+3-3|}{3\sqrt{3} \times \sqrt{1+1+1}} \\ &= \frac{|-3|}{3\sqrt{3} \times \sqrt{3}} \\ &= \tfrac{3}{9} \\ &= \tfrac{1}{3} \\ \therefore\ \theta &= \cos^{-1}\left(\tfrac{1}{3}\right) \\ &\approx 70.5^\circ \end{aligned}$$

6 The direction vector is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$\therefore\ 2\sqrt{10}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ has length 20 units.

So, the velocity vector is $\begin{pmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{pmatrix}$ or $2\sqrt{10}\,(3\mathbf{i} - \mathbf{j})$.

7 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + t\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$

a When $t = 0$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix}$

$\therefore$ the initial position of the zip-liner is $(-10, 5, 12)$.

b The velocity vector is $\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$.

c The speed of the zip-liner is $\left|\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}\right| = \sqrt{6^2 + 14^2 + (-0.4)^2}$

$\approx 15.2\ \text{m s}^{-1}$

d $z = 12 - 0.4t$

When $z = 0$, $12 - 0.4t = 0$

$\therefore\ 0.4t = 12$

$\therefore\ t = 30$

$\therefore$ it takes 30 seconds to reach the end of the line.

e When $t = 30$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + 30\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$

$= \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + \begin{pmatrix} 180 \\ 420 \\ -12 \end{pmatrix}$

$= \begin{pmatrix} 170 \\ 425 \\ 0 \end{pmatrix}$

$\therefore$ the endpoint of the line is $(170, 425, 0)$.

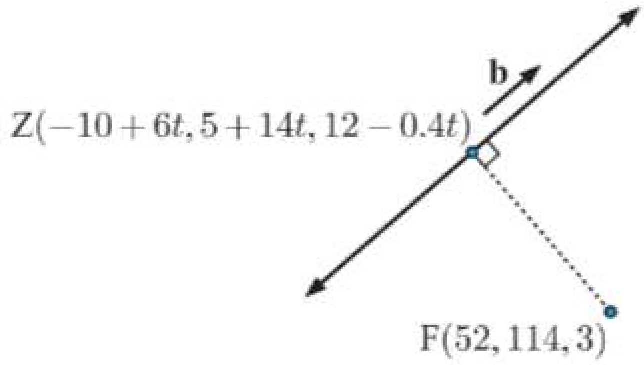
f The zip-line has direction vector $\mathbf{b} = \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$.

The zip-liner has position $Z(-10 + 6t,\ 5 + 14t,\ 12 - 0.4t)$ at time t.

The friend is watching from $F(52, 144, 3)$.

$$\therefore\ \overrightarrow{FZ} = \begin{pmatrix} -10 + 6t - 52 \\ 5 + 14t - 144 \\ 12 - 0.4t - 3 \end{pmatrix} = \begin{pmatrix} 6t - 62 \\ 14t - 139 \\ 9 - 0.4t \end{pmatrix}$$

If Z is the closest point on the line to F, then $\overrightarrow{FZ} \perp \mathbf{b}$.



$$\therefore\ \overrightarrow{FZ} \bullet \mathbf{b} = 0$$

$$\therefore\ \begin{pmatrix} 6t - 62 \\ 14t - 139 \\ 9 - 0.4t \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix} = 0$$

$$\therefore\ 6(6t - 62) + 14(14t - 139) - 0.4(9 - 0.4t) = 0$$

$$\therefore\ 36t - 372 + 196t - 1946 - 3.6 + 0.16t = 0$$

$$\therefore\ 232.16t = 2321.6$$

$$\therefore\ t = 10$$

Substituting $t = 10$ into the parametric equations, the foot of the perpendicular is $(50, 145, 8)$.

$\therefore$ the position of the zip-liner when he is closest to his friend is $(50, 145, 8)$.

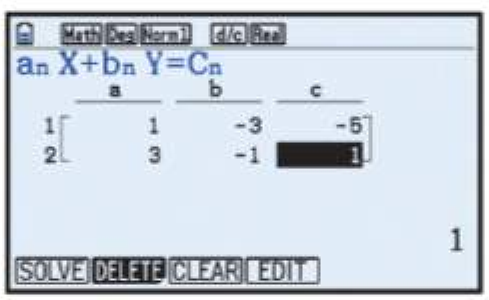
8 **a** Lines (AB) and (AC) meet at A.

$$\therefore\ \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 4 + t \\ -1 + 3t \end{pmatrix} = \begin{pmatrix} -1 + 3u \\ u \end{pmatrix}$$

$$\therefore\ t - 3u = -5 \quad .... (1)$$

$$\text{and}\quad 3t - u = 1 \quad .... (2)$$



Solving (1) and (2) simultaneously using technology gives $t = 1,\ u = 2$.

So, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\therefore$ A is at $(5, 2)$.

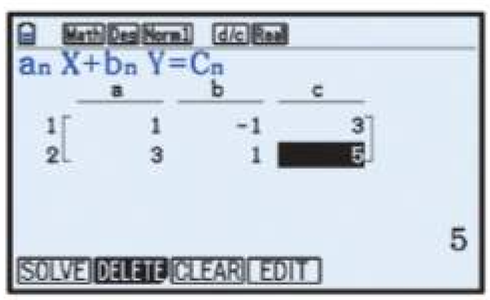
Lines (AB) and (BC) meet at B.

$$\therefore\ \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} 4 + t \\ -1 + 3t \end{pmatrix} = \begin{pmatrix} 7 + s \\ 4 - s \end{pmatrix}$$

$$\therefore\ t - s = 3 \quad .... (1)$$

$$\text{and}\quad 3t + s = 5 \quad .... (2)$$



Solving (1) and (2) simultaneously using technology gives $t = 2,\ s = -1$.

So, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$\therefore$ B is at $(6, 5)$.

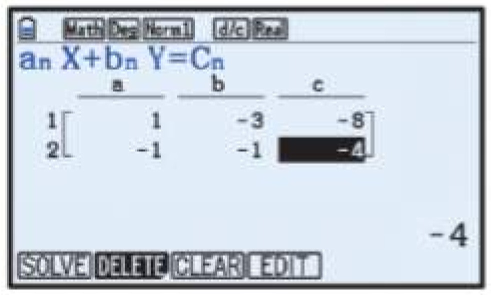
Lines (BC) and (AC) meet at C.

$$\therefore\ \begin{pmatrix}7\\4\end{pmatrix}+s\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}+u\begin{pmatrix}3\\1\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}7+s\\4-s\end{pmatrix}=\begin{pmatrix}-1+3u\\u\end{pmatrix}$$

$$\therefore\ s-3u=-8\ \ \text{.... (1)} \quad \text{and} \quad -s-u=-4\ \ \text{.... (2)}$$

Solving (1) and (2) simultaneously using technology gives $s=1,\ u=3$.



So, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}8\\3\end{pmatrix}$

$\therefore$ C is at (8, 3).

b $\overrightarrow{AB}=\begin{pmatrix}1\\3\end{pmatrix}$, so $|\overrightarrow{AB}|=\sqrt{1+9}=\sqrt{10}$ units

$\overrightarrow{BC}=\begin{pmatrix}2\\-2\end{pmatrix}$, so $|\overrightarrow{BC}|=\sqrt{4+4}=\sqrt{8}$ units

$\overrightarrow{AC}=\begin{pmatrix}3\\1\end{pmatrix}$, so $|\overrightarrow{AC}|=\sqrt{9+1}=\sqrt{10}$ units

c Triangle ABC is isosceles.

9 **a** Jet skis A and B have direction vectors $-24\mathbf{i}+7\mathbf{j}$ and $-3\mathbf{i}-9\mathbf{j}$ respectively.

$(-24\mathbf{i}+7\mathbf{j})\bullet(-3\mathbf{i}-9\mathbf{j})=-24(-3)+7(-9)=9\neq 0$

$\therefore$ the direction vectors are not perpendicular, so the jet skis are not travelling at right angles to one another.

b Jet ski A has direction vector $-24\mathbf{i}+7\mathbf{j}$ and is travelling at $10\ \text{m s}^{-1}$.

So its velocity vector is $\dfrac{10}{|-24\mathbf{i}+7\mathbf{j}|}(-24\mathbf{i}+7\mathbf{j})=\frac{10}{25}\begin{pmatrix}-24\\7\end{pmatrix}=\begin{pmatrix}-9.6\\2.8\end{pmatrix}$.

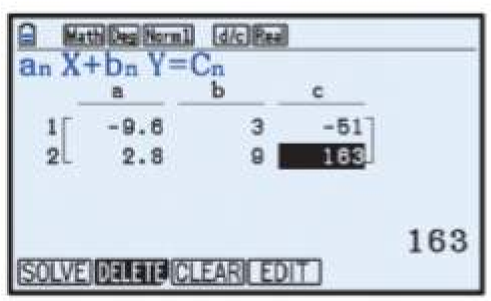
c The paths of the jet skis intersect when $\begin{pmatrix}150\\-30\end{pmatrix}+t_A\begin{pmatrix}-9.6\\2.8\end{pmatrix}=\begin{pmatrix}99\\133\end{pmatrix}+t_B\begin{pmatrix}-3\\-9\end{pmatrix}$,

where t_A and t_B are the times at which jet ski A and jet ski B respectively reach the intersection point.

$$\therefore\ 150-9.6t_A=99-3t_B \quad \text{and} \quad -30+2.8t_A=133-9t_B$$

$$\therefore\ -9.6t_A+3t_B=-51\ \ \text{.... (1)} \quad \text{and} \quad 2.8t_A+9t_B=163\ \ \text{.... (2)}$$

Solving (1) and (2) simultaneously using technology gives $t_A=10,\ t_B=15$.



Using jet ski A, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 150 \\ -30 \end{pmatrix} + 10\begin{pmatrix} -9.6 \\ 2.8 \end{pmatrix} = \begin{pmatrix} 54 \\ -2 \end{pmatrix}$

Using jet ski B, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 99 \\ 133 \end{pmatrix} + 15\begin{pmatrix} -3 \\ -9 \end{pmatrix} = \begin{pmatrix} 54 \\ -2 \end{pmatrix}$ ✓

$\therefore$ the paths intersect at $(54, -2)$.

d Jet ski B passes through $(54, -2)$ when $t = 15$.

Now, when $t = 15$, jet ski A is at $\begin{pmatrix} 150 \\ -30 \end{pmatrix} + 15\begin{pmatrix} -9.6 \\ 2.8 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$.

$\therefore$ the distance between the jet skis is $\sqrt{(54-6)^2 + (-2-12)^2} = \sqrt{2500} = 50$ m.

e The distance between jet ski A's starting point $(150, -30)$ and jet ski B's position at time t, $(99 - 3t,\ 133 - 9t)$, is given by

$$\begin{aligned} D &= \sqrt{(99 - 3t - 150)^2 + (133 - 9t - (-30))^2} \\ &= \sqrt{(-51 - 3t)^2 + (163 - 9t)^2} \\ &= \sqrt{2601 + 306t + 9t^2 + 26\,569 - 2934t + 81t^2} \\ &= \sqrt{90t^2 - 2628t + 29\,170} \end{aligned}$$

Now D is minimised when $D^2 = 90t^2 - 2628t + 29\,170$ is minimised.

This occurs when $t = -\dfrac{-2628}{2 \times 90} = 14.6$

Jet ski B is closest to jet ski A's starting point after 14.6 seconds.

10 At time t, object P is at $(-2 - 3t,\ 5 + 2t)$, and object Q is at $(-1 - 2t,\ 22 - t)$.

$\therefore$ at time t, the distance between them is

$$\begin{aligned} D &= \sqrt{[(-2 - 3t) - (-1 - 2t)]^2 + [(5 + 2t) - (22 - t)]^2} \\ &= \sqrt{(-1 - t)^2 + (-17 + 3t)^2} \\ &= \sqrt{1 + 2t + t^2 + 289 - 102t + 9t^2} \\ &= \sqrt{10t^2 - 100t + 290} \end{aligned}$$

Now D is minimised when $D^2 = 10t^2 - 100t + 290$ is minimised.

This occurs when $t = -\dfrac{-100}{2 \times 10} = 5$.

When $t = 5$, $D = \sqrt{10(5)^2 - 100(5) + 290} = \sqrt{40} = 2\sqrt{10}$.

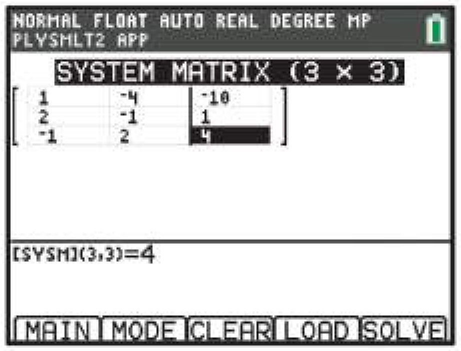
The shortest distance between the objects is $2\sqrt{10} \approx 6.32$ cm after 5 seconds.

11 **a** L_1 and L_2 meet where $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 7 \end{pmatrix} + s\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

$\therefore\ 2 + t = -8 + 4s, \quad -1 + 2t = s, \quad$ and $\quad 3 - t = 7 - 2s$

$\therefore\ t - 4s = -10 \ \ .... \ (1), \quad 2t - s = 1 \ \ .... \ (2), \quad$ and $\quad -t + 2s = 4 \ \ .... \ (3)$

Solving (1), (2), and (3) simultaneously using technology gives $t = 2, \ s = 3$.



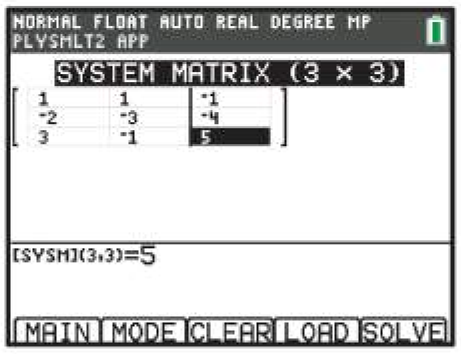
Substituting $t = 2$ into L_1, we find that L_1 and L_2 intersect at $(2 + 2, \ -1 + 2(2), \ 3 - 2)$ which is $(4, 3, 1)$.

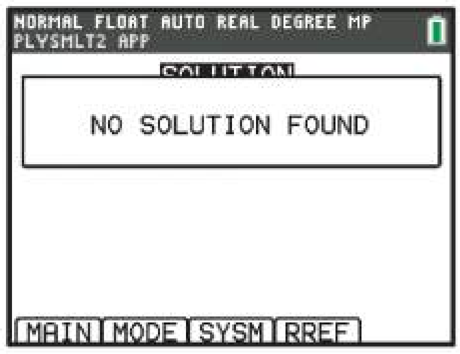
b L_1 and L_2 meet where $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$.

$\therefore\ 3 + t = 2 - s, \quad 5 - 2t = 1 + 3s, \quad$ and $\quad -1 + 3t = 4 + s$

$\therefore\ t + s = -1 \ \ .... \ (1), \quad -2t - 3s = -4 \ \ .... \ (2), \quad$ and $\quad 3t - s = 5 \ \ .... \ (3)$

Solving (1), (2), and (3) simultaneously using technology gives no solutions.





$\therefore$ the lines do not intersect.

Since $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, we conclude that the lines are skew.

12 **a** Haley's drone has direction vector $\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix}$ and speed 2.5 m s^{-1}.

So its velocity vector is $\dfrac{2.5}{\sqrt{(-2)^2 + 0^2 + 1.5^2}}\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix} = \dfrac{2.5}{2.5}\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix}$.

It takes off from $(5, 1, 0)$, so its position vector after t seconds is given by

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix}, \quad t \geqslant 0.$$

Liam's drone has direction vector $\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}$ and speed $1.5\ \text{m s}^{-1}$.

So its velocity vector is $\dfrac{1.5}{\sqrt{1^2+1^2+0.5^2}}\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix} = \dfrac{1.5}{1.5}\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}$.

It takes off from $(-12, -3, 0)$, so its position vector after t seconds is given by

$$\mathbf{r} = \begin{pmatrix} -12 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}, \quad t \geqslant 0.$$

b After 2 seconds, Haley's drone is at $(5-2(2),\ 1,\ 1.5(2))$, which is $(1, 1, 3)$, and Liam's drone is at $(-12+2,\ -3+2,\ 0.5(2))$, which is $(-10, -1, 1)$.

So their drones are $\sqrt{(-10-1)^2+(-1-1)^2+(1-3)^2} = \sqrt{129} \approx 11.4$ m apart after 2 seconds.

c The paths of the drones intersect when $\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix} = \begin{pmatrix} -12 \\ -3 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}$, where t_1 and t_2 are the times at which Haley and Liam's drones respectively reach the intersection point.

$$\therefore\ 5-2t_1 = -12+t_2, \qquad 1 = -3+t_2, \qquad \text{and} \qquad 1.5t_1 = 0.5t_2$$
$$\therefore\ -2t_1 - t_2 = -17 \ \ .... (1), \qquad t_2 = 4 \ \ .... (2), \qquad \text{and} \qquad 1.5t_1 - 0.5t_2 = 0 \ \ .... (3)$$

Substituting (2) into (1) gives $-2t_1 - 4 = -17$
$\therefore\ -2t_1 = -13$
$\therefore\ t_1 = 6.5$

Substituting into (3) gives $1.5(6.5) - 0.5(4) = 7.75$ ✗
So the paths of the drones do not intersect.

d At time t, Haley's drone is at $(5-2t,\ 1,\ 1.5t)$, and Liam's drone is at $(-12+t,\ -3+t,\ 0.5t)$.

$\therefore$ at time t, the distance between them is

$$\begin{aligned} D &= \sqrt{[(-12+t)-(5-2t)]^2 + [(-3+t)-1]^2 + [0.5t-1.5t]^2} \\ &= \sqrt{(-17+3t)^2+(-4+t)^2+(-t)^2} \\ &= \sqrt{289-102t+9t^2+16-8t+t^2+t^2} \\ &= \sqrt{11t^2-110t+305} \end{aligned}$$

Now D is minimised when $D^2 = 11t^2-110t+305$ is minimised.

This occurs when $t = -\dfrac{-110}{2\times 11} = 5$.

When $t=5$, $D = \sqrt{11(5)^2-110(5)+305} = \sqrt{30}$.

The drones are closest to one another after 5 seconds, when they are $\sqrt{30} \approx 5.48$ m apart.

Chapter 11

COMPLEX NUMBERS

EXERCISE 11A

1 **a** $\sqrt{-9}$
$= \sqrt{9} \times \sqrt{-1}$
$= 3i$

b $\sqrt{-64}$
$= \sqrt{64} \times \sqrt{-1}$
$= 8i$

c $\sqrt{-121}$
$= \sqrt{121} \times \sqrt{-1}$
$= 11i$

d $\sqrt{-\frac{1}{4}}$
$= \sqrt{\frac{1}{4}} \times \sqrt{-1}$
$= \frac{1}{2}i$

e $\sqrt{-5}$
$= \sqrt{5} \times \sqrt{-1}$
$= i\sqrt{5}$

f $\sqrt{-17}$
$= \sqrt{17} \times \sqrt{-1}$
$= i\sqrt{17}$

g $\sqrt{-8}$
$= \sqrt{8} \times \sqrt{-1}$
$= i\sqrt{8}$
$= 2\sqrt{2}i$

h $-\sqrt{-36}$
$= -\sqrt{36} \times \sqrt{-1}$
$= -6i$

i $-\sqrt{-11}$
$= -\sqrt{11} \times \sqrt{-1}$
$= -i\sqrt{11}$

j $-\sqrt{-\frac{1}{2}}$
$= -\sqrt{\frac{1}{2}} \times \sqrt{-1}$
$= -\frac{1}{\sqrt{2}}i$

2 **a** $x^2 = 25$
$\therefore\ x = \pm\sqrt{25}$
$\therefore\ x = \pm 5$

b $x^2 = -25$
$\therefore\ x = \pm\sqrt{-25}$
$\therefore\ x = \pm\sqrt{25} \times \sqrt{-1}$
$\therefore\ x = \pm 5i$

c $x^2 = 5$
$\therefore\ x = \pm\sqrt{5}$

d $x^2 = -5$
$\therefore\ x = \pm\sqrt{-5}$
$\therefore\ x = \pm\sqrt{5} \times \sqrt{-1}$
$\therefore\ x = \pm i\sqrt{5}$

e $4x^2 = 9$
$\therefore\ x^2 = \frac{9}{4}$
$\therefore\ x = \pm\sqrt{\frac{9}{4}}$
$\therefore\ x = \pm\frac{3}{2}$

f $4x^2 = -9$
$\therefore\ x^2 = -\frac{9}{4}$
$\therefore\ x = \pm\sqrt{-\frac{9}{4}}$
$\therefore\ x = \pm\sqrt{\frac{9}{4}} \times \sqrt{-1}$
$\therefore\ x = \pm\frac{3}{2}i$

3 **a** $x^2 - 10x + 29 = 0$
$\therefore\ x = \dfrac{10 \pm \sqrt{100 - 4 \times 1 \times 29}}{2}$
$\therefore\ x = \dfrac{10 \pm \sqrt{-16}}{2}$
$\therefore\ x = 5 \pm \sqrt{-4}$
$\therefore\ x = 5 \pm 2i$

b $x^2 + 6x + 25 = 0$
$\therefore\ x = \dfrac{-6 \pm \sqrt{36 - 4 \times 1 \times 25}}{2}$
$\therefore\ x = \dfrac{-6 \pm \sqrt{-64}}{2}$
$\therefore\ x = -3 \pm \sqrt{-16}$
$\therefore\ x = -3 \pm 4i$

c $x^2 + 14x + 50 = 0$

$$\therefore \quad x = \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 50}}{2}$$
$$\therefore \quad x = \frac{-14 \pm \sqrt{-4}}{2}$$
$$\therefore \quad x = -7 \pm \sqrt{-1}$$
$$\therefore \quad x = -7 \pm i$$

d $x^2 - 3x + 5 = 0$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times 1 \times 5}}{2}$$
$$\therefore \quad x = \frac{3 \pm \sqrt{-11}}{2}$$
$$\therefore \quad x = \tfrac{3}{2} \pm \tfrac{\sqrt{11}}{2} i$$

e $x^2 + 4 = x$

$$\therefore \quad x^2 - x + 4 = 0$$
$$\therefore \quad x = \frac{1 \pm \sqrt{1 - 4 \times 1 \times 4}}{2}$$
$$\therefore \quad x = \frac{1 \pm \sqrt{-15}}{2}$$
$$\therefore \quad x = \tfrac{1}{2} \pm \tfrac{\sqrt{15}}{2} i$$

f $3x^2 + 6x + 5 = 0$

$$\therefore \quad x = \frac{-6 \pm \sqrt{36 - 4 \times 3 \times 5}}{2 \times 3}$$
$$\therefore \quad x = \frac{-6 \pm \sqrt{-24}}{6}$$
$$\therefore \quad x = \frac{-6 \pm 2\sqrt{6} i}{6}$$
$$\therefore \quad x = -1 \pm \tfrac{\sqrt{6}}{3} i$$

g $2x^2 + 5 = 6x$

$$\therefore \quad 2x^2 - 6x + 5 = 0$$
$$\therefore \quad x = \frac{6 \pm \sqrt{36 - 4 \times 2 \times 5}}{4}$$
$$\therefore \quad x = \frac{6 \pm \sqrt{-4}}{4}$$
$$\therefore \quad x = \frac{3 \pm \sqrt{-1}}{2}$$
$$\therefore \quad x = \tfrac{3}{2} \pm \tfrac{1}{2} i$$

h $x^2 - 2\sqrt{3}x + 4 = 0,$

$$\therefore \quad x = \frac{2\sqrt{3} \pm \sqrt{12 - 4 \times 1 \times 4}}{2}$$
$$\therefore \quad x = \frac{2\sqrt{3} \pm \sqrt{-4}}{2}$$
$$\therefore \quad x = \sqrt{3} \pm \sqrt{-1}$$
$$\therefore \quad x = \sqrt{3} \pm i$$

i $2x + \dfrac{1}{x} = 1,$

$$\therefore \quad 2x^2 + 1 = x$$
$$\therefore \quad 2x^2 - x + 1 = 0$$
$$\therefore \quad x = \frac{1 \pm \sqrt{1 - 4 \times 2 \times 1}}{4}$$
$$\therefore \quad x = \frac{1 \pm \sqrt{-7}}{4}$$
$$\therefore \quad x = \tfrac{1}{4} \pm i\tfrac{\sqrt{7}}{4}$$

EXERCISE 11B

1

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$3+2i$	3	2	$3-2i$
$5-i$	5	-1	$5+i$
3	3	0	3
$4i$	0	4	$-4i$
0	0	0	0
$-3+4i$	-3	4	$-3-4i$
$-7-2i$	-7	-2	$-7+2i$
$-11i$	0	-11	$11i$
$i\sqrt{3}$	0	$\sqrt{3}$	$-i\sqrt{3}$
$1-i\sqrt{2}$	1	$-\sqrt{2}$	$1+i\sqrt{2}$

2 **a** 3 and 0 have imaginary part 0.
$\therefore$ 3 and 0 are real.

b $4i$, $-11i$, and $i\sqrt{3}$ have real part 0, and non-zero imaginary part.
$\therefore$ $4i$, $-11i$, and $i\sqrt{3}$ are purely imaginary.

3 $z=-1+4i, \quad w=6-5i$

a $\mathcal{Re}(z)=-1$ **b** $\mathcal{Im}(w)=-5$ **c** $z^*=-1-4i$

d $w^*=6+5i$ **e** $\mathcal{Im}(z^*)=-4$ **f** $\mathcal{Re}(w^*)=6$

4 If z is purely imaginary, then $z=bi$ where $b\in\mathbb{R}$.

$$\begin{aligned}\therefore\ z^* &= (bi)^*\\ &= (0+bi)^*\\ &= 0-bi\\ &= -bi \quad \text{which is also purely imaginary.}\end{aligned}$$

5 Let $z=a+bi$ where $a, b\in\mathbb{R}$

$\therefore\ z^*=a-bi$

$\therefore\ (z^*)^*=a+bi=z$

EXERCISE 11C

1 $z=5-2i, \quad w=2+i$

a $$\begin{aligned}&z+w\\ &=(5-2i)+(2+i)\\ &=7-i\end{aligned}$$

b $$\begin{aligned}&2z\\ &=2(5-2i)\\ &=10-4i\end{aligned}$$

c $$\begin{aligned}iw&=i(2+i)\\ &=2i+i^2\\ &=-1+2i\end{aligned}$$

d $$\begin{aligned}&z-w\\ &=(5-2i)-(2+i)\\ &=5-2i-2-i\\ &=3-3i\end{aligned}$$

e $$\begin{aligned}&2z-3w\\ &=2(5-2i)-3(2+i)\\ &=10-4i-6-3i\\ &=4-7i\end{aligned}$$

f $$\begin{aligned}&zw\\ &=(5-2i)(2+i)\\ &=10+5i-4i-2i^2\\ &=12+i\end{aligned}$$

g $w^2 = (2+i)^2$
$= 4 + 4i + i^2$
$= 3 + 4i$

h $z^2 = (5-2i)^2$
$= 25 - 20i + 4i^2$
$= 21 - 20i$

2 $z = 1+i, \quad w = -2+3i$

a $z + 2w$
$= (1+i) + 2(-2+3i)$
$= 1 + i - 4 + 6i$
$= -3 + 7i$

b z^2
$= (1+i)^2$
$= 1 + 2i + i^2$
$= 2i$

c $z^3 = z^2 \times z$
$= 2i(1+i)$ {using **b**}
$= 2i + 2i^2$
$= -2 + 2i$

d $iz = i(1+i)$
$= i + i^2$
$= -1 + i$

e $w^2 = (-2+3i)^2$
$= 4 - 12i + 9i^2$
$= -5 - 12i$

f zw
$= (1+i)(-2+3i)$
$= -2 + 3i - 2i + 3i^2$
$= -5 + i$

g z^2w
$= 2i(-2+3i)$ {using **b**}
$= -4i + 6i^2$
$= -6 - 4i$

h izw
$= i(-5+i)$ {using **f**}
$= -5i + i^2$
$= -1 - 5i$

3 $(a + \cancel{bi}) + (a - \cancel{bi}) = 2a$ which is real

4 **a**

$i^0 = 1$	$i^4 = 1$	$i^8 = 1$	$i^{-1} = -i$
$i^1 = i$	$i^5 = i$	$i^9 = i$	$i^{-2} = -1$
$i^2 = -1$	$i^6 = -1$		$i^{-3} = i$
$i^3 = -i$	$i^7 = -i$		$i^{-4} = 1$
			$i^{-5} = -i$

b $i^{4n+3} = -i$ where n is any integer

5 **a** $\sqrt{-4} \times \sqrt{-9} = \sqrt{-1} \times \sqrt{4} \times \sqrt{-1} \times \sqrt{9}$
$= -1 \times \sqrt{36}$
$= -6$

b No, $\sqrt{-4} \times \sqrt{-9} = -6 \neq \sqrt{36}$

6 $z = 2-i, \quad w = 1+3i$

a $\dfrac{z}{w} = \left(\dfrac{2-i}{1+3i}\right) \times \left(\dfrac{1-3i}{1-3i}\right)$
$= \dfrac{2 - 6i - i + 3i^2}{1 - 9i^2}$
$= \dfrac{-1-7i}{10}$
$= -\frac{1}{10} - \frac{7}{10}i$

b $\dfrac{i}{z} = \left(\dfrac{i}{2-i}\right) \times \left(\dfrac{2+i}{2+i}\right)$
$= \dfrac{2i + i^2}{4 - i^2}$
$= \dfrac{-1+2i}{5}$
$= -\frac{1}{5} + \frac{2}{5}i$

c $\dfrac{w}{iz} = \dfrac{1+3i}{i(2-i)}$
$= \dfrac{1+3i}{2i-i^2}$
$= \left(\dfrac{1+3i}{1+2i}\right) \times \left(\dfrac{1-2i}{1-2i}\right)$
$= \dfrac{1-2i+3i-6i^2}{1-4i^2}$
$= \dfrac{7+i}{5}$
$= \frac{7}{5} + \frac{1}{5}i$

d $\dfrac{z^2}{w-i} = \dfrac{(2-i)^2}{1+3i-i}$
$= \dfrac{4-4i+i^2}{1+2i}$
$= \left(\dfrac{3-4i}{1+2i}\right) \times \left(\dfrac{1-2i}{1-2i}\right)$
$= \dfrac{3-6i-4i+8i^2}{1-4i^2}$
$= \dfrac{-5-10i}{5}$
$= -1-2i$

7 **a** $\dfrac{i}{1-2i} = \left(\dfrac{i}{1-2i}\right) \times \left(\dfrac{1+2i}{1+2i}\right)$
$= \dfrac{i+2i^2}{1-4i^2}$
$= \dfrac{-2+i}{5}$
$= -\frac{2}{5} + \frac{1}{5}i$

b $\dfrac{i(2-i)}{3-2i} = \dfrac{2i-i^2}{3-2i}$
$= \left(\dfrac{1+2i}{3-2i}\right) \times \left(\dfrac{3+2i}{3+2i}\right)$
$= \dfrac{3+2i+6i+4i^2}{9-4i^2}$
$= \dfrac{-1+8i}{13}$
$= -\frac{1}{13} + \frac{8}{13}i$

c $\dfrac{1}{2-i} - \dfrac{2}{2+i} = \left(\dfrac{1}{2-i}\right) \times \left(\dfrac{2+i}{2+i}\right) - \left(\dfrac{2}{2+i}\right) \times \left(\dfrac{2-i}{2-i}\right)$
$= \dfrac{2+i-2(2-i)}{(2-i)(2+i)}$
$= \dfrac{2+i-4+2i}{4-i^2}$
$= \dfrac{-2+3i}{5}$
$= -\frac{2}{5} + \frac{3}{5}i$

d $\dfrac{i}{x+i} - \dfrac{i}{x-i} = \left(\dfrac{i}{x+i}\right) \times \left(\dfrac{x-i}{x-i}\right) - \left(\dfrac{i}{x-i}\right) \times \left(\dfrac{x+i}{x+i}\right)$
$= \dfrac{i(x-i)-i(x+i)}{x^2-i^2}$
$= \dfrac{xi-i^2-xi-i^2}{x^2-i^2}$
$= \dfrac{2}{x^2+1}$

8 $z = 2+i, \quad w = -1+2i$

a $4z-3w = 4(2+i)-3(-1+2i)$
$= 8+4i+3-6i$
$= 11-2i$
$\therefore \; \mathcal{Im}\,(4z-3w) = -2$

b $zw = (2+i)(-1+2i)$
$= -2+4i-i+2i^2$
$= -4+3i$
$\therefore \; \mathcal{Re}\,(zw) = -4$

c

$$\begin{aligned} iz^2 &= i(2+i)^2 \\ &= i(4+4i+i^2) \\ &= i(3+4i) \\ &= 3i+4i^2 \\ &= -4+3i \end{aligned}$$

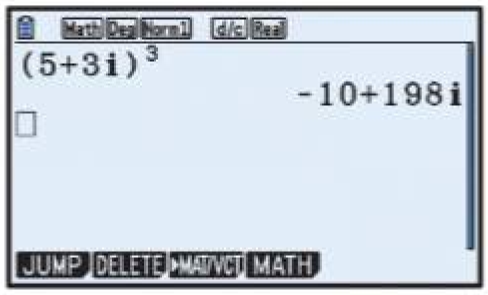
$$\therefore\ \mathscr{Im}\,(iz^2) = 3$$

d

$$\begin{aligned} \frac{z}{w} &= \left(\frac{2+i}{-1+2i}\right) \times \left(\frac{-1-2i}{-1-2i}\right) \\ &= \frac{-2-4i-i-2i^2}{1-4i^2} \\ &= \frac{0-5i}{5} \\ &= -i \end{aligned}$$

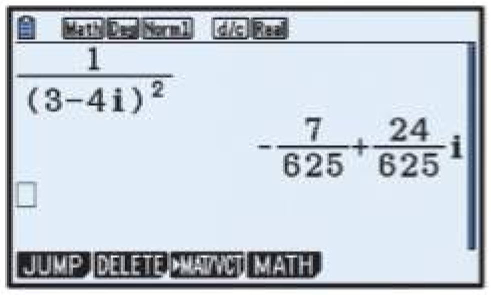
$$\therefore\ \mathscr{Re}\left(\frac{z}{w}\right) = 0$$

9 a



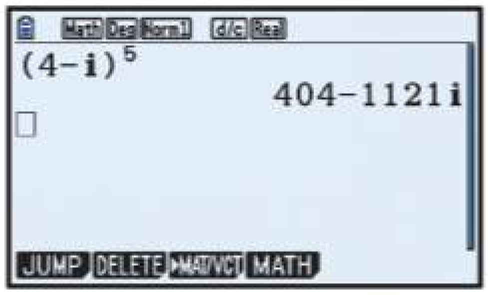
$(5+3i)^3 = -10+198i$

b



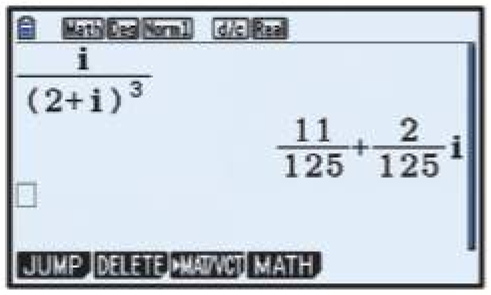
$\dfrac{1}{(3-4i)^2} = -\frac{7}{625} + \frac{24}{625}i$

c



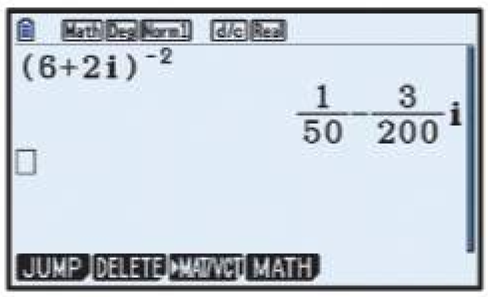
$(4-i)^5 = 404-1121i$

d



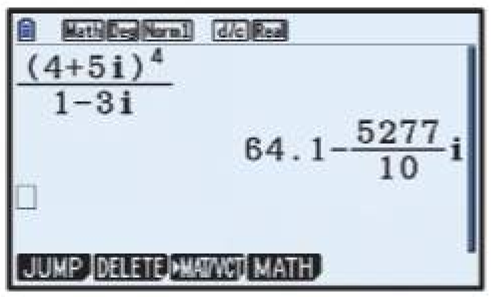
$\dfrac{i}{(2+i)^3} = \frac{11}{125} + \frac{2}{125}i$

e



$(6+2i)^{-2} = \frac{1}{50} - \frac{3}{200}i$

f



$\dfrac{(4+5i)^4}{1-3i} = \frac{641}{10} - \frac{5277}{10}i$

10

$$\begin{aligned} z &= \frac{3i}{\sqrt{2}-i} + 1 \\ &= \left(\frac{3i}{\sqrt{2}-i}\right) \times \left(\frac{\sqrt{2}+i}{\sqrt{2}+i}\right) + 1 \\ &= \frac{3i\sqrt{2}+3i^2}{2-i^2} + 1 \\ &= \frac{3i\sqrt{2}-3}{3} + \frac{3}{3} \\ &= \frac{3i\sqrt{2}}{3} \\ &= 0 + i\sqrt{2} \end{aligned}$$

11 **a** $w = \dfrac{z-1}{z^*+1} = \dfrac{a+bi-1}{a-bi+1} = \dfrac{(a-1)+bi}{(a+1)-bi}$

$$\therefore\ w = \left(\frac{(a-1)+bi}{(a+1)-bi}\right) \times \left(\frac{(a+1)+bi}{(a+1)+bi}\right)$$

$$= \frac{(a^2-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2}$$

$$= \frac{a^2-b^2-1}{(a+1)^2+b^2} + \frac{2ab}{(a+1)^2+b^2}\,i$$

b w is purely imaginary when $a^2-b^2-1=0$ and $2ab \neq 0$

$\therefore\ a^2-b^2=1$ and $a \neq 0,\ b \neq 0$

EXERCISE 11D

1 **a** $a+bi = 3-i$

Equating real and imaginary parts, $a=3$ and $b=-1$.

b $a+bi = i-\sqrt{2}$

$= -\sqrt{2}+i$

Equating real and imaginary parts, $a=-\sqrt{2}$ and $b=1$.

c $a+bi=0$

$=0+0i$

Equating real and imaginary parts, $a=0$ and $b=0$.

2 **a** $2x+3yi = -x-6i$

Equating real and imaginary parts,

$2x=-x$ and $3y=-6$

$\therefore\ 3x=0$ and $y=-2$

$\therefore\ x=0$ and $y=-2$

b $x^2+xi = 4-2i$

Equating real and imaginary parts,

$x^2=4$ and $x=-2$

$\therefore\ x=\pm 2$ and $x=-2$

$\therefore\ x=-2$

c $(x+yi)(2-i)=8+i$

$$\therefore\ x+yi = \left(\frac{8+i}{2-i}\right)\times\left(\frac{2+i}{2+i}\right)$$

$$\therefore\ x+yi = \frac{16+8i+2i+i^2}{4-i^2}$$

$$\therefore\ x+yi = \frac{15+10i}{5}$$

$$\therefore\ x+yi = 3+2i$$

Equating real and imaginary parts,

$x=3$ and $y=2$.

d $(3+2i)(x+yi)=-i$

$$\therefore\ x+yi = \left(\frac{-i}{3+2i}\right)\times\left(\frac{3-2i}{3-2i}\right)$$

$$\therefore\ x+yi = \frac{-3i+2i^2}{9-4i^2}$$

$$\therefore\ x+yi = \frac{-2-3i}{13}$$

$$\therefore\ x+yi = -\tfrac{2}{13}-\tfrac{3}{13}i$$

Equating real and imaginary parts,

$x=-\frac{2}{13}$ and $y=-\frac{3}{13}$.

e $2(x+yi)=x-yi$

$\therefore\ 2x+2yi = x-yi$

Equating real and imaginary parts, $2x=x$ and $2y=-y$

$\therefore\ x=0$ and $3y=0$

$\therefore\ x=0$ and $y=0$

f
$$(x+2i)(y-i) = -4-7i$$
$$\therefore\ xy - xi + 2yi - 2i^2 = -4-7i$$
$$\therefore\ (xy+2) + (2y-x)i = -4-7i$$
Equating real and imaginary parts,
$xy + 2 = -4$ and $2y - x = -7$
$\therefore\ xy = -6$ and $x = 2y + 7$
$\therefore\ (2y+7)y = -6$
$\therefore\ 2y^2 + 7y = -6$
$\therefore\ 2y^2 + 7y + 6 = 0$
$\therefore\ (2y+3)(y+2) = 0$
$\therefore\ y = -\frac{3}{2}$ or $y = -2$
When $y = -\frac{3}{2}$, $x = 2(-\frac{3}{2}) + 7 = 4$
and when $y = -2$, $x = 2(-2) + 7 = 3$
$\therefore\ x = 4$ and $y = -\frac{3}{2}$
or $x = 3$ and $y = -2$

g
$$(x+i)(3-yi) = 1+13i$$
$$\therefore\ 3x - xyi + 3i - yi^2 = 1+13i$$
$$\therefore\ (3x+y) + (3-xy)i = 1+13i$$
Equating real and imaginary parts,
$3x + y = 1$ and $3 - xy = 13$
$\therefore\ y = 1 - 3x$ and $xy = -10$
$\therefore\ x(1-3x) = -10$
$\therefore\ x - 3x^2 = -10$
$\therefore\ 0 = 3x^2 - x - 10$
$\therefore\ 0 = (3x+5)(x-2)$
$\therefore\ x = -\frac{5}{3}$ or $x = 2$
When $x = -\frac{5}{3}$, $y = 1 - 3(-\frac{5}{3}) = 6$
and when $x = 2$, $y = 1 - 3 \times 2 = -5$
$\therefore\ x = -\frac{5}{3}$ and $y = 6$
or $x = 2$ and $y = -5$

h
$$(x+yi)(2+i) = 2x - (y+1)i$$
$$\therefore\ 2x + xi + 2yi + yi^2 = 2x + (-y-1)i$$
$$\therefore\ (2x-y) + (x+2y)i = 2x + (-y-1)i$$
Equating real and imaginary parts, $2x - y = 2x$ and $x + 2y = -y - 1$
$\therefore\ -y = 0$ and $x = -3y - 1$
$\therefore\ y = 0$ and $x = -3y - 1$
$\therefore\ x = -3(0) - 1$
$\therefore\ x = -1$
$\therefore\ x = -1$ and $y = 0$

3 $3z + 17i = iz + 11$
$\therefore\ z(3-i) = 11 - 17i$
$$\therefore\ z = \left(\frac{11-17i}{3-i}\right) \times \left(\frac{3+i}{3+i}\right)$$
$$= \frac{33 + 11i - 51i - 17i^2}{9 - i^2}$$
$$= \frac{50 - 40i}{10}$$
$$= 5 - 4i$$

4 $3(m + ni) = n - 2mi - (1 - 2i)$
$\therefore\ 3m + 3ni = (n-1) + (2-2m)i$
Equating real and imaginary parts,
$3m = n - 1$ and $3n = 2 - 2m$
$\therefore\ n = 3m + 1$ and $3n = 2 - 2m$
$\therefore\ 3(3m+1) = 2 - 2m$
$\therefore\ 9m + 3 = 2 - 2m$
$\therefore\ 11m = -1$
$\therefore\ m = -\frac{1}{11}$ and $n = 3(-\frac{1}{11}) + 1 = \frac{8}{11}$

5

$$(a+bi)^2 = -16-30i$$
$$\therefore\ a^2+2abi+b^2i^2 = -16-30i$$
$$\therefore\ (a^2-b^2)+2abi = -16-30i$$

Equating real and imaginary parts, $a^2-b^2=-16$ and $2ab=-30$

$$\therefore\ a^2-b^2=-16 \quad \text{and} \quad b=-\frac{15}{a}$$
$$\therefore\ a^2-\left(-\frac{15}{a}\right)^2=-16$$
$$\therefore\ a^2-\frac{225}{a^2}=-16$$
$$\therefore\ a^4+16a^2-225=0$$
$$\therefore\ (a^2+25)(a^2-9)=0$$
$$\therefore\ a^2=9 \quad \{a^2+25>0\}$$
$$\therefore\ a=3 \quad \text{and} \quad b=-\tfrac{15}{3}=-5$$
$$\text{or} \quad a=-3 \quad \text{and} \quad b=-\tfrac{15}{-3}=5$$

6 Let $z=a+bi$, then $(a+bi)^2 = 1+i+\left(\dfrac{58}{9(3-7i)}\right)\times\left(\dfrac{3+7i}{3+7i}\right)$

$$\therefore\ a^2+2abi+b^2i^2 = 1+i+\frac{58(3+7i)}{9(9-49i^2)}$$
$$\therefore\ (a^2-b^2)+2abi = 1+i+\frac{58(3+7i)}{9\times 58}$$
$$\therefore\ (a^2-b^2)+2abi = 1+i+\frac{3+7i}{9}$$
$$\therefore\ (a^2-b^2)+2abi = \tfrac{4}{3}+\tfrac{16}{9}i$$

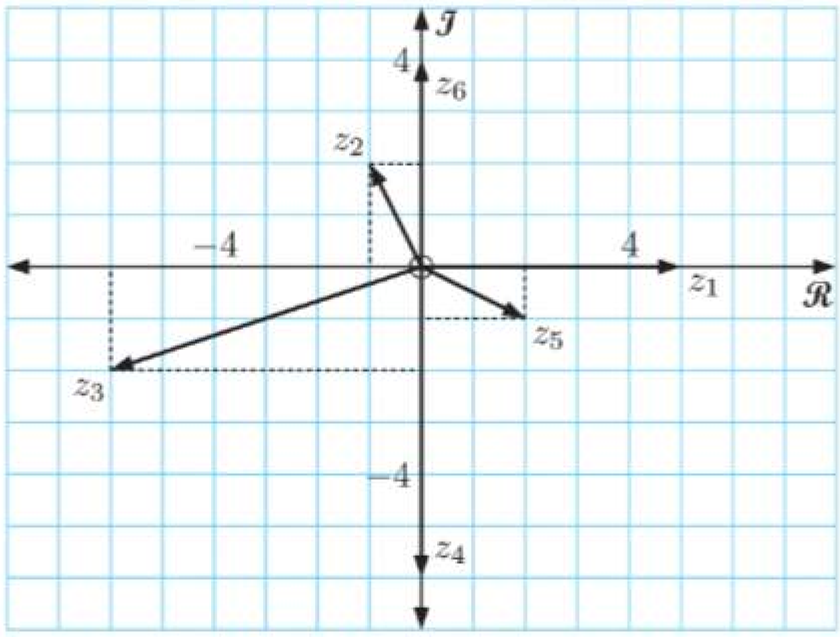
Equating real and imaginary parts, $a^2-b^2=\frac{4}{3}$ and $2ab=\frac{16}{9}$

$$\therefore\ a^2-b^2=\tfrac{4}{3} \quad \text{and} \quad b=\frac{8}{9a}$$
$$\therefore\ a^2-\left(\frac{8}{9a}\right)^2=\tfrac{4}{3}$$
$$\therefore\ a^2-\frac{64}{81a^2}=\tfrac{4}{3}$$
$$\therefore\ 81a^4-108a^2-64=0$$
$$\therefore\ (9a^2-16)(9a^2+4)=0$$
$$\therefore\ a^2=\tfrac{16}{9} \quad \{9a^2+4>0\}$$
$$\therefore\ a=\tfrac{4}{3} \quad \text{and} \quad b=\frac{8}{9(\frac{4}{3})}=\tfrac{2}{3}$$
$$\text{or} \quad a=-\tfrac{4}{3} \quad \text{and} \quad b=\frac{8}{9(-\frac{4}{3})}=-\tfrac{2}{3}$$

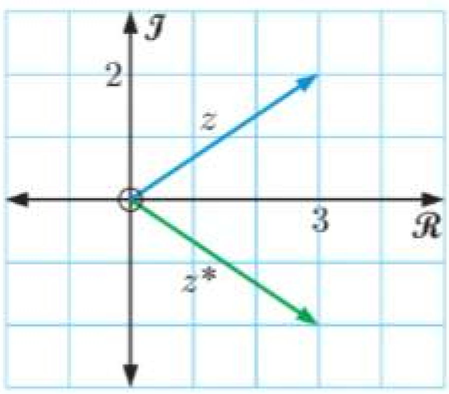
So, $z=\frac{4}{3}+\frac{2}{3}i$ or $z=-\frac{4}{3}-\frac{2}{3}i$.

EXERCISE 11E

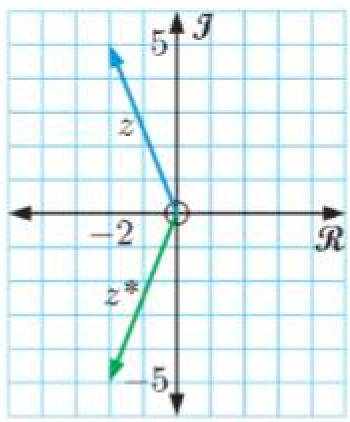
1



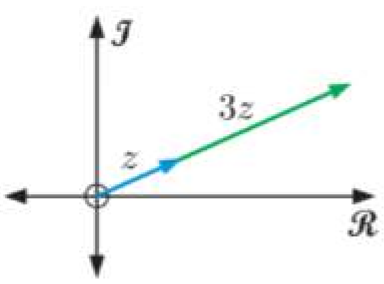
2 **a**



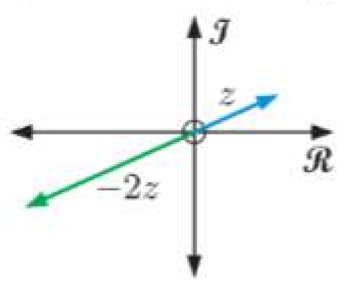
b



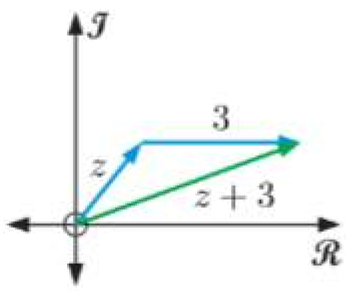
3 **a** $3z$ is parallel to z and 3 times its length.



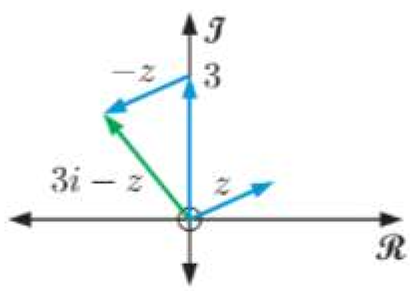
b $-2z$ is parallel to z, in the opposite direction, and twice its length.



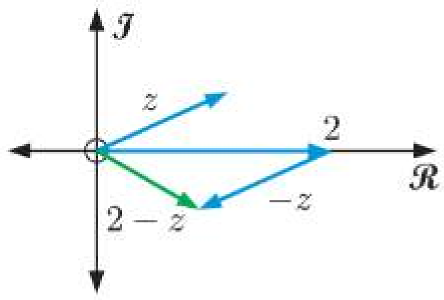
c Add 3 to z.



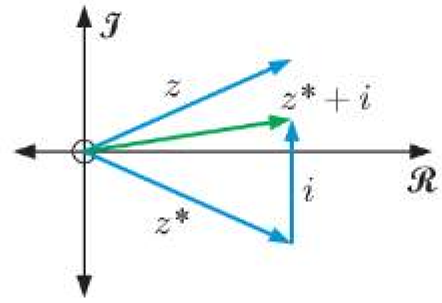
d Add $-z$ to $3i$.



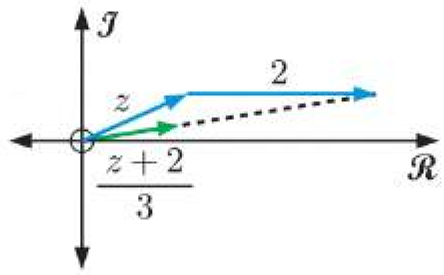
e Add $-z$ to 2.



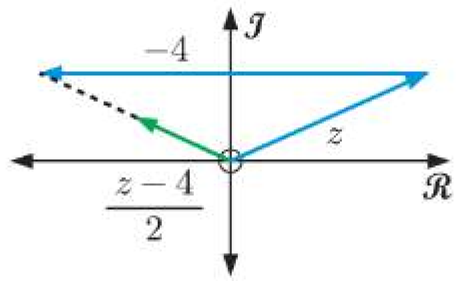
f Reflect z in a horizontal line through the start of z and then add i.



g Add z and 2 and find the vector $\frac{1}{3}$ of the length of the result.

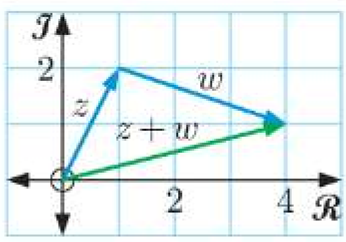


h Add z and -4 and find the vector $\frac{1}{2}$ of the length of the result.



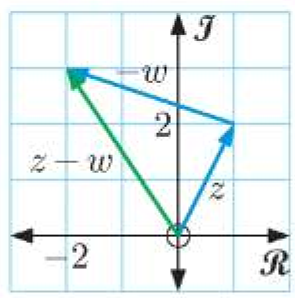
4 $z = 1 + 2i, \quad w = 3 - i$

a



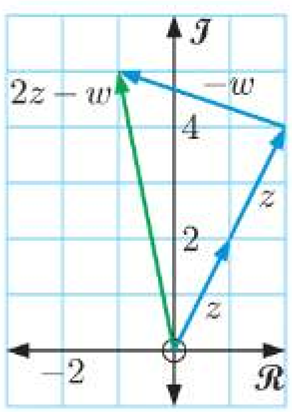
$$\begin{aligned} & z + w \\ &= (1+2i) + (3-i) \\ &= 4 + i \end{aligned}$$

b



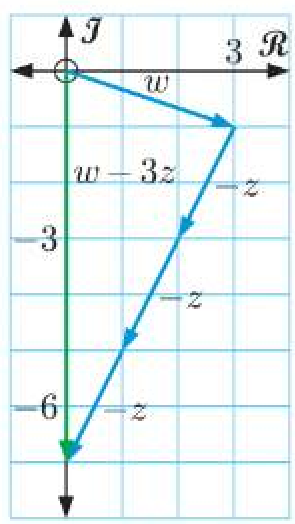
$$\begin{aligned} & z - w \\ &= (1+2i) - (3-i) \\ &= 1 + 2i - 3 + i \\ &= -2 + 3i \end{aligned}$$

c



$$\begin{aligned} & 2z - w \\ &= 2(1+2i) - (3-i) \\ &= 2 + 4i - 3 + i \\ &= -1 + 5i \end{aligned}$$

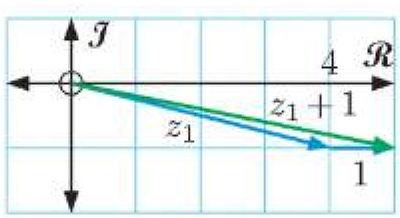
d



$$\begin{aligned} & w - 3z \\ &= (3-i) - 3(1+2i) \\ &= 3 - i - 3 - 6i \\ &= -7i \end{aligned}$$

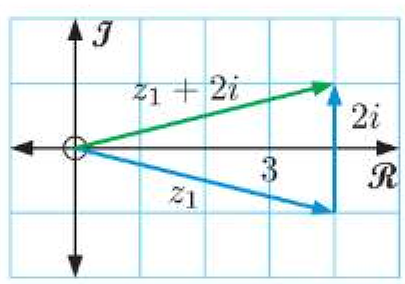
5 $z_1 = 4 - i, \quad z_2 = 2 + 3i$

a



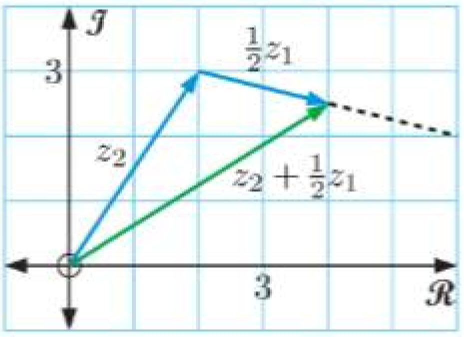
$$\begin{aligned} z_1 + 1 &= 4 - i + 1 \\ &= 5 - i \end{aligned}$$

b



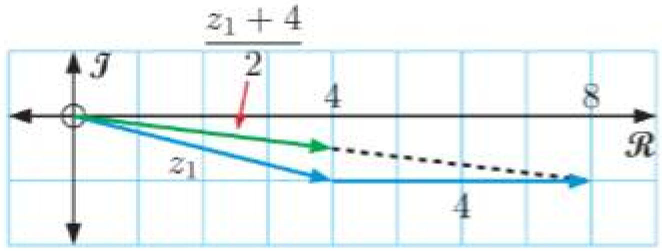
$$\begin{aligned} z_1 + 2i &= 4 - i + 2i \\ &= 4 + i \end{aligned}$$

c



$$\begin{aligned} z_2 + \tfrac{1}{2}z_1 &= (2+3i) + \tfrac{1}{2}(4-i) \\ &= 2 + 3i + 2 - \tfrac{1}{2}i \\ &= 4 + \tfrac{5}{2}i \end{aligned}$$

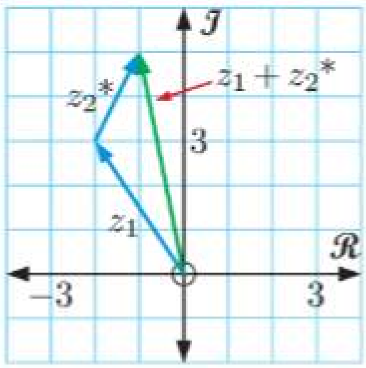
d



$$\begin{aligned} \frac{z_1+4}{2} &= \frac{4-i+4}{2} \\ &= \frac{8-i}{2} \\ &= 4 - \tfrac{1}{2}i \end{aligned}$$

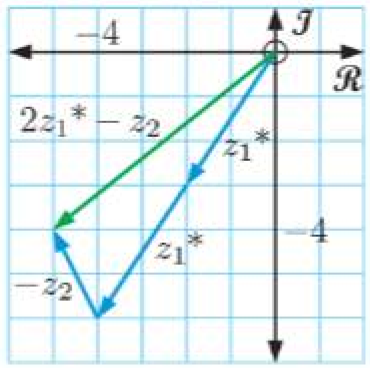
6 $z_1 = -2+3i, \quad z_2 = 1-2i$

a



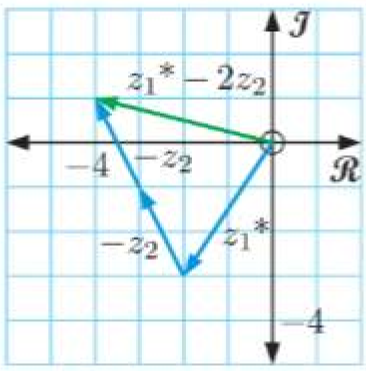
$$\begin{aligned} z_1 + {z_2}^* &= (-2+3i) + (1+2i) \\ &= -1 + 5i \end{aligned}$$

b



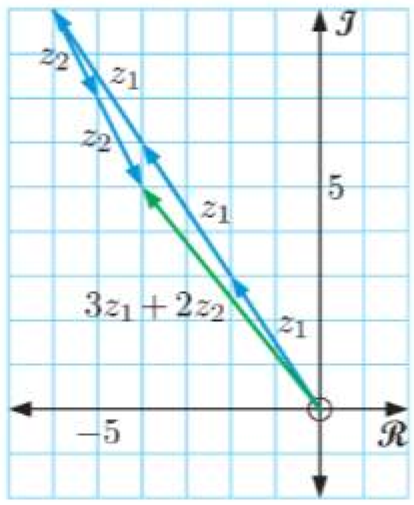
$$\begin{aligned} 2{z_1}^* - z_2 &= 2(-2-3i) - (1-2i) \\ &= -4 - 6i - 1 + 2i \\ &= -5 - 4i \end{aligned}$$

c



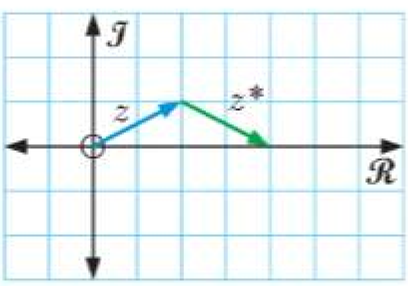
$$\begin{aligned} {z_1}^* - 2z_2 &= (-2-3i) - 2(1-2i) \\ &= -2 - 3i - 2 + 4i \\ &= -4 + i \end{aligned}$$

d



$$\begin{aligned} 3z_1 + 2z_2 &= 3(-2+3i) + 2(1-2i) \\ &= -6 + 9i + 2 - 4i \\ &= -4 + 5i \end{aligned}$$

7 **a**



Suppose $z = a + bi$, where $a, b \in \mathbb{R}$

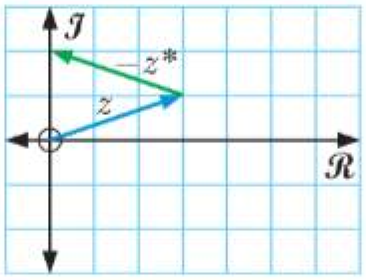
$\therefore\ z^* = a - bi$

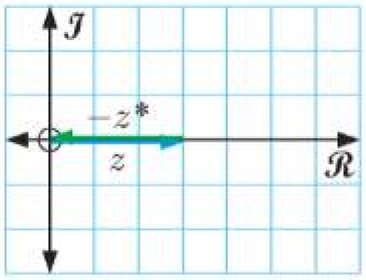
and $z + z^* = a + bi + a - bi$

$= 2a$, which is real (since a is real)

$\therefore\ z + z^*$ is always real for any complex number z.

b





Suppose $z = a + bi$, where $a, b \in \mathbb{R}$

$\therefore \; z^* = a - bi$

and $z - z^* = (a + bi) - (a - bi)$

$= a + bi - a + bi$

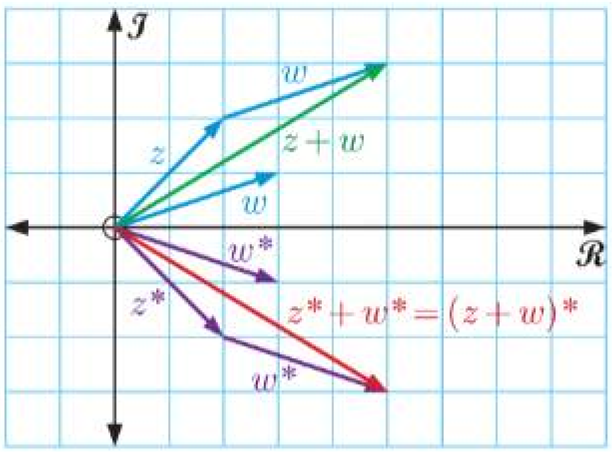
$= 2bi$

Since b is real, $z - z^*$ is purely imaginary for $b \neq 0$.

If $b = 0$ then $z - z^* = 0$.

$\therefore$ $z - z^*$ is purely imaginary, unless z is real, then $z - z^* = 0$.

8 **a**



Let $z = a + bi$ and $w = c + di$

$\therefore \; (z + w)^* = (a + bi + c + di)^*$

$= ((a + c) + (b + d)i)^*$

$= (a + c) - (b + d)i$

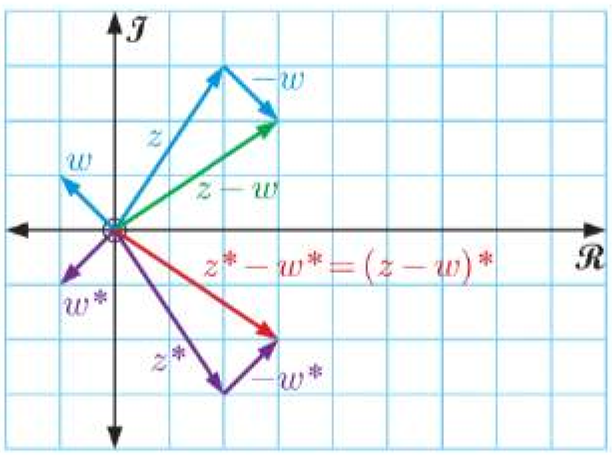
$= a + c - bi - di$

$= (a - bi) + (c - di)$

$= z^* + w^*$

$\therefore \; (z + w)^* = z^* + w^*$ for all complex z, w

b



Let $z = a + bi$ and $w = c + di$

$\therefore \; (z - w)^* = ((a + bi) - (c + di))^*$

$= (a + bi - c - di)^*$

$= ((a - c) + (b - d)i)^*$

$= (a - c) - (b - d)i$

$= a - c - bi + di$

$= (a - bi) - (c - di)$

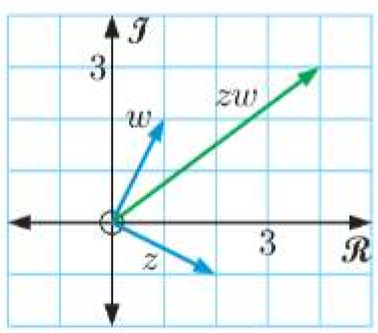
$= z^* - w^*$

$\therefore \; (z - w)^* = z^* - w^*$ for all complex z, w

9 **a** $zw = (2 - i)(1 + 2i)$

$= 2 + 4i - i + 2$

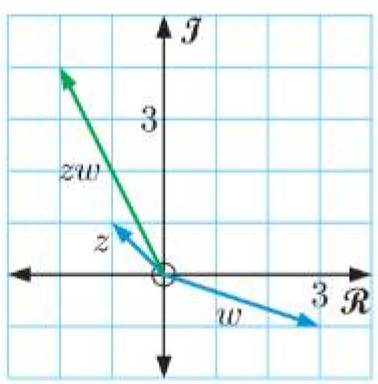
$= 4 + 3i$



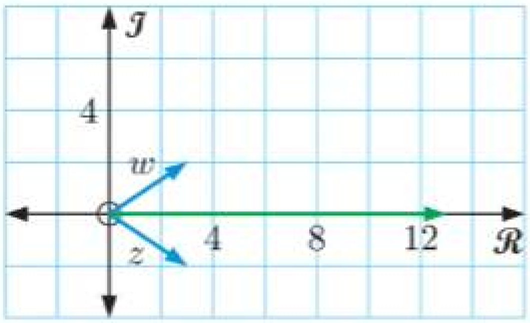
b $zw = (-1 + i)(3 - i)$

$= -3 + i + 3i + 1$

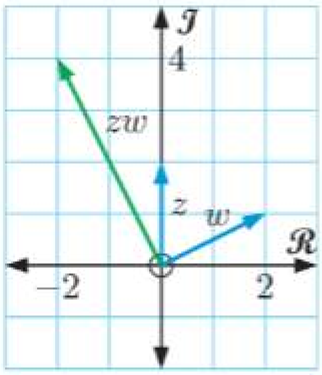
$= -2 + 4i$



c $zw = (3-2i)(3+2i)$
$= 9 + 6i - 6i + 4$
$= 13$



d $zw = (2i)(2+i)$
$= 4i - 2$
$= -2 + 4i$



EXERCISE 11F

1 **a** $|3-4i|$
$= \sqrt{3^2 + (-4)^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

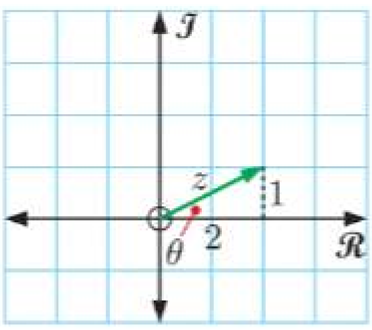
b $|5+12i|$
$= \sqrt{5^2+12^2}$
$= \sqrt{25+144}$
$= \sqrt{169}$
$= 13$

c $|-8+2i|$
$= \sqrt{(-8)^2+2^2}$
$= \sqrt{64+4}$
$= \sqrt{68}$
$= 2\sqrt{17}$

d $|3i|$
$= \sqrt{0^2+3^2}$
$= \sqrt{9}$
$= 3$

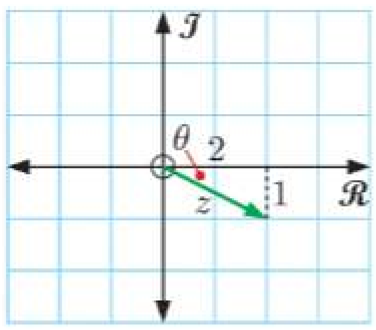
e $|-4|$
$= \sqrt{(-4)^2+0^2}$
$= \sqrt{16}$
$= 4$

2 **a** $|z| = \sqrt{2^2+1^2}$
$= \sqrt{5}$



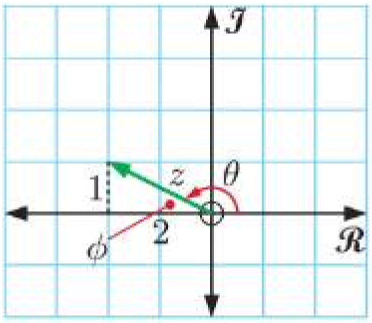
$\tan\theta = \frac{1}{2}$
$\therefore \ \theta \approx 0.464$
$\therefore \ \arg z \approx 0.464$

b $|z| = \sqrt{2^2+(-1)^2}$
$= \sqrt{5}$



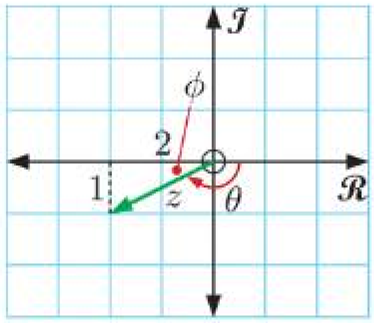
$\tan\theta = \frac{1}{2}$
$\therefore \ \theta \approx 0.464$
$\therefore \ \arg z \approx -0.464$

c $|z| = \sqrt{(-2)^2 + 1^2}$
$= \sqrt{5}$



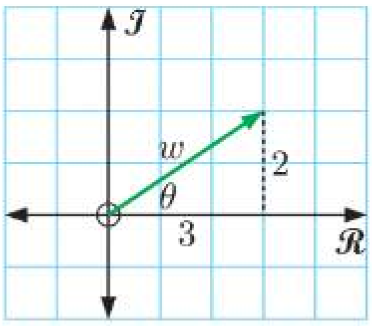
$\tan\phi = \frac{1}{2}$
$\therefore\ \phi \approx 0.464$
But $\theta = \pi - \phi$
$\therefore\ \arg z \approx 2.68$

d $|z| = \sqrt{(-2)^2 + (-1)^2}$
$= \sqrt{5}$



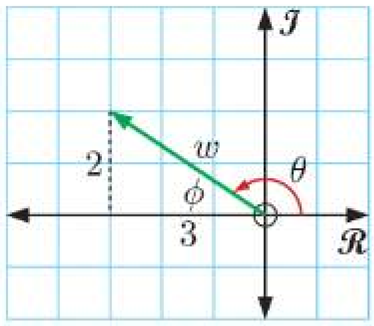
$\tan\phi = \frac{1}{2}$
$\therefore\ \phi \approx 0.464$
But $\theta = -\pi + \phi$
$\therefore\ \arg z \approx -2.68$

3 **a** $|w| = \sqrt{3^2 + 2^2}$
$= \sqrt{13}$



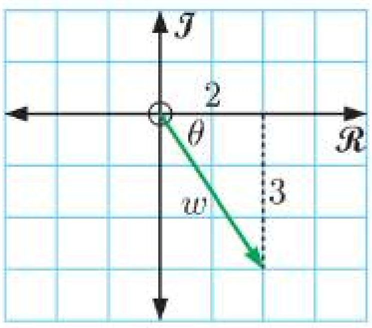
$\tan\theta = \frac{2}{3}$
$\therefore\ \theta \approx 0.588$
$\therefore\ \arg w \approx 0.588$

b $|w| = \sqrt{(-3)^2 + 2^2}$
$= \sqrt{13}$



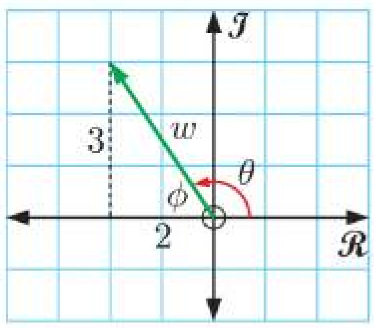
$\tan\phi = \frac{2}{3}$
$\therefore\ \phi \approx 0.588$
But $\theta = \pi - \phi$
$\therefore\ \arg w \approx 2.55$

c $|w| = \sqrt{2^2 + (-3)^2}$
$= \sqrt{13}$



$\tan\theta = \frac{3}{2}$
$\therefore\ \theta \approx 0.983$
$\therefore\ \arg w \approx -0.983$

d $|w| = \sqrt{(-2)^2 + 3^2}$
$= \sqrt{13}$



$\tan\phi = \frac{3}{2}$
$\therefore\ \phi \approx 0.983$
But $\theta = \pi - \phi$
$\therefore\ \arg w \approx 2.16$

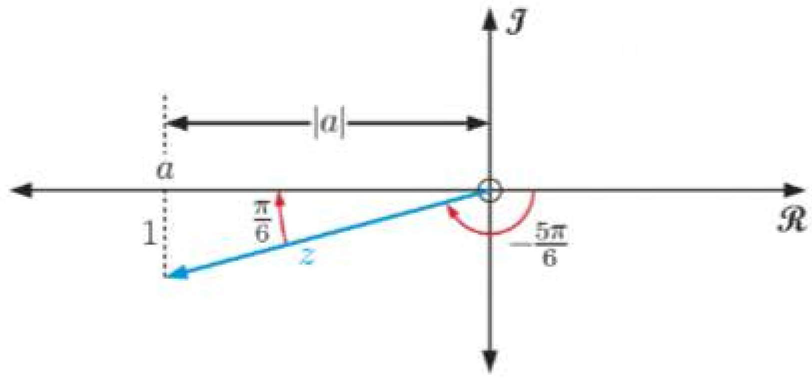
4 $\tan\left(\frac{\pi}{6}\right) = \dfrac{1}{|a|}$

$\therefore \; |a| = \dfrac{1}{\tan(\frac{\pi}{6})}$

$\therefore \; a = \pm\sqrt{3}$

but from the Argand diagram, a is negative

$\therefore \; a = -\sqrt{3}$



5 **a**
$$\begin{aligned} &|z| \\ &= |2+i| \\ &= \sqrt{2^2+1^2} \\ &= \sqrt{5} \end{aligned}$$

b
$$\begin{aligned} &|z^*| \\ &= |2-i| \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{5} \end{aligned}$$

c
$$\begin{aligned} &|z^*|^2 \\ &= (\sqrt{5})^2 \quad \{\text{from } \mathbf{b}\} \\ &= 5 \end{aligned}$$

d
$$\begin{aligned} &zz^* \\ &= (2+i)(2-i) \\ &= 4-2i+2i-i^2 \\ &= 4+1 \\ &= 5 \end{aligned}$$

e
$$\begin{aligned} &|zw| \\ &= |(2+i)(-1+3i)| \\ &= \left|-2+6i-i+3i^2\right| \\ &= |-5+5i| \\ &= \sqrt{(-5)^2+5^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

f
$$\begin{aligned} &|z||w| \\ &= |2+i||-1+3i| \\ &= \sqrt{2^2+1^2}\sqrt{(-1)^2+3^2} \\ &= \sqrt{5}\times\sqrt{10} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

g
$$\begin{aligned} &\left|\frac{z}{w}\right| \\ &= \left|\frac{2+i}{-1+3i}\right| \\ &= \left|\left(\frac{2+i}{-1+3i}\right)\times\left(\frac{-1-3i}{-1-3i}\right)\right| \\ &= \left|\frac{-2-6i-i-3i^2}{(-1)^2-(3i)^2}\right| \\ &= \left|\frac{-2+3-7i}{10}\right| \\ &= \left|\tfrac{1}{10}-\tfrac{7}{10}i\right| \\ &= \sqrt{\left(\tfrac{1}{10}\right)^2+\left(\tfrac{-7}{10}\right)^2} \\ &= \sqrt{\tfrac{1+49}{100}} \\ &= \sqrt{\tfrac{50}{100}} \\ &= \tfrac{1}{\sqrt{2}} \end{aligned}$$

h
$$\begin{aligned} &\frac{|z|}{|w|} \\ &= \frac{|2+i|}{|-1+3i|} \\ &= \frac{\sqrt{2^2+1^2}}{\sqrt{(-1)^2+3^2}} \\ &= \frac{\sqrt{5}}{\sqrt{10}} \\ &= \sqrt{\tfrac{5}{10}} \\ &= \tfrac{1}{\sqrt{2}} \end{aligned}$$

i
$$\begin{aligned} z^2 &= (2+i)^2 \\ &= 4+4i+i^2 \\ &= 3+4i \\ \therefore \; |z^2| &= \sqrt{3^2+4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

j
$$\begin{aligned} &|z|^2 \\ &= (\sqrt{5})^2 \quad \{\text{from } \mathbf{a}\} \\ &= 5 \end{aligned}$$

k
$$\begin{aligned} z^3 &= z^2 \times z \\ &= (3+4i)(2+i) \quad \{\text{from } \mathbf{i}\} \\ &= 6+3i+8i+4i^2 \\ &= 2+11i \end{aligned}$$
$$\begin{aligned} \therefore\ |z^3| &= \sqrt{2^2+11^2} \\ &= \sqrt{4+121} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

l
$$\begin{aligned} |z|^3 &= (\sqrt{5})^3 \quad \{\text{from } \mathbf{a}\} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

6 **a**
$$\begin{aligned} z &= \cos\theta + i\sin\theta \\ \therefore\ |z| &= \sqrt{\cos^2\theta + \sin^2\theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

b
$$\begin{aligned} z &= r(\cos\theta + i\sin\theta), \quad r \in \mathbb{R} \\ &= r\cos\theta + ri\sin\theta \\ \therefore\ |z| &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{r^2} \\ &= |r| \end{aligned}$$

7 **a**
$$\begin{aligned} |3z| &= |3||z| \\ &= 3 \times 2 \\ &= 6 \end{aligned}$$

b
$$\begin{aligned} |-2z| &= |-2||z| \\ &= 2 \times 2 \\ &= 4 \end{aligned}$$

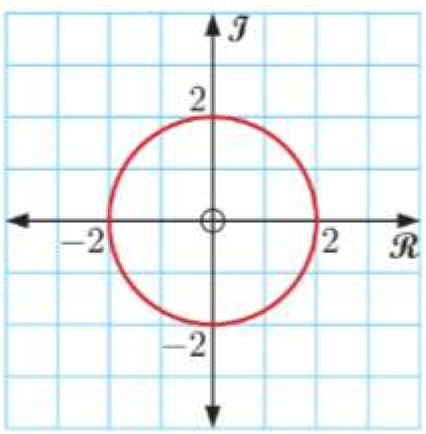
c
$$\begin{aligned} |(2-i)z| &= |2-i||z| \\ &= \sqrt{4+1} \times 2 \\ &= 2\sqrt{5} \end{aligned}$$

d
$$\begin{aligned} |iz| &= |i||z| \\ &= 1 \times 2 \\ &= 2 \end{aligned}$$

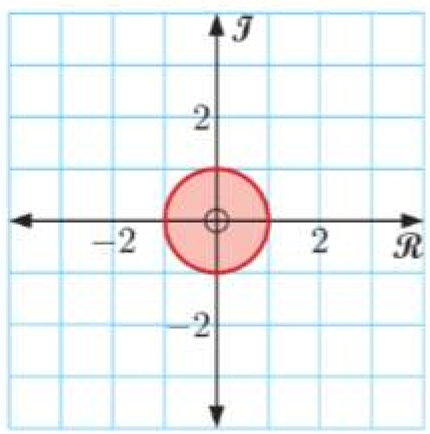
e
$$\begin{aligned} \left|\frac{1}{z}\right| &= \frac{|1|}{|z|} \\ &= \tfrac{1}{2} \end{aligned}$$

f
$$\begin{aligned} \left|\frac{z^2}{3i}\right| &= \frac{|z^2|}{|3i|} \\ &= \frac{|z|^2}{|3||i|} \\ &= \frac{2^2}{3} \\ &= \tfrac{4}{3} \end{aligned}$$

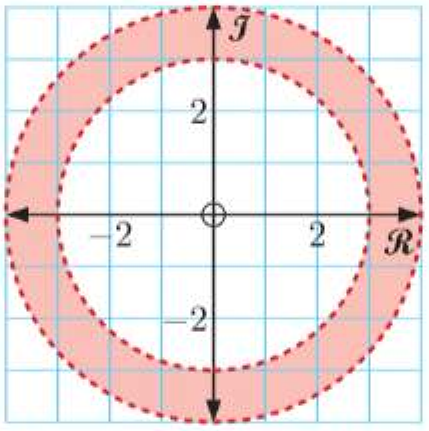
8 **a** $|z| = 2$



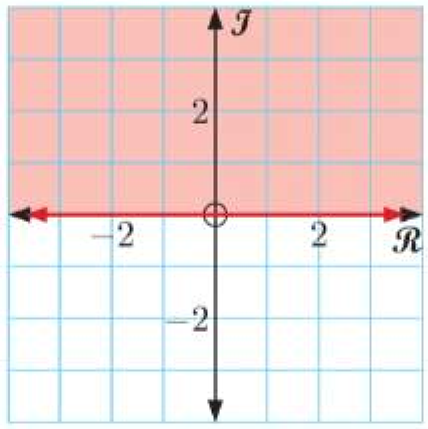
b $|z| \leqslant 1$



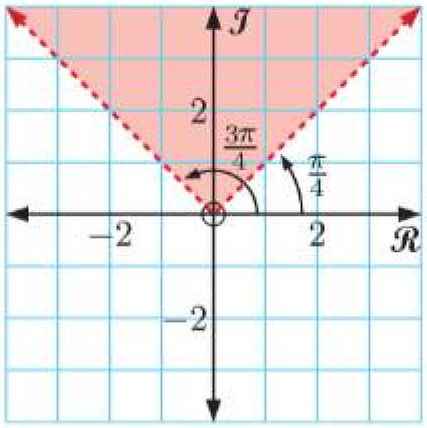
c $3 < |z| < 4$



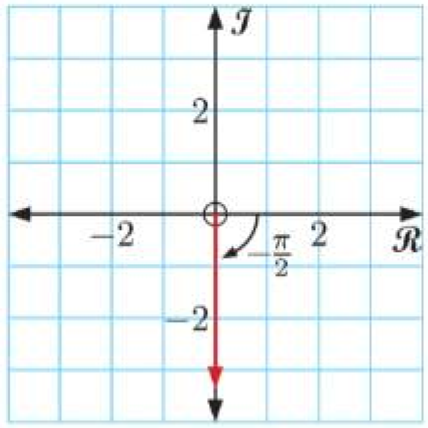
d $0 \leqslant \arg z \leqslant \pi$



e $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$

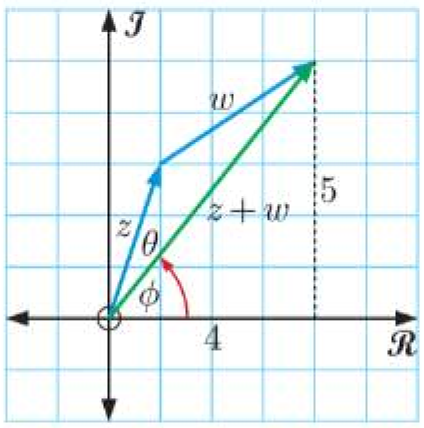


f $\arg z = -\frac{\pi}{2}$



9 $z = 1 + 3i, \quad w = 3 + 2i$

a $z + w = (1 + 3i) + (3 + 2i)$
$= 4 + 5i$



b $|z + w| = \sqrt{4^2 + 5^2}$
$= \sqrt{41}$

$\tan \phi = \frac{5}{4}$
$\therefore \quad \phi \approx 0.896$

$\therefore \quad \arg(z + w) \approx 0.896$

c $|z| = \sqrt{1^2 + 3^2} = \sqrt{10}, \quad |w| = \sqrt{3^2 + 2^2} = \sqrt{13}, \quad$ and $\quad |z + w| = \sqrt{41}$

Now $\quad |w|^2 = |z|^2 + |z + w|^2 - 2|z||z + w| \cos \theta \qquad$ {cosine rule}

$\therefore \quad \left(\sqrt{13}\right)^2 = \left(\sqrt{10}\right)^2 + \left(\sqrt{41}\right)^2 - 2\sqrt{10}\sqrt{41} \cos \theta$

$\therefore \quad 13 = 10 + 41 - 2\sqrt{10}\sqrt{41} \cos \theta$

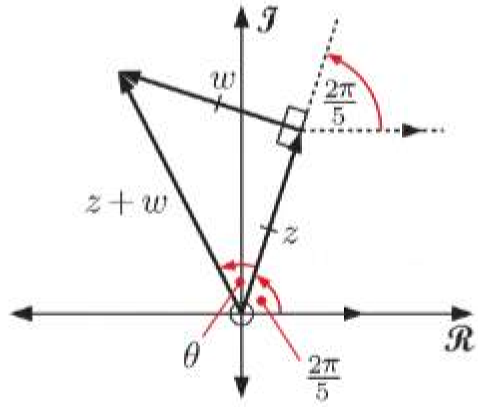
$\therefore \quad \cos \theta = \dfrac{38}{2\sqrt{410}}$

$\therefore \quad \theta \approx 0.353$

d $\arg z = \tan^{-1}\left(\frac{3}{1}\right) \approx 1.249$

$\arg(z + w) + \theta \approx 0.896 + 0.353$
$\approx 1.249 \quad \checkmark$

10 $|z| = 3, \quad \arg z = \frac{2\pi}{5}$



a $|w| = |z| = 3$

b $\arg w = \arg z + \frac{\pi}{2}$
$= \frac{2\pi}{5} + \frac{\pi}{2}$
$= \frac{9\pi}{10}$

c $|z+w|^2 = |z|^2 + |w|^2$ {Pythagoras}
$= 3^2 + 3^2$
$= 18$

$\therefore \quad |z+w| = \sqrt{18}$ $\{|z+w| > 0\}$
$= 3\sqrt{2}$

d $\tan\theta = \dfrac{3}{3} = 1$

$\therefore \quad \theta = \frac{\pi}{4}$

So, $\arg(z+w) = \arg z + \frac{\pi}{4}$
$= \frac{2\pi}{5} + \frac{\pi}{4}$
$= \frac{13\pi}{20}$

11 **a** $w = \dfrac{z+i}{z-i}$, where $z = a+bi$, $a, b \in \mathbb{R}$

$$\therefore \quad w = \frac{a+bi+i}{a+bi-i}$$
$$= \frac{a+(b+1)i}{a+(b-1)i}$$
$$= \left(\frac{a+(b+1)i}{a+(b-1)i}\right) \times \left(\frac{a-(b-1)i}{a-(b-1)i}\right)$$
$$= \frac{a^2 - a(b-1)i + a(b+1)i - (b+1)(b-1)i^2}{a^2-(b-1)^2i^2}$$
$$= \frac{a^2 - \cancel{abi} + ai + \cancel{abi} + ai + b^2 - 1}{a^2+(b-1)^2}$$
$$= \left(\frac{a^2+b^2-1}{a^2+(b-1)^2}\right) + \left(\frac{2a}{a^2+(b-1)^2}\right)i$$

b $\mathcal{Re}(w) = \dfrac{a^2+b^2-1}{a^2+(b-1)^2} = \dfrac{a^2+b^2-1}{a^2+b^2-2b+1}$

Since $|z| = 1$, $\sqrt{a^2+b^2} = 1 \quad \therefore \quad a^2+b^2 = 1$

$\therefore \quad \mathcal{Re}(w) = \dfrac{1-1}{1-2b+1} = 0$ provided $b \neq 1$

If $b = 1$, then $\mathcal{Re}(w)$ is undefined.

12 $\dfrac{50}{z^*} - \dfrac{10}{z} = 2 + 9i$ where $z = a + bi$, $a, b \in \mathbb{R}$

$\therefore\ 50z - 10z^* = (2+9i)(|z|^2)$ {multiply both sides by $zz^* = |z|^2$}

$\therefore\ 50(a+bi) - 10(a-bi) = (2+9i)(40)$ {$|z| = 2\sqrt{10}$ $\therefore$ $|z|^2 = 40$}

$\therefore\ 50a + 50bi - 10a + 10bi = 80 + 360i$

$\therefore\ 40a + 60bi = 80 + 360i$

Equating real and imaginary parts, $40a = 80$ and $60b = 360$

$\therefore\ a = 2$ and $b = 6$

$\therefore\ z = 2 + 6i$

EXERCISE 11G

1 **a** A(3, 6), B(−1, 2), $z = 3+6i$, $w = -1+2i$

i
$$\begin{aligned} z - w &= (3+6i) - (-1+2i) \\ &= 4 + 4i \\ \therefore\ |z-w| &= \sqrt{4^2+4^2} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$
$\therefore$ AB $= 4\sqrt{2}$ units

ii
$$\begin{aligned} \frac{z+w}{2} &= \frac{(3+6i)+(-1+2i)}{2} \\ &= \frac{2+8i}{2} \\ &= 1 + 4i \end{aligned}$$
$\therefore$ the midpoint of [AB] is (1, 4).

b A(−4, 7), B(1, −3), $z = -4+7i$, $w = 1-3i$

i
$$\begin{aligned} z - w &= (-4+7i) - (1-3i) \\ &= -5 + 10i \\ \therefore\ |z-w| &= \sqrt{(-5)^2+10^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$
$\therefore$ AB $= 5\sqrt{5}$ units

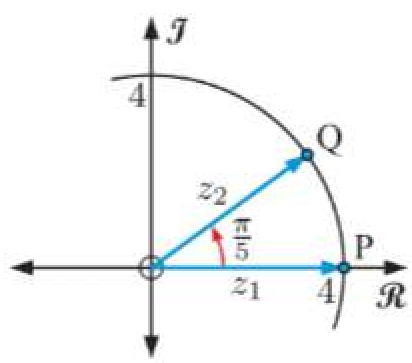
ii
$$\begin{aligned} \frac{z+w}{2} &= \frac{(-4+7i)+(1-3i)}{2} \\ &= \frac{-3+4i}{2} \\ &= -\tfrac{3}{2} + 2i \end{aligned}$$
$\therefore$ the midpoint of [AB] is $(-\frac{3}{2}, 2)$.

2 **a** P and Q both lie on a circle of radius 4 units, centred at the origin O.

P lies on the horizontal axis, so P is (4, 0).

Q makes an angle of $\frac{\pi}{5}$ with the real axis, so Q is $(4\cos\frac{\pi}{5},\ 4\sin\frac{\pi}{5})$.

$z_1 \equiv \overrightarrow{OP}$, and $z_2 \equiv \overrightarrow{OQ}$, so $|z_1 - z_2|$ is the distance between P and Q.



$$\begin{aligned} \text{PQ} &= \sqrt{(4\cos\tfrac{\pi}{5} - 4)^2 + (4\sin\tfrac{\pi}{5} - 0)^2} \\ &= \sqrt{16\cos^2(\tfrac{\pi}{5}) - 32\cos\tfrac{\pi}{5} + 16 + 16\sin^2(\tfrac{\pi}{5})} \\ &= \sqrt{16 - 32\cos\tfrac{\pi}{5} + 16} \qquad \{\cos^2\theta + \sin^2\theta = 1\} \\ &= \sqrt{32 - 32\cos\tfrac{\pi}{5}} \end{aligned}$$

$\therefore\ |z_1 - z_2| = \sqrt{32 - 32\cos\frac{\pi}{5}}$

b Perimeter of triangle OPQ

$= \text{OP} + \text{OQ} + \text{PQ}$

$= |z_1| + |z_2| + |z_1 - z_2|$

$= 4 + 4 + \sqrt{32 - 32\cos\frac{\pi}{5}}$ {from **a**}

$= 8 + \sqrt{32 - 32\cos\frac{\pi}{5}}$ units

Area of triangle OPQ

$= \frac{1}{2} \times \text{OP} \times \text{OQ} \times \sin \text{P}\widehat{\text{O}}\text{Q}$

$= \frac{1}{2} \times 4 \times 4 \times \sin\frac{\pi}{5}$

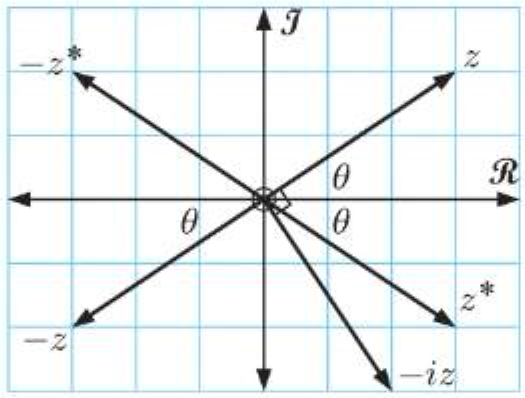
$= 8\sin\frac{\pi}{5}$ units^2

3 **a** $z = x + yi \mapsto z^* = x - yi$
Reflection in the real axis.

b $z = x + yi \mapsto -z = -x - yi$
Rotation of π about O.

c $z = x + yi \mapsto -z^* = -x + yi$
Reflection in the imaginary axis.

d $z = x + yi \mapsto -iz = y - xi$
Clockwise rotation of $\frac{\pi}{2}$ about O.



4 **a** **i** $z_1 \equiv \overrightarrow{\text{OA}}$ and $z_2 \equiv \overrightarrow{\text{OB}}$

$\therefore\ |z_1 - z_2| = \text{BA}$
$= 2$

ii $z_3 \equiv \overrightarrow{\text{OC}}$ and $z_6 \equiv \overrightarrow{\text{OF}}$

$|z_3 - z_6| = \text{FC}$
$= 2 + 2 = 4$

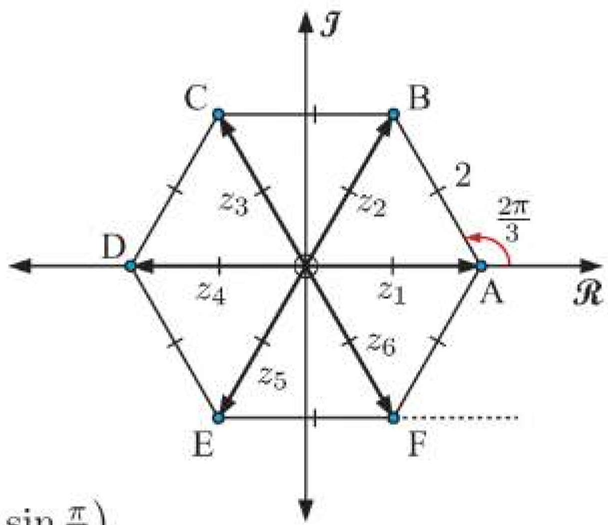
iii $z_2 \equiv \overrightarrow{\text{OB}}$ and $z_4 \equiv \overrightarrow{\text{OD}}$

$\therefore\ |z_2 - z_4| = \text{DB}$

Now $\text{A}\widehat{\text{O}}\text{B} = \frac{\pi}{3}$, so B is $\left(2\cos\frac{\pi}{3},\ 2\sin\frac{\pi}{3}\right)$

which is $(1, \sqrt{3})$, and D is $(-2, 0)$.

$\therefore\ |z_2 - z_4| = \sqrt{(-2-1)^2 + (0-\sqrt{3})^2}$
$= \sqrt{9+3}$
$= \sqrt{12}$
$= 2\sqrt{3}$



iv $z_2 - z_1$ corresponds to $\overrightarrow{\text{AB}}$

$\therefore\ \arg(z_2 - z_1) = \frac{2\pi}{3}$

v $z_4 - z_6$ corresponds to $\overrightarrow{\text{FD}}$

Now $\triangle\text{FOD}$ is isosceles with $\text{F}\widehat{\text{O}}\text{D} = \frac{2\pi}{3}$, so $\text{O}\widehat{\text{F}}\text{D} = \dfrac{\pi - \frac{2\pi}{3}}{2}$
$= \frac{\pi}{6}$

$\therefore\ \arg(z_4 - z_6) = \frac{2\pi}{3} + \frac{\pi}{6}$
$= \frac{5\pi}{6}$

b $|z_4 - i| = |-2 - i|$
$= \sqrt{(-2)^2 + (-1)^2}$
$= \sqrt{5}$

The point representing z_4 is $\sqrt{5}$ units from the point representing i.

ACTIVITY 1 THE TRIANGLE INEQUALITY

1 **a** $|z_1 + z_2| = |3 + 4i + 6i|$
$= |3 + 10i|$
$= \sqrt{3^2 + 10^2}$
$= \sqrt{109}$

$|z_1| + |z_2| = |3 + 4i| + |6i|$
$= \sqrt{3^2 + 4^2} + \sqrt{0^2 + 6^2}$
$= \sqrt{9 + 16} + \sqrt{36}$
$= \sqrt{25} + \sqrt{36}$
$= 5 + 6$
$= 11$

$11 = \sqrt{121} > \sqrt{109}$

$\therefore \quad |z_1 + z_2| \leqslant |z_1| + |z_2|$ ✓

b $|z_1 + z_2| = |2 - 3i + 4 + 7i|$
$= |6 + 4i|$
$= \sqrt{6^2 + 4^2}$
$= \sqrt{36 + 16}$
$= \sqrt{52}$

$|z_1| + |z_2| = |2 - 3i| + |4 + 7i|$
$= \sqrt{2^2 + (-3)^2} + \sqrt{4^2 + 7^2}$
$= \sqrt{4 + 9} + \sqrt{16 + 49}$
$= \sqrt{13} + \sqrt{65}$

$\sqrt{52} < \sqrt{65}$

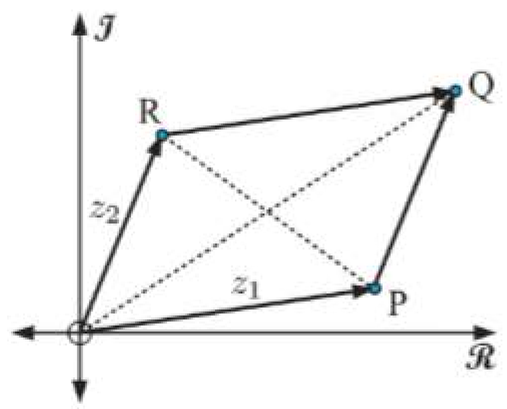
$\therefore \quad \sqrt{52} < \sqrt{65} + \sqrt{13}$

$\therefore \quad |z_1 + z_2| \leqslant |z_1| + |z_2|$ ✓

2 **a** Opposite sides in a parallelogram are parallel and have the same length, and so the vectors representing opposite sides are equal.

$\therefore \quad \overrightarrow{RQ} = \overrightarrow{OP}, \quad \overrightarrow{PQ} = \overrightarrow{OR}$

So, $\overrightarrow{RQ}$ represents z_1 and $\overrightarrow{PQ}$ represents z_2.



i $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$
which represents
$z_1 + z_2$

ii $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$
which represents
$-z_1 + z_2$ or
$z_2 - z_1$

b The sum of the lengths of two sides of a triangle is greater than the length of the remaining side.

So, in triangle OPQ, $\quad OP + PQ > OQ$

$\therefore \quad |z_1| + |z_2| > |z_1 + z_2| \qquad$ {from **a i**}

The case of equality $|z_1| + |z_2| = |z_1 + z_2|$ occurs if P lies on [OQ].

In this case, $\overrightarrow{OP} = k\,\overrightarrow{OQ}$ for some $k \geqslant 0$. So, z_1 and z_2 must satisfy $z_1 = kz_2$ for some $k \geqslant 0$.

Note: If O lies on [PQ] and P and Q are not O, then

$OQ < PQ$ and so $|z_1 + z_2| < |z_2|$

$\therefore \quad |z_1 + z_2| < |z_1| + |z_2|$

So, $|z_1 + z_2| = |z_1| + |z_2|$ is impossible.

c In triangle OPR, $\text{PR} + \text{OP} > \text{OR}$

$$\therefore \ |z_2 - z_1| + |z_1| > |z_2| \qquad \{\text{from } \mathbf{a\ ii}\}$$

$$\therefore \ |z_2 - z_1| > |z_2| - |z_1|$$

The case of equality $|z_2 - z_1| = |z_2| - |z_1|$ occurs if P lies on [OR]. In this case, $\overrightarrow{\text{OP}} \parallel \overrightarrow{\text{OR}}$ and $\overrightarrow{\text{OP}} = k\,\overrightarrow{\text{OR}}$ for some $0 \leqslant k \leqslant 1$. So, z_1 and z_2 must satisfy $z_1 = kz_2$ for some $0 \leqslant k \leqslant 1$.

Note: If R lies on [OP] and R is not O or P, then

$\text{OR} < \text{OP}$ and so $|z_2| < |z_1|$

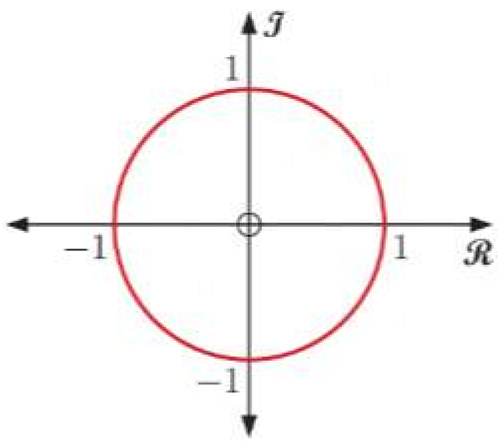
$$\therefore \ |z_2| - |z_1| < 0$$

But $|z_2 - z_1| > 0$, so $|z_2 - z_1| = |z_2| - |z_1|$ is impossible.

A similar situation occurs if O lies on [PR] and P and R are not O.

ACTIVITY 2 LOCUS

1 **a**



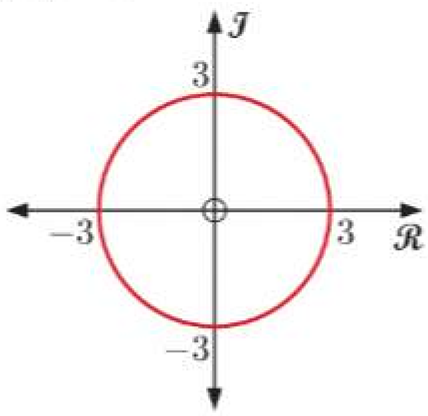
b $z = x + yi, \quad |z| = 1$

$$\therefore \ |z| = |x + yi| = \sqrt{x^2 + y^2}$$

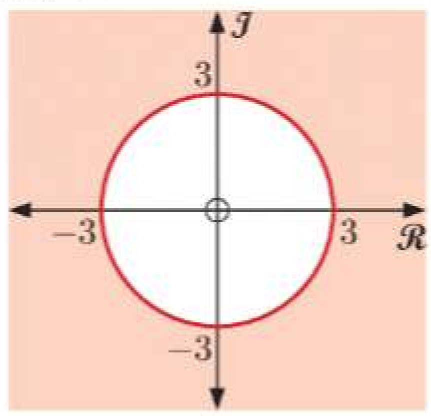
$$\therefore \ \sqrt{x^2 + y^2} = 1$$

or $x^2 + y^2 = 1$ which is the equation of the unit circle.

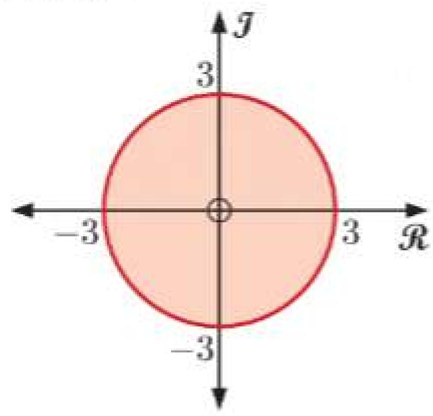
2 **a** $|z| = 3$



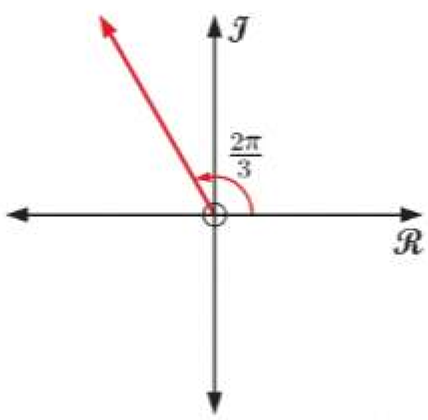
b $|z| \geqslant 3$



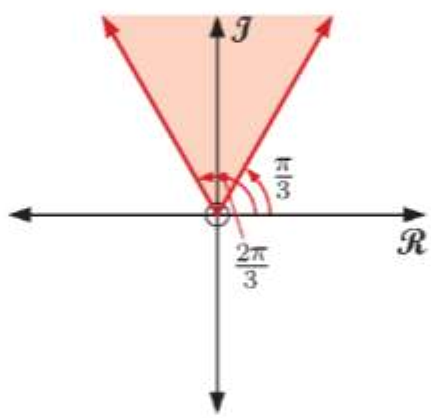
c $|z| \leqslant 3$



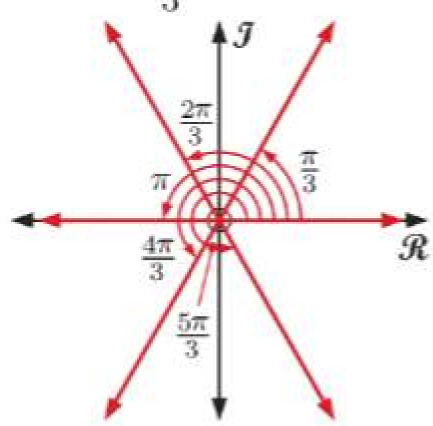
d $\arg z = \frac{2\pi}{3}$



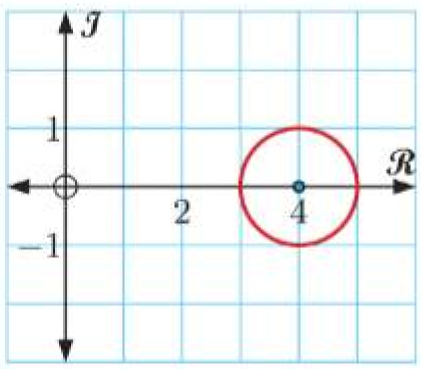
e $\frac{\pi}{3} \leqslant \arg z \leqslant \frac{2\pi}{3}$



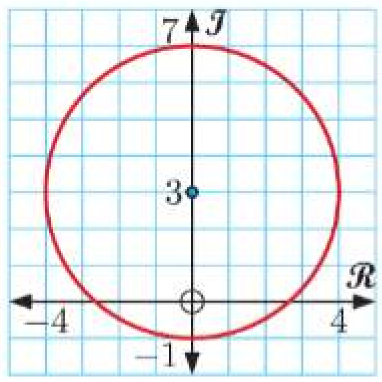
f $\arg z = \dfrac{k\pi}{3}, \ k \in \mathbb{Z}$



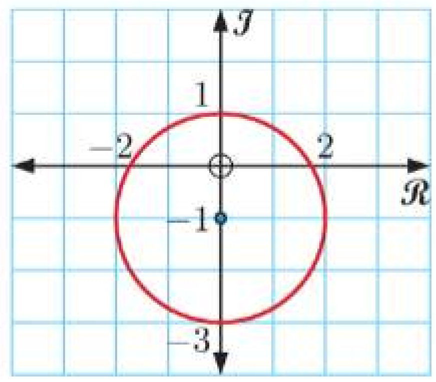
3 **a** $|z-4|=1$



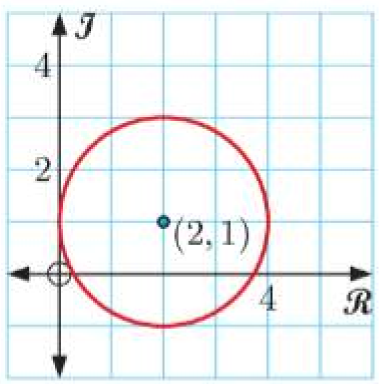
b $|z-3i|=4$



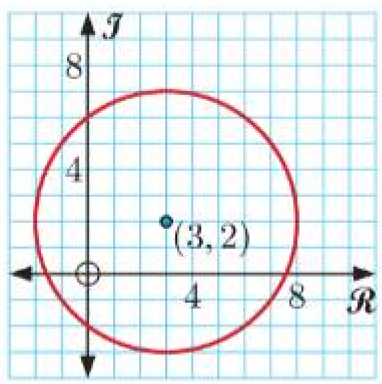
c $|z+i|=2$



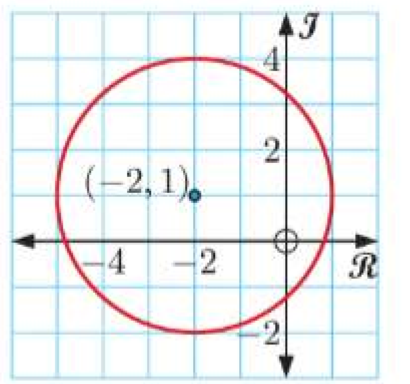
d $|z-(2+i)|=2$



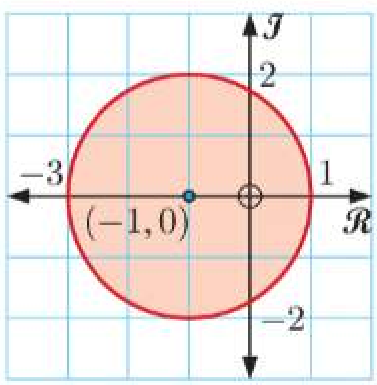
e $|z-(3+2i)|=5$



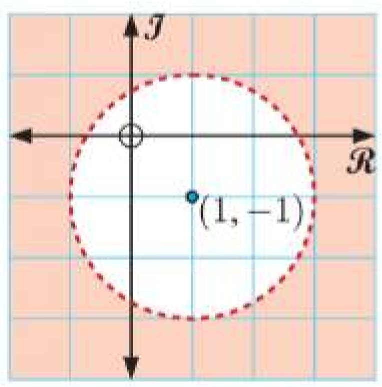
f $|z-(-2+i)|=3$



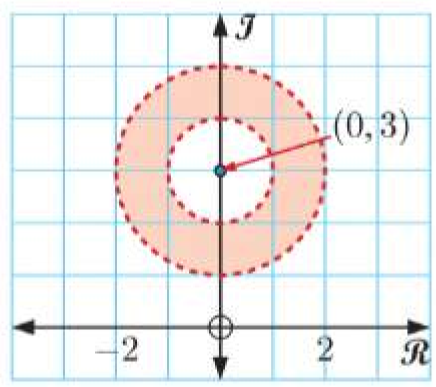
g $|z+1| \leqslant 2$



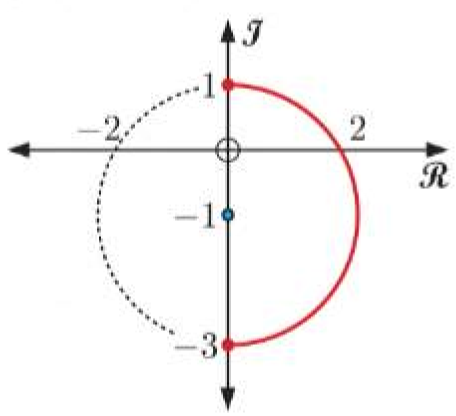
h $|z-(1-i)|>2$



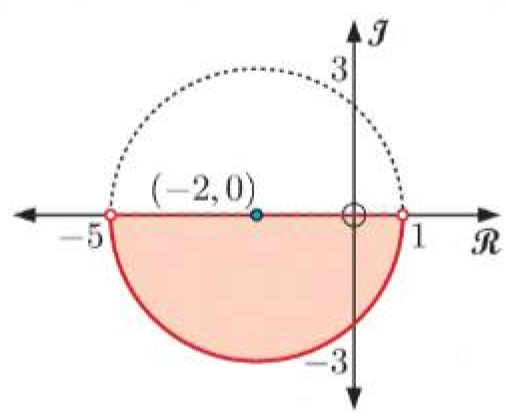
i $1<|z-3i|<2$



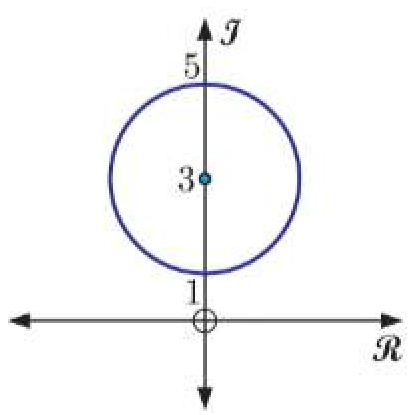
4 **a** $\{z \mid |z+i|=2 \text{ and } \mathcal{Re}(z) \geqslant 0\}$



b $\{z \mid |z+2| \leqslant 3 \text{ and } \mathcal{Im}(z) < 0\}$

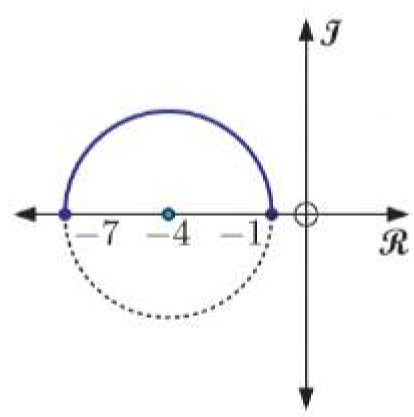


5 **a**



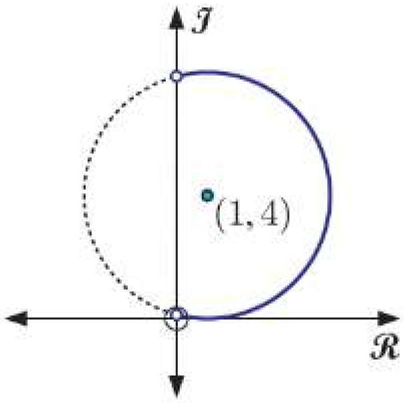
$\{z \mid |z-3i|=2\}$

b



$\{z \mid |z+4|=3 \text{ and } \mathcal{Im}(z) \geqslant 0\}$

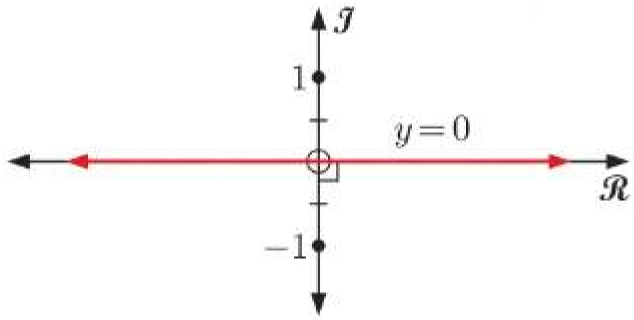
c



$\{z \mid |z-(1+4i)| = 4 \quad \text{and} \quad \mathcal{R}e(z) > 0\}$

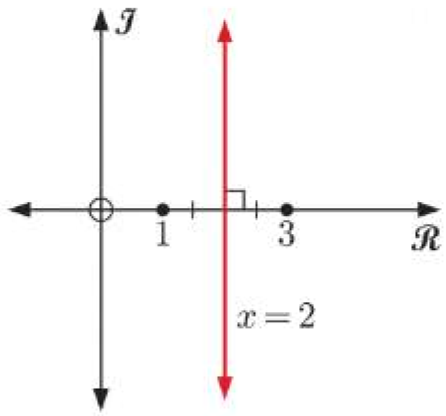
6 **a** $|z+i| = |z-i|$

The point representing z is equidistant from A(0, 1) and B(0, -1), so z lies on the perpendicular bisector of [AB].
This is the line $y = 0$.



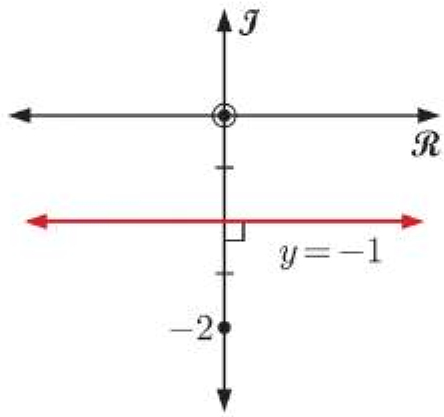
b $|z-1| = |z-3|$

The point representing z is equidistant from A(1, 0) and B(3, 0), so z lies on the perpendicular bisector of [AB].
This is the line $x = 2$.



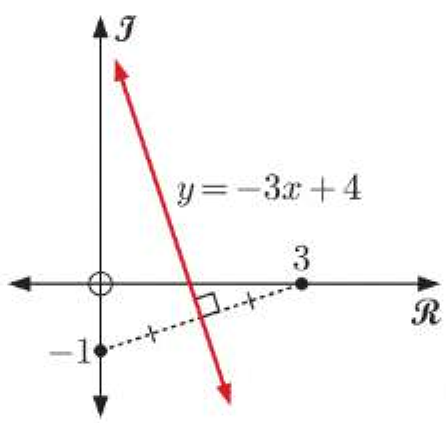
c $|z+2i| = |z|$

The point representing z is equidistant from A(0, -2) and B(0, 0), so z lies on the perpendicular bisector of [AB].
This is the line $y = -1$.



d $|z-3| = |z+i|$

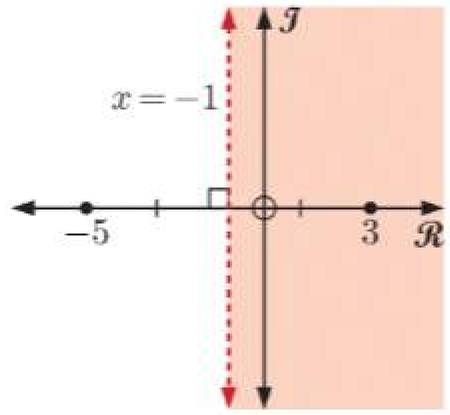
The point representing z is equidistant from A(3, 0) and B(0, -1), so z lies on the perpendicular bisector of [AB].
This is the line $y = -3x + 4$.



e $|z-3| < |z+5|$

The point representing z is closer to A(3, 0) than B(−5, 0), so z lies to the right of the perpendicular bisector of [AB].

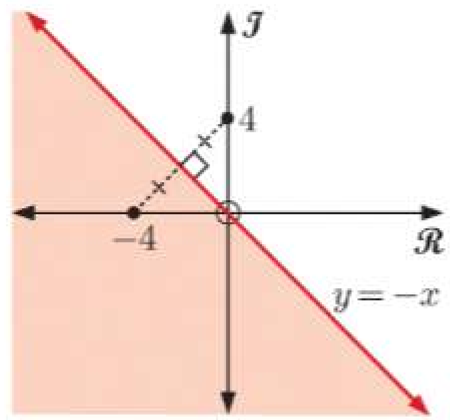
This is the region $x > -1$.



f $|z-4i| \geqslant |z+4|$

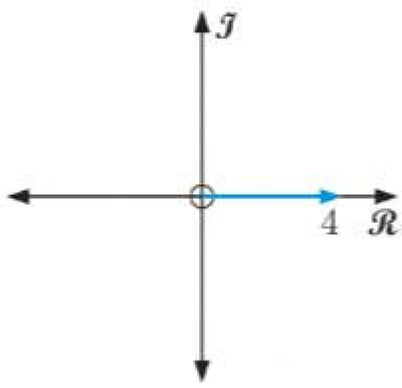
The point representing z is either equidistant from A(0, 4) and B(−4, 0), or closer to B than A. So z lies on or below the perpendicular bisector of [AB].

This is the region $y \leqslant -x$.



EXERCISE 11H.1

1 **a**

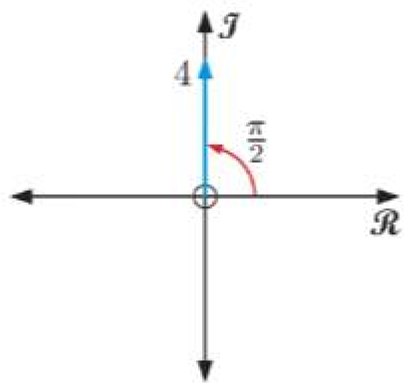


$|4| = 4$

$\arg(4) = 0$

$\therefore\ 4 = 4\operatorname{cis}0$

b

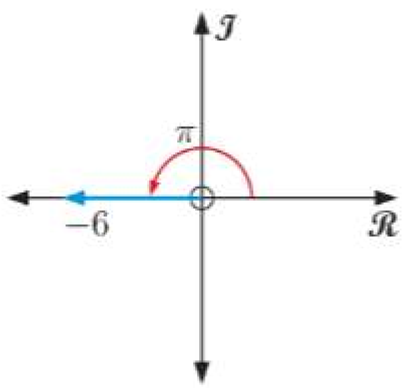


$|4i| = 4$

$\arg(4i) = \frac{\pi}{2}$

$\therefore\ 4i = 4\operatorname{cis}\frac{\pi}{2}$

c

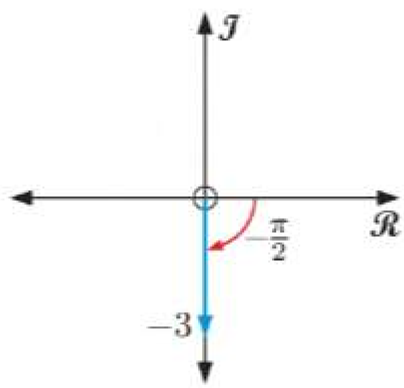


$|-6| = 6$

$\arg(-6) = \pi$

$\therefore\ -6 = 6\operatorname{cis}\pi$

d

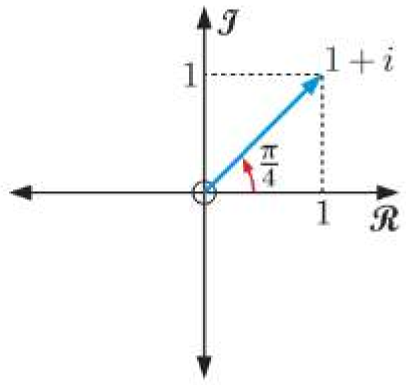


$|-3i| = 3$

$\arg(-3i) = -\frac{\pi}{2}$

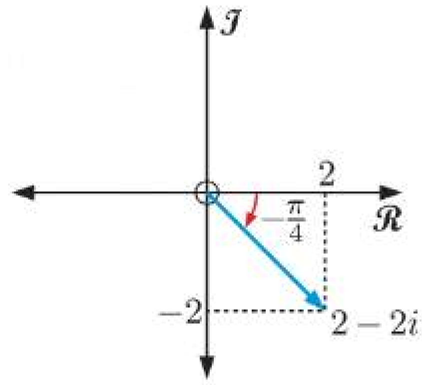
$\therefore\ -3i = 3\operatorname{cis}\left(-\frac{\pi}{2}\right)$

e



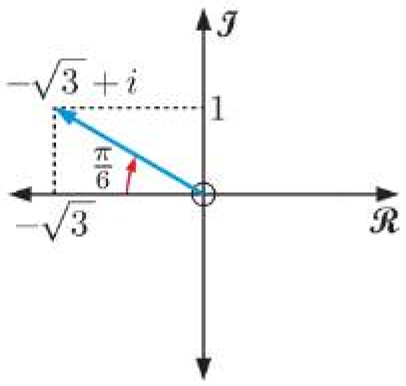
$$|1+i| = \sqrt{1+1} = \sqrt{2}$$
$$\arg(1+i) = \tfrac{\pi}{4}$$
$$\therefore\ 1+i = \sqrt{2}\operatorname{cis}\tfrac{\pi}{4}$$

f



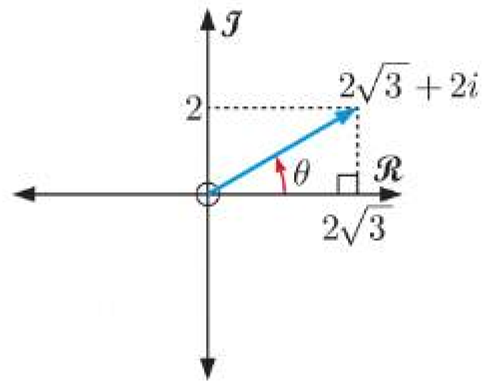
$$|2-2i| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$
$$\arg(2-2i) = -\tfrac{\pi}{4}$$
$$\therefore\ 2-2i = 2\sqrt{2}\operatorname{cis}\left(-\tfrac{\pi}{4}\right)$$

g



$$\left|-\sqrt{3}+i\right| = \sqrt{(\sqrt{3})^2+1^2}$$
$$= \sqrt{4} = 2$$
$$\arg(-\sqrt{3}+i) = \pi - \tfrac{\pi}{6}$$
$$= \tfrac{5\pi}{6}$$
$$\therefore\ -\sqrt{3}+i = 2\operatorname{cis}\tfrac{5\pi}{6}$$

h

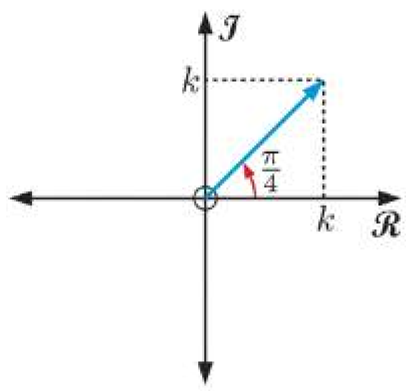


$$\left|2\sqrt{3}+2i\right| = \sqrt{(2\sqrt{3})^2+2^2}$$
$$= \sqrt{12+4} = \sqrt{16} = 4$$
$$\tan\theta = \tfrac{2}{2\sqrt{3}}$$
$$\therefore\ \tan\theta = \tfrac{1}{\sqrt{3}}$$
$$\therefore\ \theta = \tfrac{\pi}{6}$$
$$\arg(2\sqrt{3}+2i) = \tfrac{\pi}{6}$$
$$\therefore\ 2\sqrt{3}+2i = 4\operatorname{cis}\tfrac{\pi}{6}$$

2 $z = 0 = 0 + 0i$ cannot be written in polar form. The vector representing z has length zero, and an argument is not defined (no angle can be formed with the positive x-axis).

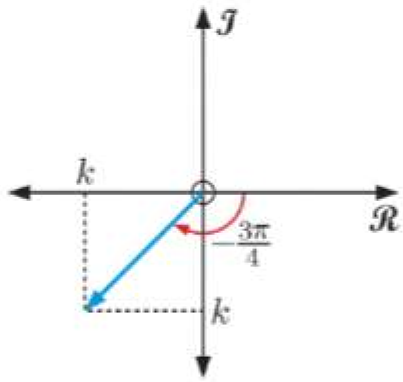
3 If $k = 0$ it is not possible. {from **2**}

If $k > 0$, $|z| = \sqrt{k^2+k^2}$
$$= \sqrt{2k^2}$$
$$= k\sqrt{2}$$
$$\arg z = \tfrac{\pi}{4}$$
$$\therefore\ z = k\sqrt{2}\operatorname{cis}\tfrac{\pi}{4}$$



If $k < 0$, $\quad |z| = \sqrt{k^2 + k^2}$
$= \sqrt{2k^2}$
$= |k|\sqrt{2}$

Since $k < 0$, $\quad |z| = -k\sqrt{2}$
$\arg z = -\frac{3\pi}{4}$
$\therefore \quad z = -k\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$



4 **a** $2\,\text{cis}\,\frac{\pi}{2}$
$= 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
$= 2(0 + i \times 1)$
$= 2i$

b $8\,\text{cis}\,\frac{\pi}{4}$
$= 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
$= 8\left(\frac{1}{\sqrt{2}} + i \times \frac{1}{\sqrt{2}}\right)$
$= 4\sqrt{2} + 4\sqrt{2}i$

c $4\,\text{cis}\,\frac{\pi}{6}$
$= 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
$= 4\left(\frac{\sqrt{3}}{2} + i \times \frac{1}{2}\right)$
$= 2\sqrt{3} + 2i$

d $3\,\text{cis}\,0$
$= 3(\cos 0 + i\sin 0)$
$= 3(1 + i \times 0)$
$= 3$

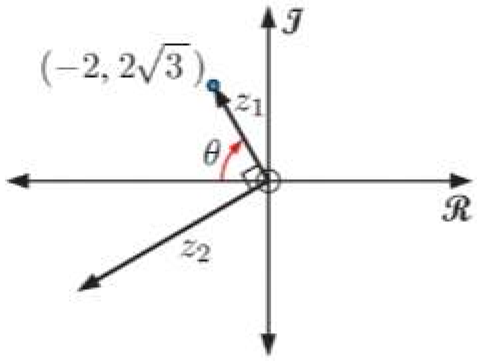
e $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$
$= \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$
$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i \times -\frac{1}{\sqrt{2}}\right)$
$= 1 - i$

f $\sqrt{3}\,\text{cis}\,\frac{2\pi}{3}$
$= \sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= \sqrt{3}\left(-\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$
$= -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

g $5\,\text{cis}\,\pi$
$= 5(\cos\pi + i\sin\pi)$
$= 5(-1 + i \times 0)$
$= -5$

h $10\,\text{cis}\left(-\frac{2\pi}{3}\right)$
$= 10\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$
$= 10\left(-\frac{1}{2} + i \times -\frac{\sqrt{3}}{2}\right)$
$= -5 - 5\sqrt{3}i$

5 **a** From the diagram, $z_1 = -2 + 2\sqrt{3}i$.



$$\begin{aligned}\text{Now} \quad |z_1| &= \sqrt{(-2)^2 + (2\sqrt{3})^2}\\ &= \sqrt{4+12}\\ &= \sqrt{16}\\ &= 4 \qquad \{|z_1| \geqslant 0\}\end{aligned}$$

$$\text{and} \quad \tan\theta = \tfrac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\therefore \quad \theta = \tfrac{\pi}{3}$$

$$\begin{aligned}\therefore \quad \arg z_1 &= \pi - \tfrac{\pi}{3}\\ &= \tfrac{2\pi}{3}\end{aligned}$$

So, $z_1 = 4\operatorname{cis}\frac{2\pi}{3}$.

b **i** z_2 has twice the length of z_1

$\therefore \quad |z_2| = 2 \times 4 = 8 \qquad \{|z_1| = 4\}$

$$\begin{aligned}\text{From the diagram,} \quad \arg z_2 &= \arg z_1 - \tfrac{3\pi}{2}\\ &= \tfrac{2\pi}{3} - \tfrac{3\pi}{2}\\ &= -\tfrac{5\pi}{6}\end{aligned}$$

So, $z_2 = 8\operatorname{cis}\left(-\frac{5\pi}{6}\right)$.

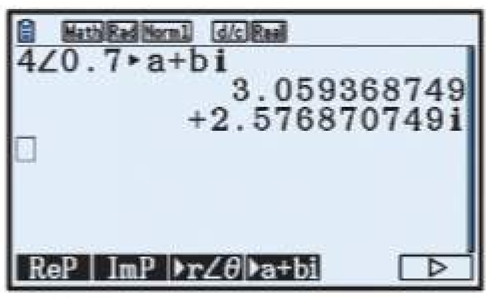
ii
$$\begin{aligned}z_2 &= 8\operatorname{cis}\left(-\tfrac{5\pi}{6}\right) \qquad \{\text{from } \mathbf{b}\ \mathbf{i}\}\\ &= 8\left[\cos\left(-\tfrac{5\pi}{6}\right) + i\sin\left(-\tfrac{5\pi}{6}\right)\right]\\ &= 8\left(-\tfrac{\sqrt{3}}{2} + i \times -\tfrac{1}{2}\right)\\ &= -4\sqrt{3} - 4i\end{aligned}$$

6 **a**
$$\begin{aligned}&\operatorname{cis}\tfrac{3\pi}{4}\\ &= \cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\\ &= -\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\end{aligned}$$

b
$$\begin{aligned}&\operatorname{cis} 0\\ &= \cos 0 + i\sin 0\\ &= 1\end{aligned}$$

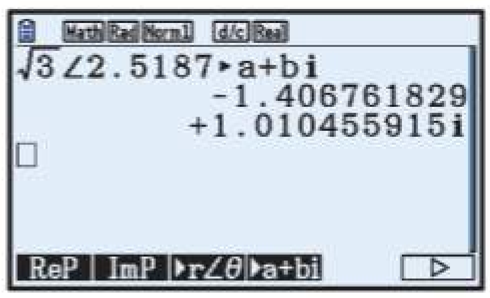
c
$$\begin{aligned}&|\operatorname{cis}\theta|\\ &= |\cos\theta + i\sin\theta|\\ &= \sqrt{\cos^2\theta + \sin^2\theta}\\ &= \sqrt{1} = 1\end{aligned}$$

7 **a**



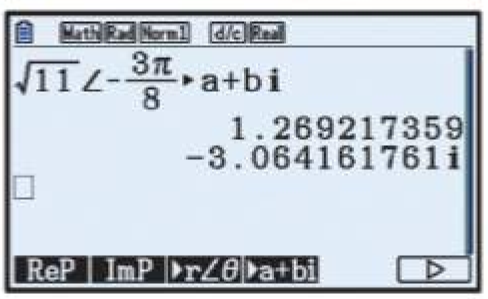
$4\operatorname{cis}(0.7) \approx 3.06 + 2.58i$

b



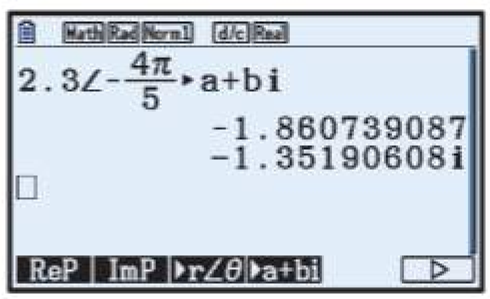
$\sqrt{3}\operatorname{cis}(2.5187) \approx -1.41 + 1.01i$

c



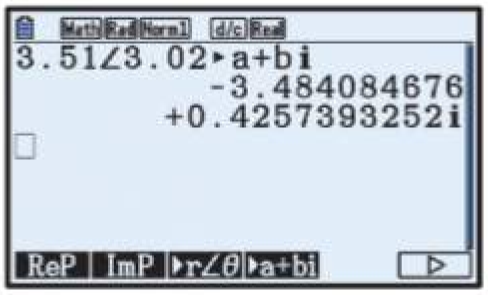
$\sqrt{11}\operatorname{cis}\left(-\frac{3\pi}{8}\right) \approx 1.27 - 3.06i$

d



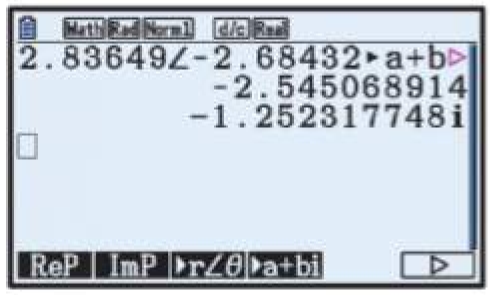
$2.3\operatorname{cis}\left(-\frac{4\pi}{5}\right) \approx -1.86 - 1.35i$

e



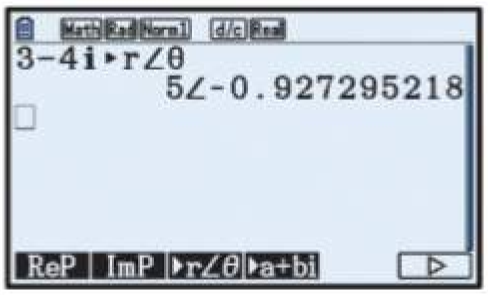
$3.51\,\text{cis}(3.02) \approx -3.48 + 0.426i$

f



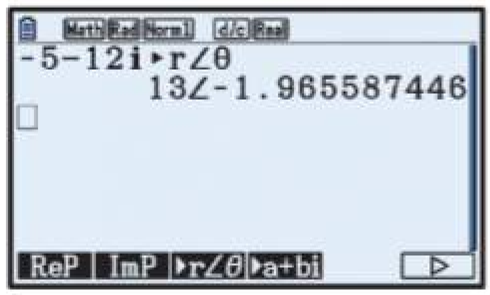
$2.836\,49\,\text{cis}(-2.684\,32) \approx -2.55 - 1.25i$

8 **a**



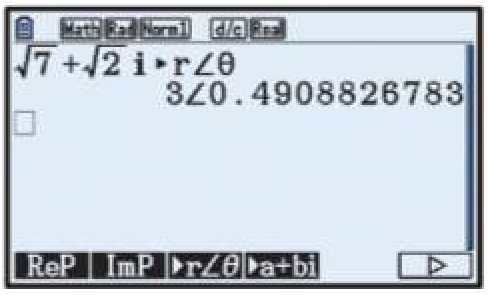
$3 - 4i \approx 5\,\text{cis}(-0.927)$

b



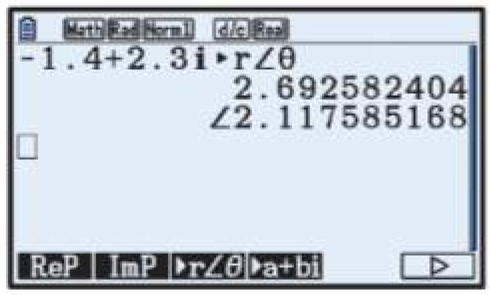
$-5 - 12i \approx 13\,\text{cis}(-1.97)$

c



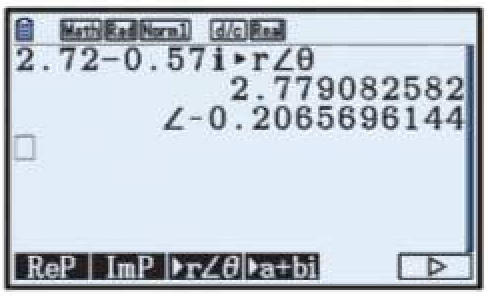
$\sqrt{7} + \sqrt{2}i \approx 3\,\text{cis}(0.491)$

d



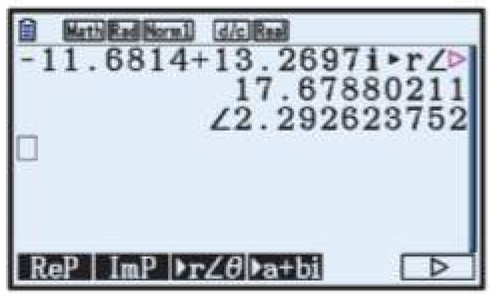
$-1.4 + 2.3i \approx 2.69\,\text{cis}(2.12)$

e



$2.72 - 0.57i \approx 2.78\,\text{cis}(-0.207)$

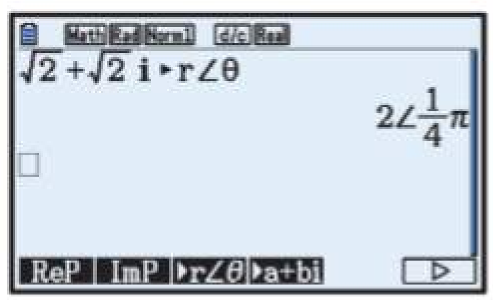
f



$-11.6814 + 13.2697i \approx 17.7\,\text{cis}(2.29)$

9 **a**

$$\begin{aligned}
&3\,\text{cis}\,\tfrac{\pi}{4} + \text{cis}\left(-\tfrac{3\pi}{4}\right)\\
&= 3\left[\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right] + \left[\cos\left(-\tfrac{3\pi}{4}\right) + i\sin\left(-\tfrac{3\pi}{4}\right)\right]\\
&= 3\left[\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right] + \left[-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right]\\
&= \tfrac{3}{\sqrt{2}} + \tfrac{3}{\sqrt{2}}i - \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\\
&= \tfrac{2}{\sqrt{2}} + \tfrac{2}{\sqrt{2}}i\\
&= \sqrt{2} + \sqrt{2}i\\
&= 2\,\text{cis}\,\tfrac{\pi}{4} \quad \{\text{using technology}\}
\end{aligned}$$



b $\quad 2\operatorname{cis}\frac{2\pi}{3} + 5\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

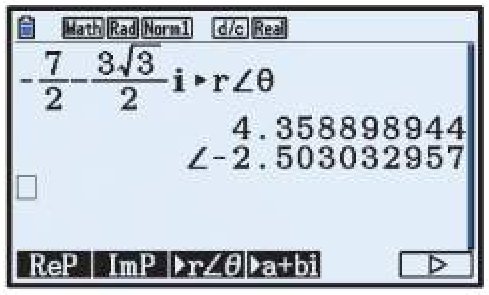
$= 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] + 5\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

$= 2\left[-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right] + 5\left[-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right]$

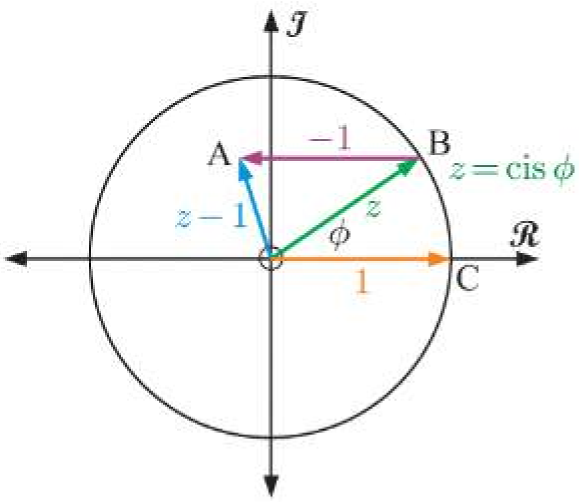
$= -1 + \sqrt{3}i - \frac{5}{2} - \frac{5\sqrt{3}}{2}\,i$

$= -\frac{7}{2} - \frac{3\sqrt{3}}{2}\,i$

$\approx 4.36\operatorname{cis}(-2.50)$ {using technology}

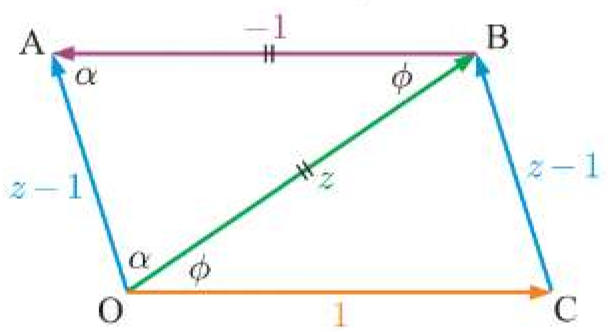


10 **a**



$|z| = 1$, so z ends on the unit circle.

$z - 1$ is the vector $\overrightarrow{OA}$ shown.



Now OABC is a parallelogram.

$\widehat{ABO} = \phi$ {alternate angles}

$OB = AB = 1$

$\therefore$ $\triangle$ABO is isosceles.

$\therefore\ B\widehat{A}O = A\widehat{O}B = \alpha$

$\therefore\ 2\alpha + \phi = \pi$

$\therefore\ \alpha = \dfrac{\pi - \phi}{2}$

$\therefore\ \arg(z-1) = \dfrac{\pi-\phi}{2} + \phi$

$= \frac{\pi}{2} - \dfrac{\phi}{2} + \phi$

$= \frac{\pi}{2} + \dfrac{\phi}{2}$ .... (1)

{since ϕ is acute, $-\pi < \frac{\pi}{2} + \dfrac{\phi}{2} \leqslant \pi$}

Using the cosine rule in $\triangle$ABO:

$OA^2 = 1^2 + 1^2 - 2(1)(1)\cos\phi$

$\therefore\ OA^2 = 2 - 2\cos\phi$

$\therefore\ OA^2 = 2 - 2\left(1 - 2\sin^2\left(\frac{\phi}{2}\right)\right)$

$\therefore\ OA^2 = 2 - 2 + 4\sin^2\left(\frac{\phi}{2}\right)$

$\therefore\ OA^2 = 4\sin^2\left(\frac{\phi}{2}\right)$

$\therefore\ OA = 2\sin\frac{\phi}{2}$ {as $OA > 0$}

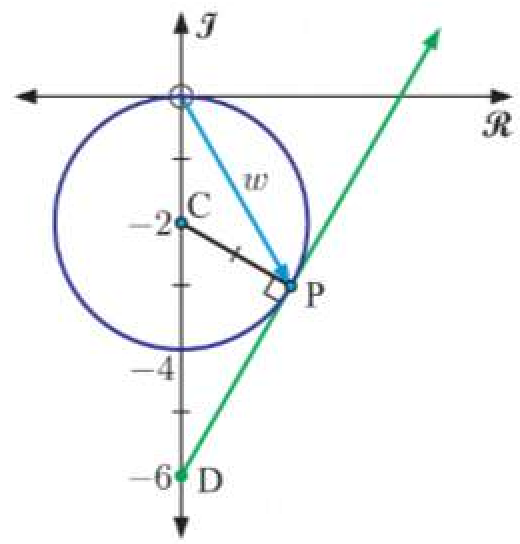
$\therefore\ |z-1| = 2\sin\frac{\phi}{2}$ .... (2)

b $\quad z - 1 = 2\sin\frac{\phi}{2}\operatorname{cis}\left(\frac{\pi}{2} + \frac{\phi}{2}\right)$ {using (1) and (2) in **a**}

$(z-1)^* = 2\sin\frac{\phi}{2}\operatorname{cis}\left(-\frac{\pi}{2} - \frac{\phi}{2}\right)$

11 **a** $|z+2i|=2$

b **i** Defining points C and D as shown, we construct the right angled triangle CDP where CD $=4$, CP $=2$, and DP $=\sqrt{4^2-2^2}=2\sqrt{3}$.



$$\cos(C\widehat{D}P)=\tfrac{2\sqrt{3}}{4}$$
$$=\tfrac{\sqrt{3}}{2}$$

Using the triangle ODP, we have that

$$\begin{aligned}OP^2&=6^2+(2\sqrt{3})^2-2\times 6\times 2\sqrt{3}\times\cos(O\widehat{D}P)\quad\{\text{cosine rule}\}\\&=36+12-36\\&=12\end{aligned}$$

$\therefore\ OP=2\sqrt{3}\quad\{\text{as } OP>0\}$

$\therefore\ |w|=2\sqrt{3}\quad\{\text{as } w\equiv\overrightarrow{OP}\}$

ii Using the results from **i**, $\triangle$OPD is isosceles, so $D\widehat{O}P=O\widehat{D}P$

$$O\widehat{D}P=\cos^{-1}\left(\tfrac{\sqrt{3}}{2}\right)=\tfrac{\pi}{6}$$

$$\therefore\ \arg w=-\tfrac{\pi}{2}+\tfrac{\pi}{6}=-\tfrac{\pi}{3}$$

iii $|w|=2\sqrt{3}$ and $\arg w=-\frac{\pi}{3}$, so $w=2\sqrt{3}\cos(-\frac{\pi}{3})+i\times 2\sqrt{3}\sin(-\frac{\pi}{3})$

$=\sqrt{3}-3i$

EXERCISE 11H.2

1 **a**
$$\begin{aligned}&\operatorname{cis}\theta\operatorname{cis}2\theta\\&=\operatorname{cis}(\theta+2\theta)\\&=\operatorname{cis}3\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\operatorname{cis}3\theta}{\operatorname{cis}\theta}\\&=\operatorname{cis}(3\theta-\theta)\\&=\operatorname{cis}2\theta\end{aligned}$$

c
$$\begin{aligned}&(\operatorname{cis}\theta)^3\\&=(\operatorname{cis}\theta)(\operatorname{cis}\theta)(\operatorname{cis}\theta)\\&=(\operatorname{cis}2\theta)(\operatorname{cis}\theta)\\&=\operatorname{cis}3\theta\end{aligned}$$

d
$$\begin{aligned}&\operatorname{cis}\tfrac{\pi}{18}\operatorname{cis}\tfrac{\pi}{9}\\&=\operatorname{cis}\left(\tfrac{\pi}{18}+\tfrac{\pi}{9}\right)\\&=\operatorname{cis}\tfrac{3\pi}{18}\\&=\operatorname{cis}\tfrac{\pi}{6}\\&=\cos\tfrac{\pi}{6}+i\sin\tfrac{\pi}{6}\\&=\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\end{aligned}$$

e
$$\begin{aligned}&2\operatorname{cis}\tfrac{\pi}{12}\operatorname{cis}\tfrac{\pi}{6}\\&=2\operatorname{cis}\left(\tfrac{\pi}{12}+\tfrac{\pi}{6}\right)\\&=2\operatorname{cis}\tfrac{3\pi}{12}\\&=2\operatorname{cis}\tfrac{\pi}{4}\\&=2\left(\cos\tfrac{\pi}{4}+i\sin\tfrac{\pi}{4}\right)\\&=2\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\\&=\sqrt{2}+i\sqrt{2}\end{aligned}$$

f
$$\begin{aligned}&2\operatorname{cis}\tfrac{2\pi}{5}\times 4\operatorname{cis}\tfrac{8\pi}{5}\\&=8\operatorname{cis}\left(\tfrac{2\pi}{5}+\tfrac{8\pi}{5}\right)\\&=8\operatorname{cis}\tfrac{10\pi}{5}\\&=8\operatorname{cis}2\pi\\&=8(\cos 2\pi+i\sin 2\pi)\\&=8(1)\\&=8\end{aligned}$$

g
$$\begin{aligned}&\sqrt{3}\operatorname{cis}\left(-\tfrac{\pi}{15}\right)\times 2\operatorname{cis}\tfrac{11\pi}{15}\\&=2\sqrt{3}\operatorname{cis}\left(-\tfrac{\pi}{15}+\tfrac{11\pi}{15}\right)\\&=2\sqrt{3}\operatorname{cis}\tfrac{10\pi}{15}\\&=2\sqrt{3}\operatorname{cis}\tfrac{2\pi}{3}\\&=2\sqrt{3}\left(\cos\tfrac{2\pi}{3}+i\sin\tfrac{2\pi}{3}\right)\\&=2\sqrt{3}\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\\&=-\sqrt{3}+3i\end{aligned}$$

h
$$\begin{aligned}&\frac{\operatorname{cis}\frac{\pi}{2}}{\operatorname{cis}\frac{\pi}{3}}\\&=\operatorname{cis}\left(\tfrac{\pi}{2}-\tfrac{\pi}{3}\right)\\&=\operatorname{cis}\tfrac{\pi}{6}\\&=\cos\tfrac{\pi}{6}+i\sin\tfrac{\pi}{6}\\&=\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\end{aligned}$$

i
$$\begin{aligned}&\frac{4\operatorname{cis}\frac{\pi}{12}}{2\operatorname{cis}\frac{7\pi}{12}}\\&=2\operatorname{cis}\left(\tfrac{\pi}{12}-\tfrac{7\pi}{12}\right)\\&=2\operatorname{cis}\left(-\tfrac{6\pi}{12}\right)\\&=2\operatorname{cis}\left(-\tfrac{\pi}{2}\right)\\&=2\left[\cos\left(-\tfrac{\pi}{2}\right)+i\sin\left(-\tfrac{\pi}{2}\right)\right]\\&=2(-i)\\&=-2i\end{aligned}$$

j
$$\begin{aligned}&\frac{\sqrt{2}\operatorname{cis}\frac{3\pi}{20}}{\operatorname{cis}\frac{9\pi}{10}}\\&=\sqrt{2}\operatorname{cis}\left(\tfrac{3\pi}{20}-\tfrac{9\pi}{10}\right)\\&=\sqrt{2}\operatorname{cis}\left(-\tfrac{15\pi}{20}\right)\\&=\sqrt{2}\operatorname{cis}\left(-\tfrac{3\pi}{4}\right)\\&=\sqrt{2}\left[\cos\left(-\tfrac{3\pi}{4}\right)+i\sin\left(-\tfrac{3\pi}{4}\right)\right]\\&=\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)\\&=-1-i\end{aligned}$$

k
$$\begin{aligned}&\frac{\sqrt{32}\operatorname{cis}\frac{\pi}{8}}{\sqrt{2}\operatorname{cis}\left(-\frac{7\pi}{8}\right)}\\&=\tfrac{\sqrt{32}}{\sqrt{2}}\operatorname{cis}\left[\tfrac{\pi}{8}-\left(-\tfrac{7\pi}{8}\right)\right]\\&=\sqrt{16}\operatorname{cis}\tfrac{8\pi}{8}\\&=4\operatorname{cis}\pi\\&=4(\cos\pi+i\sin\pi)\\&=4(-1)\\&=-4\end{aligned}$$

l
$$\begin{aligned}&\left(\sqrt{2}\operatorname{cis}\tfrac{\pi}{8}\right)^4\\&=\sqrt{2}\operatorname{cis}\tfrac{\pi}{8}\times\sqrt{2}\operatorname{cis}\tfrac{\pi}{8}\times\sqrt{2}\operatorname{cis}\tfrac{\pi}{8}\times\sqrt{2}\operatorname{cis}\tfrac{\pi}{8}\\&=(\sqrt{2})^4\operatorname{cis}\left(\tfrac{\pi}{8}+\tfrac{\pi}{8}+\tfrac{\pi}{8}+\tfrac{\pi}{8}\right)\\&=4\operatorname{cis}\tfrac{\pi}{2}\\&=4\left(\cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}\right)\\&=4i\end{aligned}$$

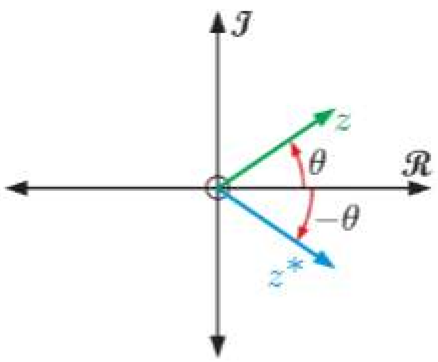
2 **a** $\begin{aligned} &\operatorname{cis} 17\pi \\ &= \operatorname{cis}(\pi + 8 \times 2\pi) \\ &= \operatorname{cis}\pi \\ &= -1 \end{aligned}$

b $\begin{aligned} &\operatorname{cis}(-37\pi) \\ &= \operatorname{cis}(\pi - 19 \times 2\pi) \\ &= \operatorname{cis}\pi \\ &= -1 \end{aligned}$

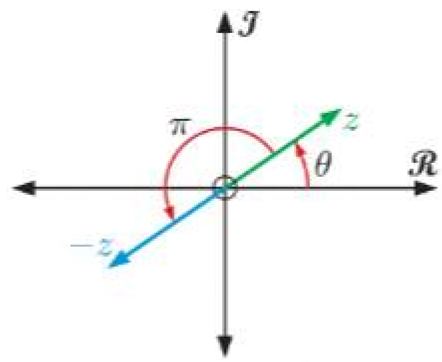
c $\begin{aligned} &\operatorname{cis}\tfrac{91\pi}{3} \\ &= \operatorname{cis}\left(\tfrac{\pi}{3} + 15 \times 2\pi\right) \\ &= \operatorname{cis}\tfrac{\pi}{3} \\ &= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \end{aligned}$

3 **a** $\begin{aligned} z &= 2\operatorname{cis}\theta \\ |z| &= 2 \\ \arg z &= \theta \end{aligned}$

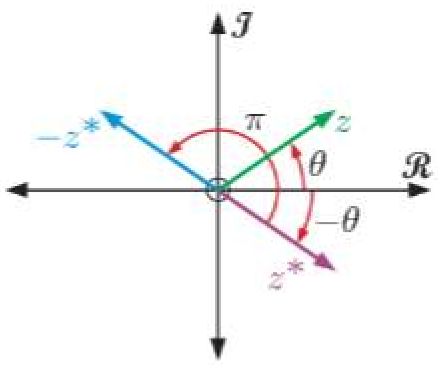
b **i** $z^* = 2\operatorname{cis}(-\theta)$



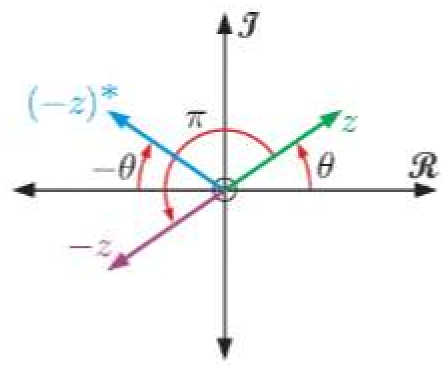
ii $-z = 2\operatorname{cis}(\theta + \pi)$



iii $-z^* = 2\operatorname{cis}(\pi - \theta)$



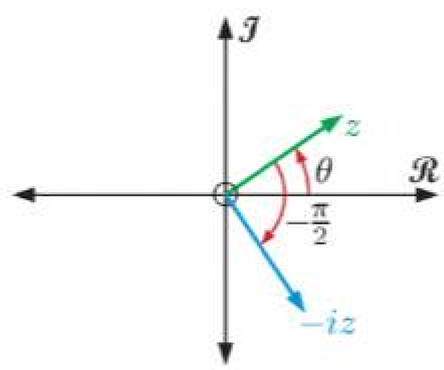
iv $(-z)^* = 2\operatorname{cis}(\pi - \theta)$



4 **a** $\begin{aligned} -i &= 1\operatorname{cis}\left(-\tfrac{\pi}{2}\right) \\ &= \operatorname{cis}\left(-\tfrac{\pi}{2}\right) \end{aligned}$

b $\begin{aligned} -iz &= \operatorname{cis}\left(-\tfrac{\pi}{2}\right) \times r\operatorname{cis}\theta \\ &= r\operatorname{cis}\left(\theta - \tfrac{\pi}{2}\right) \end{aligned}$

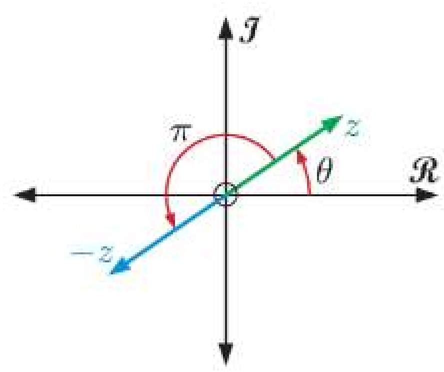
c z is rotated clockwise through $\frac{\pi}{2}$ about the origin.



5 **a** $-1 = \operatorname{cis}\pi$

b $\begin{aligned} -z &= \operatorname{cis}\pi \times r\operatorname{cis}\theta \\ &= r\operatorname{cis}(\theta + \pi) \end{aligned}$

c z is rotated anticlockwise through π about the origin.



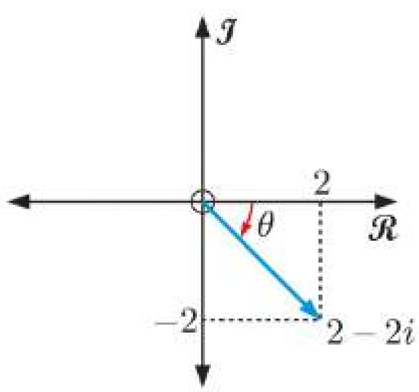
6 z^* is the reflection of z in the real axis.

$\therefore \quad |z^*| = |z| = r \quad$ and $\quad \arg(z^*) = -\arg z = -\theta$

$\therefore \quad z^* = r\,\text{cis}(-\theta)$

EXERCISE 11H.3

1 **a**



If $z = 2-2i$, then $|z| = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$

Now $\tan\theta = \dfrac{2}{2} = 1$

$\therefore \quad \theta = \frac{\pi}{4}$

$\therefore \quad \arg z = -\frac{\pi}{4}$

$\therefore \quad z = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

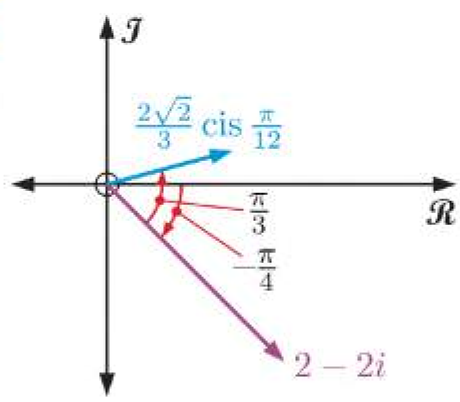
b $z \times \frac{1}{3}\,\text{cis}\,\frac{\pi}{3}$

$= 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right) \times \frac{1}{3}\,\text{cis}\,\frac{\pi}{3}$

$= \frac{2\sqrt{2}}{3}\,\text{cis}\left(-\frac{\pi}{4}+\frac{\pi}{3}\right)$

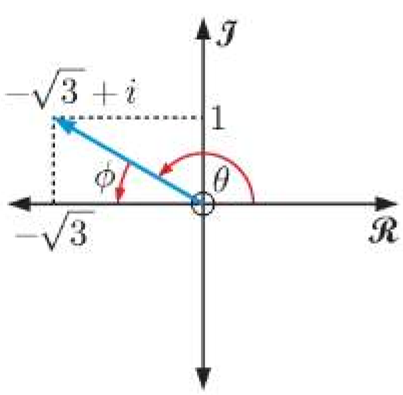
$= \frac{2\sqrt{2}}{3}\,\text{cis}\,\frac{\pi}{12}$

c



d When z is multiplied by $\frac{1}{3}\,\text{cis}\,\frac{\pi}{3}$, it is dilated with scale factor $\frac{1}{3}$, then rotated anticlockwise through $\frac{\pi}{3}$ about the origin.

2 **a**



If $z = -\sqrt{3}+i$, then $|z| = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{4} = 2$

Now $\tan\phi = \frac{1}{\sqrt{3}}$

$\therefore \quad \phi = \frac{\pi}{6}$

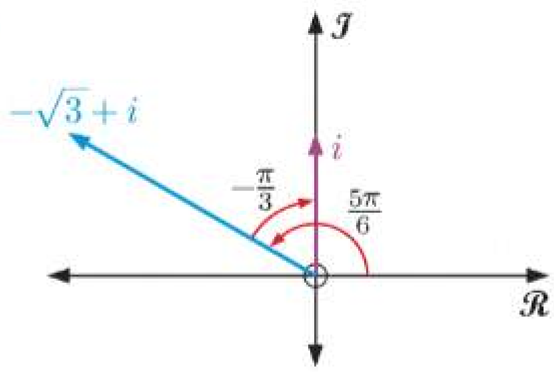
But $\theta = \pi - \phi$

$\therefore \quad \arg z = \frac{5\pi}{6}$

$\therefore \quad z = 2\,\text{cis}\,\frac{5\pi}{6}$

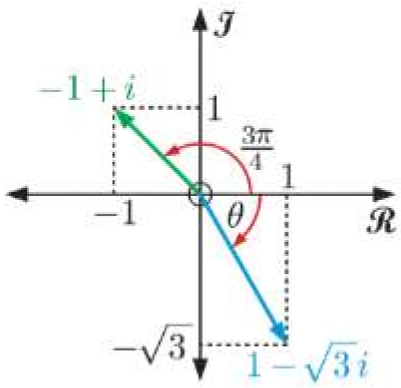
b $z \times \frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$
$= 2\operatorname{cis}\frac{5\pi}{6} \times \frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$
$= \operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)$
$= \operatorname{cis}\frac{\pi}{2}$
$= i$

c



d When z is multiplied by $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$, it is dilated with scale factor $\frac{1}{2}$, then rotated clockwise through $\frac{\pi}{3}$ about the origin.

3 **a**



If $z = -1 + i$, then $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\arg z = \frac{3\pi}{4}$

$\therefore\ z = \sqrt{2}\operatorname{cis}\frac{3\pi}{4}$

If $w = 1 - \sqrt{3}i$, then $|w| = \sqrt{1^2 + (-\sqrt{3})^2}$
$= \sqrt{4} = 2$

Now $\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$
$\therefore\ \theta = \frac{\pi}{3}$
$\therefore\ \arg w = -\frac{\pi}{3}$ $\{\mathcal{R}e\,(z) > 0,\ \mathcal{I}m\,(z) < 0\}$
$\therefore\ w = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

b $zw = \sqrt{2}\operatorname{cis}\frac{3\pi}{4} \times 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$
$= 2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$
$= 2\sqrt{2}\operatorname{cis}\frac{5\pi}{12}$

c When z is multiplied by w, it is dilated with scale factor 2, then rotated clockwise through $\frac{\pi}{3}$ about the origin.

4 **a** $\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$
$= \operatorname{cis}\frac{\pi}{12}$
$= \operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$
$= \operatorname{cis}\frac{\pi}{3} \times \operatorname{cis}\left(-\frac{\pi}{4}\right)$ $\{\operatorname{cis}(\theta+\phi) = \operatorname{cis}\theta \times \operatorname{cis}\phi\}$
$= \left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] \times \left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$
$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$
$= \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i + \frac{\sqrt{3}}{2\sqrt{2}}i - \frac{\sqrt{3}}{2\sqrt{2}}i^2$
$= \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)$

Equating real parts: $\cos\frac{\pi}{12} = \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$

Equating imaginary parts: $\sin\frac{\pi}{12} = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}$

$= \operatorname{cis}\frac{11\pi}{12}$

$= \operatorname{cis}\left(\frac{3\pi}{12}+\frac{8\pi}{12}\right)$

$= \operatorname{cis}\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$

$= \operatorname{cis}\frac{\pi}{4}\times\operatorname{cis}\frac{2\pi}{3}$ $\quad\{\operatorname{cis}(\theta+\phi)=\operatorname{cis}\theta\times\operatorname{cis}\phi\}$

$= \left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\times\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$= \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$= -\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}i-\frac{1}{2\sqrt{2}}i+\frac{\sqrt{3}}{2\sqrt{2}}i^2$

$= \left(-\frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\right)$

Equating real parts: $\quad\cos\frac{11\pi}{12}=\left(\frac{-1-\sqrt{3}}{2\sqrt{2}}\right)\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{-\sqrt{2}-\sqrt{6}}{4}$

Equating imaginary parts: $\quad\sin\frac{11\pi}{12}=\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}$

5 **a** Let $z=R\operatorname{cis}\theta$ and $w=r\operatorname{cis}\phi,\quad w\neq 0$

$$\frac{z}{w}=\frac{R\operatorname{cis}\theta}{r\operatorname{cis}\phi}=\frac{R}{r}\operatorname{cis}(\theta-\phi)$$

$$\therefore\ \left|\frac{z}{w}\right|=\frac{R}{r}=\frac{|z|}{|w|}$$

and $\arg\left(\frac{z}{w}\right)=\theta-\phi=\arg z-\arg w$ if $w\neq 0$

b Let $z=r\operatorname{cis}\theta$ and $n\in\mathbb{Z}^+$

$$\therefore\ z^n=\underbrace{r\operatorname{cis}\theta\times r\operatorname{cis}\theta\times\ldots.\times r\operatorname{cis}\theta}_{n\text{ times}}$$

$$=r^n\operatorname{cis}(\underbrace{\theta+\theta+\ldots.+\theta}_{n\text{ times}})$$

$$=r^n\operatorname{cis}(n\theta)$$

$$\therefore\ |z^n|=r^n=|z|^n$$

and $\arg(z^n)=n\theta=n\arg z$ for any $n\in\mathbb{Z}^+$.

6 $z=2\operatorname{cis}\frac{\pi}{5},\quad w=5\operatorname{cis}\left(-\frac{2\pi}{7}\right)$

a $|z|=2$ and $\arg z=\frac{\pi}{5}$

b $|w|=5$ and $\arg w=-\frac{2\pi}{7}$

c $|zw|=|z||w|$ and $\arg zw=\arg z+\arg w$

$=2\times 5$ $\quad\quad = \frac{\pi}{5}+(-\frac{2\pi}{7})$

$=10$ $\quad\quad = -\frac{3\pi}{35}$

d Using **5 a**, $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$ and $\arg\left(\frac{z}{w}\right)=\arg z-\arg w$

$=\frac{2}{5}$ $\quad\quad =\frac{\pi}{5}-(-\frac{2\pi}{7})$

$=\frac{17\pi}{35}$

e Using **5 b**, $\left|z^4\right| = |z|^4 = 2^4 = 16$ and $\arg(z^4) = 4\arg z = 4 \times \frac{\pi}{5} = \frac{4\pi}{5}$

f Using **5 b**, $\left|w^3\right| = |w|^3 = 5^3 = 125$ and $\arg(w^3) = 3\arg w = 3 \times (-\frac{2\pi}{7}) = -\frac{6\pi}{7}$

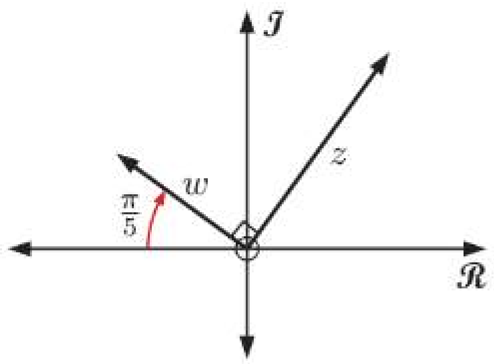
7 $|z| = 3$ and $|w| = 2$

a $\arg z = \pi - \frac{\pi}{5} - \frac{\pi}{2} = \frac{3\pi}{10}$

$\therefore\ z = 3\operatorname{cis}\frac{3\pi}{10}$

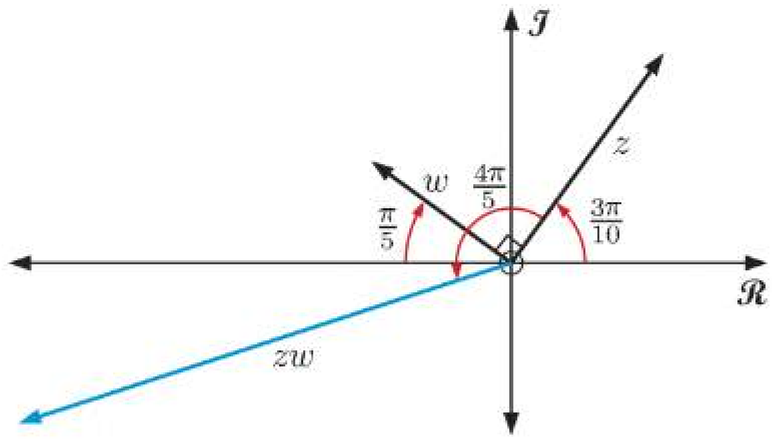
$\arg w = \pi - \frac{\pi}{5} = \frac{4\pi}{5}$

$\therefore\ w = 2\operatorname{cis}\frac{4\pi}{5}$



b When z is multiplied by w, it is dilated with scale factor 2, then rotated anticlockwise through $\frac{4\pi}{5}$ about the origin.

c



8 **a**

$$\begin{aligned} z &= 3\operatorname{cis}\theta \\ \therefore\ -z &= (-1) \times 3\operatorname{cis}\theta \\ &= \operatorname{cis}\pi \times 3\operatorname{cis}\theta \quad \text{or} \quad \operatorname{cis}(-\pi) \times 3\operatorname{cis}\theta \\ &= 3\operatorname{cis}(\theta \pm \pi) \end{aligned}$$

$\therefore\ |-z| = 3$

$\therefore\ \arg(-z) = \theta - \pi \quad \{\text{since } \theta \text{ is acute, } -\pi < \theta - \pi < \pi\}$

b

$$\begin{aligned} z^2 &= 3\operatorname{cis}\theta \times 3\operatorname{cis}\theta \\ &= 9\operatorname{cis}2\theta \end{aligned}$$

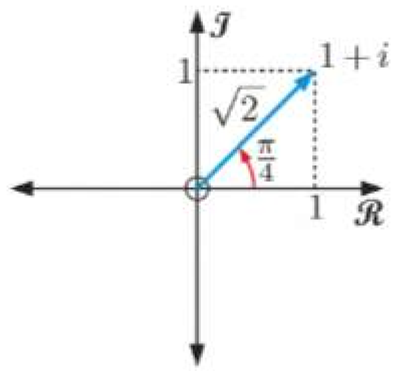
$\therefore\ \left|z^2\right| = 9$ and $\arg(z^2) = 2\theta \quad \{\text{since } \theta \text{ is acute, } -\pi < 2\theta \leqslant \pi\}$

c

$$\begin{aligned} iz &= i \times 3\operatorname{cis}\theta \\ &= \operatorname{cis}\tfrac{\pi}{2} \times 3\operatorname{cis}\theta \\ &= 3\operatorname{cis}\left(\theta + \tfrac{\pi}{2}\right) \end{aligned}$$

$\therefore\ |iz| = 3$ and $\arg(iz) = \theta + \frac{\pi}{2} \quad \{\text{since } \theta \text{ is acute, } -\pi < \theta + \frac{\pi}{2} \leqslant \pi\}$

d

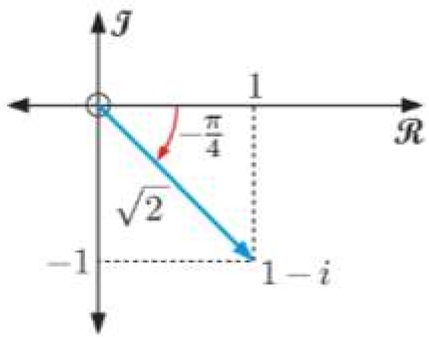


$$1+i=\sqrt{2}\operatorname{cis}\tfrac{\pi}{4}$$
$$\therefore\ (1+i)z=\sqrt{2}\operatorname{cis}\tfrac{\pi}{4}\times 3\operatorname{cis}\theta$$
$$=3\sqrt{2}\operatorname{cis}\left(\theta+\tfrac{\pi}{4}\right)$$
$$\therefore\ |(1+i)z|=3\sqrt{2}\quad\text{and}\quad\arg[(1+i)z]=\theta+\tfrac{\pi}{4}$$
$$\{\text{since } \theta \text{ is acute, } -\pi<\theta+\tfrac{\pi}{2}\leqslant\pi\}$$

e

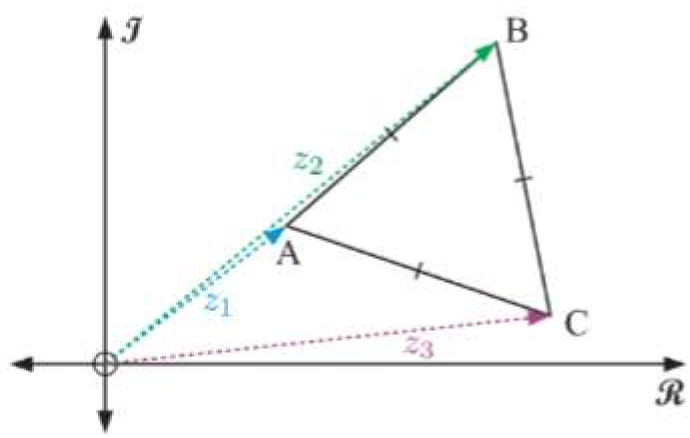
$$\frac{z}{i}=\frac{3\operatorname{cis}\theta}{\operatorname{cis}\frac{\pi}{2}}$$
$$=3\operatorname{cis}\left(\theta-\tfrac{\pi}{2}\right)$$
$$\therefore\ \left|\frac{z}{i}\right|=3\quad\text{and}\quad\arg\left(\frac{z}{i}\right)=\theta-\tfrac{\pi}{2}\qquad\{\text{since } \theta \text{ is acute, } -\pi<\theta-\tfrac{\pi}{2}\leqslant\pi\}$$

f



$$1-i=\sqrt{2}\operatorname{cis}\left(-\tfrac{\pi}{4}\right)$$
$$\therefore\ \frac{z}{1-i}=\frac{3\operatorname{cis}\theta}{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}$$
$$=\tfrac{3}{\sqrt{2}}\operatorname{cis}\left(\theta+\tfrac{\pi}{4}\right)$$
$$\therefore\ \left|\frac{z}{1-i}\right|=\tfrac{3}{\sqrt{2}}$$
$$\text{and}\quad\arg\left(\frac{z}{1-i}\right)=\theta+\tfrac{\pi}{4}$$
$$\{\text{since } \theta \text{ is acute, } -\pi<\theta+\tfrac{\pi}{4}\leqslant\pi\}$$

9 **a**



$$z_2-z_1=\overrightarrow{\text{AB}}$$
$$z_3-z_2=\overrightarrow{\text{BC}}$$

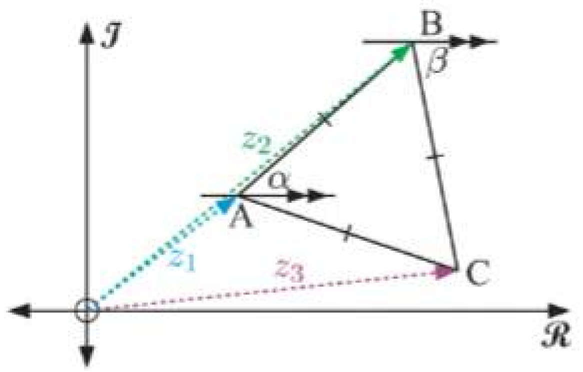
b $\left|\dfrac{z_2 - z_1}{z_3 - z_2}\right| = \dfrac{|z_2 - z_1|}{|z_3 - z_2|}$

$= \dfrac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|}$

But $\triangle$ABC is equilateral

$\therefore\ |\overrightarrow{AB}| = |\overrightarrow{BC}|$

$\therefore\ \left|\dfrac{z_2 - z_1}{z_3 - z_2}\right| = 1$



Let $\arg(z_2 - z_1) = \alpha$

and $\arg(z_3 - z_2) = -\beta$ as shown

$\therefore\ \arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$

$= \alpha - (-\beta)$

$= \alpha + \beta$

But $\widehat{ABC} = \frac{\pi}{3}$ since the triangle is equilateral

$\therefore\ \alpha + \beta + \frac{\pi}{3} = \pi$ {co-interior angles}

$\therefore\ \alpha + \beta = \frac{2\pi}{3}$

$\therefore\ \arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right) = \frac{2\pi}{3}$

c From **b**, $\dfrac{z_2 - z_1}{z_3 - z_2} = 1 \operatorname{cis} \frac{2\pi}{3}$

$\therefore\ \left(\dfrac{z_2 - z_1}{z_3 - z_2}\right)^3 = \left(\operatorname{cis} \frac{2\pi}{3}\right)^3$

$= \operatorname{cis} \frac{2\pi}{3} \times \operatorname{cis} \frac{2\pi}{3} \times \operatorname{cis} \frac{2\pi}{3}$

$= \operatorname{cis}\left(\frac{2\pi}{3} + \frac{2\pi}{3} + \frac{2\pi}{3}\right)$

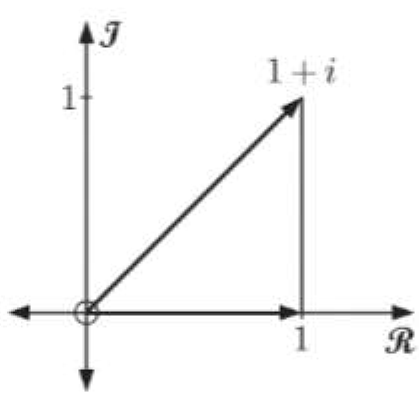
$= \operatorname{cis} 2\pi$

$= 1$

INVESTIGATION — EXPONENTIAL FORM

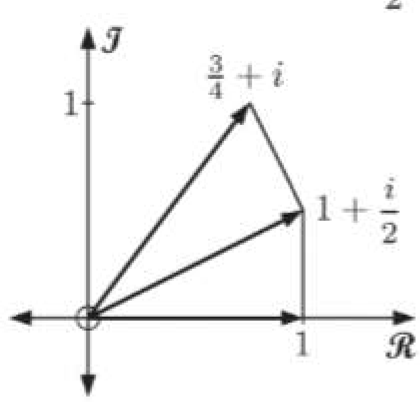
1 **a** **i** Sequence: $1,\ \left(1 + \frac{i}{1}\right)$

$= 1,\ 1 + i$



ii Sequence: $1,\ \left(1 + \frac{i}{2}\right),\ \left(1 + \frac{i}{2}\right)^2$

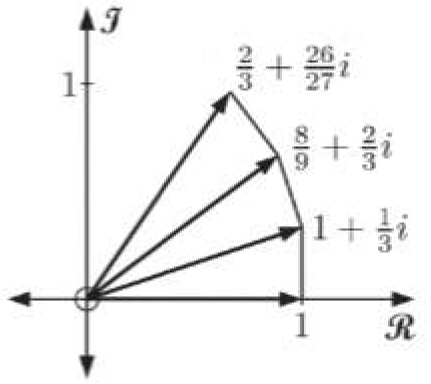
$= 1,\ 1 + \frac{i}{2},\ \frac{3}{4} + i$



For the following parts, we have calculated the real and imaginary parts using technology.

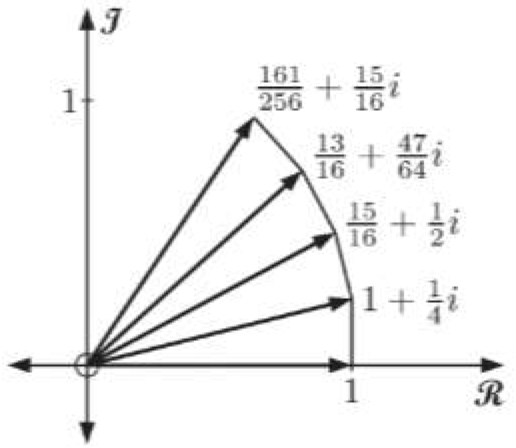
iii Sequence: $1,\ \left(1+\frac{i}{3}\right),\ \left(1+\frac{i}{3}\right)^2,\ \left(1+\frac{i}{3}\right)^3$

$= 1,\ 1+\frac{1}{3}i,\ \frac{8}{9}+\frac{2}{3}i,\ \frac{2}{3}+\frac{26}{27}i$



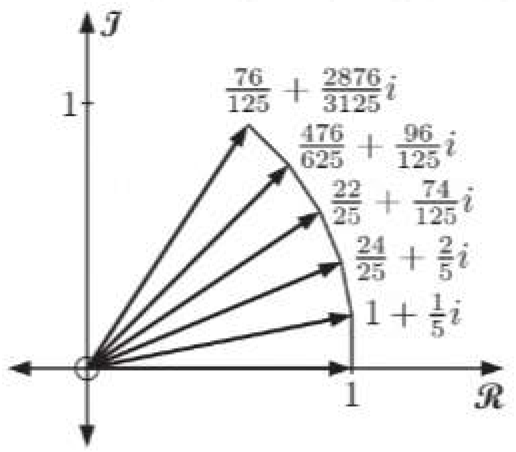
iv Sequence: $1,\ \left(1+\frac{i}{4}\right),\ \left(1+\frac{i}{4}\right)^2,\ \left(1+\frac{i}{4}\right)^3,\ \left(1+\frac{i}{4}\right)^4$

$= 1,\ 1+\frac{1}{4}i,\ \frac{15}{16}+\frac{1}{2}i,\ \frac{13}{16}+\frac{47}{64}i,\ \frac{161}{256}+\frac{15}{16}i$



v Sequence: $1,\ \left(1+\frac{i}{5}\right),\ \left(1+\frac{i}{5}\right)^2,\ \left(1+\frac{i}{5}\right)^3,\ \left(1+\frac{i}{5}\right)^4,\ \left(1+\frac{i}{5}\right)^5$

$= 1,\ 1+\frac{1}{5}i,\ \frac{24}{25}+\frac{2}{5}i,\ \frac{22}{25}+\frac{74}{125}i,\ \frac{476}{625}+\frac{96}{125}i,\ \frac{76}{125}+\frac{2876}{3125}i$



b **i** As n gets large, the line segments joining successive complex numbers begin to form an arc of the unit circle.

ii $\cos 1^c \approx 0.540\,30$ and $\sin 1^c \approx 0.841\,47$ {using technology}

iii From the software, $\left(1+\frac{i}{500}\right)^{500} \approx 0.5408 + 0.8423i$

iv As $n \to \infty$, $\left(1+\frac{i}{n}\right)^n \to (\cos 1^c) + (\sin 1^c)i = e^i$ {Euler}

2 **a** **i** As n gets large, the line segments joining successive complex numbers begin to form an arc of the unit circle, about twice as large as the arc in question **1**.

ii $\cos 2^c \approx -0.416\,15$ and $\sin 2^c \approx 0.909\,30$ {using technology}

iii As $n \to \infty$, $\left(1+\frac{2i}{n}\right)^n \to (\cos 2^c) + (\sin 2^c)i = e^{2i}$ {Euler}

b **i** As $n \to \infty$, $\left(1+\dfrac{i\frac{\pi}{2}}{n}\right)^n \to 0 + 1 \times i = e^{i\frac{\pi}{2}}$ {Euler}

ii Multiplying 1 by $e^{i\frac{\pi}{2}}$ rotates it anticlockwise about the origin by $\frac{\pi}{2}$, similarly to how multiplying z by $\operatorname{cis}\frac{\pi}{2}$ rotates it anticlockwise about the origin by $\frac{\pi}{2}$.

3 **a** The effect of *real* compound growth is to increase the size or modulus of a quantity. So, when we multiply a complex number z by e^r, the vector representing z is *enlarged* with scale factor e^r. This means the *modulus* of z is multiplied by e^r.

b The effect of *imaginary* compound growth is to rotate the quantity in the complex plane. So, when we multiply a complex number z by $\operatorname{cis}\theta = e^{i\theta}$, z is *rotated* anticlockwise through angle θ about the origin. This means the *argument* of z is increased by θ.

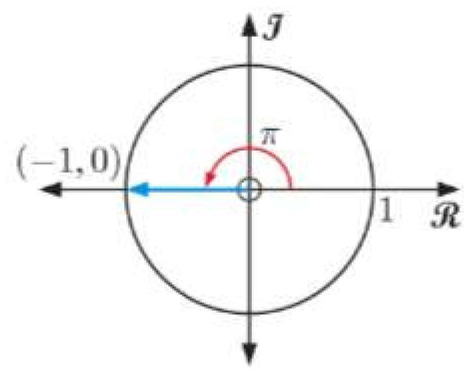
4 When z is multiplied by w, the vector representing z is enlarged with scale factor $|w|$, and is rotated anticlockwise through angle ϕ about the origin.

EXERCISE 11I

Note: We assume that all arguments are in $]-\pi, \pi]$.

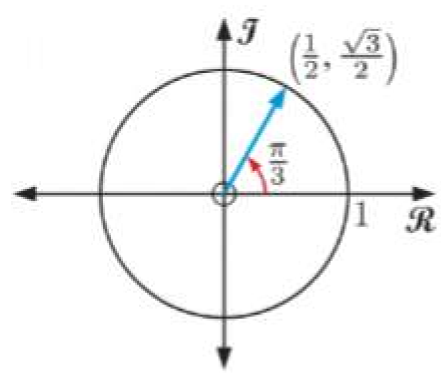
1 **a**
$$\begin{aligned}&e^{i\pi}\\&= \cos\pi + i\sin\pi\\&= -1\end{aligned}$$

We expand $\operatorname{cis}\pi$ using a unit circle diagram.



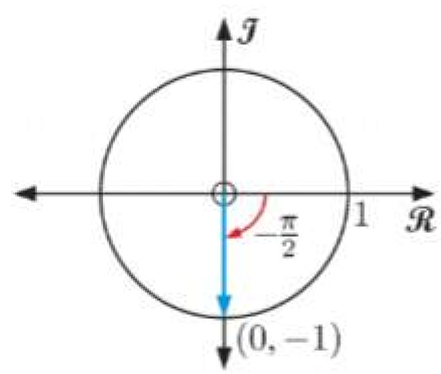
b
$$\begin{aligned}&e^{i\frac{\pi}{3}}\\&= \cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\\&= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\end{aligned}$$

We expand $\operatorname{cis}\frac{\pi}{3}$ using a unit circle diagram.



c
$$\begin{aligned}&e^{-i\frac{\pi}{2}}\\&= \cos\left(-\tfrac{\pi}{2}\right) + i\sin\left(-\tfrac{\pi}{2}\right)\\&= -i\end{aligned}$$

We expand $\operatorname{cis}\left(-\frac{\pi}{2}\right)$ using a unit circle diagram.



2 **a** $|-1+i| = \sqrt{2}, \quad \arg(-1+i) = \frac{3\pi}{4}$

$\therefore \quad -1+i = \sqrt{2}\operatorname{cis}\frac{3\pi}{4}$ (polar form)

$= \sqrt{2}e^{i\frac{3\pi}{4}}$ (exponential form)

b $3\operatorname{cis}\left(-\frac{\pi}{6}\right) = 3\cos\left(-\frac{\pi}{6}\right) + 3i\sin\left(-\frac{\pi}{6}\right)$

$= \frac{3\sqrt{3}}{2} - \frac{3}{2}i$ (Cartesian form)

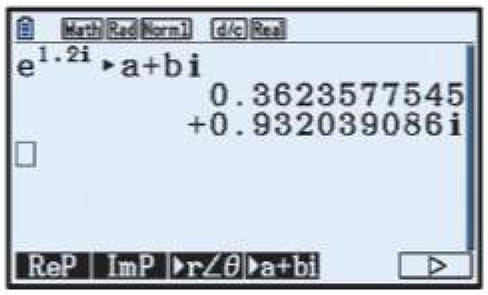
$= 3e^{-i\frac{\pi}{6}}$ (exponential form)

c $2e^{i\frac{2\pi}{3}} = 2\cos\frac{2\pi}{3} + 2i\sin\frac{2\pi}{3}$

$= -1 + i\sqrt{3}$ (Cartesian form)

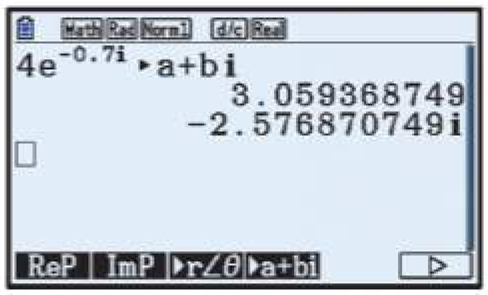
$= 2\operatorname{cis}\frac{2\pi}{3}$ (polar form)

3 **a**



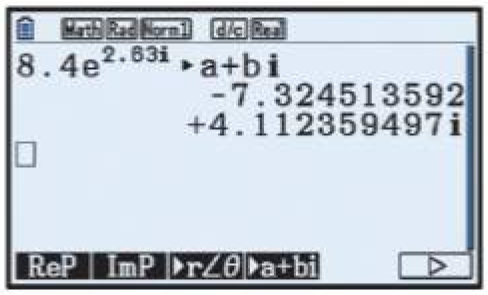
$e^{1.2i} \approx 0.362 + 0.932i$

b



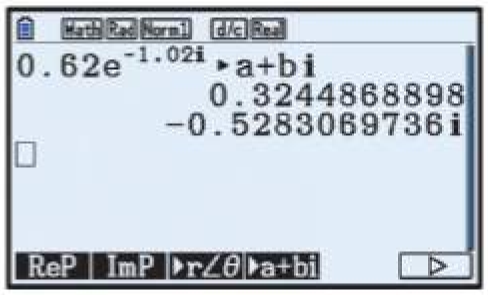
$4e^{-0.7i} \approx 3.06 - 2.58i$

c



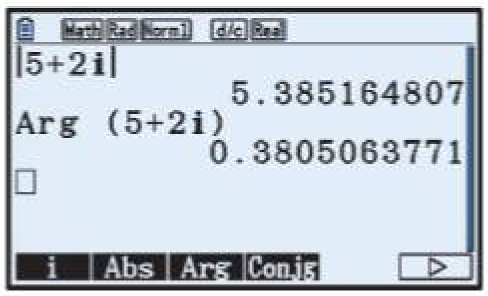
$8.4e^{2.63i} \approx -7.32 + 4.11i$

d



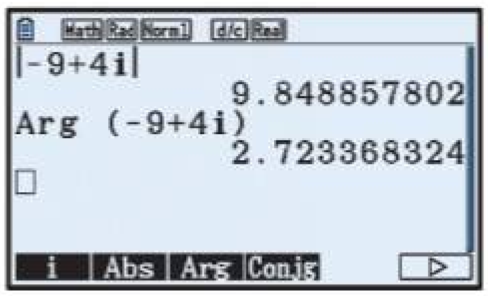
$0.62e^{-1.02i} \approx 0.324 - 0.528i$

4 **a**



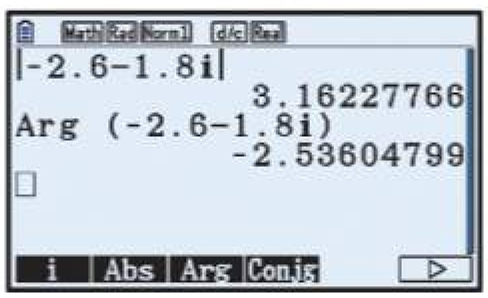
$5 + 2i \approx 5.39e^{0.381i}$

b



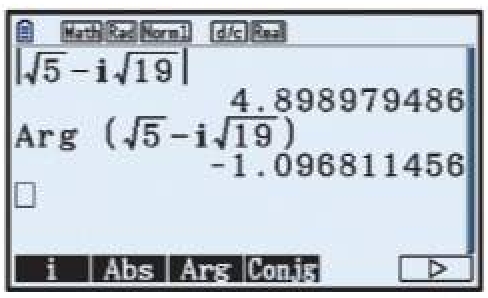
$-9 + 4i \approx 9.85e^{2.72i}$

c



$-2.6 - 1.8i \approx 3.16e^{-2.54i}$

d



$\sqrt{5} - i\sqrt{19} \approx 4.90e^{-1.10i}$

5 **a** $\operatorname{cis}\theta\operatorname{cis}\phi = e^{i\theta}e^{i\phi}$

$= e^{i(\theta+\phi)}$

$= \operatorname{cis}(\theta+\phi)$

b $\dfrac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \dfrac{e^{i\theta}}{e^{i\phi}}$

$= e^{i(\theta-\phi)}$

$= \operatorname{cis}(\theta-\phi)$

c $\begin{aligned}(\text{cis}\,\theta)^n &= (e^{i\theta})^n \\ &= e^{in\theta} \\ &= \text{cis}\,n\theta\end{aligned}$

d $\begin{aligned}\text{cis}(\theta+2k\pi) &= e^{i(\theta+2k\pi)} \\ &= e^{i\theta+i2k\pi} \\ &= e^{i\theta}e^{i2k\pi} \\ &= \text{cis}\,\theta\times(e^{i\pi})^{2k} \\ &= \text{cis}\,\theta\times(-1)^{2k} && \{\text{using } \mathbf{1\ a}\} \\ &= \text{cis}\,\theta && \{k\in\mathbb{Z}\}\end{aligned}$

6 $z=\text{cis}\,\theta$

a $\begin{aligned}z^3 &= (\text{cis}\,\theta)^3 \\ &= (e^{i\theta})^3 \\ &= e^{3i\theta} \\ &= \text{cis}\,3\theta\end{aligned}$

$\therefore\ \arg(z^3)=3\theta$

b $\begin{aligned}\sqrt{z} &= (\text{cis}\,\theta)^{\frac{1}{2}} \\ &= (e^{i\theta})^{\frac{1}{2}} \\ &= e^{\frac{i\theta}{2}} \\ &= \text{cis}\,\tfrac{\theta}{2}\end{aligned}$

$\therefore\ \arg(\sqrt{z})=\frac{\theta}{2}$

c $\begin{aligned}iz^4 &= \text{cis}\,\tfrac{\pi}{2}\times(\text{cis}\,\theta)^4 \\ &= e^{i\frac{\pi}{2}}\times(e^{i\theta})^4 \\ &= e^{i\frac{\pi}{2}}\times e^{4i\theta} \\ &= e^{i(4\theta+\frac{\pi}{2})} \\ &= \text{cis}\left(4\theta+\tfrac{\pi}{2}\right)\end{aligned}$

$\therefore\ \arg(iz^4)=4\theta+\frac{\pi}{2}$

d $\begin{aligned}-iz^{\frac{2}{5}} &= \text{cis}\left(-\tfrac{\pi}{2}\right)\times(\text{cis}\,\theta)^{\frac{2}{5}} \\ &= e^{-i\frac{\pi}{2}}\times(e^{i\theta})^{\frac{2}{5}} \\ &= e^{-i\frac{\pi}{2}}\times e^{i\frac{2\theta}{5}} \\ &= e^{i(\frac{2\theta}{5}-\frac{\pi}{2})} \\ &= \text{cis}\left(\tfrac{2}{5}\theta-\tfrac{\pi}{2}\right)\end{aligned}$

$\therefore\ \arg(-iz^{\frac{2}{5}})=\frac{2}{5}\theta-\frac{\pi}{2}$

7 $z=2e^{i\theta},\quad w=3e^{i\phi}$

a $\begin{aligned}\sqrt{w} &= (3e^{i\phi})^{\frac{1}{2}} \\ &= \sqrt{3}e^{i\frac{\phi}{2}} \\ &= \sqrt{3}\,\text{cis}\,\tfrac{\phi}{2}\end{aligned}$

$\therefore\ |\sqrt{w}|=\sqrt{3},\quad \arg(\sqrt{w})=\frac{\phi}{2}$

b $\begin{aligned}z^4w &= (2e^{i\theta})^4\times 3e^{i\phi} \\ &= 16e^{4i\theta}\times 3e^{i\phi} \\ &= 48e^{i(4\theta+\phi)} \\ &= 48\,\text{cis}(4\theta+\phi)\end{aligned}$

$\therefore\ |z^4w|=48,\quad \arg(z^4w)=4\theta+\phi$

c $\begin{aligned}\frac{z}{w^2} &= 2e^{i\theta}\times(3e^{i\phi})^{-2} \\ &= 2e^{i\theta}\times\tfrac{1}{9}e^{-2i\phi} \\ &= \tfrac{2}{9}e^{i(\theta-2\phi)} \\ &= \tfrac{2}{9}\,\text{cis}(\theta-2\phi)\end{aligned}$

$\therefore\ \left|\frac{z}{w^2}\right|=\frac{2}{9},\quad \arg\left(\frac{z}{w^2}\right)=\theta-2\phi$

d $\begin{aligned}w^3\times\sqrt{z} &= (3e^{i\phi})^3\times(2e^{i\theta})^{\frac{1}{2}} \\ &= 27e^{3i\phi}\times\sqrt{2}e^{i\frac{\theta}{2}} \\ &= 27\sqrt{2}\times e^{i(3\phi+\frac{\theta}{2})} \\ &= 27\sqrt{2}\,\text{cis}(3\phi+\tfrac{\theta}{2})\end{aligned}$

$\therefore\ |w^3\times\sqrt{z}|=27\sqrt{2},$

$\arg(w^3\times\sqrt{z})=3\phi+\frac{\theta}{2}$

8 **a** $e^{i} = \cos(1) + i\sin(1) \quad \{\theta = 1\}$
$\approx 0.540 + 0.841i$

b $e^{-i} = \cos(-1) + i\sin(-1) \quad \{\theta = -1\}$
$\approx 0.540 - 0.841i$

c $3^{i} = (e^{\ln 3})^{i}$
$= e^{i\ln 3}$
$= \cos(\ln 3) + i\sin(\ln 3)$
$\approx 0.455 + 0.891i$

d $i = 0 + 1i$
$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
$= e^{i\frac{\pi}{2}}$
$\therefore\ i^{i} = (e^{i\frac{\pi}{2}})^{i}$
$= e^{i^2\frac{\pi}{2}}$
$= e^{-\frac{\pi}{2}}$
≈ 0.208

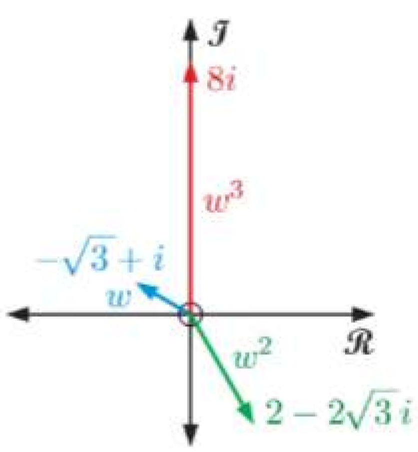
9 **a**
$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2}$$
$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \qquad \{\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta\}$$
$$= \frac{2\cos\theta}{2} = \cos\theta \quad \text{as required}$$

b
$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta - (\cos(-\theta) + \sin(-\theta))}{2}$$
$$= \frac{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}{2i} \qquad \{\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta\}$$
$$= \frac{2i\sin\theta}{2i} = \sin\theta \quad \text{as required}$$

10 **a** $w = 2e^{\frac{5\pi}{6}i}$
$= 2\operatorname{cis}\frac{5\pi}{6}$
$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= -\sqrt{3} + i$

$w^2 = \left(2e^{\frac{5\pi}{6}i}\right)^2$
$= 4e^{\frac{5\pi}{3}i}$
$= 4\operatorname{cis}\frac{5\pi}{3}$
$= 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$
$= 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= 2 - 2\sqrt{3}i$

$w^3 = \left(2e^{\frac{5\pi}{6}i}\right)^3$
$= 8e^{\frac{5\pi}{2}i}$
$= 8\operatorname{cis}\frac{5\pi}{2}$
$= 8\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$
$= 8(0 + i)$
$= 8i$



b $w^k = \left(2e^{\frac{5\pi}{6}i}\right)^k$

$= 2^k e^{\frac{5\pi k}{6}i}$

$= 2^k \operatorname{cis} \frac{5\pi k}{6}$

$= 2^k \left(\cos \frac{5\pi k}{6} + i \sin \frac{5\pi k}{6}\right)$

Now w^k is purely imaginary if $\cos \frac{5\pi k}{6} = 0$

$$\therefore \frac{5\pi k}{6} = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

$$\therefore k = \frac{3}{5} + \frac{6n}{5}, \quad n \in \mathbb{Z}$$

$$\therefore k = \frac{3(1+2n)}{5}, \quad n \in \mathbb{Z}$$

So, the smallest integer $k > 3$ such that w^k is purely imaginary is $k = 9$ when $n = 7$.

11

$$Ae^{i\frac{\pi}{7}} + \sqrt{2}Be^{-i\frac{\pi}{4}} = e^{i\frac{\pi}{5}}$$

$$\therefore A \operatorname{cis} \tfrac{\pi}{7} + \sqrt{2}B \operatorname{cis}\left(-\tfrac{\pi}{4}\right) = \operatorname{cis} \tfrac{\pi}{5}$$

$$\therefore A\left(\cos \tfrac{\pi}{7} + i \sin \tfrac{\pi}{7}\right) + \sqrt{2}B\left(\cos(-\tfrac{\pi}{4}) + i \sin(-\tfrac{\pi}{4})\right) = \cos \tfrac{\pi}{5} + i \sin \tfrac{\pi}{5}$$

$$\therefore A \cos \tfrac{\pi}{7} + Ai \sin \tfrac{\pi}{7} + \sqrt{2}B\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) = \cos \tfrac{\pi}{5} + i \sin \tfrac{\pi}{5}$$

$$\therefore A \cos \tfrac{\pi}{7} + Ai \sin \tfrac{\pi}{7} + B - Bi = \cos \tfrac{\pi}{5} + i \sin \tfrac{\pi}{5}$$

$$\therefore (A \cos \tfrac{\pi}{7} + B) + i(A \sin \tfrac{\pi}{7} - B) = \cos \tfrac{\pi}{5} + i \sin \tfrac{\pi}{5}$$

Equating real and imaginary parts, we get

$A \cos \frac{\pi}{7} + B = \cos \frac{\pi}{5}$ .... (1)

and $A \sin \frac{\pi}{7} - B = \sin \frac{\pi}{5}$ .... (2)

Adding (1) and (2) gives $A \cos \frac{\pi}{7} + A \sin \frac{\pi}{7} = \cos \frac{\pi}{5} + \sin \frac{\pi}{5}$

$$\therefore A\left(\cos \tfrac{\pi}{7} + \sin \tfrac{\pi}{7}\right) = \cos \tfrac{\pi}{5} + \sin \tfrac{\pi}{5}$$

$$\therefore A = \frac{\cos \frac{\pi}{5} + \sin \frac{\pi}{5}}{\cos \frac{\pi}{7} + \sin \frac{\pi}{7}} \quad .... (3)$$

From (2), $B = A \sin \frac{\pi}{7} - \sin \frac{\pi}{5}$

$$= \left(\frac{\cos \frac{\pi}{5} + \sin \frac{\pi}{5}}{\cos \frac{\pi}{7} + \sin \frac{\pi}{7}}\right) \sin \tfrac{\pi}{7} - \sin \tfrac{\pi}{5} \qquad \{\text{using (3)}\}$$

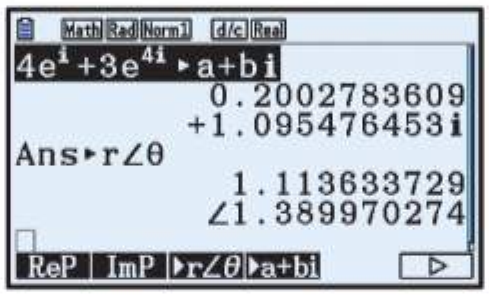
$$= \frac{\sin \frac{\pi}{7} \cos \frac{\pi}{5} + \sin \frac{\pi}{7} \sin \frac{\pi}{5} - \cos \frac{\pi}{7} \sin \frac{\pi}{5} - \sin \frac{\pi}{7} \sin \frac{\pi}{5}}{\cos \frac{\pi}{7} + \sin \frac{\pi}{7}}$$

$$= \frac{\sin \frac{\pi}{7} \cos \frac{\pi}{5} - \cos \frac{\pi}{7} \sin \frac{\pi}{5}}{\cos \frac{\pi}{7} + \sin \frac{\pi}{7}}$$

EXERCISE 11J

1 **a** $4\cos(3t+1) + 3\cos(3t+4) = \mathcal{Re}\,(4\,\text{cis}(3t+1) + 3\,\text{cis}(3t+4))$
$$= \mathcal{Re}\left(4e^{(3t+1)i} + 3e^{(3t+4)i}\right)$$

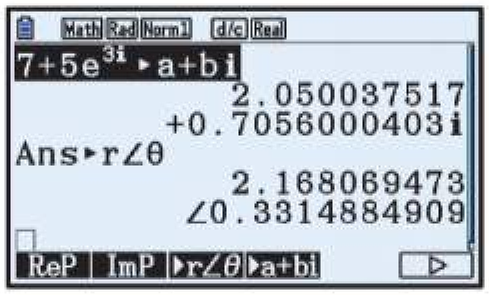
Now $\quad 4e^{(3t+1)i} + 3e^{(3t+4)i}$
$$= e^{3ti}(4e^{i} + 3e^{4i})$$
$$\approx e^{3ti}(0.200 + 1.10i)$$
$$\approx e^{3ti}(1.11e^{1.39i})$$
$$\approx 1.11e^{(3t+1.39)i}$$



So, $\quad 4\cos(3t+1) + 3\cos(3t+4) \approx \mathcal{Re}\left(1.11e^{(3t+1.39)i}\right)$
$$\approx 1.11\cos(3t+1.39)$$

b $7\cos 2t + 5\cos(2t+3) = \mathcal{Re}\,(7\,\text{cis}\,2t + 5\,\text{cis}(2t+3))$
$$= \mathcal{Re}\left(7e^{2ti} + 5e^{(2t+3)i}\right)$$

Now $\quad 7e^{2ti} + 5e^{(2t+3)i}$
$$= e^{2ti}(7 + 5e^{3i})$$
$$\approx e^{2ti}(2.05 + 0.706i)$$
$$\approx e^{2ti}(2.17e^{0.331i})$$
$$\approx 2.17e^{(2t+0.331)i}$$

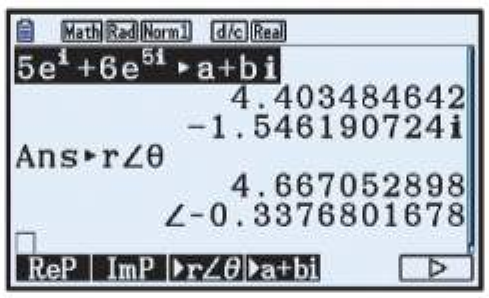


So, $\quad 7\cos 2t + 5\cos(2t+3) \approx \mathcal{Re}\left(2.17e^{(2t+0.331)i}\right)$
$$\approx 2.17\cos(2t+0.331)$$

2 **a** $5e^{(t+1)i} + 6e^{(t+5)i} = 5\,\text{cis}(t+1) + 6\,\text{cis}(t+5)$
$$= 5(\cos(t+1) + i\sin(t+1)) + 6(\cos(t+5) + i\sin(t+5))$$
$$= 5\cos(t+1) + 6\cos(t+5) + i(5\sin(t+1) + 6\sin(t+5))$$

$\therefore \quad \mathcal{Im}\left(5e^{(t+1)i} + 6e^{(t+5)i}\right) = 5\sin(t+1) + 6\sin(t+5)$

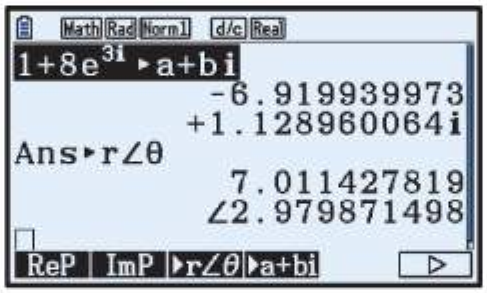
b $\quad 5e^{(t+1)i} + 6e^{(t+5)i}$
$$= e^{ti}(5e^{i} + 6e^{5i})$$
$$\approx e^{ti}(4.40 - 1.55i)$$
$$\approx e^{ti}(4.67e^{-0.338i})$$
$$\approx 4.67e^{(t-0.338)i}$$



So, $\quad 5\sin(t+1) + 6\sin(t+5) = \mathcal{Im}\left(5e^{(t+1)i} + 6e^{(t+5)i}\right) \qquad$ {from **a**}
$$\approx \mathcal{Im}\left(4.67e^{(t-0.338)i}\right)$$
$$\approx 4.67\sin(t-0.338)$$

3 **a** $\sin 4t + 8\sin(4t+3) = \mathcal{I}m\,(\operatorname{cis} 4t + 8\operatorname{cis}(4t+3))$
$= \mathcal{I}m\,(e^{4ti} + 8e^{(4t+3)i})$

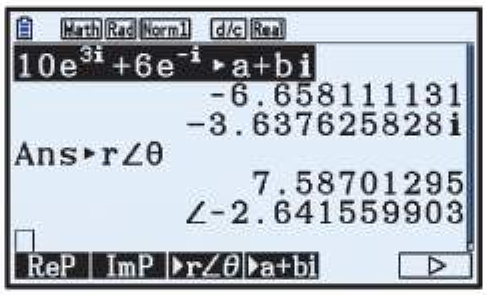
Now $\quad e^{4ti} + 8e^{(4t+3)i}$
$= e^{4ti}(1 + 8e^{3i})$
$\approx e^{4ti}(-6.92 + 1.13i)$
$\approx e^{4ti}(7.01e^{2.98i})$
$\approx 7.01e^{(4t+2.98)i}$



So, $\quad \sin 4t + 8\sin(4t+3) \approx \mathcal{I}m\,(7.01e^{(4t+2.98)i})$
$\approx 7.01\sin(4t + 2.98)$

b $10\sin(2t+3) + 6\sin(2t-1) = \mathcal{I}m\,(10\operatorname{cis}(2t+3) + 6\operatorname{cis}(2t-1))$
$= \mathcal{I}m\,(10e^{(2t+3)i} + 6e^{(2t-1)i})$

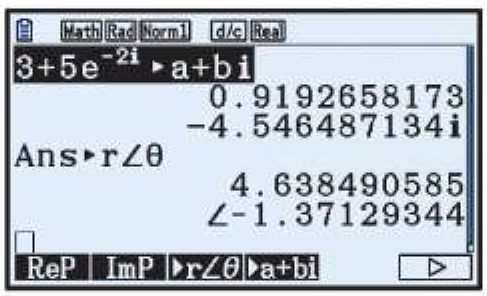
Now $\quad 10e^{(2t+3)i} + 6e^{(2t-1)i}$
$= e^{2ti}(10e^{3i} + 6e^{-i})$
$\approx e^{2ti}(-6.66 - 3.64i)$
$\approx e^{2ti}(7.59e^{-2.64i})$
$\approx 7.59e^{(2t-2.64)i}$



So, $\quad 10\sin(2t+3) + 6\sin(2t-1) \approx \mathcal{I}m\,(7.59e^{(2t-2.64)i})$
$\approx 7.59\sin(2t - 2.64)$

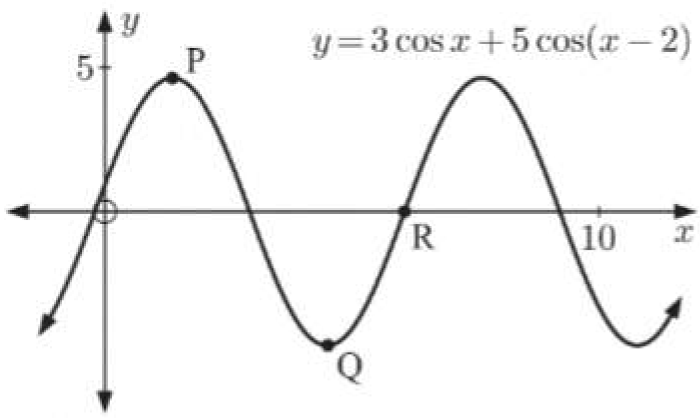
4 **a** $3\cos x + 5\cos(x-2) = \mathcal{R}e\,(3\operatorname{cis} x + 5\operatorname{cis}(x-2))$
$= \mathcal{R}e\,(3e^{xi} + 5e^{(x-2)i})$

Now $\quad 3e^{xi} + 5e^{(x-2)i}$
$= e^{xi}(3 + 5e^{-2i})$
$\approx e^{xi}(0.919 - 4.55i)$
$\approx e^{xi}(4.64e^{-1.37i})$
$\approx 4.64e^{(x-1.37)i}$



So, $\quad 3\cos x + 5\cos(x-2) \approx \mathcal{R}e\,(4.64e^{(x-1.37)i})$
$\approx 4.64\cos(x - 1.37)$

b



$y = 3\cos x + 5\cos(x-2) \approx 4.64\cos(x - 1.37)$

The maximum value is ≈ 4.64.

For $x > 0$, the first maximum occurs when

$$\begin{aligned} \cos(x - 1.37) &\approx 1 \\ \therefore\ x - 1.37 &\approx 0 \\ \therefore\ x &\approx 1.37 \end{aligned}$$

$\therefore$ P is at $(1.37, 4.64)$.

The minimum value is ≈ -4.64.

For $x > 0$, the first minimum occurs when

$$\begin{aligned} \cos(x - 1.37) &\approx -1 \\ \therefore\ x - 1.37 &\approx \pi \\ \therefore\ x &\approx 1.37 + \pi \\ \therefore\ x &\approx 4.51 \end{aligned}$$

$\therefore$ Q is at $(4.51, -4.64)$.

The second positive x-intercept occurs when

$$\begin{aligned} \cos(x - 1.37) &\approx 0 \\ \therefore\ x - 1.37 &\approx \tfrac{3\pi}{2} \\ \therefore\ x &\approx 1.37 + \tfrac{3\pi}{2} \\ \therefore\ x &\approx 6.08 \end{aligned}$$

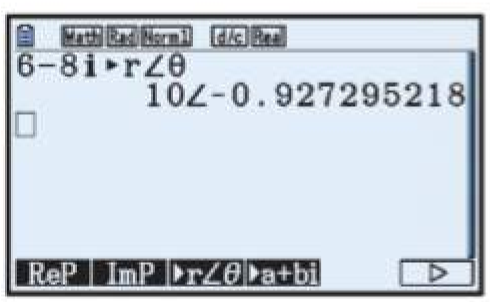
$\therefore$ R is at $(6.08, 0)$.

5

$$\begin{aligned} 6\cos x + 8\sin x &= 6\cos x + 8\cos\left(x - \tfrac{\pi}{2}\right) \\ &= \mathcal{R}e\left(6\operatorname{cis} x + 8\operatorname{cis}\left(x - \tfrac{\pi}{2}\right)\right) \\ &= \mathcal{R}e\left(6e^{xi} + 8e^{(x-\frac{\pi}{2})i}\right) \end{aligned}$$

Now $6e^{xi} + 8e^{(x-\frac{\pi}{2})i}$

$$\begin{aligned} &= e^{xi}(6 + 8e^{-\frac{\pi}{2}i}) \\ &= e^{xi}(6 - 8i) \\ &\approx e^{xi}(10e^{-0.927i}) \\ &\approx 10e^{(x-0.927)i} \end{aligned}$$



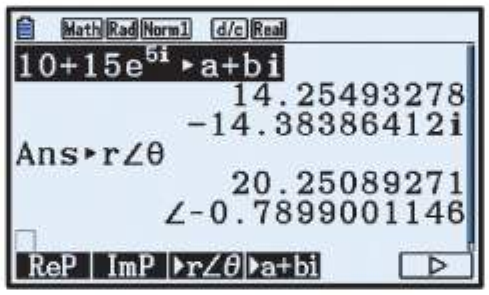
So, $6\cos x + 8\sin x \approx \mathcal{R}e\left(10e^{(x-0.927)i}\right)$

$$\approx 10\cos(x - 0.927)$$

6 $V_A(t) = 10\cos(30t)$ and $V_B(t) = 15\cos(30t+5)$

a $$\begin{aligned} V(t) &= V_A(t) + V_B(t) \\ &= 10\cos(30t) + 15\cos(30t+5) \\ &= \mathcal{R}e\,(10\operatorname{cis}(30t) + 15\operatorname{cis}(30t+5)) \\ &= \mathcal{R}e\,(10e^{30ti} + 15e^{(30t+5)i}) \end{aligned}$$

Now $$\begin{aligned} & 10e^{30ti} + 15e^{(30t+5)i} \\ &= e^{30ti}(10 + 15e^{5i}) \\ &\approx e^{30ti}(14.3 - 14.4i) \\ &\approx e^{30ti}(20.3e^{-0.790i}) \\ &\approx 20.3e^{(30t-0.790)i} \end{aligned}$$



So, $$\begin{aligned} V(t) &\approx \mathcal{R}e\,(20.3e^{(30t-0.790)i}) \\ &\approx 20.3\cos(30t - 0.790) \end{aligned}$$

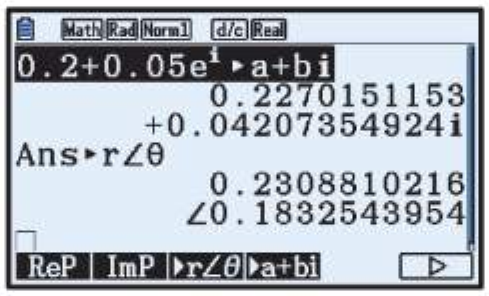
b $V(t)$ has maximum value ≈ 20.3 when $\cos(30t - 0.790) \approx 1$.

$\therefore$ the highest voltage in the circuit is about 20.3 volts.

7 $H_1(t) = 0.2\sin 2t$ and $H_2(t) = 0.05\sin(2t+1)$

a $$\begin{aligned} H(t) &= H_1(t) + H_2(t) \\ &= 0.2\sin 2t + 0.05\sin(2t+1) \\ &= \mathcal{I}m\,(0.2\operatorname{cis} 2t + 0.05\operatorname{cis}(2t+1)) \\ &= \mathcal{I}m\,(0.2e^{2ti} + 0.05e^{(2t+1)i}) \end{aligned}$$

Now $$\begin{aligned} & 0.2e^{2ti} + 0.05e^{(2t+1)i} \\ &= e^{2ti}(0.2 + 0.05e^{i}) \\ &\approx e^{2ti}(0.227 + 0.0421i) \\ &\approx e^{2ti}(0.231e^{0.183i}) \\ &\approx 0.231e^{(2t+0.183)i} \end{aligned}$$



So, $$\begin{aligned} H(t) &\approx \mathcal{I}m\,(0.231e^{(2t+0.183)i}) \\ &\approx 0.231\sin(2t + 0.183) \end{aligned}$$

b $H(t) = 0$ when $\sin(2t + 0.183) \approx 0$

Now $0 \leqslant t \leqslant 10$

$\therefore\ 0 \leqslant 2t \leqslant 20$

$\therefore\ 0.183 \leqslant 2t + 0.183 \leqslant 20.183$

So, $2t + 0.183 \approx \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$

$\therefore\ t \approx 1.48, 3.05, 4.62, 6.19, 7.76, 9.33$

$\therefore$ the wave is not affecting Gracie at all at about 1.48, 3.05, 4.62, 6.19, 7.76, and 9.33 seconds.

c **i** $H(t)$ has maximum value ≈ 0.231.

This first occurs when $\sin(2t + 0.183) \approx 1$

$$\therefore \quad 2t + 0.183 \approx \tfrac{\pi}{2}$$

$$\therefore \quad t \approx \frac{\frac{\pi}{2} - 0.183}{2}$$

$$\therefore \quad t \approx 0.694$$

$\therefore$ the wave affecting Gracie has maximum height at about 0.694 seconds.

ii $H_1(0.694) \approx 0.197$

$\therefore$ the percentage of the wave caused by the speed boat is about $\dfrac{0.197}{0.231} \times 100\% \approx 85.2\%$.

REVIEW SET 11A

1 **a**
$$\sqrt{-49} = \sqrt{49} \times \sqrt{-1} = 7i$$

b
$$\sqrt{-23} = \sqrt{23} \times \sqrt{-1} = i\sqrt{23}$$

c
$$-\sqrt{-\tfrac{1}{9}} = -\sqrt{\tfrac{1}{9}} \times \sqrt{-1} = -\tfrac{1}{3}i$$

2 **a** $x^2 - 6x + 11 = 0$

$$\therefore \quad x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 11}}{2}$$

$$\therefore \quad x = \frac{6 \pm \sqrt{-8}}{2}$$

$$\therefore \quad x = 3 \pm \sqrt{-2}$$

$$\therefore \quad x = 3 \pm i\sqrt{2}$$

b $x^2 + 3 = 2\sqrt{2}x$

$$\therefore \quad x^2 - 2\sqrt{2}x + 3 = 0$$

$$\therefore \quad x = \frac{2\sqrt{2} \pm \sqrt{8 - 4 \times 1 \times 3}}{2}$$

$$\therefore \quad x = \frac{2\sqrt{2} \pm \sqrt{-4}}{2}$$

$$\therefore \quad x = \sqrt{2} \pm \sqrt{-1}$$

$$\therefore \quad x = \sqrt{2} \pm i$$

c $4x^2 + 3x + 1 = 0$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9 - 4 \times 4 \times 1}}{2 \times 4}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{-7}}{8}$$

$$\therefore \quad x = -\tfrac{3}{8} \pm \tfrac{\sqrt{7}}{8}i$$

3 $z = 6 - 2i, \quad w = 3 + 7i$

a $z^* = 6 + 2i$

b $w^* = 3 - 7i$

$$\therefore \quad \mathcal{Im}(w^*) = -7$$

c
$$z - 3w = 6 - 2i - 3(3 + 7i) = 6 - 2i - 9 - 21i = -3 - 23i$$

$$\therefore \quad \mathcal{Re}(z - 3w) = -3$$

4 $z = 3 + i, \quad w = -2 - i$

a $$\begin{aligned} 2z - 3w &= 2(3+i) - 3(-2-i) \\ &= 6 + 2i + 6 + 3i \\ &= 12 + 5i \end{aligned}$$

b $$\begin{aligned} \frac{z^*}{w} &= \left(\frac{3-i}{-2-i}\right) \times \left(\frac{-2+i}{-2+i}\right) \\ &= \frac{-6+3i+2i-i^2}{4-i^2} \\ &= \frac{-5+5i}{5} \\ &= -1+i \end{aligned}$$

c $$\begin{aligned} \frac{w}{w^*} &= \left(\frac{-2-i}{-2+i}\right) \times \left(\frac{-2-i}{-2-i}\right) \\ &= \frac{4+4i+i^2}{4-i^2} \\ &= \frac{3+4i}{5} \\ &= \tfrac{3}{5} + \tfrac{4}{5}i \end{aligned}$$

5 $$\begin{aligned} &\frac{3}{i+\sqrt{3}} + \sqrt{3} \\ &= \left(\frac{3}{\sqrt{3}+i}\right) \times \left(\frac{\sqrt{3}-i}{\sqrt{3}-i}\right) + \sqrt{3} \\ &= \frac{3\sqrt{3}-3i}{3-i^2} + \sqrt{3} \\ &= \frac{3\sqrt{3}-3i}{4} + \frac{4\sqrt{3}}{4} \\ &= \tfrac{7\sqrt{3}}{4} - \tfrac{3}{4}i \end{aligned}$$

$\therefore$ the real part is $\frac{7\sqrt{3}}{4}$, and the imaginary part is $-\frac{3}{4}$.

6 $$\begin{aligned} 2z - 1 &= iz - i \\ \therefore\ (2-i)z &= 1 - i \\ \therefore\ z &= \left(\frac{1-i}{2-i}\right) \times \left(\frac{2+i}{2+i}\right) \\ &= \frac{2+i-2i-i^2}{4-i^2} \\ &= \frac{3-i}{5} \\ \therefore\ z &= \tfrac{3}{5} - \tfrac{1}{5}i \end{aligned}$$

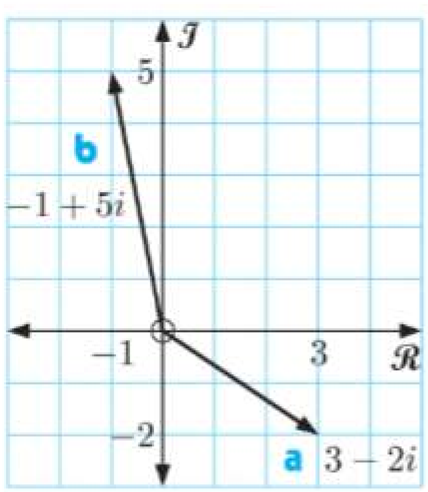
7 If $\dfrac{2-3i}{2a+bi} = 3+2i$ then $2a + bi = \left(\dfrac{2-3i}{3+2i}\right) \times \left(\dfrac{3-2i}{3-2i}\right)$

$$\begin{aligned} \therefore\ 2a+bi &= \frac{6-4i-9i+6i^2}{9-4i^2} \\ \therefore\ 2a+bi &= \frac{-13i}{13} \\ \therefore\ 2a+bi &= 0 - i \end{aligned}$$

Equating real and imaginary parts, $2a = 0$ and $b = -1$

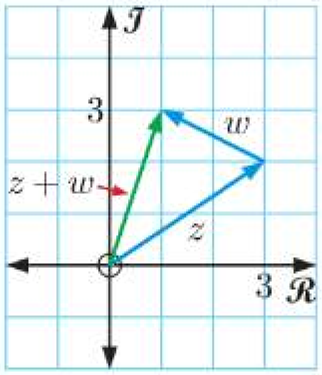
$\therefore\ a = 0$ and $b = -1$

8

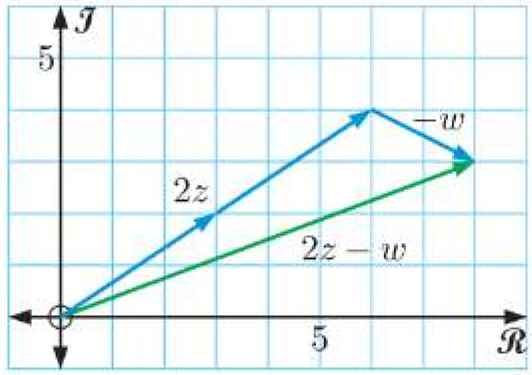


9 $z = 3 + 2i, \quad w = -2 + i$

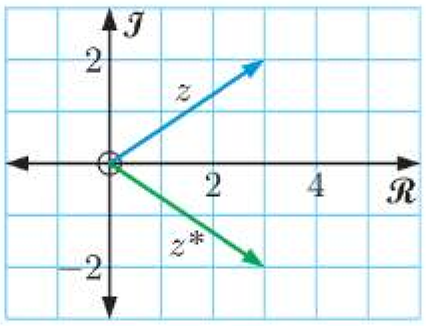
a
$$\begin{aligned} & z + w \\ &= 3 + 2i + (-2 + i) \\ &= 3 + 2i - 2 + i \\ &= 1 + 3i \end{aligned}$$



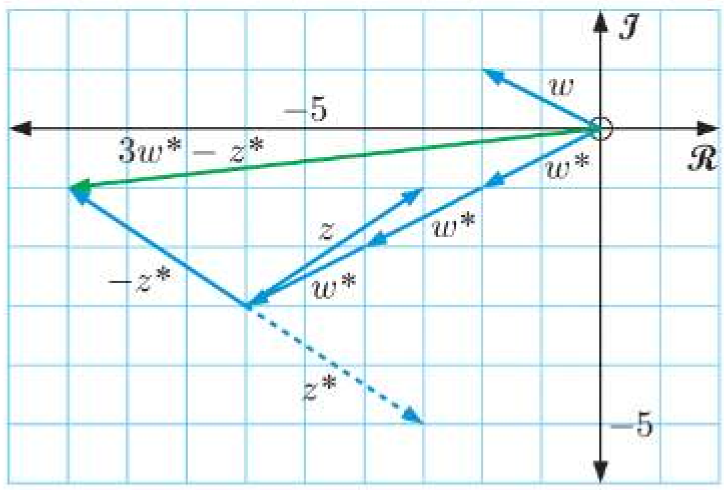
b
$$\begin{aligned} & 2z - w \\ &= 2(3 + 2i) - (-2 + i) \\ &= 6 + 4i + 2 - i \\ &= 8 + 3i \end{aligned}$$



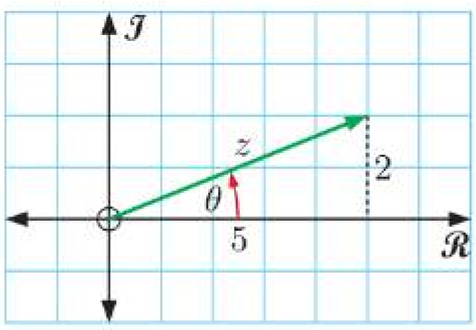
c
$$\begin{aligned} & z^* \\ &= (3 + 2i)^* \\ &= 3 - 2i \end{aligned}$$



d
$$\begin{aligned} & 3w^* - z^* \\ &= 3(-2 - i) - (3 - 2i) \\ &= -6 - 3i - 3 + 2i \\ &= -9 - i \end{aligned}$$



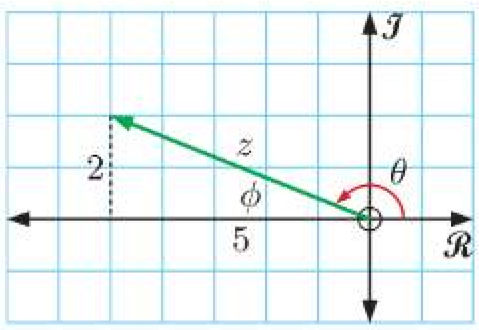
10 **a** $|\,z\,| = \sqrt{5^2 + 2^2}$
$= \sqrt{29}$



$\tan\theta = \frac{2}{5}$

$\therefore \ \arg z \approx 0.381$

b $|\,z\,| = \sqrt{(-5)^2 + 2^2}$
$= \sqrt{29}$



$\tan\phi = \frac{2}{5}$

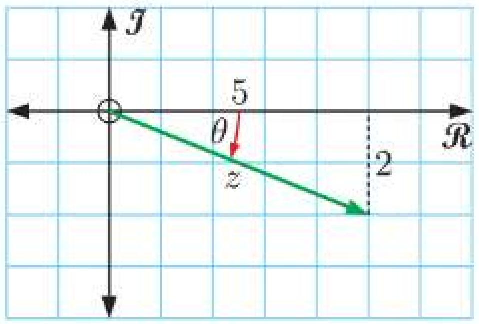
$\therefore \ \phi \approx 0.381$

But $\theta = \pi - \phi$

≈ 2.76

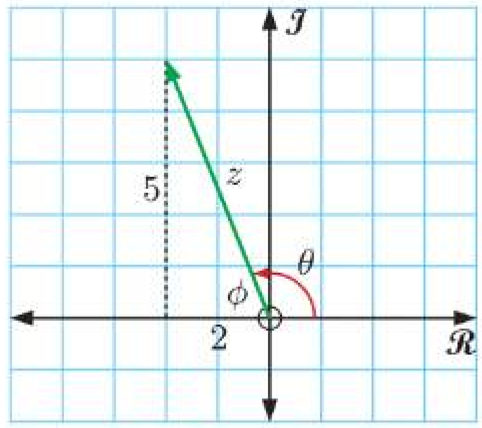
$\therefore \ \arg z \approx 2.76$

c $|z| = \sqrt{5^2 + (-2)^2}$
$= \sqrt{29}$



$\tan\theta = \frac{2}{5}$
$\therefore\ \theta \approx 0.381$
$\therefore\ \arg z \approx -0.381$

d $|z| = \sqrt{(-2)^2 + 5^2}$
$= \sqrt{29}$



$\tan\phi = \frac{5}{2}$
$\therefore\ \phi \approx 1.19$
But $\theta = \pi - \phi$
$\therefore\ \arg z \approx 1.95$

11 **a** $|5z| = |5||z|$
$= 5 \times 5$
$= 25$

b $|-4z^*| = |-4||z^*|$
$= 4|z|$
$= 4 \times 5$
$= 20$

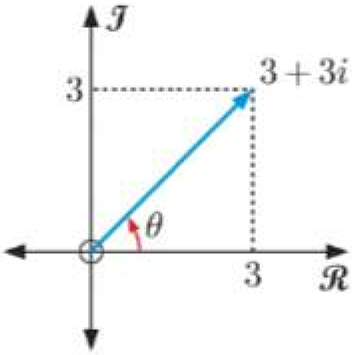
c $|(3+i)z| = |3+i||z|$
$= \sqrt{3^2 + 1^2} \times 5$
$= 5\sqrt{10}$

d $\left|\dfrac{i}{z}\right| = \dfrac{|i|}{|z|}$
$= \frac{1}{5}$

e $\left|\dfrac{3}{z^2}\right| = \dfrac{|3|}{|z^2|}$
$= \dfrac{3}{|z|^2}$
$= \dfrac{3}{5^2}$
$= \frac{3}{25}$

f $\left|\dfrac{3+4i}{z}\right| = \dfrac{|3+4i|}{|z|}$
$= \dfrac{\sqrt{3^2+4^2}}{5}$
$= \dfrac{5}{5}$
$= 1$

12 **a**



$$|3+3i| = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$$

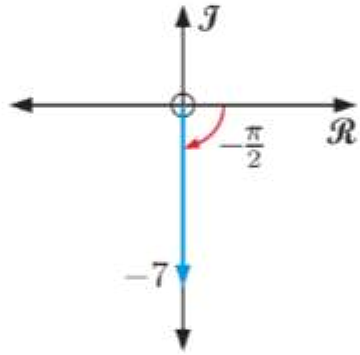
$$\tan\theta = \frac{3}{3} = 1$$

$$\therefore\ \theta = \tfrac{\pi}{4}$$

$$\arg(3+3i) = \tfrac{\pi}{4}$$

$$\therefore\ 3+3i = 3\sqrt{2}\operatorname{cis}\tfrac{\pi}{4}$$

b

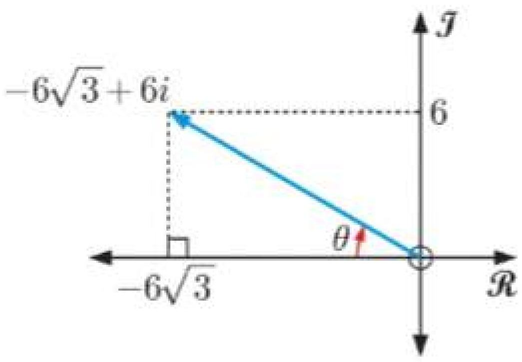


$$|-7i| = 7$$

$$\arg(-7i) = -\tfrac{\pi}{2}$$

$$\therefore\ -7i = 7\operatorname{cis}\left(-\tfrac{\pi}{2}\right)$$

c



$$\left|-6\sqrt{3}+6i\right| = \sqrt{(6\sqrt{3})^2+6^2} = \sqrt{108+36} = \sqrt{144} = 12$$

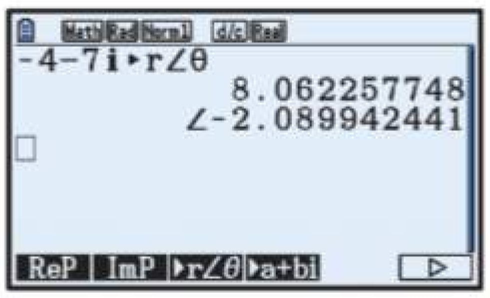
$$\tan\theta = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore\ \theta = \tfrac{\pi}{6}$$

$$\arg(-6\sqrt{3}+6i) = \pi - \tfrac{\pi}{6} = \tfrac{5\pi}{6}$$

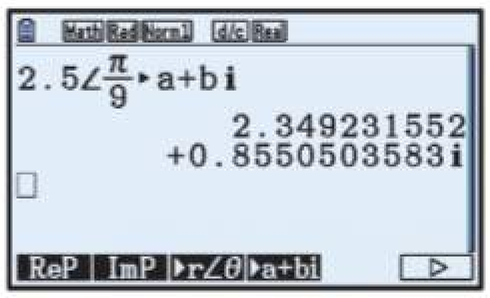
$$\therefore\ -6\sqrt{3}+6i = 12\operatorname{cis}\tfrac{5\pi}{6}$$

13 **a**



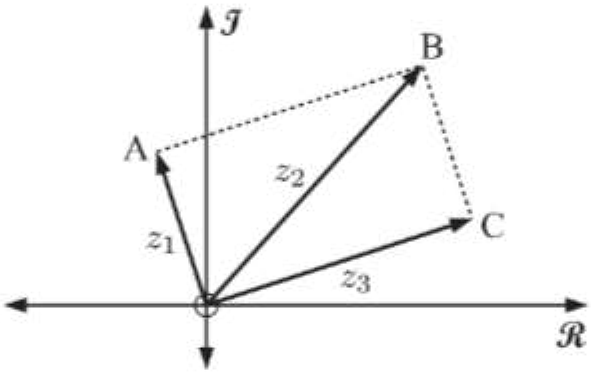
$-4-7i \approx 8.06\operatorname{cis}(-2.09)$

b



$2.5\operatorname{cis}\frac{\pi}{9} \approx 2.35 + 0.855i$

14 $z_1 = \operatorname{cis}\frac{3\pi}{5}$, $\quad z_3 = \sqrt{3}\operatorname{cis}\phi$



a

$$\phi = \arg z_3 = \arg z_1 - \tfrac{\pi}{2} = \tfrac{3\pi}{5} - \tfrac{\pi}{2} = \tfrac{\pi}{10}$$

b $z_3 - z_2$ corresponds to $\overrightarrow{BC}$

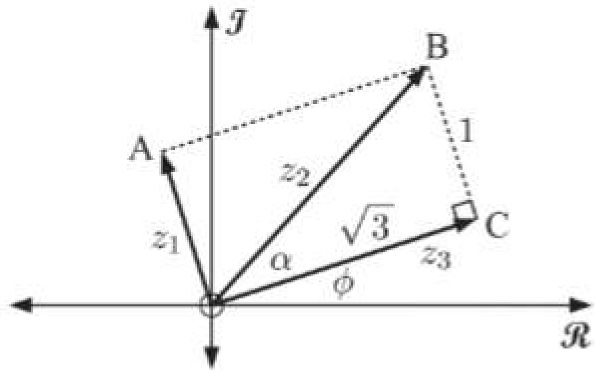
$$\therefore\ \arg(z_3 - z_2) = \arg z_3 - \tfrac{\pi}{2} = \tfrac{\pi}{10} - \tfrac{\pi}{2} = -\tfrac{4\pi}{10} = -\tfrac{2\pi}{5}$$

c $|z_2| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$, so $\alpha = \frac{\pi}{6}$

$\therefore \arg(z_2) = \phi + \alpha = \frac{\pi}{10} + \frac{\pi}{6} = \frac{4\pi}{15}$

$\therefore z_2 = 2\operatorname{cis}\frac{4\pi}{15}$



15 $w = \dfrac{1+z}{1+z^*}$

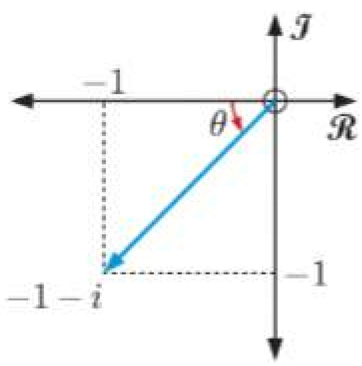
$= \left(\dfrac{1+\operatorname{cis}\phi}{1+\operatorname{cis}(-\phi)}\right) \times \dfrac{\operatorname{cis}\phi}{\operatorname{cis}\phi}$

$= \dfrac{(1+\operatorname{cis}\phi)\operatorname{cis}\phi}{\operatorname{cis}\phi + \operatorname{cis}0}$

$= \dfrac{(1+\operatorname{cis}\phi)\operatorname{cis}\phi}{1+\operatorname{cis}\phi}$

$= \operatorname{cis}\phi$

16 **a**



If $z = -1 - i$, then $|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

Now $\tan\theta = \dfrac{1}{1} = 1$

$\therefore \theta = \frac{\pi}{4}$

$\therefore \arg z = -\frac{3\pi}{4}$

$\therefore z = \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

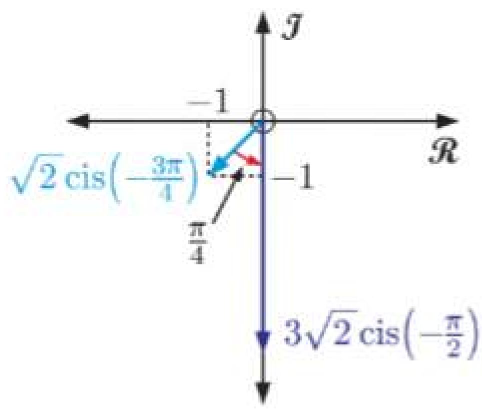
b $z \times 3\operatorname{cis}\frac{\pi}{4}$

$= \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right) \times 3\operatorname{cis}\frac{\pi}{4}$

$= 3\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4} + \frac{\pi}{4}\right)$

$= 3\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{2}\right)$

c



d When z is multiplied by $3\operatorname{cis}\frac{\pi}{4}$, it is dilated with scale factor 3, then rotated anticlockwise through $\frac{\pi}{4}$ about the origin.

17 $z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \text{cis}\,\frac{\pi}{6}$ and $z_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \text{cis}\,\frac{\pi}{4}$

$$\begin{aligned}
\therefore\quad \left(\frac{z_1}{z_2}\right)^3 &= \left(\frac{\text{cis}\,\frac{\pi}{6}}{\text{cis}\,\frac{\pi}{4}}\right)^3 \\
&= \left[\text{cis}\left(\tfrac{\pi}{6} - \tfrac{\pi}{4}\right)\right]^3 \\
&= \left[\text{cis}\left(-\tfrac{\pi}{12}\right)\right]^3 \\
&= \left[e^{-i\frac{\pi}{12}}\right]^3 \\
&= e^{-i\frac{\pi}{4}} \\
&= \cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right) \\
&= \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i
\end{aligned}$$

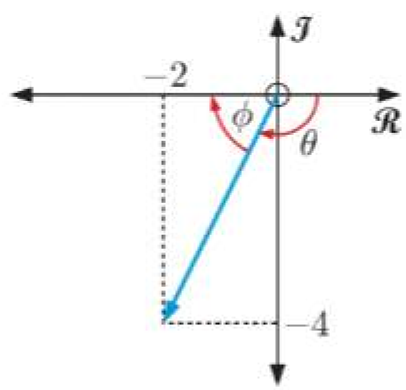
18 $z = 4 + i, \quad w = 2 - 3i$

a
$$\begin{aligned}
&2w^* - iz \\
&= 2(2+3i) - i(4+i) \\
&= 4 + 6i - 4i - i^2 \\
&= 5 + 2i
\end{aligned}$$

b
$$\begin{aligned}
|\,w - z^*\,| &= |\,(2-3i) - (4-i)\,| \\
&= |\,2 - 3i - 4 + i\,| \\
&= |\,-2 - 2i\,| \\
&= \sqrt{(-2)^2 + (-2)^2} \\
&= \sqrt{8} \\
&= 2\sqrt{2}
\end{aligned}$$

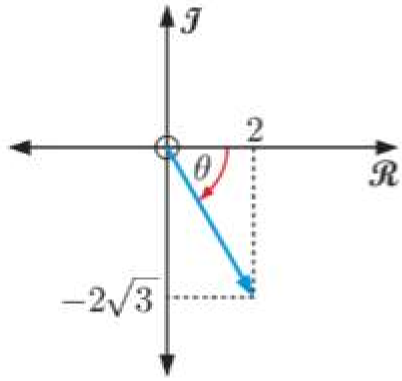
c
$$\begin{aligned}
|\,z^{10}\,| &= |\,z\,|^{10} \\
&= |\,4+i\,|^{10} \\
&= \left(\sqrt{16+1}\right)^{10} \\
&= \left(\sqrt{17}\right)^{10} \\
&= 17^5
\end{aligned}$$

d
$$\begin{aligned}
\arg(w - z) &= \arg\left[(2-3i) - (4+i)\right] \\
&= \arg(-2 - 4i)
\end{aligned}$$



$$\begin{aligned}
\tan\phi &= \tfrac{4}{2} = 2 \\
\therefore\quad \phi &\approx 1.11 \\
\text{But}\quad \theta &= \pi - \phi \\
&\approx 2.03 \\
\therefore\quad \arg(w - z) &\approx -2.03
\end{aligned}$$

19 $2-2\sqrt{3}i$ has modulus $\sqrt{2^2+(-2\sqrt{3})^2}=\sqrt{16}=4$



$$\begin{aligned}
\text{Now}\quad \tan\theta &= \tfrac{2\sqrt{3}}{2}\\
&= \sqrt{3}\\
\therefore\ \theta &= \tfrac{\pi}{3}\\
\therefore\ \arg(2-2\sqrt{3}i) &= -\tfrac{\pi}{3}\\
\therefore\ 2-2\sqrt{3}i &= 4\operatorname{cis}\left(-\tfrac{\pi}{3}\right)\\
&= 4e^{-i\frac{\pi}{3}}\\
\therefore\ (2-2\sqrt{3}i)^n &= \left(4e^{-i\frac{\pi}{3}}\right)^n\\
&= 4^n e^{-i\frac{n\pi}{3}}\\
&= 4^n\operatorname{cis}\left(-\tfrac{n\pi}{3}\right)
\end{aligned}$$

which is purely imaginary if $\cos\left(-\frac{n\pi}{3}\right)=0$

$$\begin{aligned}
\therefore\ \cos\tfrac{n\pi}{3} &= 0\\
\therefore\ \tfrac{n\pi}{3} &= \tfrac{\pi}{2}+k\pi,\quad k\in\mathbb{Z}\\
\therefore\ n &= 3(k+\tfrac{1}{2}),\quad k\in\mathbb{Z}
\end{aligned}$$

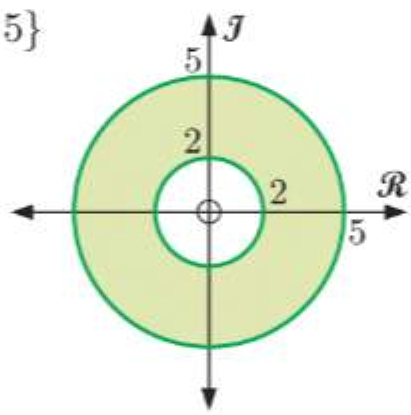
20 $z=4\operatorname{cis}\theta$

a
$$\begin{aligned}
z^3 &= (4\operatorname{cis}\theta)^3\\
&= (4e^{i\theta})^3\\
&= 4^3e^{3\theta i}\\
&= 64\operatorname{cis}3\theta
\end{aligned}$$
$\therefore\ |z^3|=64$ and $\arg(z^3)=3\theta$

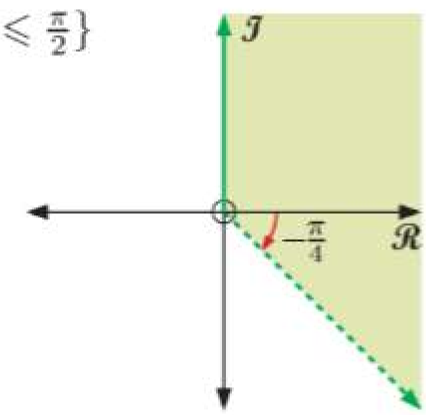
b
$$\begin{aligned}
\frac{1}{z} &= \frac{1}{4\operatorname{cis}\theta}\\
&= \frac{1}{4e^{i\theta}}\\
&= \tfrac{1}{4}e^{-i\theta}\\
&= \tfrac{1}{4}\operatorname{cis}(-\theta)
\end{aligned}$$
$\therefore\ \left|\frac{1}{z}\right|=\frac{1}{4}$ and $\arg\left(\frac{1}{z}\right)=-\theta$

c
$$\begin{aligned}
z^* &= 4\operatorname{cis}(-\theta)\\
\therefore\ iz^* &= \left(\operatorname{cis}\tfrac{\pi}{2}\right)(4\operatorname{cis}(-\theta))\\
&= 4\operatorname{cis}\left(\tfrac{\pi}{2}-\theta\right)
\end{aligned}$$
$\therefore\ |iz^*|=4$ and $\arg(iz^*)=\frac{\pi}{2}-\theta$

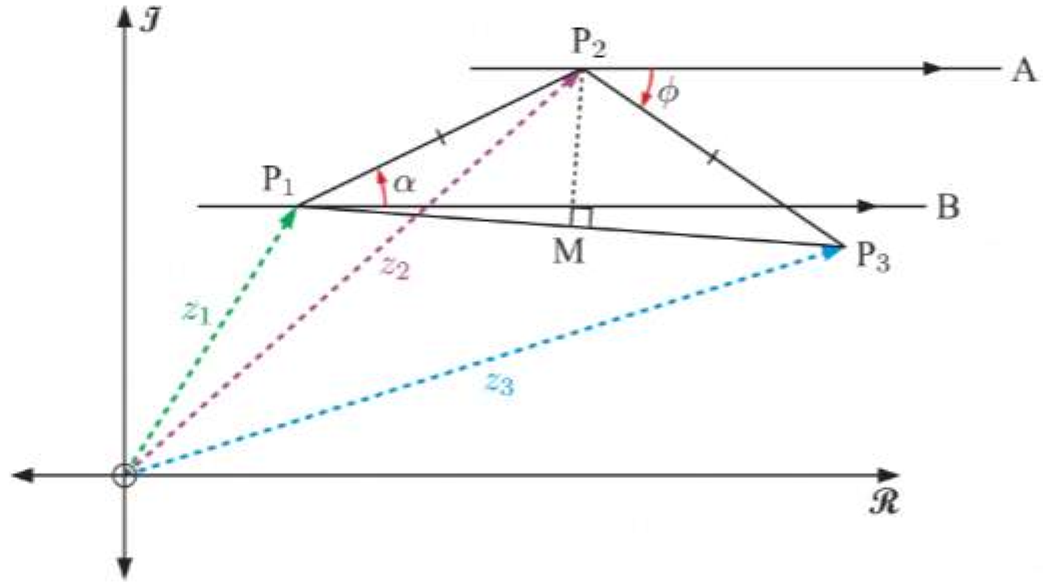
21 **a** $\{z\mid 2\leqslant |z|\leqslant 5\}$



b $\{z\mid -\frac{\pi}{4}<\arg z\leqslant\frac{\pi}{2}\}$



22



a $\arg(z_2 - z_1) = \alpha$, and suppose $\arg(z_3 - z_2) = -\phi$ as shown.

Let M be the midpoint of P_1 and P_3, so the distance $P_1M = \dfrac{\sqrt{3}P_1P_2}{2}$.

$\therefore \sin(P_1\widehat{P_2}M) = \dfrac{\frac{\sqrt{3}}{2}P_1P_2}{P_1P_2} = \dfrac{\sqrt{3}}{2}$, so $P_1\widehat{P_2}M = \frac{\pi}{3}$ and $P_1\widehat{P_2}P_3 = \frac{2\pi}{3}$

$\therefore \alpha + \frac{2\pi}{3} + \phi = \pi$ $\{[P_1B] \parallel [P_2A]$, co-interior angles$\}$

$\therefore -\phi = \alpha - \frac{\pi}{3}$

$\therefore \arg(z_3 - z_2) = \alpha - \frac{\pi}{3}$ as required.

b $(z_2 - z_1) \equiv \overrightarrow{P_1P_2}$ and $(z_3 - z_2) \equiv \overrightarrow{P_2P_3}$

$$\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = \frac{|z_2 - z_1|}{|z_3 - z_2|}$$
$$= \frac{|\overrightarrow{P_1P_2}|}{|\overrightarrow{P_2P_3}|}$$
$$= 1 \qquad \{\triangle P_1P_2P_3 \text{ is isosceles with } P_1P_2 = P_2P_3\}$$

and $$\arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$$
$$= \alpha - (\alpha - \tfrac{\pi}{3}) \qquad \{\text{from } \mathbf{a}\}$$
$$= \tfrac{\pi}{3}$$

23 **a** $e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$
$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

b $5e^{-i\pi} = 5(\cos(-\pi) + i\sin(-\pi))$
$= 5(-1 + i \times 0)$
$= -5$

24 $zw = 4e^{i\frac{\pi}{4}} \times 3e^{-i\frac{\pi}{3}}$
$= 12e^{i(\frac{\pi}{4} - \frac{\pi}{3})}$
$= 12e^{-i\frac{\pi}{12}}$
$= 12\operatorname{cis}\left(-\frac{\pi}{12}\right)$

25 $\sin 3t + 4\sin\left(3t - \frac{3\pi}{4}\right) = \mathcal{Im}\left(\text{cis}\,3t + 4\,\text{cis}\left(3t - \frac{3\pi}{4}\right)\right)$

$$= \mathcal{Im}\left(e^{3ti} + 4e^{(3t-\frac{3\pi}{4})i}\right)$$

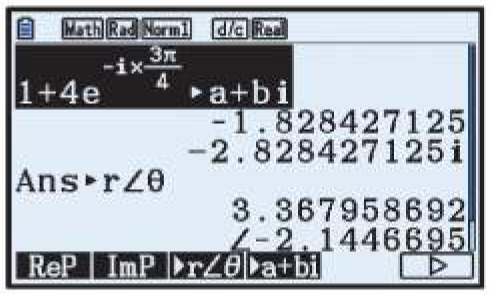
Now $e^{3ti} + 4e^{(3t-\frac{3\pi}{4})i}$

$= e^{3ti}(1 + 4e^{-i\frac{3\pi}{4}})$

$\approx e^{3ti}(-1.83 - 2.83i)$

$\approx e^{3ti}(3.37e^{-2.14i})$

$\approx 3.37e^{(3t-2.14)i}$



So, $\sin 3t + 4\sin\left(3t - \frac{3\pi}{4}\right) \approx \mathcal{Im}\left(3.37e^{(3t-2.14)i}\right)$

$$\approx 3.37\sin(3t - 2.14)$$

REVIEW SET 11B

1 **a** $2x^2 = -18$

$\therefore\ x^2 = -9$

$\therefore\ x = \pm\sqrt{-9}$

$\therefore\ x = \pm\sqrt{9} \times \sqrt{-1}$

$\therefore\ x = \pm 3i$

b $x^2 - 5x + 7 = 0$

$\therefore\ x = \dfrac{5 \pm \sqrt{25 - 4 \times 1 \times 7}}{2}$

$\therefore\ x = \dfrac{5 \pm \sqrt{-3}}{2}$

$\therefore\ x = \frac{5}{2} \pm \frac{\sqrt{3}}{2}i$

c $-2x^2 + 6x - 9 = 0$

$\therefore\ x = \dfrac{-6 \pm \sqrt{36 - 4 \times (-2) \times (-9)}}{2 \times (-2)}$

$\therefore\ x = \dfrac{-6 \pm \sqrt{-36}}{-4}$

$\therefore\ x = \dfrac{-6 \pm 6i}{-4}$

$\therefore\ x = \frac{3}{2} \pm \frac{3}{2}i$

2 **a** $(-3 - 8i)^* = -3 + 8i$

b $\left(\frac{1}{2} + \frac{5}{2}i\right)^* = \frac{1}{2} - \frac{5}{2}i$

c $(2 + i)(4 - 5i) = 8 - 10i + 4i - 5i^2$

$= 13 - 6i$

$\therefore\ [(2 + i)(4 - 5i)]^* = 13 + 6i$

3 $z = 2 + 3i, \quad w = -6 + i$

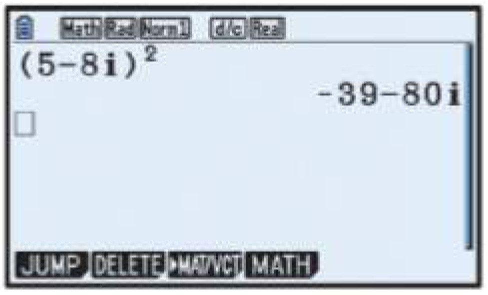
a $$\begin{aligned} z + 3w &= 2 + 3i + 3(-6 + i) \\ &= 2 + 3i - 18 + 3i \\ &= -16 + 6i \end{aligned}$$

b $$\begin{aligned} zw &= (2 + 3i)(-6 + i) \\ &= -12 + 2i - 18i + 3i^2 \\ &= -15 - 16i \end{aligned}$$

c $$\begin{aligned} \frac{w}{z} &= \left(\frac{-6 + i}{2 + 3i}\right) \times \left(\frac{2 - 3i}{2 - 3i}\right) \\ &= \frac{-12 + 18i + 2i - 3i^2}{4 - 9i^2} \\ &= \frac{-9 + 20i}{13} \\ &= -\tfrac{9}{13} + \tfrac{20}{13}i \end{aligned}$$

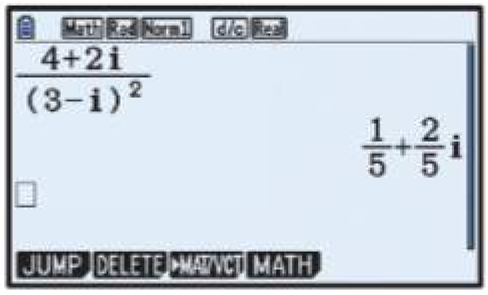
d $$\begin{aligned} \frac{z^*}{i} &= \left(\frac{2 - 3i}{i}\right) \times \left(\frac{-i}{-i}\right) \\ &= \frac{-2i + 3i^2}{-i^2} \\ &= -3 - 2i \end{aligned}$$

4 **a**



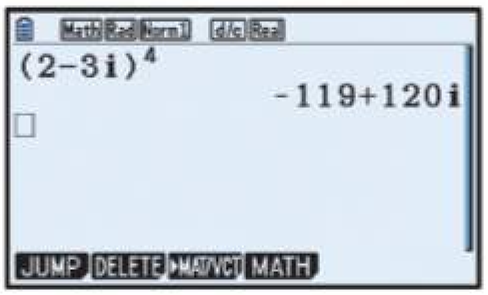
$(5 - 8i)^2 = -39 - 80i$

b



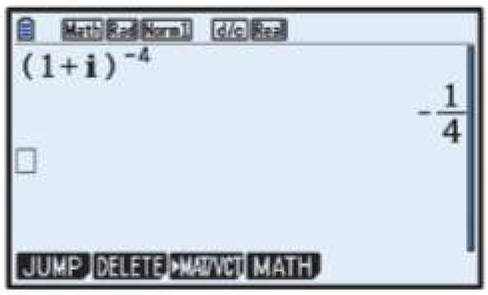
$\dfrac{4 + 2i}{(3 - i)^2} = \tfrac{1}{5} + \tfrac{2}{5}i$

c



$(2 - 3i)^4 = -119 + 120i$

d



$(1 + i)^{-4} = -\tfrac{1}{4}$

5 $$\begin{aligned} & 2z + w = i \\ \therefore\ & 6z + 3w = 0 + 3i \\ \text{and}\ & z - 3w = 7 - 10i \\ \text{Adding,}\ & 7z = 7 - 7i \\ \therefore\ & z = 1 - i \\ \text{Now,}\ & 2z + w = i \\ \therefore\ & z + w = i - z \\ & = i - (1 - i) \\ & = i - 1 + i \\ & = -1 + 2i \end{aligned}$$

6 **a** $x + yi = 0$

Equating real and imaginary parts, $x = 0$ and $y = 0$.

b
$$(3 - 2i)(x + i) = 17 + yi$$
$$3x + 3i - 2xi - 2i^2 = 17 + yi$$
$$(3x + 2) + (3 - 2x)i = 17 + yi$$

Equating real and imaginary parts, $3x + 2 = 17$ and $3 - 2x = y$

$\therefore\ 3x = 15$ and $y = 3 - 2x$

$\therefore\ x = 5$ and $y = -7$

c
$$(x + yi)^2 = x - yi$$
$$\therefore\ x^2 + 2xyi + y^2i^2 = x - yi$$
$$\therefore\ (x^2 - y^2) + 2xyi = x - yi$$

Equating real and imaginary parts, $x^2 - y^2 = x$ and $2xy = -y$

$\therefore\ 2xy + y = 0$

$\therefore\ y(2x + 1) = 0$

$\therefore\ y = 0$ or $x = -\frac{1}{2}$

When $y = 0$, $x^2 = x$

$\therefore\ x^2 - x = 0$

$\therefore\ x(x - 1) = 0$

$\therefore\ x = 0$ or 1

and when $x = -\frac{1}{2}$, $(-\frac{1}{2})^2 - y^2 = -\frac{1}{2}$

$\therefore\ \frac{1}{4} - y^2 = -\frac{1}{2}$

$\therefore\ y^2 = \frac{3}{4}$

$\therefore\ y = \pm\frac{\sqrt{3}}{2}$

The possible solutions are:

x	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
y	0	0	$\frac{\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{2}$

d
$$(3x + 2yi)(1 - i) = (3y + 1)i - x$$
$$\therefore\ 3x - 3xi + 2yi - 2yi^2 = 3yi + i - x$$
$$\therefore\ (3x + 2y) + (2y - 3x)i = -x + (3y + 1)i$$

Equating real and imaginary parts, $3x + 2y = -x$ and $2y - 3x = 3y + 1$

$\therefore\ 4x = -2y$ and $-1 - 3x = y$

$\therefore\ 4x = -2(-1 - 3x)$

$\therefore\ 4x = 2 + 6x$

$\therefore\ -2x = 2$

$\therefore\ x = -1$ and $y = -1 - 3(-1) = 2$

7 Letting $z = a + bi$ gives $(a+bi)^2 = 5 - 12i$

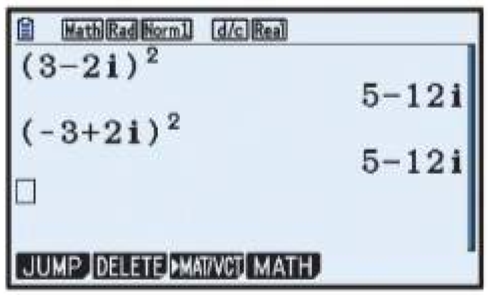
$$\therefore \ a^2 + 2abi + b^2i^2 = 5 - 12i$$
$$\therefore \ (a^2 - b^2) + 2abi = 5 - 12i$$

Equating real and imaginary parts, $a^2 - b^2 = 5$ and $2ab = -12$

$$\therefore \ a^2 - b^2 = 5 \quad \text{and} \quad b = -\frac{6}{a}$$

So, $a^2 - \left(-\frac{6}{a}\right)^2 = 5$

$$\therefore \ a^2 - \frac{36}{a^2} = 5$$
$$\therefore \ a^4 - 5a^2 - 36 = 0$$
$$\therefore \ (a^2 - 9)(a^2 + 4) = 0$$
$$\therefore \ a^2 = 9 \quad \{a^2 + 4 > 0\}$$
$$\therefore \ a = 3 \quad \text{and} \quad b = -\frac{6}{3} = -2$$
$$\text{or} \quad a = -3 \quad \text{and} \quad b = \frac{-6}{-3} = 2$$
$$\therefore \ z = 3 - 2i \quad \text{or} \quad z = -3 + 2i$$



8 $(kz)^* = (ka + kbi)^*$
$$= ka - kbi$$
$$= k(a - bi)$$
$$= kz^*$$

9 Let $z = a + bi$ and $w = c + di$ where $b \neq 0$ and $d \neq 0$

Now $z + w = (a + c) + (b + d)i$ and $zw = (a + bi)(c + di)$
$$= (ac - bd) + (bc + ad)i$$

As $z + w$ is real, $b + d = 0$ and as zw is real, $bc + ad = 0$ .... (2)

$$\therefore \ b = -d \quad .... \ (1)$$

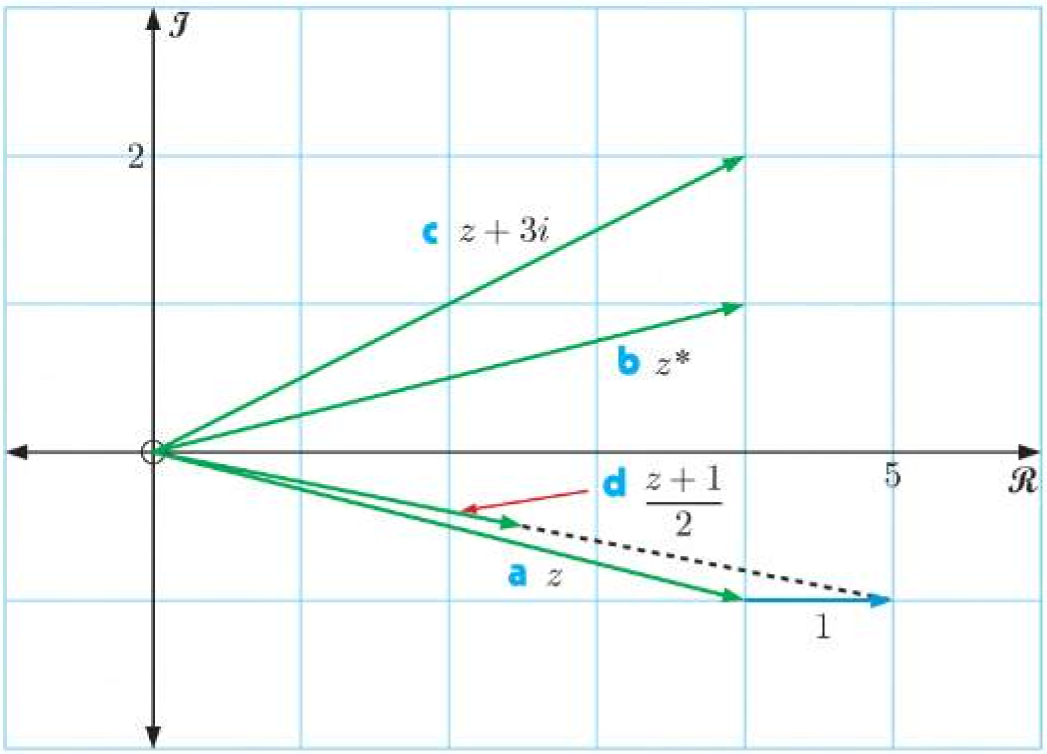
Substituting (1) into (2), $-dc + ad = 0$

$$\therefore \ d(a - c) = 0$$

But $d \neq 0$, $\therefore \ a = c$ and $b = -d$ {from (1)}

$$\therefore \ z^* = a - bi = c + di = w$$

10



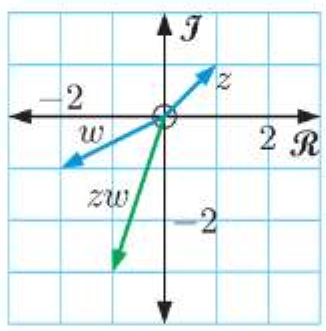
11 **a** $|3z|$
$= |3||z|$
$= 3 \times 4$
$= 12$

b $|2iz|$
$= |2i||z|$
$= 2 \times 4$
$= 8$

c $\left|\frac{i+1}{z}\right|$
$= \frac{|i+1|}{|z|}$
$= \frac{\sqrt{1^2+1^2}}{4}$
$= \frac{\sqrt{2}}{4}$

d $|(3-i)z|$
$= |3-i||z|$
$= 4 \times \sqrt{3^2+(-1)^2}$
$= 4\sqrt{10}$

12 $zw = (1+i)(-2-i)$
$= -2 - i - 2i + 1$
$= -1 - 3i$



13 $z = -1+3i, \quad w = 3+i$

a $|z| = \sqrt{(-1)^2+3^2}$
$= \sqrt{10}$

b $|w| = \sqrt{3^2+1^2}$
$= \sqrt{10}$

c $|z^*|^2 = |z|^2$
$= (\sqrt{10})^2$
$= 10$

d $zz^* = |z|^2$
$= (\sqrt{10})^2$
$= 10$

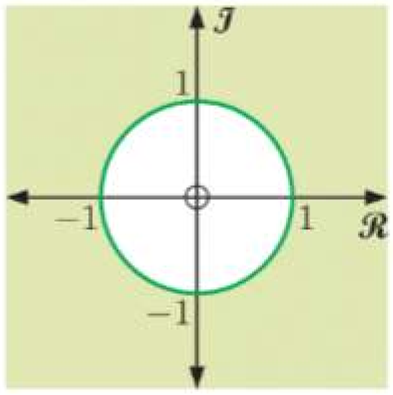
e $|zw| = |z||w|$
$= \sqrt{10} \times \sqrt{10}$
$= 10$

f $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
$= \frac{\sqrt{10}}{\sqrt{10}}$
$= 1$

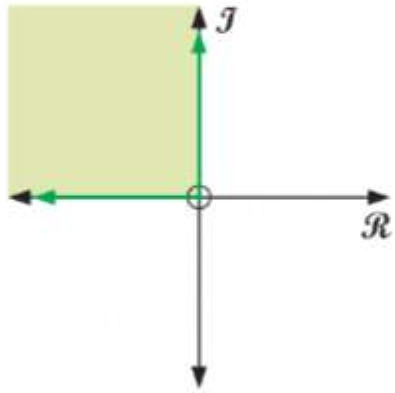
g $|z^2| = |z|^2$
$= (\sqrt{10})^2$
$= 10$

h $|z^3| = |z|^3$
$= (\sqrt{10})^3$
$= 10\sqrt{10}$

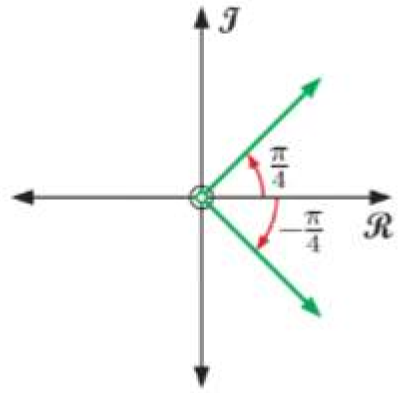
14 **a** $|z| \geqslant 1$



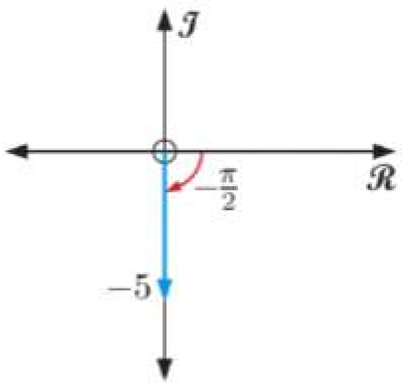
b $\frac{\pi}{2} \leqslant \arg z \leqslant \pi$



c $\arg z = \pm\frac{\pi}{4}$

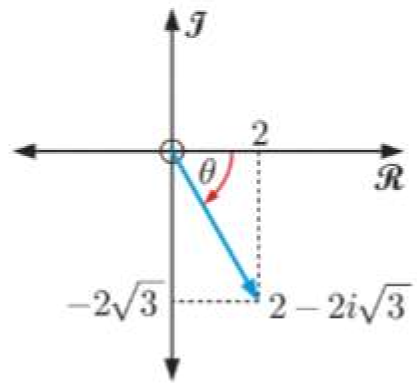


15 **a**



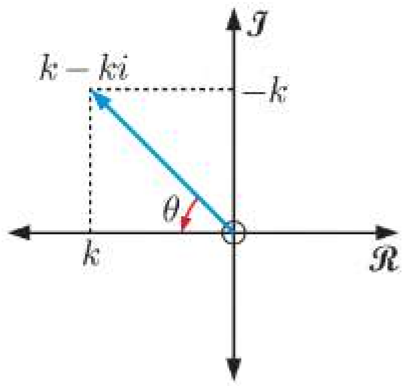
$$|-5i| = 5$$
$$\arg(-5i) = -\tfrac{\pi}{2}$$
$$\therefore \quad -5i = 5\operatorname{cis}\left(-\tfrac{\pi}{2}\right)$$

b



$$\left|2-2i\sqrt{3}\right| = \sqrt{2^2+(-2\sqrt{3})^2}$$
$$= \sqrt{16}$$
$$= 4$$
$$\tan\theta = \tfrac{2\sqrt{3}}{2}$$
$$= \sqrt{3}$$
$$\therefore \quad \theta = \tfrac{\pi}{3}$$
$$\therefore \quad \arg(2-2i\sqrt{3}) = -\tfrac{\pi}{3}$$
$$\therefore \quad 2-2i\sqrt{3} = 4\operatorname{cis}\left(-\tfrac{\pi}{3}\right)$$

c



$$\begin{aligned} |\,k - ki\,| &= \sqrt{k^2 + (-k)^2} \\ &= \sqrt{2k^2} \\ &= |\,k\,|\,\sqrt{2} \end{aligned}$$

Since $k < 0$, $\quad |\,k - ki\,| = -k\sqrt{2}$

$$\begin{aligned} \tan\theta &= \frac{k}{k} \\ &= 1 \qquad \{k \neq 0\} \\ \therefore\ \theta &= \tfrac{\pi}{4} \\ \therefore\ \arg(k - ki) &= \pi - \tfrac{\pi}{4} \\ &= \tfrac{3\pi}{4} \\ \therefore\ k - ki &= -k\sqrt{2}\operatorname{cis}\tfrac{3\pi}{4} \end{aligned}$$

which is in polar form since $k < 0$

16 **a** $$\begin{aligned} 4\operatorname{cis}\tfrac{\pi}{2} &= 4\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right) \\ &= 4(0 + i \times 1) \\ &= 4i \end{aligned}$$

b $$\begin{aligned} \sqrt{2}\operatorname{cis}\tfrac{3\pi}{4} &= \sqrt{2}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right) \\ &= \sqrt{2}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) \\ &= -1 + i \end{aligned}$$

c $$\begin{aligned} 8\operatorname{cis}\left(-\tfrac{\pi}{3}\right) &= 8\left(\cos\left(-\tfrac{\pi}{3}\right) + i\sin\left(-\tfrac{\pi}{3}\right)\right) \\ &= 8\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) \\ &= 4 - 4\sqrt{3}i \end{aligned}$$

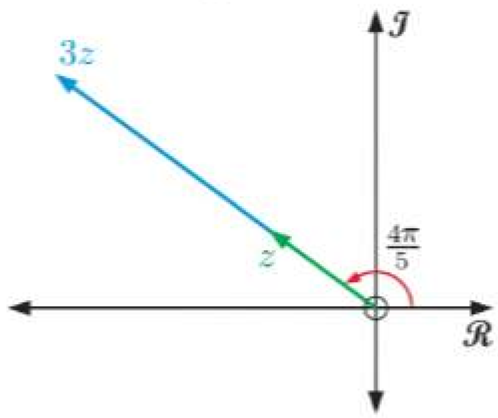
17 **a** $$\begin{aligned} \operatorname{cis}\tfrac{\pi}{5}\operatorname{cis}\tfrac{2\pi}{3} &= \operatorname{cis}\left(\tfrac{\pi}{5} + \tfrac{2\pi}{3}\right) \\ &= \operatorname{cis}\tfrac{13\pi}{15} \end{aligned}$$

b $$\begin{aligned} \frac{3\operatorname{cis}\left(-\frac{\pi}{10}\right)}{\operatorname{cis}\frac{\pi}{4}} &= 3\operatorname{cis}\left(-\tfrac{\pi}{10} - \tfrac{\pi}{4}\right) \\ &= 3\operatorname{cis}\left(-\tfrac{7\pi}{20}\right) \end{aligned}$$

c $$\begin{aligned} \operatorname{cis}\tfrac{73\pi}{3} &= \operatorname{cis}\left(\tfrac{\pi}{3} + \tfrac{72\pi}{3}\right) \\ &= \operatorname{cis}\left(\tfrac{\pi}{3} + 24\pi\right) \\ &= \operatorname{cis}\tfrac{\pi}{3} \end{aligned}$$

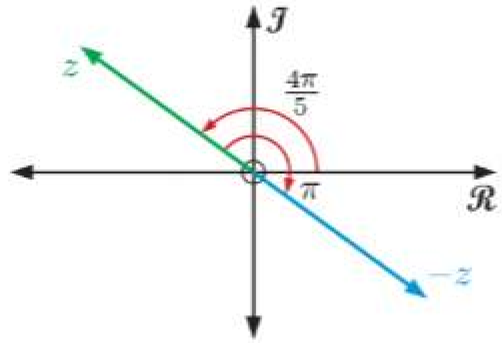
18 $z = r\operatorname{cis}\frac{4\pi}{5}, \quad r > 0$

a $3z = 3r\operatorname{cis}\frac{4\pi}{5}$

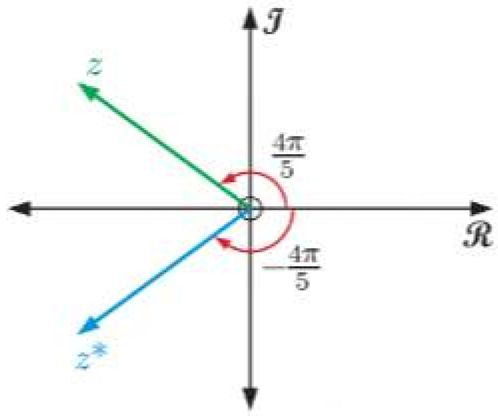


b $-z = r\operatorname{cis}\left(\frac{4\pi}{5} - \pi\right)$

$= r\operatorname{cis}\left(-\frac{\pi}{5}\right)$



c $z^* = r\operatorname{cis}\left(-\frac{4\pi}{5}\right)$



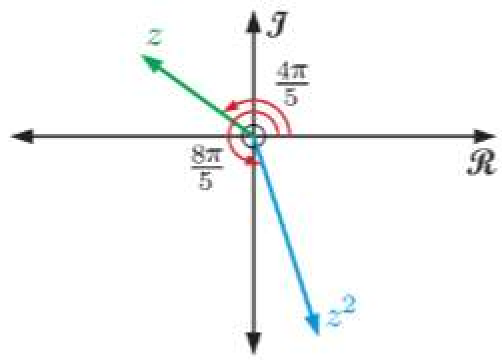
d $z^2 = \left[r\operatorname{cis}\frac{4\pi}{5}\right]^2$

$= r\operatorname{cis}\frac{4\pi}{5} \times r\operatorname{cis}\frac{4\pi}{5}$

$= r^2\operatorname{cis}\left(\frac{4\pi}{5} + \frac{4\pi}{5}\right)$

$= r^2\operatorname{cis}\frac{8\pi}{5}$

$= r^2\operatorname{cis}\left(-\frac{2\pi}{5}\right)$

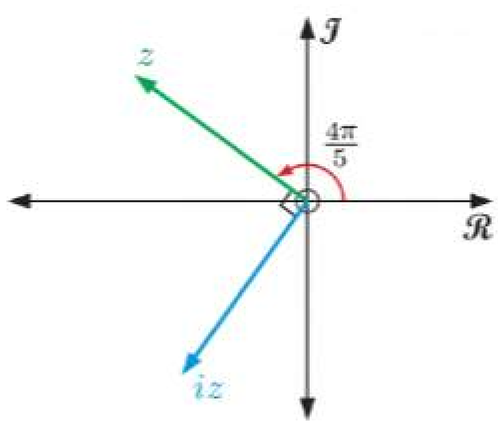


e $iz = \operatorname{cis}\frac{\pi}{2} \times r\operatorname{cis}\frac{4\pi}{5}$

$= r\operatorname{cis}\left(\frac{\pi}{2} + \frac{4\pi}{5}\right)$

$= r\operatorname{cis}\left(\frac{13\pi}{10}\right)$

$= r\operatorname{cis}\left(-\frac{7\pi}{10}\right)$



19 $$z = (1+bi)^2 \quad \text{has argument } \tfrac{\pi}{3}$$

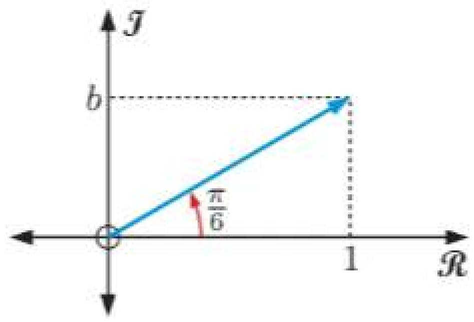
$$\therefore\ (1+bi)^2 = re^{i\frac{\pi}{3}} \quad \text{where } r > 0$$

$$\therefore\ 1+bi = \left(re^{i\frac{\pi}{3}}\right)^{\frac{1}{2}}$$

$$= r^{\frac{1}{2}}e^{i\frac{\pi}{6}} \quad \text{which has argument } \tfrac{\pi}{6}$$

$$\therefore\ \tan\tfrac{\pi}{6} = \frac{b}{1}$$

$$\therefore\ b = \tfrac{1}{\sqrt{3}}$$

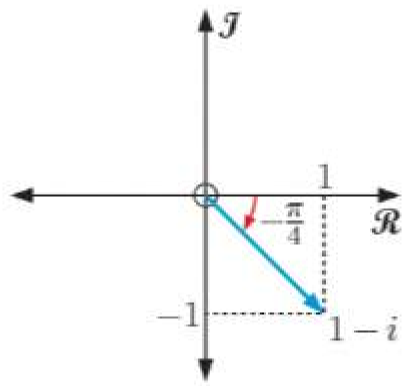


20 $$1 - i = \sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right)$$

$$\therefore\ (1-i)z = \sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right) \times 2\sqrt{2}\,\text{cis}\,\alpha$$

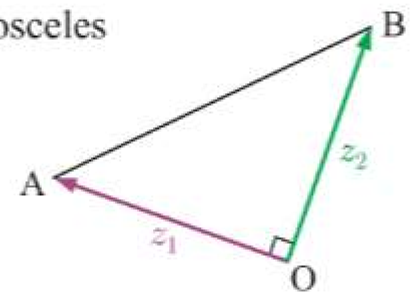
$$= 4\,\text{cis}\left(\alpha - \tfrac{\pi}{4}\right)$$

$$\therefore\ \arg[(1-i)z] = \alpha - \tfrac{\pi}{4}$$



21 **a** $$\left|\frac{{z_1}^2}{{z_2}^2}\right| = \frac{|z_1|^2}{|z_2|^2} \qquad \text{But } |z_1| = |z_2| \text{ since the triangle is isosceles}$$

$$\therefore\ \left|\frac{{z_1}^2}{{z_2}^2}\right| = 1$$



Also, $$\arg\left(\frac{{z_1}^2}{{z_2}^2}\right) = \arg({z_1}^2) - \arg({z_2}^2)$$

$$= 2\arg z_1 - 2\arg z_2 \qquad \{\text{using } \textbf{Exercise 11H.3 5 b}\}$$

$$= 2(\arg z_1 - \arg z_2)$$

$$= 2 \times \tfrac{\pi}{2} \quad \text{since } z_1 \text{ and } z_2 \text{ are perpendicular, and } \arg z_1 > \arg z_2$$

$$= \pi$$

b $$\frac{{z_1}^2}{{z_2}^2} = \text{cis}\,\pi = -1$$

$$\therefore\ {z_1}^2 = -{z_2}^2$$

$$\therefore\ {z_1}^2 + {z_2}^2 = 0$$

22 **a** $$\tan\theta = \frac{2}{2} = 1$$

$$\therefore\ \theta = \tfrac{\pi}{4}$$

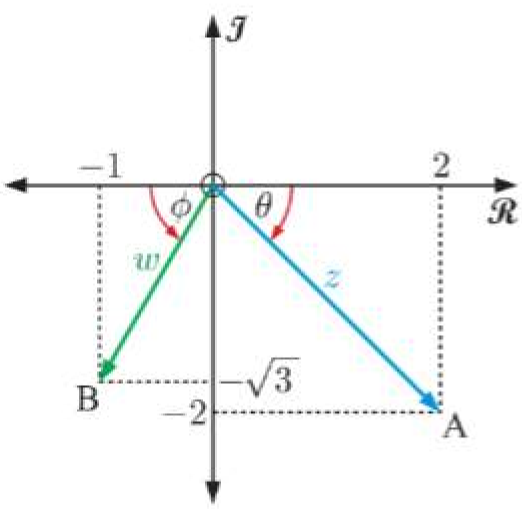
$$\tan\phi = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\therefore\ \phi = \tfrac{\pi}{3}$$

$$\therefore\ \text{A}\widehat{\text{O}}\text{B} = \pi - \phi - \theta$$

$$= \pi - \tfrac{\pi}{3} - \tfrac{\pi}{4}$$

$$= \tfrac{5\pi}{12}$$



b $\arg z = -\theta = -\frac{\pi}{4}$

$\arg w = -\pi + \phi = -\frac{2\pi}{3}$

$\therefore\ \arg(zw) = \arg z + \arg w$

$= -\frac{\pi}{4} - \frac{2\pi}{3}$

$= -\frac{11\pi}{12}$

c $|z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$

Now, the real powers of $\frac{z}{kw}$ all lie on a circle centred at O.

$\therefore\ \left|\frac{z}{kw}\right| = 1$

$\therefore\ \frac{|z|}{|kw|} = 1$

$\therefore\ \frac{|z|}{k|w|} = 1 \qquad \{k > 0\}$

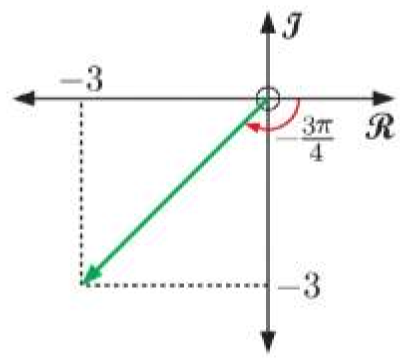
$\therefore\ \frac{2\sqrt{2}}{2k} = 1$

$\therefore\ k = \sqrt{2}$

23 **a** $|-3-3i| = 3\sqrt{2}, \quad \arg(-3-3i) = -\frac{3\pi}{4}$

$\therefore\ -3-3i = 3\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ (polar form)

$= 3\sqrt{2}e^{-i\frac{3\pi}{4}}$ (exponential form)



b $5\operatorname{cis}\left(-\frac{5\pi}{6}\right) = 5\cos\left(-\frac{5\pi}{6}\right) + 5i\sin\left(-\frac{5\pi}{6}\right)$

$= -\frac{5\sqrt{3}}{2} - \frac{5}{2}i$ (Cartesian form)

$= 5e^{-i\frac{5\pi}{6}}$ (exponential form)

c $7e^{i\frac{\pi}{2}} = 7\cos\frac{\pi}{2} + 7i\sin\frac{\pi}{2}$

$= 7i$ (Cartesian form)

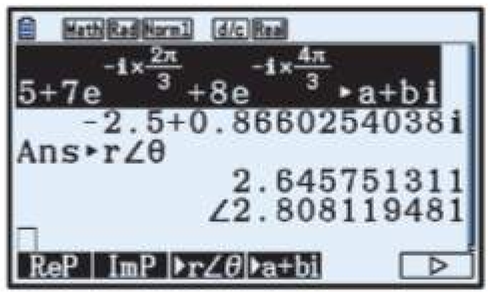
$= 7\operatorname{cis}\frac{\pi}{2}$ (polar form)

24 **a** $I(t) = 5\sin 10t + 7\sin\left(10t - \frac{2\pi}{3}\right) + 8\sin\left(10t - \frac{4\pi}{3}\right)$

$$= \mathcal{I}m\left(5\operatorname{cis} 10t + 7\operatorname{cis}\left(10t - \tfrac{2\pi}{3}\right) + 8\operatorname{cis}\left(10t - \tfrac{4\pi}{3}\right)\right)$$

$$= \mathcal{I}m\left(5e^{10ti} + 7e^{(10t-\frac{2\pi}{3})i} + 8e^{(10t-\frac{4\pi}{3})i}\right)$$

Now $\quad 5e^{10ti} + 7e^{(10t-\frac{2\pi}{3})i} + 8e^{(10t-\frac{4\pi}{3})i}$



$$= e^{10ti}\left(5 + 7e^{-i\frac{2\pi}{3}} + 8e^{-i\frac{4\pi}{3}}\right)$$

$$\approx e^{10ti}(-2.5 + 0.866i)$$

$$\approx e^{10ti}(2.65e^{2.81i})$$

$$\approx 2.65e^{(10t+2.81)i}$$

So, $\quad I(t) \approx \mathcal{I}m\left(2.65e^{(10t+2.81)i}\right)$

$$\approx 2.65\sin(10t + 2.81)$$

$$\approx 2.65\cos\left(10t + 2.81 - \tfrac{\pi}{2}\right)$$

$$\approx 2.65\cos(10t + 1.24)$$

b $I(t)$ has maximum value ≈ 2.65 when $\cos(10t + 1.24) \approx 1$.

Now $\quad t \geqslant 0$

$\therefore \quad 10t + 1.24 \geqslant 1.24$

So, the first maximum occurs when $\quad 10t + 1.24 \approx 2\pi$

$$\therefore \quad 10t \approx 2\pi - 1.24$$

$$\therefore \quad t \approx \frac{2\pi - 1.24}{10} \approx 0.505$$

$\therefore$ the highest current flowing through the circuit is about 2.65 milliamperes which occurs at about 0.505 ms.

Chapter 12
MATRICES

EXERCISE 12A

1 **a** $\begin{pmatrix} 5 & 1 & 0 & 2 \end{pmatrix}$ has 1 row and 4 columns.
$\therefore$ its order is 1×4.

b $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ has 2 rows and 1 column.
$\therefore$ its order is 2×1.

c $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ has 2 rows and 2 columns.
$\therefore$ its order is 2×2.

d $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$ has 3 rows and 3 columns.
$\therefore$ its order is 3×3.

2 $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ -3 & 1 \end{pmatrix}$

a -1 is in row 1, column 2 $\therefore$ $a_{12} = -1$.
b 0 is in row 2, column 1 $\therefore$ $a_{21} = 0$.
c -3 is in row 3, column 1 $\therefore$ $a_{31} = -3$.
d 1 is in row 3, column 2 $\therefore$ $a_{32} = 1$.

3 **a** The quantities matrix $\mathbf{Q} = \begin{array}{cccc} & & & \\ \end{array}\begin{pmatrix} 2 & 1 & 6 & 1 \end{pmatrix}$
$\qquad\qquad\qquad\qquad\qquad\qquad$ bread butter eggs cream

b The prices matrix $\mathbf{P} = \begin{array}{cl} \begin{pmatrix} 1.95 \\ 2.35 \\ 0.45 \\ 2.95 \end{pmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{butter} \\ \leftarrow \text{eggs} \\ \leftarrow \text{cream} \end{array} \end{array}$

c $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.45) + (1 \times 2.95)$ represents the total cost of the groceries.

4 **a**

The production level matrix $\mathbf{P} = \begin{array}{cl} \begin{array}{ccc} 200\text{ g} & 300\text{ g} & 500\text{ g} \\ \downarrow & \downarrow & \downarrow \end{array} & \\ \begin{pmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{pmatrix} & \begin{array}{l} \leftarrow \text{week 1} \\ \leftarrow \text{week 2} \\ \leftarrow \text{week 3} \\ \leftarrow \text{week 4} \end{array} \end{array}$

b $\mathbf{P}$ has 4 rows and 3 columns.
$\therefore$ its order is 4×3.

5

$$\begin{array}{cl} \begin{array}{cccc} \text{pies} & \text{pasties} & \text{rolls} & \text{buns} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} & \\ \begin{pmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{pmatrix} & \begin{array}{l} \leftarrow \text{Friday} \\ \leftarrow \text{Saturday} \\ \leftarrow \text{Sunday} \\ \leftarrow \text{Monday} \end{array} \end{array} \quad \text{(in dozens)}$$

EXERCISE 12B

1 **a** $a_{12} = 2$ and $b_{12} = -2$, so $a_{12} \neq b_{12}$

$\therefore$ $\mathbf{A} \neq \mathbf{B}$ as elements a_{12} and b_{12} are not equal.

b $\mathbf{A}$ has order 2×2 and $\mathbf{C}$ has order 2×3.

$\therefore$ $\mathbf{A} \neq \mathbf{C}$ as $\mathbf{A}$ and $\mathbf{C}$ do not have the same order.

2 **a** $\begin{pmatrix} x & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & y \end{pmatrix}$

Equating corresponding elements:

$x = 0$ and $1 = y$

$\therefore$ $x = 0$, $y = 1$

b $\begin{pmatrix} -2 & x \\ x-1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ y & y+1 \end{pmatrix}$

Equating corresponding elements:

$x = 3$, $x - 1 = y$, and $3 = y + 1$

$\therefore$ $x = 3$, $y = 2$

Checking in $x - 1 = y$, $3 - 1 = 2$ ✓

c $\begin{pmatrix} x & x^2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} y & 4 \\ 3 & y+1 \end{pmatrix}$

Equating corresponding elements:

$x = y$, $x^2 = 4$ and $-1 = y + 1$

$\therefore$ $x = \pm 2$ and $y = -2$

But $x = y$ $\therefore$ $x = -2$, $y = -2$

d $\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} -y & x \\ x & -y \end{pmatrix}$

Equating corresponding elements:

$x = -y$ and $y = x$

$\therefore$ $x = 0$, $y = 0$

EXERCISE 12C

1 **a**
$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3+6 & 4+(-3) \\ 5+(-2) & 2+1 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 1 \\ 3 & 3 \end{pmatrix} \end{aligned}$$

b
$$\begin{aligned} \mathbf{A} + \mathbf{B} + \mathbf{C} &= \begin{pmatrix} 9 & 1 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 9+(-3) & 1+7 \\ 3+(-4) & 3+(-2) \end{pmatrix} \\ &= \begin{pmatrix} 6 & 8 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

c
$$\begin{aligned} \mathbf{B} + \mathbf{C} &= \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 6+(-3) & -3+7 \\ -2+(-4) & 1+(-2) \end{pmatrix} \\ &= \begin{pmatrix} 3 & 4 \\ -6 & -1 \end{pmatrix} \end{aligned}$$

d
$$\begin{aligned} \mathbf{C} + \mathbf{B} - \mathbf{A} &= \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -3+6-3 & 7+(-3)-4 \\ -4+(-2)-5 & -2+1-2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -11 & -3 \end{pmatrix} \end{aligned}$$

2 **a** $\mathbf{P}+\mathbf{Q}$

$$= \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix} + \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} 3+17 & 5+(-4) & -11+3 \\ 10+(-2) & 2+8 & 6+(-8) \\ -2+3 & -1+(-4) & 7+11 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{pmatrix}$$

b $\mathbf{P}-\mathbf{Q}$

$$= \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix} - \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} 3-17 & 5-(-4) & -11-3 \\ 10-(-2) & 2-8 & 6-(-8) \\ -2-3 & -1-(-4) & 7-11 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{pmatrix}$$

c $$\mathbf{Q}-\mathbf{P} = \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix} - \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 17-3 & -4-5 & 3-(-11) \\ -2-10 & 8-2 & -8-6 \\ 3-(-2) & -4-(-1) & 11-7 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{pmatrix}$$

3 **a** Friday

$$\begin{array}{cl} \begin{pmatrix} 85 \\ 92 \\ 52 \end{pmatrix} & \begin{array}{l} \text{men} \\ \text{women} \\ \text{children} \end{array} \end{array}$$

Saturday

$$\begin{array}{cl} \begin{pmatrix} 102 \\ 137 \\ 49 \end{pmatrix} & \begin{array}{l} \text{men} \\ \text{women} \\ \text{children} \end{array} \end{array}$$

b Total for Friday and Saturday

$$= \begin{pmatrix} 85 \\ 92 \\ 52 \end{pmatrix} + \begin{pmatrix} 102 \\ 137 \\ 49 \end{pmatrix}$$

$$= \begin{pmatrix} 85+102 \\ 92+137 \\ 52+49 \end{pmatrix} = \begin{array}{cl} \begin{pmatrix} 187 \\ 229 \\ 101 \end{pmatrix} & \begin{array}{l} \text{men} \\ \text{women} \\ \text{children} \end{array} \end{array}$$

$\therefore$ over the two nights, a total of 187 men, 229 women, and 101 children were served.

4

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

a **i** Cost price

$$\begin{array}{cl} \begin{pmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{pmatrix} & \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} \end{array}$$

ii Selling price

$$\begin{array}{cl} \begin{pmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{pmatrix} & \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} \end{array}$$

b In order to find David's profit/loss for each type of share, we subtract the cost price matrix from the selling price matrix.

c Profit/loss matrix $= \begin{pmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{pmatrix} - \begin{pmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{pmatrix}$

$$= \begin{pmatrix} 1.79 - 1.72 \\ 28.75 - 27.85 \\ 1.33 - 0.92 \\ 2.25 - 2.53 \\ 3.51 - 3.56 \end{pmatrix}$$

$$= \begin{array}{c} \begin{pmatrix} 0.07 \\ 0.90 \\ 0.41 \\ -0.28 \\ -0.05 \end{pmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}$$

5 **a**
$$\begin{array}{cc} \text{Lou} & \text{Rose} \end{array}$$
$$\begin{pmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{pmatrix} \begin{array}{l} \text{fridges} \\ \text{stoves} \\ \text{microwaves} \end{array}$$

b
$$\begin{array}{cc} \text{Lou} & \text{Rose} \end{array}$$
$$\begin{pmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{pmatrix} \begin{array}{l} \text{fridges} \\ \text{stoves} \\ \text{microwaves} \end{array}$$

c Total sales for November and December

$$= \begin{pmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{pmatrix} + \begin{pmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{pmatrix}$$

$$= \begin{pmatrix} 23+18 & 19+25 \\ 17+7 & 29+13 \\ 31+36 & 24+19 \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{pmatrix} \begin{array}{l} \text{fridges} \\ \text{stoves} \\ \text{microwaves} \end{array}$$

6 **a** $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 2+-1 & 1+2 \\ 3+2 & -1+3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$$

$\mathbf{B} + \mathbf{A} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} -1+2 & 2+1 \\ 2+3 & 3+(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$$

b $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ because the addition of numbers is commutative.

7 **a** $$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \left(\begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}\right) + \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -1+3 & 0+4 \\ 1+(-1) & 5+(-2) \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 3 \\ -1 & 6 \end{pmatrix}$$

$$\mathbf{A}+(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix} + \left(\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}\right)$$
$$= \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 3+4 & 4+(-1) \\ -1+(-1) & -2+3 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 7 & 3 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 6 & 3 \\ -1 & 6 \end{pmatrix}$$

b Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

$$\therefore\ (\mathbf{A}+\mathbf{B})+\mathbf{C} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) + \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$
$$= \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$
$$= \begin{pmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{pmatrix}$$

and $$\mathbf{A}+(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix}\right)$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p+w & q+x \\ r+y & s+z \end{pmatrix}$$
$$= \begin{pmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{pmatrix}$$
$$= (\mathbf{A}+\mathbf{B})+\mathbf{C}$$

EXERCISE 12D

1 **a** $2\mathbf{B} = 2\begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$

$= \begin{pmatrix} 2\times 6 & 2\times 12 \\ 2\times 24 & 2\times 6 \end{pmatrix}$

$= \begin{pmatrix} 12 & 24 \\ 48 & 12 \end{pmatrix}$

b $\frac{1}{3}\mathbf{B} = \frac{1}{3}\begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{3}\times 6 & \frac{1}{3}\times 12 \\ \frac{1}{3}\times 24 & \frac{1}{3}\times 6 \end{pmatrix}$

$= \begin{pmatrix} 2 & 4 \\ 8 & 2 \end{pmatrix}$

c $\frac{1}{12}\mathbf{B} = \frac{1}{12}\begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{12}\times 6 & \frac{1}{12}\times 12 \\ \frac{1}{12}\times 24 & \frac{1}{12}\times 6 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{pmatrix}$

d $-\frac{1}{2}\mathbf{B} = -\frac{1}{2}\begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2}\times 6 & -\frac{1}{2}\times 12 \\ -\frac{1}{2}\times 24 & -\frac{1}{2}\times 6 \end{pmatrix}$

$= \begin{pmatrix} -3 & -6 \\ -12 & -3 \end{pmatrix}$

2 **a** $\mathbf{A} + \mathbf{B}$

$= \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2+1 & 3+2 & 5+1 \\ 1+1 & 6+2 & 4+3 \end{pmatrix}$

$= \begin{pmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{pmatrix}$

b $\mathbf{A} - \mathbf{B}$

$= \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2-1 & 3-2 & 5-1 \\ 1-1 & 6-2 & 4-3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}$

c $2\mathbf{A} + \mathbf{B}$

$= 2\begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2\times 2 & 2\times 3 & 2\times 5 \\ 2\times 1 & 2\times 6 & 2\times 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 4 & 6 & 10 \\ 2 & 12 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 4+1 & 6+2 & 10+1 \\ 2+1 & 12+2 & 8+3 \end{pmatrix}$

$= \begin{pmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{pmatrix}$

d $3\mathbf{A} - \mathbf{B}$

$= 3\begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3\times 2 & 3\times 3 & 3\times 5 \\ 3\times 1 & 3\times 6 & 3\times 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 6 & 9 & 15 \\ 3 & 18 & 12 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 6-1 & 9-2 & 15-1 \\ 3-1 & 18-2 & 12-3 \end{pmatrix}$

$= \begin{pmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{pmatrix}$

3 The matrix is $12\mathbf{A} = 12\begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 12\times 1 \\ 12\times 4 \\ 12\times 2 \\ 12\times 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 48 \\ 24 \\ 12 \end{pmatrix}$

4 **a** Weekdays

$$\mathbf{A} = \begin{pmatrix} 75 \\ 27 \\ 102 \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{Blu-ray} \\ \leftarrow \text{games} \end{array}$$

Weekends

$$\mathbf{B} = \begin{pmatrix} 136 \\ 43 \\ 129 \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{Blu-ray} \\ \leftarrow \text{games} \end{array}$$

b $5\mathbf{A} + 2\mathbf{B}$

$$= 5\begin{pmatrix} 75 \\ 27 \\ 102 \end{pmatrix} + 2\begin{pmatrix} 136 \\ 43 \\ 129 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \times 75 \\ 5 \times 27 \\ 5 \times 102 \end{pmatrix} + \begin{pmatrix} 2 \times 136 \\ 2 \times 43 \\ 2 \times 129 \end{pmatrix}$$

$$= \begin{pmatrix} 375 \\ 135 \\ 510 \end{pmatrix} + \begin{pmatrix} 272 \\ 86 \\ 258 \end{pmatrix}$$

$$= \begin{pmatrix} 375 + 272 \\ 135 + 86 \\ 510 + 258 \end{pmatrix}$$

$$= \begin{pmatrix} 647 \\ 221 \\ 768 \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD} \\ \leftarrow \text{Blu-ray} \\ \leftarrow \text{games} \end{array}$$

This matrix represents the total weekly average rentals.

5 **a** Annual order $= 12 \times$ monthly order

$$= 12\begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \times 30 & 12 \times 40 & 12 \times 40 & 12 \times 60 \\ 12 \times 50 & 12 \times 40 & 12 \times 30 & 12 \times 75 \\ 12 \times 40 & 12 \times 40 & 12 \times 50 & 12 \times 50 \\ 12 \times 10 & 12 \times 20 & 12 \times 20 & 12 \times 15 \end{pmatrix}$$

$$= \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 360 & 480 & 480 & 720 \\ 600 & 480 & 360 & 900 \\ 480 & 480 & 600 & 600 \\ 120 & 240 & 240 & 180 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array}$$

b **i** Increase of 15% $= 1.15\begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix}$

$$= \begin{pmatrix} 1.15\times 30 & 1.15\times 40 & 1.15\times 40 & 1.15\times 60 \\ 1.15\times 50 & 1.15\times 40 & 1.15\times 30 & 1.15\times 75 \\ 1.15\times 40 & 1.15\times 40 & 1.15\times 50 & 1.15\times 50 \\ 1.15\times 10 & 1.15\times 20 & 1.15\times 20 & 1.15\times 15 \end{pmatrix}$$

$$\approx \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array}$$

rounded to the nearest whole number

ii Decrease of 15% $= 0.85\begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix}$

$$= \begin{pmatrix} 0.85\times 30 & 0.85\times 40 & 0.85\times 40 & 0.85\times 60 \\ 0.85\times 50 & 0.85\times 40 & 0.85\times 30 & 0.85\times 75 \\ 0.85\times 40 & 0.85\times 40 & 0.85\times 50 & 0.85\times 50 \\ 0.85\times 10 & 0.85\times 20 & 0.85\times 20 & 0.85\times 15 \end{pmatrix}$$

$$\approx \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array}$$

rounded to the nearest whole number

EXERCISE 12E

1 **a** $\mathbf{A} + 2\mathbf{A} = 3\mathbf{A}$

b $3\mathbf{B} - 3\mathbf{B} = \mathbf{O}$

c $\mathbf{C} - 2\mathbf{C} = -\mathbf{C}$

d $-\mathbf{B} + \mathbf{B} = \mathbf{O}$

e $2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$

f $-(\mathbf{A} + \mathbf{B}) = -\mathbf{A} - \mathbf{B}$

g $\begin{aligned} &-(2\mathbf{A} - \mathbf{C}) \\ &= -2\mathbf{A} + \mathbf{C} \end{aligned}$

h $\begin{aligned} &3\mathbf{A} - (\mathbf{B} - \mathbf{A}) \\ &= 3\mathbf{A} - \mathbf{B} + \mathbf{A} \\ &= 4\mathbf{A} - \mathbf{B} \end{aligned}$

i $\begin{aligned} &\mathbf{A} + 2\mathbf{B} - (\mathbf{A} - \mathbf{B}) \\ &= \mathbf{A} + 2\mathbf{B} - \mathbf{A} + \mathbf{B} \\ &= 3\mathbf{B} \end{aligned}$

2 **a**
$$\begin{aligned} \mathbf{X} + \mathbf{B} &= \mathbf{A} \\ \therefore\ \mathbf{X} + \mathbf{B} + (-\mathbf{B}) &= \mathbf{A} + (-\mathbf{B}) \\ \therefore\ \mathbf{X} + \mathbf{O} &= \mathbf{A} - \mathbf{B} \\ \therefore\ \mathbf{X} &= \mathbf{A} - \mathbf{B} \end{aligned}$$

b
$$\begin{aligned} \mathbf{B} + \mathbf{X} &= \mathbf{C} \\ \therefore\ \mathbf{B} + \mathbf{X} + (-\mathbf{B}) &= \mathbf{C} + (-\mathbf{B}) \\ \therefore\ \mathbf{O} + \mathbf{X} &= \mathbf{C} - \mathbf{B} \\ \therefore\ \mathbf{X} &= \mathbf{C} - \mathbf{B} \end{aligned}$$

c
$$\begin{aligned} 4\mathbf{B}+\mathbf{X} &= 2\mathbf{C} \\ \therefore\ 4\mathbf{B}+\mathbf{X}+(-4\mathbf{B}) &= 2\mathbf{C}+(-4\mathbf{B}) \\ \therefore\ \mathbf{O}+\mathbf{X} &= 2\mathbf{C}-4\mathbf{B} \\ \therefore\ \mathbf{X} &= 2\mathbf{C}-4\mathbf{B} \end{aligned}$$

d
$$\begin{aligned} 2\mathbf{X} &= \mathbf{A} \\ \therefore\ \tfrac{1}{2}(2\mathbf{X}) &= \tfrac{1}{2}\mathbf{A} \\ \therefore\ 1\mathbf{X} &= \tfrac{1}{2}\mathbf{A} \\ \therefore\ \mathbf{X} &= \tfrac{1}{2}\mathbf{A} \end{aligned}$$

e
$$\begin{aligned} 3\mathbf{X} &= \mathbf{B} \\ \therefore\ \tfrac{1}{3}(3\mathbf{X}) &= \tfrac{1}{3}\mathbf{B} \\ \therefore\ 1\mathbf{X} &= \tfrac{1}{3}\mathbf{B} \\ \therefore\ \mathbf{X} &= \tfrac{1}{3}\mathbf{B} \end{aligned}$$

f
$$\begin{aligned} \mathbf{A}-\mathbf{X} &= \mathbf{B} \\ \therefore\ \mathbf{A}-\mathbf{X}+\mathbf{X} &= \mathbf{B}+\mathbf{X} \\ \therefore\ \mathbf{A}+\mathbf{O} &= \mathbf{B}+\mathbf{X} \\ \therefore\ \mathbf{A} &= \mathbf{B}+\mathbf{X} \\ \text{and}\quad \mathbf{A}+(-\mathbf{B}) &= \mathbf{B}+\mathbf{X}+(-\mathbf{B}) \\ \therefore\ \mathbf{A}-\mathbf{B} &= \mathbf{X}+\mathbf{O} \\ \therefore\ \mathbf{A}-\mathbf{B} &= \mathbf{X} \\ \therefore\ \mathbf{X} &= \mathbf{A}-\mathbf{B} \end{aligned}$$

g
$$\begin{aligned} \tfrac{1}{2}\mathbf{X} &= \mathbf{C} \\ \therefore\ 2(\tfrac{1}{2}\mathbf{X}) &= 2\mathbf{C} \\ \therefore\ 1\mathbf{X} &= 2\mathbf{C} \\ \therefore\ \mathbf{X} &= 2\mathbf{C} \end{aligned}$$

h
$$\begin{aligned} 2(\mathbf{X}+\mathbf{A}) &= \mathbf{B} \\ \therefore\ \tfrac{1}{2}\,[2(\mathbf{X}+\mathbf{A})] &= \tfrac{1}{2}\mathbf{B} \\ \therefore\ 1(\mathbf{X}+\mathbf{A}) &= \tfrac{1}{2}\mathbf{B} \\ \therefore\ \mathbf{X}+\mathbf{A} &= \tfrac{1}{2}\mathbf{B} \\ \therefore\ \mathbf{X}+\mathbf{A}+(-\mathbf{A}) &= \tfrac{1}{2}\mathbf{B}+(-\mathbf{A}) \\ \therefore\ \mathbf{X}+\mathbf{O} &= \tfrac{1}{2}\mathbf{B}-\mathbf{A} \\ \therefore\ \mathbf{X} &= \tfrac{1}{2}\mathbf{B}-\mathbf{A} \end{aligned}$$

i
$$\begin{aligned} \mathbf{A}-4\mathbf{X} &= \mathbf{C} \\ \therefore\ \mathbf{A}-4\mathbf{X}+4\mathbf{X} &= \mathbf{C}+4\mathbf{X} \\ \therefore\ \mathbf{A}+\mathbf{O} &= \mathbf{C}+4\mathbf{X} \\ \therefore\ \mathbf{A} &= \mathbf{C}+4\mathbf{X} \\ \text{and}\quad \mathbf{A}+(-\mathbf{C}) &= \mathbf{C}+4\mathbf{X}+(-\mathbf{C}) \\ \therefore\ \mathbf{A}-\mathbf{C} &= 4\mathbf{X}+\mathbf{O} \\ \therefore\ \mathbf{A}-\mathbf{C} &= 4\mathbf{X} \\ \therefore\ \tfrac{1}{4}(\mathbf{A}-\mathbf{C}) &= \tfrac{1}{4}(4\mathbf{X}) \\ \therefore\ \tfrac{1}{4}(\mathbf{A}-\mathbf{C}) &= 1\mathbf{X} \\ \therefore\ \mathbf{X} &= \tfrac{1}{4}(\mathbf{A}-\mathbf{C}) \end{aligned}$$

3 **a**
$$\begin{aligned} \text{If}\quad \tfrac{1}{3}\mathbf{X} &= \mathbf{M} \\ \text{then}\quad 3(\tfrac{1}{3}\mathbf{X}) &= 3\mathbf{M} \\ \therefore\ \mathbf{X} &= 3\mathbf{M} \\ &= 3\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix} \end{aligned}$$

b
$$\begin{aligned} \text{If}\quad 4\mathbf{X} &= \mathbf{N} \\ \text{then}\quad \tfrac{1}{4}(4\mathbf{X}) &= \tfrac{1}{4}\mathbf{N} \\ \therefore\ \mathbf{X} &= \tfrac{1}{4}\mathbf{N} \\ \therefore\ \mathbf{X} &= \tfrac{1}{4}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \end{aligned}$$

c If $\mathbf{A} - 2\mathbf{X} = 3\mathbf{B}$

then $\mathbf{A} - 2\mathbf{X} + 2\mathbf{X} = 3\mathbf{B} + 2\mathbf{X}$

$$\therefore \mathbf{A} = 3\mathbf{B} + 2\mathbf{X}$$
$$\therefore \mathbf{A} + (-3\mathbf{B}) = 3\mathbf{B} + 2\mathbf{X} + (-3\mathbf{B})$$
$$\therefore \mathbf{A} - 3\mathbf{B} = 2\mathbf{X} + \mathbf{O}$$
$$\therefore \mathbf{A} - 3\mathbf{B} = 2\mathbf{X}$$
$$\therefore \tfrac{1}{2}(\mathbf{A} - 3\mathbf{B}) = \tfrac{1}{2}(2\mathbf{X})$$
$$\therefore \tfrac{1}{2}(\mathbf{A} - 3\mathbf{B}) = 1\mathbf{X}$$
$$\therefore \mathbf{X} = \tfrac{1}{2}(\mathbf{A} - 3\mathbf{B})$$
$$= \tfrac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}\right)$$
$$= \tfrac{1}{2}\begin{pmatrix} -2 & -12 \\ 2 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -6 \\ 1 & -\frac{1}{2} \end{pmatrix}$$

EXERCISE 12F.1

1 **a** $\begin{pmatrix} 3 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = (3 \times 5) + ((-1) \times 4)$
$= 15 - 4$
$= 11$

b $\begin{pmatrix} 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} = (1 \times 5) + (3 \times 1) + (2 \times 7)$
$= 5 + 3 + 14$
$= 22$

c $\begin{pmatrix} 6 & -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \end{pmatrix} = (6 \times 1) + ((-1) \times 0) + (2 \times (-1)) + (3 \times 4)$
$= 6 + 0 - 2 + 12$
$= 16$

2 **a** $\begin{pmatrix} w & x & y & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (w \times 1) + (x \times 1) + (y \times 1) + (z \times 1)$
$= w + x + y + z$

b $\frac{1}{4}(w + x + y + z)$, which is the average of w, x, y, and z.

This can be represented as $\begin{pmatrix} w & x & y & z \end{pmatrix}\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$.

3 **a** $\mathbf{Q} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 27 & 35 & 39 \end{pmatrix}$

b total cost $= \mathbf{PQ} = \begin{pmatrix} 27 & 35 & 39 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$= (27 \times 4) + (35 \times 3) + (39 \times 2)$
$= 108 + 105 + 78$
$= 291$

$\therefore$ total cost is \$291

4 **a** $\mathbf{P} = \begin{pmatrix} 10 & 6 & 3 & 1 \end{pmatrix}$

$\mathbf{N} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 2 \end{pmatrix}$

b total points $= \mathbf{PN} = \begin{pmatrix} 10 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \\ 2 \end{pmatrix}$

$= (10 \times 3) + (6 \times 2) + (3 \times 4) + (1 \times 2)$
$= 30 + 12 + 12 + 2$
$= 56$

So, the total number of points awarded is 56.

EXERCISE 12F.2

1 $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$ which is 1 row $\times$ 3 columns,

$\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ which is 2 rows $\times$ 3 columns.

AB cannot be found because the number of columns in **A** does not equal the number of rows in **B**.

2 **A** is $2 \times n$ and **B** is $m \times 3$.

a We can find **AB** if the number of columns in **A** equals the number of rows in **B**.
This occurs when $n = m$.

b If **AB** can be found its order is 2×3.

c **BA** cannot be found because the number of columns in **B** does not equal the number of rows in **A**.

3 **a** **A** is 2×2 and **B** is 1×2 $\therefore$ **AB** does not exist.
not equal

b **B** is 1×2 and **A** is 2×2
✓

$\therefore$ **BA** is 1×2

$\mathbf{BA} = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 5 \times 2 + 6 \times 3 & 5 \times 1 + 6 \times 4 \end{pmatrix}$
$= \begin{pmatrix} 10 + 18 & 5 + 24 \end{pmatrix}$
$= \begin{pmatrix} 28 & 29 \end{pmatrix}$

4 **a** **A** is 1×3 and **B** is 3×1, $\therefore$ **AB** is 1×1
✓

$\mathbf{AB} = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = (2 \times 1) + (0 \times 4) + (3 \times 2) = 2 + 0 + 6 = 8$

b $\mathbf{B}$ is 3×1 and $\mathbf{A}$ is 1×3, $\quad \therefore$ $\mathbf{BA}$ is 3×3

$$\mathbf{BA} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 \times 2 & 1 \times 0 & 1 \times 3 \\ 4 \times 2 & 4 \times 0 & 4 \times 3 \\ 2 \times 2 & 2 \times 0 & 2 \times 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{pmatrix}$$

5 **a** $\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$ is 2×2 by 2×2 $\quad \therefore$ resultant matrix is 2×2

$$= \begin{pmatrix} 3 \times (-1) + 1 \times 2 & 3 \times 1 + 1 \times (-1) \\ (-1) \times (-1) + 0 \times 2 & (-1) \times 1 + 0 \times (-1) \end{pmatrix} = \begin{pmatrix} -3+2 & 3-1 \\ 1+0 & -1+0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

b $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -1 \end{pmatrix}$ is 2×2 by 2×2 $\quad \therefore$ resultant matrix is 2×2

$$= \begin{pmatrix} 2 \times 1 + 1 \times 2 & 2 \times (-3) + 1 \times (-1) \\ 0 \times 1 + (-1) \times 2 & 0 \times (-3) + (-1) \times (-1) \end{pmatrix} = \begin{pmatrix} 2+2 & -6-1 \\ 0-2 & 0+1 \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ -2 & 1 \end{pmatrix}$$

c $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ is 1×3 by 3×3 $\quad \therefore$ resultant matrix is 1×3

$$= \begin{pmatrix} 1 \times 2 + 2 \times 0 + 1 \times 1 & 1 \times 3 + 2 \times 1 + 1 \times 0 & 1 \times 1 + 2 \times 0 + 1 \times 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2+0+1 & 3+2+0 & 1+0+2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 5 & 3 \end{pmatrix}$$

d $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is 3×3 by 3×1 $\quad \therefore$ resultant matrix is 3×1

$$= \begin{pmatrix} 1 \times 2 + 0 \times 3 + (-1) \times 4 \\ (-1) \times 2 + 1 \times 3 + 0 \times 4 \\ 0 \times 2 + (-1) \times 3 + 1 \times 4 \end{pmatrix} = \begin{pmatrix} 2+0-4 \\ -2+3+0 \\ 0-3+4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

6 **a**

The quantities matrix $\mathbf{Q} = \begin{array}{c} \begin{array}{cc} \text{Pentex} & \text{Rollerball} \end{array} \\ \begin{pmatrix} 32 & 24 \\ 25 & 16 \\ 13 & 9 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Blue} \\ \text{Black} \\ \text{Red} \end{array}$

	Brand	
Colour	*Pentex*	*Rollerball*
Blue	32	24
Black	25	16
Red	13	9

b The price matrix $\mathbf{P} = \begin{pmatrix} 1.19 \\ 1.55 \end{pmatrix} \begin{array}{l} \text{Pentex} \\ \text{Rollerball} \end{array}$

c $\mathbf{Q}$ is 3×2 and $\mathbf{P}$ is 2×1 $\therefore$ $\mathbf{QP}$ is 3×1

$$\mathbf{QP} = \begin{pmatrix} 32 & 24 \\ 25 & 16 \\ 13 & 9 \end{pmatrix} \begin{pmatrix} 1.19 \\ 1.55 \end{pmatrix}$$

$$= \begin{pmatrix} 32 \times 1.19 + 24 \times 1.55 \\ 25 \times 1.19 + 16 \times 1.55 \\ 13 \times 1.19 + 9 \times 1.55 \end{pmatrix}$$

$$= \begin{pmatrix} 75.28 \\ 54.55 \\ 29.42 \end{pmatrix}$$

The matrix $\mathbf{QP}$ represents the total sales of each colour of pen for the week.

d Total revenue $= \$(75.28 + 54.55 + 29.42)$
$= \$159.25$

7 **a**

$$\mathbf{P} = \begin{array}{cl} \begin{pmatrix} 12.5 \\ 9.5 \end{pmatrix} & \begin{array}{l} \text{adults} \\ \text{children} \end{array} \end{array} \qquad \mathbf{N} = \begin{array}{cl} \begin{array}{cc} \text{adults} & \text{children} \end{array} & \\ \begin{pmatrix} 2375 & 5156 \\ 2502 & 3612 \end{pmatrix} & \begin{array}{l} \text{first day} \\ \text{second day} \end{array} \end{array}$$

b $\mathbf{N}$ is 2×2 and $\mathbf{P}$ is 2×1 $\therefore$ $\mathbf{NP}$ is 2×1

$$\mathbf{NP} = \begin{pmatrix} 2375 & 5156 \\ 2502 & 3612 \end{pmatrix} \begin{pmatrix} 12.5 \\ 9.5 \end{pmatrix}$$

$$= \begin{pmatrix} 2375 \times 12.5 + 5156 \times 9.5 \\ 2502 \times 12.5 + 3612 \times 9.5 \end{pmatrix}$$

$$= \begin{array}{cl} \begin{pmatrix} 78\,669.5 \\ 65\,589 \end{pmatrix} & \begin{array}{l} \text{income from day 1} \\ \text{income from day 2} \end{array} \end{array}$$

$\mathbf{NP}$ represents the total income on each day in pounds.

c Total income $= £(78\,669.50 + 65\,589)$
$= £144\,258.50$

8

	Hammer	*Screwdriver*	*Can of paint*
Store A	\$14	\$6	\$38
Store B	\$12	\$4	\$44

a

$$\mathbf{R} = \begin{array}{cl} \begin{array}{cc} \text{me} & \text{friend} \end{array} & \\ \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} & \begin{array}{l} \text{hammers} \\ \text{screwdrivers} \\ \text{cans of paint} \end{array} \end{array}$$

b

$$\mathbf{P} = \begin{array}{cl} \begin{array}{ccc} \text{hammer} & \text{screwdriver} & \text{paint} \end{array} & \\ \begin{pmatrix} 14 & 6 & 38 \\ 12 & 4 & 44 \end{pmatrix} & \begin{array}{l} \text{store A} \\ \text{store B} \end{array} \end{array}$$

c **P** is 2×3 and **R** is 3×2, $\therefore$ **PR** is 2×2

$$\mathbf{PR} = \begin{pmatrix} 14 & 6 & 38 \\ 12 & 4 & 44 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \times 1 + 6 \times 1 + 38 \times 2 & 14 \times 1 + 6 \times 2 + 38 \times 3 \\ 12 \times 1 + 4 \times 1 + 44 \times 2 & 12 \times 1 + 4 \times 2 + 44 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 14 + 6 + 76 & 14 + 12 + 114 \\ 12 + 4 + 88 & 12 + 8 + 132 \end{pmatrix}$$

$$= \begin{pmatrix} 96 & 140 \\ 104 & 152 \end{pmatrix}$$

d **i** My costs at Store A are \$96. **ii** My friend's costs at Store B are \$152.

e No, the elements of **PR** only tell us that, if *all* the items are to be bought at one store, it is cheapest to do so at store A for both you and your friend. However, the cheapest way is to buy paint from store A, and hammers and screwdrivers from store B.

INVESTIGATION 1 MATRIX MULTIPLICATION

1 **AB** and **BA**

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 \times (-1) + 0 \times 0 & 1 \times 1 + 0 \times 3 \\ 1 \times (-1) + 2 \times 0 & 1 \times 1 + 2 \times 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 7 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 \times 1 + 1 \times 1 & -1 \times 0 + 1 \times 2 \\ 0 \times 1 + 3 \times 1 & 0 \times 0 + 3 \times 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 6 \end{pmatrix} \neq \mathbf{AB}$$

So no, for a pair of 2×2 matrices **A** and **B**, **AB** is not necessarily equal to **BA**.

2 **AO** and **OA**

$$\mathbf{AO} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a \times 0 + b \times 0 & a \times 0 + b \times 0 \\ c \times 0 + d \times 0 & c \times 0 + d \times 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{O}$$

$$\mathbf{OA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \times a + 0 \times c & 0 \times b + 0 \times d \\ 0 \times a + 0 \times c & 0 \times b + 0 \times d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{O} = \mathbf{AO}$$

3 **a** $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \times 0 + 0 \times 0 & 1 \times 0 + 0 \times 1 \\ 0 \times 0 + 0 \times 0 & 0 \times 0 + 0 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{O}$$

b $\mathbf{AB} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

$= \begin{pmatrix} 4\times 1 + -2\times 2 & 4\times(-3)+(-2)\times(-6) \\ -2\times 1+1\times 2 & -2\times(-3)+1\times(-6) \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{O}$

4 **a** Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 2 \\ 0 & 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$\mathbf{B}+\mathbf{C} = \begin{pmatrix} -3 & 2 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\therefore\ \mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1\times(-2)+1\times 1 & 1\times 1+1\times 3 \\ 2\times(-2)+3\times 1 & 2\times 1+3\times 3 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 11 \end{pmatrix}$$

Now $\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1\times(-3)+1\times 0 & 1\times 2+1\times 2 \\ 2\times(-3)+3\times 0 & 2\times 2+3\times 2 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -6 & 10 \end{pmatrix}$

and $\mathbf{AC} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1\times 1+1\times 1 & 1\times(-1)+1\times 1 \\ 2\times 1+3\times 1 & 2\times(-1)+3\times 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}$

$$\therefore\ \mathbf{AB}+\mathbf{AC} = \begin{pmatrix} -3 & 4 \\ -6 & 10 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 11 \end{pmatrix} = \mathbf{A}(\mathbf{B}+\mathbf{C})$$

b $\mathbf{B}+\mathbf{C} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} p+w & q+x \\ r+y & s+z \end{pmatrix}$

$$\therefore\ \mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p+w & q+x \\ r+y & s+z \end{pmatrix} = \begin{pmatrix} a(p+w)+b(r+y) & a(q+x)+b(s+z) \\ c(p+w)+d(r+y) & c(q+x)+d(s+z) \end{pmatrix}$$

Now $\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$

and $\mathbf{AC} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$

$$\therefore\ \mathbf{AB}+\mathbf{AC} = \begin{pmatrix} ap+br+aw+by & aq+bs+ax+bz \\ cp+dr+cw+dy & cq+ds+cx+dz \end{pmatrix}$$

$$= \begin{pmatrix} a(p+w)+b(r+y) & a(q+x)+b(s+z) \\ c(p+w)+d(r+y) & c(q+x)+d(s+z) \end{pmatrix}$$

$$= \mathbf{A}(\mathbf{B}+\mathbf{C})$$

c Using **A**, **B**, and **C** as defined in **a**, we have

$$\mathbf{AB} = \begin{pmatrix} -3 & 4 \\ -6 & 10 \end{pmatrix}$$

$$\therefore\ (\mathbf{AB})\mathbf{C} = \begin{pmatrix} -3 & 4 \\ -6 & 10 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3\times 1 + 4\times 1 & -3\times(-1) + 4\times 1 \\ -6\times 1 + 10\times 1 & -6\times(-1) + 10\times 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 4 & 16 \end{pmatrix}$$

Now $\mathbf{BC} = \begin{pmatrix} -3 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -3\times 1 + 2\times 1 & -3\times(-1) + 2\times 1 \\ 0\times 1 + 2\times 1 & 0\times(-1) + 2\times 1 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 2 & 2 \end{pmatrix}$$

$$\therefore\ \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 5 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times(-1) + 1\times 2 & 1\times 5 + 1\times 2 \\ 2\times(-1) + 3\times 2 & 2\times 5 + 3\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 4 & 16 \end{pmatrix} = (\mathbf{AB})\mathbf{C}$$

d Using **A**, **B**, and **C** as defined in **b**, we have

$$\mathbf{AB} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

$$\therefore\ (\mathbf{AB})\mathbf{C} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$$= \begin{pmatrix} w(ap+br)+y(aq+bs) & x(ap+br)+z(aq+bs) \\ w(cp+dr)+y(cq+ds) & x(cp+dr)+z(cq+ds) \end{pmatrix}$$

Now $\mathbf{BC} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{pmatrix}$

$$\therefore\ \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{pmatrix}$$

$$= \begin{pmatrix} a(pw+qy)+b(rw+sy) & a(px+qz)+b(rx+sz) \\ c(pw+qy)+d(rw+sy) & c(px+qz)+d(rx+sz) \end{pmatrix}$$

$$= \begin{pmatrix} w(ap+br)+y(aq+bs) & x(ap+br)+z(aq+bs) \\ w(cp+dr)+y(cq+ds) & x(cp+dr)+z(cq+ds) \end{pmatrix}$$

$$= (\mathbf{AB})\mathbf{C}$$

5 **a** If $w = z = 1$ and $x = y = 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a\times 1 + b\times 0 & a\times 0 + b\times 1 \\ c\times 1 + d\times 0 & c\times 0 + d\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

b Letting $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we saw in **a** that $\mathbf{AI} = \mathbf{A}$.

Now $\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1\times a + 0\times c & 1\times b + 0\times d \\ 0\times a + 1\times c & 0\times b + 1\times d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{A}$

So, the matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$.

6 **a**
$$\begin{aligned}\mathbf{A}^2 = \mathbf{AA} &= \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2\times 2 + 1\times 3 & 2\times 1 + 1\times(-2) \\ 3\times 2 + (-2)\times 3 & 3\times 1 + (-2)\times(-2) \end{pmatrix} \\ &= \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\end{aligned}$$

b
$$\begin{aligned}\mathbf{A}^3 &= \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 5\times 5 + (-1)\times 2 & 5\times(-1) + (-1)\times 4 \\ 2\times 5 + 4\times 2 & 2\times(-1) + 4\times 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 23 & -9 \\ 18 & 14 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 23\times 5 + (-9)\times 2 & 23\times(-1) + (-9)\times 4 \\ 18\times 5 + 14\times 2 & 18\times(-1) + 14\times 4 \end{pmatrix} \\ &= \begin{pmatrix} 97 & -59 \\ 118 & 38 \end{pmatrix}\end{aligned}$$

c $\mathbf{A}$ is 3×2, so $\mathbf{A}^2$ gives $3\times \underset{\uparrow}{2}$ by $\underset{\uparrow}{3}\times 2$ $\quad\therefore\ \mathbf{A}^2$ does not exist.

not equal

d We can square a matrix if it has dimensions $n\times n$, where n is a positive integer.

7 $\mathbf{I}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 0\times 0 & 1\times 0 + 0\times 1 \\ 0\times 1 + 1\times 0 & 0\times 0 + 1\times 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

Now, using **4 d** we know that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ for any 2×2 matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

$\therefore\ \mathbf{I}^3 = (\mathbf{II})\mathbf{I} = \mathbf{I}^2\mathbf{I} = \mathbf{II} = \mathbf{I}^2 = \mathbf{I}$.

EXERCISE 12F.3

1 **a**
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+\mathbf{I}) \\ &= \mathbf{A}^2 + \mathbf{AI} \\ &= \mathbf{A}^2 + \mathbf{A}\end{aligned}$$

b
$$\begin{aligned}&(\mathbf{B}+2\mathbf{I})\mathbf{B} \\ &= \mathbf{B}^2 + 2\mathbf{IB} \\ &= \mathbf{B}^2 + 2\mathbf{B}\end{aligned}$$

c
$$\begin{aligned}&\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}) \\ &= \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{AI} \\ &= \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}\end{aligned}$$

d
$$\begin{aligned}&\mathbf{A}(\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I}) \\ &= \mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{AI} \\ &= \mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{A}\end{aligned}$$

e
$$\begin{aligned}&(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) \\ &= (\mathbf{A}+\mathbf{B})\mathbf{C} + (\mathbf{A}+\mathbf{B})\mathbf{D} \\ &= \mathbf{AC} + \mathbf{BC} + \mathbf{AD} + \mathbf{BD}\end{aligned}$$

f
$$\begin{aligned}&(\mathbf{A}+\mathbf{B})^2\\&=(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})\\&=(\mathbf{A}+\mathbf{B})\mathbf{A}+(\mathbf{A}+\mathbf{B})\mathbf{B}\\&=\mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2\end{aligned}$$

g
$$\begin{aligned}&(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})\\&=(\mathbf{A}+\mathbf{B})\mathbf{A}-(\mathbf{A}+\mathbf{B})\mathbf{B}\\&=\mathbf{A}^2+\mathbf{BA}-\mathbf{AB}-\mathbf{B}^2\end{aligned}$$

h
$$\begin{aligned}&(\mathbf{A}+\mathbf{I})^2\\&=(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I})\\&=(\mathbf{A}+\mathbf{I})\mathbf{A}+(\mathbf{A}+\mathbf{I})\mathbf{I}\\&=\mathbf{A}^2+\mathbf{IA}+\mathbf{AI}+\mathbf{I}^2\\&=\mathbf{A}^2+\mathbf{A}+\mathbf{A}+\mathbf{I}\\&=\mathbf{A}^2+2\mathbf{A}+\mathbf{I}\end{aligned}$$

i
$$\begin{aligned}&(3\mathbf{I}-\mathbf{B})^2\\&=(3\mathbf{I}-\mathbf{B})(3\mathbf{I}-\mathbf{B})\\&=(3\mathbf{I}-\mathbf{B})3\mathbf{I}-(3\mathbf{I}-\mathbf{B})\mathbf{B}\\&=9\mathbf{I}^2-3\mathbf{BI}-3\mathbf{IB}+\mathbf{B}^2\\&=9\mathbf{I}-3\mathbf{B}-3\mathbf{B}+\mathbf{B}^2\\&=9\mathbf{I}-6\mathbf{B}+\mathbf{B}^2\end{aligned}$$

2 **a** $\mathbf{A}^2=2\mathbf{A}-\mathbf{I}$

$$\begin{aligned}\mathbf{A}^3&=\mathbf{A}\times\mathbf{A}^2\\&=\mathbf{A}(2\mathbf{A}-\mathbf{I})\\&=2\mathbf{A}^2-\mathbf{AI}\\&=2(2\mathbf{A}-\mathbf{I})-\mathbf{A}\\&=4\mathbf{A}-2\mathbf{I}-\mathbf{A}\\&=3\mathbf{A}-2\mathbf{I}\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}^4&=\mathbf{A}\times\mathbf{A}^3\\&=\mathbf{A}(3\mathbf{A}-2\mathbf{I})\\&=3\mathbf{A}^2-2\mathbf{AI}\\&=3(2\mathbf{A}-\mathbf{I})-2\mathbf{A}\\&=6\mathbf{A}-3\mathbf{I}-2\mathbf{A}\\&=4\mathbf{A}-3\mathbf{I}\end{aligned}$$

b $\mathbf{B}^2=2\mathbf{I}-\mathbf{B}$

$$\begin{aligned}\mathbf{B}^3&=\mathbf{B}\times\mathbf{B}^2\\&=\mathbf{B}(2\mathbf{I}-\mathbf{B})\\&=2\mathbf{BI}-\mathbf{B}^2\\&=2\mathbf{B}-(2\mathbf{I}-\mathbf{B})\\&=2\mathbf{B}-2\mathbf{I}+\mathbf{B}\\&=3\mathbf{B}-2\mathbf{I}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}^4&=\mathbf{B}\times\mathbf{B}^3\\&=\mathbf{B}(3\mathbf{B}-2\mathbf{I})\\&=3\mathbf{B}^2-2\mathbf{BI}\\&=3(2\mathbf{I}-\mathbf{B})-2\mathbf{B}\\&=6\mathbf{I}-3\mathbf{B}-2\mathbf{B}\\&=6\mathbf{I}-5\mathbf{B}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}^5&=\mathbf{B}\times\mathbf{B}^4\\&=\mathbf{B}(6\mathbf{I}-5\mathbf{B})\\&=6\mathbf{BI}-5\mathbf{B}^2\\&=6\mathbf{B}-5(2\mathbf{I}-\mathbf{B})\\&=6\mathbf{B}-10\mathbf{I}+5\mathbf{B}\\&=11\mathbf{B}-10\mathbf{I}\end{aligned}$$

c $\mathbf{C}^2=4\mathbf{C}-3\mathbf{I}$

$$\begin{aligned}\mathbf{C}^3&=\mathbf{C}\times\mathbf{C}^2\\&=\mathbf{C}(4\mathbf{C}-3\mathbf{I})\\&=4\mathbf{C}^2-3\mathbf{CI}\\&=4(4\mathbf{C}-3\mathbf{I})-3\mathbf{C}\\&=16\mathbf{C}-12\mathbf{I}-3\mathbf{C}\\&=13\mathbf{C}-12\mathbf{I}\end{aligned}$$

$$\begin{aligned}\mathbf{C}^5&=\mathbf{C}^2\times\mathbf{C}^3\\&=(4\mathbf{C}-3\mathbf{I})(13\mathbf{C}-12\mathbf{I})\\&=(4\mathbf{C}-3\mathbf{I})13\mathbf{C}-(4\mathbf{C}-3\mathbf{I})12\mathbf{I}\\&=52\mathbf{C}^2-39\mathbf{IC}-48\mathbf{CI}+36\mathbf{I}^2\\&=52(4\mathbf{C}-3\mathbf{I})-39\mathbf{C}-48\mathbf{C}+36\mathbf{I}\\&=208\mathbf{C}-156\mathbf{I}-87\mathbf{C}+36\mathbf{I}\\&=121\mathbf{C}-120\mathbf{I}\end{aligned}$$

3 **a** If $\mathbf{A}^2=\mathbf{I}$:

i
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+2\mathbf{I})\\&=\mathbf{A}^2+2\mathbf{AI}\\&=\mathbf{I}+2\mathbf{A}\end{aligned}$$

ii
$$\begin{aligned}&(\mathbf{A}-\mathbf{I})^2\\&=(\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{I})\\&=(\mathbf{A}-\mathbf{I})\mathbf{A}-(\mathbf{A}-\mathbf{I})\mathbf{I}\\&=\mathbf{A}^2-\mathbf{IA}-\mathbf{AI}+\mathbf{I}^2\\&=\mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{I}\\&=2\mathbf{I}-2\mathbf{A}\end{aligned}$$

iii
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+3\mathbf{I})^2\\&=\mathbf{A}(\mathbf{A}+3\mathbf{I})(\mathbf{A}+3\mathbf{I})\\&=\mathbf{A}[(\mathbf{A}+3\mathbf{I})\mathbf{A}+(\mathbf{A}+3\mathbf{I})3\mathbf{I}]\\&=\mathbf{A}(\mathbf{A}^2+3\mathbf{IA}+3\mathbf{AI}+9\mathbf{I}^2)\\&=\mathbf{A}(\mathbf{I}+3\mathbf{A}+3\mathbf{A}+9\mathbf{I})\\&=\mathbf{A}(10\mathbf{I}+6\mathbf{A})\\&=10\mathbf{AI}+6\mathbf{A}^2\\&=10\mathbf{A}+6\mathbf{I}\end{aligned}$$

b If $\mathbf{A}^3 = \mathbf{I}$,

$$\begin{aligned}
\mathbf{A}^2(\mathbf{A}+\mathbf{I})^2 &= \mathbf{A}^2(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I}) \\
&= \mathbf{A}^2[(\mathbf{A}+\mathbf{I})\mathbf{A} + (\mathbf{A}+\mathbf{I})\mathbf{I}] \\
&= \mathbf{A}^2(\mathbf{A}^2 + \mathbf{IA} + \mathbf{AI} + \mathbf{I}^2) \\
&= \mathbf{A}^2(\mathbf{A}^2 + \mathbf{A} + \mathbf{A} + \mathbf{I}) \\
&= \mathbf{A}^2(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) \\
&= \mathbf{A}^4 + 2\mathbf{A}^3 + \mathbf{A}^2\mathbf{I} \\
&= \mathbf{A}(\mathbf{A}^3) + 2\mathbf{I} + \mathbf{A}^2\mathbf{I} \\
&= \mathbf{AI} + 2\mathbf{I} + \mathbf{A}^2 \\
&= \mathbf{A}^2 + \mathbf{A} + 2\mathbf{I}
\end{aligned}$$

c If $\mathbf{A}^2 = \mathbf{O}$:

i

$$\begin{aligned}
&\mathbf{A}(2\mathbf{A} - 3\mathbf{I}) \\
&= 2\mathbf{A}^2 - 3\mathbf{AI} \\
&= 2\mathbf{O} - 3\mathbf{A} \\
&= -3\mathbf{A}
\end{aligned}$$

ii

$$\begin{aligned}
&\mathbf{A}(\mathbf{A}+2\mathbf{I})(\mathbf{A}-\mathbf{I}) \\
&= \mathbf{A}[(\mathbf{A}+2\mathbf{I})\mathbf{A} - (\mathbf{A}+2\mathbf{I})\mathbf{I}] \\
&= \mathbf{A}(\mathbf{A}^2 + 2\mathbf{IA} - \mathbf{AI} - 2\mathbf{I}^2) \\
&= \mathbf{A}(\mathbf{O} + 2\mathbf{A} - \mathbf{A} - 2\mathbf{I}) \\
&= \mathbf{A}(\mathbf{O} + \mathbf{A} - 2\mathbf{I}) \\
&= \mathbf{A}(\mathbf{A} - 2\mathbf{I}) \\
&= \mathbf{A}^2 - 2\mathbf{AI} \\
&= \mathbf{O} - 2\mathbf{A} \\
&= -2\mathbf{A}
\end{aligned}$$

iii

$$\begin{aligned}
&\mathbf{A}(\mathbf{A}+\mathbf{I})^3 \\
&= \mathbf{A}(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I}) \\
&= (\mathbf{A}^2+\mathbf{AI})(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I}) \\
&= (\mathbf{O}+\mathbf{A})(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I}) \\
&= \mathbf{A}(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I}) \\
&= (\mathbf{A}^2+\mathbf{AI})(\mathbf{A}+\mathbf{I}) \\
&= (\mathbf{O}+\mathbf{A})(\mathbf{A}+\mathbf{I}) \\
&= \mathbf{A}(\mathbf{A}+\mathbf{I}) \\
&= \mathbf{A}^2 + \mathbf{AI} \\
&= \mathbf{O} + \mathbf{A} \\
&= \mathbf{A}
\end{aligned}$$

4 **a** $\mathbf{A}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \\ \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \mathbf{A}$

b

$$\begin{aligned}
&\mathbf{A}^2 = \mathbf{A} \\
\therefore\ &\mathbf{A}^2 - \mathbf{A} = \mathbf{O} \\
\therefore\ &\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O} \\
\therefore\ &\mathbf{A} = \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O} \\
\therefore\ &\mathbf{A} = \mathbf{O} \text{ or } \mathbf{I}
\end{aligned}$$

The argument contains a false step. $\mathbf{AB} = \mathbf{O}$ does not imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.
This is a property of real numbers that does not hold for matrices. Therefore it is false to say that if $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} - \mathbf{I} = \mathbf{O}$.

c Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\mathbf{A}^2 = \mathbf{A}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\therefore \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Equating corresponding elements:

$a^2+bc = a \quad \therefore \quad bc = a(1-a) \quad .... (1)$

$ab+bd = b \quad \therefore \quad b(a+d-1) = 0 \quad .... (2)$

$ac+cd = c \quad \therefore \quad c(a+d-1) = 0 \quad .... (3)$

$bc+d^2 = d \quad \therefore \quad bc = d(1-d) \quad .... (4)$

If $a+d-1 \neq 0$: then from (2) and (3), $b = c = 0$.

$\therefore$ from (1) and (4), $a = 0$ or 1 and $d = 0$ or 1

$\therefore$ $a = 0,\ d = 0$ or $a = 1,\ d = 1$ or $a = 0,\ d = 1$ or $a = 1,\ d = 0$
where the last two cases are not possible as $a+d \neq 1$.

So, if $a = 0,\ d = 0$ then $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and if $a = 1,\ d = 1$ then $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $a+d-1 = 0$: then $d = 1-a$ and $c = \dfrac{a-a^2}{b}$

So $\mathbf{A}$ is $\begin{pmatrix} a & b \\ \dfrac{a-a^2}{b} & 1-a \end{pmatrix}$, provided $b \neq 0$.

5 Suppose $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\therefore \mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ $\mathbf{A} \neq \mathbf{O}$ but $\mathbf{A}^2 = \mathbf{O}$. So, "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a false statement.

6 **a** Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} = a\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\therefore \begin{pmatrix} 1+(-2) & 2+4 \\ -1+(-2) & -2+4 \end{pmatrix} = \begin{pmatrix} a & 2a \\ -a & 2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1 & 6 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ -a & 2a+b \end{pmatrix}$$

$$\therefore a+b = -1 \text{ and } 2a = 6$$

$$\therefore a = 3 \text{ and } b = -4$$

Checking for consistency: $-a = -3$ ✓ $2a+b = 6+(-4) = 2$ ✓

$\therefore$ $\mathbf{A}^2 = 3\mathbf{A} - 4\mathbf{I}$

b Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} = a\begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\therefore \begin{pmatrix} 9+2 & 3+(-2) \\ 6+(-4) & 2+4 \end{pmatrix} = \begin{pmatrix} 3a & a \\ 2a & -2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 11 & 1 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 3a+b & a \\ 2a & -2a+b \end{pmatrix}$$

$$\therefore\ 3a+b=11 \quad \text{and} \quad a=1$$

$$\therefore\ a=1 \quad \text{and} \quad b=8$$

Checking for consistency: $2a = 2(1) = 2$ ✓ $\quad -2a+b = -2(1)+8 = 6$ ✓

$\therefore\ \mathbf{A}^2 = \mathbf{A} + 8\mathbf{I}$

7 **a**

$$\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$$

$$\therefore \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} = p\begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1+(-2) & 2+(-6) \\ -1+3 & -2+9 \end{pmatrix} = \begin{pmatrix} p & 2p \\ -p & -3p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1 & -4 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} p+q & 2p \\ -p & -3p+q \end{pmatrix}$$

$$\therefore\ p+q=-1 \quad \text{and} \quad 2p=-4$$

$$\therefore\ p=-2 \quad \text{and} \quad q=1$$

Checking for consistency: $-p = -(-2) = 2$ ✓ $\quad -3p+q = -3(-2)+1 = 7$ ✓

$\therefore\ \mathbf{A}^2 = -2\mathbf{A} + \mathbf{I}$

b
$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 \\ &= \mathbf{A}(-2\mathbf{A}+\mathbf{I}) \\ &= -2\mathbf{A}^2 + \mathbf{AI} \\ &= -2(-2\mathbf{A}+\mathbf{I}) + \mathbf{A} \\ &= 4\mathbf{A} - 2\mathbf{I} + \mathbf{A} \\ &= 5\mathbf{A} - 2\mathbf{I} \end{aligned}$$

c
$$\begin{aligned} \mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\ &= \mathbf{A}(5\mathbf{A}-2\mathbf{I}) \\ &= 5\mathbf{A}^2 - 2\mathbf{AI} \\ &= 5(-2\mathbf{A}+\mathbf{I}) - 2\mathbf{A} \\ &= -10\mathbf{A} + 5\mathbf{I} - 2\mathbf{A} \\ &= -12\mathbf{A} + 5\mathbf{I} \end{aligned}$$

EXERCISE 12G.1

1
$$\begin{aligned} \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix} &= \begin{pmatrix} 15-12 & -30+30 \\ 6-6 & -12+15 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\ &= 3\mathbf{I} \end{aligned}$$

$$\therefore \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix} \times \tfrac{1}{3}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix} = \mathbf{I}$$

$$\therefore \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}^{-1} = \tfrac{1}{3}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{pmatrix}$$

2
$$\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6+4 & 12-12 \\ 2-2 & 4+6 \end{pmatrix}$$
$$= \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$
$$= 10\mathbf{I}$$
$$\therefore \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} \times \tfrac{1}{10}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} = \mathbf{I}$$
$$\therefore \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}^{-1} = \tfrac{1}{10}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix}$$

3 **a** $|\mathbf{A}| = 3(4) - 7(2)$
$= 12 - 14$
$= -2$

b $|\mathbf{A}| = (-1)(-2) - 3(1)$
$= 2 - 3$
$= -1$

c $|\mathbf{A}| = 0 - 0$
$= 0$

d $|\mathbf{A}| = 1 - 0$
$= 1$

4 **a** $\det \mathbf{B} = 3(4) - (-2)(7)$
$= 12 + 14$
$= 26$

b $\det \mathbf{B} = 3(2) - 0$
$= 6 - 0$
$= 6$

c $\det \mathbf{B} = 0 - 1$
$= -1$

d $\det \mathbf{B} = a(a) - (-a)(1)$
$= a^2 + a$

5 $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$

a $|\mathbf{A}| = 2(7) - 3(4)$
$= 14 - 12$
$= 2$

b $\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 7 & -3 \\ -4 & 2 \end{pmatrix}$
$= \begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ -2 & 1 \end{pmatrix}$

c
$$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2(\frac{7}{2}) + 3(-2) & 2(-\frac{3}{2}) + 3(1) \\ 4(\frac{7}{2}) + 7(-2) & 4(-\frac{3}{2}) + 7(1) \end{pmatrix}$$
$$= \begin{pmatrix} 7-6 & -3+3 \\ 14-14 & -6+7 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{7}{2}(2) - \frac{3}{2}(4) & \frac{7}{2}(3) - \frac{3}{2}(7) \\ -2(2) + 1(4) & -2(3) + 1(7) \end{pmatrix}$$
$$= \begin{pmatrix} 7-6 & \frac{21}{2} - \frac{21}{2} \\ -4+4 & -6+7 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \mathbf{I}$$

6 **a** $\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}^{-1} = \dfrac{1}{2(5)-4(-1)} \begin{pmatrix} 5 & -4 \\ -(-1) & 2 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 5 & -4 \\ 1 & 2 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}^{-1} = \dfrac{1}{1(-1)-0(1)} \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

c $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}^{-1}$ does not exist, since $ad - bc = 2(2) - 4(1) = 0$

d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \dfrac{1}{1(1)-0(0)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}^{-1} = \dfrac{1}{5(2)-0(-1)} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$

f $\begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}^{-1}$ does not exist, since $ad - bc = 3(-10) - 5(-6) = 0$

g $\begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix}^{-1} = \dfrac{1}{(-1)(7)-2(4)} \begin{pmatrix} 7 & -2 \\ -4 & -1 \end{pmatrix} = -\frac{1}{15} \begin{pmatrix} 7 & -2 \\ -4 & -1 \end{pmatrix}$

h $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}^{-1} = \dfrac{1}{3(2)-4(-1)} \begin{pmatrix} 2 & -4 \\ -(-1) & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}$

i $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}^{-1} = \dfrac{1}{(-1)(3)-(-1)(2)} \begin{pmatrix} 3 & -(-1) \\ -2 & -1 \end{pmatrix} = -\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix}$

j $\begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix}^{-1}$ does not exist, since $ad - bc = 4(5) - 10(2) = 0$

k $\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}^{-1} = \dfrac{1}{\frac{1}{2}(\frac{1}{2})-0(-\frac{1}{2})} \begin{pmatrix} \frac{1}{2} & 0 \\ -(-\frac{1}{2}) & \frac{1}{2} \end{pmatrix} = 4\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$

l $\begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix}^{-1} = \dfrac{1}{0(0)-\frac{1}{2}(-2)} \begin{pmatrix} 0 & -\frac{1}{2} \\ -(-2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$

7 **a** $\mathbf{A}^{-1} = \dfrac{1}{2k-(-6)} \begin{pmatrix} 2 & -1 \\ 6 & k \end{pmatrix}$

$= \begin{pmatrix} \dfrac{2}{2k+6} & \dfrac{-1}{2k+6} \\ \dfrac{6}{2k+6} & \dfrac{k}{2k+6} \end{pmatrix}$

$\mathbf{A}^{-1}$ exists provided that $2k + 6 \neq 0$

$\therefore \; k \neq -3$

b $\mathbf{A}^{-1} = \dfrac{1}{3k} \begin{pmatrix} k & 1 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{3} & \dfrac{1}{3k} \\ 0 & \dfrac{3}{3k} \end{pmatrix}$

$\mathbf{A}^{-1}$ exists provided that $3k \neq 0$

$\therefore \; k \neq 0$

c $\mathbf{A}^{-1} = \dfrac{1}{k(k+1)-2}\begin{pmatrix} k & -2 \\ -1 & k+1 \end{pmatrix}$

$= \begin{pmatrix} \dfrac{k}{k^2+k-2} & \dfrac{-2}{k^2+k-2} \\ \dfrac{-1}{k^2+k-2} & \dfrac{k+1}{k^2+k-2} \end{pmatrix}$

$\mathbf{A}^{-1}$ exists provided that

$$k^2+k-2 \neq 0$$
$$\therefore\ (k+2)(k-1) \neq 0$$
$$\therefore\ k \neq -2 \text{ or } 1$$

d $\mathbf{A}^{-1} = \dfrac{1}{k(k-2)+3k}\begin{pmatrix} k & -k \\ 3 & k-2 \end{pmatrix}$

$= \begin{pmatrix} \dfrac{k}{k^2+k} & \dfrac{-k}{k^2+k} \\ \dfrac{3}{k^2+k} & \dfrac{k-2}{k^2+k} \end{pmatrix}$

$\mathbf{A}^{-1}$ exists provided that

$$k^2+k \neq 0$$
$$\therefore\ k(k+1) \neq 0$$
$$\therefore\ k \neq 0 \text{ or } -1$$

e $\mathbf{A}^{-1} = \dfrac{1}{k^2-2k(k-1)}\begin{pmatrix} 1 & 1-k \\ -2k & k^2 \end{pmatrix}$

$= \begin{pmatrix} \dfrac{1}{2k-k^2} & \dfrac{1-k}{2k-k^2} \\ \dfrac{-2k}{2k-k^2} & \dfrac{k^2}{2k-k^2} \end{pmatrix}$

$\mathbf{A}^{-1}$ exists provided that

$$2k-k^2 \neq 0$$
$$\therefore\ k(2-k) \neq 0$$
$$\therefore\ k \neq 0 \text{ or } 2$$

f $\mathbf{A}^{-1} = \dfrac{1}{3k(k+1)-2(k^2+2)}\begin{pmatrix} 3k & -2 \\ -k^2-2 & k+1 \end{pmatrix}$

$= \dfrac{1}{3k^2+3k-2k^2-4}\begin{pmatrix} 3k & -2 \\ -k^2-2 & k+1 \end{pmatrix}$

$= \begin{pmatrix} \dfrac{3k}{k^2+3k-4} & \dfrac{-2}{k^2+3k-4} \\ \dfrac{-k^2-2}{k^2+3k-4} & \dfrac{k+1}{k^2+3k-4} \end{pmatrix}$

$\mathbf{A}^{-1}$ exists provided that

$$k^2+3k-4 \neq 0$$
$$\therefore\ (k+4)(k-1) \neq 0$$
$$\therefore\ k \neq -4 \text{ or } 1$$

INVESTIGATION 2 PROPERTIES OF MATRIX DETERMINANTS

1 **a** $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$

$$\begin{aligned}\therefore\ \det \mathbf{A} &= 2(-1)-(-1)(-1) \\ &= -2-1 \\ &= -3\end{aligned}$$

b $-\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{aligned}\therefore\ \det(-\mathbf{A}) &= (-2)(1)-1(1) \\ &= -2-1 \\ &= -3\end{aligned}$$

c $2\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & -2 \end{pmatrix}$

$$\begin{aligned}\therefore\ \det(2\mathbf{A}) &= 4(-2)-(-2)(-2) \\ &= -8-4 \\ &= -12\end{aligned}$$

d $3\mathbf{A} = \begin{pmatrix} 6 & -3 \\ -3 & -3 \end{pmatrix}$

$$\begin{aligned}\therefore\ \det(3\mathbf{A}) &= 6(-3)-(-3)(-3) \\ &= -18-9 \\ &= -27\end{aligned}$$

3 If $\mathbf{A}$ is a 2×2 matrix and k is a scalar, then $\det(k\mathbf{A}) = k^2(\det \mathbf{A})$.

4 $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $k\mathbf{A} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$, where k, a, b, c, and d are scalars.

$\therefore$ $\det \mathbf{A} = ad - bc$ and
$$\begin{aligned} \det(k\mathbf{A}) &= ka(kd) - kb(kc) \\ &= k^2ad - k^2bc \\ &= k^2(ad - bc) \\ &= k^2(\det \mathbf{A}) \quad \text{for any } 2 \times 2 \text{ matrix } \mathbf{A} \text{ and scalar } k. \end{aligned}$$

5 **a** $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\therefore \det \mathbf{A} = 1(4) - 2(3) = 4 - 6 = -2$$

b $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

$$\therefore \det \mathbf{B} = (-1)(1) - 2(0) = -1$$

c
$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times (-1) + 2 \times 0 & 1 \times 2 + 2 \times 1 \\ 3 \times (-1) + 4 \times 0 & 3 \times 2 + 4 \times 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 4 \\ -3 & 10 \end{pmatrix} \end{aligned}$$

$$\therefore \det(\mathbf{AB}) = (-1)(10) - 4(-3) = -10 + 12 = 2$$

7 In **5**, we observe that $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$.
So, we predict that for all square matrices $\mathbf{A}$ and $\mathbf{B}$, $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$.

8 $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, where a, b, c, d, p, q, r, and s are scalars

$$\therefore \mathbf{AB} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

So, $\det \mathbf{A} = ad - bc$, $\det \mathbf{B} = ps - qr$, and $\det(\mathbf{AB}) = (ap + br)(cq + ds) - (aq + bs)(cp + dr)$

Now
$$\begin{aligned} \det \mathbf{A} \times \det \mathbf{B} &= (ad - bc)(ps - qr) \\ &= adps - adqr - bcps + bcqr \end{aligned}$$

and
$$\begin{aligned} \det(\mathbf{AB}) &= \cancel{acpq} + adps + bcqr + \cancel{bdrs} - (\cancel{acpq} + adqr + bcps + \cancel{bdrs}) \\ &= adps + bcqr - adqr - bcps \end{aligned}$$

$\therefore$ $\det \mathbf{A} \times \det \mathbf{B} = \det(\mathbf{AB})$ for any 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

EXERCISE 12G.2

1 $\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}\begin{pmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{pmatrix}$

$$= \begin{pmatrix} 2(-11)+0(-1)+3(8) & 2(9)+0(1)+3(-6) & 2(15)+0(1)+3(-10) \\ 1(-11)+5(-1)+2(8) & 1(9)+5(1)+2(-6) & 1(15)+5(1)+2(-10) \\ 1(-11)+(-3)(-1)+1(8) & 1(9)+(-3)(1)+1(-6) & 1(15)+(-3)(1)+1(-10) \end{pmatrix}$$

$$= \begin{pmatrix} -22+24 & 18-18 & 30-30 \\ -11-5+16 & 9+5-12 & 15+5-20 \\ -11+3+8 & 9-3-6 & 15-3-10 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

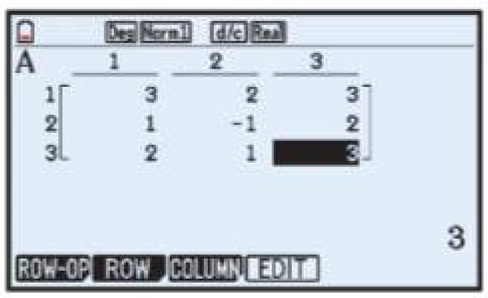
$$= 2\mathbf{I}$$

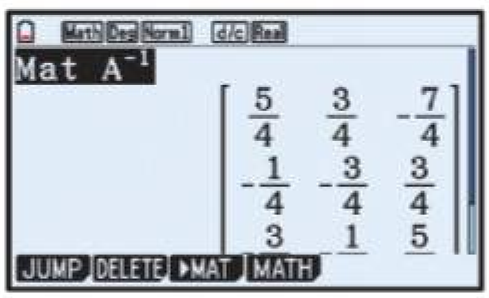
$$\therefore \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix} \times \tfrac{1}{2}\begin{pmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{pmatrix} = \mathbf{I}$$

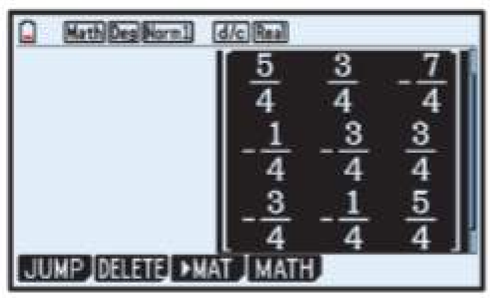
$$\therefore \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}^{-1} = \tfrac{1}{2}\begin{pmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{pmatrix}$$

2 **a**





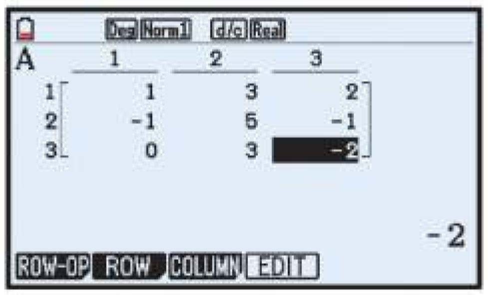


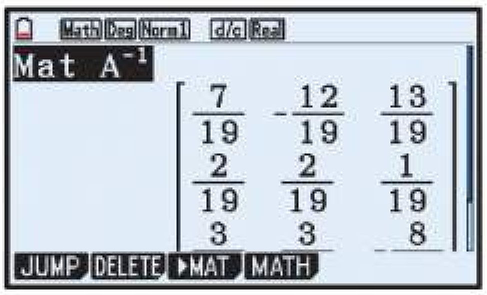
So, $\begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{pmatrix}$.

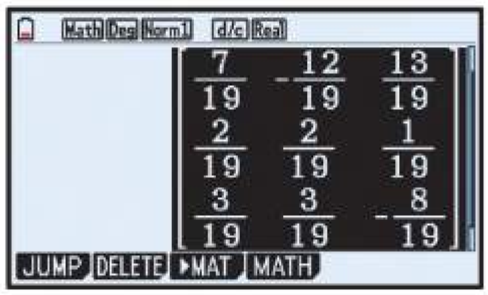
Now $\begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{pmatrix} = \begin{pmatrix} \frac{15}{4}-\frac{2}{4}-\frac{9}{4} & \frac{9}{4}-\frac{6}{4}-\frac{3}{4} & -\frac{21}{4}+\frac{6}{4}+\frac{15}{4} \\ \frac{5}{4}+\frac{1}{4}-\frac{6}{4} & \frac{3}{4}+\frac{3}{4}-\frac{2}{4} & -\frac{7}{4}-\frac{3}{4}+\frac{10}{4} \\ \frac{10}{4}-\frac{1}{4}-\frac{9}{4} & \frac{6}{4}-\frac{3}{4}-\frac{3}{4} & -\frac{14}{4}+\frac{3}{4}+\frac{15}{4} \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \checkmark$$

b







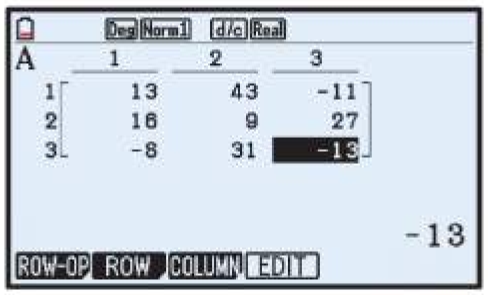
So, $\begin{pmatrix} 1 & 3 & 2 \\ -1 & 5 & -1 \\ 0 & 3 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{7}{19} & -\frac{12}{19} & \frac{13}{19} \\ \frac{2}{19} & \frac{2}{19} & \frac{1}{19} \\ \frac{3}{19} & \frac{3}{19} & -\frac{8}{19} \end{pmatrix}$.

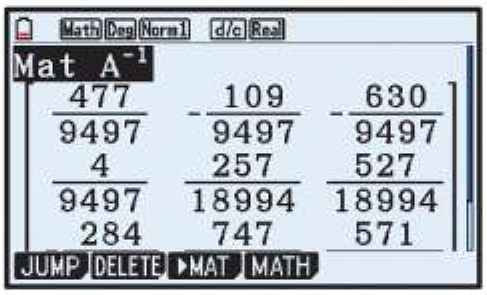
Now $\begin{pmatrix} 1 & 3 & 2 \\ -1 & 5 & -1 \\ 0 & 3 & -2 \end{pmatrix}\begin{pmatrix} \frac{7}{19} & -\frac{12}{19} & \frac{13}{19} \\ \frac{2}{19} & \frac{2}{19} & \frac{1}{19} \\ \frac{3}{19} & \frac{3}{19} & -\frac{8}{19} \end{pmatrix}$

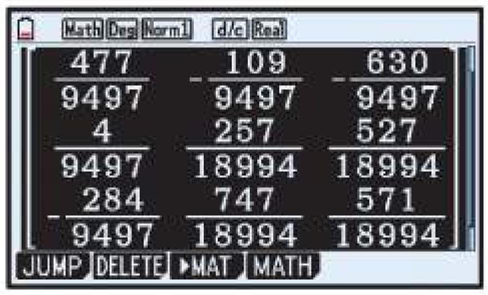
$$= \begin{pmatrix} \frac{7}{19}+\frac{6}{19}+\frac{6}{19} & -\frac{12}{19}+\frac{6}{19}+\frac{6}{19} & \frac{13}{19}+\frac{3}{19}-\frac{16}{19} \\ -\frac{7}{19}+\frac{10}{19}-\frac{3}{19} & \frac{12}{19}+\frac{10}{19}-\frac{3}{19} & -\frac{13}{19}+\frac{5}{19}+\frac{8}{19} \\ \frac{0}{19}+\frac{6}{19}-\frac{6}{19} & \frac{0}{19}+\frac{6}{19}-\frac{6}{19} & \frac{0}{19}+\frac{3}{19}+\frac{16}{19} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \checkmark$$

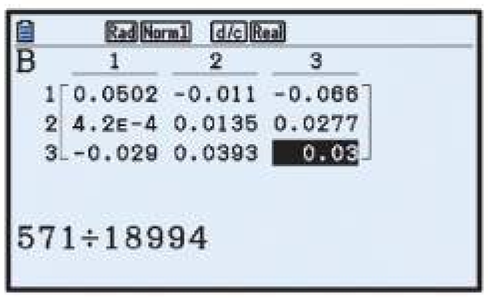
c

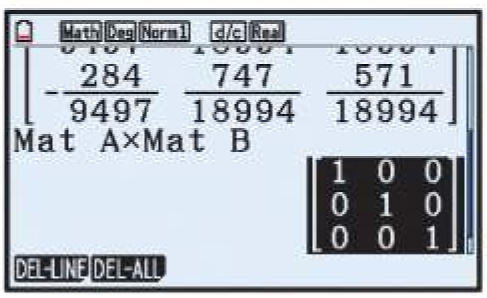






So, $\begin{pmatrix} 13 & 43 & -11 \\ 16 & 9 & 27 \\ -8 & 31 & -13 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{477}{9497} & -\frac{109}{9497} & -\frac{630}{9497} \\ \frac{4}{9497} & \frac{257}{18\,994} & \frac{527}{18\,994} \\ -\frac{284}{9497} & \frac{747}{18\,994} & \frac{571}{18\,994} \end{pmatrix}$.

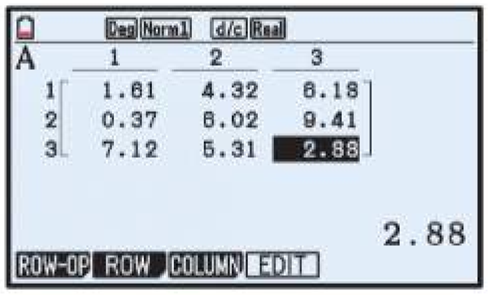


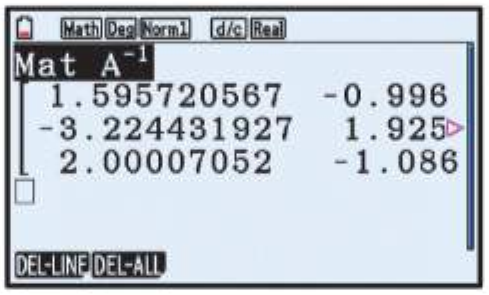


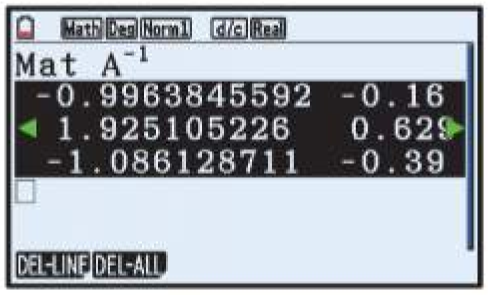
Checking using technology, we see that

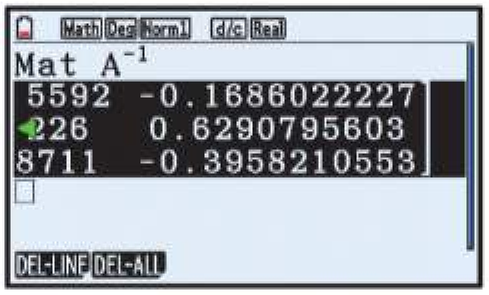
$$\begin{pmatrix} 13 & 43 & -11 \\ 16 & 9 & 27 \\ -8 & 31 & -13 \end{pmatrix}\begin{pmatrix} \frac{477}{9497} & -\frac{109}{9497} & -\frac{630}{9497} \\ \frac{4}{9497} & \frac{257}{18\,994} & \frac{527}{18\,994} \\ -\frac{284}{9497} & \frac{747}{18\,994} & \frac{571}{18\,994} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \checkmark$$

d

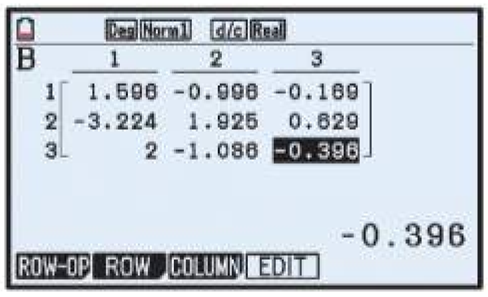


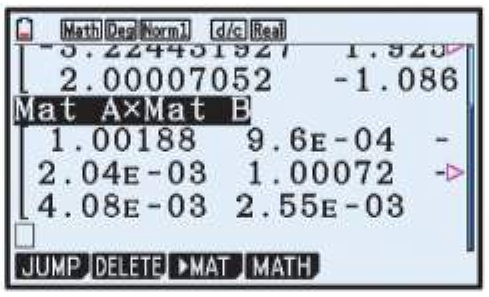


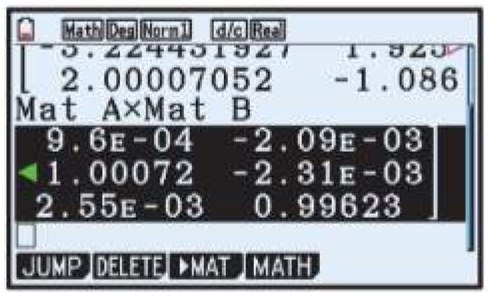




So, $\begin{pmatrix} 1.61 & 4.32 & 6.18 \\ 0.37 & 6.02 & 9.41 \\ 7.12 & 5.31 & 2.88 \end{pmatrix}^{-1} \approx \begin{pmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{pmatrix}$





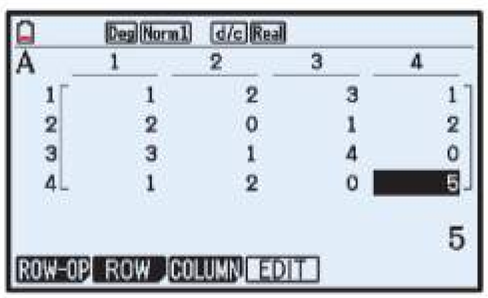


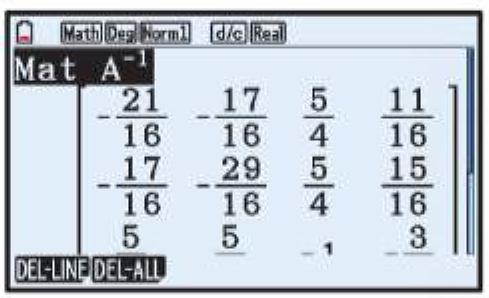
Checking using technology, we see that

$$\begin{pmatrix} 1.61 & 4.32 & 6.18 \\ 0.37 & 6.02 & 9.41 \\ 7.12 & 5.31 & 2.88 \end{pmatrix} \begin{pmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1.002 & 0.001 & -0.002 \\ 0.002 & 1.001 & -0.002 \\ 0.004 & 0.003 & 0.996 \end{pmatrix} \approx \mathbf{I} \quad \checkmark$$

e





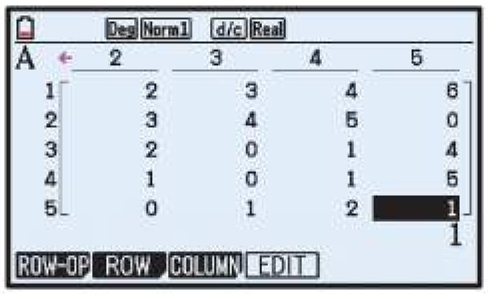
So, $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{21}{16} & -\frac{17}{16} & \frac{5}{4} & \frac{11}{16} \\ -\frac{17}{16} & -\frac{29}{16} & \frac{5}{4} & \frac{15}{16} \\ \frac{5}{4} & \frac{5}{4} & -1 & -\frac{3}{4} \\ \frac{11}{16} & \frac{15}{16} & -\frac{3}{4} & -\frac{5}{16} \end{pmatrix}.$

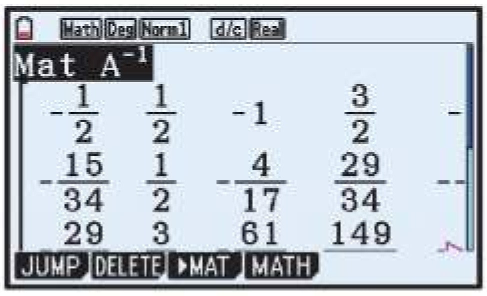
Now $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} -\frac{21}{16} & -\frac{17}{16} & \frac{5}{4} & \frac{11}{16} \\ -\frac{17}{16} & -\frac{29}{16} & \frac{5}{4} & \frac{15}{16} \\ \frac{5}{4} & \frac{5}{4} & -1 & -\frac{3}{4} \\ \frac{11}{16} & \frac{15}{16} & -\frac{3}{4} & -\frac{5}{16} \end{pmatrix}$

$$= \begin{pmatrix} -\frac{21}{16} - \frac{34}{16} + \frac{15}{4} + \frac{11}{16} & -\frac{17}{16} - \frac{58}{16} + \frac{15}{4} + \frac{15}{16} & \frac{5}{4} + \frac{10}{4} - 3 - \frac{3}{4} & \frac{11}{16} + \frac{30}{16} - \frac{9}{4} - \frac{5}{16} \\ -\frac{42}{16} - \frac{0}{16} + \frac{5}{4} + \frac{22}{16} & -\frac{34}{16} - \frac{0}{16} + \frac{5}{4} + \frac{30}{16} & \frac{10}{4} + \frac{0}{4} - 1 - \frac{6}{4} & \frac{22}{16} + \frac{0}{16} - \frac{3}{4} - \frac{10}{16} \\ -\frac{63}{16} - \frac{17}{16} + \frac{20}{4} + \frac{0}{16} & -\frac{51}{16} - \frac{29}{16} + \frac{20}{4} + \frac{0}{16} & \frac{15}{4} + \frac{5}{4} - 4 - \frac{0}{4} & \frac{33}{16} + \frac{15}{16} - \frac{12}{4} - \frac{0}{16} \\ -\frac{21}{16} - \frac{34}{16} + \frac{0}{4} + \frac{55}{16} & -\frac{17}{16} - \frac{58}{16} + \frac{0}{4} + \frac{75}{16} & \frac{5}{4} + \frac{10}{4} - 0 - \frac{15}{4} & \frac{11}{16} + \frac{30}{16} - \frac{0}{4} - \frac{25}{16} \end{pmatrix}$$

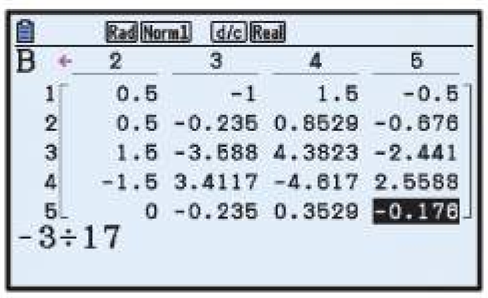
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \checkmark$$

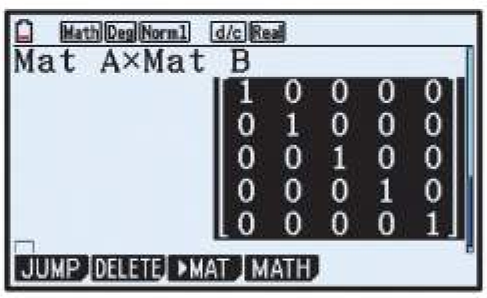
f





So, $\begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -1 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{15}{34} & \frac{1}{2} & -\frac{4}{17} & \frac{29}{34} & -\frac{23}{34} \\ -\frac{29}{34} & \frac{3}{2} & -\frac{61}{17} & \frac{149}{34} & -\frac{83}{34} \\ \frac{39}{34} & -\frac{3}{2} & \frac{58}{17} & -\frac{157}{34} & \frac{87}{34} \\ \frac{1}{17} & 0 & -\frac{4}{17} & \frac{6}{17} & -\frac{3}{17} \end{pmatrix}.$





Checking using technology, we see that

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -1 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{15}{34} & \frac{1}{2} & -\frac{4}{17} & \frac{29}{34} & -\frac{23}{34} \\ -\frac{29}{34} & \frac{3}{2} & -\frac{61}{17} & \frac{149}{34} & -\frac{83}{34} \\ \frac{39}{34} & -\frac{3}{2} & \frac{58}{17} & -\frac{157}{34} & \frac{87}{34} \\ \frac{1}{17} & 0 & -\frac{4}{17} & \frac{6}{17} & -\frac{3}{17} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \checkmark$$

EXERCISE 12H

1 **a** $\begin{cases} 3x - y = 8 \\ 2x + 3y = 6 \end{cases}$ can be written as $\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

b $\begin{cases} 4x - 3y = 11 \\ 3x + 2y = -5 \end{cases}$ can be written as $\begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}$

c $\begin{cases} 3a - b = 6 \\ 2a + 7b = -4 \end{cases}$ can be written as $\begin{pmatrix} 3 & -1 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

2 **a** $\begin{cases} 2x - y = 6 \\ x + 3y = 14 \end{cases}$ can be written as $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \end{pmatrix}$

$$\therefore \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ 14 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ 14 \end{pmatrix}$$

$$= \tfrac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 14 \end{pmatrix}$$

$$= \tfrac{1}{7}\begin{pmatrix} 18+14 \\ -6+28 \end{pmatrix} = \begin{pmatrix} \frac{32}{7} \\ \frac{22}{7} \end{pmatrix}$$

and so $x = \frac{32}{7}$, $y = \frac{22}{7}$

b $\begin{cases} 5x - 4y = 5 \\ 2x + 3y = -13 \end{cases}$ can be written as $\begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$

$$\therefore \begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

$$= \tfrac{1}{23}\begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

$$= \tfrac{1}{23}\begin{pmatrix} 15+(-52) \\ -10+(-65) \end{pmatrix} = \begin{pmatrix} -\frac{37}{23} \\ -\frac{75}{23} \end{pmatrix}$$

and so $x = -\frac{37}{23}$, $y = -\frac{75}{23}$

c $\begin{cases} x - 2y = 7 \\ 5x + 3y = -2 \end{cases}$ can be written as $\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$

$$\therefore \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

$$= \tfrac{1}{13}\begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

$$= \tfrac{1}{13}\begin{pmatrix} 21 + (-4) \\ -35 + (-2) \end{pmatrix} = \begin{pmatrix} \frac{17}{13} \\ -\frac{37}{13} \end{pmatrix}$$

and so $x = \frac{17}{13}$, $y = -\frac{37}{13}$

d $\begin{cases} 3x + 5y = 4 \\ 2x - y = 11 \end{cases}$ can be written as $\begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$

$$\therefore \begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$= \tfrac{1}{-13}\begin{pmatrix} -1 & -5 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$= -\tfrac{1}{13}\begin{pmatrix} -4 + (-55) \\ -8 + 33 \end{pmatrix} = \begin{pmatrix} \frac{59}{13} \\ -\frac{25}{13} \end{pmatrix}$$

and so $x = \frac{59}{13}$, $y = -\frac{25}{13}$

e $\begin{cases} 4x - 7y = 8 \\ 3x - 5y = 0 \end{cases}$ can be written as $\begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$= \tfrac{1}{1}\begin{pmatrix} -5 & 7 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -40 + 0 \\ -24 + 0 \end{pmatrix}$$

and so $x = -40$, $y = -24$

f $\begin{cases} 7x+11y=18 \\ 11x-7y=-11 \end{cases}$ can be written as $\begin{pmatrix} 7 & 11 \\ 11 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -11 \end{pmatrix}$

$$\therefore \begin{pmatrix} 7 & 11 \\ 11 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 7 & 11 \\ 11 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ 11 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 18 \\ -11 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ 11 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 18 \\ -11 \end{pmatrix}$$

$$= \tfrac{1}{-170}\begin{pmatrix} -7 & -11 \\ -11 & 7 \end{pmatrix}\begin{pmatrix} 18 \\ -11 \end{pmatrix}$$

$$= -\tfrac{1}{170}\begin{pmatrix} -126+121 \\ -198-77 \end{pmatrix}$$

$$= -\tfrac{1}{170}\begin{pmatrix} -5 \\ -275 \end{pmatrix} = \begin{pmatrix} \frac{1}{34} \\ \frac{55}{34} \end{pmatrix}$$

and so $x = \frac{1}{34}$, $y = \frac{55}{34}$

3 a i $\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$,

where $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}$

and $\det \mathbf{A} = -2-(-12)$
$= 10$

ii As $\det \mathbf{A} \neq 0$, the system has a unique solution.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$= \tfrac{1}{10}\begin{pmatrix} -1 & 3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$= \tfrac{1}{10}\begin{pmatrix} 25 \\ -10 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -1 \end{pmatrix}$$

$\therefore x = \frac{5}{2}$, $y = -1$

b i $\begin{pmatrix} 2 & k \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$,

where $\mathbf{A} = \begin{pmatrix} 2 & k \\ 4 & -1 \end{pmatrix}$

and $\det \mathbf{A} = -2-4k$

ii The system has a unique solution if
$\det \mathbf{A} \neq 0$
$\therefore -2-4k \neq 0$
$\therefore k \neq -\frac{1}{2}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & k \\ 4 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$= \frac{1}{-2-4k}\begin{pmatrix} -1 & -k \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$= \frac{1}{-2-4k}\begin{pmatrix} -8-11k \\ -10 \end{pmatrix}$$

$\therefore x = \dfrac{8+11k}{2+4k}$, $y = \dfrac{5}{1+2k}$, $k \neq -\frac{1}{2}$

is the unique solution.

iii When $k = -\frac{1}{2}$, the equations are

$\begin{cases} 2x - \frac{1}{2}y = 8 \\ 4x - y = 11 \end{cases}$ or $\begin{cases} 4x-y=16 \\ 4x-y=11 \end{cases}$

So, we have no solutions (as the lines are parallel and so do not meet).

4 a $\begin{cases} x-y-z=2 \\ x+y+3z=7 \\ 9x-y-3z=-1 \end{cases}$ has matrix equation $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$

b $\begin{cases} 2x+y-z=3 \\ y+2z=6 \\ x-y+z=13 \end{cases}$ has matrix equation $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 13 \end{pmatrix}$

c $\begin{cases} a+b-c=7 \\ a-b+c=6 \\ 2a+b-3c=-2 \end{cases}$ has matrix equation $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -2 \end{pmatrix}$

5 **a** $\mathbf{AB} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{pmatrix}\begin{pmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{pmatrix}$

$$= \begin{pmatrix} 8-1-6 & 14-2-12 & -6+1+5 \\ -4-2+6 & -7-4+12 & 3+2-5 \\ 0-6+6 & 0-12+12 & 0+6-5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ $\mathbf{AB} = \mathbf{I}$ and so $\mathbf{A} = \mathbf{B}^{-1}$

b $\begin{cases} 4a+7b-3c=-8 \\ -a-2b+c=3 \\ 6a+12b-5c=-15 \end{cases}$ has matrix equation $\begin{pmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \\ -15 \end{pmatrix}$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{pmatrix}^{-1}\begin{pmatrix} -8 \\ 3 \\ -15 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{pmatrix}\begin{pmatrix} -8 \\ 3 \\ -15 \end{pmatrix} \qquad \{\mathbf{B}^{-1} = \mathbf{A}\}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -16+3+15 \\ 8+6-15 \\ 0+18-15 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad \therefore\ a=2,\ b=-1,\ c=3$$

6 **a** $\mathbf{MN} = \begin{pmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 15+3-14 & 10-3-7 & 15+6-21 \\ -3-3+6 & -2+3+3 & -3-6+9 \\ -9-1+10 & -6+1+5 & -9-2+15 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4\mathbf{I}$$

$\therefore$ $(\frac{1}{4}\mathbf{M})\mathbf{N} = \mathbf{I}$ and so $\frac{1}{4}\mathbf{M} = \mathbf{N}^{-1}$

b $\begin{cases} 3u+2v+3w=18 \\ u-v+2w=6 \\ 2u+v+3w=16 \end{cases}$ has matrix equation $\begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 18 \\ 6 \\ 16 \end{pmatrix}$

$$\therefore \begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 18 \\ 6 \\ 16 \end{pmatrix}=\tfrac{1}{4}\begin{pmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{pmatrix}\begin{pmatrix} 18 \\ 6 \\ 16 \end{pmatrix} \qquad \{\mathbf{N}^{-1}=\tfrac{1}{4}\mathbf{M}\}$$

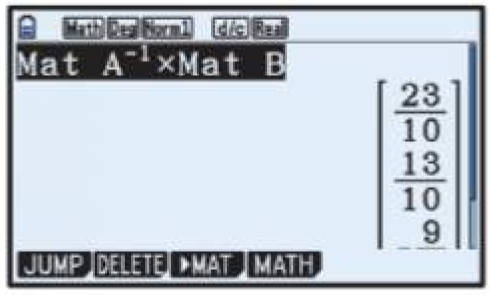
$$\therefore \begin{pmatrix} u \\ v \\ w \end{pmatrix}=\tfrac{1}{4}\begin{pmatrix} 90+18-112 \\ -18-18+48 \\ -54-6+80 \end{pmatrix}=\tfrac{1}{4}\begin{pmatrix} -4 \\ 12 \\ 20 \end{pmatrix}=\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

$\therefore \ u=-1, \ v=3, \ w=5$

7 **a** $\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 14 \\ -8 \\ 13 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ -8 \\ 13 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{23}{10} \\ \frac{13}{10} \\ -\frac{9}{2} \end{pmatrix} \quad \{\text{using technology}\}$$

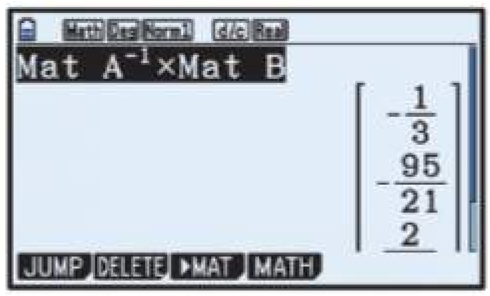


$\therefore \ x=\frac{23}{10}, \ y=\frac{13}{10}, \ z=-\frac{9}{2}$

b $\begin{pmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 4 \\ -6 \\ 17 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ -6 \\ 17 \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{3} \\ -\frac{95}{21} \\ \frac{2}{21} \end{pmatrix} \quad \{\text{using technology}\}$$

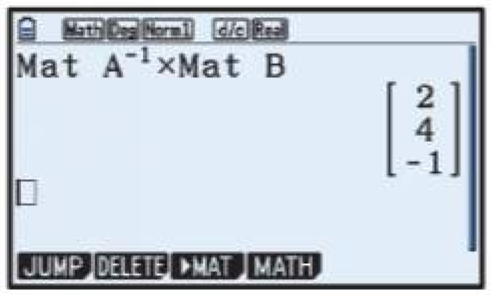


$\therefore \ x=-\frac{1}{3}, \ y=-\frac{95}{21}, \ z=\frac{2}{21}$

c $\begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 15 \\ 7 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \quad \{\text{using technology}\}$$

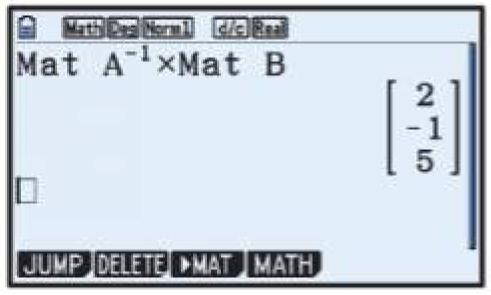


$\therefore\ x = 2,\ y = 4,\ z = -1$

d $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \quad \{\text{using technology}\}$$

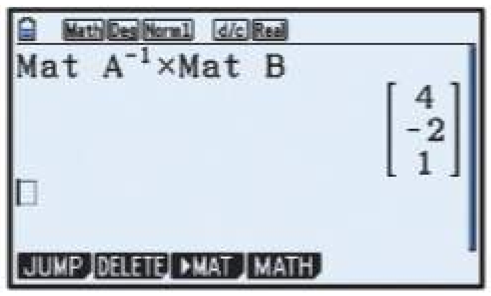


$\therefore\ x = 2,\ y = -1,\ z = 5$

e $\begin{pmatrix} 1 & 4 & 11 \\ 1 & 6 & 17 \\ 1 & 4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 4 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4 & 11 \\ 1 & 6 & 17 \\ 1 & 4 & 8 \end{pmatrix}^{-1}\begin{pmatrix} 7 \\ 9 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \quad \{\text{using technology}\}$$

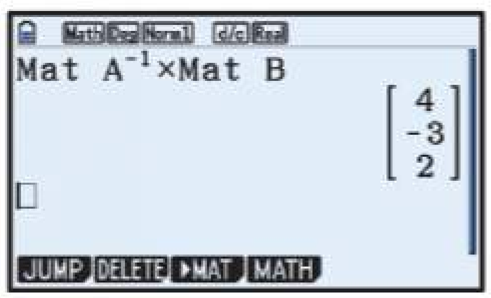


$\therefore\ x = 4,\ y = -2,\ z = 1$

f $\begin{pmatrix} 2 & -1 & 3 \\ 2 & -2 & -5 \\ 3 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 4 \\ 10 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -2 & -5 \\ 3 & 2 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 17 \\ 4 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \quad \{\text{using technology}\}$$

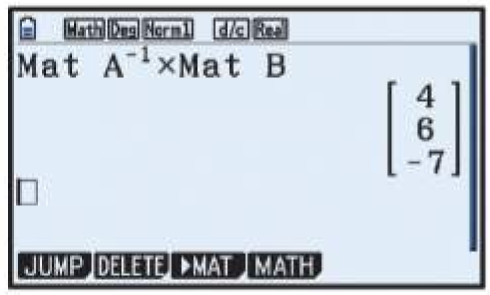


$\therefore\ x = 4,\ y = -3,\ z = 2$

g $\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 7 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ -23 \\ 62 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 7 & 1 & -4 \end{pmatrix}^{-1}\begin{pmatrix} 23 \\ -23 \\ 62 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix} \quad \text{\{using technology\}}$$

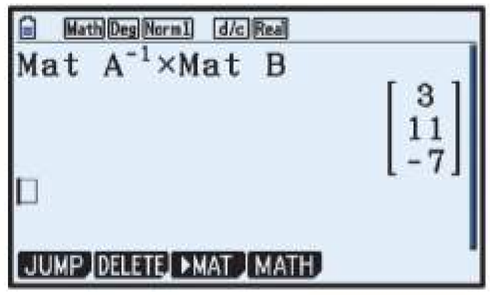


$\therefore \quad x = 4, \ \ y = 6, \ \ z = -7$

h $\begin{pmatrix} 10 & -1 & 4 \\ 7 & 3 & -5 \\ 13 & -17 & 23 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 89 \\ -309 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 & -1 & 4 \\ 7 & 3 & -5 \\ 13 & -17 & 23 \end{pmatrix}^{-1}\begin{pmatrix} -9 \\ 89 \\ -309 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 11 \\ -7 \end{pmatrix} \quad \text{\{using technology\}}$$

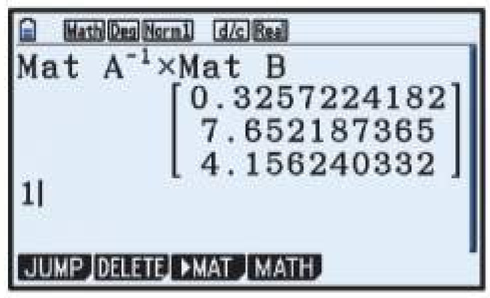


$\therefore \quad x = 3, \ \ y = 11, \ \ z = -7$

i $\begin{pmatrix} 1.3 & 2.7 & -3.1 \\ 2.8 & -0.9 & 5.6 \\ 6.1 & 1.4 & -3.2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8.2 \\ 17.3 \\ -0.6 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.3 & 2.7 & -3.1 \\ 2.8 & -0.9 & 5.6 \\ 6.1 & 1.4 & -3.2 \end{pmatrix}^{-1}\begin{pmatrix} 8.2 \\ 17.3 \\ -0.6 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.326 \\ 7.65 \\ 4.16 \end{pmatrix} \quad \text{\{using technology\}}$$



$\therefore \quad x \approx 0.326, \ \ y \approx 7.65, \ \ z \approx 4.16$

8 **a** If x is the cost of one football, y is the cost of one baseball, and z is the cost of one basketball, then the total cost for each school can be described by

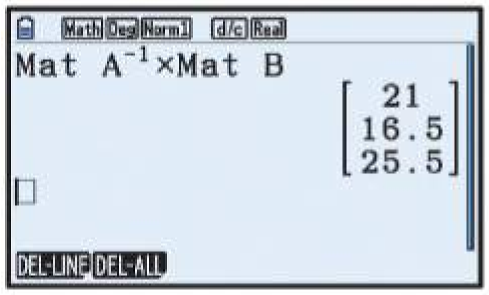
$$\begin{cases} 2x + y + 3z = 135 \\ 3x + 2y + z = 121.50 \\ \quad 5x + 2z = 156 \end{cases}.$$

b The system of equations can be written as

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 5 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 135 \\ 121.50 \\ 156 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 5 & 0 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 135 \\ 121.50 \\ 156 \end{pmatrix}$$

$$= \begin{pmatrix} 21 \\ 16.5 \\ 25.5 \end{pmatrix} \quad \{\text{using technology}\}$$



So, one football costs \$21, one baseball costs \$16.50, and one basketball costs \$25.50.

Now the cost of 4 footballs and 5 baseballs is $4 \times \$21 + 5 \times \$16.50 = \$166.50$

$\therefore$ amount left for basketballs is $\$470 - \$166.50 = \$303.50$

One basketball costs \$25.50 and $\dfrac{303.50}{25.50} \approx 11.9$

$\therefore$ East Park International School will be able to purchase 11 basketballs.

9

Company	*Managers*	*Clerks*	*Labourers*	*Total salary bill*
Xenon	2	3	8	€903 000
Xanda	1	5	4	€749 000
Xylon	1	2	11	€865 000

a If x, y, and z represent the salaries (in thousands of euros) for managers, clerks, and labourers respectively, then the information can be represented by

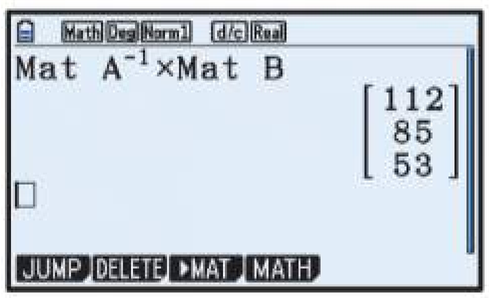
$$\begin{cases} 2x + 3y + 8z = 903 \\ x + 5y + 4z = 749 \\ x + 2y + 11z = 865 \end{cases}.$$

b The system of equations can be written as

$$\begin{pmatrix} 2 & 3 & 8 \\ 1 & 5 & 4 \\ 1 & 2 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 903 \\ 749 \\ 865 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 1 & 5 & 4 \\ 1 & 2 & 11 \end{pmatrix}^{-1}\begin{pmatrix} 903 \\ 749 \\ 865 \end{pmatrix}$$

$$= \begin{pmatrix} 112 \\ 85 \\ 53 \end{pmatrix} \quad \{\text{using technology}\}$$



So, the salaries are: manager €112 000, clerk €85 000, and labourer €53 000.

c The total salary bill for Xudu is $3 \times €112\,000 + 8 \times €85\,000 + 37 \times €53\,000$

$= €2\,977\,000$

10

	Mix A	*Mix B*	*Mix C*
Cashews	5	2	6
Macadamias	3	4	1
Brazil nuts	2	4	3
Cost	\$12.50	\$12.40	\$11.70

a Let x be the cost in dollars of 1 kg of cashews,
y be the cost in dollars of 1 kg of macadamias, and
z be the cost in dollars of 1 kg of Brazil nuts.

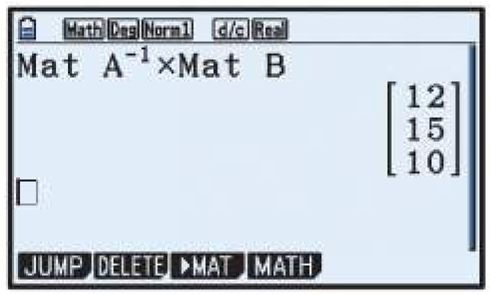
The cost of 1 kg of mix A is $0.5x + 0.3y + 0.2z = 12.5$,
the cost of 1 kg of mix B is $0.2x + 0.4y + 0.4z = 12.4$,
the cost of 1 kg of mix C is $0.6x + 0.1y + 0.3z = 11.7$.

The system of equations can be written as

$$\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.5 \\ 12.4 \\ 11.7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.1 & 0.3 \end{pmatrix}^{-1} \begin{pmatrix} 12.5 \\ 12.4 \\ 11.7 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 15 \\ 10 \end{pmatrix} \quad \{\text{using technology}\}$$



So, the cost of 1 kg of cashews is \$12, the cost of 1 kg of macadamias is \$15 and the cost of 1 kg of Brazil nuts is \$10.

b The cost of 400 g cashews, 200 g macadamias, and 400 g Brazil nuts
$= 0.4 \times 12 + 0.2 \times 15 + 0.4 \times 10$ dollars
$= \$11.80$

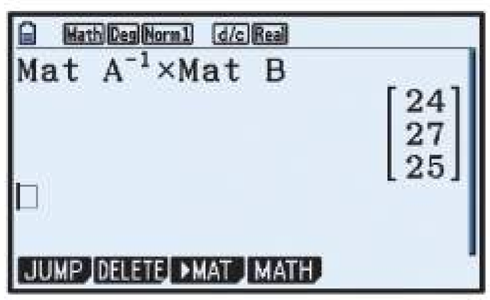
11 The number of students who study Chemistry is $\frac{1}{3}p + \frac{1}{3}q + \frac{2}{5}r = 27$,
the number of students who study Physics is $\frac{1}{2}p + \frac{2}{3}q + \frac{1}{5}r = 35$,
the number of students who study Geography is $\frac{1}{4}p + \frac{1}{3}q + \frac{3}{5}r = 30$

The system of equations can be written as

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 27 \\ 35 \\ 30 \end{pmatrix}$$

$$\therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{5} \end{pmatrix}^{-1} \begin{pmatrix} 27 \\ 35 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ 27 \\ 25 \end{pmatrix} \quad \{\text{using technology}\}$$



There are 24 students in class P, 27 students in class Q, and 25 students in class R.

12 The asteroid's path is given by $x^2 + axy + by^2 + cx + dy + e = 0$,

which is $axy + by^2 + cx + dy + e = -x^2$.

It passes through the points:

$(-1.03, 2.164)$ $\therefore$ $(-1.03)(2.164)a + (2.164)^2 b - 1.03c + 2.164d + e = -(-1.03)^2$

$(-0.56, -1.868)$ $\therefore$ $(-0.56)(-1.868)a + (-1.868)^2 b - 0.56c - 1.868d + e = -(-0.56)^2$

$(0.38, -1.668)$ $\therefore$ $(0.38)(-1.668)a + (-1.668)^2 b + 0.38c - 1.668d + e = -(0.38)^2$

$(1.17, 4.876)$ $\therefore$ $(1.17)(4.876)a + (4.876)^2 b + 1.17c + 4.876d + e = -(1.17)^2$

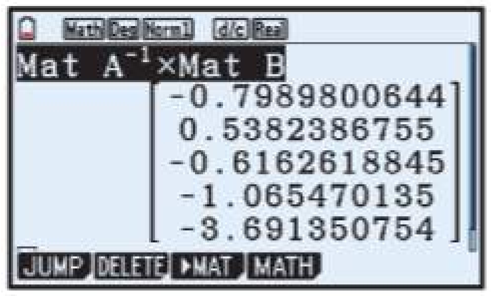
$(2.89, 1.019)$ $\therefore$ $(2.89)(1.019)a + (1.019)^2 b + 2.89c + 1.019d + e = -(2.89)^2$

The system of equations can be written as

$$\begin{pmatrix} -2.22892 & 4.682896 & -1.03 & 2.164 & 1 \\ 1.04608 & 3.489424 & -0.56 & -1.868 & 1 \\ -0.63384 & 2.782224 & 0.38 & -1.668 & 1 \\ 5.70492 & 23.775376 & 1.17 & 4.876 & 1 \\ 2.94491 & 1.038361 & 2.89 & 1.019 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -1.0609 \\ -0.3136 \\ -0.1444 \\ -1.3689 \\ -8.3521 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -2.22892 & 4.682896 & -1.03 & 2.164 & 1 \\ 1.04608 & 3.489424 & -0.56 & -1.868 & 1 \\ -0.63384 & 2.782224 & 0.38 & -1.668 & 1 \\ 5.70492 & 23.775376 & 1.17 & 4.876 & 1 \\ 2.94491 & 1.038361 & 2.89 & 1.019 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1.0609 \\ -0.3136 \\ -0.1444 \\ -1.3689 \\ -8.3521 \end{pmatrix}$$

$$\approx \begin{pmatrix} -0.799 \\ 0.538 \\ -0.616 \\ -1.07 \\ -3.69 \end{pmatrix} \quad \{\text{using technology}\}$$



So, the equation of the asteroid's orbit is approximately:

$x^2 - 0.799xy + 0.538y^2 - 0.616x - 1.07y - 3.69 = 0$

ACTIVITY CRYPTOGRAPHY

1 $\begin{pmatrix} 21 & 12 \\ 1 & 9 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 19 & 5 \\ -12 & 4 \end{pmatrix} \equiv \begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} \pmod{26}$

$\begin{pmatrix} 15 & 22 \\ 1 & 10 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 13 & 15 \\ -12 & 5 \end{pmatrix} \equiv \begin{pmatrix} 13 & 15 \\ 14 & 5 \end{pmatrix} \pmod{26}$

$\begin{pmatrix} 1 & 23 \\ 25 & 10 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 16 \\ 12 & 5 \end{pmatrix} \equiv \begin{pmatrix} 25 & 16 \\ 12 & 5 \end{pmatrix} \pmod{26}$

$\begin{pmatrix} 3 & 0 \\ 18 & 10 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 5 & 5 \end{pmatrix} \equiv \begin{pmatrix} 1 & 19 \\ 5 & 5 \end{pmatrix} \pmod{26}$

So, the whole message is 19 5 14 4 13 15 14 5 25 16 12 5 1 19 5 5, which when decoded gives SENDMONEYPLEASEE.

2 $\begin{pmatrix} 21 & 12 \\ 1 & 9 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 19 & 5 \\ -12 & 4 \end{pmatrix} \equiv \begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} \pmod{26}$

$$\begin{pmatrix} 22 & 15 \\ 18 & 25 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 20 & 8 \\ 5 & 20 \end{pmatrix} \equiv \begin{pmatrix} 20 & 8 \\ 5 & 20 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 20 & 22 \\ 2 & 21 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 18 & 15 \\ -11 & 16 \end{pmatrix} \equiv \begin{pmatrix} 18 & 15 \\ 15 & 16 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 21 & 1 \\ 2 & 25 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 19 & -6 \\ -11 & 20 \end{pmatrix} \equiv \begin{pmatrix} 19 & 20 \\ 15 & 20 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 10 & 12 \\ 0 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -13 & 15 \end{pmatrix} \equiv \begin{pmatrix} 8 & 5 \\ 13 & 15 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 23 & 1 \\ 21 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 21 & -6 \\ 8 & 15 \end{pmatrix} \equiv \begin{pmatrix} 21 & 20 \\ 8 & 15 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 8 & 1 \\ 21 & 10 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 6 & -6 \\ 8 & 5 \end{pmatrix} \equiv \begin{pmatrix} 6 & 20 \\ 8 & 5 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 15 & 2 \\ 5 & 23 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 13 & -5 \\ -8 & 18 \end{pmatrix} \equiv \begin{pmatrix} 13 & 21 \\ 18 & 18 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 3 & 6 \\ 12 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 25 \\ 25 & 25 \end{pmatrix} \pmod{26}$$

So, the whole message is

19 5 14 4 20 8 5 20 18 15 15 16 19 20 15 20 8 5
13 15 21 20 8 15 6 20 8 5 13 21 18 18 1 25 25 25

which when decoded gives SENDTHETROOPSTOTHEMOUTHOFTHEMURRAYYY.

4 **a** $\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 43 & 67 \\ 32 & 50 \end{pmatrix} \equiv \begin{pmatrix} 17 & 15 \\ 6 & 24 \end{pmatrix} \pmod{26}$

$$\begin{pmatrix} 13 & 15 \\ 14 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 41 & 69 \\ 33 & 52 \end{pmatrix} \equiv \begin{pmatrix} 15 & 17 \\ 7 & 0 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 25 & 16 \\ 12 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 66 & 107 \\ 29 & 46 \end{pmatrix} \equiv \begin{pmatrix} 14 & 3 \\ 3 & 20 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 1 & 19 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 21 & 41 \\ 15 & 25 \end{pmatrix} \equiv \begin{pmatrix} 21 & 15 \\ 15 & 25 \end{pmatrix} \pmod{26}$$

So, the coded form of SEND MONEY PLEASE is

17 15 6 24 15 17 7 0 14 3 3 20 21 15 15 25.

b The receiver needs the inverse of the encryption matrix $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 17 & 15 \\ 6 & 24 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 19 & -21 \\ -12 & 30 \end{pmatrix} \equiv \begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 15 & 17 \\ 7 & 0 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 13 & -11 \\ 14 & -21 \end{pmatrix} \equiv \begin{pmatrix} 13 & 15 \\ 14 & 5 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 14 & 3 \\ 3 & 20 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 25 & -36 \\ -14 & 31 \end{pmatrix} \equiv \begin{pmatrix} 25 & 16 \\ 12 & 5 \end{pmatrix} \pmod{26}$$

$$\begin{pmatrix} 21 & 15 \\ 15 & 25 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 27 & -33 \\ 5 & 5 \end{pmatrix} \equiv \begin{pmatrix} 1 & 19 \\ 5 & 5 \end{pmatrix} \pmod{26}$$

So, the whole message is 19 5 14 4 13 15 14 5 25 16 12 5 1 19 5 5, which when decoded gives SENDMONEYPLEASEE.

REVIEW SET 12A

1 **a** $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$

b $3\mathbf{A} = 3\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$

c $-2\mathbf{B} = -2\begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} -2 & 0 \\ 4 & -8 \end{pmatrix}$

d $\mathbf{A}-\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & -5 \end{pmatrix}$

e $\mathbf{B}-2\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} - 2\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 6 & 4 \\ 0 & -2 \end{pmatrix}$

$= \begin{pmatrix} -5 & -4 \\ -2 & 6 \end{pmatrix}$

f $3\mathbf{A}-2\mathbf{B} = 3\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -4 & 8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 6 \\ 4 & -11 \end{pmatrix}$

g $\mathbf{AB} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} -1 & 8 \\ 2 & -4 \end{pmatrix}$

h $\mathbf{BA} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 2 \\ -6 & -8 \end{pmatrix}$

i $\mathbf{A}^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{pmatrix}$

j $\mathbf{A}^2 = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 4 \\ 0 & 1 \end{pmatrix}$

k $\mathbf{ABA} = (\mathbf{AB})\mathbf{A}$

$= \begin{pmatrix} -1 & 8 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$ {using **g**}

$= \begin{pmatrix} -3 & -10 \\ 6 & 8 \end{pmatrix}$

l $(\mathbf{AB})^{-1} = \begin{pmatrix} -1 & 8 \\ 2 & -4 \end{pmatrix}^{-1}$ {using **g**}

$= \frac{1}{4-16}\begin{pmatrix} -4 & -8 \\ -2 & -1 \end{pmatrix}$

$= \frac{1}{-12}\begin{pmatrix} -4 & -8 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{pmatrix}$

2 **a** $\begin{pmatrix} a & b-2 \\ c & d \end{pmatrix} = \begin{pmatrix} -a & 3 \\ 2-c & -4 \end{pmatrix}$

Equating corresponding elements,

$a = -a$

$b - 2 = 3$

$c = 2 - c$

$d = -4$

$\therefore\ a = 0,\ b = 5,\ c = 1,\ d = -4$

b $\begin{pmatrix} 3 & 2a \\ b & -2 \end{pmatrix} + \begin{pmatrix} b & -a \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2 \\ 2 & 6 \end{pmatrix}$

$\therefore \begin{pmatrix} 3+b & a \\ b+c & -2+d \end{pmatrix} = \begin{pmatrix} a & 2 \\ 2 & 6 \end{pmatrix}$

Equating corresponding elements,

$3 + b = a$

$a = 2$

$b + c = 2$

$-2 + d = 6$

$\therefore\ a = 2,\ b = -1,\ c = 3,\ d = 8$

3 **a** $\mathbf{B} - \mathbf{Y} = \mathbf{A}$

$\therefore\ -\mathbf{Y} = \mathbf{A} - \mathbf{B}$

$\therefore\ \mathbf{Y} = -(\mathbf{A} - \mathbf{B})$

$\therefore\ \mathbf{Y} = \mathbf{B} - \mathbf{A}$

b $2\mathbf{Y} + \mathbf{C} = \mathbf{A}$

$\therefore\ 2\mathbf{Y} = \mathbf{A} - \mathbf{C}$

$\therefore\ \mathbf{Y} = \frac{1}{2}(\mathbf{A} - \mathbf{C})$

c $\mathbf{AY} = \mathbf{B}$

$\therefore\ \mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}\mathbf{B}$

provided $\mathbf{A}^{-1}$ exists

$\therefore\ \mathbf{IY} = \mathbf{A}^{-1}\mathbf{B}$

$\therefore\ \mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$

d $\mathbf{YB} = \mathbf{C}$

$\therefore\ \mathbf{YBB}^{-1} = \mathbf{CB}^{-1}$

provided $\mathbf{B}^{-1}$ exists

$\therefore\ \mathbf{YI} = \mathbf{CB}^{-1}$

$\therefore\ \mathbf{Y} = \mathbf{CB}^{-1}$

e $\mathbf{C} - \mathbf{AY} = \mathbf{B}$

$\therefore\ -\mathbf{AY} = \mathbf{B} - \mathbf{C}$

$\therefore\ \mathbf{AY} = \mathbf{C} - \mathbf{B}$

$\therefore\ \mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

provided $\mathbf{A}^{-1}$ exists

$\therefore\ \mathbf{Y} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

f $\mathbf{AY}^{-1} = \mathbf{B}$

$\therefore\ \mathbf{A}^{-1}\mathbf{AY}^{-1} = \mathbf{A}^{-1}\mathbf{B}$

provided $\mathbf{A}^{-1}$ exists

$\therefore\ \mathbf{Y}^{-1} = \mathbf{A}^{-1}\mathbf{B}$

$\therefore\ (\mathbf{Y}^{-1})^{-1} = (\mathbf{A}^{-1}\mathbf{B})^{-1}$

$\therefore\ \mathbf{Y} = \mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1}$

provided $\mathbf{Y}$ exists

$\therefore\ \mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$

4 **a** Over a 4 week period, the hens lay 4 times the number of eggs laid during 1 week, that is, $4\mathbf{L}$.

b Each fortnight, the hens lay 2 times the number of eggs laid during 1 week, that is $2\mathbf{L}$. When the eggs are collected, the number of eggs each hen loses is given by their respective position in the matrix $-2\mathbf{L}$.

5 $\mathbf{A}=\begin{pmatrix}-2 & 3\\ 4 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-7 & 9\\ 9 & -3\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}-1 & 0 & 3\\ 0 & 2 & 1\end{pmatrix}$

a
$$\begin{aligned}2\mathbf{A}-2\mathbf{B}&=2\begin{pmatrix}-2 & 3\\ 4 & -1\end{pmatrix}-2\begin{pmatrix}-7 & 9\\ 9 & -3\end{pmatrix}\\ &=\begin{pmatrix}-4 & 6\\ 8 & -2\end{pmatrix}-\begin{pmatrix}-14 & 18\\ 18 & -6\end{pmatrix}\\ &=\begin{pmatrix}10 & -12\\ -10 & 4\end{pmatrix}\end{aligned}$$

b $\mathbf{A}$ is 2×2 and $\mathbf{C}$ is 2×3 $\therefore$ $\mathbf{AC}$ is 2×3

$\checkmark$

$$\mathbf{AC}=\begin{pmatrix}-2 & 3\\ 4 & -1\end{pmatrix}\begin{pmatrix}-1 & 0 & 3\\ 0 & 2 & 1\end{pmatrix}=\begin{pmatrix}2+0 & 0+6 & -6+3\\ -4+0 & 0-2 & 12-1\end{pmatrix}=\begin{pmatrix}2 & 6 & -3\\ -4 & -2 & 11\end{pmatrix}$$

c $\mathbf{C}$ is 2×3 and $\mathbf{B}$ is 2×2 $\therefore$ $\mathbf{CB}$ is not possible.

not equal

6 **a**
$$\begin{aligned}&\mathbf{A}(\mathbf{I}-\mathbf{A})\\ &=\mathbf{AI}-\mathbf{AA}\\ &=\mathbf{A}-\mathbf{A}^2\end{aligned}$$

b
$$\begin{aligned}&(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{A})\\ &=(\mathbf{A}-\mathbf{B})\mathbf{B}+(\mathbf{A}-\mathbf{B})\mathbf{A}\\ &=\mathbf{AB}-\mathbf{B}^2+\mathbf{A}^2-\mathbf{BA}\end{aligned}$$

c
$$\begin{aligned}&(2\mathbf{A}-\mathbf{I})^2\\ &=(2\mathbf{A}-\mathbf{I})(2\mathbf{A}-\mathbf{I})\\ &=(2\mathbf{A}-\mathbf{I})(2\mathbf{A})-(2\mathbf{A}-\mathbf{I})\mathbf{I}\\ &=4\mathbf{A}^2-2\mathbf{IA}-2\mathbf{AI}+\mathbf{I}^2\\ &=4\mathbf{A}^2-2\mathbf{A}-2\mathbf{A}+\mathbf{I}\\ &=4\mathbf{A}^2-4\mathbf{A}+\mathbf{I}\end{aligned}$$

7 If $\mathbf{A}^2=5\mathbf{A}+2\mathbf{I}$,

$$\begin{aligned}\mathbf{A}^3&=\mathbf{A}\times\mathbf{A}^2\\ &=\mathbf{A}(5\mathbf{A}+2\mathbf{I})\\ &=5\mathbf{A}^2+2\mathbf{AI}\\ &=5(5\mathbf{A}+2\mathbf{I})+2\mathbf{A}\\ &=25\mathbf{A}+10\mathbf{I}+2\mathbf{A}\\ &=27\mathbf{A}+10\mathbf{I}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^4&=\mathbf{A}\times\mathbf{A}^3\\ &=\mathbf{A}(27\mathbf{A}+10\mathbf{I})\\ &=27\mathbf{A}^2+10\mathbf{AI}\\ &=27(5\mathbf{A}+2\mathbf{I})+10\mathbf{A}\\ &=135\mathbf{A}+54\mathbf{I}+10\mathbf{A}\\ &=145\mathbf{A}+54\mathbf{I}\end{aligned}$$

8 If $\mathbf{A}=\begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}$, let $\mathbf{A}^2=a\mathbf{A}+b\mathbf{I}$

$$\begin{aligned}\therefore\ \begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}&=a\begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}+b\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ \therefore\ \begin{pmatrix}4-3 & -2-2\\ 6+6 & -3+4\end{pmatrix}&=\begin{pmatrix}2a & -a\\ 3a & 2a\end{pmatrix}+\begin{pmatrix}b & 0\\ 0 & b\end{pmatrix}\\ \therefore\ \begin{pmatrix}1 & -4\\ 12 & 1\end{pmatrix}&=\begin{pmatrix}2a+b & -a\\ 3a & 2a+b\end{pmatrix}\end{aligned}$$

$$\therefore\ 1=2a+b \quad\text{and}\quad -4=-a$$
$$\therefore\ a=4 \quad\text{and}\quad b=-7$$
$$\therefore\ \mathbf{A}^2=4\mathbf{A}-7\mathbf{B}$$

Checking for consistency: $3a=3(4)=12$ ✓

and $2a+b=2(4)+(-7)=1$ ✓

9 **a** $\begin{pmatrix} 6 & 8 \\ 5 & 7 \end{pmatrix}^{-1} = \frac{1}{42-40}\begin{pmatrix} 7 & -8 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{pmatrix}$

b $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}^{-1}$ does not exist as $ad - bc = -24 - (-24) = 0$

c $\begin{pmatrix} 11 & 5 \\ -6 & -3 \end{pmatrix}^{-1} = \frac{1}{-33-(-30)}\begin{pmatrix} -3 & -5 \\ 6 & 11 \end{pmatrix} = \begin{pmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{pmatrix}$

10 $\begin{cases} x + 4y = 2 \\ kx + 3y = -6 \end{cases}$ can be written as $\begin{pmatrix} 1 & 4 \\ k & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

which is in the form $\mathbf{AX} = \mathbf{B}$ where $\det \mathbf{A} = 3 - 4k$.

The system has a unique solution if $\det \mathbf{A} \neq 0$

$\therefore\ 3 - 4k \neq 0$

$\therefore\ k \neq \frac{3}{4}$

11 **a** $\begin{cases} 3x - 4y = 2 \\ 5x + 2y = -1 \end{cases}$ can be written as

$$\begin{pmatrix} 3 & -4 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{aligned} \therefore\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & -4 \\ 5 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \frac{1}{26}\begin{pmatrix} 2 & 4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \frac{1}{26}\begin{pmatrix} 0 \\ -13 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \end{aligned}$$

$\therefore\ x = 0,\quad y = -\frac{1}{2}$

b $\begin{cases} 4x - y = 5 \\ 2x + 3y = 9 \end{cases}$ can be written as

$$\begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

$$\begin{aligned} \therefore\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 9 \end{pmatrix} \\ &= \frac{1}{14}\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 9 \end{pmatrix} \\ &= \frac{1}{14}\begin{pmatrix} 24 \\ 26 \end{pmatrix} \\ &= \begin{pmatrix} \frac{12}{7} \\ \frac{13}{7} \end{pmatrix} \end{aligned}$$

$\therefore\ x = \frac{12}{7},\quad y = \frac{13}{7}$

c $\begin{cases} 3x - y + 2z = 3 \\ 2x + 3y - z = -3 \\ x - 2y + 3z = 2 \end{cases}$ can be written as

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{14} & -\frac{5}{14} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{5}{14} & \frac{11}{14} \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \quad \{\text{using technology}\}$$

$$= \begin{pmatrix} \frac{3}{2} + \frac{3}{14} - \frac{10}{14} \\ -\frac{3}{2} - \frac{3}{2} + \frac{2}{2} \\ -\frac{3}{2} - \frac{15}{14} + \frac{22}{14} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$$\therefore \; x = 1, \quad y = -2, \quad z = -1$$

12 **a** $s(t) = at^2 + bt + c$

$s(1) = 63 \quad \therefore \; a + b + c = 63$

$s(2) = 72 \quad \therefore \; 4a + 2b + c = 72$

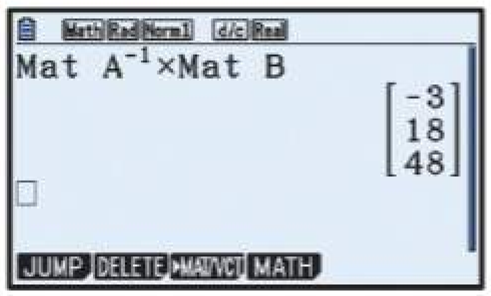
$s(7) = 27 \quad \therefore \; 49a + 7b + c = 27$

The system of equations can be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 49 & 7 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 63 \\ 72 \\ 27 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 49 & 7 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 63 \\ 72 \\ 27 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 18 \\ 48 \end{pmatrix} \quad \{\text{using technology}\}$$



$\therefore \; a = -3, \; b = 18, \; c = 48$

So, $s(t) = -3t^2 + 18t + 48$.

b $s(0) = 48 \quad \therefore$ the height of the cliff is 48 m.

c The rock reaches sea level when $s(t) = 0$

$$\therefore \; -3(t^2 - 6t - 16) = 0$$

$$\therefore \; -3(t - 8)(t + 2) = 0$$

$$\therefore \; t = 8 \text{ or } -2$$

But $t \geqslant 0$, so the rock reaches sea level after 8 seconds.

13

	Opera	*Play*	*Concert*	*Total cost*
Hung	3	2	5	€267
Quan	2	3	1	€145
Ariel	1	5	4	€230

a Let €x be the cost of an opera ticket,
€y be the cost of a play ticket,
and €z be the cost of a concert ticket.

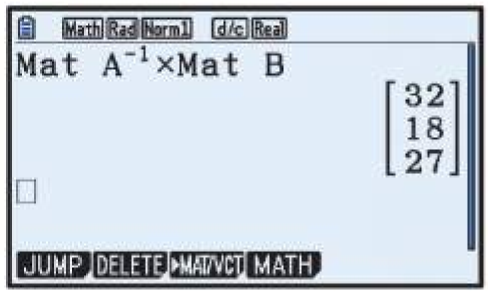
$$\therefore \begin{cases} 3x+2y+5z=267 \\ 2x+3y+z=145 \\ x+5y+4z=230 \end{cases}$$

b The system of equations can be written as

$$\begin{pmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 267 \\ 145 \\ 230 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 267 \\ 145 \\ 230 \end{pmatrix}$$

$$=\begin{pmatrix} 32 \\ 18 \\ 27 \end{pmatrix} \quad \{\text{using technology}\}$$



$\therefore$ the cost of each ticket is €32 for an opera, €18 for a play, €27 for a concert.

c Total cost $= 4 \times €32 + 1 \times €18 + 2 \times €27$
$= €128 + €18 + €54 = €200$

REVIEW SET 12B

1 **a** $\mathbf{P}+\mathbf{Q}=\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}+\begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$

$=\begin{pmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$

b $\mathbf{Q}-\mathbf{P}=\begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$

$=\begin{pmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{pmatrix}$

c $\frac{3}{2}\mathbf{P}-\mathbf{Q}=\frac{3}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}-\begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$

$=\begin{pmatrix} \frac{3}{2} & 3 \\ \frac{3}{2} & 0 \\ 3 & \frac{9}{2} \end{pmatrix}-\begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$

$=\begin{pmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{pmatrix}$

2 **a** The books currently on the shelves can be described by the matrix equation

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}4 & 2\\ 5 & 2\\ 6 & 3\end{pmatrix}-\begin{pmatrix}2 & 0\\ 1 & 1\\ 3 & 2\end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc}\text{paperback} & \text{hard cover}\end{array} \\ = & \begin{pmatrix}2 & 2\\ 4 & 1\\ 3 & 1\end{pmatrix}\begin{array}{l}\leftarrow \text{book 1}\\ \leftarrow \text{book 2}\\ \leftarrow \text{book 3}\end{array}\end{array}$$

b **i**

$$\begin{array}{cc} & \begin{array}{ccc}\text{book 1} & \text{book 2} & \text{book 3}\end{array} \\ \mathbf{C}= & \begin{pmatrix}7 & 7 & 8\\ 15 & 16 & 20\end{pmatrix}\begin{array}{l}\leftarrow \text{paperback}\\ \leftarrow \text{hard cover}\end{array}\end{array}$$

The entry in row 2, column 2 is 16. This entry corresponds to the hard cover edition of book 2, so this book has value \$16.

ii Total value of books on loan

$= \$(7\times 2+7\times 1+8\times 3+15\times 0+16\times 1+20\times 2)$

$= \$(14+7+24+0+16+40)$

$= \$101$

3 Let $\mathbf{A}$ be an $n\times n$ square matrix and $\mathbf{O}$ be the $n\times n$ zero matrix.

$\mathbf{A}=(a_{ij})$ where $i=1, 2, 3, \ldots., n$
$\quad j=1, 2, 3, \ldots., n$

$\mathbf{O}=(b_{ij})$ where $b_{ij}=0$ for all $i=1, 2, 3, \ldots., n$ and $j=1, 2, 3, \ldots., n$

If $\mathbf{C}=\mathbf{AO}$ then

$$\begin{aligned}c_{ij}&=\sum_{r=1}^{n} a_{ir}b_{rj}\\ &=\sum_{r=1}^{n} a_{ir}0 \qquad \{\text{as } b_{ij}=0 \text{ for all } i, j\}\\ &=\sum_{r=1}^{n} 0\\ &=0\end{aligned}$$

$\therefore$ $\mathbf{AO}=(c_{ij})$ where $c_{ij}=0$ for all i, j

$\therefore$ $\mathbf{AO}=\mathbf{O}$

Similarly, if $\mathbf{D}=\mathbf{OA}$ then

$$\begin{aligned}d_{ij}&=\sum_{r=1}^{n} b_{ir}a_{rj}\\ &=\sum_{r=1}^{n} 0a_{rj} \qquad \{\text{as } b_{ij}=0 \text{ for all } i, j\}\\ &=\sum_{r=1}^{n} 0\\ &=0\end{aligned}$$

$\therefore$ $\mathbf{OA}=(d_{ij})$ where $d_{ij}=0$ for all i, j

$\therefore$ $\mathbf{OA}=\mathbf{O}$

So, $\mathbf{AO}=\mathbf{OA}=\mathbf{O}$

4 **a** $2\mathbf{X} = \mathbf{B} - \mathbf{A}$

$\therefore \ \mathbf{X} = \frac{1}{2}(\mathbf{B} - \mathbf{A})$

b $3(\mathbf{A} + \mathbf{X}) = 2\mathbf{B}$

$\therefore \ \mathbf{A} + \mathbf{X} = \frac{1}{3}(2\mathbf{B})$

$\therefore \ \mathbf{X} = \frac{2}{3}\mathbf{B} - \mathbf{A}$

c $\mathbf{B} - 4\mathbf{X} = \mathbf{A}$

$\therefore \ -4\mathbf{X} = \mathbf{A} - \mathbf{B}$

$\therefore \ \mathbf{X} = -\frac{1}{4}(\mathbf{A} - \mathbf{B})$

$\therefore \ \mathbf{X} = \frac{1}{4}(\mathbf{B} - \mathbf{A})$

5 $\mathbf{A} + 2\mathbf{X} = -\mathbf{B}$

$$\begin{aligned}
\therefore \ 2\mathbf{X} &= -\mathbf{B} - \mathbf{A} \\
\therefore \ \mathbf{X} &= -\tfrac{1}{2}(\mathbf{B} + \mathbf{A}) \\
&= -\tfrac{1}{2}\left(\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\right) \\
&= -\tfrac{1}{2}\begin{pmatrix} 1 & 3 \\ 0 & -3 \end{pmatrix} \\
\therefore \ \mathbf{X} &= \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{3}{2} \end{pmatrix}
\end{aligned}$$

6 **a** $2\mathbf{B} = 2\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{pmatrix}$

b $\frac{1}{2}\mathbf{B} = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$

c $\mathbf{A}$ is 1×3 and $\mathbf{B}$ is 3×2 $\quad \therefore \ \mathbf{AB}$ is 1×2

$\checkmark$

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 12 \end{pmatrix}$$

d $\mathbf{B}$ is 3×2 and $\mathbf{A}$ is 1×3 $\quad \therefore \ \mathbf{BA}$ does not exist.

not equal

7 **a** If $\mathbf{AB} = \mathbf{B}$ then

$\mathbf{ABB}^{-1} = \mathbf{BB}^{-1}$ provided $\mathbf{B}^{-1}$ exists, that is $\det \mathbf{B} \neq 0$

$\therefore \ \mathbf{AI} = \mathbf{I}$ provided $\det \mathbf{B} \neq 0$

$\therefore \ \mathbf{A} = \mathbf{I}$ provided $\det \mathbf{B} \neq 0$

b
$$\begin{aligned}
(\mathbf{A} + \mathbf{B})^2 &= (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) \\
&= (\mathbf{A} + \mathbf{B})\mathbf{A} + (\mathbf{A} + \mathbf{B})\mathbf{B} \\
&= \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2 \\
&= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 \qquad \text{provided } \mathbf{AB} = \mathbf{BA}
\end{aligned}$$

8 $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

a
$$\begin{aligned}
\det \mathbf{A} &= 3(1) - 2(-1) \\
&= 5
\end{aligned}$$

b
$$\begin{aligned}
\det(-2\mathbf{A}) &= (-2)^2 \times \det \mathbf{A} \qquad \{\det(k\mathbf{A}) = k^2 \times \det \mathbf{A}\} \\
&= 4 \times 5 \\
&= 20
\end{aligned}$$

c $\det(\mathbf{A}^2) = \det \mathbf{A} \times \det \mathbf{A}$ $\{\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}\}$

$= 5 \times 5$

$= 25$

9 **a** $\begin{cases} x + y = 5 \\ x - 2y = 4 \end{cases}$ can be written as

$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= \tfrac{1}{-3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= -\tfrac{1}{3}\begin{pmatrix} -14 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{14}{3} \\ \frac{1}{3} \end{pmatrix}$$

$\therefore\ x = \frac{14}{3},\ \ y = \frac{1}{3}$

b $\begin{cases} 3x + 2y = 3 \\ 5x + 3y = 4 \end{cases}$ can be written as

$$\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \tfrac{1}{-1}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= -\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$\therefore\ x = -1,\ \ y = 3$

c $\begin{cases} 2x + y + z = 8 \\ 4x - 7y + 3z = 10 \\ 3x - 2y - z = 1 \end{cases}$ can be written as

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & -\frac{1}{52} & \frac{5}{26} \\ \frac{1}{4} & -\frac{5}{52} & -\frac{1}{26} \\ \frac{1}{4} & \frac{7}{52} & -\frac{9}{26} \end{pmatrix}\begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} \quad \{\text{using technology}\}$$

$$= \begin{pmatrix} \frac{8}{4} - \frac{10}{52} + \frac{5}{26} \\ \frac{8}{4} - \frac{50}{52} - \frac{1}{26} \\ \frac{8}{4} + \frac{70}{52} - \frac{9}{26} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore\ x = 2,\ \ y = 1,\ \ z = 3$

10 $$\mathbf{M} = \begin{pmatrix} k & 2 \\ 2 & k \end{pmatrix}\begin{pmatrix} k-1 & -2 \\ -3 & k \end{pmatrix}$$

$$\begin{aligned} \therefore\ \det\mathbf{M} &= \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} \times \begin{vmatrix} k-1 & -2 \\ -3 & k \end{vmatrix} \qquad \{\det(\mathbf{AB}) = \det\mathbf{A}\times\det\mathbf{B}\} \\ &= (k^2-4)(k(k-1)-6) \\ &= (k+2)(k-2)(k^2-k-6) \\ &= (k+2)^2(k-2)(k-3) \end{aligned}$$

The inverse $\mathbf{M}^{-1}$ exists provided $\det\mathbf{M} \neq 0$

$\therefore\ k \neq -2, 2, \text{ or } 3,\ k \in \mathbb{R}$

11 **a** $\begin{cases} kx + 3y = -6 \\ x + (k+2)y = 2 \end{cases}$ can be written as $\begin{pmatrix} k & 3 \\ 1 & k+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

which is in the form $\mathbf{AX} = \mathbf{B}$ where $\det\mathbf{A} = k(k+2) - 3$.

The system has a unique solution provided $\det\mathbf{A} \neq 0$

$$\begin{aligned} &\therefore\ k(k+2) - 3 \neq 0 \\ &\therefore\ k^2 + 2k - 3 \neq 0 \\ &\therefore\ (k+3)(k-1) \neq 0 \\ &\therefore\ k \neq -3 \text{ or } 1 \end{aligned}$$

b Now for $k \neq -3$ or 1,

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} k & 3 \\ 1 & k+2 \end{pmatrix}^{-1}\begin{pmatrix} -6 \\ 2 \end{pmatrix} \\ &= \frac{1}{(k+3)(k-1)}\begin{pmatrix} k+2 & -3 \\ -1 & k \end{pmatrix}\begin{pmatrix} -6 \\ 2 \end{pmatrix} \\ &= \frac{1}{(k+3)(k-1)}\begin{pmatrix} -6(k+2) - 6 \\ 6 + 2k \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{-6k-18}{(k+3)(k-1)} \\ \dfrac{6+2k}{(k+3)(k-1)} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{-6\cancel{(k+3)}}{\cancel{(k+3)}(k-1)} \\ \dfrac{2\cancel{(k+3)}}{\cancel{(k+3)}(k-1)} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{-6}{k-1} \\ \dfrac{2}{k-1} \end{pmatrix} \end{aligned}$$

$\therefore\ x = \dfrac{-6}{k-1},\ y = \dfrac{2}{k-1}$ is the unique solution, provided $k \neq -3$ or 1.

12 **a** The circle has equation $x^2 + y^2 + ax + by + c = 0$, which is $ax + by + c = -x^2 - y^2$.

It passes through the points:

$(2, 2)$ $\therefore\ 2a + 2b + c = -8$

$(-2, 4)$ $\therefore\ -2a + 4b + c = -20$

$(1, 3)$ $\therefore\ a + 3b + c = -10$

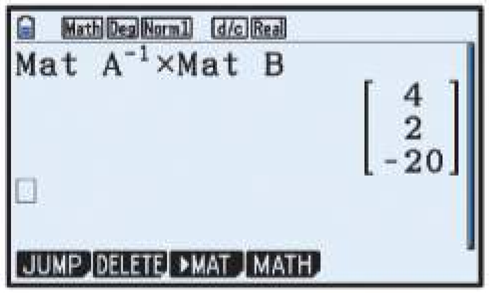
So, the system of equations for a, b, and c is $\begin{cases} 2a + 2b + c = -8 \\ -2a + 4b + c = -20 \\ a + 3b + c = -10 \end{cases}$.

b The system of equations can be written as

$$\begin{pmatrix} 2 & 2 & 1 \\ -2 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -8 \\ -20 \\ -10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ -2 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -8 \\ -20 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 2 \\ -20 \end{pmatrix} \quad \{\text{using technology}\}$$



The equation of the circle is $x^2 + y^2 + 4x + 2y - 20 = 0$.

13

Customer	*Adult*	*Student*	*Senior*	*Child*	*Total cost*
A	2	0	2	1	\$294
B	1	4	0	2	\$368
C	2	0	4	0	\$360
D	0	3	2	1	\$314

a Let $\$a$ be the cost of an adult ticket,
$\$b$ be the cost of a student ticket,
$\$c$ be the cost of a senior ticket,
and $\$d$ be the cost of a child ticket.

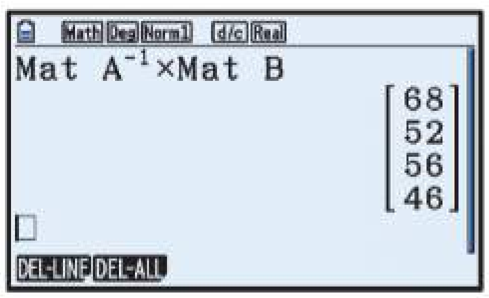
$$\therefore \begin{cases} 2a + 2c + d = 294 \\ a + 4b + 2d = 368 \\ 2a + 4c = 360 \\ 3b + 2c + d = 314 \end{cases}$$

The system of equations can be written as

$$\begin{pmatrix} 2 & 0 & 2 & 1 \\ 1 & 4 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 294 \\ 368 \\ 360 \\ 314 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 1 \\ 1 & 4 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 3 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 294 \\ 368 \\ 360 \\ 314 \end{pmatrix}$$

$$= \begin{pmatrix} 68 \\ 52 \\ 56 \\ 46 \end{pmatrix}$$



Adult tickets cost \$68, student tickets cost \$52, senior tickets cost \$56, and child tickets cost \$46.

b The cost of 2 adult tickets and 4 child tickets is $2 \times \$68 + 4 \times \$46 = \$320$.

Chapter 13

EIGENVALUES AND EIGENVECTORS

EXERCISE 13A.1

1 $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$

a
$$\begin{aligned} p(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} \lambda-2 & -1 \\ -4 & \lambda-5 \end{vmatrix} \\ &= (\lambda-2)(\lambda-5) - 4 \\ &= \lambda^2 - 7\lambda + 10 - 4 \\ &= \lambda^2 - 7\lambda + 6 \end{aligned}$$

b
$$\begin{aligned} p(\lambda) &= 0 \\ \therefore\ \lambda^2 - 7\lambda + 6 &= 0 \\ \therefore\ (\lambda-1)(\lambda-6) &= 0 \\ \therefore\ \lambda &= 1 \text{ or } 6 \end{aligned}$$
$\therefore$ the eigenvalues are 1 and 6.

2 **a** If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda-2 & 3 \\ 8 & \lambda \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda-2)\lambda - 24 &= 0 \\ \therefore\ \lambda^2 - 2\lambda - 24 &= 0 \\ \therefore\ (\lambda+4)(\lambda-6) &= 0 \\ \therefore\ \lambda &= -4 \text{ or } 6 \end{aligned}$$
$\therefore$ the eigenvalues are -4 and 6.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda-2 & -1 \\ 1 & \lambda-4 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda-2)(\lambda-4) - (-1) &= 0 \\ \therefore\ \lambda^2 - 6\lambda + 8 + 1 &= 0 \\ \therefore\ \lambda^2 - 6\lambda + 9 &= 0 \\ \therefore\ (\lambda-3)^2 &= 0 \\ \therefore\ \lambda &= 3 \end{aligned}$$
$\therefore$ the eigenvalue is 3.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda+3 & -5 \\ -2 & \lambda+1 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda+3)(\lambda+1) - 10 &= 0 \\ \therefore\ \lambda^2 + 4\lambda + 3 - 10 &= 0 \\ \therefore\ \lambda^2 + 4\lambda - 7 &= 0 \\ \therefore\ \lambda &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-7)}}{2} \\ &= \frac{-4 \pm \sqrt{44}}{2} \\ &= \frac{-4 \pm 2\sqrt{11}}{2} \\ &= -2 \pm \sqrt{11} \end{aligned}$$
$\therefore$ the eigenvalues are $-2 + \sqrt{11}$ and $-2 - 2\sqrt{11}$.

d If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda+5 & 6 \\ -1 & \lambda+1 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda+5)(\lambda+1) - (-6) &= 0 \\ \therefore\ \lambda^2 + 6\lambda + 5 + 6 &= 0 \\ \therefore\ \lambda^2 + 6\lambda + 11 &= 0 \\ \therefore\ \lambda &= \frac{-6 \pm \sqrt{6^2 - 4(1)(11)}}{2} \\ &= \frac{-6 \pm \sqrt{-8}}{2} \\ &= \frac{-6 \pm 2\sqrt{2}i}{2} \\ &= -3 \pm i\sqrt{2} \end{aligned}$$
$\therefore$ the eigenvalues are $-3 + i\sqrt{2}$ and $3 - i\sqrt{2}$.

e If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda+3 & -7 \\ 7 & \lambda-11 \end{vmatrix} = 0$

$\therefore\ (\lambda+3)(\lambda-11) - (-49) = 0$

$\therefore\ \lambda^2 - 8\lambda - 33 + 49 = 0$

$\therefore\ \lambda^2 - 8\lambda + 16 = 0$

$\therefore\ (\lambda-4)^2 = 0$

$\therefore\ \lambda = 4$

$\therefore$ the eigenvalue is 4.

f If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda-2 & -8 \\ 1 & \lambda-7 \end{vmatrix} = 0$

$\therefore\ (\lambda-2)(\lambda-7) - (-8) = 0$

$\therefore\ \lambda^2 - 9\lambda + 14 + 8 = 0$

$\therefore\ \lambda^2 - 9\lambda + 22 = 0$

$$\therefore\ \lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(22)}}{2} = \frac{9 \pm \sqrt{-7}}{2} = \frac{9 \pm i\sqrt{7}}{2} = \tfrac{9}{2} \pm \tfrac{\sqrt{7}}{2}i$$

$\therefore$ the eigenvalues are $\frac{9}{2} + \frac{\sqrt{7}}{2}i$ and $\frac{9}{2} - \frac{\sqrt{7}}{2}i$.

3 **a** $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies $\mathbf{Ix} = \mathbf{x}$ for all non-zero vectors $\mathbf{x}$.

So, $\mathbf{I}$ must only have the single eigenvalue 1.

b If $\det(\lambda\mathbf{I} - \mathbf{I}) = 0$

then $\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix} = 0$

$\therefore\ (\lambda-1)^2 = 0$

$\therefore\ \lambda = 1$

$\therefore$ the only eigenvalue of $\mathbf{I}$ is 1. ✓

4 $\mathbf{A} = \begin{pmatrix} 8 & 4 \\ 7 & 5 \end{pmatrix}$

a $\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8+4 \\ 7+5 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} = 12\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $\mathbf{A}\begin{pmatrix} -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} -4 \\ 7 \end{pmatrix} = \begin{pmatrix} -32+28 \\ -28+35 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

$\therefore$ the eigenvalues of $\mathbf{A}$ are 12 and 1.

b The eigenvalues of $\mathbf{A}$ are the solutions to $p(\lambda) = 0$.

$\therefore\ p(\lambda) = (\lambda-12)(\lambda-1)$
$= \lambda^2 - 13\lambda + 12$

5 $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -5 & 3 \end{pmatrix}$

a $$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} 1 & -3 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1+15 & -3+(-9) \\ -5+(-15) & 15+9 \end{pmatrix} \\ &= \begin{pmatrix} 16 & -12 \\ -20 & 24 \end{pmatrix} \end{aligned}$$

Now $\det(\mathbf{A}) = 1(3) - (-3)(-5) = -12$

$$\begin{aligned} \therefore\ \mathbf{A}^{-1} &= -\tfrac{1}{12}\begin{pmatrix} 3 & 3 \\ 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{12} & -\frac{1}{12} \end{pmatrix} \end{aligned}$$

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$

then $\begin{vmatrix} \lambda-1 & 3 \\ 5 & \lambda-3 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda-1)(\lambda-3) - 15 &= 0 \\ \therefore\ \lambda^2 - 4\lambda + 3 - 15 &= 0 \\ \therefore\ \lambda^2 - 4\lambda - 12 &= 0 \\ \therefore\ (\lambda+2)(\lambda-6) &= 0 \\ \therefore\ \lambda &= -2 \text{ or } 6 \end{aligned}$$

$\therefore$ the eigenvalues of $\mathbf{A}$ are -2 and 6.

If $\det(\lambda\mathbf{I} - \mathbf{A}^2) = 0$

then $\begin{vmatrix} \lambda-16 & 12 \\ 20 & \lambda-24 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda-16)(\lambda-24) - 240 &= 0 \\ \therefore\ \lambda^2 - 40\lambda + 384 - 240 &= 0 \\ \therefore\ \lambda^2 - 40\lambda + 144 &= 0 \\ \therefore\ (\lambda-4)(\lambda-36) &= 0 \\ \therefore\ \lambda &= 4 \text{ or } 36 \end{aligned}$$

$\therefore$ the eigenvalues of $\mathbf{A}^2$ are 4 and 36.

If $\det(\lambda\mathbf{I} - \mathbf{A}^{-1}) = 0$

then $\begin{vmatrix} \lambda+\frac{1}{4} & \frac{1}{4} \\ \frac{5}{12} & \lambda+\frac{1}{12} \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda+\tfrac{1}{4})(\lambda+\tfrac{1}{12}) - \tfrac{5}{48} &= 0 \\ \therefore\ \lambda^2 + \tfrac{1}{3}\lambda + \tfrac{1}{48} - \tfrac{5}{48} &= 0 \\ \therefore\ \lambda^2 + \tfrac{1}{3}\lambda - \tfrac{1}{12} &= 0 \\ \therefore\ 12\lambda^2 + 4\lambda - 1 &= 0 \\ \therefore\ (2\lambda+1)(6\lambda-1) &= 0 \\ \therefore\ \lambda &= -\tfrac{1}{2} \text{ or } \tfrac{1}{6} \end{aligned}$$

$\therefore$ the eigenvalues of $\mathbf{A}^{-1}$ are $-\frac{1}{2}$ and $\frac{1}{6}$.

6 Suppose λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{x}$.

a
$$\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \therefore\ -\mathbf{Ax} &= -\lambda\mathbf{x} \\ \therefore\ (-\mathbf{A})\mathbf{x} &= (-\lambda)\mathbf{x} \end{aligned}$$

So, if λ is an eigenvalue of $\mathbf{A}$, then $-\lambda$ is an eigenvalue of $-\mathbf{A}$.

$\therefore$ the eigenvalues of $-\mathbf{A}$ are the negatives of the eigenvalues of $\mathbf{A}$.

b
$$\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \therefore\ \mathbf{A}(\mathbf{Ax}) &= \mathbf{A}(\lambda\mathbf{x}) \\ \therefore\ \mathbf{A}^2\mathbf{x} &= \lambda(\mathbf{Ax}) \\ &= \lambda(\lambda\mathbf{x}) \\ &= \lambda^2\mathbf{x} \end{aligned}$$

So, if λ is an eigenvalue of $\mathbf{A}$, then λ^2 is an eigenvalue of $\mathbf{A}^2$.

$\therefore$ the eigenvalues of $\mathbf{A}^2$ are the squares of the eigenvalues of $\mathbf{A}$.

c
$$\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \therefore\ \mathbf{A}^{-1}(\mathbf{Ax}) &= \mathbf{A}^{-1}(\lambda\mathbf{x}) \\ \therefore\ \mathbf{Ix} &= \lambda(\mathbf{A}^{-1}\mathbf{x}) \qquad \{\text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}\} \\ \therefore\ \mathbf{x} &= \lambda(\mathbf{A}^{-1}\mathbf{x}) \qquad \{\text{since } \mathbf{Ix} = \mathbf{x}\} \\ \therefore\ \mathbf{A}^{-1}\mathbf{x} &= \frac{1}{\lambda}\mathbf{x} \end{aligned}$$

So, if λ is an eigenvalue of $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$.

$\therefore$ the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

EXERCISE 13A.2

1 **a** If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda-2 & 1 \\ -2 & \lambda-5 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda-2)(\lambda-5) - (-2) &= 0 \\ \therefore\ \lambda^2 - 7\lambda + 10 + 2 &= 0 \\ \therefore\ \lambda^2 - 7\lambda + 12 &= 0 \\ \therefore\ (\lambda-3)(\lambda-4) &= 0 \\ \therefore\ \lambda &= 3 \text{ or } 4 \end{aligned}$$

The eigenvalues are 3 and 4.

For $\lambda = 3$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ a + b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 3.

For $\lambda = 4$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2a + b = 0$$

Letting $a = t$, $t \neq 0$, then $b = -2t$

$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 4.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 2 & -1 \\ -4 & \lambda - 2 \end{vmatrix} = 0$

$$\therefore (\lambda - 2)^2 - 4 = 0$$
$$\therefore \lambda^2 - 4\lambda + 4 - 4 = 0$$
$$\therefore \lambda^2 - 4\lambda = 0$$
$$\therefore \lambda(\lambda - 4) = 0$$
$$\therefore \lambda = 0 \text{ or } 4$$

The eigenvalues are 0 and 4.

For $\lambda = 0$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -2a - b = 0$$

Letting $a = t$, $t \neq 0$, then $b = -2t$

$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 0.

For $\lambda = 4$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2a - b = 0$$

Letting $a = t$, $t \neq 0$, then $b = 2t$

$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ 2 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 4.

c If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix} \lambda-2 & -1 \\ -4 & \lambda+1 \end{vmatrix}=0$

$$\therefore\ (\lambda-2)(\lambda+1)-4=0$$
$$\therefore\ \lambda^2-\lambda-2-4=0$$
$$\therefore\ \lambda^2-\lambda-6=0$$
$$\therefore\ (\lambda+2)(\lambda-3)=0$$
$$\therefore\ \lambda=-2 \text{ or } 3$$

The eigenvalues are -2 and 3.

For $\lambda=-2$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -4 & -1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -4a-b=0$$

Letting $a=t,\ t\neq 0,$ then $b=-4t$

$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ -4 \end{pmatrix}t,\quad t\neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ -4 \end{pmatrix}t,\ t\neq 0$ is an eigenvector corresponding to the eigenvalue -2.

For $\lambda=3$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ a-b=0$$

Letting $b=t,\ t\neq 0,$ then $a=t$

$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\quad t\neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\ t\neq 0$ is an eigenvector corresponding to the eigenvalue 3.

2 **a** If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix} \lambda-5 & -3 \\ 3 & \lambda+1 \end{vmatrix}=0$

$$\therefore\ (\lambda-5)(\lambda+1)-(-9)=0$$
$$\therefore\ \lambda^2-4\lambda-5+9=0$$
$$\therefore\ \lambda^2-4\lambda+4=0$$
$$\therefore\ (\lambda-2)^2=0$$
$$\therefore\ \lambda=2$$

$\therefore$ **A** has a single eigenvalue $\lambda=2$.

b For $\lambda = 2$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 3a + 3b = 0$$

$$\therefore a + b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

c $\mathbf{Ax} = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} -5 + 3 \\ 3 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \lambda\mathbf{x}$$

3 $\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - \lambda_1 & 0 \\ 0 & \lambda - \lambda_2 \end{vmatrix} = 0$

$$\therefore (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\therefore \lambda = \lambda_1 \text{ or } \lambda_2$$

The eigenvalues are λ_1 and λ_2.

For $\lambda = \lambda_1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 - \lambda_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore (\lambda_1 - \lambda_2)b = 0$$

$$\therefore b = 0 \qquad \{\lambda_1 \neq \lambda_2\}$$

Let $a = t$, $t \neq 0$.

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue λ_1.

For $\lambda = \lambda_2$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} \lambda_2 - \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore (\lambda_2 - \lambda_1)a = 0$$

$$\therefore a = 0 \quad \{\lambda_1 \neq \lambda_2\}$$

Let $b = t$, $t \neq 0$.

$\therefore \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue λ_2.

4 $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \neq 0$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - a & 0 \\ 0 & \lambda - a \end{vmatrix} = 0$

$$\therefore (\lambda - a)^2 = 0$$

$$\therefore \lambda = a$$

$\therefore$ the only eigenvalue of $\mathbf{A}$ is a.

b For $\lambda = a$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is always true.

Let $a = s$ and $b = t$, where s and t are not both zero.

$\therefore \mathbf{x} = \begin{pmatrix} s \\ t \end{pmatrix}$, where s and t are not both zero.

Any vector of the form $\begin{pmatrix} s \\ t \end{pmatrix}$, where s and t are not both zero, is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue a.

c If $\mathbf{x}$ is a non-zero vector, then
$$\begin{aligned} \mathbf{A}\mathbf{x} &= (a\mathbf{I})\mathbf{x} \\ &= a(\mathbf{I}\mathbf{x}) \\ &= a\mathbf{x} \end{aligned}$$

Thus any non-zero vector $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue a, as in **b**.

5 **a** If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix} \lambda+1 & 1 \\ 4 & \lambda-2 \end{vmatrix}=0$

$$\therefore\ (\lambda+1)(\lambda-2)-4=0$$
$$\therefore\ \lambda^2-\lambda-2-4=0$$
$$\therefore\ \lambda^2-\lambda-6=0$$
$$\therefore\ (\lambda+2)(\lambda-3)=0$$
$$\therefore\ \lambda=-2 \text{ or } 3$$

The eigenvalues are -2 and 3.

For $\lambda=-2$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -a+b=0$$

Letting $a=t,\ t\neq 0,$ then $b=t$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\quad t\neq 0$$

Any vector of the form $\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\ t\neq 0$ is an eigenvector corresponding to the eigenvalue -2.

For $\lambda=3$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 4a+b=0$$

Letting $a=t,\ t\neq 0,$ then $b=-4t$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ -4 \end{pmatrix}t,\quad t\neq 0$$

Any vector of the form $\begin{pmatrix} 1 \\ -4 \end{pmatrix}t,\ t\neq 0$ is an eigenvector corresponding to the eigenvalue 3.

b Suppose $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

$$\begin{aligned} \therefore\ \mathbf{Ax}&=\lambda\mathbf{x} \\ \therefore\ \mathbf{A}(\mathbf{Ax})&=\mathbf{A}(\lambda\mathbf{x}) \\ \therefore\ \mathbf{A}^2\mathbf{x}&=\lambda(\mathbf{Ax}) \\ &=\lambda(\lambda\mathbf{x}) \\ &=\lambda^2\mathbf{x} \end{aligned}$$

So, $\mathbf{x}$ is also an eigenvector of $\mathbf{A}^2$ corresponding to the eigenvalue λ^2.

Thus, using **a**:

- Any vector of the form $\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\ t\neq 0$ is an eigenvector of $\mathbf{A}^2$ corresponding to the eigenvalue $(-2)^2=4$.
- Any vector of the form $\begin{pmatrix} 1 \\ -4 \end{pmatrix}t,\ t\neq 0$ is an eigenvector of $\mathbf{A}^2$ corresponding to the eigenvalue $3^2=9$.

6 $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 1 & -4 \\ 0 & \lambda - 3 \end{vmatrix} = 0$

$$\therefore (\lambda - 1)(\lambda - 3) = 0$$
$$\therefore \lambda = 1 \text{ or } 3$$

The eigenvalues of $\mathbf{A}$ are 1 and 3.

If $\det(\lambda\mathbf{I} - \mathbf{B}) = 0$ then $\begin{vmatrix} \lambda - 1 & 0 \\ -4 & \lambda - 3 \end{vmatrix} = 0$

$$\therefore (\lambda - 1)(\lambda - 3) = 0$$
$$\therefore \lambda = 1 \text{ or } 3$$

The eigenvalues of $\mathbf{B}$ are 1 and 3.

b For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 & -4 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore -2b = 0$$
$$\therefore b = 0$$

Let $a = t$, $t \neq 0$.

$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix} t$, $t \neq 0$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

For $\lambda = 3$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore 2a - 4b = 0$$
$$\therefore a - 2b = 0$$

Letting $b = t$, $t \neq 0$, then $a = 2t$

$\therefore \mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 2 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

c If $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\mathbf{Bx} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\therefore \ \mathbf{Bx} \neq \mathbf{x} \ \text{or} \ 3\mathbf{x}$$

If $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then $\mathbf{Bx} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 8+3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 11 \end{pmatrix}$$

$$\therefore \ \mathbf{Bx} \neq \mathbf{x} \ \text{or} \ 3\mathbf{x}$$

$\therefore$ the eigenvectors of **A** are *not* also eigenvectors of **B**.

7 Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 3 & -2 \\ -2 & \lambda - 3 \end{vmatrix} = 0$

$$\therefore \ (\lambda - 3)^2 - 4 = 0$$

$$\therefore \ \lambda^2 - 6\lambda + 9 - 4 = 0$$

$$\therefore \ \lambda^2 - 6\lambda + 5 = 0$$

$$\therefore \ (\lambda - 1)(\lambda - 5) = 0$$

$$\therefore \ \lambda = 1 \ \text{or} \ 5$$

The eigenvalues are 1 and 5.

For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \ \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \ -2a - 2b = 0$$

$$\therefore \ a + b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -t$

$$\therefore \ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

For $\lambda = 5$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \ \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \ 2a - 2b = 0$$

$$\therefore \ a - b = 0$$

Letting $a = s,\ s \neq 0$, then $b = s$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s,\quad s \neq 0$

$\therefore$ choosing $s = 1$, $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 5.

Now $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1 + 1 = 0$

$\therefore$ the eigenvectors of $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are mutually perpendicular.

8 $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

a Let $\mathbf{x} = \begin{pmatrix} 1 \\ -i \end{pmatrix} t,\quad t \neq 0$

$$\begin{aligned} \text{Now}\quad \mathbf{Ax} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ -ti \end{pmatrix} \\ &= \begin{pmatrix} ti \\ t \end{pmatrix} \\ &= \begin{pmatrix} i \\ 1 \end{pmatrix} t \\ &= i \begin{pmatrix} 1 \\ -i \end{pmatrix} t \end{aligned}$$

$\therefore\ \mathbf{Ax} = i\mathbf{x}$

So, any vector of the form $\begin{pmatrix} 1 \\ -i \end{pmatrix} t,\ t \neq 0$ is an eigenvector corresponding to the eigenvalue i.

b i is an eigenvalue of $\mathbf{A}$

$\therefore\ -i$ is the other eigenvalue of $\mathbf{A}$.

c For $\lambda = -i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -i - 0 & 0 - (-1) \\ 0 - 1 & -i - 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ -a - bi = 0$$

Letting $b = t,\ t \neq 0$, then $a = -ti$

$\therefore\ \mathbf{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix} t,\quad t \neq 0$

Any vector of the form $\begin{pmatrix} -i \\ 1 \end{pmatrix} t,\ t \neq 0$ is an eigenvector corresponding to the eigenvalue $-i$.

9 **a** If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix}\lambda-3 & -1\\ 2 & \lambda+1\end{vmatrix}=0$

$$\therefore\ (\lambda-3)(\lambda+1)-(-2)=0$$
$$\therefore\ \lambda^2-2\lambda-3+2=0$$
$$\therefore\ \lambda^2-2\lambda-1=0$$
$$\therefore\ \lambda=\frac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2}$$
$$=\frac{2\pm\sqrt{8}}{2}$$
$$=\frac{2\pm2\sqrt{2}}{2}$$
$$=1\pm\sqrt{2}$$

The eigenvalues are $1+\sqrt{2}$ and $1-\sqrt{2}$.

For $\lambda=1+\sqrt{2}$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\b\end{pmatrix}$

$$\therefore\ \begin{pmatrix}1+\sqrt{2}-3 & -1\\ 2 & 1+\sqrt{2}+1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ \begin{pmatrix}\sqrt{2}-2 & -1\\ 2 & 2+\sqrt{2}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ (\sqrt{2}-2)a-b=0$$

Letting $a=t,\ t\neq0,$ then $b=(\sqrt{2}-2)t$

$$\therefore\ \mathbf{x}=\begin{pmatrix}1\\ \sqrt{2}-2\end{pmatrix}t,\quad t\neq0$$

Any vector of the form $\begin{pmatrix}1\\ \sqrt{2}-2\end{pmatrix}t,\ t\neq0$ is an eigenvector corresponding to the eigenvalue $1+\sqrt{2}$.

For $\lambda=1-\sqrt{2}$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\b\end{pmatrix}$

$$\therefore\ \begin{pmatrix}1-\sqrt{2}-3 & -1\\ 2 & 1-\sqrt{2}+1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ \begin{pmatrix}-\sqrt{2}-2 & -1\\ 2 & 2-\sqrt{2}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ (-\sqrt{2}-2)a-b=0$$

Letting $a=t,\ t\neq0,$ then $b=(-\sqrt{2}-2)t$

$$\therefore\ \mathbf{x}=\begin{pmatrix}1\\ -\sqrt{2}-2\end{pmatrix}t,\quad t\neq0$$

Any vector of the form $\begin{pmatrix}1\\ -\sqrt{2}-2\end{pmatrix}t,\ t\neq0$ is an eigenvector corresponding to the eigenvalue $1-\sqrt{2}$.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 5 & 5 \\ -4 & \lambda - 1 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 5)(\lambda - 1) - (-20) = 0$$
$$\therefore\ \lambda^2 - 6\lambda + 5 + 20 = 0$$
$$\therefore\ \lambda^2 - 6\lambda + 25 = 0$$
$$\therefore\ \lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(25)}}{2}$$
$$= \frac{6 \pm \sqrt{-64}}{2}$$
$$= \frac{6 \pm 8i}{2}$$
$$= 3 \pm 4i$$

The eigenvalues are $3 + 4i$ and $3 - 4i$.

For $\lambda = 3 + 4i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 3 + 4i - 5 & 5 \\ -4 & 3 + 4i - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ \begin{pmatrix} -2 + 4i & 5 \\ -4 & 2 + 4i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -4a + (2 + 4i)b = 0$$
$$\therefore\ 2a - (1 + 2i)b = 0$$

Letting $b = t,\ t \neq 0,$ then $a = \frac{1}{2}(1 + 2i)t$
$$= \left(\tfrac{1}{2} + i\right)t$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} \frac{1}{2} + i \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} \frac{1}{2} + i \\ 1 \end{pmatrix}t,\ t \neq 0$ is an eigenvector corresponding to the eigenvalue $3 + 4i$.

For $\lambda = 3 - 4i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 3 - 4i - 5 & 5 \\ -4 & 3 - 4i - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ \begin{pmatrix} -2 - 4i & 5 \\ -4 & 2 - 4i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -4a + (2 - 4i)b = 0$$
$$\therefore\ 2a - (1 - 2i)b = 0$$

Letting $b = t,\ t \neq 0,$ then $a = \frac{1}{2}(1 - 2i)t$
$$= \left(\tfrac{1}{2} - i\right)t$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} \frac{1}{2} - i \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} \frac{1}{2} - i \\ 1 \end{pmatrix}t,\ t \neq 0$ is an eigenvector corresponding to the eigenvalue $3 - 4i$.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda+1 & 3 \\ -2 & \lambda-1 \end{vmatrix} = 0$

$$\therefore\ (\lambda+1)(\lambda-1) - (-6) = 0$$
$$\therefore\ \lambda^2 - 1 + 6 = 0$$
$$\therefore\ \lambda^2 = -5$$
$$\therefore\ \lambda = \pm i\sqrt{5}$$

The eigenvalues are $i\sqrt{5}$ and $-i\sqrt{5}$.

For $\lambda = i\sqrt{5}$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 1+i\sqrt{5} & 3 \\ -2 & -1+i\sqrt{5} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -2a + (-1+i\sqrt{5})b = 0$$

Letting $b = t$, $t \neq 0$, then $a = \frac{1}{2}(-1+i\sqrt{5})t$

$$= \left(-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}i\right)t$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2}i \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2}i \\ 1 \end{pmatrix}t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue $i\sqrt{5}$.

For $\lambda = -i\sqrt{5}$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 1-i\sqrt{5} & 3 \\ -2 & -1-i\sqrt{5} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -2a + (-1-i\sqrt{5})b = 0$$

Letting $b = t$, $t \neq 0$, then $a = \frac{1}{2}(-1-i\sqrt{5})t$

$$= \left(-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}i\right)t$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2}i \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2}i \\ 1 \end{pmatrix}t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue $-i\sqrt{5}$.

EXERCISE 13B

1 $\mathbf{P}_1 = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

$\therefore\ \det \mathbf{P}_1 = 2 - (-1) = 3$

$\therefore\ \mathbf{P}_1^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

$\therefore\ \mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1$

$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & 8 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 12 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\mathbf{P}_2 = (\mathbf{x}_2 \mid \mathbf{x}_1) = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$

$\therefore\ \det \mathbf{P}_2 = -1 - 2 = -3$

$\therefore\ \mathbf{P}_2^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}$

$\therefore\ \mathbf{P}_2^{-1}\mathbf{A}\mathbf{P}_2$

$= -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -4 & 1 \\ 8 & 1 \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} -12 & 0 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

2 $\mathbf{P}_1 = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$

$\therefore\ \det \mathbf{P}_1 = -1 - 8 = -9$

$\therefore\ \mathbf{P}_1^{-1} = -\frac{1}{9}\begin{pmatrix} 1 & -4 \\ -2 & -1 \end{pmatrix}$

$\therefore\ \mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1$

$= -\frac{1}{9}\begin{pmatrix} 1 & -4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$

$= -\frac{1}{9}\begin{pmatrix} 1 & -4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 24 \\ -6 & 6 \end{pmatrix}$

$= -\frac{1}{9}\begin{pmatrix} 27 & 0 \\ 0 & -54 \end{pmatrix}$

$= \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\mathbf{P}_2 = (\mathbf{x}_2 \mid \mathbf{x}_1) = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

$\therefore\ \det \mathbf{P}_2 = 8 - (-1) = 9$

$\therefore\ \mathbf{P}_2^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

$\therefore\ \mathbf{P}_2^{-1}\mathbf{A}\mathbf{P}_2$

$= \frac{1}{9}\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

$= \frac{1}{9}\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 24 & 3 \\ 6 & -6 \end{pmatrix}$

$= \frac{1}{9}\begin{pmatrix} 54 & 0 \\ 0 & -27 \end{pmatrix}$

$= \begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

3 **a** $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$

i If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 1 & -4 \\ -1 & \lambda + 2 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 1)(\lambda + 2) - 4 = 0$$
$$\therefore\ \lambda^2 + \lambda - 2 - 4 = 0$$
$$\therefore\ \lambda^2 + \lambda - 6 = 0$$
$$\therefore\ (\lambda - 2)(\lambda + 3) = 0$$
$$\therefore\ \lambda = 2 \text{ or } -3$$

For $\lambda_1 = 2$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} 1 & -4 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a - 4b = 0$$

Letting $b = t$, $t \neq 0$, then $a = 4t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

For $\lambda_2 = -3$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -4 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

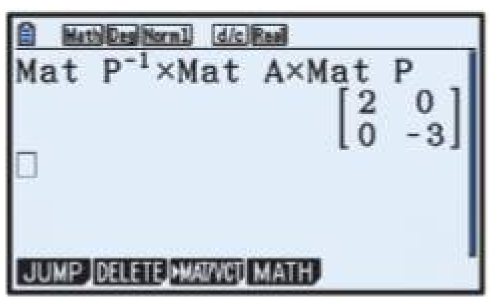
$$\therefore\ -a - b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -3$.

ii $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$



$\therefore\ \mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ {using technology}

$\therefore$ **P** diagonalises **A**.

b $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ -1 & -1 \end{pmatrix}$

i If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 5 & -6 \\ 1 & \lambda + 1 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 5)(\lambda + 1) - (-6) = 0$$
$$\therefore\ \lambda^2 - 4\lambda - 5 + 6 = 0$$
$$\therefore\ \lambda^2 - 4\lambda + 1 = 0$$
$$\therefore\ \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

For $\lambda_1 = 2 + \sqrt{3}$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -3 + \sqrt{3} & -6 \\ 1 & 3 + \sqrt{3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a + (3 + \sqrt{3})b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -(3 + \sqrt{3})t = (-3 - \sqrt{3})t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3 - \sqrt{3} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} -3 - \sqrt{3} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 2 + \sqrt{3}$.

For $\lambda_2 = 2 - \sqrt{3}$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -3 - \sqrt{3} & -6 \\ 1 & 3 - \sqrt{3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a + (3 - \sqrt{3})b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -(3 - \sqrt{3})t = (-3 + \sqrt{3})t$

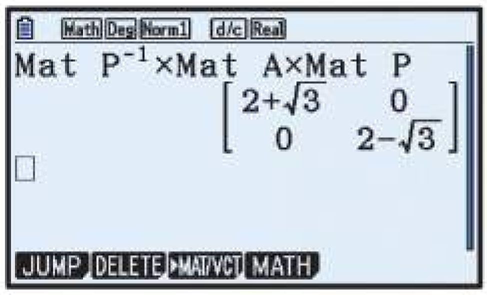
$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3 + \sqrt{3} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} -3 + \sqrt{3} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2 - \sqrt{3}$.

ii Let $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -3-\sqrt{3} & -3+\sqrt{3} \\ 1 & 1 \end{pmatrix}$

$\therefore\ \mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix}$ {using technology}

$\therefore$ **P** diagonalises **A**.



4 $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda-8 & -3 \\ -2 & \lambda-7 \end{vmatrix} = 0$

$$\therefore\ (\lambda-8)(\lambda-7) - 6 = 0$$
$$\therefore\ \lambda^2 - 15\lambda + 56 - 6 = 0$$
$$\therefore\ \lambda^2 - 15\lambda + 50 = 0$$
$$\therefore\ (\lambda-10)(\lambda-5) = 0$$
$$\therefore\ \lambda = 10 \text{ or } 5$$

For $\lambda_1 = 10$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ 2a - 3b = 0$$

Letting $b = t$, $t \neq 0$, then $2a = 3t$

$$\therefore\ a = \tfrac{3}{2}t$$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 10$.

For $\lambda_2 = 5$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -3 & -3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ -3a - 3b = 0$$
$$\therefore\ a + b = 0$$

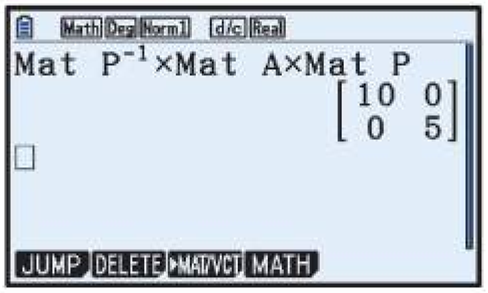
Letting $b = t$, $t \neq 0$, then $a = -t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = 5$.

b $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{pmatrix}$

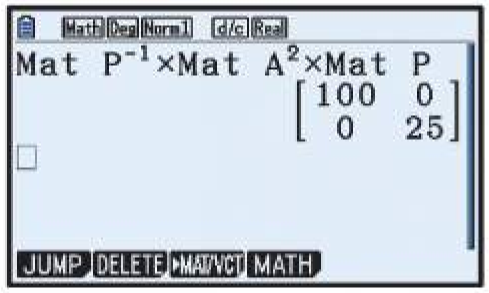
i



Using technology,

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

ii



Using technology,

$$\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} = \begin{pmatrix} 100 & 0 \\ 0 & 25 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

EXERCISE 13C

1 The matrix $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{A}$ with $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

Now $\mathbf{A}^n = \mathbf{P}\begin{pmatrix} 1^n & 0 \\ 0 & 4^n \end{pmatrix}\mathbf{P}^{-1}$

$$\therefore \mathbf{A}^5 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^5 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4^5 & 2 \times 4^5 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 2 + 4^5 & 2 - 2 \times 4^5 \\ 1 - 4^5 & 1 + 2 \times 4^5 \end{pmatrix}$$
$$= \begin{pmatrix} 342 & -682 \\ -341 & 683 \end{pmatrix}$$

2 $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 3 & -4 \\ -5 & \lambda - 2 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 3)(\lambda - 2) - 20 = 0$$
$$\therefore\ \lambda^2 - 5\lambda + 6 - 20 = 0$$
$$\therefore\ \lambda^2 - 5\lambda - 14 = 0$$
$$\therefore\ (\lambda - 7)(\lambda + 2) = 0$$
$$\therefore\ \lambda = 7 \text{ or } -2$$

For $\lambda_1 = 7$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} 4 & -4 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ 4a - 4b = 0$$
$$\therefore\ a - b = 0$$

Letting $b = t$, $t \neq 0$, then $a = t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 7$.

For $\lambda_2 = -2$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -5 & -4 \\ -5 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ -5a - 4b = 0$$

Letting $b = t$, $t \neq 0$, then $5a = -4t$

$$\therefore\ a = -\tfrac{4}{5}t$$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 5$, $\mathbf{x}_2 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -2$.

b Let $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}$.

The matrix $\mathbf{P}$ diagonalises $\mathbf{A}$ with $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix}$.

c $\mathbf{A}^n = \mathbf{P}\begin{pmatrix} 7^n & 0 \\ 0 & (-2)^n \end{pmatrix}\mathbf{P}^{-1}$

$$= \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 7^n & 0 \\ 0 & (-2)^n \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}^{-1}$$

d Using **c**, $\mathbf{A}^{13} = \mathbf{P}\begin{pmatrix} 7^{13} & 0 \\ 0 & (-2)^{13} \end{pmatrix}\mathbf{P}^{-1}$

$\therefore\ \mathbf{P}^{-1}\mathbf{A}^{13}\mathbf{P} = \begin{pmatrix} 7^{13} & 0 \\ 0 & (-2)^{13} \end{pmatrix}$

3 If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda+1 & -1 \\ -2 & \lambda-1 \end{vmatrix} = 0$

$$\therefore\ (\lambda+1)(\lambda-1) - 2 = 0$$
$$\therefore\ \lambda^2 - 1 - 2 = 0$$
$$\therefore\ \lambda^2 = 3$$
$$\therefore\ \lambda = \pm\sqrt{3}$$

For $\lambda_1 = \sqrt{3}$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} \sqrt{3}+1 & -1 \\ -2 & \sqrt{3}-1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore\ (\sqrt{3}+1)a - b = 0$

Letting $a = t$, $t \neq 0$, then $b = (\sqrt{3}+1)t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3}+1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ \sqrt{3}+1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = \sqrt{3}$.

For $\lambda_2 = -\sqrt{3}$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -\sqrt{3}+1 & -1 \\ -2 & -\sqrt{3}-1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore\ (-\sqrt{3}+1)a - b = 0$

Letting $a = t$, $t \neq 0$, then $b = (-\sqrt{3}+1)t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3}+1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -\sqrt{3}+1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -\sqrt{3}$.

The matrix $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ \sqrt{3}+1 & -\sqrt{3}+1 \end{pmatrix}$ diagonalises $\mathbf{A}$ with $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$.

$$\text{Now} \quad \mathbf{A}^n = \mathbf{P}\begin{pmatrix} (\sqrt{3})^n & 0 \\ 0 & (-\sqrt{3})^n \end{pmatrix}\mathbf{P}^{-1}$$

$$\begin{aligned}
\therefore \quad \mathbf{A}^{60} &= \mathbf{P}\begin{pmatrix} (3^{\frac{1}{2}})^{60} & 0 \\ 0 & (-3^{\frac{1}{2}})^{60} \end{pmatrix}\mathbf{P}^{-1} \\
&= \mathbf{P}\begin{pmatrix} 3^{30} & 0 \\ 0 & 3^{30} \end{pmatrix}\mathbf{P}^{-1} \\
&= 3^{30}\mathbf{PIP}^{-1} \\
&= 3^{30}\mathbf{PP}^{-1} \qquad \{\mathbf{PI} = \mathbf{P}\} \\
&= 3^{30}\mathbf{I} \qquad \{\mathbf{PP}^{-1} = \mathbf{I}\} \\
&= \begin{pmatrix} 3^{30} & 0 \\ 0 & 3^{30} \end{pmatrix}
\end{aligned}$$

4 If $\det(\lambda\mathbf{I} - \mathbf{C}) = 0$ then $\begin{vmatrix} \lambda & -2 \\ -1 & \lambda - 1 \end{vmatrix} = 0$

$$\therefore \quad \lambda(\lambda - 1) - 2 = 0$$
$$\therefore \quad \lambda^2 - \lambda - 2 = 0$$
$$\therefore \quad (\lambda - 2)(\lambda + 1) = 0$$
$$\therefore \quad \lambda = 2 \text{ or } -1$$

For $\lambda_1 = 2$, $(\lambda\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad -a + b = 0$$

Letting $b = t$, $t \neq 0$, then $a = t$

$\therefore \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

For $\lambda_2 = -1$, $(\lambda\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

gives $\begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad -a - 2b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -2t$

$\therefore \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$.

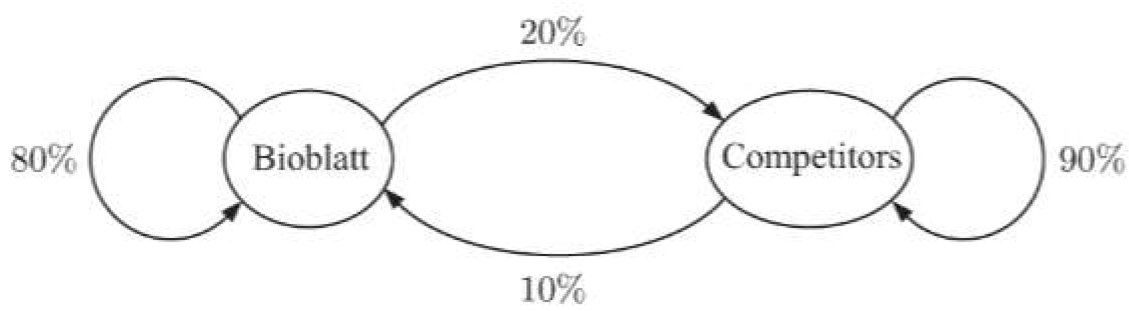
The matrix $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{C}$ with $\mathbf{P}^{-1}\mathbf{C}\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\text{Now} \quad \mathbf{C}^n = \mathbf{P}\begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}\mathbf{P}^{-1}$$

$$\begin{aligned} \therefore \quad \mathbf{C}^{2015} &= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{2015} & 0 \\ 0 & -1 \end{pmatrix}\tfrac{1}{3}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \\ &= \tfrac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{2015} & 2^{2016} \\ 1 & -1 \end{pmatrix} \\ &= \tfrac{1}{3}\begin{pmatrix} 2^{2015}-2 & 2^{2016}+2 \\ 2^{2015}+1 & 2^{2016}-1 \end{pmatrix} \end{aligned}$$

EXERCISE 13D.1

1



a $\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$

b **i** The element 0.2 means that 20% of customers who buy from Bioblatt one week will buy from its competitors the next.

ii The element 0.9 means that 90% of customers who buy from one of the competitors one week will buy from one of the competitors the next week.

c $\mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$

After one week Bioblatt has 10% of the market share.

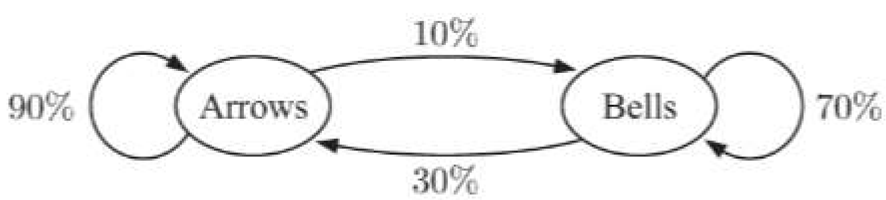
d **i** $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.17 \\ 0.83 \end{pmatrix}$

After two weeks, Bioblatt has 17% of the market share.

ii $\mathbf{s}_4 = \mathbf{T}^4\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}^4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2533 \\ 0.7467 \end{pmatrix}$

After four weeks, Bioblatt has about 25.3% of the market share.

2 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$

b $\mathbf{s}_0 = \begin{pmatrix} 600 \\ 400 \end{pmatrix}$

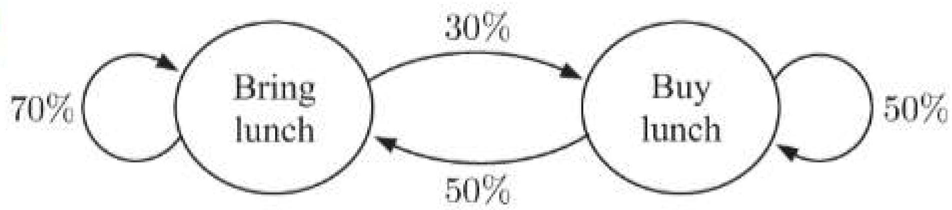
c **i** $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 600 \\ 400 \end{pmatrix} = \begin{pmatrix} 660 \\ 340 \end{pmatrix}$

We expect 660 people to shop at Arrows 1 week later.

ii $\mathbf{s}_3 = \mathbf{T}^3\mathbf{s}_0 = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}^3 \begin{pmatrix} 600 \\ 400 \end{pmatrix} = \begin{pmatrix} 717.6 \\ 282.4 \end{pmatrix}$

Rounding to the nearest person, we expect about 718 people to shop at Arrows 3 weeks later.

3 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$

b $\mathbf{s}_0 = \begin{pmatrix} 243 \\ 400 - 243 \end{pmatrix} = \begin{pmatrix} 243 \\ 157 \end{pmatrix}$

c Tuesday is 1 day after Monday.

$\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} 243 \\ 157 \end{pmatrix} = \begin{pmatrix} 248.6 \\ 151.4 \end{pmatrix}$

Rounding to the nearest student, we expect about 151 students to buy their lunch at the canteen on Tuesday.

d Friday is 4 days after Monday.

$\mathbf{s}_4 = \mathbf{T}^4\mathbf{s}_0 = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}^4 \begin{pmatrix} 243 \\ 157 \end{pmatrix} \approx \begin{pmatrix} 250 \\ 150 \end{pmatrix}$

Rounding to the nearest student, we expect about 150 students to buy their lunch at the canteen on Friday.

4

Brand bought this week

Brand bought next week		Baaah	Sheez
	Baaah	0.7	0.4
	Sheez	0.3	0.6

a The value 0.4 means that 40% of customers who bought Sheez this week will buy Baaah next week.

b $\mathbf{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$

c **i** $\mathbf{T}\mathbf{s}_1 = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.64 \\ 0.36 \end{pmatrix}$

This represents the market shares for the second week.
In the second week, Baaah's market share is 64% and Sheez's market share is 36%.

ii $\mathbf{s}_3 = \mathbf{T}^2\mathbf{s}_1 = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}^2 \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.592 \\ 0.408 \end{pmatrix}$

In the third week, Baaah's market share is 59.2%.

iii $\mathbf{s}_6 = \mathbf{T}^5\mathbf{s}_1 = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}^5 \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} \approx \begin{pmatrix} 0.572 \\ 0.428 \end{pmatrix}$

In the sixth week, Sheez's market share is about 42.8%.

5

This month

		Smoking	Not Smoking
Next month	Smoking	80%	10%
	Not Smoking	20%	90%

a $\mathbf{s}_0 = \begin{pmatrix} 62 \\ 100 - 62 \end{pmatrix} = \begin{pmatrix} 62 \\ 38 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$

i $\mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \begin{pmatrix} 62 \\ 38 \end{pmatrix} = \begin{pmatrix} 53.4 \\ 46.6 \end{pmatrix}$

ii From **i**, rounding to the nearest member, about 47 members will *not* have smoked between the January and February meetings.

iii $\mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}^2 \begin{pmatrix} 62 \\ 38 \end{pmatrix} = \begin{pmatrix} 47.38 \\ 52.62 \end{pmatrix}$

March is 2 months after January.
Rounding to the nearest member, we expect about 47 smokers and about 53 non-smokers at the March meeting.

c January the following year is 12 months after this January.

$\mathbf{s}_{12} = \mathbf{T}^{12}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}^{12} \begin{pmatrix} 62 \\ 38 \end{pmatrix} \approx \begin{pmatrix} 33.7 \\ 66.3 \end{pmatrix}$

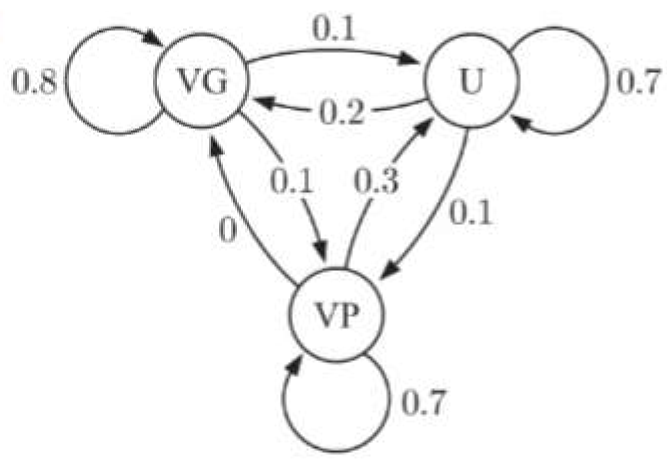
Rounding to the nearest member, we expect about 34 members to be smoking in January the following year.

6

$$\mathbf{T} = \begin{array}{c|ccc} & \text{VG} & \text{U} & \text{VP} \\ \hline \text{VG} & 0.8 & 0.2 & 0.0 \\ \text{U} & 0.1 & 0.7 & 0.3 \\ \text{VP} & 0.1 & 0.1 & 0.7 \end{array}$$

VG = Very good
U = Usable
VP = Very poor

a



b **i** In row 3, column 1 we have 0.1. This means that 10% of the land which is very good this year will be very poor next year.

ii In row 1, column 3 we have 0.0. This means that no land which is very poor this year will be very good next year.

c $$\mathbf{T}^2 = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.7 \end{pmatrix}^2 = \begin{pmatrix} 0.66 & 0.3 & 0.06 \\ 0.18 & 0.54 & 0.42 \\ 0.16 & 0.16 & 0.52 \end{pmatrix}$$

i In row 3, column 2 we have 0.16. This means that 16% of the land which is usable now will be very poor in two years' time.

ii In row 2, column 3 we have 0.42. This means that 42% of the land which is very poor now will be usable in two years' time.

d $$\mathbf{s}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

e $$\mathbf{s}_3 = \mathbf{T}^3\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.7 \end{pmatrix}^3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.132 \\ 0.456 \\ 0.412 \end{pmatrix}$$

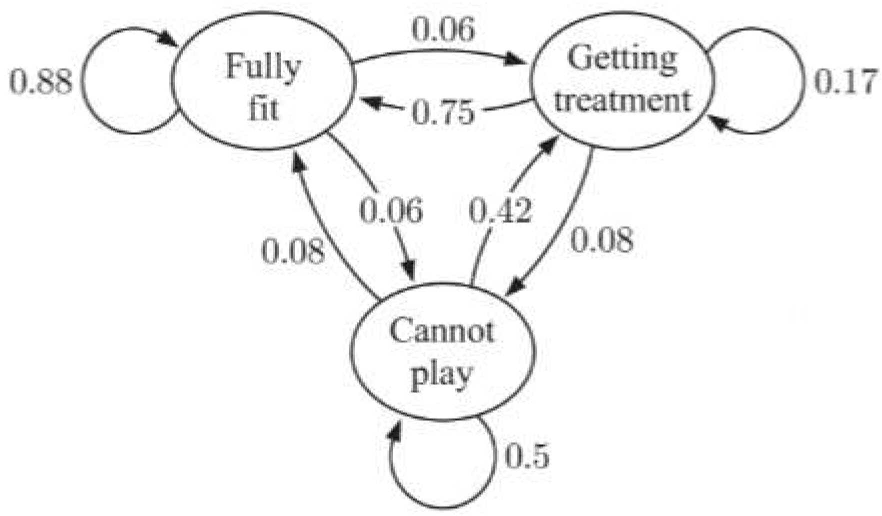
After 3 years, 13.2% is very good, 45.6% is usable, and 41.2% is very poor.

7

Condition this week

Condition next week		Fully fit	Getting treatment	Cannot play
	Fully fit	0.88	0.75	0.08
	Getting treatment	0.06	0.17	0.42
	Cannot play	0.06	0.08	0.50

a



b $$\mathbf{T} = \begin{pmatrix} 0.88 & 0.75 & 0.08 \\ 0.06 & 0.17 & 0.42 \\ 0.06 & 0.08 & 0.50 \end{pmatrix}$$

c **i** In row 1, column 1 we have 0.88. This means that 88% of players who are fully fit this week will be fully fit next week.

ii In row 2, column 3 we have 0.42. This means that 42% of players who cannot play this week will be getting treatment next week.

d $\mathbf{T}^2 = \begin{pmatrix} 0.88 & 0.75 & 0.08 \\ 0.06 & 0.17 & 0.42 \\ 0.06 & 0.08 & 0.50 \end{pmatrix}^2 = \begin{pmatrix} 0.8242 & 0.7939 & 0.4254 \\ 0.0882 & 0.1075 & 0.2862 \\ 0.0876 & 0.0986 & 0.2884 \end{pmatrix}$

In row 1, column 2 we have 0.7939. This means that 79.39% of players getting treatment this week will be fully fit in two weeks' time.

e $\mathbf{s}_0 = \begin{pmatrix} 30 \\ 6 \\ 2 \end{pmatrix}$

i $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.88 & 0.75 & 0.08 \\ 0.06 & 0.17 & 0.42 \\ 0.06 & 0.08 & 0.50 \end{pmatrix} \begin{pmatrix} 30 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 31.06 \\ 3.66 \\ 3.28 \end{pmatrix}$

Rounding to the nearest player, the coach would expect about 31 fully fit players, 4 players getting treatment, and 3 players who cannot play next week.

ii $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.8242 & 0.7939 & 0.4254 \\ 0.0882 & 0.1075 & 0.2862 \\ 0.0876 & 0.0986 & 0.2884 \end{pmatrix} \begin{pmatrix} 30 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 30.3402 \\ 3.8634 \\ 3.7964 \end{pmatrix}$

Rounding to the nearest player, the coach would expect about 30 fully fit players, 4 players getting treatment, and 4 players who cannot play in two weeks' time.

ACTIVITY 1 HIDDEN MARKOV CHAINS

1 From the **Opening Problem**, $\mathbf{T} = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$.

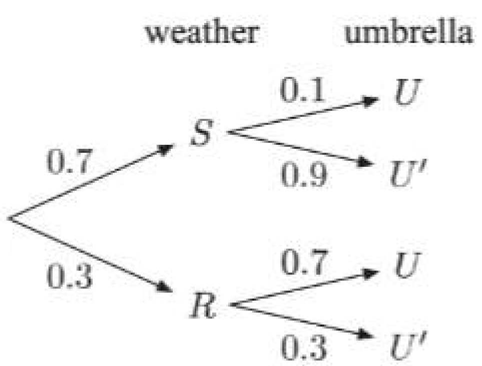
The day before you became sick, you guess that each type of weather was equally likely.

$\therefore \quad \mathbf{s}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

Now $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

On the first day of sickness, the probability it was sunny is 0.7, and the probability it was raining is 0.3.

a Let U be the event that your friend brings an umbrella, S be the event "sunny", and R be the event "rainy".



b **i** $\mathrm{P}(S \cap U) = 0.7 \times 0.1$
$= 0.07$

ii $\mathrm{P}(R \cap U) = 0.3 \times 0.7$
$= 0.21$

iii $\mathrm{P}(U) = \mathrm{P}(S \cap U) + \mathrm{P}(R \cap U)$
$= 0.07 + 0.21$ {using **i** and **ii**}
$= 0.28$

c **i** $\mathrm{P}(S \mid U) = \dfrac{\mathrm{P}(S \cap U)}{\mathrm{P}(U)}$
$= \dfrac{0.07}{0.28}$
$= 0.25$

ii $\mathrm{P}(R \mid U) = \dfrac{\mathrm{P}(R \cap U)}{\mathrm{P}(U)}$
$= \dfrac{0.21}{0.28}$
$= 0.75$

d From **c**, $\mathrm{P}(R \mid U) > \mathrm{P}(S \mid U)$
$\therefore$ it is more likely to have rained on the first day.

e $\mathrm{P}(S \mid U)$ and $\mathrm{P}(R \mid U)$ are obtained by dividing $\mathrm{P}(S \cap U)$ and $\mathrm{P}(R \cap U)$ (respectively) by the same constant $\mathrm{P}(U) = 0.28$.
Thus comparing $\mathrm{P}(S \cap U)$ and $\mathrm{P}(R \cap U)$ will give the same conclusion as comparing $\mathrm{P}(S \mid U)$ and $\mathrm{P}(R \mid U)$.

2

$$\mathbf{H} = \begin{array}{c} \begin{array}{cc} \text{sunny} & \text{rainy} \end{array} \\ \begin{pmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{umbrella} \\ \text{no umbrella} \end{array}$$

So, $\mathrm{P}(U \mid S) = 0.1$,
$\mathrm{P}(U \mid R) = 0.7$,
$\mathrm{P}(U' \mid S) = 0.9$,
and $\mathrm{P}(U' \mid R) = 0.3$

a Consider $\mathbf{Ts}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

So, on the first day, $\mathrm{P}(S) = 0.7$ and $\mathrm{P}(R) = 0.3$.

Now $\mathbf{HTs}_0 = \begin{pmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

$\mathrm{P}(U \mid S)\,\mathrm{P}(S) = \mathrm{P}(S \cap U)$ $\qquad$ $\mathrm{P}(U \mid R)\,\mathrm{P}(R) = \mathrm{P}(R \cap U)$

$$= \begin{pmatrix} \overbrace{0.1 \times 0.7} & + & \overbrace{0.7 \times 0.3} \\ \underbrace{0.9 \times 0.7} & + & \underbrace{0.3 \times 0.3} \end{pmatrix}$$

$\mathrm{P}(U' \mid S)\,\mathrm{P}(S) = \mathrm{P}(S \cap U')$ $\qquad$ $\mathrm{P}(U' \mid R)\,\mathrm{P}(R) = \mathrm{P}(R \cap U')$

$$= \begin{pmatrix} 0.28 \\ 0.72 \end{pmatrix} \begin{array}{l} \leftarrow \mathrm{P}(U) \\ \leftarrow \mathrm{P}(U') \end{array}$$

So, we can calculate the probabilities in **1 b** using $\mathbf{HTs}_0$.

b $\mathbf{T}^n\mathbf{s}_0 = \mathbf{s}_n$, which gives the probabilities that it was sunny or rainy on the nth day, given that each type of weather was initially equally likely.

So, $\mathbf{T}^n\mathbf{s}_0 = \begin{pmatrix} P(S) \\ P(R) \end{pmatrix}$ on the nth day.

Now, the matrix of conditional probabilities $\mathbf{H}$ does not depend on the previous day's weather.

So, $\mathbf{H} = \begin{pmatrix} P(U \mid S) & P(U \mid R) \\ P(U' \mid S) & P(U' \mid R) \end{pmatrix} = \begin{pmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{pmatrix}$ on the nth day

$\therefore$ $\mathbf{HT}^n\mathbf{s}_0 = \begin{pmatrix} P(U \mid S) & P(U \mid R) \\ P(U' \mid S) & P(U' \mid R) \end{pmatrix}\begin{pmatrix} P(S) \\ P(R) \end{pmatrix}$ on the nth day

$= \begin{pmatrix} P(U \mid S)\,P(S) + P(U \mid R)\,P(R) \\ P(U' \mid S)\,P(S) + P(U' \mid R)\,P(R) \end{pmatrix}$ on the nth day

$= \begin{pmatrix} P(S \cap U) + P(R \cap U) \\ P(S \cap U') + P(R \cap U') \end{pmatrix}$ on the nth day

$= \begin{pmatrix} P(U) \\ P(U') \end{pmatrix}$ on the nth day

So, we can calculate the probabilities $P(S \cap U)$, $P(R \cap U)$, and $P(U)$ after n days using $\mathbf{HT}^n\mathbf{s}_0$.

3 **a** Your friend brought an umbrella on the first day.

So, on the first day, $P(S) = P(S \mid U) = 0.25$ {from **1 c i**}
and $P(R) = P(R \mid U) = 0.75$ {from **1 c ii**}.

Now, $\mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}^2 \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.74 \\ 0.26 \end{pmatrix}$

$\therefore$ $\mathbf{HT}^2\mathbf{s}_0 = \begin{pmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{pmatrix}\begin{pmatrix} 0.74 \\ 0.26 \end{pmatrix}$

$= \begin{pmatrix} 0.1 \times 0.74 + 0.7 \times 0.26 \\ 0.9 \times 0.74 + 0.3 \times 0.26 \end{pmatrix} = \begin{pmatrix} 0.256 \\ 0.744 \end{pmatrix}$

So, on the second day, $P(S \cap U) = 0.1 \times 0.74 = 0.074$,
$P(R \cap U) = 0.7 \times 0.26 = 0.182$,
and $P(U) = 0.256$ {using **2 b**}

$\therefore$ on the second day, $P(S \mid U) = \dfrac{P(S \cap U)}{P(U)} = \dfrac{0.074}{0.256} = \dfrac{37}{128}$

and $P(R \mid U) = \dfrac{P(R \cap U)}{P(U)} = \dfrac{0.182}{0.256} = \dfrac{91}{128}$

i The probability that it rained on the first day, then was sunny on the second day given your friend brought an umbrella is $0.75 \times \frac{37}{128} = \frac{112}{512}$ which is about 0.217.

ii The probability it rained on the first day, then on the second day it was rainy given your friend brought an umbrella is $0.75 \times \frac{91}{128} = \frac{273}{512}$ which is about 0.533.

iii The probability that it was sunny on the first day, then on the second day it was sunny given your friend brought an umbrella is $0.25 \times \frac{37}{128} = \frac{37}{512}$ which is about 0.0723.

iv The probability that it was sunny on the first day, then on the second day it was rainy given your friend brought an umbrella is $0.25 \times \frac{91}{128} = \frac{91}{512}$ which is about 0.178.

b The highest probability in **a** is about 0.533 which corresponds to rainy on the first day, then rainy on the second day.

$\therefore$ the most likely sequence of weather is **D**.

4
$$\mathbf{T}^3\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}^3 \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.748 \\ 0.252 \end{pmatrix}$$

$$\begin{aligned} \therefore\ \mathbf{HT}^3\mathbf{s}_0 &= \begin{pmatrix} 0.1 & 0.7 \\ 0.9 & 0.3 \end{pmatrix} \begin{pmatrix} 0.748 \\ 0.252 \end{pmatrix} \\ &= \begin{pmatrix} 0.1 \times 0.748 + 0.7 \times 0.252 \\ 0.9 \times 0.748 + 0.3 \times 0.252 \end{pmatrix} \\ &= \begin{pmatrix} 0.2512 \\ 0.7488 \end{pmatrix} \end{aligned}$$

So, on the third day, $\mathrm{P}(S \cap U') = 0.9 \times 0.748 = 0.6732$,
$\mathrm{P}(R \cap U') = 0.3 \times 0.252 = 0.0756$,
and $\mathrm{P}(U') = 0.7488$

$$\therefore\ \mathrm{P}(S \mid U') = \frac{0.6732}{0.7488} = \frac{187}{208} \quad \text{and} \quad \mathrm{P}(R \mid U') = \frac{0.0756}{0.7488} = \frac{21}{208}$$

The probability of sunny on the first day, sunny on the second day, and rainy on the third day is

$$\begin{aligned} \mathrm{P}(\text{sunny, sunny, rainy}) &= \frac{37}{512} \times \frac{21}{208} \quad \{\text{using } \mathbf{3\ a\ iii}\} \\ &\approx 0.007\,30 \end{aligned}$$

In a similar way, $\mathrm{P}(\text{sunny, sunny, sunny}) \approx 0.0650$,
$\mathrm{P}(\text{sunny, rainy, rainy}) \approx 0.0179$,
$\mathrm{P}(\text{sunny, rainy, sunny}) \approx 0.160$,
$\mathrm{P}(\text{rainy, sunny, rainy}) \approx 0.0221$,
$\mathrm{P}(\text{rainy, sunny, sunny}) \approx 0.197$,
$\mathrm{P}(\text{rainy, rainy, rainy}) \approx 0.0538$,
and $\mathrm{P}(\text{rainy, rainy, sunny}) \approx 0.479$.

The highest probability is about 0.479 which corresponds to rainy on the first day, rainy on the second day, and sunny on the third day.
So, the most probable 3 day long sequence of weather is rainy, rainy, sunny.

INVESTIGATION STEADY STATES

2 As the number of days increases, the proportion of sunny days converges to about 0.75.

4 $\mathbf{s}_n = \mathbf{T}^n\mathbf{s}_0$

a $\mathbf{s}_{30} = \mathbf{T}^{30}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}^{30} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.750 \\ 0.250 \end{pmatrix}$

b $\mathbf{s}_{100} = \mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}^{100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.750 \\ 0.250 \end{pmatrix}$

c $\mathbf{s}_{1000} = \mathbf{T}^{1000}\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}^{1000} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.750 \\ 0.250 \end{pmatrix}$

As the number of days increases, the proportion of sunny days approaches 0.750, and the proportion of rainy days approaches 0.250.
This agrees with our observation in **2**.

EXERCISE 13D.2

1

		State when they arrive	
		Open	Closed
State when they leave	Open	30%	5%
	Closed	70%	95%

a $\mathbf{s}_0 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$ describes the current state of the system. The door is currently open 10% of the time and closed 90% of the time.

$\mathbf{T} = \begin{pmatrix} 0.3 & 0.05 \\ 0.7 & 0.95 \end{pmatrix}$ is the transition matrix which predicts how the system will change each time someone visits.

b If $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the steady state matrix then $\mathbf{Ts} = \mathbf{s}$

$$\therefore \begin{pmatrix} 0.3 & 0.05 \\ 0.7 & 0.95 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0.3a + 0.05b \\ 0.7a + 0.95b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Using either row, $0.3a + 0.05b = a$

$$\therefore 0.05b = 0.7a$$

$$\therefore b = 14a$$

Now $\mathbf{s}$ is a matrix of probabilities, so $a + b = 1$

$$\therefore a + 14a = 1$$

$$\therefore a = \tfrac{1}{15} \text{ and } b = \tfrac{14}{15}$$

So, $\mathbf{s} = \begin{pmatrix} \frac{1}{15} \\ \frac{14}{15} \end{pmatrix}$.

c In the long term the door is left open $\frac{1}{15} \times 100\% \approx 6.67\%$ of the time. This is lower than Jada's estimate of 10%, so her concern is not justified.

d $\mathbf{T}^{10}\mathbf{s}_0 = \begin{pmatrix} 0.3 & 0.05 \\ 0.7 & 0.95 \end{pmatrix}^{10} \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \approx \begin{pmatrix} 0.0667 \\ 0.933 \end{pmatrix}$

This is very close to the steady state in **b**.

2

$$\mathbf{T} = \begin{array}{c} \textit{Now} \\ \begin{array}{cc} \text{C} & \text{R} \end{array} \\ \begin{pmatrix} 0.84 & 0.21 \\ 0.16 & 0.79 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{C} \\ \text{R} \end{array} \quad \begin{array}{c} \\ \\ \textit{Next} \\ \textit{month} \end{array}$$

a $\mathbf{s}_0 = \begin{pmatrix} 425 \\ 716 \end{pmatrix}$

b $\mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0.84 & 0.21 \\ 0.16 & 0.79 \end{pmatrix}^{100} \begin{pmatrix} 425 \\ 716 \end{pmatrix} \approx \begin{pmatrix} 647.6 \\ 493.4 \end{pmatrix}$

Rounding to the nearest passenger, we predict the monthly number of passengers to be about 648 passengers for Clydes, and about 493 passengers for Roos in the long term.

c We assume that the preferences of the passengers remain the same from month to month, and that the total number of passengers travelling between the cities does not change.
The first assumption may be reasonable, but the second assumption may not be if the number of people travelling between the cities varies throughout the year.

d We assume there are $425 + 716 = 1141$ passengers each month.

Using **b**, we estimate about $\dfrac{648}{1141} \times 100\% \approx 56.8\%$ of passengers to use Clydes, and about $\dfrac{493}{1141} \times 100\% \approx 43.2\%$ of passengers to use Roos.

e If $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the steady state matrix then $\mathbf{Ts} = \mathbf{s}$

$$\therefore \begin{pmatrix} 0.84 & 0.21 \\ 0.16 & 0.79 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0.84a + 0.21b \\ 0.16a + 0.79b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Using either row, $0.84a + 0.21b = a$

$$\therefore 0.21b = 0.16a$$

$$\therefore b = \tfrac{16}{21}a$$

Now $\mathbf{s}$ is a matrix of probabilities, so $a + b = 1$

$$\therefore a + \tfrac{16}{21}a = 1$$

$$\therefore \tfrac{37}{21}a = 1$$

$$\therefore a = \tfrac{21}{37} \quad \text{and} \quad b = \tfrac{16}{37}$$

So, the steady state proportions are $\mathbf{s} = \begin{pmatrix} \frac{21}{37} \\ \frac{16}{37} \end{pmatrix}$.

3 $\mathbf{T} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{T}) = 0$ then $\begin{vmatrix} \lambda - 0.1 & -0.4 \\ -0.9 & \lambda - 0.6 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 0.1)(\lambda - 0.6) - 0.36 = 0$$
$$\therefore\ \lambda^2 - 0.7\lambda + 0.06 - 0.36 = 0$$
$$\therefore\ \lambda^2 - 0.7\lambda - 0.3 = 0$$
$$\therefore\ (\lambda - 1)(\lambda + 0.3) = 0$$
$$\therefore\ \lambda = 1 \text{ or } -0.3$$

The eigenvalues are 1 and $-\frac{3}{10}$.

b For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{T})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 0.9 & -0.4 \\ -0.9 & 0.4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 0.9a - 0.4b = 0$$
$$\therefore\ 9a - 4b = 0$$

Letting $b = t$, $t \neq 0$, then $a = \frac{4}{9}t$

$$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{x} = \begin{pmatrix} \frac{4}{9} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

For $\lambda = -0.3$, consider $(\lambda\mathbf{I} - \mathbf{T})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -0.4 & -0.4 \\ -0.9 & -0.9 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -0.4a - 0.4b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -t$

$$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue -0.3.

c If $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the steady state matrix then $\mathbf{Ts} = \mathbf{s}$.

So, $\mathbf{s}$ is an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue 1.

From **b**, $\mathbf{s} = \begin{pmatrix} \frac{4}{9} \\ 1 \end{pmatrix} k$ for some k

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{4}{9}k \\ k \end{pmatrix}$$

$$\therefore a = \tfrac{4}{9}k \quad \text{and} \quad b = k$$

Now $\mathbf{s}$ is a matrix of probabilities, so $a + b = 1$

$$\therefore \tfrac{4}{9}k + k = 1$$

$$\therefore \tfrac{13}{9}k = 1$$

$$\therefore k = \tfrac{9}{13}$$

So, $\mathbf{s} = \begin{pmatrix} \frac{4}{9} \\ 1 \end{pmatrix} \frac{9}{13} = \begin{pmatrix} \frac{4}{13} \\ \frac{9}{13} \end{pmatrix}$.

4 $\mathbf{s}_0 = \begin{pmatrix} 30 \\ 5 \\ 5 \end{pmatrix}$, $\mathbf{T} = \begin{pmatrix} 0.86 & 0.68 & 0.00 \\ 0.12 & 0.24 & 0.32 \\ 0.02 & 0.08 & 0.68 \end{pmatrix}$

a **i** $\mathbf{s}_1 = \mathbf{Ts}_0 = \begin{pmatrix} 0.86 & 0.68 & 0.00 \\ 0.12 & 0.24 & 0.32 \\ 0.02 & 0.08 & 0.68 \end{pmatrix} \begin{pmatrix} 30 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 29.2 \\ 6.4 \\ 4.4 \end{pmatrix}$

Rounding to the nearest player, we expect about 29 fully fit players, 6 players getting treatment, and 4 injured players next week.

ii $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.86 & 0.68 & 0.00 \\ 0.12 & 0.24 & 0.32 \\ 0.02 & 0.08 & 0.68 \end{pmatrix}^2 \begin{pmatrix} 30 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 29.464 \\ 6.448 \\ 4.088 \end{pmatrix}$

Rounding to the nearest player, we expect about 29 fully fit players, 6 players getting treatment, and 4 injured players after two weeks.

iii $\mathbf{s}_{100} = \mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0.86 & 0.68 & 0.00 \\ 0.12 & 0.24 & 0.32 \\ 0.02 & 0.08 & 0.68 \end{pmatrix}^{100} \begin{pmatrix} 30 \\ 5 \\ 5 \end{pmatrix} \approx \begin{pmatrix} 30.3 \\ 6.24 \\ 3.45 \end{pmatrix}$

Rounding to the nearest player, we expect about 30 fully fit players, 6 players getting treatment, and 3 injured players in the long term.

b From **a iii**, the squad can expect about 30 fully fit players each week in the long term.
So, in the long term, the squad should have enough fully fit players to field three teams of 9 each week.

c In the long term, about $\frac{30.3}{40} \times 100\% \approx 75.8\%$ of players are available.

To ensure that at least 27 players are available, there would need to be $\frac{27}{0.758} \approx 35.6$ players in the squad.

$\therefore$ to field three teams of 9 each week, the squad should have at least 36 players.

5

		This year		
		Paua	Manu	Chalk
Next year	Paua	75%	20%	15%
	Manu	15%	60%	20%
	Chalk	10%	20%	65%

a $\mathbf{T} = \begin{pmatrix} 0.75 & 0.2 & 0.15 \\ 0.15 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.65 \end{pmatrix}$

b $\mathbf{T}^2 = \begin{pmatrix} 0.75 & 0.2 & 0.15 \\ 0.15 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.65 \end{pmatrix}^2 = \begin{pmatrix} 0.6075 & 0.3 & 0.25 \\ 0.2225 & 0.43 & 0.2725 \\ 0.17 & 0.27 & 0.4775 \end{pmatrix}$

The third column represents the proportion of birds who currently live on Chalk who will live on each of the three islands in two years' time.

Of the birds currently living on Chalk, in two years' time 25% of them will be living on Paua, 27.25% will be living on Manu, and 47.75% will be living on Chalk.

c $\mathbf{s}_0 = \begin{pmatrix} 0.26 \\ 0.39 \\ 0.35 \end{pmatrix}$

i 2020 is 1 year after 2019.

$$\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.75 & 0.2 & 0.15 \\ 0.15 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.65 \end{pmatrix} \begin{pmatrix} 0.26 \\ 0.39 \\ 0.35 \end{pmatrix} = \begin{pmatrix} 0.3255 \\ 0.343 \\ 0.3315 \end{pmatrix}$$

In 2020, 32.55% of the birds will live on Paua, 34.3% will live on Manu, and 33.15% will live on Chalk.

ii 2022 is 3 years after 2019.

$$\mathbf{s}_3 = \mathbf{T}^3\mathbf{s}_0 = \begin{pmatrix} 0.75 & 0.2 & 0.15 \\ 0.15 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.65 \end{pmatrix}^3 \begin{pmatrix} 0.26 \\ 0.39 \\ 0.35 \end{pmatrix} \approx \begin{pmatrix} 0.384 \\ 0.310 \\ 0.306 \end{pmatrix}$$

In 2022, about 38.4% of the birds will live on Paua, 31.0% will live on Manu, and 30.6% will live on Chalk.

d $\mathbf{s}_{100} = \mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0.75 & 0.2 & 0.15 \\ 0.15 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.65 \end{pmatrix}^{100} \begin{pmatrix} 0.26 \\ 0.39 \\ 0.35 \end{pmatrix} \approx \begin{pmatrix} 0.412 \\ 0.299 \\ 0.289 \end{pmatrix}$

In the long term, about 41.2% of the birds will live on Paua, 29.9% will live on Manu, and 28.9% will live on Chalk.

e The birds do not stop migrating between the islands once the steady state proportions have been reached.

From the table, each year:

- $(15+10)\% = 25\%$ of birds on Paua will migrate
- $(20+20)\% = 40\%$ of birds on Manu will migrate
- $(15+20)\% = 35\%$ of birds on Chalk will migrate.

So, using **d**, once the steady state proportions have been reached, about $(0.25 \times 0.412 + 0.4 \times 0.299 + 0.35 \times 0.289) \times 100\% \approx 32.4\%$ of birds will migrate each year.

6 $\mathbf{T} = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.3 & 0.1 & 0 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0 \end{pmatrix}$

a The courier is at B, so $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

i $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.3 & 0.1 & 0 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \\ 0.1 \\ 0.5 \end{pmatrix}$

There is a 10% chance that the courier's first delivery goes to C.

ii $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.3 & 0.1 & 0 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.07 \\ 0.4 \\ 0.42 \\ 0.11 \end{pmatrix}$

There is a 7% chance that the courier's second delivery goes to A.

iii $\mathbf{s}_3 = \mathbf{T}^3\mathbf{s}_0 = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.3 & 0.1 & 0 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0 \end{pmatrix}^3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.255 \\ 0.278 \\ 0.127 \\ 0.34 \end{pmatrix}$

There is a 34% chance that the courier's third delivery goes to D.

b **i** From **a ii**, $\mathbf{s}_2 = \begin{pmatrix} 0.07 \\ 0.4 \\ 0.42 \\ 0.11 \end{pmatrix}$.

The highest value in $\mathbf{s}_2$ is 0.42 which corresponds to the courier being at C.
So, the courier is most likely to be at C after 2 deliveries.

ii From **a iii**, $\mathbf{s}_3 = \begin{pmatrix} 0.255 \\ 0.278 \\ 0.127 \\ 0.34 \end{pmatrix}$.

The highest value in $\mathbf{s}_3$ is 0.34 which corresponds to the courier being at D.
So, the courier is most likely to be at D after 3 deliveries.

c $\mathbf{s}_{100} = \mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.3 & 0.1 & 0 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0 \end{pmatrix}^{100} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.195 \\ 0.298 \\ 0.245 \\ 0.262 \end{pmatrix}$

The highest value in $\mathbf{s}_{100}$ is about 0.298 which corresponds to the courier being at B.
So, in the long run, the courier visits office B the most.

7 $\mathbf{T} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{T}) = 0$ then $\begin{vmatrix} \lambda-(1-p) & -q \\ -p & \lambda-(1-q) \end{vmatrix} = 0$

$$\therefore\ (\lambda-(1-p))(\lambda-(1-q)) - pq = 0$$
$$\therefore\ \lambda^2 - (1-q)\lambda - (1-p)\lambda + (1-p)(1-q) - pq = 0$$
$$\therefore\ \lambda^2 - (2-p-q)\lambda + 1 - q - p + pq - pq = 0$$
$$\therefore\ \lambda^2 - (2-p-q)\lambda + 1 - p - q = 0$$
$$\therefore\ (\lambda-1)(\lambda-(1-p-q)) = 0$$
$$\therefore\ \lambda = 1 \text{ or } 1-p-q$$

The eigenvalues are 1 and $1-p-q$.

b If $\mathbf{s}$ is the steady state matrix then $\mathbf{Ts} = \mathbf{s}$.
So, $\mathbf{s}$ is an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue 1.

For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{T})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} p & -q \\ -p & q \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ pa - qb = 0$$

Letting $b = t$, $t \neq 0$ then $a = \frac{q}{p}t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} \frac{q}{p} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} \frac{q}{p} \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 1.

So, $\mathbf{s} = \begin{pmatrix} \frac{q}{p} \\ 1 \end{pmatrix} k$ for some k

$$= \begin{pmatrix} \frac{q}{p}k \\ k \end{pmatrix}$$

Now $\mathbf{s}$ is a matrix of probabilities, so $\frac{q}{p}k + k = 1$

$$\therefore \frac{p+q}{p}k = 1$$

$$\therefore k = \frac{p}{p+q}$$

So, $\mathbf{s} = \begin{pmatrix} \frac{q}{p} \times \frac{p}{p+q} \\ \frac{p}{p+q} \end{pmatrix} = \begin{pmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{pmatrix}$

c In the long term, the two populations will have the proportions $\frac{q}{p+q}$ and $\frac{p}{p+q}$.

ACTIVITY 2 LESLIE MATRICES

1 a

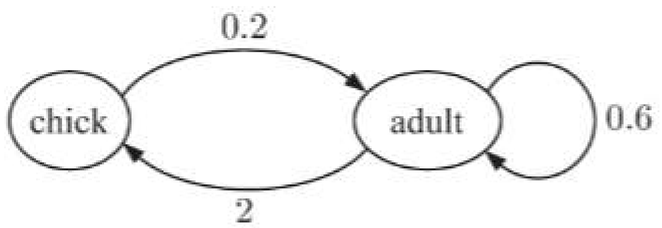
There are $4 \times 0.5 = 2$ female chicks born to each female adult.

$$\mathbf{L} = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}$$

20% of female chicks survive to adult age.

60% of female adults survive to the next year.

b



c $\mathbf{s}_0 = \begin{pmatrix} 250 \\ 300 \end{pmatrix}$

i $\mathbf{L}\mathbf{s}_0 = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}\begin{pmatrix} 250 \\ 300 \end{pmatrix} = \begin{pmatrix} 600 \\ 230 \end{pmatrix}$

Next year, there will be 600 female chicks and 230 adult female penguins.

$\mathbf{L}^2\mathbf{s}_0 = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}^2\begin{pmatrix} 250 \\ 300 \end{pmatrix} = \begin{pmatrix} 460 \\ 258 \end{pmatrix}$

In two years, there will be 460 female chicks and 258 adult female penguins.

ii $\mathbf{L}^5\mathbf{s}_0 = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}^5\begin{pmatrix} 250 \\ 300 \end{pmatrix} = \begin{pmatrix} 502.56 \\ 249.488 \end{pmatrix}$

$\mathbf{L}^{10}\mathbf{s}_0 = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}^{10}\begin{pmatrix} 250 \\ 300 \end{pmatrix} \approx \begin{pmatrix} 499.97 \\ 250.01 \end{pmatrix}$

In the long term, we predict there will be about 500 female chicks and 250 adult female penguins.

d $\mathbf{s}_0 = \begin{pmatrix} 20 \\ 250 \end{pmatrix}$

$$\therefore \quad \mathbf{L}^{10}\mathbf{s}_0 = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}^{10} \begin{pmatrix} 20 \\ 250 \end{pmatrix} \approx \begin{pmatrix} 362.82 \\ 181.44 \end{pmatrix}$$

Rounding to the nearest penguin, in the long term, we predict there will be 363 female chicks and 181 adult female penguins.
So, the population will survive but will not recover to 500 female chicks and 250 adult female penguins as in **c ii**.

e If $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the steady state matrix, then $\mathbf{L}\mathbf{s} = \mathbf{s}$

$$\therefore \quad \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 2b \\ 0.2a + 0.6b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Using either row, $2b = a$

Now $\mathbf{s}$ is a matrix of proportions, so $a + b = 1$

$$\therefore \quad 2b + b = 1$$
$$\therefore \quad 3b = 1$$
$$\therefore \quad b = \tfrac{1}{3} \quad \text{and} \quad a = \tfrac{2}{3}$$

So, $\mathbf{s} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

In the long term, $\frac{1}{3}$ of the population will be female chicks, and $\frac{2}{3}$ of the population will be adult female penguins.

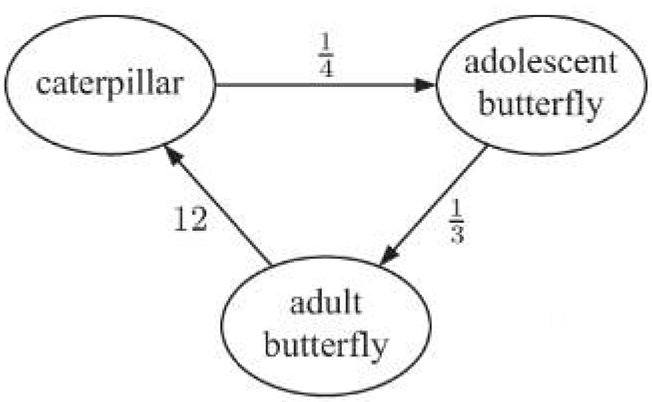
2 **a** $1 - \frac{3}{4} = \frac{1}{4}$ of female caterpillars survive to adolescence.

There are 12 female caterpillars born to each female butterfly.

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$$

$1 - \frac{2}{3} = \frac{1}{3}$ of adolescent female butterflies survive to mating age.

b



c $\mathbf{s}_0 = \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix}$

1 time period is 24 hours.

i 24 hours is 1 time period.

$$\mathbf{s}_1 = \mathbf{L}\mathbf{s}_0 = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix} = \begin{pmatrix} 96 \\ 30 \\ 5 \end{pmatrix}$$

After 24 hours, there will be 96 female caterpillars, 30 adolescent female butterflies, and 5 adult female butterflies.

ii 48 hours is 2 time periods.

$$\mathbf{s}_2 = \mathbf{L}^2\mathbf{s}_0 = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}^2 \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix} = \begin{pmatrix} 60 \\ 24 \\ 10 \end{pmatrix}$$

After 48 hours, there will be 60 female caterpillars, 24 adolescent female butterflies, and 10 adult female butterflies.

iii 72 hours is 3 time periods.

$$\mathbf{s}_3 = \mathbf{L}^3\mathbf{s}_0 = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}^3 \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix} = \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix}$$

After 72 hours, there will be 120 female caterpillars, 15 adolescent female butterflies, and 8 adult female butterflies.

d $$\mathbf{L}^4\mathbf{s}_0 = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}^4 \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix} = \begin{pmatrix} 96 \\ 30 \\ 5 \end{pmatrix} = \mathbf{s}_1$$

$$\mathbf{L}^5\mathbf{s}_0 = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}^5 \begin{pmatrix} 120 \\ 15 \\ 8 \end{pmatrix} = \begin{pmatrix} 60 \\ 24 \\ 10 \end{pmatrix} = \mathbf{s}_2$$

After $3n$ years, there will be 120 female caterpillars, 15 adolescent female butterflies, and 8 adult female butterflies.

After $3n+1$ years, there will be 96 female caterpillars, 30 adolescent female butterflies, and 5 adult female butterflies.

After $3n+2$ years, there will be 60 female caterpillars, 24 adolescent female butterflies, and 10 adult female butterflies.

REVIEW SET 13A

1 $\mathbf{A} = \begin{pmatrix} -4 & 8 \\ -1 & 3 \end{pmatrix}$

a $$\begin{aligned} p(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} \lambda+4 & -8 \\ 1 & \lambda-3 \end{vmatrix} \\ &= (\lambda+4)(\lambda-3)+8 \\ &= \lambda^2+\lambda-12+8 \\ &= \lambda^2+\lambda-4 \end{aligned}$$

b $$\begin{aligned} p(\lambda) &= 0 \\ \therefore\ \lambda^2+\lambda-4 &= 0 \\ \therefore\ \lambda &= \frac{-1 \pm \sqrt{1^2-4(1)(-4)}}{2} \\ &= \frac{-1\pm\sqrt{17}}{2} \\ &= -\tfrac{1}{2} \pm \tfrac{\sqrt{17}}{2} \end{aligned}$$

$\therefore$ the eigenvalues are $-\frac{1}{2}+\frac{\sqrt{17}}{2}$ and $-\frac{1}{2}-\frac{\sqrt{17}}{2}$.

2 $\mathbf{A} = \begin{pmatrix} 3 & -7 \\ -3 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$

a $$\begin{aligned} \mathbf{Ax} &= \begin{pmatrix} 3 & -7 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -7 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -21-21 \\ 21-3 \end{pmatrix} \\ &= \begin{pmatrix} -42 \\ 18 \end{pmatrix} \\ &= 6\begin{pmatrix} -7 \\ 3 \end{pmatrix} \\ &= 6\mathbf{x} \end{aligned}$$

$\therefore$ 6 is the eigenvalue corresponding to $\mathbf{x}$.

b If $\det(\lambda\mathbf{I}-\mathbf{A}) = 0$ then $\begin{vmatrix} \lambda-3 & 7 \\ 3 & \lambda+1 \end{vmatrix} = 0$

$$\begin{aligned} \therefore\ (\lambda-3)(\lambda+1)-21 &= 0 \\ \therefore\ \lambda^2-2\lambda-3-21 &= 0 \\ \therefore\ \lambda^2-2\lambda-24 &= 0 \\ \therefore\ (\lambda-6)(\lambda+4) &= 0 \\ \therefore\ \lambda &= 6 \text{ or } -4 \end{aligned}$$

The remaining eigenvalue is -4.

3 $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 8 & 6 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 8 & -3 \\ -8 & \lambda - 6 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 8)(\lambda - 6) - 24 = 0$$
$$\therefore\ \lambda^2 - 14\lambda + 48 - 24 = 0$$
$$\therefore\ \lambda^2 - 14\lambda + 24 = 0$$
$$\therefore\ (\lambda - 12)(\lambda - 2) = 0$$
$$\therefore\ \lambda = 12 \text{ or } 2$$

For $\lambda_1 = 12$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 4 & -3 \\ -8 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 4a - 3b = 0$$

Letting $b = t$, $t \neq 0$ then $a = \frac{3}{4}t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 4$, $\mathbf{x}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 12$.

For $\lambda_2 = 2$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -6 & -3 \\ -8 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -6a - 3b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -\frac{1}{2}t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 2$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$.

b $$\begin{aligned} \mathbf{A}\mathbf{x}_1 &= \begin{pmatrix} 8 & 3 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 24 + 12 \\ 24 + 24 \end{pmatrix} \\ &= \begin{pmatrix} 36 \\ 48 \end{pmatrix} \\ &= 12\begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \lambda_1\mathbf{x}_1 \quad \checkmark \end{aligned}$$

$$\begin{aligned} \mathbf{A}\mathbf{x}_2 &= \begin{pmatrix} 8 & 3 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -8 + 6 \\ -8 + 12 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \lambda_2\mathbf{x}_2 \quad \checkmark \end{aligned}$$

4 **a** If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 7 & -1 \\ 4 & \lambda - 3 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 7)(\lambda - 3) + 4 = 0$$
$$\therefore\ \lambda^2 - 10\lambda + 21 + 4 = 0$$
$$\therefore\ \lambda^2 - 10\lambda + 25 = 0$$
$$\therefore\ (\lambda - 5)^2 = 0$$
$$\therefore\ \lambda = 5$$

The only eigenvalue is 5.

For $\lambda = 5$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -2a - b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -\frac{1}{2}t$

$$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 2$, $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda + 3 & -2 \\ -1 & \lambda + 5 \end{vmatrix} = 0$

$$\therefore\ (\lambda + 3)(\lambda + 5) - 2 = 0$$
$$\therefore\ \lambda^2 + 8\lambda + 15 - 2 = 0$$
$$\therefore\ \lambda^2 + 8\lambda + 13 = 0$$
$$\therefore\ \lambda = \frac{-8 \pm \sqrt{8^2 - 4(1)(13)}}{2}$$
$$= \frac{-8 \pm \sqrt{12}}{2}$$
$$= \frac{-8 \pm 2\sqrt{3}}{2}$$
$$= -4 \pm \sqrt{3}$$

The eigenvalues are $-4 + \sqrt{3}$ and $-4 - \sqrt{3}$.

For $\lambda_1 = -4 + \sqrt{3}$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -4 + \sqrt{3} + 3 & -2 \\ -1 & -4 + \sqrt{3} + 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ \begin{pmatrix} -1 + \sqrt{3} & -2 \\ -1 & 1 + \sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -a + (1 + \sqrt{3})b = 0$$

Letting $b = t$, $t \neq 0$ then $a = (1+\sqrt{3})t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -4+\sqrt{3}$.

For $\lambda_2 = -4-\sqrt{3}$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -4-\sqrt{3}+3 & -2 \\ -1 & -4-\sqrt{3}+5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1-\sqrt{3} & -2 \\ -1 & 1-\sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -a + (1-\sqrt{3})b = 0$$

Letting $b = t$, $t \neq 0$ then $a = (1-\sqrt{3})t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -4-\sqrt{3}$.

5 $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda-2 & -4 \\ -5 & \lambda-3 \end{vmatrix} = 0$

$$\therefore (\lambda-2)(\lambda-3) - 20 = 0$$
$$\therefore \lambda^2 - 5\lambda + 6 - 20 = 0$$
$$\therefore \lambda^2 - 5\lambda - 14 = 0$$
$$\therefore (\lambda-7)(\lambda+2) = 0$$
$$\therefore \lambda = 7 \text{ or } -2$$

For $\lambda_1 = 7$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 5 & -4 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 5a - 4b = 0$$

Letting $b = t$, $t \neq 0$ then $a = \frac{4}{5}t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 5$, $\mathbf{x}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 7$.

For $\lambda_2 = -2$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -4 & -4 \\ -5 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -4a - 4b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -t$

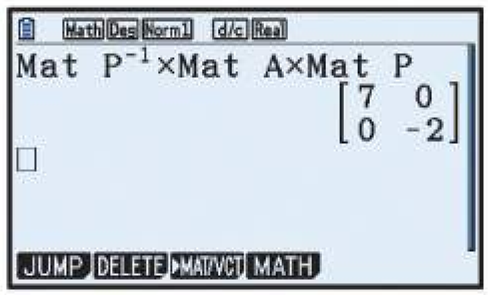
$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -2$.

b Let $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} 4 & -1 \\ 5 & 1 \end{pmatrix}$

$\therefore \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix}$ {using technology}

$\therefore$ $\mathbf{P}$ diagonalises $\mathbf{A}$.



6 The matrix $\mathbf{P} = \begin{pmatrix} -2 & -5 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{A}$ with $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -7 & 0 \\ 0 & -4 \end{pmatrix}$.

Now $\mathbf{A}^n = \mathbf{P}\begin{pmatrix} (-7)^n & 0 \\ 0 & (-4)^n \end{pmatrix}\mathbf{P}^{-1}$

$$\begin{aligned}
\therefore \mathbf{A}^6 &= \begin{pmatrix} -2 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (-7)^6 & 0 \\ 0 & (-4)^6 \end{pmatrix}\tfrac{1}{3}\begin{pmatrix} 1 & 5 \\ -1 & -2 \end{pmatrix} \\
&= \tfrac{1}{3}\begin{pmatrix} -2 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 7^6 & 0 \\ 0 & 4^6 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -1 & -2 \end{pmatrix} \\
&= \tfrac{1}{3}\begin{pmatrix} -2 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 7^6 & 5 \times 7^6 \\ -4^6 & -2 \times 4^6 \end{pmatrix} \\
&= \tfrac{1}{3}\begin{pmatrix} -2 \times 7^6 + 5 \times 4^6 & -10 \times 7^6 + 10 \times 4^6 \\ 7^6 - 4^6 & 5 \times 7^6 - 2 \times 4^6 \end{pmatrix}
\end{aligned}$$

7 If $\det(\lambda\mathbf{I} - \mathbf{C}) = 0$ then $\begin{vmatrix} \lambda - 2 & 3 \\ 8 & \lambda - 4 \end{vmatrix} = 0$

$$\begin{aligned}
\therefore (\lambda - 2)(\lambda - 4) - 24 &= 0 \\
\therefore \lambda^2 - 6\lambda + 8 - 24 &= 0 \\
\therefore \lambda^2 - 6\lambda - 16 &= 0 \\
\therefore (\lambda - 8)(\lambda + 2) &= 0 \\
\therefore \lambda &= 8 \text{ or } -2
\end{aligned}$$

For $\lambda_1 = 8$, consider $(\lambda\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 6 & 3 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 6a + 3b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -\frac{1}{2}t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 2$, $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 8$.

For $\lambda_2 = -2$, consider $(\lambda\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -4 & 3 \\ 8 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -4a + 3b = 0$$

Letting $b = t$, $t \neq 0$ then $a = \frac{3}{4}t$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} t, \quad t \neq 0$$

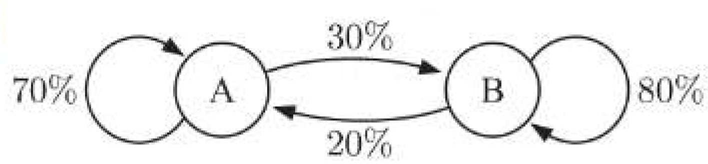
$\therefore$ choosing $t = 4$, $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -2$.

The matrix $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$ diagonalises $\mathbf{C}$ with $\mathbf{P}^{-1}\mathbf{C}\mathbf{P} = \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}$.

Now $\mathbf{C}^n = \mathbf{P}\begin{pmatrix} 8^n & 0 \\ 0 & (-2)^n \end{pmatrix}\mathbf{P}^{-1}$

$$\begin{aligned}
\therefore \mathbf{C}^{10} &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 8^{10} & 0 \\ 0 & (-2)^{10} \end{pmatrix}\left(-\tfrac{1}{10}\right)\begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix} \\
&= -\tfrac{1}{10}\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 8^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix} \\
&= -\tfrac{1}{10}\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \times 8^{10} & -3 \times 8^{10} \\ -2 \times 2^{10} & -2^{10} \end{pmatrix} \\
&= -\tfrac{1}{10}\begin{pmatrix} -4 \times 8^{10} - 6 \times 2^{10} & 3 \times 8^{10} - 3 \times 2^{10} \\ 8 \times 8^{10} - 8 \times 2^{10} & -6 \times 8^{10} - 4 \times 2^{10} \end{pmatrix} \\
&= \tfrac{1}{10}\begin{pmatrix} 4 \times 8^{10} + 6 \times 2^{10} & -3 \times 8^{10} + 3 \times 2^{10} \\ -8 \times 8^{10} + 8 \times 2^{10} & 6 \times 8^{10} + 4 \times 2^{10} \end{pmatrix}
\end{aligned}$$

8 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$

b $\mathbf{s}_0 = \begin{pmatrix} 250 \\ 250 \end{pmatrix}$

i Monday is 1 day after Sunday.

$$\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 250 \\ 250 \end{pmatrix} = \begin{pmatrix} 225 \\ 275 \end{pmatrix}$$

$\therefore$ 225 passengers will eat at restaurant A on Monday.

ii Tuesday is 2 days after Sunday.

$$\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^2 \begin{pmatrix} 250 \\ 250 \end{pmatrix} = \begin{pmatrix} 212.5 \\ 287.5 \end{pmatrix}$$

Rounding to the nearest passenger, about 288 passengers will eat at restaurant B on Tuesday.

9

		This generation	
		Smoker	Non-smoker
Next generation	Smoker	55%	5%
	Non-smoker	45%	95%

a $\mathbf{T} = \begin{pmatrix} 0.55 & 0.05 \\ 0.45 & 0.95 \end{pmatrix}$

b In row 2, column 1 we have 0.45. This means that 45% of the children of smokers will be non-smokers.

c $\mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.55 & 0.05 \\ 0.45 & 0.95 \end{pmatrix}^2 \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.225 \\ 0.775 \end{pmatrix}$

This means that 22.5% of the grandchildren of this generation will be smokers, and 77.5% will be non-smokers.

d If $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the steady state matrix then $\mathbf{T}\mathbf{s} = \mathbf{s}$

$$\therefore \begin{pmatrix} 0.55 & 0.05 \\ 0.45 & 0.95 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0.55a + 0.05b \\ 0.45a + 0.95b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Using either row, $0.55a + 0.05b = a$

$$\therefore 0.05b = 0.45a$$

$$\therefore b = 9a$$

Now $\mathbf{s}$ is a matrix of probabilities, so $a + b = 1$

$$\therefore a + 9a = 1$$

$$\therefore 10a = 1$$

$$\therefore a = \tfrac{1}{10} \text{ and } b = \tfrac{9}{10}$$

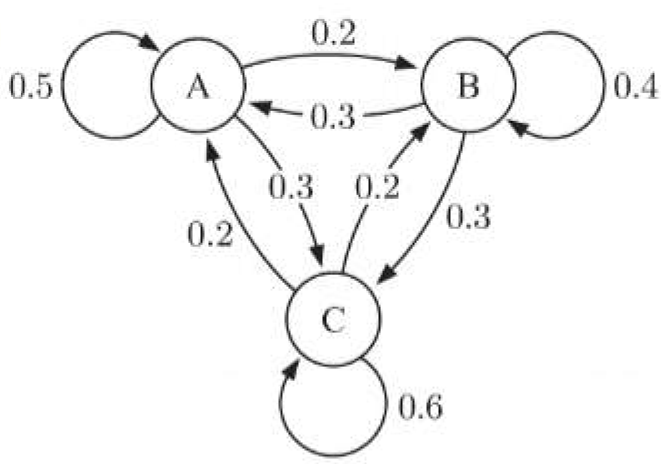
So, $\mathbf{s} = \begin{pmatrix} \frac{1}{10} \\ \frac{9}{10} \end{pmatrix}$.

In the long term, 10% are adult smokers.

10

		This time		
		A	B	C
Next time	A	0.5	0.3	0.2
	B	0.2	0.4	0.2
	C	0.3	0.3	0.6

a **i**



ii $\mathbf{T} = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.6 \end{pmatrix}$

b In row 1, column 3 we have 0.2. This means that after leaving taxi rank C, the probability that Bill will return to taxi rank A is 0.2.

c $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.6 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.33 \\ 0.28 \\ 0.39 \end{pmatrix}$$

The highest value in $\mathbf{s}_2$ is 0.39 which corresponds to Bill returning to taxi rank C.
So, Bill is most likely to return to taxi rank C to collect his third customer.

d $$\mathbf{s}_{100} = \mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.6 \end{pmatrix}^{100} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.321 \\ 0.25 \\ 0.429 \end{pmatrix}$$

The highest value in $\mathbf{s}_{100}$ is about 0.429 which corresponds to Bill returning to taxi rank C.
In the long run, Bill uses taxi rank C most often.

REVIEW SET 13B

1 **a** If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 4 & -2 \\ -3 & \lambda + 4 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 4)(\lambda + 4) - 6 = 0$$
$$\therefore\ \lambda^2 - 16 - 6 = 0$$
$$\therefore\ \lambda^2 = 22$$
$$\therefore\ \lambda = \pm\sqrt{22}$$

$\therefore$ the eigenvalues are $\sqrt{22}$ and $-\sqrt{22}$.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 3 & -18 \\ 2 & \lambda + 9 \end{vmatrix} = 0$

$$\therefore \ (\lambda - 3)(\lambda + 9) + 36 = 0$$
$$\therefore \ \lambda^2 + 6\lambda - 27 + 36 = 0$$
$$\therefore \ \lambda^2 + 6\lambda + 9 = 0$$
$$\therefore \ (\lambda + 3)^2 = 0$$
$$\therefore \ \lambda = -3$$

$\therefore$ the eigenvalue is -3.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 5 & -1 \\ 3 & \lambda - 2 \end{vmatrix} = 0$

$$\therefore \ (\lambda - 5)(\lambda - 2) + 3 = 0$$
$$\therefore \ \lambda^2 - 7\lambda + 10 + 3 = 0$$
$$\therefore \ \lambda^2 - 7\lambda + 13 = 0$$
$$\therefore \ \lambda = \frac{7 \pm \sqrt{7^2 - 4(1)(13)}}{2}$$
$$= \frac{7 \pm \sqrt{-3}}{2}$$
$$= \frac{7 \pm i\sqrt{3}}{2}$$
$$= \tfrac{7}{2} \pm \tfrac{\sqrt{3}}{2}\, i$$

$\therefore$ the eigenvalues are $\frac{7}{2} + \frac{\sqrt{3}}{2}\, i$ and $\frac{7}{2} - \frac{\sqrt{3}}{2}\, i$.

2 $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{B})$

$$= \begin{vmatrix} \lambda - 4 & 1 \\ -k & \lambda - 2 \end{vmatrix}$$
$$= (\lambda - 4)(\lambda - 2) + k$$
$$= \lambda^2 - 6\lambda + 8 + k$$

The eigenvalues of $\mathbf{B}$ are the solutions to $p(\lambda) = 0$.

Using $\lambda = 3 + \sqrt{8}$, $(3 + \sqrt{8})^2 - 6(3 + \sqrt{8}) + 8 + k = 0$

$$\therefore \ 9 + 6\sqrt{8} + 8 - 18 - 6\sqrt{8} + 8 + k = 0$$
$$\therefore \ k = -7$$

Checking with $\lambda = 3 - \sqrt{8}$, $(3 - \sqrt{8})^2 - 6(3 - \sqrt{8}) + 8 - 7$

$$= 9 - 6\sqrt{8} + 8 - 18 + 6\sqrt{8} + 1$$
$$= 0 \ \checkmark$$

3 $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 2 & -5 \\ -4 & \lambda - 3 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 2)(\lambda - 3) - 20 = 0$$
$$\therefore\ \lambda^2 - 5\lambda + 6 - 20 = 0$$
$$\therefore\ \lambda^2 - 5\lambda - 14 = 0$$
$$\therefore\ (\lambda - 7)(\lambda + 2) = 0$$
$$\therefore\ \lambda = 7 \text{ or } -2$$

For $\lambda_1 = 7$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 5 & -5 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 5a - 5b = 0$$

Letting $b = t$, $t \neq 0$ then $a = t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t,\quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 7$.

For $\lambda_2 = -2$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -4 & -5 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -4a - 5b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -\frac{5}{4}t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} \\ 1 \end{pmatrix} t,\quad t \neq 0$

$\therefore$ choosing $t = 4$, $\mathbf{x}_2 = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -2$.

b Suppose $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

$\therefore\ \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

$\therefore\ 3(\mathbf{A}\mathbf{x}) = 3(\lambda\mathbf{x})$

$\therefore\ (3\mathbf{A})\mathbf{x} = (3\lambda)\mathbf{x}$

So, $\mathbf{x}$ is also an eigenvector of $3\mathbf{A}$ corresponding to the eigenvalue 3λ.

Thus, using **a**:

- $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $3\mathbf{A}$ corresponding to the eigenvalue $\lambda_1 = 3 \times 7 = 21$.
- $\mathbf{x}_2 = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$ is an eigenvector of $3\mathbf{A}$ corresponding to the eigenvalue $\lambda_2 = 3(-2) = -6$.

4 $\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$

a Let $\mathbf{x} = \begin{pmatrix} 3+i \\ 2 \end{pmatrix} t, \quad t \neq 0$

Now $$\mathbf{Ax} = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} (3+i)t \\ 2t \end{pmatrix}$$
$$= \begin{pmatrix} 3(3+i)t - 10t \\ 2(3+i)t - 6t \end{pmatrix}$$
$$= \begin{pmatrix} 9+3i-10 \\ 6+2i-6 \end{pmatrix} t$$
$$= \begin{pmatrix} -1+3i \\ 2i \end{pmatrix} t$$
$$= i \begin{pmatrix} 3+i \\ 2 \end{pmatrix} t$$

$\therefore \quad \mathbf{Ax} = i\mathbf{x}$

So, any vector of the form $\begin{pmatrix} 3+i \\ 2 \end{pmatrix} t, \quad t \neq 0,$ is an eigenvector corresponding to the eigenvalue i.

b i is an eigenvalue of $\mathbf{A}$

$\therefore \quad -i$ is the other eigenvalue of $\mathbf{A}$.

c For $\lambda = -i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -i-3 & 0-(-5) \\ 0-2 & -i-(-3) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \begin{pmatrix} -3-i & 5 \\ -2 & 3-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad -2a + (3-i)b = 0$$

Letting $b = t, \; t \neq 0$ then $a = \frac{1}{2}(3-i)t$

$\therefore \quad \mathbf{x} = \begin{pmatrix} \frac{1}{2}(3-i) \\ 1 \end{pmatrix} t, \quad t \neq 0$

Any vector of the form $\begin{pmatrix} \frac{1}{2}(3-i) \\ 1 \end{pmatrix} t, \quad t \neq 0$ is an eigenvector corresponding to the eigenvalue $-i$.

5 **a** If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix}\lambda-2 & 9\\ -4 & \lambda+10\end{vmatrix}=0$

$$\therefore\ (\lambda-2)(\lambda+10)+36=0$$
$$\therefore\ \lambda^2+8\lambda-20+36=0$$
$$\therefore\ \lambda^2+8\lambda+16=0$$
$$\therefore\ (\lambda+4)^2=0$$
$$\therefore\ \lambda=-4$$

The only eigenvalue is -4.

For $\lambda=-4$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\b\end{pmatrix}$

$$\therefore\ \begin{pmatrix}-6 & 9\\ -4 & 6\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ -6a+9b=0$$

Letting $b=t$, $t\neq 0$ then $a=\frac{3}{2}t$

$$\therefore\ \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\1\end{pmatrix}t,\quad t\neq 0$$

$\therefore$ choosing $t=2$, $\mathbf{x}=\begin{pmatrix}3\\2\end{pmatrix}$ is an eigenvector corresponding to the eigenvalue -4.

b If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix}\lambda-5 & 2\\ -1 & \lambda+1\end{vmatrix}=0$

$$\therefore\ (\lambda-5)(\lambda+1)+2=0$$
$$\therefore\ \lambda^2-4\lambda-5+2=0$$
$$\therefore\ \lambda^2-4\lambda-3=0$$
$$\therefore\ \lambda=\frac{4\pm\sqrt{(-4)^2-4(1)(-3)}}{2}$$
$$=\frac{4\pm\sqrt{28}}{2}$$
$$=\frac{4\pm 2\sqrt{7}}{2}$$
$$=2\pm\sqrt{7}$$

The eigenvalues are $2+\sqrt{7}$ and $2-\sqrt{7}$.

For $\lambda_1=2+\sqrt{7}$, consider $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\b\end{pmatrix}$

$$\therefore\ \begin{pmatrix}2+\sqrt{7}-5 & 2\\ -1 & 2+\sqrt{7}+1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ \begin{pmatrix}-3+\sqrt{7} & 2\\ -1 & 3+\sqrt{7}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ -a+(3+\sqrt{7})b=0$$

Letting $b = t,\ t \neq 0$ then $a = (3+\sqrt{7})t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3+\sqrt{7} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} 3+\sqrt{7} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 2+\sqrt{7}$.

For $\lambda_2 = 2-\sqrt{7}$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 2-\sqrt{7}-5 & 2 \\ -1 & 2-\sqrt{7}+1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -3-\sqrt{7} & 2 \\ -1 & 3-\sqrt{7} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ -a + (3-\sqrt{7})b = 0$$

Letting $b = t,\ t \neq 0$ then $a = (3-\sqrt{7})t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3-\sqrt{7} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} 3-\sqrt{7} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2-\sqrt{7}$.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda-4 & -2 \\ 1 & \lambda-2 \end{vmatrix} = 0$

$$\therefore\ (\lambda-4)(\lambda-2) + 2 = 0$$

$$\therefore\ \lambda^2 - 6\lambda + 8 + 2 = 0$$

$$\therefore\ \lambda^2 - 6\lambda + 10 = 0$$

$$\therefore\ \lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

The eigenvalues are $3+i$ and $3-i$.

For $\lambda_1 = 3+i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 3+i-4 & -2 \\ 1 & 3+i-2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -1+i & -2 \\ 1 & 1+i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ a + (1+i)b = 0$$

Letting $b = t$, $t \neq 0$ then $a = (-1-i)t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1-i \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 3 + i$.

For $\lambda_2 = 3 - i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 3-i-4 & -2 \\ 1 & 3-i-2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore a + (1-i)b = 0$$

Letting $b = t$, $t \neq 0$ then $a = (-1+i)t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1+i \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 3 - i$.

6 $\mathbf{B} = \begin{pmatrix} 5 & 0 \\ -5 & 4 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{B}) = 0$ then $\begin{vmatrix} \lambda - 5 & 0 \\ 5 & \lambda - 4 \end{vmatrix} = 0$

$$\therefore (\lambda - 5)(\lambda - 4) = 0$$

$$\therefore \lambda = 5 \text{ or } 4$$

The eigenvalues are 5 and 4.

For $\lambda_1 = 5$, consider $(\lambda\mathbf{I} - \mathbf{B})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 5a + b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -\frac{1}{5}t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 5$, $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 5$.

For $\lambda_2 = 4$, consider $(\lambda\mathbf{I} - \mathbf{B})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -1 & 0 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -a = 0$$

$$\therefore a = 0$$

Let $b = t,\ t \neq 0$.

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 4$.

b The matrix $\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}$ diagonalises $\mathbf{B}$ with $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$.

c
$$\begin{aligned} \mathbf{B}^n &= \mathbf{P}\begin{pmatrix} 5^n & 0 \\ 0 & 4^n \end{pmatrix}\mathbf{P}^{-1} \\ &= \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & 4^n \end{pmatrix}(-1)\begin{pmatrix} 1 & 0 \\ -5 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & 4^n \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix} \end{aligned}$$

i
$$\begin{aligned} \mathbf{B}^3 &= \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 5^3 & 0 \\ 0 & 4^3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -5^3 & 0 \\ 5 \times 4^3 & 4^3 \end{pmatrix} \\ &= \begin{pmatrix} 5^3 & 0 \\ -5^4 + 5 \times 4^3 & 4^3 \end{pmatrix} \\ &= \begin{pmatrix} 125 & 0 \\ -305 & 64 \end{pmatrix} \end{aligned}$$

ii
$$\begin{aligned} \mathbf{B}^6 &= \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 5^6 & 0 \\ 0 & 4^6 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -5^6 & 0 \\ 5 \times 4^6 & 4^6 \end{pmatrix} \\ &= \begin{pmatrix} 5^6 & 0 \\ -5^7 + 5 \times 4^6 & 4^6 \end{pmatrix} \\ &= \begin{pmatrix} 15\,625 & 0 \\ -57\,645 & 4096 \end{pmatrix} \end{aligned}$$

7 $\mathbf{A} = \begin{pmatrix} 7 & 6 \\ 8 & 9 \end{pmatrix}$

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 7 & -6 \\ -8 & \lambda - 9 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 7)(\lambda - 9) - 48 = 0$$
$$\therefore\ \lambda^2 - 16\lambda + 63 - 48 = 0$$
$$\therefore\ \lambda^2 - 16\lambda + 15 = 0$$
$$\therefore\ (\lambda - 1)(\lambda - 15) = 0$$
$$\therefore\ \lambda = 1 \text{ or } 15$$

For $\lambda_1 = 1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -6 & -6 \\ -8 & -8 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -6a - 6b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

For $\lambda_2 = 15$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 8 & -6 \\ -8 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 8a - 6b = 0$$

Letting $b = t$, $t \neq 0$ then $a = \frac{3}{4}t$

$\therefore\ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 4$, $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 15$.

The matrix $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$ diagonalises $\mathbf{A}$ with $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix}$.

b $$\mathbf{A}^n = \mathbf{P}\begin{pmatrix} 1^n & 0 \\ 0 & 15^n \end{pmatrix}\mathbf{P}^{-1}$$

$$\therefore \mathbf{A}^{10} = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 15^{10} \end{pmatrix}\left(-\tfrac{1}{7}\right)\begin{pmatrix} 4 & -3 \\ -1 & -1 \end{pmatrix}$$

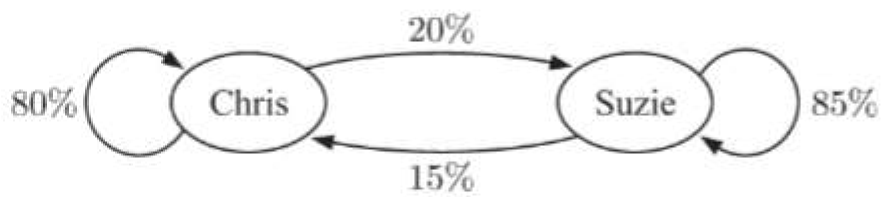
$$= \tfrac{1}{7}\begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 15^{10} \end{pmatrix}\begin{pmatrix} -4 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= \tfrac{1}{7}\begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -4 & 3 \\ 15^{10} & 15^{10} \end{pmatrix}$$

$$= \tfrac{1}{7}\begin{pmatrix} 4+3\times 15^{10} & -3+3\times 15^{10} \\ -4+4\times 15^{10} & 3+4\times 15^{10} \end{pmatrix}$$

8

		This week	
		Chris	Suzie
Next week	Chris	80%	15%
	Suzie	20%	85%

a **i** 

ii $\mathbf{T} = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}$

b $$\mathbf{T}^2 = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}^2 = \begin{pmatrix} 0.67 & 0.2475 \\ 0.33 & 0.7525 \end{pmatrix}$$

$$\mathbf{T}^3 = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}^3 = \begin{pmatrix} 0.5855 & 0.310\,875 \\ 0.4145 & 0.689\,125 \end{pmatrix}$$

c The second column of $\mathbf{T}^2$ tells us how the people who supported Suzie this week will respond in two weeks' time.
So, 24.75% will support Chris, and 75.25% will support Suzie.

d $\mathbf{s}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$

i $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.67 & 0.2475 \\ 0.33 & 0.7525 \end{pmatrix}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.501 \\ 0.499 \end{pmatrix}$

The highest value in $\mathbf{s}_2$ is 0.501 which corresponds to Chris having 50.1% of the vote.
So, we expect Chris to win the election if it is held in 2 weeks' time.

ii $\mathbf{s}_4 = \mathbf{T}^4\mathbf{s}_0 = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}^4\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.459\,172\,5 \\ 0.540\,827\,5 \end{pmatrix}$

The highest value in $\mathbf{s}_4$ is about 0.541 which corresponds to Suzie having about 54.1% of the vote.
So, we expect Suzie to win the election if it is held in 4 weeks' time.

9

This generation

Next generation		Underweight	Healthy	Overweight
	Underweight	0.43	0.18	0.09
	Healthy	0.47	0.56	0.55
	Overweight	0.10	0.26	0.36

a $\mathbf{T} = \begin{pmatrix} 0.43 & 0.18 & 0.09 \\ 0.47 & 0.56 & 0.55 \\ 0.10 & 0.26 & 0.36 \end{pmatrix}$

$$\mathbf{T}^2 = \begin{pmatrix} 0.43 & 0.18 & 0.09 \\ 0.47 & 0.56 & 0.55 \\ 0.10 & 0.26 & 0.36 \end{pmatrix}^2 = \begin{pmatrix} 0.2785 & 0.2016 & 0.1701 \\ 0.5203 & 0.5412 & 0.5483 \\ 0.2012 & 0.2572 & 0.2816 \end{pmatrix}$$

b **i** In row 2, column 1 of $\mathbf{T}$ we have 0.47. This means that 47% of underweight people of the current generation are expected to have healthy weight children.

ii In row 3, column 1 of $\mathbf{T}^2$ we have 0.2012. This means that 20.12% of underweight people of the current generation are expected to have overweight grandchildren.

c $\mathbf{s}_0 = \begin{pmatrix} 0.15 \\ 0.56 \\ 0.29 \end{pmatrix}$

i $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.43 & 0.18 & 0.09 \\ 0.47 & 0.56 & 0.55 \\ 0.10 & 0.26 & 0.36 \end{pmatrix} \begin{pmatrix} 0.15 \\ 0.56 \\ 0.29 \end{pmatrix} = \begin{pmatrix} 0.1914 \\ 0.5436 \\ 0.265 \end{pmatrix}$

In the next generation, about 19.1% of people will be underweight, 54.4% will be of healthy weight, and 26.5% will be overweight.

ii $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.2785 & 0.2016 & 0.1701 \\ 0.5203 & 0.5412 & 0.5483 \\ 0.2012 & 0.2572 & 0.2816 \end{pmatrix} \begin{pmatrix} 0.15 \\ 0.56 \\ 0.29 \end{pmatrix} = \begin{pmatrix} 0.204 \\ 0.540\,124 \\ 0.255\,876 \end{pmatrix}$

In two generations, about 20.4% of people will be underweight, 54.0% will be of healthy weight, and 25.6% will be overweight.

d $\mathbf{s}_{100} = \mathbf{T}^{100}\mathbf{s}_0 = \begin{pmatrix} 0.43 & 0.18 & 0.09 \\ 0.47 & 0.56 & 0.55 \\ 0.10 & 0.26 & 0.36 \end{pmatrix}^{100} \begin{pmatrix} 0.15 \\ 0.56 \\ 0.29 \end{pmatrix} \approx \begin{pmatrix} 0.210 \\ 0.539 \\ 0.252 \end{pmatrix}$

In the long run, about 21.0% of people will be underweight, 53.9% will be of healthy weight, and 25.2% will be overweight.

e No, it is unlikely that the transition matrix will remain constant over a number of generations, as attitudes and lifestyles will change.

10 **a** $\mathbf{s}_0 = \begin{pmatrix} 60 \\ 60 \end{pmatrix}$, $\mathbf{T} = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$

b $\mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} 60 \\ 60 \end{pmatrix} = \begin{pmatrix} 48 \\ 72 \end{pmatrix}$

After 1 week, 48 students are painting and 72 students are doing pottery.

c If $\det(\lambda\mathbf{I}-\mathbf{T})=0$ then $\begin{vmatrix}\lambda-0.6 & -0.2\\ -0.4 & \lambda-0.8\end{vmatrix}=0$

$$\therefore\ (\lambda-0.6)(\lambda-0.8)-0.08=0$$
$$\therefore\ \lambda^2-1.4\lambda+0.48-0.08=0$$
$$\therefore\ \lambda^2-1.4\lambda+0.4=0$$
$$\therefore\ (\lambda-1)(\lambda-0.4)=0$$
$$\therefore\ \lambda=1 \text{ or } 0.4$$

The eigenvalues are 1 and 0.4.

For $\lambda_1=1$, consider $(\lambda\mathbf{I}-\mathbf{T})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\b\end{pmatrix}$

$$\therefore\ \begin{pmatrix}0.4 & -0.2\\ -0.4 & 0.2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ 0.4a-0.2b=0$$

Letting $b=t$, $t\neq0$ then $a=\frac{1}{2}t$

$$\therefore\ \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}t,\quad t\neq0$$

$\therefore$ choosing $t=1$, $\mathbf{x}_1=\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1=1$.

For $\lambda_2=0.4$, consider $(\lambda\mathbf{I}-\mathbf{T})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\b\end{pmatrix}$

$$\therefore\ \begin{pmatrix}-0.2 & -0.2\\ -0.4 & -0.4\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ -0.2a-0.2b=0$$

Letting $b=t$, $t\neq0$ then $a=-t$

$$\therefore\ \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}t,\quad t\neq0$$

$\therefore$ choosing $t=1$, $\mathbf{x}_2=\begin{pmatrix}-1\\1\end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2=0.4$.

d The matrix $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{T}$ with $\mathbf{P}^{-1}\mathbf{T}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$.

$$\begin{aligned}
\text{Now} \quad \mathbf{T}^n &= \mathbf{P}\begin{pmatrix} 1^n & 0 \\ 0 & (0.4)^n \end{pmatrix}\mathbf{P}^{-1} \\
&= \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\frac{2}{5})^n \end{pmatrix}\frac{2}{3}\begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \\
&= \frac{2}{3}\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (\frac{2}{5})^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \\
&= \frac{2}{3}\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -(\frac{2}{5})^n & \frac{1}{2}(\frac{2}{5})^n \end{pmatrix} \\
&= \frac{2}{3}\begin{pmatrix} \frac{1}{2} + (\frac{2}{5})^n & \frac{1}{2} - \frac{1}{2}(\frac{2}{5})^n \\ 1 - (\frac{2}{5})^n & 1 + \frac{1}{2}(\frac{2}{5})^n \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3} + \frac{2}{3}(\frac{2}{5})^n & \frac{1}{3} - \frac{1}{3}(\frac{2}{5})^n \\ \frac{2}{3} - \frac{2}{3}(\frac{2}{5})^n & \frac{2}{3} + \frac{1}{3}(\frac{2}{5})^n \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{So,} \quad \mathbf{s}_n &= \mathbf{T}^n\mathbf{s}_0 \\
&= \begin{pmatrix} \frac{1}{3} + \frac{2}{3}(\frac{2}{5})^n & \frac{1}{3} - \frac{1}{3}(\frac{2}{5})^n \\ \frac{2}{3} - \frac{2}{3}(\frac{2}{5})^n & \frac{2}{3} + \frac{1}{3}(\frac{2}{5})^n \end{pmatrix}\begin{pmatrix} 60 \\ 60 \end{pmatrix} \\
&= \begin{pmatrix} 20 + 40(\frac{2}{5})^n \ + \ 20 - 20(\frac{2}{5})^n \\ 40 - 40(\frac{2}{5})^n \ + \ 40 + 20(\frac{2}{5})^n \end{pmatrix} \\
&= \begin{pmatrix} 40 + 20(\frac{2}{5})^n \\ 80 - 20(\frac{2}{5})^n \end{pmatrix} \\
&= \begin{pmatrix} 20\left(2 + (\frac{2}{5})^n\right) \\ 20\left(4 - (\frac{2}{5})^n\right) \end{pmatrix}
\end{aligned}$$

$\therefore$ there are $20\left(2 + (\frac{2}{5})^n\right)$ students painting after n weeks.

e As $n \to \infty$, $(\frac{2}{5})^n \to 0$

In the long term, there are $20(2+0) = 40$ students painting each week.

Chapter 14

AFFINE TRANSFORMATIONS

EXERCISE 14A

1 **a** $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$

$\therefore$ the image is $(4, 6)$.

b $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}-4\\0\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$

$\therefore$ the image is $(-1, 2)$.

c $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}5\\-1\end{pmatrix}$

$\therefore$ the image is $(5, -1)$.

d $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}0\\-3\end{pmatrix}$

$\therefore$ the image is $(0, -3)$.

e $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}h\\k\end{pmatrix}=\begin{pmatrix}3+h\\2+k\end{pmatrix}$

$\therefore$ the image is $(3+h,\ 2+k)$.

2 **a** $\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}-2\\3\end{pmatrix}$

$\therefore$ the image is $(-2, 3)$.

b $\begin{pmatrix}4\\1\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$

$\therefore$ the image is $(2, 4)$.

c $\begin{pmatrix}2\\-2\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore$ the image is $(0, 1)$.

d $\begin{pmatrix}-3\\-4\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}-5\\-1\end{pmatrix}$

$\therefore$ the image is $(-5, -1)$.

e $\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}x-2\\y+3\end{pmatrix}$

$\therefore$ the image is $(x-2,\ y+3)$.

3 **a** Let (x, y) be the object point.

$\therefore\ \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-1\\3\end{pmatrix}$

$\therefore\ x+1=-1$ and $y+2=3$

$\therefore\ x=-2$ and $y=1$

So, the object point is $(-2, 1)$.

b Let (x, y) be the object point.

$\therefore\ \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}-1\\3\end{pmatrix}$

$\therefore\ x=-1$ and $y-1=3$

$\therefore\ x=-1$ and $y=4$

So, the object point is $(-1, 4)$.

c Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\therefore \ x - 1 = -1 \quad \text{and} \quad y + 5 = 3$$
$$\therefore \ x = 0 \quad \text{and} \quad y = -2$$

So, the object point is $(0, -2)$.

d Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\therefore \ x - 2 = -1 \quad \text{and} \quad y - 3 = 3$$
$$\therefore \ x = 1 \quad \text{and} \quad y = 6$$

So, the object point is $(1, 6)$.

e Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\therefore \ x + h = -1 \quad \text{and} \quad y + k = 3$$
$$\therefore \ x = -1 - h \quad \text{and} \quad y = 3 - k$$

So, the object point is $(-1 - h, \ 3 - k)$.

4 **a** Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \ x + 4 = 0 \quad \text{and} \quad y - 1 = 0$$
$$\therefore \ x = -4 \quad \text{and} \quad y = 1$$

So, the object point is $(-4, 1)$.

b Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\therefore \ x + 4 = 3 \quad \text{and} \quad y - 1 = -2$$
$$\therefore \ x = -1 \quad \text{and} \quad y = -1$$

So, the object point is $(-1, -1)$.

c Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\therefore \ x + 4 = 7 \quad \text{and} \quad y - 1 = 4$$
$$\therefore \ x = 3 \quad \text{and} \quad y = 5$$

So, the object point is $(3, 5)$.

d Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$\therefore \; x + 4 = -1 \quad \text{and} \quad y - 1 = -5$$
$$\therefore \; x = -5 \quad \text{and} \quad y = -4$$

So, the object point is $(-5, -4)$.

e Let (a, b) be the object point.

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \; a + 4 = x \quad \text{and} \quad b - 1 = y$$
$$\therefore \; a = x - 4 \quad \text{and} \quad b = y + 1$$

So, the object point is $(x - 4, \; y + 1)$.

5 **a** Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$$\therefore \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\therefore \; 2 + e = 4 \quad \text{and} \quad 3 + f = 7$$
$$\therefore \; e = 2 \quad \text{and} \quad f = 4$$

So, the translation vector is $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

b Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$$\therefore \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\therefore \; 1 + e = -2 \quad \text{and} \quad -2 + f = 0$$
$$\therefore \; e = -3 \quad \text{and} \quad f = 2$$

So, the translation vector is $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

c Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$$\therefore \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\therefore \; -3 + e = 1 \quad \text{and} \quad 1 + f = -4$$
$$\therefore \; e = 4 \quad \text{and} \quad f = -5$$

So, the translation vector is $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

d Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$$\therefore \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore\ 5 + e = 0 \quad \text{and} \quad -2 + f = 1$$
$$\therefore\ e = -5 \quad \text{and} \quad f = 3$$

So, the translation vector is $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

e Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x-1 \\ y+3 \end{pmatrix}$$

$$\therefore\ x + e = x - 1 \quad \text{and} \quad y + f = y + 3$$
$$\therefore\ e = -1 \quad \text{and} \quad f = 3$$

So, the translation vector is $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

f Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\therefore\ x + e = 2 \quad \text{and} \quad y + f = -3$$
$$\therefore\ e = 2 - x \quad \text{and} \quad f = -3 - y$$

So, the translation vector is $\begin{pmatrix} 2-x \\ -3-y \end{pmatrix}$.

6 **a** Let (x', y') be the image point.

$$\therefore \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\overbrace{\left[\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}\right]}^{(x,\,y) \text{ is translated through } \mathbf{b}_1} + \begin{pmatrix} e_2 \\ f_2 \end{pmatrix}}_{\text{the result is translated through } \mathbf{b}_2}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} + \begin{pmatrix} e_2 \\ f_2 \end{pmatrix}$$

b (x, y) is translated through $\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} + \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} e_1 + e_2 \\ f_1 + f_2 \end{pmatrix}$.

7 Suppose $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

Now the image of (x, y) translated through $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$ is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\therefore \quad \mathbf{x}' = \mathbf{x} + \mathbf{b}$$

$$\therefore \quad \mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{x} + \mathbf{b}$$

$$\therefore \quad \mathbf{A}\mathbf{x} = \mathbf{x}$$

$$\therefore \quad \mathbf{A} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \det \mathbf{A} = 1$$

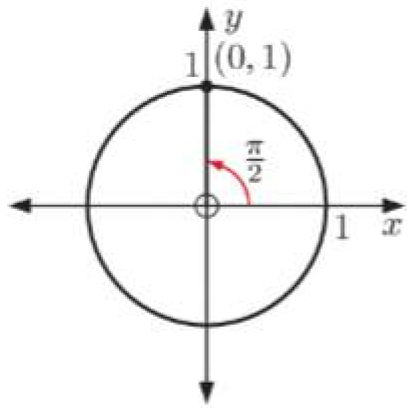
INVESTIGATION 1 ROTATIONS ABOUT THE ORIGIN

Transformation	$\mathbf{A}$	$\lvert\mathbf{A}\rvert$
Anticlockwise rotation about O through $\frac{\pi}{2}$.	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	1
Clockwise rotation about O through $\frac{\pi}{2}$.	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	1
Rotation about O through π.	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1
Rotation about O through 0.	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1

Each transformation matrix has determinant 1.

EXERCISE 14B

1 **a**

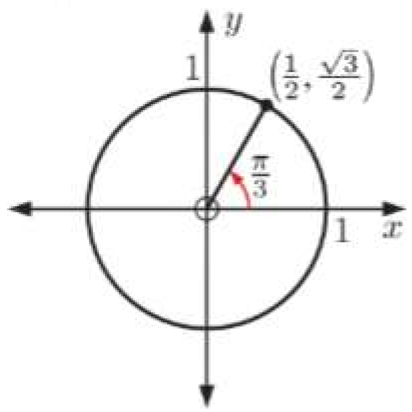


$\theta = \frac{\pi}{2}$ anticlockwise

$\therefore \quad \cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b

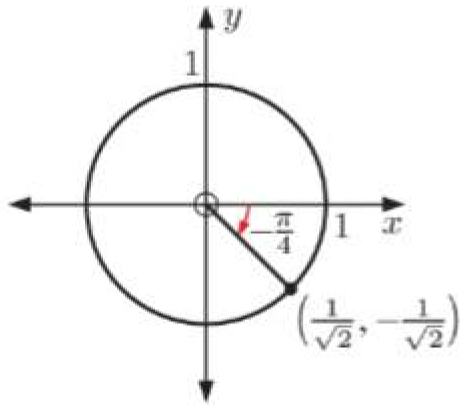


$\theta = \frac{\pi}{3}$ anticlockwise

$\therefore \quad \cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

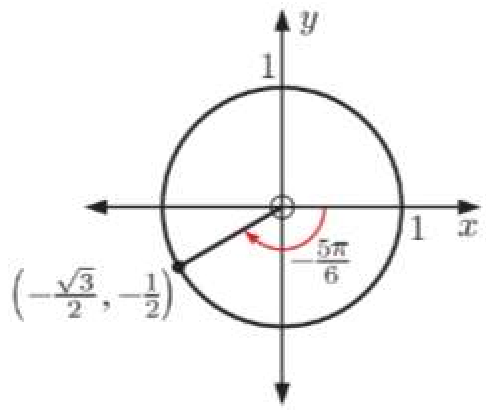
c



$\theta = \frac{\pi}{4}$ clockwise

$$\therefore \quad \cos\theta = \cos\left(-\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}},$$
$$\sin\theta = \sin\left(-\tfrac{\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}$$
$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

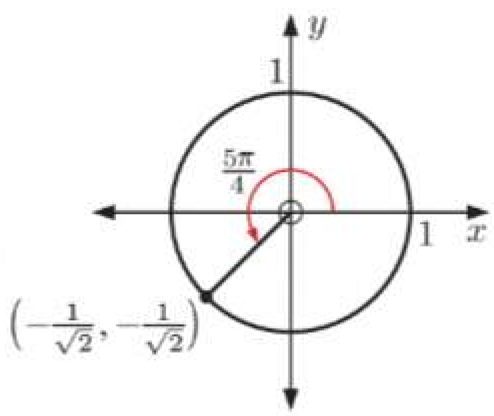
d



$\theta = \frac{5\pi}{6}$ clockwise

$$\therefore \quad \cos\theta = \cos\left(-\tfrac{5\pi}{6}\right) = -\tfrac{\sqrt{3}}{2},$$
$$\sin\theta = \sin\left(-\tfrac{5\pi}{6}\right) = -\tfrac{1}{2}$$
$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\left(-\frac{5\pi}{6}\right) & -\sin\left(-\frac{5\pi}{6}\right) \\ \sin\left(-\frac{5\pi}{6}\right) & \cos\left(-\frac{5\pi}{6}\right) \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

e



$\theta = \frac{5\pi}{4}$ anticlockwise

$$\therefore \quad \cos\theta = \cos\tfrac{5\pi}{4} = -\tfrac{1}{\sqrt{2}},$$
$$\sin\theta = \sin\tfrac{5\pi}{4} = -\tfrac{1}{\sqrt{2}}$$
$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\frac{5\pi}{4} & -\sin\frac{5\pi}{4} \\ \sin\frac{5\pi}{4} & \cos\frac{5\pi}{4} \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

2 **a** $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ where $|\,\mathbf{A}\,| = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $|\,\mathbf{A}\,| = 1$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$, so θ is in quadrant 4.

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{1}{\sqrt{2}}\right)$$
$$= \tfrac{\pi}{4}$$

$\therefore$ the transformation is a clockwise rotation about O through $\frac{\pi}{4}$.

b $\mathbf{A} = \begin{pmatrix} -\frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{pmatrix}$ where $|\mathbf{A}| = \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1.$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $|\mathbf{A}| = 1$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = -\frac{5}{13}$ and $\sin\theta = -\frac{12}{13}$, so θ is in quadrant 3.

$\therefore\ \theta = \cos^{-1}\left(-\frac{5}{13}\right)$

$\approx 1.97^c$

$\therefore$ the transformation is a clockwise rotation about O through about 1.97^c.

3 **a** $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the transformation matrix for an anticlockwise rotation about O through angle θ.

We replace θ by $(-\theta)$ in $\mathbf{A}$.

$\therefore$ the transformation matrix for an anticlockwise rotation about O through angle $-\theta$

$$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \{\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta\}$$

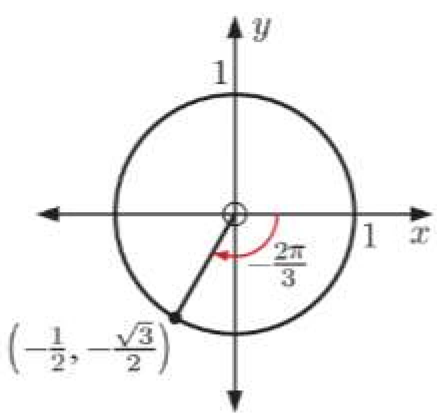
b $$\mathbf{A}^{-1} = \frac{1}{\cos^2\theta - (-\sin^2\theta)} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \{\cos^2\theta + \sin^2\theta = 1\}$$

c The inverse of the transformation matrix for an anticlockwise rotation about O through θ is the matrix for an anticlockwise rotation about O through $-\theta$.

4 **a** $\theta = -\frac{2\pi}{3}$ anticlockwise

$\therefore\ \cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$



$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos\left(-\frac{2\pi}{3}\right) & -\sin\left(-\frac{2\pi}{3}\right) \\ \sin\left(-\frac{2\pi}{3}\right) & \cos\left(-\frac{2\pi}{3}\right) \end{pmatrix}$$

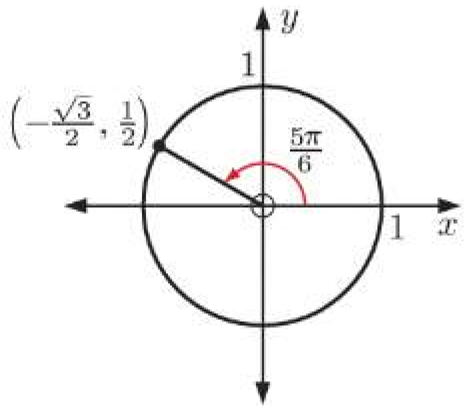
$$= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{2} - \frac{\sqrt{3}}{2} \\ -\frac{5\sqrt{3}}{2} + \frac{1}{2} \end{pmatrix}$$

$\therefore$ the image is $\left(\dfrac{-5-\sqrt{3}}{2}, \dfrac{-5\sqrt{3}+1}{2}\right)$.

b $\theta = \frac{5\pi}{6}$ anticlockwise



$\therefore \quad \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos \frac{5\pi}{6} & -\sin \frac{5\pi}{6} \\ \sin \frac{5\pi}{6} & \cos \frac{5\pi}{6} \end{pmatrix}$$

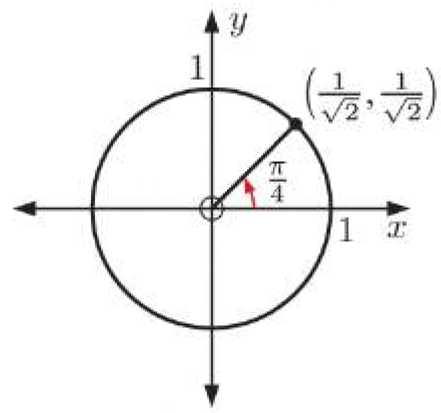
$$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3\sqrt{3}}{2} - \frac{1}{2} \\ -\frac{3}{2} - \frac{\sqrt{3}}{2} \end{pmatrix}$$

$\therefore$ the image is $\left(\dfrac{3\sqrt{3}-1}{2}, \dfrac{-3-\sqrt{3}}{2}\right)$.

c $\theta = \frac{\pi}{4}$ anticlockwise



$\therefore \quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}$$

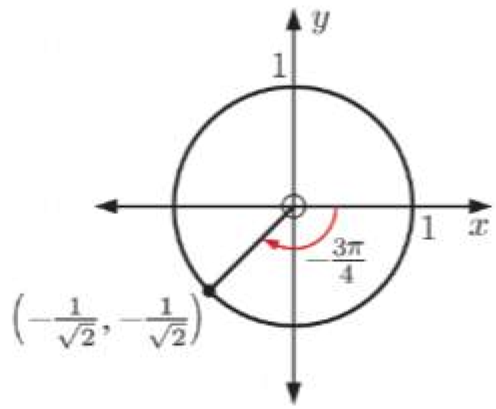
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ the image is $\left(-\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$.

d $\theta = -\frac{3\pi}{4}$ anticlockwise



$\therefore$ $\cos\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$\therefore \ \mathbf{A} = \begin{pmatrix} \cos\left(-\frac{3\pi}{4}\right) & -\sin\left(-\frac{3\pi}{4}\right) \\ \sin\left(-\frac{3\pi}{4}\right) & \cos\left(-\frac{3\pi}{4}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ the image is $\left(-\frac{3}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$.

INVESTIGATION 2 REFLECTIONS

Transformation	**A**	$\lvert\mathbf{A}\rvert$
Reflection in the x-axis.	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	-1
Reflection in the y-axis.	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	-1
Reflection in the line $y = x$.	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	-1
Reflection in the line $y = -x$.	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	-1

Each transformation matrix has determinant -1.

EXERCISE 14C

1 **a** For a reflection in the y-axis, $\alpha = \frac{\pi}{2}$ $\therefore$ $2\alpha = \pi$

$\therefore$ $\cos 2\alpha = -1$ and $\sin 2\alpha = 0$.

$$\therefore \ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

b For a reflection in the line $y = -x$, $m = \tan\alpha = -1$

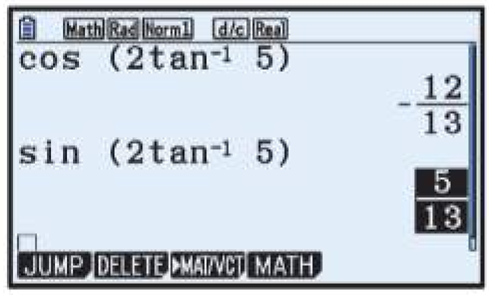
$\therefore\ \alpha = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore\ 2\alpha = -\frac{\pi}{2}$

$\therefore\ \cos 2\alpha = 0$ and $\sin 2\alpha = -1$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

c For a reflection in the line $y = 5x$, $m = \tan\alpha = 5$



$\therefore\ \alpha = \tan^{-1}(5)$

$\therefore\ \cos 2\alpha = -\frac{12}{13}$ and $\sin 2\alpha = \frac{5}{13}$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$$

d For a reflection in the line $y = \sqrt{3}x$, $m = \tan\alpha = \sqrt{3}$

$\therefore\ \alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\therefore\ 2\alpha = \frac{2\pi}{3}$

$\therefore\ \cos 2\alpha = -\frac{1}{2}$ and $\sin 2\alpha = \frac{\sqrt{3}}{2}$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

2 **a** $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ where $|\,\mathbf{A}\,| = -\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = -1.$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $|\,\mathbf{A}\,| = -1$, $\mathbf{A}$ is a reflection matrix where $\cos 2\alpha = \frac{1}{\sqrt{2}}$ and $\sin 2\alpha = \frac{1}{\sqrt{2}}$.

$\therefore\ \tan 2\alpha = 1$ and $0 < 2\alpha < \frac{\pi}{2}$

$\therefore\ 2\alpha = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore\ \alpha = \frac{\pi}{8}$

$\therefore\ \tan\alpha = \tan\frac{\pi}{8}$

$\therefore$ the transformation is a reflection in the line $y = \left(\tan\frac{\pi}{8}\right)x$.

b $\mathbf{A} = \begin{pmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{pmatrix}$ where $|\mathbf{A}| = -\left(\frac{15}{17}\right)^2 - \left(\frac{8}{17}\right)^2 = -1.$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $|\mathbf{A}| = -1$, $\mathbf{A}$ is a reflection matrix where

$\cos 2\alpha = -\frac{15}{17}$ and $\sin 2\alpha = \frac{8}{17}$.

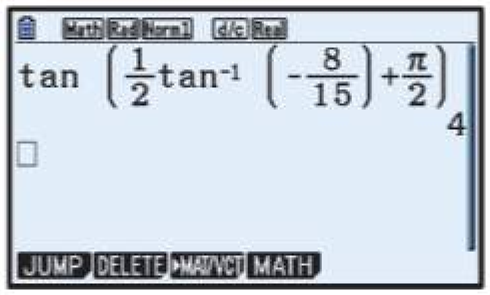
$\therefore\ \tan 2\alpha = -\frac{8}{15}$ and $\frac{\pi}{2} < 2\alpha < \pi$

$\therefore\ \tan(2\alpha - \pi) = -\frac{8}{15},\ \ -\frac{\pi}{2} < 2\alpha - \pi < 0$

$\therefore\ 2\alpha - \pi = \tan^{-1}\left(-\frac{8}{15}\right)$

$\therefore\ \alpha = \frac{1}{2}\tan^{-1}\left(-\frac{8}{15}\right) + \frac{\pi}{2}$

$\therefore\ \tan\alpha = 4$



$\therefore$ the transformation is a reflection in the line $y = 4x$.

3 Let $\mathbf{A} = \begin{pmatrix} s & s \\ t+\sqrt{2} & t \end{pmatrix}$ where $s, t \in \mathbb{R}$.

a If the transformation is a rotation about O then $|\mathbf{A}| = 1$

$\therefore\ st - s(t+\sqrt{2}) = 1$

$\therefore\ st - st - \sqrt{2}s = 1$

$\therefore\ s = -\frac{1}{\sqrt{2}}$

In $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$ $\therefore\ \theta = \frac{3\pi}{4}$

$\therefore\ t = -\frac{1}{\sqrt{2}}$ and $t + \sqrt{2} = -\frac{1}{\sqrt{2}} + \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$

So, the transformation could be an anticlockwise rotation about O through $\frac{3\pi}{4}$.

b If the transformation is a reflection in $y = (\tan\alpha)x$ then $|\mathbf{A}| = -1$

$\therefore\ st - s(t+\sqrt{2}) = -1$

$\therefore\ st - st - \sqrt{2}s = -1$

$\therefore\ s = \frac{1}{\sqrt{2}}$

In $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$, $\cos 2\alpha = \frac{1}{\sqrt{2}}$ and $\sin 2\alpha = \frac{1}{\sqrt{2}}$

$\therefore\ t = -\frac{1}{\sqrt{2}}$ and $2\alpha = \frac{\pi}{4}$

$\therefore\ \alpha = \frac{\pi}{8}$

So, the transformation could be a reflection in the line $y = \left(\tan\frac{\pi}{8}\right)x$.

4 For a reflection in the line $y = (\tan\alpha)x$,

$$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \quad \text{where} \quad |\mathbf{A}| = -1$$

$$\therefore \quad \mathbf{A}^{-1} = \tfrac{1}{-1}\begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \qquad \{-\cos^2 2\alpha - \sin^2 2\alpha = -1\}$$

$$= \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

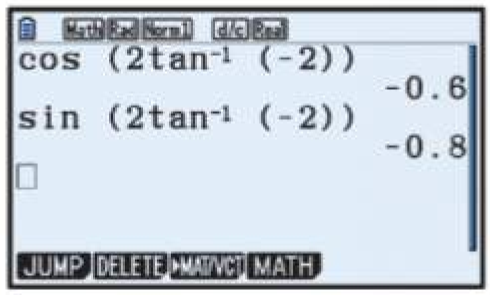
$$= \mathbf{A}$$

A is its own inverse. If an object is reflected in $y = (\tan\alpha)x$, then the resulting image is reflected in $y = (\tan\alpha)x$, then we obtain the original object.

5 For a reflection in the line $y = -2x$, $m = \tan\alpha = -2$

$\therefore \quad \alpha = \tan^{-1}(-2)$

$\therefore \quad \cos 2\alpha = -\frac{3}{5}$ and $\sin 2\alpha = -\frac{4}{5}$



$\therefore$ the transformation matrix

$$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

a $\mathbf{x}' = \mathbf{A}\begin{pmatrix} -4 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} \\ \frac{22}{5} \end{pmatrix}$$

$\therefore \quad (-4, 2)$ has image $\left(\frac{4}{5}, \frac{22}{5}\right)$.

b $\mathbf{x}' = \mathbf{A}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore \quad (3, -1)$ has image $(-1, -3)$.

c $\mathbf{x}' = \mathbf{A}\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$\therefore \quad (-2, 4)$ has image $(-2, 4)$.

EXERCISE 14D

1 **a** $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$ is a horizontal stretch with scale factor $\frac{1}{2}$.

b $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}$ is a vertical stretch with scale factor 2.

c $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$ is a horizontal stretch with scale factor 3.

d $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\mathbf{x}$ is a vertical stretch with scale factor $\frac{1}{4}$.

2 **a** A horizontal stretch with scale factor $\frac{5}{2}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & 1 \end{pmatrix}$.

b $$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ 4 \end{pmatrix}$$

$\therefore$ $(3, 4)$ has image $\left(\frac{15}{2}, 4\right)$.

c $\mathbf{A}^{-1} = \frac{1}{\frac{5}{2}}\begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore$ $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$ is a horizontal stretch with scale factor $\frac{2}{5}$.

3 **a** A vertical stretch with scale factor $\frac{3}{4}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$.

b $$\mathbf{x}' = \mathbf{A}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix}$$

$\therefore$ $(-1, 2)$ has image $\left(-1, \frac{3}{2}\right)$.

c $\mathbf{A}^{-1} = \frac{1}{\frac{3}{4}}\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$

$\therefore$ $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$ is a vertical stretch with scale factor $\frac{4}{3}$.

EXERCISE 14E

1 **a** $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x}$ is an enlargement with scale factor 3.

b $\mathbf{x}' = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\mathbf{x}$ is a reduction with scale factor $\frac{1}{4}$.

2 **a** An enlargement with scale factor $\frac{4}{3}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$.

b $$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$\therefore$ $(3, 6)$ has image $(4, 8)$.

c $\mathbf{A}^{-1} = \frac{1}{\frac{16}{9}}\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$

$\therefore$ $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$ is a reduction with scale factor $\frac{3}{4}$.

3 **a** A horizontal stretch with scale factor k has the transformation matrix $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

Now $\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} kx \\ y \end{pmatrix}$

b A vertical stretch with scale factor k has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

Now $\mathbf{x}'' = \mathbf{A}\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} kx \\ y \end{pmatrix}$

$= \begin{pmatrix} kx \\ ky \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

This is the transformation matrix for an enlargement with scale factor k.
So, a horizontal stretch with scale factor k followed by a vertical stretch with scale factor k, is equivalent to an enlargement with scale factor k.

EXERCISE 14F

1 **a** A reflection in the line $y = x$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For an anticlockwise rotation about O through $\frac{\pi}{3}$, we have $\theta = \frac{\pi}{3}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$.

$\therefore$ the rotation has transformation matrix $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

b **i** $\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

$\therefore$ **AB** has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = (-1)(1) = -1$

$\therefore$ the composition of these transformations is a reflection.

If the reflection is in the line $y = (\tan\alpha)x$ then

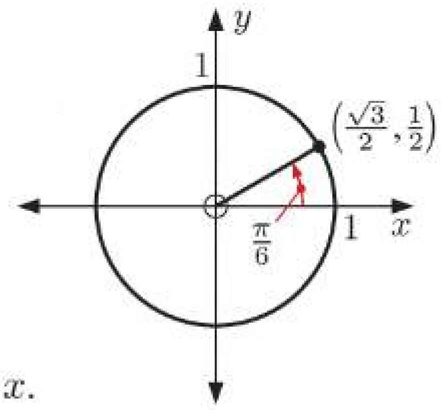
$\cos 2\alpha = \frac{\sqrt{3}}{2}$ and $\sin 2\alpha = \frac{1}{2}$

$\therefore\ 2\alpha = \frac{\pi}{6}$ $\quad \{0 < 2\alpha < \frac{\pi}{2}\}$

$\therefore\ \alpha = \frac{\pi}{12}$

$\therefore\ \tan\alpha = \tan\frac{\pi}{12}$

$\therefore\ \mathbf{x}' = \mathbf{ABx}$ is a reflection in the line $y = \left(\tan\frac{\pi}{12}\right)x$.



ii $\mathbf{BA} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$\therefore$ **BA** has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}||\mathbf{A}| = (1)(-1) = -1$

$\therefore$ the composition of these transformations is a reflection.

If the reflection is in the line $y = (\tan\alpha)x$ then

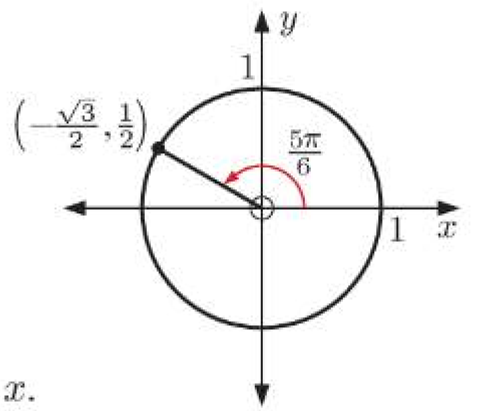
$\cos 2\alpha = -\frac{\sqrt{3}}{2}$ and $\sin 2\alpha = \frac{1}{2}$

$\therefore\ 2\alpha = \frac{5\pi}{6}$ $\quad \{\frac{\pi}{2} < 2\alpha < \pi\}$

$\therefore\ \alpha = \frac{5\pi}{12}$

$\therefore\ \tan\alpha = \tan\frac{5\pi}{12}$

$\therefore\ \mathbf{x}' = \mathbf{BAx}$ is a reflection in the line $y = \left(\tan\frac{5\pi}{12}\right)x$.



c In general for 2×2 matrices **A** and **B**, $\mathbf{AB} \neq \mathbf{BA}$, so the corresponding linear transformations are different.

2 **a** A reflection in the x-axis has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

An anticlockwise rotation of $\frac{\pi}{2}$ about O has the transformation matrix $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This is the transformation matrix for a reflection in the line $y = x$.

$\therefore$ the composite transformation is a reflection in the line $y = x$.

b For an anticlockwise rotation through $\frac{2\pi}{3}$ about O, we have $\theta = \frac{2\pi}{3}$, $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$.

$\therefore$ the rotation has the transformation matrix $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

For the reflection in the line $y = -x$, $\tan\alpha = -1$

$$\therefore \alpha = -\tfrac{\pi}{4}$$
$$\therefore 2\alpha = -\tfrac{\pi}{2}$$

So, $\cos 2\alpha = 0$ and $\sin 2\alpha = -1$.

$\therefore$ the reflection has the transformation matrix $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

From **1 b ii**, this is the transformation matrix for a reflection in the line $y = \left(\tan\frac{5\pi}{12}\right)x$.

$\therefore$ the composite transformation is a reflection in the line $y = \left(\tan\frac{5\pi}{12}\right)x$.

c For a reflection in the line $y = \sqrt{3}x$, $\tan\alpha = \sqrt{3}$

$$\therefore \alpha = \tfrac{\pi}{3}$$
$$\therefore 2\alpha = \tfrac{2\pi}{3}$$

So, $\cos 2\alpha = -\frac{1}{2}$ and $\sin 2\alpha = \frac{\sqrt{3}}{2}$.

$\therefore$ the reflection has the transformation matrix $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

A reflection in the y-axis has the transformation matrix $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

From **1 a**, this is the transformation matrix for an anticlockwise rotation about O through $\frac{\pi}{3}$.

$\therefore$ the composite transformation is an anticlockwise rotation about O through $\frac{\pi}{3}$.

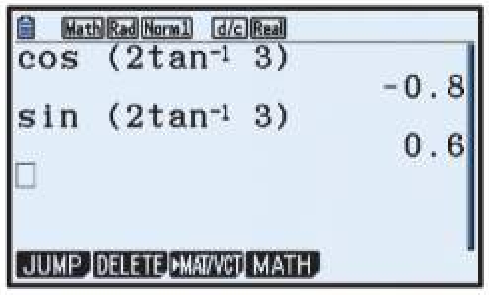
d A reflection in the line $y = x$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For a reflection in the line $y = 3x$, $\tan\alpha = 3$

$\therefore \cos 2\alpha = \cos(2\tan^{-1} 3) = -\frac{4}{5}$

and $\sin 2\alpha = \sin(2\tan^{-1} 3) = \frac{3}{5}$

$\therefore$ the reflection has transformation matrix $\mathbf{B} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$.



Now $\mathbf{BA} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$\therefore$ **BA** has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}||\mathbf{A}| = (-1)(-1) = 1$

$\therefore$ the composition of these transformations is a rotation.

If the angle of rotation is θ, $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$, so θ is in quadrant 1.

$\therefore \theta = \cos^{-1}\left(\frac{3}{5}\right)$

$\approx 0.927^c$

$\therefore$ the composite transformation is an anticlockwise rotation about O through about 0.927^c.

3 **a** A horizontal stretch with scale factor k has the transformation matrix $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

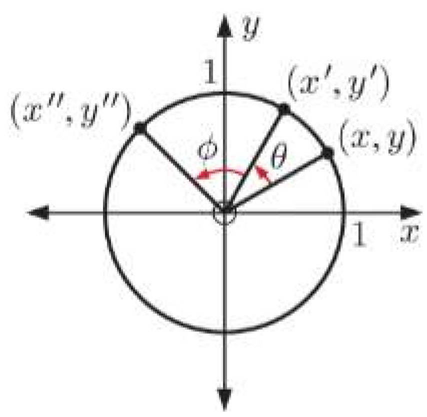
A vertical stretch with scale factor k has the transformation matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

From **Exercise 14E 3 c**, $\mathbf{BA} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ is the transformation matrix for an enlargement with scale factor k.

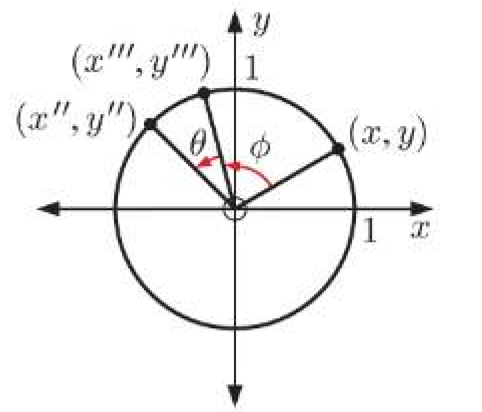
Now $\mathbf{AB} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

So, the composite transformation is the same if the order of transformations is reversed.

b When (x, y) is rotated about O through θ, then rotated about O through ϕ, it has been rotated about O through $\theta + \phi$.



When $(x,\ y)$ is rotated about O through ϕ, then rotated about O through θ, it has been rotated about O through $\phi+\theta$ or $\theta+\phi$.



So, the composite transformation is a rotation about O through $\theta+\phi$ which is the same if the order of transformations is reversed.

c A reflection in the x-axis has the transformation matrix $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

A rotation about O through θ has the transformation matrix $\mathbf{B}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.

$$\text{Now}\quad \mathbf{BA}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
$$=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$
$$=\begin{pmatrix}\cos\left(2\left(\frac{\theta}{2}\right)\right) & \sin\left(2\left(\frac{\theta}{2}\right)\right)\\ \sin\left(2\left(\frac{\theta}{2}\right)\right) & -\cos\left(2\left(\frac{\theta}{2}\right)\right)\end{pmatrix}$$

which is the transformation matrix for a reflection in the line $y=\left(\tan\frac{\theta}{2}\right)x$.

$$\text{Also}\quad \mathbf{AB}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$
$$=\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}$$
$$=\begin{pmatrix}\cos(-\theta) & \sin(-\theta)\\ \sin(-\theta) & -\cos(-\theta)\end{pmatrix}\qquad \{\cos(-\theta)=\cos\theta,\ \ \sin(-\theta)=-\sin\theta\}$$
$$=\begin{pmatrix}\cos\left(2\left(-\frac{\theta}{2}\right)\right) & \sin\left(2\left(-\frac{\theta}{2}\right)\right)\\ \sin\left(2\left(-\frac{\theta}{2}\right)\right) & -\cos\left(2\left(-\frac{\theta}{2}\right)\right)\end{pmatrix}$$

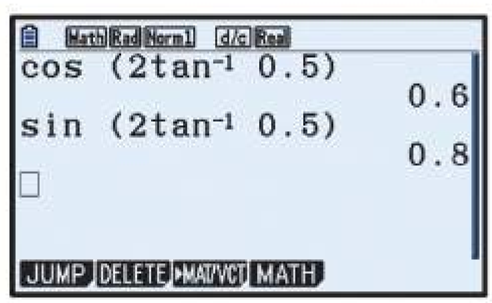
which is the transformation matrix for a reflection in the line $y=\left(\tan\left(-\frac{\theta}{2}\right)\right)x$.

So, for $\theta\neq k\pi,\ \ k\in\mathbb{Z}$, the composite transformation is different if the order of the transformations is reversed.

4 **a** For the reflection in the line $y=\frac{1}{2}x$, $\tan\alpha=\frac{1}{2}$

$\therefore\ \cos 2\alpha=\cos\left(2\tan^{-1}\frac{1}{2}\right)=\frac{3}{5}$

and $\sin 2\alpha=\sin\left(2\tan^{-1}\frac{1}{2}\right)=\frac{4}{5}$



$\therefore$ the reflection has the transformation matrix $\begin{pmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{pmatrix}$.

So, a reflection in the line $y=\frac{1}{2}x$ followed by a translation through $\begin{pmatrix}3\\ -1\end{pmatrix}$ has matrix equation $\mathbf{x}'=\begin{pmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{pmatrix}\mathbf{x}+\begin{pmatrix}3\\ -1\end{pmatrix}$.

b For an anticlockwise rotation through $\frac{\pi}{6}$ about O, we have $\theta = \frac{\pi}{6}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$.

$\therefore$ the rotation has the transformation matrix $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

So, an anticlockwise rotation through $\frac{\pi}{6}$ about O followed by a translation through $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

c A horizontal stretch with scale factor $\frac{1}{3}$ has the transformation matrix $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$.

So, a horizontal stretch with scale factor $\frac{1}{3}$ followed by a translation through $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

5 A reflection in the line $y = (\tan\alpha)x$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.

An anticlockwise rotation about O through θ has the transformation matrix $\mathbf{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

$$\begin{aligned}
\text{Now}\quad \mathbf{BA} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos 2\alpha - \sin\theta\sin 2\alpha & \cos\theta\sin 2\alpha + \sin\theta\cos 2\alpha \\ \sin\theta\cos 2\alpha + \cos\theta\sin 2\alpha & \sin\theta\sin 2\alpha - \cos\theta\cos 2\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos 2\alpha - \sin\theta\sin 2\alpha & \cos\theta\sin 2\alpha + \sin\theta\cos 2\alpha \\ \cos\theta\sin 2\alpha + \sin\theta\cos 2\alpha & -(\cos\theta\cos 2\alpha - \sin\theta\sin 2\alpha) \end{pmatrix}
\end{aligned}$$

$\therefore$ **BA** has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

$$\begin{aligned}
\text{with}\quad |\mathbf{BA}| &= -(\cos\theta\cos 2\alpha - \sin\theta\sin 2\alpha)^2 - (\cos\theta\sin 2\alpha + \sin\theta\cos 2\alpha)^2 \\
&= -(\cos^2\theta\cos^2 2\alpha - \cancel{2\cos\theta\sin\theta\cos 2\alpha\sin 2\alpha} + \sin^2\theta\sin^2 2\alpha) \\
&\qquad - (\cos^2\theta\sin^2 2\alpha + \cancel{2\cos\theta\sin\theta\cos 2\alpha\sin 2\alpha} + \sin^2\theta\sin^2 2\alpha) \\
&= -\cos^2\theta\cos^2 2\alpha - \sin^2\theta\sin^2 2\alpha - \cos^2\theta\sin^2 2\alpha - \sin^2\theta\sin^2 2\alpha \\
&= -\cos^2\theta(\cos^2 2\alpha + \sin^2 2\alpha) - \sin^2\theta(\cos^2 2\alpha + \sin^2 2\alpha) \\
&= -\cos^2\theta - \sin^2\theta \\
&= -(\cos^2\theta + \sin^2\theta) \\
&= -1
\end{aligned}$$

$\therefore$ the composition of these transformations is a reflection in another line through O.

6 **a** An enlargement with scale factor 2 has the transformation matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

So, a translation through $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ followed by an enlargement with scale factor 2 has matrix equation $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\left(\mathbf{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right)$.

b $$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\therefore \mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ -4 \end{pmatrix}$ which is of the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

This is the affine transformation composed of an enlargement with scale factor k followed by a translation through $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$.

c The composition of a translation through **c** followed by a linear transformation with transformation matrix **A** has matrix equation

$\mathbf{x}' = \mathbf{A}(\mathbf{x} + \mathbf{c})$

$\therefore \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{c}$ which is an affine transformation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

7 Let the unknown transformation have the transformation matrix **A**.

A clockwise rotation about O through $\frac{\pi}{2}$ has the transformation matrix $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From **4 a**, a reflection in $y = \frac{1}{2}x$ has the transformation matrix $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$$\begin{aligned} \therefore \mathbf{A} &= \mathbf{B}^{-1}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \\ &= \frac{1}{1} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \end{aligned}$$

From **2 d**, this is the transformation matrix for a reflection in the line $y = 3x$.

$\therefore$ the unknown transformation is a reflection in the line $y = 3x$.

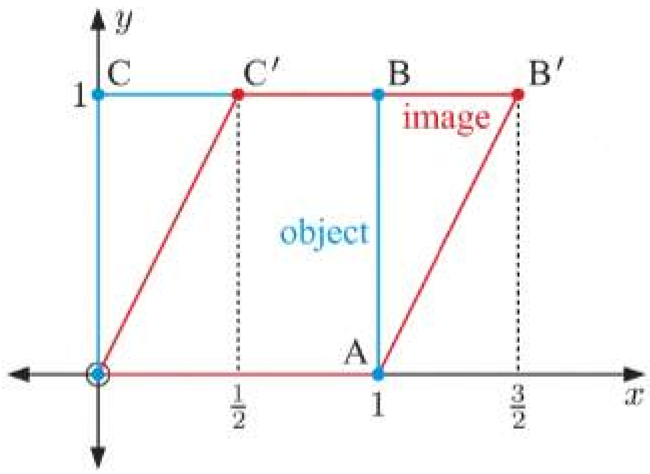
INVESTIGATION 3 PROPERTIES OF LINEAR TRANSFORMATIONS

1 For any matrix $\mathbf{A}$, $\mathbf{A0} = \mathbf{0}$.

$\therefore$ the point $(0, 0)$ is invariant under any linear transformation.

2 **a** The shear has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$.

Using **1**, O$(0, 0)$ is invariant under the shear.



$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\therefore$ A$(1, 0)$ is invariant under the shear.

$$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

$\therefore$ the image of B$(1, 1)$ is B$'(\frac{3}{2}, 1)$.

$$\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$\therefore$ the image of C$(0, 1)$ is C$'(\frac{1}{2}, 1)$.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ 0 & \lambda - 1 \end{vmatrix} = 0$

$$\therefore (\lambda - 1)^2 = 0$$

$$\therefore \lambda = 1$$

$\therefore$ the transformation matrix has eigenvalue 1.

For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -\tfrac{1}{2}b = 0$$

$$\therefore b = 0$$

Let $a = t$, $t \neq 0$

$$\therefore \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}t, \quad t \neq 0$$

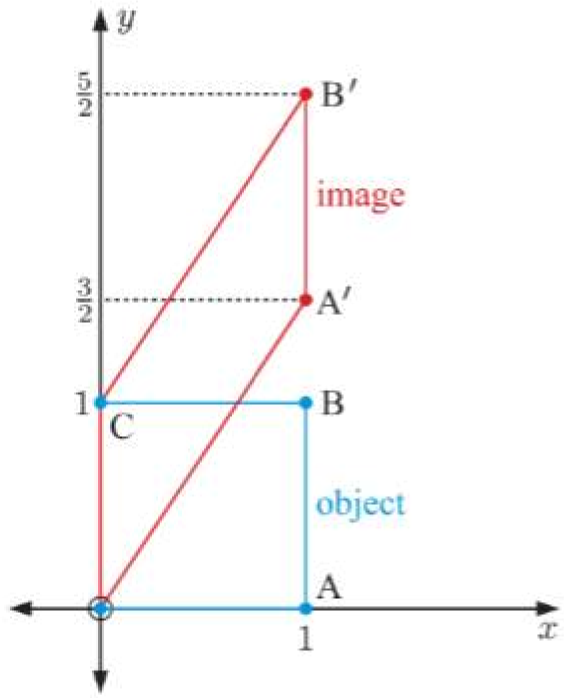
Any vector of the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 1.

c If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1, then $\mathbf{Ax} = \mathbf{x}$.

$\therefore$ $\mathbf{x}$ is invariant under the shear.

3 **a** The shear has transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$.

Using **1**, O(0, 0) is invariant under the shear.



$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

$\therefore$ the image of A(1, 0) is A′(1, $\frac{3}{2}$).

$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \end{pmatrix}$

$\therefore$ the image of B(1, 1) is B′(1, $\frac{5}{2}$).

$\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ C(0, 1) is invariant under the shear.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 1 & 0 \\ -\frac{3}{2} & \lambda - 1 \end{vmatrix} = 0$

$$\therefore (\lambda - 1)^2 = 0$$
$$\therefore \lambda = 1$$

$\therefore$ the transformation matrix has eigenvalue 1.

For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 & 0 \\ -\frac{3}{2} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore -\tfrac{3}{2}a = 0$$
$$\therefore a = 0$$

Let $b = t$, $t \neq 0$

$\therefore \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$, $t \neq 0$

Any vector of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 1.

c If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1, then $\mathbf{Ax} = \mathbf{x}$.

$\therefore$ $\mathbf{x}$ is invariant under the shear.

This agrees with our answer to **2 c**.

4 a i The transformation matrix is $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}$.

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 5 & -4 \\ -3 & \lambda - 1 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 5)(\lambda - 1) - 12 = 0$$
$$\therefore\ \lambda^2 - 6\lambda + 5 - 12 = 0$$
$$\therefore\ \lambda^2 - 6\lambda - 7 = 0$$
$$\therefore\ (\lambda - 7)(\lambda + 1) = 0$$
$$\therefore\ \lambda = 7 \text{ or } -1$$

$\therefore$ the transformation matrix has eigenvalues 7 and -1.

For $\lambda = 7$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 2a - 4b = 0$$
$$\therefore\ a - 2b = 0$$

Letting $b = t,\ t \neq 0$ then $a = 2t$

$\therefore\ \mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t,\ t \neq 0$

Any vector of the form $\begin{pmatrix} 2 \\ 1 \end{pmatrix} t,\ t \neq 0$ is an eigenvector corresponding to the eigenvalue 7.

For $\lambda = -1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -6 & -4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -3a - 2b = 0$$

Letting $b = t,\ t \neq 0$ then $a = -\frac{2}{3}t$

$\therefore\ \mathbf{x} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} t,\ t \neq 0$

Any vector of the form $\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} t,\ t \neq 0$ is an eigenvector corresponding to the eigenvalue -1.

ii The transformation matrix is $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = 0$

$$\therefore \ (\lambda - 1)^2 - 4 = 0$$
$$\therefore \ (\lambda - 1)^2 = 4$$
$$\therefore \ \lambda - 1 = \pm 2$$
$$\therefore \ \lambda = 3 \text{ or } -1$$

$\therefore$ the transformation matrix has eigenvalues 3 and -1.

For $\lambda = 3$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \ \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \ 2a - 2b = 0$$
$$\therefore \ a - b = 0$$

Letting $a = t$, $t \neq 0$ then $b = t$

$\therefore \ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \ t \neq 0$

Any vector of the form $\begin{pmatrix} 1 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 3.

For $\lambda = -1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \ \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \ -2a - 2b = 0$$
$$\therefore \ a + b = 0$$

Letting $b = t$, $t \neq 0$ then $a = -t$

$\therefore \ \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \ t \neq 0$

Any vector of the form $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue -1.

b **i** If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 7, then $\mathbf{Ax} = 7\mathbf{x}$ which is also an eigenvector corresponding to the eigenvalue 7.

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1, then $\mathbf{Ax} = -\mathbf{x}$ which is also an eigenvector corresponding to the eigenvalue -1.

ii If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3, then $\mathbf{Ax} = 3\mathbf{x}$ which is also an eigenvector corresponding to the eigenvalue 3.

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1, then $\mathbf{Ax} = -\mathbf{x}$ which is also an eigenvector corresponding to the eigenvalue -1.

In each case, the eigenvectors corresponding to λ are transformed to eigenvectors corresponding to λ.

5 $\mathbf{x}' = \mathbf{A}\mathbf{x}$ moves the point P to the point P$'$.

$\therefore\ \mathbf{A}(\overrightarrow{OP}) = \overrightarrow{OP'}$

($\Rightarrow$) If P$'$ lies on (OP), then $\overrightarrow{OP'} = \lambda(\overrightarrow{OP})$ for some $\lambda \in \mathbb{R}$.

$\therefore\ \mathbf{A}(\overrightarrow{OP}) = \overrightarrow{OP'} = \lambda(\overrightarrow{OP})$

So, $\overrightarrow{OP}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

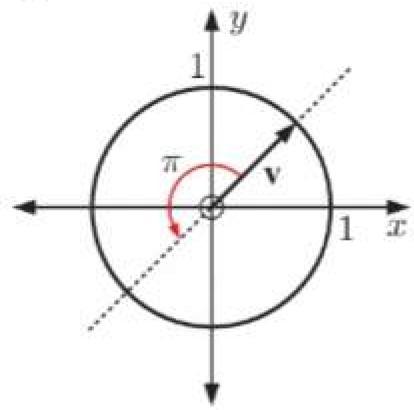
($\Leftarrow$) If $\overrightarrow{OP}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{A}(\overrightarrow{OP}) = \lambda(\overrightarrow{OP})$.

But $\mathbf{A}(\overrightarrow{OP}) = \overrightarrow{OP'}$, so $\overrightarrow{OP'} = \lambda(\overrightarrow{OP})$.

$\therefore$ P$'$ lies on the line (OP).

6 $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

a The image point with position vector $\mathbf{v}'$ can be located by a scalar multiple of $\mathbf{v}$ if it lies on the line with direction vector $\mathbf{v}$ which passes through the origin.
The only rotations which make this possible are when $\theta = k\pi,\ k \in \mathbb{Z}$.



b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = 0$

$$\therefore\ (\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

$$\therefore\ \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$$

$$\therefore\ \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\begin{aligned}\therefore\ \lambda &= \frac{2\cos\theta \pm \sqrt{(-2\cos\theta)^2 - 4(1)(1)}}{2} \\ &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\ &= \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2} \\ &= \cos\theta \pm \sqrt{-\sin^2\theta} \\ &= \cos\theta \pm i\sin\theta \\ &= \cos(\pm\theta) + i\sin(\pm\theta) \\ &= e^{\pm i\theta}\end{aligned}$$

$\therefore$ the eigenvalues of $\mathbf{A}$ are $e^{\pm i\theta}$.

c As $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, $\mathbf{Av} = e^{\pm i\theta}\mathbf{v}$.

But $\mathbf{v}' = \mathbf{Av}$, so $\mathbf{v}' = e^{\pm i\theta}\mathbf{v}$.

So, $\mathbf{v}'$ is a scalar multiple of $\mathbf{v}$ if $e^{\pm i\theta}$ is real.

This occurs when $\sin\theta = 0$

$\therefore\ \theta = k\pi,\ k \in \mathbb{Z}$ as in **a**.

If k is even, $e^{\pm i\theta} = 1$ so 1 is a repeated eigenvalue.

If k is odd, $e^{\pm i\theta} = -1$ so -1 is a repeated eigenvalue.

d For the repeated eigenvalue 1, $\theta = k\pi$, k even.
This is equivalent to a rotation about O through 0.

So, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

For any non-zero vector $\mathbf{x}$, $\mathbf{Ax} = \mathbf{Ix} = \mathbf{x}$.
So, every non-zero vector in the plane is an eigenvector corresponding to the repeated eigenvalue 1.

For the repeated eigenvalue -1, $\theta = k\pi$, k odd.
This is equivalent to a rotation about O through π.

So, $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$.

For any non-zero vector $\mathbf{x}$, $\mathbf{Ax} = -\mathbf{Ix} = -\mathbf{x}$.
So, every non-zero vector in the plane is an eigenvector corresponding to the repeated eigenvalue -1.

ACTIVITY 1 REPEATED AFFINE TRANSFORMATIONS

1 **a** The transformation matrix $\mathbf{A} = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $|\mathbf{A}| = (0.6)^2 + (0.8)^2 = 1$.

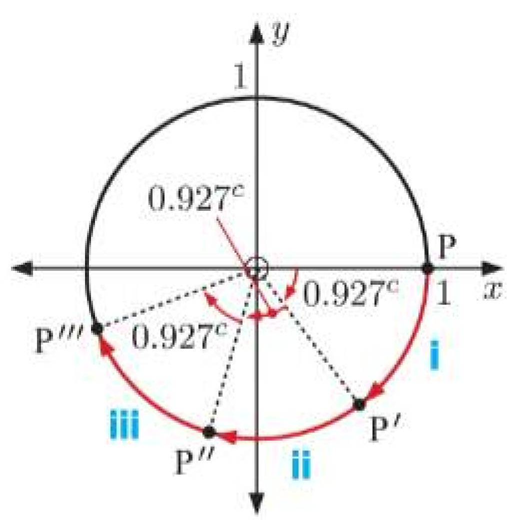
$\therefore$ the transformation is a rotation.

If the angle of rotation is θ, $\cos\theta = 0.6$ and $\sin\theta = -0.8$, so θ is in quadrant 4.

$\therefore\ \theta = \sin^{-1}(-0.8)$
$\approx -0.927^c$

$\therefore$ the transformation is a clockwise rotation about O through about 0.927^c.

b



c We predict that applying the transformation to successive image points will trace out the unit circle.

2 $\mathbf{A}_1 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}$

a $$\mathbf{A}_1\mathbf{A}_2 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{9}{20} & -\frac{9\sqrt{3}}{20} \\ \frac{9\sqrt{3}}{20} & \frac{9}{20} \end{pmatrix} \approx \begin{pmatrix} 0.45 & -0.779 \\ 0.779 & 0.45 \end{pmatrix}$$

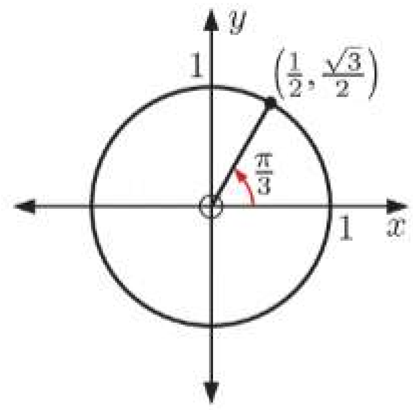
b $\mathbf{A}_1$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $|\mathbf{A}| = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

$\therefore$ $\mathbf{A}_1$ is the transformation matrix for a rotation.

If the angle of rotation is θ, $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, so θ is in quadrant 1.

$$\therefore \ \theta = \cos^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{3}$$

$\therefore$ $\mathbf{A}_1$ is the transformation matrix for an anticlockwise rotation about O through $\frac{\pi}{3}$.



$\mathbf{A}_2$ is the transformation matrix for a reduction with scale factor 0.9.

So, the composite transformation $\mathbf{x}' = \mathbf{A}_1\mathbf{A}_2\mathbf{x}$ is a reduction with scale factor 0.9 followed by an anticlockwise rotation about O through $\frac{\pi}{3}$.

c We predict that the point $P(1, 0)$ and its images will trace out an anticlockwise spiral which spirals *towards* the origin.

e **i** $\mathbf{A}_2 = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}$ is the transformation matrix for an enlargement with scale factor 1.1.

We predict that the point $P(1, 0)$ and its images will trace out an anticlockwise spiral which spirals *away* from the origin.

3 $\mathbf{A}_1 = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}$

a $$\mathbf{A}_1\mathbf{A}_2 = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix}\begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.54 & 0.72 \\ -0.72 & 0.54 \end{pmatrix}$$

b From **1 a**, $\mathbf{A}_1$ is the transformation matrix for a clockwise rotation about O through about 0.927^c.

So, the composite transformation $\mathbf{x}' = \mathbf{A}_1\mathbf{A}_2\mathbf{x}$ is a reduction with scale factor 0.9 followed by a clockwise rotation about O through about 0.927^c.

c We predict that the point $\mathrm{P}(1, 0)$ and its images will trace out a clockwise spiral which spirals *towards* the origin.

e **i** We predict that the point $\mathrm{P}(1, 0)$ and its images will trace out a clockwise spiral which spirals *away* from the origin.

4 $\mathbf{x}' = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & k \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

a When $k = 0.6$, the point $\mathrm{P}(1, 0)$ and its images appear to trace out a circle.

b For $k = 0.55$, the point $\mathrm{P}(1, 0)$ and its images appear to trace out a spiral which spirals inwards.

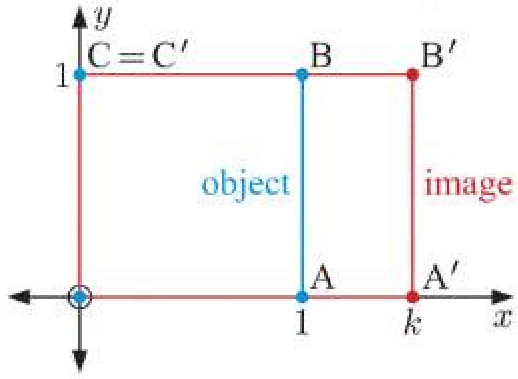
For $k = 0.65$, the point $\mathrm{P}(1, 0)$ and its images appear to trace out a spiral which spirals outwards.

INVESTIGATION 4 AREA

1 **a** Translations, rotations, and reflections will not change the size of an object.

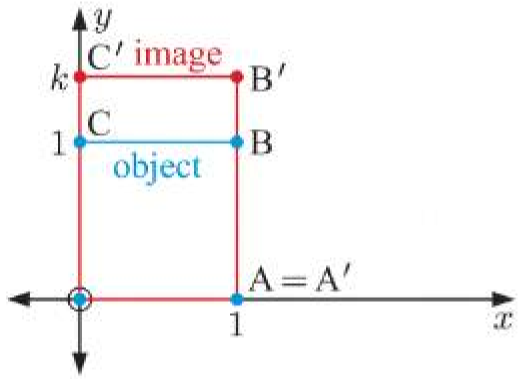
For a horizontal stretch with scale factor k,

$$\frac{\text{area of image}}{\text{area of object}} = \frac{k \times 1}{1} = k.$$



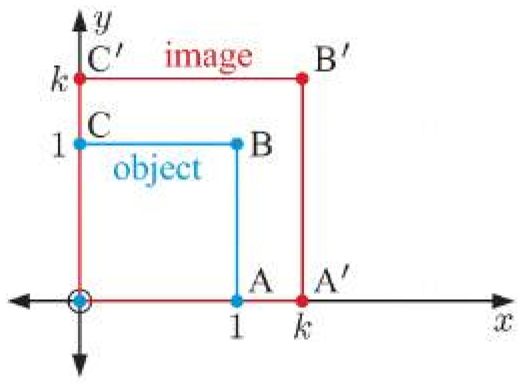
For a vertical stretch with scale factor k,

$$\frac{\text{area of image}}{\text{area of object}} = \frac{1 \times k}{1} = k.$$



For an enlargement with scale factor k,

$$\frac{\text{area of image}}{\text{area of object}} = \frac{k \times k}{1} = k^2.$$



b

		det $\mathbf{A}$
Translation through $\begin{pmatrix} e \\ f \end{pmatrix}$	$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$	1
Rotation about O through θ	$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	1
Reflection in $y = (\tan\alpha)x$	$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-1
Horizontal stretch with scale factor k	$\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	k
Vertical stretch with scale factor k	$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	k
Enlargement with scale factor k	$\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	k^2

The transformations which will not change the size of an object satisfy $\det \mathbf{A} = 1$ or -1.

The transformations which will change the size of an object satisfy $|\det \mathbf{A}| = \dfrac{\text{area of image}}{\text{area of object}}$.

2 **a** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

i $\mathbf{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ the image of O(0, 0) is O′(0, 0).

$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore$ the image of A(1, 0) is A′(1, 2).

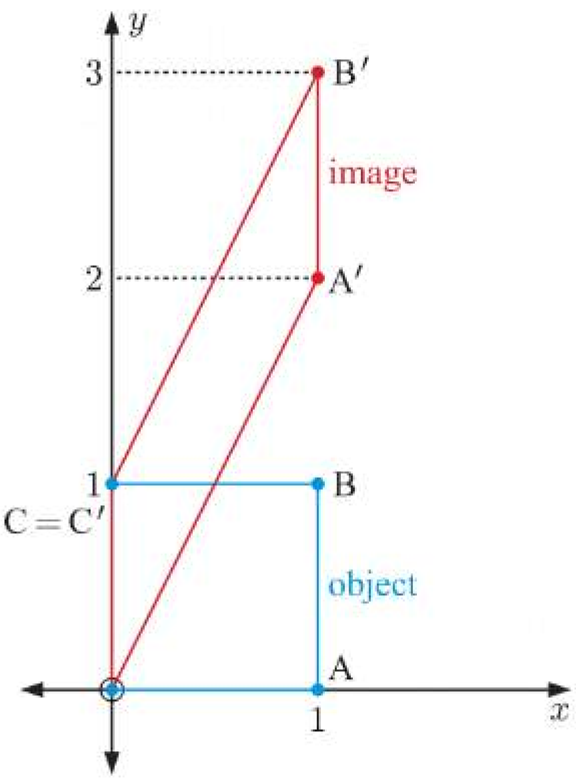
$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\therefore$ the image of B(1, 1) is B′(1, 3).

$\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ the image of C(0, 1) is C′(0, 1).

ii



iii $\det \mathbf{A} = 1 - 0 = 1$

iv O′A′B′C′ is a parallelogram with base 1 unit and height 1 unit.

$\therefore \dfrac{\text{area of O}'\text{A}'\text{B}'\text{C}'}{\text{area of OABC}} = \dfrac{1 \times 1}{1} = 1$

b $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

i $\mathbf{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ the image of O(0, 0) is O′(0, 0).

$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\therefore$ the image of A(1, 0) is A′(2, 0).

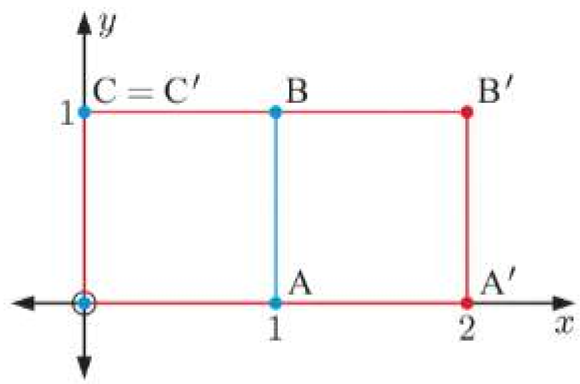
$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ the image of B(1, 1) is B′(2, 1).

$\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ the image of C(0, 1) is C′(0, 1).

ii



iii $\det \mathbf{A} = 2 - 0 = 2$

iv $\dfrac{\text{area of O}'\text{A}'\text{B}'\text{C}'}{\text{area of OABC}} = \dfrac{2}{1} = 2$

c $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

i $\mathbf{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ the image of O(0, 0) is O′(0, 0).

$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\therefore$ the image of A(1, 0) is A′(0, 2).

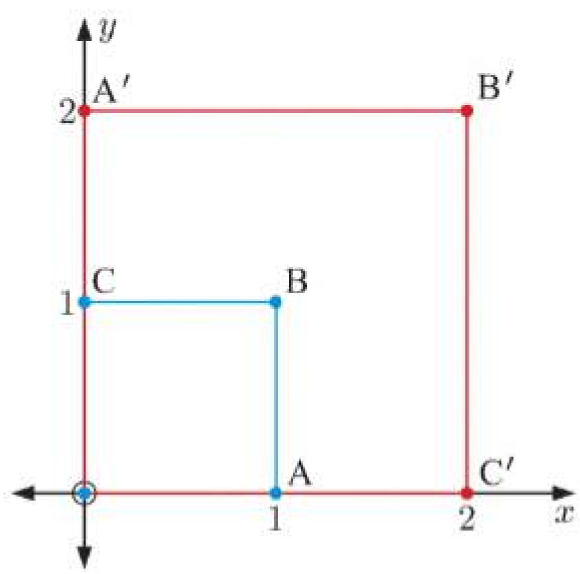
$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\therefore$ the image of B(1, 1) is B′(2, 2).

$\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\therefore$ the image of C(0, 1) is C′(2, 0).

ii



iii $\det \mathbf{A} = 0 - 4 = -4$

iv $\dfrac{\text{area of O}'\text{A}'\text{B}'\text{C}'}{\text{area of OABC}} = \dfrac{2 \times 2}{1} = 4$

d $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$

i $\mathbf{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ the image of O(0, 0) is O$'$(0, 0).

$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\therefore$ the image of A(1, 0) is A$'$(2, -1).

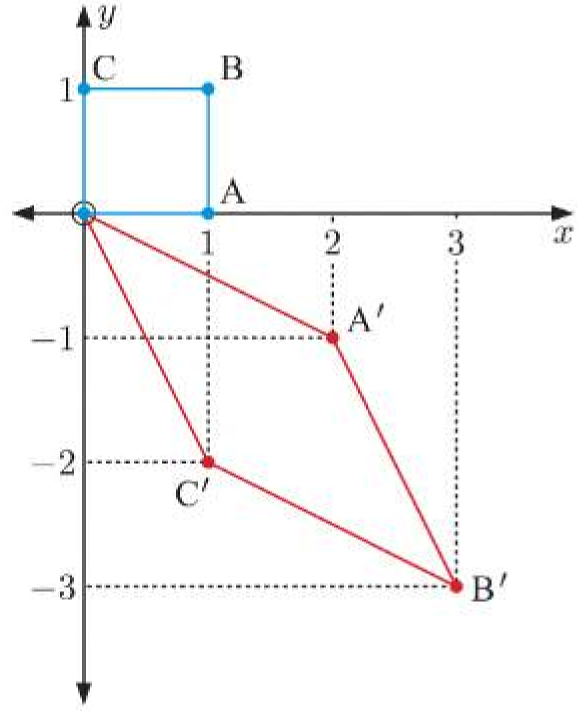
$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$\therefore$ the image of B(1, 1) is B$'$(3, -3).

$\mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\therefore$ the image of C(0, 1) is C$'$(1, -2).

ii



iii $\det \mathbf{A} = -4 - (-1) = -3$

iv $\text{O}'\text{A}' = \sqrt{(2-0)^2 + (-1-0)^2} = \sqrt{4+1} = \sqrt{5}$ units

$\text{A}'\text{B}' = \sqrt{(3-2)^2 + (-3-(-1))^2} = \sqrt{1+4} = \sqrt{5}$ units

$\text{B}'\text{C}' = \sqrt{(1-3)^2 + (-2-(-3))^2} = \sqrt{4+1} = \sqrt{5}$ units

$\text{C}'\text{O}' = \sqrt{(0-1)^2 + (0-(-2))^2} = \sqrt{1+4} = \sqrt{5}$ units

$\therefore$ O$'$A$'$B$'$C$'$ is a rhombus.

Now $\text{O}'\text{B}' = \sqrt{(3-0)^2 + (-3-0)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$ units and $\text{A}'\text{C}' = \sqrt{(1-2)^2 + (-2-(-1))^2} = \sqrt{1+1} = \sqrt{2}$ units

$$\therefore \frac{\text{area of O}'\text{A}'\text{B}'\text{C}'}{\text{area of OABC}} = \frac{\frac{3\sqrt{2}\times\sqrt{2}}{2}}{1} = 3$$

3 For the affine transformation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\dfrac{\text{area of image}}{\text{area of object}} = |\det \mathbf{A}|$.

EXERCISE 14G

1 **a** $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ with $\det \mathbf{A} = 3 - 0 = 3$

$$\therefore \frac{\text{area of image}}{\text{area of object}} = |3| = 3$$

b $\mathbf{A} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ with $\det \mathbf{A} = 0 - 4 = -4$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |-4| = 4$

c $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ with $\det \mathbf{A} = 6 + 1 = 7$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |7| = 7$

d $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ -2 & -4 \end{pmatrix}$ with $\det \mathbf{A} = -16 + 6 = -10$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |-10| = 10$

2 **a** **i** A translation through $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ has transformation equation $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

ii $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with $\det \mathbf{A} = 1 - 0 = 1$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |1| = 1$

b **i** A rotation about O through $\frac{\pi}{3}$ has transformation equation $\mathbf{x}' = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}\mathbf{x}$

$\therefore \quad \mathbf{x}' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\mathbf{x}$

ii $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ with $\det \mathbf{A} = \frac{1}{4} + \frac{3}{4} = 1$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |1| = 1$

c **i** A reflection in the x-axis has transformation equation $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x}$.

ii $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $\det \mathbf{A} = -1 - 0 = -1$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |-1| = 1$

d **i** A horizontal stretch with scale factor 2 has transformation equation $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$.

ii $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ with $\det \mathbf{A} = 2 - 0 = 2$

$\therefore \quad \dfrac{\text{area of image}}{\text{area of object}} = |2| = 2$

e **i** A vertical stretch with scale factor $\frac{1}{3}$ has transformation equation $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\mathbf{x}$.

ii $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ with $\det \mathbf{A} = \frac{1}{3} - 0 = \frac{1}{3}$

$\therefore \dfrac{\text{area of image}}{\text{area of object}} = \left|\frac{1}{3}\right| = \frac{1}{3}$

f **i** An enlargement with scale factor $\frac{4}{3}$ has transformation equation $\mathbf{x}' = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}\mathbf{x}$.

ii $\mathbf{A} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$ with $\det \mathbf{A} = \frac{16}{9} - 0 = \frac{16}{9}$

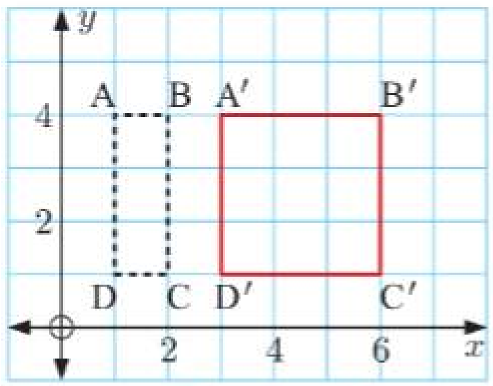
$\therefore \dfrac{\text{area of image}}{\text{area of object}} = \left|\frac{16}{9}\right| = \frac{16}{9}$

3 $\mathbf{A} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ with $\det \mathbf{A} = 1 - 0 = 1$

$\therefore$ area of image $= |1| \times$ area of object
$=$ area of object

So, a shear will not change the area of an object.

4 **a** The transformation mapping rectangle ABCD onto the square A′B′C′D′ is a horizontal stretch with scale factor 3.



b The transformation equation is $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$.

c $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ with $\det \mathbf{A} = 3 - 0 = 3$

Now $\dfrac{\text{area of A}'\text{B}'\text{C}'\text{D}'}{\text{area of ABCD}} = \dfrac{3 \times 3}{1 \times 3} = 3 = |\det \mathbf{A}|$ ✓

5 A vertical stretch with scale factor $1\frac{1}{2}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$.

a $\mathbf{A}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\therefore$ A$(-1, 0)$ has image A′$(-1, 0)$.

$\mathbf{A}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

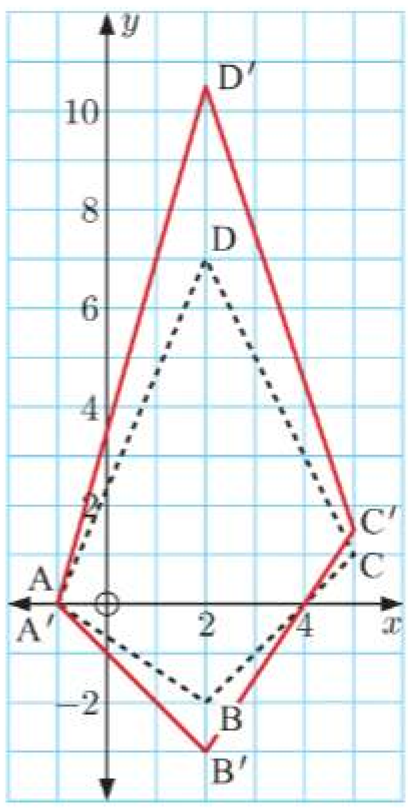
$\therefore$ B$(2, -2)$ has image B′$(2, -3)$.

$\mathbf{A}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix}$

$\therefore$ C$(5, 1)$ has image C′$(5, \frac{3}{2})$.

$\mathbf{A}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{21}{2} \end{pmatrix}$

$\therefore$ D$(2, 7)$ has image D′$(2, \frac{21}{2})$.



b Area of ABCD = area of $\triangle$ABD + area of $\triangle$BCD

$$= \tfrac{1}{2} \times 9 \times 3 + \tfrac{1}{2} \times 9 \times 3$$
$$= 27 \text{ units}^2$$

c The transformation equation is $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\mathbf{x}$.

d Area of image = $|\det \mathbf{A}| \times$ area of object

$$= \left|\tfrac{3}{2} - 0\right| \times 27 \qquad \{\text{using } \mathbf{b}\}$$
$$= \tfrac{3}{2} \times 27$$
$$= \tfrac{81}{2}$$
$$= 40.5 \text{ units}^2$$

6 A vertical stretch with scale factor $\frac{5}{3}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}$.

a The circle $x^2 + y^2 = 9$ has axes intercepts A(3, 0), B(0, 3), C(−3, 0), and D(0, −3).

$$\mathbf{A}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$\therefore$ A(3, 0) has image A′(3, 0).

$$\mathbf{A}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$\therefore$ B(0, 3) has image B′(0, 5).

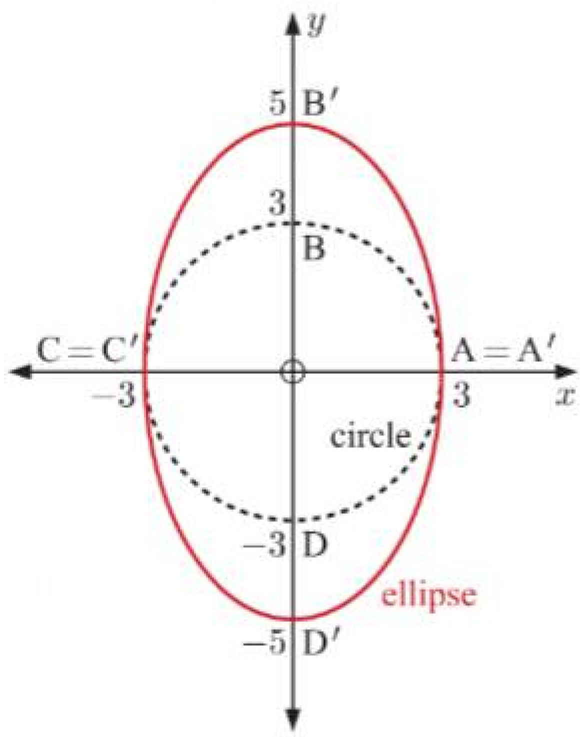
$$\mathbf{A}\begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$\therefore$ C(−3, 0) has image C′(−3, 0).

$$\mathbf{A}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$\therefore$ D(0, −3) has image D′(0, −5).

The image is an ellipse with axes intercepts A′, B′, C′, and D′.



b area of circle $= \pi \times 3^2$

$$= 9\pi \text{ units}^2$$

$\therefore$ area of image $= |\det \mathbf{A}| \times 9\pi$

$$= \left|\tfrac{5}{3} - 0\right| \times 9\pi$$
$$= 15\pi \text{ units}^2$$

7 A horizontal stretch with scale factor $\frac{b}{a}$, $a, b > 0$, has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix}$.

a The circle $x^2 + y^2 = a^2$ has axes intercepts A$(a, 0)$, B$(0, a)$, C$(-a, 0)$, and D$(0, -a)$.

$$\mathbf{A}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$\therefore$ A$(a, 0)$ has image A$'(b, 0)$.

$$\mathbf{A}\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

$\therefore$ B$(0, a)$ has image B$'(0, a)$.

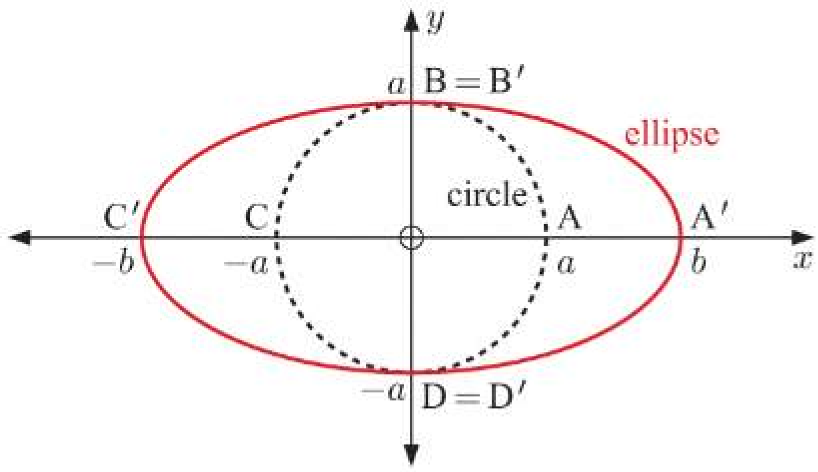
$$\mathbf{A}\begin{pmatrix} -a \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -a \\ 0 \end{pmatrix} = \begin{pmatrix} -b \\ 0 \end{pmatrix}$$

$\therefore$ C$(-a, 0)$ has image C$'(-b, 0)$.

$$\mathbf{A}\begin{pmatrix} 0 \\ -a \end{pmatrix} = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -a \end{pmatrix} = \begin{pmatrix} 0 \\ -a \end{pmatrix}$$

$\therefore$ D$(0, -a)$ has image D$'(0, -a)$.

The image is an ellipse with axes intercepts A$'$, B$'$, C$'$, and D$'$.



b area of circle $= \pi \times a^2$

$= \pi a^2$ units^2

$\therefore$ area of image $= |\det \mathbf{A}| \times \pi a^2$

$= \left|\frac{b}{a} - 0\right| \times \pi a^2$

$= \pi ab$ units^2

8 $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

a $\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{b} = \begin{pmatrix} 2 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ P$(1, 1)$ has image P$'(0, 0)$.

$\mathbf{A}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \mathbf{b} = \begin{pmatrix} 2 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

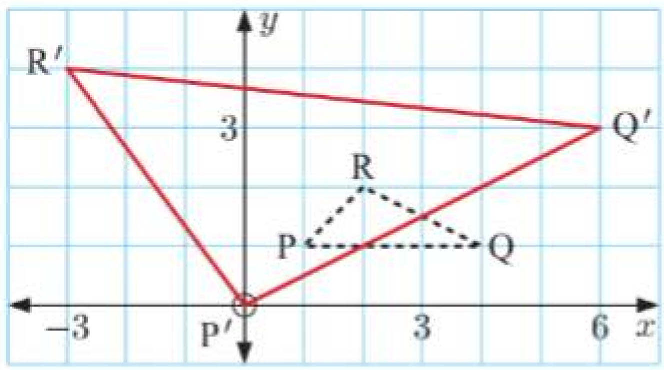
$= \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

$\therefore$ Q$(4, 1)$ has image Q$'(6, 3)$.

$$\mathbf{A}\begin{pmatrix}2\\2\end{pmatrix}+\mathbf{b}=\begin{pmatrix}2 & -5\\1 & 3\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}3\\-4\end{pmatrix}$$
$$=\begin{pmatrix}-6\\8\end{pmatrix}+\begin{pmatrix}3\\-4\end{pmatrix}$$
$$=\begin{pmatrix}-3\\4\end{pmatrix}$$

$\therefore$ R(2, 2) has image R′(−3, 4).

b



c Area of $\triangle$PQR $= \frac{1}{2}\times 3\times 1$
$= \frac{3}{2}$ units^2

Now area of $\triangle P'Q'R' = |\det \mathbf{A}| \times$ area of $\triangle$PQR
$= |6+5| \times \frac{3}{2}$
$= \frac{33}{2} = 16\frac{1}{2}$ units^2

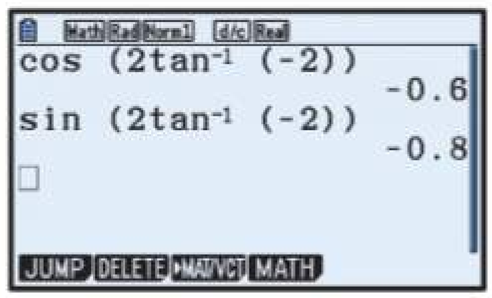
9 **a** A vertical stretch with scale factor k has the transformation matrix $\begin{pmatrix}1 & 0\\0 & k\end{pmatrix}$.

b A vertical stretch with scale factor 2 has the transformation matrix $\mathbf{A}=\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$.

For the reflection in the line $y=-2x$, $\tan\alpha=-2$

$\therefore$ $\cos 2\alpha = \cos(2\tan^{-1}(-2)) = -\frac{3}{5}$

and $\sin 2\alpha = \sin(2\tan^{-1}(-2)) = -\frac{4}{5}$



$\therefore$ the reflection has the transformation matrix

$$\mathbf{B}=\begin{pmatrix}-\frac{3}{5} & -\frac{4}{5}\\-\frac{4}{5} & \frac{3}{5}\end{pmatrix}.$$

So, a vertical stretch with scale factor 2 followed by a reflection in the line $y=-2x$ has the transformation matrix $\mathbf{BA}=\begin{pmatrix}-\frac{3}{5} & -\frac{4}{5}\\-\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}=\begin{pmatrix}-\frac{3}{5} & -\frac{8}{5}\\-\frac{4}{5} & \frac{6}{5}\end{pmatrix}$.

c $\det(\mathbf{BA}) = -\frac{18}{25}-\frac{32}{25}=-\frac{50}{25}=-2$

$\therefore$ area of image $= |-2| \times$ area of object
$= 2\times$ area of object

So, the combination of transformations doubles area.

10 **a** A vertical stretch with scale factor $\frac{1}{2}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

A reflection in the line $y = -x$ has the transformation matrix $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

An enlargement with scale factor 3 has the transformation matrix $\mathbf{C} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

$\therefore$ this sequence of transformations has the transformation matrix

$$\begin{aligned} \mathbf{D} &= \mathbf{CBA} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{3}{2} \\ -3 & 0 \end{pmatrix} \end{aligned}$$

b $\mathbf{D}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ O(0, 0) has image O$'$(0, 0).

$\mathbf{D}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -3 \end{pmatrix}$

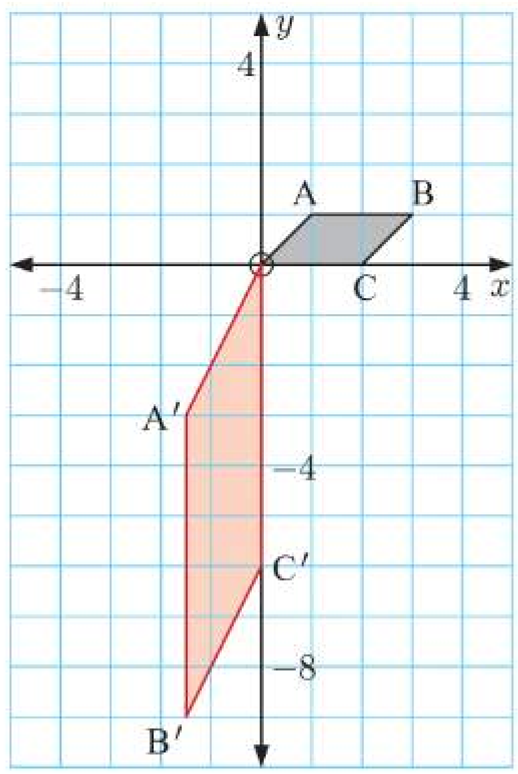
$\therefore$ A(1, 1) has image A$'(-\frac{3}{2}, -3)$.

$\mathbf{D}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -9 \end{pmatrix}$

$\therefore$ B(3, 1) has image B$'(-\frac{3}{2}, -9)$.

$\mathbf{D}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$

$\therefore$ C(2, 0) has image C$'(0, -6)$.



c
$$\begin{aligned} \text{area of OABC} &= 2 \times 1 \\ &= 2 \text{ units}^2 \end{aligned}$$

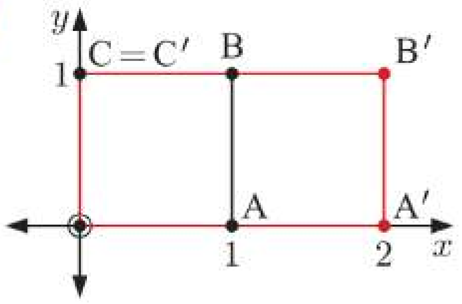
$$\begin{aligned} \therefore \ \text{area of image} &= |\det \mathbf{D}| \times 2 \\ &= \left|0 - \tfrac{9}{2}\right| \times 2 \\ &= 9 \text{ units}^2 \end{aligned}$$

11 The order in which transformations are performed does not affect the area of the image. This is because $\det(\mathbf{BA}) = \det(\mathbf{AB})$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

ACTIVITY 2 SENSE

1 **a** $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

i

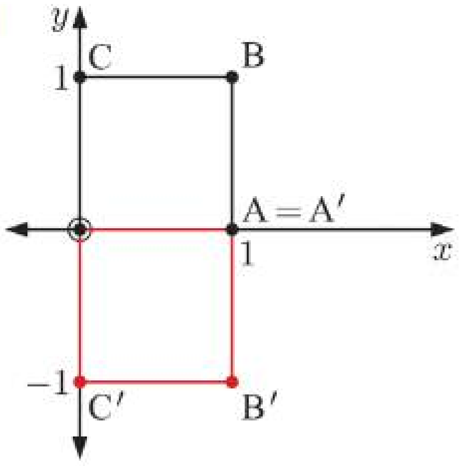


ii $\det \mathbf{A} = 2 - 0 = 2$

iii The labelling O′A′B′C′ is also anticlockwise, so sense has been preserved.

b $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

i

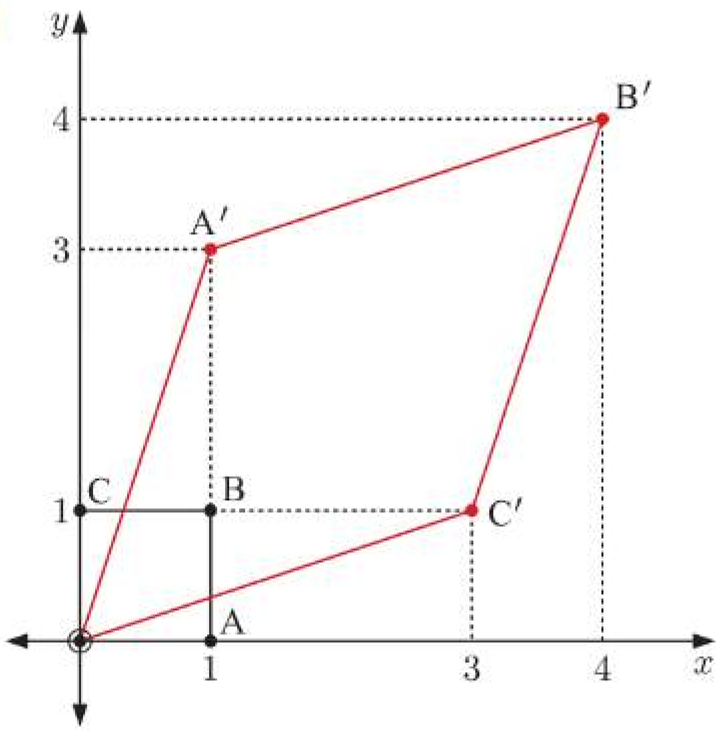


ii $\det \mathbf{A} = -1 - 0 = -1$

iii The labelling O′A′B′C′ is clockwise, so sense has been reversed.

c $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

i

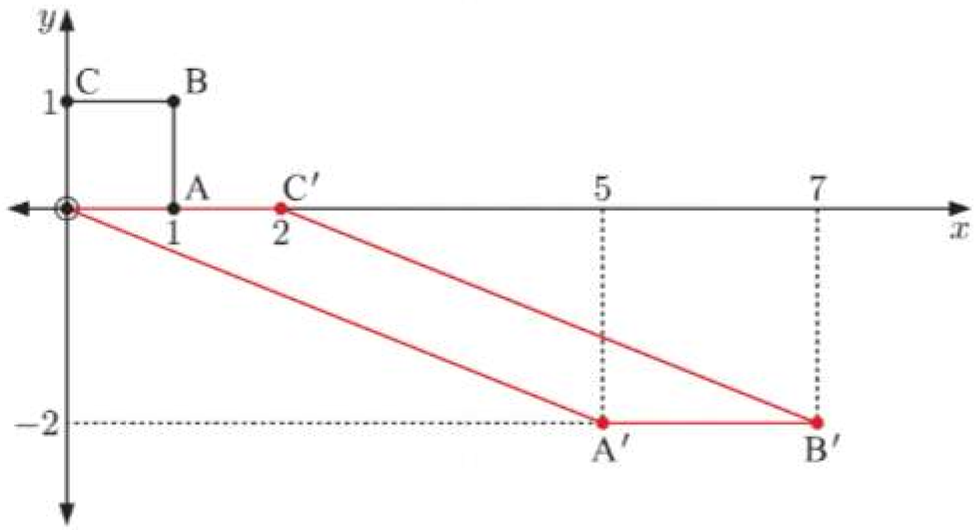


ii $\det \mathbf{A} = 1 - 9 = -8$

iii The labelling O′A′B′C′ is clockwise, so sense has been reversed.

d $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -2 & 0 \end{pmatrix}$

i



ii $\det \mathbf{A} = 0 + 4 = 4$

iii The labelling O′A′B′C′ is also anticlockwise, so sense has been preserved.

2 For a linear transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$,

- sense is preserved if $\det \mathbf{A} > 0$
- sense is reversed if $\det \mathbf{A} < 0$.

3 **a** A rotation has the transformation matrix $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\det \mathbf{A} = 1$.
Since $\det \mathbf{A} > 0$, sense is preserved.

b A reflection has the transformation matrix $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ with $\det \mathbf{A} = -1$.
Since $\det \mathbf{A} < 0$, sense is reversed.

c An enlargement has the transformation matrix $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ with $\det \mathbf{A} = k^2$.
Since $\det \mathbf{A} > 0$, sense is preserved.

d A horizontal stretch has the transformation matrix $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 0$, with $\det \mathbf{A} = k$.

A vertical stretch has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 0$, with $\det \mathbf{A} = k$.
In both cases, $\det \mathbf{A} > 0$, so sense is preserved.

e For a rotation, $\det \mathbf{A} = 1$ {from **a**}
For a reflection, $\det \mathbf{B} = -1$ {from **b**}
$\therefore$ for a rotation followed by a reflection, $\det(\mathbf{BA}) = \det \mathbf{B} \times \det \mathbf{A} = -1$.
Since $\det(\mathbf{BA}) < 0$, sense is reversed.

f For a reflection, $\det \mathbf{A} = -1$ {from **b**}
For n reflections, where n is even, $\det(\mathbf{A}^n) = (\det \mathbf{A})^n = (-1)^n = 1$.
Since $\det(\mathbf{A}^n) > 0$, sense is preserved.

g For a reflection, $\det \mathbf{A} = -1$ {from **b**}
For n reflections, where n is odd, $\det(\mathbf{A}^n) = (\det \mathbf{A})^n = (-1)^n = -1$.
Since $\det(\mathbf{A}^n) < 0$, sense is reversed.

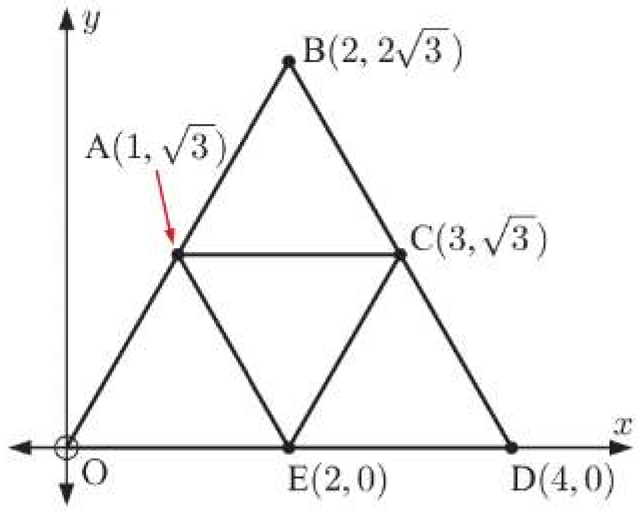
ACTIVITY 3 FRACTAL GEOMETRY

PART 1: SIERPIŃSKI'S TRIANGLE

1 **a** A reduction with scale factor $\frac{1}{2}$ maps $\triangle$OBD to $\triangle$OAE.

b A translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ maps $\triangle$OAE to $\triangle$ECD.

c A translation through $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ maps $\triangle$OAE to $\triangle$ABC.



2 **a** f_1 is the transformation mapping $\triangle$OBD to $\triangle$OAE.

From **1 a**, this is a reduction with scale factor $\frac{1}{2}$.

$\therefore$ f_1 has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\mathbf{x}$.

b f_2 is the transformation mapping $\triangle$OBD to $\triangle$ECD.

From **1 a** and **1 b**, this is a reduction with scale factor $\frac{1}{2}$ followed by a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$\therefore$ f_2 has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

c f_3 is the transformation mapping $\triangle$OBD to $\triangle$ABC.

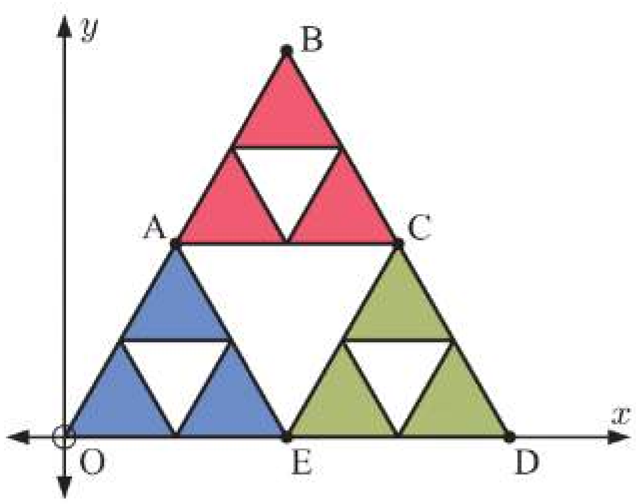
From **1 a** and **1 c**, this is a reduction with scale factor $\frac{1}{2}$ followed by a translation through $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

$\therefore$ f_3 has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

3 **a** **i** A reduction with scale factor $\frac{1}{2}$ maps the first iteration to the blue region.

ii A reduction with scale factor $\frac{1}{2}$ followed by a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ maps the first iteration to the green region.

iii A reduction with scale factor $\frac{1}{2}$ followed by a translation through $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ maps the first iteration to the red region.



b The transformations in **a i**, **a ii**, and **a iii** are the same as the transformations f_1, f_2, and f_3 respectively.

5 **a** Area of $\triangle$OBD $= \frac{1}{2} \times 4 \times 2\sqrt{3}$
$= 4\sqrt{3}$ units^2

b $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ for transformations f_1, f_2, and f_3, and $\det \mathbf{A} = \frac{1}{4} - 0 = \frac{1}{4}$.

So, in the first iteration, $\triangle$OAE, $\triangle$ECD, and $\triangle$ABC each have area $\frac{1}{4} \times 4\sqrt{3}$ units^2.

$\therefore$ area of first iteration $= 3 \times \frac{1}{4} \times 4\sqrt{3}$
$= \frac{3}{4} \times 4\sqrt{3}$ units^2

Similarly, in the second iteration, the blue, green, and red regions each have area $\frac{1}{4} \times \frac{3}{4} \times 4\sqrt{3}$ units^2.

$\therefore$ area of second iteration $= 3 \times \frac{1}{4} \times \frac{3}{4} \times 4\sqrt{3}$
$= \left(\frac{3}{4}\right)^2 \times 4\sqrt{3}$ units^2

In general, the area of nth iteration $= \left(\frac{3}{4}\right)^n \times 4\sqrt{3}$ units^2.

Thus, the areas of the iterations of the Sierpiński triangle form a geometric sequence with first term $4\sqrt{3}$ and common ratio $\frac{3}{4}$.

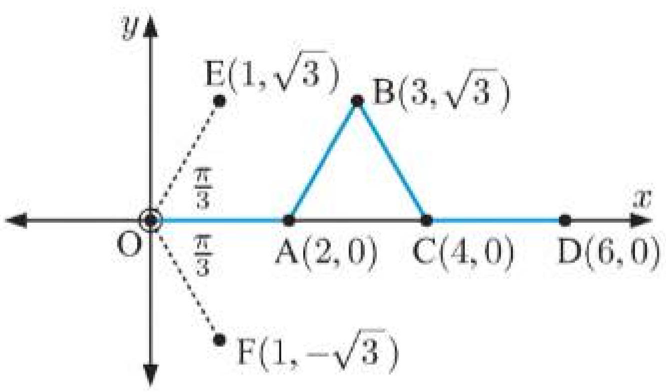
c As $n \to \infty$, $\left(\frac{3}{4}\right)^n \to 0$

$\therefore$ $\left(\frac{3}{4}\right)^n \times 4\sqrt{3} \to 0$

So, the complete Sierpiński triangle has no area.

PART 2: VON KOCH'S CURVE

1



a A reduction with scale factor $\frac{1}{3}$ maps [OD] to [OA].

b An anticlockwise rotation about O through $\frac{\pi}{3}$ maps [OA] to [OE].

c A translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ maps [OE] to [AB].

d A clockwise rotation about O through $\frac{\pi}{3}$ maps [OA] to [OF].

e A translation through $\begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}$ maps [OF] to [BC].

f A translation through $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ maps [OA] to [CD].

2 **a** f_1 is the transformation mapping [OD] to [OA].

From **1 a**, this is a reduction with scale factor $\frac{1}{3}$.

$\therefore$ f_1 has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\mathbf{x}$.

b f_2 is the transformation mapping [OD] to [AB].

From **1 a**, **1 b**, and **1 c**, this is a reduction with scale factor $\frac{1}{3}$, then an anticlockwise rotation about O through $\frac{\pi}{3}$, then a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$$\therefore \quad f_2 \text{ has matrix equation } \quad \mathbf{x}' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} & -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

c f_3 is the transformation mapping [OD] to [BC].

From **1 a**, **1 d**, and **1 e**, this is a reduction with scale factor $\frac{1}{3}$, then a clockwise rotation about O through $\frac{\pi}{3}$, then a translation through $\begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}$.

$$\therefore \quad f_3 \text{ has matrix equation } \quad \mathbf{x}' = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}$$

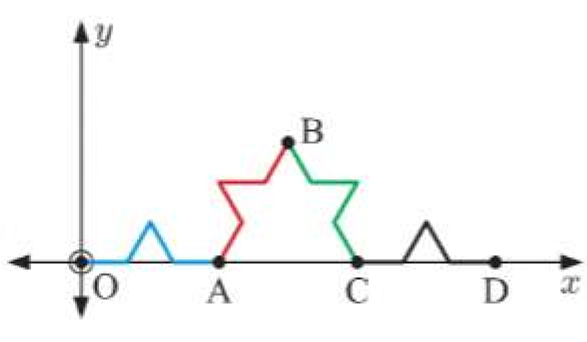
$$= \begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}.$$

d f_4 is the transformation mapping [OD] to [CD].

From **1 a** and **1 f**, this is a reduction with scale factor $\frac{1}{3}$ followed by a translation through $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

$\therefore$ f_4 has matrix equation $\mathbf{x}' = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

3



a **i** A reduction with scale factor $\frac{1}{3}$ maps the first iteration to the blue curve.

ii A reduction with scale factor $\frac{1}{3}$, then an anticlockwise rotation about O through $\frac{\pi}{3}$, then a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ maps the first iteration to the red curve.

iii A reduction with scale factor $\frac{1}{3}$, then a clockwise rotation about O through $\frac{\pi}{3}$, then a translation through $\begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}$ maps the first iteration to the green curve.

iv A reduction with scale factor $\frac{1}{3}$ followed by a translation through $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ maps the first iteration to the black curve.

b The transformations in **a i**, **a ii**, **a iii**, and **a iv** are the same as the transformations f_1, f_2, f_3, and f_4 respectively.

5 **a** Length of [OD] $= 6$ units

In the first iteration, [OA], [AB], [BC], and [CD] each have length $\frac{1}{3} \times 6$ units.

$$\therefore \text{ length of first iteration} = 4 \times \tfrac{1}{3} \times 6 = \tfrac{4}{3} \times 6 \text{ units}$$

Similarly, in the second iteration, the blue, red, green, and black curves each have length $\frac{1}{3} \times \frac{4}{3} \times 6$ units.

$$\therefore \text{ length of second iteration} = 4 \times \tfrac{1}{3} \times \tfrac{4}{3} \times 6 = \left(\tfrac{4}{3}\right)^2 \times 6 \text{ units}$$

In general, the length of the nth iteration $= 6\left(\frac{4}{3}\right)^n$ units.

b As $n \to \infty$, $\left(\frac{4}{3}\right)^n \to \infty$

$\therefore \left(\frac{4}{3}\right)^n \times 6 \to \infty$

So, the complete von Koch curve has infinite length.

REVIEW SET 14A

1 **a** $\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

$\therefore$ the image is $(-2, 6)$.

b $\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

$\therefore$ the image is $(-5, 4)$.

c $\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

$\therefore$ the image is $(-4, -1)$.

d $\begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -5 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$

$\therefore$ the image is $(-8, -4)$.

2 **a** Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$\therefore x - 2 = 1$ and $y + 3 = 5$

$\therefore x = 3$ and $y = 2$

So, the object point is $(3, 2)$.

b Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\therefore \ x - 2 = 0 \quad \text{and} \quad y + 3 = -1$$
$$\therefore \ x = 2 \quad \text{and} \quad y = -4$$

So, the object point is $(2, -4)$.

c Let (x, y) be the object point.

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$$

$$\therefore \ x - 2 = -6 \quad \text{and} \quad y + 3 = -3$$
$$\therefore \ x = -4 \quad \text{and} \quad y = -6$$

So, the object point is $(-4, -6)$.

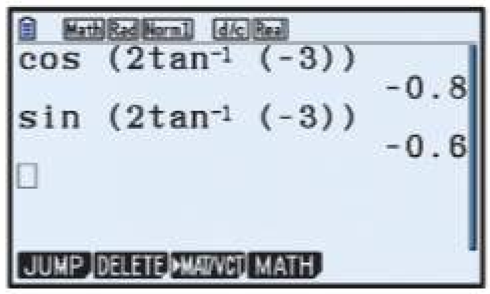
3 **a** For an anticlockwise rotation about O through $\theta = \frac{4\pi}{3}$, $\cos\theta = -\frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$.

$\therefore$ the transformation matrix is $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

b For a reflection in the line $y = -3x$, $\tan\alpha = -3$

$\therefore \ \cos 2\alpha = \cos(2\tan^{-1}(-3)) = -\frac{4}{5}$

and $\sin 2\alpha = \sin(2\tan^{-1}(-3)) = -\frac{3}{5}$



$\therefore$ the transformation matrix $\mathbf{A} = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$.

c A vertical stretch with scale factor $\frac{4}{3}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$.

4 **a** $\mathbf{A} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ where $|\mathbf{A}| = -\left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -1$

Since $\mathbf{A}$ has form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $|\mathbf{A}| = -1$, $\mathbf{A}$ is a reflection matrix where $\cos 2\alpha = -\frac{3}{5}$ and $\sin 2\alpha = \frac{4}{5}$.

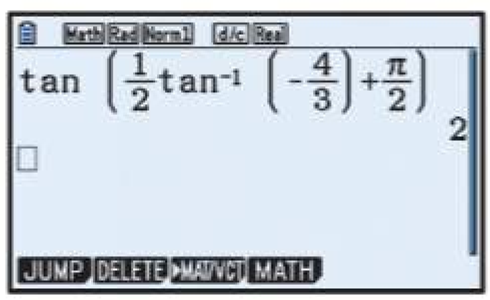
$\therefore \ \tan 2\alpha = -\frac{4}{3}$ and $\frac{\pi}{2} < 2\alpha < \pi$.

$\therefore \ \tan(2\alpha - \pi) = -\frac{4}{3}, \quad -\frac{\pi}{2} < 2\alpha - \pi < 0$

$\therefore \ 2\alpha - \pi = \tan^{-1}\left(-\frac{4}{3}\right)$

$\therefore \ \alpha = \frac{1}{2}\tan^{-1}\left(-\frac{4}{3}\right) + \frac{\pi}{2}$

$\therefore \ \tan\alpha = 2$



$\therefore$ the transformation is a reflection in the line $y = 2x$.

b $\mathbf{A} = \begin{pmatrix} \frac{12}{13} & -\frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$ where $|\mathbf{A}| = \left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2 = 1$

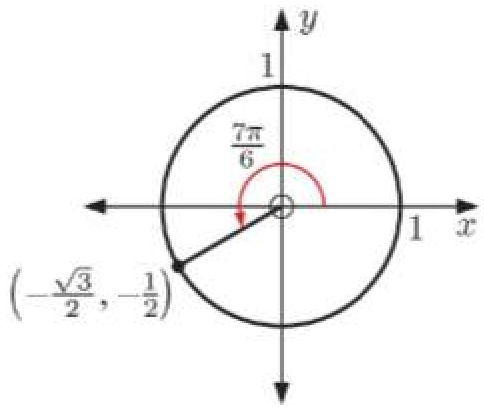
Since $\mathbf{A}$ has form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $|\mathbf{A}| = 1$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = \frac{12}{13}$ and $\sin\theta = \frac{5}{13}$, so θ is in quadrant 1.

$\therefore \theta = \cos^{-1}\left(\frac{12}{13}\right)$
$\approx 0.395^c$

$\therefore$ the transformation is an anticlockwise rotation about O through about 0.395^c.

5 $\theta = \frac{7\pi}{6}$ anticlockwise



$\therefore \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{7\pi}{6} = -\frac{1}{2}$

$\therefore \mathbf{A} = \begin{pmatrix} \cos\frac{7\pi}{6} & -\sin\frac{7\pi}{6} \\ \sin\frac{7\pi}{6} & \cos\frac{7\pi}{6} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{3\sqrt{3}}{2} + 2 \\ -\frac{3}{2} + 2\sqrt{3} \end{pmatrix}$$

$\therefore (3, -4)$ has image $\left(-\frac{3\sqrt{3}}{2} + 2, \ -\frac{3}{2} + 2\sqrt{3}\right)$.

6 **a** $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\mathbf{x}$ is a vertical stretch with scale factor 5.

b $\mathbf{x}' = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$ is a horizontal stretch with scale factor $\frac{2}{3}$.

c $\mathbf{x}' = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\mathbf{x}$ is a reduction with scale factor $\frac{1}{4}$.

7 **a** A horizontal stretch with scale factor $\frac{7}{2}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \end{pmatrix}$.

b $\mathbf{A}\begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \end{pmatrix}$
$= \begin{pmatrix} 21 \\ -4 \end{pmatrix}$

$\therefore (6, -4)$ has image $(21, -4)$.

c $\mathbf{A}^{-1} = \frac{1}{\frac{7}{2}}\begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{2} \end{pmatrix}$
$= \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$ is a horizontal stretch with scale factor $\frac{2}{7}$.

8 **a** For a reflection in the line $y = -\frac{1}{\sqrt{3}}x$, $\tan\alpha = -\frac{1}{\sqrt{3}}$

$$\therefore \ \alpha = -\tfrac{\pi}{6}$$

$$\therefore \ 2\alpha = -\tfrac{\pi}{3}$$

So, $\cos 2\alpha = \frac{1}{2}$ and $\sin 2\alpha = -\frac{\sqrt{3}}{2}$.

$\therefore$ the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1-2\sqrt{3} \\ \sqrt{3}-2 \end{pmatrix}$$

$\therefore$ $(-2, 4)$ has image $(-1-2\sqrt{3}, \ \sqrt{3}-2)$.

b An enlargement with scale factor $\frac{9}{4}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix}$.

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{9}{2} \\ 9 \end{pmatrix}$$

$\therefore$ $(-2, 4)$ has image $\left(-\frac{9}{2}, 9\right)$.

9 **a** For an anticlockwise rotation through $\frac{\pi}{6}$, we have $\theta = \frac{\pi}{6}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

$\therefore$ the rotation has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

A reflection in the line $y = x$ has the transformation matrix $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$\therefore$ **BA** has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}|\,|\mathbf{A}| = (-1)(1) = -1$

$\therefore$ the composition of these transformations is a reflection.

If the reflection is in the line $y = (\tan\alpha)x$ then

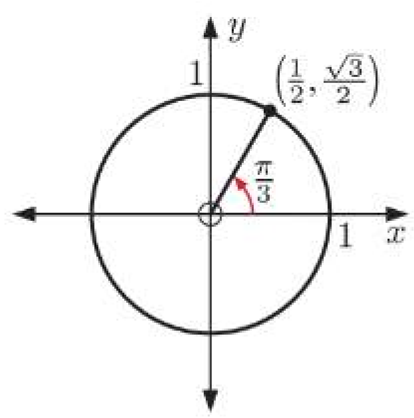
$\cos 2\alpha = \frac{1}{2}$ and $\sin 2\alpha = \frac{\sqrt{3}}{2}$

$\therefore$ $2\alpha = \frac{\pi}{3}$ $\quad \{0 < 2\alpha < \frac{\pi}{2}\}$

$\therefore$ $\alpha = \frac{\pi}{6}$

$\therefore$ $\tan\alpha = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\therefore$ the composite transformation is a reflection in the line $y = \frac{1}{\sqrt{3}}x$.



b For a reflection in the line $y = -\sqrt{3}x$, $\tan\alpha = -\sqrt{3}$

$$\therefore \alpha = -\tfrac{\pi}{3}$$
$$\therefore 2\alpha = -\tfrac{2\pi}{3}$$

So, $\cos 2\alpha = -\frac{1}{2}$ and $\sin 2\alpha = -\frac{\sqrt{3}}{2}$.

$\therefore$ the reflection has the transformation matrix $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

A reflection in the x-axis has the transformation matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

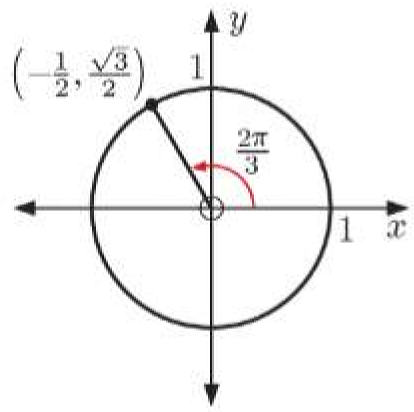
Now $\mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$\therefore$ **BA** has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}||\mathbf{A}| = (-1)(-1) = 1$

$\therefore$ the composition of these transformations is a rotation.

If the angle of rotation is θ, $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, so θ is in quadrant 2.

$\therefore \theta = \cos^{-1}(-\frac{1}{2})$
$= \frac{2\pi}{3}$



$\therefore$ the composite transformation is an anticlockwise rotation about O through $\frac{2\pi}{3}$.

10 **a** A vertical stretch with scale factor 4 has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

A reflection in the line $y = -x$ has the transformation matrix $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix}$

$\therefore$ the composition of these transformations has matrix equation $\mathbf{x}' = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix}\mathbf{x}$.

b $\mathbf{BA}\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$
$= \begin{pmatrix} -20 \\ 3 \end{pmatrix}$

$\therefore$ $(-3, 5)$ has image $(-20, 3)$.

11 **a** **i** A horizontal stretch with scale factor 5 has the transformation matrix $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$.

ii $\dfrac{\text{area of image}}{\text{area of object}} = |\det \mathbf{A}| = |5 - 0| = 5$

b **i** A reflection in the y-axis has transformation matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

ii $\dfrac{\text{area of image}}{\text{area of object}} = |\det \mathbf{A}| = |-1 - 0| = 1$

c **i** A reduction with scale factor $\frac{1}{4}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$.

ii $\dfrac{\text{area of image}}{\text{area of object}} = |\det \mathbf{A}| = \left|\frac{1}{16} - 0\right| = \frac{1}{16}$

12 $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

a
$$\begin{aligned}\mathbf{A}\begin{pmatrix} -1 \\ -2 \end{pmatrix} + \mathbf{b} &= \begin{pmatrix} 3 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 0 \end{pmatrix}\end{aligned}$$

$\therefore$ A$(-1, -2)$ has image A$'(-5, 0)$.

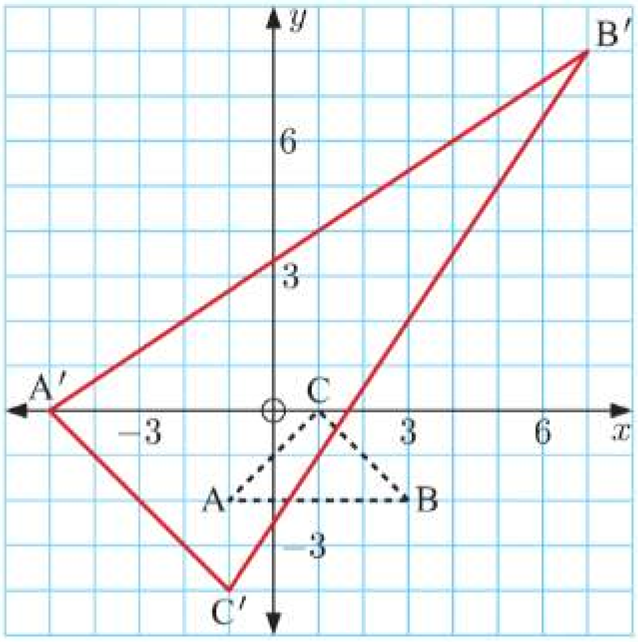
$$\begin{aligned}\mathbf{A}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \mathbf{b} &= \begin{pmatrix} 3 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ 14 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 8 \end{pmatrix}\end{aligned}$$

$\therefore$ B$(3, -2)$ has image B$'(7, 8)$.

$$\begin{aligned}\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{b} &= \begin{pmatrix} 3 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -4 \end{pmatrix}\end{aligned}$$

$\therefore$ C$(1, 0)$ has image C$'(-1, -4)$.

b



c Area of $\triangle ABC = \frac{1}{2} \times 4 \times 2$
$= 4$ units^2

Now, area of $\triangle A'B'C' = |\det \mathbf{A}| \times$ area of $\triangle ABC$
$= |-12 + 2| \times 4$
$= 40$ units^2

REVIEW SET 14B

1 **a** $\begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$\therefore$ the image is $(7, 3)$.

b $\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$\therefore$ the image is $(4, -2)$.

c $\begin{pmatrix} -7 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

$\therefore$ the image is $(-2, -5)$.

2 **a** Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$\therefore \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\therefore 1 + e = 4$ and $3 + f = 5$
$\therefore e = 3$ and $f = 2$

So, the translation vector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

b Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$\therefore \begin{pmatrix} -2 \\ -1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$\therefore -2 + e = 4$ and $-1 + f = -2$

$\therefore e = 6$ and $f = -1$

So, the translation vector is $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

c Let $\begin{pmatrix} e \\ f \end{pmatrix}$ be the translation vector.

$\therefore \begin{pmatrix} 8 \\ -1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$

$\therefore 8 + e = -1$ and $-1 + f = -4$

$\therefore e = -9$ and $f = -3$

So, the translation vector is $\begin{pmatrix} -9 \\ -3 \end{pmatrix}$.

3 **a** A clockwise rotation of $\frac{\pi}{6}$ is equivalent to an anticlockwise rotation of $\theta = -\frac{\pi}{6}$.

$\therefore \cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$

$\therefore$ the transformation matrix is $\mathbf{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

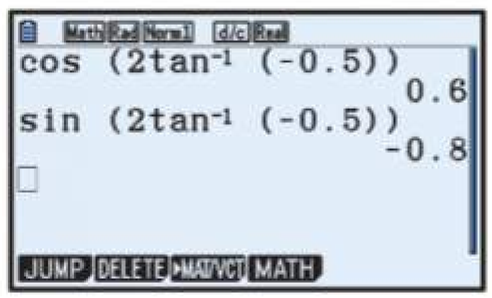
b A horizontal stretch with scale factor $\frac{5}{3}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{5}{3} & 0 \\ 0 & 1 \end{pmatrix}$.

c For a reflection in the line $y = -\frac{1}{2}x$, $\tan\alpha = -\frac{1}{2}$

$\therefore \cos 2\alpha = \cos\left(2\tan^{-1}\left(-\frac{1}{2}\right)\right) = \frac{3}{5}$

and $\sin 2\alpha = \sin\left(2\tan^{-1}\left(-\frac{1}{2}\right)\right) = -\frac{4}{5}$

$\therefore$ the transformation matrix is $\mathbf{A} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{pmatrix}$.



4 **a** $\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$ is the transformation matrix for an enlargement with scale factor $\frac{4}{3}$.

b $\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $|\mathbf{A}| = \left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$.

$\therefore$ the transformation is a rotation.

If the angle of rotation is θ, $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$, so θ is in quadrant 1.

$\therefore \theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\approx 0.464^c$

$\therefore$ the transformation is an anticlockwise rotation about O through about 0.464^c.

c $\begin{pmatrix} -\frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{A}| = -\left(\frac{2}{\sqrt{13}}\right)^2 - \left(\frac{3}{\sqrt{13}}\right)^2 = -1$

$\therefore$ the transformation is a reflection where $\cos 2\alpha = -\frac{2}{\sqrt{13}}$ and $\sin 2\alpha = -\frac{3}{\sqrt{13}}$

$\therefore$ $\tan 2\alpha = \frac{3}{2}$ where $-\pi < 2\alpha < -\frac{\pi}{2}$

$\therefore$ $\tan(2\alpha + \pi) = \frac{3}{2}$, $0 < 2\alpha + \pi < \frac{\pi}{2}$

$\therefore$ $2\alpha + \pi = \tan^{-1}\left(\frac{3}{2}\right)$

$\therefore$ $\alpha = \frac{1}{2}\tan^{-1}\left(\frac{3}{2}\right) - \frac{\pi}{2}$

$\therefore$ $\tan\alpha \approx -1.87$

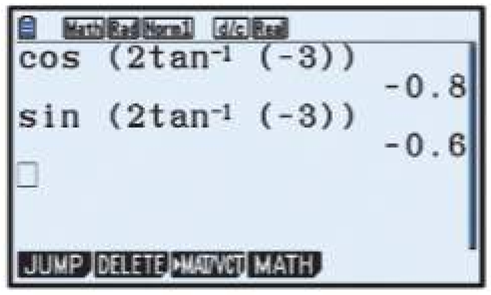
$\therefore$ the transformation is a reflection in the line $y \approx -1.87x$.

5 For a reflection in the line $y = -3x$, $\tan\alpha = -3$

$\therefore$ $m = \tan\alpha = -3$

$\therefore$ $\cos 2\alpha = \cos(2\tan^{-1}(-3)) = -\frac{4}{5}$

and $\sin 2\alpha = \sin(2\tan^{-1}(-3)) = -\frac{3}{5}$



$\therefore$ the transformation matrix is $\mathbf{A} = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$.

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5} - 3 \\ \frac{3}{5} + 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{11}{5} \\ \frac{23}{5} \end{pmatrix}$$

$\therefore$ $(-1, 5)$ has image $\left(-\frac{11}{5}, \frac{23}{5}\right)$.

6 **a** A vertical stretch with scale factor $\frac{3}{5}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{pmatrix}$.

b $\mathbf{A}\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -4 \\ -2 \end{pmatrix}$

$$= \begin{pmatrix} -4 \\ -\frac{6}{5} \end{pmatrix}$$

$\therefore$ $(-4, -2)$ has image $\left(-4, -\frac{6}{5}\right)$.

c $\mathbf{A}^{-1} = \frac{1}{\frac{3}{5}}\begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}$$

$\therefore$ $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$ is a vertical stretch with scale factor $\frac{5}{3}$.

7 **a** **i** For a reflection in the line $y = 2x$, $\tan\alpha = 2$

$\therefore\ \cos 2\alpha = \cos(2\tan^{-1} 2) = -\frac{3}{5}$

and $\sin 2\alpha = \sin(2\tan^{-1} 2) = \frac{4}{5}$

$\therefore$ the transformation matrix is $\mathbf{A} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$\therefore$ A(3, 1) has image B(−1, 3).

ii $\text{OA} = \sqrt{(3-0)^2 + (1-0)^2}$
$= \sqrt{9+1}$
$= \sqrt{10}$ units

$\text{OB} = \sqrt{(-1-0)^2 + (3-0)^2}$
$= \sqrt{1+9}$
$= \sqrt{10}$ units
$= \text{OA}$

b A reflection in the line $y = (\tan\alpha)x$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.

Let P have coordinates (x, y).

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} x\cos 2\alpha + y\sin 2\alpha \\ x\sin 2\alpha - y\cos 2\alpha \end{pmatrix}$$

$\therefore$ $\text{P}(x, y)$ has image $\text{P}'(x\cos 2\alpha + y\sin 2\alpha,\ x\sin 2\alpha - y\cos 2\alpha)$.

Now $\text{OP} = \sqrt{(x-0)^2 + (y-0)^2}$
$= \sqrt{x^2 + y^2}$ units

and $\text{OP}' = \sqrt{(x\cos 2\alpha + y\sin 2\alpha - 0)^2 + (x\sin 2\alpha - y\cos 2\alpha - 0)^2}$

$$= \sqrt{\begin{array}{l} x^2\cos^2 2\alpha + \cancel{2xy\cos 2\alpha\sin 2\alpha} + y^2\sin^2 2\alpha \\ \quad + x^2\sin^2 2\alpha - \cancel{2xy\cos 2\alpha\sin 2\alpha} + y^2\cos^2 2\alpha \end{array}}$$
$$= \sqrt{x^2(\cos^2 2\alpha + \sin^2 2\alpha) + y^2(\sin^2 2\alpha + \cos^2 2\alpha)}$$
$$= \sqrt{x^2 + y^2} \text{ units}$$
$$= \text{OP}$$

8 **a** For a reflection in the line $y = \frac{1}{\sqrt{3}}x$, $\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore\ \alpha = \frac{\pi}{6}$

$\therefore\ 2\alpha = \frac{\pi}{3}$

So, $\cos 2\alpha = \frac{1}{2}$ and $\sin 2\alpha = \frac{\sqrt{3}}{2}$

$\therefore$ the reflection has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

For an anticlockwise rotation about O through $-\frac{\pi}{6}$, we have $\theta = -\frac{\pi}{6}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = -\frac{1}{2}$

$\therefore$ the rotation has transformation matrix $\mathbf{B} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

$\therefore$ **BA** has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}|\,|\mathbf{A}| = (1)(-1) = -1$.

$\therefore$ the composition of these transformations is a reflection.

If the reflection is in the line $y = (\tan\alpha)x$ then

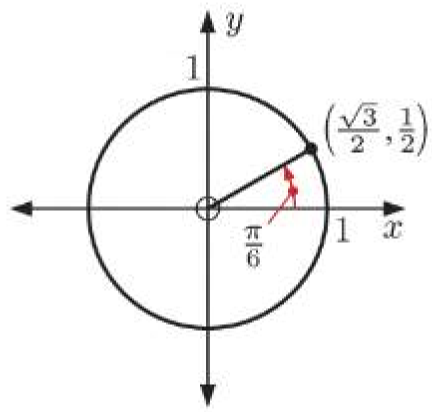
$\cos 2\alpha = \frac{\sqrt{3}}{2}$ and $\sin 2\alpha = \frac{1}{2}$

$\therefore$ $2\alpha = \frac{\pi}{6}$ $\quad \{0 < 2\alpha < \frac{\pi}{2}\}$

$\therefore$ $\alpha = \frac{\pi}{12}$

$\therefore$ $\tan\alpha = \tan\frac{\pi}{12}$

$\therefore$ the composite transformation is a reflection in the line $y = \left(\tan\frac{\pi}{12}\right)x$.



b A reflection in the x-axis has the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A reflection in the y-axis has the transformation matrix $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

This is the transformation matrix for a rotation about O through π.

$\therefore$ the composite transformation is a rotation about O through π.

9 **a** An enlargement with scale factor 3 has the transformation matrix $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

So, an enlargement with scale factor 3 followed by a translation through $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ has matrix equation $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

b A horizontal stretch with scale factor 6 has the transformation matrix $\mathbf{A} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$.

So, a translation through $\begin{pmatrix} 2 \\ -7 \end{pmatrix}$ followed by a horizontal stretch with scale factor 6 has matrix equation

$$\begin{aligned}\mathbf{x}' &= \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\left(\mathbf{x} + \begin{pmatrix} 2 \\ -7 \end{pmatrix}\right) \\ &= \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 12 \\ -7 \end{pmatrix}\end{aligned}$$

10 **a** For an anticlockwise rotation about O through $\frac{3\pi}{4}$, we have $\theta = \frac{3\pi}{4}$, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore$ the transformation matrix $\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

An enlargement with scale factor $\frac{3}{2}$ has the transformation matrix $\mathbf{B} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$.

b
$$\begin{aligned}
\mathbf{x}' = \mathbf{AB}\begin{pmatrix} -4 \\ 5 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} -4 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -6 \\ \frac{15}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{6}{\sqrt{2}} - \frac{15}{2\sqrt{2}} \\ -\frac{6}{\sqrt{2}} - \frac{15}{2\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{2\sqrt{2}} \\ -\frac{27}{2\sqrt{2}} \end{pmatrix}
\end{aligned}$$

$\therefore$ $(-4, 5)$ has image $\left(-\frac{3}{2\sqrt{2}}, -\frac{27}{2\sqrt{2}}\right)$.

c
$$\mathbf{AB} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \\ \frac{3}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \end{pmatrix}$$

and

$$\mathbf{BA} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \\ \frac{3}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \end{pmatrix}$$

So, $\mathbf{AB} = \mathbf{BA}$. This means that the order in which the transformations are applied does not matter.

11 A horizontal stretch with scale factor $\frac{3}{2}$ has the transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix}$.

a The image will be an ellipse.

b
$$\begin{aligned}
\text{area of image} &= |\det \mathbf{A}| \times \text{area of object} \\
&= \left|\tfrac{3}{2} - 0\right| \times 4\pi \\
&= 6\pi \text{ cm}^2
\end{aligned}$$

12 For an anticlockwise rotation about O through $\frac{2\pi}{3}$, we have $\theta = \frac{2\pi}{3}$, $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ the rotation has the transformation matrix $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

A horizontal stretch with scale factor 4 has the transformation matrix $\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

$\therefore$ the composite transformation has the transformation matrix

$$\begin{aligned} \mathbf{C} &= \mathbf{BA} \\ &= \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -2 & -2\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

a $\mathbf{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & -2\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$

$\therefore$ P(1, 0) has image $\mathrm{P}'\left(-2, \frac{\sqrt{3}}{2}\right)$.

$\mathbf{C}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & -2\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -8+2\sqrt{3} \\ 2\sqrt{3}+\frac{1}{2} \end{pmatrix}$

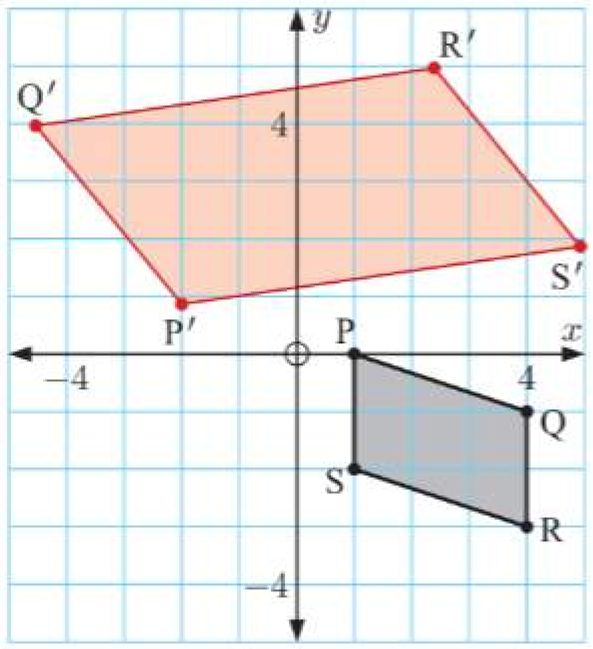
$\therefore$ Q(4, −1) has image $\mathrm{Q}'\left(-8+2\sqrt{3},\ 2\sqrt{3}+\frac{1}{2}\right)$.

$\mathbf{C}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 & -2\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -8+6\sqrt{3} \\ 2\sqrt{3}+\frac{3}{2} \end{pmatrix}$

$\therefore$ R(4, −3) has image $\mathrm{R}'\left(-8+6\sqrt{3},\ 2\sqrt{3}+\frac{3}{2}\right)$.

$\mathbf{C}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 & -2\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2+4\sqrt{3} \\ \frac{\sqrt{3}}{2}+1 \end{pmatrix}$

$\therefore$ S(1, −2) has image $\mathrm{S}'\left(-2+4\sqrt{3},\ \frac{\sqrt{3}}{2}+1\right)$.



b Area of PQRS $= 2 \times 3$
$= 6$ units^2

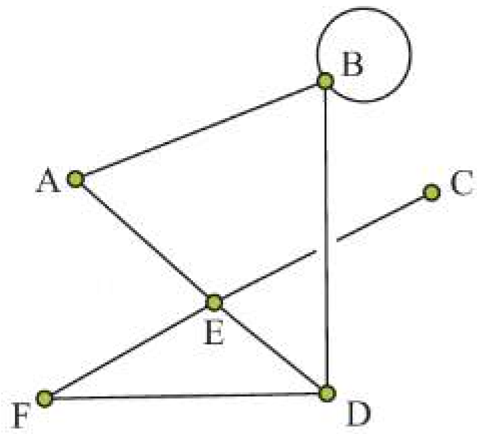
Now, area of P′Q′R′S′ $= |\det \mathbf{C}| \times$ area of PQRS
$= |1+3| \times 6$
$= 24$ units^2

Chapter 15

GRAPH THEORY

EXERCISE 15A

1 **a** The graph has 6 vertices and 8 edges.



b **i** A is connected by edges to B and E.

$\therefore$ B and E are adjacent to A.

ii D is connected by edges to B, E, and F.

$\therefore$ B, E, and F are adjacent to D.

c **i** CE shares a vertex with AE, DE, and EF.

$\therefore$ AE, DE, and EF are adjacent to CE.

ii DF shares a vertex with BD, DE, and EF.

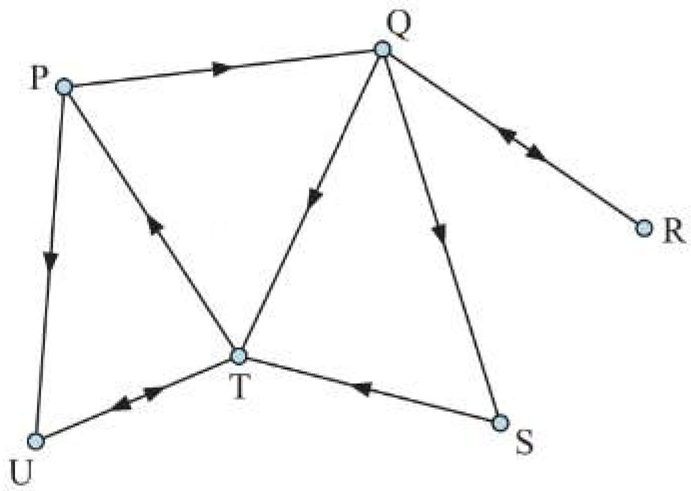
$\therefore$ BD, DE, and EF are adjacent to DF.

d

Vertex	*Degree*
A	2
B	4
C	1
D	3
E	4
F	2

The loop at B contributes 2 to the degree of B.

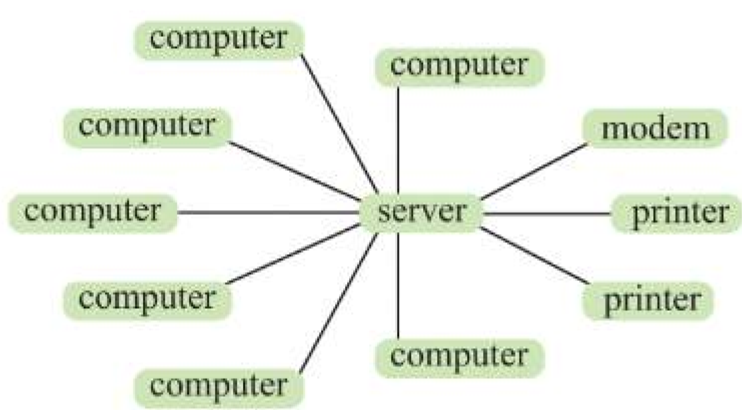
2 **a** There is 1 edge coming in to P (TP), and 2 edges going out from P (PQ and PU).
So, P has in degree 1 and out degree 2.



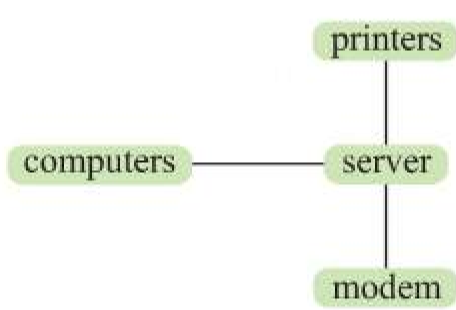
Vertex	*in degree*	*out degree*
P	1	2
Q	2	3
R	1	1
S	1	1
T	3	2
U	2	1

b $S \to T \to P \to Q \to R$

3 **a**

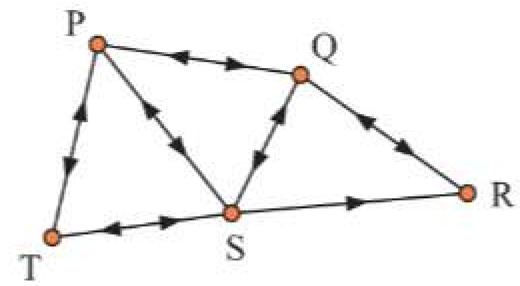


b The groups of components are computers, printers, the server, and the modem.

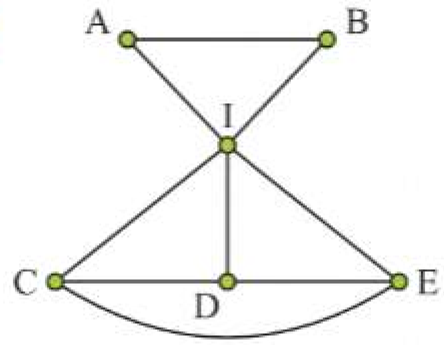


4 **a** The graph is directed as S is connected to R, but R is not connected to S.

b



5 **a**



b **i** There are 2 vertices directly connected to vertex B.

∴ 2 people can communicate directly with B.

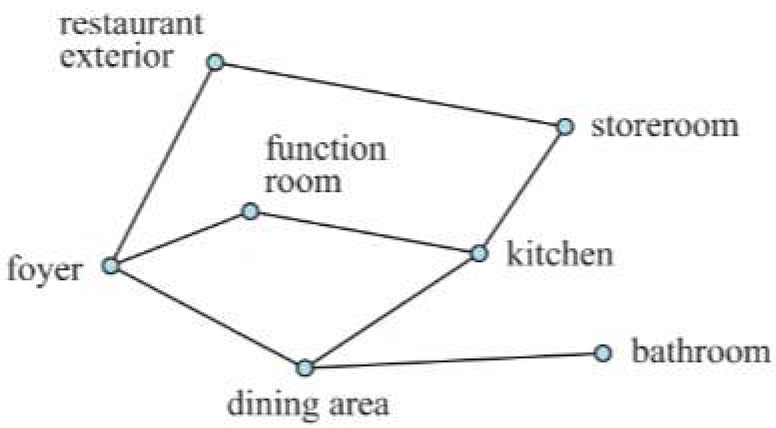
ii There are 3 vertices directly connected to vertex E.

∴ 3 people can communicate directly with E.

iii There are 5 vertices directly connected to vertex I.

∴ 5 people can communicate directly with I.

6 **a**



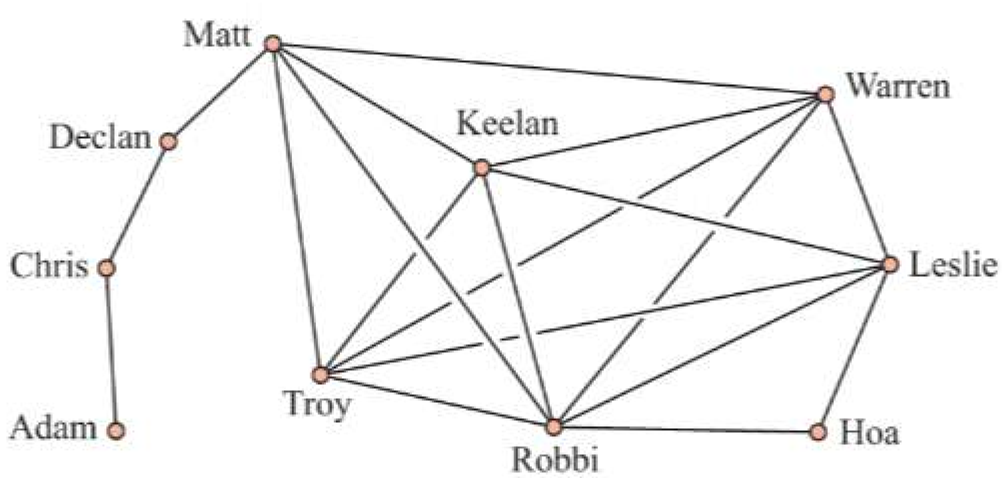
b **i** There are 3 vertices directly connected to the vertex representing the kitchen.

∴ there are 3 rooms connected to the kitchen.

ii There are 3 vertices directly connected to the vertex representing the dining area.

∴ there are 3 rooms connected to the dining area.

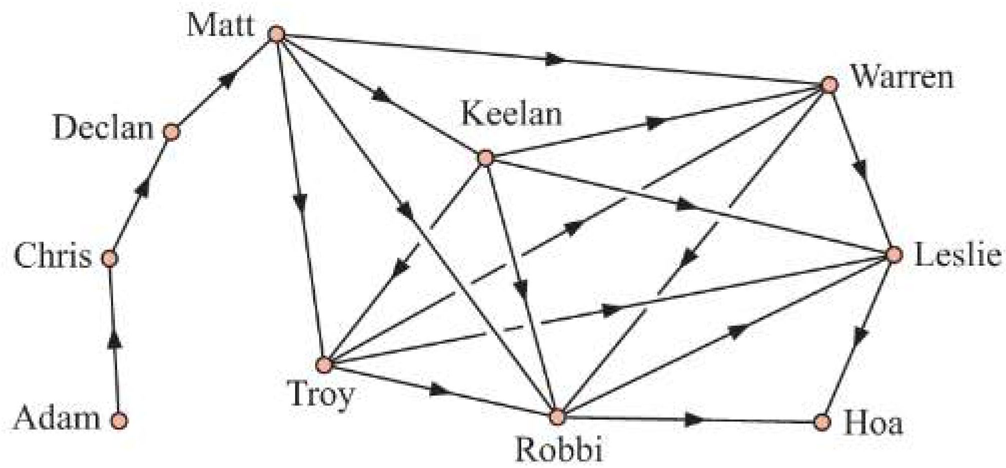
7 **a**



Member	*Weight* (kg)
Declan	82
Matt	75
Adam	94
Warren	68
Keelan	71
Hoa	57
Leslie	63
Chris	86
Robbi	66
Troy	70

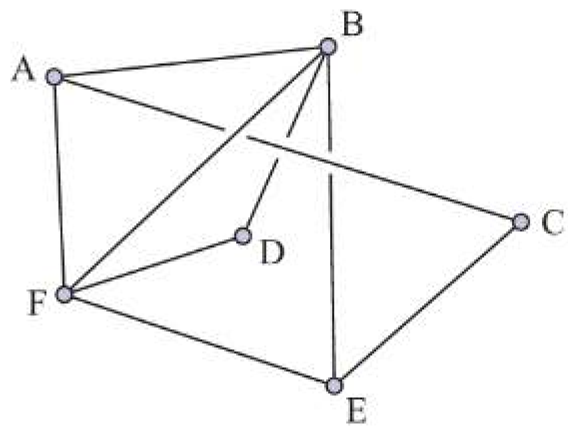
b The vertex corresponding to Warren has degree 5. This means that Warren can compete against 5 other members.

c



d The vertex corresponding to Keelan has in degree 1 and out degree 4. This means that, of his 5 potential opponents, Keelan weighs less than 1 of them (Matt), and weighs more than 4 of them (Troy, Robbi, Leslie, and Warren).

8 **a** **i** The graph has 9 edges.



ii

Vertex	*Degree*
A	3
B	4
C	2
D	2
E	3
F	4

$$\text{Sum of the degree of the vertices} = 3 + 4 + 2 + 2 + 3 + 4 = 18$$

b $d = 2e$, every edge contributes 2 to the total sum of the degrees.

c The sum of the degree of the vertices $= 1 + 2 + 2 + 3 + 4 + 5 + 5 = 22$

$\therefore\ d = 22$, so $e = 11$.

The graph has 11 edges.

d If the undirected graph has an odd number of vertices with odd degree, then the sum of the degrees of the vertices d will be odd.

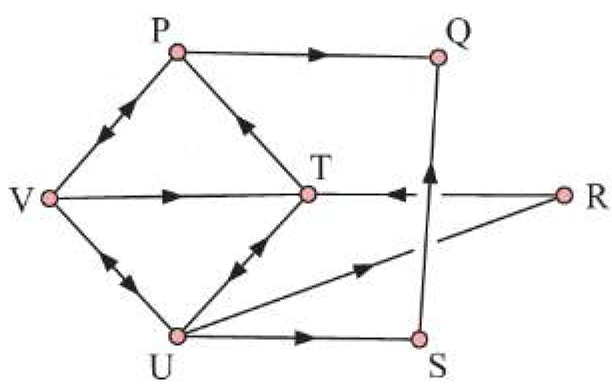
However, $d = 2e$ which must be even.

$\therefore$ an undirected graph must have an even number of vertices of odd degree.

9 No, the degree of a vertex is not necessarily equal to the number of adjacent vertices in an undirected graph. There may be loops, or more than one edge between the same pair of vertices.

10 **a**

Vertex	*in degree*	*out degree*
P	2	2
Q	2	0
R	1	1
S	1	1
T	3	2
U	2	4
V	2	3



b **i** sum of the in degrees $= 2+2+1+1+3+2+2 = 13$

ii sum of the out degrees $= 2+0+1+1+2+4+3 = 13$

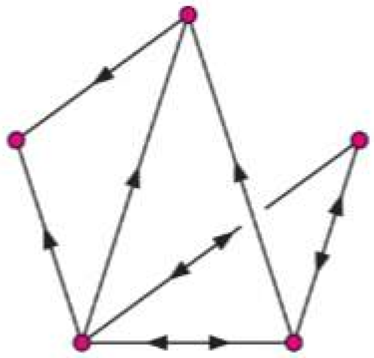
In a directed graph, each movement *to* a vertex has a corresponding movement *from* another vertex. So, the sums of the in and out degrees will be the same.

11 **a** The smallest values for the five distinct out degrees are 0, 1, 2, 3, and 4. These have sum 10, so the sum of the in degrees is also 10.

10 is divisible by 5, so each of the 5 vertices can have in degree $10 \div 5 = 2$.

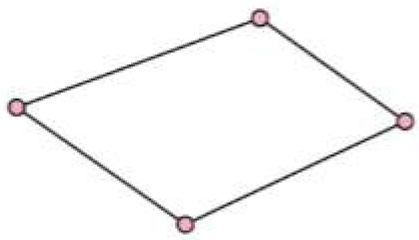
So, the smallest value of k is 2.

b



EXERCISE 15B

1 **a**



i The graph has no loops, and there is a maximum of one edge joining any pair of distinct vertices.

$\therefore$ the graph is simple.

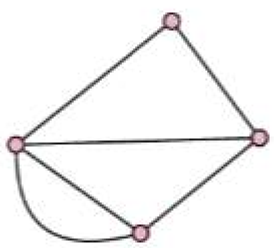
ii It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is connected.

iii Not every vertex is connected to every other vertex by an edge.

$\therefore$ the graph is not complete.

b



i There are two edges joining a pair of distinct vertices.

$\therefore$ the graph is not simple.

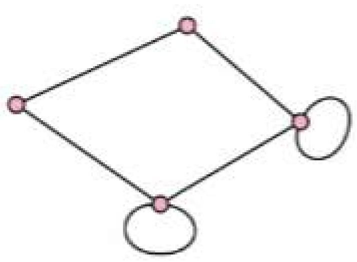
ii It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is connected.

iii Not every vertex is connected to every other vertex by an edge.

$\therefore$ the graph is not complete.

c



i The graph contains loops.

$\therefore$ the graph is not simple.

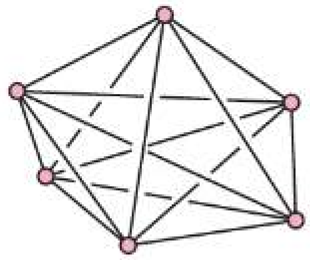
ii It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is connected.

iii Not every vertex is connected to every other vertex by an edge.

$\therefore$ the graph is not complete.

d



i The graph has no loops, and there is a maximum of one edge joining any pair of distinct vertices.

$\therefore$ the graph is simple.

ii It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is connected.

iii Every vertex is connected to every other vertex by an edge.

$\therefore$ the graph is complete.

2 **a** **i** It takes 3 minutes to run from T to S.

ii It takes $8+9=17$ minutes to run from P to R via Q.

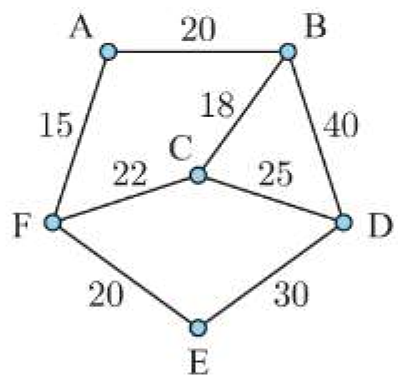
b

	P	Q	R	S	T
P	–	8	–	7	6
Q	8	–	9	–	3
R	–	9	–	5	–
S	7	–	5	–	3
T	6	3	–	3	–

3

	A	B	C	D	E	F
A	–	20	–	–	–	15
B	20	–	18	40	–	–
C	–	18	–	25	–	22
D	–	40	25	–	30	–
E	–	–	–	30	–	20
F	15	–	22	–	20	–

a



b B has degree 3. This means that there are 3 railway stations connected to station B (stations A, C, and D).

c **i** It is possible to travel from every vertex to every other vertex by following edges.

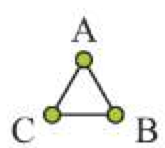
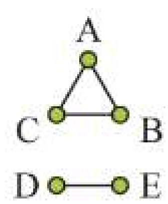
$\therefore$ the graph is connected.

ii Not every vertex is connected to every other vertex by an edge.

$\therefore$ the graph is not complete.

d The cheapest way to travel from A to D is $A \to B \to D$, with cost £20 + £40 = £60.

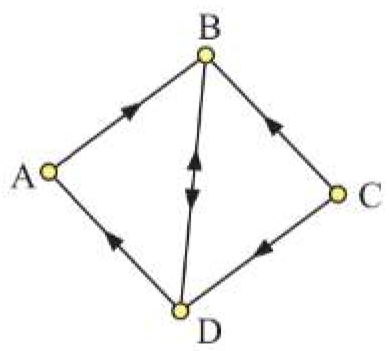
4 **a** The statement "If A is connected then B must be connected." is false.

For example,  is a subgraph of  but the latter is not connected.

b The statement "If B is complete then A must be simple." is true.

A complete graph is simple by definition. A subset of a complete graph will contain no loops and a maximum of one edge joining any pair of distinct vertices. So, the subgraph must also be simple.

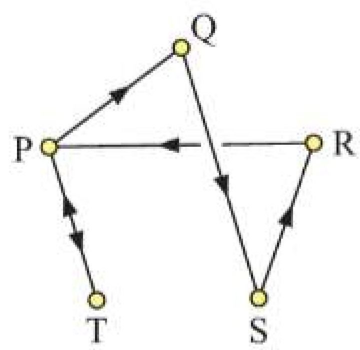
5 **a**



For example, it is not possible to travel from B to C.

$\therefore$ the graph is not strongly connected.

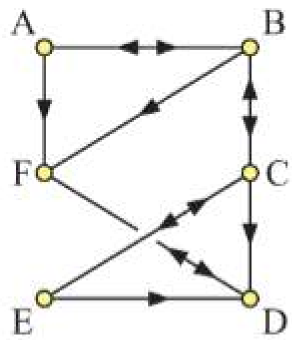
b



It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is strongly connected.

c

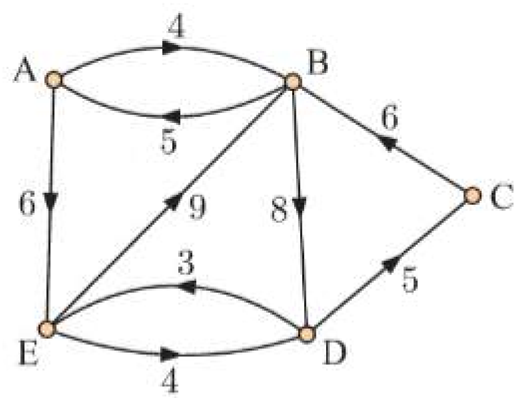


For example, it is not possible to travel from D to E.

$\therefore$ the graph is not strongly connected.

6 **a** It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is strongly connected.

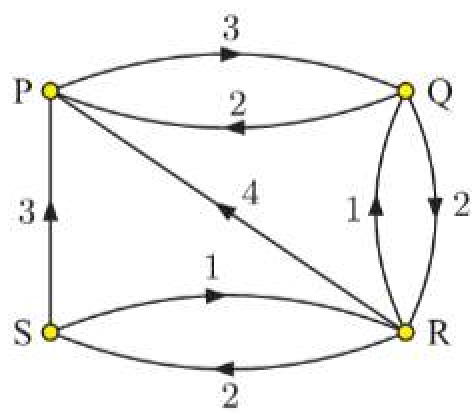


b

	A	B	C	D	E
A	–	4	–	–	6
B	5	–	–	8	–
C	–	6	–	–	–
D	–	–	5	–	3
E	–	9	–	4	–

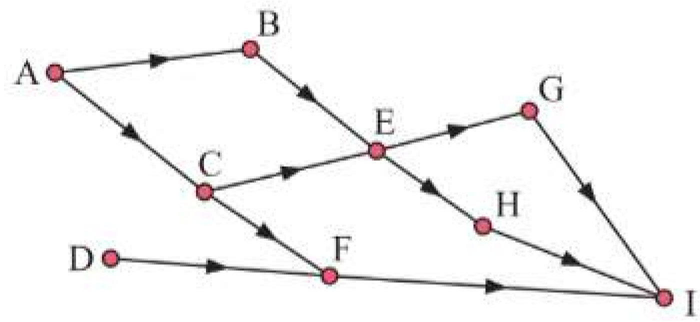
7

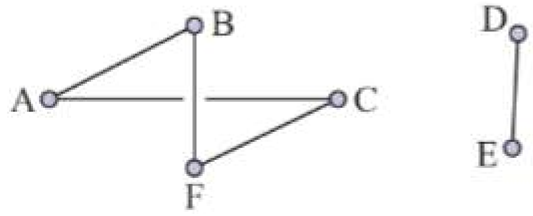
	P	Q	R	S
P	–	3	–	–
Q	2	–	2	–
R	4	1	–	2
S	3	–	1	–



8 **a** It is not possible to travel from every vertex to every other vertex since we cannot travel from right to left on the graph. So, the graph is not strongly connected.

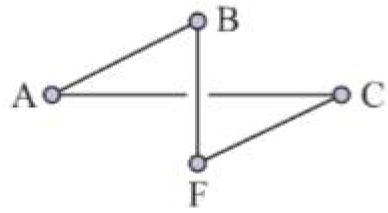
b A minimum of 2 chairlifts must be added so that the graph is strongly connected. For example, chairlifts could be added from I to A, and A to D.



9 

a For example, it is not possible to travel from A to D.

$\therefore$ the graph is disconnected.

b 

10 **a** For a graph to be connected with the minimum number of edges, it must take the form alongside.

For a graph with n vertices, a minimum of $n-1$ edges are required.

b In a complete graph with n vertices, each vertex is connected to every other vertex by an edge.

$\therefore$ each vertex has degree $n-1$.

$\therefore$ the sum of the degrees of the vertices $= n(n-1)$

From **Exercise 15A** question **8**, $d = 2e = n(n-1)$

$$\therefore \quad e = \frac{n(n-1)}{2}$$

So, there are $\dfrac{n(n-1)}{2}$ edges.

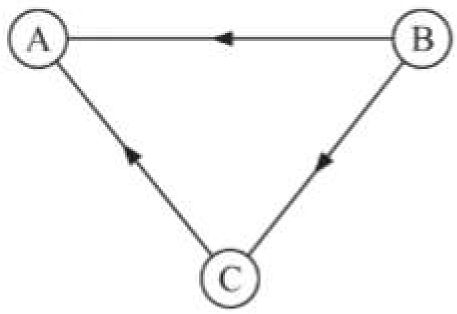
c From **a** and **b**, $n-1 \leqslant e \leqslant \dfrac{n(n-1)}{2}$

$$\therefore \quad 2n-2 \leqslant 2e \leqslant n(n-1)$$

$$\therefore \quad 2n-2 \leqslant 2e \leqslant n^2-n$$

ACTIVITY 1 THE PAGERANK ALGORITHM

1 **a** We have $\mathrm{L(A)} = 0$, $\mathrm{L(B)} = 2$, $\mathrm{L(C)} = 1$



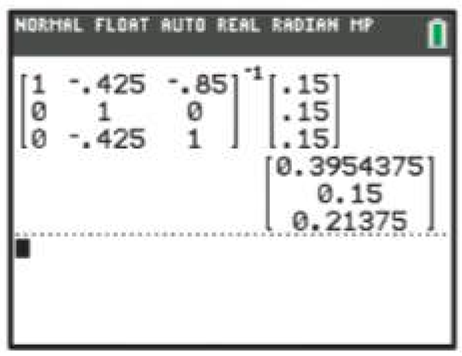
So, $\mathrm{PR(A)} = 0.15 + 0.85\left(\dfrac{\mathrm{PR(B)}}{2} + \dfrac{\mathrm{PR(C)}}{1}\right)$

$\mathrm{PR(B)} = 0.15 + 0.85(0)$

$\mathrm{PR(C)} = 0.15 + 0.85\left(\dfrac{\mathrm{PR(B)}}{2}\right)$

We rearrange this to give the system:

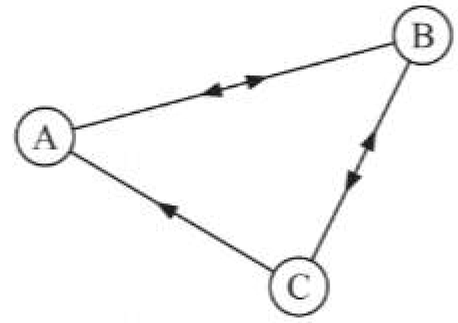
$$\mathrm{PR(A)} - 0.425\,\mathrm{PR(B)} - 0.85\,\mathrm{PR(C)} = 0.15$$
$$\mathrm{PR(B)} = 0.15$$
$$-0.425\,\mathrm{PR(B)} + \mathrm{PR(C)} = 0.15$$



Solving the equations simultaneously gives

$\mathrm{PR(A)} \approx 0.395$, $\mathrm{PR(B)} = 0.15$, and $\mathrm{PR(C)} \approx 0.214$.

b We have $L(A) = 1,\ L(B) = 2,\ L(C) = 2$

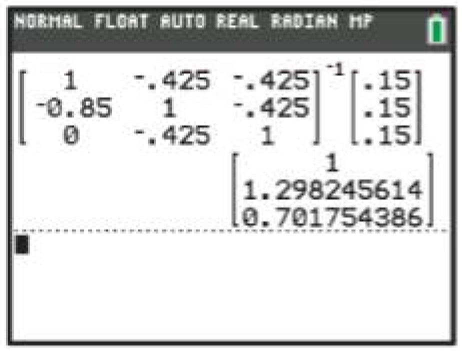


So, $PR(A) = 0.15 + 0.85\left(\frac{PR(B)}{2} + \frac{PR(C)}{2}\right)$

$PR(B) = 0.15 + 0.85\left(\frac{PR(A)}{1} + \frac{PR(C)}{2}\right)$

$PR(C) = 0.15 + 0.85\left(\frac{PR(B)}{2}\right)$

We rearrange this to give the system:

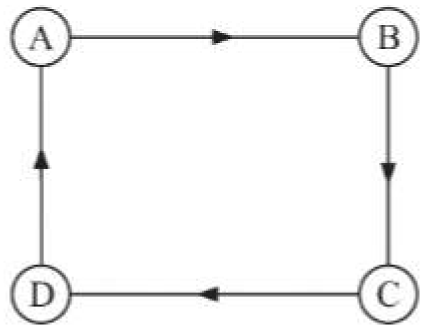


$$PR(A) - 0.425\,PR(B) - 0.425\,PR(C) = 0.15$$
$$-0.85\,PR(A) + PR(B) - 0.425\,PR(C) = 0.15$$
$$-0.425\,PR(B) + PR(C) = 0.15$$

Solving the equations simultaneously gives

$PR(A) = 1,\ PR(B) \approx 1.30,$ and $PR(C) \approx 0.702$.

c We have $L(A) = 1,\ L(B) = 1,\ L(C) = 1,\ L(D) = 1$



So, $PR(A) = 0.15 + 0.85\left(\frac{PR(D)}{1}\right)$

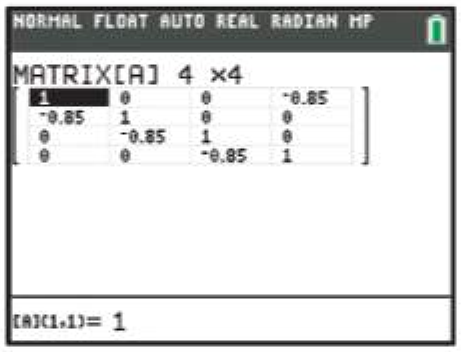
$PR(B) = 0.15 + 0.85\left(\frac{PR(A)}{1}\right)$

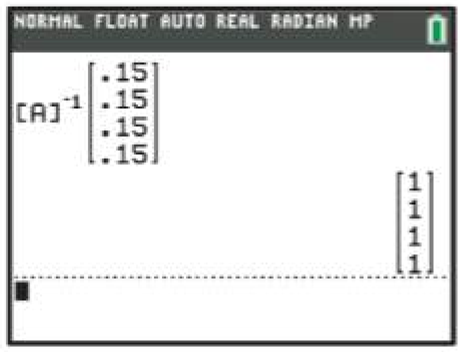
$PR(C) = 0.15 + 0.85\left(\frac{PR(B)}{1}\right)$

$PR(D) = 0.15 + 0.85\left(\frac{PR(C)}{1}\right)$

We rearrange this to give the system:

$$PR(A) - 0.85\,PR(D) = 0.15$$
$$-0.85\,PR(A) + PR(B) = 0.15$$
$$-0.85\,PR(B) + PR(C) = 0.15$$
$$-0.85\,PR(C) + PR(D) = 0.15$$

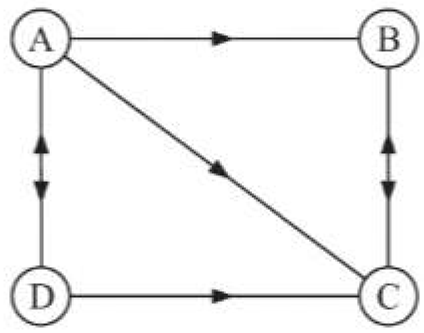




Solving the equations simultaneously gives

$PR(A) = 1,\ PR(B) = 1,\ PR(C) = 1,$ and $PR(D) = 1$.

d We have $L(A) = 3,\ L(B) = 1,\ L(C) = 1,\ L(D) = 2$



So, $PR(A) = 0.15 + 0.85\left(\frac{PR(D)}{2}\right)$

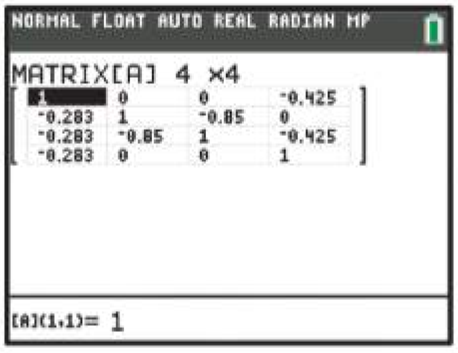
$PR(B) = 0.15 + 0.85\left(\frac{PR(A)}{3} + \frac{PR(C)}{1}\right)$

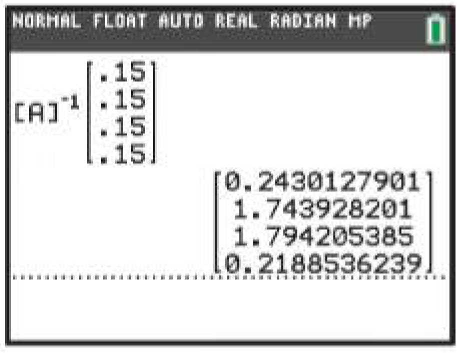
$PR(C) = 0.15 + 0.85\left(\frac{PR(A)}{3} + \frac{PR(B)}{1} + \frac{PR(D)}{2}\right)$

$PR(D) = 0.15 + 0.85\left(\frac{PR(A)}{3}\right)$

We rearrange this to give the system:

$$\begin{aligned} \mathrm{PR(A)} - 0.425\ \mathrm{PR(D)} &= 0.15 \\ -\frac{0.85}{3}\ \mathrm{PR(A)} + \mathrm{PR(B)} - 0.85\ \mathrm{PR(C)} &= 0.15 \\ -\frac{0.85}{3}\ \mathrm{PR(A)} - 0.85\ \mathrm{PR(B)} + \mathrm{PR(C)} - 0.425\ \mathrm{PR(D)} &= 0.15 \\ -\frac{0.85}{3}\ \mathrm{PR(A)} + \mathrm{PR(D)} &= 0.15 \end{aligned}$$

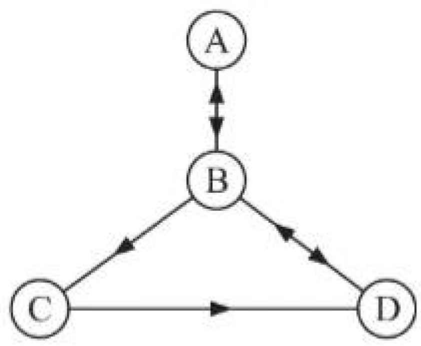




Solving the equations simultaneously gives

$\mathrm{PR(A)} \approx 0.243$, $\mathrm{PR(B)} \approx 1.74$, $\mathrm{PR(C)} \approx 1.79$, and $\mathrm{PR(D)} \approx 0.219$.

e We have $L(A) = 1$, $L(B) = 3$, $L(C) = 1$, $L(D) = 1$

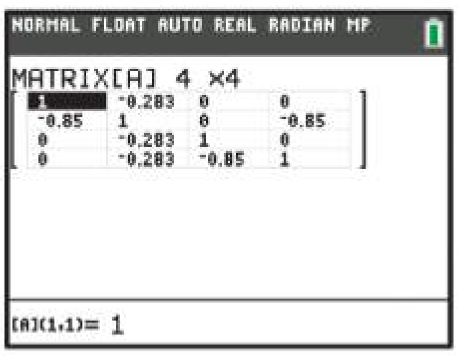


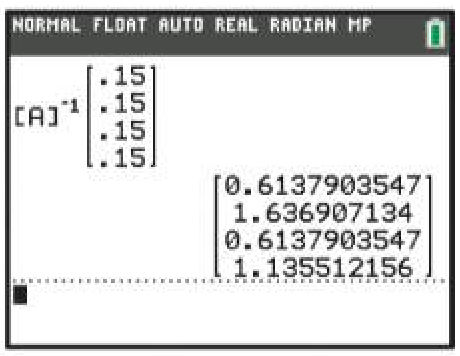
So,

$$\begin{aligned} \mathrm{PR(A)} &= 0.15 + 0.85\left(\frac{\mathrm{PR(B)}}{3}\right) \\ \mathrm{PR(B)} &= 0.15 + 0.85\left(\frac{\mathrm{PR(A)}}{1} + \frac{\mathrm{PR(D)}}{1}\right) \\ \mathrm{PR(C)} &= 0.15 + 0.85\left(\frac{\mathrm{PR(B)}}{3}\right) \\ \mathrm{PR(D)} &= 0.15 + 0.85\left(\frac{\mathrm{PR(B)}}{3} + \frac{\mathrm{PR(C)}}{1}\right) \end{aligned}$$

We rearrange this to give the system:

$$\begin{aligned} \mathrm{PR(A)} - \frac{0.85}{3}\ \mathrm{PR(B)} &= 0.15 \\ -0.85\ \mathrm{PR(A)} + \mathrm{PR(B)} - 0.85\ \mathrm{PR(D)} &= 0.15 \\ -\frac{0.85}{3}\ \mathrm{PR(B)} + \mathrm{PR(C)} &= 0.15 \\ -\frac{0.85}{3}\ \mathrm{PR(B)} - 0.85\ \mathrm{PR(C)} + \mathrm{PR(D)} &= 0.15 \end{aligned}$$

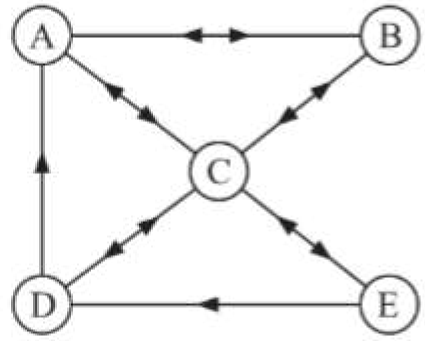




Solving the equations simultaneously gives

$\mathrm{PR(A)} \approx 0.614$, $\mathrm{PR(B)} \approx 1.64$, $\mathrm{PR(C)} \approx 0.614$, and $\mathrm{PR(D)} \approx 1.14$.

f We have $L(A) = 2$, $L(B) = 2$, $L(C) = 4$, $L(D) = 2$, $L(E) = 2$



So, $PR(A) = 0.15 + 0.85\left(\frac{PR(B)}{2} + \frac{PR(C)}{4} + \frac{PR(D)}{2}\right)$

$PR(B) = 0.15 + 0.85\left(\frac{PR(A)}{2} + \frac{PR(C)}{4}\right)$

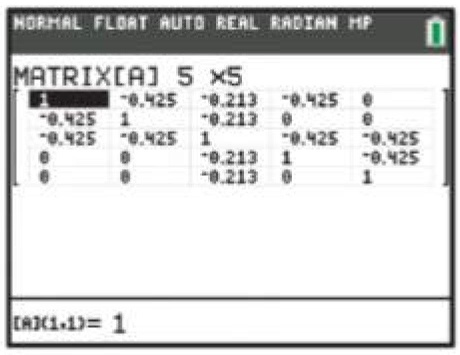
$PR(C) = 0.15 + 0.85\left(\frac{PR(A)}{2} + \frac{PR(B)}{2} + \frac{PR(D)}{2} + \frac{PR(E)}{2}\right)$

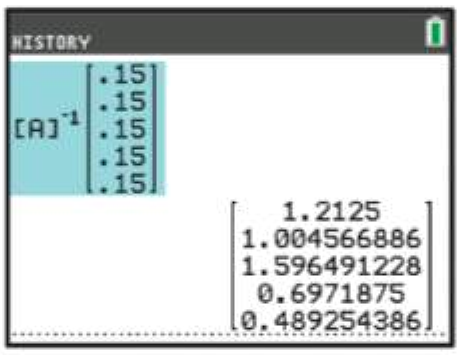
$PR(D) = 0.15 + 0.85\left(\frac{PR(C)}{4} + \frac{PR(E)}{2}\right)$

$PR(E) = 0.15 + 0.85\left(\frac{PR(C)}{4}\right)$

We rearrange this to give the system:

$$\begin{aligned}
PR(A) - 0.425\,PR(B) - 0.2125\,PR(C) - 0.425\,PR(D) &= 0.15\\
-0.425\,PR(A) + PR(B) - 0.2125\,PR(C) &= 0.15\\
-0.425\,PR(A) - 0.425\,PR(B) + PR(C) - 0.425\,PR(D) - 0.425\,PR(E) &= 0.15\\
-0.2125\,PR(C) + PR(D) - 0.425\,PR(E) &= 0.15\\
-0.2125\,PR(C) + PR(E) &= 0.15
\end{aligned}$$





Solving the equations simultaneously gives

$PR(A) \approx 1.21$, $PR(B) \approx 1.00$, $PR(C) \approx 1.60$, $PR(D) \approx 0.697$, and $PR(E) \approx 0.489$.

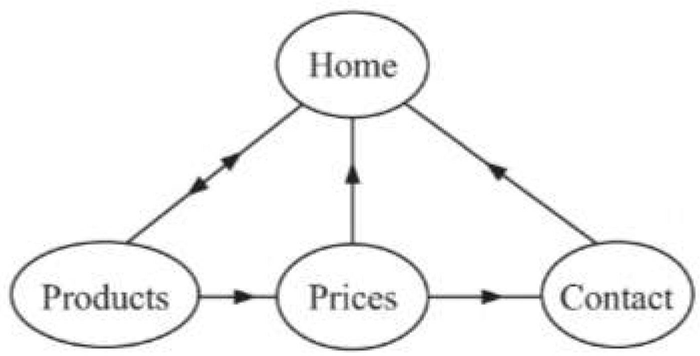
2 **a** We have $L(\text{Home}) = 1$, $L(\text{Products}) = 2$, $L(\text{Prices}) = 2$, $L(\text{Contact}) = 1$

So, $PR(\text{Home}) = 0.15 + 0.85\left(\frac{PR(\text{Products})}{2} + \frac{PR(\text{Prices})}{2} + \frac{PR(\text{Contact})}{1}\right)$

$PR(\text{Products}) = 0.15 + 0.85\left(\frac{PR(\text{Home})}{1}\right)$

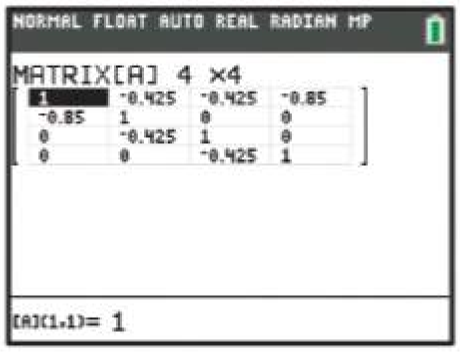
$PR(\text{Prices}) = 0.15 + 0.85\left(\frac{PR(\text{Products})}{2}\right)$

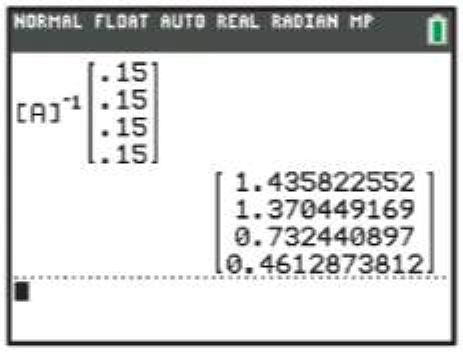
$PR(\text{Contact}) = 0.15 + 0.85\left(\frac{PR(\text{Prices})}{2}\right)$



We rearrange this to give the system:

$$\begin{aligned}
PR(\text{Home}) - 0.425\,PR(\text{Products}) - 0.425\,PR(\text{Prices}) - 0.85\,PR(\text{Contact}) &= 0.15\\
-0.85\,PR(\text{Home}) + PR(\text{Products}) &= 0.15\\
-0.425\,PR(\text{Products}) + PR(\text{Prices}) &= 0.15\\
-0.425\,PR(\text{Prices}) + PR(\text{Contact}) &= 0.15
\end{aligned}$$





Solving the equations simultaneously gives

PR(Home) ≈ 1.44, PR(Products) ≈ 1.37, PR(Prices) ≈ 0.732, and PR(Contact) ≈ 0.461.

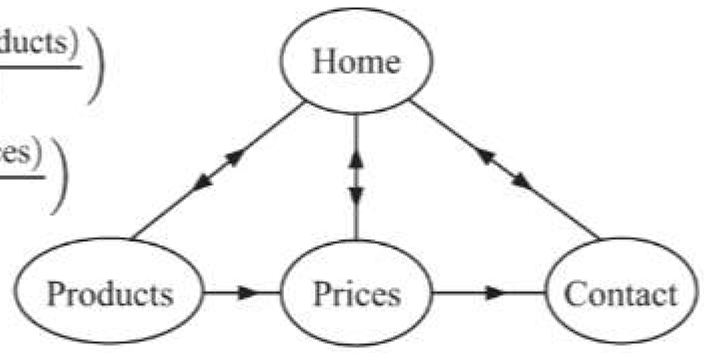
b We have L(Home) $= 3$, L(Products) $= 2$, L(Prices) $= 2$, L(Contact) $= 1$

$$\text{So, } \text{PR(Home)} = 0.15 + 0.85\left(\frac{\text{PR(Products)}}{2} + \frac{\text{PR(Prices)}}{2} + \frac{\text{PR(Contact)}}{1}\right)$$

$$\text{PR(Products)} = 0.15 + 0.85\left(\frac{\text{PR(Home)}}{3}\right)$$

$$\text{PR(Prices)} = 0.15 + 0.85\left(\frac{\text{PR(Home)}}{3} + \frac{\text{PR(Products)}}{2}\right)$$

$$\text{PR(Contact)} = 0.15 + 0.85\left(\frac{\text{PR(Home)}}{3} + \frac{\text{PR(Prices)}}{2}\right)$$



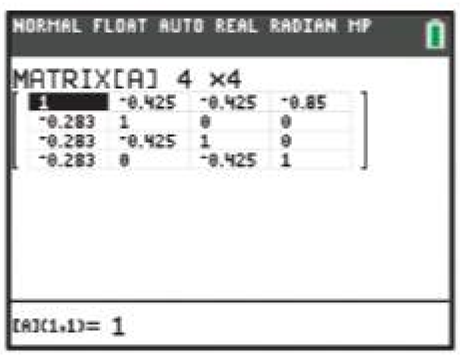
We rearrange this to give the system:

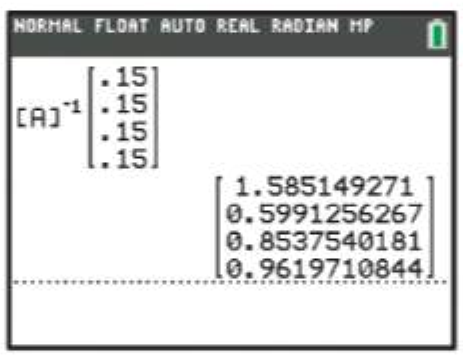
$$\text{PR(Home)} - 0.425\,\text{PR(Products)} - 0.425\,\text{PR(Prices)} - 0.85\,\text{PR(Contact)} = 0.15$$

$$-\frac{0.85}{3}\,\text{PR(Home)} + \text{PR(Products)} = 0.15$$

$$-\frac{0.85}{3}\,\text{PR(Home)} - 0.425\,\text{PR(Products)} + \text{PR(Prices)} = 0.15$$

$$-\frac{0.85}{3}\,\text{PR(Home)} - 0.425\,\text{PR(Prices)} + \text{PR(Contact)} = 0.15$$





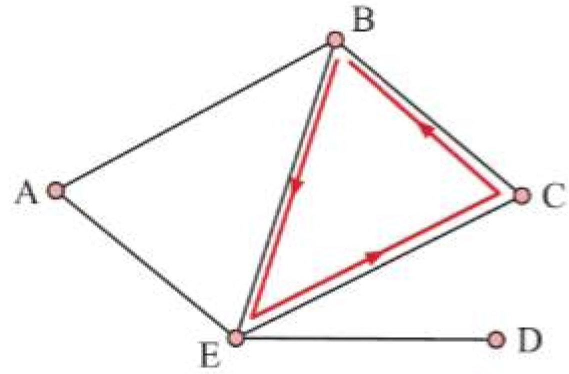
Solving the equations simultaneously gives

PR(Home) ≈ 1.59, PR(Products) ≈ 0.599, PR(Prices) ≈ 0.854, and PR(Contact) ≈ 0.962.

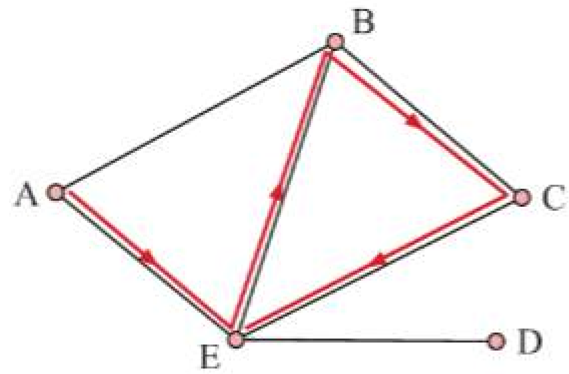
c Adding links from the Home page to the Prices and Contact pages increases the PageRank of those pages, which in turn increases the PageRank of the Home page, since those pages link back to the Home page.

EXERCISE 15C

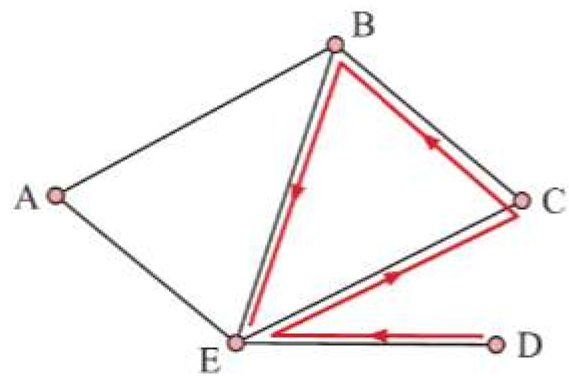
1 **a** B → E → C → B is a walk as there is an edge between each successive pair of vertices.



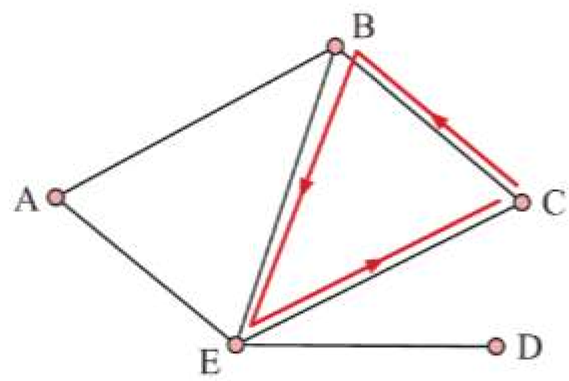
b A → E → B → C → E is a trail as it is a walk in which no edge is repeated.



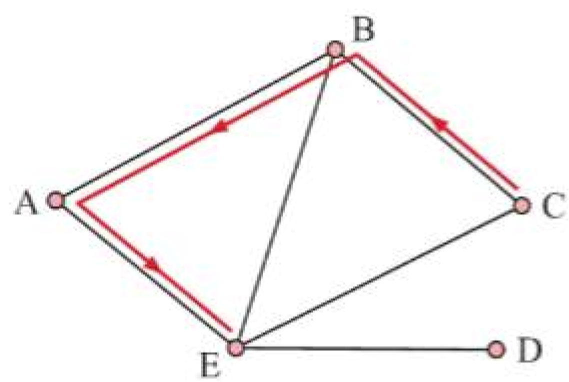
c D → E → C → B → E is *not* a path as the vertex E is repeated.



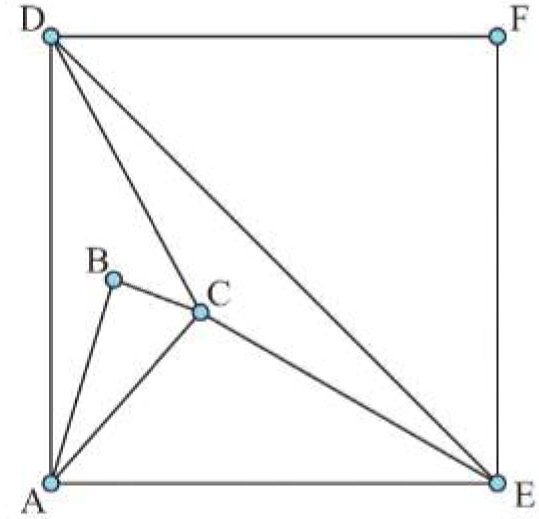
d E → C → B → E is a circuit as no edges are repeated, and it starts and finishes at the same vertex.



e C → B → A → E is *not* a cycle as it does not start and finish at the same vertex.



2 These are **examples only**.



a A → C → D is a path of length 2 from A to D.

b A → B → C → D is a path of length 3 from A to D.

c A → B → C → E → D is a path of length 4 from A to D.

d B → A → C → E → A → D is a trail which is not a path, of length 5 from B to D.

e A → B → C → D → E → A is a cycle of length 5.

f A cycle of length 7 is not possible as it would require 7 distinct vertices, but the graph only has 6 vertices.

g A → B → C → D → E → F → D → A is a circuit which is not a cycle, of length 7.

h A → D → F → E → A → B → C → D → E → C → A is a circuit of length 10.

3 **a** The statement "Every path is a trail." is true. A trail only requires that no edge is repeated.

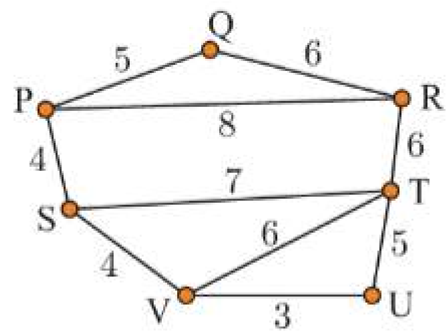
b The statement "Every trail is a path." is false. A trail can have repeated vertices, but a path cannot.

c The statement "Every cycle is a circuit." is true. A cycle is a circuit by definition.

d The statement "No circuit is a path." is true. Every circuit has a repeated vertex.

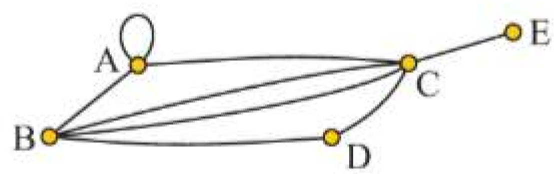
4 **a**, **b**

Path	*Weight*
P → Q → R → T	$5+6+6=17$
P → R → T	$8+6=14$
P → S → T	$4+7=11$
P → S → V → T	$4+4+6=14$
P → S → V → U → T	$4+4+3+5=16$



c Q → P → S → V → U is the trail of lowest total weight. It has weight $5+4+4+3=16$.

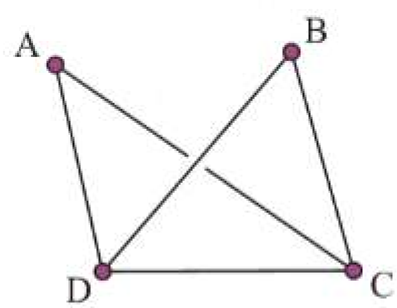
5 **a** E → C → A → A → B → C → B → D → C is a trail which includes every edge.



b A circuit which includes every edge is not possible. For example, we cannot include edge EC without traversing it twice.

EXERCISE 15D.1

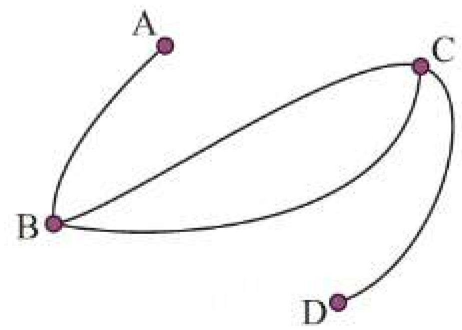
1 **a** There are 4 vertices, so the adjacency matrix has order 4×4.
The vertices are in order A, B, C, D.
The adjacency matrix is



$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

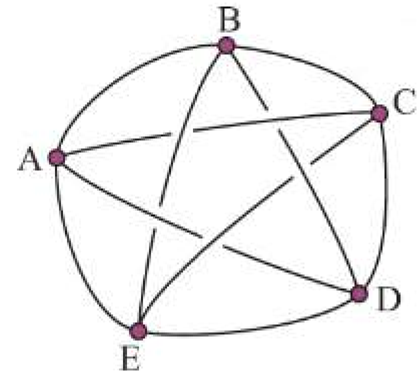
b There are 4 vertices, so the adjacency matrix has order 4×4.
The vertices are in order A, B, C, D.
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



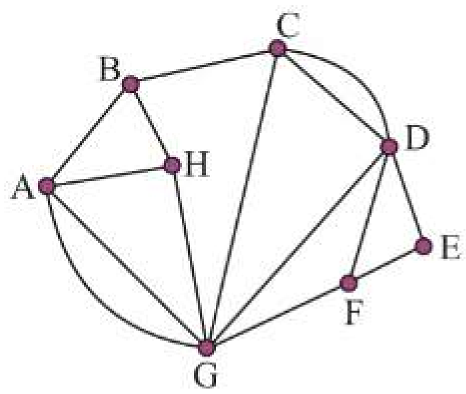
c There are 5 vertices, so the adjacency matrix has order 5×5.
The vertices are in order A, B, C, D, E.
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$



d There are 8 vertices, so the adjacency matrix has order 8×8.
The vertices are in order A, B, C, D, E, F, G, H.
The adjacency matrix is

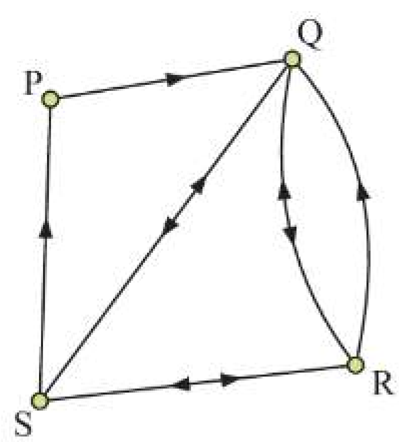
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



2 Terms opposite the leading diagonal correspond to the same pair of vertices.
In an undirected graph we can move between a pair of vertices in either direction, and in the same number of ways.
So, the adjacency matrix for an undirected graph is always symmetric about the leading diagonal.

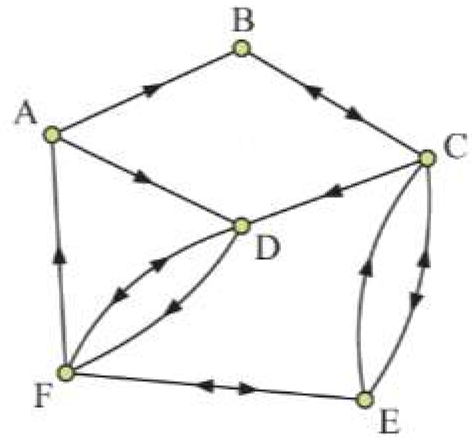
3 **a** There are 4 vertices, so the adjacency matrix has order 4×4.
The vertices are in order P, Q, R, S.
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



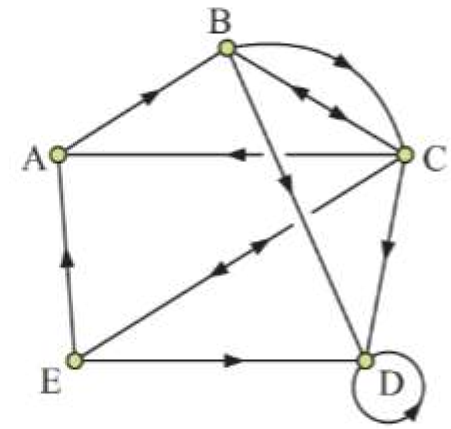
b There are 6 vertices, so the adjacency matrix has order 6×6.
The vertices are in order A, B, C, D, E, F.
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



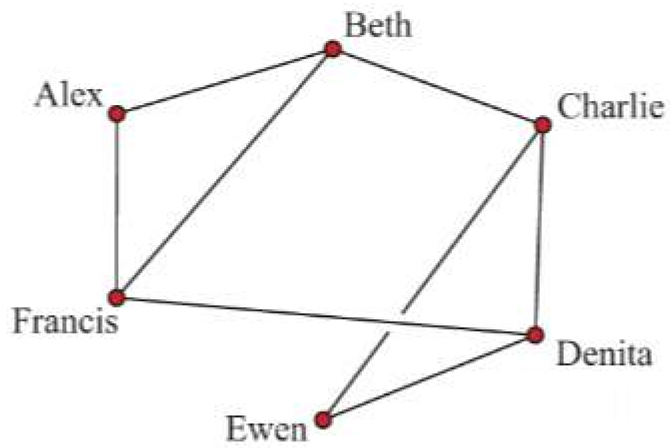
c There are 5 vertices, so the adjacency matrix has order 5×5.
The vertices are in order A, B, C, D, E.
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$



4 **a** There are 6 vertices, so the adjacency matrix has order 6×6.
The vertices are in order Alex, Beth, Charlie, Denita, Ewen, Francis.
The adjacency matrix is

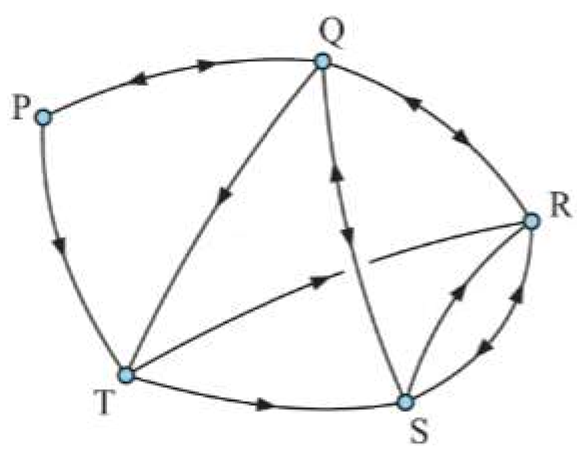
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$



b The adjacency matrix is symmetric about the leading diagonal which indicates that the graph is undirected.

5 **a** There are 5 vertices, so the adjacency matrix has order 5×5.
The vertices are in order P, Q, R, S, T.
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

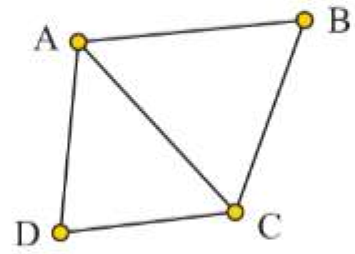


b The sum of the 2nd row of the adjacency matrix is $1+0+1+1+1=4$.
This is the number of 1-step routes starting at Q.

c The sum of the 3rd column of the adjacency matrix is $0+1+0+2+1=4$.
This is the number of 1-step routes ending at R.

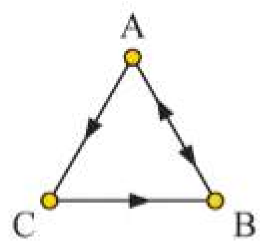
6 **a** For the adjacency matrix
$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

the graph is



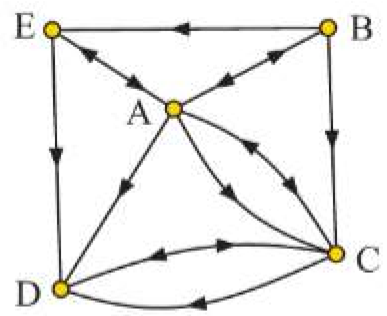
b For the adjacency matrix
$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

the graph is



c For the adjacency matrix
$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

the graph is

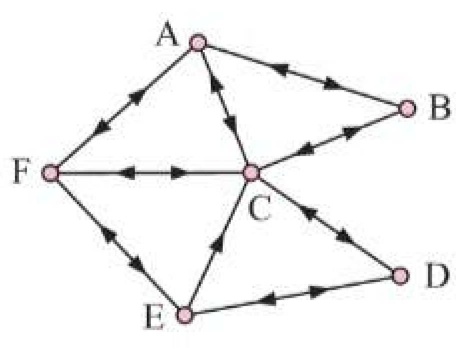


7 **a** The adjacency matrix shows that there is a 1-step route from E to C but no 1-step route from C to E.

$\therefore$ the street from E to C is a one-way street.

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

b

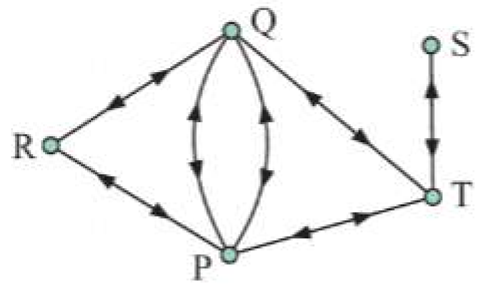


8 **a** The element in the 2nd row, 4th column of the adjacency matrix is 0, so there are no 1-step routes from Q to S.
$\therefore$ there is no direct route from Q to S.

$$\begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

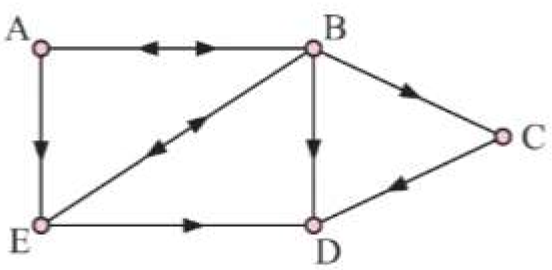
b From the adjacency matrix, there is a 1-step route from Q to T, and a 1-step route from T to S.
So, a 2-step route from Q to S is $Q \to T \to S$.

c



INVESTIGATION MULTI-STEP ROUTES

1



has adjacency matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$.

a There is one 2-step route from E to A $\{E \to B \to A\}$, so the shaded entry is 1.

b

Finishing vertex (columns); *Starting vertex* (rows)

	A	B	C	D	E
A	1	1	1	2	1
B	0	2	0	2	1
C	0	0	0	0	0
D	0	0	0	0	0
E	1	0	1	1	1

2 $\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$ $\mathbf{A}^2$ is the adjacency matrix for the 2-step routes.

3 We predict that $\mathbf{A}^3$ is the adjacency matrix for the 3-step routes and $\mathbf{A}^4$ is the adjacency matrix for the 4-step routes.

For example, $\mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 1 & 2 & \boxed{3} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \end{pmatrix}$

The shaded entry tells us there are three 3-step routes from B to D.
These are $B \to A \to E \to D$, $B \to A \to B \to D$, and $B \to E \to B \to D$.

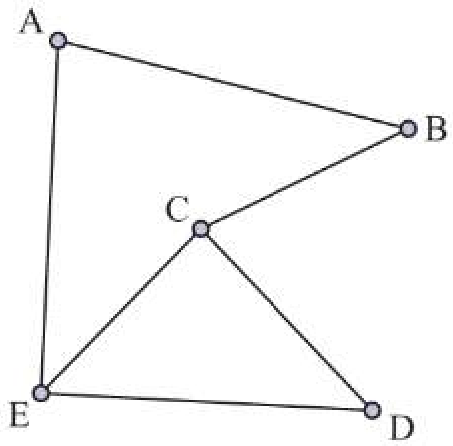
EXERCISE 15D.2

1 **a** There are 5 vertices, so the adjacency matrix has order 5×5.

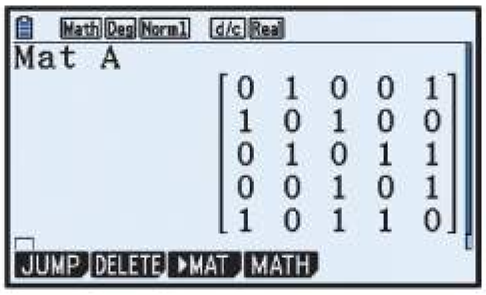
The vertices are in order A, B, C, D, E.

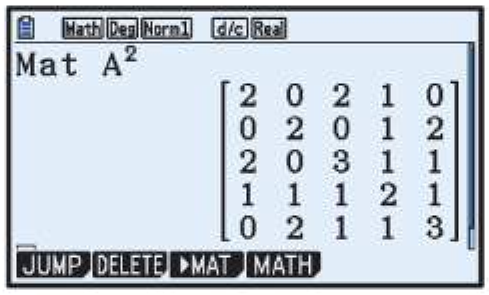
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$



b





Using technology, $\mathbf{A}^2 = \begin{pmatrix} 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 2 & 0 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 & 3 \end{pmatrix}$

c The number in the 5th row and 2nd column of $\mathbf{A}^2$ is 2.

$\therefore$ there are two 2-step routes from E to B.

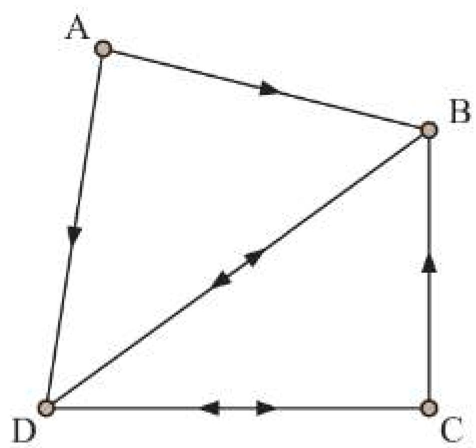
d The two 2-step routes from E to B are E $\rightarrow$ C $\rightarrow$ B and E $\rightarrow$ A $\rightarrow$ B.

2 **a** There are 4 vertices, so the adjacency matrix has order 4×4.

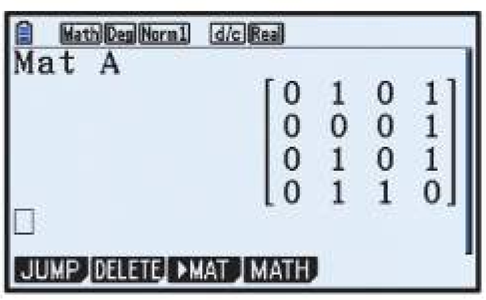
The vertices are in order A, B, C, D.

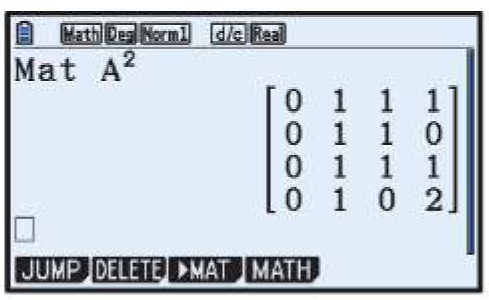
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$



b



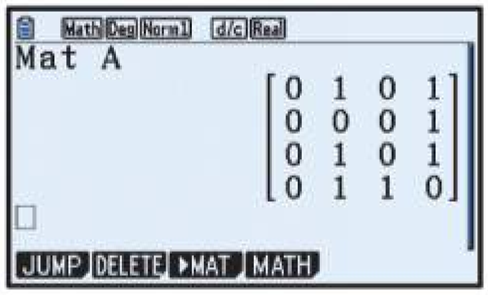


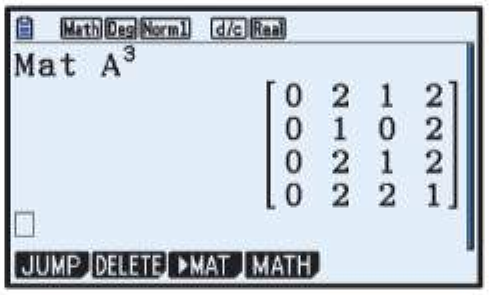
Using technology, $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$

c **i** The 2 in row 4, column 4 signifies that there are two 2-step routes from D to D.

ii The value in row 1, column 2 is 1 which is the number of 2-step routes from A to B.
The value in row 2, column 1 is 0 which is the number of 2-step routes from B to A.
Because the paths are directed, a 2-step route from A to B does not necessarily correspond to a 2-step route from B to A, so the values are different.

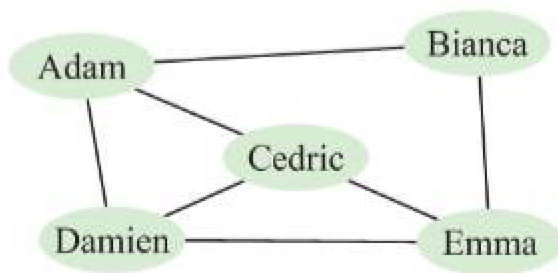
d





Using technology, $\mathbf{A}^3 = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$ This shows the number of 3-step routes between each of the vertices.

3 **a** There are 5 vertices, so the adjacency matrix has order 5×5.
The vertices are in order Adam, Bianca, Cedric, Damien, Emma.
The adjacency matrix is



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

b Using technology,

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 3 & 1 \\ 3 & 0 & 1 & 1 & 3 \end{pmatrix}$$

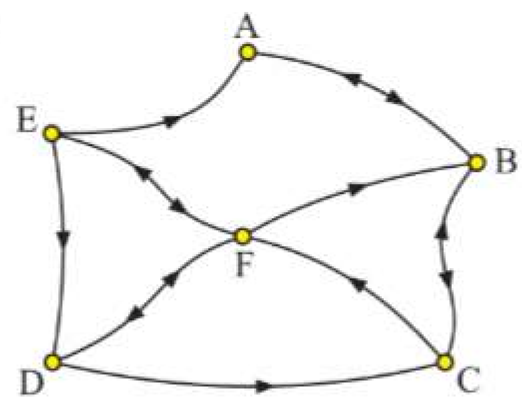
$\mathbf{A}^2$ shows the number of "mutual friends" between each pair of class members.

$$\mathbf{A}^3 = \begin{pmatrix} 2 & 6 & 7 & 7 & 2 \\ 6 & 0 & 2 & 2 & 6 \\ 7 & 2 & 4 & 5 & 7 \\ 7 & 2 & 5 & 4 & 7 \\ 2 & 6 & 7 & 7 & 2 \end{pmatrix}$$

$\mathbf{A}^3$ shows the number of ways class members are connected via 3 friendships.

c In $\mathbf{A}$, there is a 1 in columns 2 and 3 of row 1, indicating that Adam is friends with both Bianca and Cedric, and a 0 in row 2, column 3 (and in row 3, column 2) indicating that Bianca and Cedric do not like each other.

4 **a** There are 6 vertices, so the adjacency matrix has order 6×6.
The vertices are in order A, B, C, D, E, F.
The adjacency matrix is



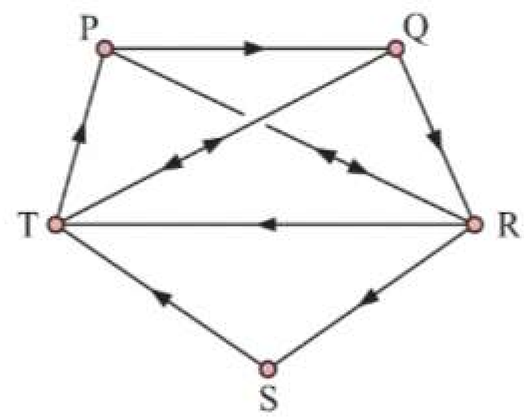
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

b Using technology, $\mathbf{A}^3 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 0 & 3 \\ 3 & 1 & 3 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 & 1 & 3 \\ 0 & 6 & 1 & 2 & 2 & 3 \end{pmatrix}$

c The number in the 5th row and 3rd column of $\mathbf{A}^3$ is 3.
$\therefore$ there are three 3-step routes from E to C.
They are $E \to A \to B \to C$, $E \to F \to B \to C$, and $E \to F \to D \to C$.

d The number in the 6th row and 2nd column is 6.
$\therefore$ there are six 3-step routes from F to B.
For a route to be a path, no vertex is visited more than once.
The only routes which are paths are $F \to D \to C \to B$ and $F \to E \to A \to B$.

5 **a** There are 5 vertices, so the adjacency matrix has order 5×5.
The vertices are in order P, Q, R, S, T.
The adjacency matrix is



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

b Using technology:

i $\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}$

ii $\mathbf{A}^3 = \begin{pmatrix} 3 & 3 & 1 & 1 & 2 \\ 1 & 3 & 3 & 0 & 2 \\ 2 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 2 & 3 \end{pmatrix}$

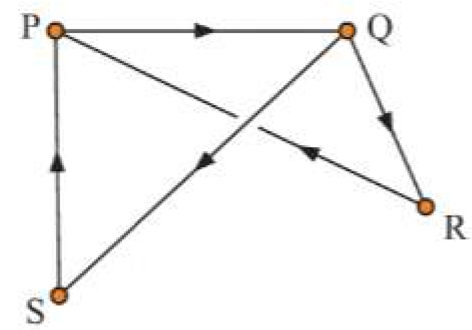
iii $\mathbf{A}^4 = \begin{pmatrix} 3 & 5 & 6 & 1 & 5 \\ 5 & 3 & 4 & 3 & 6 \\ 6 & 5 & 4 & 3 & 6 \\ 3 & 1 & 1 & 2 & 3 \\ 4 & 6 & 4 & 1 & 4 \end{pmatrix}$

c We consider the values in the 4th row and 4th column of $\mathbf{A}^2$, $\mathbf{A}^3$, and $\mathbf{A}^4$ which correspond to the 2-step, 3-step, and 4-step routes respectively, starting from Sam and returning to Sam. This value is 0 in $\mathbf{A}^2$ and $\mathbf{A}^3$, which indicates that there are no 2-step nor 3-step routes from Sam back to Sam. This value is 2 in $\mathbf{A}^4$, which indicates that there are two 4-step routes from Sam back to Sam.

The two 4-step routes are $S \to T \to P \to R \to S$ and $S \to T \to Q \to R \to S$.

So, if Sam starts a rumour, it must pass through at least 3 other people before it gets back to him.

6 **a** There are 4 vertices, so the adjacency matrix has order 4×4.



The vertices are in order P, Q, R, S.

The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

b Using technology, $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $\mathbf{A}^3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

These matrices show the number of 2-step and 3-step routes between each of the locations.

c **i** We consider the values in the 4th row and 3rd column of $\mathbf{A}^2$ and $\mathbf{A}^3$ which correspond to the 2-step and 3-step routes respectively from S to R.

This value is 0 in $\mathbf{A}^2$ and 1 in $\mathbf{A}^3$, which indicates that there are no 2-step routes but one 3-step route from S to R.

This 3-step route is $S \to P \to Q \to R$.

So, the least number of train rides needed to travel from S to R is 3 train rides.

ii We consider the values in the 3rd row and 4th column of $\mathbf{A}^2$ and $\mathbf{A}^3$, which correspond to the 2-step and 3-step routes respectively from R to S.

This value is 0 in $\mathbf{A}^2$ and 1 in $\mathbf{A}^3$, which indicates that there are no 2-step routes but one 3-step route from R to S.

This 3-step route is $R \to P \to Q \to S$.

So, the least number of train rides needed to travel from R to S is 3 train rides.

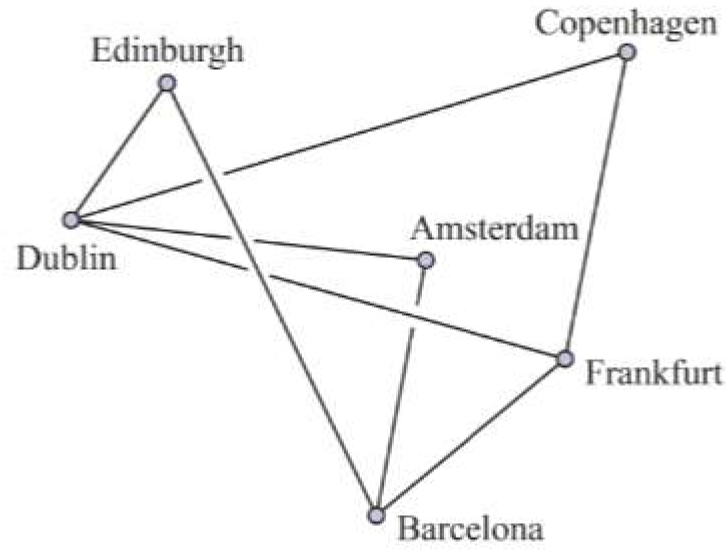
iii In order to find the least number of train rides needed to travel from S to R and back to S, we add the least number of train rides needed to travel from S to R, to the least number of train rides needed to travel from R to S.

From **i** and **ii**, the least number of train rides needed to travel from S to R and back to S is $3 + 3 = 6$ train rides.

7 **a** There are 6 vertices, so the adjacency matrix has order 6×6.

The vertices are in order Amsterdam, Barcelona, Copenhagen, Dublin, Edinburgh, Frankfurt.

The adjacency matrix is



$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

b Using technology:

i $\mathbf{A}^2 = \begin{pmatrix} 2 & 0 & 1 & 0 & 2 & 2 \\ 0 & 3 & 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 2 & 3 \end{pmatrix}$

ii $\mathbf{A}^4 = \begin{pmatrix} 13 & 1 & 8 & 3 & 13 & 15 \\ 1 & 19 & 8 & 22 & 1 & 4 \\ 8 & 8 & 9 & 10 & 8 & 10 \\ 3 & 22 & 10 & 27 & 3 & 8 \\ 13 & 1 & 8 & 3 & 13 & 15 \\ 15 & 4 & 10 & 8 & 15 & 19 \end{pmatrix}$

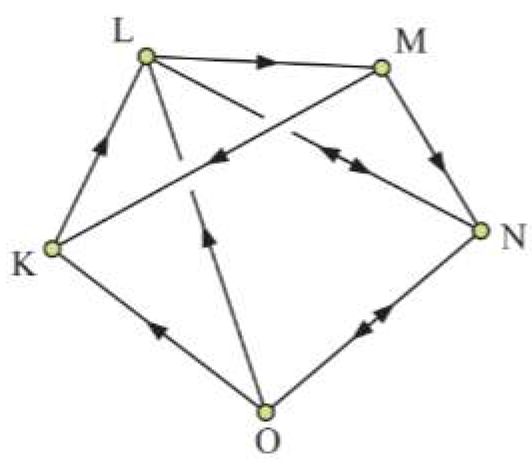
c There are only two combinations of places Arabella could visit:

- Dublin, Edinburgh, Barcelona
- Dublin, Frankfurt, Barcelona

Notice that she could visit these cities in the reverse order to the order listed here.

d The value in the first row and first column of $\mathbf{A}^4$ is 13, so there are 13 4-step routes starting and finishing at Amsterdam. However, this includes routes which revisit cities, such as $A \to D \to A \to D \to A$, and routes which visit the same combinations of cities, such as $A \to D \to F \to B \to A$ and $A \to B \to F \to D \to A$.

8 **a** $$\mathbf{A} = \begin{array}{c} \\ K \\ L \\ M \\ N \\ O \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} K & L & M & N & O \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

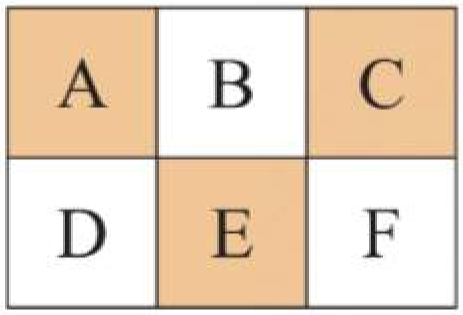


b Using technology, $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 & 1 \end{pmatrix}$

c Whenever there is a 0 in $\mathbf{A} + \mathbf{A}^2$, it is not possible to travel between the vertices in two steps or less. So, it is not possible to travel from vertices K to K, K to O, or M to M in either one or two steps.

d If the street connecting O and N was blocked, people at K, L, M, and N would be unable to visit O. It would also take longer for people at N to get to K.

9 **a** $$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$



b Using technology, $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 0 \\ 1 & 3 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 1 & 1 & 2 \end{pmatrix}$

There are 0s in the matrix $\mathbf{A} + \mathbf{A}^2$, which indicates that there are some pairs of squares on the board for which it is not possible to move between in at most two moves.

c Using technology, $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 2 & 6 & 1 & 5 & 2 & 3 \\ 6 & 3 & 6 & 2 & 8 & 2 \\ 1 & 6 & 2 & 3 & 2 & 5 \\ 5 & 2 & 3 & 2 & 6 & 1 \\ 2 & 8 & 2 & 6 & 3 & 6 \\ 3 & 2 & 5 & 1 & 6 & 2 \end{pmatrix}$

There are no 0s in the matrix $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$, which indicates that it is possible to move between any two squares in at most three moves.

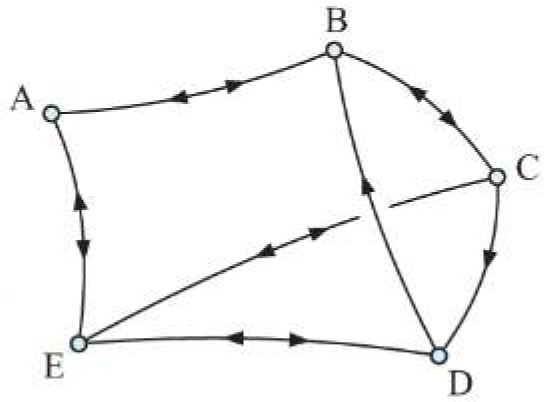
d If a counter can now be moved diagonally as well, the adjacency matrix becomes:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Using technology, $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 3 & 3 & 2 & 3 & 3 & 2 \\ 3 & 5 & 3 & 3 & 5 & 3 \\ 2 & 3 & 3 & 2 & 3 & 3 \\ 3 & 3 & 2 & 3 & 3 & 2 \\ 3 & 5 & 3 & 3 & 5 & 3 \\ 2 & 3 & 3 & 2 & 3 & 3 \end{pmatrix}$

Since there are no 0s in the matrix $\mathbf{A} + \mathbf{A}^2$, it is now possible to move between any two squares in at most two moves.

10 **a** $\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$



b Using technology, $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 & 3 \end{pmatrix}$

There are no 0s in the matrix $\mathbf{A} + \mathbf{A}^2$, so it is possible to move between any two towns in at most 2 trips.

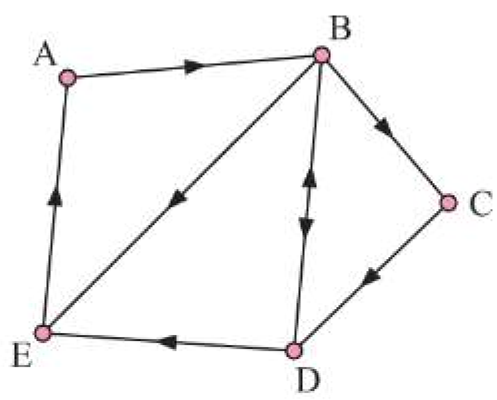
c **i** If the service from C to D is cancelled, then the new adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 3 \end{pmatrix}.$$

The 0 in $\mathbf{A} + \mathbf{A}^2$ indicates it will not be possible to travel from B to D in at most 2 trips.

ii If the service from C to B, or from C to E, is cancelled, then it will still be possible to travel between any two towns in at most 2 trips.

11 **a** $\mathbf{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$



b Using technology, $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ 2 & 3 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

This shows the number of ways a signal can be sent between stations in at most 3 steps.

c $\mathbf{A}+\frac{1}{2}\mathbf{A}^2+\frac{1}{4}\mathbf{A}^3=\begin{pmatrix} 0.25 & 1.25 & 0.5 & 0.75 & 0.75 \\ 0.75 & 1 & 1.25 & 1.75 & 2 \\ 0.25 & 0.5 & 0.25 & 1.25 & 0.75 \\ 0.75 & 1.5 & 0.5 & 0.75 & 1.75 \\ 1 & 0.5 & 0.25 & 0.25 & 0.25 \end{pmatrix}$

This could represent the strength of a signal between two stations.

d **i** If BE is blocked, this means that an electrical signal cannot be transmitted from A to A, or E to E in at most 3 steps.

ii If D is no longer able to send signals to B, this means that an electrical signal cannot be transmitted from C to B, C to C, D to C, or D to D in at most 3 steps.

e Considering the matrix $\mathbf{A}+\frac{1}{2}\mathbf{A}^2+\frac{1}{4}\mathbf{A}^3$, the scenario in **d i** would mean that the values in row 1, column 1 and row 5, column 5 would both be 0.

Similarly, the scenario in **d ii** would mean that the values in row 3, column 2; row 3, column 3; row 4, column 3, and row 4, column 4 would all be 0.

The values in row 1, column 1 and row 5, column 5 are very small to begin with, so the effect of **d i** is relatively small.

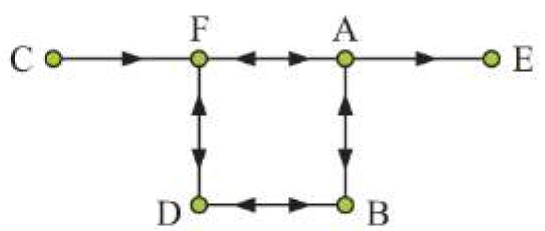
The effect in **d ii** creates more serious problems.

12 **a** No, **A** is not symmetric about the leading diagonal. For example, the value in row 1, column 5 is not equal to the value in row 5, column 1.

This means there are some one-way doorways.

$$\mathbf{A}=\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

b



c **i** C cannot be accessed from any other vertex, so the museum's entrance must be represented by the letter C.

ii The most likely vertex representing the shop is A, as a person must pass through A in order to reach the exit.

iii There is no route from E to any other vertex, which suggests that the museum's exit is represented by the letter E.

d **i** $\mathbf{A}+\mathbf{A}^2=\begin{pmatrix} 2 & 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 2 \end{pmatrix}$ $\quad\mathbf{A}^3=\begin{pmatrix} 0 & 4 & 0 & 0 & 2 & 4 \\ 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 2 \\ 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 \end{pmatrix}$

ii The 0s in row 3 of $\mathbf{A}+\mathbf{A}^2$ show that you cannot travel from the entrance (at C) to B, C, or E by passing through at most 2 areas.

The non-zero values in row 3 of $\mathbf{A}^3$ show that you can travel to areas B, E, and F from C by passing through 3 areas.

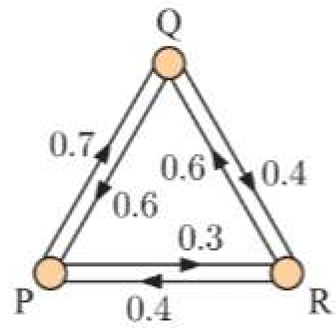
Hence the dinosaur room must be in either B or E.

iii The dinosaur room must be represented by the letter B. It cannot be E as E is the exit.

EXERCISE 15E

1 **a** From Q, trains move to R with probability 0.4, so they move to P with probability 0.6.

From R, trains move to P with probability 0.4, so they move to Q with probability 0.6.



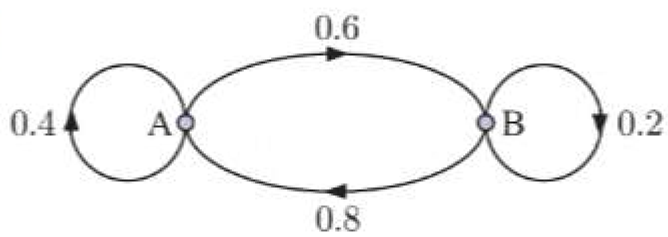
b $\mathbf{T} = \begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{pmatrix}$

c The 0.3 in row 1, column 3 of $\mathbf{T}$ indicates that a train at station P will go to station R with probability 0.3.

d $\mathbf{s}_0\mathbf{T}^3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0.22 & 0.546 & 0.234 \end{pmatrix}$

Three stops from now, a bus currently at P has probability 0.22 of being at P, probability 0.546 of being at Q, and probability 0.234 of being at R.

2 **a**



b $\mathbf{T} = \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix}$

c The initial state matrix $\mathbf{s}_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

$\mathbf{s}_0\mathbf{T}^4 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix}^4 = \begin{pmatrix} 0.5824 & 0.4176 \end{pmatrix}$

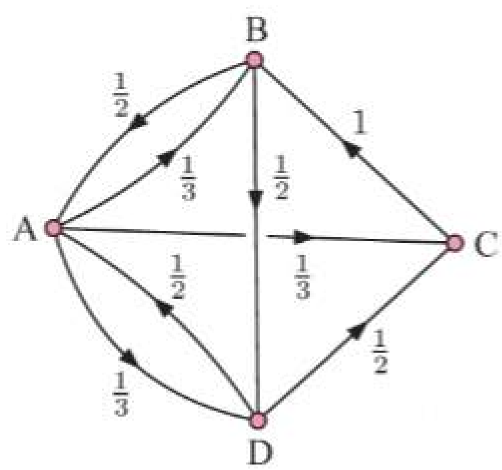
So, the probability that Maria is at station A after jogging along 4 tracks, is 0.5824.

3 **a** From A there are 3 trails, so each is chosen with probability $\frac{1}{3}$.

From B there are 2 trails, so each is chosen with probability $\frac{1}{2}$.

From C there is 1 trail, so it is chosen with probability 1.

From D there are 2 trails, so each is chosen with probability $\frac{1}{2}$.



b $\mathbf{T} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$

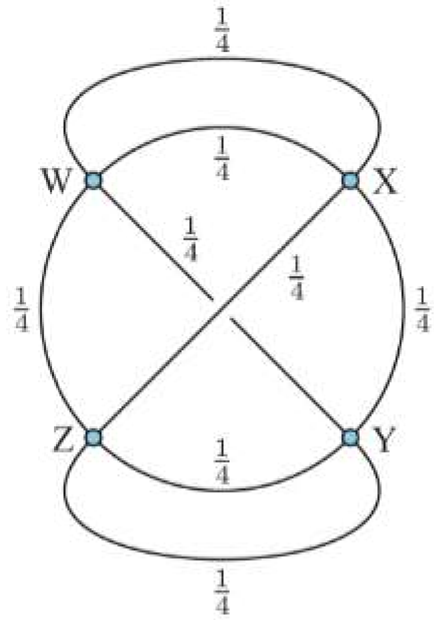
c The initial state matrix is $\mathbf{s}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$

$$\mathbf{s}_0\mathbf{T}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{4} & \frac{5}{18} & \frac{7}{36} & \frac{5}{18} \end{pmatrix}$$

So, the probability that Prisha will be on platform D after travelling along 3 trails, is $\frac{5}{18}$.

The possible routes are A → B → A → D, A → C → B → D, and A → D → A → D.

4 **a** There are 4 tunnels at each junction, so when the mouse reaches a junction, each tunnel is chosen with probability $\frac{1}{4}$.



Notice that there are 2 tunnels joining W and X, and 2 tunnels joining Y and Z.

So, the transition matrix
$$\mathbf{T} = \begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

b **i** $$\mathbf{s}_0\mathbf{T}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

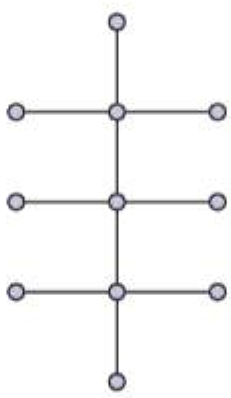
The mouse is most likely to be at junction X. No matter which tunnel the mouse takes first, it is always possible to return to X after 2 tunnels.

ii $\mathbf{s}_0\mathbf{T}^{100} \approx \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

In the long term, the mouse will visit each junction equally often.

EXERCISE 15F

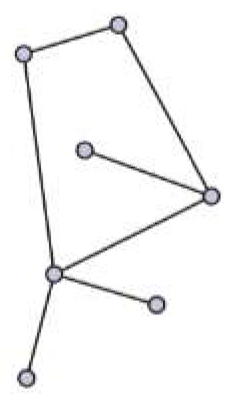
1 **a**



The graph is connected and simple, and it has no cycles.

∴ the graph is a tree.

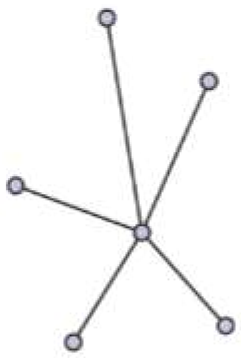
b



The graph has a cycle.

∴ the graph is not a tree.

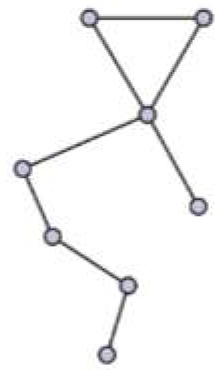
c



The graph is connected and simple, and it has no cycles.

$\therefore$ the graph is a tree.

d

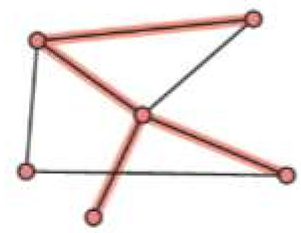


The graph has a cycle.

$\therefore$ the graph is not a tree.

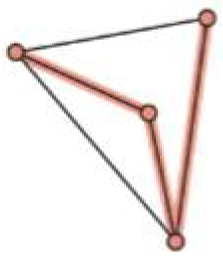
2 **a** The subgraph is connected and simple, and it has no cycles.

$\therefore$ the subgraph is a tree.

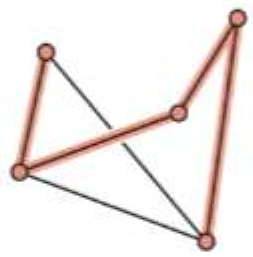


b The subgraph is not a spanning tree, as it does not contain all the vertices of the graph.

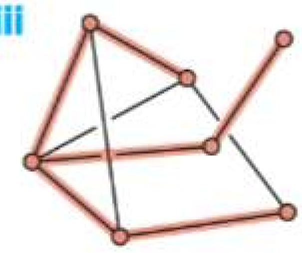
3 **a** **i**



ii



iii



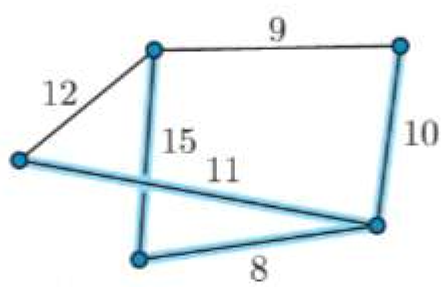
b

	Number of vertices	*Number of edges in spanning tree*
i	4	3
ii	5	4
iii	7	6

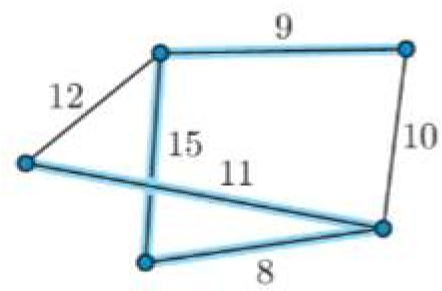
c *If a graph has n vertices, its spanning tree has $n-1$ edges.*

The first edge connects two vertices, and each additional edge adds exactly one vertex to the tree. We therefore need $n-1$ edges to connect all n vertices.

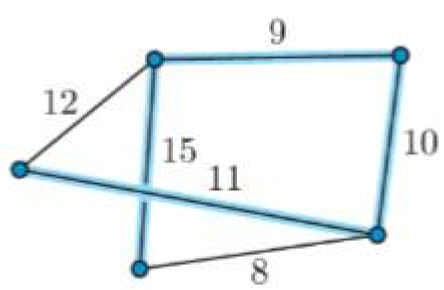
4 The spanning trees are:



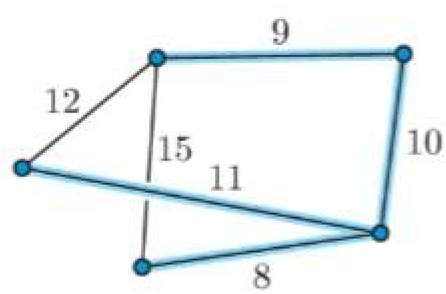
weight $= 10 + 11 + 8 + 15 = 44$



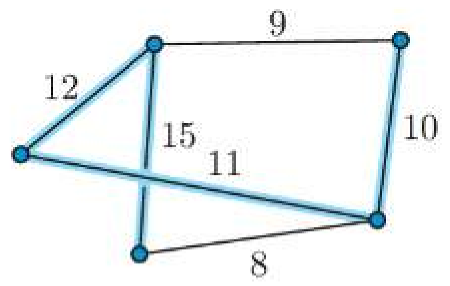
weight $= 9 + 15 + 8 + 11 = 43$



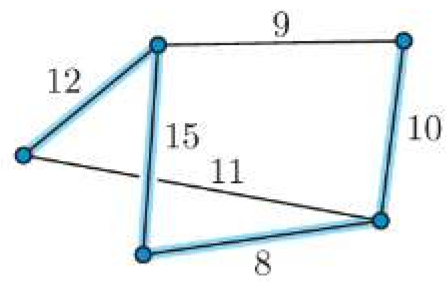
weight $= 9 + 10 + 11 + 15 = 45$



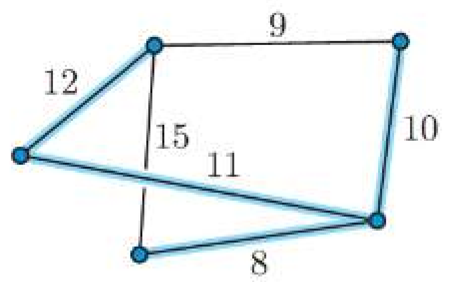
weight $= 9 + 10 + 11 + 8 = 38$



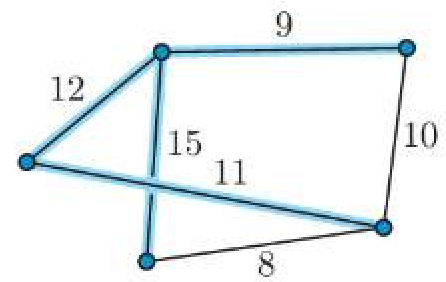
weight $= 10 + 11 + 12 + 15 = 48$



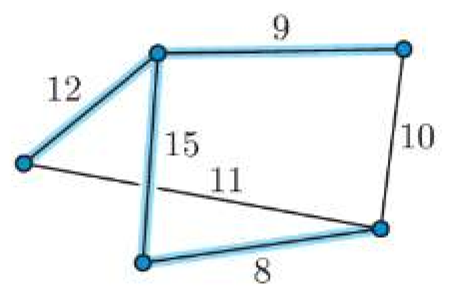
weight $= 10 + 8 + 15 + 12 = 45$



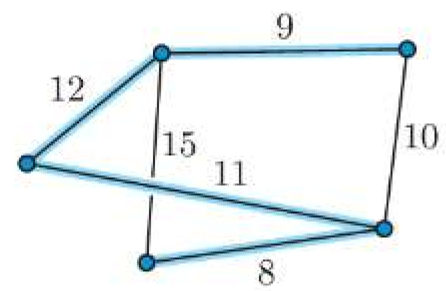
weight $= 10 + 8 + 11 + 12 = 41$



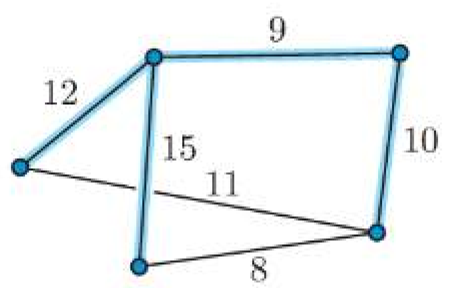
weight $= 9 + 12 + 11 + 15 = 47$



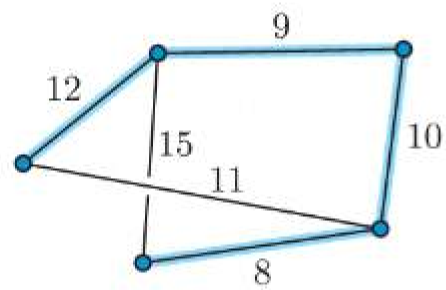
weight $= 9 + 12 + 15 + 8 = 44$



weight $= 9 + 12 + 11 + 8 = 40$



weight $= 10 + 9 + 12 + 15 = 46$

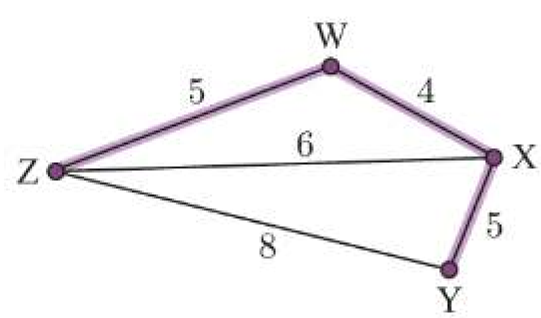


weight $= 8 + 10 + 9 + 12 = 39$

5 No, disconnected graphs do not have a spanning tree.

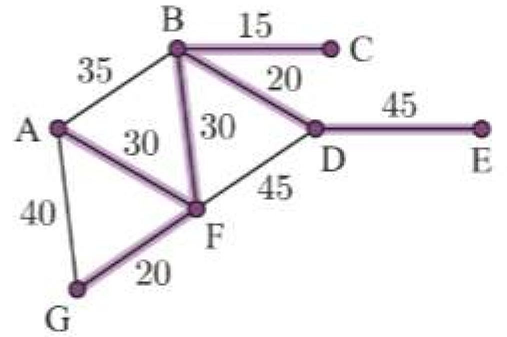
EXERCISE 15G.1

1 **a** **i** There are 4 vertices, so we require 3 edges. Edge WX has the least weight. We then choose XY and ZW.



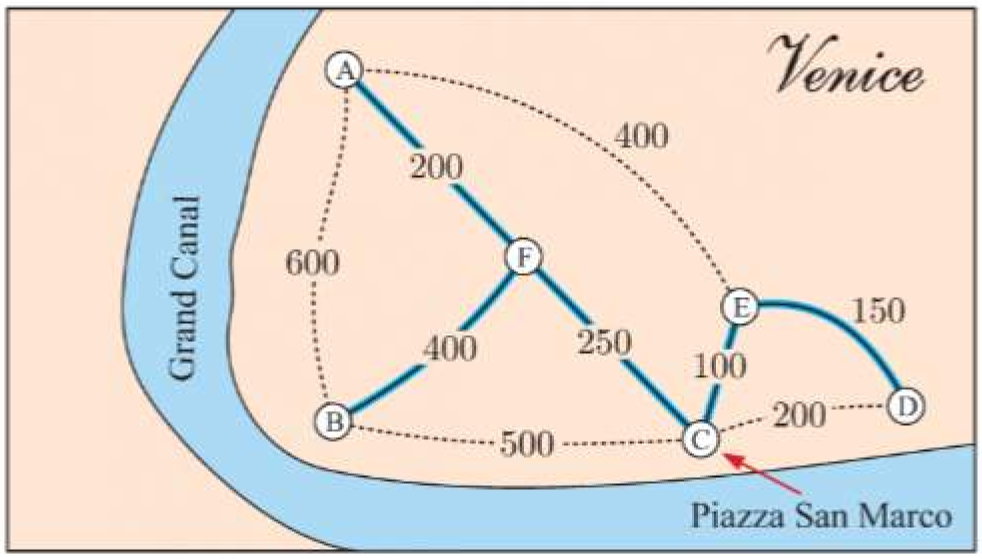
ii Weight $= 4 + 5 + 5 = 14$ units

b **i** There are 7 vertices, so we require 6 edges.
Edge BC has the least weight. We then choose BD, FG, BF, AF, and DE.



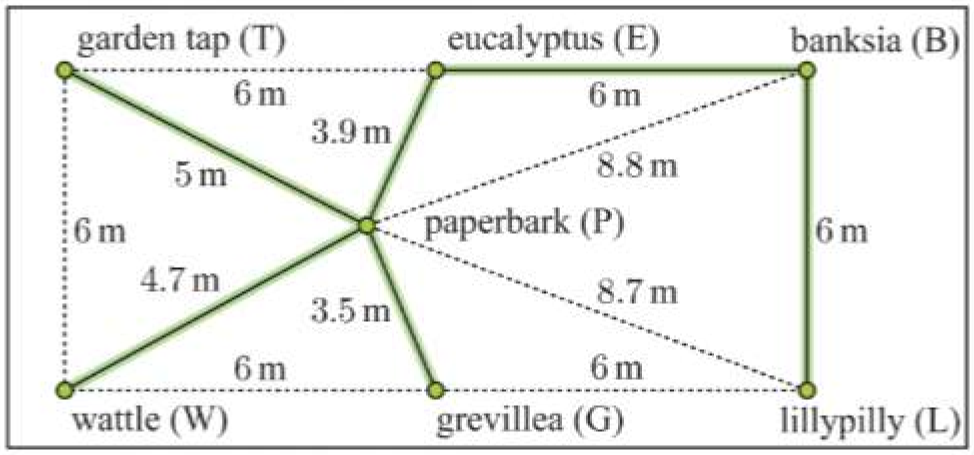
ii Weight $= 15 + 20 + 20 + 30 + 30 + 45 = 160$ units

2 There are 6 vertices, so we require 5 edges.
Edge CE has the least weight. We then choose DE, AF, CF, and BF.
So, the authorities raise platforms over AF, BF, CF, CE, and DE.



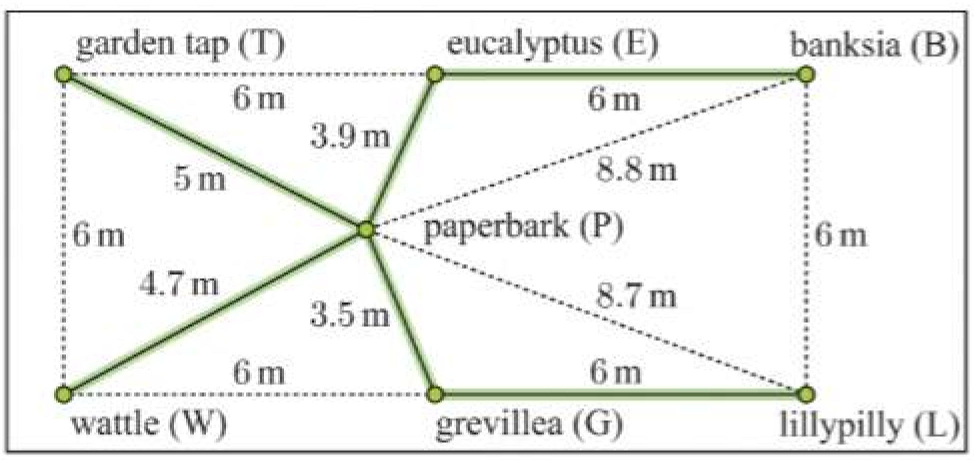
3 **a** There are 7 vertices, so we require 6 edges.
Edge PG has the least weight.
We then choose PE, PW, PT, EB, and BL.
The minimum length of pipe needed is

$3.5 + 3.9 + 4.7 + 5 + 6 + 6$
$= 29.1$ m.

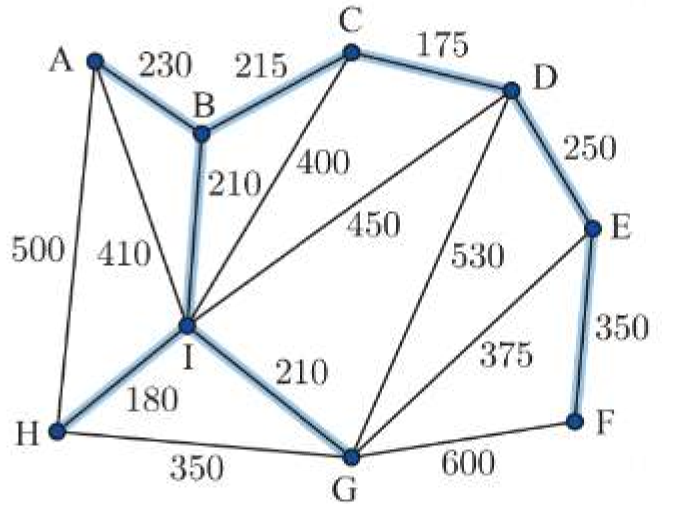


b There is not a unique network of pipes that will give this length. For example, GL could have been chosen for the last edge instead of BL.

c GL is closer to the tap than BL, so GL should be included to maximise water pressure.

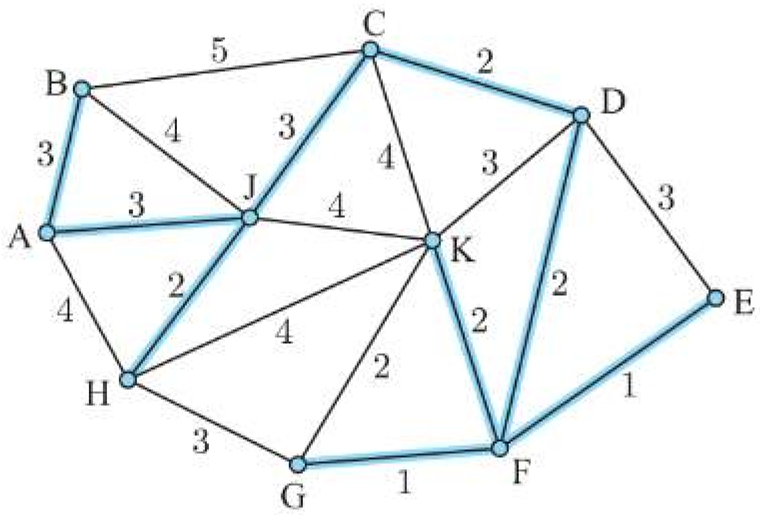


4 **a** There are 9 vertices, so we require 8 edges. Edge CD has the least weight. We then choose HI, BI, GI, BC, AB, DE, and EF.



b Total length $= 175 + 180 + 210 + 210 + 215 + 230 + 250 + 350$
$= 1820$ m

5 **a**



There are 10 vertices, so we require 9 edges.

EF is an edge with least weight.

We then choose FG, CD, DF, FK, HJ, AJ, CJ, and AB.

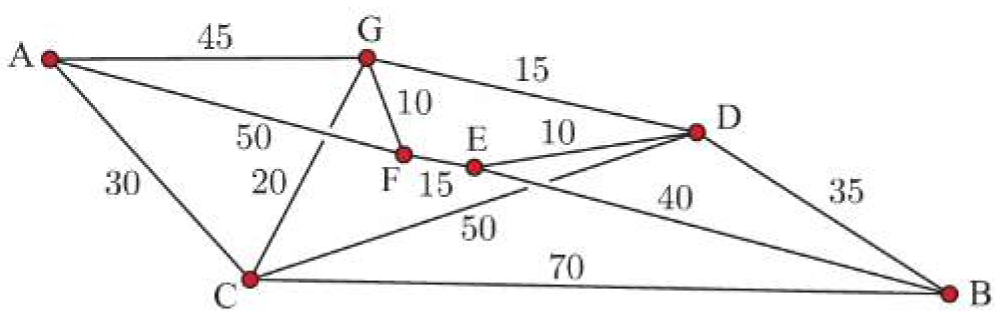
So, the trails which should be paved are EF, FG, CD, DF, FK, HJ, AJ, CJ, and AB.

b Length of paving needed $= 1 + 1 + 2 + 2 + 2 + 2 + 3 + 3 + 3$
$= 19$ km

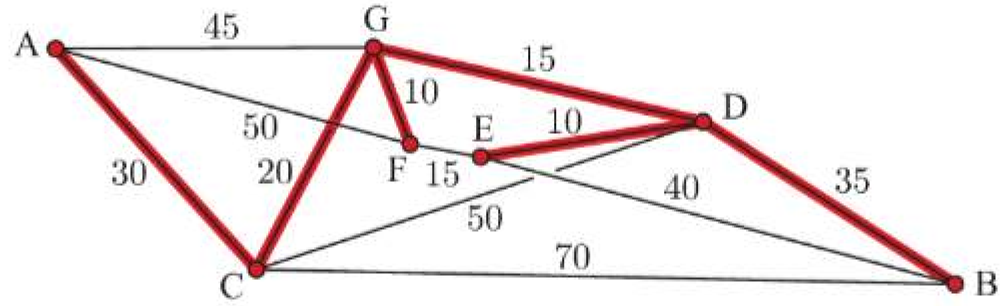
6

	A	B	C	D	E	F	G
A			30			50	45
B			70	35	40		
C	30	70		50			20
D		35	50		10		15
E		40		10		15	
F	50				15		10
G	45		20	15		10	

a



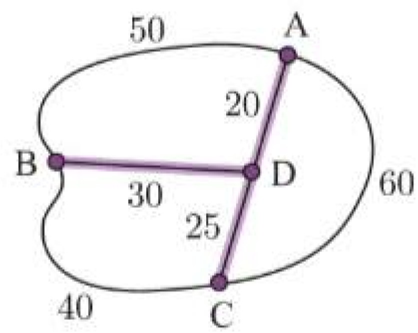
b There are 7 vertices, so we require 6 edges.
Edge DE is an edge with least weight. We then choose FG, DG, CG, AC, and BD.



EXERCISE 15G.2

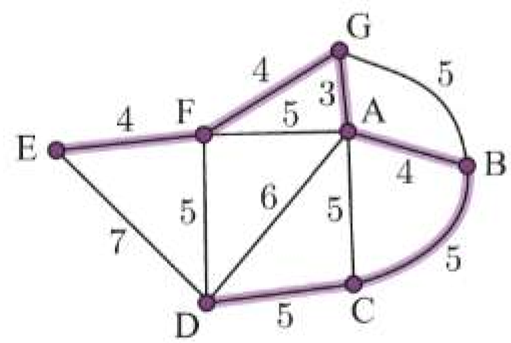
1 **a** **i** We choose a vertex at random, say vertex A.
The vertex nearest A is D, so we choose edge AD.
We then choose CD and BD.

ii weight $= 20 + 25 + 30 = 75$ units



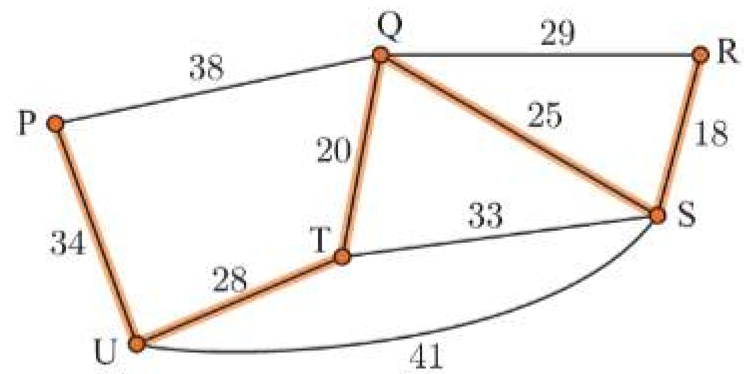
b **i** We choose a vertex at random, say vertex A.
The vertex nearest A is G, so we choose edge AG.
We then choose AB, FG, EF, BC, and CD.
Note: Other solutions are possible.

ii weight $= 3 + 4 + 4 + 4 + 5 + 5 = 25$ units

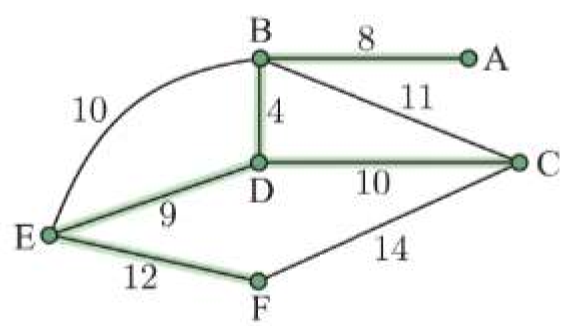


2 **a** We choose a vertex at random, say vertex P.
The vertex nearest P is U, so we choose edge PU. We then choose UT, TQ, QS, and SR.

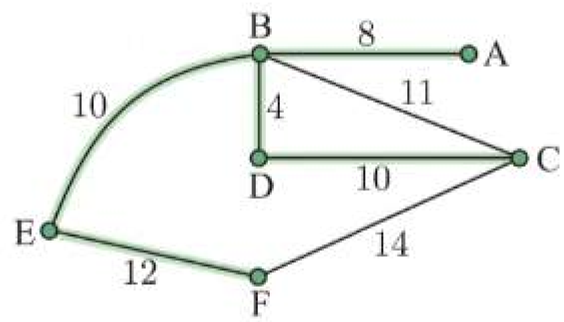
b length of cable $= 34 + 28 + 20 + 25 + 18$
$= 125$ m



3 **a** We choose a vertex at random, say vertex A.
The vertex nearest A is B, so we choose edge AB.
We then choose BD, DE, DC, and EF.
So, roads AB, BD, CD, DE, and EF should be cleared.
Minimum total length $= 8 + 4 + 9 + 10 + 12$
$= 43$ km.



b If DE should not be cleared, then the roads AB, BD, CD, BE, and EF should be cleared.
This is the same as in **a**, but with BE instead of DE.



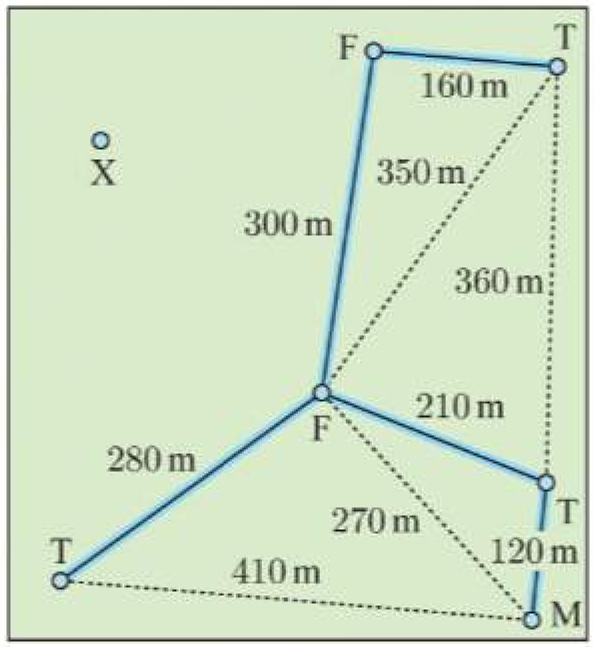
4 **a** Distances for the park are shown alongside. Distances that are clearly inefficient have been omitted.

Using Prim's algorithm, we find the minimum spanning tree shown.

Minimum pipe length

$= 120 + 210 + 280 + 300 + 160$

$= 1070$ m



Scale: 1 cm ≡ 100 m

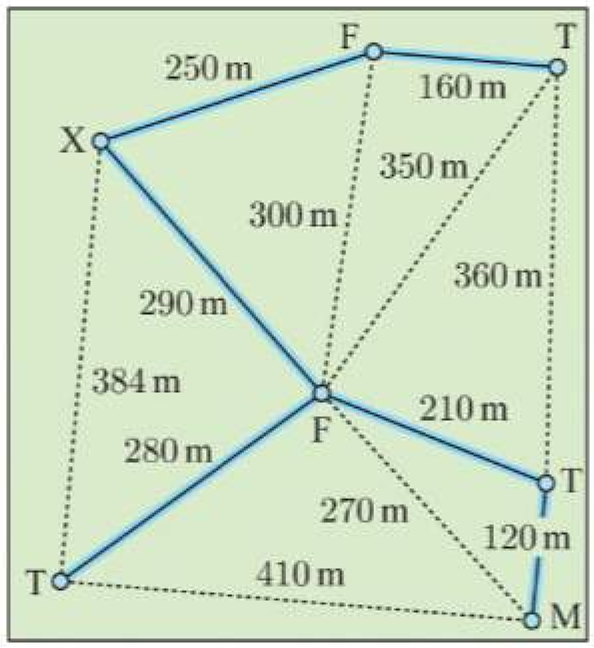
b Distances from X have now been included.

Using Prim's algorithm, we find the minimum spanning tree shown.

Minimum pipe length

$= 120 + 210 + 280 + 290 + 250 + 160$

$= 1310$ m



Scale: 1 cm ≡ 100 m

c We assume it is possible to dig in straight lines between the fountains and taps, and that it is equally difficult to dig everywhere.

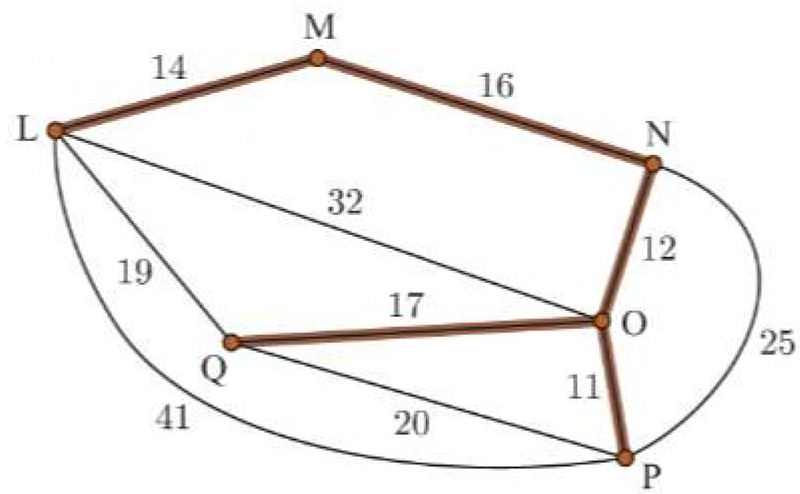
5 **a** We choose a vertex at random, say vertex L.

The vertex nearest L is M, so we choose edge LM. We then choose MN, NO, OP, and OQ.

Minimum cost

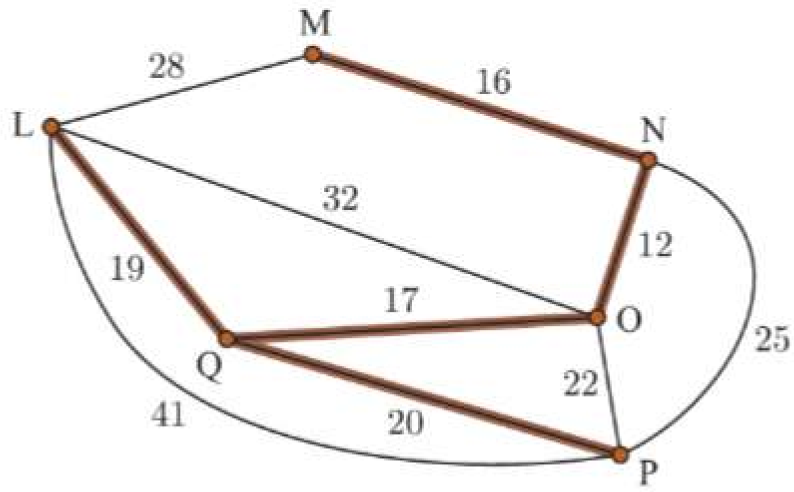
$= 14 + 16 + 12 + 11 + 17$

$= \$70$ million



b The updated costs are shown alongside.
We now choose edges LQ, QO, ON, NM, and QP.

Minimum cost
$= 19 + 17 + 12 + 16 + 20$
$= \$84$ million

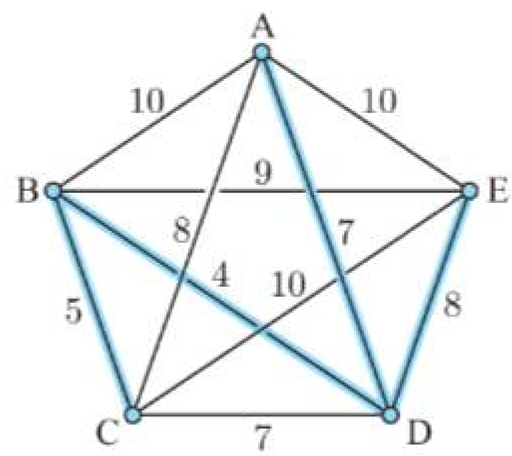


6 **a** The graph is complete as there is a weight for every edge from every vertex to every other vertex.

b We start with vertex A.

	1	3	4	2	
	A	B	C	D	E
A	–	10	8	7	10
B	10	–	5	4	9
C	8	5	–	7	10
D	7	4	7	–	8
E	10	9	10	8	–

So, the edges in the minimum spanning tree are AD, BD, BC, and DE.



7 **a** We start with vertex A.

	1	4	3		2
	A	B	C	D	E
A	–	43	25	40	16
B	43	–	10	27	30
C	25	10	–	62	20
D	40	27	62	–	50
E	16	30	20	50	–

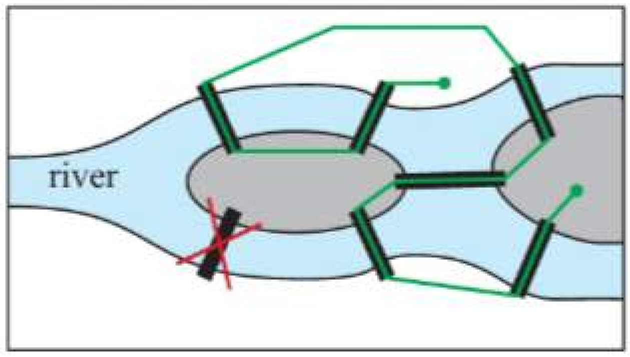
So, the edges in the minimum spanning tree are AE, CE, BC, and BD.

b Minimum cost $= \$16\,000 + \$20\,000 + \$10\,000 + \$27\,000$
$= \$73\,000$

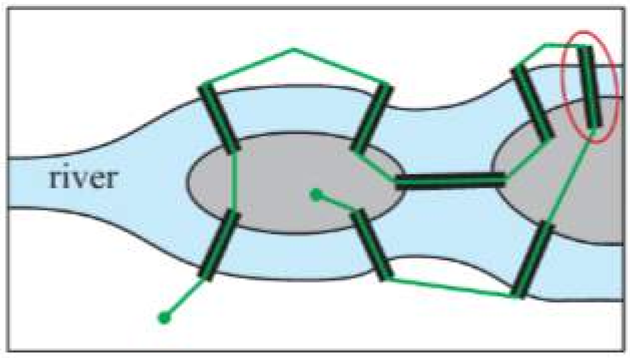
ACTIVITY 2 THE BRIDGES OF KÖNIGSBERG

1 No, a tour cannot be made with crosses each bridge exactly once.

2 **a** Any bridge can be removed. An example is shown alongside.

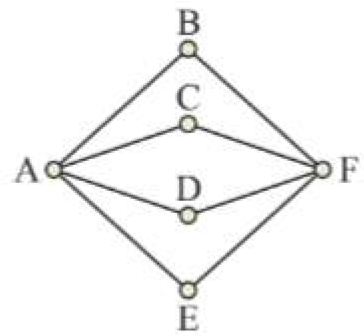


b Any bridge between the islands or banks can be added. An example is shown alongside.

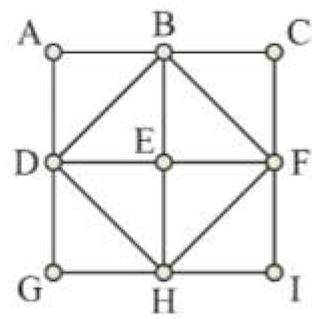


EXERCISE 15H

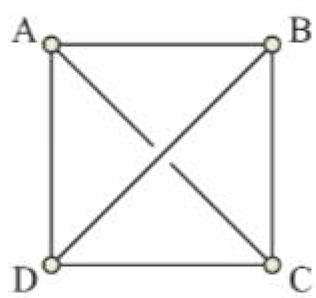
1 **a** All vertices have even degree, so the graph is Eulerian.
An Eulerian circuit is
$A \to B \to F \to E \to A \to C \to F \to D \to A$.



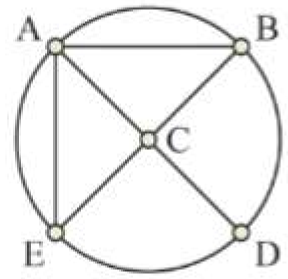
b The graph contains more than two vertices of odd degree, so the graph is neither Eulerian nor semi-Eulerian.



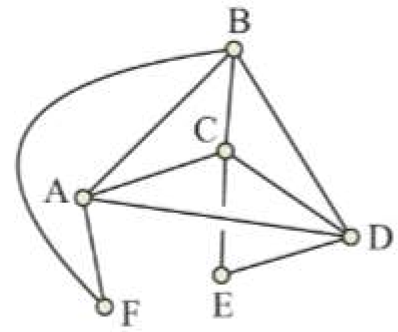
c The graph contains more than two vertices of odd degree, so the graph is neither Eulerian nor semi-Eulerian.



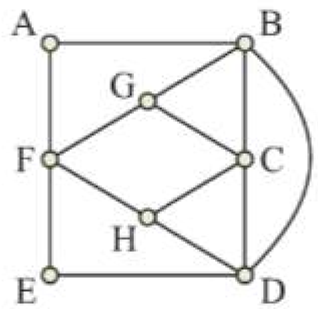
d The graph contains two vertices (A and D) of odd degree, so the graph is semi-Eulerian.
An Eulerian trail is
A → B → D → C → B → A → C → E → A → E → D.



e All vertices have even degree, so the graph is Eulerian.
An Eulerian circuit is
A → B → F → A → C → B → D → E → C → D → A.

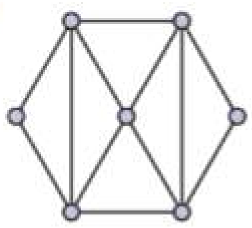


f The graph contains two vertices (G and H) of odd degree, so the graph is semi-Eulerian.
An Eulerian trail is
G → F → H → D → E → F → A → B → D → C → B → G → C → H.

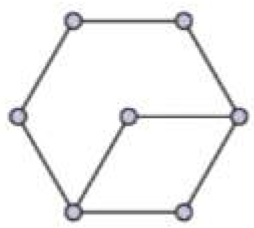


2 **Note:** These are examples only.

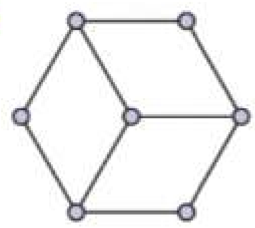
a



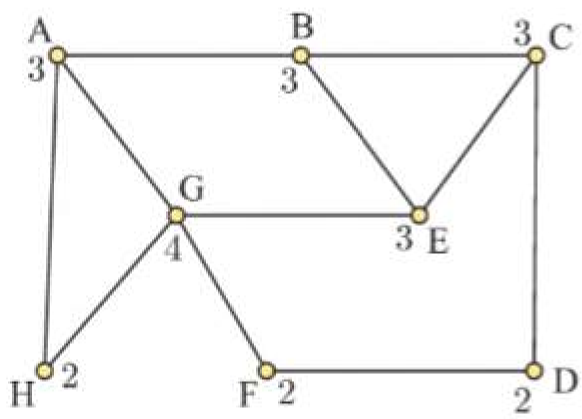
b



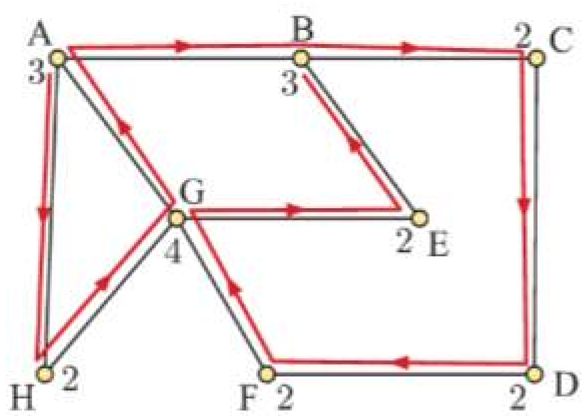
c



3 **a** More than two vertices (A, B, C, and E) have odd degree.
∴ the graph is neither Eulerian nor semi-Eulerian.



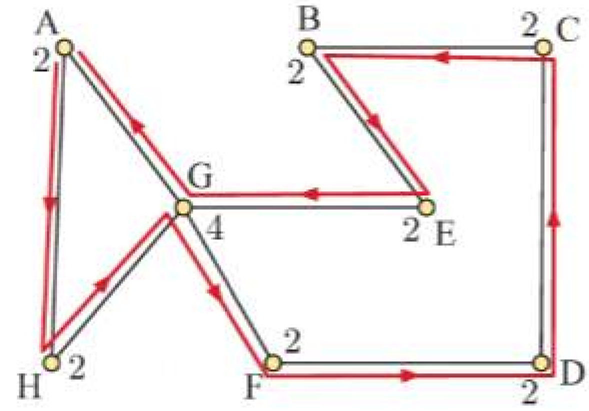
b The subgraph formed by removing the edge CE, has exactly two vertices (A and B) of odd degree.
∴ the subgraph is semi-Eulerian.
A → H → G → A → B → C → D → F → G → E → B is an Eulerian trail.



c The subgraph formed by removing edges CE *and* AB, has no vertices of odd degree.

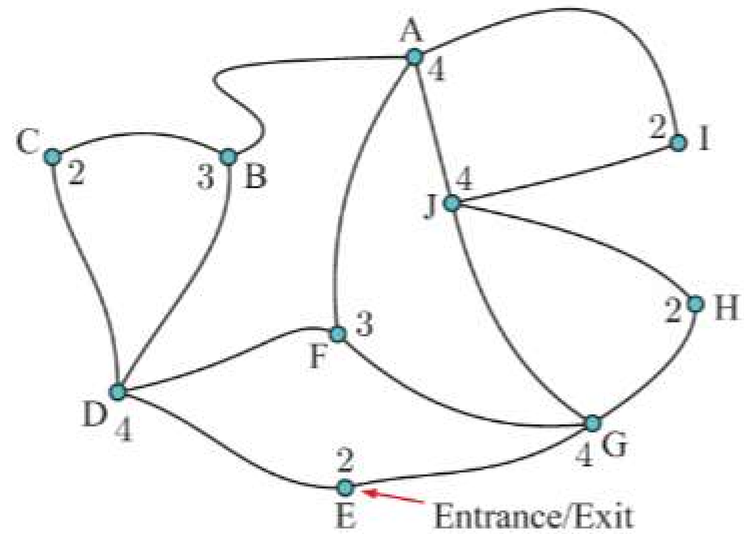
∴ the subgraph is Eulerian.

A → H → G → F → D → C → B → E → G → A is an Eulerian circuit.



4 **a** No, the graph contains vertices (B and F) of odd degree, so it is not Eulerian.

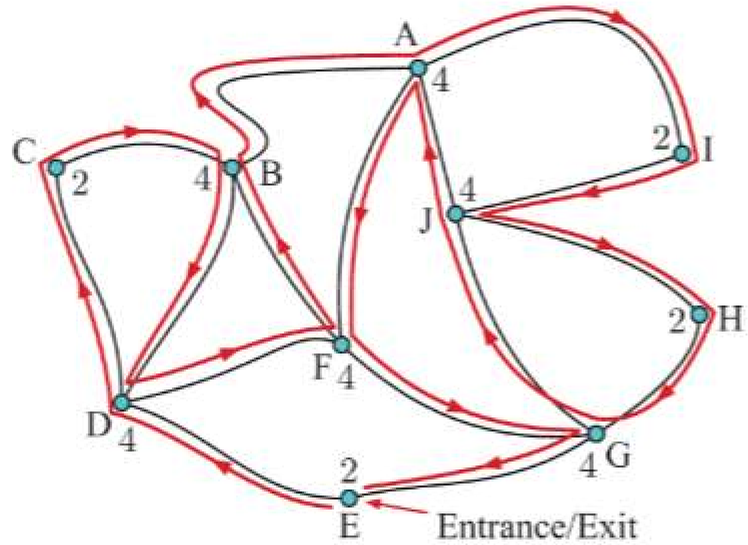
∴ a tour which starts at E, traverses every track exactly once, and returns to E does not exist.



b If a track BF is added, the graph has no vertices of odd degree.

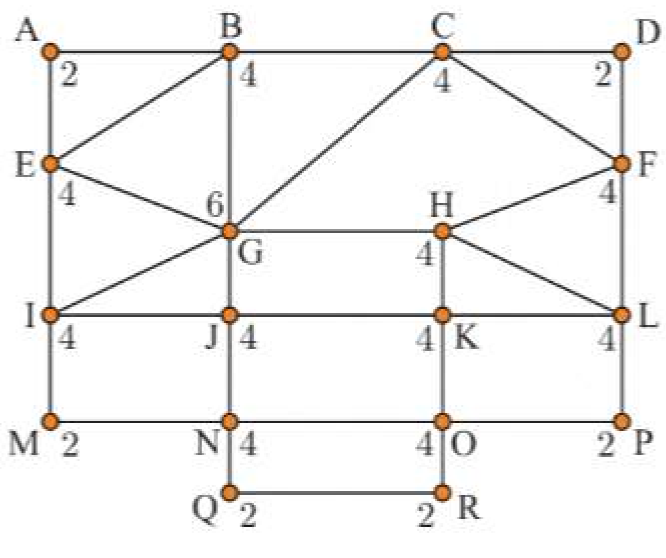
∴ the graph would be Eulerian.

E → D → C → B → D → F → B → A → I → J → H → G → J → A → F → G → E is an Eulerian circuit which starts and ends at E.



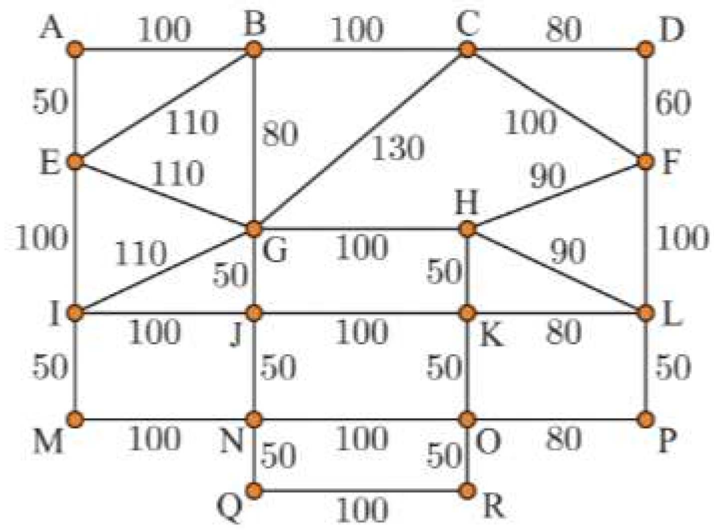
5 **a** The graph does not contain any vertices of odd degree, so it is Eulerian.

∴ there exists a route which starts and ends at A, and visits every street exactly once.

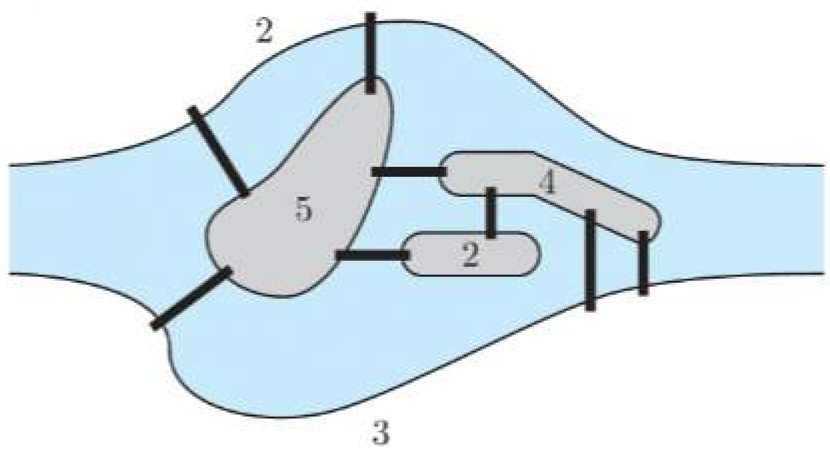


b In any such route, each street is traversed exactly once, so the total distance travelled is the same regardless of the order of the vertices.

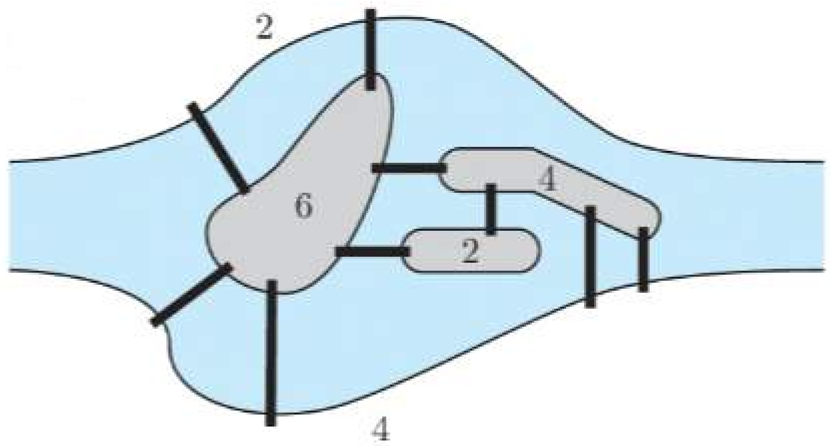
c The length of the route is the sum of all distances shown on the graph, which is 2570 m or 2.57 km.

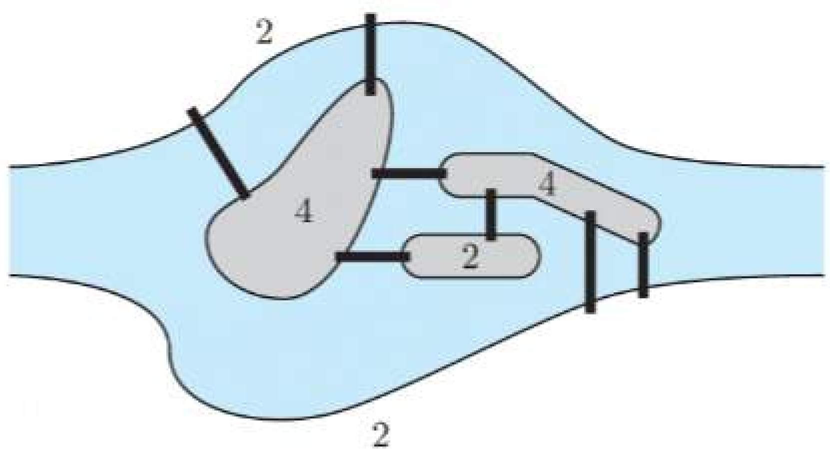


6 **a** Let each island and the northern and southern banks be the vertices of the graph.
There are vertices of odd degree, so the graph is not Eulerian.
$\therefore$ the graph does not contain an Eulerian circuit.

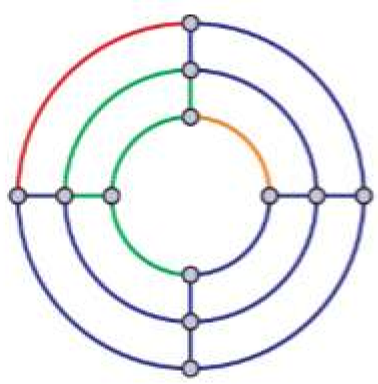


b Yes, adding a bridge between the largest island and southern bank, *or* removing the existing bridge between the largest island and southern bank, will create an Eulerian circuit.





7 **a** 4 pen strokes are needed to draw the diagram. An example is shown alongside.



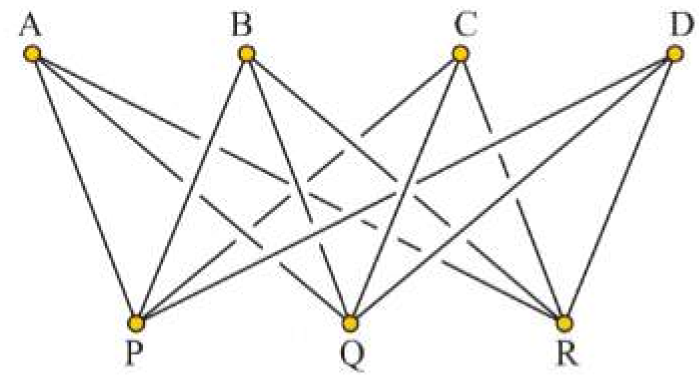
b If a graph has 2 vertices of odd degree, it is semi-Eulerian, and the graph can be drawn with a single pen stroke.

This graph has 8 vertices of odd degree, so we could make the graph semi-Eulerian by adding 3 new edges to the graph. Equivalently, we can think of adding an edge between two vertices as lifting the pen at one vertex and moving it to the other.

So, an additional 3 pen strokes are required to complete the diagram, making 4 in total.

8 **a** This graph of $k_{4,\,3}$ has 4 vertices (A, B, C, and D) of odd degree.

So $k_{4,\,3}$ is neither Eulerian nor semi-Eulerian.



b The graph of $k_{m,\,n}$ has m vertices of degree n, and n vertices of degree m.

$\therefore$ $k_{m,\,n}$ is Eulerian if m and n are both even.

In this case the sum of the degrees of the vertices $d = mn + nm$

$$\therefore\ 2e = 2mn$$
$$\therefore\ e = mn$$

There are mn edges, so the length of the Eulerian circuit is mn.

c A semi-Eulerian graph has exactly two vertices of odd degree.

$\therefore$ $k_{m,\,n}$ is semi-Eulerian if $m = n = 1$, or if one of m, n is odd and the other is 2.

EXERCISE 15I

1 **a** There are no vertices of odd degree.

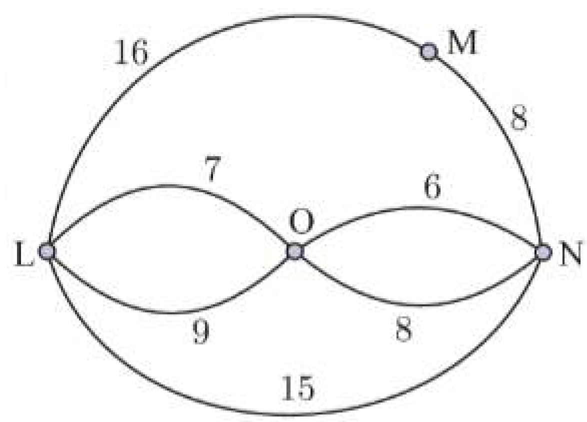
$\therefore$ the graph is Eulerian.

b The minimum distance to be travelled is the sum of the distances of the edges, which is

$16 + 8 + 15 + 7 + 6 + 8 + 9 = 69$ km.

A possible route is

L → M → N → L → O → N → O → L.

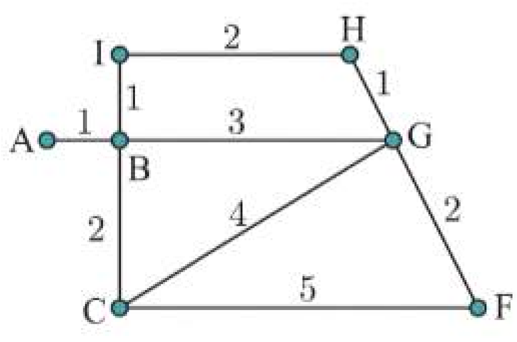


2 **a** There are two vertices (A and C) of odd degree, so the graph is semi-Eulerian.

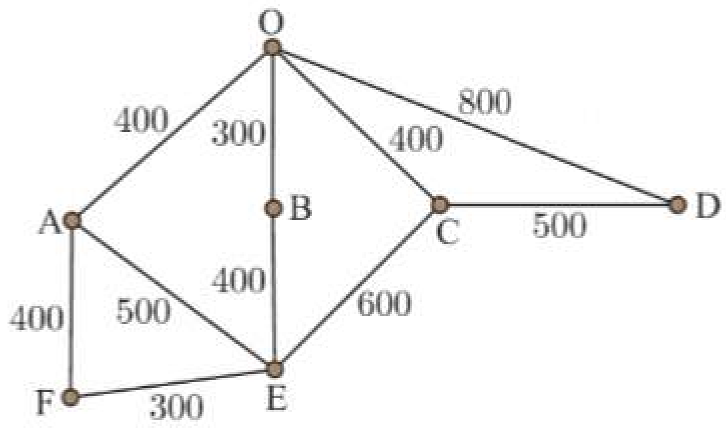
b We have to travel between A and C twice.

The sum of the lengths of all the roads is 21 km, and the shortest path from A to C is $1 + 2 = 3$ km.

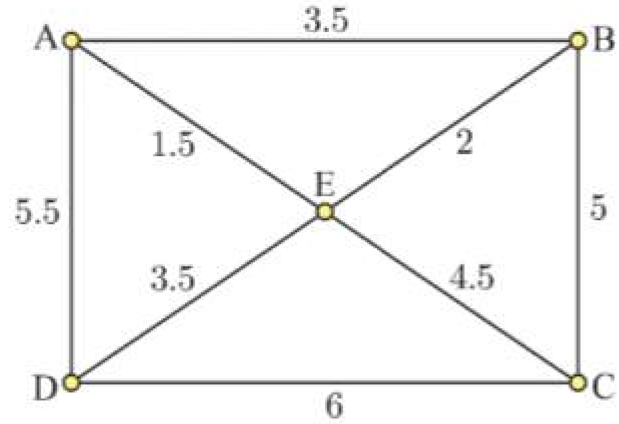
So, the shortest distance the snowplough must travel is $21 + 3 = 24$ km.



3 There are two vertices (A and C) of odd degree, so we need to travel between A and C twice.
The sum of the lengths of the tunnels is 4600 m, and the shortest path from A to C is $400 + 400 = 800$ m.
So, the shortest distance the worker can walk is $4600 + 800 = 5400$ m or 5.4 km.
A possible route is O → D → C → O → A → F → E → A → O → B → E → C → O.



4 **a** The four vertices of odd degree are A, B, C, and D. They can be paired as AB and CD, AC and BD, or AD and BC.

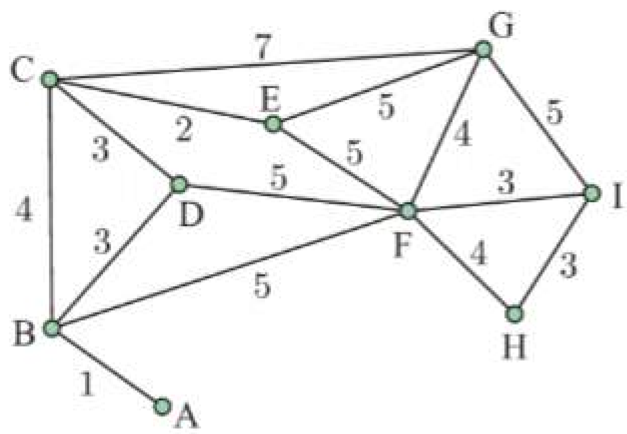


b

Pairing	*Shortest path*	*Weight*	*Total weight*
AB CD	A → B C → D	3.5 6	9.5
AC BD	A → E → C B → E → D	6 5.5	11.5
AD BC	A → E → D B → C	5 5	10

The sum of the lengths of the roads is 31.5 km, and the shortest distance to connect the vertices of odd degree is 9.5 km.
So, the shortest distance the procession has to march is $31.5 + 9.5 = 41$ km.

5 The vertices of odd degree are A, D, E, and I.

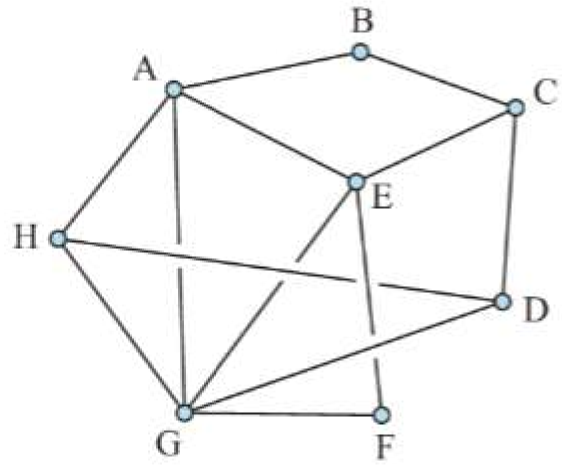


Pairing	*Shortest path*	*Weight*	*Total weight*
AD EI	A → B → D E → F → I	4 8	12
AE DI	A → B → C → E D → F → I	7 8	15
AI DE	A → B → F → I D → C → E	9 5	14

The sum of the lengths of the roads is 59 units, and the shortest distance to connect the vertices of odd degree is 12 units.
So, the shortest distance Peter needs to walk is $59 + 12 = 71$ units.
Peter must traverse A → B → D and E → F → I twice.
An example route is:
A → B → C → D → B → D → F → E → C → G → E → F → G → I → H → F → I → F → B → A

EXERCISE 15J

1 **a** A → B → C → E → F → G → D → H → A is a Hamiltonian cycle starting and finishing at A.

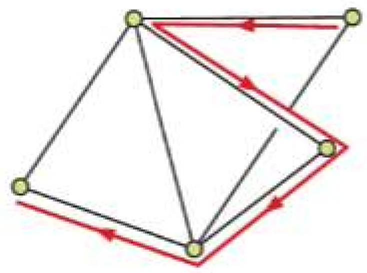


b **i** E → F → G → D → C → B → A → H is a Hamiltonian path which starts at E and finishes at H.

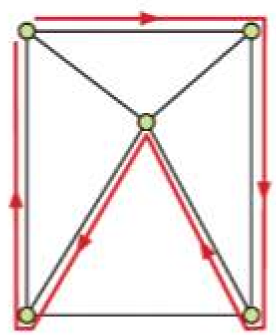
ii E → F → G → A → H → D → C → B is a Hamiltonian path which starts at E and finishes at B.

c The graph is Hamiltonian since it contains a Hamiltonian cycle (in **a**).

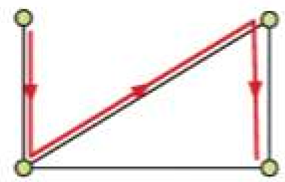
2 **a** The graph contains a Hamiltonian path, but not a Hamiltonian cycle.
∴ the graph is semi-Hamiltonian.



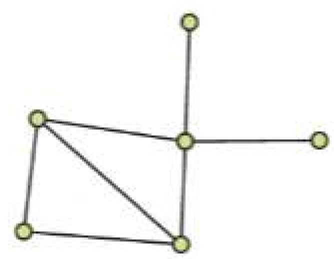
b The graph contains a Hamiltonian cycle.
∴ the graph is Hamiltonian.



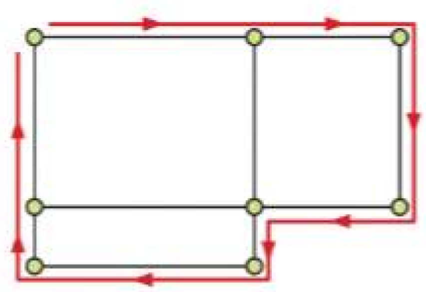
c The graph contains a Hamiltonian path, but not a Hamiltonian cycle.
∴ the graph is semi-Hamiltonian.



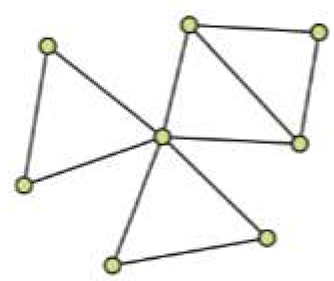
d The graph does not contain a Hamiltonian cycle nor a Hamiltonian path.
∴ the graph is neither Hamiltonian nor semi-Hamiltonian.



e The graph contains a Hamiltonian cycle.
∴ the graph is Hamiltonian.

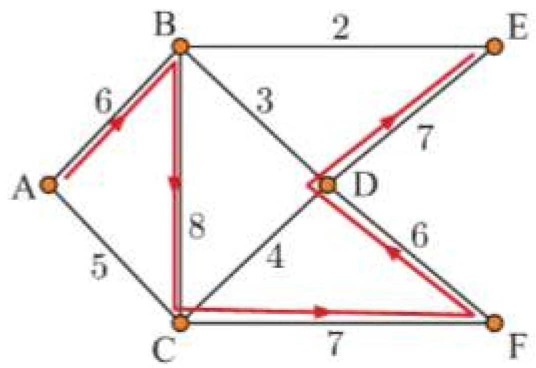


f The graph does not contain a Hamiltonian cycle nor a Hamiltonian path.
∴ the graph is neither Hamiltonian nor semi-Hamiltonian.



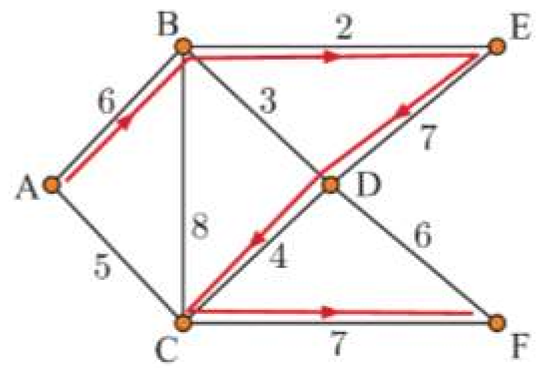
3 **a** All the Hamiltonian cycles which start at A are:

A → B → C → F → D → E



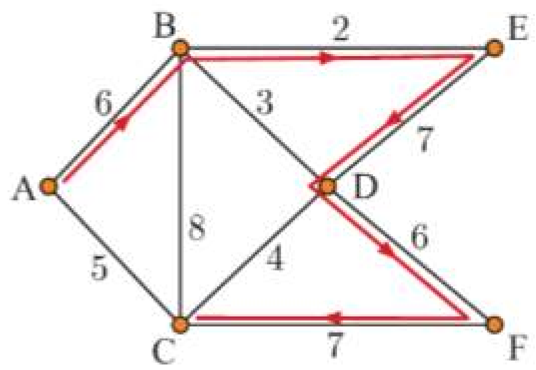
weight $= 6+8+7+6+7 = 34$

A → B → E → D → C → F



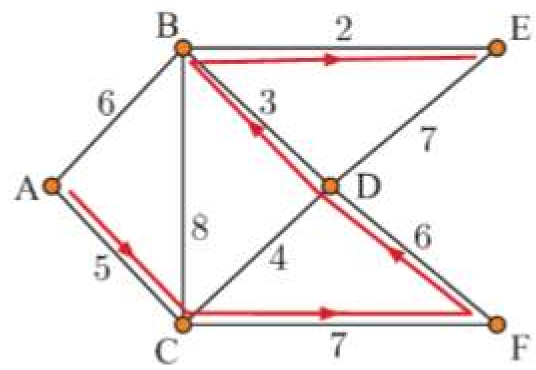
weight $= 6+2+7+4+7 = 26$

A → B → E → D → F → C



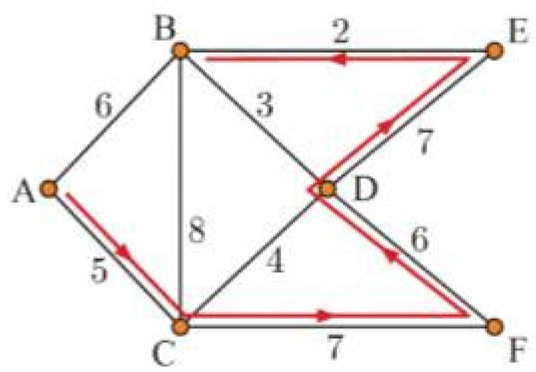
weight $= 6+2+7+6+7 = 28$

A → C → F → D → B → E



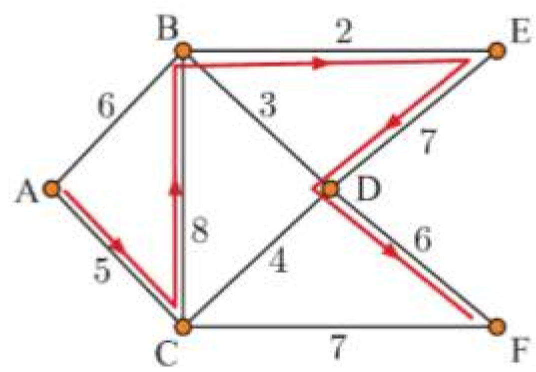
weight $= 5+7+6+3+2 = 23$

A → C → F → D → E → B



weight $= 5+7+6+7+2 = 27$

A → C → B → E → D → F

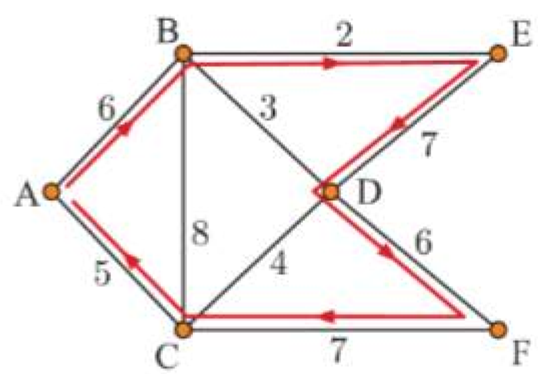


weight $= 5+8+2+7+6 = 28$

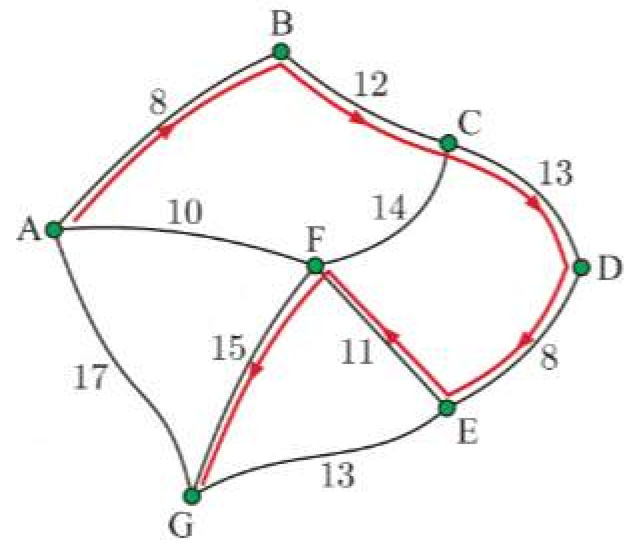
b A → C → F → D → B → E is the Hamiltonian path with the lowest total weight, which is 23.

c A → B → E → D → F → C → A is a Hamiltonian cycle.

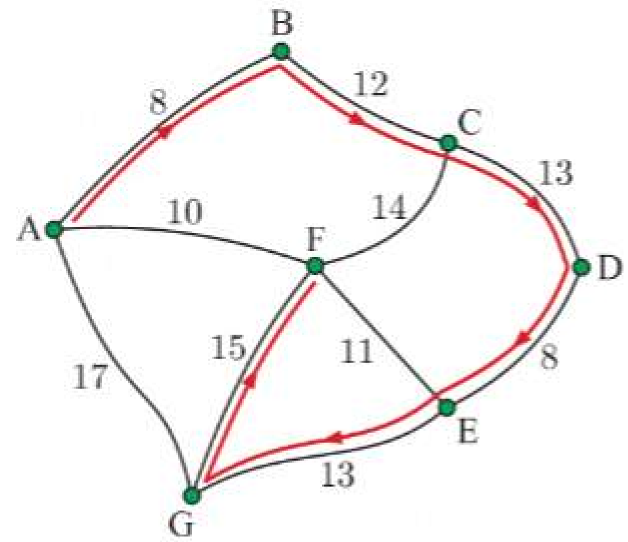
∴ the graph is Hamiltonian.



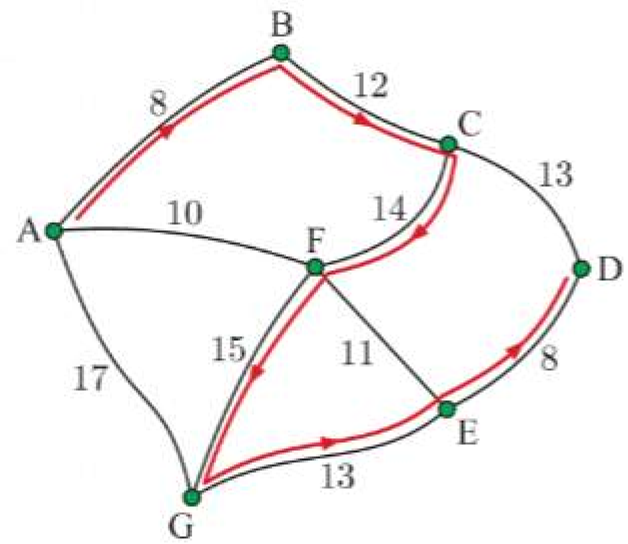
4 **a** The path $A \to B \to C \to D \to E \to F \to G$ has total weight
$8+12+13+8+11+15=67$.
$\therefore$ this path is 67 km long.



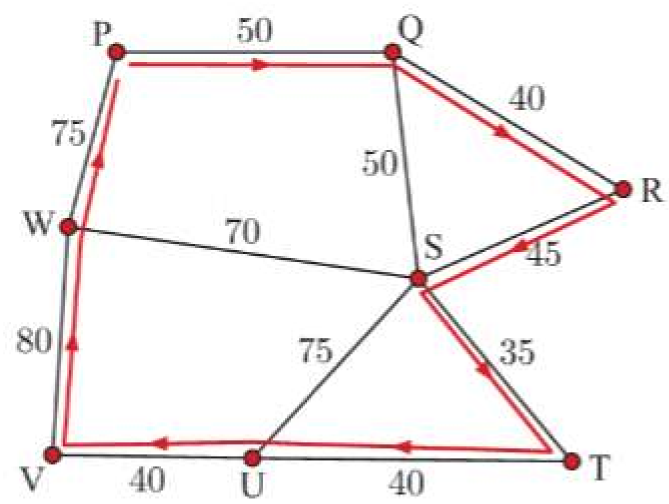
b **i** $A \to B \to C \to D \to E \to G \to F$ is a Hamiltonian path starting at A and ending at F.
The length of this path is
$8+12+13+8+13+15=69$ km.



ii $A \to B \to C \to F \to G \to E \to D$ is a Hamiltonian path starting at A and ending at D.
The length of this path is
$8+12+14+15+13+8=70$ km.

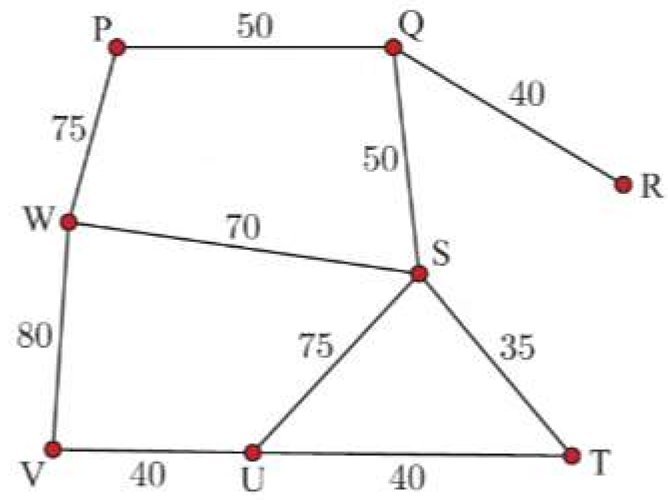


5 **a** $P \to Q \to R \to S \to T \to U \to V \to W \to P$ is a Hamiltonian cycle which starts and ends at P.
The length of this path is
$50+40+45+35+40+40+80+75$
$=405$ km.



b **i** The edge QR would need to be traversed twice.

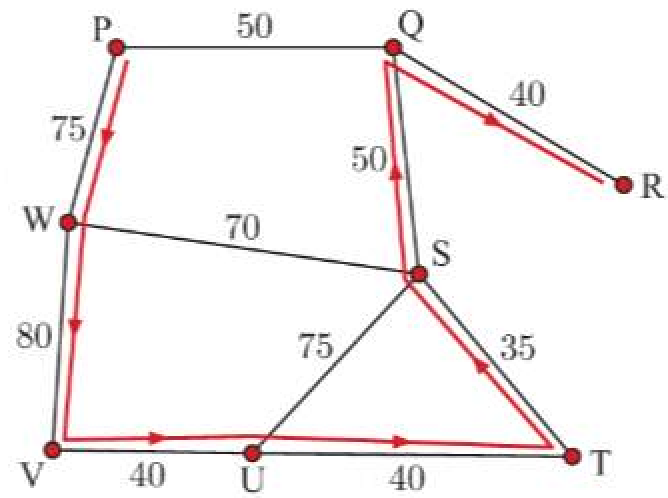
∴ no, it is no longer possible for the supplier to complete a Hamiltonian cycle.



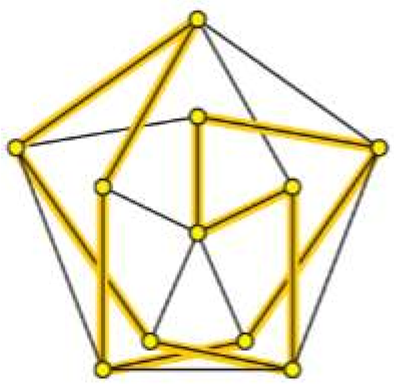
ii The Hamiltonian path needs to finish at R.

P → W → V → U → T → S → Q → R

is a Hamiltonian path which the supplier can take.

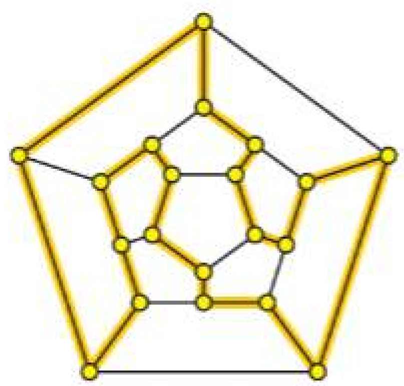


6 **a** A Hamiltonian cycle is:



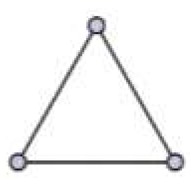
b It is not possible to find a Hamiltonian cycle.

c A Hamiltonian cycle is:

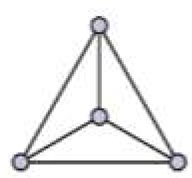


7 **Note:** These are only examples.

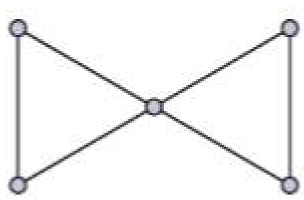
a



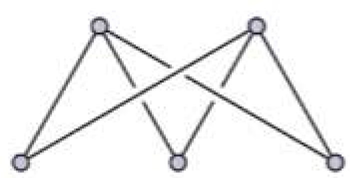
b



c



d

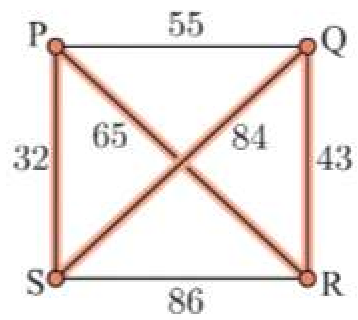


8 **a** Each vertex is connected to every other vertex.
It is always possible to travel to a vertex that has not been visited, and return to the starting vertex immediately, without travelling along the same edge (if $n \geqslant 3$).

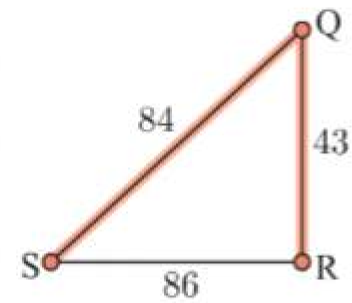
b k_2 is  which only has one edge and no Hamiltonian cycle.

EXERCISE 15K.1

1 **a** From P, the nearest vertex is S, so we choose edge PS. We then choose SQ, then QR.
All vertices have now been visited, so we add RP to complete the Hamiltonian cycle $\text{P} \to \text{S} \to \text{Q} \to \text{R} \to \text{P}$.
So, the upper bound for the TSP is $32 + 84 + 43 + 65 = 224$.



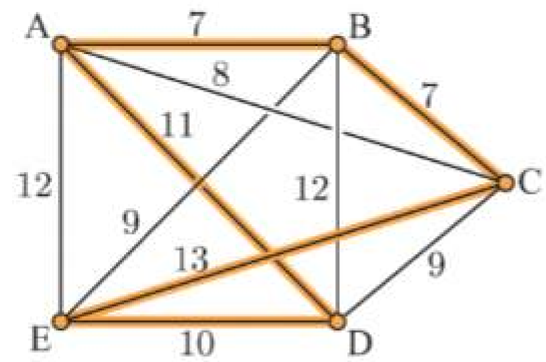
b We find a minimum spanning tree for the graph with vertex P and all edges connected to it deleted.
The minimum spanning tree has weight $43 + 84 = 127$.
The two shortest deleted edges have weight 32 and 55.
So, the lower bound for the TSP is $127 + 32 + 55 = 214$.



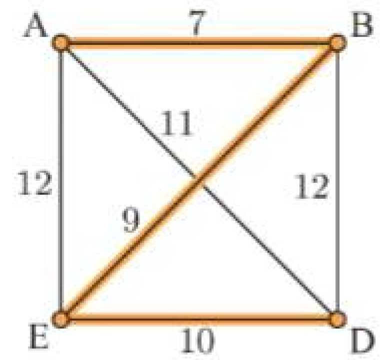
c **i** $214 \leqslant 216 \leqslant 224$ ✓

ii The Hamiltonian cycle with minimum weight is $\text{P} \to \text{Q} \to \text{R} \to \text{S} \to \text{P}$.

2 **a** From C, the nearest vertex is B, so we choose edge CB.
We then choose BA, then AD, then DE.
All vertices have now been visited, so we add EC to complete the Hamiltonian cycle $\text{C} \to \text{B} \to \text{A} \to \text{D} \to \text{E} \to \text{C}$.
So, the upper bound for the TSP is
$7 + 7 + 11 + 10 + 13 = 48$.

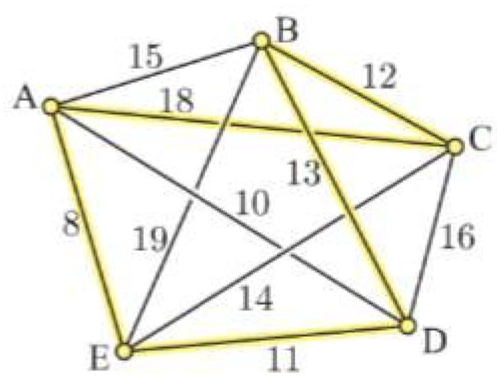


b We find a minimum spanning tree for the graph with vertex C and all edges connected to it deleted.
The minimum spanning tree has weight
$7 + 9 + 10 = 26$.
The two shortest deleted edges have weight 7 and 8.
So, the lower bound for the TSP is $26 + 7 + 8 = 41$.



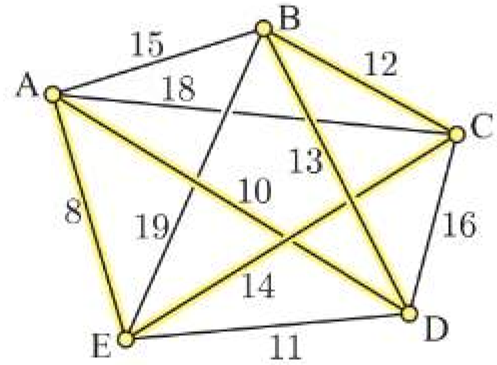
c The Hamitonian cycle with minimum weight is $\text{C} \to \text{A} \to \text{B} \to \text{E} \to \text{D} \to \text{C}$, with weight 43.

3 **a** **i** The Hamiltonian cycle is
$\text{A} \to \text{E} \to \text{D} \to \text{B} \to \text{C} \to \text{A}$.
So, the upper bound $= 8 + 11 + 13 + 12 + 18$
$= 62$.



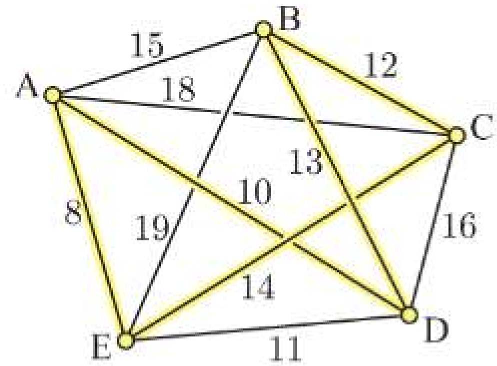
ii The Hamiltonian cycle is
B → C → E → A → D → B.

So, the upper bound $= 12 + 14 + 8 + 10 + 13$
$= 57.$



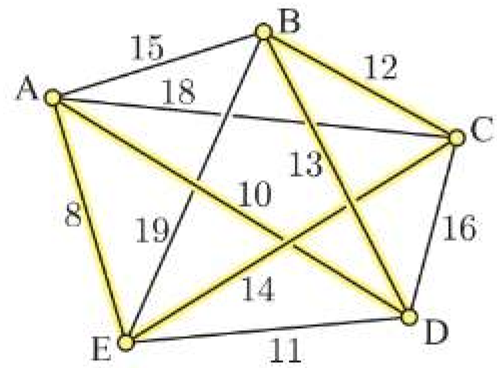
iii The Hamiltonian cycle is
C → B → D → A → E → C.

So, the upper bound $= 12 + 13 + 10 + 8 + 14$
$= 57.$



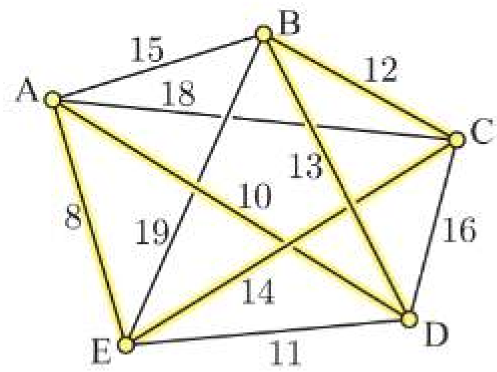
iv The Hamiltonian cycle is
D → A → E → C → B → D.

So, the upper bound $= 10 + 8 + 14 + 12 + 13$
$= 57.$



v The Hamiltonian cycle is
E → A → D → B → C → E.

So, the upper bound $= 8 + 10 + 13 + 12 + 14$
$= 57.$

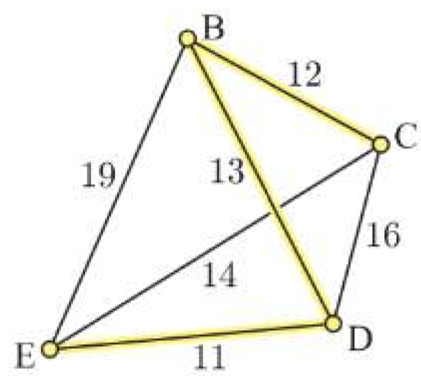


The best upper bound is 57.

b **i** With vertex A and its edges deleted, the minimum spanning tree has weight $11 + 12 + 13 = 36$.

The two shortest deleted edges have weight 8 and 10.

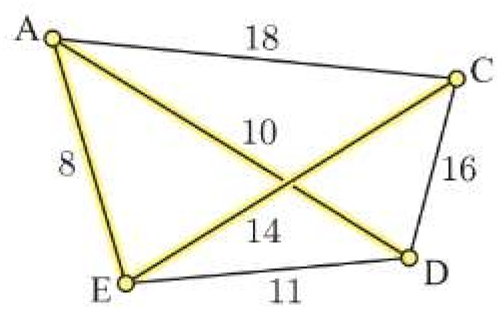
So, the lower bound $= 36 + 8 + 10 = 54$.



ii With vertex B and its edges deleted, the minimum spanning tree has weight $8 + 10 + 14 = 32$.

The two shortest deleted edges have weight 12 and 13.

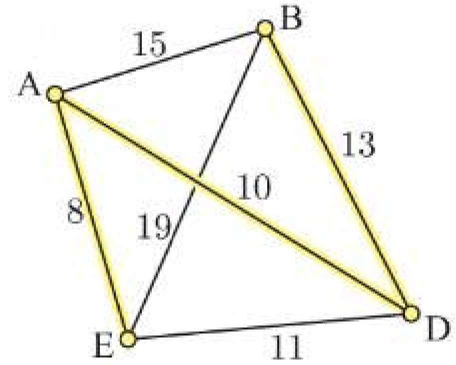
So, the lower bound $= 32 + 12 + 13 = 57$.



iii With vertex C and its edges deleted, the minimum spanning tree has weight $8 + 10 + 13 = 31$.

The two shortest deleted edges have weight 12 and 14.

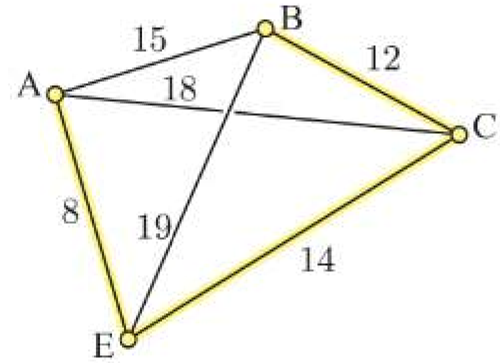
So, the lower bound $= 31 + 12 + 14 = 57$.



iv With vertex D and its edges deleted, the minimum spanning tree has weight $8 + 12 + 14 = 34$.

The two shortest deleted edges have weight 10 and 11.

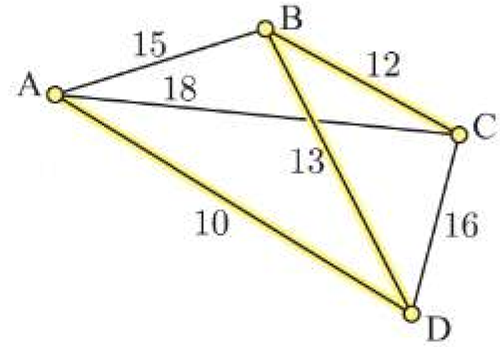
So, the lower bound $= 34 + 10 + 11 = 55$.



v With vertex E and its edges deleted, the minimum spanning tree has weight $10 + 12 + 13 = 35$.

The two shortest deleted edges have weight 8 and 11.

So, the lower bound $= 35 + 8 + 11 = 54$.



The best lower bound is 57.

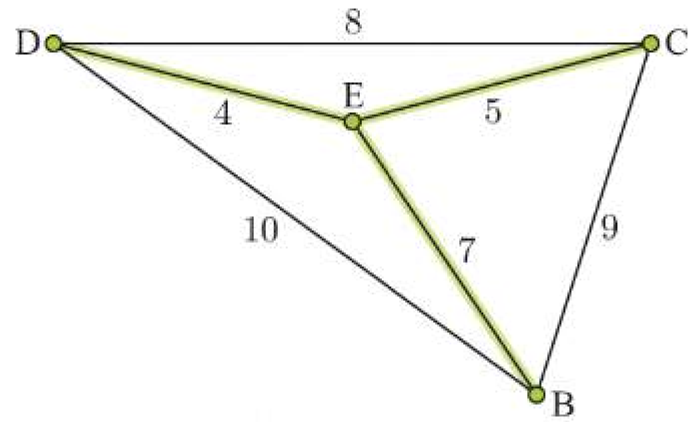
c Since lower bound $=$ upper bound $= 57$, the Hamiltonian cycle of minimum weight must have weight 57.

An example is $C \to B \to D \to A \to E \to C$.

4 **a** With vertex A and its edges deleted, the minimum spanning tree has weight $4 + 5 + 7 = 16$.

The two shortest deleted edges have weight 7 and 8.

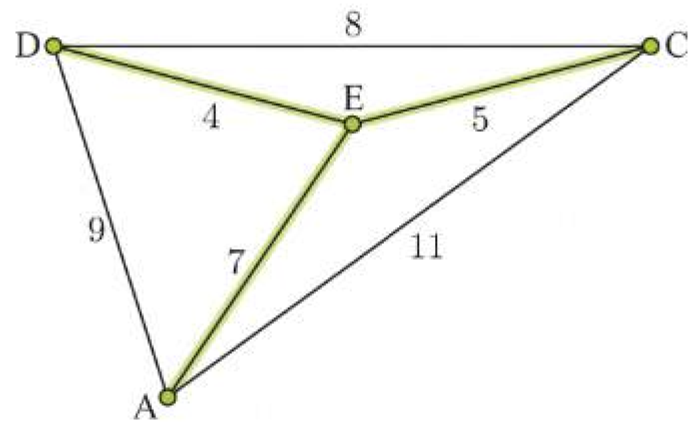
So, the lower bound $= 16 + 7 + 8 = 31$.



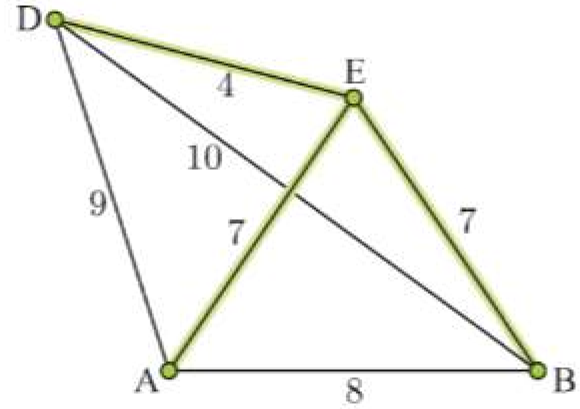
With vertex B and its edges deleted, the minimum spanning tree has weight $4 + 5 + 7 = 16$.

The two shortest deleted edges have weight 7 and 8.

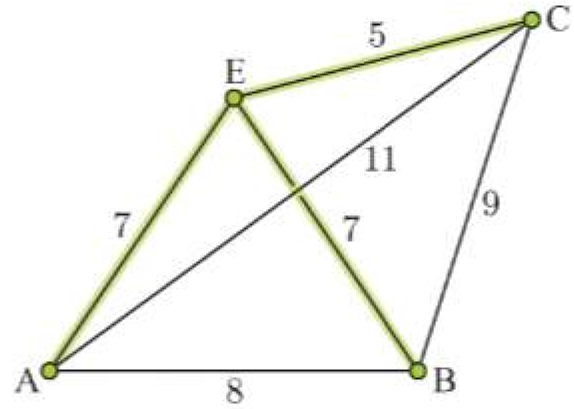
So, the lower bound $= 16 + 7 + 8 = 31$.



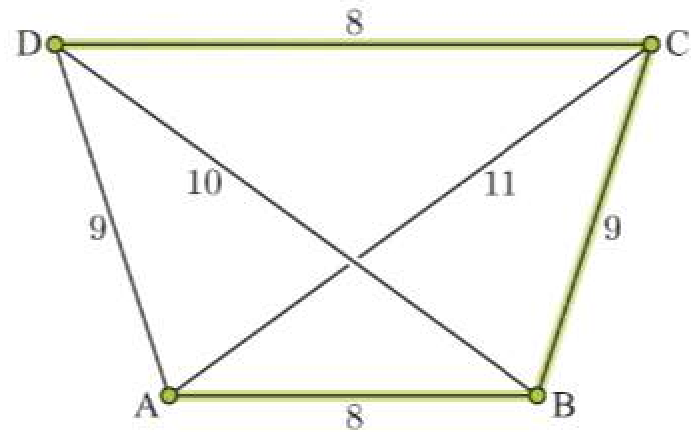
With vertex C and its edges deleted, the minimum spanning tree has weight
$4+7+7=18$.
The two shortest deleted edges have weight 5 and 8.
So, the lower bound $= 18+5+8=31$.



With vertex D and its edges deleted, the minimum spanning tree has weight
$5+7+7=19$.
The two shortest deleted edges have weight 4 and 8.
So, the lower bound $= 19+4+8=31$.



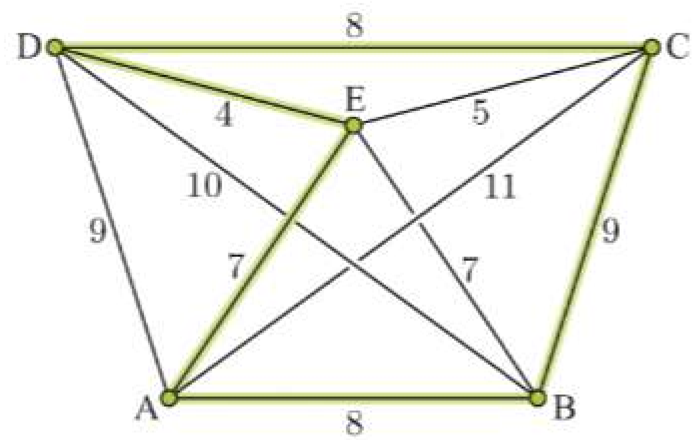
With vertex E and its edges deleted, the minimum spanning tree has weight
$8+8+9=25$.
The two shortest deleted edges have weight 4 and 5.
So, the lower bound $= 25+4+5=34$.



The best lower bound is 34.

b The Hamiltonian cycle is
A → E → D → C → B → A.

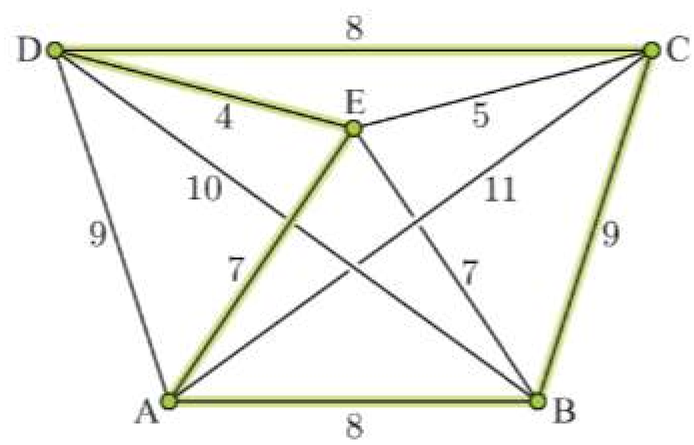
So, the upper bound $= 7+4+8+9+8$
$= 36$.



c **i** We would choose to base ourselves at town E, as the roads between E and the other towns are the shortest on the graph.

ii The Hamiltonian cycle is
E → D → C → B → A → E.

So, the upper bound $= 4+8+9+8+7$
$= 36$.



5

	1	6	4	3	2	5
	A	B	C	D	E	F
A		16	13	11	7	8
B	16		10	5	12	10
C	13	10		4	7	9
D	11	5	4		6	7
E	7	12	7	6		9
F	8	10	9	7	9	

From A, the nearest town is E.
From E, the nearest unvisited town is D.
From D, the nearest unvisited town is C.
From C, the nearest unvisited town is F.
From F, we must visit B, then return to A.

So, the Hamiltonian cycle is $A \to E \to D \to C \to F \to B \to A$, with total distance $7 + 6 + 4 + 9 + 10 + 16 = 52$ km.

6

Bordeaux								
870	Calais							
641	543	Dijon						
550	751	192	Lyons					
649	1067	507	316	Marseille				
457	421	297	445	761	Orléans			
247	625	515	431	733	212	Poitiers		
519	803	244	59	309	392	421	St-Etienne	
244	996	726	535	405	582	435	582	Toulouse

a To compare the distances from a particular town, we must look at the values in the row *and* column corresponding to the town.

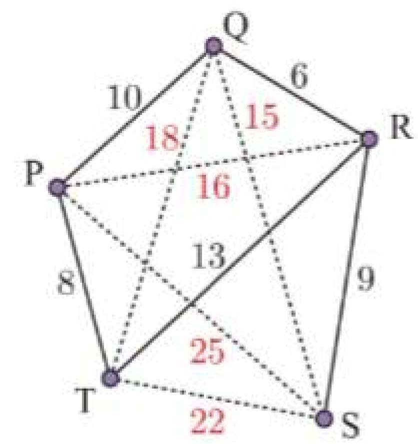
i From Toulouse, the nearest town is Bordeaux.
From Bordeaux, the nearest unvisited town is Poitiers.
Continuing this process,we visit Orléans, then Dijon, then Lyons, then St-Etienne, then Marseille, then Calais, before returning to Toulouse.
The Hamiltonian cycle is $T \to B \to P \to O \to D \to L \to S \to M \to C \to T$.

ii From Calais, the nearest town is Orléans.
From Orléans, the nearest unvisited town is Poitiers.
Continuing this process, we visit Bordeaux, then Toulouse, then Marseille, then St-Etienne, then Lyons, then Dijon, before returning to Calais.
The Hamiltonian cycle is $C \to O \to P \to B \to T \to M \to S \to L \to D \to C$.

b It makes no difference whether the company representative is based in Toulouse or Calais. The cycles given in **a** are not necessarily optimal, and any cycle starting and ending in Calais could also be travelled starting and ending at Toulouse.

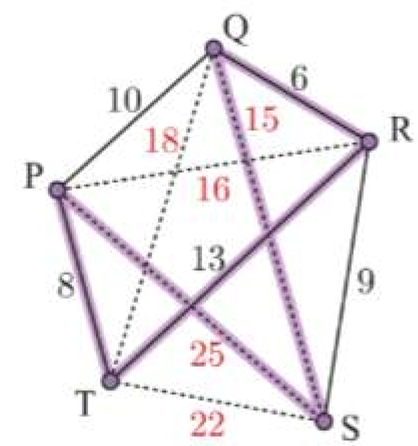
EXERCISE 15K.2

1 **a** The shortest distance from Q to T is $10+8=18$ units.
The shortest distance from Q to S is $6+9=15$ units.
The shortest distance from P to R is $10+6=16$ units.
The shortest distance from P to S is $10+6+9=25$ units.
The shortest distance from S to T is $9+13=22$ units.



b Starting from vertex P, the nearest neighbour algorithm on the transformed graph gives the Hamiltonian cycle $P \to T \to R \to Q \to S \to P$.

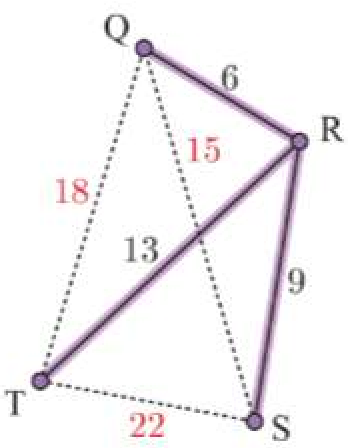
So, an upper bound for the TSP is
$8+13+6+15+25=67$.



c Deleting vertex P and the edges connected to it, we find the minimum spanning tree. The minimum spanning tree has weight 28.

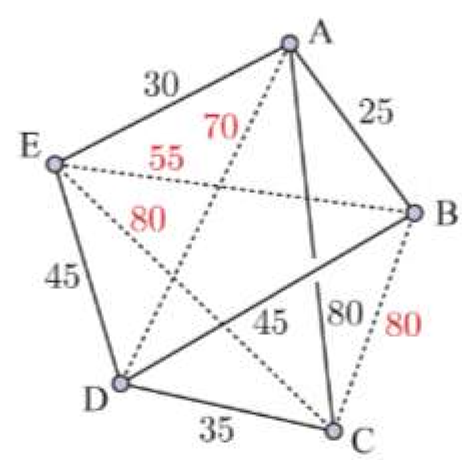
The two shortest deleted edges have weight 8 and 10.

So, a lower bound for the TSP is $28+8+10=46$.



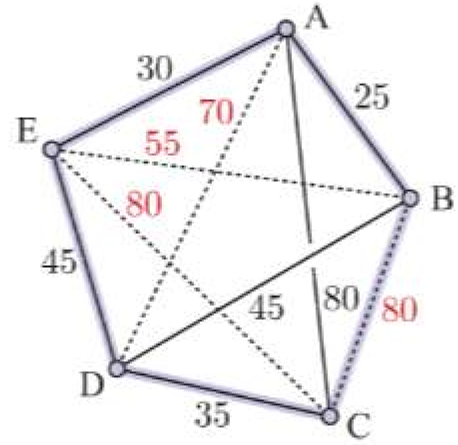
d The Hamiltonian cycle of minimum weight is $P \to Q \to R \to S \to R \to T \to P$ with weight $10+6+9+9+13+8=55$, and $46 \leqslant 55 \leqslant 67$ ✓

2 **a** The cheapest route from E to B is £30 + £25 = £55.
The cheapest route from A to D is £25 + £45 = £70.
The cheapest route from B to C is £45 + £35 = £80.
The cheapest route from E to C is £45 + £35 = £80.



b **i** Starting from vertex B, the nearest neighbour algorithm on the transformed graph gives the Hamiltonian cycle B → A → E → D → C → B.

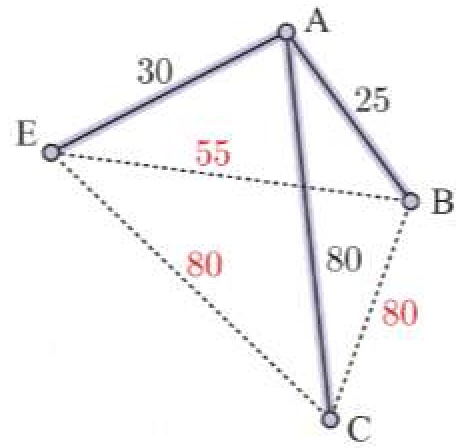
So, an upper bound for the TSP is
$25 + 30 + 45 + 35 + 80 = 215$.



ii Deleting vertex D and the edges connected to it, we find the minimum spanning tree. The minimum spanning tree has weight 135.

The two shortest deleted edges have weight 35 and 45.

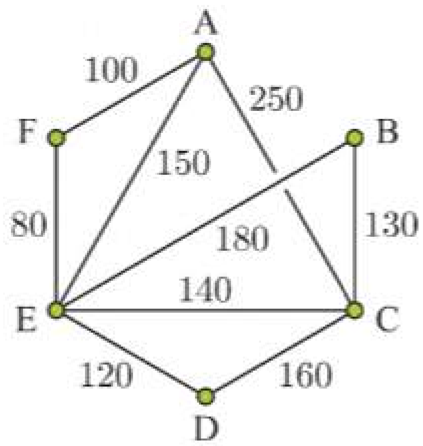
So, a lower bound for the TSP is $135 + 35 + 45 = 215$.



iii From **i** and **ii**, the cheapest route for the salesman must be £215.
From **i**, a possible route is B → A → E → D → C → D → B.
Town D is visited more than once.

3 **a** 3 cities (C, E, and F) are directly connected to A.

b



c The shortest distance between A and B is $150 + 180 = 330$ km.
The shortest distance between A and D is $150 + 120 = 270$ km.
The shortest distance between B and D is $130 + 160 = 290$ km.
The shortest distance between B and F is $180 + 80 = 260$ km.
The shortest distance between C and F is $140 + 80 = 220$ km.
The shortest distance between D and F is $120 + 80 = 200$ km.

	A	B	C	D	E	F
A	0	330	250	270	150	100
B	330	0	130	290	180	260
C	250	130	0	160	140	220
D	270	290	160	0	120	200
E	150	180	140	120	0	80
F	100	260	220	200	80	0

d **i** Starting with D, the nearest neighbour algorithm gives the Hamiltonian cycle $D \to E \to F \to A \to C \to B \to D$.

So, an upper bound for the TSP is $120 + 80 + 100 + 250 + 130 + 290 = 970$.

ii Deleting vertex D and the edges connected to it, the minimum spanning tree has edges EF, AF, BC, and CE.

The minimum spanning tree has weight $80 + 100 + 130 + 140 = 450$.

The two shortest deleted edges have weight 120 and 160.

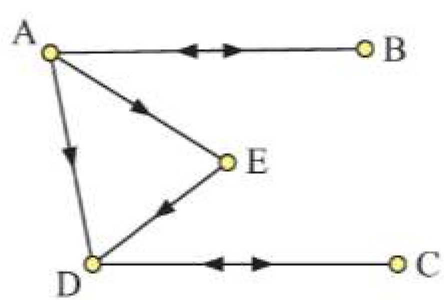
So, a lower bound for the TSP is $450 + 120 + 160 = 730$.

iii An example is $D \to E \to F \to A \to E \to B \to C \to D$ with total distance $120 + 80 + 100 + 150 + 180 + 130 + 160 = 920$ km.

Trudi must visit E twice in this route.

REVIEW SET 15A

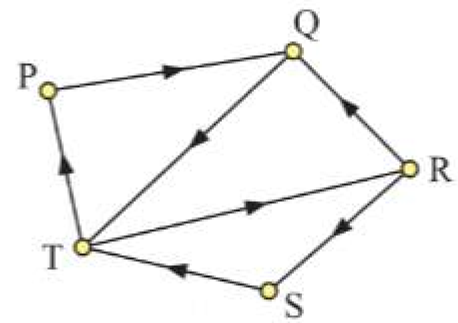
1 **a**



For example, it is not possible to travel from D to E.

$\therefore$ the graph is not strongly connected.

b



It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is strongly connected.

2 **a** **i** The graph has no loops, and there is a maximum of one edge joining any pair of distinct vertices.

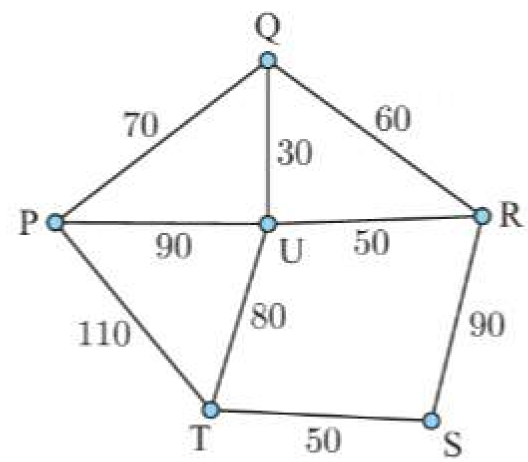
$\therefore$ the graph is simple.

ii It is possible to travel from every vertex to every other vertex by following edges.

$\therefore$ the graph is connected.

iii Not every vertex is connected to every other vertex by an edge.

$\therefore$ the graph is not complete.

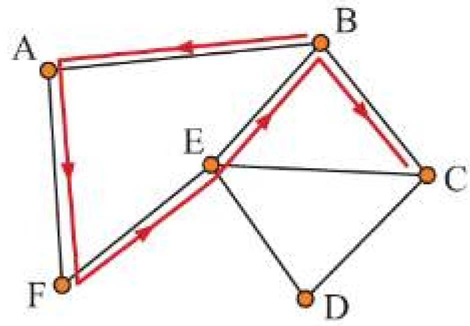


b

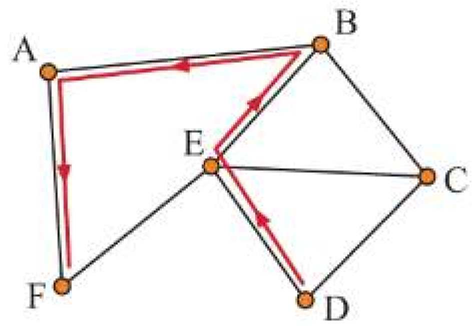
	P	Q	R	S	T	U
P	–	70	–	–	110	90
Q	70	–	60	–	–	30
R	–	60	–	90	–	50
S	–	–	90	–	50	–
T	110	–	–	50	–	80
U	90	30	50	–	80	–

c The quickest route from P to S is $P \to T \to S$, with time $110 + 50 = 160$ minutes.

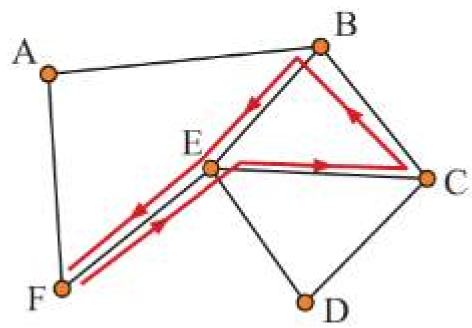
3 **a** $B \to A \to F \to E \to B \to C$ is a trail as it is a walk in which no edge is repeated.



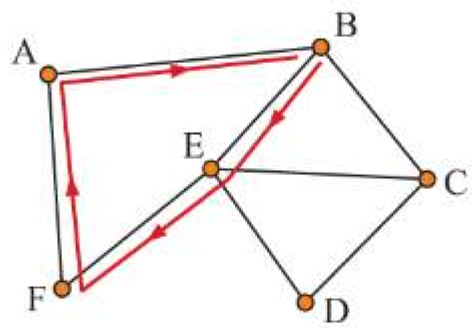
b $D \to E \to B \to A \to F$ is a path as it is a walk in which no edge or vertex is repeated.



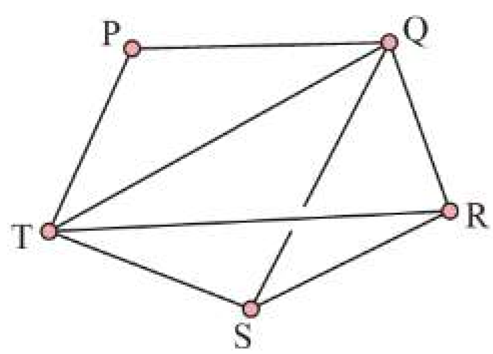
c $F \to E \to C \to B \to E \to F$ is not a circuit as the edge EF is repeated.



d $B \to E \to F \to A \to B$ is a cycle as it is a walk which starts and ends at the same vertex, with no repeated vertices or edges (apart from the starting vertex).



4 **a** There are 5 vertices, so the adjacency matrix has order 5×5.
The vertices are in order P, Q, R, S, T.
The adjacency matrix is



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

b $$\mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 2 & 2 & 1 \\ 1 & 4 & 2 & 2 & 3 \\ 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 \\ 1 & 3 & 2 & 2 & 4 \end{pmatrix}$$

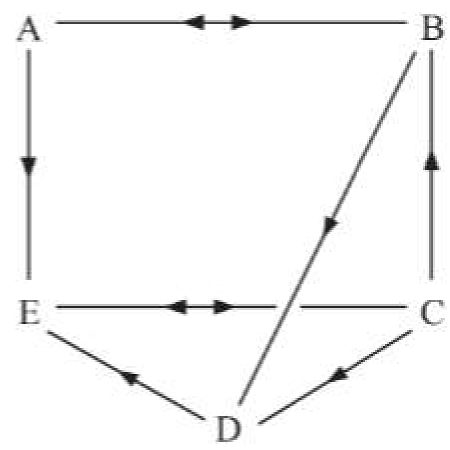
c The number of 2-step routes from T to Q is given by the value in row 5, column 2 of $\mathbf{A}^2$, which is 3.
So there are three 2-step routes from T to Q.

d The three 2-step routes from T to Q are $T \to P \to Q$, $T \to R \to Q$, $T \to S \to Q$.

5 **a** There are 5 vertices, so the adjacency matrix has order 5×5.

The vertices are in order A, B, C, D, E.

The adjacency matrix is



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

b **i** $$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 1 & 3 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 2 \\ 1 & 3 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

ii No, it is not possible to move items between any two areas of the warehouse in at most three steps. The 0 in row 4, column 1 of $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$ indicates that it is not possible to move items from D to A in at most three steps.

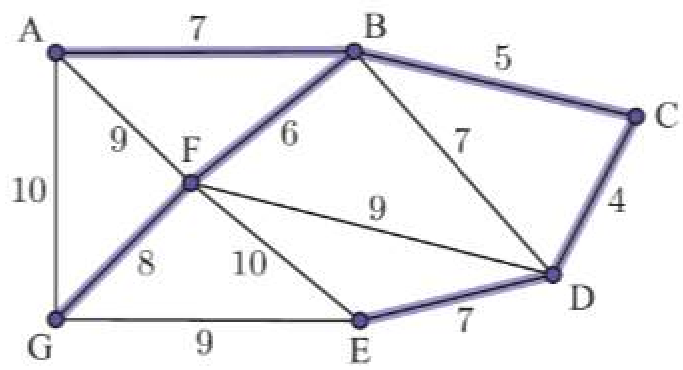
c **i** If the belt between C and B is shut down, the new adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

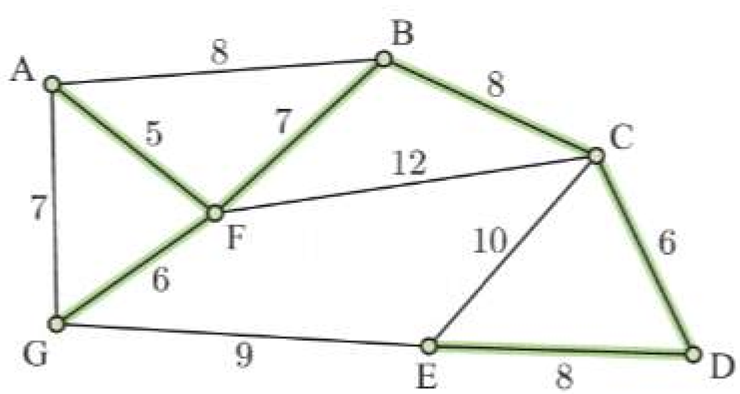
$$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 \end{pmatrix}$$

ii There are now six 0s in the matrix $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$ which indicates that it is not possible to move items from C, D, or E to A or B at all.

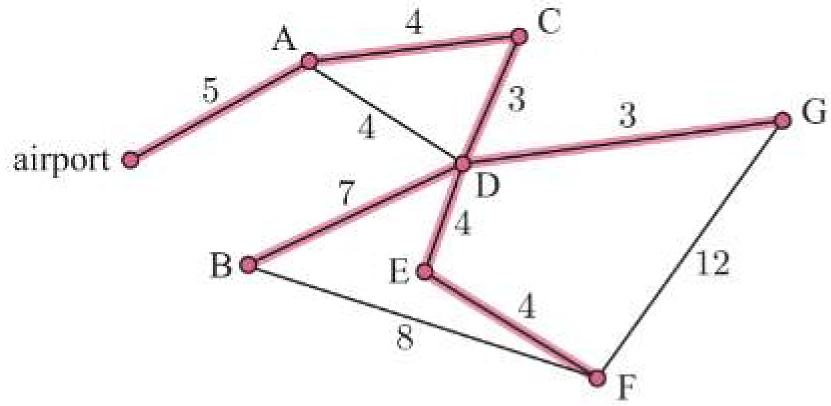
6 **a** We choose a vertex at random, say vertex A. The vertex nearest A is B, so we choose edge AB. We then choose BC, CD, BF, DE, and FG.



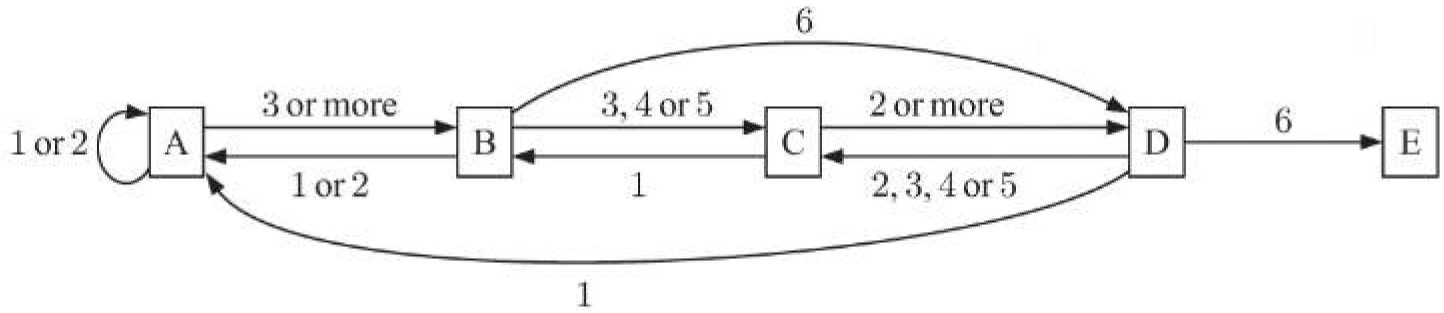
b We choose a vertex at random, say vertex A. The vertex nearest A is F, so we choose edge AF. We then choose FG, FB, BC, CD, and DE.



7 Using Kruskal's algorithm:
Edge CD is an edge with least weight. We then choose DG, DE, EF, AC, airport-A, and BD.



8



a From A, players stay at A with probability $\frac{2}{6} = \frac{1}{3}$, and move to B with probability $\frac{4}{6} = \frac{2}{3}$.

From B, players move to A with probability $\frac{2}{6} = \frac{1}{3}$, move to C with probability $\frac{3}{6} = \frac{1}{2}$, and move to D with probability $\frac{1}{6}$.

From C, players move to B with probability $\frac{1}{6}$, and move to D with probability $\frac{5}{6}$.

From D, players move to A with probability $\frac{1}{6}$, move to C with probability $\frac{4}{6} = \frac{2}{3}$, and move to E with probability $\frac{1}{6}$.

$$\mathbf{T} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ \frac{1}{6} & 0 & \frac{2}{3} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

b Players start at A, so the initial state matrix $\mathbf{s}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$\mathbf{s}_0\mathbf{T}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ \frac{1}{6} & 0 & \frac{2}{3} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^3$$

$$\approx \begin{pmatrix} 0.204 & 0.278 & 0.185 & 0.315 & 0.0185 \end{pmatrix}$$

i P(still at square A) ≈ 0.204

Possible routes are:

A → A → A → A, A → A → B → A, A → B → A → A, A → B → D → A.

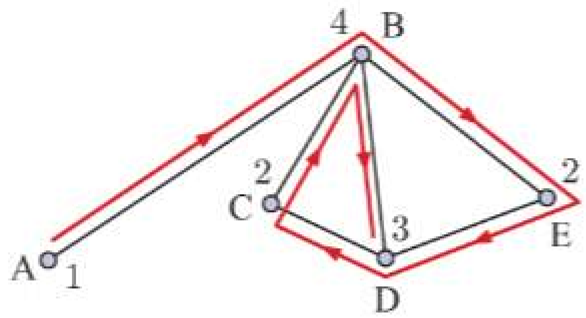
ii P(has reached square E) ≈ 0.0185

The only possible route is A → B → D → E.

9 **a** Exactly two vertices (A and D) have odd degree.

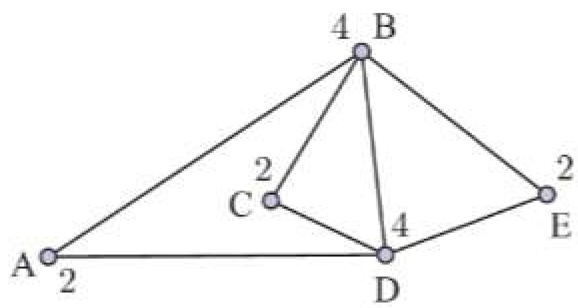
$\therefore$ the graph is semi-Eulerian.

b A $\rightarrow$ B $\rightarrow$ E $\rightarrow$ D $\rightarrow$ C $\rightarrow$ B $\rightarrow$ D is an Eulerian trail.



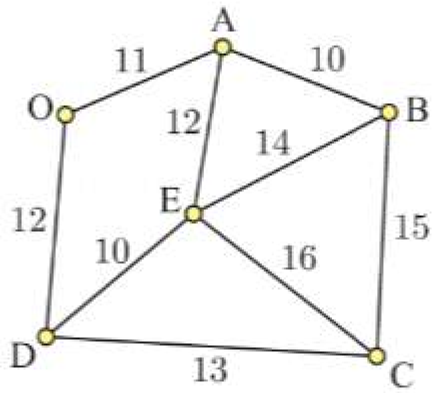
c If an edge AD is added, there are no vertices of odd degree.

$\therefore$ the graph is Eulerian.



10 There are 4 vertices (A, B, C, D) of odd degree.

Pairing	*Shortest path*	*Weight*	*Total weight*
AB CD	A $\rightarrow$ B C $\rightarrow$ D	10 13	23
AC BD	A $\rightarrow$ B $\rightarrow$ C B $\rightarrow$ E $\rightarrow$ D	25 24	49
AD BC	A $\rightarrow$ E $\rightarrow$ D B $\rightarrow$ C	22 15	37

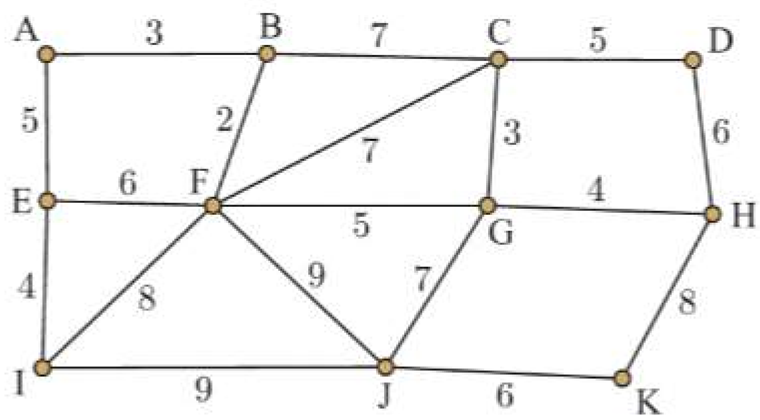


So, we must traverse A $\rightarrow$ B and C $\rightarrow$ D twice.

A possible route is O $\rightarrow$ A $\rightarrow$ E $\rightarrow$ B $\rightarrow$ A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ E $\rightarrow$ D $\rightarrow$ C $\rightarrow$ D $\rightarrow$ O, with distance 136 units.

11 There are 4 vertices (B, E, H, I) of odd degree.

Pairing	*Shortest path*	*Weight*	*Total weight*
BE HI	B $\rightarrow$ A $\rightarrow$ E H $\rightarrow$ G $\rightarrow$ F $\rightarrow$ I	8 17	25
BH EI	B $\rightarrow$ F $\rightarrow$ G $\rightarrow$ H E $\rightarrow$ I	11 4	15
BI EH	B $\rightarrow$ F $\rightarrow$ I E $\rightarrow$ F $\rightarrow$ G $\rightarrow$ H	10 15	25

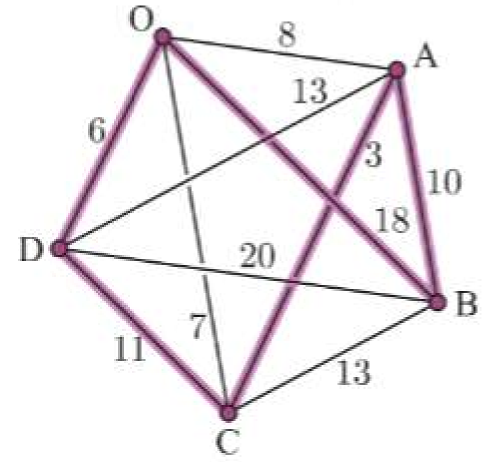


The total sum of the times on the graph is 104 minutes, and the shortest way to connect the vertices of odd degree is 15 minutes.

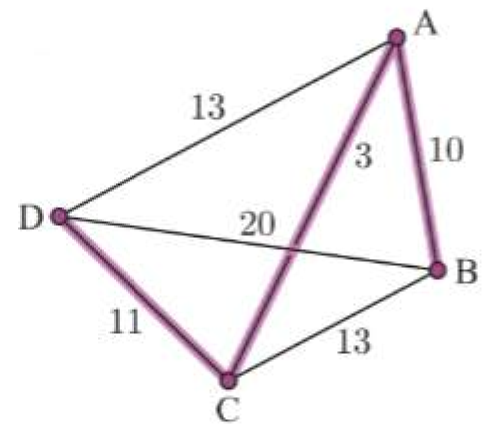
So, the shortest time Jim could take is $104 + 15 = 119$ minutes, which is less than 2 hours.

A possible route is A $\rightarrow$ B $\rightarrow$ F $\rightarrow$ G $\rightarrow$ H $\rightarrow$ G $\rightarrow$ F $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ H $\rightarrow$ K $\rightarrow$ J $\rightarrow$ G $\rightarrow$ C $\rightarrow$ F $\rightarrow$ J $\rightarrow$ I $\rightarrow$ F $\rightarrow$ E $\rightarrow$ I $\rightarrow$ E $\rightarrow$ A.

12 **a** The Hamiltonian cycle is $O \to D \to C \to A \to B \to O$.
So, the upper bound for the TSP is
$6 + 11 + 3 + 10 + 18 = 48$.



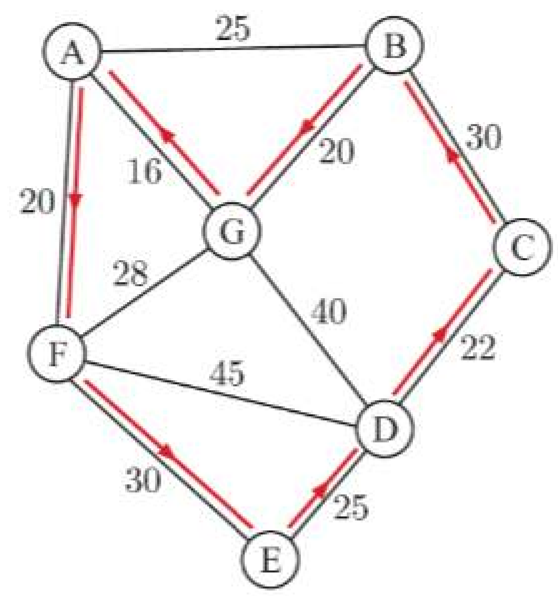
b We find a minimum spanning tree for the graph with vertex O and all edges connected to it deleted.
The minimum spanning tree has weight $3 + 10 + 11 = 24$.
The two shortest deleted edges have weight 6 and 7.
So, the lower bound for the TSP is $24 + 6 + 7 = 37$.



13 **a** Starting on island A, visiting each island once and then returning to A would be a Hamiltonian cycle.

b John could take the route
$A \to F \to E \to D \to C \to B \to G \to A$ which is a Hamiltonian cycle.

c The total travel time for this route is
$20 + 30 + 25 + 22 + 30 + 20 + 16 = 163$ minutes
(or 2 hours 43 minutes).

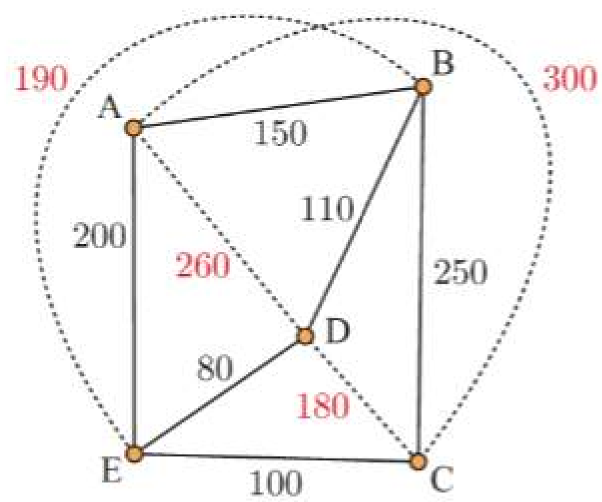


14 **a** The shortest distance from A to C is
$200 + 100 = 300$ km.

The shortest distance from A to D is
$150 + 110 = 260$ km.

The shortest distance from B to E is
$110 + 80 = 190$ km.

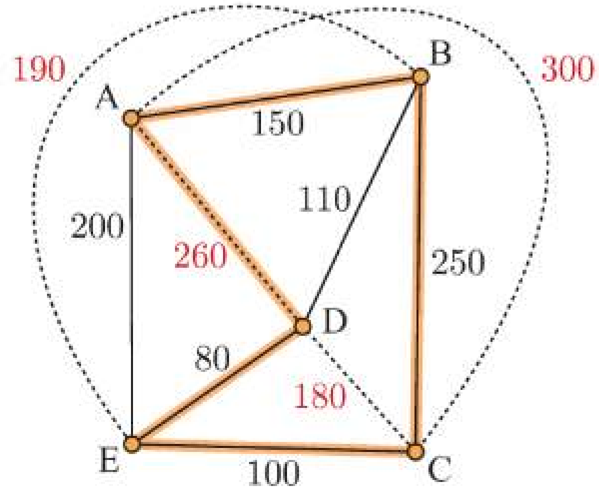
The shortest distance from C to D is
$100 + 80 = 180$ km.



b Starting from D, the nearest neighbout algorithm on the transformed graph gives the Hamiltonian cycle D $\rightarrow$ E $\rightarrow$ C $\rightarrow$ B $\rightarrow$ A $\rightarrow$ D.

So, an upper bound for the TSP is

$80 + 100 + 250 + 150 + 260 = 840$ km.



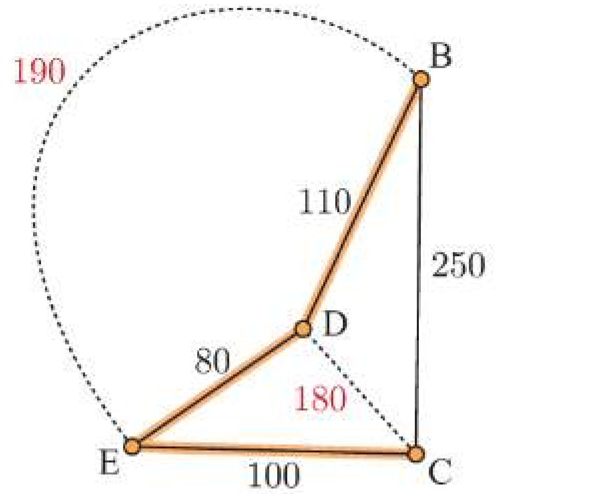
c Deleting vertex A and the edges connected to it, we find the minimum spanning tree. The minimum spanning tree has weight 290.

The two shortest deleted edges have weight 150 and 200.

So, a lower bound for the TSP is

$290 + 150 + 200 = 640$ km.

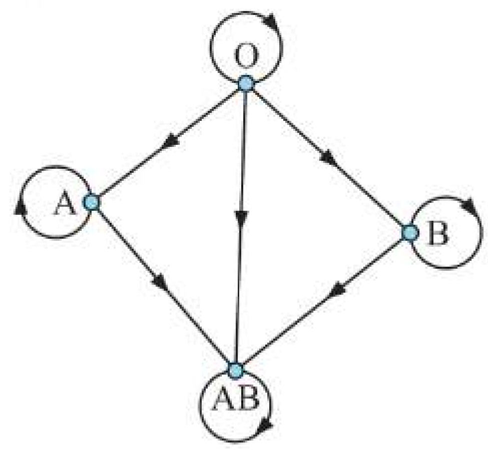
The van must travel at least 640 km, which is greater than 600 km. So, the driver must refuel during the trip.



REVIEW SET 15B

1 **a** **i** There is a direct route from A to A, and from A to AB. So, Lily can donate a kidney to blood types A and AB.

ii There is a direct route from A to A, and from O to A. So, Lily can receive a kidney from blood types A and O.

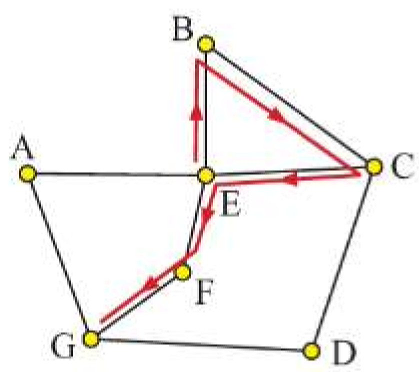


b The out degree of vertex B is 2. This means that people with type B blood can donate kidneys to 2 different blood types.

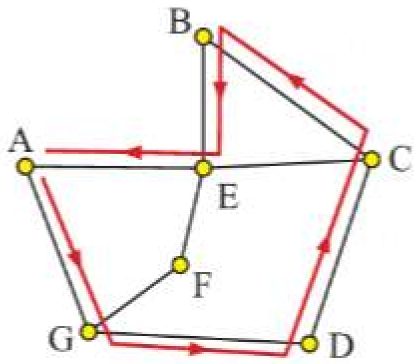
c **i** There is a direct route from O to every other vertex. So, blood type O is described as the "universal donor".

ii There is a direct route to AB from every other vertex. So, blood type AB is described as the "universal recipient".

2 **a** E $\rightarrow$ B $\rightarrow$ C $\rightarrow$ E $\rightarrow$ F $\rightarrow$ G is a trail, as no edges are repeated, but not a path, as the vertex E is repeated.



b $A \to G \to D \to C \to B \to E \to A$ is a cycle of length 6.



3 For a simple connected graph to have as many edges as possible, we consider the complete graphs k_n.

k_n has $\dfrac{n(n-1)}{2}$ edges.

Hence, we require the lowest integer n such that $\dfrac{n(n-1)}{2} \geqslant 500$.

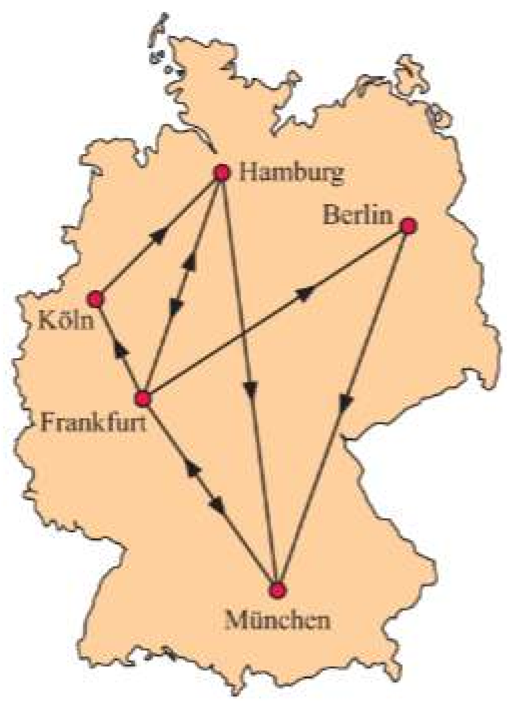
Using technology, $\dfrac{n(n-1)}{2} \geqslant 500$ for $n \approx 32.1$.

So, the fewest number of vertices required is 33 vertices.

4 **a** 9 different routes are offered.

b

$$\mathbf{A} = \begin{array}{c} \\ B \\ F \\ H \\ K \\ M \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} B & F & H & K & M \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$



c **i**

$$\mathbf{A} + \mathbf{A}^2 = \begin{array}{c} \\ B \\ F \\ H \\ K \\ M \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} B & F & H & K & M \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

There are 5 zeros in $\mathbf{A} + \mathbf{A}^2$, so there are 5 journeys which cannot be made using at most one stopover.

ii From **c i**, we know that 2 trips in insufficient to travel between any two cities.

$$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{array}{c} \\ B \\ F \\ H \\ K \\ M \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} B & F & H & K & M \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 3 & 5 & 4 & 3 & 6 \\ 2 & 4 & 3 & 2 & 5 \\ 1 & 2 & 2 & 1 & 2 \\ 1 & 3 & 2 & 1 & 3 \end{pmatrix} \end{array}$$

There are no zeros in $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$, so 3 trips are needed to travel between any two cities.

d If the route between Frankfurt and Köln is made "two-way", then the new adjacency matrix is

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{ccccc} B & F & H & K & M \end{array} \\ \begin{array}{c} B \\ F \\ H \\ K \\ M \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\therefore \quad \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 2 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

So, the number of journeys which cannot be made with at most one stopover reduces to 3.

If a route from Köln to Berlin is created, the new adjacency matrix is

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{ccccc} B & F & H & K & M \end{array} \\ \begin{array}{c} B \\ F \\ H \\ K \\ M \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\therefore \quad \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

So, the number of journeys which cannot be made with at most one stopover reduces to 4.

So, making the route between Frankfurt and Köln "two-way" would be the better decision.

5 $$\mathbf{A} = \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

a It is not possible to travel from town C to town D without going through another town. This is shown by the element 0 in row 3, column 4 of $\mathbf{A}$.

b $$\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

c $$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 3 & 8 & 1 & 3 \\ 3 & 3 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 2 & 4 & 1 & 1 \end{pmatrix}$$

To get from B to C, C to D, or D to D requires going through 2 other towns.

Going from C to C requires going through 3 other towns. By contrast, there are many ways to get from A to B.

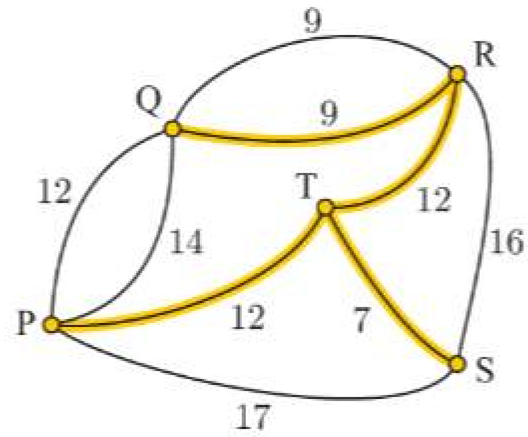
d **i** $\mathbf{A}^4 = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 1 & 5 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 \end{pmatrix}$

The element in row 2, column 2 tells us there are 5 ways to travel from town B and return to town B in exactly 4 trips.

ii Yes, the route $B \to A \to D \to C \to B$ starts and ends at town B and visits the other 3 towns exactly once.

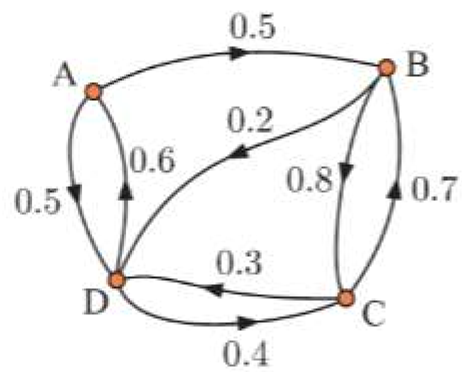
6 Edge ST has the least weight. We then choose QR, RT, and PT.

This minimum spanning tree has weight
$7 + 9 + 12 + 12 = 40$.



7 **a** It is possible to travel from every vertex to every other vertex by following edges.
$\therefore$ the graph is strongly connected.

b From C, the guard moves to D with probability 0.3, so the guard moves to B with probability 0.7.
From D, the guard moves to A with probability 0.6, so the guard moves to C with probability 0.4.



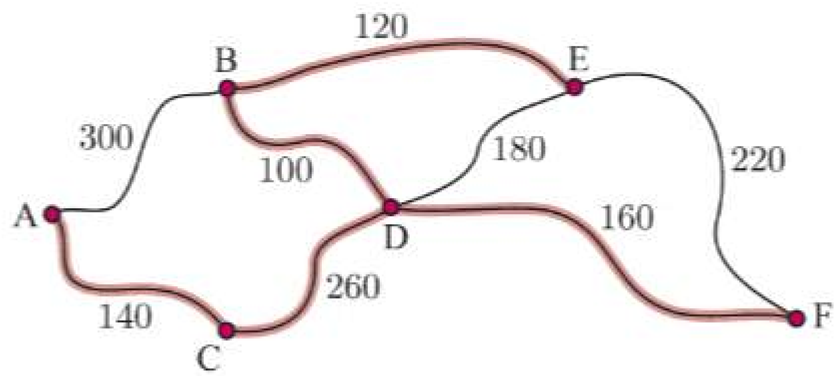
c $\mathbf{T} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.7 & 0 & 0.3 \\ 0.6 & 0 & 0.4 & 0 \end{pmatrix}$

d The initial state matrix is $\mathbf{s}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$

$$\mathbf{s}_0\mathbf{T}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.7 & 0 & 0.3 \\ 0.6 & 0 & 0.4 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0.144 & 0.116 & 0.544 & 0.196 \end{pmatrix}$$

So, there is probability 0.116 that the guard will return to B after visiting exactly two other exhibits.

8 Using Kruskal's algorithm to find the minimum spanning tree:
The edge BD has the least weight. We then choose BE, AC, DF, and CD.



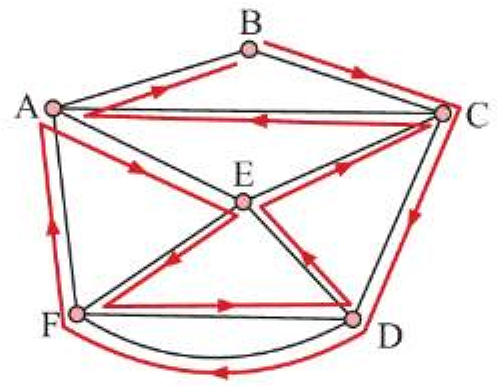
9 **a** We start with vertex A.

	1	4	3	2	5
	A	B	C	D	E
A	–	640	450	320	350
B	640	–	230	410	360
C	450	230	–	220	300
D	320	410	220	–	420
E	350	360	300	420	–

The edges in the minimum spanning tree are AD, CD, BC, and CE.
Total length $= 320 + 220 + 230 + 300 = 1070$ km.

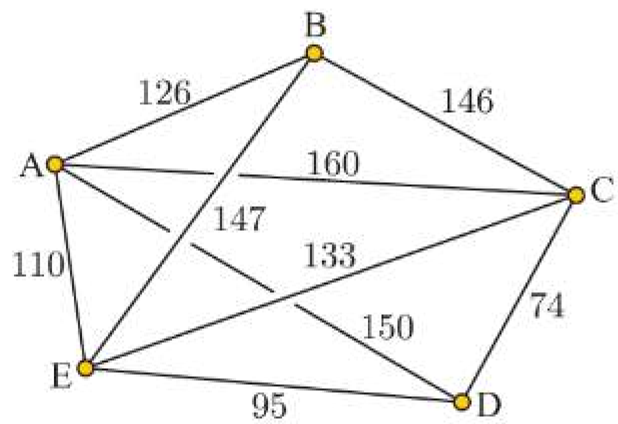
b The most popular flight routes should be considered. For example, there should be more flights to and from the most populated town.

10 **a** There are no vertices of odd degree.
$\therefore$ the graph is Eulerian.



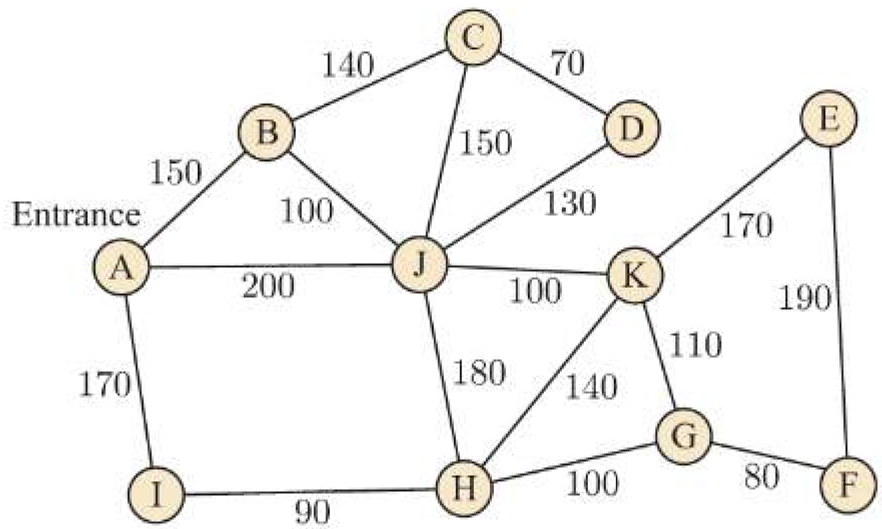
b B → C → D → F → A → E → F → D → E → C → A → B is an Eulerian circuit which starts and ends at B.

11 There are 2 vertices (B and D) of odd degree.
The shortest path connecting B and D is B → C → D, so these edges must be traversed twice.
A possible route is A → B → C → D → C → B → E → D → A → C → E → A, with distance 1361 units.



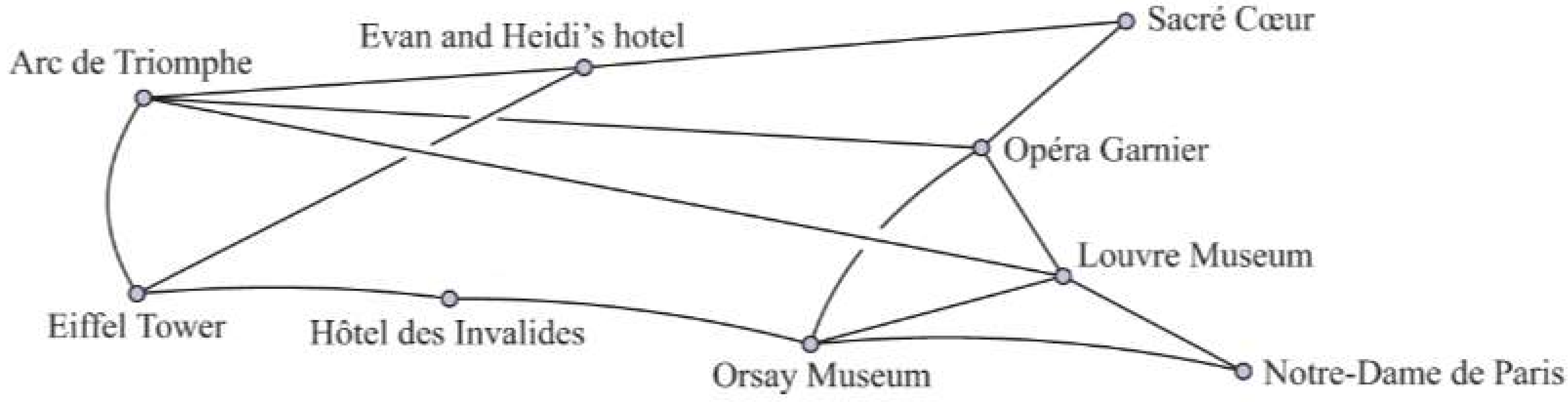
12 There are 4 vertices (A, B, C, G) of odd degree.

Pairing	*Shortest path*	*Weight*	*Total weight*
AB CG	A → B C → J → K → G	150 360	510
AC BG	A → B → C B → J → K → G	290 310	600
AG BC	A → I → H → G B → C	360 140	500



So, Yuto must traverse A → I → H → G and B → C twice.
An example route is A → B → C → D → J → C → B → J → A → I → H → G → K → H → J → K → E → F → G → H → I → A
Its length is 2770 m or 2.77 km.

13

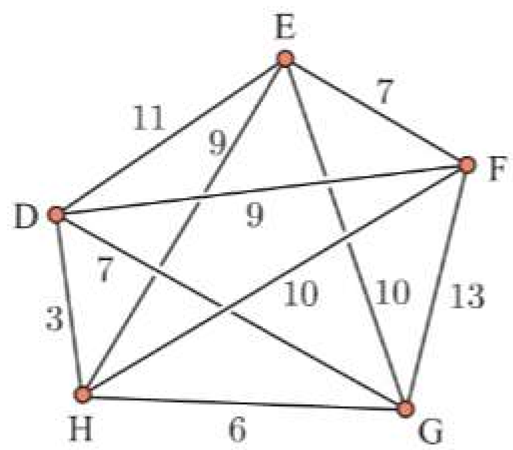


a A possible route which starts and ends at their hotel and visits each landmark exactly once is:
Evan and Heidi's hotel → Sacré Cœur → Opéra Garnier → Louvre Museum → Notre-Dame de Paris → Orsay Museum → Hôtel des Invalides → Eiffel Tower → Arc de Triomphe → Evan and Heidi's hotel

b **i** The graph contains vertices of odd degree, so it is not Eulerian.
$\therefore$ Heidi cannot find a suitable route which traverses each road exactly once and starts and finishes at their hotel.

ii Their hotel and the Eiffel Tower are the only vertices of odd degree, so there is a route which starts at their hotel and ends at the Eiffel Tower.
A possible route is:
Evan and Heidi's hotel → Arc de Triomphe → Opéra Garnier → Sacré Cœur → Evan and Heidi's hotel → Eiffel Tower → Arc de Triomphe → Louvre Museum → Opéra Garnier → Orsay Museum → Notre-Dame de Paris → Louvre Museum → Orsay Museum → Hôtel des Invalides → Eiffel Tower

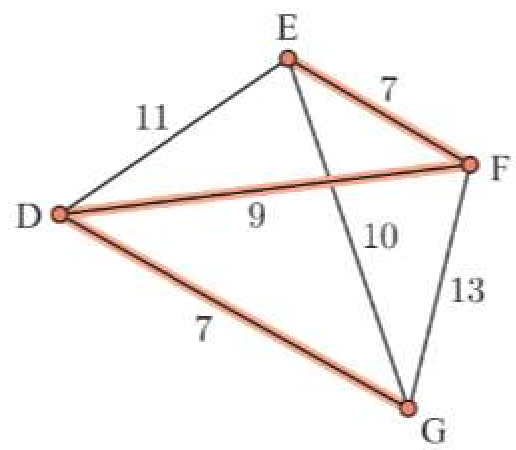
14 **a** **i** Starting from H, the Hamiltonian cycle is
H → D → G → E → F → H.
So, an upper bound for the TSP is
$3+7+10+7+10 = 37$ minutes.



ii Starting from D, the Hamiltonian cycle is
D → H → G → E → F → D.
So, an upper bound for the TSP is
$3+6+10+7+9 = 35$ minutes.

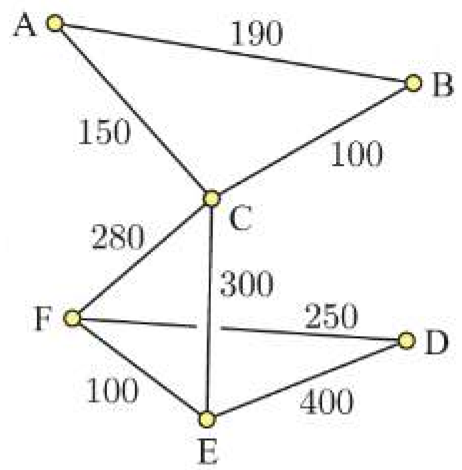
The best upper bound is 35 minutes.

b Deleting vertex H and the edges connected to it, we find the minimum spanning tree. The minimum spanning tree has weight 23.
The two shortest deleted edges have weight 3 and 6.
So, a lower bound for the TSP is
$23+3+6 = 32$ minutes.



c A Hamiltonian cycle of minimum weight is D → H → G → E → F → D, with time 35 minutes.

15



a The cheapest route from A to D is $\$150 + \$280 + \$250 = \$680$.
The cheapest route from A to E is $\$150 + \$300 = \$450$.
The cheapest route from B to D is $\$100 + \$280 + \$250 = \$630$.
The cheapest route from B to E is $\$100 + \$300 = \$400$.
The cheapest route from B to F is $\$100 + \$280 = \$380$.
The cheapest route from D to E is $\$250 + \$100 = \$350$.

	A	B	C	D	E	F
A	0	190	150	680	450	430
B	190	0	100	630	400	380
C	150	100	0	530	300	280
D	680	630	530	0	350	250
E	450	400	300	350	0	100
F	430	380	280	250	100	0

b **i** From the graph, she must visit C twice.

ii Starting from D, the nearest neighbour algorithm gives the Hamiltonian cycle $D \to F \to E \to C \to B \to A \to D$.
So, an upper bound for the TSP is $250 + 100 + 300 + 100 + 190 + 680 = \$1620$.

iii Deleting vertex A and the edges connected to it, the minimum spanning tree has edges BC, EF, DF, and CF.
The minimum spanning tree has weight $100 + 100 + 250 + 280 = 730$.
The two shortest deleted edges have weight 150 and 190.
So, a lower bound for the TSP is $730 + 150 + 190 = \$1070$.

iv The cheapest route is $D \to F \to C \to A \to B \to C \to E \to F \to D$, with total cost \$1620. Towns C and F are visited twice.

v The cheapest route from C to D is now \$400.
The cheapest route from A to D is now $\$150 + \$400 = \$550$.
The cheapest route from B to D is now $\$100 + \$400 = \$500$.
So, the table of lowest costs is shown.

	A	B	C	D	E	F
A	0	190	150	550	450	430
B	190	0	100	500	400	380
C	150	100	0	400	300	280
D	550	500	400	0	350	250
E	450	400	300	350	0	100
F	430	380	280	250	100	0

Starting from D, the nearest neighbour algorithm gives the Hamiltonian cycle $D \to F \to E \to C \to B \to A \to D$.
So, an upper bound for the TSP is $250 + 100 + 300 + 100 + 190 + 550 = \$1490$.
So, the route $D \to F \to E \to C \to B \to A \to C \to D$ is \$130 cheaper than the route in **b iv**.

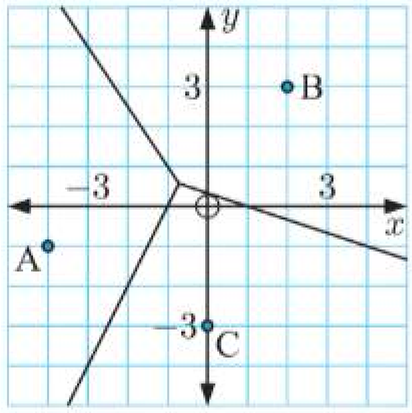
Chapter 16

VORONOI DIAGRAMS

EXERCISE 16A

1 **a** The diagram contains:

- **i** 3 cells
- **ii** 3 edges
- **iii** 1 vertex.



b **i** $(-1, 2)$ lies in cell B, so P is closest to site B.

ii A has coordinates $(-4, -1)$, so
$$\begin{aligned}PA &= \sqrt{(-4-(-1))^2+(-1-2)^2}\\ &= \sqrt{(-3)^2+(-3)^2}\\ &= \sqrt{18} = 3\sqrt{2} \approx 4.24 \text{ units}\end{aligned}$$

B has coordinates $(2, 3)$, so
$$\begin{aligned}PB &= \sqrt{(2-(-1))^2+(3-2)^2}\\ &= \sqrt{3^2+1^2}\\ &= \sqrt{10} \approx 3.16 \text{ units}\end{aligned}$$

C has coordinates $(0, -3)$, so
$$\begin{aligned}PC &= \sqrt{(0-(-1))^2+(-3-2)^2}\\ &= \sqrt{1^2+(-5)^2}\\ &= \sqrt{26} \approx 5.10 \text{ units}\end{aligned}$$

So, $PB < PA$ and $PB < PC$ ✓

c **i** $(-1, 0)$ lies on the edge adjacent to cells A and C, so Q is equally closest to sites A and C.

ii
$$\begin{aligned}QA &= \sqrt{(-4-(-1))^2+(-1-0)^2}\\ &= \sqrt{(-3)^2+(-1)^2}\\ &= \sqrt{10} \approx 3.16 \text{ units}\end{aligned}$$

$$\begin{aligned}QB &= \sqrt{(2-(-1))^2+(3-0)^2}\\ &= \sqrt{3^2+3^2}\\ &= \sqrt{18} = 3\sqrt{2} \approx 4.24 \text{ units}\end{aligned}$$

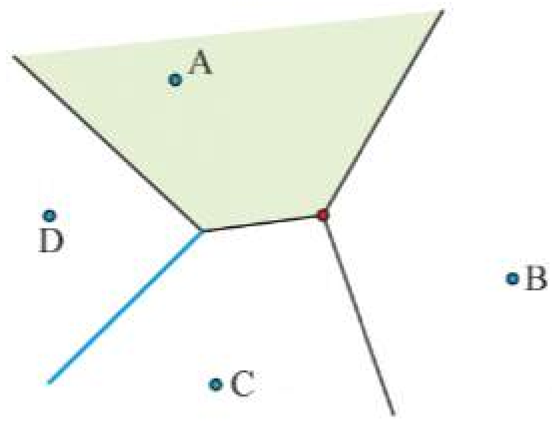
$$\begin{aligned}QC &= \sqrt{(0-(-1))^2+(-3-0)^2}\\ &= \sqrt{1^2+(-3)^2}\\ &= \sqrt{10} \approx 3.16 \text{ units}\end{aligned}$$

So, $QA = QC$, $QA < QB$, and $QC < QB$ ✓

2 **a** The green cell is cell A, so points which lie in this cell are closest to site A.

b The blue edge is adjacent to cells C and D, so points which lie on this edge are equally closest to sites C and D.

c The red vertex is where cells A, B, and C meet, so it is equally closest to sites A, B, and C.

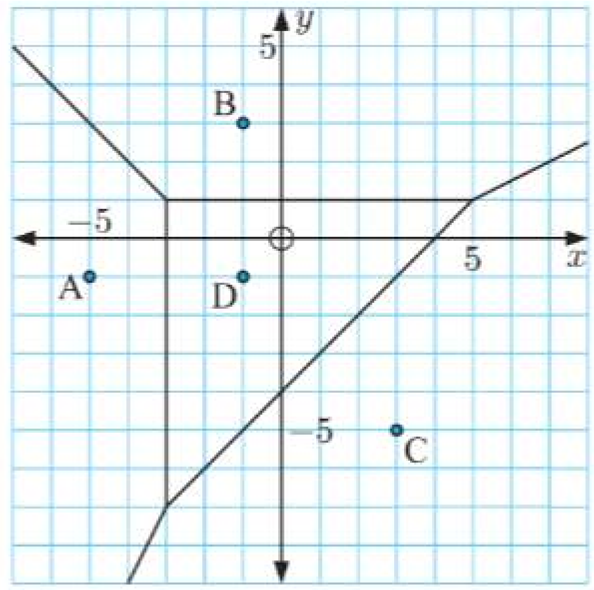


3 **a** **i** $(2, 3)$ lies in cell B, so it is closest to site B.

ii $(-1, -4)$ lies in cell D, so it is closest to site D.

iii $(6, 0)$ lies in cell C, so it is closest to site C.

iv $(-4, -3)$ lies in cell A, so it is closest to site A.



b **i** Let $(-3, 0)$ be the point P.

A has coordinates $(-5, -1)$, so
$$\begin{aligned} PA &= \sqrt{(-5-(-3))^2+(-1-0)^2} \\ &= \sqrt{(-2)^2+(-1)^2} \\ &= \sqrt{5} \text{ units} \end{aligned}$$

D has coordinates $(-1, -1)$, so
$$\begin{aligned} PD &= \sqrt{(-1-(-3))^2+(-1-0)^2} \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{5} \text{ units} \\ &= PA \quad \checkmark \end{aligned}$$

ii Let $(-3, 2)$ be the point Q.

$$\begin{aligned} QA &= \sqrt{(-5-(-3))^2+(-1-2)^2} \\ &= \sqrt{(-2)^2+(-3)^2} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

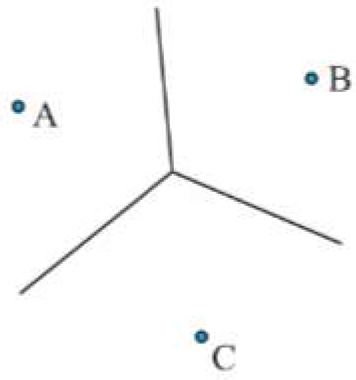
$$\begin{aligned} QD &= \sqrt{(-1-(-3))^2+(-1-2)^2} \\ &= \sqrt{2^2+(-3)^2} \\ &= \sqrt{13} \text{ units} \\ &= QA \quad \checkmark \end{aligned}$$

c $(-3, 0)$ is equally *closest* to sites A and D, and so lies on the edge adjacent to cells A and D. $(-3, 2)$ is equally *close* to sites A and D, but is *closest* to site B, and so does not lie on an edge.

d Cell D is a triangle with base 8 units and height 8 units.

$$\begin{aligned} \therefore \text{ area of cell D} &= \tfrac{1}{2} \times 8 \times 8 \\ &= 32 \text{ units}^2 \end{aligned}$$

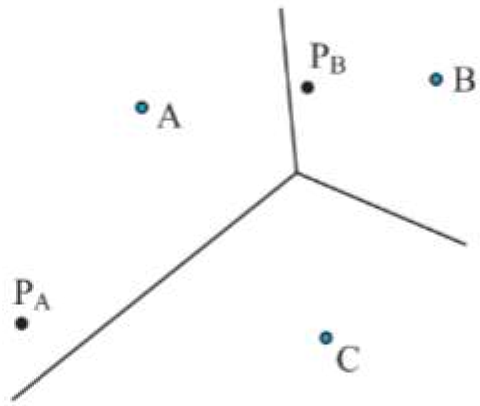
4 **a** The statement "P_A is closer to A than to any other site" is true, as P_A lies in cell A.



b The statement "P_B is closer to B than to C" is true, as P_B lies in cell B.

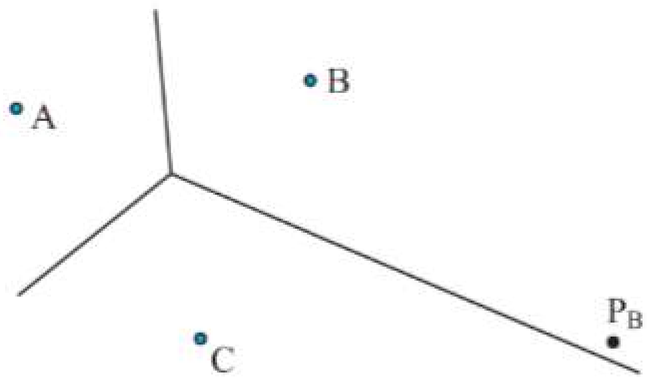
c The statement "A is closer to P_A than to P_B" is not necessarily true.

For example:



d The statement "B is closer to P_B than to C" is not necessarily true.

For example:



5 If the circle passes through another site, then P is equidistant from that site and X, and therefore is not within the interior of cell X.

If another site lies within the circle, then P is closer to that site than to X, and therefore is not within the interior of cell X.

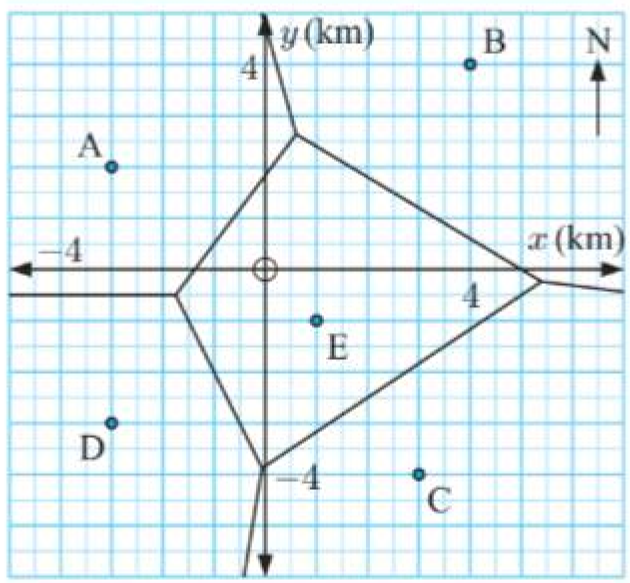
Since P lies within the interior of cell X, this circle cannot contain any other sites.

6 **a** **i** $(0, 0)$ lies within cell E, so the holding post office is post office E.

ii $(4, 1)$ lies within cell B, so the holding post office is post office B.

iii $(-2, 0)$ lies within cell A, so the holding post office is post office A.

iv $(3, -2)$ lies within cell C, so the holding post office is post office C.



b One building lies on the edge adjacent to cells D and E, and the other lies on the edge adjacent to cells A and E.

Consider a building located at $(-1, -2)$ which lies on the edge adjacent to cells D and E.

A building 2.5 km north of $(-1, -2)$ is located at $(-1, \ -2+2.5)$, which is $(-1, 0.5)$ or $(-1, \frac{1}{2})$.

Now $(-1, \frac{1}{2})$ lies on the edge adjacent to cells A and E.

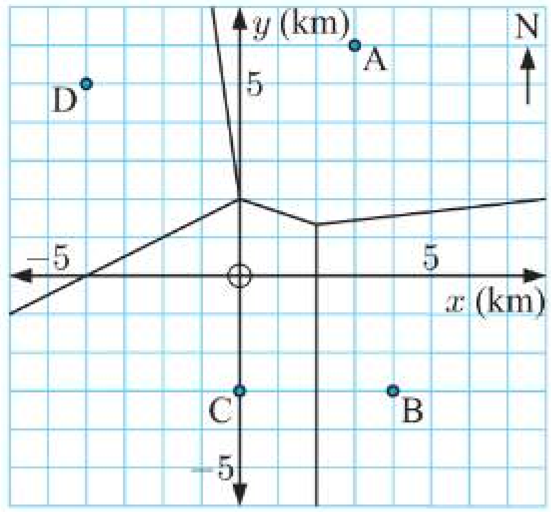
So the two buildings are located at $(-1, -2)$ and $(-1, \frac{1}{2})$.

7 **a** **i** $(-3, -2)$ lies in cell C, so the nearest public school is school C.

ii $(6, 2)$ lies in cell A, so the nearest public school is school A.

iii $(-2, 4)$ lies in cell D, so the nearest public school is school D.

iv $(5, 1)$ lies in cell B, so the nearest public school is school B.



b **i** $(-1, 4)$ lies in cell D, so Bailey lives in school D's catchment zone.

ii 1 km to the east of $(-1, 4)$ is $(-1+1,\ 4)$ which is $(0, 4)$. This is in cell A rather than cell D, and so Bailey will move into a new catchment zone.

iii The distance from Bailey's home at $(0, 4)$ to school A at $(3, 6)$ is

$$\sqrt{(3-0)^2+(6-4)^2} = \sqrt{13}$$
$$\approx 3.61 \text{ km.}$$

The distance from Bailey's home at $(0, 4)$ to school D at $(-4, 5)$ is

$$\sqrt{(-4-0)^2+(5-4)^2} = \sqrt{17}$$
$$\approx 4.12 \text{ km.}$$

So, Bailey's new school, school A, is now closer to Bailey than his old school, school D.

c **i** Lizzy lives at a point adjacent to cells A, C, and D, which is $(0, 2)$.

ii The distance from Lizzy's house at $(0, 2)$ to school C at $(0, -3)$ is 5 km.
Since Lizzy's house is equally closest to schools A, C, and D, Lizzy lives 5 km from each of these schools.

8 A vertex of a Voronoi diagram is equally closest to at least 3 sites (whose cells meet at that vertex). The circle's edge passes through site X, which is one of the sites V is closest to, and since every point on the edge is equidistant from the centre, the other closest sites must also lie on the edge of the circle.
So, the circle must pass through at least two other sites.

EXERCISE 16B

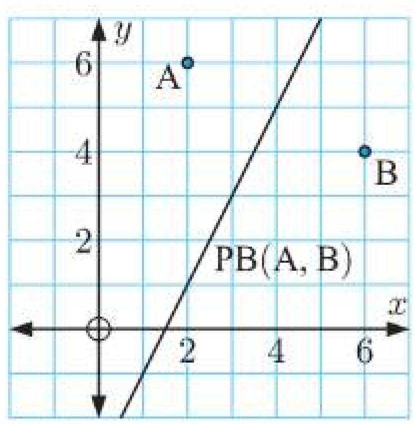
1 **a** A(2, 6), B(6, 4)

The midpoint of [AB] is $\left(\dfrac{2+6}{2}, \dfrac{6+4}{2}\right)$

or $(4, 5)$.

The gradient of [AB] is $\dfrac{4-6}{6-2} = \dfrac{-2}{4} = -\dfrac{1}{2}$.

So, PB(A, B) has gradient 2 and passes through $(4, 5)$.



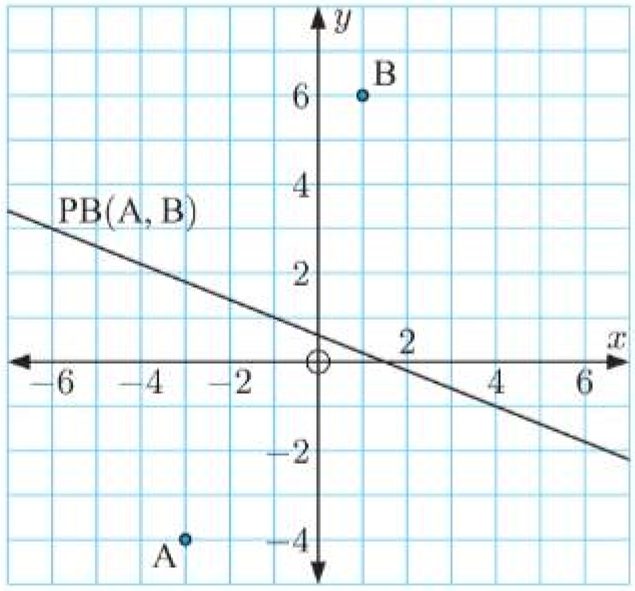
b A$(-3, -4)$, B$(1, 6)$

The midpoint of [AB] is $\left(\frac{-3+1}{2}, \frac{-4+6}{2}\right)$

or $(-1, 1)$.

The gradient of [AB] is $\frac{6--4}{1--3} = \frac{10}{4} = \frac{5}{2}$.

So, PB(A, B) has gradient $-\frac{2}{5}$ and passes through $(-1, 1)$.



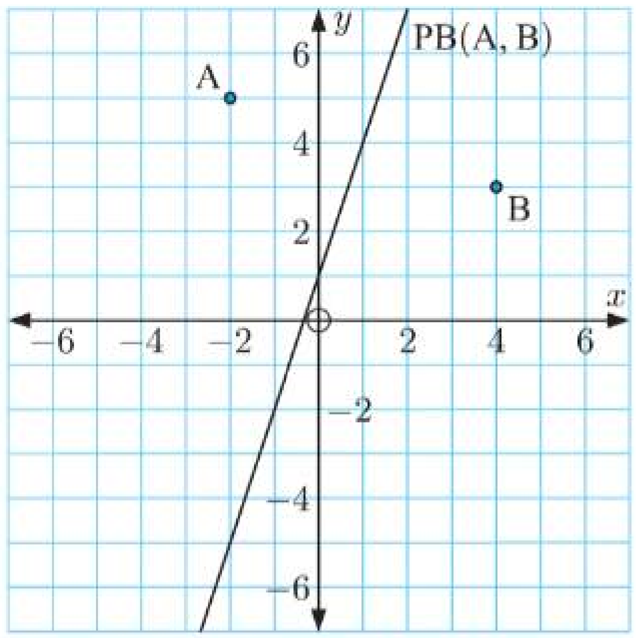
2 **a** A$(-2, 5)$, B$(4, 3)$

The midpoint of [AB] is $\left(\frac{-2+4}{2}, \frac{5+3}{2}\right)$

or $(1, 4)$.

The gradient of [AB] is $\frac{3-5}{4--2} = \frac{-2}{6} = -\frac{1}{3}$.

So, PB(A, B) has gradient 3 and passes through $(1, 4)$.



b The Voronoi edge has gradient 3 and passes through $(1, 4)$.

$\therefore$ its equation is $3x - y = 3(1) - 1(4)$

or $y = 3x + 1$

c **i** When $x = -2$, $y = 3(-2) + 1 = -5$

$\therefore$ $(-2, -5)$ is a point on the line $y = 3x + 1$.

$\therefore$ $(-2, -5)$ lies on the Voronoi edge.

ii The distance from A$(-2, 5)$ to $(-2, -5)$ is $\sqrt{(-2-(-2))^2 + (-5-5)^2} = 10$ units

and the distance from B$(4, 3)$ to $(-2, -5)$ is $\sqrt{(-2-4)^2 + (-5-3)^2} = \sqrt{36+64}$

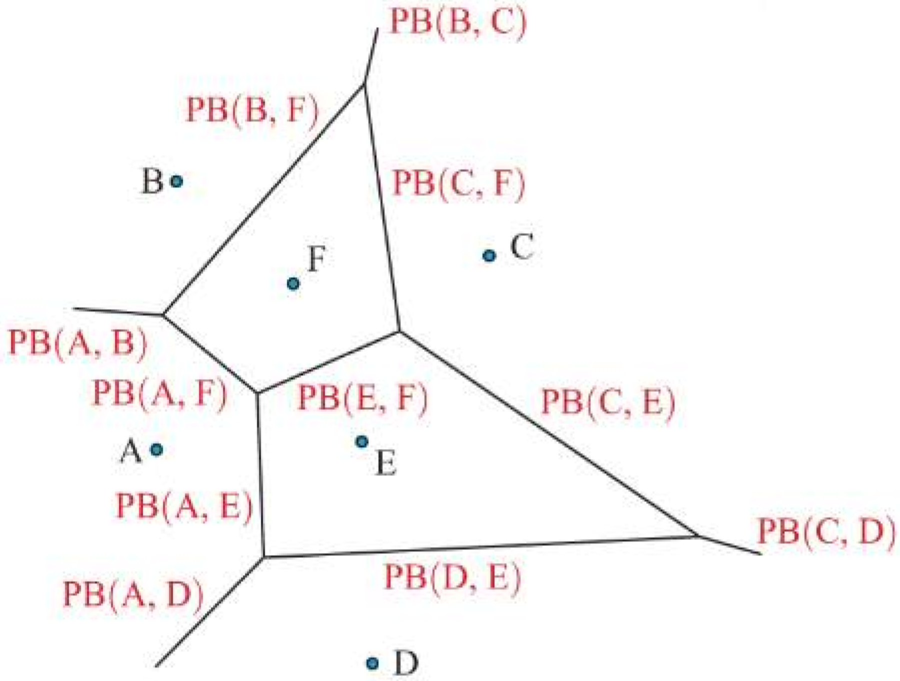
$= 10$ units ✓

d **i** $(0, 0)$ lies in cell B, so it is closest to site B.

ii $(3, 6)$ lies in cell B, so it is closest to site B.

iii $(-4, -5)$ lies in cell A, so it is closest to site A.

3



4 **a** A(4, 7), B(8, 3), C(0, −5)

The midpoint of [AB] is $\left(\frac{4+8}{2}, \frac{7+3}{2}\right)$ or $(6, 5)$.

The gradient of [AB] is $\frac{3-7}{8-4} = \frac{-4}{4} = -1$.

So, PB(A, B) has gradient 1 and passes through $(6, 5)$.

The midpoint of [AC] is $\left(\frac{4+0}{2}, \frac{7+-5}{2}\right)$ or $(2, 1)$.

The gradient of [AC] is $\frac{-5-7}{0-4} = \frac{-12}{-4} = 3$.

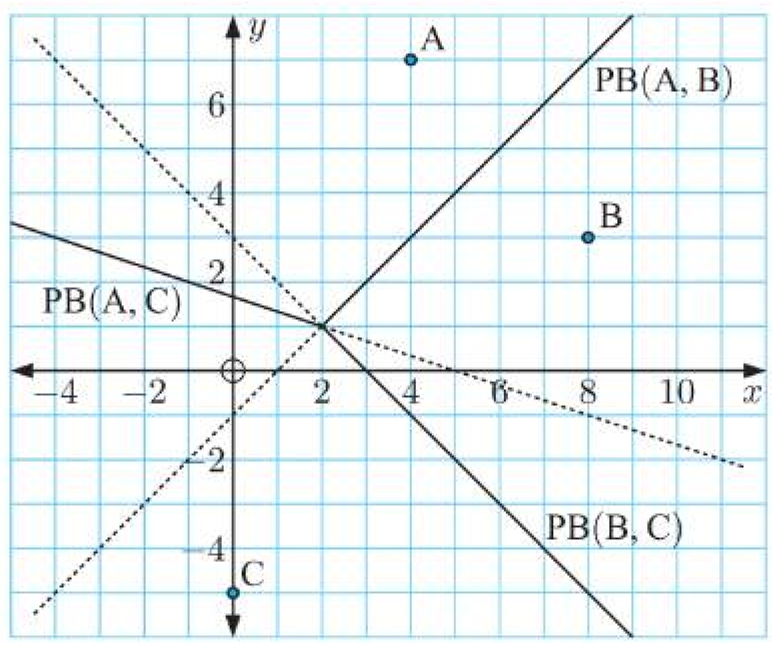
So, PB(A, C) has gradient $-\frac{1}{3}$ and passes through $(2, 1)$.

The midpoint of [BC] is $\left(\frac{8+0}{2}, \frac{3+-5}{2}\right)$ or $(4, -1)$.

The gradient of [BC] is $\frac{-5-3}{0-8} = \frac{-8}{-8} = 1$.

So, PB(B, C) has gradient -1 and passes through $(4, -1)$.

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



b A$(-1, 4)$, B$(5, 2)$, C$(-5, -4)$

The midpoint of [AB] is $\left(\frac{-1+5}{2}, \frac{4+2}{2}\right)$ or $(2, 3)$.

The gradient of [AB] is $\frac{2-4}{5--1} = \frac{-2}{6} = -\frac{1}{3}$.

So, PB(A, B) has gradient 3 and passes through $(2, 3)$.

The midpoint of [AC] is $\left(\frac{-1+-5}{2}, \frac{4+-4}{2}\right)$ or $(-3, 0)$.

The gradient of [AC] is $\frac{-4-4}{-5--1} = \frac{-8}{-4} = 2$.

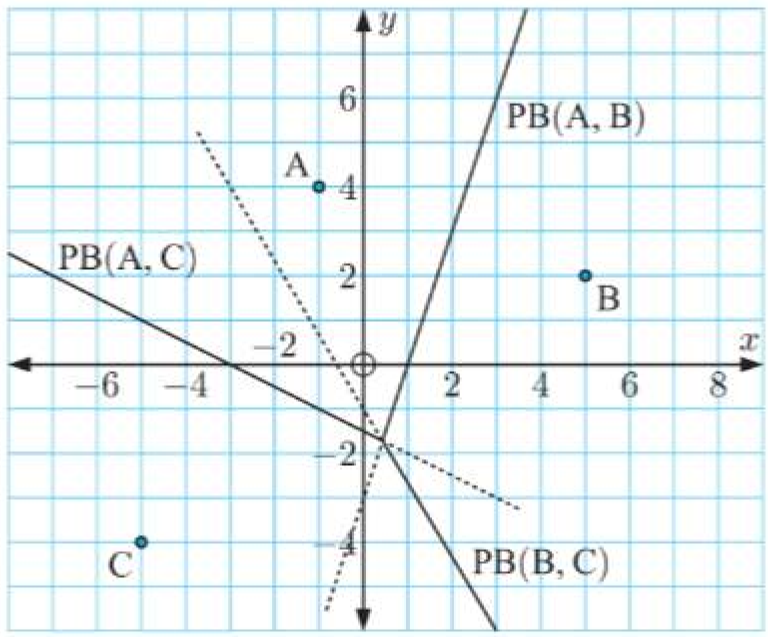
So, PB(A, C) has gradient $-\frac{1}{2}$ and passes through $(-3, 0)$.

The midpoint of [BC] is $\left(\frac{5+-5}{2}, \frac{2+-4}{2}\right)$ or $(0, -1)$.

The gradient of [BC] is $\frac{-4-2}{-5-5} = \frac{-6}{-10} = \frac{3}{5}$.

So, PB(B, C) has gradient $-\frac{5}{3}$ and passes through $(0, -1)$.

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



5 **a** A$(-10, 9)$, B$(10, 13)$, C$(-2, -7)$

The midpoint of [AB] is $\left(\frac{-10+10}{2}, \frac{9+13}{2}\right)$ or $(0, 11)$.

The gradient of [AB] is $\frac{13-9}{10--10} = \frac{4}{20} = \frac{1}{5}$.

So, PB(A, B) has gradient -5 and passes through $(0, 11)$.

The midpoint of [AC] is $\left(\frac{-10+-2}{2}, \frac{9+-7}{2}\right)$ or $(-6, 1)$.

The gradient of [AC] is $\frac{-7-9}{-2--10} = \frac{-16}{8} = -2$.

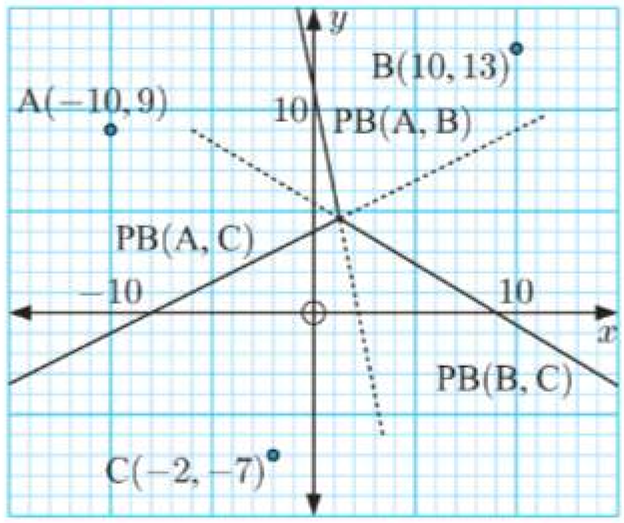
So, PB(A, C) has gradient $\frac{1}{2}$ and passes through $(-6, 1)$.

The midpoint of [BC] is $\left(\frac{10+-2}{2}, \frac{13+-7}{2}\right)$ or $(4, 3)$.

The gradient of [BC] is $\dfrac{-7-13}{-2-10} = \dfrac{-20}{-12} = \dfrac{5}{3}$.

So, PB(B, C) has gradient $-\frac{3}{5}$ and passes through $(4, 3)$.

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



b PB(A, B) has gradient -5 and passes through $(0, 11)$.

$\therefore$ its equation is $5x + y = 5(0) + 1(11)$

or $y = -5x + 11$

PB(A, C) has gradient $\frac{1}{2}$ and passes through $(-6, 1)$.

$\therefore$ its equation is $x - 2y = 1(-6) - 2(1)$

which is $2y = x + 8$

or $y = \frac{1}{2}x + 4$

PB(B, C) has gradient $-\frac{3}{5}$ and passes through $(4, 3)$.

$\therefore$ its equation is $3x + 5y = 3(4) + 5(3)$

which is $5y = -3x + 27$

or $y = -\frac{3}{5}x + \frac{27}{5}$

c The vertex is the point at which all 3 edges intersect, so we equate the equations found in **b**.

PB(A, B) and PB(A, C) intersect where $-5x + 11 = \frac{1}{2}x + 4$

$\therefore\ 10x - 22 = -x - 8$

$\therefore\ 11x = 14$

$\therefore\ x = \frac{14}{11}$

When $x = \frac{14}{11}$, $y = -5(\frac{14}{11}) + 11$

$= \frac{51}{11}$

So, the vertex is $V\left(\frac{14}{11}, \frac{51}{11}\right)$.

Check: Using the equation of PB(B, C), when $x = \frac{14}{11}$, $y = -\frac{3}{5}\left(\frac{14}{11}\right) + \frac{27}{5} = \frac{51}{11}$ ✓

$VA = \sqrt{\left(-10 - \frac{14}{11}\right)^2 + \left(9 - \frac{51}{11}\right)^2}$

$= \sqrt{\left(-\frac{124}{11}\right)^2 + \left(\frac{48}{11}\right)^2}$

$= \sqrt{\frac{17\,680}{121}} \approx 12.1$ units

$VB = \sqrt{\left(10 - \frac{14}{11}\right)^2 + \left(13 - \frac{51}{11}\right)^2}$

$= \sqrt{\left(\frac{96}{11}\right)^2 + \left(\frac{92}{11}\right)^2}$

$= \sqrt{\frac{17\,680}{121}} \approx 12.1$ units

$$\text{VC} = \sqrt{\left(-2 - \tfrac{14}{11}\right)^2 + \left(-7 - \tfrac{51}{11}\right)^2}$$
$$= \sqrt{\left(-\tfrac{36}{11}\right)^2 + \left(-\tfrac{128}{11}\right)^2}$$
$$= \sqrt{\tfrac{17\,680}{121}} \approx 12.1 \text{ units}$$

So, the vertex V is equidistant from A, B, and C.

d **i** $(-2, 8)$ lies in cell A, so it is closest to site A.

ii $(5, 5)$ lies in cell B, so it is closest to site B.

iii $(2, -3)$ lies in cell C, so it is closest to site C.

6 **a** $A(5, 3)$, $B(4, -2)$, $C(-4, -6)$

The midpoint of [AB] is $\left(\frac{5+4}{2}, \frac{3+-2}{2}\right)$ or $\left(\frac{9}{2}, \frac{1}{2}\right)$.

The gradient of [AB] is $\frac{-2-3}{4-5} = \frac{-5}{-1} = 5$.

So, PB(A, B) has gradient $-\frac{1}{5}$ and passes through $\left(\frac{9}{2}, \frac{1}{2}\right)$.

The midpoint of [AC] is $\left(\frac{5+-4}{2}, \frac{3+-6}{2}\right)$ or $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

The gradient of [AC] is $\frac{-6-3}{-4-5} = \frac{-9}{-9} = 1$.

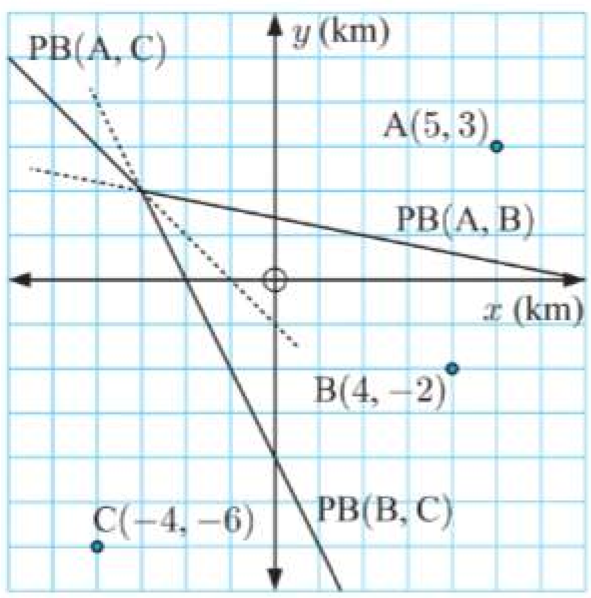
So, PB(A, C) has gradient -1 and passes through $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

The midpoint of [BC] is $\left(\frac{4+-4}{2}, \frac{-2+-6}{2}\right)$ or $(0, -4)$.

The gradient of [BC] is $\frac{-6--2}{-4-4} = \frac{-4}{-8} = \frac{1}{2}$.

So, PB(B, C) has gradient -2 and passes through $(0, -4)$.

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



b **i** $(3, 1)$ lies in cell A, so they are closest to store A.

ii $(-2, 2)$ lies in cell A, so they are closest to store A.

iii $(-5, -1)$ lies in cell C, so they are closest to store C.

c **i** Amanda must live at the vertex of the Voronoi diagram.
Using the information found in **a**:

PB(A, B) has equation $x + 5y = 1\left(\frac{9}{2}\right) + 5\left(\frac{1}{2}\right)$

which is $5y = -x + 7$

or $y = -\frac{1}{5}x + \frac{7}{5}$

PB(A, C) has equation $x + y = \frac{1}{2} - \frac{3}{2}$

or $y = -x - 1$

PB(B, C) has equation $2x + y = 2(0) - 4$

or $y = -2x - 4$

PB(A, C) and PB(B, C) intersect where $-x - 1 = -2x - 4$

$\therefore\ x = -3$

When $x = -3$, $y = -(-3) - 1 = 2$

So, Amanda lives at $(-3, 2)$.

Check: Using the equation of PB(A, B), when $x = -3$, $y = -\frac{1}{5}(-3) + \frac{7}{5} = 2$ ✓

ii The distance from Amanda's house at $(-3, 2)$ to A$(5, 3)$ is

$\sqrt{(5 - (-3))^2 + (3 - 2)^2}$

$= \sqrt{8^2 + 1^2}$

$= \sqrt{65} \approx 8.06$ km

Amanda's house is equidistant from all 3 stores, so she is about 8.06 km from each store.

7 **a** The Voronoi diagram must have an edge missing because sites A and D are in the same cell.

b The missing edge is the perpendicular bisector of A$(-5, 3)$ and D$(-3, -3)$.

The midpoint of [AD] is $\left(\frac{-5 + -3}{2}, \frac{3 + -3}{2}\right)$ or $(-4, 0)$.

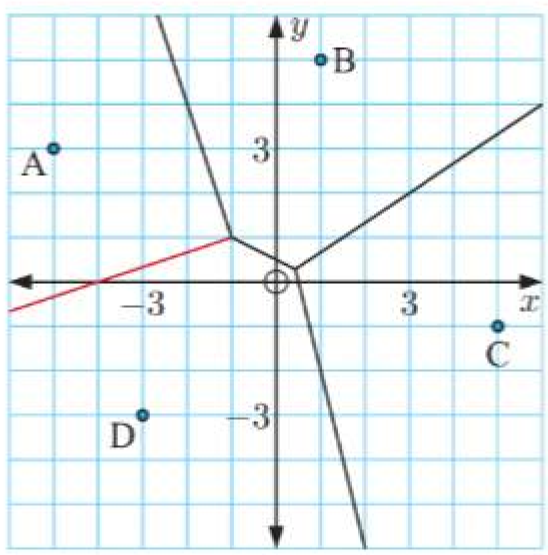
The gradient of [AD] is $\frac{-3 - 3}{-3 - -5} = \frac{-6}{2} = -3$.

So, PB(A, D) has gradient $\frac{1}{3}$ and passes through $(-4, 0)$.

$\therefore$ its equation is $x - 3y = 1(-4) - 3(0)$

which is $3y = -x + 4$

or $y = \frac{1}{3}x + \frac{4}{3}$



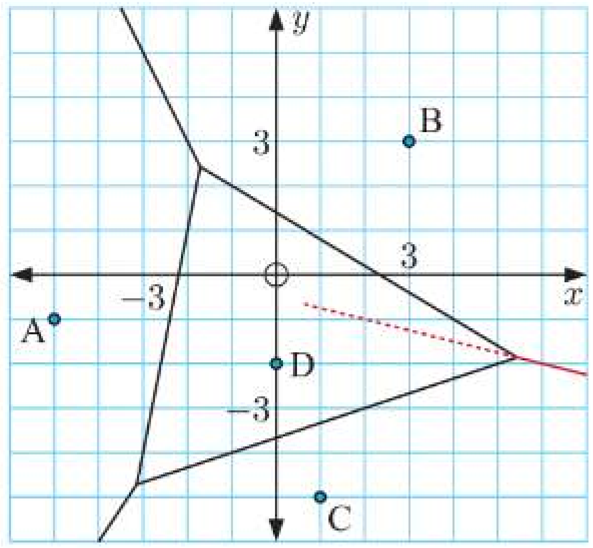
8 **a** Sites $B(3, 3)$ and $C(1, -5)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of [BC].

The midpoint of [BC] is $\left(\frac{3+1}{2}, \frac{3+-5}{2}\right)$ or $(2, -1)$.

The gradient of [BC] is $\frac{-5-3}{1-3} = \frac{-8}{-2} = 4$.

So, PB(B, C) has gradient $-\frac{1}{4}$ and passes through $(2, -1)$.

$\therefore$ its equation is $x + 4y = 1(2) + (-1)$
or $x + 4y + 2 = 0$



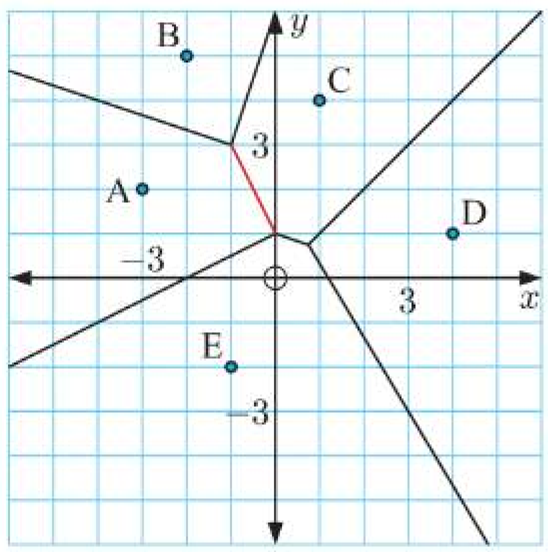
b Sites $A(-3, 2)$ and $C(1, 4)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of [AC].

The midpoint of [AC] is $\left(\frac{-3+1}{2}, \frac{2+4}{2}\right)$ or $(-1, 3)$.

The gradient of [AC] is $\frac{4-2}{1--3} = \frac{2}{4} = \frac{1}{2}$.

So, PB(A, C) has gradient -2 and passes through $(-1, 3)$.

$\therefore$ its equation is $2x + y = 2(-1) + 3$
or $2x + y - 1 = 0$



9 **a** Sites A and D are in the same cell, as are sites C and E.
So, there are 2 missing edges.

b The missing edges are the perpendicular bisector of $A(-3, 4)$ and $D(-1, -2)$, and the perpendicular bisector of $C(5, -4)$ and $E(1, 0)$.

The midpoint of [AD] is $\left(\frac{-3+-1}{2}, \frac{4+-2}{2}\right)$ or $(-2, 1)$.

The gradient of [AD] is $\frac{-2-4}{-1--3} = \frac{-6}{2} = -3$.

So, PB(A, D) has gradient $\frac{1}{3}$ and passes through $(-2, 1)$.

$\therefore$ its equation is $x - 3y = 1(-2) - 3(1)$
which is $3y = x + 5$
or $y = \frac{1}{3}x + \frac{5}{3}$

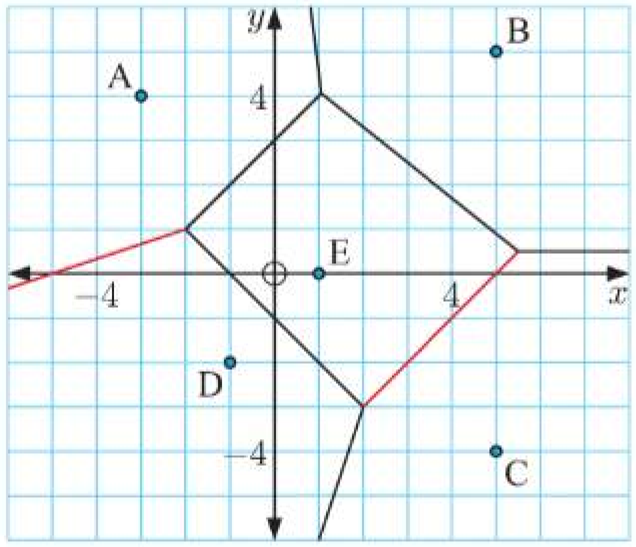
The midpoint of [CE] is $\left(\frac{5+1}{2}, \frac{-4+0}{2}\right)$ or $(3, -2)$.

The gradient of [CE] is $\frac{0--4}{1-5} = \frac{4}{-4} = -1$.

So, PB(C, E) has gradient 1 and passes through $(3, -2)$.

$\therefore$ its equation is $x - y = 1(3) - (-2)$

or $y = x - 5$



c **i** $(-4, 1)$ lies in cell A, so it is closest to site A.

ii $(2, 3)$ lies in cell E, so it is closest to site E.

d **i** $(4, -1)$ lies on the edge adjacent to cells C and E, so it is equally closest to sites C and E.

ii $(-2, 1)$ lies on the vertex adjacent to cells A, D, and E, so it is equally closest to sites A, D, and E.

10 **a** The blue edge has gradient 3 and passes through the point $(-1, 0)$.

$\therefore$ the equation of the blue edge is $3x - y = 3(-1) - 0$

or $y = 3x + 3$

b Sites $A(-3, 4)$ and $B(4, 5)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of [AB].

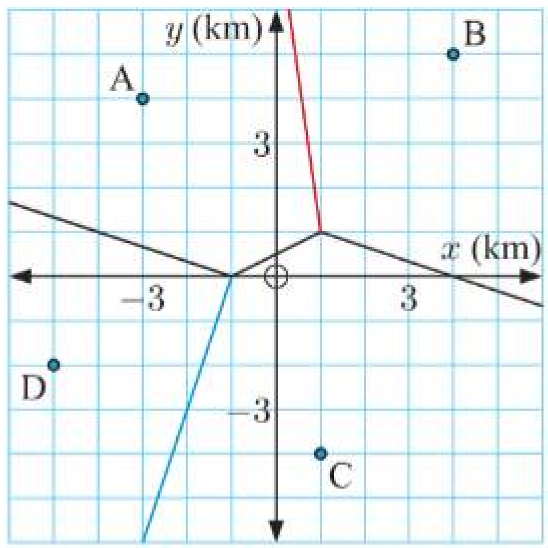
The midpoint of [AB] is $\left(\frac{-3+4}{2}, \frac{4+5}{2}\right)$ or $\left(\frac{1}{2}, \frac{9}{2}\right)$.

The gradient of [AB] is $\frac{5-4}{4--3} = \frac{1}{7}$.

So, PB(A, B) has gradient -7 and passes through $\left(\frac{1}{2}, \frac{9}{2}\right)$.

$\therefore$ its equation is $7x + y = 7(\frac{1}{2}) + \frac{9}{2}$

or $y = -7x + 8$



c **i** $(1, 3)$ lies in cell B, so Julie is nearest to campsite B.

ii The distance from $(1, 3)$ to $B(4, 5)$ is $\sqrt{(4-1)^2 + (5-3)^2}$

$= \sqrt{3^2 + 2^2}$

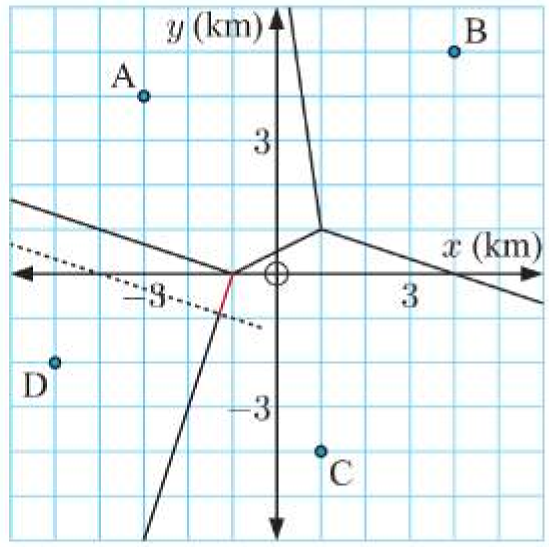
$= \sqrt{13} \approx 3.61$ km

So, Julie is about 3.61 km from campsite B.

d Simon is equally closest to campsites C and D, so he is on the edge adjacent to cells C and D.

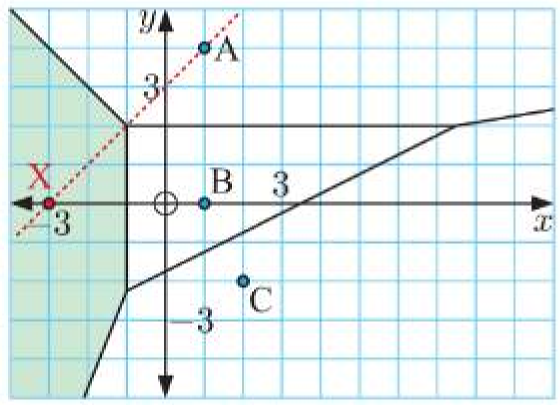
He is less than 1 km south of cell A, so we construct a line which is 1 km south of the edge adjacent to cells A and D.

Simon is therefore somewhere on the line segment shown alongside in red.



11 **a** The Voronoi diagram must have a site missing as the shaded cell in the diagram alongside currently has no site.

b Let the missing site be X. X must lie on the x-axis because site B lies on the x-axis, and the edge adjacent to sites B and X is perpendicular to the x-axis.



c PB(A, X) has gradient -1, so [AX] has gradient 1.

If we draw the line (AX), X must be the point where the line cuts the x-axis (from **b**). We observe that X has coordinates $(-3, 0)$.

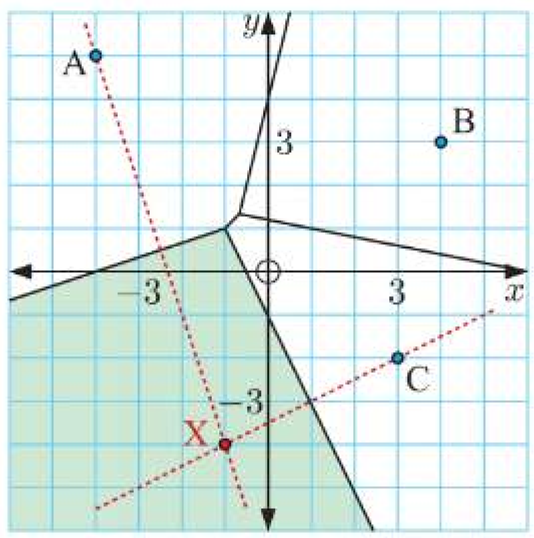
12 **a** The missing site X must lie in the shaded cell, as this cell currently has no site.

Now PB(A, X) has gradient $\frac{1}{3}$, and PB(C, X) has gradient -2.

$\therefore$ [AX] has gradient -3, and [CX] has gradient $\frac{1}{2}$.

If we draw lines (AX) and (CX) through A and D respectively, their intersection point must be site X.

We observe that X has coordinates $(-1, -4)$.



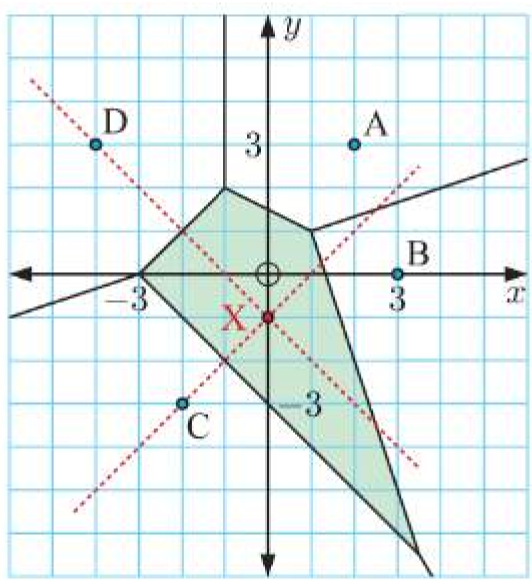
b The missing site X must lie in the shaded cell, as this cell currently has no site.

Now PB(C, X) has gradient -1, and PB(D, X) has gradient 1.

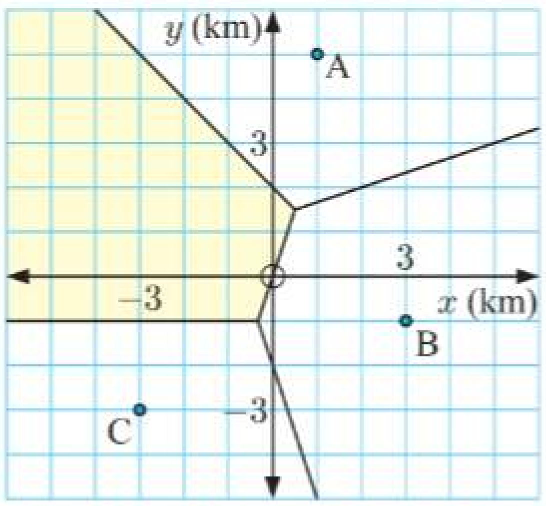
$\therefore$ [CX] has gradient 1, and [DX] has gradient -1.

If we draw lines (CX) and (DX) through C and D respectively, their intersection point must be site X.

We observe that X has coordinates $(0, -1)$.



13 **a**



b The distance from $(0, -1)$ to $(-2, 1)$ is $\sqrt{(-2-0)^2 + (1-(-1))^2} = \sqrt{8} \approx 2.83$ km.

The distance from $(0, -1)$ to $B(3, -1)$ is $\sqrt{(3-0)^2 + (-1-(-1))^2} = 3$ km.

So, if the missing vet clinic was at $(-2, 1)$, then its cell would include $(0, -1)$.

But $(0, -1)$ is in cell B.

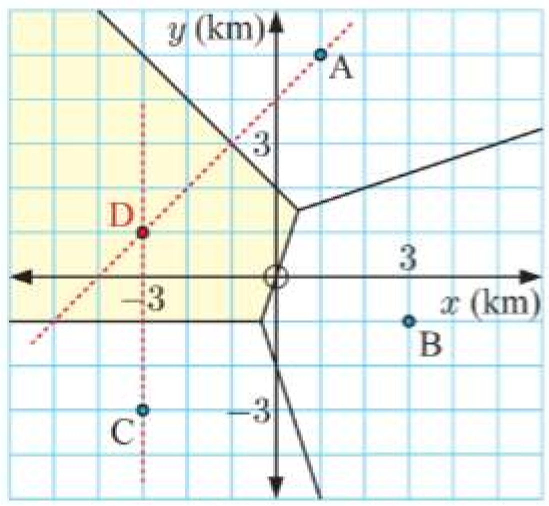
$\therefore$ the missing vet clinic is not at $(-2, 1)$.

c PB(A, D) has gradient -1, and PB(C, D) is horizontal.

$\therefore$ [AD] has gradient 1, and [CD] is vertical.

If we draw lines (AD) and (CD) through A and D respectively, their intersection must be site D.

We observe that the missing vet clinic D is at $(-3, 1)$.



d **i** $(-1, 4)$ lies in cell A, so it is closest to vet clinic A.

ii $(1, -5)$ lies on the edge adjacent to cells B and C, so it is equally closest to vet clinics B and C.

e Larissa lives equally closest to vet clinics A and D, so Larissa's house must lie on the edge adjacent to cells A and D.

The shortest possible distance to either vet clinic is the point at which PB(A, D) and [AD] meet.

From the construction lines drawn in **c**, we observe that this point is $(-1, 3)$.

The distance from $(-1, 3)$ to $A(1, 5)$ is $\sqrt{(1-(-1))^2 + (5-3)^2} = \sqrt{8} \approx 2.83$ km.

So, the shortest distance Larissa's house could be from vet clinics A and D is $2\sqrt{2} \approx 2.83$ km.

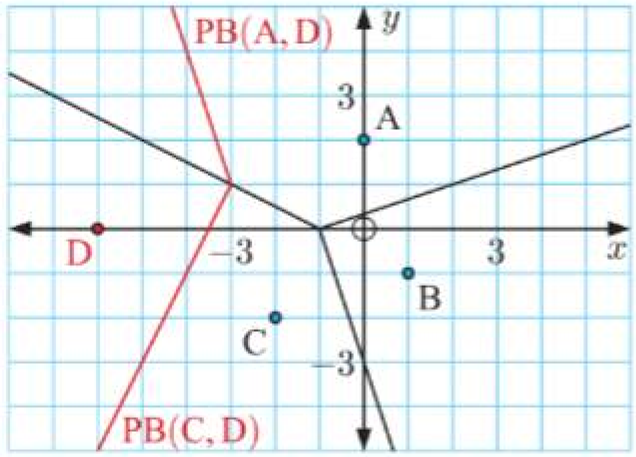
EXERCISE 16C

1 **a** Site D lies in the existing cell C.

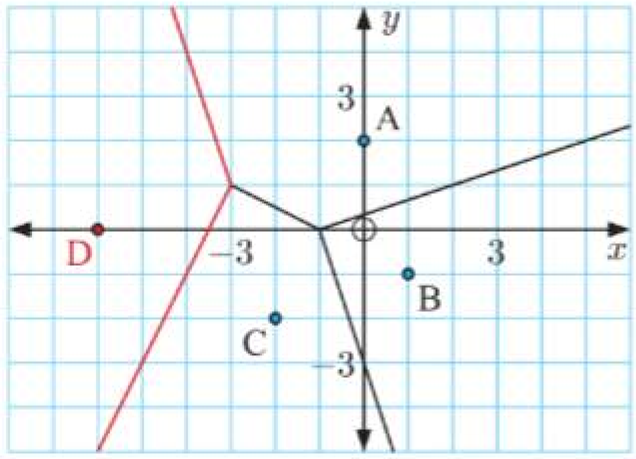
b Cells A and C will be affected, as they both contain points which are closest to D.

c We construct PB(C, D) within cell C, which creates a new vertex at $(-3, 1)$.

Cell A is adjacent to $(-3, 1)$, so we construct PB(A, D) from $(-3, 1)$ through cell A.



We then remove the edge segment from the original Voronoi diagram which now lies within cell D, giving us the Voronoi diagram which includes site D.



2 **a** Adding site D will affect all cells as cells A, B, and C each contain points which are closer to site D than they are to sites A, B, and C respectively.

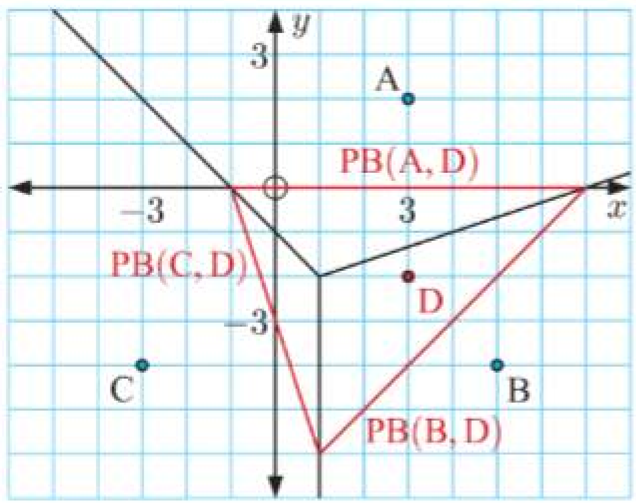
b We construct PB(A, D) within cell A, which creates new vertices at $(-1, 0)$ and $(7, 0)$.

Cell C is adjacent to $(-1, 0)$, so we construct PB(C, D) from $(-1, 0)$ through cell C.

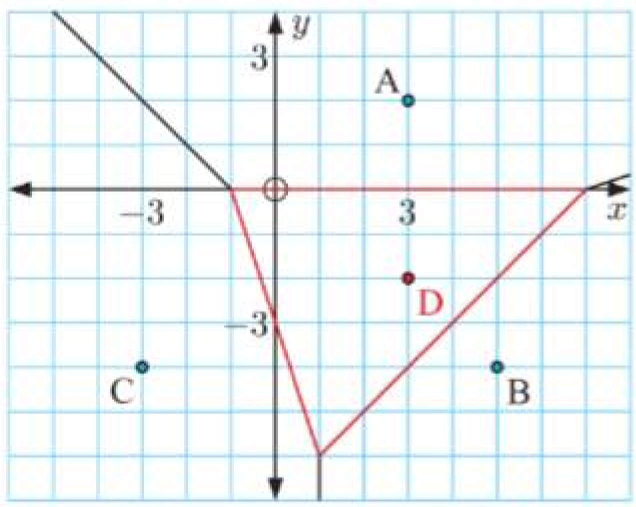
This creates a new vertex at $(1, -6)$.

Cell B is adjacent to $(1, -6)$, so we construct PB(B, D) from $(1, -6)$ through cell B.

This connects us back to $(7, 0)$.



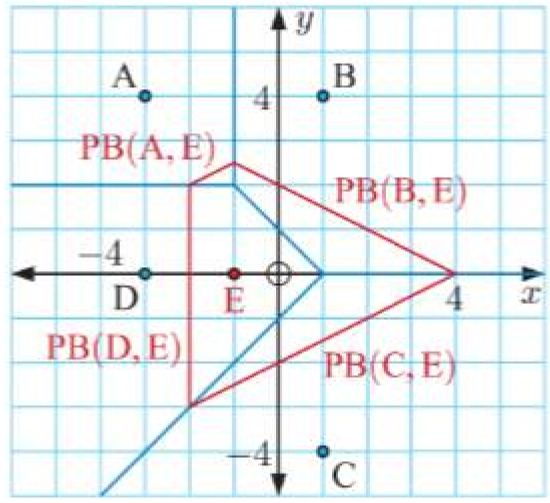
We then remove the segments of edges from the original Voronoi diagram which now lie within cell D, giving us the Voronoi diagram which includes site D.



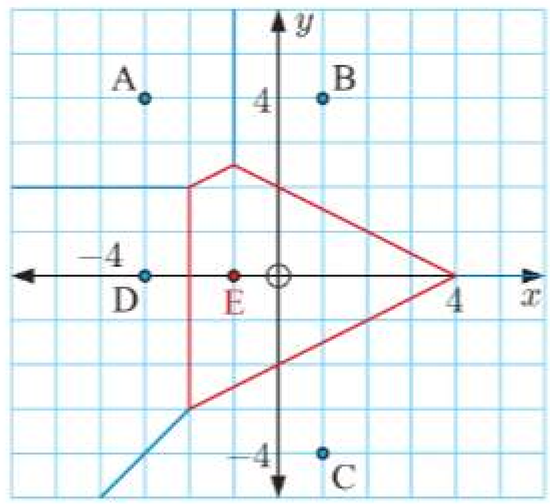
c From the diagram in **b**, cell D is a triangle with base 8 units and height 6 units.

$\therefore$ area of cell D $= \frac{1}{2} \times 8 \times 6 = 24$ units^2.

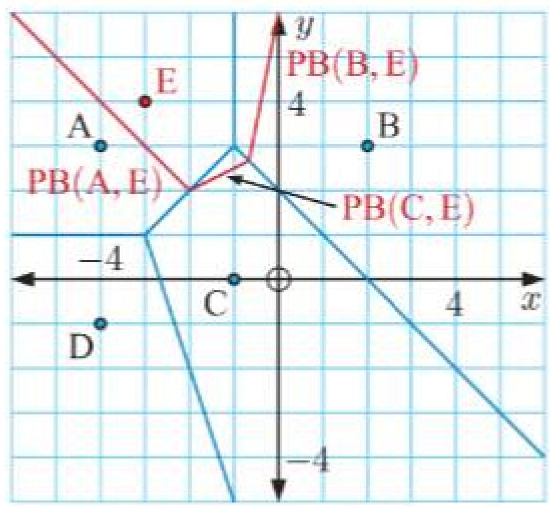
3 **a** We construct PB(A, E), PB(B, E), PB(C, E), and PB(D, E) within the original cells A, B, C, and D respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram which includes site E.

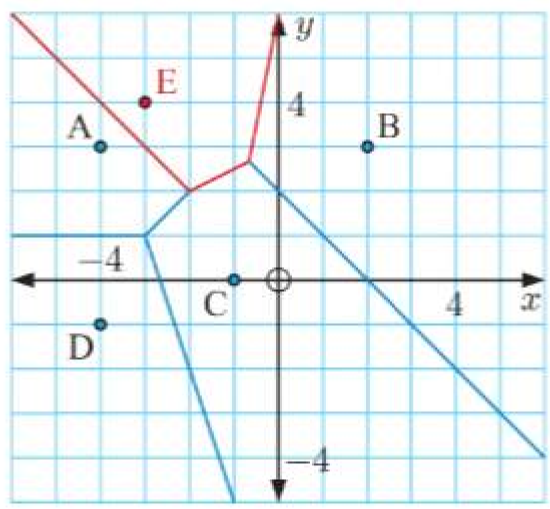


b We construct PB(A, E), PB(B, E), and PB(C, E) within the original cells A, B, and C respectively.



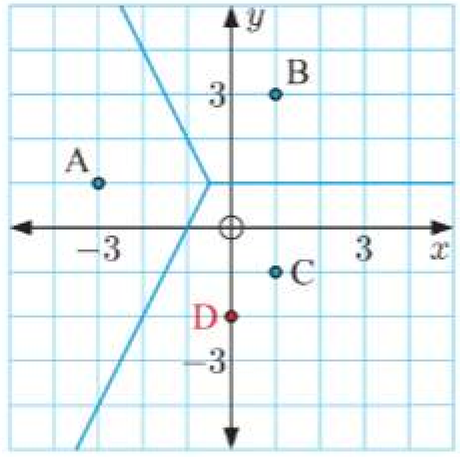
We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram which includes site E.

Note: We do not need to construct PB(D, E) as there are no points in cell D which are now closest to site E.



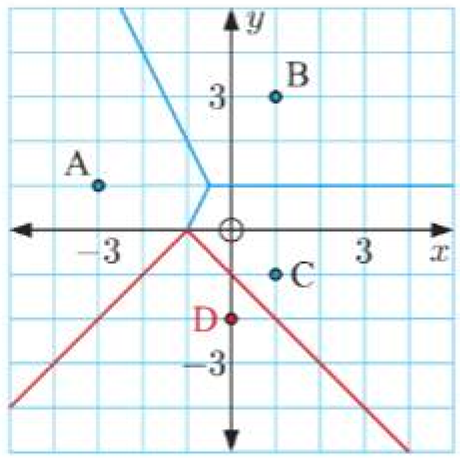
4 **a** A(-3, 1), B(1, 3), C(1, -1), D(0, -2)

We draw PB(A, B), PB(A, C), and PB(B, C), then remove segments which do not form part of the Voronoi diagram.



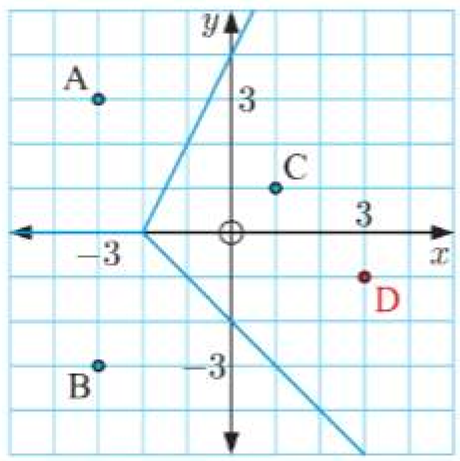
Given this Voronoi diagram, we now construct PB(A, D) and PB(C, D) within the original cells A and C respectively.

Then we remove the segment of the edge from the original Voronoi diagram which now lies within cell D, giving us the Voronoi diagram for A, B, C, and D.



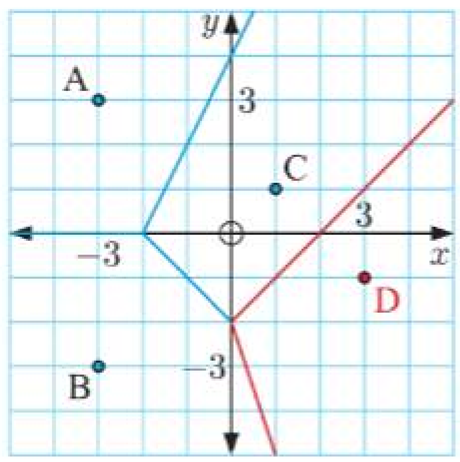
b A(-3, 3), B(-3, -3), C(1, 1), D(3, -1)

We draw PB(A, B), PB(A, C), and PB(B, C), then remove segments which do not form part of the Voronoi diagram.



Given this Voronoi diagram, we now construct PB(B, D) and PB(C, D) within the original cells B and C respectively.

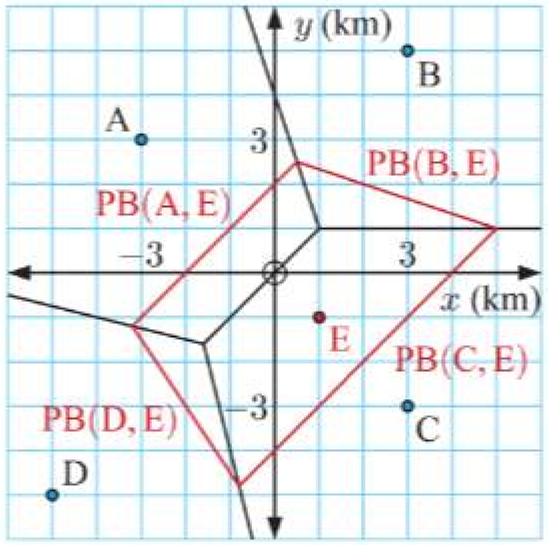
Then we remove the segment of the edge from the original Voronoi diagram which now lies within cell D, giving us the Voronoi diagram for A, B, C, and D.



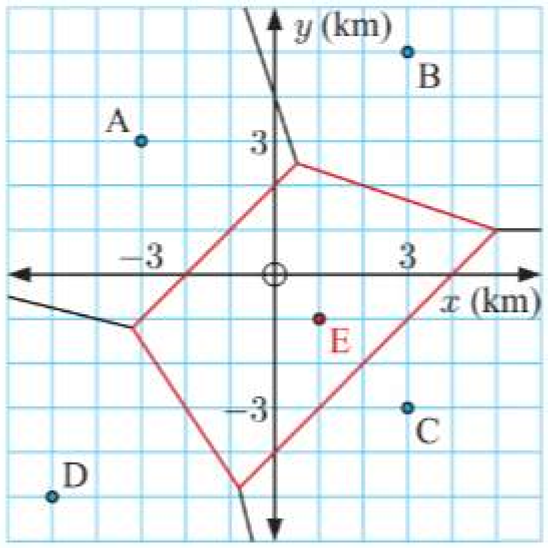
5 **a** **i** $(-2, -1)$ lies in cell A, so it is closest to ATM A.

ii $(5, 2)$ lies in cell B, so it is closest to ATM B.

b **i** We construct PB(A, E), PB(B, E), PB(C, E), and PB(D, E) within the original cells A, B, C, and D respectively.



We then remove the segments of edges which now lie within cell E, giving us the Voronoi diagram which includes site E.



ii Yes, there are residents whose nearest ATM has changed from D to E, as part of the original cell D now lies in cell E.

iii If Morris is equally closest to ATMs B, C, and E, then he is at the vertex adjacent to cells B, C, and E.

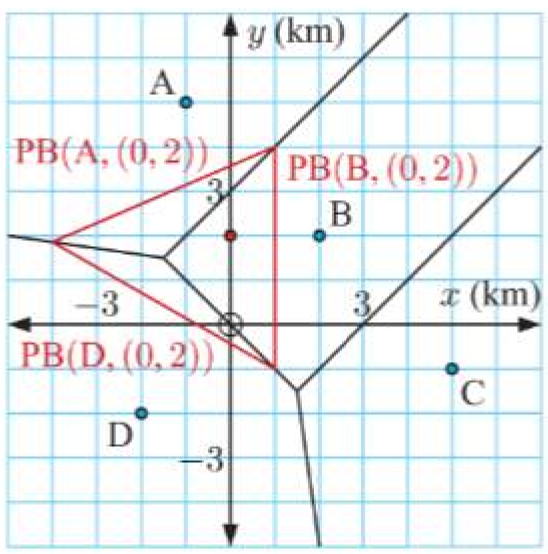
We observe from the Voronoi diagram in **i** that this is the point $(5, 1)$.

Morris is therefore located at $(5, 1)$.

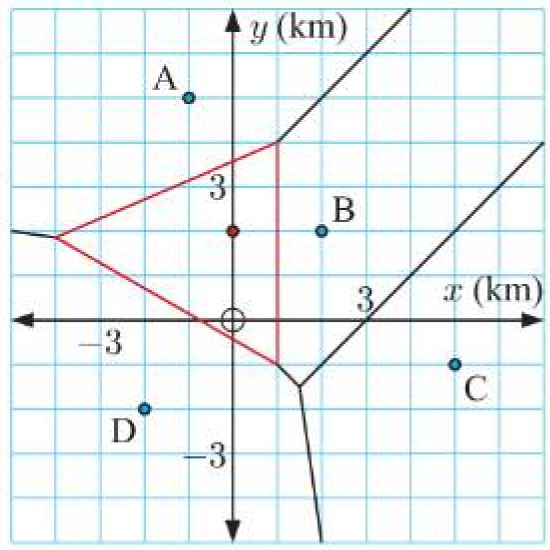
6 **a** **i** $(-3, 2)$ lies in cell A, so polling booth A is the closest.

ii $(1, -4)$ lies in cell D, so polling booth D is the closest.

b **i** We construct PB(A, (0, 2)), PB(B, (0, 2)), and PB(D, (0, 2)) within the original cells A, B, and D respectively.



We then remove the segments of edges which now lie within the new cell, giving us the Voronoi diagram which includes the polling booth at $(0, 2)$.



ii We see from the Voronoi diagram in **i** that cell C will be unaffected, as it does not contain any points which are closer to site $(0, 2)$ than to site C.

iii From the diagram in **i**, the new cell is a triangle with base 5 units and height 5 units.

$\therefore$ area of new cell $= \frac{1}{2} \times 5 \times 5 = 12.5$ km^2.

EXERCISE 16D

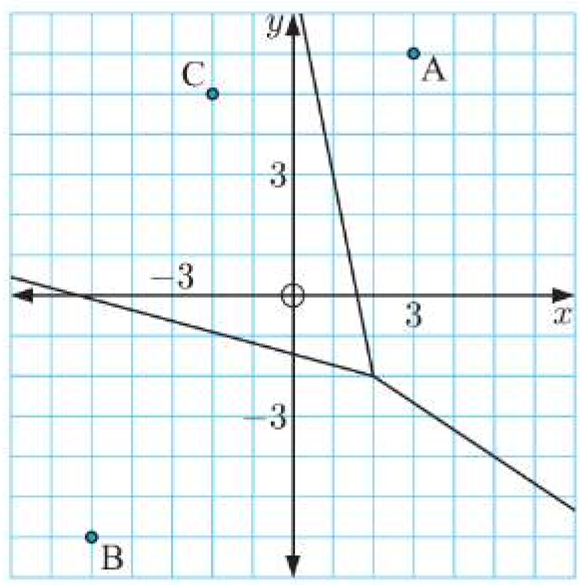
1

Location	*Temperature* (°C)
A	28.4
B	25.6
C	27.3

a $(1, 0)$ is closest to C, so we estimate a temperature of 27.3°C at 3 pm at $(1, 0)$.

b $(-3, -1)$ is closest to B, so we estimate a temperature of 25.6°C at 3 pm at $(-3, -1)$.

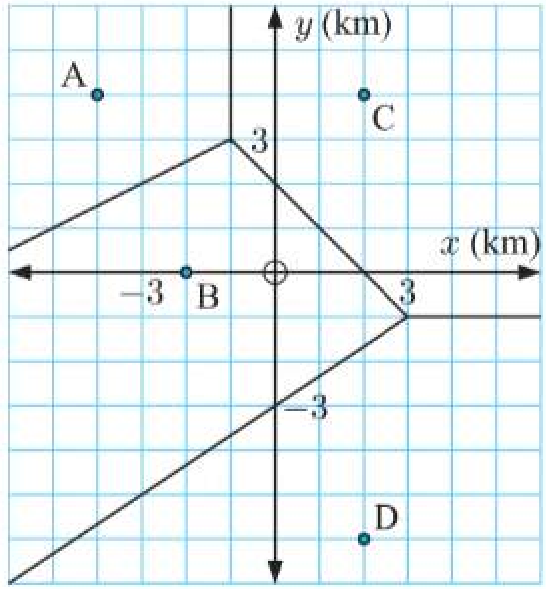
c $(5, -2)$ is closest to A, so we estimate a temperature of 28.4°C at 3 pm at $(5, -2)$.



2

Location	*Elevation* (m)
A(−4, 4)	57
B(−2, 0)	48
C(2, 4)	55
D(2, −6)	36

a We draw PB(A, B), PB(A, C), PB(B, C), PB(B, D), and PB(C, D), then remove segments which do not form part of the Voronoi diagram.



b **i** $(0, 1)$ is closest to B, so we estimate an elevation of 48 m at $(0, 1)$.

ii $(-4, 2)$ is closest to A, so we estimate an elevation of 57 m at $(-4, 2)$.

iii $(3, -4)$ is closest to D, so we estimate an elevation of 36 m at $(3, -4)$.

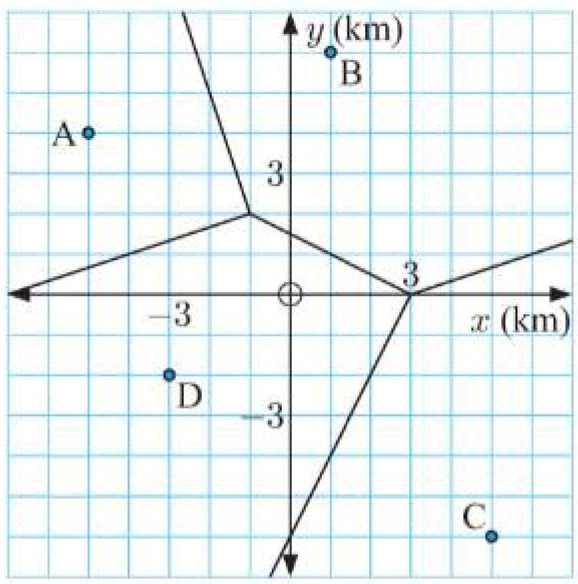
3

Location	*Snowfall* (inches)
A	7
B	5.5
C	12.2
D	9.3

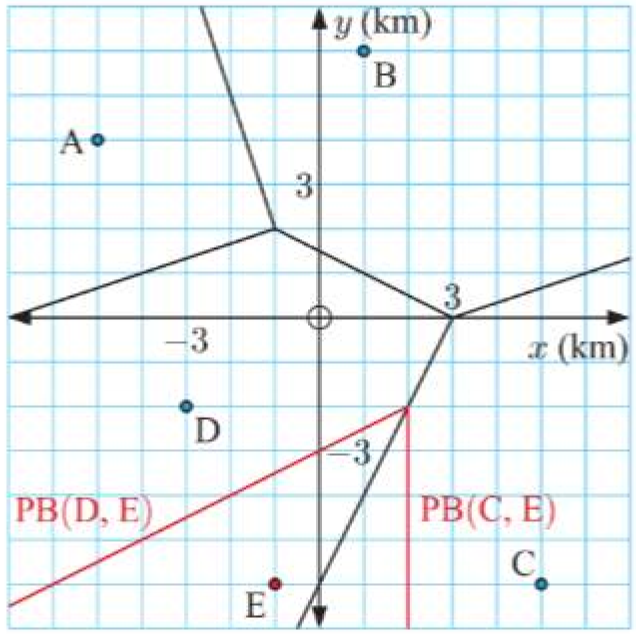
a **i** $(-1, -4)$ is closest to D, so we estimate that 9.3 inches of snow were received at $(-1, -4)$.

ii $(2, 1)$ is closest to B, so we estimate that 5.5 inches of snow were received at $(2, 1)$.

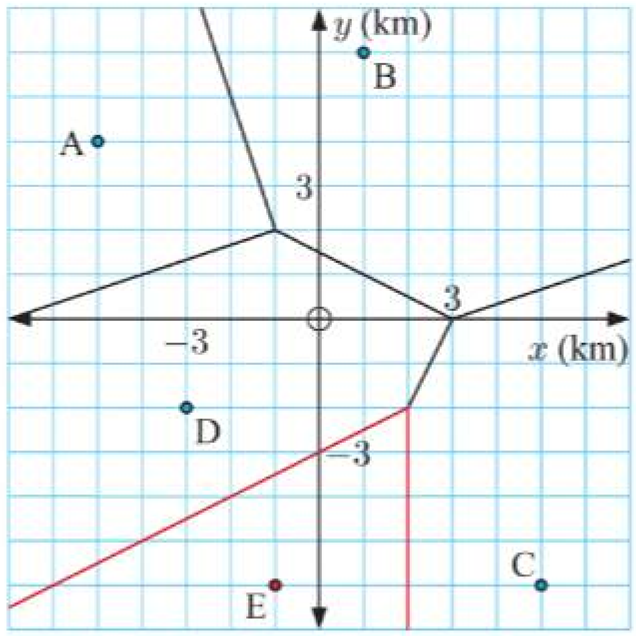
iii $(1, -4)$ is equally closest to C and D, so we estimate that $\dfrac{12.2+9.3}{2} = 10.75$ inches of snow were received at $(1, -4)$.



b **i** We construct PB(C, E) and PB(D, E) within the original cells C and D respectively.



We then remove the edge segment from the original Voronoi diagram which now lies within cell E, giving us the Voronoi diagram which includes location E.



ii Both $(-1, -4)$ and $(1, -4)$ from **a i** and **a iii** respectively are now closest to location E.

So, we would now estimate 10.6 inches of snow at both locations.

EXERCISE 16E

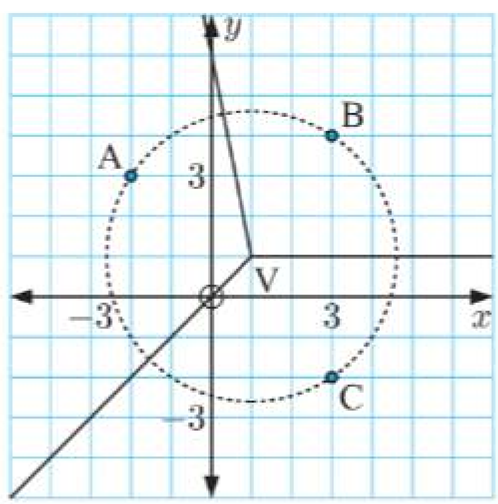
1 **a** The Voronoi diagram has vertex $V(1, 1)$.

V is equidistant from A, B, and C.

A has coordinates $(-2, 3)$, so

$$\begin{aligned} VA &= \sqrt{(-2-1)^2 + (3-1)^2} \\ &= \sqrt{(-3)^2 + 2^2} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

So the largest empty circle has centre $V(1, 1)$ and radius $\sqrt{13}$ units.



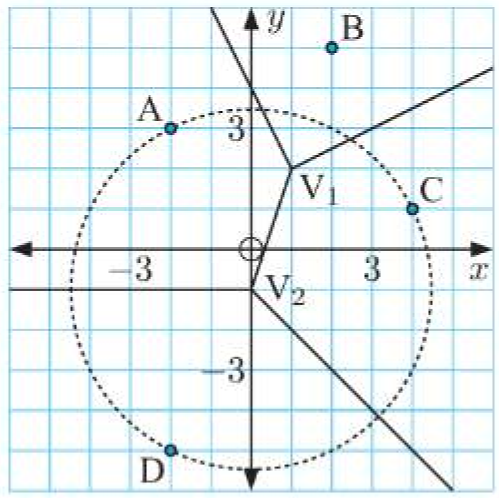
b The Voronoi diagram has vertices $V_1(1, 2)$ and $V_2(0, -1)$.

V_1 is equidistant from A, B, and C, and V_2 is equidistant from A, C, and D.

A has coordinates $(-2, 3)$.

$$V_1A = \sqrt{(-2-1)^2 + (3-2)^2}$$
$$= \sqrt{(-3)^2 + 1^2}$$
$$= \sqrt{10} \text{ units}$$

$$V_2A = \sqrt{(-2-0)^2 + (3-(-1))^2}$$
$$= \sqrt{(-2)^2 + 4^2}$$
$$= \sqrt{20} = 2\sqrt{5} \text{ units}$$



So, the largest empty circle has centre $V_2(0, -1)$ and radius $2\sqrt{5}$ units.

2 **a** The Voronoi diagram has vertices $V_1(20, 30)$, $V_2(20, 10)$, and $V_3(8, 2)$.

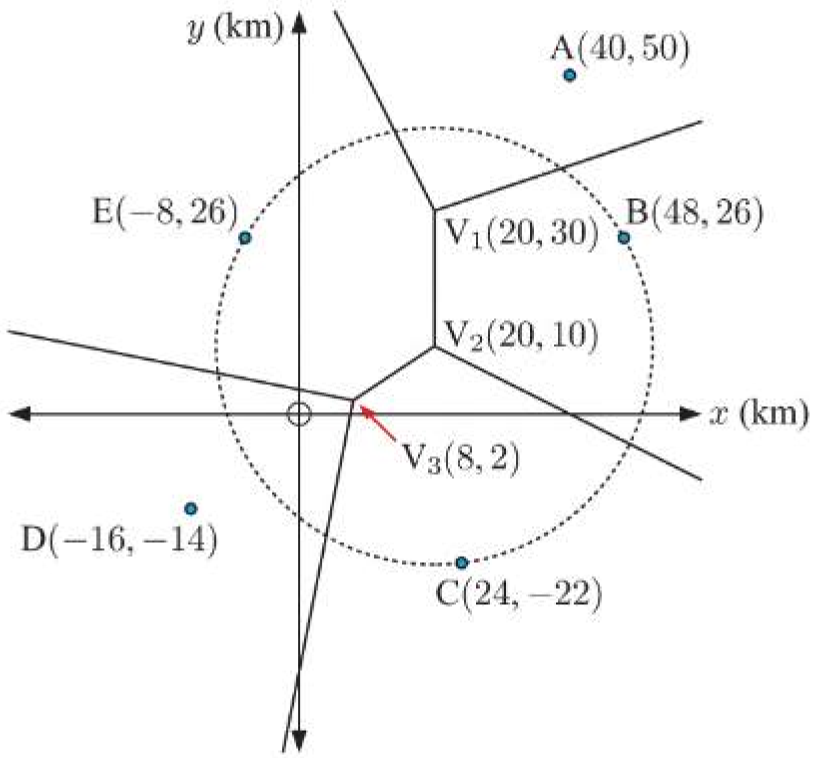
V_1 is equidistant from A, B, and E, V_2 is equidistant from B, C, and E, and V_3 is equidistant from C, D, and E.

E has coordinates $(-8, 26)$.

$$V_1E = \sqrt{(-8-20)^2 + (26-30)^2}$$
$$= \sqrt{(-28)^2 + (-4)^2}$$
$$= \sqrt{800} = 20\sqrt{2} \text{ km}$$

$$V_2E = \sqrt{(-8-20)^2 + (26-10)^2}$$
$$= \sqrt{(-28)^2 + 16^2}$$
$$= \sqrt{1040} = 4\sqrt{65} \text{ km}$$

$$V_3E = \sqrt{(-8-8)^2 + (26-2)^2}$$
$$= \sqrt{(-16)^2 + 24^2}$$
$$= \sqrt{832} = 8\sqrt{13} \text{ km}$$



So, the largest empty circle has centre $V_2(20, 10)$, and radius $4\sqrt{65}$ km.

$\therefore$ the rubbish dump should be established at $(20, 10)$.

b From **a**, the largest empty circle has radius $4\sqrt{65}$ km.

$\therefore$ the dump is $4\sqrt{65} \approx 32.2$ km from the nearest town.

c $V_2(20, 10)$ is adjacent to cells B, C, and E.

So towns B, C, and E are closest to the rubbish dump.

d To show that the location found in **a** is preferable, we must show that there is at least one town which is less than ≈ 32.2 km from $(25, 15)$.

The distance from $(25, 15)$ to $B(48, 26)$ is $\sqrt{(48-25)^2 + (26-15)^2}$
$$= \sqrt{23^2 + 11^2}$$
$$= \sqrt{650} \approx 25.5 \text{ km}$$

So, B is closer to $(25, 15)$ than the answer found in **a** is to any other town.

3 Let P lie in cell X. The largest empty circle centred at P would touch X. As P lies within cell X, the largest empty circle does not touch any other site, meaning that a larger empty circle could be created with centre closer to the edge of the cell.

$\therefore$ the circle centred at P is not the largest empty circle.

$\therefore$ the largest empty circle cannot lie within a cell.

4 **a** **i** A$(-15, 10)$, D$(-3, -18)$

The midpoint of [AD] is $\left(\frac{-15+-3}{2}, \frac{10+-18}{2}\right)$ or $(-9, -4)$.

The gradient of [AD] is $\frac{-18-10}{-3--15} = \frac{-28}{12} = -\frac{7}{3}$.

So, PB(A, D) has gradient $\frac{3}{7}$ and passes through $(-9, -4)$.

$\therefore$ its equation is $3x - 7y = 3(-9) - 7(-4)$

which is $7y = 3x - 1$

or $y = \frac{3}{7}x - \frac{1}{7}$

ii C$(15, 0)$, D$(-3, -18)$

The midpoint of [CD] is $\left(\frac{15+-3}{2}, \frac{0+-18}{2}\right)$ or $(6, -9)$.

The gradient of [CD] is $\frac{-18-0}{-3-15} = \frac{-18}{-18} = 1$.

So, PB(C, D) has gradient -1 and passes through $(6, -9)$.

$\therefore$ its equation is $x + y = 6 + (-9)$

or $y = -x - 3$

b PB(A, D) and PB(C, D) intersect where $\frac{3}{7}x - \frac{1}{7} = -x - 3$

$\therefore\ 3x - 1 = -7x - 21$

$\therefore\ 10x = -20$

$\therefore\ x = -2$

When $x = -2$, $y = -(-2) - 3 = -1$

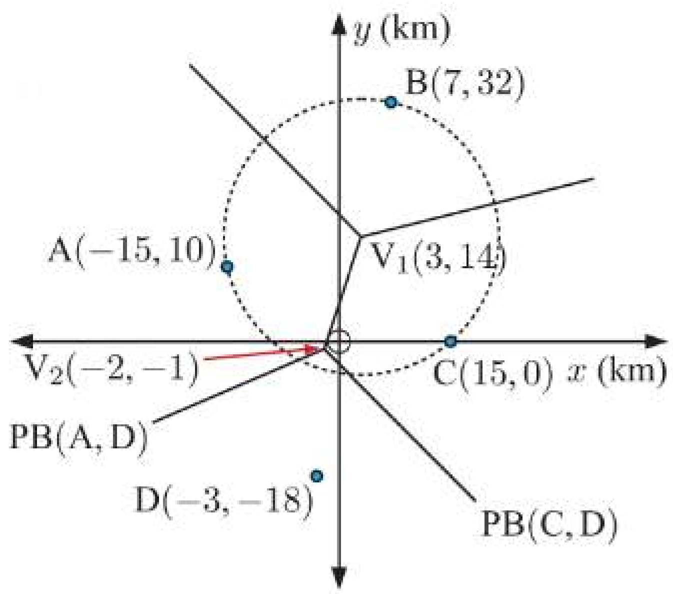
So, V_2 has coordinates $(-2, -1)$.

c From **b**, the Voronoi diagram has vertices $V_1(3, 14)$ and $V_2(-2, -1)$.

V_1 is adjacent to cells A, B, and C, and V_2 is adjacent to vertices A, C, and D.

$$V_1C = \sqrt{(15-3)^2 + (0-14)^2} = \sqrt{12^2 + (-14)^2} = \sqrt{340} \text{ km}$$

$$V_2C = \sqrt{(15-(-2))^2 + (0-(-1))^2} = \sqrt{17^2 + 1^2} = \sqrt{290} \text{ km}$$



So, the largest empty circle has centre $V_1(3, 14)$, and radius $\sqrt{340}$ km.

$\therefore$ the optimal position for a toxic waste dump is $(3, 14)$.

5 A$(-5, -10)$, B$(11, 18)$, C$(5, -12)$

a The midpoint of [AB] is $\left(\frac{-5+11}{2}, \frac{-10+18}{2}\right)$ or $(3, 4)$.

The gradient of [AB] is $\frac{18--10}{11--5} = \frac{28}{16} = \frac{7}{4}$.

So, PB(A, B) has gradient $-\frac{4}{7}$ and passes through $(3, 4)$.

$\therefore$ its equation is $4x + 7y = 4(3) + 7(4)$

which is $7y = -4x + 40$

or $y = -\frac{4}{7}x + \frac{40}{7}$

The midpoint of [AC] is $\left(\frac{-5+5}{2}, \frac{-10+-12}{2}\right)$ or $(0, -11)$.

The gradient of [AC] is $\frac{-12--10}{5--5} = \frac{-2}{10} = -\frac{1}{5}$.

So, PB(A, C) has gradient 5 and passes through $(0, -11)$.

$\therefore$ its equation is $5x - y = 5(0) - (-11)$

or $y = 5x - 11$

The midpoint of [BC] is $\left(\frac{11+5}{2}, \frac{18+-12}{2}\right)$ or $(8, 3)$.

The gradient of [BC] is $\frac{-12-18}{5-11} = \frac{-30}{-6} = 5$.

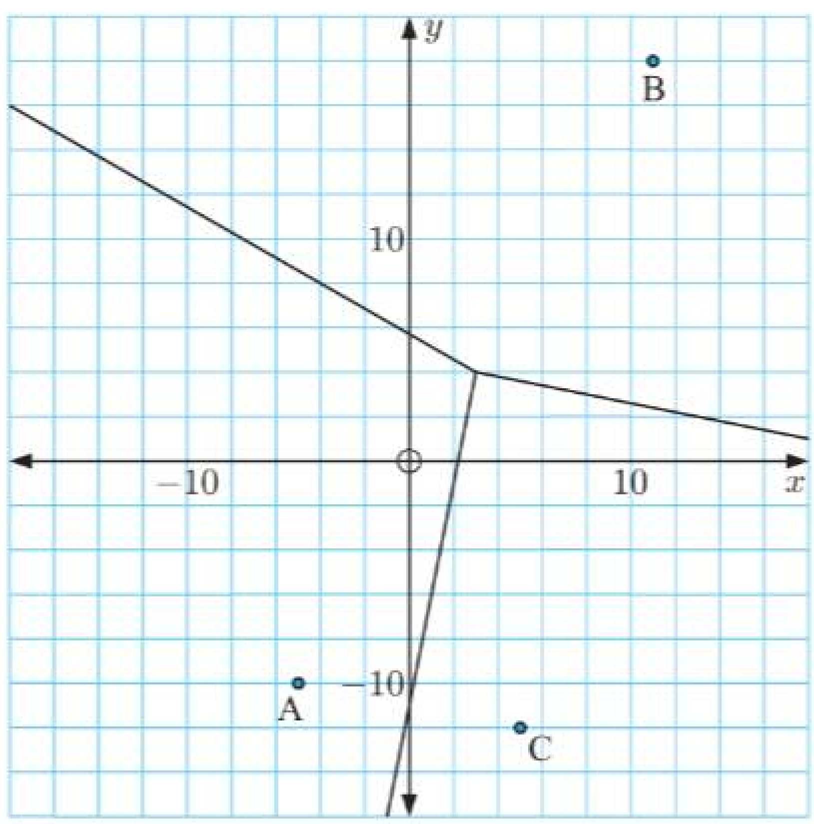
So, PB(B, C) has gradient $-\frac{1}{5}$ and passes through $(8, 3)$.

$\therefore$ its equation is $x + 5y = 1(8) + 5(3)$

which is $5y = -x + 23$

or $y = -\frac{1}{5}x + \frac{23}{5}$

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C), then remove segments which do not form part of the Voronoi diagram.



b PB(A, C) and PB(B, C) intersect where $5x - 11 = -\frac{1}{5}x + \frac{23}{5}$

$$\therefore\ 25x - 55 = -x + 23$$
$$\therefore\ 26x = 78$$
$$\therefore\ x = 3$$

When $x = 3$, $y = 5(3) - 11 = 4$, so the point of intersection is $(3, 4)$.

Check: Using the equation of PB(A, B), when $x = 3$, $y = -\frac{4}{7}(3) + \frac{40}{7} = 4$ ✓

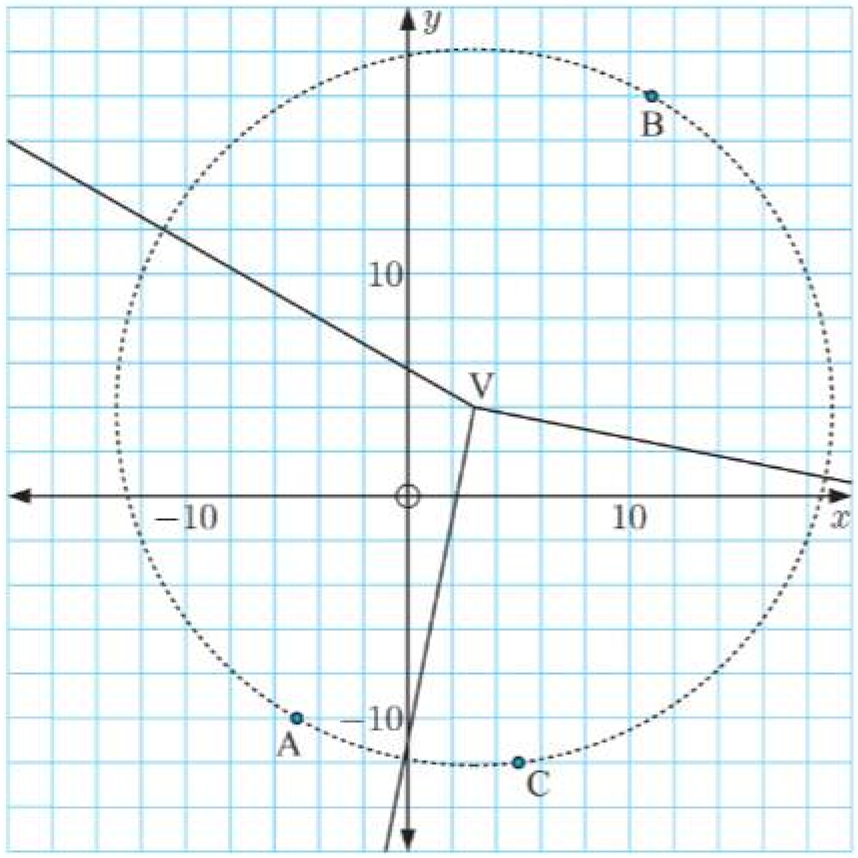
So, $(3, 4)$ is the vertex of the Voronoi diagram.

c The Voronoi diagram has vertex V(3, 4).

V is equidistant from A, B, and C.

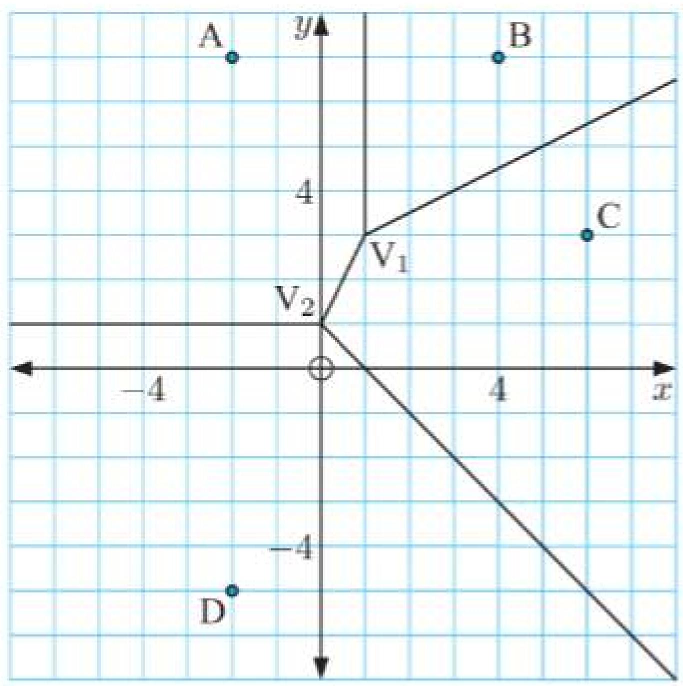
$$\text{VA} = \sqrt{(-5-3)^2 + (-10-4)^2}$$
$$= \sqrt{(-8)^2 + (-14)^2}$$
$$= \sqrt{260} = 2\sqrt{65} \text{ units}$$

So, the largest empty circle has centre V(3, 4) and radius $2\sqrt{65}$ units.



6 A(−2, 7), B(4, 7), C(6, 3), D(−2, −5)

a We draw PB(A, B), PB(A, C), PB(A, D), PB(B, C), and PB(C, D), then remove segments which do not form part of the Voronoi diagram.



b From the Voronoi diagram in **a**, we observe that the vertices are $V_1(1, 3)$ and $V_2(0, 1)$.

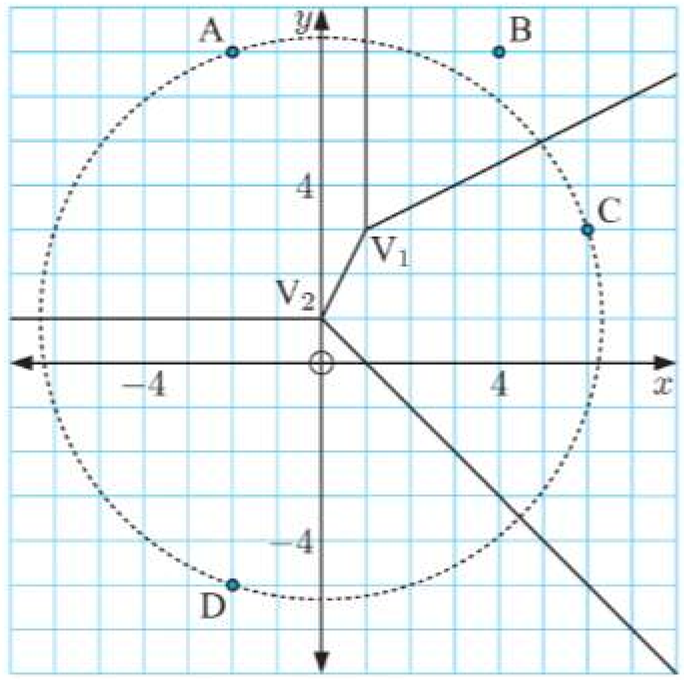
c The Voronoi diagram has vertices $V_1(1, 3)$ and $V_2(0, 1)$.

V_1 is equidistant from A, B, and C, and V_2 is equidistant from A, C, and D.

$$\begin{aligned} V_1A &= \sqrt{(-2-1)^2 + (7-3)^2} \\ &= \sqrt{(-3)^2 + 4^2} \\ &= \sqrt{25} = 5 \text{ units} \end{aligned}$$

$$\begin{aligned} V_2A &= \sqrt{(-2-0)^2 + (7-1)^2} \\ &= \sqrt{(-2)^2 + 6^2} \\ &= \sqrt{40} = 2\sqrt{10} \text{ units} \end{aligned}$$

So, the largest empty circle has centre $V_2(0, 1)$ and radius $2\sqrt{10} \approx 6.32$ units.



7 **a** **i** $(3, 4)$ lies in cell B, so station B is the closest to $(3, 4)$.

ii $(-3, -3)$ lies in cell E, so station E is the closest to $(-3, -3)$.

b PB(C, E) and PB(D, E) intersect at V_3.

$C(3, -1)$, $D(-3, -9)$, $E(-5, -1)$

The midpoint of [CE] is $\left(\frac{3 + -5}{2}, \frac{-1 + -1}{2}\right)$ or $(-1, -1)$.

Now [CE] is horizontal.

So, PB(C, E) is vertical and passes through $(-1, -1)$.

$\therefore$ its equation is $x = -1$.

The midpoint of [DE] is $\left(\frac{-3 + -5}{2}, \frac{-9 + -1}{2}\right)$ or $(-4, -5)$.

The gradient of [DE] is $\frac{-1 - -9}{-5 - -3} = \frac{8}{-2} = -4$.

So, PB(D, E) has gradient $\frac{1}{4}$ and passes through $(-4, -5)$.

$\therefore$ its equation is $x - 4y = 1(-4) - 4(-5)$

which is $4y = x - 16$

or $y = \frac{1}{4}x - 4$

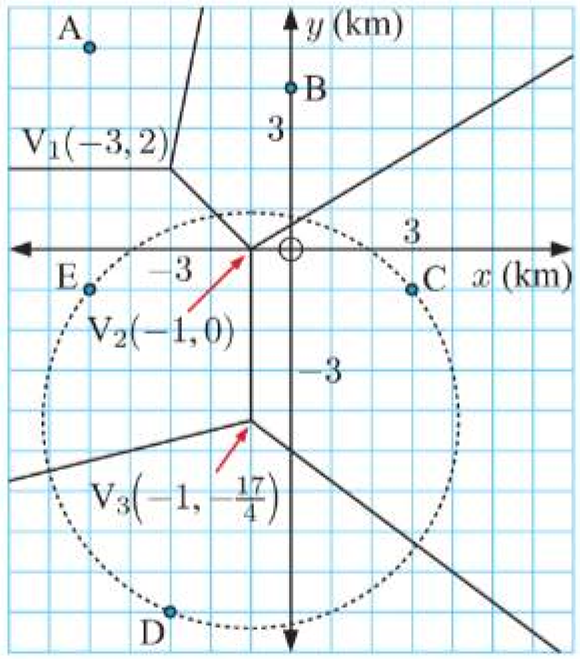
So, PB(C, E) and PB(D, E) intersect where $x = -1$.

When $x = -1$, $y = \frac{1}{4}(-1) - 4 = -\frac{17}{4}$.

$\therefore$ V_3 has coordinates $\left(-1, -\frac{17}{4}\right)$.

c **i** The Voronoi diagram has vertices $V_1(-3, 2)$, $V_2(-1, 0)$, and $V_3\left(-1, -\frac{17}{4}\right)$.

V_1 is equidistant from stations A, B, and E, V_2 is equidistant from stations B, C, and E, and V_3 is equidistant from stations C, D, and E.



$$\begin{aligned} V_1E &= \sqrt{(-5-(-3))^2 + (-1-2)^2} \\ &= \sqrt{(-2)^2 + (-3)^2} \\ &= \sqrt{13} \approx 3.61 \text{ km} \end{aligned}$$

$$\begin{aligned} V_2E &= \sqrt{(-5-(-1))^2 + (-1-0)^2} \\ &= \sqrt{(-4)^2 + (-1)^2} \\ &= \sqrt{17} \approx 4.12 \text{ km} \end{aligned}$$

$$\begin{aligned} V_3E &= \sqrt{(-5-(-1))^2 + \left(-1-\left(-\frac{17}{4}\right)\right)^2} \\ &= \sqrt{(-4)^2 + \left(\frac{13}{4}\right)^2} \\ &= \sqrt{\frac{425}{16}} = \frac{5\sqrt{17}}{4} \approx 5.15 \text{ km} \end{aligned}$$

So, the largest empty circle has centre $V_3\left(-1, -\frac{17}{4}\right)$ and radius $\frac{5\sqrt{17}}{4} \approx 5.15$ km.

$\therefore$ the new police station should be built at $\left(-1, -\frac{17}{4}\right)$.

ii Before the new police station is built, $(-3, -3)$ lies in cell E, so its closest station is E.

The distance from $(-3, -3)$ to $E(-5, -1)$ is
$$\begin{aligned} &\sqrt{(-5-(-3))^2 + (-1-(-3))^2} \\ &= \sqrt{(-2)^2 + 2^2} \\ &= \sqrt{8} \approx 2.82 \text{ km} \end{aligned}$$

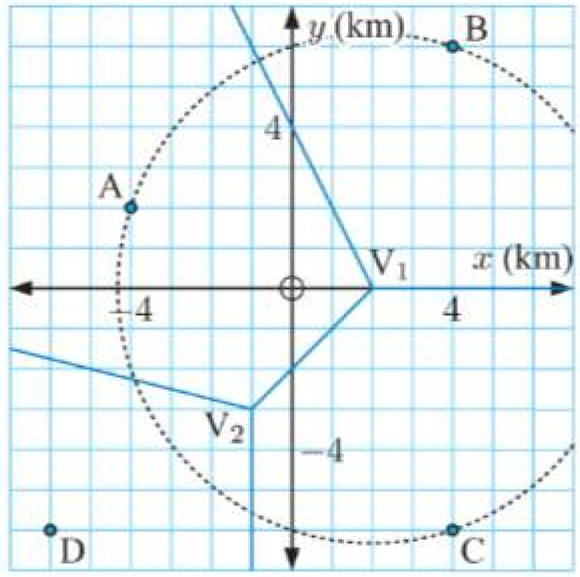
The distance from $(-3, -3)$ to $V_3\left(-1, -\frac{17}{4}\right)$ is
$$\begin{aligned} &\sqrt{(-1-(-3))^2 + \left(-\frac{17}{4}-(-3)\right)^2} \\ &= \sqrt{2^2 + \left(-\frac{5}{4}\right)^2} \\ &= \sqrt{\frac{89}{16}} \approx 2.36 \text{ km} \end{aligned}$$

So, the new station will now be the closest station to $(-3, -3)$.

8 **a** The Voronoi diagram has vertices $V_1(2,\ 0)$ and $V_2(-1,\ -3)$.

V_1 is adjacent to cells A, B, and C, and V_2 is adjacent to cells A, C, and D.

A has coordinates $(-4,\ 2)$.



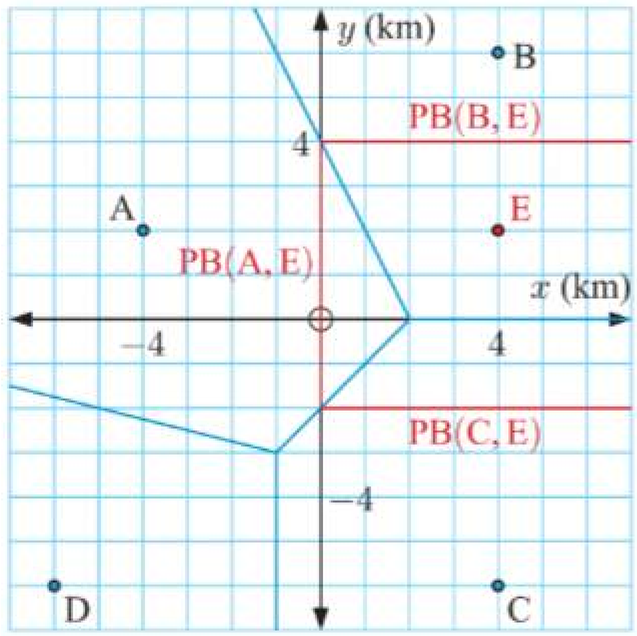
$$\begin{aligned} V_1A &= \sqrt{(-4-2)^2+(2-0)^2} \\ &= \sqrt{(-6)^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \text{ km} \end{aligned}$$

$$\begin{aligned} V_2A &= \sqrt{(-4-(-1))^2+(2-(-3))^2} \\ &= \sqrt{(-3)^2+5^2} \\ &= \sqrt{34} \text{ km} \end{aligned}$$

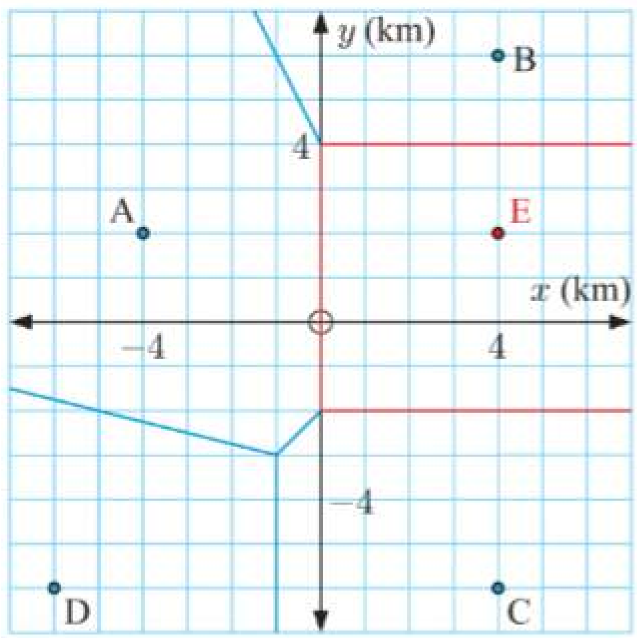
So, the largest empty circle has centre $V_1(2,\ 0)$ and radius $2\sqrt{10}$ km.

$\therefore$ the optimal position for Brigette's store is $(2,\ 0)$, which is $2\sqrt{10} \approx 6.32$ km from the nearest competitor.

b **i** Let E be the site with coordinates $(4,\ 2)$. We construct PB(A, E), PB(B, E), and PB(C, E) within the original cells A, B, and C respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram with the new store at $(4,\ 2)$.



ii One vertex of the original Voronoi diagram remains, $V_2(-1, -3)$, and so is still $\sqrt{34}$ km from its nearest stores.

The new Voronoi diagram also has vertices $V_3(0, 4)$ and $V_4(0, -2)$.

V_3 is adjacent to A, B, and E, and V_4 is adjacent to A, C, and E.

$$\begin{aligned} V_3A &= \sqrt{(-4-0)^2 + (2-4)^2} \\ &= \sqrt{(-4)^2 + (-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \text{ km} \end{aligned}$$

$$\begin{aligned} V_4A &= \sqrt{(-4-0)^2 + (2-(-2))^2} \\ &= \sqrt{(-4)^2 + 4^2} \\ &= \sqrt{32} = 4\sqrt{2} \text{ km} \end{aligned}$$

So, the largest empty circle has centre $V_2(-1, -3)$ and radius $\sqrt{34}$ km.

$\therefore$ the new optimal position for Brigette's store is $(-1, -3)$.

9 **a** To determine which fire station is closest to a particular point, we first draw a Voronoi diagram.

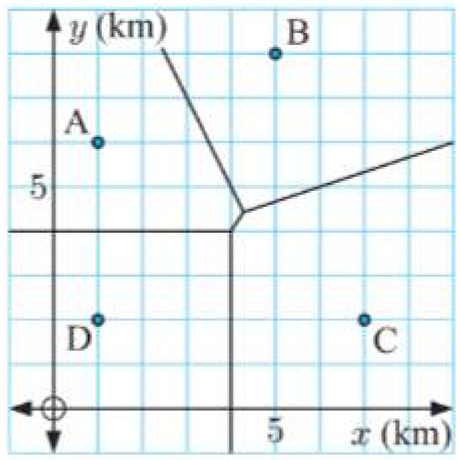
$A(1, 6)$, $B(5, 8)$, $C(7, 2)$, $D(1, 2)$

We draw PB(A, B), PB(A, C), PB(A, D), PB(B, C), and PB(C, D), then remove segments which do not form part of the Voronoi diagram.

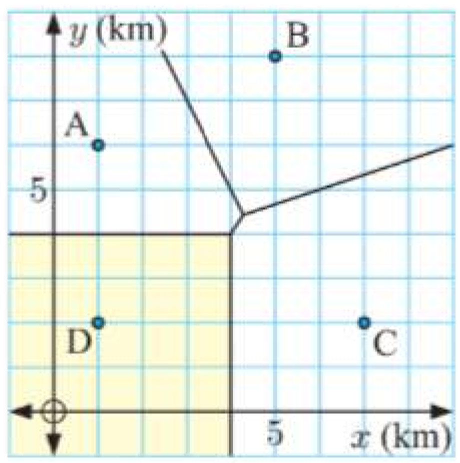
We observe from the Voronoi diagram that:

i $(6, 3)$ lies in cell C, so fire station C is closest to $(6, 3)$.

ii $(4, 6)$ lies in cell B, so fire station B is closest to $(4, 6)$.



b The region closest to fire station D is shaded in the Voronoi diagram alongside.



c We could create a Voronoi diagram like in **a**.

d The vertex adjacent to cells A, C, and D is $V_1(4, 4)$.

The vertex adjacent to cells A, B, and C is where PB(A, B) and PB(B, C) intersect.

PB(A, B) has gradient -2 and passes through $(4, 5)$.

$$\begin{aligned} \therefore \text{ its equation is } \quad 2x + y &= 2(4) + 5 \\ \text{or} \quad y &= -2x + 13 \end{aligned}$$

PB(B, C) has gradient $\frac{1}{3}$ and passes through $(6, 5)$.

$\therefore$ its equation is $x - 3y = 1(6) - 3(5)$

which is $3y = x + 9$

or $y = \frac{1}{3}x + 3$

PB(A, B) intersects PB(B, C) where $-2x + 13 = \frac{1}{3}x + 3$

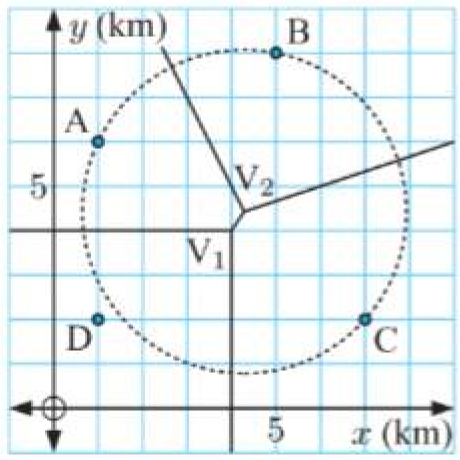
$$\therefore \quad -\tfrac{7}{3}x = -10$$

$$\therefore \quad x = \tfrac{30}{7}$$

When $x = \frac{30}{7}$, $y = \frac{1}{3}\left(\frac{30}{7}\right) + 3 = \frac{31}{7}$.

So, V_2 has coordinates $\left(\frac{30}{7}, \frac{31}{7}\right)$.

$$\begin{aligned} V_1A &= \sqrt{(1-4)^2 + (6-4)^2} \\ &= \sqrt{(-3)^2 + 2^2} \\ &= \sqrt{13} \approx 3.61 \text{ km} \end{aligned}$$



$$\begin{aligned} V_2A &= \sqrt{\left(1 - \tfrac{30}{7}\right)^2 + \left(6 - \tfrac{31}{7}\right)^2} \\ &= \sqrt{\left(-\tfrac{23}{7}\right)^2 + \left(\tfrac{11}{7}\right)^2} \\ &= \sqrt{\tfrac{650}{49}} \approx 3.64 \text{ km} \end{aligned}$$

So, the largest empty circle has centre $V_2\left(\frac{30}{7}, \frac{31}{7}\right)$ and radius $\sqrt{13}$ km.

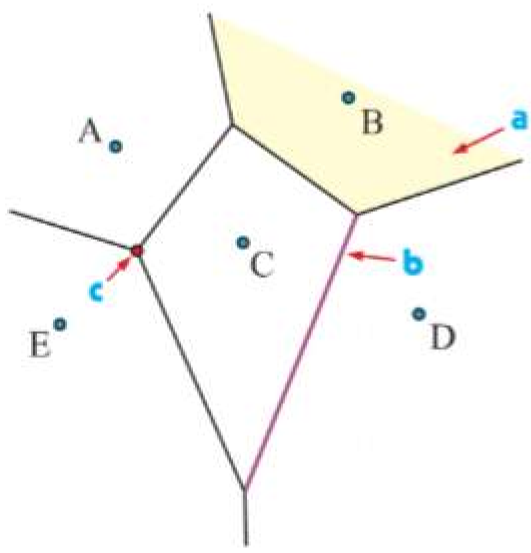
$\therefore$ the new station should be built at $\left(\frac{30}{7}, \frac{31}{7}\right)$.

REVIEW SET 16A

1 **a** The parts closest to site B are in the yellow area, which is cell B.

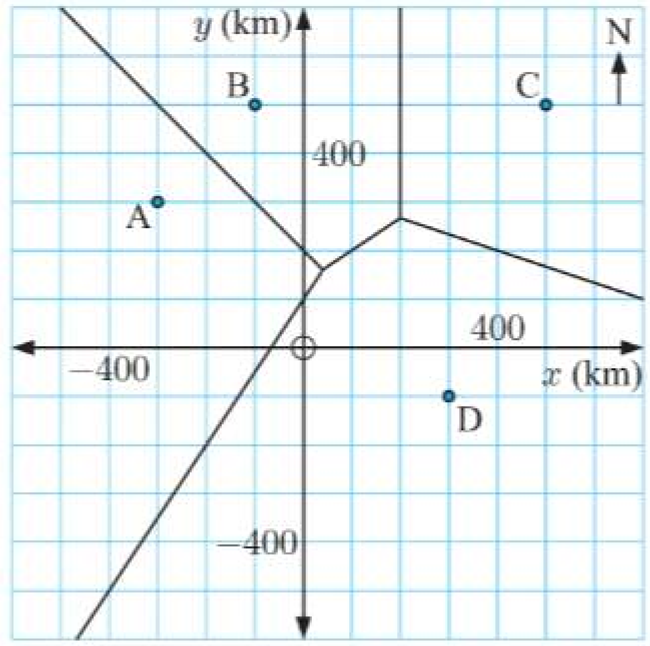
b The parts equally closest to C and D are on the purple line, which is the edge adjacent to cells C and D.

c The part equally closest to A, C, and E is the red point, which is the vertex adjacent to cells A, C, and E.



2 **a** **i** $(300, 400)$ lies in cell C, so the nearest airport is Airport C.

ii $(-100, 0)$ lies in cell A, so the nearest airport is Airport A.



b **i** $(100, 200)$ lies on the edge adjacent to cells B and D, so it is equally closest to Airports B and D.

ii The distance from $(100, 200)$ to $D(300, -100)$ is

$$\sqrt{(300-100)^2+(-100-200)^2}$$
$$= \sqrt{200^2+(-300)^2}$$
$$= \sqrt{130\,000} \approx 361 \text{ km}$$

So, the aeroplane is about 361 km from Airports B and D.

iii $(400, 200)$ lies on the edge adjacent to cells C and D, and is $400 - 100 = 300$ km east of $(100, 200)$.

So, the aeroplane must travel more than 300 km east before it is closest to Airport C.

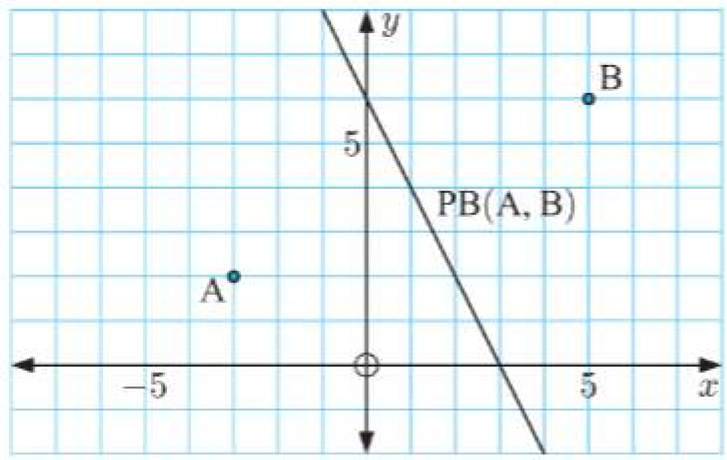
3 **a** $A(-3, 2)$, $B(5, 6)$

The midpoint of [AB] is $\left(\frac{-3+5}{2}, \frac{2+6}{2}\right)$ or $(1, 4)$.

The gradient of [AB] is $\frac{6-2}{5--3} = \frac{4}{8} = \frac{1}{2}$.

So, PB(A, B) has gradient -2 and passes through $(1, 4)$.

We plot sites A and B, then draw PB(A, B) to give us the Voronoi diagram.



b PB(A, B) has gradient -2 and passes through $(1, 4)$.

$\therefore$ its equation is $2x + y = 2(1) + 4$
or $y = -2x + 6$

c **i** When $x = 3$, $y = -2(3) + 6 = 0$

$\therefore$ $(3, 0)$ is a point on the line $y = -2x + 6$.

$\therefore$ $(3, 0)$ lies on the edge.

ii The distance from $(3, 0)$ to $A(-3, 2)$ is $\sqrt{(-3-3)^2+(2-0)^2}$
$$= \sqrt{(-6)^2+2^2}$$
$$= \sqrt{40} = 2\sqrt{10} \text{ units}$$

The distance from $(3, 0)$ to $B(5, 6)$ is $\sqrt{(5-3)^2+(6-0)^2}$
$$= \sqrt{2^2+6^2}$$
$$= \sqrt{40} = 2\sqrt{10} \text{ units} \quad \checkmark$$

d **i** $(-1, 7)$ lies in cell A, so it is closest to site A.

ii $(2, -5)$ lies in cell A, so it is closest to site A.

4 A(0, 20), B(20, 0), C(50, 10)

a The midpoint of [AB] is $\left(\frac{0+20}{2}, \frac{20+0}{2}\right)$ or $(10, 10)$.

The gradient of [AB] is $\frac{0-20}{20-0} = \frac{-20}{20} = -1$.

So, PB(A, B) has gradient 1 and passes through $(10, 10)$.

The midpoint of [AC] is $\left(\frac{0+50}{2}, \frac{20+10}{2}\right)$ or $(25, 15)$.

The gradient of [AC] is $\frac{10-20}{50-0} = \frac{-10}{50} = -\frac{1}{5}$.

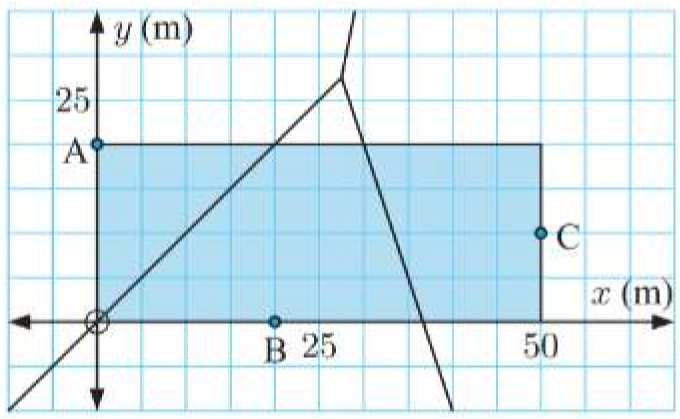
So, PB(A, C) has gradient 5 and passes through $(25, 15)$.

The midpoint of [BC] is $\left(\frac{20+50}{2}, \frac{0+10}{2}\right)$ or $(35, 5)$.

The gradient of [BC] is $\frac{10-0}{50-20} = \frac{10}{30} = \frac{1}{3}$.

So, PB(B, C) has gradient -3 and passes through $(35, 5)$.

We draw PB(A, B), PB(A, C), and PB(B, C), then remove segments which do not form part of the Voronoi diagram.



b There is no point in the pool equidistant from all three exits, as the vertex of the Voronoi diagram lies outside the pool.

c **i** $(35, 10)$ lies in cell C, so Jenny is closest to exit C.

ii The distance from $(35, 10)$ to $C(50, 10)$ is $\sqrt{(50-35)^2 + (10-10)^2}$

$= \sqrt{15^2} = 15$ m

So, Jenny is 15 m away from exit C.

d Area of the pool $= 50 \times 20 = 1000$ m^2

The region of the pool which is within cell:

i A is a triangle with base 20 m and height 20 m.

So the area is $\frac{1}{2} \times 20 \times 20 = 200$ m^2.

$\therefore$ $\frac{200}{1000} = \frac{1}{5}$ of the pool is closest to exit A.

ii B is a trapezium which is 20 m high, and has parallel sides 10 m and $35 + 5(\frac{1}{3}) = \frac{110}{3}$ m long.

So the area is $\left(\dfrac{10 + \frac{110}{3}}{2}\right) \times 20 = \frac{1400}{3}$ m^2.

$\therefore \dfrac{\frac{1400}{3}}{1000} = \frac{7}{15}$ of the pool is closest to exit B.

iii Using **i** and **ii**, the proportion of the pool that is closest to exit C is $1 - \frac{1}{5} - \frac{7}{15} = \frac{5}{15} = \frac{1}{3}$.

5 **a** The Voronoi diagram must have an edge missing because sites C and D currently lie within the same cell.

b $C(-3, -3)$, $D(-2, 0)$

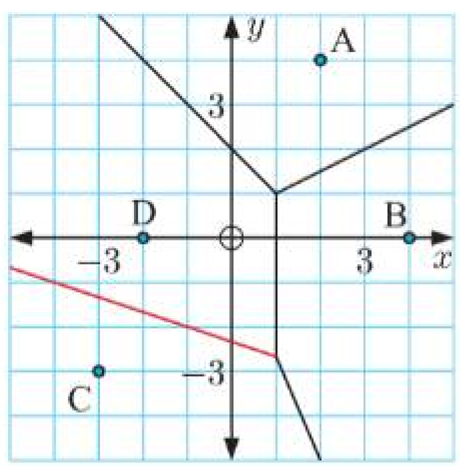
The midpoint of [CD] is $\left(\dfrac{-3 + -2}{2}, \dfrac{-3 + 0}{2}\right)$ or $\left(-\frac{5}{2}, -\frac{3}{2}\right)$.

The gradient of [CD] is $\dfrac{0 - -3}{-2 - -3} = \dfrac{3}{1} = 3$.

So, PB(C, D) has gradient $-\frac{1}{3}$ and passes through $\left(-\frac{5}{2}, -\frac{3}{2}\right)$.

$\therefore$ its equation is $x + 3y = 1\left(-\frac{5}{2}\right) + 3\left(-\frac{3}{2}\right)$

or $x + 3y + 7 = 0$



6 **a** The blue edge has gradient $\frac{1}{3}$ and passes through $(4, 0)$.

$\therefore$ its equation is $x - 3y = 1(4) - 3(0)$

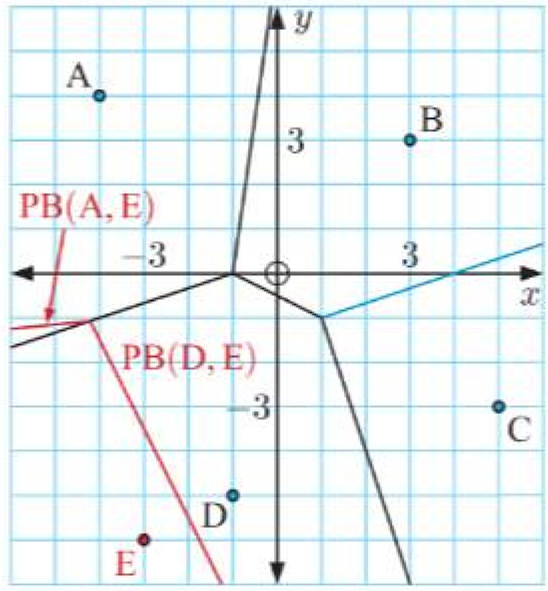
which is $3y = x - 4$

or $y = \frac{1}{3}x - \frac{4}{3}$

b **i** $(2, -1)$ lies in cell C, so it is closest to site C.

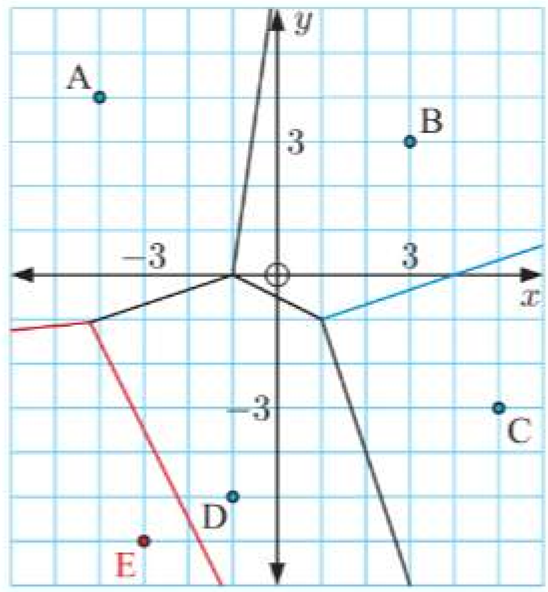
ii $(-5, -2)$ lies in cell D, so it is closest to site D.

c We construct PB(A, E) and PB(D, E) within the original cells A and D respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram which includes site E.

Note: We do not need to construct PB(B, E) or PB(C, E), as there are no points in cells B or C which are now closest to site E.



d The addition of site E affects our answer to **b ii**, as $(-5, -2)$ now lies within cell E, and so is now closest to site E.

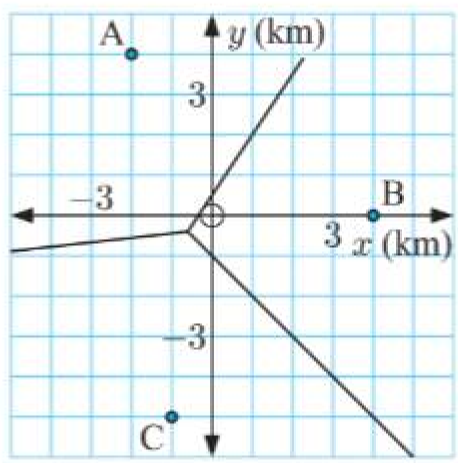
7

Location	*Wind speed* (km h^{-1})
A	14
B	11
C	19

a $(0, 0)$ is closest to B, so we estimate a wind speed of 11 km h^{-1} at $(0, 0)$.

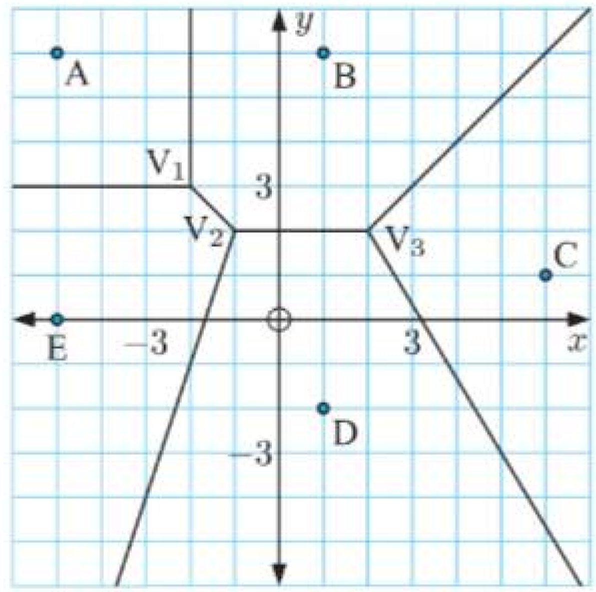
b $(-1, 3)$ is closest to A, so we estimate a wind speed of 14 km h^{-1} at $(-1, 3)$.

c $(-4, -2)$ is closest to C, so we estimate a wind speed of 19 km h^{-1} at $(-4, -2)$.



8 **a** A(−5, 6), B(1, 6), C(6, 1), D(1, −2), E(−5, 0)

We draw PB(B, C), PB(B, D), PB(C, D), and PB(D, E), then remove segments which do not form part of the Voronoi diagram.



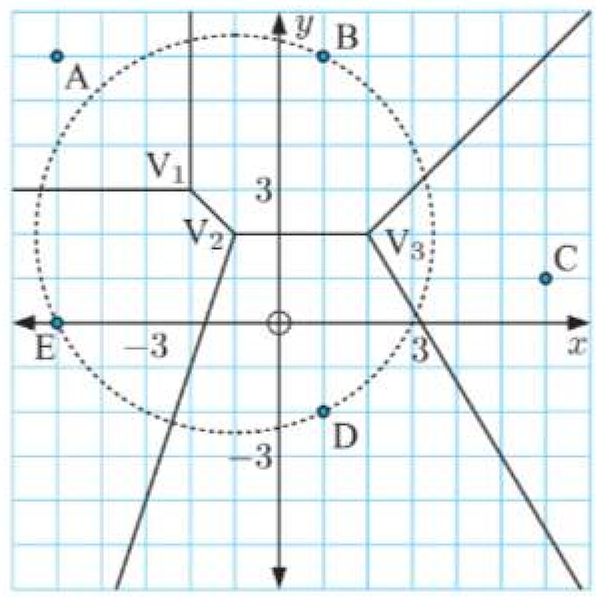
b We observe from the Voronoi diagram in **a** that:

- the vertex adjacent to A, B, and E is $V_1(-2, 3)$
- the vertex adjacent to B, D, and E is $V_2(-1, 2)$
- the vertex adjacent to B, C, and D is $V_3(2, 2)$.

c
$$\begin{aligned} V_1B &= \sqrt{(1-(-2))^2+(6-3)^2} \\ &= \sqrt{3^2+3^2} \\ &= \sqrt{18} = 3\sqrt{2} \text{ units} \end{aligned}$$

$$\begin{aligned} V_2B &= \sqrt{(1-(-1))^2+(6-2)^2} \\ &= \sqrt{2^2+4^2} \\ &= \sqrt{20} = 2\sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} V_3B &= \sqrt{(1-2)^2+(6-2)^2} \\ &= \sqrt{(-1)^2+4^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$



So, the largest empty circle has centre $V_2(-1, 2)$ and radius $2\sqrt{5}$ units.

9 **a** **i** The midpoint of [BC] is $\left(\dfrac{14+22}{2}, \dfrac{7+-9}{2}\right)$

or $(18, -1)$.

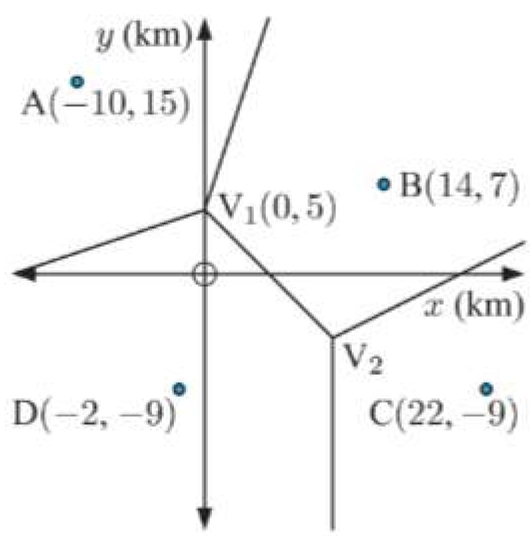
The gradient of [BC] is $\dfrac{-9-7}{22-14} = \dfrac{-16}{8} = -2$.

So, PB(B, C) has gradient $\frac{1}{2}$ and passes through $(18, -1)$.

$\therefore$ its equation is $x - 2y = 1(18) - 2(-1)$

which is $2y = x - 20$

or $y = \frac{1}{2}x - 10$



ii The midpoint of [CD] is $\left(\frac{22+(-2)}{2}, \frac{-9+(-9)}{2}\right)$ or $(10, -9)$.

The gradient of [CD] is horizontal.

So, PB(C, D) is vertical and passes through $(10, -9)$.

$\therefore$ its equation is $x = 10$.

b PB(B, C) and PB(C, D) intersect where $x = 10$

When $x = 10$, $y = \frac{1}{2}(10) - 10 = -5$

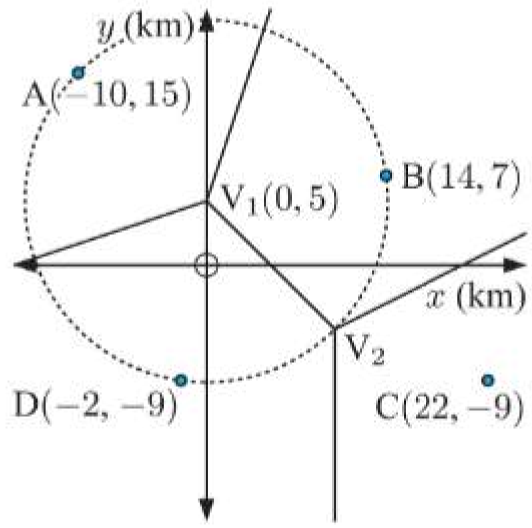
So, the vertex V_2 has coordinates $(10, -5)$.

c **i** The Voronoi diagram has vertices $V_1(0, 5)$ and $V_2(10, -5)$.

V_1 is adjacent to cells A, B, and D, and V_2 is adjacent to cells B, C, and D.

$$\begin{aligned} V_1D &= \sqrt{(-2-0)^2 + (-9-5)^2} \\ &= \sqrt{(-2)^2 + (-14)^2} \\ &= \sqrt{200} = 10\sqrt{2} \text{ km} \end{aligned}$$

$$\begin{aligned} V_2D &= \sqrt{(-2-10)^2 + (-9-(-5))^2} \\ &= \sqrt{(-12)^2 + (-4)^2} \\ &= \sqrt{160} = 4\sqrt{10} \text{ km} \end{aligned}$$



So, the largest empty circle has centre $V_1(0, 5)$ and radius $10\sqrt{2}$ km.

$\therefore$ the observatory should be built at $V_1(0, 5)$.

ii From **i**, the observatory will be $10\sqrt{2} \approx 14.1$ km from the nearest towns.

iii From **i**, the observatory will be equally closest to towns A, B, and D.

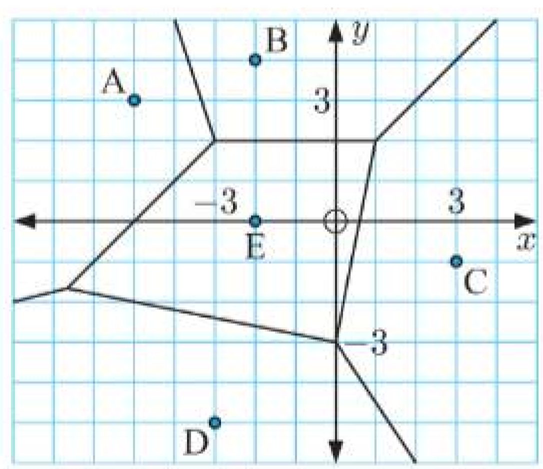
REVIEW SET 16B

1 **a** **i** $(1, 0)$ lies in cell C, so it is closest to site C.

ii $(-4, -2)$ lies in cell E, so it is closest to site E.

iii $(3, -5)$ lies in cell C, so it is closest to site C.

iv $(-3, 2)$ lies on the vertex adjacent to cells A, B, and E, so it is equally closest to sites A, B, and E.



b There are no points equally closest to sites B and D, as there is no edge or vertex which is adjacent to both cell B and cell D.

c The vertex which is adjacent to cells B, C, and E is $(1, 2)$.

So, $(1, 2)$ is equally closest to sites B, C, and E.

2 A point on an edge is equally closest to two sites. If the circle passes through a site, then the radius of the circle is the distance to that closest site.

$\therefore$ the radius would also be the distance to the other site.

$\therefore$ the circle passes through two sites.

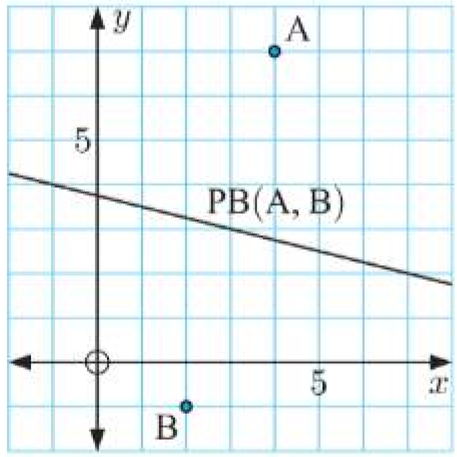
3 **a** A(4, 7), B(2, −1)

The midpoint of [AB] is $\left(\frac{4+2}{2}, \frac{7+-1}{2}\right)$

or (3, 3).

The gradient of [AB] is $\frac{-1-7}{2-4} = \frac{-8}{-2} = 4$.

So, PB(A, B) has gradient $-\frac{1}{4}$, and passes through (3, 3).



b A(−5, 0), B(1, 6), C(7, 4)

The midpoint of [AB] is $\left(\frac{-5+1}{2}, \frac{0+6}{2}\right)$ or (−2, 3).

The gradient of [AB] is $\frac{6-0}{1--5} = \frac{6}{6} = 1$.

So, PB(A, B) has gradient −1 and passes through (−2, 3).

The midpoint of [AC] is $\left(\frac{-5+7}{2}, \frac{0+4}{2}\right)$ or (1, 2).

The gradient of [AC] is $\frac{4-0}{7--5} = \frac{4}{12} = \frac{1}{3}$.

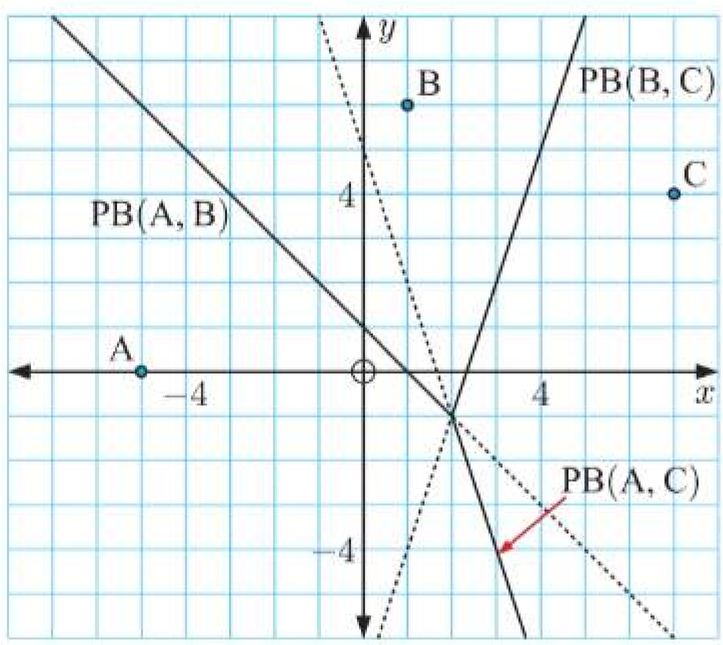
So, PB(A, C) has gradient −3 and passes through (1, 2).

The midpoint of [BC] is $\left(\frac{1+7}{2}, \frac{6+4}{2}\right)$ or (4, 5).

The gradient of [BC] is $\frac{4-6}{7-1} = \frac{-2}{6} = -\frac{1}{3}$.

So, PB(B, C) has gradient 3 and passes through (4, 5).

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



4 **a** The Voronoi diagram must have a site missing as there is a cell without a site.

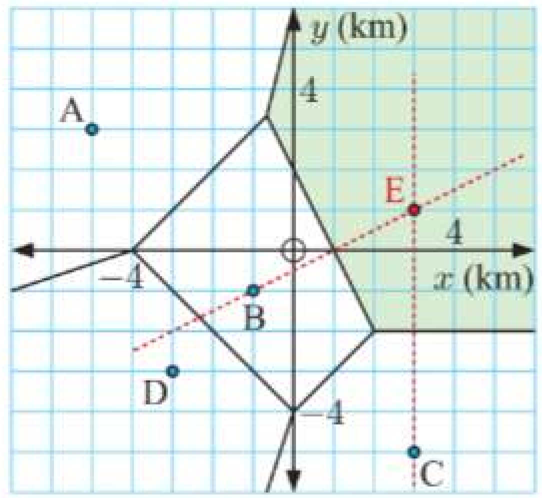
b The missing site E must lie in the shaded cell.

Now PB(B, E) has gradient -2, and PB(C, E) is horizontal.

$\therefore$ [BE] has gradient $\frac{1}{2}$, and [CE] is vertical.

If we draw lines (BE) and (CE) through B and C respectively, their intersection point must be E.

We observe that E has coordinates $(3, 1)$.



c **i** $(-3, 2)$ lies in cell A, so Elizabeth is closest to taxi rank A.

ii The distance from $(-3, 2)$ to $A(-5, 3)$ is $\sqrt{(-5-(-3))^2+(3-2)^2}$

$$= \sqrt{(-2)^2+1^2}$$
$$= \sqrt{5}\text{ km}$$

Elizabeth can walk at 5 km h^{-1}, so it will take $\frac{\sqrt{5}}{5} \approx 0.447\,21$ hours, or about 26 minutes 50 seconds, for her to walk to taxi rank A.

We assume that she can walk in a straight line to A, and is walking at a constant speed of 5 km h^{-1}.

d **i** $(-2, -2)$ lies on the edge adjacent to cells B and D, so Albert is equally closest to taxi ranks B and D.

ii Albert might consider:

- whether he can walk there in a straight line
- the price
- his direction of travel once he gets into the taxi.

5 **a** $A(-6, -2)$, $B(-4, 2)$, $C(4, 4)$

The midpoint of [AB] is $\left(\frac{-6+-4}{2}, \frac{-2+2}{2}\right)$ or $(-5, 0)$.

The gradient of [AB] is $\frac{2--2}{-4-(-6)} = \frac{4}{2} = 2$.

So, PB(A, B) has gradient $-\frac{1}{2}$ and passes through $(-5, 0)$.

The midpoint of [AC] is $\left(\frac{-6+4}{2}, \frac{-2+4}{2}\right)$ or $(-1, 1)$.

The gradient of [AC] is $\frac{4--2}{4--6} = \frac{6}{10} = \frac{3}{5}$.

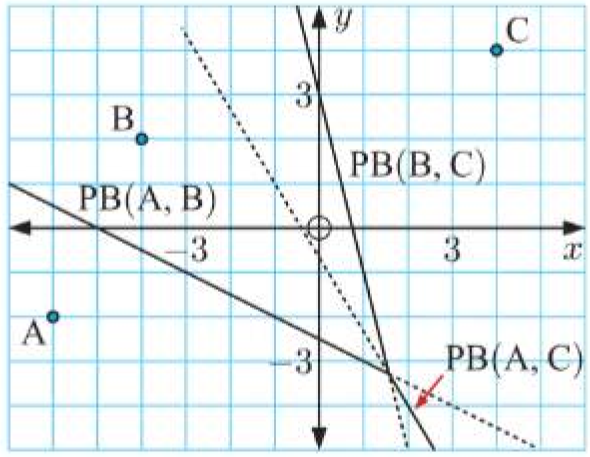
So, PB(A, C) has gradient $-\frac{5}{3}$ and passes through $(-1, 1)$.

The midpoint of [BC] is $\left(\frac{-4+4}{2}, \frac{2+4}{2}\right)$ or $(0, 3)$.

The gradient of [BC] is $\frac{4-2}{4--4} = \frac{2}{8} = \frac{1}{4}$.

So, PB(B, C) has gradient -4 and passes through $(0, 3)$.

We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



b **i** $(1, 2)$ lies in cell C, so it is closest to site C.

ii $(-1, -5)$ lies in cell A, so it is closest to site A.

iii $(-3, -1)$ lies on the edge adjacent to cells A and B, so it is equally closest to sites A and B.

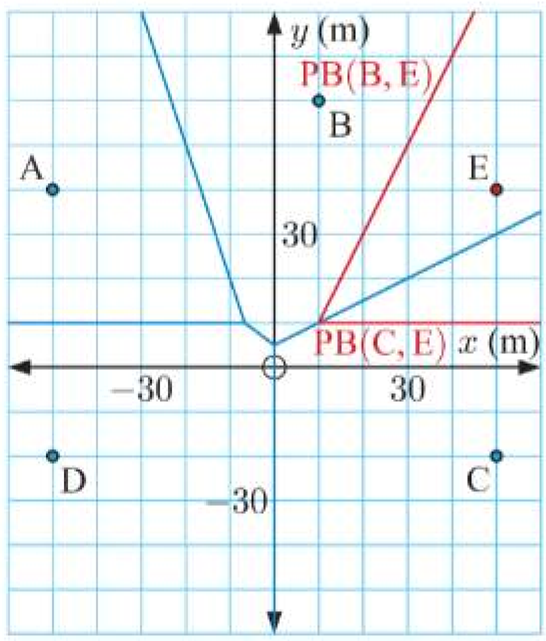
6 **a** $(20, 20)$ lies in cell B, so Boris is closest to Bin B.

The distance from $(20, 20)$ to $B(10, 60)$ is

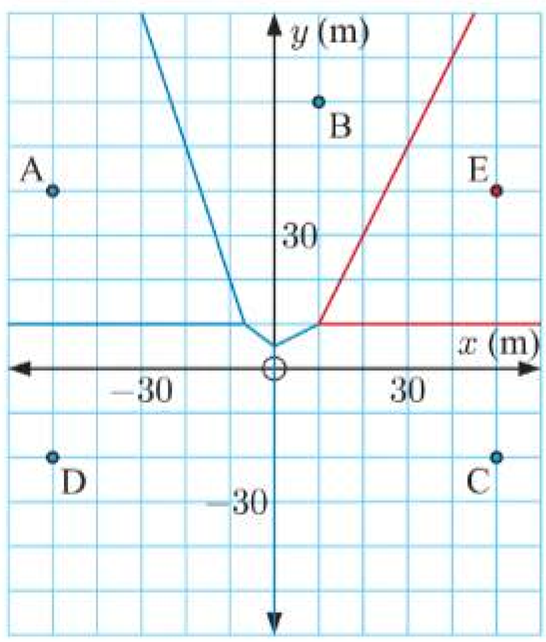
$$\sqrt{(10-20)^2+(60-20)^2}$$
$$=\sqrt{(-10)^2+40^2}$$
$$=\sqrt{1700}=10\sqrt{17}\approx 41.2 \text{ m}$$

So, Boris is about 41.2 m from the closest bin.

b **i** We construct PB(B, E) and PB(C, E) within the original cells B and E respectively.

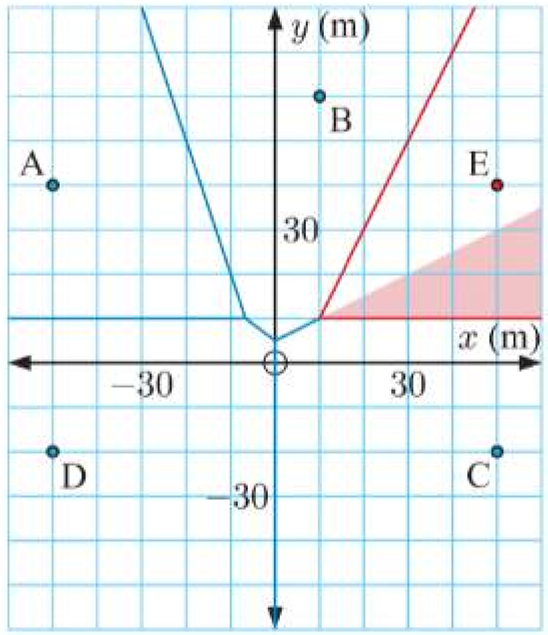


We then remove the segments of edges which now lie within cell E, giving us the Voronoi diagram which includes site E.



ii Cells B and C have been affected by the introduction of the new bin at E.

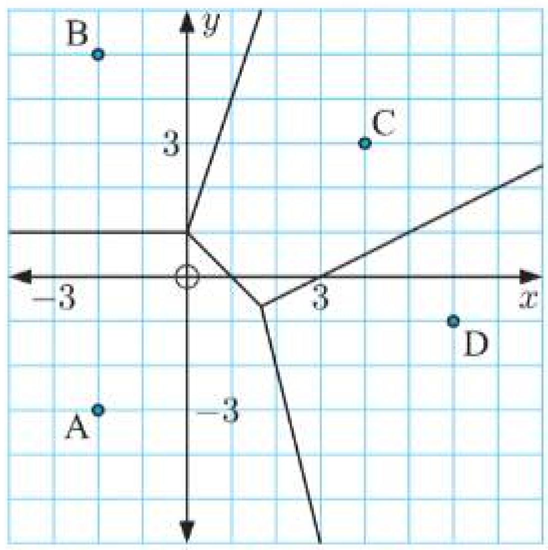
iii The intersection between the original cell C and the current cell E is shaded in red, which is the area of park which used to be closest to Bin C and is now closest to Bin E.



7

Location	*Depth* (m)
A(−2, −3)	7.5
B(−2, 5)	9.2
C(4, 3)	6.1
D(6, −1)	6.9

a We plot sites A, B, C, and D on a set of axes. We draw PB(A, B), PB(A, C), PB(A, D), PB(B, C), and PB(C, D), then remove segments which do not form part of the Voronoi diagram.



b **i** (−1, 3) is closest to B, so we estimate the depth of the lake to be 9.2 m at (−1, 3).

ii (4, 0) is closest to D, so we estimate the depth of the lake to be 6.9 m at (4, 0).

8 **a** **i** (−100, 200) lies in cell E, so it is closest to parking lot E.

ii (0, −400) lies in cell D, so it is closest to parking lot D.

b **i** The Voronoi diagram has vertices $V_1(-100, 300)$, $V_2(0, 200)$, and $V_3(-100, -100)$.

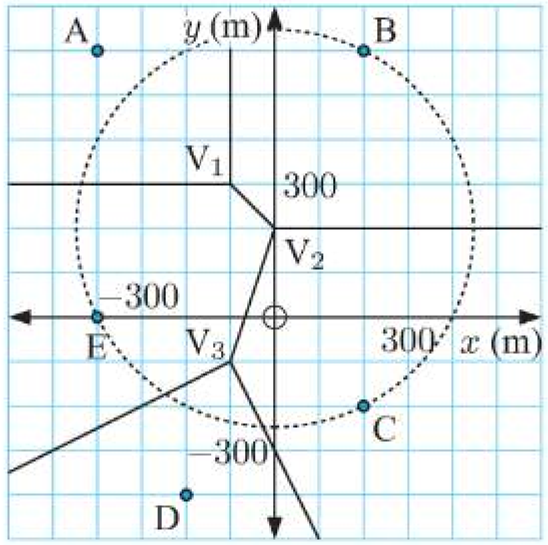
V_1 is adjacent to cells A, B, and E, V_2 is adjacent to cells B, C, and E, and V_3 is adjacent to cells C, D, and E.

E has coordinates (−400, 0).

$$V_1E = \sqrt{(-400-(-100))^2+(0-300)^2}$$
$$= \sqrt{(-300)^2+(-300)^2}$$
$$= \sqrt{180\,000} = 300\sqrt{2} \text{ m}$$

$$V_2E = \sqrt{(-400-0)^2+(0-200)^2}$$
$$= \sqrt{(-400)^2+(-200)^2}$$
$$= \sqrt{200\,000} = 200\sqrt{5} \text{ m}$$

$$V_3E = \sqrt{(-400-(-100))^2+(0-(-100))^2}$$
$$= \sqrt{(-300)^2+100^2}$$
$$= \sqrt{100\,000} = 100\sqrt{10} \text{ m}$$

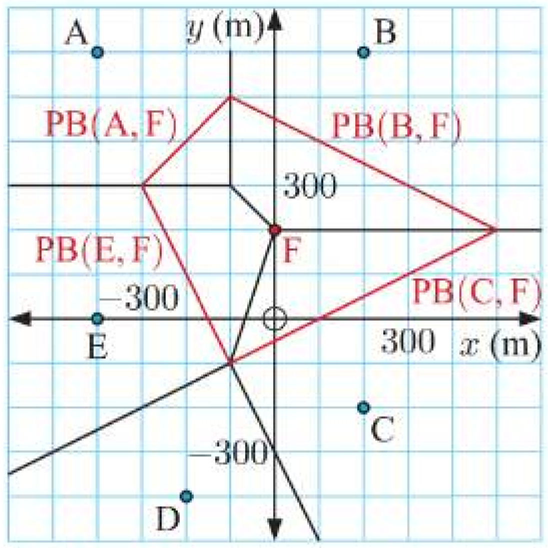


So, the largest empty circle has centre $V_2(0, 200)$ and radius $200\sqrt{5}$ m.

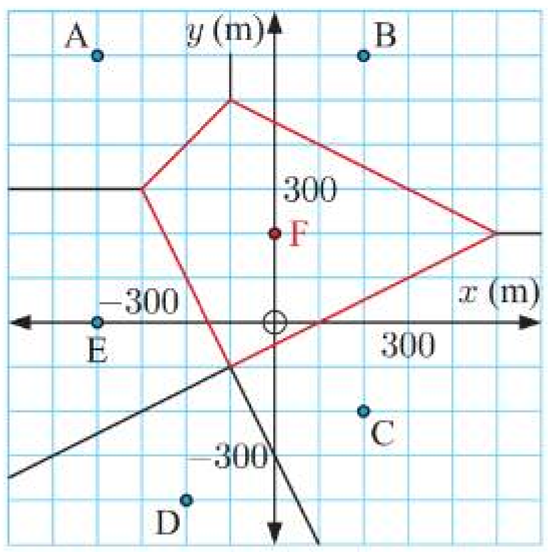
$\therefore$ the optimal position for the new parking lot F is $(0, 200)$.

ii From **i**, the new parking lot at $F(0, 200)$ is $200\sqrt{5} \approx 447$ m from the closest existing parking lots.

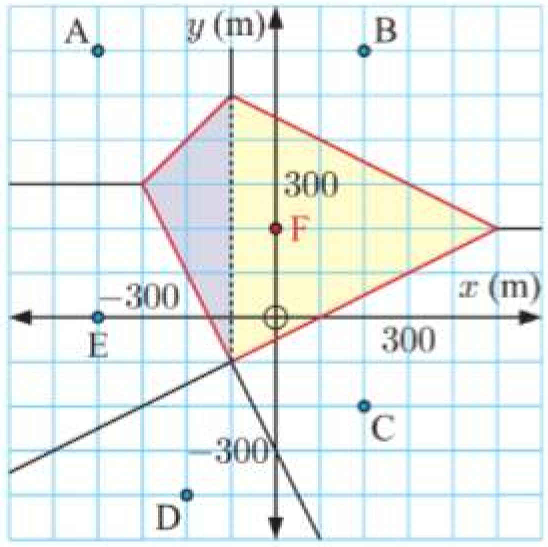
iii We construct PB(A, F), PB(B, F), PB(C, F), and PB(E, F) within the original cells A, B, C, and E respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell F, giving us the Voronoi diagram with the new parking lot added at F.



iv The region closest to the new parking lot is cell F, which can be divided into two triangles. One has base length 600 m and height 200 m, and the other has base length 600 m and height 600 m.



$\therefore$ area of cell F

$= \frac{1}{2} \times 600 \times 200 + \frac{1}{2} \times 600 \times 600$

$= 240\,000 \text{ m}^2$

c **i** From the diagram in **b iii**, we observe that $(200, 200)$ is in cell F.

$\therefore$ Joseph is closest to the new parking lot F.

ii The distance between $(200, 200)$ and $F(0, 200)$ is

$\sqrt{(0-200)^2 + (200-200)^2}$

$= \sqrt{(-200)^2 + 0^2}$

$= 200$ m

We observe that $(200, 200)$ was previously on the edge adjacent to cells B and C.

The distance between $(200, 200)$ and $B(200, 600)$ is

$\sqrt{(200-200)^2 + (600-200)^2}$

$= \sqrt{0^2 + 400^2}$

$= 400$ m

So, Joseph is now $400 - 200 = 200$ m closer to a parking lot.

Chapter 17

INTRODUCTION TO DIFFERENTIAL CALCULUS

EXERCISE 17A.1

1 **a**

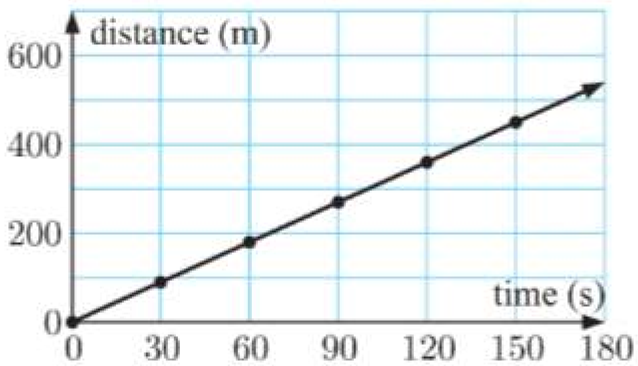
Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

+90 +90 +90 +90 +90

The distance travelled increases by the same amount each time interval.

$\therefore$ the jogger is travelling at a constant speed.

b

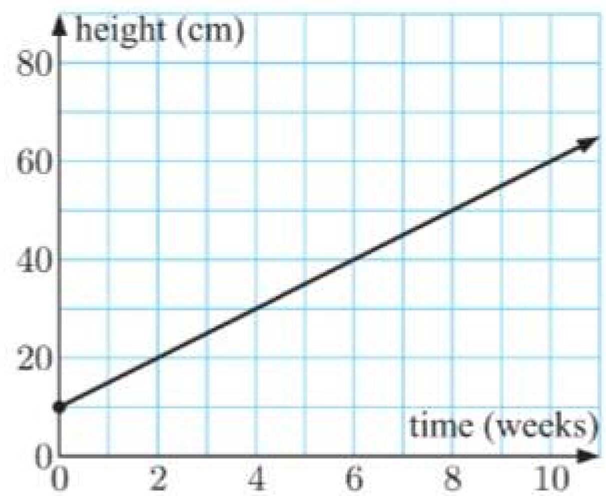


c speed $= \dfrac{(90-0)\text{ m}}{(30-0)\text{ s}}$

$= 3$ m per s

2 **a** The graph of height against time is a straight line.

$\therefore$ the rate of change in height is constant.

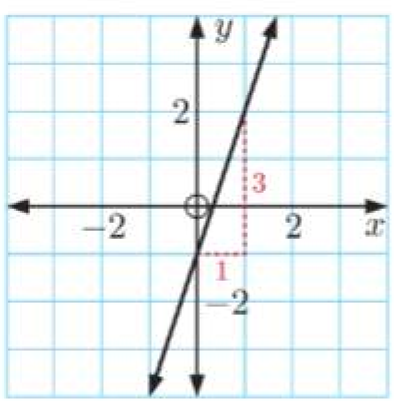


b rate of change $= \dfrac{(60-10)\text{ cm}}{(10-0)\text{ weeks}}$

$= \dfrac{50\text{ cm}}{10\text{ weeks}}$

$= 5$ cm per week

3 **a**

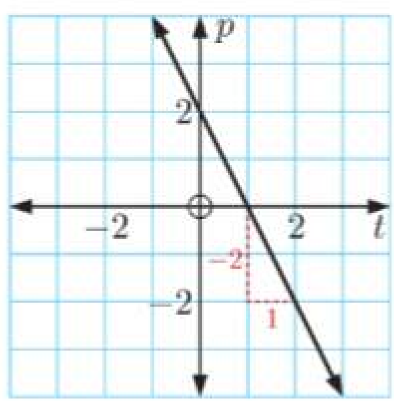


rate of change = gradient of line

$= \dfrac{3}{1}$

$= 3$

b

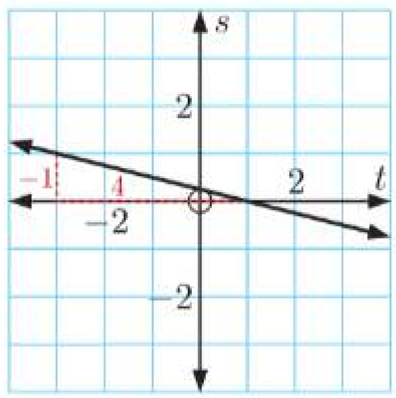


rate of change = gradient of line

$= \dfrac{-2}{1}$

$= -2$

c



rate of change = gradient of line

$= \frac{-1}{4}$

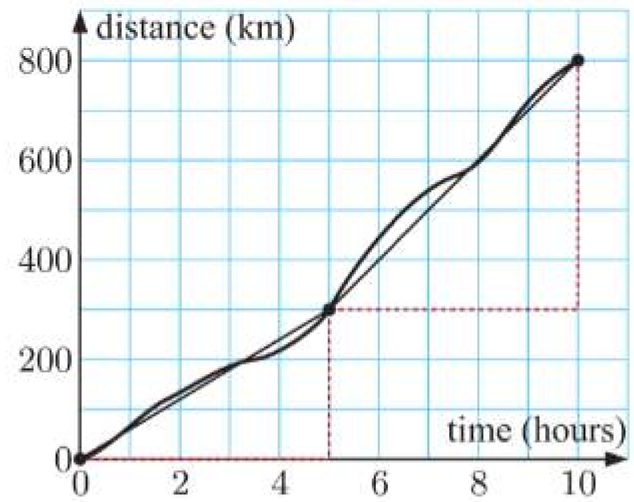
$= -\frac{1}{4}$

4 For the function $f(x) = \frac{5}{2}x - 3$, the gradient is $\frac{5}{2}$, so the rate of change is $\frac{5}{2}$.

EXERCISE 17A.2

1 **a** The graph of distance against time is not a straight line.

$\therefore$ Aileen did not travel at a constant speed.



b **i** average speed from $t = 0$ to $t = 5$ h

$= \frac{(300 - 0) \text{ km}}{(5 - 0) \text{ h}}$

$= 60$ km per hour

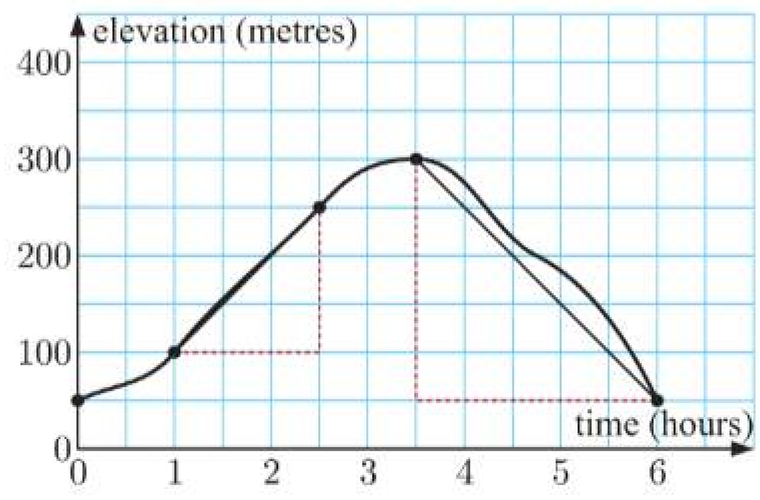
ii average speed from $t = 5$ h to $t = 10$ h

$= \frac{(800 - 300) \text{ km}}{(10 - 5) \text{ h}}$

$= \frac{500}{5}$ km per hour

$= 100$ km per hour

2 **a** average rate of change from $t = 1$ h to $t = 2.5$ h



$= \frac{(250 - 100) \text{ m}}{(2.5 - 1) \text{ h}}$

$= \frac{150}{1.5}$ m per hour

$= 100$ m per hour

b average rate of change from $t = 3.5$ h to $t = 6$ h

$= \frac{(50 - 300) \text{ m}}{(6 - 3.5) \text{ h}}$

$= \frac{-250}{2.5}$ m per hour

$= -100$ m per hour

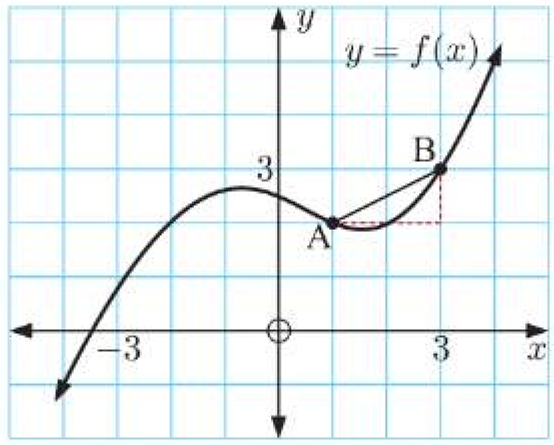
$= 100$ m per hour (downwards)

3 **a** average rate of change in $f(x)$ from A to B

$= \dfrac{f(b)-f(a)}{b-a}$

$= \dfrac{3-2}{3-1}$

$= \frac{1}{2}$

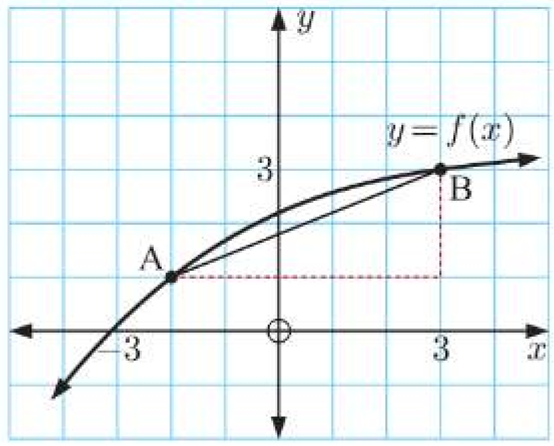


b average rate of change in $f(x)$ from A to B

$= \dfrac{f(b)-f(a)}{b-a}$

$= \dfrac{3-1}{3-(-2)}$

$= \frac{2}{5}$

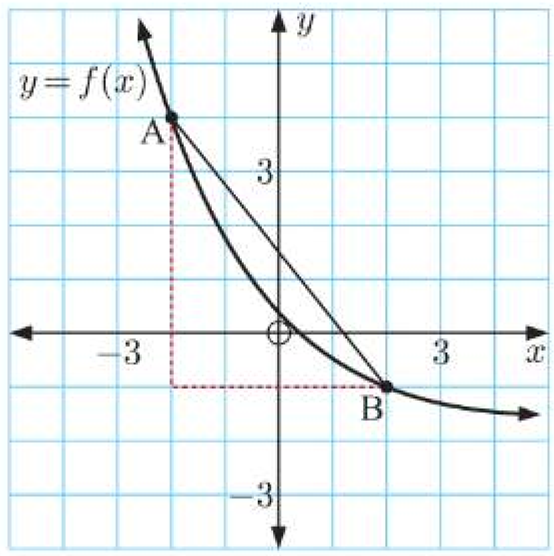


c average rate of change in $f(x)$ from A to B

$= \dfrac{f(b)-f(a)}{b-a}$

$= \dfrac{-1-4}{2-(-2)}$

$= -\frac{5}{4}$



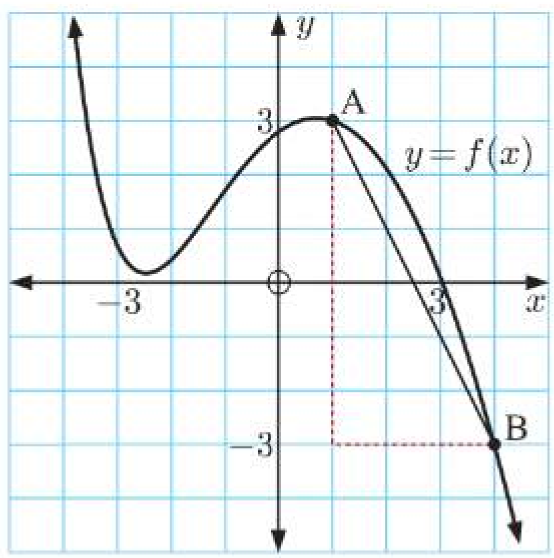
d average rate of change in $f(x)$ from A to B

$= \dfrac{f(b)-f(a)}{b-a}$

$= \dfrac{-3-3}{4-1}$

$= \dfrac{-6}{3}$

$= -2$

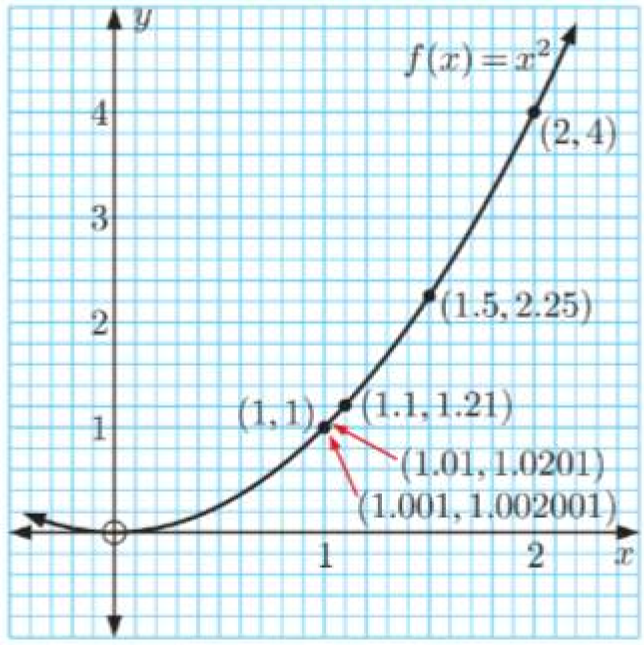


4 **a** **i** average rate of change in $f(x)$ from $x = 1$ to $x = 2$

$= \frac{f(2) - f(1)}{2 - 1}$

$= \frac{4 - 1}{1}$

$= 3$



ii average rate of change in $f(x)$ from $x = 1$ to $x = 1.5$

$= \frac{f(1.5) - f(1)}{1.5 - 1}$

$= \frac{2.25 - 1}{0.5}$

$= 2.5$

iii average rate of change in $f(x)$ from $x = 1$ to $x = 1.1$

$= \frac{f(1.1) - f(1)}{1.1 - 1}$

$= \frac{1.21 - 1}{0.1}$

$= 2.1$

iv average rate of change in $f(x)$ from $x = 1$ to $x = 1.01$

$= \frac{f(1.01) - f(1)}{1.01 - 1}$

$= \frac{1.0201 - 1}{0.01}$

$= 2.01$

v average rate of change in $f(x)$ from $x = 1$ to $x = 1.001$

$= \frac{f(1.001) - f(1)}{1.001 - 1}$

$= \frac{1.002\,001 - 1}{0.001}$

$= 2.001$

b The average rate of change approaches 2.

INVESTIGATION 1 INSTANTANEOUS SPEED

2

t	gradient of [FM]
4	30
3	25
2.5	22.5
2.1	20.5
2.01	20.05

3 As M *approaches* F, the gradient of [FM] approaches 20. However, when M *reaches* F, the gradient is undefined since we cannot divide by zero.

4 As t approaches 2 from the right, the gradient of [FM] approaches 20.
We suspect that the instantaneous speed of the ball bearing when $t = 2$ seconds is $20\ \text{m s}^{-1}$.

5

t	gradient of [FM]
0	10
1.5	17.5
1.9	19.5
1.99	19.95

6 As t approaches 2 from the left, the gradient of [FM] approaches 20.

The instantaneous speed of the ball bearing when $t = 2$ seconds appears to be 20 m s^{-1}, which agrees with our result in **4**.

EXERCISE 17B

1 **a** The tangent at A has gradient

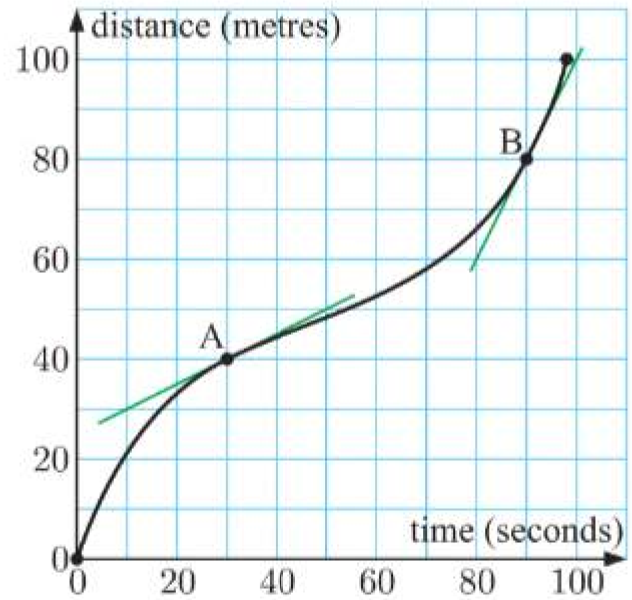
$\frac{50-40}{50-30} = \frac{10}{20} = \frac{1}{2}$.

$\therefore$ the swimmer's instantaneous speed after 30 seconds is 0.5 m s^{-1}.

b The tangent at B has gradient

$\frac{80-60}{90-80} = \frac{20}{10} = 2$.

$\therefore$ the swimmer's instantaneous speed after 90 seconds is 2 m s^{-1}.



2 **a** Initially at time 0 hours, the volume is 8 thousands of litres.
So, there were 8000 L in the tank originally.

b At time 1 hour, the volume is 3 thousands of litres.
So, after 1 hour there were 3000 L in the tank.

c The tangent at time $t = 0$ hours passes through $(0, 8)$ and $(0.75, 0)$.

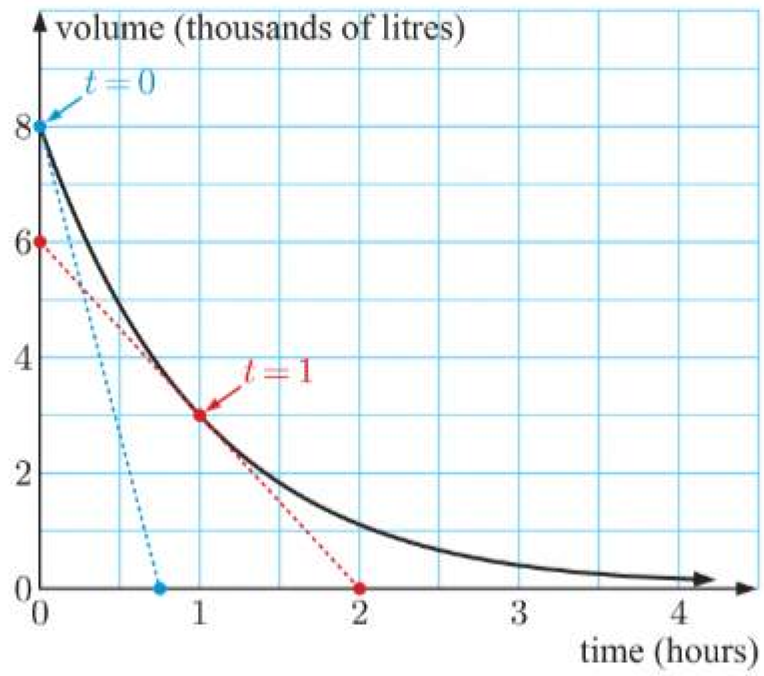
$\therefore$ initial rate of water loss

$=$ gradient of tangent at time 0 hours

$\approx \frac{(0-8) \text{ thousand L}}{(0.75-0) \text{ hours}}$

≈ -10.667 thousand L per hour

$\therefore$ initially the rate of water loss was about 10 700 L per hour.



d The tangent at time $t = 1$ passes through $(0, 6)$ and $(2, 0)$.

$\therefore$ rate of water loss after 1 hour = gradient of tangent at time 1 hour

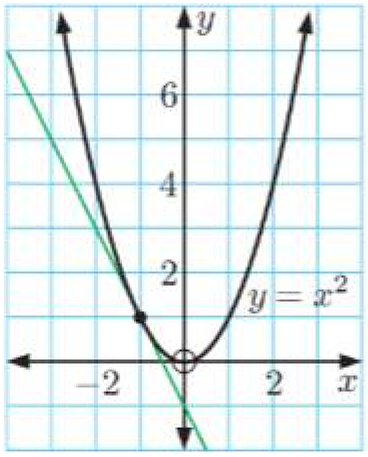
$$\approx \frac{(0-6) \text{ thousand L}}{(2-0) \text{ hours}}$$

$$\approx -3 \text{ thousand L per hour}$$

$\therefore$ after 1 hour the rate of water loss was about 3000 L per hour.

e The rate at which the tank is leaking water is decreasing.

3 **a, b**



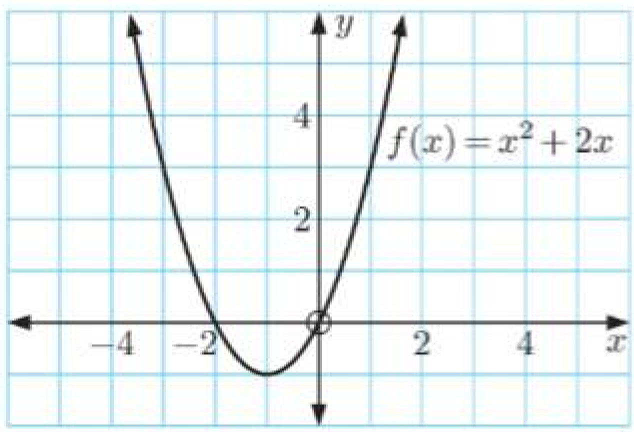
c The tangent at $x = -1$ has gradient $\dfrac{1-(-1)}{-1-0} = -2$.

$\therefore$ the instantaneous rate of change in $y = x^2$ when $x = -1$ is -2.

EXERCISE 17C.1

1 **a** From the graph, we see that

as $x \to 1^-$, $f(x) \to 3^-$,
and as $x \to 1^+$, $f(x) \to 3^+$.



b As $x \to 1$ from either direction, $f(x) \to 3$.

$\therefore \lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (x^2 + 2x) = 3$

c **i** $x^2 + 2x$ can be made as close as we like to 0 by making x sufficiently close to -2.

$\therefore \lim\limits_{x \to -2} (x^2 + 2x) = 0$

ii $x^2 + 2x$ can be made as close as we like to -1 by making x sufficiently close to -1.

$\therefore \lim\limits_{x \to -1} (x^2 + 2x) = -1$

2 **a** As $x \to 3$, $x + 4 \to 7$

$\therefore \lim\limits_{x \to 3} (x + 4) = 7$

b As $x \to -1$, $5 - 2x \to 7$

$\therefore \lim\limits_{x \to -1} (5 - 2x) = 7$

c As $x \to 4$, $3x - 1 \to 11$

$\therefore \lim\limits_{x \to 4} (3x - 1) = 11$

d As $x \to 2$, $5x^2 - 3x + 2 \to 5(4) - 3(2) + 2 = 16$

$\therefore \lim\limits_{x \to 2} (5x^2 - 3x + 2) = 16$

e As $h \to 0$, $h^2 \to 0$ and $1 - h \to 1$

$\therefore \lim\limits_{h \to 0} h^2(1 - h) = 0 \times 1 = 0$

f As $x \to 0$, $x^2 + 5 \to 5$

$\therefore \lim\limits_{x \to 0} (x^2 + 5) = 5$

3 **a** $\lim_{x\to 0} 5 = 5$ **b** $\lim_{h\to 2} 7 = 7$ **c** $\lim_{x\to 0} c = c$ (when c is a constant)

4 **a** $\frac{x^2-3x}{x}$ can be made as close as we like to -2 by making x sufficiently close to 1.

$\therefore \lim_{x\to 1} \frac{x^2-3x}{x} = -2$

b $\frac{h^2+5h}{h}$ can be made as close as we like to 7 by making h sufficiently close to 2.

$\therefore \lim_{h\to 2} \frac{h^2+5h}{h} = 7$

c $\frac{x-1}{x+1}$ can be made as close as we like to -1 by making x sufficiently close to 0.

$\therefore \lim_{x\to 0} \frac{x-1}{x+1} = -1$

5 $\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1$ {since $x \neq 0$}
$= 1$

6 **a**

x	$\frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.999 99	3.999 99

x	$\frac{x^2-4}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.000 01	4.000 01

b $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

c $\lim_{x\to 2} \frac{x^2-4}{x-2}$
$= \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$
$= \lim_{x\to 2} (x+2)$ {since $x \neq 2$}
$= 4$

7 **a** $\lim_{x\to 0} \frac{x^2-3x}{x}$
$= \lim_{x\to 0} \frac{\cancel{x}(x-3)}{\cancel{x}}$
$= \lim_{x\to 0} (x-3)$ {since $x \neq 0$}
$= -3$

b $\lim_{x\to 0} \frac{x^2+5x}{x}$
$= \lim_{x\to 0} \frac{\cancel{x}(x+5)}{\cancel{x}}$
$= \lim_{x\to 0} (x+5)$ {since $x \neq 0$}
$= 5$

c $\lim_{x\to 0} \frac{2x^2-x}{x}$
$= \lim_{x\to 0} \frac{\cancel{x}(2x-1)}{\cancel{x}}$
$= \lim_{x\to 0} (2x-1)$ {since $x \neq 0$}
$= -1$

d $\lim_{h\to 0} \frac{2h^2+6h}{h}$
$= \lim_{h\to 0} \frac{2\cancel{h}(h+3)}{\cancel{h}}$
$= \lim_{h\to 0} 2(h+3)$ {since $h \neq 0$}
$= 6$

e $\lim_{h\to 0} \frac{3h^2-4h}{h}$
$= \lim_{h\to 0} \frac{\cancel{h}(3h-4)}{\cancel{h}}$
$= \lim_{h\to 0} (3h-4)$ {since $h \neq 0$}
$= -4$

f $\lim_{h\to 0} \frac{h^3-8h}{h}$
$= \lim_{h\to 0} \frac{\cancel{h}(h^2-8)}{\cancel{h}}$
$= \lim_{h\to 0} (h^2-8)$ {since $h \neq 0$}
$= -8$

g $\lim\limits_{x\to1}\dfrac{x^2-x}{x-1}$

$=\lim\limits_{x\to1}\dfrac{x\cancel{(x-1)}}{\cancel{x-1}}$

$=\lim\limits_{x\to1} x \quad \{\text{since } x\neq1\}$

$=1$

h $\lim\limits_{x\to2}\dfrac{x^2-2x}{x-2}$

$=\lim\limits_{x\to2}\dfrac{x\cancel{(x-2)}}{\cancel{x-2}}$

$=\lim\limits_{x\to2} x \quad \{\text{since } x\neq2\}$

$=2$

i $\lim\limits_{x\to3}\dfrac{x^2-x-6}{x-3}$

$=\lim\limits_{x\to3}\dfrac{(x+2)\cancel{(x-3)}}{\cancel{x-3}}$

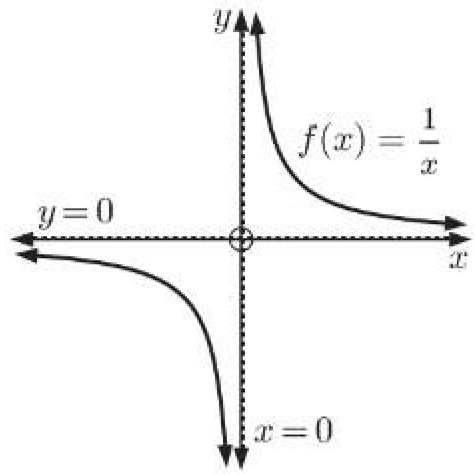
$=\lim\limits_{x\to3}(x+2) \quad \{\text{since } x\neq3\}$

$=5$

EXERCISE 17C.2

1 **a** $f(x)=\dfrac{1}{x}$

i As $x\to0^-$, $f(x)\to-\infty$
As $x\to0^+$, $f(x)\to\infty$
As $x\to-\infty$, $f(x)\to0^-$
As $x\to\infty$, $f(x)\to0^+$
The vertical asymptote is $x=0$.
The horizontal asymptote is $y=0$.

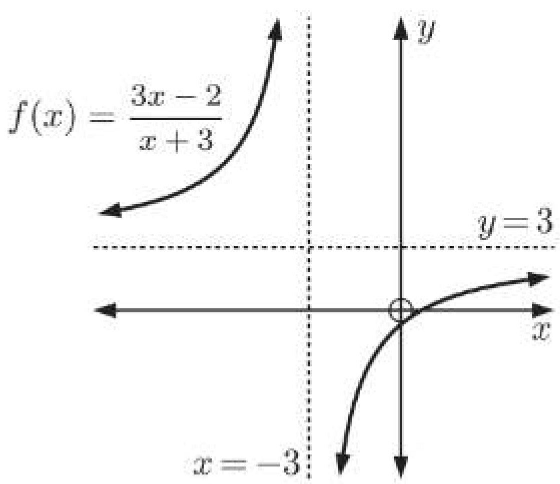


ii $\lim\limits_{x\to-\infty} f(x)=0$,

$\lim\limits_{x\to\infty} f(x)=0$

b $f(x)=\dfrac{3x-2}{x+3}$

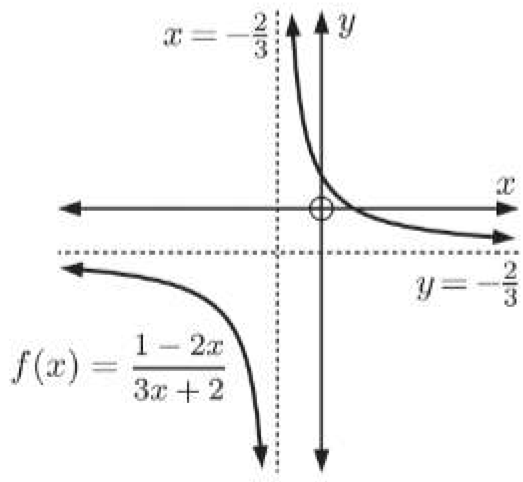
i As $x\to-3^-$, $f(x)\to\infty$
As $x\to-3^+$, $f(x)\to-\infty$
As $x\to-\infty$, $f(x)\to3^+$
As $x\to\infty$, $f(x)\to3^-$
The vertical asymptote is $x=-3$.
The horizontal asymptote is $y=3$.



ii $\lim\limits_{x\to-\infty} f(x)=3$,

$\lim\limits_{x\to\infty} f(x)=3$

c $f(x) = \dfrac{1-2x}{3x+2}$



i As $x \to -\frac{2}{3}^-$, $f(x) \to -\infty$

As $x \to -\frac{2}{3}^+$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to -\frac{2}{3}^-$

As $x \to \infty$, $f(x) \to -\frac{2}{3}^+$

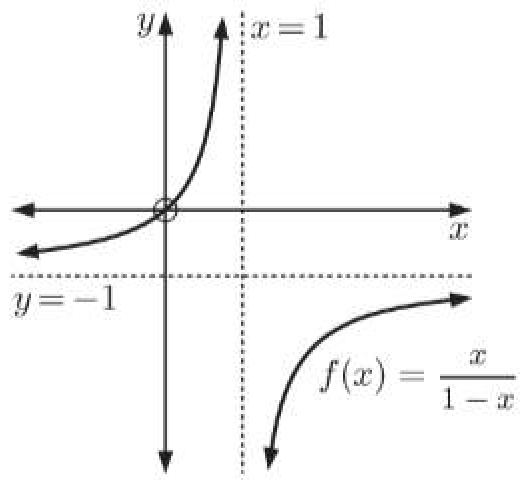
The vertical asymptote is $x = -\frac{2}{3}$.

The horizontal asymptote is $y = -\frac{2}{3}$.

ii $\lim\limits_{x \to -\infty} f(x) = -\frac{2}{3}$,

$\lim\limits_{x \to \infty} f(x) = -\frac{2}{3}$

d $f(x) = \dfrac{x}{1-x}$



i As $x \to 1^-$, $f(x) \to \infty$

As $x \to 1^+$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to -1^+$

As $x \to \infty$, $f(x) \to -1^-$

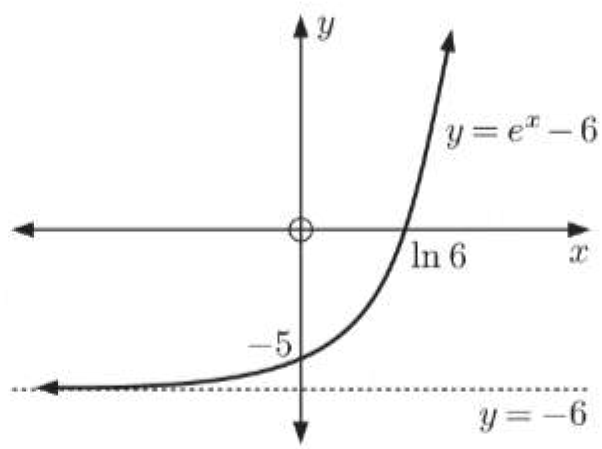
The vertical asymptote is $x = 1$.

The horizontal asymptote is $y = -1$.

ii $\lim\limits_{x \to -\infty} f(x) = -1$,

$\lim\limits_{x \to \infty} f(x) = -1$

2 **a**



b **i** As $x \to -\infty$, $e^x - 6 \to -6^+$

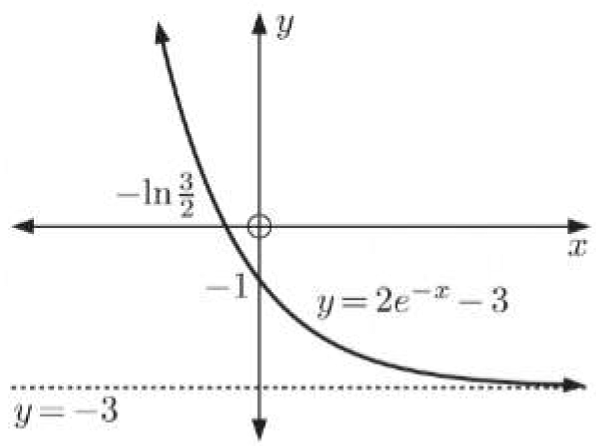
$\therefore \lim\limits_{x \to -\infty} (e^x - 6) = -6$

$\therefore$ the function has horizontal asymptote $y = -6$.

ii As $x \to \infty$, $e^x - 6 \to \infty$

$\therefore \lim\limits_{x \to \infty} (e^x - 6)$ does not exist.

3 We sketch the graph of $y = 2e^{-x} - 3$:



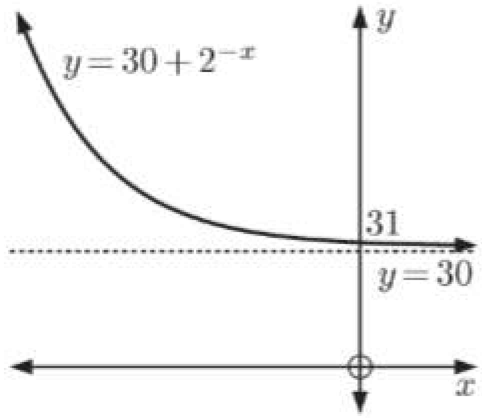
a As $x \to -\infty$, $2e^{-x} - 3 \to \infty$

$\therefore \lim\limits_{x \to -\infty} (2e^{-x} - 3)$ does not exist.

b As $x \to \infty$, $2e^{-x} - 3 \to -3^+$

$\therefore \lim\limits_{x \to \infty} (2e^{-x} - 3) = -3$.

We sketch the graph of $y = 30 + 2^{-x}$:



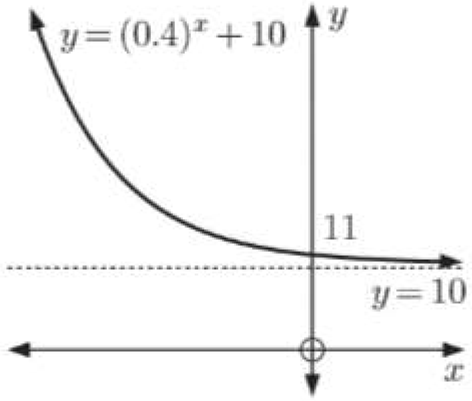
c As $x \to -\infty$, $30 + 2^{-x} \to \infty$

$\therefore \lim\limits_{x \to -\infty} (30 + 2^{-x})$ does not exist.

d As $x \to \infty$, $30 + 2^{-x} \to 30^+$

$\therefore \lim\limits_{x \to \infty} (30 + 2^{-x}) = 30.$

We sketch the graph of $y = (0.4)^x + 10$:



e As $x \to -\infty$, $(0.4)^x + 10 \to \infty$

$\therefore \lim\limits_{x \to -\infty} ((0.4)^x + 10)$ does not exist.

f As $x \to \infty$, $(0.4)^x + 10 \to 10^+$

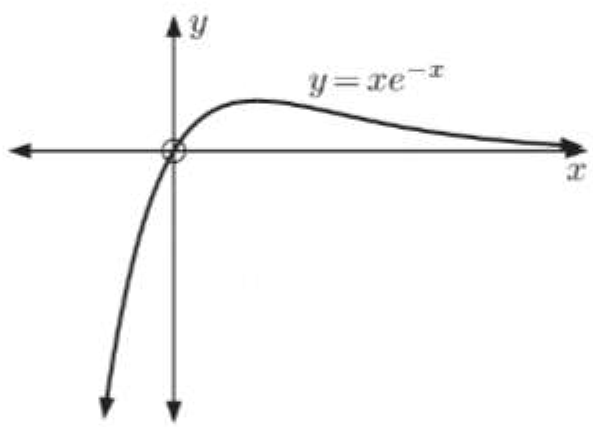
$\therefore \lim\limits_{x \to \infty} ((0.4)^x + 10) = 10.$

4 **a**

x	xe^{-x}
10	$\approx 0.000\,454$
50	$\approx 9.64 \times 10^{-21}$
100	$\approx 3.72 \times 10^{-42}$
200	$\approx 2.77 \times 10^{-85}$

b We predict that $\lim\limits_{x \to \infty} xe^{-x} = 0$.

c



As $x \to \infty$, $y \to 0^+$

The graph supports our prediction in **b**.

INVESTIGATION 2 LIMITS IN NUMBER SEQUENCES

$x_n = 0.333....3$ where there are n 3s after the decimal point, $n \in \mathbb{Z}^+$.

1

n	x_n	$3x_n$	$1 - 3x_n$
1	0.3	0.9	0.1
2	0.33	0.99	0.01
3	0.333	0.999	0.001
4	0.3333	0.9999	0.0001
5	0.333 33	0.999 99	0.000 01
10	0.333 333 333 3	0.999 999 999 9	0.000 000 000 1

2 $(1-3x_{100})$ has 99 0s between the decimal point and the 1.

3 $(1-3x_n)$ has $(n-1)$ 0s between the decimal point and the 1.

4 $\lim\limits_{n\to\infty}(1-3x_n)=0$ since the number of 0s between the decimal point and the 1 approaches infinity, and the number approaches zero.

5
$$\begin{aligned}\lim_{n\to\infty}(1-3x_n)&=0\\ \therefore\ \lim_{n\to\infty}1-\lim_{n\to\infty}3x_n&=0\\ \therefore\ 1-3\lim_{n\to\infty}x_n&=0\\ \therefore\ 3\lim_{n\to\infty}x_n&=1\\ \therefore\ \lim_{n\to\infty}x_n&=\tfrac{1}{3}\end{aligned}$$

EXERCISE 17D

1 **a** M has x-coordinate $3+h$ and lies on the graph of $f(x)=x^2$.
$\therefore$ its y-coordinate is $(3+h)^2$.

b The gradient of [FM] $=\dfrac{y_M-y_F}{x_M-x_F}$

$$\begin{aligned}&=\frac{(3+h)^2-9}{(3+h)-3}\\ &=\frac{\cancel{9}+6h+h^2-\cancel{9}}{h}\\ &=\frac{6h+h^2}{h}\\ &=\frac{\cancel{h}(6+h)}{\cancel{h}}\\ &=6+h \quad \text{provided } h\neq 0\end{aligned}$$

c **i** M has x-coordinate $3+h$.
$\therefore$ at the point $(4, 16)$, $3+h=4$
$\therefore$ $h=1$
The gradient of [FM] is $6+h$. {from **b**}
$\therefore$ the gradient of [FM] at $(4, 16)$ is $6+1=7$.

ii M has x-coordinate $3+h$.
$\therefore$ at the point $(3.5, 12.25)$,
$3+h=3.5$
$\therefore$ $h=0.5$
The gradient of [FM] is $6+h$. {from **b**}
$\therefore$ the gradient of [FM] at $(3.5, 12.25)$ is $6+0.5=6.5$.

iii M has x-coordinate $3+h$.
$\therefore$ at the point $(3.1, 9.61)$,
$3+h=3.1$
$\therefore$ $h=0.1$
The gradient of [FM] is $6+h$. {from **b**}
$\therefore$ the gradient of [FM] at $(3.1, 9.61)$ is $6+0.1=6.1$.

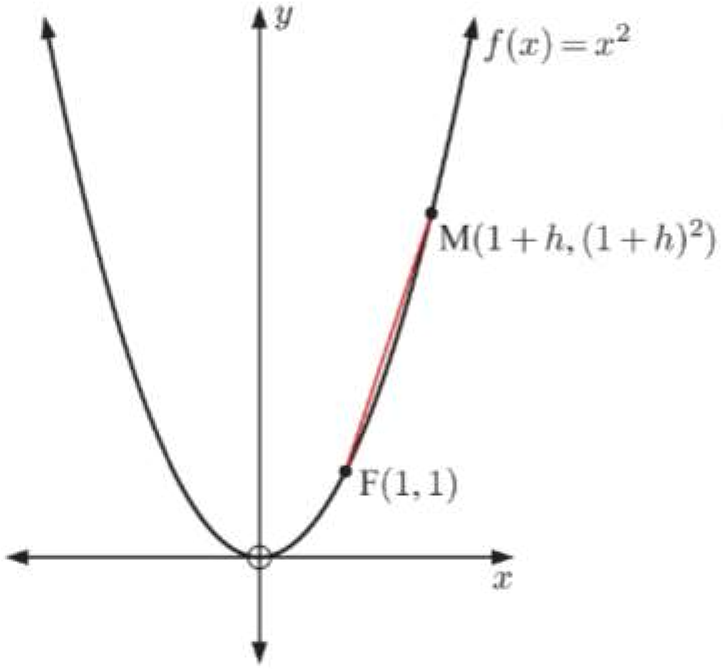
iv M has x-coordinate $3+h$.
$\therefore$ at the point $(3.01, 9.0601)$,
$3+h=3.01$
$\therefore$ $h=0.01$
The gradient of [FM] is $6+h$. {from **b**}
$\therefore$ the gradient of [FM] at $(3.01, 9.0601)$ is $6+0.01=6.01$.

d Using limit theory, the gradient of the tangent to $f(x) = x^2$ at the point $(3, 9)$ is

$$\begin{aligned}\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\ &= \lim_{h \to 0} \frac{6h + h^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\ &= \lim_{h \to 0} (6+h) \qquad \{\text{as } h \neq 0\} \\ &= 6\end{aligned}$$

2 **a** **i** At $x = 1$, $f(1) = 1^2 = 1$.

Let F be the point $(1, 1)$ and M have x-coordinate $1 + h$, so M is $(1+h, (1+h)^2)$.

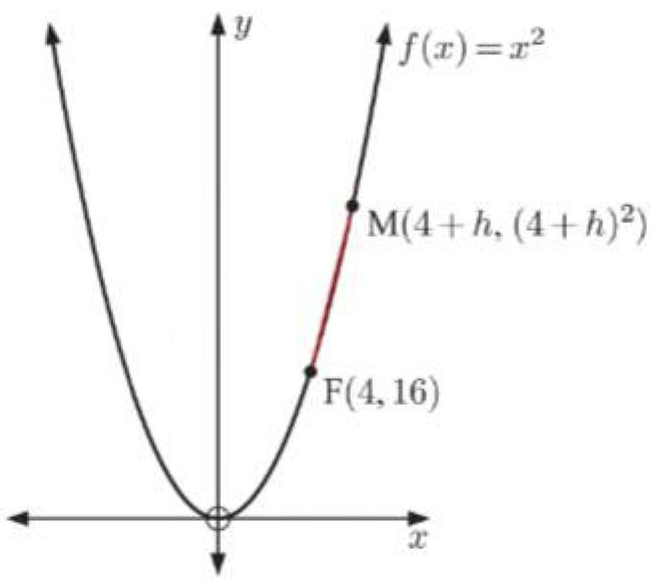


The gradient of the tangent at F

$$\begin{aligned}&= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} \\ &= \lim_{h \to 0} (2+h) \qquad \{\text{as } h \neq 0\} \\ &= 2\end{aligned}$$

ii At $x = 4$, $f(4) = 4^2 = 16$.

Let F be the point $(4, 16)$ and M have x-coordinate $4 + h$, so M is $(4+h, (4+h)^2)$.



The gradient of the tangent at F

$$\begin{aligned}&= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h \to 0} \frac{(4+h)^2 - 4^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{16} + 8h + h^2 - \cancel{16}}{h} \\ &= \lim_{h \to 0} \frac{8h + h^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(8+h)}{\cancel{h}} \\ &= \lim_{h \to 0} (8+h) \qquad \{\text{as } h \neq 0\} \\ &= 8\end{aligned}$$

b The gradient of the tangent to $f(x) = x^2$ at the point where $x = 2$

$$= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$
$$= \lim_{h\to 0} \frac{4h + h^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (4+h) \qquad \{\text{as } h \neq 0\}$$
$$= 4$$

The gradient of the tangent to $f(x) = x^2$ at the point where $x = 3$

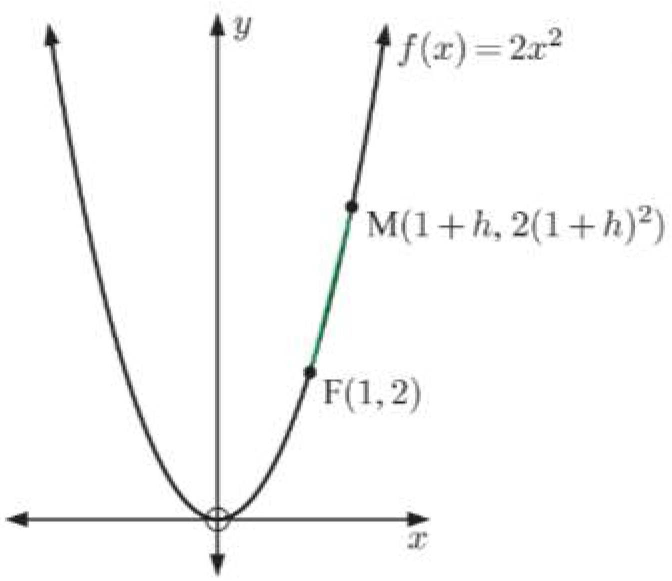
$$= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h}$$
$$= \lim_{h\to 0} \frac{6h + h^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (6+h) \qquad \{\text{as } h \neq 0\}$$
$$= 6$$

Using the results from **a**, the table is:

x-coordinate	*Gradient of tangent to $f(x) = x^2$*
1	2
2	4
3	6
4	8

c The gradient of the tangent is equal to twice the x-coordinate in each case in **b**. So, we predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$ will be $2a$.

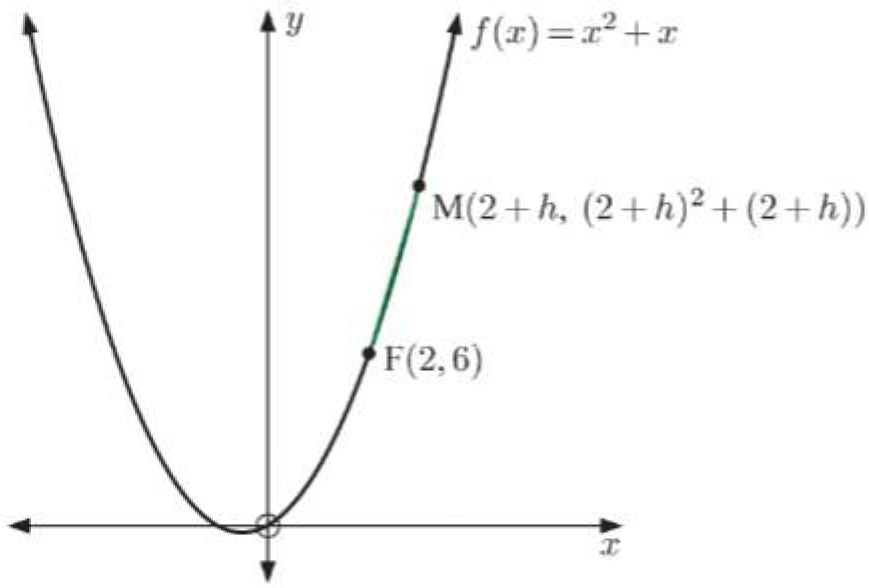
3 **a** Let F be the point $(1, 2)$ and M have x-coordinate $1 + h$, so M is $(1 + h, 2(1 + h)^2)$.



The gradient of the tangent at F

$$= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{2(1+h)^2 - 2(1)^2}{h}$$
$$= \lim_{h\to 0} \frac{2(1+2h+h^2) - 2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{2} + 4h + 2h^2 - \cancel{2}}{h}$$
$$= \lim_{h\to 0} \frac{4h + 2h^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(4+2h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (4+2h) \qquad \{\text{as } h \neq 0\}$$
$$= 4$$

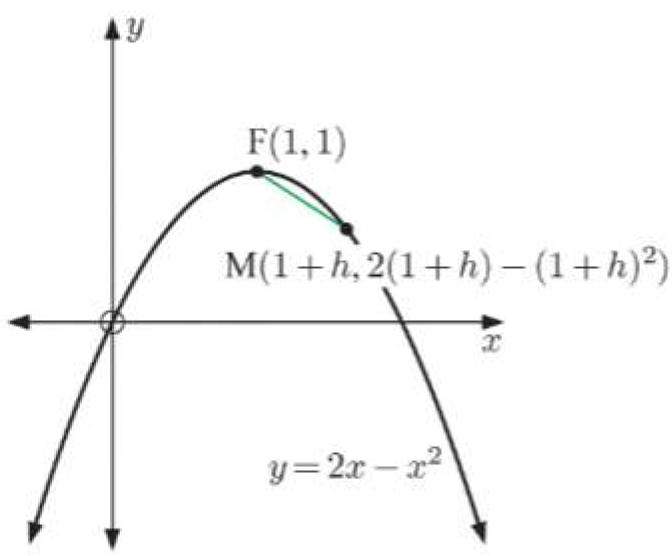
b Let F be the point $(2, 6)$ and M have x-coordinate $2+h$,
so M is $(2+h,\ (2+h)^2+(2+h))$.



The gradient of the tangent at F

$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{(2+h)^2+(2+h)-(2^2+2)}{h}$$
$$= \lim_{h\to 0} \frac{4+4h+h^2+2+h-6}{h}$$
$$= \lim_{h\to 0} \frac{5h+h^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(5+h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (5+h) \qquad \{\text{as } h \neq 0\}$$
$$= 5$$

c At $x=1$, $f(1)=2(1)-1^2=1$.
Let F be the point $(1, 1)$ and M have x-coordinate $1+h$,
so M is $\left(1+h,\ 2(1+h)-(1+h)^2\right)$.

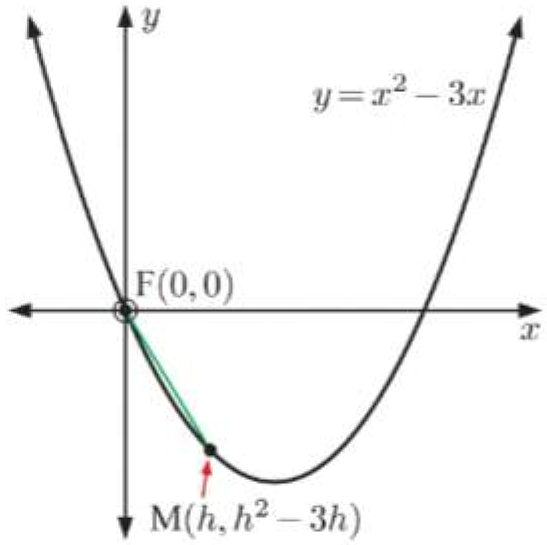


The gradient of the tangent at F

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0} \frac{2(1+h)-(1+h)^2-1}{h}$$
$$= \lim_{h\to 0} \frac{2+2h-(1+2h+h^2)-1}{h}$$
$$= \lim_{h\to 0} \frac{2+2h-1-2h-h^2-1}{h}$$
$$= \lim_{h\to 0} \frac{-h^{\cancel{2}}}{\cancel{h}}$$
$$= \lim_{h\to 0} -h \qquad \{\text{as } h \neq 0\}$$
$$= 0$$

d At $x = 0$, $f(0) = 0^2 - 3(0) = 0$.

Let F be the point $(0, 0)$ and M have x-coordinate $0 + h$, so M is $\left(h, \ h^2 - 3h\right)$.



The gradient of the tangent at F

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 3h - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h-3)}{\cancel{h}}$$

$$= \lim_{h \to 0} (h - 3) \qquad \{\text{as } h \neq 0\}$$

$$= -3$$

4 **a** $(1+h)^3 = (1+h)^2(1+h)$

$$= (1 + 2h + h^2)(1+h)$$

$$= 1 + h + 2h + 2h^2 + h^2 + h^3$$

$$= 1 + 3h + 3h^2 + h^3$$

b **i** M has x-coordinate $1 + h$ and lies on the graph of $y = x^3$, so M is $(1+h, \ (1+h)^3)$.

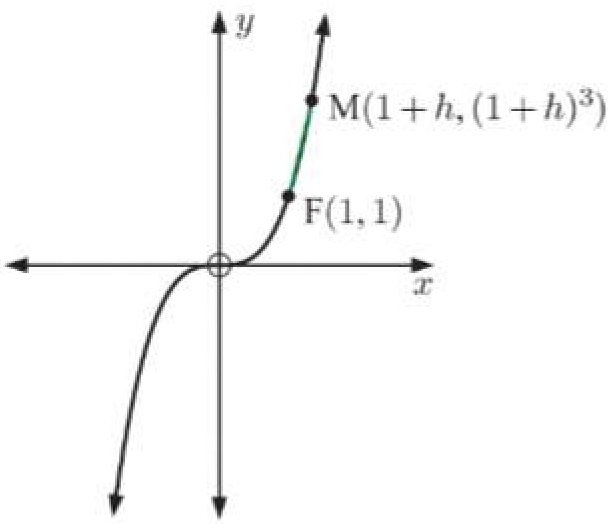
The gradient of [FM] $= \dfrac{y_M - y_F}{x_M - x_F}$

$$= \frac{(1+h)^3 - 1}{(1+h) - 1}$$

$$= \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \qquad \{\text{using } \mathbf{a}\}$$

$$= \frac{3h + 3h^2 + h^3}{h}$$

ii



The gradient of the tangent at F

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h} \qquad \{\text{from } \mathbf{b}\}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3 + 3h + h^2)}{\cancel{h}}$$

$$= \lim_{h \to 0} (3 + 3h + h^2) \qquad \{\text{as } h \neq 0\}$$

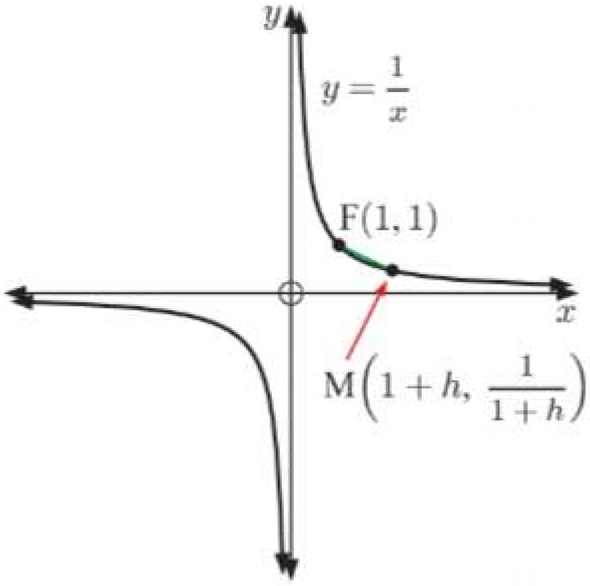
$$= 3$$

5 **a** $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{x - (x+h)}{x(x+h)}$

$$= \frac{-h}{x(x+h)}$$

b **i** At $x = 1$, $y = \frac{1}{1} = 1$.

Let F be the point $(1, 1)$ and M have x-coordinate $1 + h$, so M is $\left(1+h, \frac{1}{1+h}\right)$.



The gradient of the tangent at F

$$= \lim_{h \to 0} \frac{y_M - y_F}{x_M - x_F}$$

$$= \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{1(1+h)}}{h} \qquad \{\text{using } \textbf{a} \text{ with } x = 1\}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(1+h)}$$

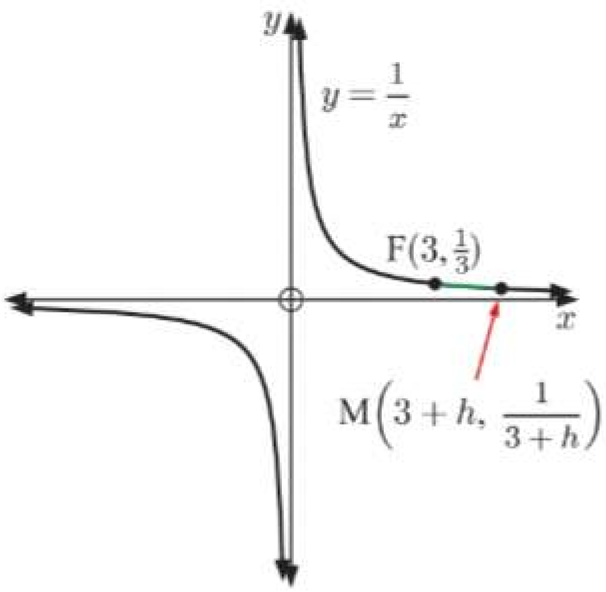
$$= \lim_{h \to 0} \frac{-1}{1+h} \qquad \{\text{as } h \neq 0\}$$

$$= \frac{-1}{1}$$

$$= -1$$

ii At $x = 3$, $y = \frac{1}{3}$.

Let F be the point $(3, \frac{1}{3})$ and M have x-coordinate $3 + h$, so M is $\left(3+h, \frac{1}{3+h}\right)$.



The gradient of the tangent at F

$$= \lim_{h \to 0} \frac{y_M - y_F}{x_M - x_F}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{3 + h - 3}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{3(3+h)}}{h} \qquad \{\text{using } \textbf{a} \text{ with } x = 3\}$$

$$= \lim_{h \to 0} \frac{-h}{3h(3+h)}$$

$$= \lim_{h \to 0} \frac{-1}{3(3+h)} \qquad \{\text{as } h \neq 0\}$$

$$= -\tfrac{1}{9}$$

EXERCISE 17E

1 **a** $f(0) = 4$

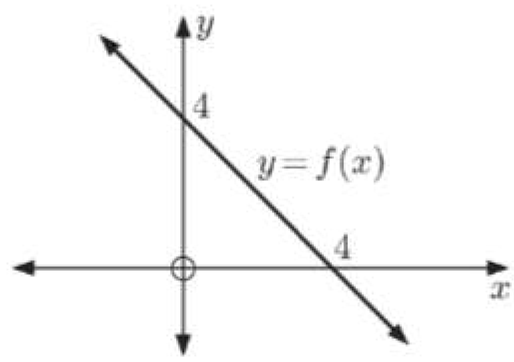
b $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$.

Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.

$f(x)$ goes through $(0, 4)$ and $(4, 0)$, so it has

gradient $= \frac{0-4}{4-0} = -1$

$\therefore\ f'(0) = -1$



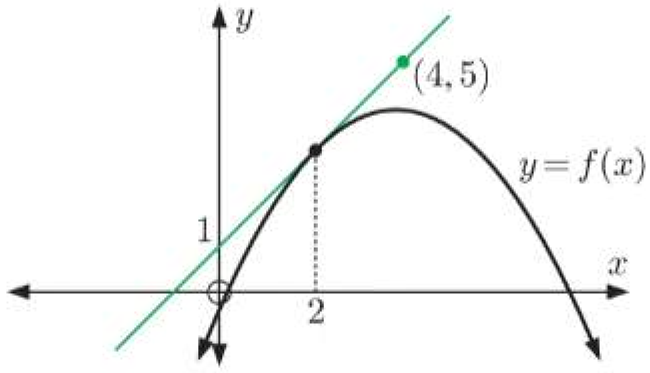
2 The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 2$.

The tangent passes through $(0, 1)$ and $(4, 5)$.

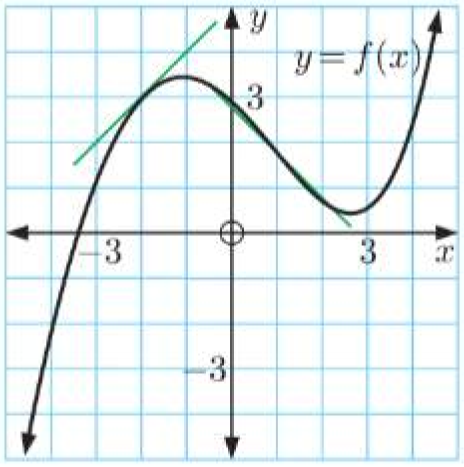
$\therefore \; f'(2) = \text{gradient of the tangent}$

$$= \frac{5-1}{4-0}$$

$$= 1$$



3



a $f(3)$ is above the x-axis, so $f(3)$ is positive.

b $f'(1)$ is the gradient of the tangent to $f(x)$ at the point where $x = 1$. Since the curve is decreasing at $x = 1$, then $f'(1)$ is negative.

c $f(-4)$ is below the x-axis, so $f(-4)$ is negative.

d $f'(-2)$ is the gradient of the tangent to $f(x)$ at the point where $x = -2$. Since the curve is increasing at $x = -2$, then $f'(-2)$ is positive.

4 **a** **i** $f'(x) = 2x + 1$

$f'(-2) = 2(-2) + 1$

$= -3$

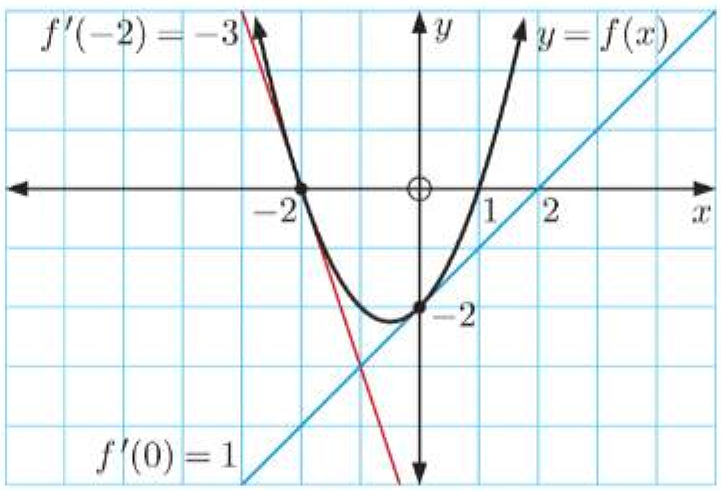
The gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -3.

ii $f'(0) = 2(0) + 1$

$= 1$

The gradient of the tangent to $y = f(x)$ at the point where $x = 0$ is 1.

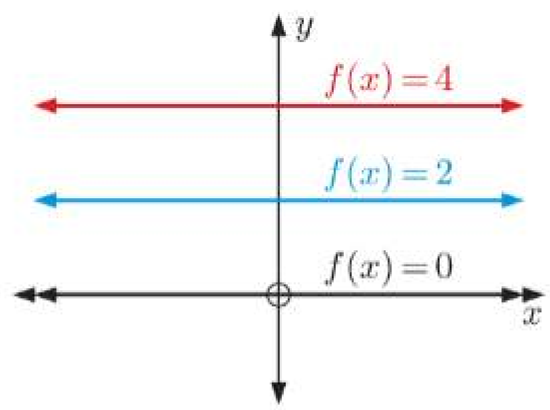
b



INVESTIGATION 3 GRADIENT FUNCTIONS

1 **a** $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$ are all horizontal lines and hence all have gradient 0.

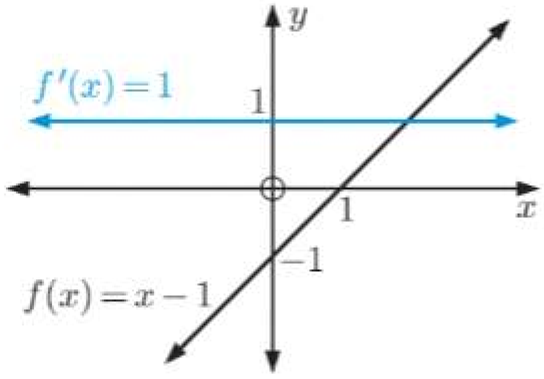
b Yes, the gradient is constant for all values of x.



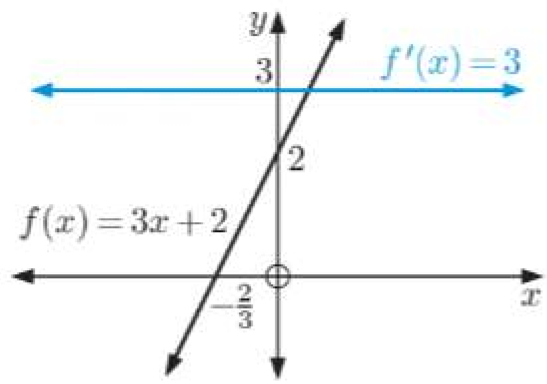
2 **a** The gradient of $f(x) = mx + c$ is m.

b The gradient m is constant for all values of x.

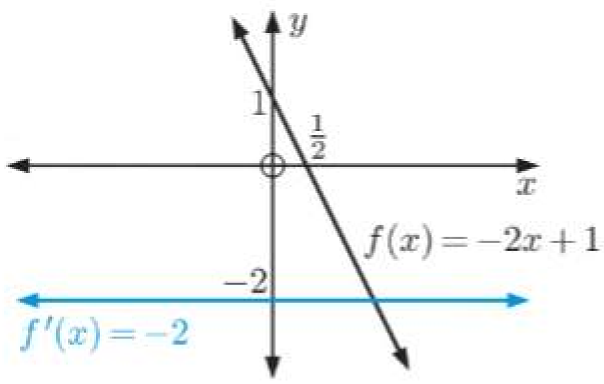
c **i**



ii



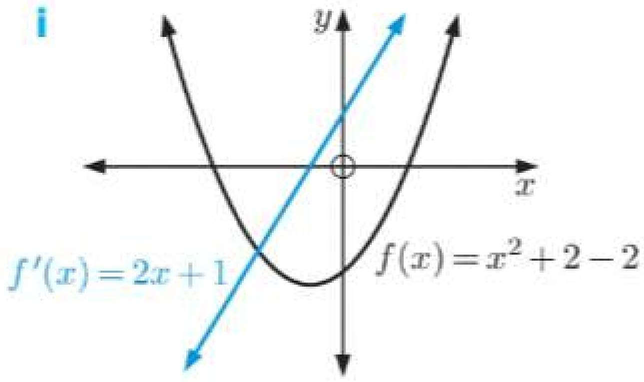
iii



$f'(x)$ is constant for all x.

3 **a** $f'(x)$ is a linear function.

b **i**



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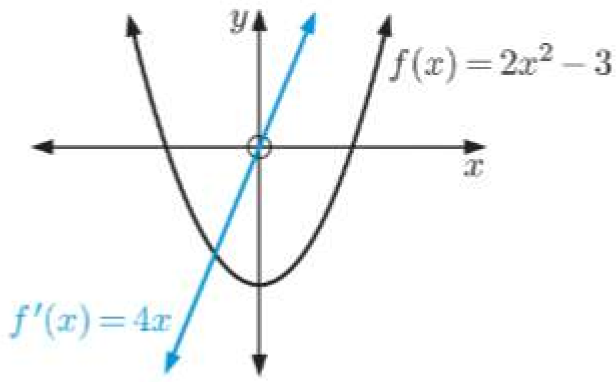
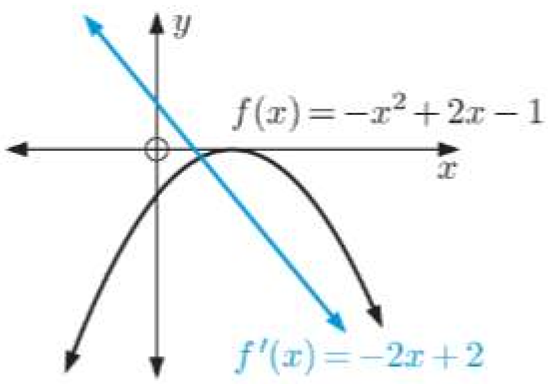
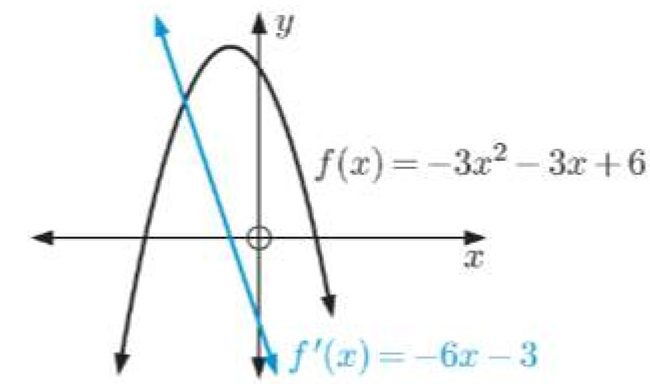


iii

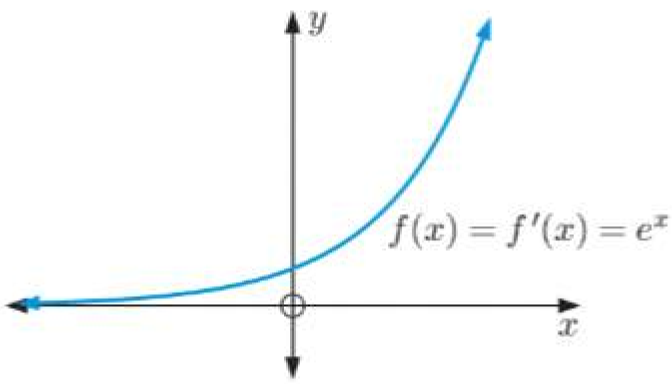


iv



c The gradient functions $f'(x)$ in **b** are all linear functions.

4 **a**



b The gradient function is
$f'(x)=f(x)=e^x$.

EXERCISE 17F

1 **a** $f(x) = 1$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{1-1}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \qquad \{\text{as } h \neq 0\} \\ &= 0 \end{aligned}$$

b $f(x) = x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} \\ &= \lim_{h\to 0} 1 \qquad \{\text{as } h \neq 0\} \\ &= 1 \end{aligned}$$

c $f(x) = 2x - 1$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[2(x+h)-1]-[2x-1]}{h} \\ &= \lim_{h\to 0} \frac{\cancel{2x}+2h-\cancel{1}-\cancel{2x}+\cancel{1}}{h} \\ &= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}} \\ &= \lim_{h\to 0} 2 \qquad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

d $f(x) = 3 - x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3-(x+h)]-[3-x]}{h} \\ &= \lim_{h\to 0} \frac{\cancel{3}-\cancel{x}-h-\cancel{3}+\cancel{x}}{h} \\ &= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}} \\ &= \lim_{h\to 0} -1 \qquad \{\text{as } h \neq 0\} \\ &= -1 \end{aligned}$$

2 **a** $y = f(x) = x^2 + 2$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[(x+h)^2+2]-[x^2+2]}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{2}-\cancel{x^2}-\cancel{2}}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\ &= \lim_{h\to 0} (2x+h) \qquad \{\text{as } h \neq 0\} \\ &= 2x \end{aligned}$$

b $y = f(x) = 3 - x^2$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3-(x+h)^2]-[3-x^2]}{h} \\ &= \lim_{h\to 0} \frac{3-(x^2+2xh+h^2)-3+x^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{3}-\cancel{x^2}-2xh-h^2-\cancel{3}+\cancel{x^2}}{h} \\ &= \lim_{h\to 0} \frac{-2xh-h^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}} \\ &= \lim_{h\to 0} (-2x-h) \qquad \{\text{as } h \neq 0\} \\ &= -2x \end{aligned}$$

c $y = f(x) = 2x^2 + x$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[2(x+h)^2 + (x+h)] - [2x^2 + x]}{h} \\
&= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) + \cancel{x} + h - 2x^2 - \cancel{x}}{h} \\
&= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + h - \cancel{2x^2}}{h} \\
&= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h + 1)}{\cancel{h}} \\
&= \lim_{h \to 0} (4x + 2h + 1) \qquad \{\text{as } h \neq 0\} \\
&= 4x + 1
\end{aligned}$$

d $y = f(x) = -x^2 + 5x - 3$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[-(x+h)^2 + 5(x+h) - 3] - [-x^2 + 5x - 3]}{h} \\
&= \lim_{h \to 0} \frac{-(x^2 + 2xh + h^2) + \cancel{5x} + 5h - \cancel{3} + x^2 - \cancel{5x} + \cancel{3}}{h} \\
&= \lim_{h \to 0} \frac{-\cancel{x^2} - 2xh - h^2 + 5h + \cancel{x^2}}{h} \\
&= \lim_{h \to 0} \frac{-2xh - h^2 + 5h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(-2x - h + 5)}{\cancel{h}} \\
&= \lim_{h \to 0} (-2x - h + 5) \qquad \{\text{as } h \neq 0\} \\
&= -2x + 5
\end{aligned}$$

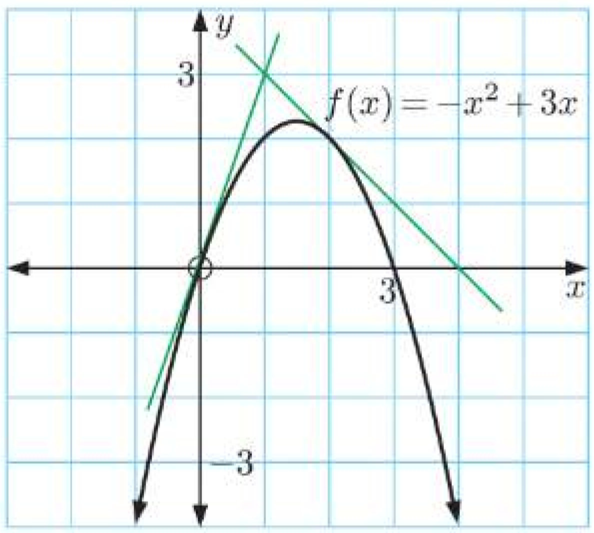
3 **a** $f(x) = 3x^2 - 1$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\
&= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - \cancel{1} - 3x^2 + \cancel{1}}{h} \\
&= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h) \qquad \{\text{as } h \neq 0\} \\
&= 6x
\end{aligned}$$

b $f'(2) = 6 \times 2$
$= 12$

The gradient of the tangent to $y = f(x)$ at the point where $x = 2$ is 12.

4 **a**



i The tangent to $f(x) = -x^2 + 3x$ at the point where $x = 0$ has gradient $\approx \frac{3-0}{1-0} \approx 3$.

ii The tangent to $f(x) = -x^2 + 3x$ at the point where $x = 2$ has gradient $\approx \frac{0-3}{4-1} \approx -1$.

b
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-(x+h)^2 + 3(x+h) - (-x^2 + 3x)}{h} \\ &= \lim_{h \to 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{3x} + 3h + \cancel{x^2} - \cancel{3x}}{h} \\ &= \lim_{h \to 0} \frac{-2xh - h^2 + 3h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(-2x - h + 3)}{\cancel{h}} \\ &= \lim_{h \to 0} (-2x - h + 3) \qquad \{\text{as } h \neq 0\} \\ &= -2x + 3 \end{aligned}$$

c
$$\begin{aligned} f'(0) &= -2(0) + 3 \\ &= 3 \end{aligned} \qquad \begin{aligned} f'(2) &= -2(2) + 3 \\ &= -1 \end{aligned}$$

Both values are the same as the estimates in **a**.

5 **a** $f(x) = \frac{1}{2}x^2 - x - 2$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\left[\frac{1}{2}(x+h)^2 - (x+h) - 2\right] - \left[\frac{1}{2}x^2 - x - 2\right]}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{2}(x^2 + 2xh + h^2) - \cancel{x} - h - \cancel{2} - \frac{1}{2}x^2 + \cancel{x} + \cancel{2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{\tfrac{1}{2}x^2} + xh + \frac{1}{2}h^2 - h - \cancel{\tfrac{1}{2}x^2}}{h} \\ &= \lim_{h \to 0} \frac{xh + \frac{1}{2}h^2 - h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(x + \frac{1}{2}h - 1)}{\cancel{h}} \\ &= \lim_{h \to 0} (x + \tfrac{1}{2}h - 1) \qquad \{\text{as } h \neq 0\} \\ &= x - 1 \end{aligned}$$

b **i** The gradient of the tangent where $x = -2$ is $f'(-2) = -2 - 1 = -3$.

ii The tangent has gradient 4 when
$$\begin{aligned} f'(x) &= 4 \\ \therefore\ x - 1 &= 4 \\ \therefore\ x &= 5 \end{aligned}$$

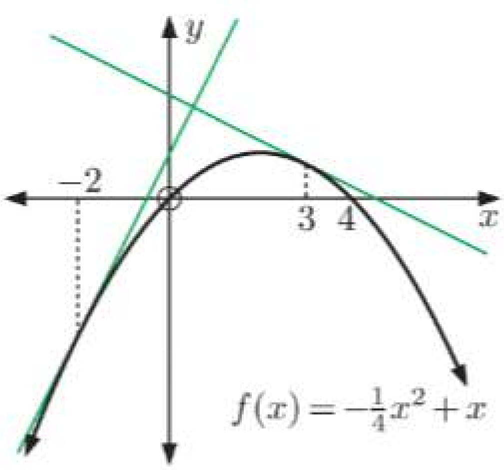
and
$$\begin{aligned} f(5) &= \tfrac{1}{2}(5)^2 - 5 - 2 \\ &= \tfrac{25}{2} - 7 \\ &= \tfrac{11}{2} \end{aligned}$$

The tangent has gradient 4 at the point $(5, \frac{11}{2})$.

6 **a** $f(x) = -\frac{1}{4}x^2 + x$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\left[-\frac{1}{4}(x+h)^2 + (x+h)\right] - \left[-\frac{1}{4}x^2 + x\right]}{h} \\ &= \lim_{h \to 0} \frac{\cancel{-\frac{1}{4}x^2} - \frac{1}{2}xh - \frac{1}{4}h^2 + \cancel{x} + h + \cancel{\frac{1}{4}x^2} - \cancel{x}}{h} \\ &= \lim_{h \to 0} \frac{-\frac{1}{2}xh - \frac{1}{4}h^2 + h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(-\frac{1}{2}x - \frac{1}{4}h + 1)}{\cancel{h}} \\ &= \lim_{h \to 0} \left(-\tfrac{1}{2}x - \tfrac{1}{4}h + 1\right) \qquad \{\text{as } h \neq 0\} \\ &= -\tfrac{1}{2}x + 1 \end{aligned}$$

b



The illustrated tangents are perpendicular if the product of their gradients is -1.

One tangent passes through the point where $x = -2$ and the other tangent passes through the point where $x = 3$.

The tangent at $x = -2$ has gradient
$$\begin{aligned} f'(-2) &= -\tfrac{1}{2}(-2) + 1 \\ &= 2 \end{aligned}$$

and the tangent at $x = 3$ has gradient
$$\begin{aligned} f'(3) &= -\tfrac{1}{2}(3) + 1 \\ &= -\tfrac{1}{2} \end{aligned}$$

Since $2 \times (-\frac{1}{2}) = -1$, the two tangents are perpendicular.

7 $y = x^3$

a Using technology:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	27	12	3	0	3	12	27

b $(x+h)^3 = (x+h)(x+h)^2$
$= (x+h)(x^2+2xh+h^2)$
$= x^3+2x^2h+xh^2+x^2h+2xh^2+h^3$
$= x^3+3x^2h+3xh^2+h^3$

c $y = f(x) = x^3$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h} && \{\text{using } \mathbf{b}\}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}\\ &= \lim_{h\to 0}(3x^2+3xh+h^2) && \{\text{as } h\neq 0\}\\ &= 3x^2\end{aligned}$$

Check:

When $x=-3$, $\frac{dy}{dx} = 3(-3)^2 = 27$ ✓ When $x=-2$, $\frac{dy}{dx} = 3(-2)^2 = 12$ ✓

When $x=-1$, $\frac{dy}{dx} = 3(-1)^2 = 3$ ✓ When $x=0$, $\frac{dy}{dx} = 3(0)^2 = 0$ ✓

When $x=1$, $\frac{dy}{dx} = 3(1)^2 = 3$ ✓ When $x=2$, $\frac{dy}{dx} = 3(2)^2 = 12$ ✓

When $x=3$, $\frac{dy}{dx} = 3(3)^2 = 27$ ✓

8 $y = \frac{1}{x}$

a Using technology:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$-\frac{1}{9}$	$-\frac{1}{4}$	-1	undefined	-1	$-\frac{1}{4}$	$-\frac{1}{9}$

b $y = f(x) = \frac{1}{x}$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &= \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h} && \{\text{using the result in } \textbf{Exercise 17D}, \text{ question } \mathbf{5\ a}\}\\ &= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}x(x+h)}\\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} && \{\text{as } h\neq 0\}\\ &= -\frac{1}{x^2}\end{aligned}$$

Check:

When $x=-3$, $\dfrac{dy}{dx}=-\dfrac{1}{(-3)^2}=-\frac{1}{9}$ ✓

When $x=-2$, $\dfrac{dy}{dx}=-\dfrac{1}{(-2)^2}=-\frac{1}{4}$ ✓

When $x=-1$, $\dfrac{dy}{dx}=-\dfrac{1}{(-1)^2}=-1$ ✓

When $x=0$, $\dfrac{dy}{dx}=-\dfrac{1}{0^2}$ which is undefined ✓

When $x=1$, $\dfrac{dy}{dx}=-\dfrac{1}{1^2}=-1$ ✓

When $x=2$, $\dfrac{dy}{dx}=-\dfrac{1}{2^2}=-\frac{1}{4}$ ✓

When $x=3$, $\dfrac{dy}{dx}=-\dfrac{1}{3^2}=-\frac{1}{9}$ ✓

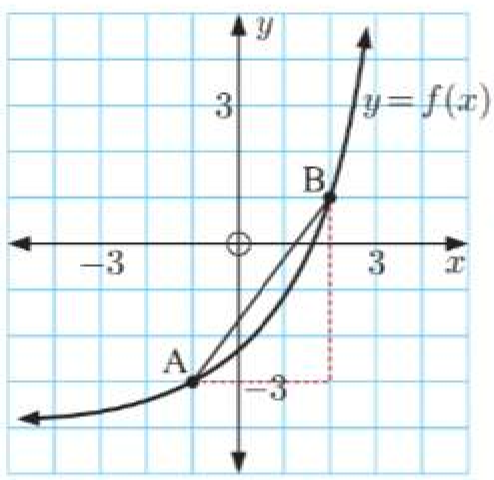
9 **a**

$f(x)$	$f'(x)$
x^1	1
x^2	$2x$
x^3	$3x^2$
x^{-1}	$-\dfrac{1}{x^2}$
x^0	0

b If $f(x)=x^n$, then $f'(x)=nx^{n-1}$.

REVIEW SET 17A

1



average rate of change in $f(x)$ from A to B

$=\dfrac{f(b)-f(a)}{b-a}$

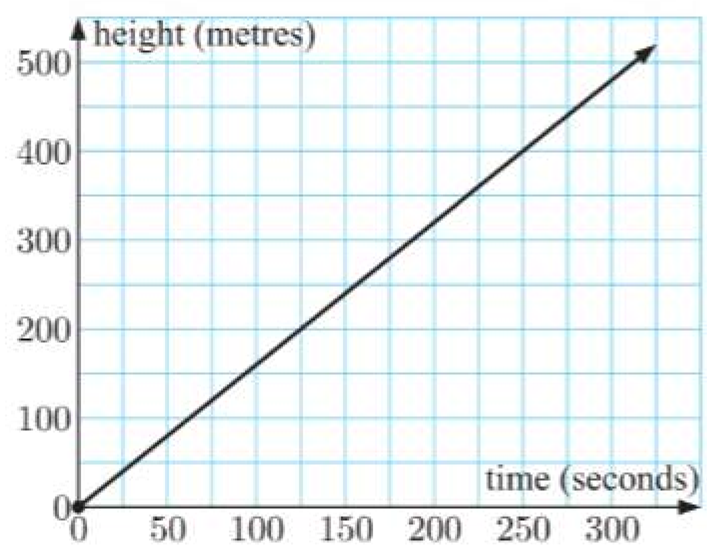
$=\dfrac{1-(-3)}{2-(-1)}$

$=\frac{4}{3}$

2 **a** The graph of height against time is a straight line.

$\therefore$ the height increases by the same amount each time interval.

$\therefore$ the ski-lift is increasing in height at a constant rate.



b rate of change $=\dfrac{(400-0)\text{ m}}{(250-0)\text{ s}}$

$=1.6\text{ m s}^{-1}$

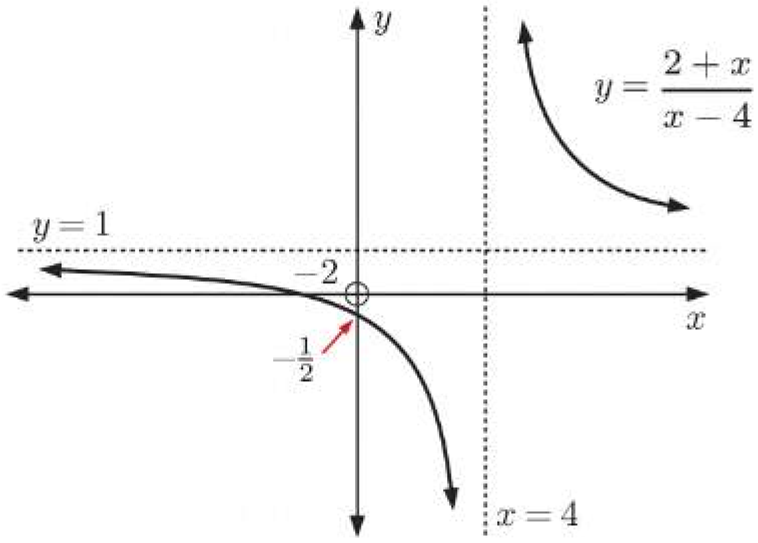
3 **a** We can make $6x-7$ as close as we like to -1 by making x sufficiently close to 1.

$\therefore \lim\limits_{x\to 1}(6x-7)=-1$

b $$\lim_{h\to 0}\frac{2h^2-h}{h}=\lim_{h\to 0}\frac{\cancel{h}(2h-1)}{\cancel{h}}$$
$$=\lim_{h\to 0}(2h-1)\qquad \{\text{as } h\neq 0\}$$
$$=-1$$

c $$\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}}$$
$$=\lim_{x\to 4}(x+4)\qquad \{\text{as } x\neq 4\}$$
$$=8$$

4 **a**



b As $x\to 4^-$, $y\to -\infty$
As $x\to 4^+$, $y\to \infty$
As $x\to -\infty$, $y\to 1^-$
As $x\to \infty$, $y\to 1^+$
The vertical asymptote is $x=4$.
The horizontal asymptote is $y=1$.

c $\lim\limits_{x\to -\infty}\dfrac{2+x}{x-4}=1$, $\lim\limits_{x\to \infty}\dfrac{2+x}{x-4}=1$

5 **a** $$f(x)=2x^2$$
$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)^2-2x^2}{h}$$
$$=\frac{2(x^2+2xh+h^2)-2x^2}{h}$$
$$=\frac{\cancel{2x^2}+4xh+2h^2-\cancel{2x^2}}{h}$$
$$=\frac{\cancel{h}(4x+2h)}{\cancel{h}}$$
$$=4x+2h \quad \text{provided } h\neq 0$$

b If $x=3$ then $\dfrac{f(3+h)-f(3)}{h}=4(3)+2h$ {using **a**}
$=12+2h$

When $h=0.1$,
$$\frac{f(3+h)-f(3)}{h}=12+2(0.1)$$
$$=12+0.2$$
$$=12.2$$

When $h=0.01$,
$$\frac{f(3+h)-f(3)}{h}=12+2(0.01)$$
$$=12+0.02$$
$$=12.02$$

When $h = 0.001$,

$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 12 + 2(0.001)\\ &= 12 + 0.002\\ &= 12.002\end{aligned}$$

When $h = 0.0001$,

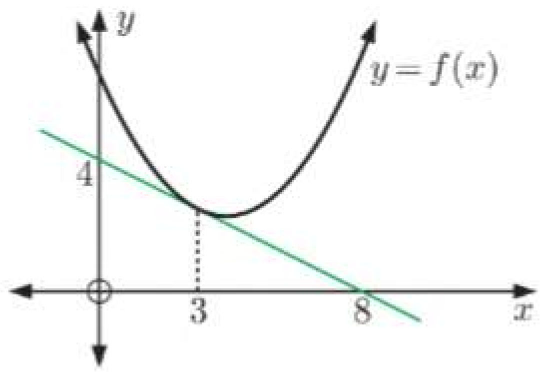
$$\begin{aligned}\frac{f(3+h)-f(3)}{h} &= 12 + 2(0.0001)\\ &= 12 + 0.0002\\ &= 12.0002\end{aligned}$$

h	$\frac{f(3+h)-f(3)}{h}$
0.1	12.2
0.01	12.02
0.001	12.002
0.0001	12.0002

c $$\begin{aligned}\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0}(12+2h)\\ &= 12\end{aligned}$$

The gradient of the tangent to $y = 2x^2$ at the point $(3, 18)$ is 12.

6



The tangent to $y = f(x)$ at the point where $x = 3$ has gradient $\frac{4-0}{0-8} = -\frac{1}{2}$.

$\therefore \quad f'(3) = -\frac{1}{2}$

7 **a** $f(x) = x^2 + 2x$

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{[(x+h)^2 + 2(x+h)] - [x^2+2x]}{h}\\ &= \lim_{h\to 0}\frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{x^2} - \cancel{2x}}{h}\\ &= \lim_{h\to 0}\frac{2xh + h^2 + 2h}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(2x+h+2)}{\cancel{h}}\\ &= \lim_{h\to 0}(2x+h+2) \qquad \{\text{as } h \neq 0\}\\ &= 2x + 2\end{aligned}$$

b $y = f(x) = 4 - 3x^2$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{[4-3(x+h)^2]-[4-3x^2]}{h}\\ &= \lim_{h\to 0}\frac{4-3(x^2+2xh+h^2)-4+3x^2}{h}\\ &= \lim_{h\to 0}\frac{\cancel{4}-\cancel{3x^2}-6xh-3h^2-\cancel{4}+\cancel{3x^2}}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(-6x-3h)}{\cancel{h}}\\ &= \lim_{h\to 0}(-6x-3h) \qquad \{\text{as } h\neq 0\}\\ &= -6x\end{aligned}$$

8 **a** $y = f(x) = 2x^2 - 1$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\left[2(x+h)^2-1\right]-\left[2x^2-1\right]}{h}\\ &= \lim_{h\to 0}\frac{2(x^2+2xh+h^2)-1-2x^2+1}{h}\\ &= \lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2-\cancel{1}-\cancel{2x^2}+\cancel{1}}{h}\\ &= \lim_{h\to 0}\frac{4xh+2h^2}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(4x+2h)}{\cancel{h}}\\ &= \lim_{h\to 0}(4x+2h) \qquad \{\text{as } h\neq 0\}\\ &= 4x\end{aligned}$$

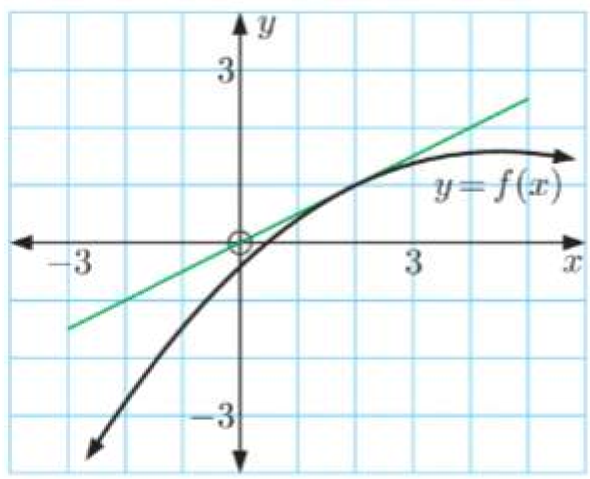
b The gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$ is $4 \times 4 = 16$.

c The gradient of the tangent is equal to -12 when $4x = -12$

$\therefore\ x = -3$

REVIEW SET 17B

1



The tangent at $x = 2$ has gradient $\frac{1-0}{2-0} = \frac{1}{2}$.

$\therefore$ the instantaneous rate of change in $f(x)$ at $x = 2$ is $\frac{1}{2}$.

2 **a** average rate of change in temperature from 7 am to noon

$= \frac{(20-10)\ °\text{C}}{(6-1)\ \text{h}}$

$= \frac{10}{5}\ °\text{C per h}$

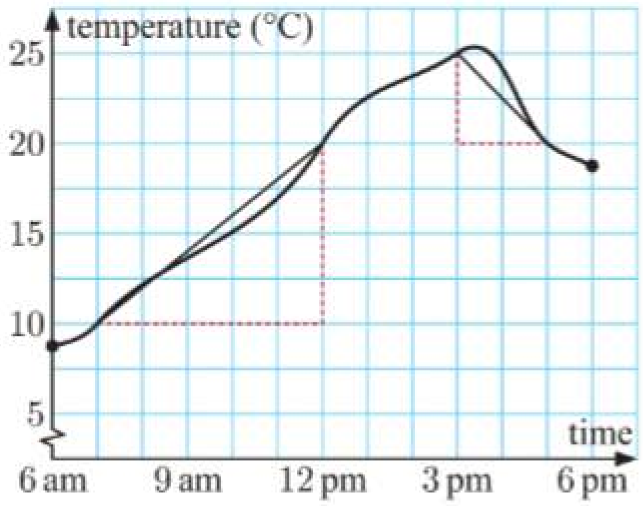
$= 2°\text{C per h}$

b average rate of change in temperature from 3 pm to 5 pm

$= \frac{(20-25)\ °\text{C}}{(11-9)\ \text{h}}$

$= \frac{-5}{2}\ °\text{C per h}$

$= -2.5°\text{C per h}$ or $-2\frac{1}{2}°\text{C per h}$



3 **a** $\lim_{h \to 0} \frac{h^3 - 3h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h^2-3)}{\cancel{h}}$

$= \lim_{h \to 0} (h^2 - 3)$ {as $h \neq 0$}

$= -3$

b $\lim_{x \to 1} \frac{3x^2 - 3x}{x - 1} = \lim_{x \to 1} \frac{3x\cancel{(x-1)}}{\cancel{(x-1)}}$

$= \lim_{x \to 1} 3x$ {as $x \neq 1$}

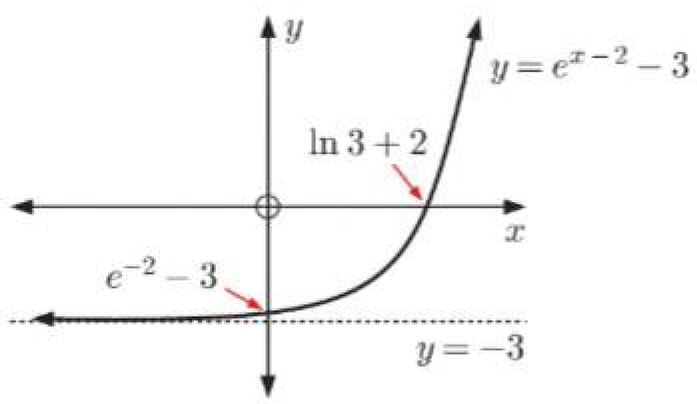
$= 3$

c $\lim_{x \to 2} \frac{x^2 - 3x + 2}{2 - x} = \lim_{x \to 2} \frac{(x-1)\cancel{(x-2)}}{-\cancel{(x-2)}}$

$= \lim_{x \to 2} -(x-1)$ {as $x \neq 2$}

$= -1$

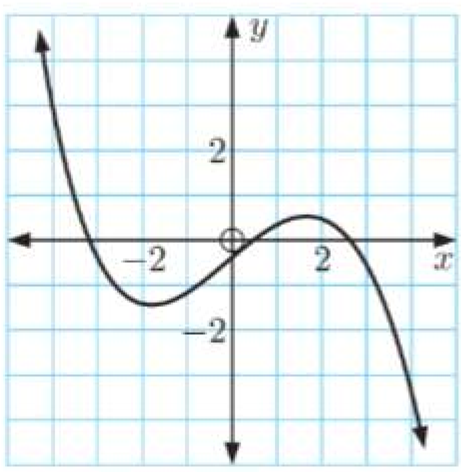
4 **a**



b **i** $\lim_{x \to -\infty} (e^{x-2} - 3) = -3$

ii $\lim_{x \to \infty} (e^{x-2} - 3)$ does not exist

5



a $f(-1)$ is below the x-axis, so $f(-1)$ is negative.

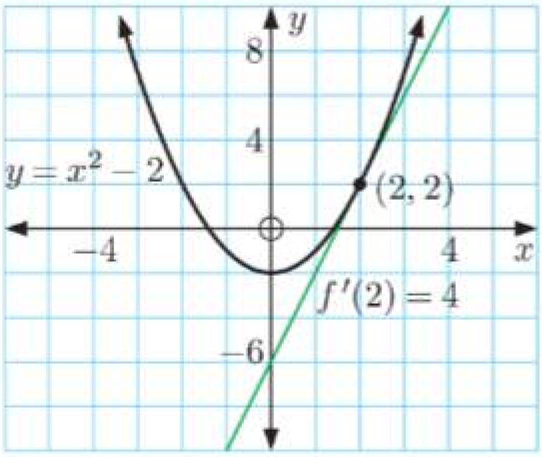
b $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$. Since the curve is increasing at $x = 0$, $f'(0)$ is positive.

c $f(2)$ is above the x-axis, so $f(2)$ is positive.

d $f'(3)$ is the gradient of the tangent to $f(x)$ at the point where $x = 3$. Since the curve is decreasing at $x = 3$, $f'(3)$ is negative.

6 **a, b**

x	-3	-2	-1	0	1	2	3
$f(x) = x^2 - 2$	7	2	-1	-2	-1	2	7



c The tangent to $f(x) = x^2 - 2$ at the point where $x = 2$ has gradient $\dfrac{6-(-2)}{3-1} = \dfrac{8}{2} = 4.$

$\therefore$ the instantaneous rate of change in $f(x) = x^2 - 2$ when $x = 2$ is 4.

7 $y = f(x) = x^2 + 5x - 2$

a
$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + 5(x+h) - 2 - (x^2 + 5x - 2)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{5x} + 5h - \cancel{2} - \cancel{x^2} - \cancel{5x} + \cancel{2}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 5h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x + h + 5)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x + h + 5) \qquad \{\text{as } h \neq 0\} \\
&= 2x + 5
\end{aligned}$$

b The tangent to the graph has gradient -3 when $\dfrac{dy}{dx} = -3$

$$\begin{aligned}
\therefore\ 2x + 5 &= -3 \\
\therefore\ 2x &= -8 \\
\therefore\ x &= -4
\end{aligned}$$

When $x = -4$, $y = (-4)^2 + 5(-4) - 2 = 16 - 20 - 2 = -6$

So, the point on the graph at which the tangent has gradient -3 is $(-4, -6)$.

8 $f(t) = 452 - 4.8t^2$ metres, $0 \leqslant t \leqslant 3$ seconds

a **i** $f(1) = 452 - 4.8(1)^2$
$= 447.2$

The jumper is 447.2 m above ground level after 1 second.

ii $f(2) = 452 - 4.8(2)^2$
$= 432.8$

The jumper is 432.8 m above ground level after 2 seconds.

b $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\left[452 - 4.8(t+h)^2\right] - \left[452 - 4.8t^2\right]}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{452} - \cancel{4.8t^2} - 9.6th - 4.8h^2 - \cancel{452} + \cancel{4.8t^2}}{h}$

$= \lim\limits_{h \to 0} \dfrac{-9.6th - 4.8h^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(-9.6t - 4.8h)}{\cancel{h}}$

$= \lim\limits_{h \to 0} (-9.6t - 4.8h) \qquad \{\text{as } h \neq 0\}$

$= -9.6t$

c The speed of the jumper is equal to the rate of change in the jumper's altitude which is given by $f'(t)$.

i $f'(1) = -9.6(1)$
$= -9.6$

The jumper's speed was $9.6\ \text{m}\,\text{s}^{-1}$ after 1 second.
(The negative sign indicates the jumper is moving downwards.)

ii $f'(2) = -9.6(2)$
$= -19.2$

The jumper's speed was $19.2\ \text{m}\,\text{s}^{-1}$ after 2 seconds.
(The negative sign indicates the jumper is moving downwards.)

Chapter 18

RULES OF DIFFERENTIATION

INVESTIGATION 1 — RULES FOR DIFFERENTIATION

1 **a** **i** $y = f(x) = 3x^2$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} \\ &= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} \\ &= \lim_{h \to 0} (6x + 3h) \qquad \{\text{as } h \neq 0\} \\ &= 6x \end{aligned}$$

ii $y = f(x) = 5x^2$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{5(x+h)^2 - 5x^2}{h} \\ &= \lim_{h \to 0} \frac{5(x^2 + 2xh + h^2) - 5x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{5x^2} + 10xh + 5h^2 - \cancel{5x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(10x + 5h)}{\cancel{h}} \\ &= \lim_{h \to 0} (10x + 5h) \qquad \{\text{as } h \neq 0\} \\ &= 10x \end{aligned}$$

iii $y = f(x) = -2x^2$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-2(x+h)^2 - (-2x^2)}{h} \\ &= \lim_{h \to 0} \frac{-2(x^2 + 2xh + h^2) + 2x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{-2x^2} - 4xh - 2h^2 + \cancel{2x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(-4x - 2h)}{\cancel{h}} \\ &= \lim_{h \to 0} (-4x - 2h) \qquad \{\text{as } h \neq 0\} \\ &= -4x \end{aligned}$$

b If $f(x) = cx^n$ where c is a constant, then $f'(x) = cnx^{n-1}$.

c **i** For $y = 4x^4$, using the software we find:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	-432	-128	-16	0	16	128	432

Now, using **b**, $\frac{dy}{dx} = 4 \times 4x^{4-1} = 16x^3$.

Check:

When $x = -3$, $\frac{dy}{dx} = 16 \times (-3)^3 = -432$ ✓

When $x = -2$, $\frac{dy}{dx} = 16 \times (-2)^3 = -128$ ✓

When $x = -1$, $\frac{dy}{dx} = 16 \times (-1)^3 = -16$ ✓

When $x = 0$, $\frac{dy}{dx} = 16 \times 0^3 = 0$ ✓

When $x = 1$, $\frac{dy}{dx} = 16 \times 1^3 = 16$ ✓

When $x = 2$, $\frac{dy}{dx} = 16 \times 2^3 = 128$ ✓

When $x = 3$, $\frac{dy}{dx} = 16 \times 3^3 = 432$ ✓

ii For $y = -\frac{2}{x} = -2x^{-1}$, using the software we find:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$0.\overline{2} = \frac{2}{9}$	0.5	2	undefined	2	0.5	$0.\overline{2} = \frac{2}{9}$

Now, using **b**, $\frac{dy}{dx} = -2 \times (-1)x^{-1-1} = 2x^{-2} = \frac{2}{x^2}$.

Check:

When $x = -3$, $\frac{dy}{dx} = \frac{2}{(-3)^2} = \frac{2}{9}$ ✓

When $x = -2$, $\frac{dy}{dx} = \frac{2}{(-2)^2} = \frac{2}{4} = \frac{1}{2}$ ✓

When $x = -1$, $\frac{dy}{dx} = \frac{2}{(-1)^2} = \frac{2}{1} = 2$ ✓

When $x = 0$, $\frac{dy}{dx} = \frac{2}{0^2}$ which is undefined ✓

When $x = 1$, $\frac{dy}{dx} = \frac{2}{1^2} = 2$ ✓

When $x = 2$, $\frac{dy}{dx} = \frac{2}{2^2} = \frac{2}{4} = \frac{1}{2}$ ✓

When $x = 3$, $\frac{dy}{dx} = \frac{2}{3^2} = \frac{2}{9}$ ✓

iii For $y = \frac{3}{x^2} = 3x^{-2}$, using the software we find:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$0.\overline{2} = \frac{2}{9}$	0.75	6	undefined	-6	-0.75	$-0.\overline{2} = -\frac{2}{9}$

Now, using **b**, $\frac{dy}{dx} = 3 \times (-2)x^{-2-1} = -6x^{-3} = -\frac{6}{x^3}$.

Check:

When $x=-3$, $\dfrac{dy}{dx} = -\dfrac{6}{(-3)^3} = -\frac{6}{-27} = \frac{2}{9}$ ✓

When $x=-2$, $\dfrac{dy}{dx} = -\dfrac{6}{(-2)^3} = -\frac{6}{-8} = \frac{3}{4}$ ✓

When $x=-1$, $\dfrac{dy}{dx} = -\dfrac{6}{(-1)^3} = -\frac{6}{-1} = 6$ ✓

When $x=0$, $\dfrac{dy}{dx} = -\dfrac{6}{0^3}$ which is undefined ✓

When $x=1$, $\dfrac{dy}{dx} = -\dfrac{6}{1^3} = -\frac{6}{1} = -6$ ✓

When $x=2$, $\dfrac{dy}{dx} = -\dfrac{6}{2^3} = -\frac{6}{8} = -\frac{3}{4}$ ✓

When $x=3$, $\dfrac{dy}{dx} = -\dfrac{6}{3^3} = -\frac{6}{27} = -\frac{2}{9}$ ✓

2 **a** **i** $f(x) = x^2 + 3x$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h+3)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x+h+3) \qquad \{\text{as } h \neq 0\} \\
&= 2x+3
\end{aligned}$$

ii $f(x) = x^3 - 2x^2$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - 2(x+h)^2 - (x^3 - 2x^2)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x^2+2xh+h^2) - 2(x^2+2xh+h^2) - (x^3-2x^2)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 - \cancel{x^3} + \cancel{2x^2}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 4x - 2h)}{\cancel{h}} \qquad \{\text{as } h \neq 0\} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 4x - 2h) \\
&= 3x^2 - 4x
\end{aligned}$$

b If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

EXERCISE 18A

1 **a** $f(x) = x^3$
$\therefore\ f'(x) = 3x^2$

b $f(x) = x^8$
$\therefore\ f'(x) = 8x^7$

c $f(x) = x^{11}$
$\therefore\ f'(x) = 11x^{10}$

d $f(x) = 6x$
$\therefore\ f'(x) = 6(1)$
$= 6$

e $f(x) = 2x^3$
$\therefore\ f'(x) = 2(3x^2)$
$= 6x^2$

f $f(x) = 7x^2$
$\therefore\ f'(x) = 7(2x)$
$= 14x$

g $f(x) = 3x^5$
$\therefore\ f'(x) = 3(5x^4)$
$= 15x^4$

h $f(x) = 5x^6$
$\therefore\ f'(x) = 5(6x^5)$
$= 30x^5$

i $f(x) = 5x - 2$
$\therefore\ f'(x) = 5(1)$
$= 5$

j $f(x) = x^2 - 3$
$\therefore\ f'(x) = 2x$

k $f(x) = x^2 + x$
$\therefore\ f'(x) = 2x + 1$

l $f(x) = x^2 + 3x - 5$
$\therefore\ f'(x) = 2x + 3(1)$
$= 2x + 3$

m $f(x) = 2x^2 + x - 1$
$\therefore\ f'(x) = 2(2x) + 1$
$= 4x + 1$

n $f(x) = 3x^2 - 7x + 8$
$\therefore\ f'(x) = 3(2x) - 7(1)$
$= 6x - 7$

o $f(x) = 4 - 2x^2$
$\therefore\ f'(x) = -2(2x)$
$= -4x$

p $f(x) = \frac{1}{2}x^4 - 6x^2$
$\therefore\ f'(x) = \frac{1}{2}(4x^3) - 6(2x)$
$= 2x^3 - 12x$

q $f(x) = x^3 - 4x^2 + 6x$
$\therefore\ f'(x) = 3x^2 - 4(2x) + 6(1)$
$= 3x^2 - 8x + 6$

r $f(x) = 2x^3 + x - 1$
$\therefore\ f'(x) = 2(3x^2) + (1)$
$= 6x^2 + 1$

s $f(x) = 7 - x - 4x^3$
$\therefore\ f'(x) = -1 - 4(3x^2)$
$= -1 - 12x^2$

t $f(x) = \frac{1}{5}x^3 - \frac{7}{2}x^2 - 2$
$\therefore\ f'(x) = \frac{1}{5}(3x^2) - \frac{7}{2}(2x)$
$= \frac{3}{5}x^2 - 7x$

2 **a** Let $f(x) = \dfrac{1}{x^2}$
$= x^{-2}$
$\therefore\ f'(x) = -2x^{-3}$
$= -\dfrac{2}{x^3}$

b Let $f(x) = \dfrac{1}{x^5}$
$= x^{-5}$
$\therefore\ f'(x) = -5x^{-6}$
$= -\dfrac{5}{x^6}$

c Let $f(x) = \dfrac{1}{3x}$
$= \frac{1}{3}x^{-1}$
$\therefore\ f'(x) = \frac{1}{3}(-1x^{-2})$
$= -\frac{1}{3}x^{-2}$
$= -\dfrac{1}{3x^2}$

d Let $f(x) = \dfrac{4}{x^3}$
$= 4x^{-3}$
$\therefore\ f'(x) = 4(-3x^{-4})$
$= -12x^{-4}$
$= -\dfrac{12}{x^4}$

e Let $f(x) = -\dfrac{7}{x^4}$
$= -7x^{-4}$
$\therefore\ f'(x) = -7(-4x^{-5})$
$= 28x^{-5}$
$= \dfrac{28}{x^5}$

f Let
$$\begin{aligned} f(x) &= 2x + \frac{3}{x^2} \\ &= 2x + 3x^{-2} \\ \therefore\ f'(x) &= 2(1) + 3(-2x^{-3}) \\ &= 2 - 6x^{-3} \\ &= 2 - \frac{6}{x^3} \end{aligned}$$

g Let
$$\begin{aligned} f(x) &= x^2 - \frac{6}{x} \\ &= x^2 - 6x^{-1} \\ \therefore\ f'(x) &= 2x - 6(-1x^{-2}) \\ &= 2x + 6x^{-2} \\ &= 2x + \frac{6}{x^2} \end{aligned}$$

h Let
$$\begin{aligned} f(x) &= \frac{2}{x^2} + \frac{9}{x^4} \\ &= 2x^{-2} + 9x^{-4} \\ \therefore\ f'(x) &= 2(-2x^{-3}) + 9(-4x^{-5}) \\ &= -4x^{-3} - 36x^{-5} \\ &= -\frac{4}{x^3} - \frac{36}{x^5} \end{aligned}$$

i Let
$$\begin{aligned} f(x) &= 3x - \frac{1}{x} + \frac{2}{x^2} \\ &= 3x - x^{-1} + 2x^{-2} \\ \therefore\ f'(x) &= 3(1) - (-1x^{-2}) + 2(-2x^{-3}) \\ &= 3 + x^{-2} - 4x^{-3} \\ &= 3 + \frac{1}{x^2} - \frac{4}{x^3} \end{aligned}$$

j Let
$$\begin{aligned} f(x) &= 5 - \frac{8}{x^2} + \frac{1}{2x^3} \\ &= 5 - 8x^{-2} + \tfrac{1}{2}x^{-3} \\ \therefore\ f'(x) &= -8(-2x^{-3}) + \tfrac{1}{2}(-3x^{-4}) \\ &= 16x^{-3} - \tfrac{3}{2}x^{-4} \\ &= \frac{16}{x^3} - \frac{3}{2x^4} \end{aligned}$$

k Let
$$\begin{aligned} f(x) &= \frac{x^3 + 4}{x} \\ &= \frac{x^3}{x} + \frac{4}{x} \\ &= x^2 + 4x^{-1} \\ \therefore\ f'(x) &= 2x + 4(-1x^{-2}) \\ &= 2x - 4x^{-2} \\ &= 2x - \frac{4}{x^2} \end{aligned}$$

l Let
$$\begin{aligned} f(x) &= \frac{2x - 5}{x^2} \\ &= \frac{2x}{x^2} - \frac{5}{x^2} \\ &= \frac{2}{x} - \frac{5}{x^2} \\ &= 2x^{-1} - 5x^{-2} \\ \therefore\ f'(x) &= 2(-1x^{-2}) - 5(-2x^{-3}) \\ &= -2x^{-2} + 10x^{-3} \\ &= -\frac{2}{x^2} + \frac{10}{x^3} \end{aligned}$$

3 **a**
$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{\frac{1}{2}} \\ \therefore\ f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \sqrt[3]{x} \\ &= x^{\frac{1}{3}} \\ \therefore\ f'(x) &= \tfrac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}} \end{aligned}$$

c
$$\begin{aligned} f(x) &= 2x - \sqrt{x} \\ &= 2x - x^{\frac{1}{2}} \\ \therefore\ f'(x) &= 2(1) - \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= 2 - \frac{1}{2\sqrt{x}} \end{aligned}$$

d
$$\begin{aligned} f(x) &= x\sqrt{x} \\ &= x^{\frac{3}{2}} \\ \therefore\ f'(x) &= \tfrac{3}{2}x^{\frac{1}{2}} \\ &= \tfrac{3}{2}\sqrt{x} \end{aligned}$$

e
$$\begin{aligned} f(x) &= \frac{1}{\sqrt{x}} \\ &= x^{-\frac{1}{2}} \\ \therefore\ f'(x) &= -\tfrac{1}{2}x^{-\frac{3}{2}} \\ &= -\frac{1}{2x\sqrt{x}} \end{aligned}$$

f
$$\begin{aligned} f(x) &= -\frac{1}{x\sqrt{x}} \\ &= -x^{-\frac{3}{2}} \\ \therefore\ f'(x) &= -(-\tfrac{3}{2}x^{-\frac{5}{2}}) \\ &= \frac{3}{2x^2\sqrt{x}} \end{aligned}$$

g
$$\begin{aligned} f(x) &= 2x^2 - \frac{3}{\sqrt{x}} \\ &= 2x^2 - 3x^{-\frac{1}{2}} \\ \therefore\ f'(x) &= 2(2x) - 3(-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= 4x + \tfrac{3}{2}x^{-\frac{3}{2}} \\ &= 4x + \frac{3}{2x\sqrt{x}} \end{aligned}$$

h
$$\begin{aligned} f(x) &= \frac{\sqrt{x} - 4}{x} \\ &= \frac{\sqrt{x}}{x} - \frac{4}{x} \\ &= \frac{x^{\frac{1}{2}}}{x} - \frac{4}{x} \\ &= x^{-\frac{1}{2}} - 4x^{-1} \\ \therefore\ f'(x) &= -\tfrac{1}{2}x^{-\frac{3}{2}} - 4(-x^{-2}) \\ &= -\frac{1}{2x\sqrt{x}} + \frac{4}{x^2} \end{aligned}$$

i
$$\begin{aligned} f(x) &= \frac{x+5}{\sqrt{x}} \\ &= \frac{x}{\sqrt{x}} + \frac{5}{\sqrt{x}} \\ &= \frac{x}{x^{\frac{1}{2}}} + 5x^{-\frac{1}{2}} \\ &= x^{\frac{1}{2}} + 5x^{-\frac{1}{2}} \\ \therefore\ f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} + 5(-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= \frac{1}{2\sqrt{x}} - \frac{5}{2x\sqrt{x}} \end{aligned}$$

j
$$\begin{aligned} f(x) &= \frac{7 - x^2}{\sqrt{x}} \\ &= \frac{7}{\sqrt{x}} - \frac{x^2}{\sqrt{x}} \\ &= 7x^{-\frac{1}{2}} - \frac{x^2}{x^{\frac{1}{2}}} \\ &= 7x^{-\frac{1}{2}} - x^{\frac{3}{2}} \\ \therefore\ f'(x) &= 7(-\tfrac{1}{2}x^{-\frac{3}{2}}) - \tfrac{3}{2}x^{\frac{1}{2}} \\ &= -\frac{7}{2x\sqrt{x}} - \frac{3\sqrt{x}}{2} \end{aligned}$$

k
$$\begin{aligned} f(x) &= 2x - \frac{3}{x^2\sqrt{x}} \\ &= 2x - \frac{3}{x^{\frac{5}{2}}} \\ &= 2x - 3x^{-\frac{5}{2}} \\ \therefore\ f'(x) &= 2(1) - 3(-\tfrac{5}{2}x^{-\frac{7}{2}}) \\ &= 2 + \tfrac{15}{2}x^{-\frac{7}{2}} \\ &= 2 + \frac{15}{2x^3\sqrt{x}} \end{aligned}$$

l
$$\begin{aligned} f(x) &= \frac{x^2 - x + 2}{\sqrt[3]{x}} \\ &= \frac{x^2}{\sqrt[3]{x}} - \frac{x}{\sqrt[3]{x}} + \frac{2}{\sqrt[3]{x}} \\ &= \frac{x^2}{x^{\frac{1}{3}}} - \frac{x}{x^{\frac{1}{3}}} + \frac{2}{x^{\frac{1}{3}}} \\ &= x^{\frac{5}{3}} - x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} \\ \therefore\ f'(x) &= \tfrac{5}{3}x^{\frac{2}{3}} - \tfrac{2}{3}x^{-\frac{1}{3}} + 2(-\tfrac{1}{3}x^{-\frac{4}{3}}) \\ &= \tfrac{5}{3}x^{\frac{2}{3}} - \tfrac{2}{3}x^{-\frac{1}{3}} - \tfrac{2}{3}x^{-\frac{4}{3}} \\ &= \frac{5\sqrt[3]{x^2}}{3} - \frac{2}{3\sqrt[3]{x}} - \frac{2}{3x\sqrt[3]{x}} \end{aligned}$$

4 **a**
$$\begin{aligned} y &= \pi x^2 \\ \therefore\ \frac{dy}{dx} &= \pi(2x) \\ &= 2\pi x \end{aligned}$$

b
$$\begin{aligned} y &= 2.5x^3 - 1.4x^2 - 1.3 \\ \therefore\ \frac{dy}{dx} &= 2.5(3x^2) - 1.4(2x) \\ &= 7.5x^2 - 2.8x \end{aligned}$$

c
$$\begin{aligned} y &= 3x^2 - \frac{8}{x^2} \\ &= 3x^2 - 8x^{-2} \\ \therefore\ \frac{dy}{dx} &= 3(2x) - 8(-2x^{-3}) \\ &= 6x + 16x^{-3} \\ &= 6x + \frac{16}{x^3} \end{aligned}$$

d
$$\begin{aligned} y &= (x+1)(x-2) \\ &= x^2 - x - 2 \\ \therefore\ \frac{dy}{dx} &= 2x - 1 \end{aligned}$$

e
$$\begin{aligned} y &= (2x+1)(3x-2) \\ &= 6x^2 - x - 2 \\ \therefore\ \frac{dy}{dx} &= 12x - 1 \end{aligned}$$

f
$$\begin{aligned} y &= (5-x)^2 \\ &= 25 - 10x + x^2 \\ \therefore\ \frac{dy}{dx} &= -10(1) + 2x \\ &= 2x - 10 \end{aligned}$$

g
$$\begin{aligned} y &= (2x-1)^2 \\ &= 4x^2 - 4x + 1 \\ \therefore\ \frac{dy}{dx} &= 4(2x) - 4(1) \\ &= 8x - 4 \end{aligned}$$

h
$$\begin{aligned} y &= x(x+1)(2x-5) \\ &= x(2x^2 - 3x - 5) \\ &= 2x^3 - 3x^2 - 5x \\ \therefore\ \frac{dy}{dx} &= 2(3x^2) - 3(2x) - 5(1) \\ &= 6x^2 - 6x - 5 \end{aligned}$$

i
$$\begin{aligned} y &= \frac{(x-3)^2}{\sqrt{x}} \\ &= \frac{x^2-6x+9}{\sqrt{x}} \\ &= \frac{x^2}{\sqrt{x}} - \frac{6x}{\sqrt{x}} + \frac{9}{\sqrt{x}} \\ &= \frac{x^2}{x^{\frac{1}{2}}} - \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}} \\ &= x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{3}{2}x^{\frac{1}{2}} - 6(\tfrac{1}{2}x^{-\frac{1}{2}}) + 9(-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= \tfrac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} - \tfrac{9}{2}x^{-\frac{3}{2}} \\ &= \tfrac{3}{2}\sqrt{x} - \frac{3}{\sqrt{x}} - \frac{9}{2x\sqrt{x}} \end{aligned}$$

5 **a**
$$\begin{aligned} y &= \tfrac{1}{2}t^4 - \tfrac{1}{3}t \\ \therefore\ \frac{dy}{dt} &= \tfrac{1}{2}(4t^3) - \tfrac{1}{3}(1) \\ &= 2t^3 - \tfrac{1}{3} \end{aligned}$$

b
$$\begin{aligned} y &= 7 - \frac{6}{\sqrt{t}} \\ &= 7 - 6t^{-\frac{1}{2}} \\ \therefore\ \frac{dy}{dt} &= -6(-\tfrac{1}{2}t^{-\frac{3}{2}}) \\ &= 3t^{-\frac{3}{2}} \\ &= \frac{3}{t\sqrt{t}} \end{aligned}$$

c
$$\begin{aligned} T &= \sqrt[3]{t} - \frac{2}{t^2} \\ &= t^{\frac{1}{3}} - 2t^{-2} \\ \therefore\ \frac{dT}{dt} &= \tfrac{1}{3}t^{-\frac{2}{3}} - 2(-2t^{-3}) \\ &= \tfrac{1}{3}t^{-\frac{2}{3}} + 4t^{-3} \\ &= \frac{1}{3\sqrt[3]{t^2}} + \frac{4}{t^3} \end{aligned}$$

d
$$\begin{aligned} P &= \frac{5}{u} - 10u\sqrt{u} \\ &= 5u^{-1} - 10u^{\frac{3}{2}} \\ \therefore\ \frac{dP}{du} &= 5(-u^{-2}) - 10(\tfrac{3}{2}u^{\frac{1}{2}}) \\ &= -5u^{-2} - 15u^{\frac{1}{2}} \\ &= -\frac{5}{u^2} - 15\sqrt{u} \end{aligned}$$

6 **a**
$$\begin{aligned} y &= x^2 \\ \therefore\ \frac{dy}{dx} &= 2x \end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = 2(2) = 4$

So, the tangent has gradient 4.

b
$$\begin{aligned} y &= x^3 - 5x + 2 \\ \therefore\ \frac{dy}{dx} &= 3x^2 - 5(1) \\ &= 3x^2 - 5 \end{aligned}$$

At the point $(3, 14)$,

$\dfrac{dy}{dx} = 3(3)^2 - 5 = 22$

So, the tangent has gradient 22.

c
$$\begin{aligned} y &= \frac{8}{x^2} \\ &= 8x^{-2} \\ \therefore \frac{dy}{dx} &= 8(-2x^{-3}) \\ &= -16x^{-3} \\ &= -\frac{16}{x^3} \end{aligned}$$

At the point $(9, \frac{8}{81})$,

$\frac{dy}{dx} = -\frac{16}{9^3} = -\frac{16}{729}$.

So, the tangent has gradient $-\frac{16}{729}$.

d
$$\begin{aligned} y &= 2x^2 - 3x + 7 \\ \therefore \frac{dy}{dx} &= 2(2x) - 3(1) \\ &= 4x - 3 \end{aligned}$$

When $x = -1$,

$$\begin{aligned} \frac{dy}{dx} &= 4(-1) - 3 \\ &= -7 \end{aligned}$$

So, the tangent has gradient -7.

e
$$\begin{aligned} y &= 3\sqrt{x} \\ &= 3x^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= 3(\tfrac{1}{2}x^{-\frac{1}{2}}) \\ &= \frac{3}{2\sqrt{x}} \end{aligned}$$

At the point $(1, 3)$,

$\frac{dy}{dx} = \frac{3}{2\sqrt{1}} = \frac{3}{2}$.

So, the tangent has gradient $\frac{3}{2}$.

f
$$\begin{aligned} y &= 2x - \frac{5}{x} \\ &= 2x - 5x^{-1} \\ \therefore \frac{dy}{dx} &= 2(1) - 5(-1x^{-2}) \\ &= 2 + \frac{5}{x^2} \end{aligned}$$

At the point $(2, \frac{3}{2})$,

$$\begin{aligned} \frac{dy}{dx} &= 2 + \frac{5}{2^2} \\ &= 2 + \tfrac{5}{4} \\ &= \tfrac{13}{4} \end{aligned}$$

So, the tangent has gradient $\frac{13}{4}$.

g
$$\begin{aligned} y &= \frac{x^2 - 4}{x^2} \\ &= \frac{x^2}{x^2} - \frac{4}{x^2} \\ &= 1 - 4x^{-2} \\ \therefore \frac{dy}{dx} &= -4(-2x^{-3}) \\ &= \frac{8}{x^3} \end{aligned}$$

At the point $(4, \frac{3}{4})$,

$\frac{dy}{dx} = \frac{8}{4^3} = \frac{8}{64} = \frac{1}{8}$.

So, the tangent has gradient $\frac{1}{8}$.

h
$$\begin{aligned} y &= \frac{x^3 - 4x - 8}{x^2} \\ &= \frac{x^3}{x^2} - \frac{4x}{x^2} - \frac{8}{x^2} \\ &= x - 4x^{-1} - 8x^{-2} \\ \therefore \frac{dy}{dx} &= 1 - 4(-1x^{-2}) - 8(-2x^{-3}) \\ &= 1 + \frac{4}{x^2} + \frac{16}{x^3} \end{aligned}$$

When $x = -1$,

$$\begin{aligned} \frac{dy}{dx} &= 1 + \frac{4}{(-1)^2} + \frac{16}{(-1)^3} \\ &= -11 \end{aligned}$$

So, the tangent has gradient -11.

7
$$\begin{aligned} f(x) &= x^3 + ax + 5 \\ \therefore f'(x) &= 3x^2 + a \end{aligned}$$

Now $f'(1) = 10$, so $3(1)^2 + a = 10$

$$\begin{aligned} \therefore 3 + a &= 10 \\ \therefore a &= 7 \end{aligned}$$

8 $$f(x) = -3x^2 + ax + b$$
$$\therefore \; f'(x) = -6x + a$$

Now $f'(-3) = 9$, so $-6(-3) + a = 9$
$$\therefore \; 18 + a = 9$$
$$\therefore \; a = -9$$

Also, $f(-3) = 8$, so $-3(-3)^2 + (-9)(-3) + b = 8$
$$\therefore \; -27 + 27 + b = 8$$
$$\therefore \; b = 8$$

9 $$f(x) = 2x^2 + a + \frac{b}{x}$$
$$= 2x^2 + a + bx^{-1}$$
$$\therefore \; f'(x) = 4x - bx^{-2}$$
$$= 4x - \frac{b}{x^2}$$

Now $f'(1) = -2$, so $4(1) - \frac{b}{1^2} = -2$
$$\therefore \; 4 - b = -2$$
$$\therefore \; b = 6$$

Also, $f(1) = 11$, so $2(1)^2 + a + \frac{6}{1} = 11$
$$\therefore \; 2 + a + 6 = 11$$
$$\therefore \; 8 + a = 11$$
$$\therefore \; a = 3$$

10 $$f(x) = ax + \frac{b}{x}, \quad f(3) = 5, \quad \text{and} \quad f'(1) = 5$$
$$= ax + bx^{-1}$$
$$\therefore \; f'(x) = a + b(-x^{-2})$$
$$= a - \frac{b}{x^2}$$

But $f'(1) = 5$, so $a - \frac{b}{(1)^2} = 5$
$$\therefore \; a - b = 5$$
$$\therefore \; a = b + 5 \quad .... \; (*)$$

and $f(3) = 5$, so $a(3) + \frac{b}{3} = 5$
$$\therefore \; 3a + \frac{b}{3} = 5$$
$$\therefore \; 3(b+5) + \frac{b}{3} = 5 \qquad \{\text{using } (*)\}$$
$$\therefore \; 3b + 15 + \frac{b}{3} = 5$$
$$\therefore \; \tfrac{10}{3}b = -10$$
$$\therefore \; b = -3$$

and so $a = -3 + 5$
$$= 2$$

11 $$\begin{aligned} y &= 4x - \frac{3}{x} \\ &= 4x - 3x^{-1} \\ \therefore \frac{dy}{dx} &= 4 + 3x^{-2} \\ &= 4 + \frac{3}{x^2} \end{aligned}$$

$\frac{dy}{dx}$ is the gradient function of $y = 4x - \frac{3}{x}$ from which the gradient of the tangent at any point can be found. It is also the instantaneous rate of change of y with respect to x.

12 $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$

a The domain of $f(x)$ is $\{x \mid x > 0\}$.

b $$\begin{aligned} f(x) &= \sqrt{x} - \frac{4}{\sqrt{x}} \\ &= x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} \\ \therefore f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}} \\ &= \frac{1}{2\sqrt{x}} + \frac{2}{x\sqrt{x}} \end{aligned}$$

c The domain of $f'(x)$ is $\{x \mid x > 0\}$.

d $$\begin{aligned} f'(1) &= \frac{1}{2\sqrt{1}} + \frac{2}{1\sqrt{1}} \\ &= \tfrac{1}{2} + 2 \\ &= \tfrac{5}{2} \\ &= 2.5 \end{aligned}$$

The gradient of the tangent to the curve $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$ at $x = 1$ is 2.5.

13 **a** $$\begin{aligned} S &= 2t^2 + 4t \text{ m} \\ \therefore \frac{dS}{dt} &= 4t + 4 \text{ m s}^{-1} \end{aligned}$$

$\frac{dS}{dt}$ is the instantaneous rate of change in position at time t. It is the velocity function of the car.

b When $t = 3$, $$\begin{aligned} \frac{dS}{dt} &= 4(3) + 4 \\ &= 16 \end{aligned}$$

The instantaneous rate of change in position at time $t = 3$ seconds is 16 m s^{-1}, or the velocity of the car at $t = 3$ seconds is 16 m s^{-1}.

14 $$\begin{aligned} C &= 1785 + 3x + 0.002x^2 \text{ pounds} \\ \therefore \frac{dC}{dx} &= 3 + 0.002(2x) \\ &= 3 + 0.004x \text{ pounds per toaster} \end{aligned}$$

When $x = 1000$, $$\begin{aligned} \frac{dC}{dx} &= 3 + 0.004(1000) \\ &= 7 \end{aligned}$$

When 1000 toasters are being produced each week, the cost of production is increasing by £7 per toaster.

EXERCISE 18B.1

1 **a** $g(x) = x^2$ and $f(x) = 2x + 7$

$\therefore\ g(f(x)) = g(2x+7)$
$= (2x+7)^2$

b $g(x) = 2x + 7$ and $f(x) = x^2$

$\therefore\ g(f(x)) = g(x^2)$
$= 2(x^2) + 7$
$= 2x^2 + 7$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$

$\therefore\ g(f(x)) = g(3-4x)$
$= \sqrt{3-4x}$

d $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

$\therefore\ g(f(x)) = g(\sqrt{x})$
$= 3 - 4\sqrt{x}$

e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2 + 3$

$\therefore\ g(f(x)) = g(x^2+3)$
$= \dfrac{2}{x^2+3}$

f $g(x) = x^2 + 3$ and $f(x) = \dfrac{2}{x}$

$\therefore\ g(f(x)) = g\left(\dfrac{2}{x}\right)$
$= \left(\dfrac{2}{x}\right)^2 + 3$
$= \dfrac{4}{x^2} + 3$

2 **Note:** There may be other answers.

a $g(f(x)) = (3x+10)^3$

If we let $f(x) = 3x + 10$ then
$g(f(x)) = (f(x))^3$
$\therefore\ g(x) = x^3$ and $f(x) = 3x + 10$

b $g(f(x)) = (7-2x)^5$

If we let $f(x) = 7 - 2x$ then
$g(f(x)) = (f(x))^5$
$\therefore\ g(x) = x^5$ and $f(x) = 7 - 2x$

c $g(f(x)) = \dfrac{1}{2x+4}$

If we let $f(x) = 2x + 4$ then
$g(f(x)) = \dfrac{1}{f(x)}$
$\therefore\ g(x) = \dfrac{1}{x}$ and $f(x) = 2x + 4$

d $g(f(x)) = \sqrt{x^2 - 3x}$

If we let $f(x) = x^2 - 3x$ then
$g(f(x)) = \sqrt{f(x)}$
$\therefore\ g(x) = \sqrt{x}$ and $f(x) = x^2 - 3x$

e $g(f(x)) = \dfrac{1}{(5x-1)^4}$

If we let $f(x) = 5x - 1$ then
$g(f(x)) = \dfrac{1}{(f(x))^4}$
$\therefore\ g(x) = \dfrac{1}{x^4}$ and $f(x) = 5x - 1$

f $g(f(x)) = \dfrac{10}{(3x-x^2)^3}$

If we let $f(x) = 3x - x^2$ then
$g(f(x)) = \dfrac{10}{(f(x))^3}$
$\therefore\ g(x) = \dfrac{10}{x^3}$ and $f(x) = 3x - x^2$

INVESTIGATION 2 DIFFERENTIATING COMPOSITE FUNCTIONS

1
$$\begin{aligned} y &= (2x+1)^2 \\ &= 4x^2+4x+1 \\ \therefore \frac{dy}{dx} &= 4(2x)+4(1) \\ &= 8x+4 \\ &= 2 \times 2(2x+1)^1 \end{aligned}$$

2
$$\begin{aligned} y &= (3x+1)^2 \\ &= 9x^2+6x+1 \\ \therefore \frac{dy}{dx} &= 9(2x)+6(1) \\ &= 18x+6 \\ &= 3 \times 2(3x+1)^1 \end{aligned}$$

3
$$\begin{aligned} y &= (ax+1)^2 \\ &= a^2x^2+2ax+1 \\ \therefore \frac{dy}{dx} &= a^2(2x)+2a(1) \\ &= 2a^2x+2a \\ &= a \times 2(ax+1)^1 \end{aligned}$$

4 **a**
$$\begin{aligned} y &= u^2 \\ \therefore \frac{dy}{du} &= 2u \end{aligned}$$

b $u = ax+1, \quad y = (ax+1)^2$

i $\dfrac{du}{dx} = a$

ii
$$\begin{aligned} \frac{dy}{du} &= 2u \\ &= 2(ax+1) \\ &= 2ax+2 \end{aligned}$$

iii
$$\begin{aligned} \frac{dy}{du} \times \frac{du}{dx} &= (2ax+2) \times a \\ &= a \times 2(ax+1)^1 \end{aligned}$$

iv Our answer to **iii** is the same as the result in **3**.

c If $y = u^2$ where u is a function of x, we suspect that $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

5
$$\begin{aligned} y &= (x^2+3x)^2 \\ &= (x^2)^2 + 2(x^2)(3x) + (3x)^2 \\ &= x^4+6x^3+9x^2 \\ \therefore \frac{dy}{dx} &= 4x^3+6(3x^2)+9(2x) \\ &= 4x^3+18x^2+18x \\ &= 2(2x^3+9x^2+9x) \\ &= 2(x^2+3x)(2x+3) \quad .... \ (*) \end{aligned}$$

Now, consider $y = u^2$ where $u = x^2+3x$. $\therefore \dfrac{du}{dx} = 2x+3$

Comparing with $(*)$, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Our answer agrees with the rule we suggested in **4 c**.

6 **a**
$$\begin{aligned} y &= (2x+1)^3 \\ &= (2x+1)(4x^2+4x+1) \\ &= 8x^3+12x^2+6x+1 \\ \therefore \frac{dy}{dx} &= 24x^2+24x+6 \end{aligned}$$

b $u = 2x+1, \quad y = u^3$

i $\dfrac{du}{dx} = 2$

ii
$$\begin{aligned} \frac{dy}{du} &= 3u^2 \\ &= 3(2x+1)^2 \end{aligned}$$

iii
$$\begin{aligned} \frac{dy}{du} \times \frac{du}{dx} &= 3(2x+1)^2 \times 2 \\ &= 3(4x^2+4x+1) \times 2 \\ &= (12x^2+12x+3) \times 2 \\ &= 24x^2+24x+6 \end{aligned}$$

iv Our answer to **iii** is the same as the result in **a**.

7 If y is a function of u, and u is a function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

EXERCISE 18B.2

1 **a** $\dfrac{1}{(2x-1)^2} = u^{-2}$ where $u = 2x-1$

b $\sqrt{x^2-3x} = u^{\frac{1}{2}}$ where $u = x^2-3x$

c $\dfrac{2}{\sqrt{2-x^2}} = 2u^{-\frac{1}{2}}$ where $u = 2-x^2$

d $\sqrt[3]{x^3-x^2} = u^{\frac{1}{3}}$ where $u = x^3-x^2$

e $\dfrac{4}{(3-x)^3} = 4u^{-3}$ where $u = 3-x$

f $\dfrac{10}{x^2-3} = 10u^{-1}$ where $u = x^2-3$

2 **a**
$$y = (2x+3)^2$$
$$\therefore\ y = u^2 \quad \text{where} \quad u = 2x+3$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= 2u(2)\\ &= 4u\\ &= 4(2x+3)\\ &= 8x+12\end{aligned}$$

b
$$\begin{aligned}y &= (2x+3)^2\\ &= 4x^2+12x+9\\ \therefore\ \frac{dy}{dx} &= 4(2x)+12(1)\\ &= 8x+12\end{aligned}$$

3 **a**
$$y = (4x-5)^2$$
$$\therefore\ y = u^2 \quad \text{where} \quad u = 4x-5$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= 2u(4)\\ &= 8u\\ &= 8(4x-5)\end{aligned}$$

b
$$y = \frac{1}{5-2x}$$
$$\therefore\ y = u^{-1} \quad \text{where} \quad u = 5-2x$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= -u^{-2}(-2)\\ &= 2u^{-2}\\ &= 2(5-2x)^{-2}\end{aligned}$$

c
$$y = \sqrt{3x-x^2}$$
$$\therefore\ y = u^{\frac{1}{2}} \quad \text{where} \quad u = 3x-x^2$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= \tfrac{1}{2}u^{-\frac{1}{2}}(3-2x)\\ &= \tfrac{1}{2}(3x-x^2)^{-\frac{1}{2}}(3-2x)\end{aligned}$$

d
$$y = (1-3x)^4$$
$$\therefore\ y = u^4 \quad \text{where} \quad u = 1-3x$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= 4u^3(-3)\\ &= -12u^3\\ &= -12(1-3x)^3\end{aligned}$$

e
$$y = 6(5-x)^3$$
$$\therefore\ y = 6u^3 \quad \text{where} \quad u = 5-x$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= 6(3u^2)(-1)\\ &= -18u^2\\ &= -18(5-x)^2\end{aligned}$$

f
$$y = \sqrt[3]{2x^3-x^2}$$
$$\therefore\ y = u^{\frac{1}{3}} \quad \text{where} \quad u = 2x^3-x^2$$
$$\begin{aligned}\text{Now}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}\\ &= \tfrac{1}{3}u^{-\frac{2}{3}}(2(3x^2)-2x)\\ &= \tfrac{1}{3}(2x^3-x^2)^{-\frac{2}{3}}(6x^2-2x)\end{aligned}$$

g $y = \dfrac{6}{(5x-4)^2}$

$\therefore\ y = 6u^{-2}$ where $u = 5x - 4$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 6(-2u^{-3})(5)$

$= -60u^{-3}$

$= -60(5x-4)^{-3}$

h $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

$\therefore\ y = 2u^3$ where $u = x^2 - 2x^{-1}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 2(3u^2)(2x - 2(-1x^{-2}))$

$= 6u^2(2x + 2x^{-2})$

$= 6(x^2 - 2x^{-1})^2(2x + 2x^{-2})$

$= 6\left(x^2 - \dfrac{2}{x}\right)^2\left(2x + \dfrac{2}{x^2}\right)$

4 **a** $y = \sqrt{1-x^2}$

$\therefore\ y = u^{\frac{1}{2}}$ where $u = 1 - x^2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= \frac{1}{2}u^{-\frac{1}{2}}(-2x)$

$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$= -x(1-x^2)^{-\frac{1}{2}}$

At $x = \frac{1}{2}$, $\dfrac{dy}{dx} = -\frac{1}{2}(1-(\frac{1}{2})^2)^{-\frac{1}{2}}$

$= -\frac{1}{2}(\frac{3}{4})^{-\frac{1}{2}}$

$= -\frac{1}{\sqrt{3}}$

$\therefore$ the gradient of the tangent to $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$ is $-\frac{1}{\sqrt{3}}$.

b $y = (3x+2)^6$

$\therefore\ y = u^6$ where $u = 3x + 2$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 6u^5(3)$

$= 18u^5$

$= 18(3x+2)^5$

At $x = -1$, $\dfrac{dy}{dx} = 18(3(-1)+2)^5$

$= 18(-1)^5$

$= -18$

$\therefore$ the gradient of the tangent to $y = (3x+2)^6$ at $x = -1$ is -18.

c $y = \dfrac{1}{(2x-1)^4}$

$\therefore\ y = u^{-4}$ where $u = 2x - 1$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= -4u^{-5}(2)$

$= -8u^{-5}$

$= -8(2x-1)^{-5}$

At $x = 1$, $\dfrac{dy}{dx} = -8(2(1)-1)^{-5}$

$= -8$

$\therefore$ the gradient of the tangent to $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$ is -8.

d $y = 6 \times \sqrt[3]{1-2x}$

$\therefore\ y = 6u^{\frac{1}{3}}$ where $u = 1 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 6(\frac{1}{3}u^{-\frac{2}{3}})(-2)$

$= -4u^{-\frac{2}{3}}$

$= -4(1-2x)^{-\frac{2}{3}}$

At $x = 0$, $\dfrac{dy}{dx} = -4(1-2(0))^{-\frac{2}{3}}$

$= -4$

$\therefore$ the gradient of the tangent to $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$ is -4.

e $$y = \frac{4}{x+2\sqrt{x}}$$

$\therefore \; y = 4u^{-1}$ where $u = x + 2x^{\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$= 4(-1u^{-2})(1 + 2(\tfrac{1}{2}x^{-\frac{1}{2}}))$$
$$= -4u^{-2}(1 + x^{-\frac{1}{2}})$$
$$= -4\left(x + 2x^{\frac{1}{2}}\right)^{-2}(1 + x^{-\frac{1}{2}})$$

At $x = 4$,

$$\frac{dy}{dx} = -4\left(4 + 2(4)^{\frac{1}{2}}\right)^{-2}(1 + 4^{-\frac{1}{2}})$$
$$= -4(8)^{-2}\left(\tfrac{3}{2}\right)$$
$$= -\tfrac{3}{32}$$

$\therefore$ the gradient of the tangent to

$y = \dfrac{4}{x+2\sqrt{x}}$ at $x = 4$ is $-\frac{3}{32}$.

f $$y = \left(x + \frac{1}{x}\right)^3$$

$\therefore \; y = u^3$ where $u = x + x^{-1}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$= 3u^2(1 - x^{-2})$$
$$= 3(x + x^{-1})^2(1 - x^{-2})$$

At $x = 1$, $\dfrac{dy}{dx} = 3(1 + 1^{-1})^2(1 - 1^{-2})$

$$= 3(4)(0)$$
$$= 0$$

$\therefore$ the gradient of the tangent to

$y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$ is 0.

5 $y = f(x) = (2x - b)^a$

$\therefore \; y = u^a$ where $u = 2x - b$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$= au^{a-1}(2)$$
$$= 2au^{a-1}$$
$$= 2a(2x - b)^{a-1}$$

But $f'(x) = 24x^2 - 24x + 6$ {given}

$\therefore \; 2a(2x - b)^{a-1} = 24x^2 - 24x + 6$

$\therefore \; a - 1 = 2$ {the derivative has degree 2}

$\therefore \; a = 3$

So, $6(2x - b)^2 = 24x^2 - 24x + 6$

$\therefore \; 6(4x^2 - 4xb + b^2) = 24x^2 - 24x + 6$

$\therefore \; 24x^2 - 24xb + 6b^2 = 24x^2 - 24x + 6$

Equating coefficients of x, $-24b = -24$

$\therefore \; b = 1$

So, $a = 3$ and $b = 1$.

6 $y = \dfrac{a}{\sqrt{1+bx}}$

$\therefore\ y = au^{-\frac{1}{2}}$ where $u = 1 + bx$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= a(-\frac{1}{2}u^{-\frac{3}{2}})(b)$

$= -\frac{1}{2}abu^{-\frac{3}{2}}$

$= -\frac{1}{2}ab(1+bx)^{-\frac{3}{2}}$

When $x = 3$, $y = 1$, and $\dfrac{dy}{dx} = -\frac{1}{8}$

$\therefore\ 1 = \dfrac{a}{\sqrt{1+b(3)}}$ and $-\frac{1}{8} = -\frac{1}{2}ab(1+b(3))^{-\frac{3}{2}}$

$\therefore\ a = \sqrt{1+3b}$ .... $(*)$

$\therefore\ \frac{1}{4} = ab(1+3b)^{-\frac{3}{2}}$

$= \dfrac{\cancel{\sqrt{1+3b}}(b)}{\cancel{\sqrt{1+3b}}(1+3b)}$ {using $(*)$}

$\therefore\ \frac{1}{4} = \dfrac{b}{1+3b}$

$\therefore\ 1 + 3b = 4b$

$\therefore\ b = 1$

$\therefore\ a = \sqrt{1+3(1)}$ {substituting $b = 1$ into $(*)$}

$= 2$

So, $a = 2$ and $b = 1$.

7 $y = f(x) = 3\left(ax - \dfrac{b}{x}\right)^3$

$\therefore\ y = 3u^3$ where $u = ax - bx^{-1}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 3(3u^2)(a - b(-1x^{-2}))$

$= 9u^2(a + bx^{-2})$

$= 9(ax - bx^{-1})^2(a + bx^{-2})$

$f\left(\frac{3}{2}\right) = 3$ {given}

$\therefore\ 3\left(a\left(\frac{3}{2}\right) - \dfrac{b}{\frac{3}{2}}\right)^3 = 3$

$\therefore\ \left(\frac{3}{2}a - \frac{2}{3}b\right)^3 = 1$

$\therefore\ \frac{3}{2}a - \frac{2}{3}b = 1$

$\therefore\ \frac{3}{2}a = \frac{2}{3}b + 1$

$\therefore\ a = \frac{2}{3}\left(\frac{2}{3}b + 1\right)$

$\therefore\ a = \frac{4}{9}b + \frac{2}{3}$ .... $(*)$

Also, $f'\left(\frac{3}{2}\right) = 30$

$$\therefore \frac{dy}{dx} = 30 \quad \text{when} \quad x = \frac{3}{2}$$

$$\therefore 9\left(a\left(\tfrac{3}{2}\right) - b\left(\tfrac{3}{2}\right)^{-1}\right)^2 \left(a + b\left(\tfrac{3}{2}\right)^{-2}\right) = 30$$

$$\therefore \left(\tfrac{3}{2}a - \tfrac{2}{3}b\right)^2 \left(a + \tfrac{4}{9}b\right) = \tfrac{10}{3}$$

$$\therefore \left(\tfrac{3}{2}\left(\tfrac{4}{9}b + \tfrac{2}{3}\right) - \tfrac{2}{3}b\right)^2 \left(\left(\tfrac{4}{9}b + \tfrac{2}{3}\right) + \tfrac{4}{9}b\right) = \tfrac{10}{3} \qquad \{\text{using } (*)\}$$

$$\therefore \left(\tfrac{2}{\cancel{3}}\cancel{b} + 1 - \tfrac{2}{\cancel{3}}\cancel{b}\right)^2 \left(\tfrac{8}{9}b + \tfrac{2}{3}\right) = \tfrac{10}{3}$$

$$\therefore \tfrac{8}{9}b + \tfrac{2}{3} = \tfrac{10}{3}$$

$$\therefore \tfrac{8}{9}b = \tfrac{8}{3}$$

$$\therefore b = 3$$

$$\therefore a = \tfrac{4}{9}(3) + \tfrac{2}{3} \qquad \{\text{substituting } b = 3 \text{ into } (*)\}$$

$$= 2$$

So, $a = 2$ and $b = 3$.

INVESTIGATION 3 THE PRODUCT RULE

1 $u(x) = x, \quad v(x) = x, \quad f(x) = u(x)\,v(x) = x^2$

a $f'(x) = 2x$ **b** $u'(x) = 1, \quad v'(x) = 1$ **c** $f'(x) \neq u'(x)\,v'(x)$

2 $u(x) = x, \quad v(x) = \sqrt{x} = x^{\frac{1}{2}}, \quad f(x) = x\sqrt{x} = x^{\frac{3}{2}}$

a $f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \dfrac{3}{2\sqrt{x}}$ **b** $u'(x) = 1, \quad v'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ **c** $f'(x) \neq u'(x)\,v'(x)$

3

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)\,v(x) + u(x)\,v'(x)$
x^2	$2x$	x	x	1	1	$2x$
$x^{\frac{3}{2}}$	$\frac{3}{2}\sqrt{x}$	x	$\sqrt{x}$	1	$\dfrac{1}{2\sqrt{x}}$	$\frac{3}{2}\sqrt{x}$
$x(x+1)$	$2x+1$	x	$x+1$	1	1	$2x+1$
$(x-1)(2-x^2)$	$-3x^2+2x+2$	$x-1$	$2-x^2$	1	$-2x$	$-3x^2+2x+2$

4 If $f(x) = u(x)\,v(x)$ then $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$.

EXERCISE 18C

1 **a** $f(x) = x(x-1)$ is the product of $u(x) = x$ and $v(x) = x - 1$

$\therefore\ u'(x) = 1$ and $v'(x) = 1$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$= 1(x-1) + x(1)$

$= x - 1 + x$

$= 2x - 1$

b $f(x) = 2x(x+1)$ is the product of $u(x) = 2x$ and $v(x) = x + 1$

$\therefore\ u'(x) = 2$ and $v'(x) = 1$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$= 2(x+1) + 2x(1)$

$= 2x + 2 + 2x$

$= 4x + 2$

c $f(x) = x^2\sqrt{x+1}$ is the product of $u(x) = x^2$ and $v(x) = (x+1)^{\frac{1}{2}}$

$\therefore\ u'(x) = 2x$ and $v'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}(1)$ {chain rule}

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$= 2x(x+1)^{\frac{1}{2}} + x^2(\frac{1}{2}(x+1)^{-\frac{1}{2}})$

$= 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}x^2(x+1)^{-\frac{1}{2}}$

d $f(x) = (x+3)(x-1)$ is the product of $u(x) = x + 3$ and $v(x) = x - 1$

$\therefore\ u'(x) = 1$ and $v'(x) = 1$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$= 1(x-1) + (x+3)(1)$

$= x - 1 + x + 3$

$= 2x + 2$

e $f(x) = x\sqrt{x^2-1}$ is the product of $u(x) = x$ and $v(x) = (x^2-1)^{\frac{1}{2}}$

$\therefore\ u'(x) = 1$ and $v'(x) = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)$ {chain rule}

$= x(x^2-1)^{-\frac{1}{2}}$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$= 1(x^2-1)^{\frac{1}{2}} + x(x(x^2-1)^{-\frac{1}{2}})$

$= (x^2-1)^{\frac{1}{2}} + x^2(x^2-1)^{-\frac{1}{2}}$

f $f(x) = x(x+1)^2$ is the product of $u(x) = x$ and $v(x) = (x+1)^2$

$\therefore\ u'(x) = 1$ and $v'(x) = 2(x+1)(1)$ {chain rule}

$= 2x + 2$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$= 1(x+1)^2 + x(2x+2)$

$= (x+1)^2 + 2x(x+1)$

2 **a** $y = x^2(2x-1)$ is the product of $u = x^2$ and $v = 2x - 1$

$\therefore\ u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 2x(2x-1) + x^2(2)$

$= 2x(2x-1) + 2x^2$

b $y = 4x(2x+1)^3$ is the product of $u = 4x$ and $v = (2x+1)^3$

$\therefore\ u' = 4$ and $v' = 3(2x+1)^2(2)$ {chain rule}

$= 6(2x+1)^2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 4(2x+1)^3 + 4x\big(6(2x+1)^2\big)$

$= 4(2x+1)^3 + 24x(2x+1)^2$

c $y = x^2\sqrt{3-x}$ is the product of $u = x^2$ and $v = (3-x)^{\frac{1}{2}}$

$\therefore\ u' = 2x$ and $v' = \frac{1}{2}(3-x)^{-\frac{1}{2}}(-1)$ {chain rule}

$= -\frac{1}{2}(3-x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 2x(3-x)^{\frac{1}{2}} + x^2(-\frac{1}{2}(3-x)^{-\frac{1}{2}})$

$= 2x(3-x)^{\frac{1}{2}} - \frac{1}{2}x^2(3-x)^{-\frac{1}{2}}$

d $y = \sqrt{x}(x-3)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-3)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x-3)(1)$ {chain rule}

$= 2(x-3)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + x^{\frac{1}{2}}(2(x-3))$

$= \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$

e $y = 5x^2(3x^2-1)^2$ is the product of $u = 5x^2$ and $v = (3x^2-1)^2$

$\therefore\ u' = 5(2x)$ and $v' = 2(3x^2-1)(3(2x))$ {chain rule}

$= 10x \qquad = 12x(3x^2-1)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 10x(3x^2-1)^2 + 5x^2\big(12x(3x^2-1)\big)$

$= 10x(3x^2-1)^2 + 60x^3(3x^2-1)$

f $y = \sqrt{x}(x-x^2)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-x^2)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(x-x^2)^2(1-2x)$ {chain rule}

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + x^{\frac{1}{2}}\big(3(x-x^2)^2(1-2x)\big)$

$= \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$

3 **a** $y = x^4(1-2x)^2$ is the product of $u = x^4$ and $v = (1-2x)^2$

$\therefore\ u' = 4x^3$ and $v' = 2(1-2x)(-2)$ {chain rule}

$= -4(1-2x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 4x^3(1-2x)^2 + x^4(-4(1-2x))$

$= 4x^3(1-2x)^2 - 4x^4(1-2x)$

At $x=-1$, $\dfrac{dy}{dx} = 4(-1)^3(1-2(-1))^2 - 4(-1)^4(1-2(-1))$

$= -4(9) - 4(3)$

$= -48$

$\therefore$ the gradient of the tangent to $y = x^4(1-2x)^2$ at $x=-1$ is -48.

b $y = \sqrt{x}(x^2-x+1)^2$ is the product of

$u = x^{\frac{1}{2}}$ and $v = (x^2-x+1)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x^2-x+1)(2x-1)$ {chain rule}

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= \frac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + x^{\frac{1}{2}}\big(2(x^2-x+1)(2x-1)\big)$

At $x=4$, $\dfrac{dy}{dx} = \frac{1}{2}(4)^{-\frac{1}{2}}(4^2-4+1)^2 + 4^{\frac{1}{2}}\big(2(4^2-4+1)(2(4)-1)\big)$

$= \frac{1}{4}(169) + 2(2(13)(7))$

$= \frac{169}{4} + 364$

$= 406\frac{1}{4}$

$\therefore$ the gradient of the tangent to $y = \sqrt{x}(x^2-x+1)^2$ at $x=4$ is $406\frac{1}{4}$.

c $y = x\sqrt{1-2x}$ is the product of $u = x$ and $v = (1-2x)^{\frac{1}{2}}$

$$\therefore \quad u' = 1 \quad \text{and} \quad v' = \tfrac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) \qquad \{\text{chain rule}\}$$
$$= -(1-2x)^{-\frac{1}{2}}$$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$= 1(1-2x)^{\frac{1}{2}} + x(-(1-2x)^{-\frac{1}{2}})$$
$$= (1-2x)^{\frac{1}{2}} - x(1-2x)^{-\frac{1}{2}}$$

At $x = -4$, $\dfrac{dy}{dx} = (1-2(-4))^{\frac{1}{2}} - (-4)(1-2(-4))^{-\frac{1}{2}}$

$$= 3 + 4\left(\tfrac{1}{3}\right)$$
$$= \tfrac{13}{3}$$

$\therefore$ the gradient of the tangent to $y = x\sqrt{1-2x}$ at $x = -4$ is $\frac{13}{3}$.

d $y = x^3\sqrt{5-x^2}$ is the product of $u = x^3$ and $v = (5-x^2)^{\frac{1}{2}}$

$$\therefore \quad u' = 3x^2 \quad \text{and} \quad v' = \tfrac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) \qquad \{\text{chain rule}\}$$
$$= -x(5-x^2)^{-\frac{1}{2}}$$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$= 3x^2(5-x^2)^{\frac{1}{2}} + x^3(-x(5-x^2)^{-\frac{1}{2}})$$
$$= 3x^2(5-x^2)^{\frac{1}{2}} - x^4(5-x^2)^{-\frac{1}{2}}$$

At $x = 1$, $\dfrac{dy}{dx} = 3(1)^2(5-1^2)^{\frac{1}{2}} - 1^4(5-1^2)^{-\frac{1}{2}}$

$$= 3(2) - 1\left(\tfrac{1}{2}\right)$$
$$= \tfrac{11}{2}$$

$\therefore$ the gradient of the tangent to $y = x^3\sqrt{5-x^2}$ at $x = 1$ is $\frac{11}{2}$.

4 **a** $y = \sqrt{x}(3-x)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (3-x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(3-x)(-1)$ {chain rule}

$= -2(3-x)$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}
&= \tfrac{1}{2}x^{-\frac{1}{2}}(3-x)^2 + x^{\frac{1}{2}}(-2(3-x)) \\
&= \tfrac{1}{2}x^{-\frac{1}{2}}(3-x)^2 - 2x^{\frac{1}{2}}(3-x) \\
&= (3-x)\left[\frac{1}{2\sqrt{x}}(3-x) - 2\sqrt{x}\right] \\
&= (3-x)\left[\frac{3-x}{2\sqrt{x}} - 2\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}}\right] \\
&= (3-x)\left(\frac{3-x-4x}{2\sqrt{x}}\right) \\
&= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}
\end{aligned}$$

b The tangent is horizontal when its gradient is zero.

$\therefore\ \frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}} = 0$

$\therefore\ (3-x)(3-5x) = 0$

$\therefore\ x = 3$ or $\frac{3}{5}$

c $\frac{dy}{dx}$ is defined if its denominator is greater than zero.

$\therefore\ 2\sqrt{x} > 0$

$\therefore\ x > 0$

$\therefore$ the domain of $\frac{dy}{dx}$ is $\{x \mid x > 0\}$ and the domain of the original function is $\{x \mid x \geqslant 0\}$.

$\frac{dy}{dx}$ is undefined when $x = 0$.

5 $y = -2x^2(x+4)$ is the product of $u = -2x^2$ and $v = x + 4$

$\therefore\ u' = -2(2x)$ and $v' = 1$

$= -4x$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}
&= -4x(x+4) + (-2x^2)(1) \\
&= -4x^2 - 16x - 2x^2 \\
&= -6x^2 - 16x
\end{aligned}$$

$\frac{dy}{dx} = 10$ when $-6x^2 - 16x = 10$

$\therefore\ 6x^2 + 16x + 10 = 0$

$\therefore\ 3x^2 + 8x + 5 = 0$

$\therefore\ (3x+5)(x+1) = 0$

$\therefore\ x = -1$ or $-\frac{5}{3}$

6 $y = (x+3)(x-2)^2$ is the product of $u = x+3$ and $v = (x-2)^2$

$\therefore\ u' = 1$ and $v' = 2(x-2)$
$\qquad = 2x - 4$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\frac{dy}{dx} &= u'v + uv' \\ &= 1(x-2)^2 + (x+3)(2x-4) \\ &= x^2 - 4x + 4 + 2x^2 - 4x + 6x - 12 \\ &= 3x^2 - 2x - 8\end{aligned}$$

$\dfrac{dy}{dx} = -7$ when $3x^2 - 2x - 8 = -7$

$$\begin{aligned}\therefore\ 3x^2 - 2x - 1 &= 0 \\ \therefore\ (3x+1)(x-1) &= 0 \\ \therefore\ x = 1 \text{ or } x &= -\tfrac{1}{3}\end{aligned}$$

7 $f(x) = ax\sqrt{1-x}$ is the product of $u(x) = ax$ and $v(x) = (1-x)^{\frac{1}{2}}$

$\therefore\ u'(x) = a$ and $v'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$ {chain rule}
$\qquad = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$$\begin{aligned} &= a(1-x)^{\frac{1}{2}} + ax(-\tfrac{1}{2}(1-x)^{-\frac{1}{2}}) \\ &= a(1-x)^{\frac{1}{2}} - \tfrac{1}{2}ax(1-x)^{-\frac{1}{2}}\end{aligned}$$

a The tangent to $f(x) = ax\sqrt{1-x}$ has gradient 0 when $f'(x) = 0$

$$\begin{aligned}\therefore\ a(1-x)^{\frac{1}{2}} - \tfrac{1}{2}ax(1-x)^{-\frac{1}{2}} &= 0 \\ \therefore\ a\sqrt{1-x} - \frac{ax}{2\sqrt{1-x}} &= 0 \\ \therefore\ a\sqrt{1-x} &= \frac{ax}{2\sqrt{1-x}} \\ \therefore\ 2a(1-x) &= ax \\ \therefore\ 2 - 2x &= x \qquad \{a \neq 0\} \\ \therefore\ 3x &= 2 \\ \therefore\ x &= \tfrac{2}{3}\end{aligned}$$

b The tangent to $f(x) = ax\sqrt{1-x}$ has gradient a when $f'(x) = a$

$$\therefore\ a(1-x)^{\frac{1}{2}} - \tfrac{1}{2}ax(1-x)^{-\frac{1}{2}} = a$$

$$\therefore\ a\sqrt{1-x} - \frac{ax}{2\sqrt{1-x}} = a$$

$$\therefore\ \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = 1 \qquad \{a \neq 0\}$$

$$\therefore\ \frac{2(1-x)-x}{2\sqrt{1-x}} = 1$$

$$\therefore\ 2 - 3x = 2\sqrt{1-x}$$

$$\therefore\ (2-3x)^2 = 4(1-x)$$

{squaring both sides assuming both sides are $\geqslant 0$}

$$\therefore\ 4 - 12x + 9x^2 = 4 - 4x$$

$$\therefore\ 9x^2 - 8x = 0$$

$$\therefore\ x(9x-8) = 0$$

$$\therefore\ x = 0 \text{ or } \tfrac{8}{9}$$

Check: If $x = 0$, $f'(0) = a(1-0)^{\frac{1}{2}} - \frac{1}{2}a(0)(1-0)^{-\frac{1}{2}} = a$ ✓

If $x = \frac{8}{9}$, $f'\left(\frac{8}{9}\right) = a\left(1-\frac{8}{9}\right)^{\frac{1}{2}} - \frac{1}{2}a\left(\frac{8}{9}\right)\left(1-\frac{8}{9}\right)^{-\frac{1}{2}} = \frac{1}{3}a - \frac{4}{3}a = -a$ ✗

So, $x = 0$.

8 $f(x) = x^2\sqrt{x^2+a}$ is the product of $u(x) = x^2$ and $v(x) = (x^2+a)^{\frac{1}{2}}$

$$\therefore\ u'(x) = 2x \text{ and } v'(x) = \tfrac{1}{2}(x^2+a)^{-\frac{1}{2}}(2x) \quad \{\text{chain rule}\}$$
$$= x(x^2+a)^{-\frac{1}{2}}$$

Now $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$ {product rule}

$$= 2x(x^2+a)^{\frac{1}{2}} + x^2\left(x(x^2+a)^{-\frac{1}{2}}\right)$$

$$= 2x\sqrt{x^2+a} + \frac{x^3}{\sqrt{x^2+a}}$$

and $f'(-2) = -\frac{34}{3}$

$$\therefore\ 2(-2)\sqrt{(-2)^2+a} + \frac{(-2)^3}{\sqrt{(-2)^2+a}} = -\frac{34}{3}$$

$$\therefore\ -4\sqrt{a+4} - \frac{8}{\sqrt{a+4}} = -\frac{34}{3}$$

$$\therefore\ -4(a+4) - 8 = -\tfrac{34}{3}\sqrt{a+4}$$

$$\therefore\ -4(a+4) + \tfrac{34}{3}\sqrt{a+4} - 8 = 0$$

$$\therefore\ -12(a+4) + 34\sqrt{a+4} - 24 = 0$$

$$\therefore\ -12X^2 + 34X - 24 = 0 \qquad \{\text{letting } X = \sqrt{a+4}\}$$

$$\therefore\ 6X^2 - 17X + 12 = 0$$

$$\therefore\ (3X-4)(2X-3) = 0$$

$$\therefore\ X = \tfrac{4}{3} \text{ or } \tfrac{3}{2}$$

$\therefore \sqrt{a+4} = \frac{4}{3}$ or $\sqrt{a+4} = \frac{3}{2}$

$\therefore a+4 = \frac{16}{9}$ or $a+4 = \frac{9}{4}$

$\therefore a = -\frac{20}{9}$ or $a = -\frac{7}{4}$

EXERCISE 18D

1 **a** $y = \dfrac{1+3x}{2-x}$ is a quotient with

$u = 1+3x$ and $v = 2-x$

$\therefore u' = 3$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{3(2-x) - (1+3x)(-1)}{(2-x)^2}$$

$$= \frac{6 - 3x + 1 + 3x}{(2-x)^2}$$

$$= \frac{7}{(2-x)^2}$$

b $y = \dfrac{x^2}{2x+1}$ is a quotient with

$u = x^2$ and $v = 2x+1$

$\therefore u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{2x(2x+1) - x^2(2)}{(2x+1)^2}$$

$$= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2}$$

$$= \frac{2x^2 + 2x}{(2x+1)^2}$$

c $y = \dfrac{x}{x^2-3}$ is a quotient with

$u = x$ and $v = x^2 - 3$

$\therefore u' = 1$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{1(x^2-3) - x(2x)}{(x^2-3)^2}$$

$$= \frac{x^2 - 3 - 2x^2}{(x^2-3)^2}$$

$$= \frac{-x^2 - 3}{(x^2-3)^2}$$

d $y = \dfrac{\sqrt{x}}{1-2x}$ is a quotient with

$u = x^{\frac{1}{2}}$ and $v = 1-2x$

$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) - \sqrt{x}(-2)}{(1-2x)^2}$$

$$= \frac{\frac{1-2x}{2\sqrt{x}} + 2\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}}}{(1-2x)^2}$$

$$= \frac{1 - 2x + 4x}{2\sqrt{x}(1-2x)^2}$$

$$= \frac{2x+1}{2\sqrt{x}(1-2x)^2}$$

e $y = \dfrac{x^2-3}{3x-x^2}$ is a quotient with

$u = x^2 - 3$ and $v = 3x - x^2$

$\therefore u' = 2x$ and $v' = 3 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2}$$

$$= \frac{6x^2 - 2x^3 - (3x^2 - 2x^3 - 9 + 6x)}{(3x-x^2)^2}$$

$$= \frac{3x^2 - 6x + 9}{(3x-x^2)^2}$$

f $y = \dfrac{x}{\sqrt{1-3x}}$ is a quotient with $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$\therefore\; u' = 1$ and $v' = \frac{1}{2}(1-3x)^{-\frac{1}{2}}(-3)$ {chain rule}

$= -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{1(1-3x)^{\frac{1}{2}} - x(-\frac{3}{2}(1-3x)^{-\frac{1}{2}})}{(\sqrt{1-3x})^2}$$

$$= \frac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x}$$

$$= \frac{\sqrt{1-3x} + \frac{3x}{2\sqrt{1-3x}}}{1-3x}$$

$$= \frac{\sqrt{1-3x} \times \frac{2\sqrt{1-3x}}{2\sqrt{1-3x}} + \frac{3x}{2\sqrt{1-3x}}}{1-3x}$$

$$= \frac{2(1-3x)+3x}{2\sqrt{1-3x}(1-3x)}$$

$$= \frac{2-3x}{2(1-3x)^{\frac{3}{2}}}$$

2 **a** $\dfrac{x+1}{3-x}$ is a quotient with $u = x+1$ and $v = 3-x$

$\therefore\; u' = 1$ and $v' = -1$

Now $\dfrac{d}{dx}\left(\dfrac{x+1}{3-x}\right) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{1(3-x) - (x+1)(-1)}{(3-x)^2}$$

$$= \frac{3 - \cancel{x} + \cancel{x} + 1}{(3-x)^2}$$

$$= \frac{4}{(3-x)^2}$$

b $\dfrac{3x}{x^2-1}$ is a quotient with $u = 3x$ and $v = x^2-1$

$\therefore\; u' = 3$ and $v' = 2x$

Now $\dfrac{d}{dx}\left(\dfrac{3x}{x^2-1}\right) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{3(x^2-1) - (3x)(2x)}{(x^2-1)^2}$$

$$= \frac{3x^2 - 3 - 6x^2}{(x^2-1)^2}$$

$$= \frac{-3x^2-3}{(x^2-1)^2}$$

c $\dfrac{x^3}{2x-1}$ is a quotient with $u = x^3$ and $v = 2x - 1$

$\therefore\ u' = 3x^2$ and $v' = 2$

$$\text{Now } \frac{d}{dx}\left(\frac{x^3}{2x-1}\right) = \frac{u'v - uv'}{v^2}$$

$$= \frac{3x^2(2x-1) - x^3(2)}{(2x-1)^2}$$

$$= \frac{6x^3 - 3x^2 - 2x^3}{(2x-1)^2}$$

$$= \frac{4x^3 - 3x^2}{(2x-1)^2}$$

d $\dfrac{4x}{\sqrt{x-5}}$ is a quotient with $u = 4x$ and $v = (x-5)^{\frac{1}{2}}$

$\therefore\ u' = 4$ and $v' = \frac{1}{2}(x-5)^{-\frac{1}{2}}(1)$ {chain rule}

$= \frac{1}{2}(x-5)^{-\frac{1}{2}}$

$$\text{Now } \frac{d}{dx}\left(\frac{4x}{\sqrt{x-5}}\right) = \frac{u'v - uv'}{v^2}$$

$$= \frac{4(x-5)^{\frac{1}{2}} - 4x(\frac{1}{2}(x-5)^{-\frac{1}{2}})}{\left((x-5)^{\frac{1}{2}}\right)^2}$$

$$= \frac{4(x-5)^{\frac{1}{2}} - 2x(x-5)^{-\frac{1}{2}}}{x-5} \times \frac{(x-5)^{\frac{1}{2}}}{(x-5)^{\frac{1}{2}}}$$

$$= \frac{4(x-5) - 2x}{(x-5)^{\frac{3}{2}}}$$

$$= \frac{4x - 20 - 2x}{(x-5)^{\frac{3}{2}}}$$

$$= \frac{2x - 20}{(x-5)^{\frac{3}{2}}}$$

e $\dfrac{\sqrt{x}}{3-x^2}$ is a quotient with $u = x^{\frac{1}{2}}$ and $v = 3 - x^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2x$

$$\text{Now } \frac{d}{dx}\left(\frac{\sqrt{x}}{3-x^2}\right) = \frac{u'v - uv'}{v^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(3-x^2) - x^{\frac{1}{2}}(-2x)}{(3-x^2)^2}$$

$$= \frac{\frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} + 2x^{\frac{3}{2}}}{(3-x^2)^2}$$

$$= \frac{\frac{3}{2\sqrt{x}} + \frac{3x\sqrt{x}}{2}}{(3-x^2)^2} \times \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{3x^2 + 3}{2\sqrt{x}(3-x^2)^2}$$

f $-\dfrac{x^2}{\sqrt{x^2+3}}$ is a quotient with $u = -x^2$ and $v = (x^2+3)^{\frac{1}{2}}$

$\therefore\ u' = -2x$ and $v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times (2x)$ {chain rule}

$= x(x^2+3)^{-\frac{1}{2}}$

$$\text{Now}\quad \frac{d}{dx}\left(-\frac{x^2}{\sqrt{x^2+3}}\right) = \frac{u'v-uv'}{v^2}$$

$$= \frac{(-2x)(x^2+3)^{\frac{1}{2}} - (-x^2)(x(x^2+3)^{-\frac{1}{2}})}{\left((x^2+3)^{\frac{1}{2}}\right)^2}$$

$$= \frac{(-2x)\sqrt{x^2+3} + \frac{x^3}{\sqrt{x^2+3}}}{x^2+3} \times \frac{\sqrt{x^2+3}}{\sqrt{x^2+3}}$$

$$= \frac{(-2x)(x^2+3)+x^3}{(x^2+3)^{\frac{3}{2}}}$$

$$= \frac{-2x^3-6x+x^3}{(x^2+3)^{\frac{3}{2}}}$$

$$= \frac{-x^3-6x}{(x^2+3)^{\frac{3}{2}}}$$

3 **a** $y = \dfrac{x}{1-2x}$ is a quotient with

$u = x$ and $v = 1-2x$

$\therefore\ u' = 1$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v-uv'}{v^2}$ {quotient rule}

$$= \frac{1(1-2x)-x(-2)}{(1-2x)^2}$$

$$= \frac{1}{(1-2x)^2}$$

At $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{(1-2(1))^2}$

$$= \frac{1}{(-1)^2}$$

$$= 1$$

$\therefore$ the gradient of the tangent $= 1$

b $y = \dfrac{x^3}{x^2+1}$ is a quotient with

$u = x^3$ and $v = x^2+1$

$\therefore\ u' = 3x^2$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v-uv'}{v^2}$ {quotient rule}

$$= \frac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2}$$

$$= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2}$$

$$= \frac{x^4+3x^2}{(x^2+1)^2}$$

At $x = -1$, $\dfrac{dy}{dx} = \dfrac{(-1)^4+3(-1)^2}{((-1)^2+1)^2}$

$$= \frac{4}{4}$$

$$= 1$$

$\therefore$ the gradient of the tangent $= 1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ is a quotient with

$u = x^{\frac{1}{2}}$ and $v = 2x + 1$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{\frac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}$$

At $x = 4$, $\dfrac{dy}{dx} = \dfrac{\frac{1}{2\sqrt{4}}(2(4)+1) - 2\sqrt{4}}{(2(4)+1)^2}$

$$= \frac{\frac{9}{4} - 4}{81}$$

$$= \frac{-\frac{7}{4}}{81}$$

$$= -\tfrac{7}{324}$$

$\therefore$ the gradient of the tangent $= -\frac{7}{324}$

d $y = \dfrac{x^2}{\sqrt{x^2+5}}$ is a quotient with

$u = x^2$ and $v = (x^2+5)^{\frac{1}{2}}$

$\therefore\ u' = 2x$ and $v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x)$

$$= x(x^2+5)^{-\frac{1}{2}}$$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{2x\sqrt{x^2+5} - x^2\left(\frac{x}{\sqrt{x^2+5}}\right)}{(\sqrt{x^2+5})^2}$$

$$= \frac{2x\sqrt{x^2+5} - \frac{x^3}{\sqrt{x^2+5}}}{x^2+5}$$

At $x = -2$,

$$\frac{dy}{dx} = \frac{2(-2)\sqrt{(-2)^2+5} - \frac{(-2)^3}{\sqrt{(-2)^2+5}}}{(-2)^2+5}$$

$$= \frac{-4(3) - \frac{-8}{3}}{9}$$

$$= \frac{-\frac{28}{3}}{9}$$

$$= -\tfrac{28}{27}$$

$\therefore$ the gradient of the tangent $= -\frac{28}{27}$

4 $f(x) = \dfrac{x}{\sqrt{x-1}}$ is a quotient with

$u(x) = x$ and $v(x) = (x-1)^{\frac{1}{2}}$

$\therefore\ u'(x) = 1$ and $v'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1)$ {chain rule}

$$= \frac{1}{2\sqrt{x-1}}$$

Now $f'(x) = \dfrac{u'(x)\,v(x) + u(x)\,v'(x)}{[v(x)]^2}$ {quotient rule}

$$= \frac{1\sqrt{x-1} - x\left(\frac{1}{2\sqrt{x-1}}\right)}{(\sqrt{x-1})^2}$$

$$= \frac{\sqrt{x-1} - \frac{x}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{\sqrt{x-1} \times \frac{2\sqrt{x-1}}{2\sqrt{x-1}} - \frac{x}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{2(x-1) - x}{2\sqrt{x-1}(x-1)}$$

$$= \frac{x-2}{2(x-1)^{\frac{3}{2}}}$$

Check:
$$\begin{aligned} f(x) &= \frac{x}{\sqrt{x-1}} \\ &= \frac{x-1}{\sqrt{x-1}} + \frac{1}{\sqrt{x-1}} \\ &= (x-1)^{\frac{1}{2}} + (x-1)^{-\frac{1}{2}} \\ \therefore\ f'(x) &= \tfrac{1}{2}(x-1)^{-\frac{1}{2}}(1) + (-\tfrac{1}{2}(x-1)^{-\frac{3}{2}}(1)) \qquad \text{\{chain rule\}} \\ &= \frac{1}{2\sqrt{x-1}} - \frac{1}{2(x-1)^{\frac{3}{2}}} \\ &= \frac{1}{2\sqrt{x-1}} \times \frac{(x-1)}{(x-1)} - \frac{1}{2(x-1)^{\frac{3}{2}}} \\ &= \frac{x-1-1}{2(x-1)^{\frac{3}{2}}} \\ &= \frac{x-2}{2(x-1)^{\frac{3}{2}}} \quad \checkmark \end{aligned}$$

5 **a** $y = \dfrac{2\sqrt{x}}{1-x}$ is a quotient with $u = 2x^{\frac{1}{2}}$ and $v = 1-x$

$\therefore\ u' = x^{-\frac{1}{2}}$ and $v' = -1$

Now
$$\begin{aligned} \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \qquad \text{\{quotient rule\}} \\ &= \frac{\frac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2} \times \left(\frac{\sqrt{x}}{\sqrt{x}}\right) \\ &= \frac{(1-x)+2x}{\sqrt{x}(1-x)^2} \\ &= \frac{x+1}{\sqrt{x}(1-x)^2} \quad \text{as required} \end{aligned}$$

b **i** $\dfrac{dy}{dx} = 0$ when $x+1=0$ $\therefore$ $x=-1$.

However $\dfrac{dy}{dx}$ is not defined for $x \leqslant 0$ because of the $\sqrt{x}$ term. Hence $\dfrac{dy}{dx}$ never equals 0.

ii $\dfrac{dy}{dx}$ is undefined when $\sqrt{x}$ is undefined or $\sqrt{x}(1-x)^2 = 0$

$\therefore$ when $x \leqslant 0$ and when $x = 1$

6 **a** $y = \dfrac{x^2+6}{2x+1}$ is a quotient with $u = x^2+6$ and $v = 2x+1$

$\therefore\ u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{2x(2x+1)-(x^2+6)(2)}{(2x+1)^2}$$

$$= \frac{4x^2+2x-2x^2-12}{(2x+1)^2}$$

$$= \frac{2x^2+2x-12}{(2x+1)^2} \quad \text{as required}$$

b **i** $\dfrac{dy}{dx} = 0$ when $2x^2+2x-12=0$

$\therefore\ x^2+x-6=0$

$\therefore\ (x+3)(x-2)=0$

$\therefore\ x=-3$ or $x=2$

ii $\dfrac{dy}{dx}$ is undefined when $(2x+1)^2=0$

$\therefore\ 2x+1=0$

$\therefore\ x=-\frac{1}{2}$

7 **a** $y = \dfrac{x^2-3x+1}{x+2}$ is a quotient with $u = x^2-3x+1$ and $v = x+2$

$\therefore\ u' = 2x-3$ and $v' = 1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{(2x-3)(x+2)-(x^2-3x+1)(1)}{(x+2)^2}$$

$$= \frac{2x^2+4x-3x-6-x^2+3x-1}{(x+2)^2}$$

$$= \frac{x^2+4x-7}{(x+2)^2} \quad \text{as required}$$

b **i** $\dfrac{dy}{dx} = 0$ when $x^2+4x-7=0$

$$\therefore\ x = \frac{-4 \pm \sqrt{4^2-4(1)(-7)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{44}}{2}$$

$$= -2 \pm \sqrt{11}$$

ii $\dfrac{dy}{dx}$ is undefined when $(x+2)^2=0$

$\therefore\ x=-2$

INVESTIGATION 4 — THE DERIVATIVE OF b^x

1 $y = 2^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	0.6931	0.6931
0.5	1.4142	0.9803	0.6931
1	2	1.3863	0.6931
1.5	2.8284	1.9605	0.6931
2	4	2.7726	0.6931

2 **a** $y = 3^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	1.0986	1.0986
0.5	1.7321	1.9029	1.0986
1	3	3.2958	1.0986
1.5	5.1962	5.7086	1.0986
2	9	9.8875	1.0986

b $y = 5^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	1.6094	1.6094
0.5	2.2361	3.5988	1.6094
1	5	8.0472	1.6094
1.5	11.1803	17.9941	1.6094
2	25	40.2359	1.6094

c $y = (0.5)^x$

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	-0.6931	-0.6931
0.5	0.7071	-0.4901	-0.6931
1	0.5	-0.3466	-0.6931
1.5	0.3536	-0.2451	-0.6931
2	0.25	-0.1733	-0.6931

3 From **2 a**, **b**, and **c**, we can see that $\frac{dy}{dx} \div y$ is always equal to the value of $\frac{dy}{dx}$ at $x = 0$.

So, if $f(x) = b^x$, then $\frac{f'(x)}{b^x} = f'(0)$

$$\therefore \quad f'(x) = f'(0) \times b^x$$

EXERCISE 18E

1 **a** If $f(x) = e^{4x}$
then $f'(x) = e^{4x}(4)$
$= 4e^{4x}$

b If $f(x) = e^x + 3$
then $f'(x) = e^x + 0$ {addition rule}
$= e^x$

c If $f(x) = e^{-2x}$
then $f'(x) = e^{-2x}(-2)$
$= -2e^{-2x}$

d If $f(x) = e^{\frac{x}{2}}$
then $f'(x) = e^{\frac{x}{2}}\left(\frac{1}{2}\right)$
$= \frac{1}{2}e^{\frac{x}{2}}$

e If $f(x) = 2e^{-\frac{x}{2}}$
then $f'(x) = 2e^{-\frac{x}{2}}\left(-\frac{1}{2}\right)$
$= -e^{-\frac{x}{2}}$

f If $f(x) = 1 - 2e^{-x}$
then $f'(x) = 0 - 2e^{-x}(-1)$
{addition rule}
$= 2e^{-x}$

g If $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
then $f'(x) = 4e^{\frac{x}{2}}\left(\frac{1}{2}\right) - 3e^{-x}(-1)$
{addition rule}
$= 2e^{\frac{x}{2}} + 3e^{-x}$

h If $f(x) = \dfrac{e^x + e^{-x}}{2}$
$= \frac{1}{2}e^x + \frac{1}{2}e^{-x}$
then $f'(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}(-1)$
$= \dfrac{e^x - e^{-x}}{2}$

i If $f(x) = e^{-x^2}$
then $f'(x) = e^{-x^2}(-2x)$
$= -2xe^{-x^2}$

j If $f(x) = e^{\frac{1}{x}}$
then $f'(x) = e^{\frac{1}{x}}\left(-\dfrac{1}{x^2}\right)$
$= -\dfrac{e^{\frac{1}{x}}}{x^2}$

k If $f(x) = 10(1 + e^{2x})$
$= 10 + 10e^{2x}$
then $f'(x) = 0 + 10e^{2x}(2)$
{addition rule}
$= 20e^{2x}$

l If $f(x) = 20(1 - e^{-2x})$
$= 20 - 20e^{-2x}$
then $f'(x) = 0 - 20e^{-2x}(-2)$
{addition rule}
$= 40e^{-2x}$

m If $f(x) = e^{2x+1}$
then $f'(x) = e^{2x+1}(2)$
$= 2e^{2x+1}$

n If $f(x) = e^{\frac{x}{4}}$
then $f'(x) = e^{\frac{x}{4}}\left(\frac{1}{4}\right)$
$= \frac{1}{4}e^{\frac{x}{4}}$

o If $f(x) = e^{1-2x^2}$
then $f'(x) = e^{1-2x^2}(-4x)$
$= -4xe^{1-2x^2}$

p If $f(x) = e^{-0.02x}$
then $f'(x) = e^{-0.02x} \times (-0.02)$
$= -0.02e^{-0.02x}$

2 **a** If $f(x) = xe^x$
then $f'(x) = 1e^x + xe^x$
{product rule}
$= e^x + xe^x$

b If $f(x) = x^3e^{-x}$
then $f'(x) = 3x^2e^{-x} + x^3(e^{-x})(-1)$
{product rule}
$= 3x^2e^{-x} - x^3e^{-x}$

c If $f(x) = \dfrac{e^x}{x}$

then $f'(x) = \dfrac{e^x x - e^x(1)}{x^2}$ {quotient rule}

$= \dfrac{xe^x - e^x}{x^2}$

d If $f(x) = \dfrac{x}{e^x}$

then $f'(x) = \dfrac{1e^x - xe^x}{(e^x)^2}$ {quotient rule}

$= \dfrac{e^x(1-x)}{(e^x)^2}$

$= \dfrac{1-x}{e^x}$

e If $f(x) = x^2e^{3x}$

then $f'(x) = 2xe^{3x} + x^2e^{3x}(3)$ {product rule}

$= 2xe^{3x} + 3x^2e^{3x}$

f If $f(x) = \dfrac{e^x}{\sqrt{x}}$

then $f'(x) = \dfrac{e^x\sqrt{x} - \frac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2}$ {quotient rule}

$= \dfrac{e^x\sqrt{x} \times \left(\frac{\sqrt{x}}{\sqrt{x}}\right) - \frac{e^x}{2\sqrt{x}}}{x}$

$= \dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g If $f(x) = 20xe^{-0.5x}$

then $f'(x) = 20e^{-0.5x} + 20xe^{-0.5x}(-0.5)$ {product rule}

$= 20e^{-0.5x} - 10xe^{-0.5x}$

h If $f(x) = \dfrac{e^x + 2}{e^{-x} + 1}$

then $f'(x) = \dfrac{e^x(e^{-x}+1) - (e^x+2)(e^{-x})(-1)}{(e^{-x}+1)^2}$ {quotient rule}

$= \dfrac{1 + e^x + 1 + 2e^{-x}}{(e^{-x}+1)^2}$

$= \dfrac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$

3 **a** $y = (2 + e^x)^4$

$= u^4$ where $u = 2 + e^x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 4u^3\dfrac{du}{dx}$

$= 4(2+e^x)^3 \times e^x$

$= 4e^x(2+e^x)^3$

b $y = \sqrt{e^x - 1}$

$= u^{\frac{1}{2}}$ where $u = e^x - 1$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= \frac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dx}$

$= \frac{1}{2}(e^x - 1)^{-\frac{1}{2}} \times e^x$

$= \dfrac{e^x}{2\sqrt{e^x - 1}}$

c $y = (e^x + e^{-x})^{\frac{3}{2}}$

$= u^{\frac{3}{2}}$ where $u = e^x + e^{-x}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= \frac{3}{2}u^{\frac{1}{2}}\frac{du}{dx}$

$= \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}} \times (e^x + e^{-x}(-1))$

$= \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}}(e^x - e^{-x})$

d $y = \frac{1}{\sqrt{e^{2x}+2}}$

$= u^{-\frac{1}{2}}$ where $u = e^{2x} + 2$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= -\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx}$

$= -\frac{1}{2}(e^{2x}+2)^{-\frac{3}{2}} \times e^{2x}(2)$

$= -e^{2x}(e^{2x}+2)^{-\frac{3}{2}}$

4 **a** $y = (e^x + 2)^4$

$= u^4$ where $u = e^x + 2$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= 4u^3\frac{du}{dx}$

$= 4(e^x+2)^3 \times e^x$

$= 4e^x(e^x+2)^3$

At $x = 0$, $\frac{dy}{dx} = 4e^0(e^0+2)^3$

$= 108$

$\therefore$ gradient of tangent $= 108$

b $y = \frac{1}{2 - e^{-x}}$

$= u^{-1}$ where $u = 2 - e^{-x}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= -u^{-2}\frac{du}{dx}$

$= -\frac{1}{(2-e^{-x})^2} \times (-e^{-x}(-1))$

$= -\frac{e^{-x}}{(2-e^{-x})^2}$

At $x = 0$, $\frac{dy}{dx} = -\frac{e^0}{(2-e^0)^2} = -1$

$\therefore$ gradient of tangent $= -1$

c $y = \sqrt{e^{2x}+10}$

$= u^{\frac{1}{2}}$ where $u = e^{2x} + 10$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx}$

$= \frac{1}{2\sqrt{e^{2x}+10}} \times e^{2x}(2)$

$= \frac{e^{2x}}{\sqrt{e^{2x}+10}}$

At $x = \ln 3$, $\frac{dy}{dx} = \frac{e^{2\ln 3}}{\sqrt{e^{2\ln 3}+10}}$

$= \frac{e^{\ln 3^2}}{\sqrt{e^{\ln 3^2}+10}}$

$= \frac{3^2}{\sqrt{3^2+10}} = \frac{9}{\sqrt{19}}$

$\therefore$ gradient of tangent $= \frac{9}{\sqrt{19}}$

d $y = \frac{2-x}{e^{3x}}$

$\therefore \frac{dy}{dx} = \frac{(-1)e^{3x} - (2-x)e^{3x}(3)}{(e^{3x})^2}$ {quotient rule}

$= \frac{-e^{3x} - 3(2-x)e^{3x}}{e^{6x}}$

At $x = 1$, $\frac{dy}{dx} = \frac{-e^{3(1)} - 3(2-1)e^{3(1)}}{e^{6(1)}}$

$= \frac{-e^3 - 3e^3}{e^6}$

$= \frac{-4e^3}{e^6}$

$= -\frac{4}{e^3}$

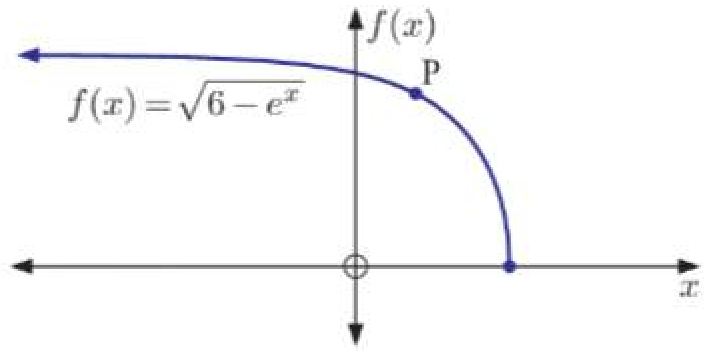
$\therefore$ gradient of tangent $= -\frac{4}{e^3}$

5 **a** Since $e^x > 0$ for all x, $6 - e^x < 6$

$\therefore \sqrt{6 - e^x} < \sqrt{6}$

But $\sqrt{6 - e^x}$ cannot be negative.

$\therefore$ the range of $f(x) = \sqrt{6 - e^x}$ is $\{y \mid 0 \leqslant y < \sqrt{6}\}$.



b **i** P has x-coordinate $\ln 2$.

$$\begin{aligned}\therefore\ y &= \sqrt{6 - e^{\ln 2}}\\ &= \sqrt{6-2}\\ &= 2 \qquad \{y \geqslant 0\}\end{aligned}$$

So, P is at $(\ln 2, 2)$.

ii
$$f(x) = \sqrt{6 - e^x} = (6 - e^x)^{\frac{1}{2}}$$

$$\begin{aligned}\therefore\ f'(x) &= \tfrac{1}{2}(6 - e^x)^{-\frac{1}{2}} \times (-e^x) \qquad \{\text{chain rule}\}\\ &= -\tfrac{1}{2}e^x(6 - e^x)^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\text{Now}\quad f'(\ln 2) &= -\tfrac{1}{2}e^{\ln 2}(6 - e^{\ln 2})^{-\frac{1}{2}}\\ &= -\tfrac{1}{2} \times 2 \times \frac{1}{\sqrt{6-2}}\\ &= -1 \times \frac{1}{\sqrt{4}}\\ &= -1 \times \tfrac{1}{2}\\ &= -\tfrac{1}{2}\end{aligned}$$

$\therefore$ the gradient of the tangent at P is $-\frac{1}{2}$.

6 $f(x) = e^{kx} + x \quad \therefore\ f'(x) = ke^{kx} + 1$

Now $f'(0) = -8$, so $ke^0 + 1 = -8$

$\therefore\ k = -9$

7 **a**
$$\begin{aligned}y &= 2^x\\ &= (e^{\ln 2})^x\\ &= e^{x\ln 2}\\ \therefore\ \frac{dy}{dx} &= e^{x\ln 2} \times \ln 2\\ &= e^{\ln 2^x} \times \ln 2\\ &= 2^x \ln 2\end{aligned}$$

b
$$\begin{aligned}y &= b^x\\ &= (e^{\ln b})^x\\ &= e^{x\ln b}\\ \therefore\ \frac{dy}{dx} &= e^{x\ln b} \times \ln b\\ &= e^{\ln b^x} \times \ln b\\ &= b^x \times \ln b\end{aligned}$$

c **i**
$$\begin{aligned}y &= 5^x\\ \therefore\ \frac{dy}{dx} &= 5^x \times \ln 5 \qquad \{\text{using } \mathbf{b}\}\\ &= 5^x \ln 5\end{aligned}$$

ii
$$\begin{aligned}y &= 8 \times 10^x\\ \therefore\ \frac{dy}{dx} &= 8 \times 10^x \times \ln 10 \qquad \{\text{using } \mathbf{b}\}\\ &= 8 \times 10^x \ln 10\end{aligned}$$

8 $f(x) = x^2e^{-x}$ is the product of $u(x) = x^2$ and $v(x) = e^{-x}$

$\therefore\ u'(x) = 2x$ and $v'(x) = -e^{-x}$

$$\begin{aligned}\therefore\ f'(x) &= u'(x)\,v(x) + u(x)\,v'(x) \qquad \{\text{product rule}\}\\ &= 2x(e^{-x}) + x^2(-e^{-x})\\ &= 2xe^{-x} - x^2e^{-x}\end{aligned}$$

The tangent to $f(x) = x^2e^{-x}$ is horizontal at point P.

$\therefore$ the gradient of the tangent at point P is zero.

$\therefore\ f'(x) = 0$ at point P.

Now $f'(x) = 0$ when $2xe^{-x} - x^2e^{-x} = 0$

$$\therefore \; xe^{-x}(2-x) = 0$$

$$\therefore \; x = 0 \text{ or } 2 - x = 0 \qquad \{\text{as } e^{-x} > 0 \text{ for all } x\}$$

$$\therefore \; x = 0 \text{ or } x = 2$$

$f(0) = 0^2e^0$ and $f(2) = 2^2e^{-2}$

$= 0$ $\qquad = 4e^{-2}$

$\qquad = \dfrac{4}{e^2}$

So, the possible coordinates of P are $(0,\ 0)$ and $\left(2,\ \dfrac{4}{e^2}\right)$.

9 $S(x) = \frac{1}{2}(e^x - e^{-x}), \quad C(x) = \frac{1}{2}(e^x + e^{-x})$

a $[C(x)]^2 - [S(x)]^2$

$$= \left[\tfrac{1}{2}(e^x + e^{-x})\right]^2 - \left[\tfrac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \tfrac{1}{4}(e^x + e^{-x})^2 - \tfrac{1}{4}(e^x - e^{-x})^2$$

$$= \tfrac{1}{4}(e^{2x} + 2e^0 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2e^0 + e^{-2x})$$

$$= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$= \cancel{\tfrac{1}{4}e^{2x}} + \tfrac{1}{2} + \cancel{\tfrac{1}{4}e^{-2x}} - \cancel{\tfrac{1}{4}e^{2x}} + \tfrac{1}{2} - \cancel{\tfrac{1}{4}e^{-2x}}$$

$$= 1$$

b $\dfrac{d}{dx}[S(x)] = \dfrac{d}{dx}\left[\tfrac{1}{2}(e^x - e^{-x})\right]$

$$= \frac{d}{dx}\left[\tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}\right]$$

$$= \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}$$

$$= \tfrac{1}{2}(e^x + e^{-x})$$

$$= C(x)$$

c $\dfrac{d}{dx}[C(x)] = \dfrac{d}{dx}\left[\tfrac{1}{2}(e^x + e^{-x})\right]$

$$= \frac{d}{dx}\left[\tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}\right]$$

$$= \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}$$

$$= \tfrac{1}{2}(e^x - e^{-x})$$

$$= S(x)$$

d $\dfrac{d}{dx}[T(x)] = \dfrac{d}{dx}\left[\dfrac{S(x)}{C(x)}\right]$

$$= \frac{S'(x)\,C(x) - S(x)\,C'(x)}{[C(x)]^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{C(x)\,C(x) - S(x)\,S(x)}{[C(x)]^2} \qquad \{\text{using } \textbf{b} \text{ and } \textbf{c}\}$$

$$= \frac{[C(x)]^2 - [S(x)]^2}{[C(x)]^2}$$

$$= \frac{1}{[C(x)]^2} \qquad \{\text{using } \textbf{a}\}$$

INVESTIGATION 6 THE DERIVATIVE OF $\ln x$

1 The gradient function has a vertical asymptote $x = 0$, and as x increases, it approaches its horizontal asymptote $y = 0$.

2 We predict that for $y = \ln x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

3

x	$f'(x)$
0.25	4
0.5	2
1	1
2	0.5
3	0.3333
4	0.25
5	0.2

It appears that $f'(x) = \dfrac{1}{x}$, which agrees with our prediction in **2**.

EXERCISE 18F

1 **a**

$$\begin{aligned} y &= \ln(7x) \\ &= \ln 7 + \ln x \\ \therefore \frac{dy}{dx} &= 0 + \frac{1}{x} = \frac{1}{x} \end{aligned}$$

or

$$\begin{aligned} y &= \ln(7x) \\ \therefore \frac{dy}{dx} &= \frac{7}{7x} \\ &= \frac{1}{x} \end{aligned}$$

b

$$\begin{aligned} y &= \ln(2x+1) \\ \therefore \frac{dy}{dx} &= \frac{2}{2x+1} \end{aligned}$$

c

$$\begin{aligned} y &= \ln(x - x^2) \\ \therefore \frac{dy}{dx} &= \frac{1-2x}{x-x^2} \end{aligned}$$

d

$$\begin{aligned} y &= 3 - 2\ln x \\ \therefore \frac{dy}{dx} &= 0 - 2\left(\frac{1}{x}\right) \\ &= -\frac{2}{x} \end{aligned}$$

e

$$\begin{aligned} y &= x^2 \ln x \\ \therefore \frac{dy}{dx} &= 2x\ln x + x^2\left(\frac{1}{x}\right) \quad \{\text{product rule}\} \\ &= 2x\ln x + x \end{aligned}$$

f

$$\begin{aligned} y &= \frac{\ln x}{2x} \\ \therefore \frac{dy}{dx} &= \frac{\left(\frac{1}{x}\right)2x - \ln x \times 2}{(2x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{2 - 2\ln x}{4x^2} \\ &= \frac{1-\ln x}{2x^2} \end{aligned}$$

g

$$\begin{aligned} y &= e^x \ln x \\ \therefore \frac{dy}{dx} &= e^x \ln x + e^x\left(\frac{1}{x}\right) \quad \{\text{product rule}\} \\ &= e^x \ln x + \frac{e^x}{x} \end{aligned}$$

h

$$\begin{aligned} y &= (\ln x)^2 \\ \therefore \frac{dy}{dx} &= 2(\ln x)^1\left(\frac{1}{x}\right) \quad \{\text{chain rule}\} \\ &= \frac{2\ln x}{x} \end{aligned}$$

i
$$y = \sqrt{\ln x} = (\ln x)^{\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = \tfrac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\right) \quad \{\text{chain rule}\}$$
$$= \frac{1}{2x\sqrt{\ln x}}$$

j
$$y = e^{-x}\ln x$$
$$\therefore \frac{dy}{dx} = e^{-x}(-1)\ln x + e^{-x}\left(\frac{1}{x}\right)$$
{product rule}
$$= \frac{e^{-x}}{x} - e^{-x}\ln x$$

k
$$y = \sqrt{x}\ln(2x) = x^{\frac{1}{2}}\ln(2x)$$
$$\therefore \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}\ln(2x) + x^{\frac{1}{2}}\left(\frac{2}{2x}\right)$$
{product rule}
$$= \frac{1}{2\sqrt{x}}\ln(2x) + \sqrt{x}\left(\frac{1}{x}\right)$$
$$= \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

l
$$y = -4\ln(1-x)$$
$$\therefore \frac{dy}{dx} = -4\left(\frac{-1}{1-x}\right)$$
$$= \frac{4}{1-x}$$

m
$$y = \frac{2\sqrt{x}}{\ln x} = \frac{2x^{\frac{1}{2}}}{\ln x}$$
$$\therefore \frac{dy}{dx} = \frac{x^{-\frac{1}{2}}\ln x - 2x^{\frac{1}{2}}\left(\frac{1}{x}\right)}{(\ln x)^2}$$
{quotient rule}
$$= \frac{\frac{1}{\sqrt{x}}\ln x - 2\sqrt{x}\left(\frac{1}{x}\right)}{(\ln x)^2}$$
$$= \frac{\frac{1}{\sqrt{x}}\ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2}$$
$$= \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$$

n
$$y = x\ln(x^2+1)$$
$$\therefore \frac{dy}{dx} = 1\big(\ln(x^2+1)\big) + x\left(\frac{2x}{x^2+1}\right)$$
{product rule}
$$= \ln(x^2+1) + \frac{2x^2}{x^2+1}$$

o
$$y = \frac{\ln x}{x^2}$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)x^2 - \ln x(2x)}{(x^2)^2}$$
{quotient rule}
$$= \frac{x - 2x\ln x}{x^4}$$
$$= \frac{1 - 2\ln x}{x^3}$$

2
$$f(x) = \ln(kx) = \ln k + \ln x$$
$$\therefore f'(x) = 0 + \frac{1}{x} = \frac{1}{x}$$

The derivative does not depend on k because we can write $\ln(kx)$ as $\ln k + \ln x$, where k is a constant.

3 **a**
$$y = x\ln 5$$
$$\therefore \frac{dy}{dx} = \ln 5$$

b
$$y = \ln(x^3) = 3\ln x$$
$\{\ln(a^n) = n\ln a\}$
$$\therefore \frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$$

c
$$y = \ln(x^4 + x)$$
$$\therefore \frac{dy}{dx} = \frac{4x^3+1}{x^4+x}$$

d
$$y = \ln(10-5x)$$
$$\therefore \frac{dy}{dx} = \frac{-5}{10-5x} = \frac{1}{x-2}$$

e
$$y = [\ln(2x+1)]^3$$
$$\therefore \frac{dy}{dx} = 3[\ln(2x+1)]^2 \times \frac{2}{2x+1} \quad \{\text{chain rule}\}$$
$$= \frac{6}{2x+1}[\ln(2x+1)]^2$$

f
$$y = \frac{\ln(4x)}{x}$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2} \quad \{\text{quotient rule}\}$$
$$= \frac{1 - \ln(4x)}{x^2}$$

g
$$y = \ln\left(\frac{1}{x}\right) = \ln(x^{-1})$$
$$= -\ln x \qquad \{\ln(a^n) = n\ln a\}$$
$$\therefore \frac{dy}{dx} = -\frac{1}{x}$$

h
$$y = \frac{1}{\ln x} = (\ln x)^{-1}$$
$$\therefore \frac{dy}{dx} = -1(\ln x)^{-2} \times \frac{1}{x} \qquad \{\text{chain rule}\}$$
$$= \frac{-1}{x(\ln x)^2}$$

4 **a**
$$y = \ln\sqrt{1-2x}$$
$$= \ln\left((1-2x)^{\frac{1}{2}}\right)$$
$$= \tfrac{1}{2}\ln(1-2x) \quad \{\ln(a^n) = n\ln a\}$$
$$\therefore \frac{dy}{dx} = \tfrac{1}{2} \times \frac{-2}{1-2x}$$
$$= \frac{-1}{1-2x}$$
$$= \frac{1}{2x-1}$$

b
$$y = \ln\left(\frac{1}{2x+3}\right)$$
$$= \ln\left((2x+3)^{-1}\right)$$
$$= -\ln(2x+3) \quad \{\ln(a^n) = n\ln a\}$$
$$\therefore \frac{dy}{dx} = -\frac{2}{2x+3}$$

c
$$y = \ln(e^x\sqrt{x})$$
$$= \ln(e^x) + \ln(x^{\frac{1}{2}}) \qquad \{\ln(ab) = \ln a + \ln b\}$$
$$= \ln(e^x) + \tfrac{1}{2}\ln x \qquad \{\ln(a^n) = n\ln a\}$$
$$= x + \tfrac{1}{2}\ln x \qquad \{\ln e^a = a\}$$
$$\therefore \frac{dy}{dx} = 1 + \tfrac{1}{2}\left(\frac{1}{x}\right)$$
$$= 1 + \frac{1}{2x}$$

d
$$y = \ln(x\sqrt{2-x})$$
$$= \ln x + \ln\left((2-x)^{\frac{1}{2}}\right)$$
$$= \ln x + \tfrac{1}{2}\ln(2-x)$$
$$\{\ln(a^n) = n\ln a\}$$
$$\therefore \frac{dy}{dx} = \frac{1}{x} + \tfrac{1}{2}\left(\frac{-1}{2-x}\right)$$
$$= \frac{1}{x} - \frac{1}{2(2-x)}$$

e
$$y = \ln\left(\frac{x+3}{x-1}\right)$$
$$= \ln(x+3) - \ln(x-1)$$
$$\left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\}$$
$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{1}{x-1}$$

f
$$\begin{aligned} y &= \ln\left(\frac{x^2}{3-x}\right) \\ &= \ln(x^2) - \ln(3-x) \qquad \{\ln\left(\tfrac{a}{b}\right) = \ln a - \ln b\} \\ &= 2\ln x - \ln(3-x) \qquad \{\ln(a^n) = n\ln a\} \\ \therefore\ \frac{dy}{dx} &= 2\left(\frac{1}{x}\right) - \frac{-1}{3-x} \\ &= \frac{2}{x} + \frac{1}{3-x} \end{aligned}$$

5 **a**
$$\begin{aligned} f(x) &= \ln\big((3x-4)^3\big) \\ &= 3\ln(3x-4) \\ &\qquad \{\ln(a^n) = n\ln a\} \\ \therefore\ f'(x) &= 3 \times \frac{3}{3x-4} = \frac{9}{3x-4} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \ln\big(x(x^2+1)\big) \\ &= \ln x + \ln(x^2+1) \\ &\qquad \{\ln(ab) = \ln a + \ln b\} \\ \therefore\ f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \ln\left(\frac{x^2+2x}{x-5}\right) \\ &= \ln(x^2+2x) - \ln(x-5) \qquad \{\ln\left(\tfrac{a}{b}\right) = \ln a - \ln b\} \\ \therefore\ f'(x) &= \frac{2x+2}{x^2+2x} - \frac{1}{x-5} \end{aligned}$$

6
$$\begin{aligned} y &= x\ln x \\ \therefore\ \frac{dy}{dx} &= 1\ln x + x \times \frac{1}{x} \qquad \{\text{product rule}\} \\ &= \ln x + 1 \end{aligned}$$

At $x = e$, $\dfrac{dy}{dx} = \ln e + 1$
$$= 1 + 1$$
$$= 2$$

$\therefore$ gradient of tangent $= 2$

7 $f(x) = a\ln(bx^2)$

Now $f(e) = 3$, $\therefore\ 3 = a\ln(be^2)$
$$\begin{aligned} \therefore\ a &= \frac{3}{\ln(be^2)} \\ &= \frac{3}{\ln b + \ln(e^2)} \\ &= \frac{3}{\ln b + 2} \quad .... \ (1) \end{aligned}$$

Now $f'(x) = a \times \dfrac{2bx}{bx^2} = \dfrac{2a}{x}$

$f'(1) = 6$, so $2a = 6$

$\therefore\ a = 3$ .... (2)

$$\therefore\ 3 = \frac{3}{\ln b + 2} \qquad \{\text{equating (1) and (2)}\}$$

$\therefore\ \ln b + 2 = 1$

$\therefore\ \ln b = -1$

$\therefore\ b = e^{-1}$

So, $a = 3$, $b = \dfrac{1}{e}$.

8 $f(x) = ax\ln(bx)$

Now $f(1) = 12$, $\therefore$ $12 = a\ln b$ .... $(*)$

$$\text{Now } f'(x) = a\ln(bx) + ax \times \frac{b}{bx} \qquad \{\text{product rule}\}$$
$$= a\ln(bx) + a$$

$f'(1) = 16$, so $a\ln b + a = 16$

$\therefore$ $12 + a = 16$ {using $(*)$}

$\therefore$ $a = 4$

Substituting $a = 4$ into $(*)$ gives $12 = 4\ln b$

$\therefore$ $\ln b = 3$

$\therefore$ $b = e^3$

So, $a = 4$, $b = e^3$.

9 $y = \ln(15 - x^2)$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{-2x}{15 - x^2}$

Now $\dfrac{dy}{dx} = 1$ when $\dfrac{-2x}{15 - x^2} = 1$

$\therefore$ $-2x = 15 - x^2$

$\therefore$ $x^2 - 2x - 15 = 0$

$\therefore$ $(x+3)(x-5) = 0$

$\therefore$ $x = -3$ or 5

But $y = \ln(15 - x^2)$ is undefined for $x = 5$.

When $x = -3$, $y = \ln(15 - (-3)^2)$

$= \ln(15 - 9)$

$= \ln 6$

$\therefore$ the tangent at $(-3, \ln 6)$ has gradient 1.

INVESTIGATION 7 DERIVATIVES OF $\sin x$ AND $\cos x$

1 We predict that if $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$.

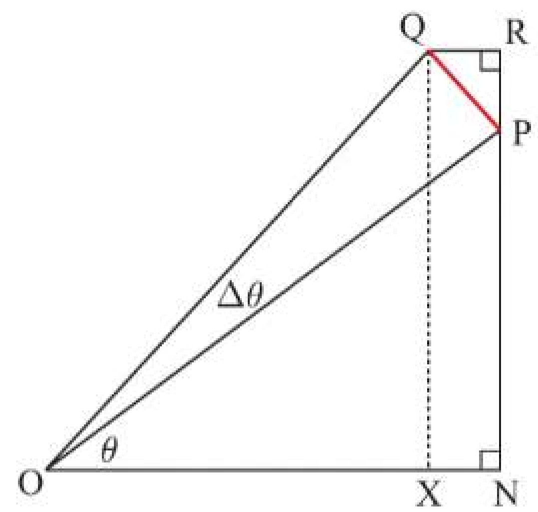
2 We predict that if $y = \cos x$, then $\dfrac{dy}{dx} = -\sin x$.

3 **a** $\sin(\theta + \Delta\theta) = \dfrac{\text{QX}}{\text{OQ}} = \text{NR}$ {OQ = 1, QX = NR}

$\sin\theta = \dfrac{\text{NP}}{\text{OP}} = \text{NP}$ {OP = 1}

$\therefore$ $\sin(\theta + \Delta\theta) - \sin\theta = \text{NR} - \text{NP}$

$= \text{PR}$



b **i** For very small $\Delta\theta$, the line segment [PQ] is a good approximation of the arc PQ.

$\therefore$ as Q approaches P, the arc PQ resembles line segment [PQ].

ii arc PQ $= 1 \times \Delta\theta$ $\quad \{l = r\theta\}$

$= \Delta\theta$

$\therefore$ as Q approaches P, PQ $\approx \Delta\theta$.

iii $Q\widehat{P}O = \frac{\pi}{2} - \frac{\Delta\theta}{2}$ $\quad$ {isosceles triangle}

As $\Delta\theta \to 0$, $Q\widehat{P}O \to \frac{\pi}{2}$

$\therefore$ as Q approaches P, $Q\widehat{P}O$ approaches a right angle.

iv As Q approaches P, $Q\widehat{P}O$ approaches a right angle $\quad$ {from **iii**}

$$\begin{aligned}\therefore\ Q\widehat{P}R &\approx \pi - \tfrac{\pi}{2} - O\widehat{P}N \quad \{\text{angles on a line}\}\\ &\approx \pi - \tfrac{\pi}{2} - (\pi - \tfrac{\pi}{2} - \theta)\\ &\approx \pi - \tfrac{\pi}{2} - \pi + \tfrac{\pi}{2} + \theta\\ &\approx \theta\end{aligned}$$

c For small $\Delta\theta$, $\cos\theta \approx \frac{\text{PR}}{\text{PQ}}$

$$\approx \frac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta} \quad \{\text{using a, b ii}\}$$

$$\therefore \lim_{\Delta\theta \to 0} \cos\theta = \lim_{\Delta\theta \to 0} \frac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta}$$

$$\therefore \cos\theta = \frac{d}{d\theta}(\sin\theta)$$

EXERCISE 18G

1 **a**
$$\begin{aligned} y &= \sin 2x\\ \therefore \frac{dy}{dx} &= (\cos 2x) \times 2\\ &= 2\cos 2x\end{aligned}$$

b
$$\begin{aligned} y &= \sin x + \cos x\\ \therefore \frac{dy}{dx} &= \cos x - \sin x\end{aligned}$$

c
$$\begin{aligned} y &= \cos 3x - \sin x\\ \therefore \frac{dy}{dx} &= (-\sin 3x) \times 3 - \cos x\\ &= -3\sin 3x - \cos x\end{aligned}$$

d
$$\begin{aligned} y &= \sin(x+1)\\ \therefore \frac{dy}{dx} &= (\cos(x+1)) \times 1\\ &= \cos(x+1)\end{aligned}$$

e
$$\begin{aligned} y &= \cos(3-2x)\\ \therefore \frac{dy}{dx} &= (-\sin(3-2x)) \times (-2)\\ &= 2\sin(3-2x)\end{aligned}$$

f
$$\begin{aligned} y &= \tan 5x\\ \therefore \frac{dy}{dx} &= \frac{1}{\cos^2 5x} \times 5\\ &= \frac{5}{\cos^2 5x}\end{aligned}$$

g
$$\begin{aligned} y &= \sin\tfrac{x}{2} - 3\cos x\\ \therefore \frac{dy}{dx} &= \left(\cos\tfrac{x}{2}\right)\left(\tfrac{1}{2}\right) - 3(-\sin x)\\ &= \tfrac{1}{2}\cos\tfrac{x}{2} + 3\sin x\end{aligned}$$

h
$$\begin{aligned} y &= 4\sin x - \cos 2x\\ \therefore \frac{dy}{dx} &= 4\cos x + (\sin 2x) \times 2\\ &= 4\cos x + 2\sin 2x\end{aligned}$$

i $y = \frac{1}{2}\cos 6x - 5\sin 4x$

$\therefore \frac{dy}{dx} = \frac{1}{2}(-\sin 6x) \times 6 - 5(\cos 4x) \times 4$

$= -3\sin 6x - 20\cos 4x$

2 **a** $y = x^2 + \cos x$

$\therefore \frac{dy}{dx} = 2x - \sin x$

b $y = \tan x - 3\sin x$

$\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} - 3\cos x$

c $y = e^x \cos x$

$\therefore \frac{dy}{dx} = e^x \cos x + e^x(-\sin x)$

{product rule}

$= e^x \cos x - e^x \sin x$

d $y = e^{-x} \sin x$

$\therefore \frac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$

{product rule}

e $y = \ln(\sin x)$

$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x}$

f $y = e^{2x}\tan x$

$\therefore \frac{dy}{dx} = 2e^{2x}\tan x + e^{2x} \times \frac{1}{\cos^2 x}$

{product rule, chain rule}

$= 2e^{2x}\tan x + \frac{e^{2x}}{\cos^2 x}$

g $y = 3\tan \pi x$

$\therefore \frac{dy}{dx} = 3 \times \frac{1}{\cos^2 \pi x} \times \pi$ {chain rule}

$= \frac{3\pi}{\cos^2 \pi x}$

h $y = \cos \frac{x}{2}$

$\therefore \frac{dy}{dx} = \left(-\sin \frac{x}{2}\right) \times \left(\frac{1}{2}\right)$ {chain rule}

$= -\frac{1}{2}\sin \frac{x}{2}$

i $y = 3\tan 2x$

$\therefore \frac{dy}{dx} = 3 \times \frac{1}{\cos^2 2x} \times 2$ {chain rule}

$= \frac{6}{\cos^2 2x}$

j $y = x\cos x$

$\therefore \frac{dy}{dx} = 1 \times \cos x + x(-\sin x)$

{product rule}

$= \cos x - x\sin x$

k $y = \frac{\sin x}{x}$

$\therefore \frac{dy}{dx} = \frac{(\cos x)(x) - \sin x \times 1}{x^2}$

{quotient rule}

$= \frac{x\cos x - \sin x}{x^2}$

l $y = x\tan x$

$\therefore \frac{dy}{dx} = (1)\tan x + (x)\frac{1}{\cos^2 x}$

{product rule}

$= \tan x + \frac{x}{\cos^2 x}$

m $y = e^{\cos\sqrt{x}}$

$\therefore \frac{dy}{dx} = e^{\cos\sqrt{x}}(-\sin\sqrt{x} \times \frac{1}{2}x^{-\frac{1}{2}})$ {chain rule}

$= -\frac{\sin\sqrt{x}\, e^{\cos\sqrt{x}}}{2\sqrt{x}}$

n $$y = \frac{2\sqrt{x}}{\tan x}$$

$$\therefore \frac{dy}{dx} = \frac{2 \times \frac{1}{2}x^{-\frac{1}{2}} \times \tan x - 2\sqrt{x} \times \frac{1}{\cos^2 x}}{(\tan x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{\frac{\tan x}{\sqrt{x}} - \frac{2\sqrt{x}}{\cos^2 x}}{\tan^2 x}$$

$$= \frac{\frac{\sin x}{\cos x} - \frac{2x}{\cos^2 x}}{\sqrt{x}\,\frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\sin x \cos x - 2x}{\sqrt{x}\sin^2 x}$$

o $$y = \frac{\cos x + \sin 2x}{x^3}$$

$$\therefore \frac{dy}{dx} = \frac{(-\sin x + 2\cos 2x) \times x^3 - (\cos x + \sin 2x) \times 3x^2}{x^6} \quad \{\text{quotient rule}\}$$

$$= \frac{2x\cos 2x - x\sin x - 3\cos x - 3\sin 2x}{x^4}$$

3 **a** $$y = \sin(x^2)$$

$$\therefore \frac{dy}{dx} = \left(\cos(x^2)\right) \times 2x \quad \{\text{chain rule}\}$$

$$= 2x\cos(x^2)$$

b $$y = \cos(\sqrt{x}) = \cos(x^{\frac{1}{2}})$$

$$\therefore \frac{dy}{dx} = -\sin(x^{\frac{1}{2}}) \times \tfrac{1}{2}x^{-\frac{1}{2}} \quad \{\text{chain rule}\}$$

$$= -\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$$

c $$y = \sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x)$$

$$\{\text{chain rule}\}$$

$$= -\frac{\sin x}{2\sqrt{\cos x}}$$

d $$y = \sin(e^x)$$

$$\therefore \frac{dy}{dx} = \cos(e^x) \times e^x \quad \{\text{chain rule}\}$$

$$= e^x\cos(e^x)$$

e $$y = \sin^2 x = (\sin x)^2$$

$$\therefore \frac{dy}{dx} = 2\sin x\cos x \quad \{\text{chain rule}\}$$

f $$y = \cos^3 x = (\cos x)^3$$

$$\therefore \frac{dy}{dx} = 3(\cos x)^2 \times (-\sin x)$$

$$\{\text{chain rule}\}$$

$$= -3\sin x\cos^2 x$$

g $$y = 5\cos^2 2x = 5(\cos 2x)^2$$

$$\therefore \frac{dy}{dx} = 2(5\cos 2x) \times 2(-\sin 2x)$$

$$\{\text{chain rule}\}$$

$$= -20\cos 2x\sin 2x$$

h $$y = \cos x\sin 2x$$

$$\therefore \frac{dy}{dx} = (-\sin x)\sin 2x + \cos x(2\cos 2x)$$

$$\{\text{product rule}\}$$

$$= -\sin x\sin 2x + 2\cos x\cos 2x$$

i $$y = \cos(\cos x)$$

$$\therefore \frac{dy}{dx} = -\sin(\cos x) \times (-\sin x)$$

$$\{\text{chain rule}\}$$

$$= \sin x\sin(\cos x)$$

j $$y = \cos^3 4x = (\cos 4x)^3$$

$$\therefore \frac{dy}{dx} = 3(\cos 4x)^2 \times (-4\sin 4x)$$

$$\{\text{chain rule}\}$$

$$= -12\sin 4x\cos^2 4x$$

k $$y = \frac{2}{\sin^2 4x} = 2(\sin 4x)^{-2}$$
$$\therefore \frac{dy}{dx} = -2 \times 2(\sin 4x)^{-3} \times 4\cos 4x$$
$$= -\frac{16\cos 4x}{\sin^3 4x}$$

l $$y = \tfrac{1}{3}\tan^2 2x = \tfrac{1}{3}(\tan 2x)^2$$
$$\therefore \frac{dy}{dx} = 2(\tfrac{1}{3}\tan 2x) \times \frac{2}{\cos^2 2x} \qquad \{\text{chain rule}\}$$
$$= \frac{4\tan 2x}{3\cos^2 2x}$$

4 **a** $$f(x) = \sin^3 x$$
$$= (\sin x)^3$$
$$\therefore f'(x) = 3(\sin x)^2(\cos x) \quad \{\text{chain rule}\}$$
$$= 3\sin^2 x \cos x$$
$$\therefore f'\left(\tfrac{2\pi}{3}\right) = 3\sin^2\left(\tfrac{2\pi}{3}\right)\cos\tfrac{2\pi}{3}$$
$$= 3\left(\tfrac{\sqrt{3}}{2}\right)^2\left(-\tfrac{1}{2}\right)$$
$$= -\tfrac{9}{8}$$
$\therefore$ gradient of tangent $= -\frac{9}{8}$

b $$f(x) = \tan\left(2x - \tfrac{\pi}{6}\right)$$
$$\therefore f'(x) = \frac{1}{\cos^2(2x - \frac{\pi}{6})} \times 2 \qquad \{\text{chain rule}\}$$
$$= \frac{2}{\cos^2(2x - \frac{\pi}{6})}$$
$$\therefore f'\left(\tfrac{\pi}{6}\right) = \frac{2}{\cos^2(\frac{2\pi}{6} - \frac{\pi}{6})}$$
$$= \frac{2}{\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \tfrac{8}{3}$$
$\therefore$ gradient of tangent $= \frac{8}{3}$

c $$f(x) = \cos x \sin x$$
$$\therefore f'(x) = -\sin x \sin x + \cos x \cos x \qquad \{\text{product rule}\}$$
$$= \cos^2 x - \sin^2 x$$
$$\therefore f'\left(\tfrac{\pi}{4}\right) = \cos^2\left(\tfrac{\pi}{4}\right) - \sin^2\left(\tfrac{\pi}{4}\right)$$
$$= 0$$
$\therefore$ gradient of tangent $= 0$

5 **a** $$f(x) = 2\cos^2 x + 2\sin^2 x + 1$$
$$= 2(\cos x)^2 + 2(\sin x)^2 + 1$$
$$\therefore f'(x) = 2(2\cos x)(-\sin x) + 2(2\sin x)(\cos x)$$
$$= -4\cos x \sin x + 4\sin x \cos x$$
$$= 0$$

b $f(x) = 2\cos^2 x + 2\sin^2 x + 1$
$= 2(\cos^2 x + \sin^2 x) + 1$
$= 2(1) + 1 \qquad \{\cos^2 x + \sin^2 x = 1\}$
$= 3$ which is a constant and the derivative of a constant is zero.

6 **a** $\dfrac{d}{d\theta}\left(\dfrac{\cos\theta + i\sin\theta}{e^{i\theta}}\right) = \dfrac{(-\sin\theta + i\cos\theta)\times \cancel{e^{i\theta}} - (\cos\theta + i\sin\theta)\times i\cancel{e^{i\theta}}}{(e^{i\theta})^{\cancel{2}}}$ {quotient rule}

$= \dfrac{-\sin\theta + \cancel{i\cos\theta} - \cancel{i\cos\theta} - i^2\sin\theta}{e^{i\theta}}$

$= \dfrac{-\sin\theta + \sin\theta}{e^{i\theta}}$

$= 0$

$\therefore \dfrac{\cos\theta + i\sin\theta}{e^{i\theta}}$ is a constant.

b $\dfrac{\cos 0 + i\sin 0}{e^0} = \dfrac{1+0}{1}$

$= 1$

c $\dfrac{\cos\theta + i\sin\theta}{e^{i\theta}} = 1$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

EXERCISE 18H

1 **a** $f(x) = 3x^2 - 6x + 2$

$\therefore f'(x) = 6x - 6$

$\therefore f''(x) = 6$

b $f(x) = \dfrac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1$

$\therefore f'(x) = -x^{-\frac{3}{2}}$

$f''(x) = \frac{3}{2}x^{-\frac{5}{2}}$

$= \dfrac{3}{2x^{\frac{5}{2}}} = \dfrac{3}{2x^2\sqrt{x}}$

c $f(x) = 2x^3 - 3x^2 - x + 5$

$\therefore f'(x) = 6x^2 - 6x - 1$

$\therefore f''(x) = 12x - 6$

d $f(x) = \dfrac{2-3x}{x^2} = 2x^{-2} - 3x^{-1}$

$\therefore f'(x) = -4x^{-3} + 3x^{-2}$

$\therefore f''(x) = 12x^{-4} - 6x^{-3}$

$= \dfrac{12-6x}{x^4}$

e $f(x) = (1-2x)^2$

$\therefore f'(x) = 2(1-2x)(-2)$ {chain rule}

$= -4(1-2x)$

$= -4 + 8x$

$\therefore f''(x) = 8$

f $f(x) = \dfrac{x+2}{2x-1}$

$\therefore f'(x) = \dfrac{1(2x-1) - (x+2)(2)}{(2x-1)^2}$ {quotient rule}

$= \dfrac{2x - 1 - 2x - 4}{(2x-1)^2}$

$= \dfrac{-5}{(2x-1)^2}$

$= -5(2x-1)^{-2}$

$\therefore f''(x) = 10(2x-1)^{-3}(2)$ {chain rule}

$= \dfrac{20}{(2x-1)^3}$

2 **a**
$$y = x - x^3$$
$$\therefore \frac{dy}{dx} = 1 - 3x^2$$
$$\therefore \frac{d^2y}{dx^2} = -6x$$

b
$$\begin{aligned} y &= x^2 - \frac{5}{x^2} \\ &= x^2 - 5x^{-2} \\ \therefore \frac{dy}{dx} &= 2x + 10x^{-3} \\ \therefore \frac{d^2y}{dx^2} &= 2 - 30x^{-4} \\ &= 2 - \frac{30}{x^4} \end{aligned}$$

c
$$\begin{aligned} y &= 2 - \frac{3}{\sqrt{x}} \\ &= 2 - 3x^{-\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \tfrac{3}{2}x^{-\frac{3}{2}} \\ \therefore \frac{d^2y}{dx^2} &= -\tfrac{9}{4}x^{-\frac{5}{2}} = -\frac{9}{4x^2\sqrt{x}} \end{aligned}$$

d
$$\begin{aligned} y &= \frac{4-x}{x} \\ &= 4x^{-1} - 1 \\ \therefore \frac{dy}{dx} &= -4x^{-2} \\ \therefore \frac{d^2y}{dx^2} &= 8x^{-3} \\ &= \frac{8}{x^3} \end{aligned}$$

e
$$\begin{aligned} y &= (x^2 - 3x)^2 \\ &= x^4 - 2(x^2)(3x) + (3x)^2 \\ &= x^4 - 6x^3 + 9x^2 \\ \therefore \frac{dy}{dx} &= 4x^3 - 18x^2 + 18x \\ \therefore \frac{d^2y}{dx^2} &= 12x^2 - 36x + 18 \end{aligned}$$

f
$$\begin{aligned} y &= x^2 - x + \frac{1}{1-x} \\ &= x^2 - x + (1-x)^{-1} \\ \therefore \frac{dy}{dx} &= 2x - 1 + (-1)(1-x)^{-2}(-1) \quad \{\text{chain rule}\} \\ &= 2x - 1 + (1-x)^{-2} \\ \therefore \frac{d^2y}{dx^2} &= 2 - 2(1-x)^{-3}(-1) \\ &= 2 + \frac{2}{(1-x)^3} \end{aligned}$$

g
$$\begin{aligned} y &= e^{3x} + 2x \\ \therefore \frac{dy}{dx} &= 3e^{3x} + 2 \\ \therefore \frac{d^2y}{dx^2} &= 3(3e^{3x}) = 9e^{3x} \end{aligned}$$

h
$$\begin{aligned} y &= \frac{1 - e^{-x}}{x} \\ \therefore \frac{dy}{dx} &= \frac{(e^{-x})(x) - (1 - e^{-x})(1)}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{xe^{-x} - 1 + e^{-x}}{x^2} \\ &= \frac{(x+1)e^{-x} - 1}{x^2} \\ \therefore \frac{d^2y}{dx^2} &= \frac{\left[e^{-x} - (x+1)e^{-x}\right](x^2) - \left[(x+1)e^{-x} - 1\right](2x)}{x^4} \quad \{\text{quotient rule}\} \\ &= \frac{\cancel{x^2e^{-x}} - x^3e^{-x} - \cancel{x^2e^{-x}} - 2x^2e^{-x} - 2xe^{-x} + 2x}{x^4} \\ &= \frac{x\left[-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}\right]}{x^4} \\ &= \frac{-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}}{x^3} \end{aligned}$$

i

$$y = \frac{3-x}{xe^x}$$

$$\therefore \frac{dy}{dx} = \frac{(-1)(xe^x) - (3-x)e^x(x+1)}{x^2e^{2x}} \quad \{\text{quotient rule}\}$$

$$= \frac{-xe^x - e^x(3x+3-x^2-x)}{x^2e^{2x}}$$

$$= \frac{-xe^x - 2xe^x - 3e^x + x^2e^x}{x^2e^{2x}}$$

$$= \frac{e^x(x^2-3x-3)}{x^2e^{2x}}$$

$$= \frac{x^2-3x-3}{x^2e^x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(2x-3)(x^2e^x) - (x^2-3x-3)(2xe^x+x^2e^x)}{x^4e^{2x}} \quad \{\text{quotient rule}\}$$

$$= \frac{2x^3e^x - 3x^2e^x - (2x^3e^x + x^4e^x - 6x^2e^x - 3x^3e^x - 6xe^x - 3x^2e^x)}{x^4e^{2x}}$$

$$= \frac{\cancel{2x^3e^x} - \cancel{3x^2e^x} - \cancel{2x^3e^x} - x^4e^x + 6x^2e^x + 3x^3e^x + 6xe^x + \cancel{3x^2e^x}}{x^4e^{2x}}$$

$$= \frac{-xe^x(x^3-3x^2-6x-6)}{x^4e^{2x}}$$

$$= -\frac{x^3-3x^2-6x-6}{x^3e^x}$$

3 **a**

$$f(x) = x^3 - 2x + 5$$

$$\therefore f(2) = (2)^3 - 2(2) + 5$$

$$= 9$$

b

$$f(x) = x^3 - 2x + 5$$

$$\therefore f'(x) = 3x^2 - 2$$

$$\therefore f'(2) = 3(2)^2 - 2$$

$$= 10$$

c

$$f'(x) = 3x^2 - 2 \quad \{\text{from } \mathbf{b}\}$$

$$\therefore f''(x) = 6x$$

$$\therefore f''(2) = 6(2)$$

$$= 12$$

4 **a**

$$f(x) = 2x^3 - 6x^2 + 5x + 1$$

$$\therefore f'(x) = 6x^2 - 12x + 5$$

$$\therefore f''(x) = 12x - 12$$

$f''(x) = 0$ when $12x - 12 = 0$

$$\therefore 12x = 12$$

$$\therefore x = 1$$

b

$$f(x) = x^4 - 10x^3 + 36x^2 - 72x + 108$$

$$\therefore f'(x) = 4x^3 - 30x^2 + 72x - 72$$

$$\therefore f''(x) = 12x^2 - 60x + 72$$

$f''(x) = 0$ when $12x^2 - 60x + 72 = 0$

$$\therefore x^2 - 5x + 6 = 0$$

$$\therefore (x-2)(x-3) = 0$$

$$\therefore x = 2 \text{ or } x = 3$$

5

$$f(x) = 2x^3 - x$$
$$\therefore\ f'(x) = 6x^2 - 1$$
$$\therefore\ f''(x) = 12x$$

$f(0) = 2(0)^3 - 0 = 0 \quad \therefore\ 0$

$f'(0) = 6(0)^2 - 1 = -1 \quad \therefore\ -$

$f''(0) = 12(0) = 0 \quad \therefore\ 0$

$f(-1) = 2(-1)^3 - (-1) = -1 \quad \therefore\ -$

$f'(-1) = 6(-1)^2 - 1 = 5 \quad \therefore\ +$

$f''(-1) = 12(-1) = -12 \quad \therefore\ -$

$f(1) = 2(1)^3 - 1 = 1 \quad \therefore\ +$

$f'(1) = 6(1)^2 - 1 = 5 \quad \therefore\ +$

$f''(1) = 12(1) = 12 \quad \therefore\ +$

We can fill in the table as follows:

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

6 **a**

$$f(x) = x^2 - \frac{1}{x}$$
$$\therefore\ f(1) = (1)^2 - \frac{1}{1} = 1 - 1 = 0$$

b

$$f(x) = x^2 - \frac{1}{x} = x^2 - x^{-1}$$
$$\therefore\ f'(x) = 2x - (-x^{-2}) = 2x + x^{-2} = 2x + \frac{1}{x^2}$$
$$\therefore\ f'(1) = 2(1) + \frac{1}{1^2} = 2 + 1 = 3$$

c

$$f'(x) = 2x + x^{-2} \quad \{\text{from } \mathbf{b}\}$$
$$\therefore\ f''(x) = 2 - 2x^{-3} = 2 - \frac{2}{x^3}$$
$$\therefore\ f''(1) = 2 - \frac{2}{1^3} = 2 - 2 = 0$$

7 **a**

$$f(x) = 3e^x - 2x$$
$$\therefore\ f(1) = 3e^1 - 2(1) = 3e - 2$$

b

$$f(x) = 3e^x - 2x$$
$$\therefore\ f'(x) = 3e^x - 2$$
$$\therefore\ f'(1) = 3e - 2$$

c

$$f'(x) = 3e^x - 2 \quad \{\text{from } \mathbf{b}\}$$
$$\therefore\ f''(x) = 3e^x$$
$$\therefore\ f''(1) = 3e$$

8 **a**

$$y = 3\tan x$$
$$\therefore\ \frac{dy}{dx} = \frac{3}{\cos^2 x} = 3(\cos x)^{-2}$$
$$\therefore\ \frac{d^2y}{dx^2} = -2 \times 3(\cos x)^{-3} \times -\sin x \quad \{\text{chain rule}\}$$
$$= \frac{6\sin x}{\cos^3 x}$$

b

$$y = x\sin x$$
$$\therefore\ \frac{dy}{dx} = \sin x + x(\cos x) \quad \{\text{product rule}\}$$
$$= \sin x + x\cos x$$
$$\therefore\ \frac{d^2y}{dx^2} = \cos x + \cos x + x(-\sin x) \quad \{\text{product rule}\}$$
$$= 2\cos x - x\sin x$$

c $$y = \sin^2 3x = (\sin 3x)^2$$
$$\therefore \frac{dy}{dx} = 2(\sin 3x) \times 3\cos 3x \qquad \{\text{chain rule}\}$$
$$= 6\sin 3x \cos 3x$$
$$\therefore \frac{d^2y}{dx^2} = 6\left[3\cos 3x \times \cos 3x + \sin 3x \times 3(-\sin 3x)\right]$$
$$= 18(\cos^2 3x - \sin^2 3x)$$

d $$y = \frac{\cos x - x}{x^2} = (\cos x - x)x^{-2}$$
$$\therefore \frac{dy}{dx} = (-\sin x - 1)x^{-2} + (\cos x - x)(-2x^{-3}) \qquad \{\text{product rule}\}$$
$$\therefore \frac{d^2y}{dx^2} = (-\cos x)x^{-2} + (-\sin x - 1)(-2x^{-3}) + (-\sin x - 1)(-2x^{-3}) + (\cos x - x)(6x^{-4})$$
{product rule}
$$= -\frac{\cos x}{x^2} + \frac{2\sin x + 2}{x^3} + \frac{2\sin x + 2}{x^3} + \frac{6\cos x - 6x}{x^4}$$
$$= -\frac{\cos x}{x^2} + \frac{4\sin x - 2}{x^3} + \frac{6\cos x}{x^4}$$

e $$y = e^{-x}\sin x$$
$$\therefore \frac{dy}{dx} = (-e^{-x})\sin x + e^{-x}(\cos x) \qquad \{\text{product rule}\}$$
$$= -e^{-x}\sin x + e^{-x}\cos x$$
$$\therefore \frac{d^2y}{dx^2} = -(-e^{-x}\sin x + e^{-x}\cos x) + (-e^{-x})\cos x + e^{-x}(-\sin x) \qquad \{\text{product rule}\}$$
$$= \cancel{e^{-x}\sin x} - e^{-x}\cos x - e^{-x}\cos x - \cancel{e^{-x}\sin x}$$
$$= -2e^{-x}\cos x$$

f $$y = \tan(x^2)$$
$$\therefore \frac{dy}{dx} = \frac{1}{\cos^2(x^2)} \times 2x \qquad \{\text{chain rule}\}$$
$$= \frac{2x}{\cos^2(x^2)}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{(2)[\cos^2(x^2)] - (2x)[2\cos(x^2)(2x)(-\sin(x^2))]}{[\cos^2(x^2)]^2}$$
$$= \frac{2\cos^2(x^2) + 8x^2\cos(x^2)\sin(x^2)}{\cos^4(x^2)}$$
$$= \frac{2 + 8x^2\tan(x^2)}{\cos^2(x^2)}$$
$$= \frac{2(4x^2\tan(x^2) + 1)}{\cos^2(x^2)}$$

9 $y = Ae^{kx}$

a $$\frac{dy}{dx} = Ae^{kx}(k)$$
$$= k(Ae^{kx})$$
$$= ky$$

b $$\frac{d^2y}{dx^2} = \frac{d}{dx}(kAe^{kx}) \qquad \{\text{from } \mathbf{a}\}$$
$$= kAe^{kx}(k)$$
$$= k^2Ae^{kx}$$
$$= k^2y$$

10 $f(x) = 2\sin^3 x - k\sin x, \quad f'\left(\frac{\pi}{3}\right) = \frac{3}{4}$

a $$\begin{aligned} f'(x) &= 2(3\sin^2 x)(\cos x) - k\cos x \\ &= 6\cos x\sin^2 x - k\cos x \end{aligned}$$

Now $f'\left(\frac{\pi}{3}\right) = \frac{3}{4}$

$$\begin{aligned} \therefore\ 6\cos\tfrac{\pi}{3}\sin^2\tfrac{\pi}{3} - k\cos\tfrac{\pi}{3} &= \tfrac{3}{4} \\ \therefore\ 6\left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)^2 - k\left(\tfrac{1}{2}\right) &= \tfrac{3}{4} \\ \therefore\ \tfrac{9}{4} - \frac{k}{2} &= \tfrac{3}{4} \\ \therefore\ \tfrac{6}{4} &= \frac{k}{2} \\ \therefore\ k &= 3 \end{aligned}$$

b $$\begin{aligned} f'(x) &= 6\cos x\sin^2 x - 3\cos x \qquad \{\text{from } \mathbf{a}\} \\ \therefore\ f''(x) &= 6(-\sin x)(\sin^2 x) + (6\cos x)(2\sin x\cos x) + 3\sin x \\ &= 12\cos^2 x\sin x - 6\sin^3 x + 3\sin x \end{aligned}$$

11 $$\begin{aligned} f(x) &= \tfrac{2}{3}\sin 3x \\ \therefore\ f'(x) &= \tfrac{2}{3}(\cos 3x) \times 3 \\ &= 2\cos 3x \\ \therefore\ f''(x) &= 2(-\sin 3x) \times 3 \\ &= -6\sin 3x \\ \therefore\ f''\left(\tfrac{2\pi}{9}\right) &= -6 \times \sin\left(3 \times \tfrac{2\pi}{9}\right) \\ &= -6 \times \sin\tfrac{2\pi}{3} \\ &= -6 \times \tfrac{\sqrt{3}}{2} \\ &= -3\sqrt{3} \end{aligned}$$

12 **a** $$\begin{aligned} y &= -\ln x \\ \therefore\ \frac{dy}{dx} &= -1 \times \frac{1}{x} \\ &= -x^{-1} \\ \therefore\ \frac{d^2y}{dx^2} &= -(-x^{-2}) \\ &= x^{-2} = \frac{1}{x^2} \end{aligned}$$

b $$\begin{aligned} y &= x\ln x \\ \therefore\ \frac{dy}{dx} &= 1 \times \ln x + x \times \frac{1}{x} \qquad \{\text{product rule}\} \\ &= \ln x + 1 \\ \therefore\ \frac{d^2y}{dx^2} &= \frac{1}{x} \end{aligned}$$

c $$y = (\ln x)^2$$

$$\therefore\ \frac{dy}{dx} = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$$ which is a quotient with $u = 2\ln x$ and $v = x$

$\therefore\ u' = \dfrac{2}{x}$ and $v' = 1$

$$\begin{aligned} \text{Now}\quad \frac{d^2y}{dx^2} &= \frac{\frac{2}{x} \times x - 2\ln x \times 1}{x^2} \qquad \{\text{quotient rule}\} \\ &= \frac{2 - 2\ln x}{x^2} = \frac{2}{x^2}(1 - \ln x) \end{aligned}$$

13 $y = 2e^{3x} + 5e^{4x}$

$\therefore \dfrac{dy}{dx} = 2e^{3x}(3) + 5e^{4x}(4)$ and $\dfrac{d^2y}{dx^2} = 6e^{3x}(3) + 20e^{4x}(4)$

$= 6e^{3x} + 20e^{4x}$ $\qquad$ $= 18e^{3x} + 80e^{4x}$

$$\begin{aligned}\text{Now } \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y &= (18e^{3x} + 80e^{4x}) - 7(6e^{3x} + 20e^{4x}) + 12(2e^{3x} + 5e^{4x}) \\ &= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x} \\ &= e^{3x}(18 - 42 + 24) + e^{4x}(80 - 140 + 60) \\ &= e^{3x}(0) + e^{4x}(0) \\ &= 0\end{aligned}$$

$\therefore \dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$ as required

14 If $y = \sin(2x+3)$, then $\dfrac{dy}{dx} = (\cos(2x+3))(2)$ and $\dfrac{d^2y}{dx^2} = (-2\sin(2x+3))(2)$

$= 2\cos(2x+3)$ $\qquad$ $= -4\sin(2x+3)$

$\therefore \dfrac{d^2y}{dx^2} + 4y = -4\sin(2x+3) + 4\sin(2x+3) = 0$ as required

15 $y = 2\sin x + 3\cos x$

$\therefore \dfrac{dy}{dx} = 2\cos x - 3\sin x$

$\therefore \dfrac{d^2y}{dx^2} = -2\sin x - 3\cos x$

$$\begin{aligned}\text{Now } \frac{d^2y}{dx^2} + y &= (-2\sin x - 3\cos x) + (2\sin x + 3\cos x) \\ &= 0 \quad \text{as required}\end{aligned}$$

REVIEW SET 18A

1 **a** $f(x) = 5x^3$

$\therefore f'(x) = 15x^2$

b $f(x) = x^6 - 5x$

$\therefore f'(x) = 6x^5 - 5$

c
$$\begin{aligned}f(x) &= 7x^2 - \frac{3}{x} \\ &= 7x^2 - 3x^{-1} \\ \therefore f'(x) &= 7(2x) - 3(-x^{-2}) \\ &= 14x + 3x^{-2} \\ &= 14x + \frac{3}{x^2}\end{aligned}$$

d
$$\begin{aligned}f(x) &= 3x - \frac{4}{x^2} \\ &= 3x - 4x^{-2} \\ \therefore f'(x) &= 3 - 4(-2x^{-3}) \\ &= 3 + 8x^{-3} \\ &= 3 + \frac{8}{x^3}\end{aligned}$$

e
$$\begin{aligned}f(x) &= 2x\sqrt{x} = 2x^{\frac{3}{2}} \\ \therefore f'(x) &= 2(\tfrac{3}{2}x^{\frac{1}{2}}) \\ &= 3x^{\frac{1}{2}} \\ &= 3\sqrt{x}\end{aligned}$$

f
$$\begin{aligned}f(x) &= 4\sqrt{x} - \frac{1}{\sqrt{x}} = 4x^{\frac{1}{2}} - x^{-\frac{1}{2}} \\ \therefore f'(x) &= 4(\tfrac{1}{2}x^{-\frac{1}{2}}) - (-\tfrac{1}{2}x^{-\frac{3}{2}}) \\ &= 2x^{-\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{3}{2}} \\ &= \frac{2}{\sqrt{x}} + \frac{1}{2x\sqrt{x}}\end{aligned}$$

2 **a** $y = 3x^2 - x^4$

$\therefore \dfrac{dy}{dx} = 3(2x) - 4x^3$

$= 6x - 4x^3$

b $y = \dfrac{x^3 - x}{x^2}$

$= x - x^{-1}$

$\therefore \dfrac{dy}{dx} = 1 - (-x^{-2})$

$= 1 + x^{-2}$

$= 1 + \dfrac{1}{x^2}$

c $y = x^2\sqrt{x-2}$ is the product of

$u = x^2$ and $v = (x-2)^{\frac{1}{2}}$

$\therefore u' = 2x$ and $v' = \frac{1}{2}(x-2)^{-\frac{1}{2}}(1)$ {chain rule}

$= \frac{1}{2}(x-2)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 2x(x-2)^{\frac{1}{2}} + x^2(\frac{1}{2}(x-2)^{-\frac{1}{2}})$

$= 2x(x-2)^{\frac{1}{2}} + \frac{1}{2}x^2(x-2)^{-\frac{1}{2}}$

$= 2x\sqrt{x-2} + \dfrac{x^2}{2\sqrt{x-2}}$

3 **a** $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ is a quotient with $u = x$ and $v = (x^2+1)^{\frac{1}{2}}$

$\therefore u' = 1$ and $v' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x$

$= x(x^2+1)^{-\frac{1}{2}}$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{1 \times (x^2+1)^{\frac{1}{2}} - x \times x(x^2+1)^{-\frac{1}{2}}}{\left((x^2+1)^{\frac{1}{2}}\right)^2}$$

$$= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{\sqrt{x^2+1} \times \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{(x^2+1) - x^2}{(x^2+1)\sqrt{x^2+1}}$$

$$= \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

$$= (x^2+1)^{-\frac{3}{2}}$$

b The tangent to $f(x)$ has gradient 1 when $f'(x) = 1$

$$\therefore\ (x^2+1)^{-\frac{3}{2}} = 1$$
$$\therefore\ x^2+1 = 1$$
$$\therefore\ x^2 = 0$$
$$\therefore\ x = 0$$
$$\text{and}\quad f(0) = \frac{0}{\sqrt{0^2+1}} = 0$$

$\therefore$ the tangent to $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ has gradient 1 at the point $(0, 0)$.

4 **a**
$$y = e^{x^3+2}$$
$$= e^u \quad \text{where} \quad u = x^3+2$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$= e^u \frac{du}{dx}$$
$$= e^{x^3+2} \times 3x^2$$
$$= 3x^2 e^{x^3+2}$$

b
$$y = \ln\left(\frac{x+3}{x^2}\right)$$
$$= \ln(x+3) - \ln(x^2)$$
$$\left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\}$$
$$\therefore\ \frac{dy}{dx} = \frac{1}{x+3} - \frac{2x}{x^2}$$
$$= \frac{1}{x+3} - \frac{2}{x}$$

c
$$y = x^3 e^{2x}$$
$$\therefore\ \frac{dy}{dx} = 3x^2e^{2x} + x^3e^{2x}(2) \quad \text{\{product rule\}}$$
$$= 3x^2e^{2x} + 2x^3e^{2x}$$

5 **a**
$$f(x) = -x^2 + 4x - 2$$
$$\therefore\ f'(x) = -2x + 4$$

At the point $(-3, -23)$,
$$f'(-3) = -2(-3) + 4$$
$$= 10$$

So, the gradient of the tangent is 10.

b
$$y = (2-3x)^5$$
$$\therefore\ \frac{dy}{dx} = 5(2-3x)^4 \times (-3) \quad \text{\{chain rule\}}$$
$$= -15(2-3x)^4$$

When $x = 1$, $\dfrac{dy}{dx} = -15(2-3(1))^4$
$$= -15(-1)^4$$
$$= -15$$

So, the gradient of the tangent is -15.

6 **a**
$$y = 5x - 3x^{-1}$$
$$\therefore\ \frac{dy}{dx} = 5 + 3x^{-2}$$

b
$$y = (3x^2 + \sqrt{x})^4 = \left(3x^2 + x^{\frac{1}{2}}\right)^4$$
$$\therefore\ \frac{dy}{dx} = 4\left(3x^2 + x^{\frac{1}{2}}\right)^3 (6x + \tfrac{1}{2}x^{-\frac{1}{2}}) \quad \text{\{chain rule\}}$$

c $y = (x^2+1)(1-x^2)^3$ is a product with $u = x^2+1$ and $v = (1-x^2)^3$

$$\therefore\ u' = 2x \quad \text{and} \quad v' = 3(1-x^2)^2(-2x)$$
$$= -6x(1-x^2)^2$$

$$\therefore\ \frac{dy}{dx} = 2x(1-x^2)^3 + (x^2+1)[-6x(1-x^2)^2] \quad \text{\{product rule\}}$$
$$= 2x(1-x^2)^3 - 6x(x^2+1)(1-x^2)^2$$

7 $$y = 2x^3 + 3x^2 - 10x + 3$$
$$\therefore \frac{dy}{dx} = 6x^2 + 6x - 10$$

The gradient of the tangent is 2 when $6x^2 + 6x - 10 = 2$
$$\therefore 6x^2 + 6x - 12 = 0$$
$$\therefore x^2 + x - 2 = 0$$
$$\therefore (x+2)(x-1) = 0$$
$$\therefore x = -2 \text{ or } 1$$

When $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 10(-2) + 3$
$= 19$

When $x = 1$, $y = 2(1)^3 + 3(1)^2 - 10(1) + 3$
$= -2$

So, the gradient of the tangent to $y = 2x^3 + 3x^2 - 10x + 3$ is 2 at the points $(-2, 19)$ and $(1, -2)$.

8 **a**
$$\frac{d}{dx}(\sin 5x \ln x)$$
$$= (\cos 5x)(5)\ln x + \sin 5x\left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$
$$= (5\cos 5x)\ln x + \frac{\sin 5x}{x}$$

b
$$\frac{d}{dx}(\sin x \cos 2x)$$
$$= \cos x \cos 2x + \sin x(-\sin 2x)(2) \quad \{\text{product rule}\}$$
$$= \cos x \cos 2x - 2\sin x \sin 2x$$

c
$$\frac{d}{dx}(e^{-2x}\tan x)$$
$$= -2e^{-2x}\tan x + e^{-2x} \times \frac{1}{\cos^2 x} \quad \{\text{product rule}\}$$
$$= e^{-2x}\left(\frac{1}{\cos^2 x} - 2\tan x\right)$$

9 $$y = \sin^2 x$$
$$= (\sin x)^2$$
$$\therefore \frac{dy}{dx} = 2\sin x \cos x \quad \{\text{chain rule}\}$$

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}$
$= \frac{\sqrt{3}}{2}$

$\therefore$ gradient of tangent $= \frac{\sqrt{3}}{2}$

10 **a**
$$f(x) = \frac{x^2 - 4x - 1}{e^x}$$
$$\therefore f'(x) = \frac{(2x-4)e^x - (x^2 - 4x - 1)e^x}{(e^x)^2} \quad \{\text{quotient rule}\}$$
$$= \frac{e^x(2x - 4 - x^2 + 4x + 1)}{(e^x)^2}$$
$$= \frac{-x^2 + 6x - 3}{e^x}$$

b $f'(1) = \dfrac{-1^2 + 6(1) - 3}{e^1}$

$= \dfrac{2}{e}$

$\therefore$ gradient of tangent $= \dfrac{2}{e}$

c The tangent to $y = f(x)$ is horizontal when $f'(x) = 0$.

$\therefore \; -x^2 + 6x - 3 = 0$

$\therefore \; x^2 - 6x + 3 = 0$

$\therefore \; x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$

$= \dfrac{6 \pm \sqrt{24}}{2}$

$= 3 \pm \sqrt{6}$

11 $y = 3e^x - e^{-x}$

$\therefore \; \dfrac{dy}{dx} = 3e^x + e^{-x}$

$\therefore \; \dfrac{d^2y}{dx^2} = 3e^x - e^{-x} = y$ as required

12 **a** $f(x) = (x^2 + 3)^4$

$\therefore \; f'(x) = 4(x^2 + 3)^3(2x)$ {chain rule}

$= 8x(x^2 + 3)^3$

b $g(x) = \dfrac{\sqrt{x+5}}{x^2}$ is a quotient with $u = (x+5)^{\frac{1}{2}}$ and $v = x^2$

$\therefore \; u' = \frac{1}{2}(x+5)^{-\frac{1}{2}}$ and $v' = 2x$

$\therefore \; g'(x) = \dfrac{\frac{1}{2}(x+5)^{-\frac{1}{2}}(x^2) - (x+5)^{\frac{1}{2}}(2x)}{(x^2)^2}$ {quotient rule}

$= \dfrac{\frac{1}{2}x(x+5)^{-\frac{1}{2}} - 2(x+5)^{\frac{1}{2}}}{x^3}$

c $h(x) = \dfrac{e^{4x}}{1 - 2x}$ is a quotient with $u = e^{4x}$ and $v = 1 - 2x$

$\therefore \; u' = 4e^{4x}$ and $v' = -2$

$\therefore \; h'(x) = \dfrac{4e^{4x}(1 - 2x) - e^{4x}(-2)}{(1 - 2x)^2}$ {quotient rule}

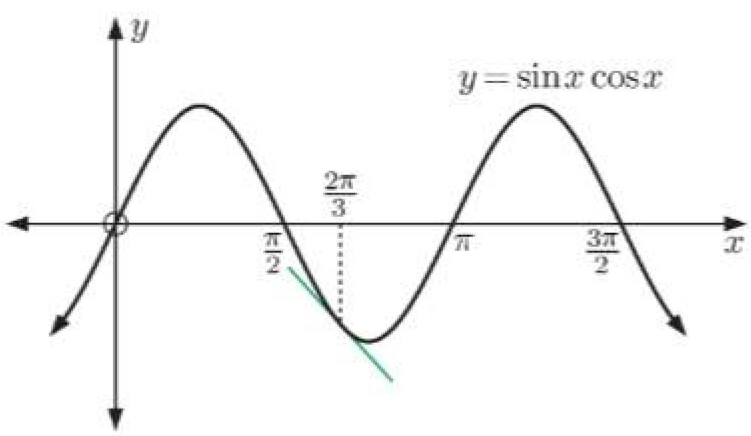
$= \dfrac{4e^{4x} - 8xe^{4x} + 2e^{4x}}{(1 - 2x)^2}$

$= \dfrac{6e^{4x} - 8xe^{4x}}{(1 - 2x)^2}$

13 **a** $y = \sin x \cos x$

$\therefore \; \dfrac{dy}{dx} = \cos x \cos x + \sin x(-\sin x)$ {product rule}

$= \cos^2 x - \sin^2 x$



b When $x = \frac{2\pi}{3}$, $\frac{dy}{dx} = \cos^2\left(\frac{2\pi}{3}\right) - \sin^2\left(\frac{2\pi}{3}\right)$

$$= \tfrac{1}{4} - \tfrac{3}{4}$$
$$= -\tfrac{1}{2}$$

$\therefore$ gradient of tangent $= -\frac{1}{2}$

14 **a**

$$\begin{aligned} f(x) &= 2\sin x + \cos 2x \\ \therefore\ f\left(\tfrac{\pi}{2}\right) &= 2\sin\tfrac{\pi}{2} + \cos\left(2 \times \tfrac{\pi}{2}\right) \\ &= 2(1) + (-1) \\ &= 1 \end{aligned}$$

b

$$\begin{aligned} f'(x) &= 2\cos x - (\sin 2x)(2) \\ &= 2\cos x - 2\sin 2x \\ \therefore\ f'\left(\tfrac{\pi}{2}\right) &= 2\cos\tfrac{\pi}{2} - 2\sin\left(2 \times \tfrac{\pi}{2}\right) \\ &= 2(0) - 2(0) \\ &= 0 \end{aligned}$$

c

$$\begin{aligned} f'(x) &= 2\cos x - 2\sin 2x \qquad \{\text{from } \mathbf{b}\} \\ \therefore\ f''(x) &= -2\sin x - (2\cos 2x)(2) \\ &= -2\sin x - 4\cos 2x \\ \therefore\ f''\left(\tfrac{\pi}{2}\right) &= -2\sin\tfrac{\pi}{2} - 4\cos\left(2 \times \tfrac{\pi}{2}\right) \\ &= -2(1) - 4(-1) \\ &= 2 \end{aligned}$$

15

$$\begin{aligned} y &= 3\tan\tfrac{x}{2} \\ \therefore\ \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(\frac{x}{2})} \times \frac{1}{2} \qquad \{\text{chain rule}\} \\ &= \frac{3}{2\cos^2(\frac{x}{2})} \end{aligned}$$

Now, when $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{3}{2\cos^2(\frac{\pi}{6})}$

$$= \frac{3}{2\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{3}{(\frac{3}{2})}$$
$$= 2$$

$\therefore$ the gradient of the tangent to $y = 3\tan\frac{x}{2}$ at $x = \frac{\pi}{3}$ is 2.

16 **a**

$$\begin{aligned} y &= \tfrac{1}{8}x^4 + \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{2}x^3 + \tfrac{1}{2}x^2 - \tfrac{1}{2}x \\ \therefore\ \frac{d^2y}{dx^2} &= \tfrac{3}{2}x^2 + x - \tfrac{1}{2} \end{aligned}$$

b

$$\begin{aligned} y &= xe^{-x} \\ \therefore\ \frac{dy}{dx} &= (1)e^{-x} + x(e^{-x})(-1) \qquad \{\text{product rule}\} \\ &= e^{-x} - xe^{-x} \\ \therefore\ \frac{d^2y}{dx^2} &= e^{-x}(-1) - \left[(1)e^{-x} + xe^{-x}(-1)\right] \\ &= -e^{-x} - e^{-x} + xe^{-x} \\ &= -2e^{-x} + xe^{-x} \end{aligned}$$

17 $f(x) = \sqrt{x}\cos 4x = x^{\frac{1}{2}}\cos 4x$

a $$\begin{aligned} f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos 4x + x^{\frac{1}{2}} \times (-\sin 4x) \times 4 \qquad \{\text{product rule}\} \\ &= \tfrac{1}{2}x^{-\frac{1}{2}}\cos 4x - 4x^{\frac{1}{2}}\sin 4x \\ &= \frac{1}{2\sqrt{x}}\cos 4x - 4\sqrt{x}\sin 4x \end{aligned}$$

$$\begin{aligned} \therefore\ f''(x) &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos 4x + \tfrac{1}{2}x^{-\frac{1}{2}} \times (-\sin 4x) \times 4 - [2x^{-\frac{1}{2}}\sin 4x + 4x^{\frac{1}{2}} \times (\cos 4x) \times 4] \\ &\qquad \{\text{product rule}\} \\ &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos 4x - 2x^{-\frac{1}{2}}\sin 4x - 2x^{-\frac{1}{2}}\sin 4x - 16x^{\frac{1}{2}}\cos 4x \\ &= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos 4x - 4x^{-\frac{1}{2}}\sin 4x - 16x^{\frac{1}{2}}\cos 4x \\ &= -\frac{1}{4x\sqrt{x}}\cos 4x - \frac{4}{\sqrt{x}}\sin 4x - 16\sqrt{x}\cos 4x \end{aligned}$$

b **i** $$\begin{aligned} f'\left(\tfrac{\pi}{16}\right) &= \frac{1}{2\sqrt{\frac{\pi}{16}}}\cos\left(4 \times \tfrac{\pi}{16}\right) - 4\sqrt{\tfrac{\pi}{16}}\sin\left(4 \times \tfrac{\pi}{16}\right) \\ &= \frac{1}{2 \times \frac{\sqrt{\pi}}{4}}\cos\tfrac{\pi}{4} - 4 \times \tfrac{\sqrt{\pi}}{4}\sin\tfrac{\pi}{4} \\ &= \frac{1}{\frac{\sqrt{\pi}}{2}} \times \tfrac{1}{\sqrt{2}} - \sqrt{\pi} \times \tfrac{1}{\sqrt{2}} \\ &= \tfrac{2}{\sqrt{\pi}} \times \tfrac{1}{\sqrt{2}} - \sqrt{\pi} \times \tfrac{1}{\sqrt{2}} \\ &= \tfrac{1}{\sqrt{2}}\left(\tfrac{2}{\sqrt{\pi}} - \sqrt{\pi}\right) \end{aligned}$$

ii $$\begin{aligned} f''\left(\tfrac{\pi}{8}\right) &= -\frac{1}{4 \times \frac{\pi}{8} \times \sqrt{\frac{\pi}{8}}}\cos\left(4 \times \tfrac{\pi}{8}\right) - \frac{4}{\sqrt{\frac{\pi}{8}}}\sin\left(4 \times \tfrac{\pi}{8}\right) - 16\sqrt{\tfrac{\pi}{8}}\cos\left(4 \times \tfrac{\pi}{8}\right) \\ &= -\frac{1}{\frac{\pi}{2} \times \frac{\sqrt{\pi}}{\sqrt{8}}}\cos\tfrac{\pi}{2} - \frac{4}{\frac{\sqrt{\pi}}{\sqrt{8}}}\sin\tfrac{\pi}{2} - 16\tfrac{\sqrt{\pi}}{\sqrt{8}}\cos\tfrac{\pi}{2} \\ &= -\frac{1}{\frac{\pi}{2} \times \frac{\sqrt{\pi}}{\sqrt{8}}}(0) - 4 \times \tfrac{\sqrt{8}}{\sqrt{\pi}}(1) - 16\,\tfrac{\sqrt{\pi}}{\sqrt{8}}(0) \\ &= -4 \times \tfrac{2\sqrt{2}}{\sqrt{\pi}} \\ &= -\tfrac{8\sqrt{2}}{\sqrt{\pi}} \end{aligned}$$

REVIEW SET 18B

1 **a** $$\begin{aligned} f(x) &= 3x^2 - 7x + 4 \\ \therefore\ f'(x) &= 3(2x) - 7(1) \\ &= 6x - 7 \end{aligned}$$

b $$\begin{aligned} f(x) &= (x+5)^2 \\ &= x^2 + 10x + 25 \\ \therefore\ f'(x) &= 2x + 10(1) \\ &= 2x + 10 \end{aligned}$$

c
$$\begin{aligned} f(x) &= 2\sqrt{x} - \frac{3}{x} \\ &= 2x^{\frac{1}{2}} - 3x^{-1} \\ \therefore\ f'(x) &= 2(\tfrac{1}{2}x^{-\frac{1}{2}}) - 3(-x^{-2}) \\ &= x^{-\frac{1}{2}} + 3x^{-2} \\ &= \frac{1}{\sqrt{x}} + \frac{3}{x^2} \end{aligned}$$

d
$$\begin{aligned} f(x) &= 6x^2\sqrt{x} \\ &= 6x^{\frac{5}{2}} \\ \therefore\ f'(x) &= 6(\tfrac{5}{2}x^{\frac{3}{2}}) \\ &= 15x^{\frac{3}{2}} \\ &= 15x\sqrt{x} \end{aligned}$$

2 **a**
$$\begin{aligned} y &= 2x^3 - 6x^2 + 7x - 4 \\ \therefore\ \frac{dy}{dx} &= 2(3x^2) - 6(2x) + 7(1) \\ &= 6x^2 - 12x + 7 \end{aligned}$$

b
$$\begin{aligned} y &= \frac{3}{x} - \frac{5}{x^3} \\ &= 3x^{-1} - 5x^{-3} \\ \therefore\ \frac{dy}{dx} &= 3(-x^{-2}) - 5(-3x^{-4}) \\ &= -3x^{-2} + 15x^{-4} \\ &= -\frac{3}{x^2} + \frac{15}{x^4} \end{aligned}$$

c
$$\begin{aligned} y &= \frac{15}{\sqrt[3]{x}} \\ &= 15x^{-\frac{1}{3}} \\ \therefore\ \frac{dy}{dx} &= 15(-\tfrac{1}{3}x^{-\frac{4}{3}}) \\ &= -\frac{5}{x^{\frac{4}{3}}} \\ &= -\frac{5}{x\sqrt[3]{x}} \end{aligned}$$

3 **a**
$$\begin{aligned} f(x) &= 7 + x - 3x^2 \\ \therefore\ f(3) &= 7 + 3 - 3(3)^2 \\ &= 7 + 3 - 27 \\ &= -17 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 7 + x - 3x^2 \\ \therefore\ f'(x) &= 1 - 6x \\ \therefore\ f'(3) &= 1 - 6(3) \\ &= 1 - 18 \\ &= -17 \end{aligned}$$

c
$$\begin{aligned} f'(x) &= 1 - 6x \qquad \{\text{from } \mathbf{b}\} \\ f''(x) &= -6 \\ \therefore\ f''(3) &= -6 \end{aligned}$$

4 **a** $y = x^3\sqrt{1-x^2}$ is the product of

$u = x^3$ and $v = (1-x^2)^{\frac{1}{2}}$

$\therefore\ u' = 3x^2$ and $v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$ {chain rule}

$= -x(1-x^2)^{-\frac{1}{2}}$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned} &= 3x^2(1-x^2)^{\frac{1}{2}} + x^3 \times \left[-x(1-x^2)^{-\frac{1}{2}}\right] \\ &= 3x^2(1-x^2)^{\frac{1}{2}} - x^4(1-x^2)^{-\frac{1}{2}} \end{aligned}$$

b $y = \dfrac{x^2 - 3x}{\sqrt{x+1}}$ is a quotient with $u = x^2 - 3x$ and $v = (x+1)^{\frac{1}{2}}$

$\therefore\ u' = 2x - 3$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{(2x-3)(x+1)^{\frac{1}{2}} - (x^2-3x) \times \frac{1}{2}(x+1)^{-\frac{1}{2}}}{(\sqrt{x+1})^2}$$

$$= \frac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$$

c $y = x^3 - x + \dfrac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$

$\therefore\ \dfrac{dy}{dx} = 3x^2 - 1 - \frac{1}{2}x^{-\frac{3}{2}}$

$= 3x^2 - 1 - \dfrac{1}{2x\sqrt{x}}$

5 **a** $y = xe^x$ is the product of $u = x$ and $v = e^x$

$\therefore\ u' = 1$ and $v' = e^x$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 1 \times e^x + x \times e^x$

$= e^x + xe^x$

b $\dfrac{dy}{dx} = e^x + xe^x = (1+x)e^x$ {from **a**}

$\dfrac{dy}{dx} = 2e$ when $(1+x)e^x = 2e$

Solving by inspection, we find $x = 1$.

When $x = 1$, $y = 1 \times e^1 = e$.

$\therefore$ the gradient of $y = xe^x$ is $2e$ at the point $(1, e)$.

6 **a** $f(x) = \ln(e^x + 3)$

$\therefore\ f'(x) = \dfrac{e^x}{e^x + 3}$

b $f(x) = \ln\left[\dfrac{(x+2)^3}{x}\right]$

$= \ln(x+2)^3 - \ln x$ $\left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\}$

$= 3\ln(x+2) - \ln x$ $\{\ln a^n = n \ln a\}$

$\therefore\ f'(x) = \dfrac{3}{x+2} - \dfrac{1}{x}$

c $$f(x) = \ln(\tan x)$$

$$\therefore\ f'(x) = \frac{\left(\frac{1}{\cos^2 x}\right)}{\tan x}$$

$$= \frac{\left(\frac{1}{\cos^2 x}\right)}{\left(\frac{\sin x}{\cos x}\right)}$$

$$= \frac{\cos x}{\cos^2 x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

7 $$y = \left(x - \frac{1}{x}\right)^4$$

$$= (x - x^{-1})^4$$

$$\therefore\ \frac{dy}{dx} = 4(x - x^{-1})^3(1 + x^{-2}) \qquad \{\text{chain rule}\}$$

$$= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)$$

When $x = 1$, $$\frac{dy}{dx} = 4\left(1 - \frac{1}{1}\right)^3\left(1 + \frac{1}{1^2}\right)$$

$$= 4 \times 0 \times 2$$

$$= 0$$

8 **a** $$f(x) = \frac{x^2 + 2}{x}$$

$$= x + 2x^{-1}$$

$$\therefore\ f'(x) = 1 - 2x^{-2}$$

$$= 1 - \frac{2}{x^2}$$

b **i** $$f'(1) = 1 - \frac{2}{1^2}$$

$$= -1$$

$\therefore$ gradient of tangent $= -1$

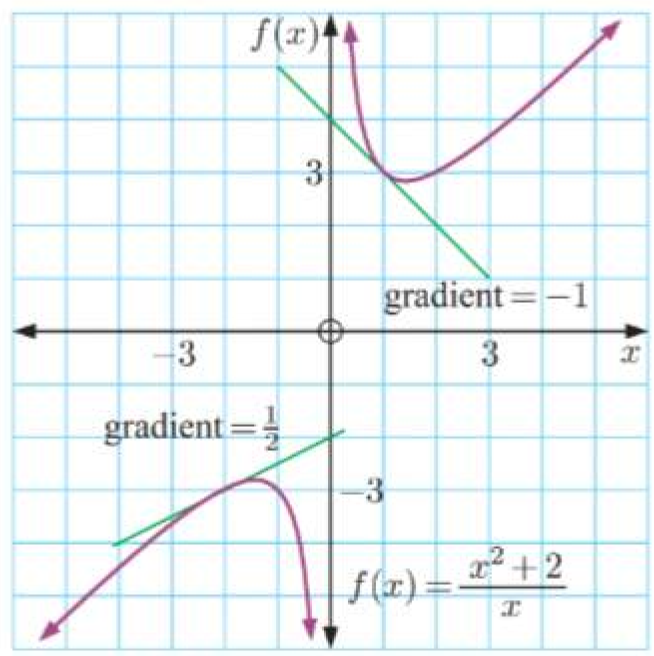
ii $$f'(-2) = 1 - \frac{2}{(-2)^2}$$

$$= 1 - \tfrac{2}{4}$$

$$= \tfrac{1}{2}$$

$\therefore$ gradient of tangent $= \frac{1}{2}$

c



9 **a** $$y = \ln(x^3 - 3x)$$

$$\therefore\ \frac{dy}{dx} = \frac{3x^2 - 3}{x^3 - 3x}$$

b $$y = \frac{e^x}{x^2}$$

$$\therefore\ \frac{dy}{dx} = \frac{e^x(x^2) - e^x(2x)}{(x^2)^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{e^x(x - 2)}{x^3}$$

c $$y = e^{2x} \sin x$$

$$\therefore\ \frac{dy}{dx} = 2e^{2x} \sin x + e^{2x} \cos x$$

{product rule}

10 **a** $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$

$\therefore\ f'(x) = 8x^3 - 12x^2 - 18x + 4$

$\therefore\ f''(x) = 24x^2 - 24x - 18$

b $f''(x) = 0$ when

$24x^2 - 24x - 18 = 0$

$\therefore\ 4x^2 - 4x - 3 = 0$

$\therefore\ (2x+1)(2x-3) = 0$

$\therefore\ x = -\frac{1}{2}$ or $x = \frac{3}{2}$

11 **a** $y = 10x - \sin 10x$

$\therefore\ \dfrac{dy}{dx} = 10 - 10\cos 10x$

b
$$\begin{aligned} y &= \ln\left(\frac{1}{\cos x}\right) \\ &= \ln[(\cos x)^{-1}] \\ \therefore\ \frac{dy}{dx} &= \frac{-(\cos x)^{-2}(-\sin x)}{(\cos x)^{-1}} \quad \{\text{chain rule}\} \\ &= \frac{\sin x \cos x}{\cos^2 x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

c
$$\begin{aligned} y &= \sin 5x \ln(2x) \\ \therefore\ \frac{dy}{dx} &= (5\cos 5x)\ln(2x) + \sin 5x \times \frac{2}{2x} \quad \{\text{product rule}\} \\ &= (5\cos 5x)\ln(2x) + \frac{\sin 5x}{x} \end{aligned}$$

12 **a**
$$\begin{aligned} y &= \frac{x^3}{x+1} \\ \therefore\ \frac{dy}{dx} &= \frac{3x^2(x+1) - x^3(1)}{(x+1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} \\ &= \frac{2x^3 + 3x^2}{(x+1)^2} \end{aligned}$$

At $x = 2$, $\dfrac{dy}{dx} = \dfrac{2(2)^3 + 3(2)^2}{(2+1)^2} = \dfrac{16+12}{9} = \frac{28}{9}$

$\therefore$ gradient of tangent $= \frac{28}{9}$

b
$$\begin{aligned} y &= \tan 4x \\ \therefore\ \frac{dy}{dx} &= \frac{1}{\cos^2 4x} \times 4 \quad \{\text{chain rule}\} \\ &= \frac{4}{\cos^2 4x} \end{aligned}$$

At $x = \frac{\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{4}{(\cos\frac{4\pi}{3})^2} = \dfrac{4}{(-\frac{1}{2})^2} = 16$

$\therefore$ gradient of tangent $= 16$

13 $f(x) = a\ln(bx)$

Now $f(e) = 12$

$$\therefore\ 12 = a\ln(be)$$
$$= a(\ln b + \ln e) \qquad \{\ln(ab) = \ln a + \ln b\}$$
$$= a(\ln b + 1)$$
$$\therefore\ a = \frac{12}{\ln b + 1} \quad .... \ (*)$$

$$f(x) = a\ln(bx)$$
$$= a(\ln b + \ln x) \qquad \{\ln(ab) = \ln a + \ln b\}$$
$$\therefore\ f'(x) = a\left(0 + \frac{1}{x}\right)$$
$$= \frac{a}{x}$$

Now $f'(2) = 2$

$$\therefore\ \frac{a}{2} = 2$$
$$\therefore\ a = 4$$

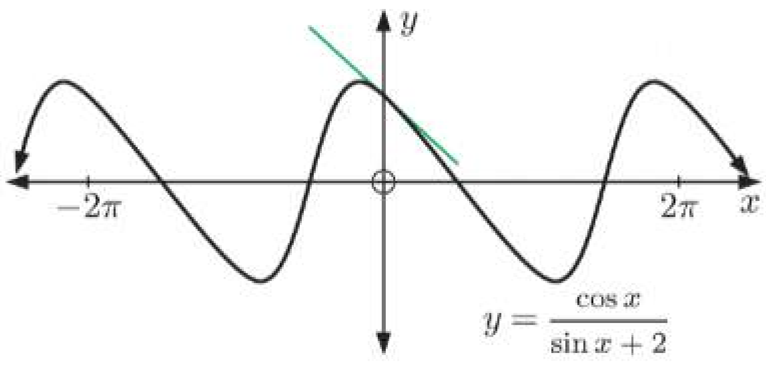
Substituting $a = 4$ into $(*)$ gives:

$$4 = \frac{12}{\ln b + 1}$$
$$\therefore\ 4\ln b + 4 = 12$$
$$\therefore\ 4\ln b = 8$$
$$\therefore\ \ln b = 2$$
$$\therefore\ b = e^2$$

So, $a = 4$ and $b = e^2$.

14 **a**

$$y = \frac{\cos x}{\sin x + 2}$$
$$\therefore\ \frac{dy}{dx} = \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2}$$
{quotient rule}



The tangent meets the graph at $x = 0$.

At $x = 0$,

$$\frac{dy}{dx} = \frac{(-\sin 0)(\sin 0 + 2) - \cos 0(\cos 0)}{(\sin 0 + 2)^2}$$
$$= \frac{0 - 1}{4}$$
$$= -\frac{1}{4}$$

$\therefore$ gradient of tangent $= -\frac{1}{4}$

b $$\frac{dy}{dx} = \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2} \qquad \{\text{from } \mathbf{a}\}$$
$$= \frac{-\sin^2 x - 2\sin x - \cos^2 x}{(\sin x + 2)^2}$$
$$= \frac{-(\sin^2 x + \cos^2 x) - 2\sin x}{(\sin x + 2)^2}$$
$$= -\frac{2\sin x + 1}{(\sin x + 2)^2}$$

A tangent of gradient $-\frac{1}{2}$ occurs when $\frac{dy}{dx} = -\frac{1}{2}$.

$$\therefore\ -\frac{2\sin x + 1}{(\sin x + 2)^2} = -\frac{1}{2}$$
$$\therefore\ \frac{2\sin x + 1}{(\sin x + 2)^2} = \frac{1}{2}$$
$$\therefore\ 2(2\sin x + 1) = (\sin x + 2)^2$$
$$\therefore\ \cancel{4\sin x} + 2 = \sin^2 x + \cancel{4\sin x} + 4$$
$$\therefore\ \sin^2 x = -2 \quad \text{which has no real solutions}$$

$\therefore$ it is impossible to draw a tangent to the graph with gradient $-\frac{1}{2}$.

15 **a** $$y = \frac{e^x}{\sqrt{x}} = \frac{e^x}{x^{\frac{1}{2}}}$$

$$\therefore\ \frac{dy}{dx} = \frac{e^x x^{\frac{1}{2}} - e^x(\frac{1}{2}x^{-\frac{1}{2}})}{\left(x^{\frac{1}{2}}\right)^2} \qquad \{\text{quotient rule}\}$$
$$= \frac{e^x\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}} - \frac{e^x}{2\sqrt{x}}}{x}$$
$$= \frac{2xe^x - e^x}{2x\sqrt{x}}$$
$$= \frac{e^x(2x-1)}{2x\sqrt{x}} \quad \text{as required}$$

b **i** $\frac{dy}{dx} = 0$ when $e^x(2x-1) = 0$
$$\therefore\ e^x = 0 \ \text{ or } \ 2x - 1 = 0$$
$$\therefore\ x = \tfrac{1}{2} \qquad \{\text{as } e^x > 0 \text{ for all } x\}$$

ii $\frac{dy}{dx}$ is undefined when $2x\sqrt{x} = 0$ or $\sqrt{x}$ is undefined
$$\therefore\ x \leqslant 0$$

16 $y = 3\sin 2x + 2\cos 2x$

$$\therefore \frac{dy}{dx} = 3 \times (\cos 2x) \times 2 + 2 \times (-\sin 2x) \times 2$$
$$= 6\cos 2x - 4\sin 2x$$

$$\therefore \frac{d^2y}{dx^2} = 6 \times (-\sin 2x) \times 2 - 4 \times (\cos 2x) \times 2$$
$$= -12\sin 2x - 8\cos 2x$$

$$\therefore 4y + \frac{d^2y}{dx^2} = 4(3\sin 2x + 2\cos 2x) + (-12\sin 2x - 8\cos 2x)$$
$$= 12\sin 2x + 8\cos 2x - 12\sin 2x - 8\cos 2x$$
$$= 0 \quad \text{as required}$$

17 **a** $f(x) = -\frac{1}{2}$ when $\dfrac{6x}{3+x^2} = -\dfrac{1}{2}$

$$\therefore 12x = -(3 + x^2)$$
$$\therefore 12x = -3 - x^2$$
$$\therefore x^2 + 12x + 3 = 0$$
$$\therefore x = \frac{-12 \pm \sqrt{(12)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{-12 \pm \sqrt{144 - 12}}{2}$$
$$= \frac{-12 \pm \sqrt{132}}{2}$$
$$= \frac{-12 \pm 2\sqrt{33}}{2}$$
$$= -6 \pm \sqrt{33}$$

b $f(x) = \dfrac{6x}{3+x^2}$

$$\therefore f'(x) = \frac{6(3+x^2) - 6x(2x)}{(3+x^2)^2} \quad \{\text{quotient rule}\}$$
$$= \frac{18 + 6x^2 - 12x^2}{(3+x^2)^2}$$
$$= \frac{18 - 6x^2}{(3+x^2)^2}$$

$f'(x) = 0$ when $\dfrac{18 - 6x^2}{(3+x^2)^2} = 0$

$$\therefore 18 - 6x^2 = 0$$
$$\therefore 6x^2 = 18$$
$$\therefore x^2 = 3$$
$$\therefore x = \pm\sqrt{3}$$

c $f'(x) = \dfrac{18-6x^2}{(3+x^2)^2}$ {from **b**}

$$\therefore\ f''(x) = \frac{(-12x)(3+x^2)^2 - (18-6x^2)\times 2(3+x^2)\times(2x)}{(3+x^2)^4}$$

$$= \frac{-12x(9+6x^2+x^4) - 4x(18-6x^2)(3+x^2)}{(3+x^2)^4}$$

$$= \frac{-108x - 72x^3 - 12x^5 - 4x(54+18x^2-18x^2-6x^4)}{(3+x^2)^4}$$

$$= \frac{-12x^5 - 72x^3 - 108x - 4x(-6x^4+54)}{(3+x^2)^4}$$

$$= \frac{-12x^5 - 72x^3 - 108x + 24x^5 - 216x}{(3+x^2)^4}$$

$$= \frac{12x^5 - 72x^3 - 324x}{(3+x^2)^4}$$

$f''(x) = 0$ when $\dfrac{12x^5 - 72x^3 - 324x}{(3+x^2)^4} = 0$

$$\therefore\ 12x^5 - 72x^3 - 324x = 0$$

$$\therefore\ x^5 - 6x^3 - 27x = 0$$

$$\therefore\ x(x^4 - 6x^2 - 27) = 0$$

$$\therefore\ x(x^2-9)(x^2+3) = 0$$

$$\therefore\ x = 0 \quad \text{or} \quad x^2 - 9 = 0$$

$$\therefore\ x^2 = 9$$

$$\therefore\ x = \pm 3$$

$$\therefore\ x = -3,\ 0,\ \text{or}\ 3$$

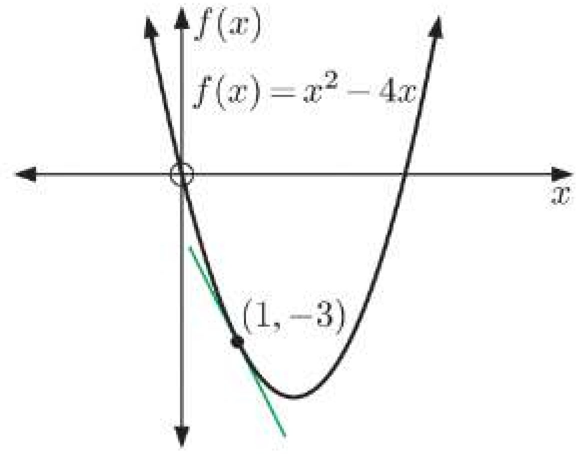
Chapter 19

PROPERTIES OF CURVES

EXERCISE 19A

1 **a** $\quad f(x) = x^2 - 4x$

$\therefore \; f'(x) = 2x - 4$



b The point of contact is $(1, -3)$.

$f'(1) = 2(1) - 4$
$\quad = -2$

So, the tangent has equation $\quad y = -2(x - 1) - 3$
$\therefore \; y = -2x - 1$

2 **a** $y = x - 2x^2 + 3$

When $x = 2$,
$y = 2 - 2(2)^2 + 3 = -3$

So, the point of contact is $(2, -3)$.

Now $\dfrac{dy}{dx} = 1 - 4x$, so at $x = 2$,
$\dfrac{dy}{dx} = 1 - 4(2) = -7$

So, the tangent has gradient -7.

The tangent has equation
$y = f'(a)(x - a) + f(a)$
$\therefore \; y = -7(x - 2) + (-3)$
$\therefore \; y = -7x + 14 - 3$
$\therefore \; y = -7x + 11$

b $y = \sqrt{x} + 1$
$\quad = x^{\frac{1}{2}} + 1$

When $x = 4$,
$y = \sqrt{4} + 1 = 3$

So, the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$, so at $x = 4$,
$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

So, the tangent has gradient $\frac{1}{4}$.

The tangent has equation
$y = f'(a)(x - a) + f(a)$
$\therefore \; y = \frac{1}{4}(x - 4) + 3$
$\therefore \; y = \frac{1}{4}x + 2$

c $y = x^3 - 5x$

When $x = 1$,
$y = 1^3 - 5(1) = -4$

So, the point of contact is $(1, -4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$, so at $x = 1$,
$\dfrac{dy}{dx} = 3(1)^2 - 5 = -2$

The tangent has equation
$y = f'(a)(x - a) + f(a)$
$\therefore \; y = -2(x - 1) + (-4)$
$\therefore \; y = -2x - 2$

d $\quad y = \dfrac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = -2x^{-\frac{3}{2}}$, so at $x = 1$,
$\dfrac{dy}{dx} = -2(1^{-\frac{3}{2}}) = -2$

The tangent has equation
$y = f'(a)(x - a) + f(a)$
$\therefore \; y = -2(x - 1) + 4$
$\therefore \; y = -2x + 6$

e $$y = \frac{3}{x} - \frac{1}{x^2}$$
$$= 3x^{-1} - x^{-2}$$

Now $\frac{dy}{dx} = -3x^{-2} + 2x^{-3}$

$= -\frac{3}{x^2} + \frac{2}{x^3}$, so at $(-1, -4)$,

$$\frac{dy}{dx} = -\frac{3}{(-1)^2} + \frac{2}{(-1)^3}$$
$$= -3 - 2$$
$$= -5$$

The tangent has equation

$y = f'(a)(x-a) + f(a)$

$\therefore \; y = -5(x+1) + (-4)$

$\therefore \; y = -5x - 9$

f $$y = 3x^2 - \frac{1}{x}$$
$$= 3x^2 - x^{-1}$$

When $x = -1$,

$$y = 3(-1)^2 - \frac{1}{(-1)} = 4$$

So, the point of contact is $(-1, 4)$.

Now $\frac{dy}{dx} = 6x + x^{-2}$

$= 6x + \frac{1}{x^2}$, so at $x = -1$,

$$\frac{dy}{dx} = 6(-1) + \frac{1}{(-1)^2} = -5$$

The tangent has equation

$y = f'(a)(x-a) + f(a)$

$\therefore \; y = -5(x+1) + 4$

$\therefore \; y = -5x - 1$

3 **a** $$y = 2x^3 + 3x^2 - 12x + 1$$

$\therefore \; \frac{dy}{dx} = 6x^2 + 6x - 12$

Horizontal tangents have gradient 0,

so $6x^2 + 6x - 12 = 0$

$\therefore \; 6(x^2 + x - 2) = 0$

$\therefore \; 6(x+2)(x-1) = 0$

$\therefore \; x = -2$ or 1

When $x = -2$,

$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1$
$= 21$

When $x = 1$,

$y = 2(1)^3 + 3(1)^2 - 12(1) + 1$
$= -6$

$\therefore$ the points of contact are $(-2, 21)$ and $(1, -6)$.

$\therefore$ the tangents are $y = 21$ and $y = -6$.

b $$y = -x^3 + 3x^2 + 9x - 4$$

$\therefore \; \frac{dy}{dx} = -3x^2 + 6x + 9$

Horizontal tangents have gradient 0,

so $-3x^2 + 6x + 9 = 0$

$\therefore \; -3(x^2 - 2x - 3) = 0$

$\therefore \; -3(x+1)(x-3) = 0$

$\therefore \; x = -1$ or 3

When $x = -1$,

$y = -(-1)^3 + 3(-1)^2 + 9(-1) - 4$
$= -9$

When $x = 3$,

$y = -3^3 + 3(3)^2 + 9(3) - 4$
$= 23$

$\therefore$ the points of contact are $(-1, -9)$ and $(3, 23)$.

$\therefore$ the tangents are $y = -9$ and $y = 23$.

c $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$= x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$= \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}$

Horizontal tangents have gradient 0, so $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}} = 0$

$\therefore \dfrac{x-1}{2x\sqrt{x}} = 0$

$\therefore x - 1 = 0$

$\therefore x = 1$

When $x = 1$, $y = \sqrt{1} + \dfrac{1}{\sqrt{1}}$

$= 2$

$\therefore$ the point of contact is $(1, 2)$.

$\therefore$ the tangent is $y = 2$.

4 $y = 2x^3 + kx^2 - 3$

a $\dfrac{dy}{dx} = 6x^2 + 2kx$

When $x = 2$, $\dfrac{dy}{dx} = 4$

$\therefore 6(2)^2 + 2k(2) = 4$

$\therefore 24 + 4k = 4$

$\therefore 4k = -20$

$\therefore k = -5$

b Since $k = -5$,

$y = 2x^3 - 5x^2 - 3$

When $x = 2$,

$y = 2(2)^3 - 5(2)^2 - 3$

$= -7$

So, the point of contact is $(2, -7)$.

The tangent has equation

$y = 4(x - 2) + (-7)$

$\therefore y = 4x - 15$

5 $y = 1 - 3x + 12x^2 - 8x^3$

$\therefore \dfrac{dy}{dx} = -3 + 24x - 24x^2$

When $x = 1$, $\dfrac{dy}{dx} = -3 + 24(1) - 24(1)^2$

$= -3$

So, the tangent at $(1, 2)$ has gradient -3.

The tangents to the curve have gradient -3 when $-3 + 24x - 24x^2 = -3$

$\therefore 24x^2 - 24x = 0$

$\therefore 24x(x - 1) = 0$

$\therefore x = 0$ or 1

So the other x-value for which the tangent to the curve has gradient -3 is $x = 0$, and when $x = 0$, $y = 1 - 3(0) + 12(0)^2 - 8(0)^3 = 1$.

$\therefore$ the tangent to the curve at $(0, 1)$ is parallel to the tangent at $(1, 2)$.

This tangent has equation $y = -3(x - 0) + 1$

$\therefore y = -3x + 1$

6 The tangent to the curve $y = x^2+ax+b$ at the point where $x = 1$ is $2x+y = 6$ or $y = -2x+6$.

$\therefore$ the tangent has gradient -2, and the point of contact is $(1, -2(1)+6)$ which is $(1, 4)$.

Now, $y = x^2 + ax + b$

$$\therefore \frac{dy}{dx} = 2x + a$$

When $x = 1$, $\dfrac{dy}{dx} = -2$

$$\therefore 2(1) + a = -2$$
$$\therefore 2 + a = -2$$
$$\therefore a = -4 \quad .... (*)$$

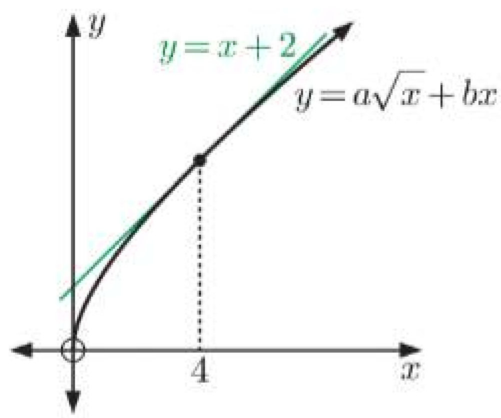
and $y = 4$

$$\therefore 1^2 + a(1) + b = 4$$
$$\therefore 1 + (-4) + b = 4 \quad \{\text{using } (*)\}$$
$$\therefore b = 7$$

So, $a = -4$ and $b = 7$.

7 The tangent to the curve $y = a\sqrt{x} + bx$ at the point where $x = 4$ is $y = x + 2$.

$\therefore$ the tangent has gradient 1, and the point of contact is $(4, 4+2)$ which is $(4, 6)$.



Now, $y = a\sqrt{x} + bx$

$$= ax^{\frac{1}{2}} + bx$$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}ax^{-\frac{1}{2}} + b$$

$$= \frac{a}{2\sqrt{x}} + b$$

When $x = 4$, $\dfrac{dy}{dx} = 1$

$$\therefore \frac{a}{2\sqrt{4}} + b = 1$$
$$\therefore \frac{a}{4} + b = 1$$
$$\therefore a + 4b = 4$$
$$\therefore a = 4 - 4b \quad .... (*)$$

and $y = 6$

$$\therefore a\sqrt{4} + b(4) = 6$$
$$\therefore 2(4 - 4b) + 4b = 6 \quad \{\text{using } (*)\}$$
$$\therefore 4 - 4b + 2b = 3$$
$$\therefore -2b = -1$$
$$\therefore b = \tfrac{1}{2}$$

Substituting $b = \frac{1}{2}$ into $(*)$ gives $a = 4 - 4\left(\frac{1}{2}\right)$
$$= 4 - 2$$
$$= 2$$

So, $a = 2$ and $b = \frac{1}{2}$.

8 $y = 2x^2 - 1$

$$\therefore \frac{dy}{dx} = 4x$$

$\therefore$ at the point where $x = a$, $\dfrac{dy}{dx} = 4a$

$\therefore$ the gradient of the tangent at the point where $x = a$ is $4a$.

Also, at $x = a$, $y = 2a^2 - 1$.

$\therefore$ the tangent has equation $y = 4a(x - a) + (2a^2 - 1)$

$$\therefore y = 4ax - 4a^2 + 2a^2 - 1$$
$$\therefore 4ax - y = 2a^2 + 1$$

9 **a**

$$\begin{aligned} f(x) &= x^2 + \frac{4}{x^2} \\ &= x^2 + 4x^{-2} \\ \therefore\ f'(x) &= 2x - 8x^{-3} \\ &= 2x - \frac{8}{x^3} \end{aligned}$$

b Horizontal tangents have gradient 0, so

$$\begin{aligned} 2x - \frac{8}{x^3} &= 0 \\ \therefore\ \frac{2x^4 - 8}{x^3} &= 0 \\ \therefore\ 2x^4 - 8 &= 0 \\ \therefore\ 2x^4 &= 8 \\ \therefore\ x^4 &= 4 \\ \therefore\ x &= \pm\sqrt{2} \end{aligned}$$

c When $x = \sqrt{2}$,

$$\begin{aligned} f(\sqrt{2}) &= (\sqrt{2})^2 + \frac{4}{(\sqrt{2})^2} \\ &= 2 + \frac{4}{2} \\ &= 4 \end{aligned}$$

$\therefore$ the horizontal tangent at $(\sqrt{2}, 4)$ is $y = 4$.

When $x = -\sqrt{2}$,

$$\begin{aligned} f(-\sqrt{2}) &= (-\sqrt{2})^2 + \frac{4}{(-\sqrt{2})^2} \\ &= 2 + \frac{4}{2} \\ &= 4 \end{aligned}$$

$\therefore$ the horizontal tangent at $(-\sqrt{2}, 4)$ is $y = 4$.

The tangents are the same line, $y = 4$.

10 The tangent to the curve $y = a\sqrt{1 - bx}$ at the point where $x = -1$ is $3x + y = 5$ or $y = -3x + 5$.

$\therefore$ the tangent has gradient -3, and the point of contact is $(-1,\ -3(-1)+5)$ which is $(-1, 8)$.

Now

$$\begin{aligned} y &= a\sqrt{1 - bx} \\ &= a(1 - bx)^{\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{2}a(1 - bx)^{-\frac{1}{2}}(-b) \\ &= -\frac{ab}{2\sqrt{1 - bx}} \end{aligned}$$

When $x = -1$, $\dfrac{dy}{dx} = -3$

$$\begin{aligned} \therefore\ -\frac{ab}{2\sqrt{1 - b(-1)}} &= -3 \\ \therefore\ ab &= 6\sqrt{1 + b} \\ \therefore\ a &= \frac{6\sqrt{1 + b}}{b} \quad \text{.... } (*) \end{aligned}$$

and

$$\begin{aligned} y &= 8 \\ \therefore\ a\sqrt{1 - b(-1)} &= 8 \\ \therefore\ \left(\frac{6\sqrt{1+b}}{b}\right)\sqrt{1 + b} &= 8 \quad \{\text{using } (*)\} \\ \therefore\ \frac{6(1 + b)}{b} &= 8 \\ \therefore\ 6 + 6b &= 8b \\ \therefore\ -2b &= -6 \\ \therefore\ b &= 3 \end{aligned}$$

Substituting $b = 3$ into $(*)$ gives

$$\begin{aligned} a &= \frac{6\sqrt{1 + 3}}{3} \\ &= 2\sqrt{4} \\ &= 4 \end{aligned}$$

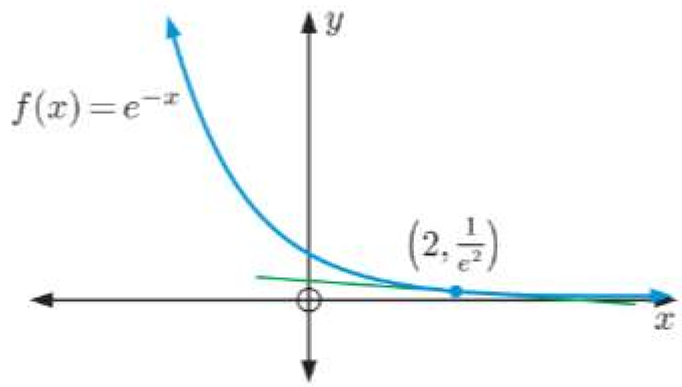
So, $a = 4$ and $b = 3$.

11 **a** $f(x) = e^{-x}$

$\therefore\ f(2) = e^{-2}$

$= \dfrac{1}{e^2}$

$\therefore$ the point of contact is $\left(2, \dfrac{1}{e^2}\right)$.



Now $f(x) = e^{-x}$ has derivative $f'(x) = -e^{-x}$

$= -\dfrac{1}{e^x}$

$\therefore$ the tangent at $\left(2, \dfrac{1}{e^2}\right)$ has gradient $-\dfrac{1}{e^2}$.

$\therefore$ the tangent has equation $y = -\dfrac{1}{e^2}(x-2) + \dfrac{1}{e^2}$

$= -\dfrac{x}{e^2} + \dfrac{2}{e^2} + \dfrac{1}{e^2}$

$= -\dfrac{x}{e^2} + \dfrac{3}{e^2}$

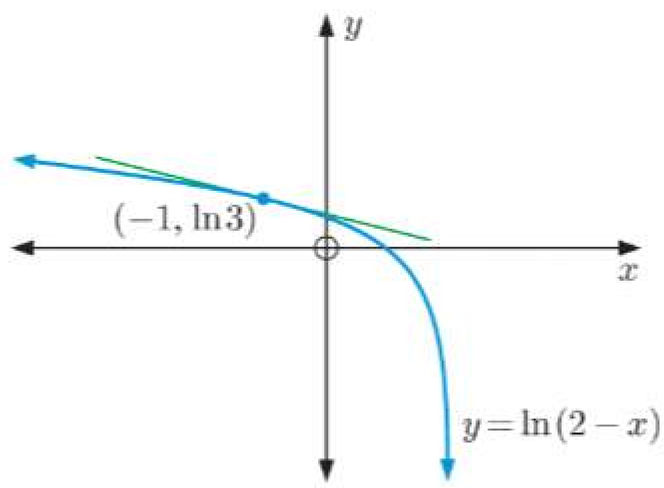
$\therefore\ y = -e^{-2}x + 3e^{-2}$

b $y = \ln(2-x)$

When $x = -1$, $y = \ln(2-(-1))$

$= \ln 3$

$\therefore$ the point of contact is $(-1, \ln 3)$.



Now $y = \ln(2-x)$ has derivative

$\dfrac{dy}{dx} = \dfrac{-1}{2-x} = \dfrac{1}{x-2}$

$\therefore$ the tangent at $(-1, \ln 3)$ has gradient

$\dfrac{1}{-1-2} = -\frac{1}{3}$

$\therefore$ the tangent has equation $y = -\frac{1}{3}(x+1) + \ln 3$

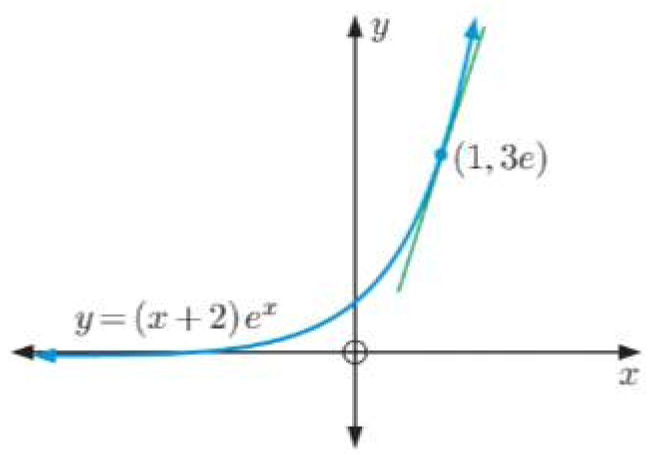
which is $y = -\frac{1}{3}x - \frac{1}{3} + \ln 3$

c $y = (x+2)e^x$

When $x = 1$, $y = (1+2)e^1$

$= 3e$

$\therefore$ the point of contact is $(1, 3e)$.



Now $y = (x+2)e^x$ has derivative

$\dfrac{dy}{dx} = (1)e^x + (x+2)e^x$ {product rule}

$\therefore$ the tangent at $(1, 3e)$ has gradient

$e^1 + (1+2)e^1 = 4e$

$\therefore$ the tangent has equation $y = 4e(x-1) + 3e$

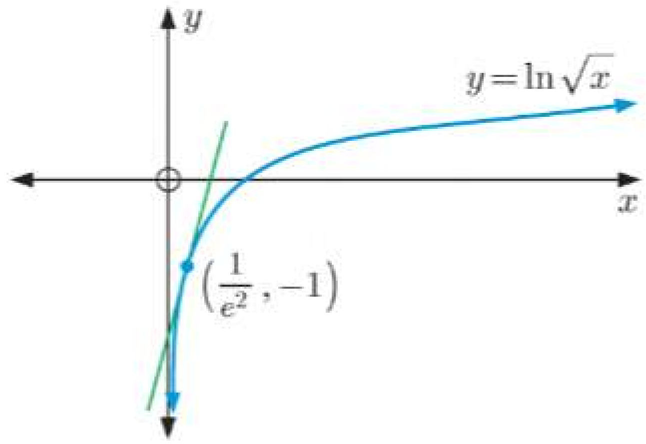
which is $y = 4ex - e$

d $y = \ln\sqrt{x}$

When $y = -1$, $\ln\sqrt{x} = -1$

$\therefore\ \sqrt{x} = e^{-1}$

$\therefore\ x = e^{-2} = \dfrac{1}{e^2}$



$\therefore$ the point of contact is $\left(\dfrac{1}{e^2}, -1\right)$.

Now $y = \ln\sqrt{x} = \ln(x^{\frac{1}{2}})$ has derivative

$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{1}{2x}$$

$\therefore$ the tangent at $\left(\dfrac{1}{e^2}, -1\right)$ has gradient $\dfrac{1}{\frac{2}{e^2}} = \dfrac{e^2}{2}$

$\therefore$ the tangent has equation $y = \dfrac{e^2}{2}\left(x - \dfrac{1}{e^2}\right) - 1$

$= \dfrac{e^2}{2}x - \frac{1}{2} - 1$

$\therefore\ y = \dfrac{e^2}{2}x - \frac{3}{2}$

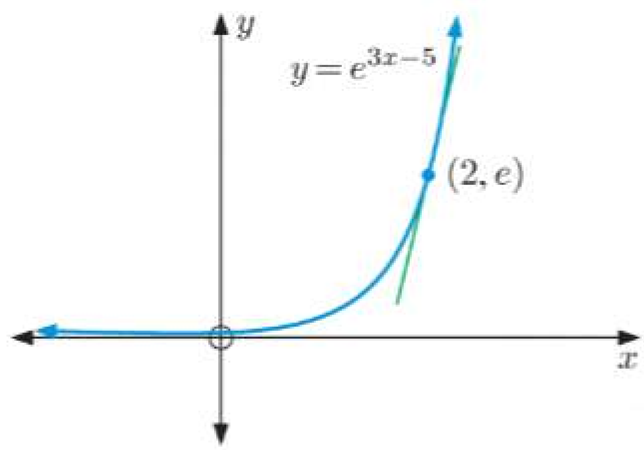
e $y = e^{3x-5}$

When $y = e$, $e^{3x-5} = e$

$\therefore\ e^{3x-6} = 1$

$\therefore\ 3x - 6 = 0 \qquad \{e^0 = 1\}$

$\therefore\ x = 2$



$\therefore$ the point of contact is $(2, e)$.

Now $y = e^{3x-5}$ has derivative $\dfrac{dy}{dx} = 3e^{3x-5}$

$\therefore$ the tangent at $(2, e)$ has gradient

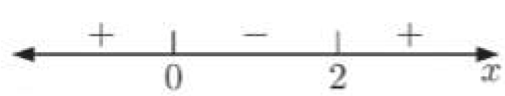
$3e^{3(2)-5} = 3e$

$\therefore$ the tangent has equation $y = 3e(x-2) + e$

which is $y = 3ex - 5e$

12 **a** $f(x) = \ln(x(x-2))$ is defined when $x(x-2) > 0$

$\therefore\ x < 0$ or $x > 2$



$\therefore$ the domain of $f(x)$ is $\{x \mid x < 0 \text{ or } x > 2\}$

b $f(x) = \ln(x(x-2))$

$= \ln x + \ln(x-2) \qquad \{\ln(ab) = \ln a + \ln b\}$

$\therefore\ f'(x) = \dfrac{1}{x} + \dfrac{1}{x-2}$

c $f(3) = \ln(3(3-2))$
$= \ln 3$

$\therefore$ the point of contact is $(3, \ln 3)$.

Now $f'(3) = \frac{1}{3} + \frac{1}{3-2}$
$= \frac{1}{3} + 1$
$= \frac{4}{3}$

$\therefore$ the tangent at $(3, \ln 3)$ has gradient $\frac{4}{3}$.

$\therefore$ the tangent has equation $y = \frac{4}{3}(x-3) + \ln 3$
which is $y = \frac{4}{3}x - 4 + \ln 3$

13 $y = x^2e^x$

When $x = 1$, $y = (1)^2e^1 = e$

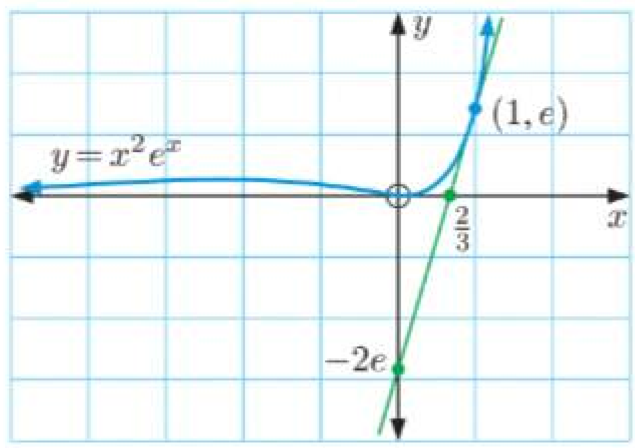
$\therefore$ the point of contact is $(1, e)$.

Now $y = x^2e^x$

$\therefore \frac{dy}{dx} = 2xe^x + x^2e^x$ {product rule}

When $x = 1$, $\frac{dy}{dx} = 2e + e = 3e$

So, the tangent has equation $y = 3e(x-1) + e$
$\therefore y = 3ex - 2e$



When $y = 0$, $0 = 3ex - 2e$
$\therefore 3ex = 2e$
$\therefore x = \frac{2}{3}$

$\therefore$ the x-intercept is $\frac{2}{3}$.

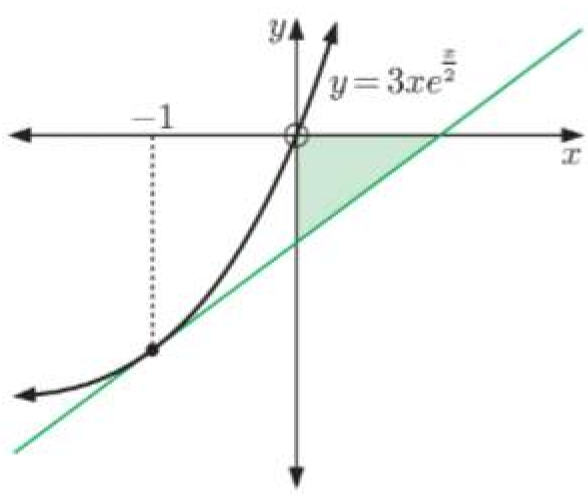
When $x = 0$, $y = -2e$

$\therefore$ the y-intercept is $-2e$.

14 $y = 3xe^{\frac{x}{2}}$

When $x = -1$, $y = 3(-1)e^{\frac{-1}{2}}$
$= -\frac{3}{\sqrt{e}}$

$\therefore$ the point of contact is $\left(-1, -\frac{3}{\sqrt{e}}\right)$.



Now $y = 3xe^{\frac{x}{2}}$

$\therefore \frac{dy}{dx} = 3e^{\frac{x}{2}} + 3x(\frac{1}{2}e^{\frac{x}{2}})$ {product rule}
$= 3e^{\frac{x}{2}} + \frac{3}{2}xe^{\frac{x}{2}}$

When $x = -1$, $\frac{dy}{dx} = 3e^{-\frac{1}{2}} - \frac{3}{2}e^{-\frac{1}{2}}$
$= \frac{3}{\sqrt{e}} - \frac{3}{2\sqrt{e}}$
$= \frac{3}{2\sqrt{e}}$

So, the tangent has equation $y = \dfrac{3}{2\sqrt{e}}(x+1) - \dfrac{3}{\sqrt{e}}$

$$= \frac{3}{2\sqrt{e}}x + \frac{3}{2\sqrt{e}} - \frac{3}{\sqrt{e}}$$

$$= \frac{3}{2\sqrt{e}}x - \frac{3}{2\sqrt{e}}$$

$$= \frac{3}{2\sqrt{e}}(x-1)$$

When $y = 0$, $0 = \dfrac{3}{2\sqrt{e}}(x-1)$

$\therefore \; x = 1$

$\therefore$ the x-intercept is 1.

When $x = 0$, $y = \dfrac{3}{2\sqrt{e}}(0-1)$

$$= -\frac{3}{2\sqrt{e}}$$

$\therefore$ the y-intercept is $-\dfrac{3}{2\sqrt{e}}$.

$\therefore$ the shaded triangle has area $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \tfrac{1}{2} \times 1 \times \frac{3}{2\sqrt{e}}$$

$$= \frac{3}{4\sqrt{e}} \text{ units}^2$$

15 **a** $y = \sin x$ has derivative $\dfrac{dy}{dx} = \cos x$

$\therefore$ the tangent at $(0, 0)$ has gradient $\cos 0 = 1$

$\therefore$ the tangent has equation $y = 1(x-0)+0$
which is $y = x$

b $y = \tan x$ has derivative $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$

$\therefore$ the tangent at $(0, 0)$ has gradient $\dfrac{1}{\cos^2 0} = 1$

$\therefore$ the tangent has equation $y = 1(x-0)+0$
which is $y = x$

c $y = \cos x$

When $x = \frac{\pi}{6}$, $y = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\therefore$ the point of contact is $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.

Now $y = \cos x$ has derivative $\dfrac{dy}{dx} = -\sin x$

$\therefore$ the tangent at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ has gradient $-\sin\frac{\pi}{6} = -\frac{1}{2}$

$\therefore$ the tangent has equation $y = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$

which is $y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$

d $y = \dfrac{1}{\sin 2x}$

When $x = \frac{\pi}{4}$, $y = \dfrac{1}{\sin \frac{\pi}{2}} = 1$

$\therefore$ the point of contact is $\left(\frac{\pi}{4}, 1\right)$.

Now $y = \dfrac{1}{\sin 2x} = (\sin 2x)^{-1}$ has derivative

$$\frac{dy}{dx} = -(\sin 2x)^{-2}(2\cos 2x) \quad \{\text{chain rule}\}$$
$$= -\frac{2\cos 2x}{\sin^2 2x}$$

$\therefore$ the tangent at $\left(\frac{\pi}{4}, 1\right)$ has gradient $-\dfrac{2\cos\frac{\pi}{2}}{\sin^2(\frac{\pi}{2})} = 0$

$\therefore$ the tangent has equation $y = 0(x - \frac{\pi}{4}) + 1$
which is $y = 1$

e $y = \cos 2x + 3\sin x$

When $x = \frac{\pi}{2}$, $y = \cos\pi + 3\sin\frac{\pi}{2}$
$= -1 + 3$
$= 2$

$\therefore$ the point of contact is $\left(\frac{\pi}{2}, 2\right)$.

Now $y = \cos 2x + 3\sin x$ has derivative $\dfrac{dy}{dx} = -2\sin 2x + 3\cos x$

$\therefore$ the tangent at $\left(\frac{\pi}{2}, 2\right)$ has gradient $-2\sin\pi + 3\cos\frac{\pi}{2} = 0$

$\therefore$ the tangent has equation $y = 0(x - \frac{\pi}{2}) + 2$
which is $y = 2$

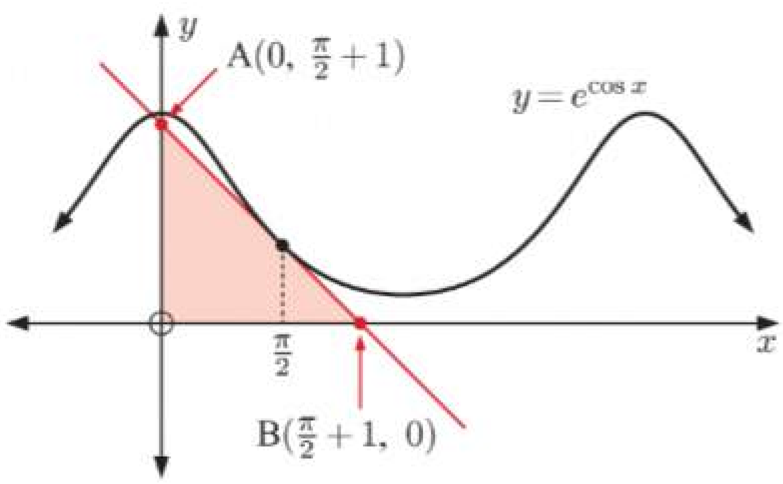
16 $y = \dfrac{\cos x}{1 + \sin x}$

$$\therefore \frac{dy}{dx} = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2} \qquad \{\text{quotient rule}\}$$
$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$
$$= \frac{-(\sin^2 x + \cos^2 x + \sin x)}{(1+\sin x)^2}$$
$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$
$$= -\frac{1}{1+\sin x} \neq 0$$

$\therefore$ there are no tangents which have gradient 0.

$\therefore$ there are no horizontal tangents to $y = \dfrac{\cos x}{1+\sin x}$.

17 $y = e^{\cos x}$



When $x = \frac{\pi}{2}$, $y = e^{\cos\frac{\pi}{2}}$
$= e^0$
$= 1$

$\therefore$ the point of contact is $(\frac{\pi}{2}, 1)$.

Now $y = e^{\cos x}$ has derivative

$\frac{dy}{dx} = -\sin x e^{\cos x}$

$\therefore$ the tangent at $(\frac{\pi}{2}, 1)$ has gradient

$-\sin\frac{\pi}{2}e^{\cos\frac{\pi}{2}} = -e^0 = -1$

$\therefore$ the tangent has equation $y = -1(x - \frac{\pi}{2}) + 1$

which is $y = -x + \frac{\pi}{2} + 1$

Let the y-intercept be A and the x-intercept be B.

When $x = 0$, $y = \frac{\pi}{2} + 1$

$\therefore$ A is $\left(0,\ \frac{\pi}{2} + 1\right)$.

When $y = 0$, $0 = -x + \frac{\pi}{2} + 1$

$\therefore$ $x = \frac{\pi}{2} + 1$

$\therefore$ B is $\left(\frac{\pi}{2} + 1,\ 0\right)$.

Area $\triangle$AOB $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2}\left(\frac{\pi}{2} + 1\right)^2$ units^2

18 Consider the tangent to $y = x^3$ at $x = 2$.

When $x = 2$, $y = 2^3 = 8$ so the point of contact is $(2, 8)$.

Now $\frac{dy}{dx} = 3x^2$ and so at $x = 2$,

$\frac{dy}{dx} = 3(2)^2 = 12$

So, the tangent at $(2, 8)$ has gradient 12 and its equation is

$12x - y = 12(2) - 8$

$\therefore$ $12x - y = 16$

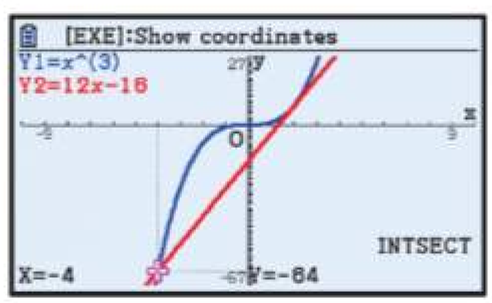
$\therefore$ $y = 12x - 16$

The tangent meets the curve where $12x - 16 = x^3$

$\therefore$ $x^3 - 12x + 16 = 0$

$\therefore$ $(x-2)^2(x+4) = 0$ $\{(x-2)^2$ must be a factor of this cubic$\}$

$\therefore$ the tangent meets the curve again when $x = -4$.



When $x = -4$, $y = (-4)^3 = -64$

$\therefore$ the tangent meets the curve again at $(-4, -64)$.

19 Consider the tangent to $y = -x^3 + 2x^2 + 1$ at $x = -1$.

When $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1 = 4$ and so the point of contact is $(-1, 4)$.

Now $\dfrac{dy}{dx} = -3x^2 + 4x$ and so at $x = -1$,

$$\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$$

So, the tangent at $(-1, 4)$ has gradient -7 and its equation is

$$-7x - y = -7(-1) - 4$$
$$\therefore\ 7x + y = -3$$
$$\therefore\ y = -7x - 3$$

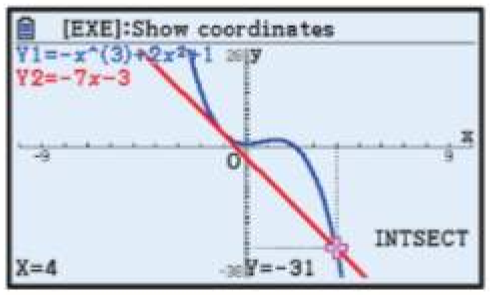
The tangent meets the curve where $-7x - 3 = -x^3 + 2x^2 + 1$

$$\therefore\ x^3 - 2x^2 - 7x - 4 = 0$$
$$\therefore\ (x+1)^2(x-4) = 0 \qquad \{(x+1)^2 \text{ must be a factor of this cubic}\}$$

$\therefore$ the tangent meets the curve again when $x = 4$.

When $x = 4$, $y = -(4)^3 + 2(4)^2 + 1$
$= -64 + 32 + 1$
$= -31$

$\therefore$ the tangent meets the curve again at $(4, -31)$.



20 Consider the tangent to $y = \dfrac{1}{x} - \dfrac{1}{x^2}$ at $x = 1$.

When $x = 1$, $y = \dfrac{1}{1} - \dfrac{1}{1^2} = 0$ and so the point of contact is $(1, 0)$.

Now $y = x^{-1} - x^{-2}$

$$\therefore\ \frac{dy}{dx} = -x^{-2} + 2x^{-3}$$
$$= -\frac{1}{x^2} + \frac{2}{x^3} \quad \text{and so at } x = 1,$$
$$\frac{dy}{dx} = -\frac{1}{1^2} + \frac{2}{1^3} = 1$$

So, the tangent at $(1, 0)$ has gradient 1 and its equation is

$$y = 1 \times (x - 1) + 0$$
$$\therefore\ y = x - 1$$

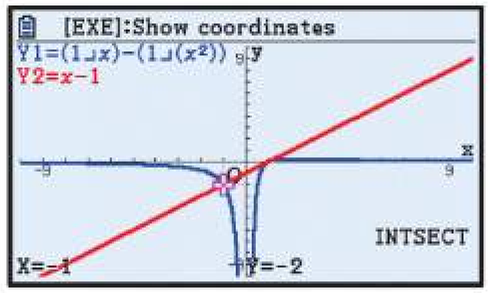
The tangent meets the curve where $x - 1 = \dfrac{1}{x} - \dfrac{1}{x^2}$

$$\therefore\ x^3 - x^2 = x - 1$$
$$\therefore\ x^3 - x^2 - x + 1 = 0$$
$$\therefore\ (x-1)^2(x+1) = 0 \qquad \{(x-1)^2 \text{ must be a factor of this cubic}\}$$

$\therefore$ the tangent meets the curve again when $x = -1$.

When $x=-1$, $y=\dfrac{1}{(-1)}-\dfrac{1}{(-1)^2}$
$$=-1-1$$
$$=-2$$

$\therefore$ the tangent meets the curve again at $(-1, -2)$.



21 **a** Consider the tangent to $y=x^2-x+9$ at $x=a$.

When $x=a$, $y=a^2-a+9$, so the point of contact is $(a,\ a^2-a+9)$.

Now $\dfrac{dy}{dx}=2x-1$ and so at $x=a$, $\dfrac{dy}{dx}=2a-1$

So, the gradient of the tangent at $(a,\ a^2-a+9)$ is $2a-1$

$\therefore$ the equation of the tangent is
$$\begin{aligned} y &= (2a-1)(x-a)+(a^2-a+9) \\ &= 2ax-2a^2-x+a+a^2-a+9 \\ &= 2ax-a^2-x+9 \\ \therefore\ y &= (2a-1)x-a^2+9 \quad .... (*) \end{aligned}$$

b This tangent passes through $(0, 0)$, so $0=-a^2+9$
$$\therefore\ a^2=9$$
$$\therefore\ a=\pm 3$$

When $a=3$, $y=(2(3)-1)x-3^2+9$ $\quad$ {from $(*)$}
$$\therefore\ y=5x$$

The tangent is $y=5x$ with point of contact $(3, 15)$.

When $a=-3$, $y=(2(-3)-1)x-(-3)^2+9$ $\quad$ {from $(*)$}
$$\therefore\ y=-7x$$

The tangent is $y=-7x$ with point of contact $(-3, 21)$.

22 **a** Consider the tangent to $y=x^2+4x$ at the point where $x=a$.

When $x=a$, $y=a^2+4a$, so the point of contact is $(a,\ a^2+4a)$.

Now $\dfrac{dy}{dx}=2x+4$, and so at $x=a$, $\dfrac{dy}{dx}=2a+4$

So, the gradient of the tangent at $(a,\ a^2+4a)$ is $2a+4$

$\therefore$ the equation of the tangent is
$$\begin{aligned} y &= (2a+4)(x-a)+(a^2+4a) \\ &= 2ax-2a^2+4x-4a+a^2+4a \\ &= 2ax+4x-a^2 \\ \therefore\ y &= (2a+4)x-a^2 \quad .... (*) \end{aligned}$$

b This tangent passes through $(1, -4)$, so $-4=(2a+4)(1)-a^2$
$$\therefore\ -4=2a+4-a^2$$
$$\therefore\ a^2-2a-8=0$$
$$\therefore\ (a+2)(a-4)=0$$
$$\therefore\ a=-2 \text{ or } 4$$

When $a=-2$, $y=(2(-2)+4)x-(-2)^2$ $\quad$ {from $(*)$}
$$\therefore\ y=(-4+4)x-4$$
$$\therefore\ y=-4$$

The tangent is $y=-4$ with point of contact $(-2, -4)$.

When $a = 4$, $y = (2(4)+4)x - (4)^2$
$\therefore$ $y = (8+4)x - 16$
$\therefore$ $y = 12x - 16$

When $x = 4$, $y = 12(4) - 16$
$= 48 - 16$
$= 32$

The tangent is $y = 12x - 16$ with point of contact $(4, 32)$.

23 Consider the tangent to $y = x^2 - 3x + 1$ at the point where $x = a$.

When $x = a$, $y = a^2 - 3a + 1$, so the point of contact is $(a,\ a^2 - 3a + 1)$.

Now $\dfrac{dy}{dx} = 2x - 3$, and so at $x = a$, $\dfrac{dy}{dx} = 2a - 3$

So, the gradient of the tangent at $(a,\ a^2 - 3a + 1)$ is $2a - 3$

$\therefore$ the equation of the tangent is
$$\begin{aligned} y &= (2a-3)(x-a) + a^2 - 3a + 1 \\ &= 2ax - 2a^2 - 3x + 3a + a^2 - 3a + 1 \\ &= 2ax - 3x - a^2 + 1 \\ \therefore\ y &= (2a-3)x - a^2 + 1 \quad .... (*) \end{aligned}$$

This tangent passes through $(1, -10)$, so
$$\begin{aligned} -10 &= (2a-3)(1) - a^2 + 1 \\ \therefore\ -10 &= 2a - 3 - a^2 + 1 \\ \therefore\ a^2 - 2a - 8 &= 0 \\ \therefore\ (a+2)(a-4) &= 0 \\ \therefore\ a &= -2 \text{ or } 4 \end{aligned}$$

When $a = -2$, $y = (2(-2)-3)x - (-2)^2 + 1$ {from $(*)$}
$\therefore$ $y = (-4-3)x - 4 + 1$
$\therefore$ $y = -7x - 3$

The tangent is $y = -7x - 3$.

When $a = 4$, $y = (2(4)-3)x - (4)^2 + 1$
$\therefore$ $y = (8-3)x - 16 + 1$
$\therefore$ $y = 5x - 15$

The tangent is $y = 5x - 15$.

24 **a** $y = e^x$

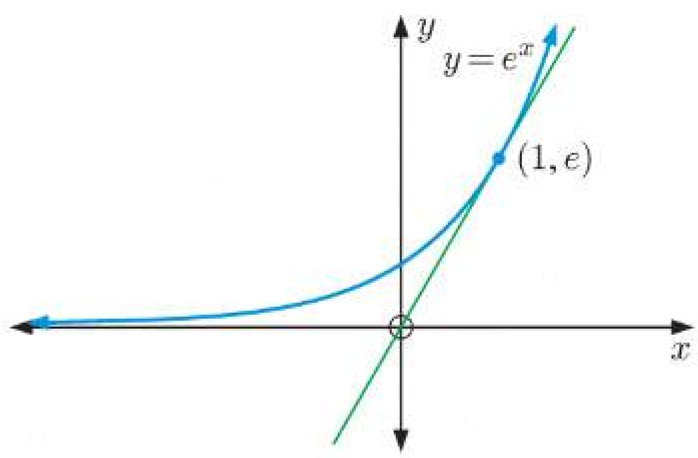
When $x = a$, $y = e^a$

$\therefore$ the point of contact is $(a,\ e^a)$.

Now $y = e^x$

$\therefore$ $\dfrac{dy}{dx} = e^x$

When $x = a$, $\dfrac{dy}{dx} = e^a$



So, the tangent has equation
$$\begin{aligned} y &= e^a(x-a) + e^a \\ &= e^a x - ae^a + e^a \\ \therefore\ y &= e^a x + e^a(1-a) \end{aligned}$$

b The tangent passes through the origin when $0 = e^a(1-a)$

$$\therefore\ 1 - a = 0 \qquad \{\text{as } e^a > 0 \text{ for all } a\}$$
$$\therefore\ a = 1$$

When $a = 1$, $y = e^1x + e^1(1-1)$
$= ex$

$\therefore$ the tangent to $y = e^x$ which passes through the origin is $y = ex$.

25 **a** $y = 2x^2$

When $x = a$, $y = 2a^2$

$\therefore$ the point of contact is $(a, 2a^2)$.

Now $y = 2x^2$

$\therefore\ \dfrac{dy}{dx} = 4x$

When $x = a$, $\dfrac{dy}{dx} = 4a$

So, the tangent has equation

$$y = 4a(x-a) + 2a^2$$
$$= 4ax - 4a^2 + 2a^2$$
$$\therefore\ y = 4ax - 2a^2$$

The tangent passes through $(1, -6)$ when $-6 = 4a(1) - 2a^2$

$$\therefore\ 2a^2 - 4a - 6 = 0$$
$$\therefore\ a^2 - 2a - 3 = 0$$
$$\therefore\ (a+1)(a-3) = 0$$
$$\therefore\ a = -1 \text{ or } 3$$

When $a = -1$, $y = 4(-1)x - 2(-1)^2$
$= -4x - 2$

When $a = 3$, $y = 4(3)x - 2(3)^2$
$= 12x - 18$

$\therefore$ the tangents to $y = 2x^2$ which passes through $(1, -6)$ are $y = -4x - 2$ and $y = 12x - 18$.

b From **a**, the point of contact is $(a, 2a^2)$.

For $y = -4x - 2$, $a = -1$, so the point of contact is $(-1, 2)$.

For $y = 12x - 18$, $a = 3$, so the point of contact is $(3, 18)$.

c The tangent to $y = 2x^2$ at a general point $(a, 2a^2)$ has equation $y = 4ax - 2a^2$.

The tangent passes through $(1, 4)$ when $4 = 4a(1) - 2a^2$

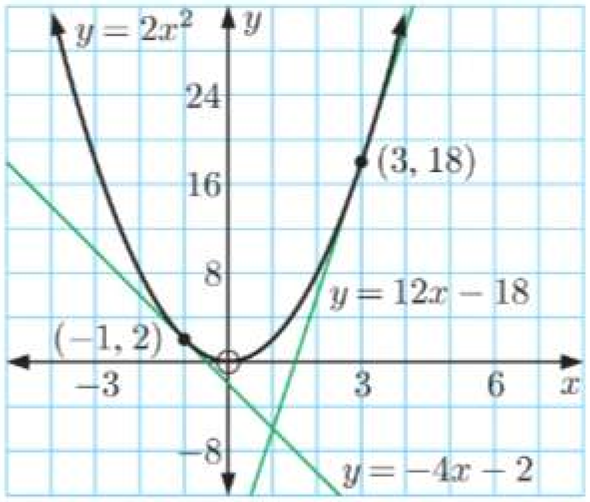
$$\therefore\ 4 = 4a - 2a^2$$
$$\therefore\ 2a^2 - 4a + 4 = 0$$
$$\therefore\ a^2 - 2a + 2 = 0$$

which has $\Delta = (-2)^2 - 4(1)(2)$
$= 4 - 8$
$= -4$

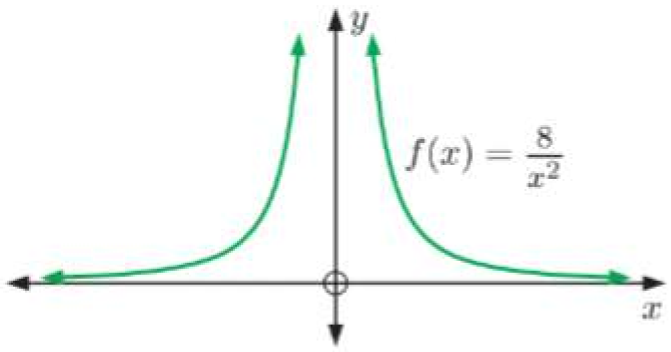
$\therefore$ there are no real solutions as $\Delta < 0$.

$\therefore$ there are no tangents to the function that pass through the point $(1, 4)$.

d



26 **a**



b $f(x) = \dfrac{8}{x^2} = 8x^{-2}$

$\therefore\ f(a) = \dfrac{8}{a^2}$

So, the point of contact is $\left(a, \dfrac{8}{a^2}\right)$.

Now $f'(x) = -16x^{-3} = -\dfrac{16}{x^3}$

$\therefore\ f'(a) = -\dfrac{16}{a^3}$

So, the gradient of the tangent at $\left(a, \dfrac{8}{a^2}\right)$ is $-\dfrac{16}{a^3}$

$\therefore$ the equation of the tangent is

$$-16x - a^3y = -16a - a^3\left(\frac{8}{a^2}\right)$$

$$\therefore\ -16x - a^3y = -16a - 8a$$

$$\therefore\ 16x + a^3y = 24a$$

c The tangent cuts the x-axis when $y = 0$

$$\therefore\ 16x = 24a$$

$$\therefore\ x = \tfrac{3}{2}a$$

$\therefore$ A is $\left(\frac{3}{2}a, 0\right)$.

The tangent cuts the y-axis when $x = 0$

$$\therefore\ a^3y = 24a$$

$$\therefore\ y = \frac{24}{a^2}$$

$\therefore$ B is $\left(0, \dfrac{24}{a^2}\right)$.

d Area of triangle OAB $= \left|\frac{1}{2} \times \left(\frac{3}{2}a\right) \times \left(\dfrac{24}{a^2}\right)\right|$

$$= \frac{18}{|a|} \text{ units}^2$$

As $a \to \infty$, $\dfrac{18}{|a|} \to 0$

$\therefore$ area $\to 0$

27 $y = 3e^{-x}$ and $y = 2 + e^x$ meet where $3e^{-x} = 2 + e^x$

$$\therefore\ 3 = 2e^x + e^{2x} \quad \{\text{multiplying both sides by } e^x\}$$
$$\therefore\ e^{2x} + 2e^x - 3 = 0$$
$$\therefore\ (e^x + 3)(e^x - 1) = 0$$
$$\therefore\ e^x = 1 \quad \{\text{as } e^x > 0\}$$
$$\therefore\ x = 0$$

Now when $x = 0$, $y = 3e^0 = 3$, so the graphs intersect at $(0, 3)$.

For $y = 2 + e^x$, $\frac{dy}{dx} = e^x$

When $x = 0$, $\frac{dy}{dx} = e^0 = 1$

$\therefore$ the gradient of the tangent at $(0, 3)$ is 1.

$\therefore$ the equation of the tangent is $y = x + 3$.

For $y = 3e^{-x}$, $\frac{dy}{dx} = -3e^{-x}$

When $x = 0$, $\frac{dy}{dx} = -3$

$\therefore$ the gradient of the tangent at $(0, 3)$ is -3.

$\therefore$ the equation of the tangent is

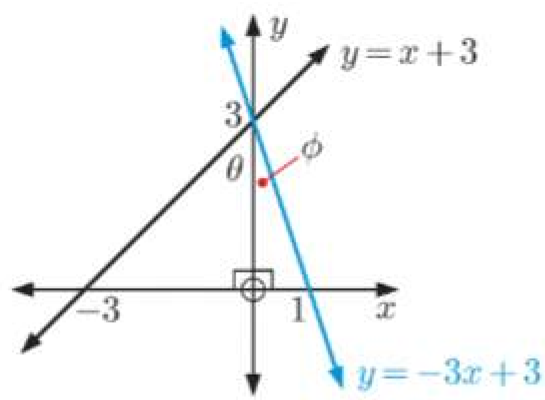
$$y = -3(x - 0) + 3$$
$$= -3x + 3$$

We graph the tangents on the same set of axes.

$\tan\theta = \frac{3}{3}$ and $\tan\phi = \frac{1}{3}$

$= 1$ $\qquad \therefore\ \phi \approx 18.43°$

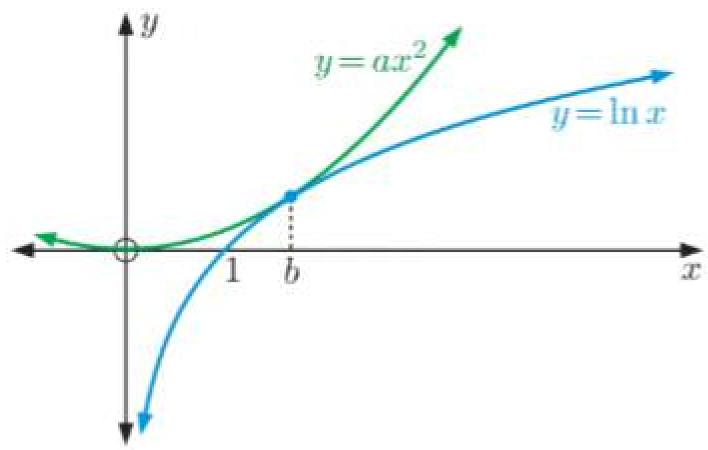
$\therefore\ \theta = 45°$



So, the acute angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection is about $45° + 18.43° \approx 63.43°$.

28 **a** $y = ax^2$, $a > 0$ touches $y = \ln x$ when $ax^2 = \ln x$

If the curves touch when $x = b$ then $ab^2 = \ln b$ .... (1)



Now for $y = ax^2$, $\frac{dy}{dx} = 2ax$ and for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$

$\therefore$ when $x = b$, $\frac{dy}{dx} = 2ab$ $\qquad \therefore$ when $x = b$, $\frac{dy}{dx} = \frac{1}{b}$

Since the curves touch each other, they share a common tangent.

$\therefore\ 2ab = \frac{1}{b}$ .... (2)

b Now $ab^2 = \frac{1}{2}$ {from (2)}

and $ab^2 = \ln b$ {from (1)}

$\therefore \ln b = \frac{1}{2}$

$\therefore b = e^{\frac{1}{2}} = \sqrt{e}$

When $x = b = \sqrt{e}$, $y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$

$\therefore$ the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c $a = \dfrac{1}{2b^2}$ {from (2)}

$\therefore a = \dfrac{1}{2(\sqrt{e})^2} = \dfrac{1}{2e}$

d The tangent has gradient $\dfrac{1}{b} = \dfrac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$\therefore$ the tangent is $\dfrac{y - \frac{1}{2}}{x - \sqrt{e}} = \dfrac{1}{\sqrt{e}}$ $\quad \therefore y - \frac{1}{2} = \dfrac{1}{\sqrt{e}}(x - \sqrt{e})$

$\therefore y - \frac{1}{2} = \dfrac{1}{\sqrt{e}}x - 1$

$\therefore y = e^{-\frac{1}{2}}x - \frac{1}{2}$

EXERCISE 19B

1 **a** $y = x^2$

Now $\dfrac{dy}{dx} = 2x$, so at $(4, 16)$,

$\dfrac{dy}{dx} = 2(4) = 8 = \dfrac{8}{1}$

$\therefore$ the normal at $(4, 16)$ has gradient $-\frac{1}{8}$.

$\therefore$ the equation of the normal is

$-x - 8y = -(4) - 8(16)$

$\therefore x + 8y = 132$

b $y = x^3 - 5x + 2$

When $x = -2$,

$y = (-2)^3 - 5(-2) + 2$

$= 4$

So, the point of contact is $(-2, 4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$, so at $x = -2$,

$\dfrac{dy}{dx} = 3(-2)^2 - 5 = 7 = \dfrac{7}{1}$

$\therefore$ the normal at $(-2, 4)$ has gradient $-\frac{1}{7}$.

$\therefore$ the equation of the normal is

$-x - 7y = -(-2) - 7(4)$

$\therefore x + 7y = 26$

c $y = \frac{5}{\sqrt{x}} - \sqrt{x}$

$= 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$, so at $(1, 4)$,

$\frac{dy}{dx} = -\frac{5}{2}(1^{-\frac{3}{2}}) - \frac{1}{2}(1^{-\frac{1}{2}})$

$= -\frac{5}{2} - \frac{1}{2}$

$= -3$

$\therefore$ the normal at $(1, 4)$ has gradient $\frac{1}{3}$.

$\therefore$ the equation of the normal is

$x - 3y = 1 - 3(4)$

$\therefore\ x - 3y = -11$

d $y = 8\sqrt{x} - \frac{1}{x^2}$

When $x = 1$,

$y = 8\sqrt{1} - \frac{1}{1^2} = 7$

So, the point of contact is $(1, 7)$.

Now $y = 8\sqrt{x} - \frac{1}{x^2}$

$= 8x^{\frac{1}{2}} - x^{-2}$

$\therefore \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$, so at $x = 1$,

$\frac{dy}{dx} = 4(1^{-\frac{1}{2}}) + 2\left(1^{-3}\right)$

$= 4 + 2$

$= 6$

$\therefore$ the normal at $(1, 7)$ has gradient $-\frac{1}{6}$.

$\therefore$ the equation of the normal is

$-x - 6y = -(1) - 6(7)$

$\therefore\ x + 6y = 43$

e $f(x) = \frac{x}{1 - 3x}$

$= x(1 - 3x)^{-1}$

$\therefore\ f'(x) = (1 - 3x)^{-1} - x(1 - 3x)^{-2}(-3)$ {product rule}

$= \frac{1}{1 - 3x} + \frac{3x}{(1 - 3x)^2}$

$\therefore\ f'(-1) = \frac{1}{1 - 3(-1)} + \frac{3(-1)}{(1 - 3(-1))^2}$

$= \frac{1}{1 + 3} - \frac{3}{(1 + 3)^2}$

$= \frac{1}{4} - \frac{3}{16}$

$= \frac{1}{16}$

$\therefore$ the normal at $(-1, -\frac{1}{4})$ has gradient -16.

$\therefore$ the equation of the normal is $16x + y = 16(-1) + (-\frac{1}{4})$

$= -16 - \frac{1}{4}$

$\therefore\ 16x + y = -\frac{65}{4}$

$\therefore\ 64x + 4y = -65$

f

$$f(x) = \frac{x^2}{1-x}$$
$$= x^2(1-x)^{-1}$$
$$\therefore\ f'(x) = 2x(1-x)^{-1} - x^2(1-x)^{-2}(-1) \qquad \{\text{product rule}\}$$
$$= \frac{2x}{1-x} + \frac{x^2}{(1-x)^2}$$
$$\therefore\ f'(2) = \frac{2(2)}{1-2} + \frac{(2)^2}{(1-2)^2}$$
$$= \frac{4}{-1} + \frac{4}{1}$$
$$= 0$$

$\therefore$ the tangent at $(2, -4)$ is a horizontal line.

$\therefore$ the normal at $(2, -4)$ is a vertical line passing through $(2, -4)$.

$\therefore$ the equation of the normal is $x = 2$.

2 **a**

$$f(x) = x^2 - \frac{8}{x}$$
$$\therefore\ f(-2) = (-2)^2 - \frac{8}{(-2)}$$
$$= 4 + 4$$
$$= 8$$

So, the point of contact is $(-2, 8)$.

Now $f(x) = x^2 - 8x^{-1}$
$$\therefore\ f'(x) = 2x + 8x^{-2}$$
$$= 2x + \frac{8}{x^2}$$
$$\therefore\ f'(-2) = 2(-2) + \frac{8}{(-2)^2}$$
$$= -4 + 2$$
$$= -2$$

$\therefore$ the tangent at $(-2, 8)$ has gradient -2.

$\therefore$ the equation of the tangent is
$$y = -2(x+2) + 8$$
$$\therefore\ y = -2x - 4 + 8$$
$$\therefore\ y = 4 - 2x$$

b

$$f(x) = x^2 - \frac{8}{x}$$
$$\therefore\ f(3) = (3)^2 - \frac{8}{3}$$
$$= 9 - \tfrac{8}{3}$$
$$= \tfrac{19}{3}$$

So, the point of contact is $(3, \frac{19}{3})$.

Now $f'(x) = 2x + \dfrac{8}{x^2}$ {from **a**}
$$\therefore\ f'(3) = 2(3) + \frac{8}{(3)^2}$$
$$= 6 + \tfrac{8}{9}$$
$$= \tfrac{62}{9}$$

$\therefore$ the normal at $(3, \frac{19}{3})$ has gradient $-\frac{9}{62}$.

$\therefore$ the equation of the normal is
$$y = -\tfrac{9}{62}(x-3) + \tfrac{19}{3}$$
$$\therefore\ y = -\tfrac{9}{62}x + \tfrac{27}{62} + \tfrac{19}{3}$$
$$\therefore\ y = -\tfrac{9}{62}x + \tfrac{1259}{186}$$

3 **a** When $x = 0$, $y = e^0 = 1$.

So, the point of contact is $(0, 1)$.

Now as $y = e^{-x}$, $\dfrac{dy}{dx} = -e^{-x}$

$\therefore$ when $x = 0$, $\dfrac{dy}{dx} = -e^0 = -1$

$\therefore$ the normal at $(0, 1)$ has gradient 1.

$\therefore$ the equation of the normal is

$y = x + 1$.

b When $x = e$, $y = \ln e = 1$.

So, the point of contact is $(e, 1)$.

Now as $y = \ln x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ when $x = e$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

$\therefore$ the normal at $(e, 1)$ has gradient $-e$.

$\therefore$ the equation of the normal is

$$y = -e(x - e) + 1$$
$$\therefore\ y = -ex + e^2 + 1$$
$$\therefore\ ex + y = e^2 + 1$$

c When $x = 1$, $y = e^{2(1)-1} = e$.

So, the point of contact is $(1, e)$.

Now as $y = e^{2x-1}$, $\dfrac{dy}{dx} = 2e^{2x-1}$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = 2e^{2(1)-1} = 2e$

$\therefore$ the normal at $(1, e)$ has gradient $-\dfrac{1}{2e}$.

$\therefore$ the equation of the normal is

$$y = -\frac{1}{2e}(x - 1) + e$$
$$\therefore\ 2ey = -(x - 1) + 2e^2$$
$$\therefore\ 2ey = -x + 1 + 2e^2$$
$$\therefore\ x + 2ey = 1 + 2e^2$$

d When $x = \frac{\pi}{3}$, $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

So, the point of contact is $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$.

Now as $y = \sin x$, $\dfrac{dy}{dx} = \cos x$

$\therefore$ when $x = \frac{\pi}{3}$, $\dfrac{dy}{dx} = \cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore$ the normal at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ has gradient -2.

$\therefore$ the equation of the normal is

$$y = -2\left(x - \tfrac{\pi}{3}\right) + \tfrac{\sqrt{3}}{2}$$
$$\therefore\ y = -2x + \tfrac{2\pi}{3} + \tfrac{\sqrt{3}}{2}$$
$$\therefore\ 2x + y = \tfrac{2\pi}{3} + \tfrac{\sqrt{3}}{2}$$

e When $x = \frac{\pi}{4}$, $y = \tan\frac{3\pi}{4} = -1$.

So, the point of contact is $\left(\frac{\pi}{4}, -1\right)$.

Now as $y = \tan 3x$, $\dfrac{dy}{dx} = \dfrac{3}{\cos^2 3x}$

$\therefore$ when $x = \frac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{3}{\left(\cos\frac{3\pi}{4}\right)^2} = \dfrac{3}{\left(-\frac{1}{\sqrt{2}}\right)^2} = 6$

$\therefore$ the normal at $\left(\frac{\pi}{4}, -1\right)$ has gradient $-\frac{1}{6}$.

$\therefore$ the equation of the normal is $y = -\frac{1}{6}\left(x - \frac{\pi}{4}\right) - 1$

$$\therefore\ y = -\tfrac{1}{6}x + \tfrac{\pi}{24} - 1$$

f When $x = \frac{\pi}{2}$, $y = \cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2}$.

So, the point of contact is $\left(\frac{\pi}{2}, -\frac{1}{2}\right)$.

Now as $y = \cos\left(2x - \frac{\pi}{3}\right)$, $\frac{dy}{dx} = -2\sin\left(2x - \frac{\pi}{3}\right)$

$\therefore$ when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = -2\sin\left(\pi - \frac{\pi}{3}\right) = -\sqrt{3}$

$\therefore$ the normal at $\left(\frac{\pi}{2}, -\frac{1}{2}\right)$ has gradient $\frac{1}{\sqrt{3}}$.

$\therefore$ the equation of the normal is $y = \frac{1}{\sqrt{3}}\left(x - \frac{\pi}{2}\right) - \frac{1}{2}$

$$\therefore \sqrt{3}\,y = x - \frac{\pi}{2} - \frac{\sqrt{3}}{2}$$

$$\therefore 2\sqrt{3}\,y = 2x - \pi - \sqrt{3}$$

$$\therefore 2x - 2\sqrt{3}\,y = \pi + \sqrt{3}$$

4
$$y = a\sqrt{x} + \frac{b}{\sqrt{x}}$$
$$= ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

When $x = 4$,
$$\frac{dy}{dx} = \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right)$$
$$= \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right)$$
$$= \frac{a}{4} - \frac{b}{16}$$
$$= \frac{4a - b}{16}$$

The gradient of the normal to the curve at $x = 4$ will be $\frac{16}{b - 4a}$.

However, the equation of the normal is $4x + y = 22$ or $y = -4x + 22$ which has gradient -4.

$$\therefore \frac{16}{b - 4a} = -4$$

$$\therefore b - 4a = -4$$

$$\therefore b = 4a - 4 \quad .... (*)$$

Also, at $x = 4$ the normal line intersects the curve.

$$\therefore a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22$$

$$\therefore 2a + \frac{b}{2} = 6$$

Consequently, $2a + \frac{4a - 4}{2} = 6$ $\{$using $(*)\}$

$$\therefore 2a + 2a - 2 = 6$$

$$\therefore 4a = 8$$

$$\therefore a = 2$$

and so $b = 4(2) - 4 = 4$ $\{$from $(*)\}$

5 When $x = 1$, $y = (1)^3 - 2(1)^2 + 1 = 0$. So the point of contact is $(1, 0)$.

Now as $y = x^3 - 2x^2 + 1$, $\dfrac{dy}{dx} = 3x^2 - 4x$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = 3(1)^2 - 4(1) = -1$

$\therefore$ the normal at $(1, 0)$ has gradient 1.

$\therefore$ the equation of the normal is $y = x - 1$.

This line meets the curve where $x - 1 = x^3 - 2x^2 + 1$

$\therefore$ $x^3 - 2x^2 - x + 2 = 0$

$\therefore$ $x = 2, 1$, or -1 {using technology}

When $x = 2$, $y = (2)^3 - 2(2)^2 + 1$
$= 8 - 8 + 1$
$= 1$

When $x = -1$, $y = (-1)^3 - 2(-1)^2 + 1$
$= -1 - 2 + 1$
$= -2$

$\therefore$ the normal meets the curve again at $(2, 1)$ and $(-1, -2)$.

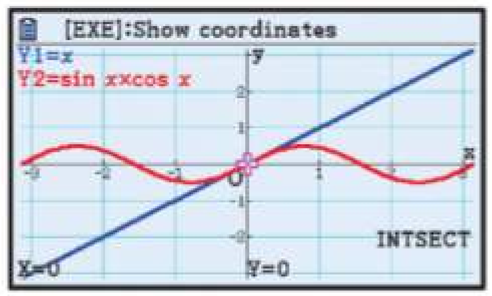
6 Let $(a, \cos a)$ be a general point on $f(x) = \cos x$.

Now $f'(x) = -\sin x$, so $f'(a) = -\sin a$

$\therefore$ the normal at $(a, \cos a)$ has gradient $\dfrac{1}{\sin a}$.

$\therefore$ the equation of the normal is $y = \dfrac{1}{\sin a}(x - a) + \cos a$.

The normal passes through the origin when



$$0 = \frac{1}{\sin a}(0 - a) + \cos a$$

$$\therefore\ 0 = -\frac{a}{\sin a} + \cos a$$

$$\therefore\ \frac{a}{\sin a} = \cos a$$

$$\therefore\ a = \sin a \cos a$$

$$\therefore\ a = 0$$

So, the normal at $(0, \cos 0)$, or $(0, 1)$, has gradient $\dfrac{1}{\sin 0}$ which is undefined. The normal is a vertical line.

$\therefore$ the equation of the normal to $f(x) = \cos x$ which passes through the origin is the vertical line $x = 0$.

7 Let $(a, \sqrt{a})$ be a general point on $y = \sqrt{x}$.

Now $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
$= \dfrac{1}{2\sqrt{x}}$

so at $x = a$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{a}}$

So, the gradient of the normal at this point is $-2\sqrt{a}$.

$\therefore$ the normal has equation $y = -2\sqrt{a}(x - a) + \sqrt{a}$.

But this normal passes through $(4, 0)$, so $0 = -2\sqrt{a}(4-a) + \sqrt{a}$

$$\therefore\ 2\sqrt{a}(4-a) - \sqrt{a} = 0$$
$$\therefore\ \sqrt{a}(8-2a-1) = 0$$
$$\therefore\ \sqrt{a}(7-2a) = 0$$
$$\therefore\ a = 0 \text{ or } \tfrac{7}{2}$$

But $a = 0$ is the end point of the function, so there is no normal here.

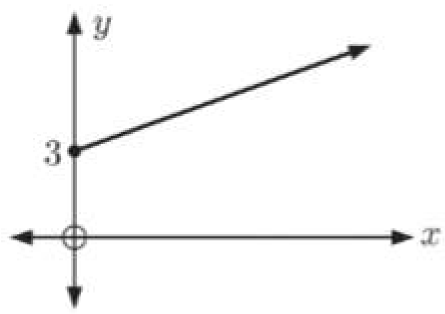
When $a = \frac{7}{2}$, $y = -2\sqrt{\frac{7}{2}}(x - \frac{7}{2}) + \sqrt{\frac{7}{2}}$

$$\therefore\ y + 2\sqrt{\tfrac{7}{2}}(x - \tfrac{7}{2}) = \sqrt{\tfrac{7}{2}}$$
$$\therefore\ \sqrt{2}y + 2\sqrt{7}(x - \tfrac{7}{2}) = \sqrt{7}$$
$$\therefore\ \sqrt{2}y + 2\sqrt{7}x - 7\sqrt{2} = \sqrt{7}$$
$$\therefore\ \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$$
$$\therefore\ 2y + 2\sqrt{14}x = 8\sqrt{14}$$
$$\therefore\ y + \sqrt{14}x = 4\sqrt{14}$$

$\therefore\ y = -\sqrt{14}x + 4\sqrt{14}$ is the normal to $y = \sqrt{x}$ from $(4, 0)$, with contact point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$.

EXERCISE 19C

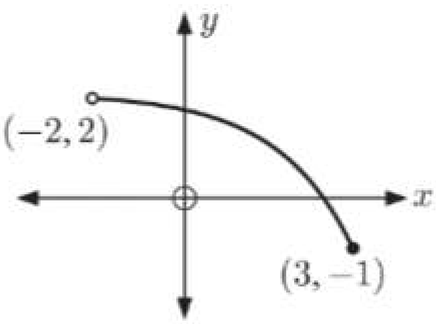
1 **a**



i The graph is increasing for $x \geqslant 0$.

ii The graph is never decreasing.

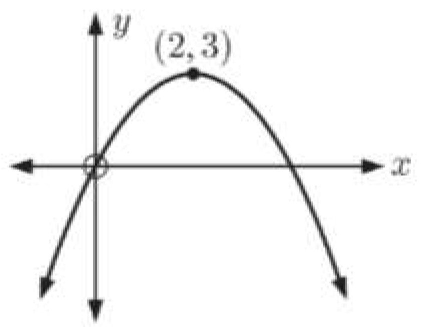
b



i The graph is never increasing.

ii The graph is decreasing for $-2 < x \leqslant 3$.

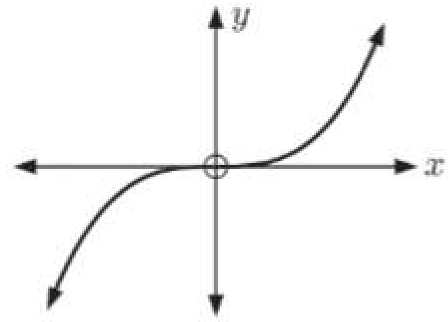
c



i The graph is increasing for $x \leqslant 2$.

ii The graph is decreasing for $x \geqslant 2$.

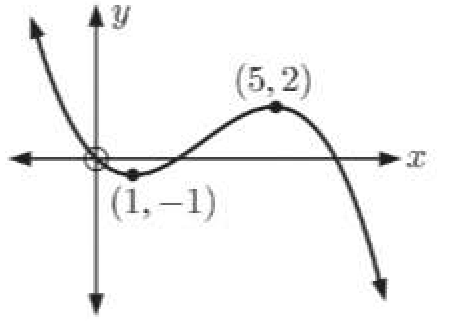
d



i The graph is increasing for all real x.

ii The graph is never decreasing.

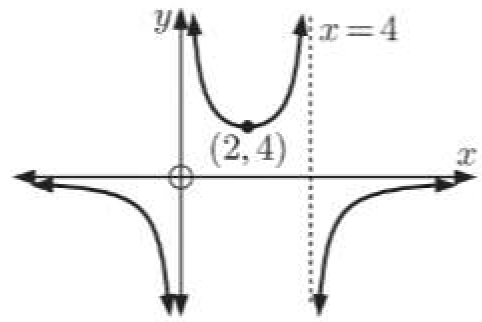
e



i The graph is increasing for $1 \leqslant x \leqslant 5$.

ii The graph is decreasing for $x \leqslant 1,\ \ x \geqslant 5$.

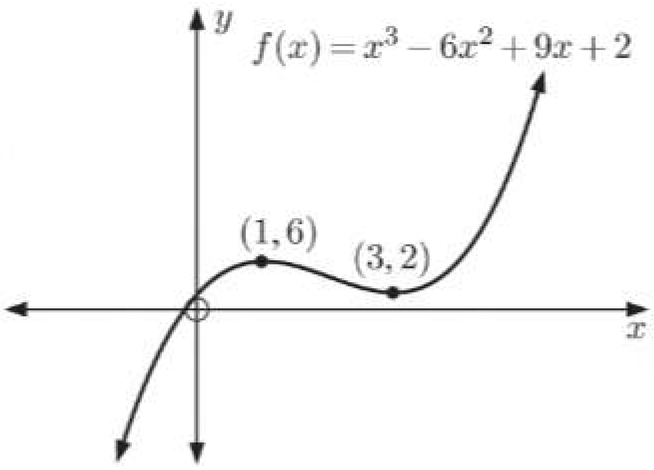
f



i The graph is increasing for $2 \leqslant x < 4,\ \ x > 4$.

ii The graph is decreasing for $x < 0,\ \ 0 < x \leqslant 2$.

2 **a**



i The function is increasing for $x \leqslant 1$ and $x \geqslant 3$.

ii The function is decreasing for $1 \leqslant x \leqslant 3$.

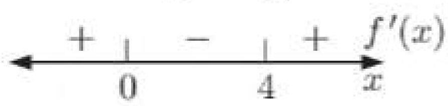
b
$$\begin{aligned} f(x) &= x^3 - 6x^2 + 9x + 2 \\ \therefore\ f'(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x-1)(x-3) \end{aligned}$$

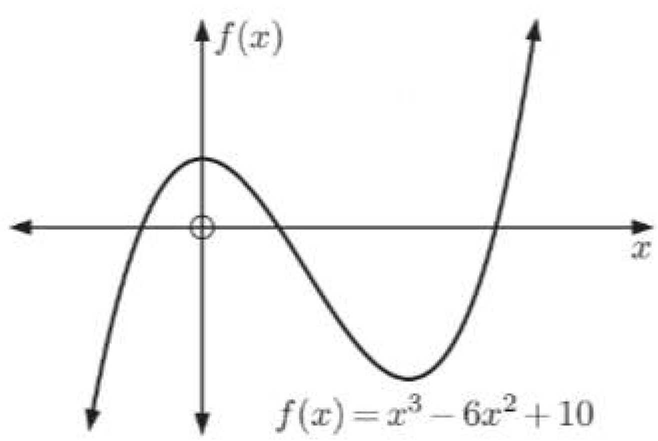
which has sign diagram:

From the sign diagram, $f(x)$ is increasing for $x \leqslant 1$ and $x \geqslant 3$, and decreasing for $1 \leqslant x \leqslant 3$.

3 **a**
$$\begin{aligned} f(x) &= x^3 - 6x^2 + 10 \\ \therefore\ f'(x) &= 3x^2 - 12x \\ &= 3x(x-4) \end{aligned}$$

which has sign diagram:



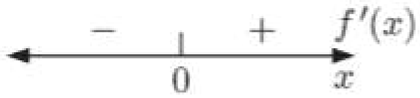


b $f(x)$ is increasing for $x \leqslant 0$ and for $x \geqslant 4$.
$f(x)$ is decreasing for $0 \leqslant x \leqslant 4$.

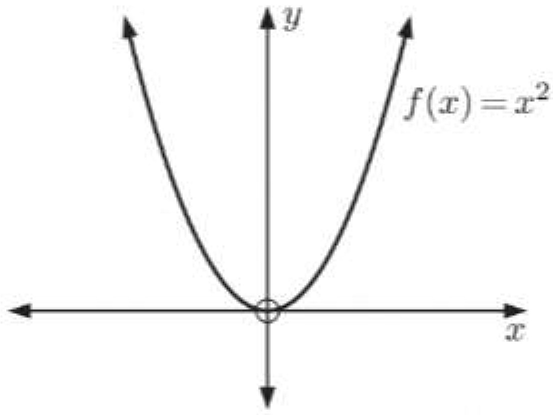
4 **a** $f(x) = x^2$

$\therefore\ f'(x) = 2x$

which has sign diagram:



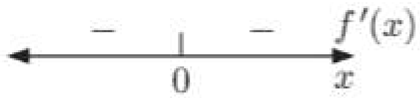
So, $f(x)$ is increasing for $x \geqslant 0$, and decreasing for $x \leqslant 0$.



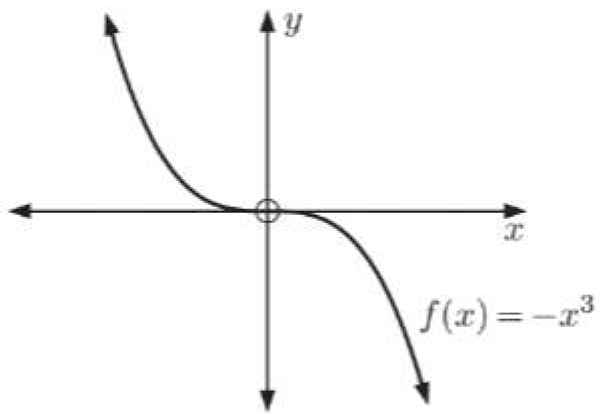
b $f(x) = -x^3$

$\therefore\ f'(x) = -3x^2$

which has sign diagram:



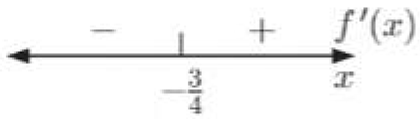
So, $f(x)$ is decreasing for all $x \in \mathbb{R}$.



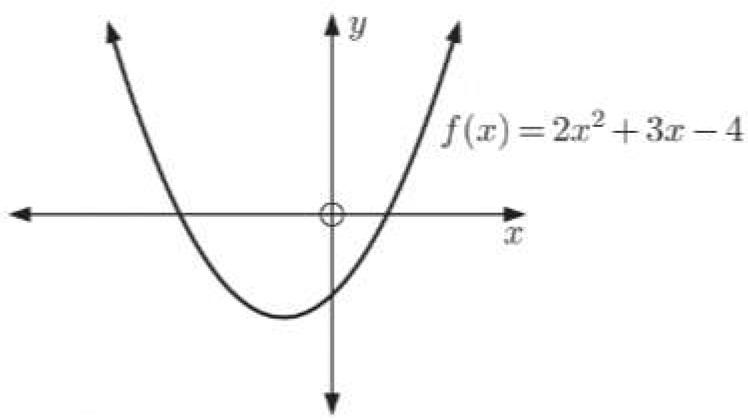
c $f(x) = 2x^2 + 3x - 4$

$\therefore\ f'(x) = 4x + 3$

which has sign diagram:



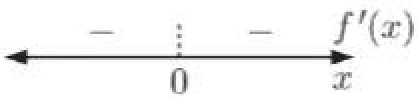
So, $f(x)$ is increasing for $x \geqslant -\frac{3}{4}$, and decreasing for $x \leqslant -\frac{3}{4}$.



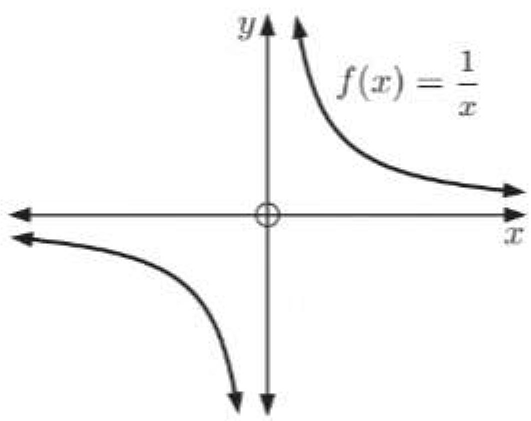
d $f(x) = \dfrac{1}{x} = x^{-1}$

$\therefore\ f'(x) = -x^{-2} = -\dfrac{1}{x^2}$

which has sign diagram:



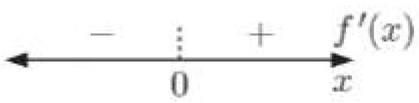
So, $f(x)$ is decreasing for all $x \neq 0$.



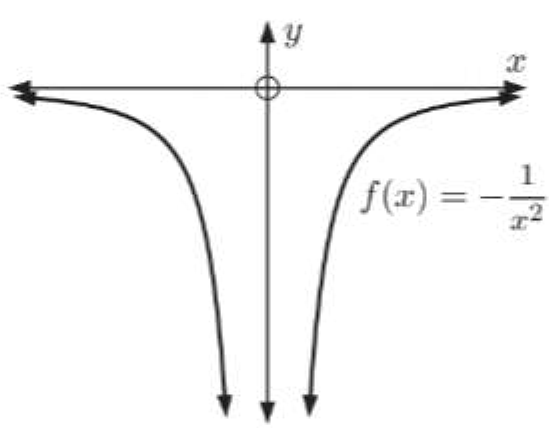
e $f(x) = -\dfrac{1}{x^2} = -x^{-2}$

$\therefore\ f'(x) = 2x^{-3} = \dfrac{2}{x^3}$

which has sign diagram:



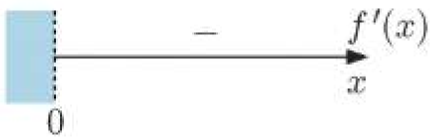
So, $f(x)$ is increasing for $x > 0$, and decreasing for $x < 0$.

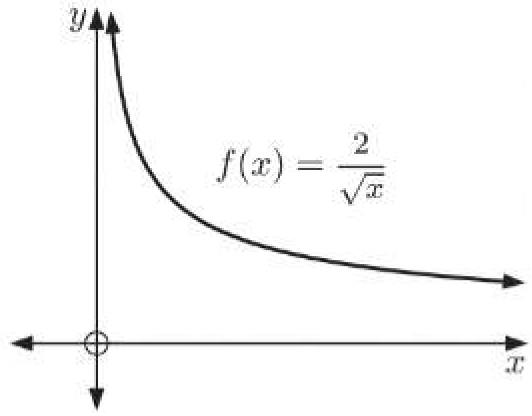


f $\quad f(x) = \dfrac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

$\therefore \quad f'(x) = -x^{-\frac{3}{2}} = -\dfrac{1}{x\sqrt{x}}$

which has sign diagram:



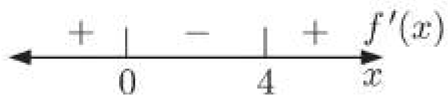


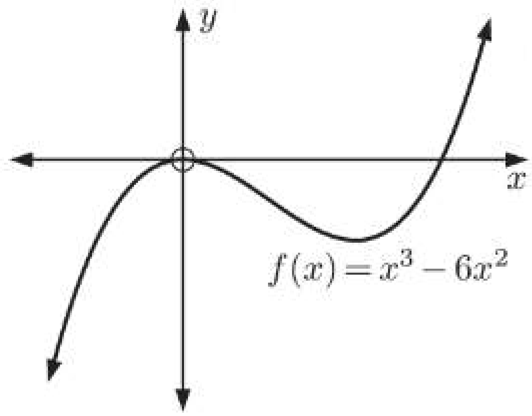
So, $f(x)$ is only defined for $x > 0$.
$f(x)$ is never increasing, but is decreasing for $x > 0$.

g $\quad f(x) = x^3 - 6x^2$

$\therefore \quad f'(x) = 3x^2 - 12x$
$\qquad\qquad = 3x(x - 4)$

which has sign diagram:



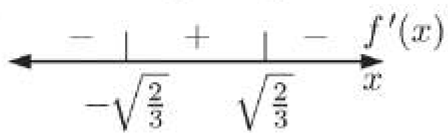


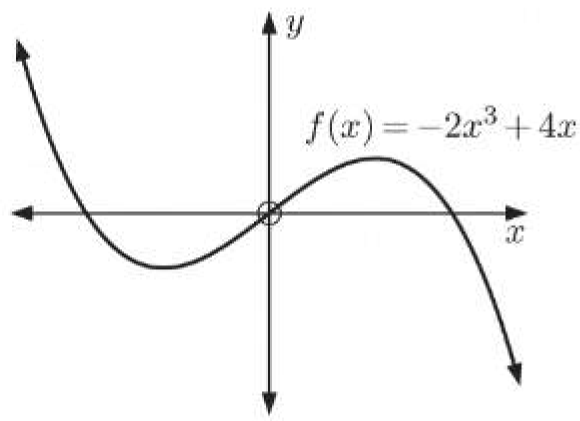
So, $f(x)$ is increasing for $x \leqslant 0$ and $x \geqslant 4$, and decreasing for $0 \leqslant x \leqslant 4$.

h $\quad f(x) = -2x^3 + 4x$

$\therefore \quad f'(x) = -6x^2 + 4$
$\qquad\qquad = -2(3x^2 - 2)$
$\qquad\qquad = -2(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$

which has sign diagram:





So, $f(x)$ is increasing for

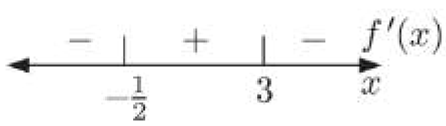
$-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$, and decreasing for

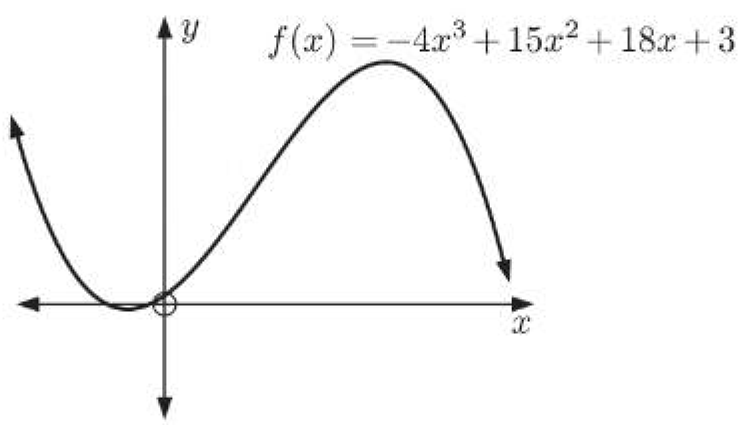
$x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$.

i $\quad f(x) = -4x^3 + 15x^2 + 18x + 3$

$\therefore \quad f'(x) = -12x^2 + 30x + 18$
$\qquad\qquad = -6(2x^2 - 5x - 3)$
$\qquad\qquad = -6(2x + 1)(x - 3)$

which has sign diagram:





So, $f(x)$ is increasing for $-\frac{1}{2} \leqslant x \leqslant 3$, and decreasing for $x \leqslant -\frac{1}{2}$ or $x \geqslant 3$.

j $f(x) = x^3 - 3x^2 + 5x + 2$

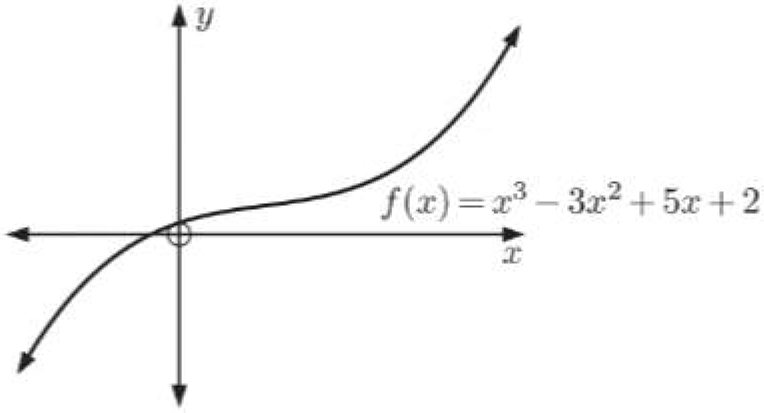
$\therefore\ f'(x) = 3x^2 - 6x + 5$

Now $\Delta = (-6)^2 - 4(3)(5) = -14 < 0$,

and $f'(0) = 5 > 0$.

$\therefore\ f'(x) > 0$ for all $x \in \mathbb{R}$.

So, $f(x)$ is increasing for all $x \in \mathbb{R}$.



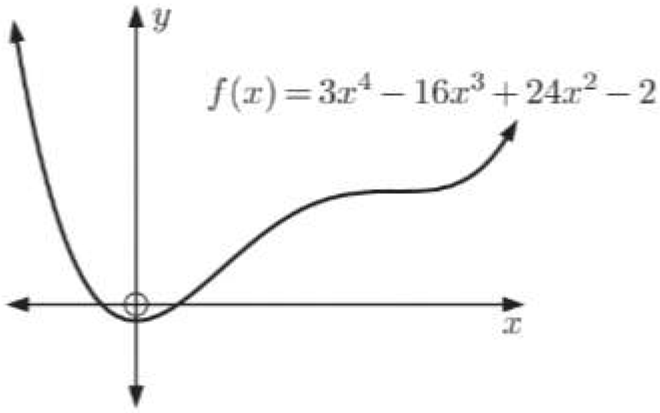
k $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$

$$\begin{aligned}\therefore\ f'(x) &= 12x^3 - 48x^2 + 48x \\ &= 12x(x^2 - 4x + 4) \\ &= 12x(x-2)^2\end{aligned}$$

which has sign diagram:

$$\underset{}{-}\ \big|_{0}\ +\ \big|_{2}\ +\qquad f'(x),\ x$$

So, $f(x)$ is increasing for $x \geqslant 0$,
and decreasing for $x \leqslant 0$.



l $f(x) = x^3 - 6x^2 + 3x - 1$

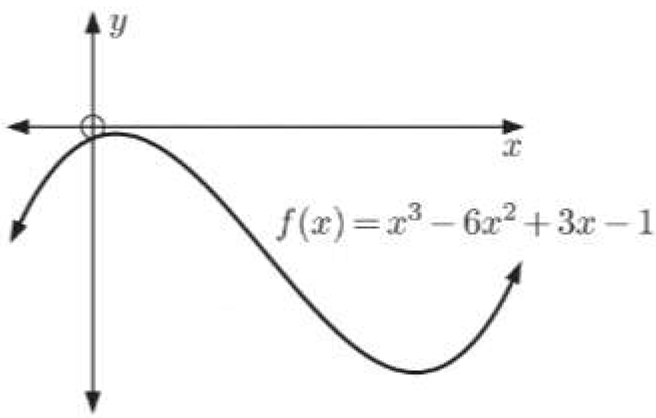
$$\begin{aligned}\therefore\ f'(x) &= 3x^2 - 12x + 3 \\ &= 3(x^2 - 4x + 1)\end{aligned}$$

$$f'(x) = 0 \text{ when } x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Sign diagram of $f'(x)$:

$$+\ \big|_{2-\sqrt{3}}\ -\ \big|_{2+\sqrt{3}}\ +\qquad f'(x),\ x$$

So, $f(x)$ is increasing for $x \leqslant 2-\sqrt{3}$ and $x \geqslant 2+\sqrt{3}$,
and decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$.

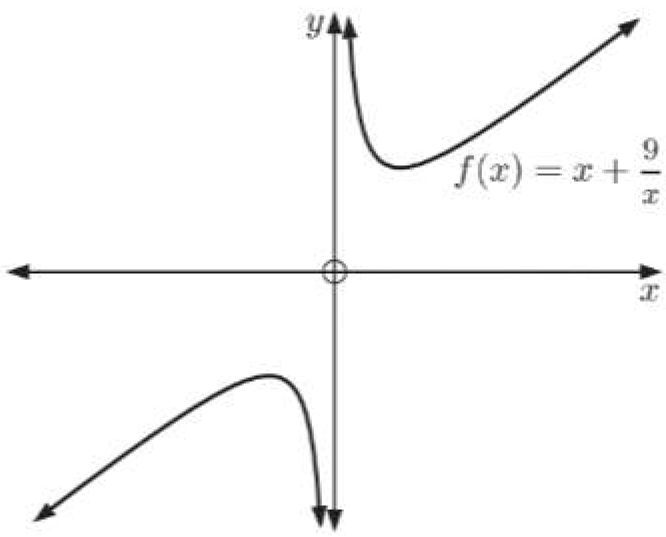


5 **a** $f(x) = x + \dfrac{9}{x}$

$$\begin{aligned} &= x + 9x^{-1} \\ \therefore\ f'(x) &= 1 - 9x^{-2} \\ &= 1 - \frac{9}{x^2} \\ &= \frac{x^2 - 9}{x^2} \\ &= \frac{(x+3)(x-3)}{x^2}\end{aligned}$$

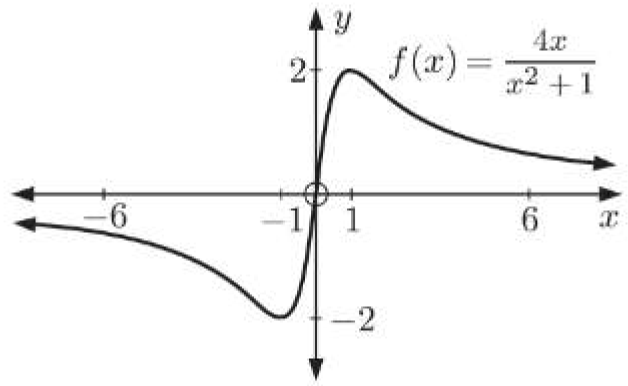
which has sign diagram:

$$+\ \big|_{-3}\ -\ \vdots_{0}\ -\ \big|_{3}\ +\qquad f'(x),\ x$$



b $y = f(x)$ is increasing for $x \leqslant -3$ and $x \geqslant 3$, and decreasing for $-3 \leqslant x < 0$ and $0 < x \leqslant 3$.

6 **a** $f(x) = \dfrac{4x}{x^2+1}$



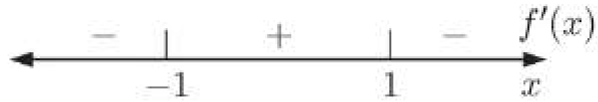
$\therefore\ f'(x) = \dfrac{4(x^2+1) - 4x(2x)}{(x^2+1)^2}$ {quotient rule}

$= \dfrac{4x^2+4-8x^2}{(x^2+1)^2}$

$= \dfrac{-4x^2+4}{(x^2+1)^2}$

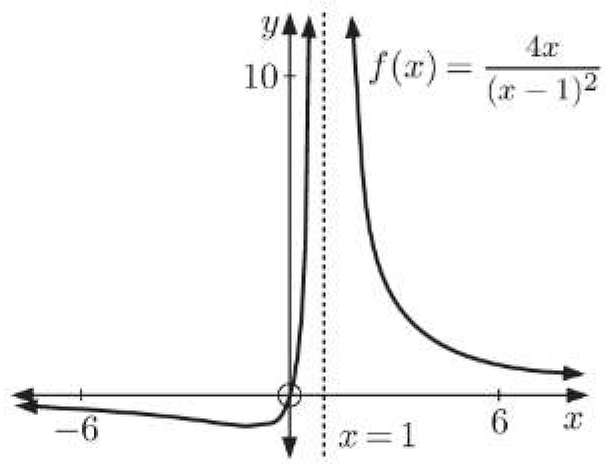
$= \dfrac{-4(x^2-1)}{(x^2+1)^2}$

$= \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$ which has sign diagram:



b $y = f(x)$ is increasing for $-1 \leqslant x \leqslant 1$, and decreasing for $x \leqslant -1$ and for $x \geqslant 1$.

7 **a** $f(x) = \dfrac{4x}{(x-1)^2}$



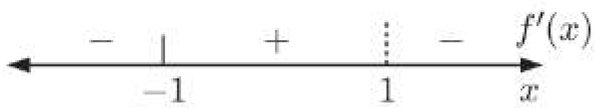
$\therefore\ f'(x) = \dfrac{4(x-1)^2 - 4x \times 2(x-1)}{(x-1)^4}$ {quotient rule}

$= \dfrac{(x-1)[4(x-1) - 8x]}{(x-1)^4}$

$= \dfrac{4x-4-8x}{(x-1)^3}$

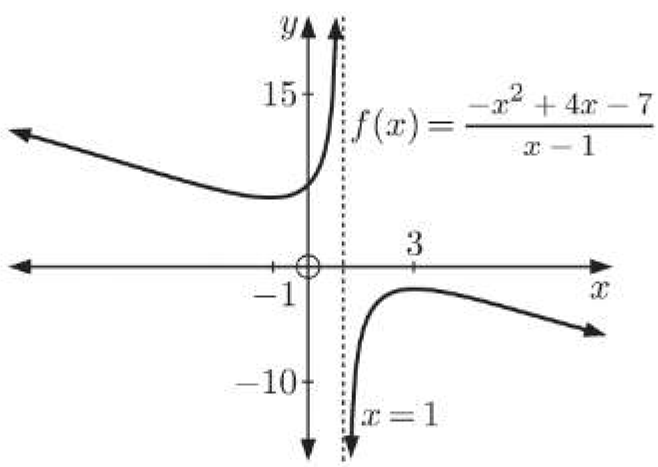
$= \dfrac{-4x-4}{(x-1)^3}$

$= \dfrac{-4(x+1)}{(x-1)^3}$ which has sign diagram:



b $y = f(x)$ is increasing for $-1 \leqslant x < 1$, and decreasing for $x \leqslant -1$ and for $x > 1$.

8 **a** $f(x) = \dfrac{-x^2+4x-7}{x-1}$



$\therefore\ f'(x) = \dfrac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2}$

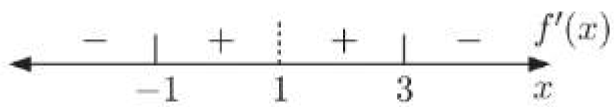
{quotient rule}

$= \dfrac{-2x^2+2x+4x-4+x^2-4x+7}{(x-1)^2}$

$= \dfrac{-x^2+2x+3}{(x-1)^2}$

$= \dfrac{-(x^2-2x-3)}{(x-1)^2}$

$= \dfrac{-(x+1)(x-3)}{(x-1)^2}$ which has sign diagram:



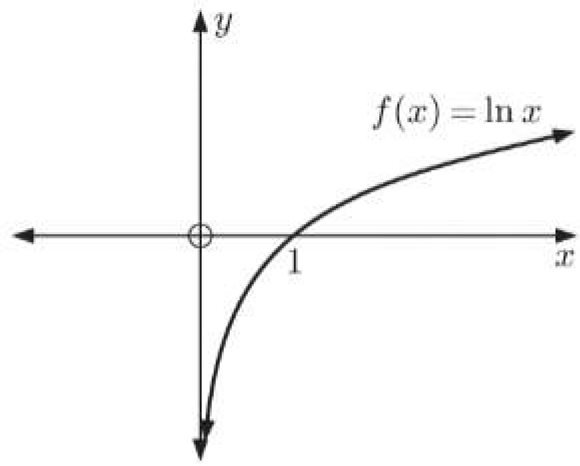
b $y = f(x)$ is increasing for $-1 \leqslant x < 1$ and for $1 < x \leqslant 3$, and decreasing for $x \leqslant -1$ and for $x \geqslant 3$.

9 Kenneth's conclusion that $f(x) = \ln x$ is decreasing for $x < 0$ is incorrect.

In this case, $f(x)$ is only defined when $x > 0$.

$\therefore$ $f'(x)$ is only defined when $x > 0$.

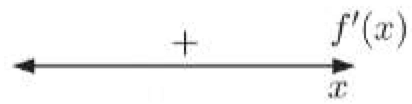
As we can see from the graph alongside, $f(x) = \ln x$ is increasing for all $x > 0$, and never decreasing.



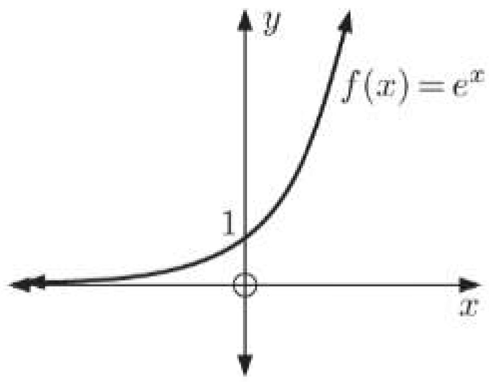
10 **a** $f(x) = e^x$

$\therefore$ $f'(x) = e^x$

which has sign diagram:



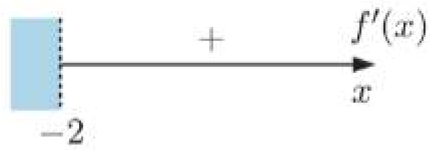
$\therefore$ $f(x)$ is increasing for all $x \in \mathbb{R}$, and never decreasing.



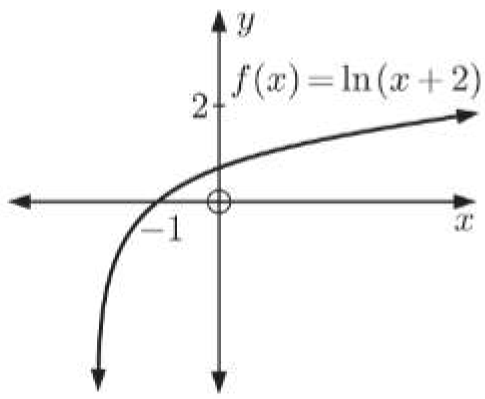
b $f(x) = \ln(x+2)$

$\therefore$ $f'(x) = \dfrac{1}{x+2}$

which has sign diagram:



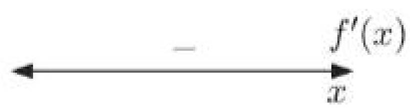
$\therefore$ $f(x)$ is increasing for $x > -2$, and never decreasing.



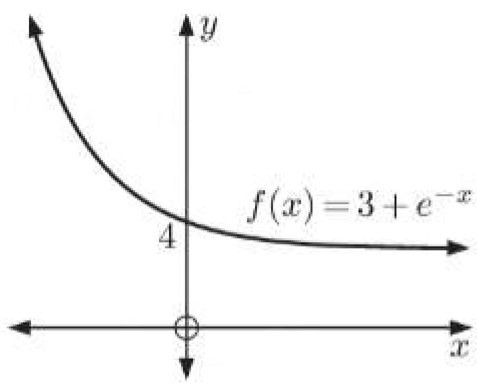
c $f(x) = 3 + e^{-x}$

$\therefore$ $f'(x) = -e^{-x}$

which has sign diagram:



$\therefore$ $f(x)$ is never increasing, and decreasing for all $x \in \mathbb{R}$.

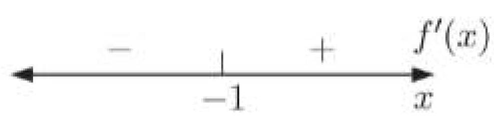


d $f(x) = xe^x$

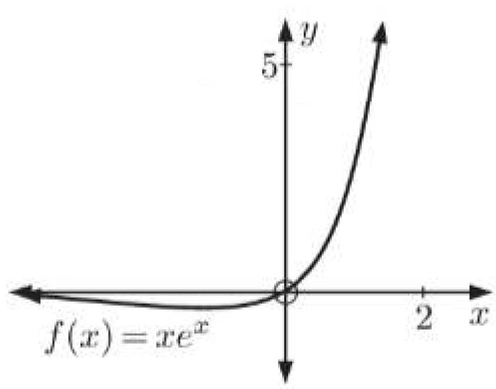
$\therefore$ $f'(x) = e^x + xe^x$ {product rule}

$= e^x(1+x)$

which has sign diagram:



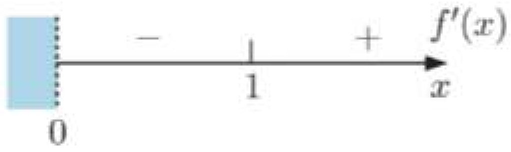
$\therefore$ $f(x)$ is increasing for $x \geqslant -1$, and decreasing for $x \leqslant -1$.

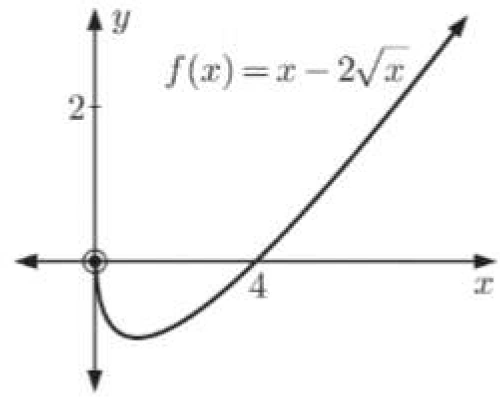


e $f(x) = x - 2\sqrt{x} = x - 2x^{\frac{1}{2}}$

$\therefore \; f'(x) = 1 - x^{-\frac{1}{2}}$

$= 1 - \dfrac{1}{\sqrt{x}}$

which has sign diagram: 



$\therefore$ $f(x)$ is increasing for $x \geqslant 1$,
and decreasing for $0 < x \leqslant 1$.

f $f(x) = x^3 \ln x$

$\therefore \; f'(x) = 3x^2 \ln x + x^3\left(\dfrac{1}{x}\right)$ {product rule}

$= 3x^2 \ln x + x^2$

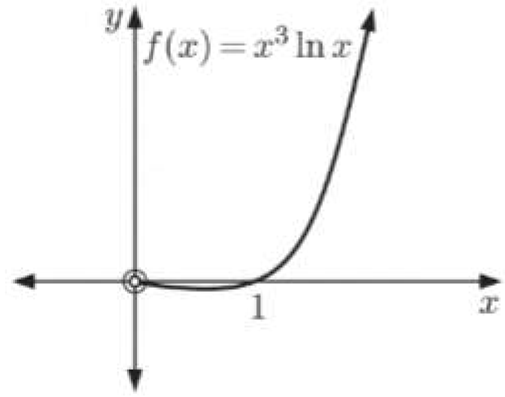
$= x^2(3\ln x + 1)$

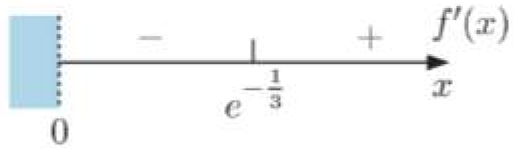
$f'(x) = 0$ when $x = 0$ or $3\ln x + 1 = 0$

$\therefore \; 3\ln x = -1$

$\therefore \; \ln x = -\frac{1}{3}$

$\therefore \; x = e^{-\frac{1}{3}}$



$\therefore$ $f'(x)$ has sign diagram: 

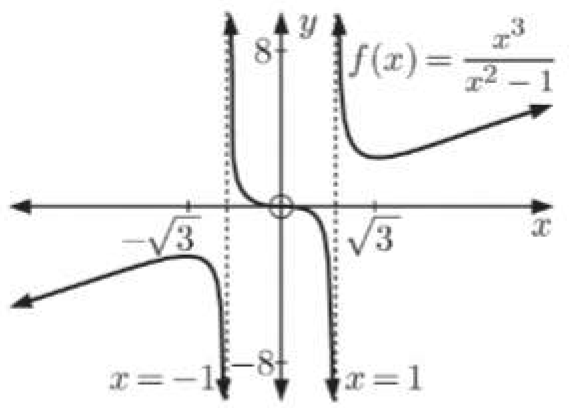
$\therefore$ $f(x)$ is increasing for $x \geqslant e^{-\frac{1}{3}}$, and decreasing for $0 < x \leqslant e^{-\frac{1}{3}}$.

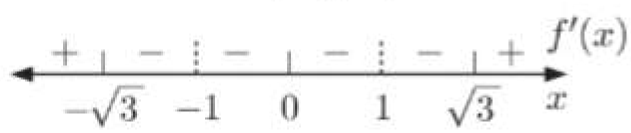
g $f(x) = \dfrac{x^3}{x^2 - 1}$

$\therefore \; f'(x) = \dfrac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2}$ {quotient rule}

$= \dfrac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2}$

$= \dfrac{x^4 - 3x^2}{(x^2-1)^2}$

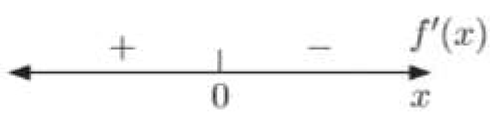


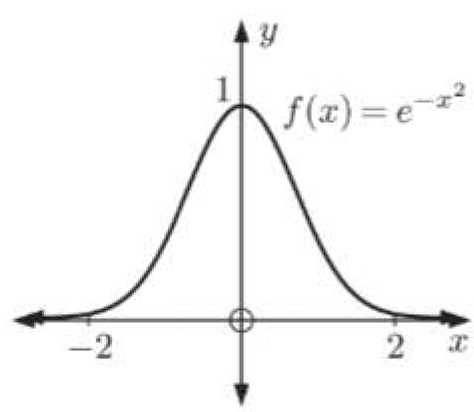
$= \dfrac{x^2(x^2-3)}{(x^2-1)^2}$ which has sign diagram: 

$\therefore$ $f(x)$ is increasing for $x \leqslant -\sqrt{3}$ and for $x \geqslant \sqrt{3}$, and decreasing for
$-\sqrt{3} \leqslant x < -1$, $-1 < x < 1$, and for $1 < x \leqslant \sqrt{3}$.

h $f(x) = e^{-x^2}$

$\therefore \; f'(x) = -2xe^{-x^2}$

which has sign diagram: 



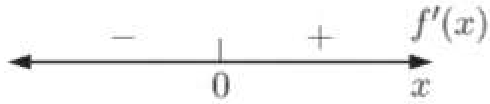
$\therefore$ $f(x)$ is increasing for $x \leqslant 0$,
and decreasing for $x \geqslant 0$.

i $f(x) = (3x^2+1)^4$

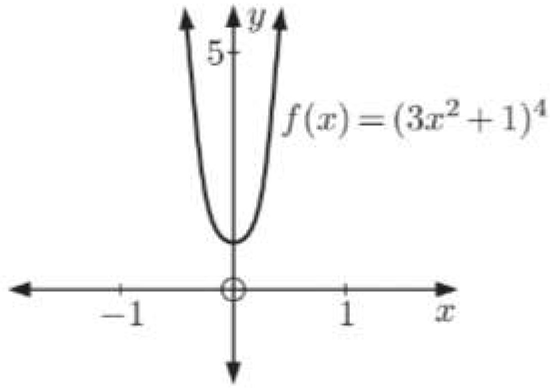
$\therefore\ f'(x) = 4(3x^2+1)^3(6x)$ {chain rule}

$= 24x(3x^2+1)^3$

which has sign diagram:



$\therefore$ $f(x)$ is increasing for $x \geqslant 0$,
and decreasing for $x \leqslant 0$.



j $f(x) = x^2 + \dfrac{4}{x-1} = x^2 + 4(x-1)^{-1}$

$\therefore\ f'(x) = 2x - 4(x-1)^{-2}(1)$ {chain rule}

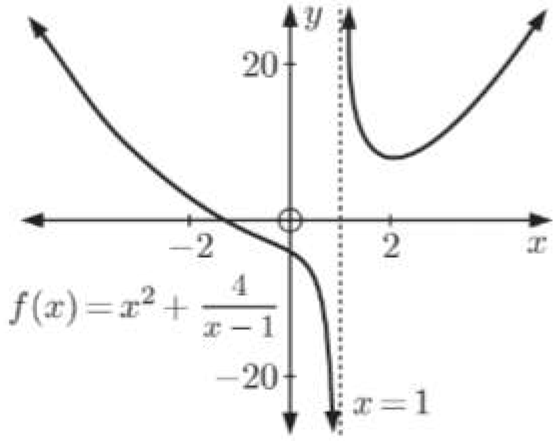
$= 2x - \dfrac{4}{(x-1)^2}$

$f'(x) = 0$ when $2x = \dfrac{4}{(x-1)^2}$

$\therefore\ x = \dfrac{2}{(x-1)^2}$

$\therefore\ x(x-1)^2 = 2$

$\therefore\ x = 2$ {using technology}



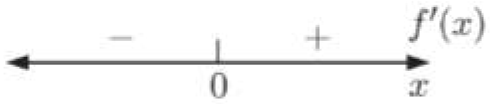
$\therefore$ $f'(x)$ has sign diagram:

$\therefore$ $f(x)$ is increasing for $x \geqslant 2$, and decreasing for $x < 1$ and for $1 < x \leqslant 2$.

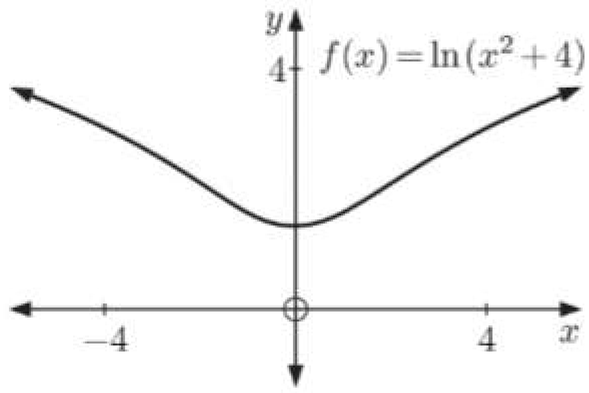
k $f(x) = \ln(x^2+4)$

$\therefore\ f'(x) = \dfrac{2x}{x^2+4}$

which has sign diagram:



$\therefore$ $f(x)$ is increasing for $x \geqslant 0$
and decreasing for $x \leqslant 0$.

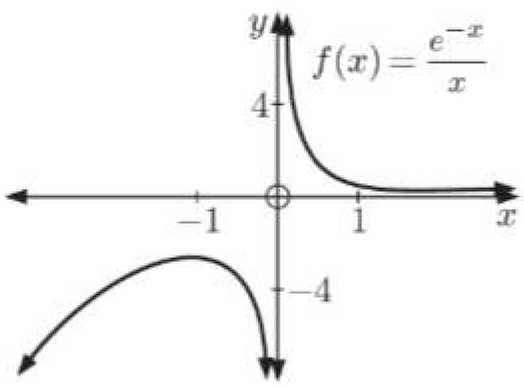


l $f(x) = \dfrac{e^{-x}}{x}$

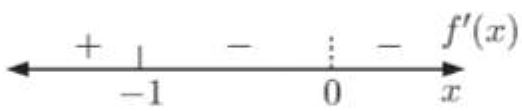
$\therefore\ f'(x) = \dfrac{(-e^{-x})x - e^{-x}(1)}{x^2}$ {quotient rule}

$= \dfrac{-xe^{-x} - e^{-x}}{x^2}$

$= \dfrac{-e^{-x}(x+1)}{x^2}$



which has sign diagram:



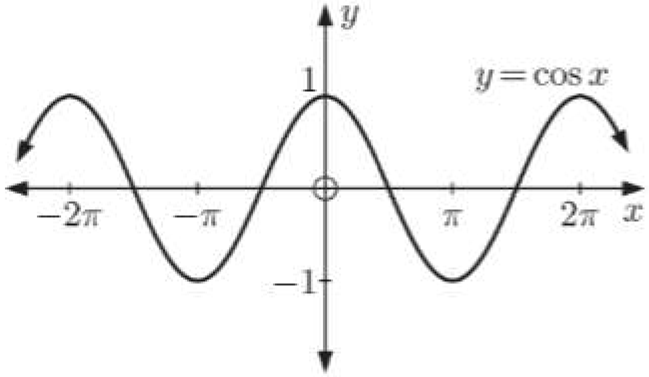
$\therefore$ $f(x)$ is increasing for $x \leqslant -1$ and decreasing for $-1 \leqslant x < 0$ and for $x > 0$.

11 **a** $y = \cos x$

$\therefore\ \dfrac{dy}{dx} = -\sin x, \quad x \in \mathbb{R}$

$-\sin x = 0$ when $x = k\pi,\ k \in \mathbb{Z}$,

$\therefore\ \dfrac{dy}{dx}$ has sign diagram:



So $y = \cos x$ is increasing for $\pi \leqslant x \leqslant 2\pi$, but this is repeated periodically every 2π units.

$\therefore\ y = \cos x$ is increasing for $\pi + 2k\pi \leqslant x \leqslant 2\pi + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore\ (2k+1)\pi \leqslant x \leqslant (2k+2)\pi, \quad k \in \mathbb{Z}$

$y = \cos x$ is decreasing for $0 \leqslant x \leqslant \pi$, but this is repeated periodically every 2π units.

$\therefore\ y = \cos x$ is decreasing for $0 + 2k\pi \leqslant x \leqslant \pi + 2k\pi, \quad k \in \mathbb{Z}$

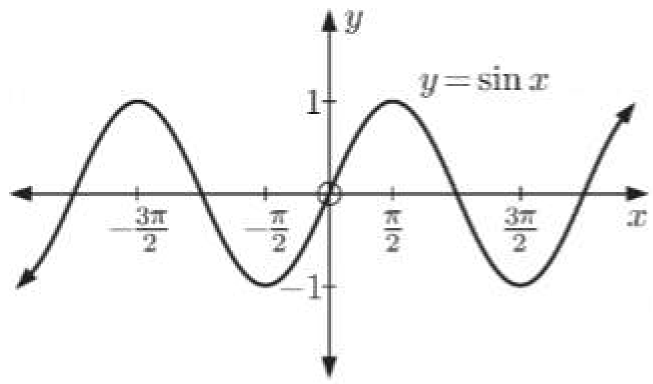
$\therefore\ 2k\pi \leqslant x \leqslant (2k+1)\pi, \quad k \in \mathbb{Z}$

b $y = \sin x$

$\therefore\ \dfrac{dy}{dx} = \cos x, \quad x \in \mathbb{R}$

$\cos x = 0$ when $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$,

$\therefore\ \dfrac{dy}{dx}$ has sign diagram:



So $y = \sin x$ is increasing for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, but this is repeated periodically every 2π units.

$\therefore\ y = \sin x$ is increasing for $-\frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore\ \dfrac{(4k-1)\pi}{2} \leqslant x \leqslant \dfrac{(4k+1)\pi}{2}, \quad k \in \mathbb{Z}$

$y = \sin x$ is decreasing for $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$, but this is repeated periodically every 2π units.

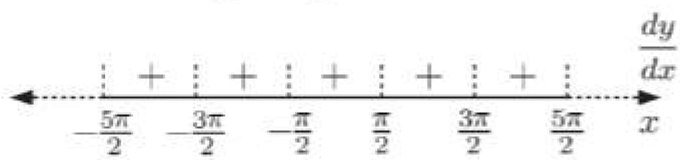
$\therefore\ y = \sin x$ is decreasing for $\frac{\pi}{2} + 2k\pi \leqslant x \leqslant \frac{3\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

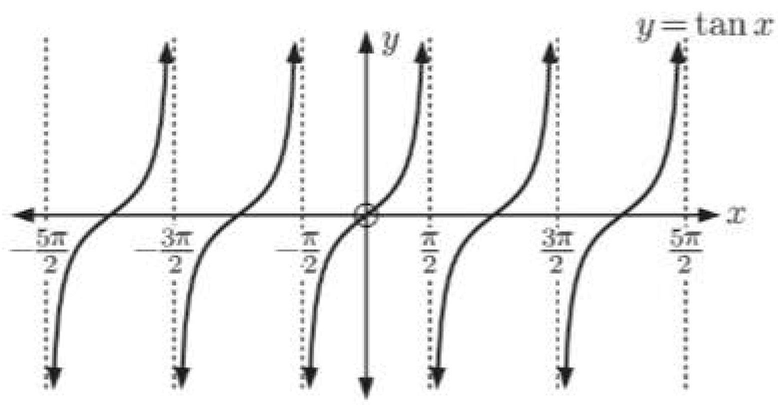
$\therefore\ \dfrac{(4k+1)\pi}{2} \leqslant x \leqslant \dfrac{(4k+3)\pi}{2}, \quad k \in \mathbb{Z}$

c $y = \tan x$

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}, \quad x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

which has sign diagram:





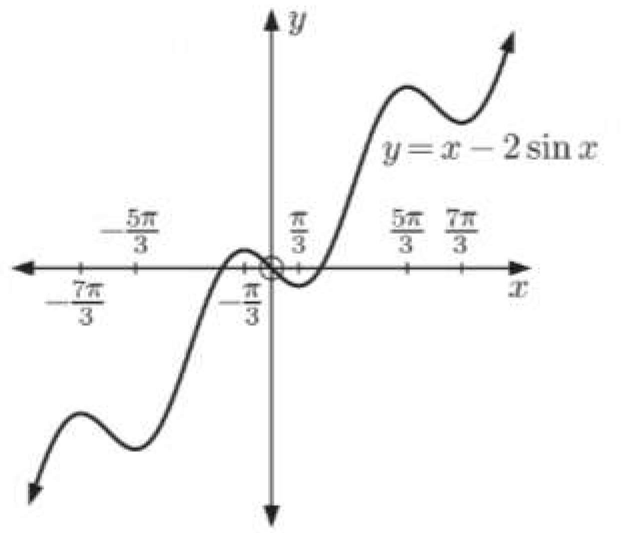
So $y = \tan x$ is increasing for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, but this is repeated periodically every π units.

$\therefore\ y = \tan x$ is increasing for $x \neq \dfrac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$

$y = \tan x$ is never decreasing.

d $y = x - 2\sin x$

$\therefore \dfrac{dy}{dx} = 1 - 2\cos x, \quad x \in \mathbb{R}$

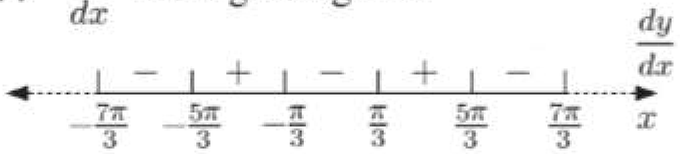


$1 - 2\cos x = 0$ when $2\cos x = 1$

$\therefore \cos x = \frac{1}{2}$

$\therefore x = \pm\frac{\pi}{3} + 2k\pi, \ k \in \mathbb{Z}$

$\therefore \dfrac{dy}{dx}$ has sign diagram:



So $y = x - 2\sin x$ is increasing for $\frac{\pi}{3} \leqslant x \leqslant \frac{5\pi}{3}$, but this interval is repeated periodically every 2π units.

$\therefore$ $y = x - 2\sin x$ is increasing for $\frac{\pi}{3} + 2k\pi \leqslant x \leqslant \frac{5\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$

$$\therefore \frac{(6k+1)\pi}{3} \leqslant x \leqslant \frac{(6k+5)\pi}{3}, \quad k \in \mathbb{Z}$$

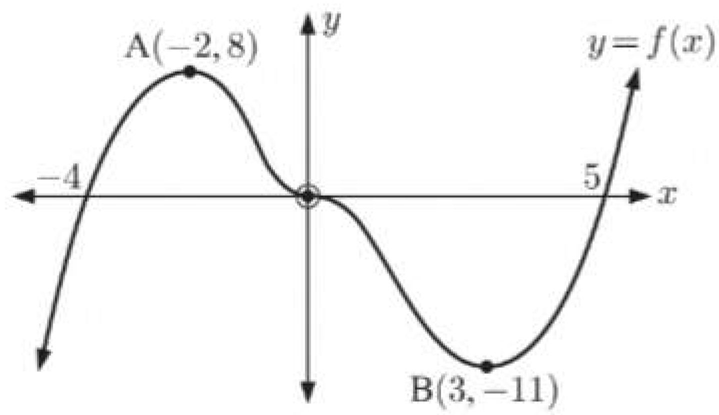
$y = x - 2\sin x$ is decreasing for $-\frac{\pi}{3} \leqslant x \leqslant \frac{\pi}{3}$, but this interval is repeated periodically every 2π units.

$\therefore$ $y = x - 2\sin x$ is decreasing for $-\frac{\pi}{3} + 2k\pi \leqslant x \leqslant \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$

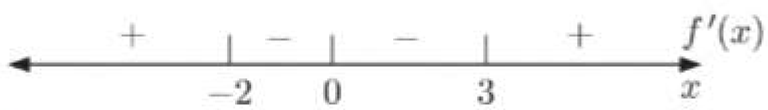
$$\therefore \frac{(6k-1)\pi}{3} \leqslant x \leqslant \frac{(6k+1)\pi}{3}, \quad k \in \mathbb{Z}$$

EXERCISE 19D

1 **a** A is a local maximum, O is a stationary inflection, B is a local minimum.



b $f'(x)$ has sign diagram:

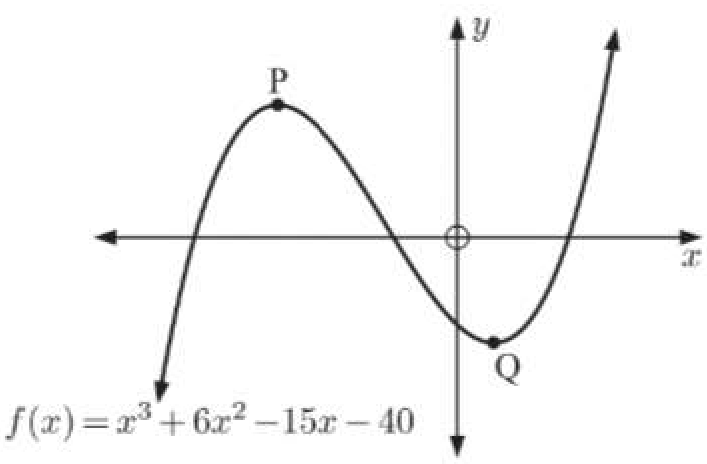


c **i** $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 3$.

ii $f(x)$ is decreasing for $-2 \leqslant x \leqslant 3$.

d $f(x)$ has sign diagram:

2

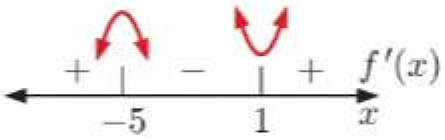


a P is a local maximum.
Q is a local minimum.

b $f(x) = x^3 + 6x^2 - 15x - 40$

$$\begin{aligned} \therefore f'(x) &= 3x^2 + 12x - 15 \\ &= 3(x^2 + 4x - 5) \\ &= 3(x-1)(x+5) \end{aligned}$$

c $f'(x)$ has sign diagram:



$\therefore$ there is a local maximum at $x = -5$
and a local minimum at $x = 1$.

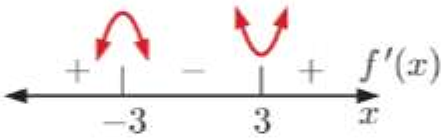
So, P has x-coordinate -5, and Q has x-coordinate 1.

$$f(-5) = (-5)^3 + 6(-5)^2 - 15(-5) - 40 = 60 \qquad f(1) = 1^3 + 6(1)^2 - 15(1) - 40 = -48$$

So, P is $(-5, 60)$ and Q is $(1, -48)$.

3 **a**
$$f(x) = \tfrac{1}{3}x^3 - 9x + 4$$
$$\therefore\ f'(x) = x^2 - 9 = (x+3)(x-3)$$

which has sign diagram:



b $f(x)$ is increasing for $x \leqslant -3$ and $x \geqslant 3$, and decreasing for $-3 \leqslant x \leqslant 3$.

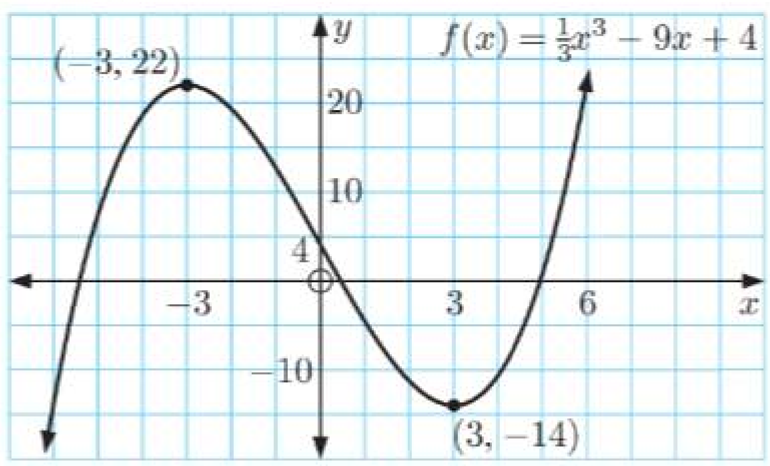
c From the sign diagram in **a**, there is a local maximum at $x = -3$, and a local minimum at $x = 3$.

$$f(-3) = \tfrac{1}{3}(-3)^3 - 9(-3) + 4 = 22 \qquad f(3) = \tfrac{1}{3}(3)^3 - 9(3) + 4 = -14$$

So, there is a local maximum at $(-3, 22)$, and a local minimum at $(3, -14)$.

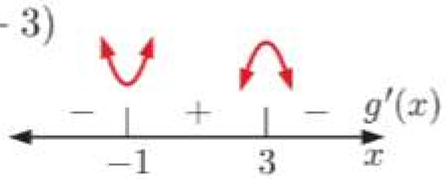
d As $x \to \infty$, $f(x) \to \infty$,
as $x \to -\infty$, $f(x) \to -\infty$.

e



4 **a**
$$g(x) = -2x^3 + 6x^2 + 18x - 7$$
$$\therefore\ g'(x) = -6x^2 + 12x + 18 = -6(x^2 - 2x - 3) = -6(x+1)(x-3)$$

which has sign diagram:



b $g(x)$ is increasing for $-1 \leqslant x \leqslant 3$, and decreasing for $x \leqslant -1$ and $x \geqslant 3$.

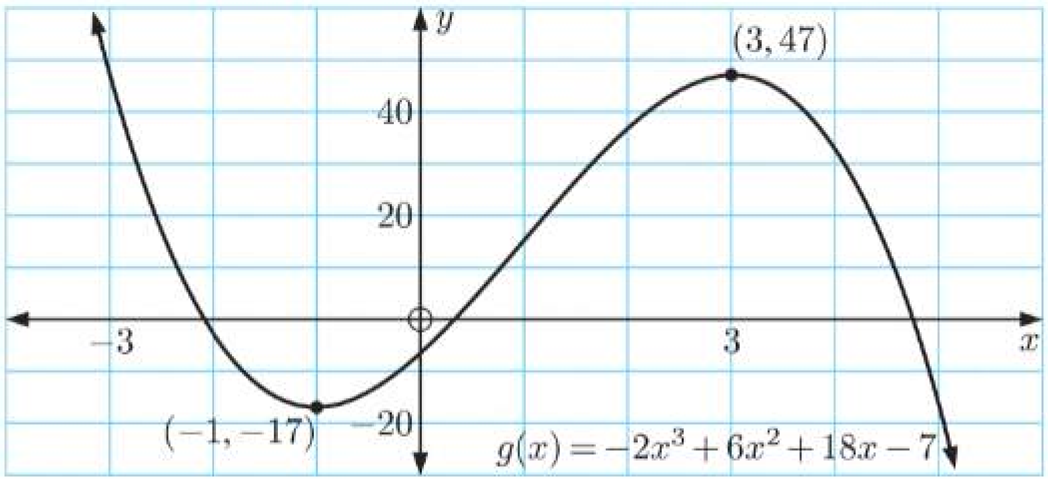
c From the sign diagram, there is a local minimum at $x = -1$, and a local maximum at $x = 3$.

$$g(-1) = -2(-1)^3 + 6(-1)^2 + 18(-1) - 7 = -17 \qquad g(3) = -2(3)^3 + 6(3)^2 + 18(3) - 7 = 47$$

So, there is a local minimum at $(-1, -17)$, and a local maximum at $(3, 47)$.

d As $x \to \infty$, $g(x) \to -\infty$, as $x \to -\infty$, $g(x) \to \infty$.

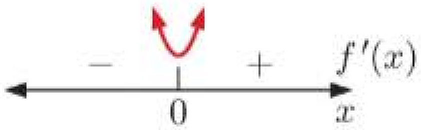
e



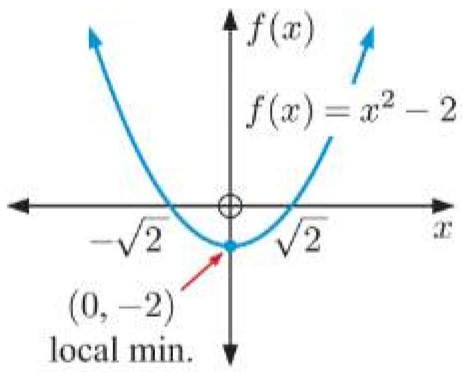
5 **a** $f(x) = x^2 - 2$

$\therefore\ f'(x) = 2x$

which has sign diagram:



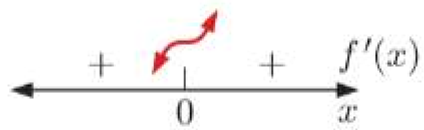
Now $f(0) = -2$, so there is a local minimum at $(0, -2)$.



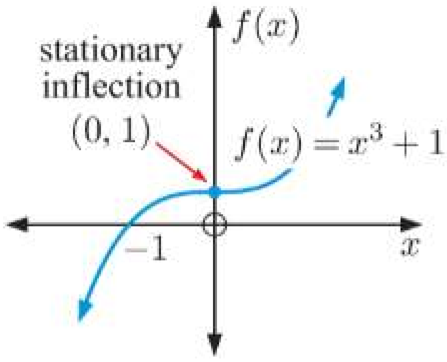
b $f(x) = x^3 + 1$

$\therefore\ f'(x) = 3x^2$

which has sign diagram:



Now $f(0) = 1$, so there is a stationary inflection at $(0, 1)$.



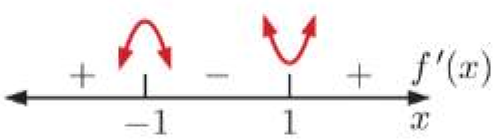
c $f(x) = x^3 - 3x + 2$

$\therefore\ f'(x) = 3x^2 - 3$

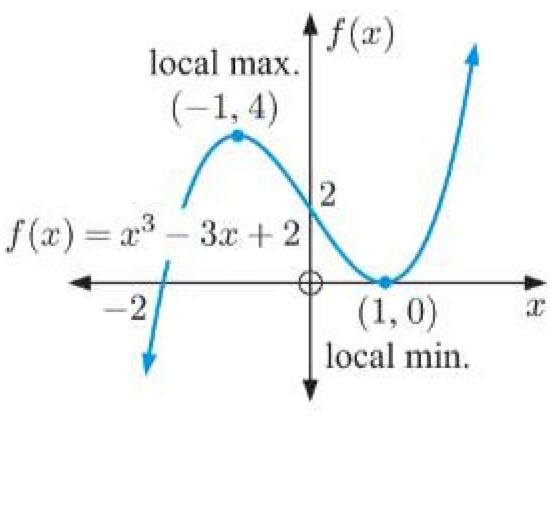
$= 3(x^2 - 1)$

$= 3(x + 1)(x - 1)$

which has sign diagram:



Now $f(-1) = 4$, $f(1) = 0$, so there is a local maximum at $(-1, 4)$, and a local minimum at $(1, 0)$.



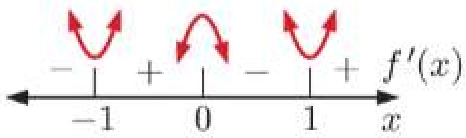
d $f(x) = x^4 - 2x^2$

$\therefore\ f'(x) = 4x^3 - 4x$

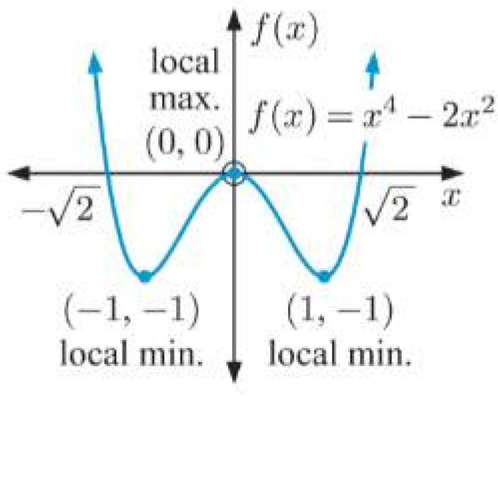
$= 4x(x^2 - 1)$

$= 4x(x + 1)(x - 1)$

which has sign diagram:



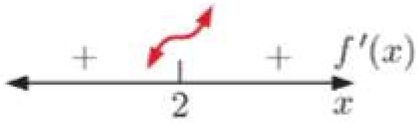
Now $f(-1) = -1$, $f(1) = -1$, $f(0) = 0$, so there are local minima at $(-1, -1)$ and $(1, -1)$, and a local maximum at $(0, 0)$.



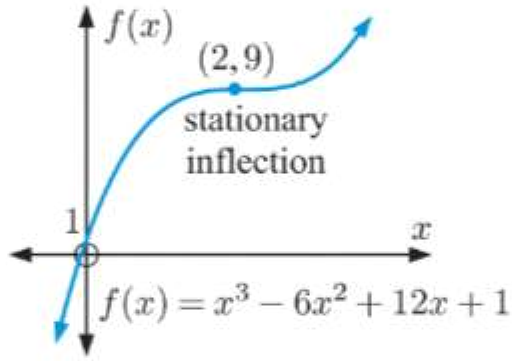
e

$$f(x) = x^3 - 6x^2 + 12x + 1$$
$$\therefore \ f'(x) = 3x^2 - 12x + 12$$
$$= 3(x^2 - 4x + 4)$$
$$= 3(x-2)^2$$

which has sign diagram:



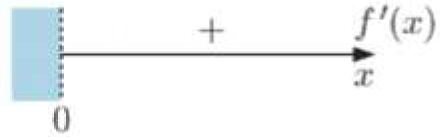
Now $f(2) = 9$, so there is a stationary inflection at $(2, 9)$.



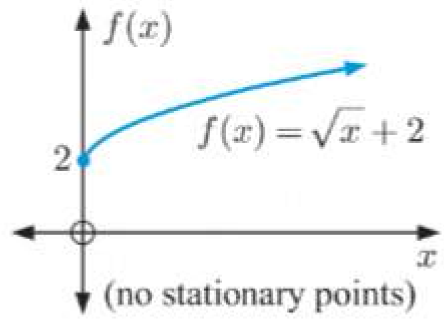
f

$$f(x) = \sqrt{x} + 2$$
$$= x^{\frac{1}{2}} + 2$$
$$\therefore \ f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x}} \neq 0$$

which has sign diagram:



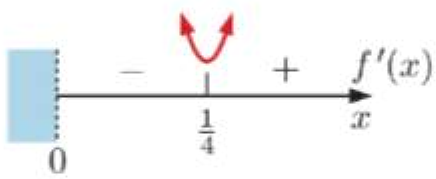
$\therefore$ there are no stationary points.



g

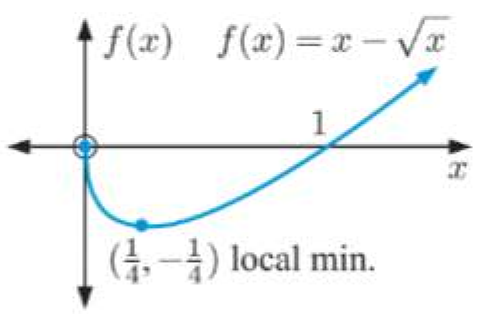
$$f(x) = x - \sqrt{x}$$
$$= x - x^{\frac{1}{2}}$$
$$\therefore \ f'(x) = 1 - \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$= 1 - \frac{1}{2\sqrt{x}}$$
$$= \frac{2\sqrt{x} - 1}{2\sqrt{x}}$$

which has sign diagram:



$f(x)$ is defined for all $x \geqslant 0$

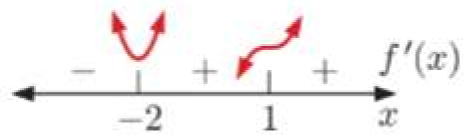
Now $f(\frac{1}{4}) = -\frac{1}{4}$, so there is a local minimum at $(\frac{1}{4}, -\frac{1}{4})$.



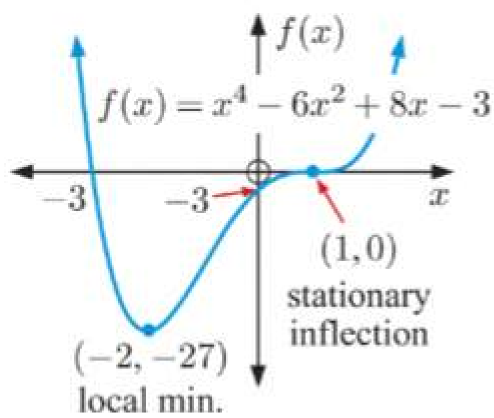
h

$$f(x) = x^4 - 6x^2 + 8x - 3$$
$$\therefore \ f'(x) = 4x^3 - 12x + 8$$
$$= 4(x^3 - 3x + 2)$$
$$= 4(x-1)(x^2 + x - 2)$$
$$= 4(x-1)(x+2)(x-1)$$

which has sign diagram:



Now $f(-2) = -27$, $f(1) = 0$, so there is a local minimum at $(-2, -27)$, and a stationary inflection at $(1, 0)$.



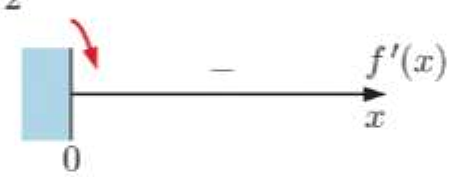
i $f(x) = 1 - x\sqrt{x}$

$= 1 - x^{\frac{3}{2}}$

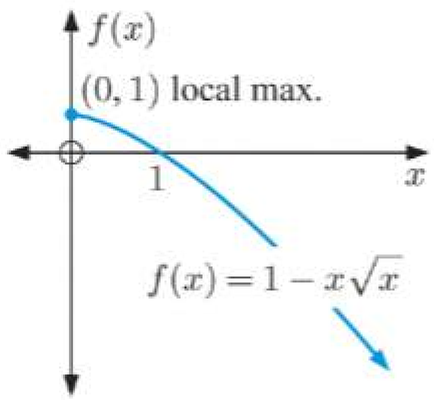
$\therefore\ f'(x) = -\frac{3}{2}x^{\frac{1}{2}}$

$= -\dfrac{3\sqrt{x}}{2}$

which has sign diagram:



$f(x)$ is only defined when $x \geqslant 0$.
Now $f(0) = 1$, so there is a local maximum at $(0, 1)$.



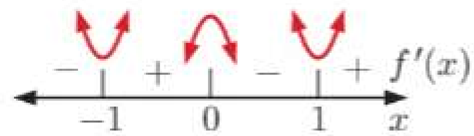
j $f(x) = x^4 - 2x^2 - 8$

$\therefore\ f'(x) = 4x^3 - 4x$

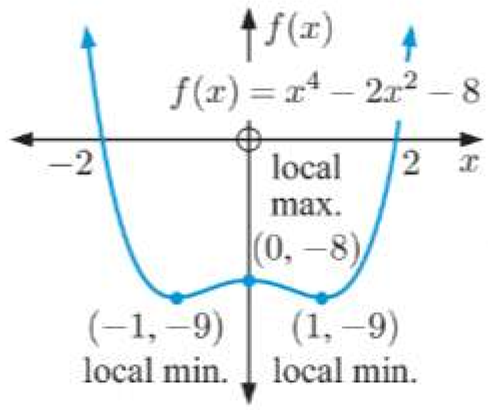
$= 4x(x^2 - 1)$

$= 4x(x+1)(x-1)$

which has sign diagram:



Now $f(-1) = -9$, $f(1) = -9$, $f(0) = -8$, so there are local minima at $(-1, -9)$ and $(1, -9)$, and a local maximum at $(0, -8)$.



6 **a** $f(x) = ax^2 + bx + c, \quad a \neq 0$

$\therefore\ f'(x) = 2ax + b$

$f(x)$ has a stationary point when $f'(x) = 0$

$\therefore\ 2ax + b = 0$

$\therefore\ x = -\dfrac{b}{2a}$

b When $a < 0$, $f(x)$ is concave down, so there is a local maximum when $a < 0$.

When $a > 0$, $f(x)$ is concave up, so there is a local minimum when $a > 0$.

7 **a** $y = xe^{-x}$

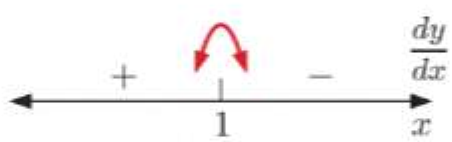
$\therefore\ \dfrac{dy}{dx} = (1)e^{-x} + x(e^{-x})(-1)$ {product rule}

$= e^{-x} - xe^{-x}$

$= e^{-x}(1 - x)$ where e^{-x} is positive for all x

So, $\dfrac{dy}{dx} = 0$ when $x = 1$.

The sign diagram of $\dfrac{dy}{dx}$ is:



When $x = 1$, $y = (1)e^{-1} = \dfrac{1}{e}$

$\therefore$ there is a local maximum at $\left(1, \dfrac{1}{e}\right)$.

b $\quad y = x^2 e^x$

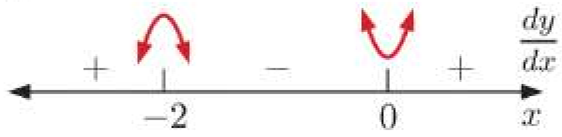
$\therefore \quad \dfrac{dy}{dx} = (2x)e^x + x^2(e^x) \qquad$ {product rule}

$\qquad = 2xe^x + x^2e^x$

$\qquad = xe^x(2 + x)$ where e^x is positive for all x

So, $\dfrac{dy}{dx} = 0$ when $x = 0$ or -2.

The sign diagram of $\dfrac{dy}{dx}$ is:



When $x = -2$, $\quad y = (-2)^2e^{-2} = \dfrac{4}{e^2}$

When $x = 0$, $\quad y = 0^2e^0 = 0$

$\therefore$ there is a local maximum at $\left(-2, \dfrac{4}{e^2}\right)$ and a local minimum at $(0, 0)$.

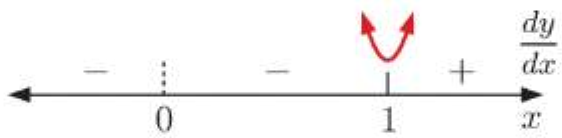
c $\quad y = \dfrac{e^x}{x}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{e^x x - e^x(1)}{x^2} \qquad$ {quotient rule}

$\qquad = \dfrac{e^x(x-1)}{x^2}$ where e^x is positive for all x

So, $\dfrac{dy}{dx} = 0$ when $x = 1$.

The sign diagram of $\dfrac{dy}{dx}$ is:



When $x = 1$, $\quad y = \dfrac{e^1}{1} = e$

$\therefore$ there is a local minimum at $(1, e)$.

d $\quad y = e^{-x}(x + 2)$

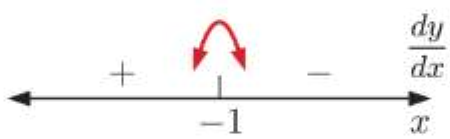
$\therefore \quad \dfrac{dy}{dx} = e^{-x}(-1)(x + 2) + e^{-x}(1) \qquad$ {product rule}

$\qquad = -e^{-x}(x + 2 - 1)$

$\qquad = -e^{-x}(x + 1)$ where $-e^{-x}$ is negative for all x

So, $\dfrac{dy}{dx} = 0$ when $x = -1$.

The sign diagram of $\dfrac{dy}{dx}$ is:



When $x = -1$, $\quad y = e^{-(-1)}(-1 + 2) = e$

$\therefore$ there is a local maximum at $(-1, e)$.

8 **a** $\quad f(x) = 2x^3 + ax^2 - 24x + 1$

$\therefore \quad f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0$, so $6(-4)^2 + 2a(-4) - 24 = 0$

$$\therefore \quad 96 - 8a - 24 = 0$$

$$\therefore \quad 72 = 8a$$

$$\therefore \quad a = 9$$

b Since $a = 9$, then $f(x) = 2x^3 + 9x^2 - 24x + 1$

$$\therefore \; f(-4) = 2(-4)^3 + 9(-4)^2 - 24(-4) + 1$$
$$= 113$$

$\therefore$ the local maximum is at $(-4, 113)$.

9 **a**

$$f(x) = x^3 + ax + b$$
$$\therefore \; f'(x) = 3x^2 + a$$

But $f'(-2) = 0$

$$\therefore \; 3(-2)^2 + a = 0$$
$$\therefore \; 12 + a = 0$$
$$\therefore \; a = -12$$

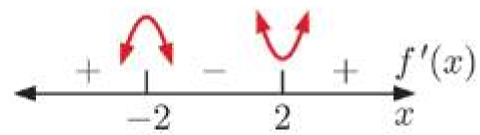
Also, $f(-2) = 3$

$$\therefore \; (-2)^3 - 12(-2) + b = 3$$
$$\therefore \; -8 + 24 + b = 3$$
$$\therefore \; b = -13$$

b Now $f(x) = x^3 - 12x - 13$

$$\therefore \; f'(x) = 3x^2 - 12$$
$$= 3(x^2 - 4)$$
$$= 3(x+2)(x-2)$$

which has sign diagram:



Now $f(2) = -29$, $f(-2) = 3$, so there is a local minimum at $(2, -29)$ and a local maximum at $(-2, 3)$.

10 **a**

$$y = \frac{e^{ax}}{bx}$$

$$\therefore \; \frac{dy}{dx} = \frac{e^{ax}(a)(bx) - e^{ax}(b)}{(bx)^2} \qquad \{\text{quotient rule}\}$$
$$= \frac{abxe^{ax} - be^{ax}}{b^2x^2}$$
$$= \frac{be^{ax}(ax-1)}{b^2x^2}$$
$$= \frac{e^{ax}(ax-1)}{bx^2} \qquad \text{.... } (*)$$

Since $\left(\frac{1}{3}, \frac{e}{2}\right)$ is a stationary point, then

when $x = \frac{1}{3}$, $\frac{dy}{dx} = 0$

Substituting $x = \frac{1}{3}$ into $(*)$ gives:

$$\therefore \; \frac{e^{\frac{a}{3}}\left(\frac{a}{3} - 1\right)}{b(\frac{1}{3})^2} = 0$$
$$\therefore \; e^{\frac{a}{3}}\left(\frac{a}{3} - 1\right) = 0 \qquad \{\text{as } b \neq 0\}$$
$$\therefore \; \frac{a}{3} - 1 = 0 \qquad \{\text{as } e^{\frac{a}{3}} > 0\}$$
$$\therefore \; \frac{a}{3} = 1$$
$$\therefore \; a = 3$$

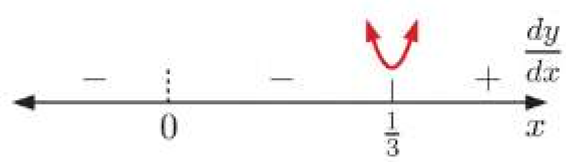
when $x = \frac{1}{3}$, $y = \frac{e}{2}$

$$\therefore \; \frac{e}{2} = \frac{e^{3(\frac{1}{3})}}{b(\frac{1}{3})}$$
$$\therefore \; \frac{e}{2} = \frac{e}{\frac{b}{3}}$$
$$\therefore \; \frac{b}{3} = 2$$
$$\therefore \; b = 6$$

So, $a = 3$ and $b = 6$.

b Since $a = 3$ and $b = 6$, then $\frac{dy}{dx} = \frac{e^{3x}(3x-1)}{6x^2}$

which has sign diagram:



$\therefore$ there is a local minimum at $\left(\frac{1}{3}, \frac{e}{2}\right)$.

11 **a** x is defined for all $x \in \mathbb{R}$, but $\ln x$ is only defined for $x > 0$.

$\therefore$ $f(x) = x \ln x$ is only defined for $x > 0$.

b $f'(x) = (1) \ln x + x\left(\frac{1}{x}\right)$ {product rule}

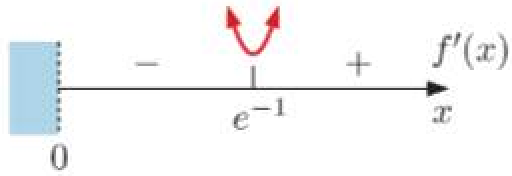
$= \ln x + 1$

$f'(x) = 0$ when $\ln x + 1 = 0$

$\therefore$ $\ln x = -1$

$\therefore$ $x = e^{-1}$

$\therefore$ $f'(x)$ has sign diagram:



$f(e^{-1}) = e^{-1} \ln(e^{-1})$

$= -\frac{1}{e}$

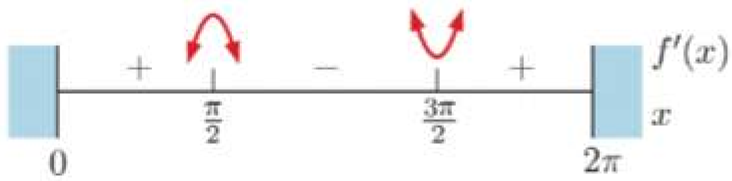
$\therefore$ there is a local minimum at $\left(\frac{1}{e}, -\frac{1}{e}\right)$.

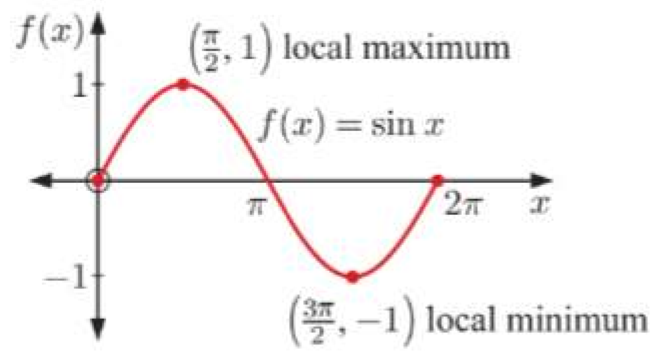
$\therefore$ the minimum value of $f(x)$ is $-\frac{1}{e}$.

12 **a** $f(x) = \sin x$, $0 \leqslant x \leqslant 2\pi$

$\therefore$ $f'(x) = \cos x$

which has sign diagram:





$f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$

$f\left(\frac{3\pi}{2}\right) = \sin \frac{3\pi}{2} = -1$

$\therefore$ there is a local maximum at $\left(\frac{\pi}{2}, 1\right)$,

and a local minimum at $\left(\frac{3\pi}{2}, -1\right)$.

b $f(x) = \cos 2x$, $0 \leqslant x \leqslant 2\pi$

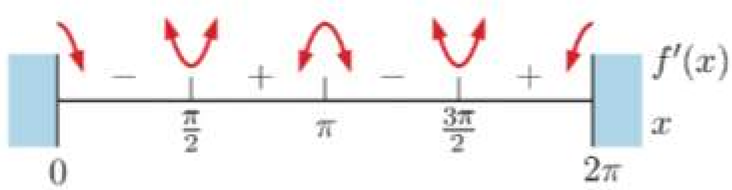
$\therefore$ $f'(x) = -2 \sin 2x$

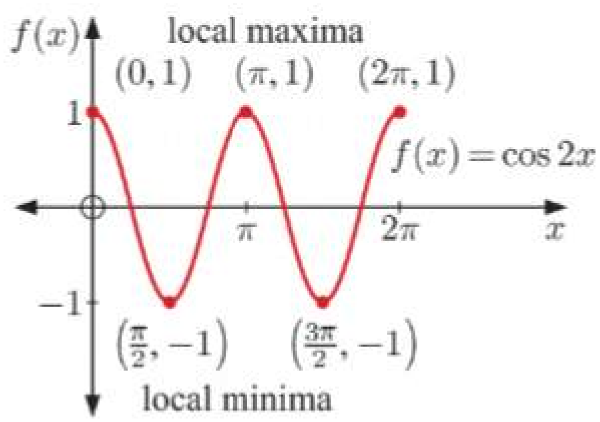
$f'(x) = 0$ when $\sin 2x = 0$

$\therefore$ $2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$\therefore$ $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

So, $f'(x)$ has sign diagram:





$f(0) = \cos 0 = 1$, $f\left(\frac{\pi}{2}\right) = \cos \pi = -1$,

$f(\pi) = \cos 2\pi = 1$, $f\left(\frac{3\pi}{2}\right) = \cos 3\pi = -1$,

$f(2\pi) = \cos 4\pi = 1$

$\therefore$ there are local maxima at $(0, 1)$, $(\pi, 1)$, and $(2\pi, 1)$, and local minima at $\left(\frac{\pi}{2}, -1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$.

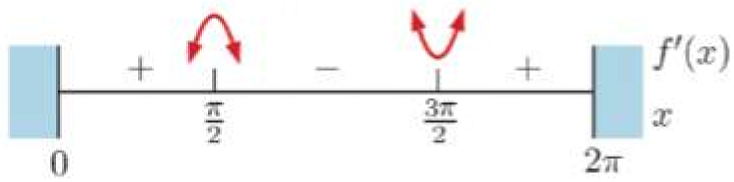
c $f(x) = e^{\sin x}, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \; f'(x) = e^{\sin x} \cos x$ where $e^{\sin x}$ is positive for all x

So, $f'(x) = 0$ when $\cos x = 0$

$\therefore \; x = \frac{\pi}{2}, \frac{3\pi}{2}$

So, $f'(x)$ has sign diagram:

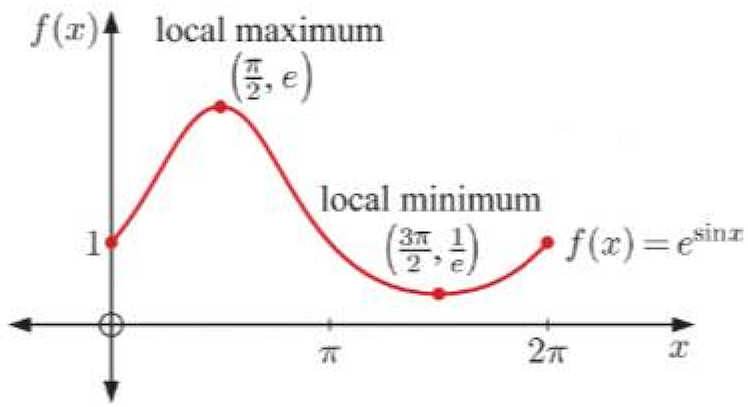


$f\left(\frac{\pi}{2}\right) = e^{\sin \frac{\pi}{2}} = e^1 = e,$

$f\left(\frac{3\pi}{2}\right) = e^{\sin \frac{3\pi}{2}} = e^{-1} = \frac{1}{e}$

$\therefore$ there is a local maximum at $\left(\frac{\pi}{2}, e\right)$,

and a local minimum at $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$.



d $f(x) = \cos x - \sin x, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \; f'(x) = -\sin x - \cos x$

$= -(\sin x + \cos x)$

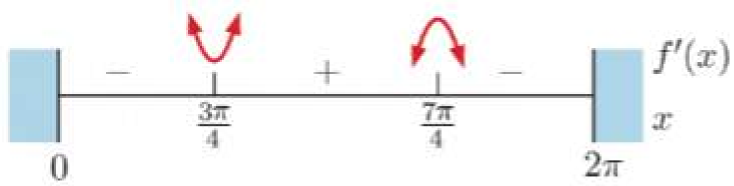
$f'(x) = 0$ when $\sin x + \cos x = 0$

$\therefore \; \sin x = -\cos x$

$\therefore \; \tan x = -1$

$\therefore \; x = \frac{3\pi}{4}, \frac{7\pi}{4}$

So, $f'(x)$ has sign diagram:



$f\left(\frac{3\pi}{4}\right) = \cos \frac{3\pi}{4} - \sin \frac{3\pi}{4}$

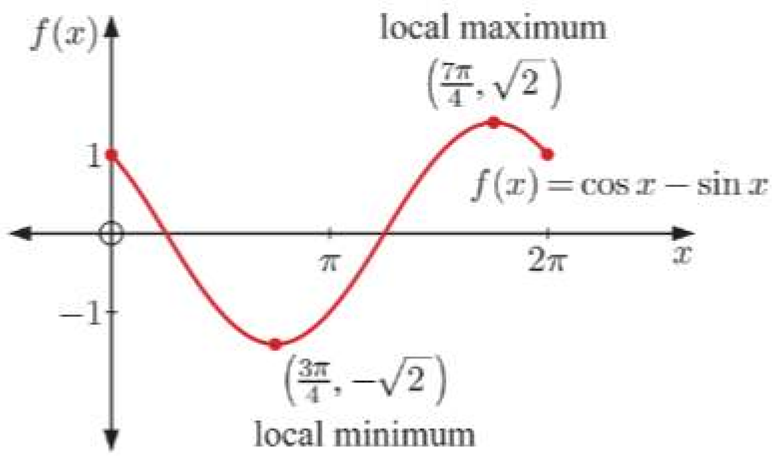
$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= -\sqrt{2}$

$f\left(\frac{7\pi}{4}\right) = \cos \frac{7\pi}{4} - \sin \frac{7\pi}{4}$

$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}$



$\therefore$ there is a local minimum at $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$, and a local maximum at $\left(\frac{7\pi}{4}, \sqrt{2}\right)$.

13 $P(x) = ax^3 + bx^2 + cx + d$

$\therefore \; P'(x) = 3ax^2 + 2bx + c \quad .... \; (1)$

Now $(0, 2)$ lies on $P(x)$, so $P(0) = 2$

$\therefore \; a(0) + b(0) + c(0) + d = 2$

$\therefore \; d = 2$

The tangent at $(0, 2)$ is $y = 9x + 2$, so $P'(0) = 9$

$\therefore \; 3a(0) + 2b(0) + c = 9$

$\therefore \; c = 9 \quad .... \; (2)$

There is a stationary point at $(-1, -7)$, so $P'(-1) = 0$.

$\therefore \; 3a(-1)^2 + 2b(-1) + c = 0 \qquad \{\text{using (1)}\}$

$\therefore \; 3a - 2b + c = 0$

So, using (2), $3a - 2b = -9 \quad .... \; (3)$

Finally, $(-1, -7)$ lies on $P(x)$

$\therefore \; a(-1)^3 + b(-1)^2 + c(-1) + d = -7$

$\therefore \; -a + b - 9 + 2 = -7$

$\therefore \; b - a = 0$

$\therefore \; a = b$

So, using (3), $3a - 2a = -9$

$\therefore \; a = -9$

$\therefore \; a = b = -9$

$\therefore \; P(x) = -9x^3 - 9x^2 + 9x + 2$

14 **a** Let $f(x) = x^3 - 12x - 2$, for $-3 \leqslant x \leqslant 5$

$\therefore \; f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x + 2)(x - 2)$

which is 0 when $x = -2$ or 2

The sign diagram of $f'(x)$ is:

$+ \quad | \quad - \quad | \quad + \qquad f'(x)$

$\qquad -2 \qquad\quad 2 \qquad\quad x$

$\therefore$ there is a local maximum at $x = -2$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
-3 (end point)	7
-2 (local maximum)	14
2 (local minimum)	-18
5 (end point)	63

The greatest of these values is 63 when $x = 5$.

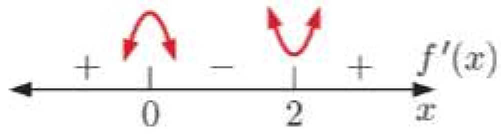
The least of these values is -18 when $x = 2$.

b Let $f(x) = 4 - 3x^2 + x^3$, for $-2 \leqslant x \leqslant 3$

$\therefore\ f'(x) = -6x + 3x^2$
$\qquad\qquad = 3x(x-2)$

which is 0 when $x = 0$ or 2

The sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
−2 (end point)	−16
0 (local maximum)	4
2 (local minimum)	0
3 (end point)	4

The greatest of these values is 4 when $x = 0$ or $x = 3$.

The least of these values is −16 when $x = -2$.

c Let $f(x) = x^2 + \dfrac{16}{x} = x^2 + 16x^{-1}$, for $1 \leqslant x \leqslant 4$

$\therefore\ f'(x) = 2x - 16x^{-2}$

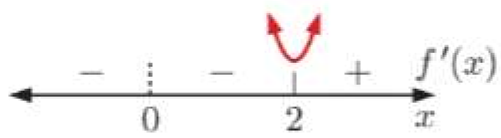
$$= 2x - \frac{16}{x^2}$$

$$= \frac{2x^3 - 16}{x^2}$$

$$= \frac{2(x^3 - 8)}{x^2}$$

which is 0 when $x = 2$

The sign diagram of $f'(x)$ is:



$\therefore$ there is a local minimum at $x = 2$.

Critical value (x)	$f(x)$
1 (end point)	17
2 (local minimum)	12
4 (end point)	20

The greatest of these values is 20 when $x = 4$.

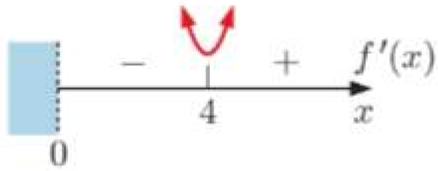
The least of these values is 12 when $x = 2$.

d Let $f(x) = x - 4\sqrt{x} = x - 4x^{\frac{1}{2}}$, for $0 \leqslant x \leqslant 5$

$$\therefore\ f'(x) = 1 - 2x^{-\frac{1}{2}}$$
$$= 1 - \frac{2}{\sqrt{x}}$$
$$= \frac{\sqrt{x} - 2}{\sqrt{x}}$$

which is 0 when $x = 4$

The sign diagram of $f'(x)$ is:



$\therefore$ there is a local minimum at $x = 4$.

Critical value (x)	$f(x)$
0 (end point)	0
4 (local minimum)	-4
5 (end point)	≈ -3.94

The greatest of these values is 0 when $x = 0$.

The least of these values is -4 when $x = 4$.

15 **a** $y = \tan x - 2x$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\cos^2 x} - 2$$

Now, when $x = -\frac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{1}{\left[\cos(-\frac{\pi}{4})\right]^2} - 2$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} - 2$$
$$= \frac{1}{(\frac{1}{2})} - 2 = 0$$

$\therefore$ $y = \tan x - 2x$ has a stationary point when $x = -\frac{\pi}{4}$.

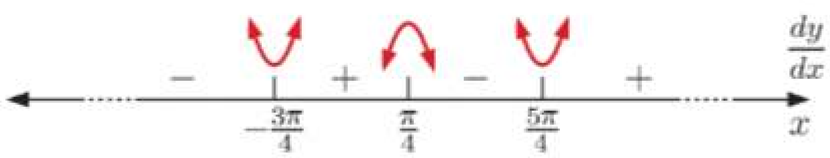
b $y = 4e^{-x} \sin x$

$$\therefore\ \frac{dy}{dx} = -4e^{-x}\sin x + 4e^{-x}\cos x \qquad \{\text{product rule}\}$$
$$= 4e^{-x}(\cos x - \sin x) \quad \text{where } 4e^{-x} \text{ is positive for all } x$$

So, $\dfrac{dy}{dx} = 0$ when $\cos x - \sin x = 0$

$$\therefore\ \cos x = \sin x$$
$$\therefore\ \tan x = 1$$
$$\therefore\ x = \tfrac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$$

So, $\dfrac{dy}{dx}$ has sign diagram:



$\therefore$ $y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

16 $f(t) = ate^{bt^2}$

$\therefore\; f'(t) = ae^{bt^2} + ate^{bt^2}(2bt)$ {product rule}

$= ae^{bt^2} + 2abt^2e^{bt^2}$

$= ae^{bt^2}(1 + 2bt^2)$

If $f(t)$ has maximum value 1 when $t = 2$, then

$f(2) = 1$ and $f'(2) = 0$

$\therefore\; a(2)e^{b(2)^2} = 1$

$\therefore\; 2ae^{4b} = 1$

$\therefore\; ae^{4b} = \frac{1}{2}$ .... (1)

$\therefore\; ae^{b(2)^2}(1 + 2b(2)^2) = 0$

$\therefore\; ae^{4b}(1 + 8b) = 0$ .... (2)

Substituting (1) into (2) gives: $\frac{1}{2}(1 + 8b) = 0$

$\therefore\; 1 + 8b = 0$

$\therefore\; 8b = -1$

$\therefore\; b = -\frac{1}{8}$

Substituting $b = -\frac{1}{8}$ into (1) gives: $ae^{4(-\frac{1}{8})} = \frac{1}{2}$

$\therefore\; ae^{-\frac{1}{2}} = \frac{1}{2}$

$\therefore\; a = \frac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$

So, $a = \dfrac{\sqrt{e}}{2}$ and $b = -\frac{1}{8}$.

17 Let $f(x) = \dfrac{\ln x}{x}$

$\therefore\; f'(x) = \dfrac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2}$ {quotient rule}

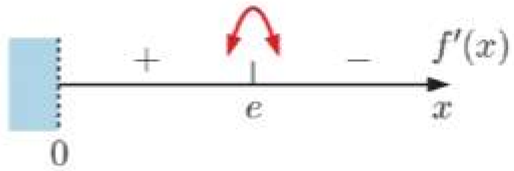
$= \dfrac{1 - \ln x}{x^2}$

$f'(x) = 0$ when $1 - \ln x = 0$

$\therefore\; \ln x = 1$

$\therefore\; x = e$

So, the sign diagram of $f'(x)$ is:



$f(e) = \dfrac{\ln e}{e} = \dfrac{1}{e}$

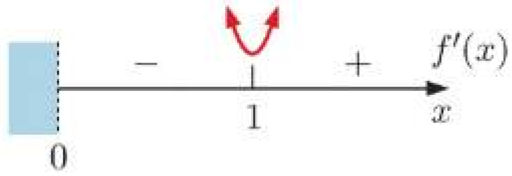
$\therefore$ there is a local maximum at $\left(e, \dfrac{1}{e}\right)$.

$\therefore\; f(x) \leqslant \dfrac{1}{e}$ for all $x > 0$, and so $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$.

18 **a** $f(x) = x - \ln x$

$\therefore\ f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$

So, the sign diagram of $f'(x)$ is:



$f(1) = 1 - \ln 1$
$= 1 - 0$
$= 1$

$\therefore\ f(x)$ has a local minimum at $(1, 1)$. This is the only turning point.

b $f(x) \geqslant 1$ for all $x > 0$

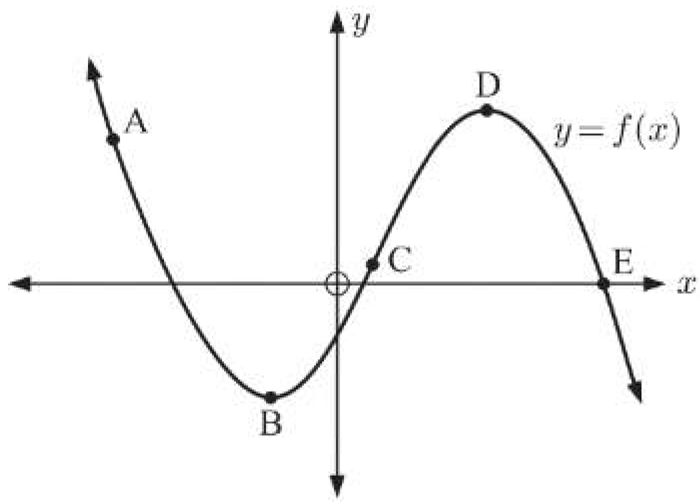
$\therefore\ x - \ln x \geqslant 1$

$\therefore\ \ln x \leqslant x - 1$ for all $x > 0$

EXERCISE 19E

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	$+$	$-$	$+$
B	$-$	0	$+$
C	$+$	$+$	0
D	$+$	0	$-$
E	0	$-$	$-$



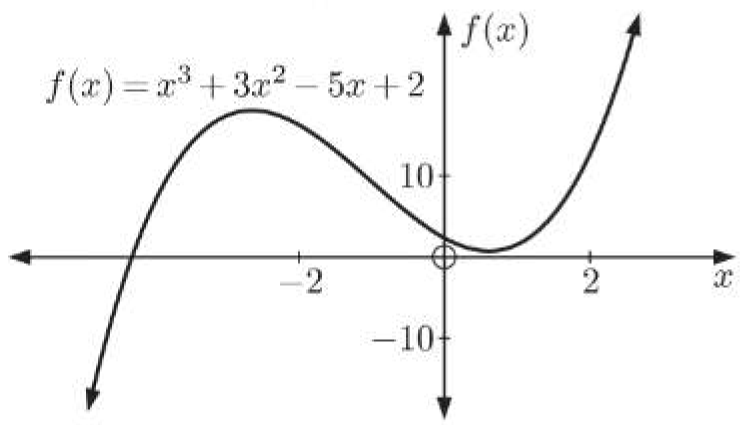
b B is a local minimum, D is a local maximum.

c The shape of $y = f(x)$ changes at C.

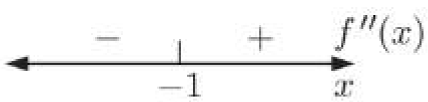
2 **a** $f(x) = x^3 + 3x^2 - 5x + 2$

$\therefore\ f'(x) = 3x^2 + 6x - 5$

$\therefore\ f''(x) = 6x + 6$



b $f''(x)$ has sign diagram:



c **i** The curve is concave up for $x \geqslant -1$.

ii The curve is concave down for $x \leqslant -1$.

3 **a** $y = 2x^2 - 3x + 4$ is a quadratic with $a = 2 > 0$.

$\therefore$ the quadratic has shape 

$\therefore$ the quadratic is concave up.

b $y = -2(x-3)(x+1)$ is a quadratic with $a = -2 < 0$.

$\therefore$ the quadratic has shape 

$\therefore$ the quadratic is concave down.

c $y = -4 - x^2 + 6x$ is a quadratic with $a = -1 < 0$.

$\therefore$ the quadratic has shape 

$\therefore$ the quadratic is concave down.

d $y = (5 - x)(1 - 2x)$
$= 5 - 10x - x + 2x^2$
$= 2x^2 - 11x + 5$

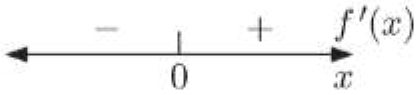
is a quadratic with $a = 2 > 0$.

$\therefore$ the quadratic has shape 

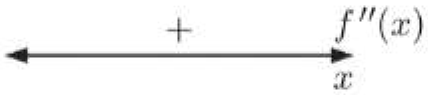
$\therefore$ the quadratic is concave up.

4 a $f(x) = x^2 + 1$

$\therefore$ $f'(x) = 2x$ which has sign diagram:



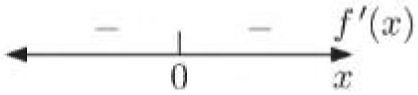
$\therefore$ $f''(x) = 2$ which has sign diagram:



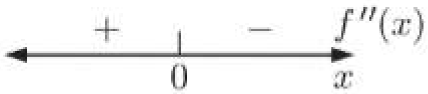
i $f(x)$ is increasing for $x \geqslant 0$.
ii $f(x)$ is decreasing for $x \leqslant 0$.
iii $f(x)$ is concave upwards for all $x \in \mathbb{R}$.
iv $f(x)$ is never concave downwards.

b $f(x) = -x^3$

$\therefore$ $f'(x) = -3x^2$ which has sign diagram:



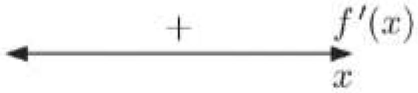
$\therefore$ $f''(x) = -6x$ which has sign diagram:



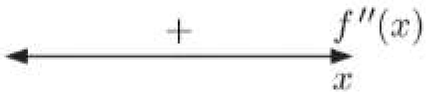
i $f(x)$ is never increasing.
ii $f(x)$ is decreasing for all $x \in \mathbb{R}$.
iii $f(x)$ is concave upwards for $x \leqslant 0$.
iv $f(x)$ is concave downwards for $x \geqslant 0$.

c $f(x) = e^x$

$\therefore$ $f'(x) = e^x$ which has sign diagram:



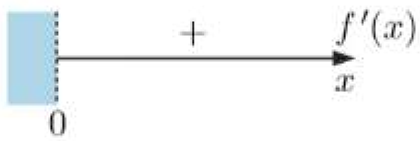
$\therefore$ $f''(x) = e^x$ which has sign diagram:



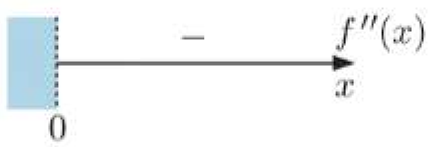
i $f(x)$ is increasing for all $x \in \mathbb{R}$.
ii $f(x)$ is never decreasing.
iii $f(x)$ is concave upwards for all $x \in \mathbb{R}$.
iv $f(x)$ is never concave downwards.

d $f(x) = \sqrt{x} - 2 = x^{\frac{1}{2}} - 2$

$\therefore$ $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ which has sign diagram:



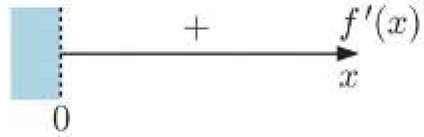
$\therefore$ $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\dfrac{1}{4x\sqrt{x}}$ which has sign diagram:



i $f(x)$ is increasing for $x > 0$.
ii $f(x)$ is never decreasing.
iii $f(x)$ is never concave upwards.
iv $f(x)$ is concave downwards for $x > 0$.

e $\quad f(x) = -\dfrac{1}{\sqrt{x}} = -x^{-\frac{1}{2}}$

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{2x\sqrt{x}}$ which has sign diagram:

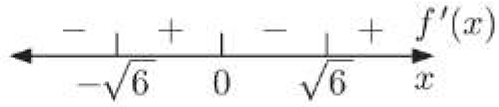


$\therefore\ f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = -\dfrac{3}{4x^2\sqrt{x}}$ which has sign diagram:

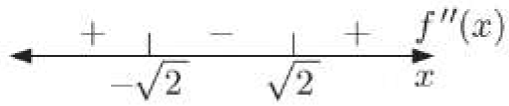
i $f(x)$ is increasing for $x > 0$.
ii $f(x)$ is never decreasing.
iii $f(x)$ is never concave upwards.
iv $f(x)$ is concave downwards for $x > 0$.

f $\quad f(x) = x^4 - 12x^2$

$$\begin{aligned}\therefore\ f'(x) &= 4x^3 - 24x \\ &= 4x(x^2-6) \\ &= 4x(x+\sqrt{6})(x-\sqrt{6})\end{aligned}$$ which has sign diagram:



$$\begin{aligned}\therefore\ f''(x) &= 12x^2 - 24 \\ &= 12(x^2-2) \\ &= 12(x+\sqrt{2})(x-\sqrt{2})\end{aligned}$$ which has sign diagram:



i $f(x)$ is increasing for $-\sqrt{6} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{6}$.
ii $f(x)$ is decreasing for $x \leqslant -\sqrt{6}$ and $0 \leqslant x \leqslant \sqrt{6}$.
iii $f(x)$ is concave upwards for all $x \leqslant -\sqrt{2}$ and $x \geqslant \sqrt{2}$.
iv $f(x)$ is concave downwards for $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$.

5 **a** $f(x) = e^{2x} - 3$

The x-intercept occurs when $f(x) = 0$

$$\begin{aligned}\therefore\ e^{2x} - 3 &= 0 \\ \therefore\ e^{2x} &= 3 \\ \therefore\ 2x &= \ln 3 \\ \therefore\ x &= \frac{\ln 3}{2} \\ &= \tfrac{1}{2}\ln 3 \\ &= \ln 3^{\frac{1}{2}} \\ &= \ln\sqrt{3}\end{aligned}$$

The y-intercept occurs when $x = 0$
$f(0) = e^0 - 3 = -2$

$\therefore$ the x-intercept is $\ln\sqrt{3}$ and the y-intercept is -2.

b $f'(x) = 2e^{2x}$

Now $e^{2x} > 0$ for all x,
so $f'(x) > 0$ for all x.

$\therefore$ the function is increasing for all x.

c $f''(x) = 2e^{2x}(2)$
$\qquad = 4e^{2x}$ which is > 0 for all x.

$\therefore\ f(x)$ is concave up for all x.

6 $f(x) = \ln(2x-1) - 3$

a The x-intercept occurs when $y = 0$

$$\begin{aligned} \therefore\ \ln(2x-1) - 3 &= 0 \\ \therefore\ \ln(2x-1) &= 3 \\ \therefore\ 2x - 1 &= e^3 \\ \therefore\ 2x &= e^3 + 1 \\ \therefore\ x &= \frac{e^3+1}{2} \approx 10.5 \end{aligned}$$

$\therefore$ the x-intercept is $\frac{e^3+1}{2}$.

b $f(0)$ cannot be found as $\ln(-1)$ is not defined.

$\therefore$ there is no y-intercept.

c $f(x) = \ln(2x-1) - 3$ is defined when $2x - 1 > 0$

$$\therefore\ 2x > 1$$
$$\therefore\ x > \tfrac{1}{2}$$

$\therefore$ domain of $f = \{x \mid x > \frac{1}{2}\}$

d $f'(x) = \dfrac{2}{2x-1}$

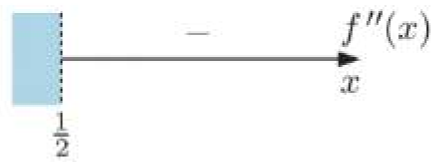
$\therefore\ f'(1) = \dfrac{2}{2(1)-1} = 2$

$\therefore$ the tangent to the curve at $x = 1$ has gradient 2.

e $f'(x) = 2(2x-1)^{-1}$

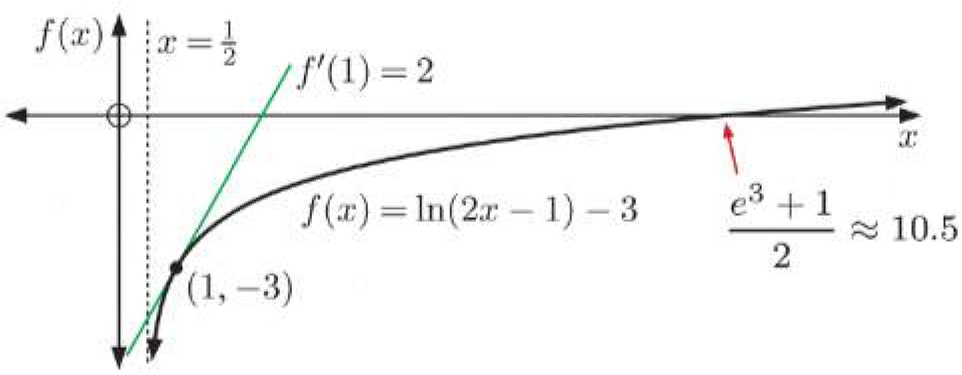
$\therefore\ f''(x) = -2(2x-1)^{-2}(2)$ {chain rule}

$= -\dfrac{4}{(2x-1)^2},\quad x > \frac{1}{2}$

$\therefore$ $f''(x)$ has sign diagram: 

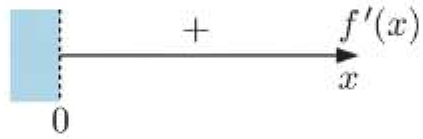
$\therefore$ $f''(x) < 0$ for all $x > \frac{1}{2}$, so $f(x)$ is concave down for all x in the domain of f.

f



7 **a** $f(x) = \ln x$ is defined when $x > 0$.

b $f'(x) = \dfrac{1}{x}$ which has sign diagram:

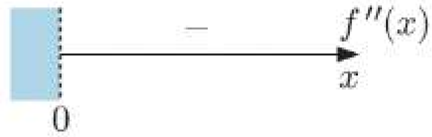


$\therefore$ $f(x)$ is increasing for $x > 0$.

$f'(x) = x^{-1}$

$\therefore$ $f''(x) = -x^{-2}$

$= -\dfrac{1}{x^2}$ which has sign diagram:



$\therefore$ $f(x)$ is concave down for $x > 0$.

c When $y = 1$, $\ln x = 1$

$\therefore$ $x = e$

So, the point of contact is $(e, 1)$.

Now $f'(e) = \dfrac{1}{e}$.

$\therefore$ the normal at $(e, 1)$ has gradient $-e$.

$\therefore$ the normal has equation $y = -e(x - e) + 1$

$\therefore$ $y = -ex + e^2 + 1$

$\therefore$ $ex + y = e^2 + 1$

8 **a** $f(x) = \dfrac{e^x}{x} \neq 0$ since $e^x \neq 0$ for all x $\therefore$ there is no x-intercept.

Also, $f(x)$ is not defined when $x = 0$ $\therefore$ there is no y-intercept.

$\therefore$ the graph of $y = f(x)$ does not have any x or y-intercepts.

b As $x \to \infty$, $e^x \to \infty$ (at a much faster rate than x)

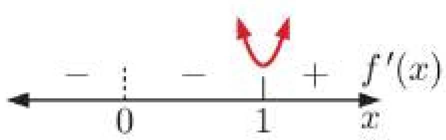
$\therefore$ as $x \to \infty$, $f(x) = \dfrac{e^x}{x} \to \infty$

As $x \to -\infty$, $e^x \to 0^+$

$\therefore$ as $x \to -\infty$, $f(x) = \dfrac{e^x}{x} \to 0^-$ $\{f(x) < 0$ for $x < 0\}$

c $f'(x) = \dfrac{e^x x - e^x(1)}{x^2}$ {quotient rule}

$= \dfrac{e^x(x-1)}{x^2}$ which has sign diagram:



Now $f(1) = \dfrac{e^1}{1} = e$

$\therefore$ $(1, e)$ is a local minimum.

d $f'(x) = \dfrac{e^x x - e^x}{x^2}$

$= \dfrac{e^x}{x} - \dfrac{e^x}{x^2}$

$\therefore\ f''(x) = \dfrac{e^x x - e^x(1)}{x^2} - \left(\dfrac{e^x x^2 - e^x(2x)}{(x^2)^2}\right)$ {quotient rule twice}

$= \dfrac{e^x x - e^x}{x^2} - \left(\dfrac{e^x x^2 - 2e^x x}{x^4}\right)$

$= \dfrac{e^x x - e^x}{x^2} - \left(\dfrac{e^x x - 2e^x}{x^3}\right)$

$= \dfrac{e^x x^2 - e^x x - (e^x x - 2e^x)}{x^3}$

$= \dfrac{e^x x^2 - e^x x - e^x x + 2e^x}{x^3}$

$= \dfrac{e^x x^2 - 2e^x x + 2e^x}{x^3}$

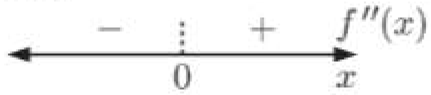
$= \dfrac{e^x(x^2 - 2x + 2)}{x^3}$

Now consider $x^2 - 2x + 2$: $\Delta = (-2)^2 - 4(1)(2)$

$= -4 < 0$

$\therefore\ x^2 - 2x + 2$ has no real roots.

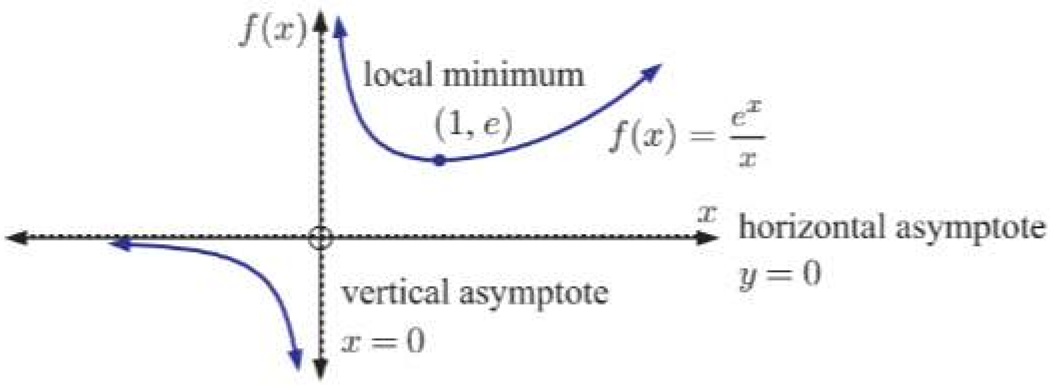
$\therefore\ f''(x)$ has sign diagram:



i $f(x)$ is concave up for $x > 0$.

ii $f(x)$ is concave down for $x < 0$.

e



f $f(-1) = \dfrac{e^{-1}}{-1} = -\dfrac{1}{e}$

$\therefore$ the point of contact is $\left(-1, -\frac{1}{e}\right)$.

Now $f'(-1) = \dfrac{e^{-1}(-1-1)}{(-1)^2}$

$= -\dfrac{2}{e}$

So, the tangent has equation $y = -\dfrac{2}{e}(x+1) - \dfrac{1}{e}$

$\therefore\ ey = -2(x+1) - 1$

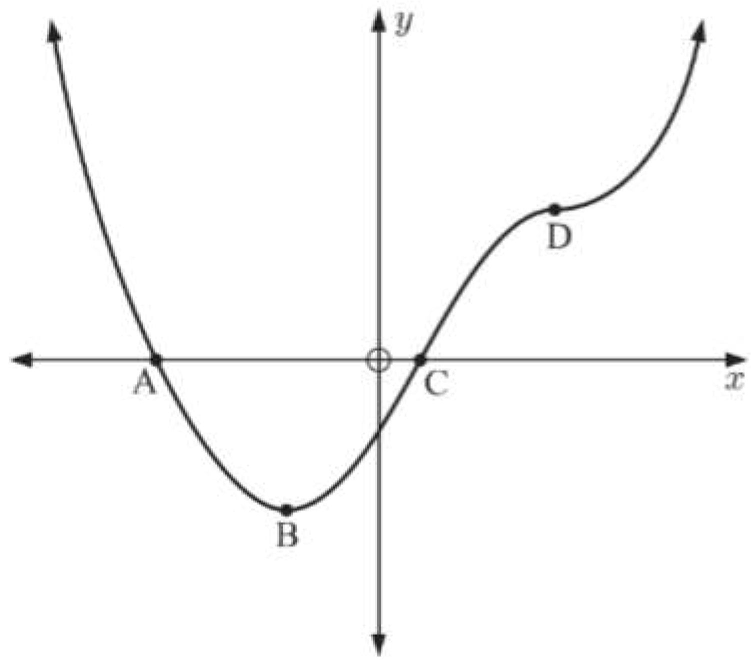
$\therefore\ ey = -2x - 2 - 1$

$\therefore\ 2x + ey = -3$

EXERCISE 19F

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	0	$-$	$+$
B	$-$	0	$+$
C	0	$+$	0
D	$+$	0	0

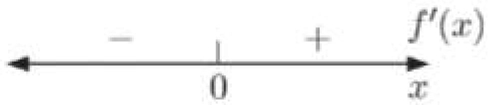


b The turning point B is a local minimum.

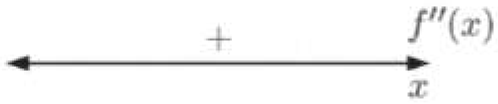
c C is a non-stationary inflection point, and D is a stationary inflection point.

2 **a** $f(x) = x^2 + 3$

$\therefore\ f'(x) = 2x$



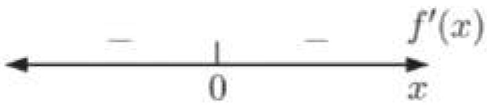
$\therefore\ f''(x) = 2$



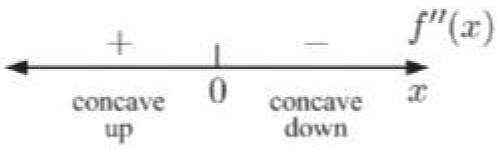
$f''(x) \neq 0$, so there are no points of inflection.

b $f(x) = 2 - x^3$

$\therefore\ f'(x) = -3x^2$



$\therefore\ f''(x) = -6x$



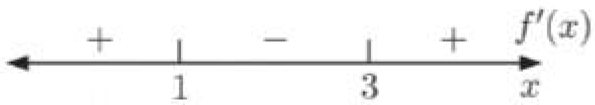
Since the sign of $f''(x)$ changes at $x = 0$, this is a point of inflection.

$f(0) = 2$ and $f'(0) = 0$

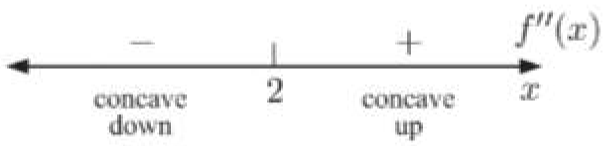
$\therefore$ $(0, 2)$ is a stationary inflection.

c $f(x) = x^3 - 6x^2 + 9x + 1$

$$\begin{aligned} \therefore\ f'(x) &= 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x-1)(x-3) \end{aligned}$$



$$\begin{aligned} \therefore\ f''(x) &= 6x - 12 \\ &= 6(x-2) \end{aligned}$$



Since the sign of $f''(x)$ changes about $x = 2$, this is a point of inflection.

$$\begin{aligned} f(2) &= 2^3 - 6(2)^2 + 9(2) + 1 \\ &= 8 - 24 + 18 + 1 \\ &= 3 \end{aligned} \qquad \text{and} \qquad f'(2) = 3(1)(-1) \neq 0$$

$\therefore$ $(2, 3)$ is a non-stationary inflection.

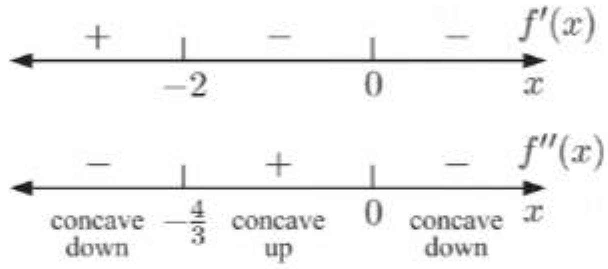
d $\quad f(x) = -3x^4 - 8x^3 + 2$

$\therefore\ f'(x) = -12x^3 - 24x^2$

$= -12x^2(x+2)$

$\therefore\ f''(x) = -36x^2 - 48x$

$= -12x(3x+4)$



Since the signs of $f''(x)$ change about $x = -\frac{4}{3}$ and $x = 0$, both of these points are points of inflection.

$f\left(-\frac{4}{3}\right) = -3\left(-\frac{4}{3}\right)^4 - 8\left(-\frac{4}{3}\right)^3 + 2 = \frac{310}{27}$

and $f'\left(-\frac{4}{3}\right) = -12\left(-\frac{4}{3}\right)^3 - 24\left(-\frac{4}{3}\right)^2 \neq 0$

$\therefore\ \left(-1\frac{1}{3},\ 11\frac{13}{27}\right)$ is a non-stationary inflection.

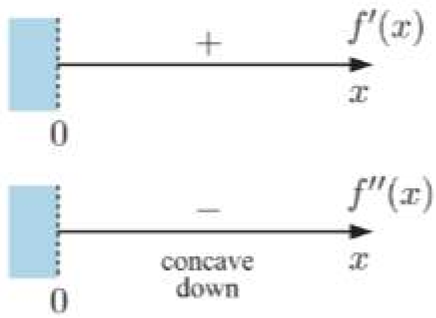
$f(0) = 2$ and $f'(0) = 0$

$\therefore\ (0,\ 2)$ is a stationary inflection.

e $\quad f(x) = 3 - \dfrac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$\therefore\ f'(x) = \frac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{2x\sqrt{x}}$

$\therefore\ f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = -\dfrac{3}{4x^2\sqrt{x}}$



$f''(x) \neq 0$, so there are no points of inflection.

f $\quad f(x) = x^3 + 6x^2 + 12x + 5$

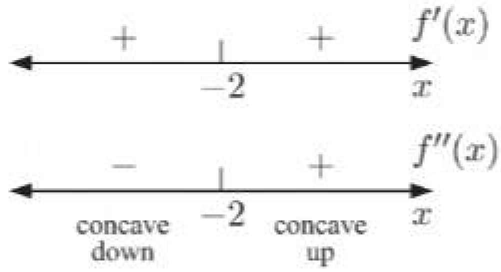
$\therefore\ f'(x) = 3x^2 + 12x + 12$

$= 3(x^2 + 4x + 4)$

$= 3(x+2)^2$

$\therefore\ f''(x) = 6x + 12$

$= 6(x+2)$



Since the sign of $f''(x)$ changes at $x = -2$, this is a point of inflection.

$f(-2) = (-2)^3 + 6(-2)^2 + 12(-2) + 5$ $\quad$ and $\quad$ $f'(-2) = 3(0)^2 = 0$

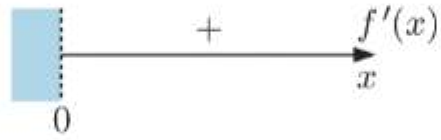
$= -8 + 24 - 24 + 5$

$= -3$

$\therefore\ (-2,\ -3)$ is a stationary inflection.

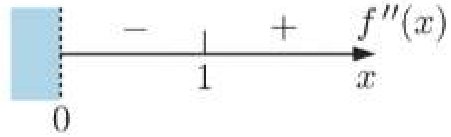
g $f(x) = x^2 + 8\sqrt{x} = x^2 + 8x^{\frac{1}{2}}$

$\therefore\ f'(x) = 2x + 4x^{-\frac{1}{2}} = 2x + \frac{4}{\sqrt{x}}$



$\therefore\ f''(x) = 2 - 2x^{-\frac{3}{2}} = 2 - \frac{2}{x\sqrt{x}}$

$= 2\left(1 - \frac{1}{x\sqrt{x}}\right)$



Since the sign of $f''(x)$ changes at $x = 1$, this is a point of inflection.

$f(1) = 1^2 + 8\sqrt{1}$ and $f'(1) = 2(1) + \frac{4}{\sqrt{1}} \neq 0$

$= 1 + 8$

$= 9$

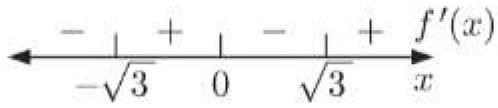
$\therefore$ $(1, 9)$ is a non-stationary inflection.

h $f(x) = x^4 - 6x^2 + 10$

$\therefore\ f'(x) = 4x^3 - 12x$

$= 4x(x^2 - 3)$

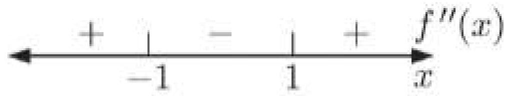
$= 4x(x + \sqrt{3})(x - \sqrt{3})$



$\therefore\ f''(x) = 12x^2 - 12$

$= 12(x^2 - 1)$

$= 12(x + 1)(x - 1)$



Since the sign of $f''(x)$ changes at $x = -1$ and $x = 1$, both of these points are points of inflection.

$f(-1) = (-1)^4 - 6(-1)^2 + 10$

$= 1 - 6 + 10$

$= 5$

and $f'(-1) = 4(-1)^3 - 12(-1) \neq 0$

$f(1) = 1^4 - 6(1)^2 + 10$

$= 1 - 6 + 10$

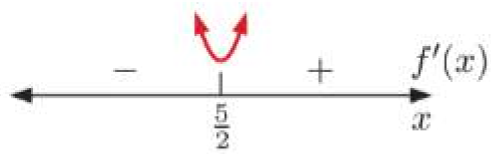
$= 5$

and $f'(1) = 4(1)^3 - 12(1) \neq 0$

$\therefore$ $(-1, 5)$ and $(1, 5)$ are non-stationary inflection points.

3 a i $f(x) = x^2 - 5x + 4$

$\therefore\ f'(x) = 2x - 5$ $\therefore$ $f'(x)$ has sign diagram:



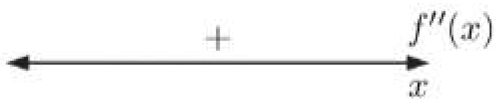
Now $f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 4$

$= \frac{25}{4} - \frac{25}{2} + 4$

$= -\frac{9}{4}$

$\therefore$ $\left(\frac{5}{2}, -\frac{9}{4}\right)$ is a local minimum.

ii $f''(x) = 2$ $\therefore$ $f''(x)$ has sign diagram:

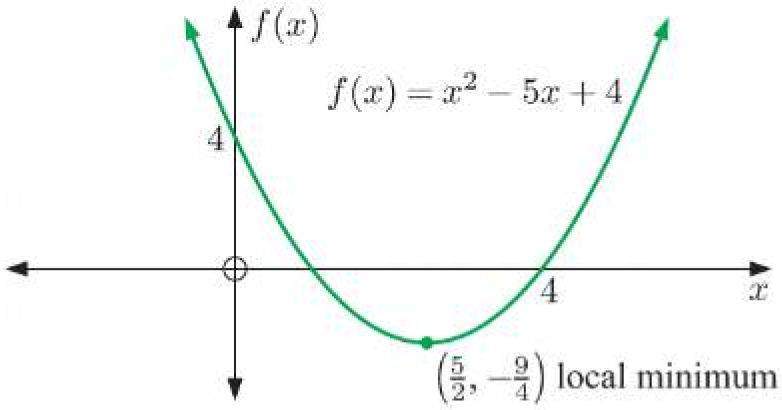


$f''(x) \neq 0$, so there are no points of inflection.

iii $f(x)$ is increasing for $x \geqslant \frac{5}{2}$, and decreasing for $x \leqslant \frac{5}{2}$.

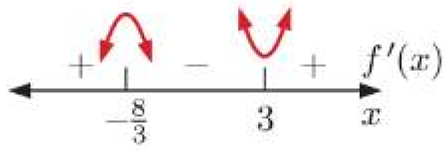
iv $f(x)$ is concave up for all $x \in \mathbb{R}$, and never concave down.

v



b **i** $f(x) = x^3 + 4x^2$

$\therefore\ f'(x) = 3x^2 + 8x \qquad \therefore\ f'(x)$ has sign diagram:



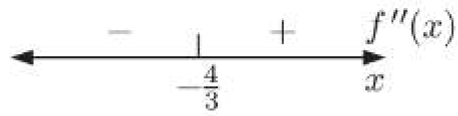
$= x(3x+8)$

Now $f(0) = 0$, $f\left(-\frac{8}{3}\right) = \left(-\frac{8}{3}\right)^3 + 4\left(-\frac{8}{3}\right)^2$

$= \frac{256}{27}$

$\therefore\ \left(-\frac{8}{3}, \frac{256}{27}\right)$ is a local maximum, $(0, 0)$ is a local minimum.

ii $f''(x) = 6x + 8 \qquad \therefore\ f''(x)$ has sign diagram:



Now $f\left(-\frac{4}{3}\right) = \left(-\frac{4}{3}\right)^3 + 4\left(-\frac{4}{3}\right)^2$ and $f'\left(-\frac{4}{3}\right) \neq 0$

$= -\frac{64}{27} + \frac{64}{9}$

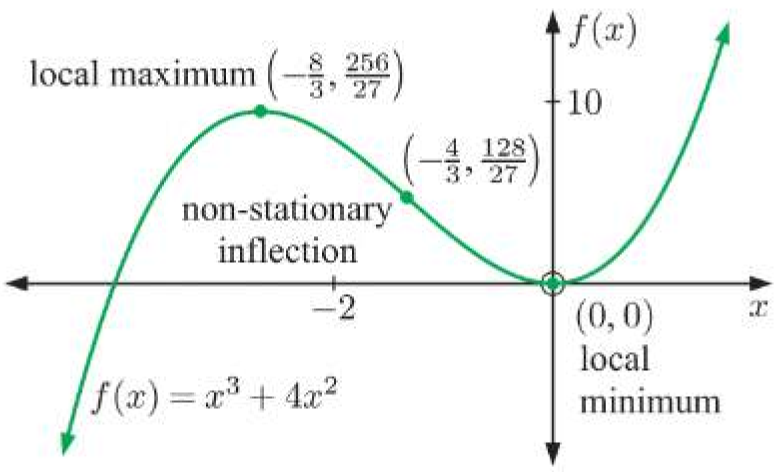
$= \frac{128}{27}$

$\therefore\ \left(-\frac{4}{3}, \frac{128}{27}\right)$ is a non-stationary inflection.

iii $f(x)$ is increasing for all $x \leqslant -\frac{8}{3}$ and $x \geqslant 0$, and decreasing for $-\frac{8}{3} \leqslant x \leqslant 0$.

iv $f(x)$ is concave up for all $x \geqslant -\frac{4}{3}$, and concave down for $x \leqslant -\frac{4}{3}$.

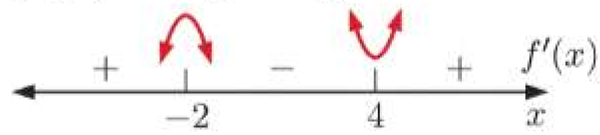
v



c **i** $f(x) = x^3 - 3x^2 - 24x + 1$

$\therefore \; f'(x) = 3x^2 - 6x - 24$
$= 3(x^2 - 2x - 8)$
$= 3(x+2)(x-4)$

$\therefore \; f'(x)$ has sign diagram:



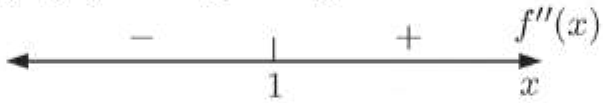
Now $f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 1$
$= -8 - 12 + 48 + 1$
$= 29$

and $f(4) = 4^3 - 3(4)^2 - 24(4) + 1$
$= 64 - 48 - 96 + 1$
$= -79$

$\therefore$ $(-2, 29)$ is a local maximum, and $(4, -79)$ is a local minimum.

ii $f''(x) = 6x - 6$
$= 6(x-1)$

$\therefore \; f''(x)$ has sign diagram:



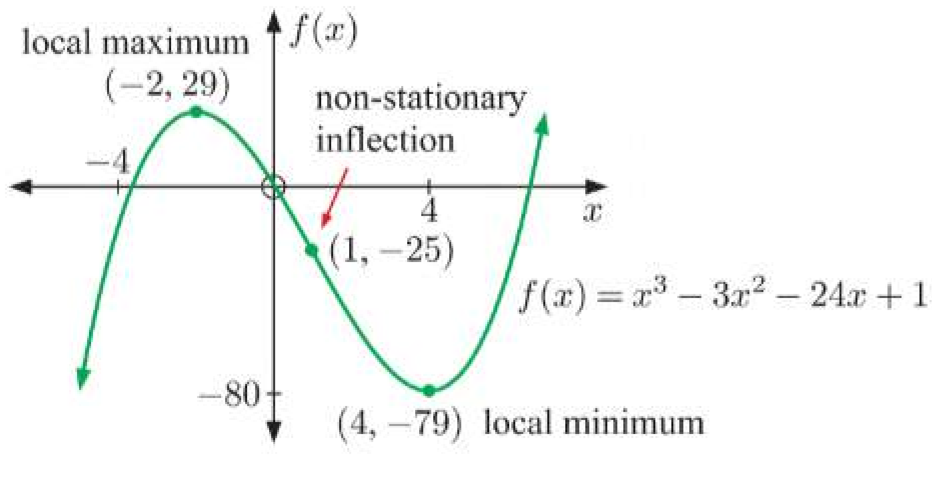
$f(1) = 1^3 - 3(1)^2 - 24(1) + 1$ and $f'(1) \neq 0$
$= 1 - 3 - 24 + 1$
$= -25$

$\therefore$ $(1, -25)$ is a non-stationary point of inflection.

iii $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 4$, and decreasing for $-2 \leqslant x \leqslant 4$.

iv $f(x)$ is concave down for $x \leqslant 1$, and concave up for $x \geqslant 1$.

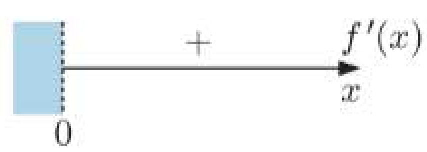
v



d **i** $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\therefore \; f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
$= \dfrac{1}{2\sqrt{x}}$

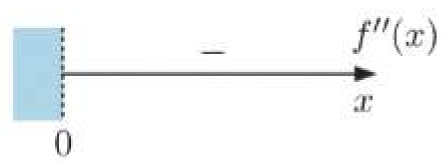
$\therefore \; f'(x)$ has sign diagram:



$\therefore$ there are no turning points.

ii $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$
$= -\dfrac{1}{4x\sqrt{x}}$

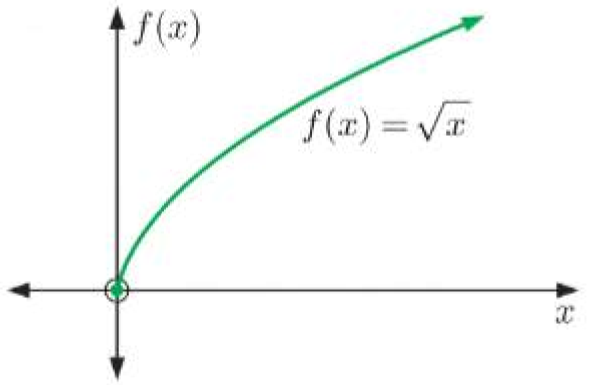
$\therefore \; f''(x)$ has sign diagram:



$f''(x) \neq 0$, so there are no points of inflection.

iii $f(x)$ is increasing for $x > 0$, and never decreasing.

iv $f(x)$ is concave down for $x > 0$, and never concave up.

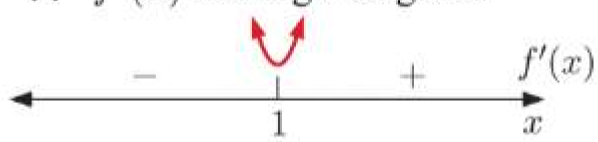
v 

e **i** $f(x) = (x-1)^4$

$\therefore\ f'(x) = 4(x-1)^3(1)$ {chain rule}

$= 4(x-1)^3$

$\therefore\ f'(x)$ has sign diagram:



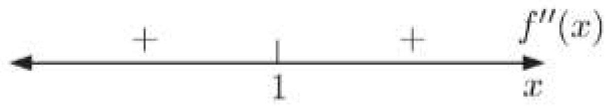
Now $f(1) = (1-1)^4 = 0$

$\therefore$ $(1, 0)$ is a local minimum.

ii $f''(x) = 12(x-1)^2(1)$ {chain rule}

$= 12(x-1)^2$

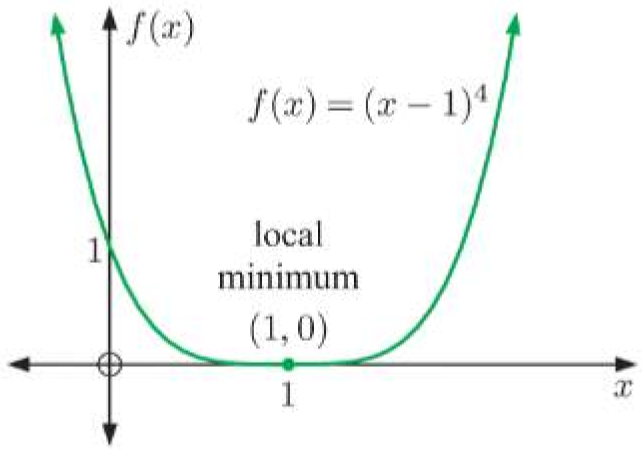
$\therefore\ f''(x)$ has sign diagram:



There is no change in sign of $f''(x)$, so there are no points of inflection.

iii $f(x)$ is increasing for $x \geqslant 1$, and decreasing for $x \leqslant 1$.

iv $f(x)$ is concave up for all $x \in \mathbb{R}$, and never concave down.

v 

f **i** $f(x) = 3x^4 + 4x^3 - 2$

$\therefore\ f'(x) = 12x^3 + 12x^2$

$= 12x^2(x+1)$

$\therefore\ f'(x)$ has sign diagram:

Now $f(-1) = 3(-1)^4 + 4(-1)^3 - 2$ and $f(0) = -2$

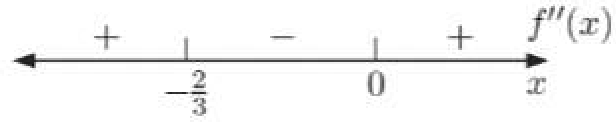
$= 3 - 4 - 2$

$= -3$

$\therefore$ $(-1, -3)$ is a local minimum, and $(0, -2)$ is a point of inflection but not a turning point.

ii $f''(x) = 36x^2 + 24x$
$= 12x(3x+2)$

$\therefore$ $f''(x)$ has sign diagram:



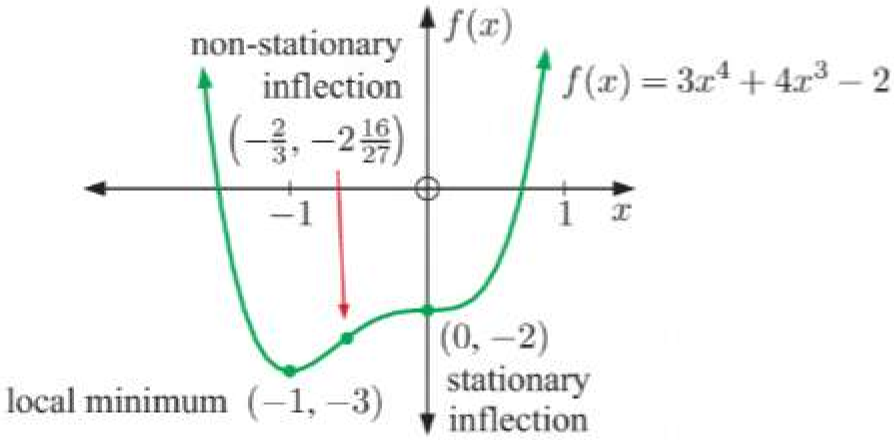
Now $f\left(-\frac{2}{3}\right) = 3\left(-\frac{2}{3}\right)^4 + 4\left(-\frac{2}{3}\right)^3 - 2$ and $f'\left(-\frac{2}{3}\right) \neq 0$
$= -\frac{70}{27}$

$\therefore$ $\left(-\frac{2}{3}, -2\frac{16}{27}\right)$ is a non-stationary inflection, and $(0, -2)$ is a stationary inflection.

iii $f(x)$ is increasing for $x \geqslant -1$, and decreasing for $x \leqslant -1$.

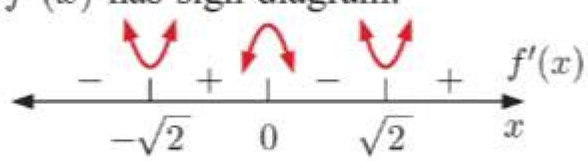
iv $f(x)$ is concave down for $-\frac{2}{3} \leqslant x \leqslant 0$, and concave up for $x \leqslant -\frac{2}{3}$ and $x \geqslant 0$.

v



g **i** $f(x) = x^4 - 4x^2 + 3$
$\therefore$ $f'(x) = 4x^3 - 8x$
$= 4x(x^2 - 2)$
$= 4x(x+\sqrt{2})(x-\sqrt{2})$

$\therefore$ $f'(x)$ has sign diagram:



Now $f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 + 3$
$= 4 - 8 + 3$
$= -1$
$f(0) = 3$
and $f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 3$
$= -1$

$\therefore$ $(-\sqrt{2}, -1)$ and $(\sqrt{2}, -1)$ are local minima, and $(0, 3)$ is a local maximum.

ii $f''(x) = 12x^2 - 8$
$= 4(3x^2 - 2)$
$= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$

$\therefore$ $f''(x)$ has sign diagram:

Now $f\left(-\sqrt{\frac{2}{3}}\right) = \left(-\sqrt{\frac{2}{3}}\right)^4 - 4\left(-\sqrt{\frac{2}{3}}\right)^2 + 3$ and $f'\left(-\sqrt{\frac{2}{3}}\right) \neq 0$
$= \frac{4}{9} - \frac{8}{3} + 3$
$= \frac{7}{9}$

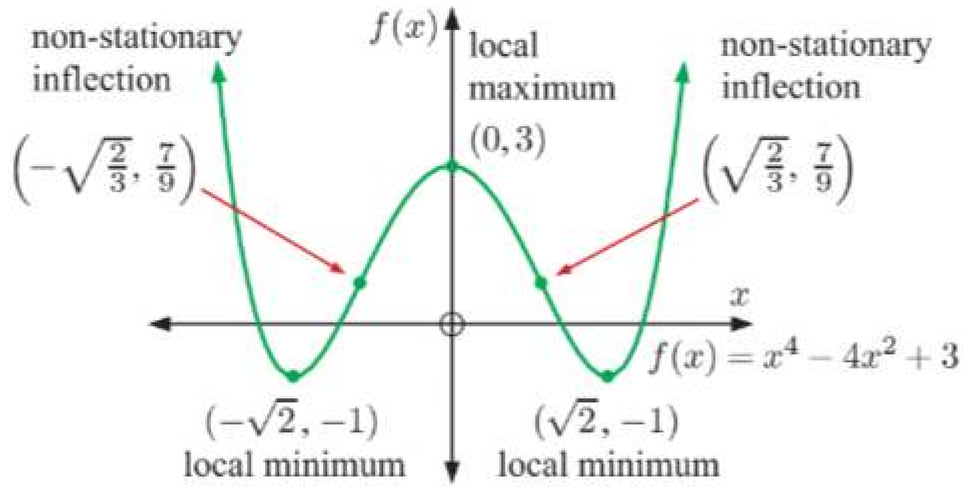
$f\left(\sqrt{\frac{2}{3}}\right) = \frac{7}{9}$ and $f'\left(\sqrt{\frac{2}{3}}\right) \neq 0$

$\therefore$ $\left(\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$ and $\left(-\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$ are non-stationary inflections.

iii $f(x)$ is increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$, and decreasing for $x \leqslant -\sqrt{2}$ and $0 \leqslant x \leqslant \sqrt{2}$.

iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$, and concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$.

v



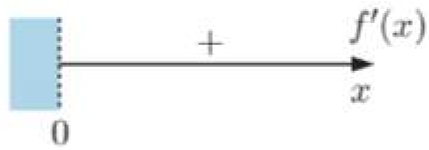
h **i** $f(x) = 3 - \dfrac{4}{\sqrt{x}} = 3 - 4x^{-\frac{1}{2}}$

$\therefore \; f'(x) = -4\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$= 2x^{-\frac{3}{2}}$

$= \dfrac{2}{x\sqrt{x}}$

$\therefore \; f'(x)$ has sign diagram:



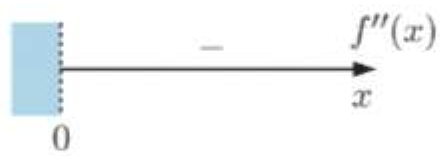
$f'(x) \neq 0$, so there are no turning points.

ii $f''(x) = 2\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$

$= -3x^{-\frac{5}{2}}$

$= -\dfrac{3}{x^2\sqrt{x}}$

$\therefore \; f''(x)$ has sign diagram:

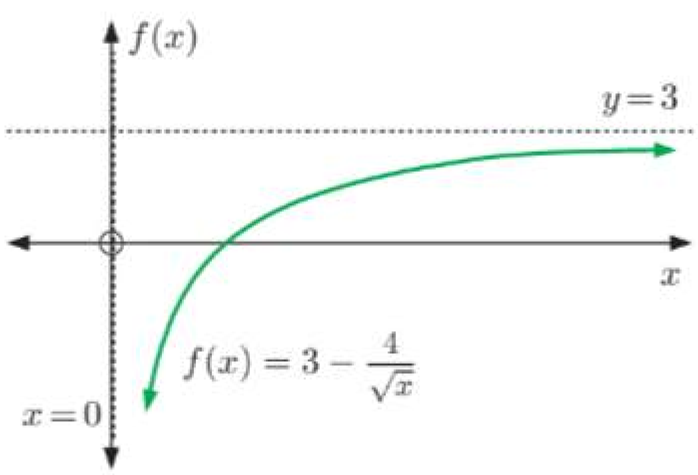


$f''(0) \neq 0$, so there are no points of inflection.

iii $f(x)$ is increasing for $x > 0$, and never decreasing.

iv $f(x)$ is concave down for $x > 0$, and never concave up.

v

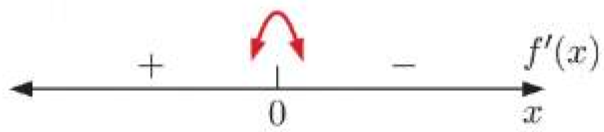


4 **a** $$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$$

$$\therefore\ f'(x) = \frac{1}{\sqrt{2\pi}}\,(-x)e^{-\frac{1}{2}x^2}$$

$$= -\frac{1}{\sqrt{2\pi}}\, xe^{-\frac{1}{2}x^2} \quad \text{where } e^{-\frac{1}{2}x^2} \text{ is positive for all } x$$

So, $f'(x)$ has sign diagram:



Now $$f(0) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}}$$

$\therefore$ $\left(0, \dfrac{1}{\sqrt{2\pi}}\right)$ is a local maximum.

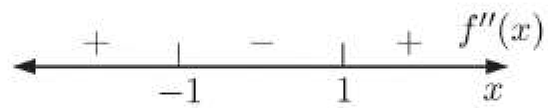
b Using the sign diagram from **a**, we see that $f(x)$ is increasing for $x \leqslant 0$, and decreasing for $x \geqslant 0$.

c $$f''(x) = -\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} + \left(-\frac{1}{\sqrt{2\pi}}\, x(-x)e^{-\frac{1}{2}x^2}\right) \qquad \{\text{product rule}\}$$

$$= -\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} + \frac{1}{\sqrt{2\pi}}\, x^2e^{-\frac{1}{2}x^2}$$

$$= -\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}(1 - x^2)$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}(x^2 - 1)$$ which has sign diagram:



Now $$f(-1) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(-1)^2} = \frac{1}{\sqrt{2\pi}\sqrt{e}} = \frac{1}{\sqrt{2e\pi}}$$ and $$f(1) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{2\pi}\sqrt{e}} = \frac{1}{\sqrt{2e\pi}}$$

Since $f'(-1) \neq 0$ and $f'(1) \neq 0$, then $\left(-1, \dfrac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \dfrac{1}{\sqrt{2e\pi}}\right)$ are non-stationary inflections.

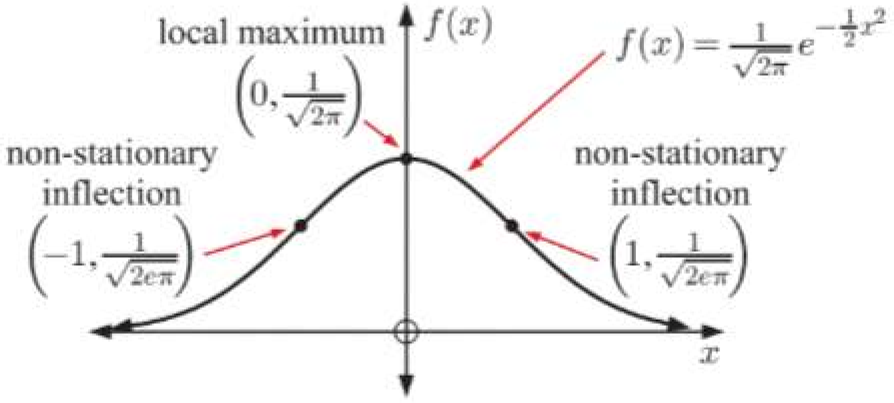
d As $x \to \infty$, $e^{-\frac{1}{2}x^2} \to 0^+$

$\therefore$ as $x \to \infty$, $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \to 0^+$

As $x \to -\infty$, $e^{-\frac{1}{2}x^2} \to 0^+$

$\therefore$ as $x \to -\infty$, $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \to 0^+$

e



5 **a** $f(x) = \cos x$

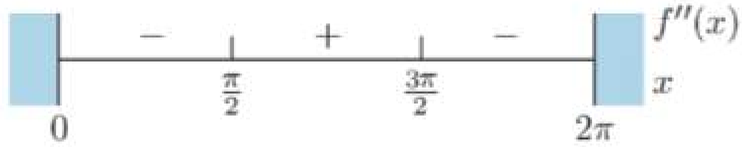
$\therefore\ f'(x) = -\sin x$

$\therefore\ f''(x) = -\cos x = -f(x)$

$\therefore$ the inflection points coincide with the x-intercepts.

b $f''(x) = -\cos x, \quad 0 \leqslant x \leqslant 2\pi$

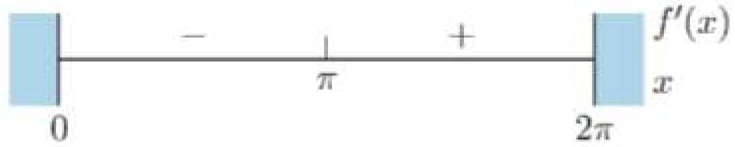
The sign diagram of $f''(x)$ is:



Since $f'(\frac{\pi}{2}) \neq 0$ and $f'(\frac{3\pi}{2}) \neq 0$, then there are non-stationary inflection points at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$.

c $f'(x) = -\sin x, \quad 0 \leqslant x \leqslant 2\pi$

The sign diagram of $f'(x)$ is:



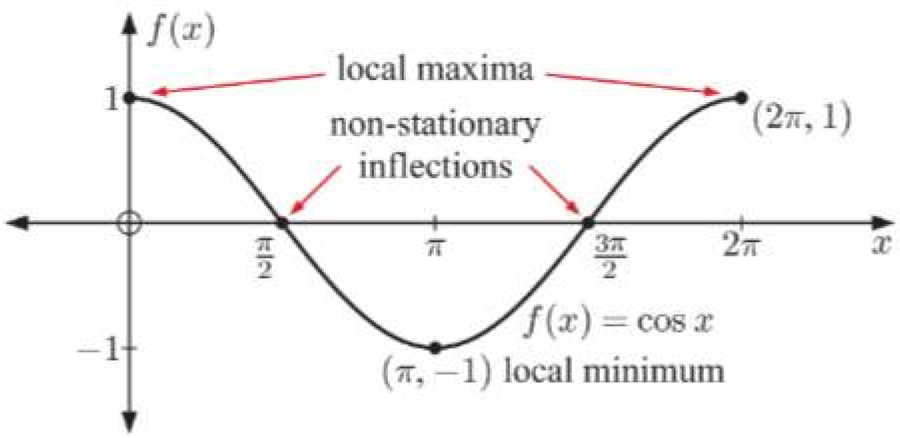
i $f(x)$ is increasing for $\pi \leqslant x \leqslant 2\pi$

ii $f(x)$ is decreasing for $0 \leqslant x \leqslant \pi$

iii $f(x)$ is concave up for $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$

iv $f(x)$ is concave down for $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

d



6 $f(t) = Ate^{-bt}, \quad t \geqslant 0, \quad A, b > 0$

a **i** $f'(t) = Ae^{-bt} + Ate^{-bt}(-b)$ $\quad$ {product rule}

$$= Ae^{-bt} - Abte^{-bt}$$
$$= Ae^{-bt}(1 - bt)$$

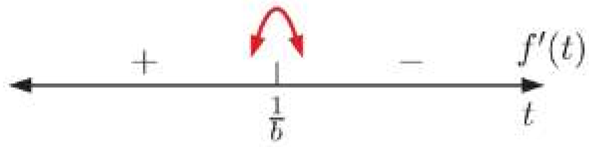
$f'(t) = 0$ when $Ae^{-bt}(1 - bt) = 0$ but $A > 0$ and $e^{-bt} > 0$

$\therefore \; 1 - bt = 0$

$\therefore \; -bt = -1$

$\therefore \; t = \dfrac{1}{b}$ $\quad$ {where $b > 0$}

$\therefore$ the sign diagram of $f'(t)$ is:



$\therefore$ there is a local maximum at $t = \dfrac{1}{b}$.

ii $f''(t) = Ae^{-bt}(-b) - [Abe^{-bt} + Abte^{-bt}(-b)]$ $\quad$ {product rule}

$$= -Abe^{-bt} - (Abe^{-bt} - Ab^2te^{-bt})$$
$$= -2Abe^{-bt} + Ab^2te^{-bt}$$
$$= Abe^{-bt}(bt - 2)$$

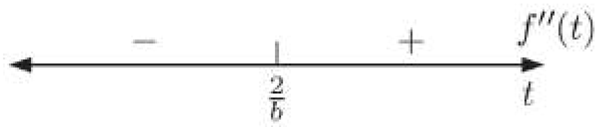
$f''(t) = 0$ when $Abe^{-bt}(bt - 2) = 0$ but $A, b > 0$ and $e^{-bt} > 0$

$\therefore \; bt - 2 = 0$

$\therefore \; bt = 2$

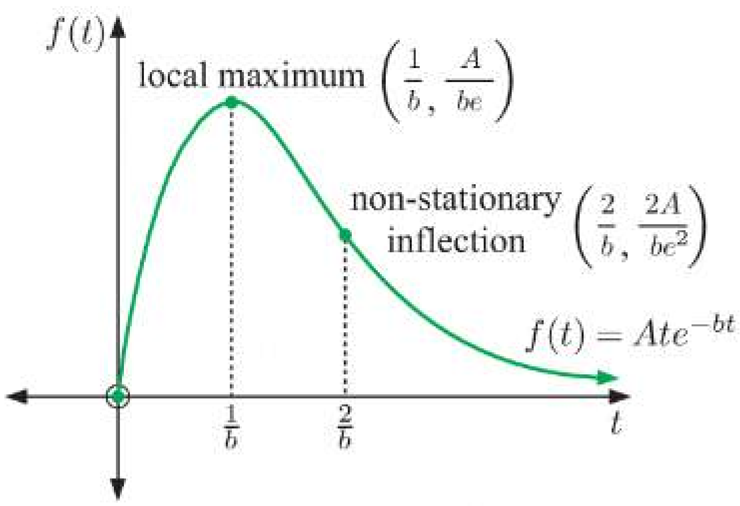
$\therefore \; t = \dfrac{2}{b}$ $\quad$ {where $b > 0$}

$\therefore$ the sign diagram of $f''(t)$ is:



Since $f'\left(\dfrac{2}{b}\right) \neq 0$, there is a non-stationary point of inflection at $t = \dfrac{2}{b}$.

b



7 $f(t) = \dfrac{L}{1 + Ce^{-kt}}, \quad t \geqslant 0, \quad L, C, k > 0$

a The y-intercept occurs when $t = 0$.

Now $f(0) = \dfrac{L}{1 + Ce^{-k(0)}} = \dfrac{L}{1 + C}$

So, the y-intercept is $\dfrac{L}{1 + C}$.

b As $t \to \infty, \quad Ce^{-kt} \to 0^+$

$\therefore \; f(t) = \dfrac{L}{1 + Ce^{-kt}} \to \dfrac{L}{1 + 0^+} = L^-$

$\therefore \; y = L$ is a horizontal asymptote.

c $$f(t) = L(1+Ce^{-kt})^{-1}$$

$$\therefore\ f'(t) = L(-1)(1+Ce^{-kt})^{-2}(-kCe^{-kt}) \qquad \{\text{chain rule}\}$$

$$= LkCe^{-kt}(1+Ce^{-kt})^{-2}$$

$$\therefore\ f''(t) = (-k)LkCe^{-kt}(1+Ce^{-kt})^{-2} + LkCe^{-kt}(-2)(1+Ce^{-kt})^{-3}(-kCe^{-kt})$$

{product rule and chain rule}

$$= -\frac{Lk^2C}{e^{kt}(1+Ce^{-kt})^2} + \frac{2Lk^2C^2}{e^{2kt}(1+Ce^{-kt})^3}$$

$f''(t) = 0$ when $\dfrac{2Lk^2C^2}{e^{2kt}(1+Ce^{-kt})^3} = \dfrac{Lk^2C}{e^{kt}(1+Ce^{-kt})^2}$

$$\therefore\ \frac{2C(Lk^2C)}{e^{kt}e^{kt}(1+Ce^{-kt})^3} = \frac{Lk^2C}{e^{kt}(1+Ce^{-kt})^2}$$

$$\therefore\ \frac{2C}{e^{kt}(1+Ce^{-kt})} = 1$$

$$\therefore\ 2C = e^{kt} + Ce^{-kt}e^{kt}$$

$$\therefore\ 2C = e^{kt} + C$$

$$\therefore\ C = e^{kt}$$

$$\therefore\ \ln C = kt$$

$$\therefore\ t = \frac{\ln C}{k}$$

Since $t \geqslant 0$ and $k > 0$, then $\dfrac{\ln C}{k} \geqslant 0 \quad \therefore\ C > 1$

$$f\left(\frac{\ln C}{k}\right) = \frac{L}{1+Ce^{-k\left(\frac{\ln C}{k}\right)}}$$

$$= \frac{L}{1+Ce^{-\ln C}}$$

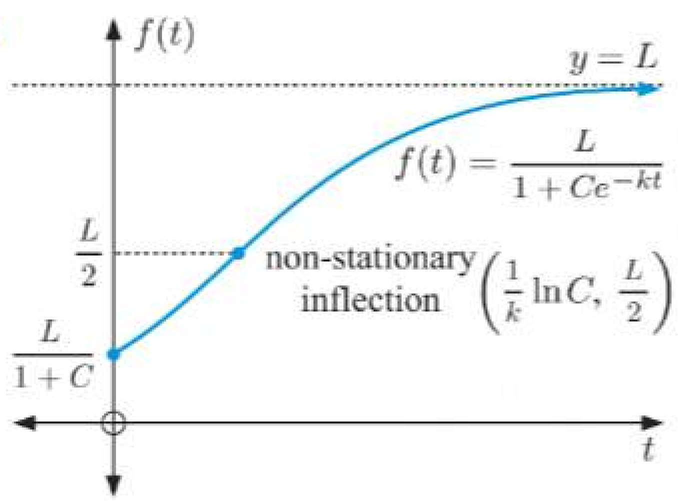
$$= \frac{L}{1+Ce^{\ln C^{-1}}}$$

$$= \frac{L}{1+CC^{-1}}$$

$$= \frac{L}{1+1} = \frac{L}{2}$$

So, if $C > 1$, there is a point of inflection with y-coordinate $\dfrac{L}{2}$.

d



REVIEW SET 19A

1 **a** $y = -2x^2$

When $x = -1$, $y = -2(-1)^2 = -2$

$\therefore$ the point of contact is $(-1, -2)$.

Now $\dfrac{dy}{dx} = -4x$

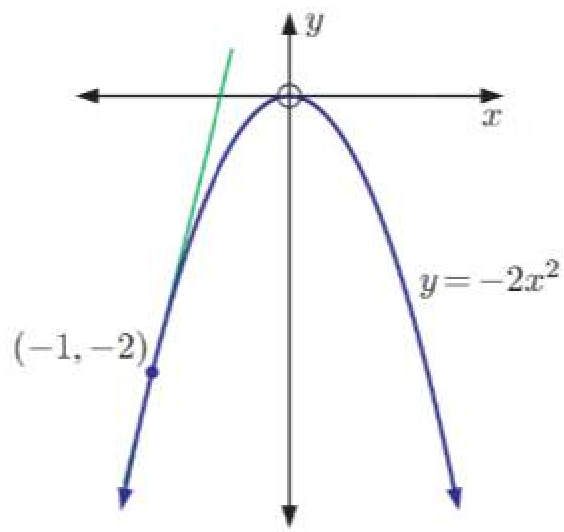
When $x = -1$, $\dfrac{dy}{dx} = -4(-1) = 4$

So, the tangent has equation $y = 4(x - (-1)) - 2$

$\therefore\ y = 4(x + 1) - 2$

$\therefore\ y = 4x + 4 - 2$

$\therefore\ y = 4x + 2$



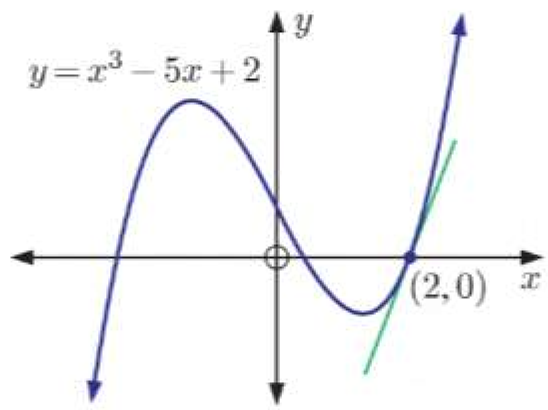
b $y = x^3 - 5x + 2$

$\therefore\ \dfrac{dy}{dx} = 3x^2 - 5$

When $x = 2$, $\dfrac{dy}{dx} = 3(2)^2 - 5$
$= 12 - 5$
$= 7$

So, the tangent has equation $y = 7(x - 2) + 0$

$\therefore\ y = 7x - 14$



c $y = \dfrac{1 - 2x}{x^2}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{(-2)x^2 - (1 - 2x)(2x)}{x^4}$ {quotient rule}

$= \dfrac{-2x^2 - 2x + 4x^2}{x^4}$

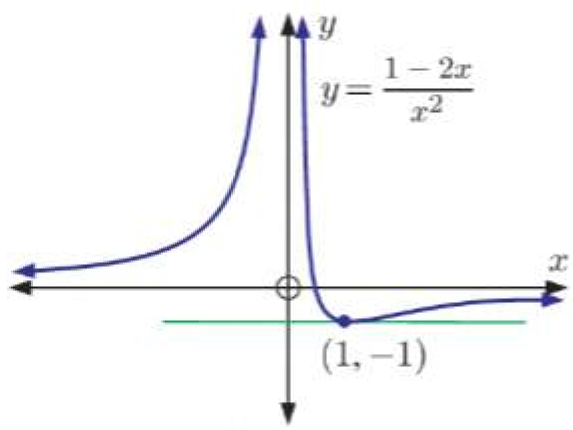
$= \dfrac{2x(x - 1)}{x^4}$

$= \dfrac{2(x - 1)}{x^3}$

When $x = 1$, $\dfrac{dy}{dx} = \dfrac{2(1 - 1)}{1^3} = 0$

So, the tangent has equation $y = 0(x - 1) - 1$

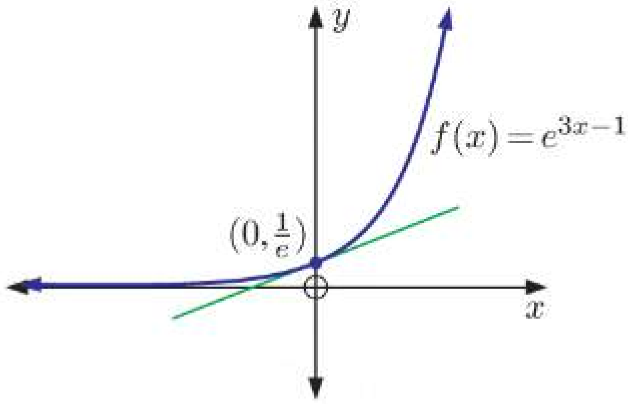
$\therefore\ y = -1$



d $f(x) = e^{3x-1}$

$\therefore\ f(0) = e^{3(0)-1}$
$= e^{-1}$
$= \dfrac{1}{e}$

$\therefore$ the point of contact is $\left(0,\ \dfrac{1}{e}\right)$.



Now $f(x) = e^{3x-1}$

$\therefore\ f'(x) = 3e^{3x-1}$

$\therefore\ f'(0) = 3e^{3(0)-1}$
$= 3e^{-1}$
$= \dfrac{3}{e}$

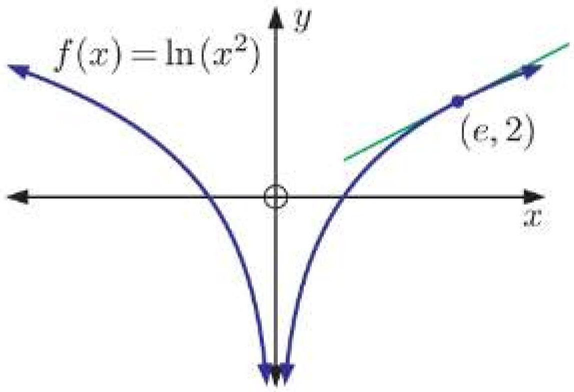
So, the tangent has equation $3x - ey = 3(0) - e\left(\dfrac{1}{e}\right)$

$\therefore\ 3x - ey = -1$

e $f(x) = \ln(x^2)$

$\therefore\ f(e) = \ln(e^2)$
$= 2$

$\therefore$ the point of contact is $(e,\ 2)$.



Now $f(x) = \ln(x^2)$
$= 2\ln x$

$\therefore\ f'(x) = \dfrac{2}{x}$

$\therefore\ f'(e) = \dfrac{2}{e}$

So, the tangent has equation $y = \dfrac{2}{e}(x - e) + 2$

$\therefore\ y = \dfrac{2}{e}x - 2 + 2$

$\therefore\ y = \dfrac{2}{e}x$

2 **a** $y = \sqrt{3x+4} = (3x+4)^{\frac{1}{2}}$

$\therefore\ \dfrac{dy}{dx} = \frac{1}{2}(3x+4)^{-\frac{1}{2}} \times 3$ {chain rule}

$= \dfrac{3}{2\sqrt{3x+4}}$, so at $(4,\ 4)$

$\dfrac{dy}{dx} = \dfrac{3}{2\sqrt{3(4)+4}}$
$= \dfrac{3}{2\sqrt{16}}$
$= \frac{3}{8}$

$\therefore$ the normal at $(4,\ 4)$ has gradient $-\frac{8}{3}$.

$\therefore$ the equation of the normal is $8x + 3y = 8(4) + 3(4)$

$\therefore\ 8x + 3y = 44$

b $y = 3e^{2x}$

When $x = 1$, $y = 3e^{2(1)} = 3e^2$

So, the point of contact is $(1, 3e^2)$.

Now $\frac{dy}{dx} = 3e^{2x} \times 2$ {chain rule}

$= 6e^{2x}$

So at $x = 1$, $\frac{dy}{dx} = 6e^{2(1)}$

$= 6e^2$

$\therefore$ the normal at $(1, 3e^2)$ has gradient $-\frac{1}{6e^2}$.

$\therefore$ the equation of the normal is $y = -\frac{1}{6e^2}(x-1) + 3e^2$

$$\therefore\ y = -\frac{1}{6e^2}x + \frac{1}{6e^2} + 3e^2 \times \frac{6e^2}{6e^2}$$

$$\therefore\ y = -\frac{1}{6e^2}x + \frac{1}{6e^2} + \frac{18e^4}{6e^2}$$

$$\therefore\ y = -\frac{1}{6e^2}x + \frac{18e^4+1}{6e^2}$$

3 $f(x) = e^{4x} + px + q$

$f(0) = e^{4(0)} + p(0) + q$

$= 1 + q$

So, the point of contact is $(0, 1+q)$.

Now $f'(x) = 4e^{4x} + p$

$\therefore\ f'(0) = 4e^{4(0)} + p$

$= 4 + p$

So, the tangent has equation $y = (4+p)(x-0) + 1 + q$

$\therefore\ y = (4+p)x + 1 + q$

But we know the tangent has equation $y = 5x - 7$.

$\therefore\ 4 + p = 5$ and $1 + q = -7$

$\therefore\ p = 1$ and $q = -8$

4 **Note:** The first print of this book erroneously repeats **Review set 18A** question **7**.

$y = 4x^3 + 6x^2 - 13x + 1$

$\therefore\ \frac{dy}{dx} = 12x^2 + 12x - 13$

The gradient of the tangent is 11 when $12x^2 + 12x - 13 = 11$

$\therefore\ 12x^2 + 12x - 24 = 0$

$\therefore\ 12(x^2 + x - 2) = 0$

$\therefore\ 12(x+2)(x-1) = 0$

$\therefore\ x = -2$ or 1

When $x = -2$,

$y = 4(-2)^3 + 6(-2)^2 - 13(-2) + 1$

$= 19$

When $x = 1$,

$y = 4(1)^3 + 6(1)^2 - 13(1) + 1$

$= -2$

So, the gradient of the tangent to $y = 4x^3 + 6x^2 - 13x + 1$ is 11 at the points $(-2, 19)$ and $(1, -2)$.

5 Let $f(x) = 2x^3 + 4x - 1$

$\therefore\ f'(x) = 6x^2 + 4$ and $\therefore\ f'(1) = 6(1)^2 + 4$

$= 10$

$\therefore$ the equation of the tangent at $(1, 5)$ is $y = 10(x-1) + 5$

which is $y = 10x - 5$

The curve meets the tangent when $2x^3 + 4x - 1 = 10x - 5$

$\therefore\ 2x^3 - 6x + 4 = 0$

$\therefore\ x^3 - 3x + 2 = 0$

$\therefore\ (x-1)^2(x+2) = 0$ {tangent at $x = 1$}

$f(-2) = 2(-2)^3 + 4(-2) - 1$

$= -16 - 8 - 1$

$= -25$

$\therefore$ the tangent meets the curve again at $(-2, -25)$.

6 **a** $y = e^{2x}$

When $x = a$, $y = e^{2a}$

So, the point of contact is (a, e^{2a}).

Now $\dfrac{dy}{dx} = 2e^{2x}$

When $x = a$, $\dfrac{dy}{dx} = 2e^{2a}$

$\therefore$ the normal at $x = a$ has gradient $-\dfrac{1}{2e^{2a}}$.

$\therefore$ the equation of the normal is $y = -\dfrac{1}{2e^{2a}}(x - a) + e^{2a}$

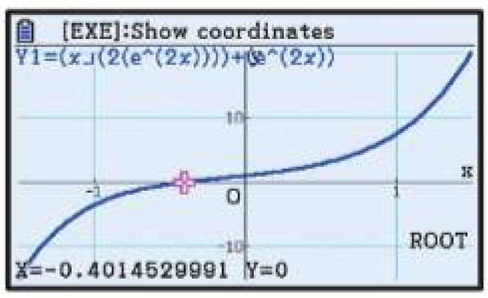
$$\therefore\ y = -\frac{1}{2e^{2a}}x + \frac{a}{2e^{2a}} + e^{2a}$$

b $y = -\dfrac{1}{2e^{2a}}x + \dfrac{a}{2e^{2a}} + e^{2a}$ passes through the origin when $x = 0$, $y = 0$

$$\therefore\ 0 = -\frac{1}{2e^{2a}}(0) + \frac{a}{2e^{2a}} + e^{2a}$$

$\therefore\ \dfrac{a}{2e^{2a}} + e^{2a} = 0$

$\therefore\ a \approx -0.401\,45$

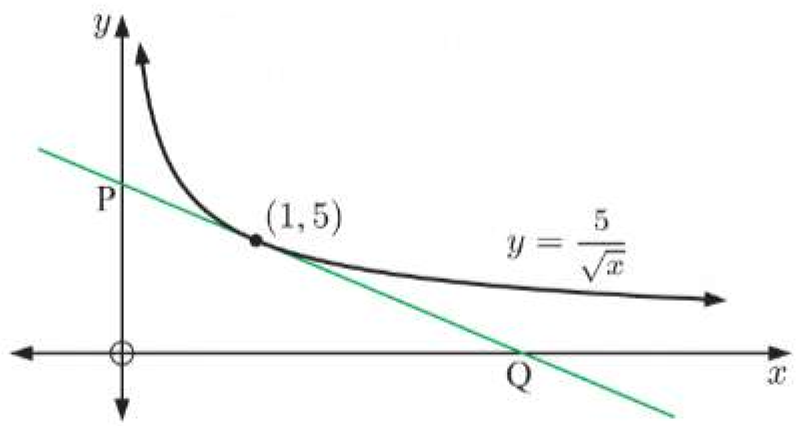


$\therefore$ the normal to $y = e^{2x}$ which passes through the origin is

$$y = -\frac{1}{2e^{2(-0.401\,45)}}x \qquad \left\{\frac{a}{2e^{2a}} + e^{2a} = 0\right\}$$

$\therefore\ y \approx -1.12x$

7 $y = \dfrac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}}$



$\therefore \dfrac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}}$

$= -\dfrac{5}{2x\sqrt{x}}$

At the point $(1, 5)$, $\dfrac{dy}{dx} = -\dfrac{5}{2(1)\sqrt{1}}$

$= -\frac{5}{2}$

$\therefore$ the gradient of the tangent at $(1, 5)$ is $-\frac{5}{2}$.

$\therefore$ the tangent has equation $y = -\frac{5}{2}(x-1)+5$

which is $y = -\frac{5}{2}x + \frac{15}{2}$

At point P, $x = 0$ $\therefore$ $y = -\frac{5}{2}(0) + \frac{15}{2} = \frac{15}{2}$

At point Q, $y = 0$ $\therefore$ $0 = -\frac{5}{2}x + \frac{15}{2}$

$\therefore$ $\frac{5}{2}x = \frac{15}{2}$

$\therefore$ $x = 3$

So, P is $(0, \frac{15}{2})$ and Q is $(3, 0)$.

8 $y = x^2\sqrt{1-x}$

When $x = -3$, $y = (-3)^2\sqrt{1-(-3)} = 18$

$\therefore$ the point of contact is $(-3, 18)$.

Now $y = x^2\sqrt{1-x} = x^2(1-x)^{\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} + x^2\left(\frac{1}{2}\right)(1-x)^{-\frac{1}{2}}(-1)$ {product rule and chain rule}

$= 2x\sqrt{1-x} - \dfrac{x^2}{2\sqrt{1-x}}$

When $x = -3$, $\dfrac{dy}{dx} = 2(-3)\sqrt{1-(-3)} - \dfrac{(-3)^2}{2\sqrt{1-(-3)}}$

$= -12 - \frac{9}{4} = -\frac{57}{4}$

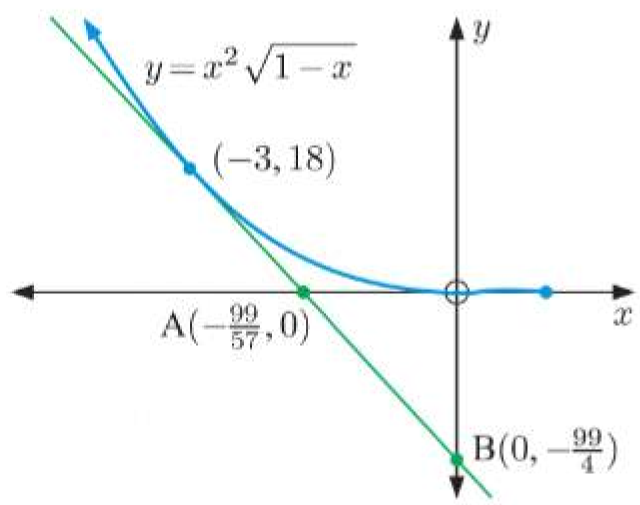
So, the tangent has equation $y = -\frac{57}{4}(x+3) + 18$

$\therefore$ $y = -\frac{57}{4}x - \frac{99}{4}$

When $y = 0$, $-\frac{57}{4}x - \frac{99}{4} = 0$

$\therefore$ $57x = -99$

$\therefore$ $x = -\frac{99}{57}$



$\therefore$ the x-intercept of the tangent is $-\frac{99}{57}$.

When $x = 0$, $y = -\frac{99}{4}$

$\therefore$ the y-intercept of the tangent is $-\frac{99}{4}$.

$\therefore$ A is $(-\frac{99}{57}, 0)$ and B is $(0, -\frac{99}{4})$.

$\therefore$ area of triangle OAB $= \frac{1}{2} \times \frac{99}{57} \times \frac{99}{4}$

$= \frac{3267}{152} \approx 21.5$ units^2

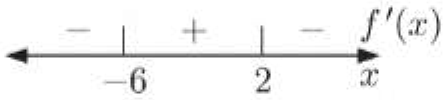
9 $f(x) = -x^3 - 6x^2 + 36x - 17$

$\therefore \ f'(x) = -3x^2 - 12x + 36$

$= -3(x^2 + 4x - 12)$

$= -3(x + 6)(x - 2)$

which has sign diagram:



a $f(x)$ is increasing for $-6 \leqslant x \leqslant 2$.

b $f(x)$ is decreasing for $x \leqslant -6$ or $x \geqslant 2$.

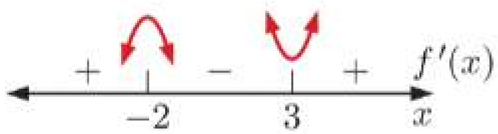
10 $f(x) = 2x^3 - 3x^2 - 36x + 7$

a $f'(x) = 6x^2 - 6x - 36$

$= 6(x^2 - x - 6)$

$= 6(x + 2)(x - 3)$

which has sign diagram:



So, there is a local maximum at $x = -2$ and a local minimum at $x = 3$.

$f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 7$
$= 51$

$f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 7$
$= -74$

There is a local maximum at $(-2, 51)$ and a local minimum at $(3, -74)$.

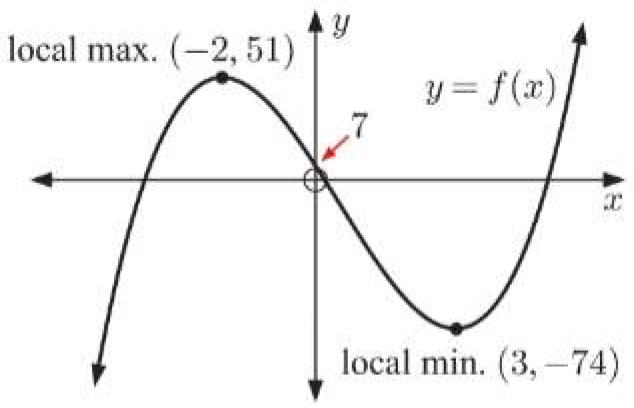
b $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 3$.

$f(x)$ is decreasing for $-2 \leqslant x \leqslant 3$.

c as $x \to \infty$, $f(x) \to \infty$

as $x \to -\infty$, $f(x) \to -\infty$

d



11 **a** $f(x) = \dfrac{3x - 2}{x + 3}$ is defined when $x + 3 \neq 0$

$\therefore \ x \neq -3$

$\therefore$ $f(x)$ has domain $\{x \mid x \neq -3\}$.

b $f(x) = 0$ when $3x - 2 = 0$

$\therefore \ 3x = 2$

$\therefore \ x = \frac{2}{3}$

$\therefore$ the x-intercept is $\frac{2}{3}$.

$f(0) = \dfrac{3(0) - 2}{0 + 3}$

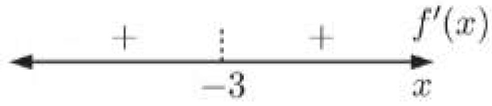
$= -\frac{2}{3}$

$\therefore$ the y-intercept is $-\frac{2}{3}$.

c $f'(x) = \dfrac{3(x+3) - (3x-2)(1)}{(x+3)^2}$ {quotient rule}

$= \dfrac{3x + 9 - 3x + 2}{(x+3)^2}$

$= \dfrac{11}{(x+3)^2}$ which has sign diagram:



d There are no values of x such that $f'(x) = 0$.

$\therefore$ $f(x)$ does not have any stationary points.

12 Let $y = x + \dfrac{32}{x^2} = x + 32x^{-2}$, $2 \leqslant x \leqslant 10$

$\therefore$ $\dfrac{dy}{dx} = 1 - 64x^{-3} = 1 - \dfrac{64}{x^3}$

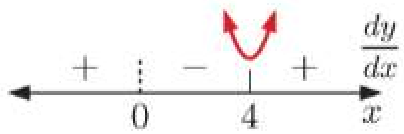
$\dfrac{dy}{dx} = 0$ when $1 - \dfrac{64}{x^3} = 0$

$\therefore$ $\dfrac{64}{x^3} = 1$

$\therefore$ $64 = x^3$

$\therefore$ $x = \sqrt[3]{64} = 4$

$\dfrac{dy}{dx}$ has sign diagram:



$\therefore$ there is a local minimum at $x = 4$.

Critical value (x)	y
2 (end point)	10
4 (local minimum)	6
10 (end point)	≈ 10.32

The greatest of these values is about 10.3 when $x = 10$.

The least of these values is 6 when $x = 4$.

13 **a** $f(x) = xe^{1-2x}$

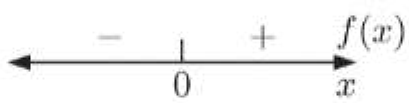
$\therefore$ $f'(x) = (1)e^{1-2x} + xe^{1-2x}(-2)$ {product rule, chain rule}

$= e^{1-2x} - 2xe^{1-2x}$

$= e^{1-2x}(1 - 2x)$

b **i** $f(x) = 0$ when $x = 0$ $\{e^{1-2x} > 0\}$

$\therefore$ $f(x)$ has sign diagram:



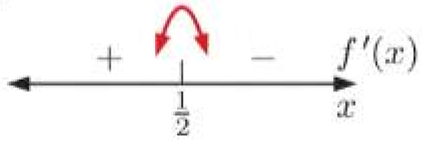
$\therefore$ $f(x) > 0$ when $x > 0$.

ii $f'(x) = e^{1-2x}(1 - 2x)$

$f'(x) = 0$ when $1 - 2x = 0$ $\{e^{1-2x} > 0\}$

$\therefore$ $x = \frac{1}{2}$

$\therefore$ $f'(x)$ has sign diagram:



$\therefore$ $f'(x) > 0$ when $x < \frac{1}{2}$.

c Stationary points corresponds to where $f'(x) = 0$.

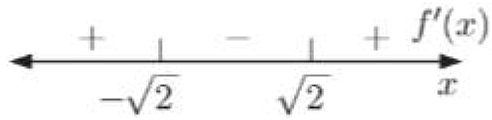
$f'(x) = 0$ when $x = \frac{1}{2}$

and $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) e^{1-2\left(\frac{1}{2}\right)}$
$= \frac{1}{2}e^0$
$= \frac{1}{2}$

$\therefore$ $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a local maximum.

14 **a** $f(x) = x^3 - 6x$

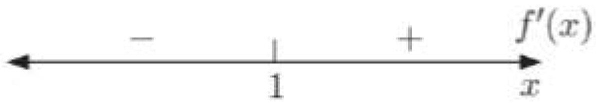
$\therefore$ $f'(x) = 3x^2 - 6$
$= 3(x^2 - 2)$
$= 3(x + \sqrt{2})(x - \sqrt{2})$ which has sign diagram:



$f(x)$ is increasing for $x \leqslant -\sqrt{2}$ and $x \geqslant \sqrt{2}$, and decreasing for $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$.

b $f(x) = e^x(x - 2)$

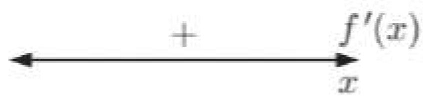
$\therefore$ $f'(x) = e^x(x - 2) + e^x(1)$ {product rule}
$= e^x(x - 1)$ which has sign diagram:



$f(x)$ is increasing for $x \geqslant 1$, and decreasing for $x \leqslant 1$.

c $f(x) = 2x - \sin x$

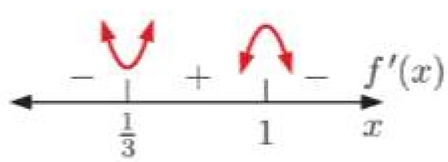
$\therefore$ $f'(x) = 2 - \cos x$ which has sign diagram:



$f(x)$ is increasing for all $x \in \mathbb{R}$.

15 **a** $f(x) = -x^3 + 2x^2 - x + 3$

$\therefore$ $f'(x) = -3x^2 + 4x - 1$
$= (1 - 3x)(x - 1)$ which has sign diagram:



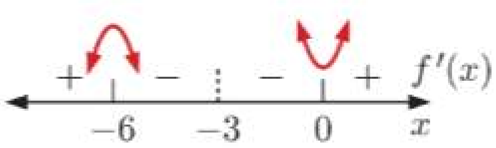
Now $f\left(\frac{1}{3}\right) = -\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 3$
$= -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} + 3$
$= \frac{77}{27}$

and $f(1) = -(1)^3 + 2(1)^2 - (1) + 3$
$= -1 + 2 - 1 + 3$
$= 3$

$\therefore$ $\left(\frac{1}{3}, \frac{77}{27}\right)$ is a local minimum, and $(1, 3)$ is a local maximum.

b $f(x) = \dfrac{x^2}{x + 3}$

$\therefore$ $f'(x) = \dfrac{2x(x + 3) - x^2(1)}{(x + 3)^2}$ {quotient rule}
$= \dfrac{2x^2 + 6x - x^2}{(x + 3)^2}$
$= \dfrac{x^2 + 6x}{(x + 3)^2}$
$= \dfrac{x(x + 6)}{(x + 3)^2}$ which has sign diagram:



$$f(-6) = \frac{(-6)^2}{(-6+3)} \qquad \text{and} \qquad f(0) = \frac{(0)^2}{0+3}$$
$$= \frac{36}{-3} = -12 \qquad\qquad = 0$$

$\therefore$ $(-6, -12)$ is a local maximum, and $(0, 0)$ is a local minimum.

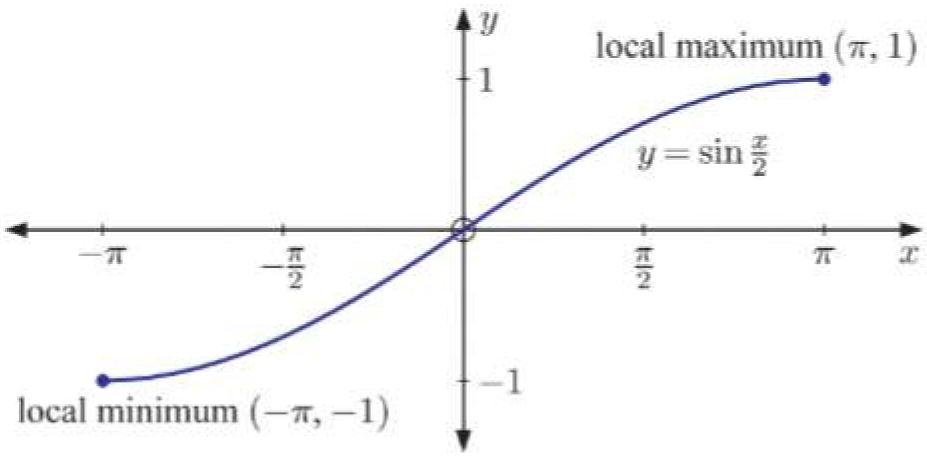
16 **a** $y = \sin\frac{x}{2}, \quad -\pi \leqslant x \leqslant \pi$

$\therefore \dfrac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2}$
$= 0$ when $x = -\pi$ or π

When $x = -\pi$, $y = \sin\left(-\frac{\pi}{2}\right)$
$= -1$

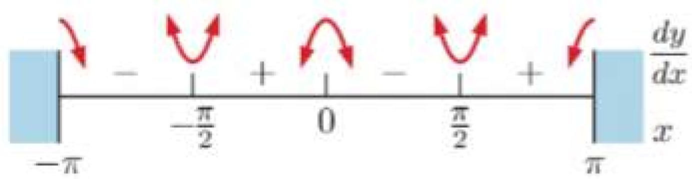
When $x = \pi$, $y = \sin\frac{\pi}{2}$
$= 1$

$\therefore$ $(-\pi, -1)$ is a local minimum, $(\pi, 1)$ is a local maximum.



b $y = \cos^2 x, \quad -\pi \leqslant x \leqslant \pi$

$\therefore \dfrac{dy}{dx} = 2\cos x(-\sin x)$
$= -2\sin x\cos x$ which has sign diagram:



When $x = -\pi$, $y = \cos^2(-\pi)$
$= 1$

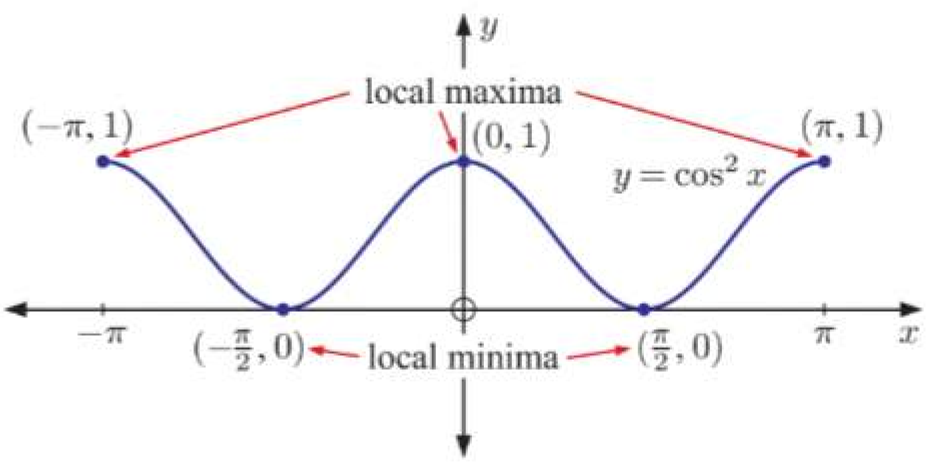
When $x = -\frac{\pi}{2}$, $y = \cos^2\left(-\frac{\pi}{2}\right)$
$= 0$

When $x = 0$, $y = \cos^2 0$
$= 1$

When $x = \frac{\pi}{2}$, $y = \cos^2\left(\frac{\pi}{2}\right)$
$= 0$

When $x = \pi$, $y = \cos^2 \pi$
$= 1$

$\therefore$ $(-\pi, 1)$, $(0, 1)$, and $(\pi, 1)$ are local maxima, $\left(-\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$ are local minima.



c $y = \tan^2 x, \quad -\pi \leqslant x \leqslant \pi$

$$\therefore \frac{dy}{dx} = 2\tan x \times \frac{1}{\cos^2 x}$$
$$= \frac{2\tan x}{\cos^2 x}$$

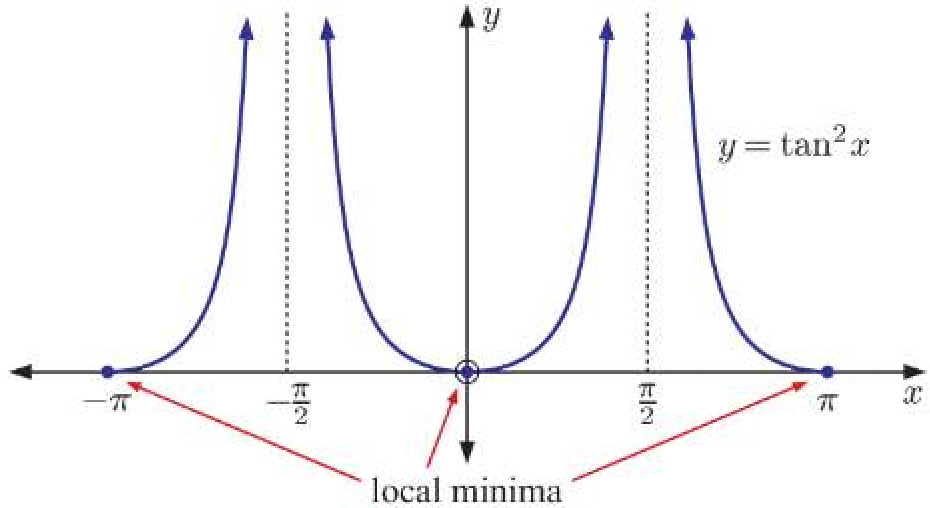
Now $\dfrac{dy}{dx} = 0$ when $\dfrac{2\tan x}{\cos^2 x} = 0$

$$\therefore \tan x = 0$$
$$\therefore x = -\pi, 0, \text{ or } \pi \qquad \{-\pi \leqslant x \leqslant \pi\}$$

When $x = -\pi$, $y = \tan^2(-\pi) = 0$ When $x = 0$, $y = \tan^2 0 = 0$

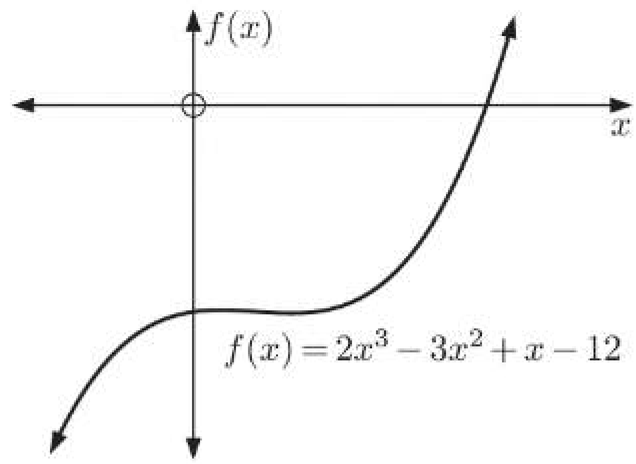
When $x = \pi$, $y = \tan^2 \pi = 0$

$\therefore$ $(-\pi, 0)$, $(0, 0)$, and $(\pi, 0)$ are local minima.

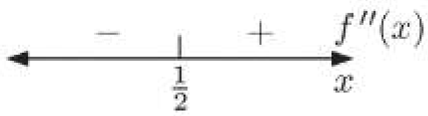


17 **a** $f(x) = 2x^3 - 3x^2 + x - 12$

$$\therefore f'(x) = 6x^2 - 6x + 1$$
$$\therefore f''(x) = 12x - 6$$



b $f''(x) = 6(2x - 1)$ has sign diagram:



c $f(x)$ is concave down for $x \leqslant \frac{1}{2}$.

d $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + \frac{1}{2} - 12$

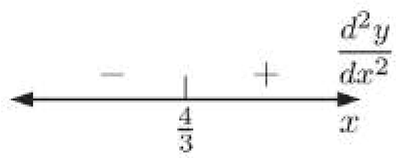
$$= \frac{1}{4} - \frac{3}{4} + \frac{1}{2} - 12$$
$$= -12$$

$\therefore$ the shape of $f(x)$ changes at $\left(\frac{1}{2}, -12\right)$.

18 **a** $y = x^3 - 4x^2 + 11$

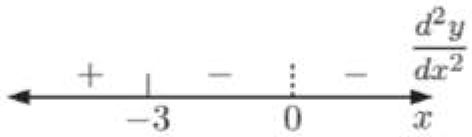
$$\therefore \frac{dy}{dx} = 3x^2 - 8x$$

$\therefore \dfrac{d^2y}{dx^2} = 6x - 8$ which has sign diagram:



The curve is concave up for $x \geqslant \frac{4}{3}$, and concave down for $x \leqslant \frac{4}{3}$.

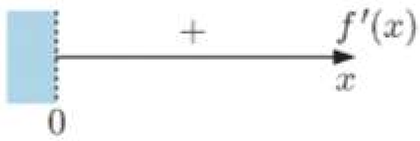
b
$$y = -\frac{x+1}{x^2}$$
$$= -x^{-1} - x^{-2}$$
$$\therefore \frac{dy}{dx} = x^{-2} + 2x^{-3}$$
$$\therefore \frac{d^2y}{dx^2} = -2x^{-3} - 6x^{-4}$$
$$= -\frac{2}{x^3} - \frac{6}{x^4}$$
$$= -\frac{2x+6}{x^4}$$ which has sign diagram:



The curve is concave up for $x \leqslant -3$, and concave down for $-3 \leqslant x < 0$ and $x > 0$.

19 **a** $f(x) = x + \ln x$ is defined for $x > 0$.

b $f'(x) = 1 + \frac{1}{x}$ which has sign diagram:



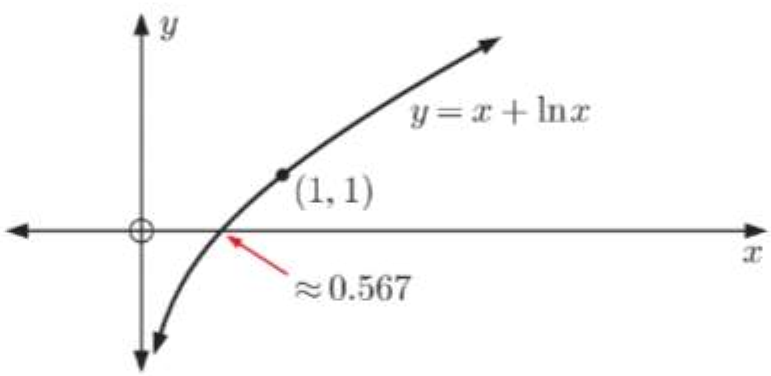
$f'(x) = 1 + x^{-1}$

$\therefore f''(x) = -x^{-2}$

$= -\frac{1}{x^2}$ which has sign diagram:

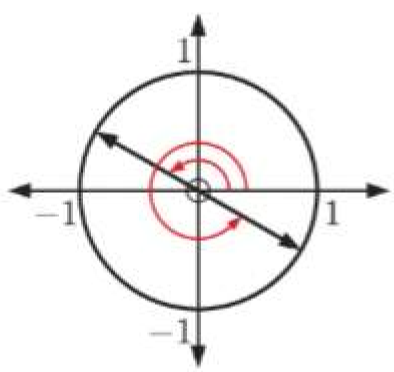
So, $f(x)$ is increasing for all $x > 0$ and is concave downwards for all $x > 0$.

c



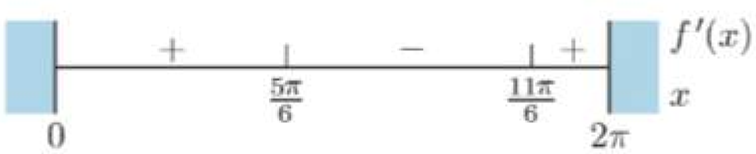
20 **a**
$$f(x) = e^{x\sqrt{3}} \sin x$$
$$\therefore f'(x) = e^{x\sqrt{3}}(\sqrt{3}) \sin x + e^{x\sqrt{3}} \cos x \qquad \{\text{chain rule and product rule}\}$$
$$= e^{x\sqrt{3}}(\cos x + \sqrt{3} \sin x)$$

b



$$f'(x) = 0$$
when $e^{x\sqrt{3}}(\cos x + \sqrt{3} \sin x) = 0$
$$\therefore \cos x + \sqrt{3} \sin x = 0 \quad \{\text{as } e^{x\sqrt{3}} > 0 \text{ for all } x\}$$
$$\therefore \sqrt{3} \sin x = -\cos x$$
$$\therefore \tan x = -\tfrac{1}{\sqrt{3}}$$
$$\therefore x = \tfrac{5\pi}{6} \text{ or } \tfrac{11\pi}{6}$$

c

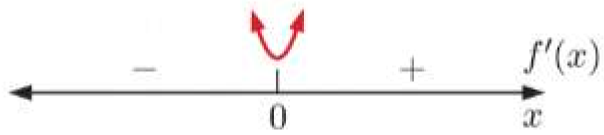


d **i** $f(x)$ is increasing for $0 \leqslant x \leqslant \frac{5\pi}{6}$ and $\frac{11\pi}{6} \leqslant x \leqslant 2\pi$.

ii $f(x)$ is decreasing for $\frac{5\pi}{6} \leqslant x \leqslant \frac{11\pi}{6}$.

21 **a** $f(x) = \ln(x^2+5)$

$\therefore\ f'(x) = \dfrac{2x}{x^2+5}$ which has sign diagram:



Now $f(0) = \ln(0^2+5)$

$= \ln 5$

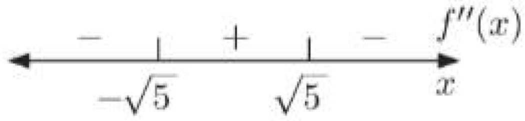
$\therefore$ $(0, \ln 5)$ is a local minimum.

b $f''(x) = \dfrac{2(x^2+5) - 2x(2x)}{(x^2+5)^2}$ {quotient rule}

$= \dfrac{2x^2+10-4x^2}{(x^2+5)^2}$

$= \dfrac{-2(x^2-5)}{(x^2+5)^2}$

$= \dfrac{-2(x+\sqrt{5})(x-\sqrt{5})}{(x^2+5)^2}$ which has sign diagram:



Now $f(-\sqrt{5}) = \ln((-\sqrt{5})^2+5)$ and $f(\sqrt{5}) = \ln((\sqrt{5})^2+5)$

$= \ln 10$ $\qquad$ $= \ln 10$

also $f'(-\sqrt{5}) = \dfrac{2(-\sqrt{5})}{(-\sqrt{5})^2+5}$ $\qquad$ $f'(\sqrt{5}) = \dfrac{2\sqrt{5}}{(\sqrt{5})^2+5}$

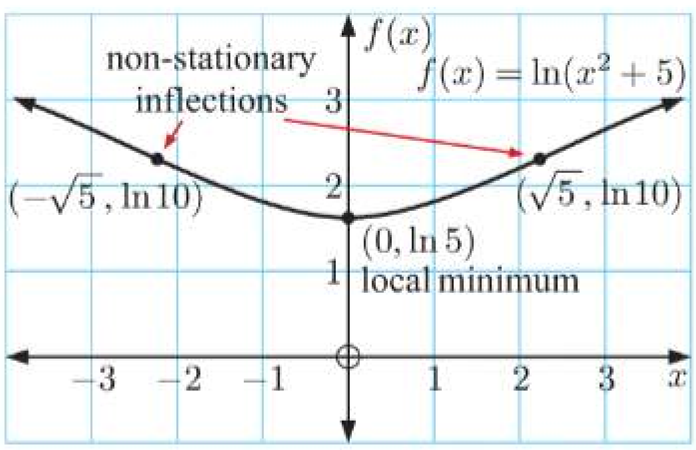
$= -\frac{2\sqrt{5}}{10} \neq 0$ $\qquad$ $= \frac{2\sqrt{5}}{10} \neq 0$

$\therefore$ $(-\sqrt{5}, \ln 10)$ and $(\sqrt{5}, \ln 10)$ are non-stationary inflections.

c $f(x)$ is increasing for $x \geqslant 0$, and decreasing for $x \leqslant 0$.

d $f(x)$ is concave up for $-\sqrt{5} \leqslant x \leqslant \sqrt{5}$, and concave down for $x \leqslant -\sqrt{5}$ and $x \geqslant \sqrt{5}$.

e



REVIEW SET 19B

1 **a** $f(x) = x^4 - 2x^2 + 7x - 3$

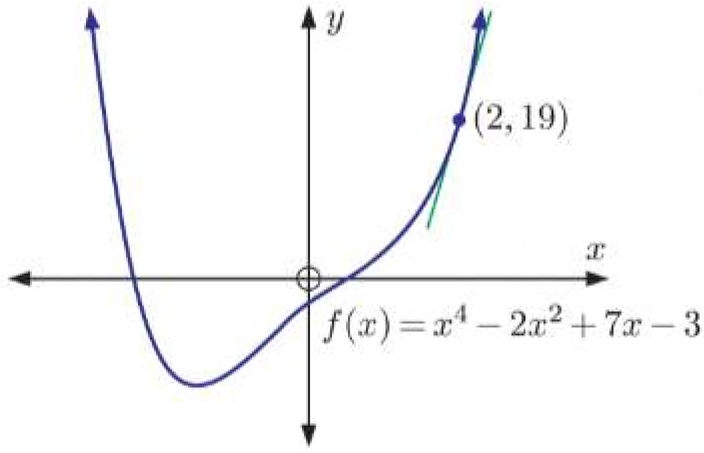
$\therefore\ f'(x) = 4x^3 - 4x + 7$

$\therefore\ f'(2) = 4(2)^3 - 4(2) + 7$
$= 32 - 8 + 7$
$= 31$

So, the tangent has equation

$y = 31(x-2) + 19$

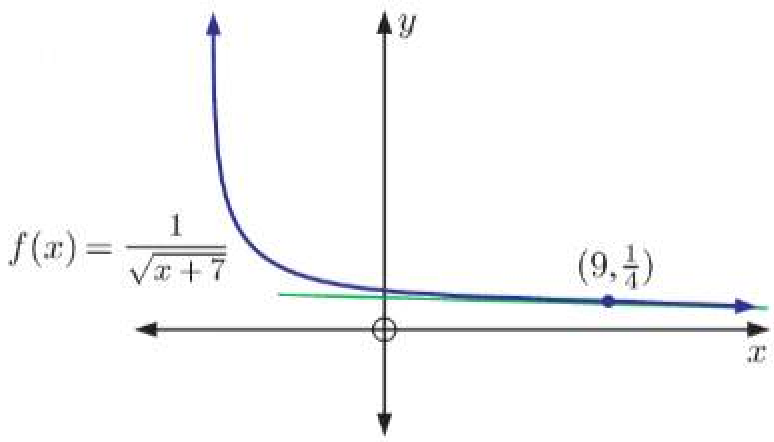
$\therefore\ y = 31x - 43$



b $f(x) = \dfrac{1}{\sqrt{x+7}} = (x+7)^{-\frac{1}{2}}$

$\therefore\ f'(x) = -\frac{1}{2}(x+7)^{-\frac{3}{2}}$
$= -\dfrac{1}{2(x+7)^{\frac{3}{2}}}$

$\therefore\ f'(9) = -\dfrac{1}{2(9+7)^{\frac{3}{2}}}$
$= -\dfrac{1}{2(16)^{\frac{3}{2}}}$
$= -\frac{1}{128}$



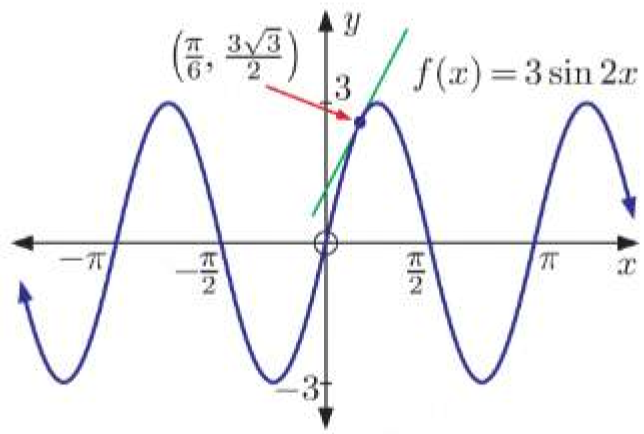
So, the tangent has equation $x + 128y = 9 + 128\left(\frac{1}{4}\right)$

$\therefore\ x + 128y = 41$

c $f(x) = 3\sin 2x$

$\therefore\ f\left(\frac{\pi}{6}\right) = 3\sin\frac{\pi}{3}$
$= 3\left(\frac{\sqrt{3}}{2}\right)$
$= \frac{3\sqrt{3}}{2}$

$\therefore$ the point of contact is $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$.



Now $f(x) = 3\sin 2x$

$\therefore\ f'(x) = 6\cos 2x$

$\therefore\ f'\left(\frac{\pi}{6}\right) = 6\cos\frac{\pi}{3}$
$= 6\left(\frac{1}{2}\right)$
$= 3$

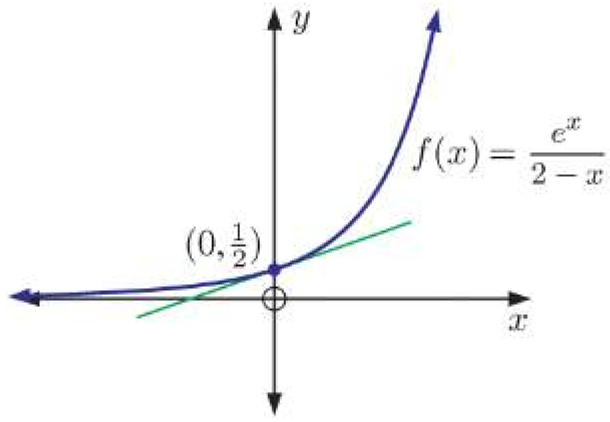
So, the tangent has equation $y = 3\left(x - \frac{\pi}{6}\right) + \frac{3\sqrt{3}}{2}$

$= 3x + \frac{3\sqrt{3}}{2} - \frac{\pi}{2}$

d $f(x) = \dfrac{e^x}{2-x}$

$\therefore \ f(0) = \dfrac{e^0}{2-0} = \frac{1}{2}$

$\therefore$ the point of contact is $(0, \frac{1}{2})$.



Now $f(x) = \dfrac{e^x}{2-x}$

$\therefore \ f'(x) = \dfrac{e^x(2-x) - e^x(-1)}{(2-x)^2}$ {quotient rule}

$= \dfrac{e^x(3-x)}{(2-x)^2}$

$\therefore \ f'(0) = \dfrac{e^0(3-0)}{(2-0)^2}$

$= \frac{3}{4}$

So, the tangent has equation $y = \frac{3}{4}(x-0) + \frac{1}{2}$

$\therefore \ y = \frac{3}{4}x + \frac{1}{2}$

2 **a** $y = \dfrac{1}{x^2} - \dfrac{2}{x} = x^{-2} - 2x^{-1}$

When $x = 1$, $y = \dfrac{1}{1^2} - \dfrac{2}{1} = 1 - 2 = -1$

So, the point of contact is $(1, -1)$.

Now as $y = x^{-2} - 2x^{-1}$, $\dfrac{dy}{dx} = -2x^{-3} + 2x^{-2}$

$= -\dfrac{2}{x^3} + \dfrac{2}{x^2}$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = -\dfrac{2}{1^3} + \dfrac{2}{1^2} = -2 + 2 = 0$

$\therefore$ the normal at $(1, -1)$ has gradient which is undefined.

So, the normal must be a vertical line.

Since the normal passes through $(1, -1)$, then the equation of the normal is $x = 1$.

b $y = x \sin x$

$\therefore \ \dfrac{dy}{dx} = (1)\sin x + x(\cos x)$ {product rule}

$= \sin x + x \cos x$

$\therefore$ when $x = 0$, $\dfrac{dy}{dx} = \sin 0 + 0 \times \cos x$

$= 0$

$\therefore$ the normal at $(0, 0)$ has gradient which is undefined.

So, the normal must be a vertical line.

Since the normal passes through $(0, 0)$, then the equation of the normal is $x = 0$.

3 $y = x\tan x$

When $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}(1)$
$= \frac{\pi}{4}$

So, the point of contact is $(\frac{\pi}{4}, \frac{\pi}{4})$.

Now $\dfrac{dy}{dx} = (1)\tan x + x\dfrac{1}{\cos^2 x}$
$= \tan x + \dfrac{x}{\cos^2 x}$

When $x = \frac{\pi}{4}$, $\dfrac{dy}{dx} = 1 + \dfrac{\frac{\pi}{4}}{\left(\frac{1}{\sqrt{2}}\right)^2}$
$= 1 + \dfrac{\frac{\pi}{4}}{\frac{1}{2}}$
$= 1 + \frac{\pi}{2}$

$\therefore$ the equation of the tangent at $(\frac{\pi}{4}, \frac{\pi}{4})$ is $y = (1+\frac{\pi}{2})(x - \frac{\pi}{4}) + \frac{\pi}{4}$
$\therefore\ y = x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8} + \frac{\pi}{4}$
$\therefore\ 2y = 2x + \pi x - \frac{\pi^2}{4}$
$\therefore\ (2+\pi)x - 2y = \frac{\pi^2}{4}$ as required

4 $y = 2x^3 + ax + b$

$\therefore\ \dfrac{dy}{dx} = 6x^2 + a$

Since the gradient of the tangent at $(-2, 33)$ is 10, then $6(-2)^2 + a = 10$
$\therefore\ 24 + a = 10$
$\therefore\ a = -14$
$\therefore\ y = 2x^3 - 14x + b$

Since the curve passes through $(-2, 33)$, then $33 = 2(-2)^3 - 14(-2) + b$
$= -16 + 28 + b$
$\therefore\ b = 21$

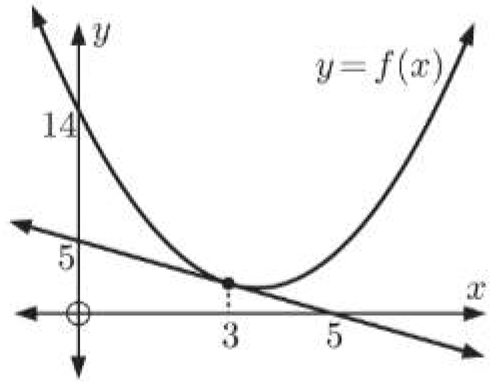
5 $y = 2 - \dfrac{7}{1+2x} = 2 - 7(1+2x)^{-1}$

$\therefore\ \dfrac{dy}{dx} = 7(1+2x)^{-2} \times 2$ {chain rule}
$= \dfrac{14}{(1+2x)^2}$

The tangent is horizontal when the gradient $\dfrac{dy}{dx} = 0$.

But $\dfrac{14}{(1+2x)^2}$ is never 0, so $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

6 **a** The tangent shown on the graph passes through $(0,\ 5)$ and $(5,\ 0)$.



$\therefore$ the gradient of the tangent is $\dfrac{0-5}{5-0} = -1$, so $f'(3) = -1$.

Also, since the tangent passes through $(0,\ 5)$, it has equation $\dfrac{y-5}{x-0} = -1$

$\therefore\ y - 5 = -x$

$\therefore\ y = -x + 5$

So when $x = 3$, $y = -3 + 5 = 2$

$\therefore$ the point of contact is $(3,\ 2)$, and hence $f(3) = 2$.

b $f(x)$ has the form $f(x) = ax^2 + bx + c$

The y-intercept is 14 $\quad \therefore\ f(0) = 14$

$\therefore\ a(0)^2 + b(0) + c = 14$

$\therefore\ c = 14$

Now $f(3) = 2$ $\quad$ {from **a**}

$\therefore\ a(3)^2 + b(3) + 14 = 2$

$\therefore\ 9a + 3b = -12$ .... (1)

Also $f'(3) = -1$

and $f'(x) = 2ax + b$

$\therefore\ 2a(3) + b = -1$

$\therefore\ 6a + b = -1$

$\therefore\ b = -6a - 1$ .... (2)

Substituting (2) into (1) gives $9a + 3(-6a - 1) = -12$

$\therefore\ 9a - 18a - 3 = -12$

$\therefore\ -9a = -9$

$\therefore\ a = 1$

Using (2), $b = -6(1) - 1$

$\therefore\ b = -7$

So, $f(x) = x^2 - 7x + 14$

7 $y = x^3 + ax^2 - 4x + 3$

a $\dfrac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x = 1$ is parallel to the line $y = 3x$, and $y = 3x$ has gradient 3.

$\therefore$ the tangent at $x = 1$ has gradient 3.

$\therefore\ 3(1)^2 + 2a(1) - 4 = 3$

$\therefore\ 3 + 2a - 4 = 3$

$\therefore\ 2a = 4$

$\therefore\ a = 2$

b Since $a = 2$, $y = x^3 + 2x^2 - 4x + 3$ and $\frac{dy}{dx} = 3x^2 + 4x - 4$

When $x = 1$, $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

and $\frac{dy}{dx} = 3(1)^2 + 4(1) - 4 = 3$

So, the point of contact is $(1, 2)$, and the tangent at $(1, 2)$ has gradient 3.

$\therefore$ the tangent has equation $y = 3(x - 1) + 2$

which is $y = 3x - 1$

c The tangent meets the curve again when $x^3 + 2x^2 - 4x + 3 = 3x - 1$

$\therefore\ x^3 + 2x^2 - 7x + 4 = 0$

$\therefore\ (x - 1)^2(x + 4) = 0$

$\{(x - 1)^2$ is a factor since the tangent to the curve is at $x = 1\}$

When $x = -4$, $y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$

$\therefore$ the tangent meets the curve again at $(-4, -13)$.

8 $y = x^2 - 4x + 2$

$\therefore\ \frac{dy}{dx} = 2x - 4$

When $x = 3$, $y = (3)^2 - 4(3) + 2 = 9 - 12 + 2 = -1$ and $\frac{dy}{dx} = 2(3) - 4 = 6 - 4 = 2$

So, the point of contact is $(3, -1)$ and the normal at $(3, -1)$ has gradient $-\frac{1}{2}$.

$\therefore$ the normal has equation $y = -\frac{1}{2}(x - 3) - 1$

$\therefore\ y = -\frac{1}{2}x + \frac{3}{2} - 1$

$\therefore\ y = -\frac{1}{2}x + \frac{1}{2}$

The normal meets the curve again when $-\frac{1}{2}x + \frac{1}{2} = x^2 - 4x + 2$

$\therefore\ -x + 1 = 2x^2 - 8x + 4$

$\therefore\ 2x^2 - 7x + 3 = 0$

$\therefore\ (2x - 1)(x - 3) = 0$

When $x = \frac{1}{2}$, $y = \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2$

$= \frac{1}{4} - 2 + 2 = \frac{1}{4}$

$\therefore$ the normal meets the curve again at $\left(\frac{1}{2}, \frac{1}{4}\right)$.

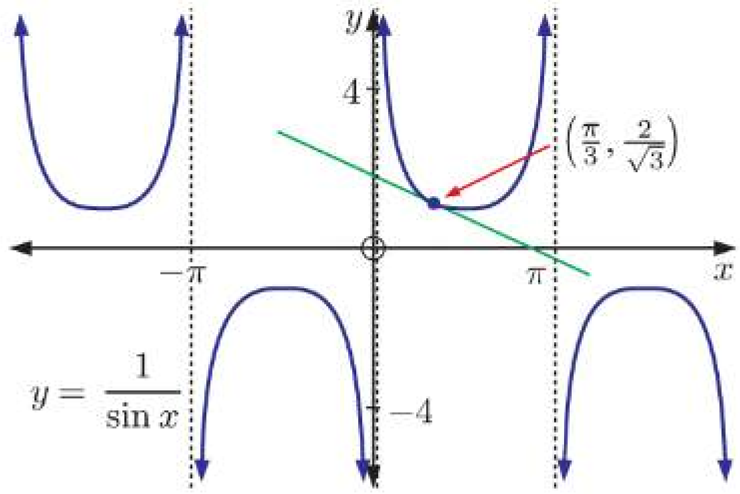
9 a

$$y = \frac{1}{\sin x}$$

When $x = \frac{\pi}{3}$, $y = \frac{1}{\sin\frac{\pi}{3}}$

$$= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$\therefore$ the point of contact is $\left(\frac{\pi}{3}, \frac{2}{\sqrt{3}}\right)$.



Now $y = \frac{1}{\sin x} = (\sin x)^{-1}$

$$\therefore \frac{dy}{dx} = -(\sin x)^{-2}\cos x \qquad \{\text{chain rule}\}$$

$$= -\frac{\cos x}{\sin^2 x}$$

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = -\frac{\cos\frac{\pi}{3}}{\sin^2(\frac{\pi}{3})}$

$$= -\frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{2}{3}$$

So, the tangent has equation $y = -\frac{2}{3}\left(x - \frac{\pi}{3}\right) + \frac{2}{\sqrt{3}}$

$$\therefore 3y = -2\left(x - \frac{\pi}{3}\right) + 2\sqrt{3}$$

$$= -2x + \frac{2\pi}{3} + 2\sqrt{3}$$

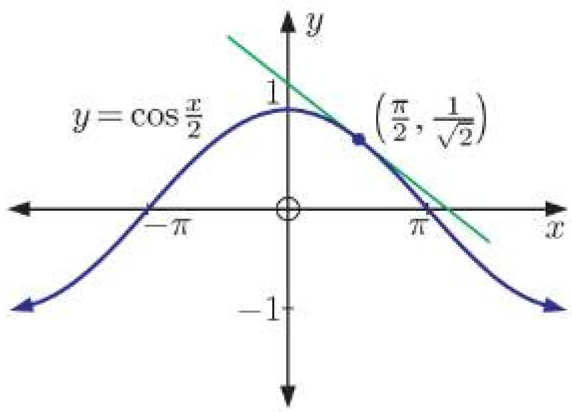
$$\therefore 2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$$

b

$$y = \cos\frac{x}{2}$$

When $x = \frac{\pi}{2}$, $y = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ the point of contact is $\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right)$.



Now $y = \cos\frac{x}{2}$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}\sin\frac{x}{2}$$

When $x = \frac{\pi}{2}$, $\frac{dy}{dx} = -\frac{1}{2}\sin\frac{\pi}{4}$

$$= -\frac{1}{2\sqrt{2}}$$

So, the tangent has equation $y = -\frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{2}\right) + \frac{1}{\sqrt{2}}$

$$\therefore 2\sqrt{2}y = -\left(x - \frac{\pi}{2}\right) + 2$$

$$= -x + \frac{\pi}{2} + 2$$

$$\therefore x + 2\sqrt{2}y = \frac{\pi}{2} + 2$$

10 $$y = \frac{ax+b}{\sqrt{x}}$$
$$= a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$
$$= \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}$$

The equation of the tangent at $x = 1$ is $2x - y = 1$
which is $y = 2x - 1$

so the gradient of the tangent is 2

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{a}{2} - \frac{b}{2} = 2$$
$$\therefore a - b = 4$$
$$\therefore a = b + 4 \quad .... \ (*)$$

Also at $x = 1$, the tangent touches the curve

$$\therefore \frac{a(1)+b}{\sqrt{1}} = 2(1) - 1$$
$$\therefore a + b = 1$$
$$\therefore b + 4 + b = 1 \quad \{\text{using } (*)\}$$
$$\therefore 2b = -3$$
$$\therefore b = -\tfrac{3}{2} \quad \text{and} \quad a = -\tfrac{3}{2} + 4 = \tfrac{5}{2}$$

11 $$f(x) = x^4 - 4x^3 - 8x^2 + 5$$
$$\therefore f'(x) = 4x^3 - 12x^2 - 16x$$
$$= 4x(x^2 - 3x - 4)$$
$$= 4x(x+1)(x-4)$$

which has sign diagram: $f'(x)$: $-$ for $x < -1$, $+$ for $-1 < x < 0$, $-$ for $0 < x < 4$, $+$ for $x > 4$ (critical values -1, 0, 4)

a $f(x)$ is increasing for $-1 \leqslant x \leqslant 0$ and $x \geqslant 4$.

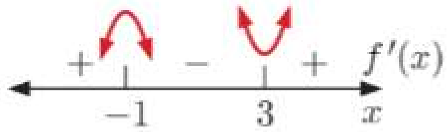
b $f(x)$ is decreasing for $x \leqslant -1$ and $0 \leqslant x \leqslant 4$.

12 $$f(x) = x^3 - 3x^2 + ax + 50$$
$$\therefore f'(x) = 3x^2 - 6x + a$$

a $f(x)$ has a stationary point at $x = 3$

$$\therefore f'(3) = 0$$
$$\therefore 3(3)^2 - 6(3) + a = 0$$
$$\therefore 27 - 18 + a = 0$$
$$\therefore a = -9$$

b Since $a = -9$, then $f(x) = x^3 - 3x^2 - 9x + 50$
and $f'(x) = 3x^2 - 6x - 9$
$= 3(x^2 - 2x - 3)$
$= 3(x + 1)(x - 3)$
which has sign diagram:



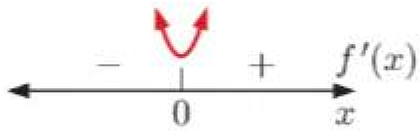
When $x = -1$, $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 50 = 55$

When $x = 3$, $f(3) = 3^3 - 3(3)^2 - 9(3) + 50 = 23$

So, there is a local maximum at $(-1, 55)$ and a local minimum at $(3, 23)$.

13 **a** $f(x) = e^x - x$

$\therefore$ $f'(x) = e^x - 1$ which has sign diagram:



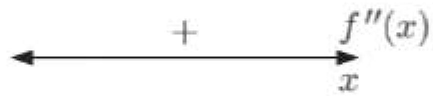
Now $f(0) = e^0 - 0 = 1$

$\therefore$ $y = f(x)$ has a local minimum at $(0, 1)$.

b As $x \to \infty$, $e^x \to \infty$ {at a much faster rate than x}

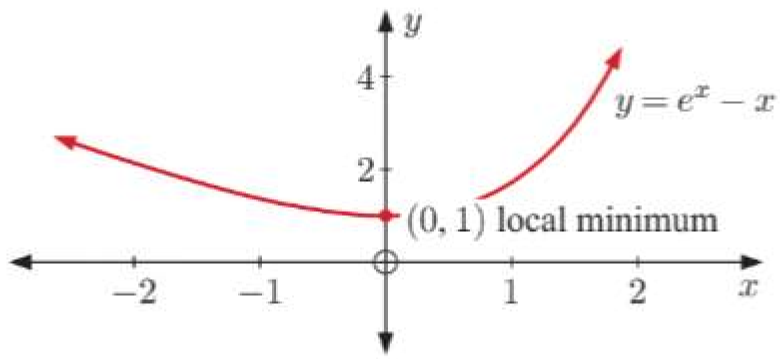
$\therefore$ as $x \to \infty$, $f(x) \to \infty$

c $f''(x) = e^x$ which has sign diagram:



$\therefore$ $f(x)$ is concave up for all $x \in \mathbb{R}$.

d



e $y = f(x)$ has a local minimum at $(0, 1)$

$\therefore$ $f(x) \geqslant 1$

$\therefore$ $e^x - x \geqslant 1$

$\therefore$ $e^x \geqslant x + 1$ for all x

14 **a** $f(x) = \dfrac{x+1}{x^2 - 2x - 8}$

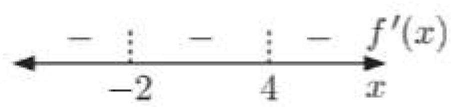
$$\therefore\ f'(x) = \frac{(1)(x^2 - 2x - 8) - (x+1)(2x-2)}{(x^2 - 2x - 8)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{x^2 - 2x - 8 - (2x^2 - 2)}{(x^2 - 2x - 8)^2}$$

$$= \frac{x^2 - 2x - 8 - 2x^2 + 2}{(x^2 - 2x - 8)^2}$$

$$= \frac{-x^2 - 2x - 6}{(x^2 - 2x - 8)^2}$$

$$= -\frac{x^2 + 2x + 6}{(x^2 - 2x - 8)^2}$$ which has sign diagram:



b $f'(x) = -\dfrac{x^2 + 2x + 1 + 5}{(x^2 - 2x - 8)^2}$

$= -\dfrac{(x+1)^2 + 5}{(x^2 - 2x - 8)^2} < 0$ for all $x \in \mathbb{R}$, $x \neq -2, 4$

$\therefore$ $f(x)$ is decreasing for all $x \in \mathbb{R}$, $x \neq -2, 4$

$\therefore$ $f(x)$ is never increasing.

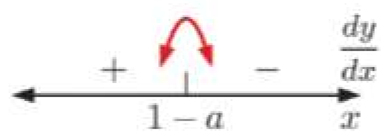
15 $$y = \frac{x+a}{e^x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(1)e^x - (x+a)e^x}{(e^x)^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{e^x - xe^x - ae^x}{e^{2x}}$$

$$= \frac{e^x(1-x-a)}{e^{2x}}$$

$$= \frac{(1-a)-x}{e^x}$$ which has sign diagram:



$\dfrac{dy}{dx} = 0$ when $x = 1-a$

When $x = 1-a$, $y = \dfrac{(1-a)+a}{e^{1-a}}$

$$= \frac{1}{e^{1-a}}$$

$$= e^{a-1}$$

$\therefore$ the stationary point of $y = \dfrac{x+a}{e^x}$ where a is a constant, is a local maximum $(1-a,\ e^{a-1})$.

16 $$f(x) = \frac{\ln(ax)}{bx}$$

$$= \frac{\ln a + \ln x}{bx}$$

$$\therefore \quad f'(x) = \frac{\left(\frac{1}{x}\right) \times bx - (\ln a + \ln x) \times b}{b^2x^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{b - b(\ln a + \ln x)}{b^2x^2}$$

$$= \frac{b(1 - \ln a - \ln x)}{b^2x^2}$$

$$= \frac{1 - \ln a - \ln x}{bx^2}$$

$f'(x) = 0$ when $\ln x = 1 - \ln a$

$$\therefore \quad \ln\left(\frac{e}{2}\right) = 1 - \ln a \qquad \left\{\left(\frac{e}{2}, \frac{2}{3e}\right) \text{ is a stationary point}\right\}$$

$$\therefore \quad \ln e - \ln 2 = 1 - \ln a$$

$$\therefore \quad 1 - \ln 2 = 1 - \ln a$$

$$\therefore \quad a = 2$$

Now $f\left(\dfrac{e}{2}\right) = \dfrac{2}{3e}$

$$\therefore \quad \frac{\ln(2 \times \frac{e}{2})}{b(\frac{e}{2})} = \frac{2}{3e}$$

$$\therefore \quad \frac{\ln e}{\frac{be}{2}} = \frac{2}{3e}$$

$$\therefore \quad 1 \times \frac{2}{be} = \frac{2}{3e}$$

$$\therefore \quad b = 3$$

17 $y = x^4 - 3x^3 + 9$

$\therefore \dfrac{dy}{dx} = 4x^3 - 9x^2$

$= x^2(4x - 9)$ which has sign diagram:

$\therefore \dfrac{d^2y}{dx^2} = 12x^2 - 18x$

$= 6x(2x - 3)$ which has sign diagram:

Since the sign of $\dfrac{d^2y}{dx^2}$ changes at $x = 0$ and $x = \frac{3}{2}$, both of these points are points of inflection.

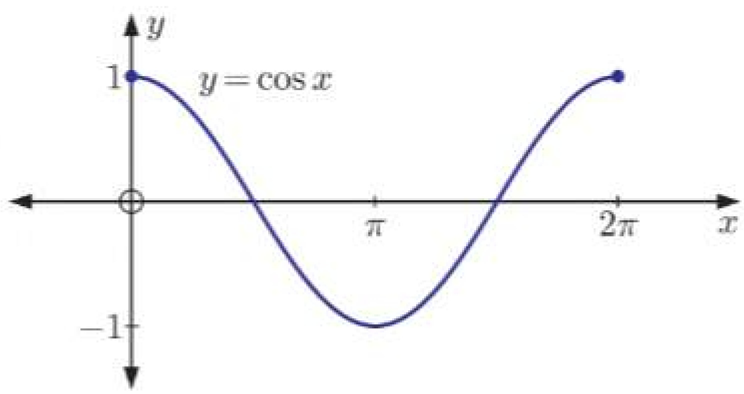
When $x = 0$, $y = (0)^4 - 3(0)^3 + 9$
$= 9$
and $\dfrac{dy}{dx} = 0$

When $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^4 - 3\left(\frac{3}{2}\right)^3 + 9$
$= \frac{81}{16} - \frac{81}{8} + 9$
$= \frac{63}{16}$
and $\dfrac{dy}{dx} \neq 0$

$\therefore$ $(0, 9)$ is a stationary inflection point, and $\left(\frac{3}{2}, \frac{63}{16}\right)$ is a non-stationary inflection point.

18 **a** $f(x) = \sqrt{\cos x}$, $0 \leqslant x \leqslant 2\pi$ is defined when $\cos x \geqslant 0$

$\therefore$ $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$



b $f(x) = \sqrt{\cos x} = (\cos x)^{\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$

$= -\dfrac{\sin x}{2\sqrt{\cos x}}$

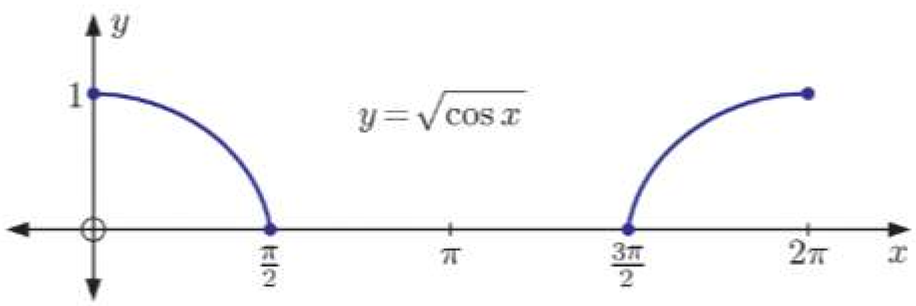
From **a**, since $f(x)$ is only defined when $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$, we only consider these values of x.

When $0 \leqslant x < \frac{\pi}{2}$, $f'(x) \leqslant 0$

When $\frac{3\pi}{2} < x \leqslant 2\pi$, $f'(x) \geqslant 0$

$\therefore$ $f(x)$ is increasing for $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$, and decreasing for $0 \leqslant x \leqslant \frac{\pi}{2}$.

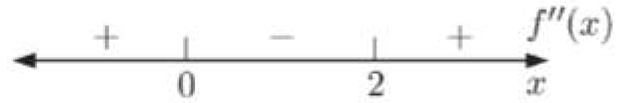
c



19 **a** $f(x) = x^4 - 4x^3 + 7$

$\therefore \; f'(x) = 4x^3 - 12x^2$
$= 4x^2(x-3)$ which has sign diagram:

$\therefore \; f''(x) = 12x^2 - 24x$
$= 12x(x-2)$ which has sign diagram:



b $f(3) = 3^4 - 4(3)^3 + 7$
$= 81 - 108 + 7$
$= -20$

$\therefore$ $(3, -20)$ is a local minimum.

c Since the signs of $f''(x)$ change about $x = 0$ and $x = 2$, these are points of inflection.

$f(0) = 7$ and $f'(0) = 0$

$\therefore$ $(0, 7)$ is a stationary inflection.

$f(2) = 2^4 - 4(2)^3 + 7$ $\quad$ and $\quad$ $f'(2) = 4(2)^3 - 12(2)^2$
$= 16 - 32 + 7$ $\qquad\qquad\qquad\qquad$ $= 32 - 48 \neq 0$
$= -9$

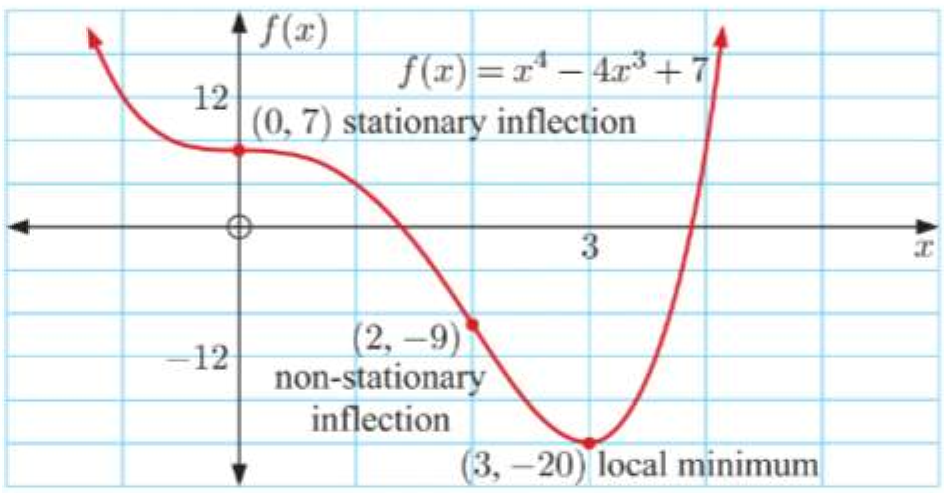
$\therefore$ $(2, -9)$ is a non-stationary inflection.

d **i** $f(x)$ is increasing for $x \geqslant 3$. $\qquad$ **ii** $f(x)$ is decreasing for $x \leqslant 3$.

iii $f(x)$ is concave up for $x \leqslant 0$ and $x \geqslant 2$.

iv $f(x)$ is concave down for $0 \leqslant x \leqslant 2$.

e



20 **a** $f(x) = \cos^2 x, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \; f'(x) = 2\cos x(-\sin x)$
$= -2\sin x \cos x$

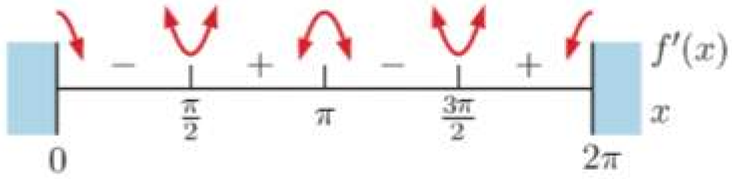
$f'(x) = 0$ when $-2\sin x \cos x = 0$

$\therefore \; \sin x = 0$ $\quad$ or $\quad$ $\cos x = 0$

$\therefore \; x = 0, \pi, 2\pi$ $\quad$ or $\quad$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore \; x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or 2π

Now $f'(x)$ has sign diagram:



$$f(0) = \cos^2 0 \qquad f\left(\tfrac{\pi}{2}\right) = \cos^2\left(\tfrac{\pi}{2}\right) \qquad f(\pi) = \cos^2 \pi$$
$$= 1 \qquad\qquad = 0 \qquad\qquad = 1$$

$$f\left(\tfrac{3\pi}{2}\right) = \cos^2\left(\tfrac{3\pi}{2}\right) \qquad f(2\pi) = \cos^2 2\pi$$
$$= 0 \qquad\qquad = 1$$

$\therefore$ there are local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$, and local minima at $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$.

b $f''(x) = -2(\cos x)\cos x - 2\sin x(-\sin x)$
$= 2\sin^2 x - 2\cos^2 x$

$f''(x) = 0$ when $2\sin^2 x - 2\cos^2 x = 0$

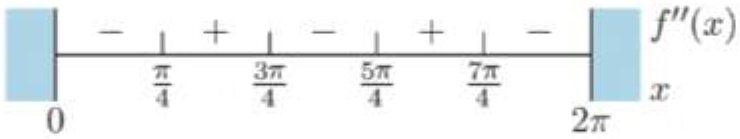
$\therefore \sin^2 x = \cos^2 x$

$\therefore \tan^2 x = 1$

$\therefore \tan x = \pm 1$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

Now $f''(x)$ has sign diagram:



$$f\left(\tfrac{\pi}{4}\right) = \cos^2\left(\tfrac{\pi}{4}\right) = \left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}$$

$$f\left(\tfrac{3\pi}{4}\right) = \cos^2\left(\tfrac{3\pi}{4}\right) = \left(-\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}$$

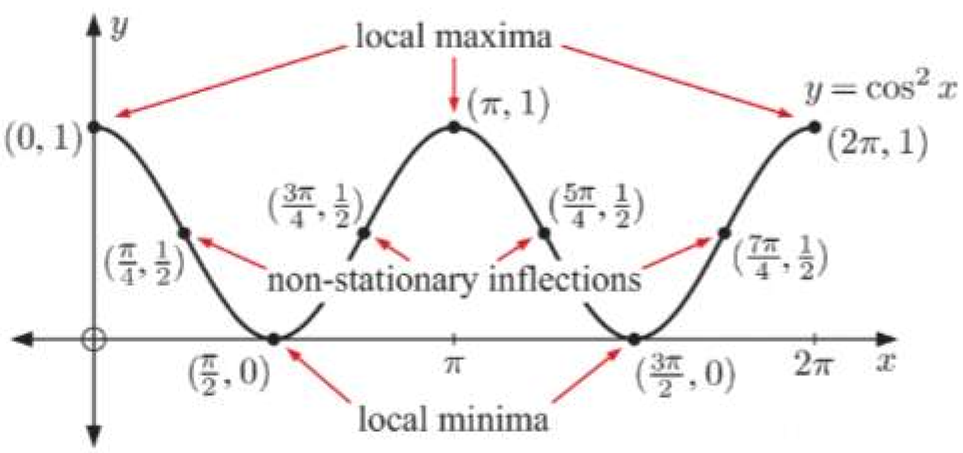
$$f\left(\tfrac{5\pi}{4}\right) = \cos^2\left(\tfrac{5\pi}{4}\right) = \left(-\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}$$

$$f\left(\tfrac{7\pi}{4}\right) = \cos^2\left(\tfrac{7\pi}{4}\right) = \left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}$$

and $f'(x) \neq 0$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

$\therefore$ there are non-stationary points of inflection at $\left(\frac{\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{3\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{5\pi}{4}, \frac{1}{2}\right)$, $\left(\frac{7\pi}{4}, \frac{1}{2}\right)$.

c



21 **a** $f(x) = \dfrac{e^x}{x-1}$

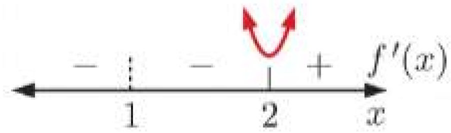
Now $f(0) = \dfrac{e^0}{-1} = -1$ so the y-intercept is -1.

b $f(x)$ is defined when $x - 1 \neq 0$

$\therefore x \neq 1$

c $f'(x) = \dfrac{e^x(x-1) - e^x(1)}{(x-1)^2}$ {quotient rule}

$= \dfrac{e^x(x-2)}{(x-1)^2}$ which has sign diagram:



$\therefore$ $f'(x) \leqslant 0$ for $x < 1$ and $1 < x \leqslant 2$ and $f'(x) \geqslant 0$ for $x \geqslant 2$

$\therefore$ $f(x)$ is decreasing for all defined values of $x \leqslant 2$, and increasing for $x \geqslant 2$.

$f''(x) = \dfrac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1(1)]}{(x-1)^4}$ {product and quotient rules}

$= \dfrac{[e^x(x-2+1)(x-1)^2] - 2e^x(x-2)(x-1)}{(x-1)^4}$

$= \dfrac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4}$

$= \dfrac{e^x(x-1)[(x-1)^2 - 2(x-2)]}{(x-1)^4}$

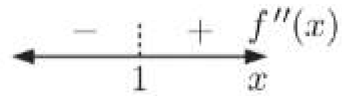
$= \dfrac{e^x(x-1)[x^2 - 2x + 1 - 2x + 4]}{(x-1)^4}$

$= \dfrac{e^x(x^2 - 4x + 5)}{(x-1)^3}$ where the quadratic term has $\Delta = (-4)^2 - 4(1)(5)$

$= 16 - 20$

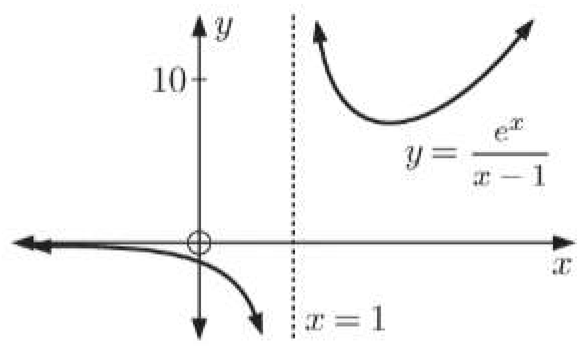
$= -4 < 0$

The sign diagram of $f''(x)$ is:



$\therefore$ $f(x)$ is concave down for all $x < 1$
and concave up for all $x > 1$.

d



e $f(2) = \dfrac{e^2}{2-1} = e^2$

Using **c**, we have a local minimum at $(2, e^2)$

$\therefore$ the tangent at $x = 2$ is horizontal with equation $y = e^2$.

Chapter 20

APPLICATIONS OF DIFFERENTIATION

EXERCISE 20A

1 **a** $P(t) = 2t^2 - 12t + 118$ thousand dollars

$P(0) = 2(0)^2 - 12(0) + 118$
$= 118$

$\therefore$ the current annual profit is \$118 000.

b $P = 2t^2 - 12t + 118$

$\therefore \dfrac{dP}{dt} = 4t - 12$ thousand dollars per year

c When $t = 8$, $\dfrac{dP}{dt} = 4(8) - 12$
$= 32 - 12$
$= 20$

This means that in 8 years from now, profits will be increasing at a rate of \$20 000 per year.

2 **a** $V = 2(50 - t)^2$ m^3

When $t = 0$, $V = 2(50)^2$
$= 2 \times 2500$
$= 5000$ m^3

When $t = 5$, $V = 2(50 - 5)^2$
$= 2(45)^2$
$= 2 \times 2025$
$= 4050$ m^3

$\therefore$ the average rate at which the water evaporates in the first 5 days is

$\dfrac{5000 - 4050}{5} = \dfrac{950}{5} = 190$ m^3 per day.

b $V = 2(50 - t)^2$

$\therefore \dfrac{dV}{dt} = 2 \times 2(50 - t)(-1)$ {chain rule}
$= -4(50 - t)$
$= 4t - 200$

When $t = 5$, $\dfrac{dV}{dt} = 4(5) - 200$
$= 20 - 200$
$= -180$

$\therefore$ the instantaneous rate at which the water is evaporating at $t = 5$ days is 180 m^3 per day.

3 **a** $Q(t) = 100 - 10\sqrt{t}$

i $Q(0) = 100 - 10\sqrt{0}$
$= 100$

ii $Q(25) = 100 - 10\sqrt{25}$
$= 50$

iii $Q(100) = 100 - 10\sqrt{100}$
$= 0$

b $Q(t) = 100 - 10\sqrt{t} = 100 - 10t^{\frac{1}{2}}$ units, $t \geqslant 0$

$\therefore\ Q'(t) = -5t^{-\frac{1}{2}}$

$= -\dfrac{5}{\sqrt{t}}$ units per year

i $Q'(25) = -\frac{5}{\sqrt{25}} = -1$

$\therefore$ when the person is aged 25 years, the quantity of the chemical is decreasing by 1 unit per year.

ii $Q'(50) = -\frac{5}{\sqrt{50}}$

$= -\frac{5}{5\sqrt{2}}$

$= -\frac{1}{\sqrt{2}}$

$\therefore$ when the person is aged 50 years, the quantity of the chemical is decreasing by $\frac{1}{\sqrt{2}}$ units per year.

c $Q'(t) = -\dfrac{5}{\sqrt{t}} < 0$ for all $t > 0$

$\therefore$ the quantity of the chemical is decreasing for all $t > 0$.

4 **a** **i** $h = 0.1t^2 + 0.15t$

$\therefore\ \dfrac{dh}{dt} = 0.2t + 0.15$ metres per year

ii When $t = 1$, $\dfrac{dh}{dt} = 0.2(1) + 0.15$

$= 0.35$ metres per year

After 1 year, the tree was growing at a rate of 0.35 metres per year.

iii The tree was growing at 75 cm per year when $\dfrac{dh}{dt} = 0.75$

$\therefore\ 0.2t + 0.15 = 0.75$ {from **i**}

$\therefore\ 0.2t = 0.6$

$\therefore\ t = 3$

It was growing at 75 cm per year after 3 years.

iv When $t = 4$, $h = 0.1(4)^2 + 0.15(4)$

$= 2.2$

After 4 years, the tree was 2.2 m tall.

b $H = 20 - \dfrac{k}{t}$ metres, $t \geqslant 4$

i From **a iv**, the tree was 2.2 m tall after 4 years, so when $t = 4$, $H = 2.2$

$\therefore\ 20 - \dfrac{k}{4} = 2.2$

$\therefore\ 17.8 = \dfrac{k}{4}$

$\therefore\ k = 71.2$

ii
$$\begin{aligned} H &= 20 - \frac{71.2}{t} \\ &= 20 - 71.2t^{-1} \\ \therefore \quad \frac{dH}{dt} &= 71.2t^{-2} \\ &= \frac{71.2}{t^2} \end{aligned}$$

iii When $t = 4$, $\frac{dh}{dt} = 0.2(4) + 0.15 = 0.95$ metres per year and $\frac{dH}{dt} = \frac{71.2}{4^2} = 4.45$ metres per year

The rate at which the tree is growing changes when it is taken out of its pot and planted in the ground, since $\frac{dh}{dt} \neq \frac{dH}{dt}$ when $t = 4$.

iv
$$\begin{aligned} t^2 &> 0 \quad \text{for all } t \geqslant 4 \\ \therefore \quad \frac{1}{t^2} &> 0 \quad \text{for all } t \geqslant 4 \\ \therefore \quad \frac{dH}{dt} = \frac{71.2}{t^2} &> 0 \quad \text{for all } t \geqslant 4 \end{aligned}$$

This means that according to the model, the tree will never stop growing.

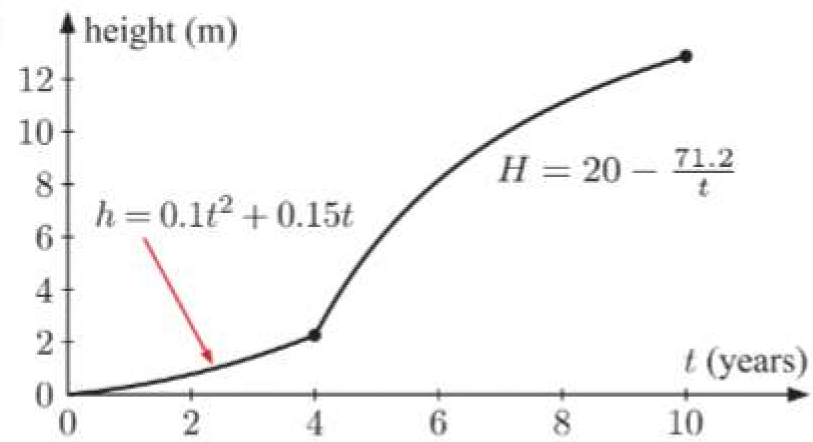
v When $t = 10$, $H = 20 - \frac{71.2}{10} = 12.88$ When $t = 20$, $H = 20 - \frac{71.2}{20} = 16.44$

When $t = 50$, $H = 20 - \frac{71.2}{50} = 18.576$

So, after 10 years, height $= 12.88$ m
after 20 years, height $= 16.44$ m
after 50 years, height $= 18.576$ m

The rate of growth is slowing as the years progress.

c



5 $C(x) = 7800 + 6x + 12x^{0.7}$ dollars

a The marginal cost function is $C'(x) = 6 + 12(0.7x^{-0.3})$
$= 6 + 8.4x^{-0.3}$ dollars per pair

b $C'(220) = 6 + 8.4(220)^{-0.3}$
$\approx \$7.67$

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) = 7800 + 6(221) + 12(221)^{0.7} - \left(7800 + 6(220) + 12(220)^{0.7}\right)$
$\approx \$7.66$

This is the actual cost of making the 221st pair of jeans.
The answer in **b** is a very good estimate.

6 **a** $C(v) = \frac{1}{5}v^2 + \dfrac{200\,000}{v}$ euros

i $C(50) = \frac{1}{5}(50)^2 + \dfrac{200\,000}{50}$
$= 500 + 4000$
$= 4500$

$\therefore$ if the average speed is 50 km h^{-1}, the total cost of the journey is 4500 euros.

ii $C(100) = \frac{1}{5}(100)^2 + \dfrac{200\,000}{100}$
$= 2000 + 2000$
$= 4000$

$\therefore$ if the average speed is 100 km h^{-1}, the total cost of the journey is 4000 euros.

b $C(v) = \frac{1}{5}v^2 + \dfrac{200\,000}{v}$ euros, $v > 0$
$= \frac{1}{5}v^2 + 200\,000v^{-1}$
$\therefore C'(v) = \frac{2}{5}v - 200\,000v^{-2}$
$= \frac{2}{5}v - \dfrac{200\,000}{v^2}$ euros per km h^{-1}

i $C'(30) = \frac{2}{5}(30) - \dfrac{200\,000}{30^2}$
$= 12 - \dfrac{2000}{9}$
≈ -210.22

$\therefore$ if the average speed is 30 km h^{-1}, the rate of change in the cost of running the train is decreasing at about 210.22 euros per km h^{-1}.

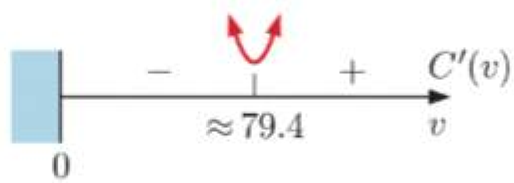
ii $C'(90) = \frac{2}{5}(90) - \dfrac{200\,000}{90^2}$
$= 36 - \dfrac{2000}{81}$
≈ 11.31

$\therefore$ if the average speed is 90 km h^{-1}, the rate of change in the cost of running the train is increasing at about 11.31 euros per km h^{-1}.

c $C(v)$ is a minimum when $C'(v) = 0$

$\therefore \frac{2}{5}v - \dfrac{200\,000}{v^2} = 0$
$\therefore \frac{2}{5}v^3 - 200\,000 = 0$
$\therefore \frac{2}{5}v^3 = 200\,000$
$\therefore v^3 = 500\,000$
$\therefore v \approx 79.4 \text{ km h}^{-1}$

$C'(v)$ has sign diagram:

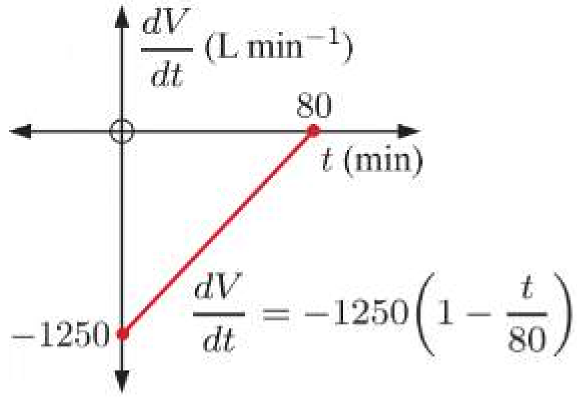


$\therefore$ the cost of running the train is a minimum when the average speed of the train is about 79.4 km h^{-1}.

7 **a** $V = 50\,000\left(1-\frac{t}{80}\right)^2$ L, $0 \leqslant t \leqslant 80$

$\therefore \frac{dV}{dt} = 100\,000\left(1-\frac{t}{80}\right)\left(-\frac{1}{80}\right)$ {chain rule}

$= -1250\left(1-\frac{t}{80}\right)$ L min^{-1}



b The outflow is fastest when $\frac{dV}{dt} = -1250\left(1-\frac{t}{80}\right)$ is smallest.

Looking at the graph in **a**, the minimum value of $\frac{dV}{dt}$ is -1250 L min^{-1} which occurs when $t = 0$.

So, the outflow is fastest at $t = 0$, when the tap was first opened.

c $\frac{dV}{dt} = -1250\left(1-\frac{t}{80}\right)$

$= -1250 + \frac{125}{8}t$

$\therefore \frac{d^2V}{dt^2} = \frac{125}{8} > 0$

This shows that the rate of change of V is constantly increasing, so the outflow is increasing at a constant rate.

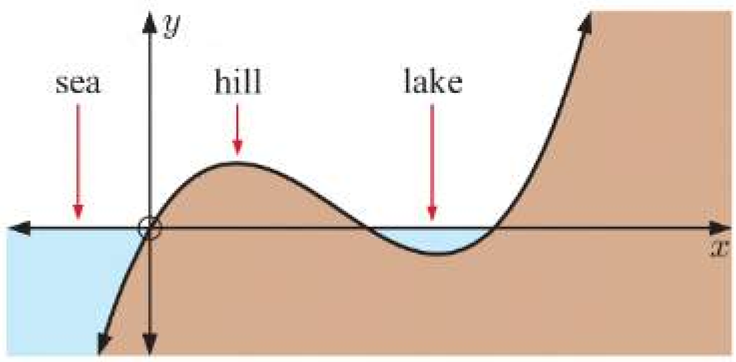
8 **a** $y = \frac{1}{10}x(x-2)(x-3)$

The edges of the lake correspond to values of x such that $y = 0$.

$\therefore \frac{1}{10}x(x-2)(x-3) = 0$

$\therefore x = 0, 2, \text{ or } 3$

From the graph, we can see that the near part of the lake is 2 km from the sea, and the furthest part is 3 km.



b $y = \frac{1}{10}x(x-2)(x-3)$

$= \frac{1}{10}x(x^2 - 5x + 6)$

$= \frac{1}{10}x^3 - \frac{1}{2}x^2 + \frac{3}{5}x$

$\therefore \frac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$

When $x = \frac{1}{2}$, $\frac{dy}{dx} = \frac{3}{10}\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{3}{5} = 0.175$

$\therefore$ the height of the hill is increasing when $x = \frac{1}{2}$ km, as the gradient is positive.

So the land is sloping upwards at this point.

When $x = 1\frac{1}{2} = \frac{3}{2}$, $\frac{dy}{dx} = \frac{3}{10}\left(\frac{3}{2}\right)^2 - \frac{3}{2} + \frac{3}{5} = -0.225$

$\therefore$ the height of the hill is decreasing when $x = 1\frac{1}{2}$ km, as the gradient is negative.

So the land is sloping downwards at this point.

This means the top of the hill is between $x = \frac{1}{2}$ km and $x = 1\frac{1}{2}$ km.

c The deepest point of the lake occurs when the slope of the land is 0, which is when $\frac{dy}{dx} = 0$.

$\therefore \quad \frac{3}{10}x^2 - x + \frac{3}{5} = 0$

$\therefore \quad x \approx 0.785 \quad \text{or} \quad 2.55 \qquad \{\text{using technology}\}$

The deepest point of the lake is a turning point which lies between the edges of the lake at $x = 2$ and $x = 3$. So we only consider the value of x which lies between 2 and 3.

When $x = 2.55$, $\quad y \approx -0.0631$ km
≈ -63.1 m

So, the deepest point of the lake is about 2.55 km from the sea, and about 63.1 m deep.

9 $W = 20e^{-0.01t}$

a **i** When $t = 0$, $\quad W = 20e^0$
$= 20$

$\therefore$ there are initially 20 grams of radioactive substance present.

ii When $t = 24$, $\quad W = 20e^{-0.01\times 24}$
$= 20e^{-0.24}$
≈ 15.7

$\therefore$ after 24 hours, there are about 15.7 grams of radioactive substance present.

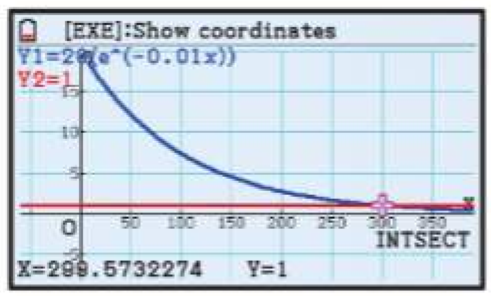
iii 1 week $= 7 \times 24 = 168$ hours

When $t = 168$, $\quad W = 20e^{-0.01\times 168}$
$= 20e^{-1.68}$
≈ 3.73

$\therefore$ after 1 week, there are about 3.73 grams of radioactive substance present.

b To find when $W = 1$, we graph $Y_1 = 20e^{-0.01x}$ and $Y_2 = 1$ on the same set of axes, and find their point of intersection.

So, the weight reaches 1 gram after about 300 hours, which is about 12 days, 12 hours.



c $\frac{dW}{dt} = -0.01 \times 20e^{-0.01t}$
$= bW \qquad \{\text{where } b = -0.01 \text{ is constant}\}$

d **i** When $t = 100$, $\quad \frac{dW}{dt} = -0.01 \times 20e^{-0.01\times 100}$
$= -0.2e^{-1}$
≈ -0.0736

$\therefore$ after 100 hours, the rate of radioactive decay is about -0.0736 grams per hour.

ii When $t = 1000$, $\quad \frac{dW}{dt} = -0.01 \times 20e^{-0.01\times 1000}$
$= -0.2e^{-10}$
$\approx -9.08 \times 10^{-6}$

$\therefore$ after 1000 hours, the rate of radioactive decay is about -9.08×10^{-6} grams per hour.

10 $v = 5\sqrt{t+1} - 5 \text{ m s}^{-1}, \quad P = 1600v \text{ kg m s}^{-1}$

a $P = 1600(5\sqrt{t+1} - 5)$
$= 8000\sqrt{t+1} - 8000 \text{ kg m s}^{-1}$

b $P = 8000(t+1)^{\frac{1}{2}} - 8000$

$\therefore \dfrac{dP}{dt} = \frac{1}{2} \times 8000(t+1)^{-\frac{1}{2}} \times 1 - 0$

$= \dfrac{4000}{\sqrt{t+1}} \text{ kg m s}^{-1}$ per second

When $t = 3$, $\dfrac{dP}{dt} = \dfrac{4000}{\sqrt{4}} = 2000$

The car's momentum is increasing by 2000 kg m s^{-1} per second after 3 seconds.

c $\dfrac{dP}{dt} = 800$

$\therefore \dfrac{4000}{\sqrt{t+1}} = 800$

$\therefore \sqrt{t+1} = 5$

$\therefore t+1 = 25$

$\therefore t = 24$

The car's momentum is increasing by 800 kg m s^{-1} per second after 24 seconds.

11 $T = 5 + 95e^{-0.12t}$ °C

a When $t = 15$, $T = 5 + 95e^{-0.12 \times 15}$
≈ 20.7

$\therefore$ after 15 minutes, the temperature of the liquid is about 20.7°C.

b $\dfrac{dT}{dt} = 0 + (-0.12)95e^{-0.12t}$
$= -0.12 \times 95e^{-0.12t}$
$= -0.12(5 + 95e^{-0.12t} - 5)$
$= c(T - 5)$ {where $c = -0.12$ is constant}

c **i** When $t = 0$, $\dfrac{dT}{dt} = -0.12 \times 95e^0$
$= -0.12 \times 95$
$= -11.4$

$\therefore$ the temperature is initially decreasing by 11.4°C per minute.

ii When $t = 10$, $\dfrac{dT}{dt} = -0.12 \times 95e^{-0.12 \times 10}$
$= -11.4e^{-1.2}$
≈ -3.43

$\therefore$ after 10 minutes, the temperature is decreasing by about 3.43°C per minute.

iii When $t = 20$, $\dfrac{dT}{dt} = -0.12 \times 95e^{-0.12 \times 20}$
$= -11.4e^{-2.4}$
≈ -1.03

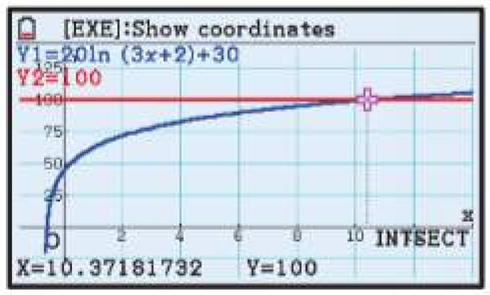
$\therefore$ after 20 minutes, the temperature is decreasing by about 1.03°C per minute.

12 **a** $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$

$\therefore H(0) = 20\ln 2 + 30$
≈ 43.9

$\therefore$ the shrub was about 43.9 cm high when it was planted.

b To find when $H(t) = 100$ cm we graph $Y_1 = 20\ln(3x+2) + 30$ and $Y_2 = 100$ on the same set of axes, and find their point of intersection.
So, the shrub will reach a height of 1 m after about 10.4 years, which is about 10 years, 5 months.



c $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$

$$\therefore\ H'(t) = 20 \times \frac{3}{3t+2} = \frac{60}{3t+2}$$

i $$H'(3) = \frac{60}{3(3)+2} = \frac{60}{11} \approx 5.45$$

$\therefore$ 3 years after being planted, the shrub is growing at about 5.45 cm per year.

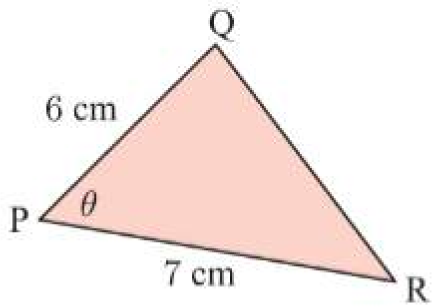
ii $$H'(10) = \frac{60}{3(10)+2} = \frac{60}{32} = 1.875$$

$\therefore$ 10 years after being planted, the shrub is growing at 1.875 cm per year.

13 Triangle PQR has area $A = \frac{1}{2} \times 6 \times 7 \times \sin\theta$

$$\therefore\ A = 21\sin\theta \text{ cm}^2$$

$$\therefore\ \frac{dA}{d\theta} = 21\cos\theta \text{ cm}^2 \text{ per radian}$$

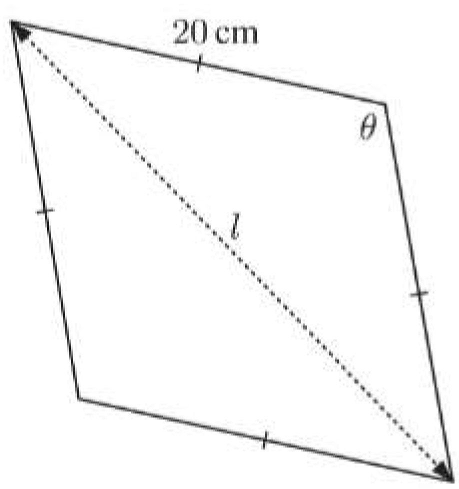


When $\theta = 45° = \frac{\pi}{4}$, $\frac{dA}{d\theta} = 21\cos\frac{\pi}{4}$
$= 21 \times \frac{1}{\sqrt{2}}$
$= \frac{21}{\sqrt{2}}$ cm^2 per radian, or $\frac{21}{\sqrt{2}} \times \frac{\pi}{180} \approx 0.259$ cm^2 per degree.

$\therefore$ the area of triangle PQR is changing at a rate of $\frac{21}{\sqrt{2}}$ cm^2 per radian (or about 0.259 cm^2 per degree) at the time when $\theta = 45°$.

14 **a** Using the cosine rule,

$$l^2 = 20^2 + 20^2 - 2 \times 20 \times 20 \times \cos\theta$$
$$\therefore\ l^2 = 400 + 400 - 800\cos\theta$$
$$\therefore\ l^2 = 800 - 800\cos\theta$$
$$\therefore\ l = \sqrt{800 - 800\cos\theta} \text{ cm} \quad \{l > 0\}$$



b $$l = (800 - 800\cos\theta)^{\frac{1}{2}}$$

$$\therefore\ \frac{dl}{d\theta} = \frac{1}{2}(800 - 800\cos\theta)^{-\frac{1}{2}}(800\sin\theta)$$
$$= \frac{400\sin\theta}{\sqrt{800 - 800\cos\theta}} \text{ cm per radian}$$

When $\theta = 120^\circ = \frac{2\pi}{3}$, $\displaystyle \frac{dl}{d\theta} = \frac{400\sin\frac{2\pi}{3}}{\sqrt{800 - 800\cos\frac{2\pi}{3}}}$

$$= \frac{400\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{800 - 800\left(-\frac{1}{2}\right)}}$$

$$= \frac{200\sqrt{3}}{\sqrt{800+400}}$$

$$= \frac{200\sqrt{3}}{\sqrt{1200}}$$

$$= \frac{200\sqrt{3}}{20\sqrt{3}}$$

$= 10$ cm per radian, or $10 \times \frac{\pi}{180} \approx 0.175$ cm per degree.

$\therefore$ l is changing at a rate of 10 cm per radian (or about 0.175 cm per degree) at the time when $\theta = 120^\circ$.

15 **a** $V(t) = 340\sin(100\pi t)$ volts

i $V(0) = 340\sin(100\pi(0))$
$= 340\sin 0$
$= 0$

$\therefore$ there are initially 0 volts in the circuit.

ii $V(0.125) = 340\sin(100\pi(0.125))$
$= 340\sin(12.5\pi)$
$= 340$

$\therefore$ there are 340 volts in the circuit after 0.125 seconds.

b $V(t) = 340\sin(100\pi t)$ volts

$\therefore$ $V'(t) = 34\,000\pi\cos(100\pi t)$ volts per second

i $V'(0.01) = 34\,000\pi\cos(100\pi(0.01))$
$= 34\,000\pi\cos\pi$
$= -34\,000\pi$

$\therefore$ after 0.01 seconds, the voltage is changing at $-34\,000\pi$ volts per second.

ii $V(t)$ is a maximum when $V'(t) = 0$.
So the voltage is changing at 0 volts per second.

16 $\displaystyle B(t) = \frac{3000}{1 + 0.5e^{-1.73t}}$

a $\displaystyle B(0) = \frac{3000}{1+0.5e^{-1.73(0)}}$

$$= \frac{3000}{1+0.5e^{0}}$$

$$= \frac{3000}{1.5}$$

$= 2000$

$\therefore$ the initial bee population is 2000.

b $B(1) = \dfrac{3000}{1 + 0.5e^{-1.73(1)}}$

≈ 2756

The percentage increase after 1 month $\approx \dfrac{2756 - 2000}{2000} \times 100\%$

$\approx 37.8\%$

$\therefore$ the percentage increase in the population after 1 month is about 37.8%.

c As $t \to \infty$, $e^{-1.73t} \to 0$

$\therefore \dfrac{3000}{1 + 0.5e^{-1.73t}} \to \dfrac{3000}{1} = 3000$

$\therefore$ the population is limited to 3000 bees.

d $B(t) = \dfrac{3000}{1 + 0.5e^{-1.73t}} = 3000(1 + 0.5e^{-1.73t})^{-1}$ bees

$\therefore B'(t) = -3000(1 + 0.5e^{-1.73t})^{-2}(0.5(-1.73)e^{-1.73t})$ {chain rule}

$= \dfrac{2595}{e^{1.73t}(1 + 0.5e^{-1.73t})^2}$ bees per month

> 0 for all t

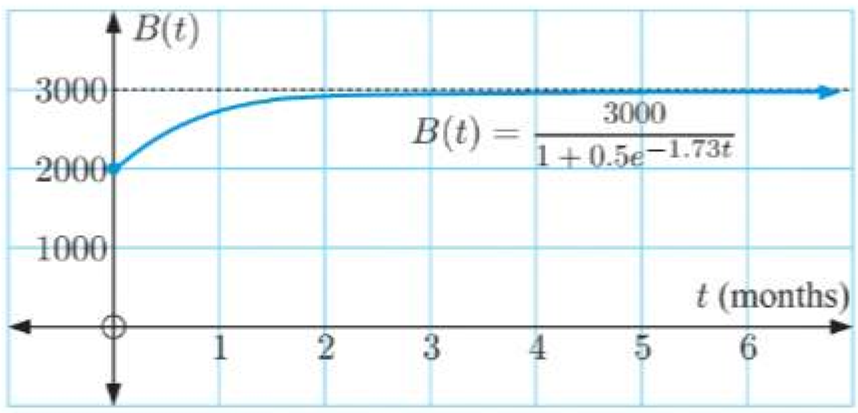
$\therefore$ the population is increasing over time.

e $B'(6) = \dfrac{2595}{e^{1.73(6)}(1 + 0.5e^{-1.73(6)})^2}$

≈ 0.0806

$\therefore$ after 6 months, the population is increasing at about 0.0806 bees per month.

f



EXERCISE 20B

1 $P(x) = -0.022x^2 + 11x - 720$

$\therefore P'(x) = -0.044x + 11$

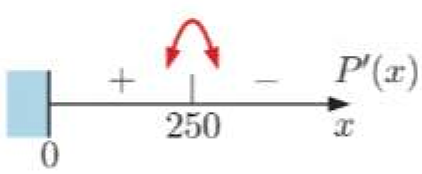
Now, $P'(x) = 0$ when $-0.044x + 11 = 0$

$\therefore 11 = 0.044x$

$\therefore x = \dfrac{11}{0.044}$

$= 250$

So, $P'(x)$ has sign diagram:



So, the profit is maximised when 250 items are made per day.

2 Production cost $C(x) = \frac{1}{4}x^2 + 8x + 20$ pounds

Selling price $p(x) = 23 - \frac{1}{2}x$ pounds per blanket

Revenue $R(x) = xp(x) = 23x - \frac{1}{2}x^2$ pounds

Profit $P(x) = \text{revenue} - \text{cost}$

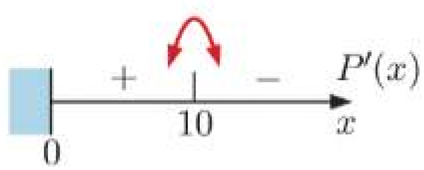
$$= (23x - \tfrac{1}{2}x^2) - (\tfrac{1}{4}x^2 + 8x + 20)$$
$$= -\tfrac{3}{4}x^2 + 15x - 20$$

$\therefore \; P'(x) = -\frac{3}{2}x + 15$

Now $P'(x) = 0$ when $-\frac{3}{2}x + 15 = 0$

$$\therefore \; x = \frac{15}{\frac{3}{2}} = 10$$

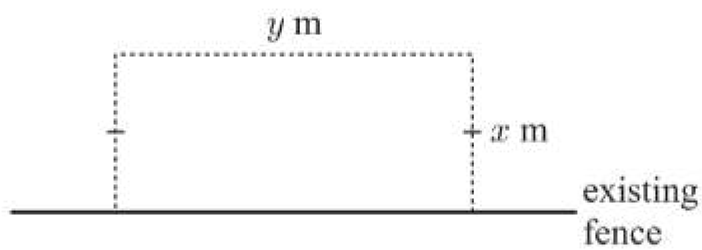
$P'(x)$ has sign diagram:



So, the profit is maximised when 10 blankets are produced per day.

3 **a** Let the remaining fence have length y m.

The total length of the fence is 60 m



$\therefore \; 2x + y = 60$

$\therefore \; y = 60 - 2x$

The area of the enclosure $A = \text{width} \times \text{length}$

$$= xy$$
$$= x(60 - 2x) \text{ m}^2$$

$\therefore$ the area of the enclosure is given by $A(x) = x(60 - 2x)$ m^2.

b $A(x) = x(60 - 2x)$

$= 60x - 2x^2$

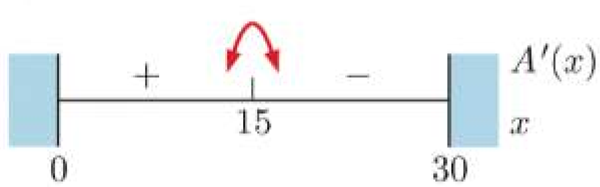
$\therefore \; A'(x) = 60 - 4x$

So, $A'(x) = 0$ when $60 - 4x = 0$

$\therefore \; 4x = 60$

$\therefore \; x = 15$

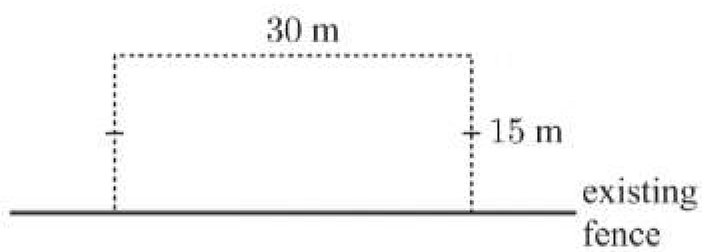
$A'(x)$ has sign diagram:



The area is maximised when $x = 15$

and $y = 60 - 2(15)$

$= 30$



The area of the enclosure is maximised by constructing a fence with width 15 m and length 30 m.

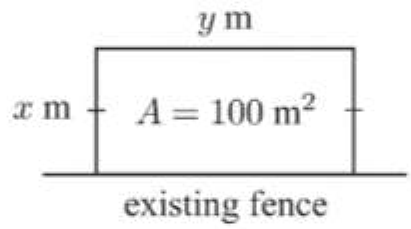
4 **a** Now $A = 100$

$\therefore\ xy = 100$

$\therefore\ y = \dfrac{100}{x}$

So, $L = 2x + y$

$\therefore\ L = 2x + \dfrac{100}{x}$



b $L = 2x + 100x^{-1}$

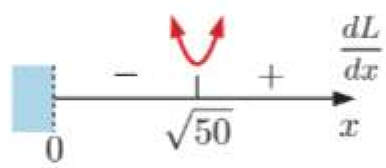
$\therefore\ \dfrac{dL}{dx} = 2 - 100x^{-2} = 2 - \dfrac{100}{x^2}$

which is 0 when $\dfrac{100}{x^2} = 2$

$\therefore\ x^2 = 50$

$\therefore\ x = \sqrt{50}\quad \{x > 0\}$

So, $\dfrac{dL}{dx}$ has sign diagram:



So, the length is a minimum when $x = \sqrt{50}$.

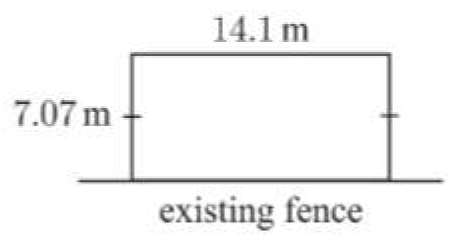
When $x = \sqrt{50}$, $L = 2\sqrt{50} + \dfrac{100}{\sqrt{50}}$

≈ 28.3

So, the minimum value of L is about 28.3 m which occurs when $x = \sqrt{50} \approx 7.07$.

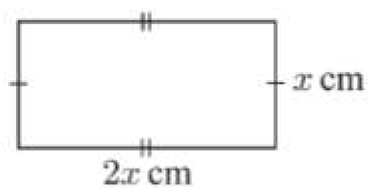
c $x = \sqrt{50} \approx 7.07$ and $y = \dfrac{100}{\sqrt{50}} \approx 14.1$

So, the optimal situation is:



5 **a** The base has dimensions in the ratio $2 : 1$.

$\therefore$ if one side is x cm, then the other side must be $2x$ cm.



$V =$ area of base $\times$ height

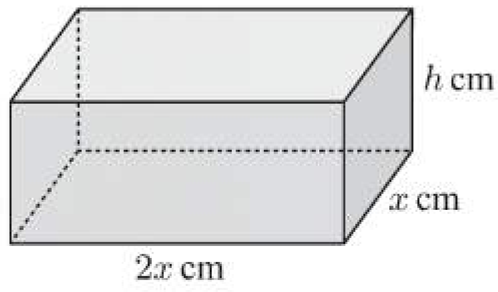
$= 2x \times x \times h$

$= 2x^2h$

but $V = 200\text{ cm}^3$, $\therefore\ 2x^2h = 200$

$\therefore\ x^2h = 100$

b $x^2h = 100$ {from **a**}



$\therefore\ h = \dfrac{100}{x^2}$

Inner surface area

$$A(x) = 2(2x \times x) + 2\left(x \times \frac{100}{x^2}\right) + 2\left(2x \times \frac{100}{x^2}\right)$$
$$= 2\left(2x^2 + \frac{100}{x} + \frac{200}{x}\right)$$
$$= 2\left(2x^2 + \frac{300}{x}\right)$$

$$\therefore\ A(x) = 4x^2 + \frac{600}{x}\ \text{cm}^2$$

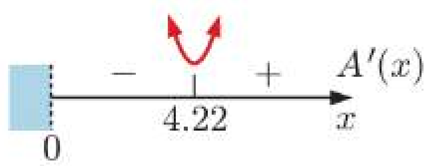
c $A(x) = 4x^2 + 600x^{-1}$

$$\therefore\ A'(x) = 8x - 600x^{-2}$$
$$= 8x - \frac{600}{x^2}$$

$A'(x) = 0$ when $8x - \dfrac{600}{x^2} = 0$

$$\therefore\ 8x = \frac{600}{x^2}$$
$$\therefore\ 8x^3 = 600$$
$$\therefore\ x^3 = 75$$
$$\therefore\ x = \sqrt[3]{75} \approx 4.22$$

$A'(x)$ has sign diagram:



So, the inner surface area of the box is a minimum when $x = \sqrt[3]{75} \approx 4.22$.

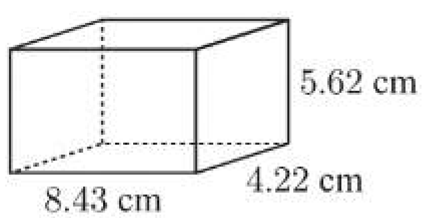
$$A(\sqrt[3]{75}) = 4(\sqrt[3]{75})^2 + \frac{600}{\sqrt[3]{75}}$$
$$\approx 213$$

So, the minimum inner surface area is about 213 cm^2, when $x = \sqrt[3]{75} \approx 4.22$.

d $x = \sqrt[3]{75} \approx 4.22$

$$\therefore\ 2x = 2\sqrt[3]{75} \approx 8.43$$
$$h = \frac{100}{x^2} = \frac{100}{(\sqrt[3]{75})^2} \approx 5.62$$

So, the optimal box shape is:



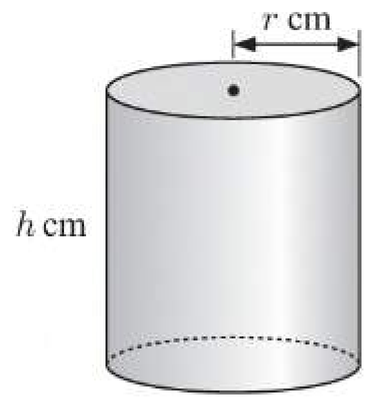
6 **a** volume of cylindrical can $= \pi r^2 h$

Now, capacity is 1 litre which is equivalent to 1000 cm^3.

So, the volume $= 1000$ cm^3

$$\therefore \quad \pi r^2 h = 1000$$

$$\therefore \quad h = \frac{1000}{\pi r^2} \text{ cm}$$



b Total surface area of cylindrical can $A = 2\pi r^2 + 2\pi r h$

$$\therefore \quad A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \qquad \{\text{using } \mathbf{a}\}$$

$$\therefore \quad A = 2\pi r^2 + \frac{2000}{r} \text{ cm}^2$$

c $$A = 2\pi r^2 + 2000r^{-1}$$

$$\therefore \quad \frac{dA}{dr} = 4\pi r - 2000r^{-2}$$

$$= 4\pi r - \frac{2000}{r^2}$$

$\frac{dA}{dr} = 0$ when $4\pi r - \frac{2000}{r^2} = 0$

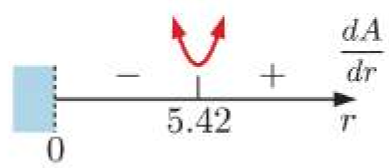
$$\therefore \quad 4\pi r = \frac{2000}{r^2}$$

$$\therefore \quad 4\pi r^3 = 2000$$

$$\therefore \quad r^3 = \frac{500}{\pi}$$

$$\therefore \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

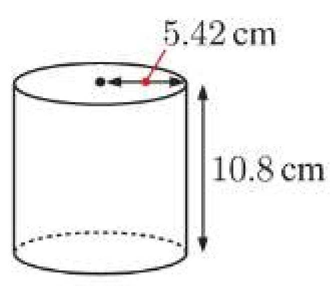
$\frac{dA}{dr}$ has sign diagram:



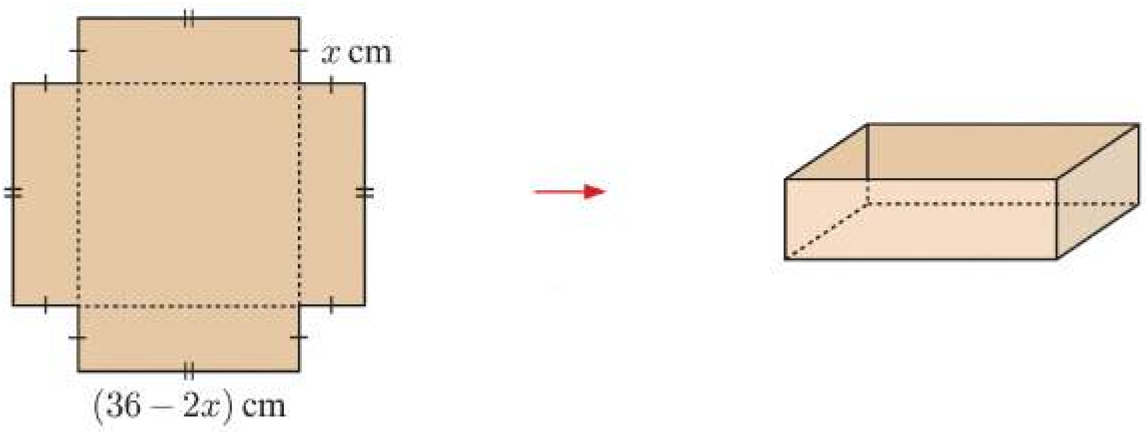
So, the total surface area is a minimum when $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$.

When $r = \sqrt[3]{\frac{500}{\pi}}$, $h = \dfrac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} \approx 10.8$

So, the can should have dimensions:



7 **a**



The volume of the container is $V(x) = \text{area of base} \times \text{height}$

$$\therefore \quad V(x) = (36-2x)^2 \times x$$

$$\therefore \quad V(x) = x(36-2x)^2 \ \text{cm}^3$$

b

$$\begin{aligned} V(x) &= x(36-2x)^2 \\ &= x(1296 - 144x + 4x^2) \\ &= 1296x - 144x^2 + 4x^3 \end{aligned}$$

$$\therefore \quad V'(x) = 1296 - 288x + 12x^2$$

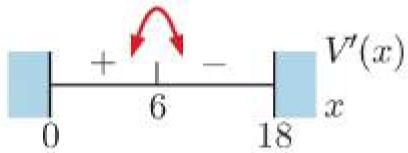
$V'(x) = 0$ when $1296 - 288x + 12x^2 = 0$

$$\therefore \quad 12(108 - 24x + x^2) = 0$$

$$\therefore \quad 12(x^2 - 24x + 108) = 0$$

$$\therefore \quad 12(x-6)(x-18) = 0$$

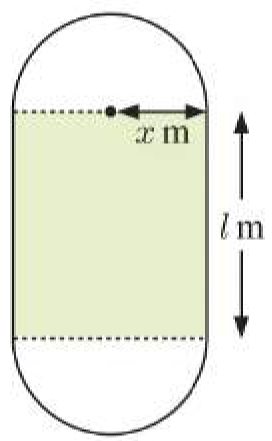
$V'(x)$ has sign diagram:



The volume is a maximum when $x = 6$.

So, $6 \text{ cm} \times 6 \text{ cm}$ squares should be cut out to produce the container of greatest capacity.

8 **a**



$$\text{Perimeter} = 2l + 2\pi x$$

$$\therefore \quad 400 = 2l + 2\pi x$$

$$\therefore \quad 2l = 400 - 2\pi x$$

$$\therefore \quad l = 200 - \pi x$$

$x > 0$ and $l > 0$ for the track to exist

$$\therefore \quad 200 - \pi x > 0$$

$$\therefore \quad \pi x < 200$$

$$\therefore \quad x < \frac{200}{\pi}$$

So, $\quad 0 < x < \dfrac{200}{\pi} \approx 63.7$

b Area of rectangle $A = 2x \times l$

$$= 2x \times (200 - \pi x)$$

$$\therefore \quad A = 400x - 2\pi x^2$$

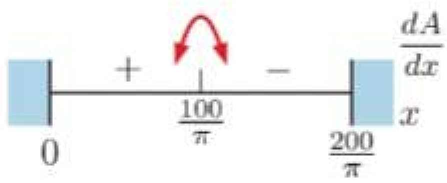
$$\therefore \quad \frac{dA}{dx} = 400 - 4\pi x$$

$\frac{dA}{dx} = 0$ when $400 - 4\pi x = 0$

$$\therefore \quad 4\pi x = 400$$

$$\therefore \quad x = \frac{100}{\pi}$$

$\frac{dA}{dx}$ has sign diagram:



The area is a maximum when $x = \frac{100}{\pi} \approx 31.8$

$$\therefore \quad l = 200 - \pi\left(\frac{100}{\pi}\right)$$

$$= 100$$

When $x = \frac{100}{\pi}$, $A = 400\left(\frac{100}{\pi}\right) - 2\pi\left(\frac{100}{\pi}\right)^2$

$$= \frac{40\,000}{\pi} - \frac{20\,000}{\pi}$$

$$= \frac{20\,000}{\pi}$$

$$\approx 6370$$

So, $l = 100$ and $x = \frac{100}{\pi} \approx 31.8$ give the maximum area $A = \frac{20\,000}{\pi} \approx 6370\ \text{m}^2$.

9 $C(x) = 4\ln x + \left(\frac{30 - x}{10}\right)^2$ pounds, $x \geqslant 10$

$$\therefore \quad C'(x) = \frac{4}{x} + 2\left(\frac{30 - x}{10}\right)\left(-\tfrac{1}{10}\right)$$

$$= \frac{4}{x} - \left(\frac{30 - x}{50}\right)$$

$$= \frac{200 - x(30 - x)}{50x}$$

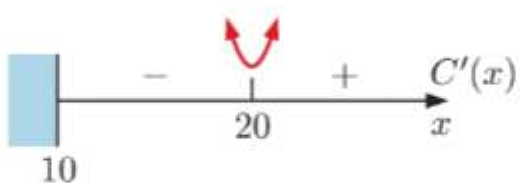
$$= \frac{x^2 - 30x + 200}{50x}$$

So, $C'(x) = 0$ when $x^2 - 30x + 200 = 0$

$$\therefore \quad (x - 10)(x - 20) = 0$$

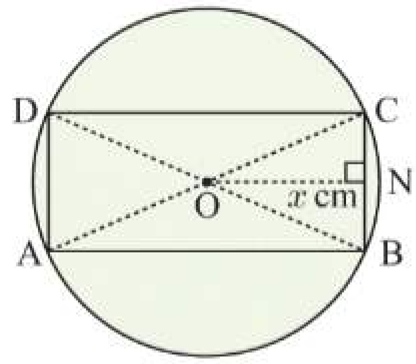
$$\therefore \quad x = 10 \text{ or } 20$$

$C'(x)$ has sign diagram:



The cost per kettle is minimised when 20 kettles are manufactured per day.

10 **a** Let CN $= y$ cm



Now $x^2 + y^2 = 5^2$ {Pythagoras}

$\therefore\ y = \sqrt{25 - x^2}$ $\{y > 0\}$

Rectangle ABCD has area $A = \text{length} \times \text{width}$

$$= 2x \times 2y$$
$$= 4xy$$
$$= 4x\sqrt{25 - x^2}\ \text{cm}^2$$

b $A = 4x\sqrt{25 - x^2} = 4x(25 - x^2)^{\frac{1}{2}}$

$$\therefore\ \frac{dA}{dx} = 4(25 - x^2)^{\frac{1}{2}} + 2x(25 - x^2)^{-\frac{1}{2}}(-2x) \qquad \text{\{product rule and chain rule\}}$$

$$= 4\sqrt{25 - x^2} - \frac{4x^2}{\sqrt{25 - x^2}}$$

$$= \frac{4(25 - x^2) - 4x^2}{\sqrt{25 - x^2}}$$

$$= \frac{100 - 4x^2 - 4x^2}{\sqrt{25 - x^2}}$$

$$= \frac{100 - 8x^2}{\sqrt{25 - x^2}}$$

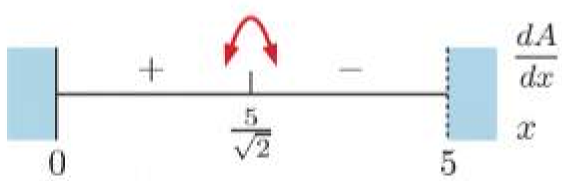
So, $\frac{dA}{dx} = 0$ when $100 - 8x^2 = 0$

$\therefore\ 8x^2 = 100$

$\therefore\ x^2 = \frac{25}{2}$

$\therefore\ x = \frac{5}{\sqrt{2}}$ {as $x > 0$}

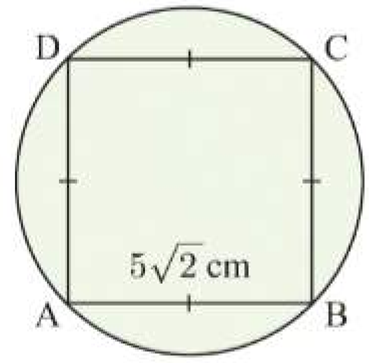
$\frac{dA}{dx}$ has sign diagram:



Area ABCD is maximised when $x = \frac{5}{\sqrt{2}}$ and $y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2}$

$$= \sqrt{25 - \frac{25}{2}}$$
$$= \sqrt{\frac{25}{2}}$$
$$= \frac{5}{\sqrt{2}}$$

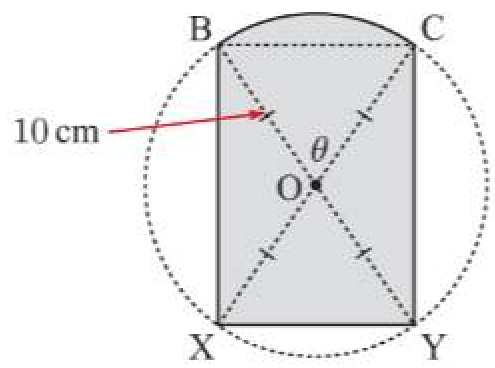
Area ABCD is maximised when it is a square with side lengths $5\sqrt{2}$ cm $\times$ $5\sqrt{2}$ cm.



11 **a** Area of sector BOC $= \frac{1}{2}\theta r^2$

$= \frac{1}{2}\theta(10)^2$

$= 50\theta \text{ cm}^2$



Now $\widehat{BOX} = \frac{\pi}{2} - \theta$ {angles on a line}

$\therefore$ area of $\triangle BOX = \frac{1}{2}ab\sin(\frac{\pi}{2} - \theta)$

$= \frac{1}{2} \times 10 \times 10 \times \sin\theta$

$\{\sin(\frac{\pi}{2} - \theta) = \sin\theta\}$

$= 50\sin\theta \text{ cm}^2$

Similarly, $\widehat{COY} = \frac{\pi}{2} - \theta$ and area of $\triangle COY = 50\sin\theta \text{ cm}^2$

Also, $\widehat{XOY} = \widehat{BOC} = \theta$ {vertically opposite angles}

$\therefore$ area of $\triangle XOY = \frac{1}{2}ab\sin\theta$

$= \frac{1}{2} \times 10 \times 10 \times \sin\theta$

$= 50\sin\theta \text{ cm}^2$

Shaded area A = area of sector BOC + area of $\triangle$BOX

+ area of $\triangle$COY + area of $\triangle$XOY

$= 50\theta + 50\sin\theta + 50\sin\theta + 50\sin\theta$

$= 50\theta + 150\sin\theta$

$= 50(\theta + 3\sin\theta) \text{ cm}^2$

b $A = 50(\theta + 3\sin\theta)$

$\therefore \frac{dA}{d\theta} = 50(1 + 3\cos\theta)$

So, $\frac{dA}{d\theta} = 0$ when $1 + 3\cos\theta = 0$

$\therefore 3\cos\theta = -1$

$\therefore \cos\theta = -\frac{1}{3}$

$\therefore \theta \approx 1.91$ $\{0 < \theta < \pi\}$

When $\theta \approx 1.91$, $A \approx 50(1.91 + 3\sin 1.91)$

≈ 237

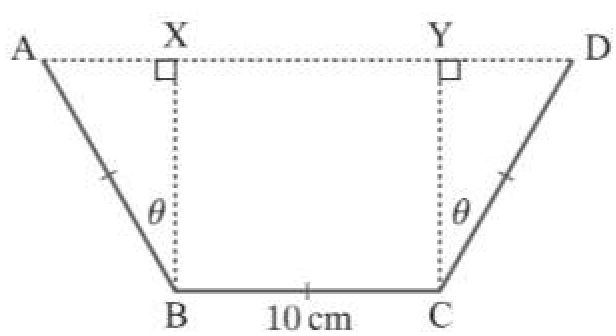
The area A has a maximum of about 237 cm^2 when $\theta \approx 1.91$.

12 **a** In $\triangle ABX$, $\cos\theta = \frac{BX}{10}$

$\therefore BX = CY = 10\cos\theta$ cm

$\sin\theta = \frac{AX}{10}$

$\therefore AX = DY = 10\sin\theta$ cm



Now, the cross-sectional area A

= area BCYX + area of $\triangle$ABX + area of $\triangle$CDY

$= 10 \times 10\cos\theta + \frac{1}{2} \times 10\sin\theta \times 10\cos\theta + \frac{1}{2} \times 10\sin\theta \times 10\cos\theta$

$= 100\cos\theta + 100\sin\theta\cos\theta$

$= 100\cos\theta(1 + \sin\theta) \text{ cm}^2$

b $$A = 100\cos\theta(1+\sin\theta)$$

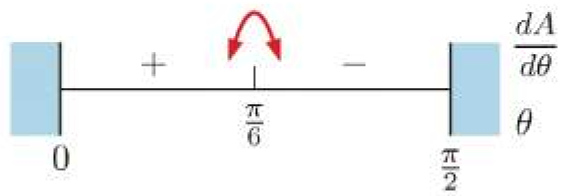
$$\therefore \quad \frac{dA}{d\theta} = 100(-\sin\theta(1+\sin\theta) + \cos\theta \times \cos\theta) \qquad \{\text{product rule}\}$$
$$= 100(-\sin\theta - \sin^2\theta + \cos^2\theta)$$
$$= 100(-\sin\theta - \sin^2\theta + 1 - \sin^2\theta)$$
$$= -100(2\sin^2\theta + \sin\theta - 1)$$
$$= -100(2\sin\theta - 1)(\sin\theta + 1)$$

$$\therefore \quad \frac{dA}{d\theta} = 0 \quad \text{when} \quad 2\sin\theta - 1 = 0 \quad \text{or} \quad \sin\theta + 1 = 0$$
$$\therefore \quad \sin\theta = \tfrac{1}{2} \quad \text{or} \quad \sin\theta = -1$$

c The gutter has maximum carrying capacity when the cross-sectional area A is maximised.

$$\frac{dA}{d\theta} = 0 \quad \text{when} \quad \sin\theta = \tfrac{1}{2} \text{ or } -1$$
$$\therefore \quad \theta = \tfrac{\pi}{6} \qquad \{0 \leqslant \theta \leqslant \tfrac{\pi}{2}\}$$

$\frac{dA}{d\theta}$ has sign diagram:



The carrying capacity is maximised when $\theta = \frac{\pi}{6}$.

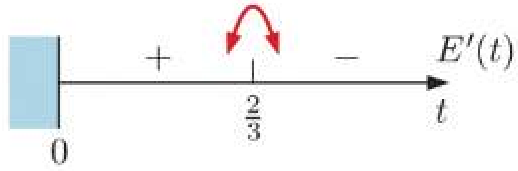
When $\theta = \frac{\pi}{6}$, $A = 100\cos\frac{\pi}{6}\left(1+\sin\frac{\pi}{6}\right)$
$$= 100 \times \tfrac{\sqrt{3}}{2} \times \tfrac{3}{2}$$
$$= 75\sqrt{3}$$
$$\approx 130$$

$\therefore$ the cross-sectional area for the maximum carrying capacity is about 130 cm^2.

13 **a** $$E(t) = 750te^{-1.5t} \text{ units}, \quad t \geqslant 0$$
$$\therefore \quad E'(t) = 750e^{-1.5t} + 750te^{-1.5t}(-1.5) \qquad \{\text{product rule}\}$$
$$= 750e^{-1.5t}(1-1.5t)$$

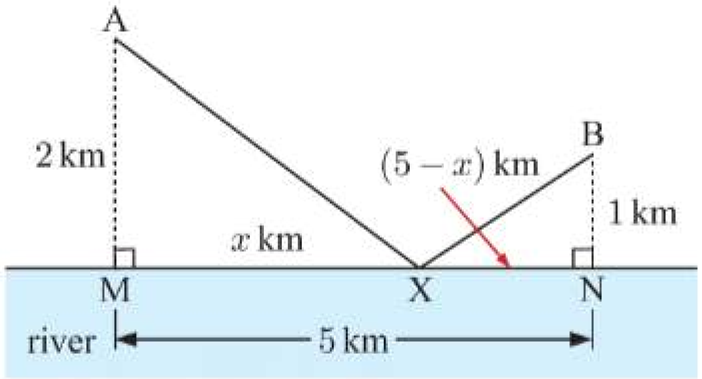
b $E'(t) = 0$ when $750e^{-1.5t}(1-1.5t) = 0$
$$\therefore \quad 1 - 1.5t = 0 \qquad \{\text{as } e^{-1.5t} > 0\}$$
$$\therefore \quad 1.5t = 1$$
$$\therefore \quad t = \tfrac{2}{3} \text{ hours} \quad (= 40 \text{ minutes})$$

$E'(t)$ has sign diagram:



The anaesthetic is most effective 40 minutes after the injection.

14



Let MX $= x$ km, so XN $= 5 - x$ km

$\therefore$ AX $= \sqrt{2^2 + x^2}$ km and XB $= \sqrt{1^2 + (5-x)^2}$ km {Pythagoras}

Let the total length of pipeline required be P km.

Now $P = \text{AX} + \text{XB}$

$$= (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\therefore \frac{dP}{dx} = \tfrac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \tfrac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(-10 + 2x) \qquad \{\text{chain rule}\}$$

$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

Now $\dfrac{dP}{dx} = 0$ when $\dfrac{x}{\sqrt{4 + x^2}} + \dfrac{x - 5}{\sqrt{x^2 - 10x + 26}} = 0$

$$\therefore \frac{x}{\sqrt{4 + x^2}} = \frac{5 - x}{\sqrt{x^2 - 10x + 26}}$$

$$\therefore \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26}$$

$$\therefore x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$

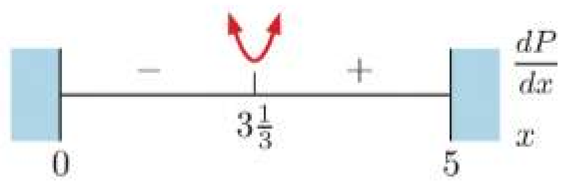
$$\therefore x^4 - 10x^3 + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - 10x^3 + x^4$$

$$\therefore -3x^2 + 40x - 100 = 0$$

$$\therefore -(3x - 10)(x - 10) = 0$$

$$\therefore x = \tfrac{10}{3} = 3\tfrac{1}{3} \qquad \{\text{as } 0 \leqslant x \leqslant 5\}$$

$\dfrac{dP}{dx}$ has sign diagram:



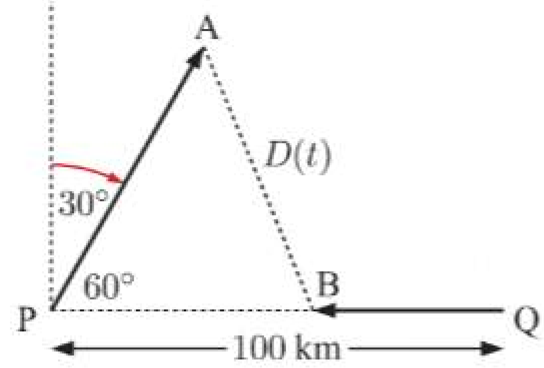
The minimum length pipeline occurs when $x = 3\frac{1}{3}$ km.

$\therefore$ X should be $3\frac{1}{3}$ km from M to minimise the total length of pipeline.

15 **a** Consider each boat's position t hours after 1:00 pm.

$PA = 12t$ and $QB = 8t$

$\therefore$ $PB = 100 - 8t$



Using the cosine rule in $\triangle PAB$,

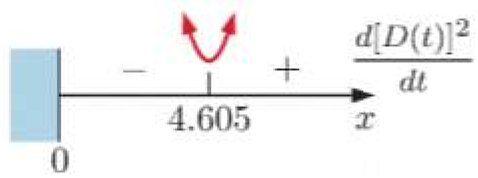
$$\begin{aligned}[D(t)]^2 &= PA^2 + PB^2 - 2\,PA \times PB\cos 60^\circ \\ &= (12t)^2 + (100-8t)^2 - 2(12t)(100-8t)(\tfrac{1}{2}) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 12t(100-8t) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\ &= 304t^2 - 2800t + 10\,000\end{aligned}$$

$\therefore$ $D(t) = \sqrt{304t^2 - 2800t + 10\,000}$ $\quad \{D(t) > 0\}$

b Now $\dfrac{d[D(t)]^2}{dt} = 608t - 2800$

$\therefore$ $\dfrac{d[D(t)]^2}{dt} = 0$ when $t = \dfrac{2800}{608} \approx 4.605$

$\dfrac{d[D(t)]^2}{dt}$ has sign diagram:



$\therefore$ $D(t)$ is a minimum when $t \approx 4.605$ hours after 1:00 pm

When $t \approx 4.605$, $[D(t)]^2 \approx 304(4.605)^2 - 2800(4.605) + 10\,000$
≈ 3550

$\therefore$ the minimum value of D^2 is about 3550.

c The ships are closest when $t \approx 4.605$ hours

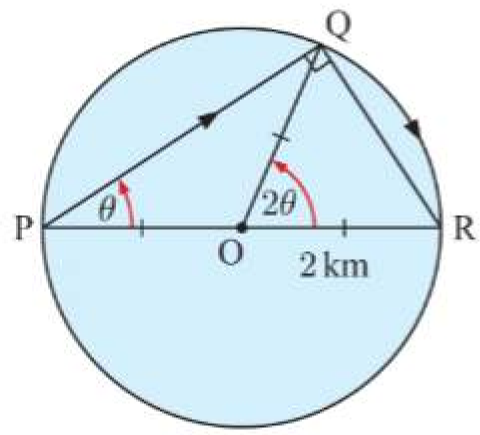
0.605 hours $\approx 0.605 \times 60$
≈ 36 minutes

$\therefore$ the ships are closest together 4 hours and 36 minutes after 1:00 pm, which is 5:36 pm.

16 **a** $P\widehat{Q}R = 90^\circ$ $\quad$ {angle in a semi-circle theorem}

$\therefore$ $\cos\theta = \dfrac{PQ}{4}$

$\therefore$ $PQ = 4\cos\theta$ km



b Hieu can row at 3 km h^{-1}.

$\therefore$ time taken to row from P to Q $= \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{4\cos\theta}{3}$

$= \frac{4}{3}\cos\theta$ hours

Now $\widehat{QOR} = 2\theta$ {angle at the centre theorem}

$\therefore$ arc QR $= 2\theta \times r$
$= 2\theta \times 2$
$= 4\theta$ km

Hieu can walk at $6\ \text{km}\,\text{h}^{-1}$.

$\therefore$ time taken to walk along arc QR $= \dfrac{\text{distance}}{\text{speed}}$
$= \dfrac{4\theta}{6}$
$= \dfrac{2\theta}{3}$ hours

$\therefore$ the time taken for Hieu's journey $T = \frac{4}{3}\cos\theta + \dfrac{2\theta}{3}$ hours where $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

c $T = \frac{4}{3}\cos\theta + \dfrac{2\theta}{3}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

$\therefore \dfrac{dT}{d\theta} = -\frac{4}{3}\sin\theta + \frac{2}{3}$

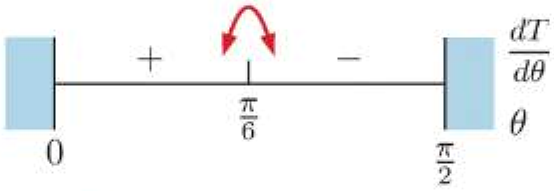
$\therefore \dfrac{dT}{d\theta} = 0$ when $-\frac{4}{3}\sin\theta + \frac{2}{3} = 0$

$\therefore \frac{4}{3}\sin\theta = \frac{2}{3}$

$\therefore \sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6} \qquad \{0 \leqslant \theta \leqslant \frac{\pi}{2}\}$

d $\dfrac{dT}{d\theta}$ has sign diagram:



e **i** From the sign diagram in **d**, the time T is a maximum when $\theta = \frac{\pi}{6}$. So the longest time taken involves Hieu rowing from P to Q at an angle of $\frac{\pi}{6}$ to the diameter of the lake, then walking from Q to R.

ii The maximum value of T occurs when $\theta = \frac{\pi}{6}$.

$\therefore$ the minimum value of T must occur at one of the end points, that is, when $\theta = 0$ or $\theta = \frac{\pi}{2}$.

When $\theta = 0$, this means Hieu must row directly from P to R.

When $\theta = \frac{\pi}{2}$, this means Hieu must walk the whole way around the shore from P to R.

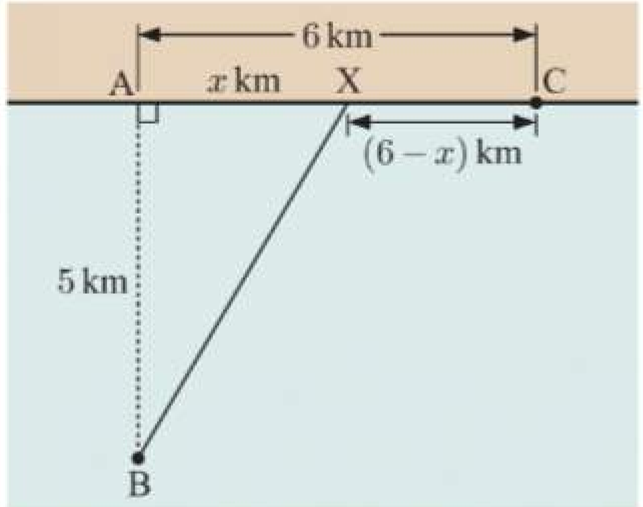
Firstly, when $\theta = 0$, $T = \frac{4}{3}\cos 0 + \dfrac{2(0)}{3} = \frac{4}{3} \approx 1.33$ hours.

Secondly, when $\theta = \frac{\pi}{2}$, $T = \frac{4}{3}\cos\frac{\pi}{2} + \dfrac{2(\frac{\pi}{2})}{3} = \frac{\pi}{3} \approx 1.05$ hours.

So, the shortest time taken involves Hieu walking the whole way around the shore from P to R.

17 **a** AC has length 6 km and X lies between A and C.

$\therefore \quad 0 \leqslant x \leqslant 6$



b Now $\quad \text{XC} = 6 - x \quad$ and $\quad \text{BX} = \sqrt{x^2 + 5^2} \quad$ {Pythagoras}

$\therefore$ the time taken to row from B to X $= \dfrac{\text{distance}}{\text{speed}} = \dfrac{\text{BX}}{8} = \dfrac{\sqrt{x^2+5^2}}{8}$ hours

and the time taken to run from X to C $= \dfrac{\text{distance}}{\text{speed}} = \dfrac{\text{XC}}{17} = \dfrac{6-x}{17}$ hours

$\therefore$ the total time $T = \dfrac{\sqrt{x^2+25}}{8} + \dfrac{6-x}{17}$ hours, $\quad 0 \leqslant x \leqslant 6$

c

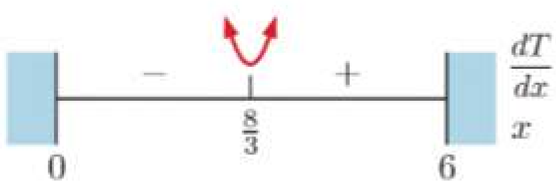
$$\begin{aligned} T &= \frac{\sqrt{x^2+25}}{8} + \frac{6-x}{17} \\ &= \tfrac{1}{8}(x^2+25)^{\frac{1}{2}} + \tfrac{6}{17} - \frac{x}{17} \end{aligned}$$

$$\begin{aligned} \therefore \quad \frac{dT}{dx} &= \tfrac{1}{16}(x^2+25)^{-\frac{1}{2}}(2x) - \tfrac{1}{17} \qquad \text{\{chain rule\}} \\ &= \frac{x}{8\sqrt{x^2+25}} - \tfrac{1}{17} \end{aligned}$$

So, $\dfrac{dT}{dx} = 0$ when $\quad \dfrac{x}{8\sqrt{x^2+25}} = \tfrac{1}{17}$

$$\begin{aligned} \therefore \quad 17x &= 8\sqrt{x^2+25} \\ \therefore \quad 289x^2 &= 64(x^2+25) \\ \therefore \quad 289x^2 &= 64x^2 + 1600 \\ \therefore \quad 225x^2 &= 1600 \\ \therefore \quad x^2 &= \tfrac{1600}{225} \\ \therefore \quad x &= \tfrac{40}{15} \qquad \{x > 0\} \\ &= \tfrac{8}{3} \approx 2.67 \end{aligned}$$

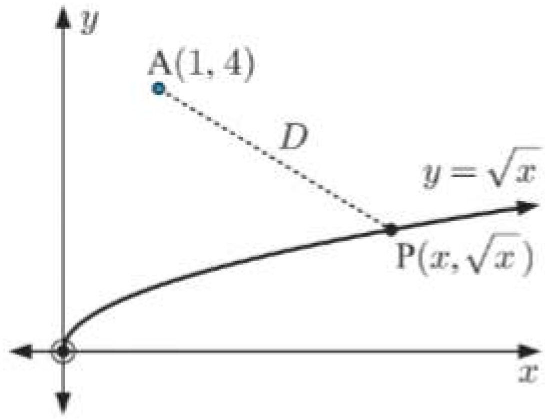
$\dfrac{dT}{dx}$ has sign diagram:



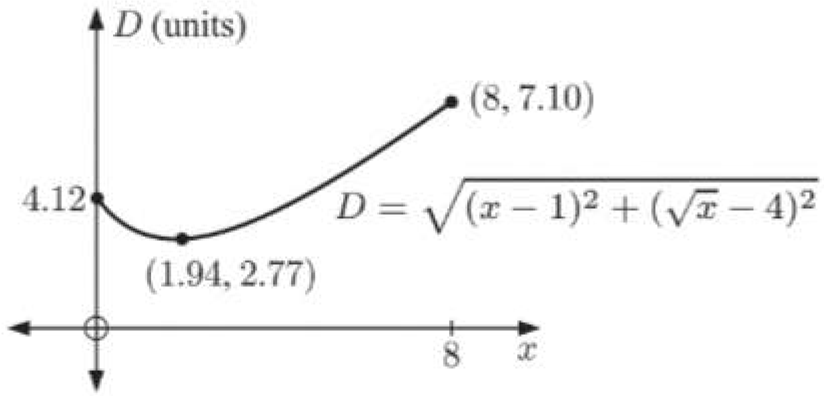
$\therefore \quad x \approx 2.67$ is the distance in km from A to X which minimises the time taken for Peter to travel from B to C.

18 **a** Using the distance formula, the distance from A(1, 4) to $P(x, \sqrt{x})$ is

$D = \sqrt{(x-1)^2 + (\sqrt{x}-4)^2}$ units



b

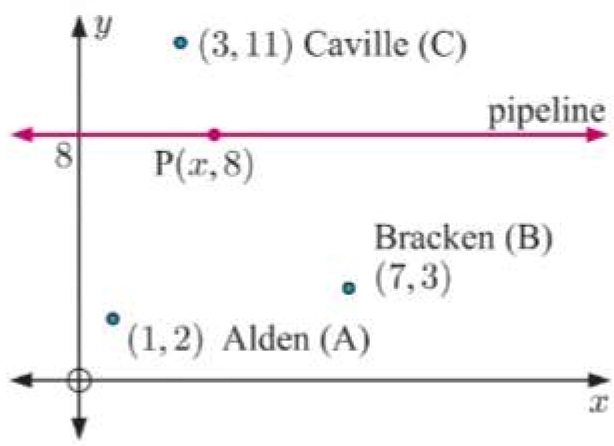


c From the graph in **b**, the minimum value of D occurs at about (1.94, 2.77), so the smallest value of D is about 2.77 units, which occurs when $x \approx 1.94$.

d Since there is a local minimum at about (1.94, 2.77), $\dfrac{dD}{dx} = 0$ when $x \approx 1.94$.

19 **a** Using the distance formula, the distance from C(3, 11) to $P(x, 8)$ is

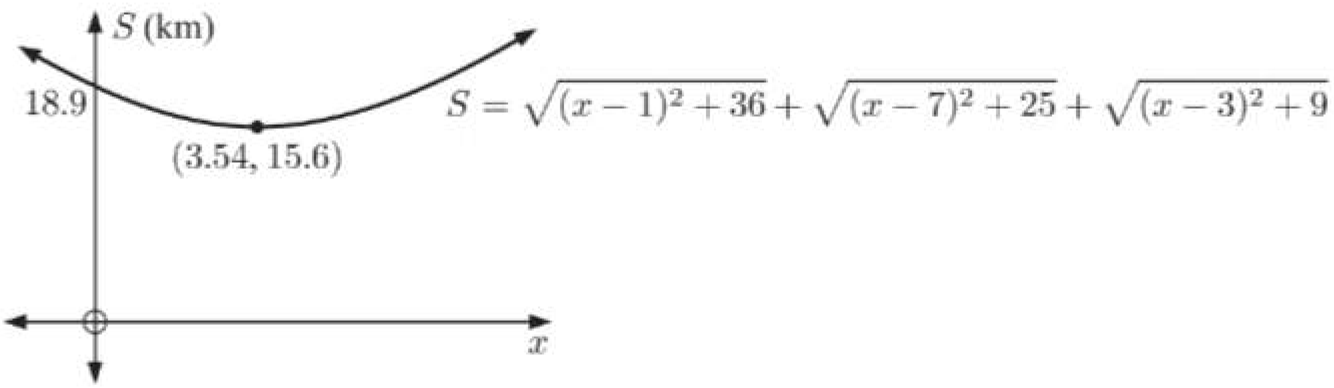
$$\begin{aligned} PC &= \sqrt{(x-3)^2 + (8-11)^2} \\ &= \sqrt{(x-3)^2 + 9} \end{aligned}$$



b Similarly,
$$\begin{aligned} PA &= \sqrt{(x-1)^2 + (8-2)^2} \\ &= \sqrt{(x-1)^2 + 36} \end{aligned}$$
and
$$\begin{aligned} PB &= \sqrt{(x-7)^2 + (8-3)^2} \\ &= \sqrt{(x-7)^2 + 25} \end{aligned}$$

c
$$\begin{aligned} S &= PA + PB + PC \\ &= \sqrt{(x-1)^2 + 36} + \sqrt{(x-7)^2 + 25} + \sqrt{(x-3)^2 + 9} \end{aligned}$$

d



e From the graph in **d**, there is a local minimum at about $(3.54, 15.6)$.

$S'(x) \leqslant 0$ when $x \leqslant 3.54$, and $S'(x) \geqslant 0$ when $x \geqslant 3.54$.

$\therefore$ $S'(x)$ has sign diagram:

f The total length of connecting pipe needed is given by $S = \text{PA} + \text{PB} + \text{PC}$.

From **e**, the minimum value of S occurs when $x \approx 3.54$.

So, P should be located at about $(3.54, 8)$.

20 $P(t) = \dfrac{50\,000}{1+1000e^{-0.5t}} = 50\,000(1+1000e^{-0.5t})^{-1}, \quad 0 \leqslant t \leqslant 25$

a $P'(t) = -50\,000(1+1000e^{-0.5t})^{-2}(-500e^{-0.5t})$

$$= \frac{25\,000\,000e^{-0.5t}}{(1+1000e^{-0.5t})^2}$$

b $P'(10) = \dfrac{25\,000\,000e^{-5}}{(1+1000e^{-5})^2}$

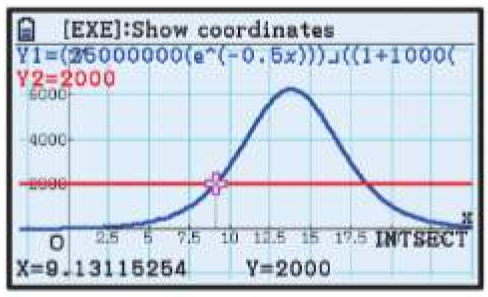
≈ 2813

$\therefore$ after 10 weeks, the population is increasing by about 2813 wasps per week.

c To find when $P'(t) = 2000$, we graph

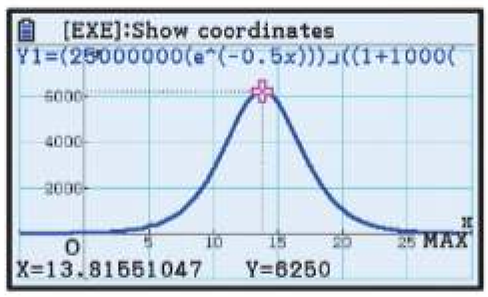
$\text{Y}_1 = \dfrac{25\,000\,000e^{-0.5x}}{(1+1000e^{-0.5x})^2}$ and $\text{Y}_2 = 2000$ on the same set of axes for $0 \leqslant x \leqslant 25$, and find their point of intersection.

So, the population is increasing at 2000 wasps per week after about 9.13 weeks, and about 18.5 weeks.



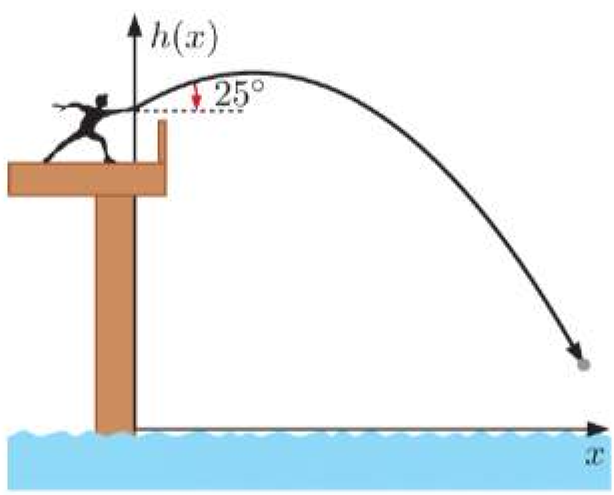
d Using technology, the maximum value of $P'(t)$ on $0 \leqslant t \leqslant 25$ is 6250, which occurs when $t \approx 13.8$.

So, the wasp population is growing fastest after about 13.8 weeks.



EXERCISE 20C

1 $h(x) = ax^2 + bx + c$ metres



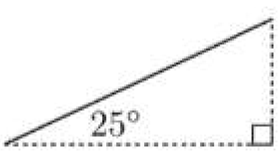
a The stone is thrown from 3 m above the water,

so $h(0) = 3$

$\therefore$ $c = 3$

b $h'(x) = 2ax + b$

c



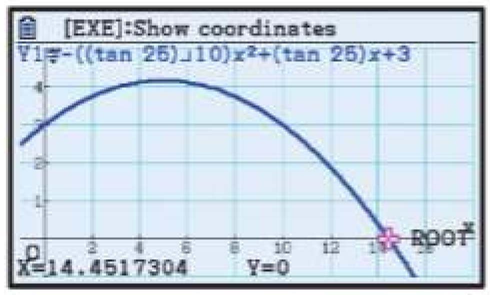
$h'(0) = \text{gradient} = \tan 25°$

$\therefore$ $b = \tan 25°$

d The stone reaches its maximum height when $x = 5$.

$$\therefore \; h'(5) = 0$$
$$\therefore \; 2a(5) + \tan 25^\circ = 0$$
$$\therefore \; 10a = -\tan 25^\circ$$
$$\therefore \; a = -\frac{\tan 25^\circ}{10}$$

$h(x) = 0$ when the stone lands in the water.



$$\therefore \; -\frac{\tan 25^\circ}{10}x^2 + (\tan 25^\circ)x + 3 = 0$$

Using technology, $x \approx 14.5$ $\{x > 0\}$

The stone lands in the water when $x \approx 14.5$.

2 $C(x) = ax^3 + bx^2 + cx + d$ euros

a The maximum output is 140 cars, so the domain of $C(x)$ is $0 \leqslant x \leqslant 140$.

The fixed costs are $C(0) = €24\,500$, so $d = 24\,500$.

Differentiating with respect to x, $C'(x) = 3ax^2 + 2bx + c$.

Now $C'(0) \approx 3280$, so $c \approx 3280$

and $C'(80) \approx 2320$, so $3a(80)^2 + 2b(80) + 3280 \approx 2320$

$$\therefore \; 19\,200a + 160b \approx -960$$
$$\therefore \; 120a + b \approx -6 \quad .... (1)$$

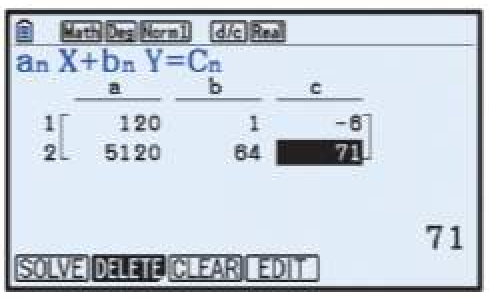
$C(80) = 294\,000$, so $a(80)^3 + b(80)^2 + c(80) + 24\,500 = 294\,000$

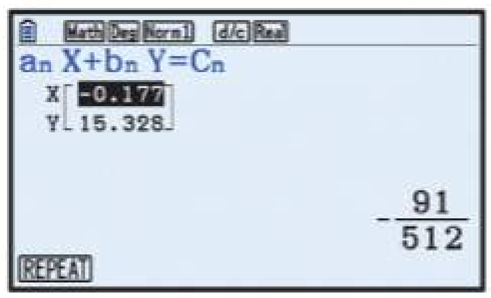
$$\therefore \; 512\,000a + 6400b + 262\,400 + 24\,500 \approx 294\,000 \quad \{c \approx 3280\}$$
$$\therefore \; 512\,000a + 6400b \approx 7100$$
$$\therefore \; 5120a + 64b \approx 71 \quad .... (2)$$

We solve (1) and (2) simultaneously using technology.

$\therefore \; a \approx -0.178$

and $b \approx 15.3$





So, $C(x) \approx -0.178x^3 + 15.3x^2 + 3280x + 24\,500$, $0 \leqslant x \leqslant 140$.

b $C(120) \approx -0.178(120)^3 + 15.3(120)^2 + 3280(120) + 24\,500$

$\approx 331\,000$

$\therefore$ it costs about €331 000 to produce the parts for 120 cars each day.

3 $P(t) = at^3 + bt^2 + ct + d$ million horses

a There were 21.5 million horses in 1900, so $P(0) = 21.5$

$$\therefore \; d = 21.5$$

b $P'(t) = 3at^2 + 2bt + c$

c $P'(t) = 0$ when the population of horses is at a local maximum or a local minimum.

The maximum number of horses occurred in 1915, when $t = 15$.

So, $P'(t) = 0$ when $t = 15$.

d $$P'(15) = 0 \quad \{\text{from } \mathbf{c}\}$$

$$\therefore \; 3a(15)^2 + 2b(15) + c = 0$$

$$\therefore \; 675a + 30b + c = 0 \quad .... \; (1)$$

The population was 26.5 million in 1915, when $t = 15$, so $P(15) = 26.5$

$$\therefore \; a(15)^3 + b(15)^2 + c(15) + 21.5 = 26.5$$

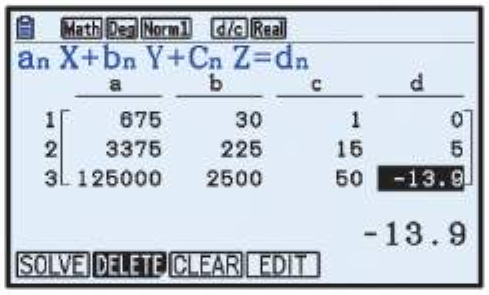
$$\therefore \; 3375a + 225b + 15c = 5 \quad .... \; (2)$$

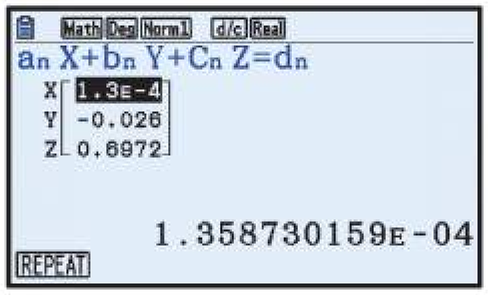
The population was 7.6 million in 1950, when $t = 50$, so $P(50) = 7.6$

$$\therefore \; a(50)^3 + b(50)^2 + c(50) + 21.5 = 7.6$$

$$\therefore \; 125\,000a + 2500b + 50c = -13.9 \quad .... \; (3)$$

We solve (1), (2), and (3) simultaneously using technology.





$\therefore \; a \approx 0.000\,136$

$b \approx -0.0263$

and $c \approx 0.697$

So, $P(t) \approx 0.000\,136t^3 - 0.0263t^2 + 0.697t + 21.5$ million horses.

e **i** In 1930, $t = 30$.

$$P(30) \approx 0.000\,136(30)^3 - 0.0263(30)^2 + 0.697(30) + 21.5$$
$$\approx 22.4$$

The population of horses in 1930 was about 22.4 million.

ii In 1960, $t = 60$.

$$P(60) \approx 0.000\,136(60)^3 - 0.0263(60)^2 + 0.697(60) + 21.5$$
$$\approx -1.98$$

The population of horses in 1960 was about -1.98 million.

f The model provided a reasonable estimate when interpolating between data points. The extrapolation in **e ii** however predicted a negative population, which is not possible.

4 $A(t) = at^3 + bt^2 + ct + d$

a There were no people at the fair outside of its opening hours.
So at 8 am, when $t = 0$, $A(0) = 0$, and at 6 pm, when $t = 10$, $A(10) = 0$.
At 10 am, when $t = 2$, the attendance was increasing at a rate of 1770 people per hour.
So $A'(2) = 1770$.
At 4 pm, when $t = 8$, the attendance was decreasing at a rate of 1800 people per hour.
So $A'(8) = -1800$.

b $A(0) = 0$

$\therefore \; d = 0$

$A(10) = 0$, so $a(10)^3 + b(10)^2 + c(10) = 0$

$$\therefore \; 1000a + 100b + 10c = 0$$

$$\therefore \; 100a + 10b + c = 0 \quad .... \; (1)$$

$A'(t) = 3at^2 + 2bt + c$

$A'(2) = 1770$, so $3a(2)^2 + 2b(2) + c = 1770$

$\therefore \ 12a + 4b + c = 1770$ .... (2)

$A'(8) = -1800$, so $3a(8)^2 + 2b(8) + c = -1800$

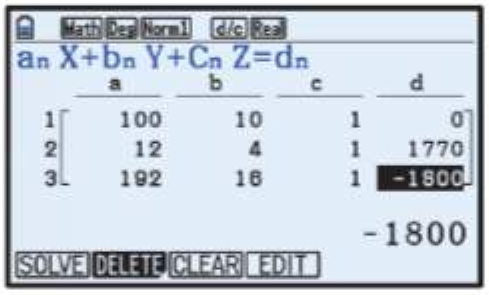
$\therefore \ 192a + 16b + c = -1800$ .... (3)

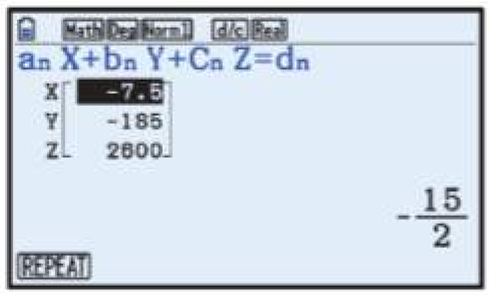
We solve (1), (2), and (3) simultaneously using technology.

$\therefore \ a = -7.5$

$b = -185$

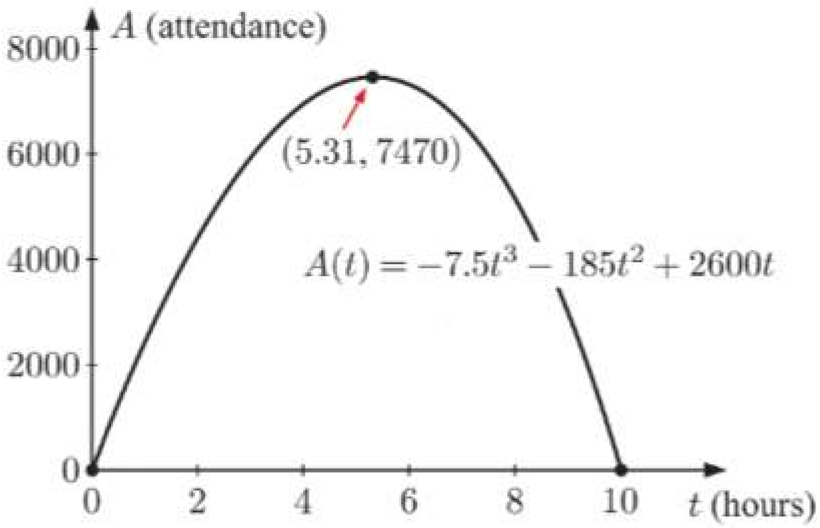
and $c = 2600$





So, $A(t) = -7.5t^3 - 185t^2 + 2600t, \quad 0 \leqslant t \leqslant 10.$

c



The mayor's model seems reasonable; the attendance was always increasing until shortly after 1 pm, then it began to decrease.

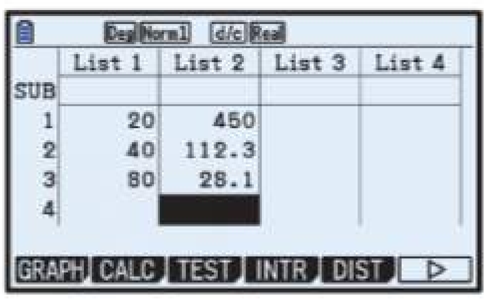
d From the graph in **c**, the maximum attendance was about 7470 people which occurred when $t \approx 5.31$, which was at about 1:19 pm.

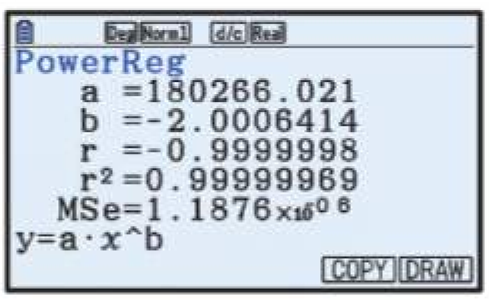
5 **a**

Distance (r cm)	20	40	80
Illuminance (I lx)	450	112.3	28.1
Ir	9000	4492	2248
Ir^2	180 000	179 680	179 840
Ir^3	3 600 000	7 187 200	14 387 200

Model **B** $\left(I = \dfrac{k}{r^2}\right)$ best fits the data as Ir^2 is approximately constant.

b





The correlation coefficient r is very close to 1, and the power is very close to -2, so it is reasonable to conclude that I is inversely proportional to r^2.

The model is $I \approx \dfrac{180\,266}{r^2}$, which agrees with our answer to **a**.

c $$\begin{aligned} I &\approx \frac{180\,266}{r^2} \\ &\approx 180\,266r^{-2} \\ \therefore \frac{dI}{dr} &\approx -360\,532r^{-3} \\ &\approx -\frac{360\,532}{r^3} \end{aligned}$$

d When $r = 1 \text{ m} = 100 \text{ cm}$, $\dfrac{dI}{dr} \approx -\dfrac{360\,532}{100^3}$
≈ -0.361

So, the illuminance is decreasing at a rate of about 0.361 lux per cm at the distance 1 m from the light.

e The surface area of a sphere is proportional to the square of its radius. The total illuminance from a point is a sphere.

Since $SA \propto r^2$ and $I \propto \dfrac{1}{r^2}$, the total illuminance is a constant.

ACTIVITY CUBIC SPLINES

2 **a** **i** $$\begin{aligned} C_i(x) &= a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \\ \therefore C_i{}'(x) &= a_i \times 3(x-x_i)^2(1) + b_i \times 2(x-x_i)(1) + c_i(1) \qquad \{\text{chain rule}\} \\ \therefore C_i{}'(x) &= 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i \end{aligned}$$

ii $$\begin{aligned} C_i{}''(x) &= 3a_i \times 2(x-x_i)(1) + 2b_i(1) \qquad \{\text{chain rule}\} \\ \therefore C_i{}''(x) &= 6a_i(x-x_i) + 2b_i \end{aligned}$$

b $$\begin{aligned} C_i(x_i) &= a_i(x_i-x_i)^3 + b_i(x_i-x_i)^2 + c_i(x_i-x_i) + d_i \\ &= d_i \\ C_i{}'(x_i) &= 3a_i(x_i-x_i)^2 + 2b_i(x_i-x_i) + c_i \\ &= c_i \\ C_i{}''(x_i) &= 6a_i(x_i-x_i) + 2b_i \\ &= 2b_i \end{aligned}$$

3
- $C_i(x_i) = C_{i-1}(x_i)$ ensures that as we transition from the $(i-1)$th cubic to the ith cubic at x_i, the curve is continuous. This requirement gives us the data point at the left end of $C_i(x)$.
- $C_i{}'(x_i) = C_{i-1}'(x_i)$ ensures that as we transition from the $(i-1)$th cubic to the ith cubic at x_i, the gradients are the same. This requirement gives us the gradient at the left end of $C_i(x)$.
- $C_i{}''(x_i) = C_{i-1}''(x_i)$ ensures that as we transition from the $(i-1)$th cubic to the ith cubic at x_i, the cubics have the same shape. This requirement gives us the curvature at the left end of $C_i(x)$.

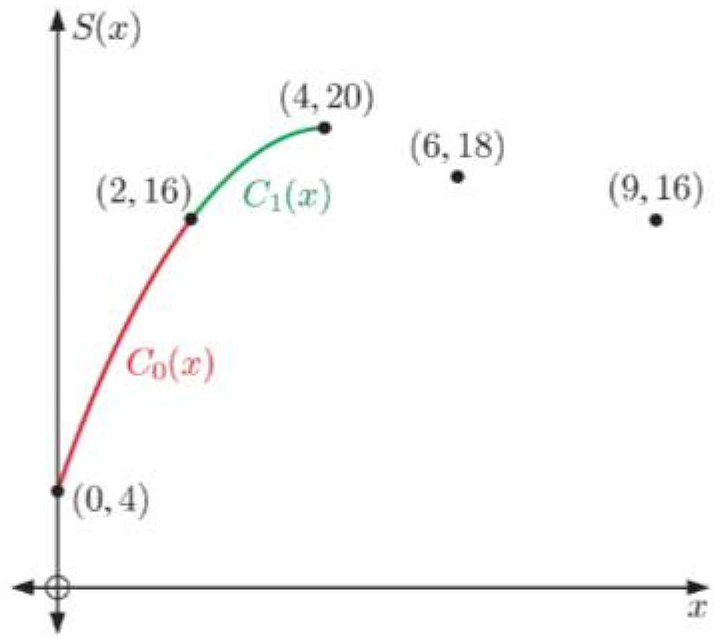
4 **a** **i** $Q(x) = -x^2 + 8x + 4$

$Q(0) = 4$ ✓

$Q(2) = -(2)^2 + 8(2) + 4$
$= -4 + 16 + 4$
$= 16$ ✓

$Q(4) = -(4)^2 + 8(4) + 4$
$= -16 + 32 + 4$
$= 20$ ✓

$\therefore$ (0, 4), (2, 16), and (4, 20) all lie on $Q(x) = -x^2 + 8x + 4$.



ii $C_0''(x_0) = 2b_0$ {from **2 b**}

Now $Q(x) = -x^2 + 8x + 4$

$\therefore Q'(x) = -2x + 8$

$\therefore Q''(x) = -2$

$\therefore Q''(x_0) = -2$

If $C_0''(x_0) = Q''(x_o)$

then $2b_0 = -2$

$\therefore b_0 = -1$

iii $C_0'(x_0) = c_0$ {from **2 b**}

$Q'(x) = -2x + 8$

$\therefore Q'(x_0) = -2x_0 + 8$
$= 8$ $\{x_0 = 0\}$

If $C_0'(x_0) = Q'(x_0)$

then $c_0 = 8$

iv $C_0(x_0) = d_0$ {from **2 b**}

If $C_0(x_0) = y_0$

then $d_0 = 4$ $\{y_0 = 4\}$

v
$$C_0(x) = a_0(x - x_0)^3 + b_0(x - x_0)^2 + c_0(x - x_0) + d_0$$
$$\therefore C_0(x_1) = a_0(x_1 - x_0)^3 + b_0(x_1 - x_0)^2 + c_0(x_1 - x_0) + d_0$$
$$= a_0(2 - 0)^3 - (2 - 0)^2 + 8(2 - 0) + 4 \quad \{x_1 = 2\}$$
$$= 8a_0 - 4 + 16 + 4$$
$$= 8a_0 + 16$$

If $C_0(x_1) = y_1$

then $8a_0 + 16 = 16$ $\{y_1 = 16\}$

$\therefore 8a_0 = 0$

$\therefore a_0 = 0$

So, $C_0(x) = -x^2 + 8x + 4, \quad 0 \leqslant x \leqslant 2$

b **i** $C_1''(x_1) = 2b_1$ {from **2 b**}

$$C_0(x) = -x^2 + 8x + 4$$
$$\therefore\ C_0'(x) = -2x + 8$$
$$\therefore\ C_0''(x) = -2$$
$$\therefore\ C_0''(x_1) = -2$$

If $C_1''(x_1) = C_0''(x_1)$

then $2b_1 = -2$

$$\therefore\ b_1 = -1$$

ii $C_1'(x_1) = c_1$ {from **2 b**}

$$C_0'(x) = -2x + 8$$
$$\begin{aligned}\therefore\ C_0'(x_1) &= -2x_1 + 8\\ &= -2(2) + 8 \quad \{x_1 = 2\}\\ &= -4 + 8\\ &= 4\end{aligned}$$

If $C_1'(x_1) = C_0'(x_1)$

then $c_1 = 4$

iii $C_1(x_1) = d_1$ {from **2 b**}

If $C_1(x_1) = y_1$

then $d_1 = 16$ $\{y_1 = 16\}$

iv
$$\begin{aligned}C_1(x) &= a_1(x - x_1)^3 + b_1(x - x_1)^2 + c_1(x - x_1) + d_1\\ \therefore\ C_1(x_2) &= a_1(x_2 - x_1)^3 + b_1(x_2 - x_1)^2 + c_1(x_2 - x_1) + d_1\\ &= a_1(4-2)^3 - (4-2)^2 + 4(4-2) + 16 \quad \{x_2 = 4\}\\ &= 8a_1 - 4 + 8 + 16\\ &= 8a_1 + 20\end{aligned}$$

If $C_1(x_2) = y_2$

then $8a_1 + 20 = 20$ $\{y_2 = 20\}$

$$\therefore\ 8a_1 = 0$$
$$\therefore\ a_1 = 0$$

So, $C_1(x) = -(x-2)^2 + 4(x-2) + 16, \quad 2 < x \leqslant 4$

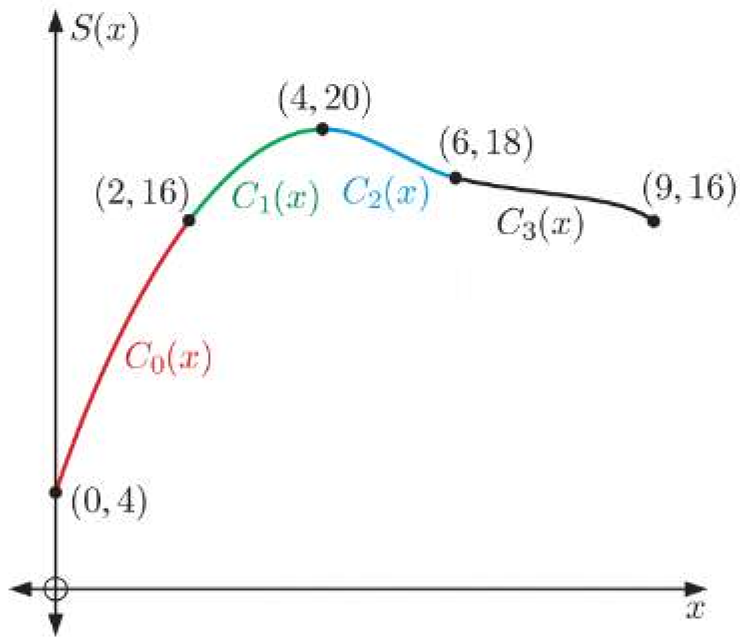
c **i** From the spreadsheet, $a_2 = \frac{1}{4}$, $b_2 = -1$, $c_2 = 0$, $d_2 = 20$

$a_3 = -\frac{7}{54}$, $b_3 = \frac{1}{2}$, $c_3 = -1$, $d_3 = 18$

Now $x_2 = 4$, $x_3 = 6$

$$\therefore\ C_2(x) = \tfrac{1}{4}(x-4)^3 - (x-4)^2 + 20,$$
$$C_3(x) = -\tfrac{7}{54}(x-6)^3 + \tfrac{1}{2}(x-6)^2 - (x-6) + 18$$

iii The cubic spline passes through each of the data points, and appears to be a good representation of the data.



5 **a** The data points $(0,\ 1)$, $(2,\ 7.39)$, $(4,\ 54.6)$, $(6,\ 403.4)$, $(8,\ 2981)$, and $(10,\ 22\,026)$ are of the form $(x,\ e^x)$ where $x = 0,\ 2,\ 4,\ 6,\ 8,\ 10$.

x	1	3.5	4.25	5.25	7.5	9
e^x	2.718	33.135	70.105	190.566	1808.042	8103.084
$S(x)$	-0.907	38.971	63.881	168.500	1895.985	7724.231

b For the x-values 3.5, 4.25, and 7.5, which are close to our original x-values, $S(x)$ approximates e^x reasonably well. For the x-values 1, 5.25, and 9, which are not as close to our original x-values, $S(x)$ is a less accurate approximation of e^x.

c In **a**, $S(x)$ predicts that $e^1 = -0.907$, which we know is absurd since $e^x > 0$ for all x.
If we add more data values, such as $(1,\ 2.718)$, to the spreadsheet, the accuracy of the approximation $S(x)$ is greatly improved.

EXERCISE 20D

1 $y = 2x^3$

a $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$ {chain rule}

$= 6x^2\,\dfrac{dx}{dt}$

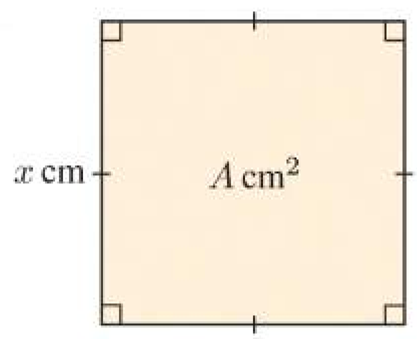
b When $x = 3$, $\dfrac{dx}{dt} = 1$, so $\dfrac{dy}{dt} = 6(3)^2 \times 1$
$= 54$

At the instant when $x = 3$, y is increasing at 54 units per second.

2 **a** $A = x^2$

b $\dfrac{dA}{dt} = \dfrac{dA}{dx} \times \dfrac{dx}{dt}$ {chain rule}

$= 2x\,\dfrac{dx}{dt}$



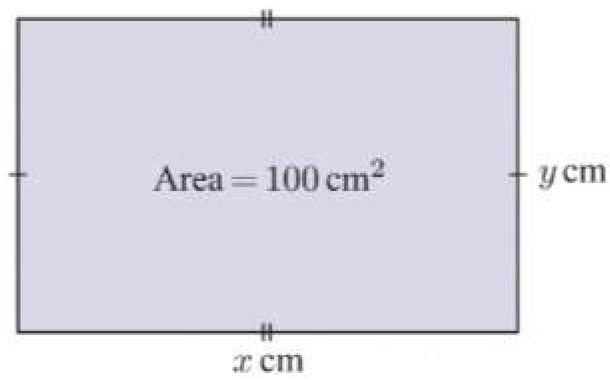
c When $x = 6$, $\dfrac{dx}{dt} = 2$ cm per second

$\therefore\ \dfrac{dA}{dt} = 2(6)(2) = 24$ cm^2 per second

$\therefore$ the area is increasing at 24 cm^2 per second.

3 **a** $xy = 100$

$\therefore\ y = \dfrac{100}{x} \quad \{x \neq 0\}$



b $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$ {chain rule}

$= (-100x^{-2}) \times \dfrac{dx}{dt}$ {chain rule}

$= -\dfrac{100}{x^2}\,\dfrac{dx}{dt}$

c For all values of t, $\dfrac{dx}{dt} = -1$ cm per minute

i When $x = 20$, $\dfrac{dy}{dt} = -\dfrac{100}{20^2}(-1)$

$\therefore\ \dfrac{dy}{dt} = \frac{1}{4}$

$= 0.25$ cm per minute

$\therefore$ the width of the rectangle is increasing at 0.25 cm per minute.

ii When $y = 4$, $x = \dfrac{100}{4} = 25$

$\therefore\ \dfrac{dy}{dt} = -\dfrac{100}{25^2}(-1)$

$= \frac{4}{25}$

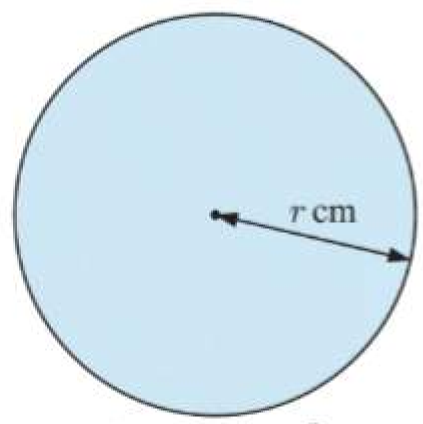
$= 0.16$ cm per minute

$\therefore$ the width of the rectangle is increasing at 0.16 cm per minute.

4 The area of the circular ripple is $A = \pi r^2$.

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt}$ {chain rule}

$= 2\pi r\,\dfrac{dr}{dt}$



Area = A cm²

Since the ripple moves out at a constant speed of 1 m s^{-1},

$\dfrac{dr}{dt} = 1$ m s^{-1}.

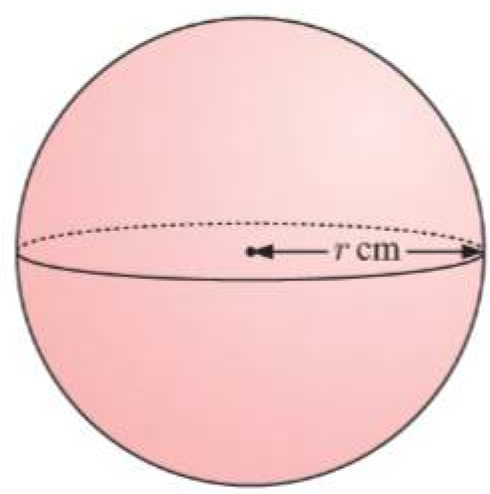
a When $r = t = 2$, $\dfrac{dA}{dt} = 2\pi \times 2 \times 1 = 4\pi$ m^2 s^{-1}

$\therefore$ the circle's area is increasing at 4π m^2 per second.

b When $r = t = 4$, $\dfrac{dA}{dt} = 2\pi \times 4 \times 1 = 8\pi$ m^2 s^{-1}

$\therefore$ the circle's area is increasing at 8π m^2 per second.

5 **a** The volume of a spherical balloon is $V = \frac{4}{3}\pi r^3$.



Volume = V cm³

b $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$ {chain rule}

$= 4\pi r^2\,\dfrac{dr}{dt}$

Particular case:

When $r = 2$ and $\dfrac{dV}{dt} = 6\pi$, $6\pi = 4\pi \times 2^2 \times \dfrac{dr}{dt}$

$\therefore\ \dfrac{dr}{dt} = \dfrac{6\pi}{16\pi}$

$= \frac{3}{8}$

$= 0.375$ m per minute

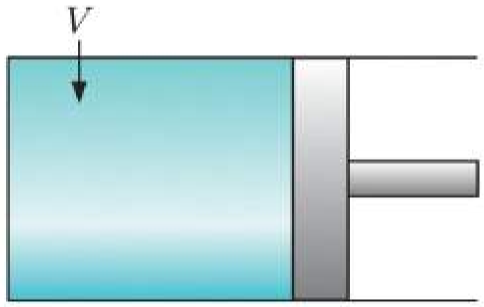
$\therefore$ the balloon's radius is increasing at 0.375 m per minute.

6 $p = \dfrac{400}{V^{1.5}} = 400V^{-\frac{3}{2}}$

$\therefore\ V^{\frac{3}{2}} = 400p^{-1} \qquad \{p \neq 0\}$

$\therefore\ V = (400p^{-1})^{\frac{2}{3}}$

$= 400^{\frac{2}{3}} \times p^{-\frac{2}{3}}$



Now, $\dfrac{dp}{dt} = \dfrac{dp}{dV} \times \dfrac{dV}{dt}$ {chain rule} and $\dfrac{dV}{dt} = \dfrac{dV}{dp} \times \dfrac{dp}{dt}$ {chain rule}

$= (-\frac{3}{2} \times 400 \times V^{-\frac{5}{2}}) \times \dfrac{dV}{dt}$ $\qquad$ $= (-\frac{2}{3} \times 400^{\frac{2}{3}} \times p^{-\frac{5}{3}}) \times \dfrac{dp}{dt}$

$= -600V^{-\frac{5}{2}}\, \dfrac{dV}{dt}$ $\qquad$ $= \left(-\frac{2}{3}\sqrt[3]{\dfrac{400^2}{p^5}}\right) \dfrac{dp}{dt}$

a When $V = 1$ and $\dfrac{dp}{dt} = 3$, $3 = -600 \times 1^{-\frac{5}{2}} \times \dfrac{dV}{dt}$

$\therefore\ \dfrac{dV}{dt} = -\frac{1}{200} = -0.005$

$\therefore$ the volume is decreasing by $0.005\ m^3\,min^{-1}$ when the volume is $1\ m^3$.

b When $p = 50$ and $\dfrac{dp}{dt} = 3$, $\dfrac{dV}{dt} = -\dfrac{2}{3}\sqrt[3]{\dfrac{400^2}{50^5}} \times 3$

$= -2 \times \frac{2}{25} = -0.16$

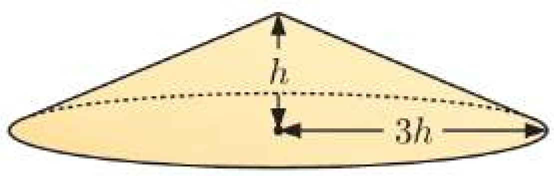
$\therefore$ the volume is decreasing by $0.16\ m^3\,min^{-1}$ when the pressure is $50\ N\,m^{-2}$

7 $V = \frac{1}{3}\pi r^2 h$ and $r = 3h$

$\therefore\ V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3$

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$ {chain rule}

$= 9\pi h^2\, \dfrac{dh}{dt}$



Particular case:

After 1 minute, $h = 20$ cm and the volume $V = 3\pi(20)^3 = 24\,000\pi\ cm^3$

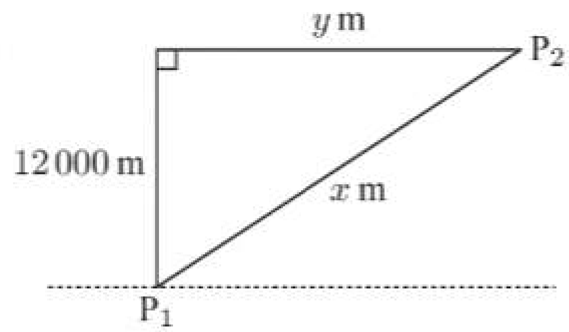
$\therefore\ \dfrac{dV}{dt} = 24\,000\pi\ cm^3$ per minute

$\therefore$ when $h = 20$ cm, $24\,000\pi = 9\pi \times 20^2 \times \dfrac{dh}{dt}$

$\therefore\ \dfrac{dh}{dt} = \dfrac{24\,000\pi}{400 \times 9\pi} = \frac{20}{3}$ cm per minute

$\therefore$ the height is rising at $\frac{20}{3}$ cm per minute.

8 Let P_1 in the diagram be the faster jet and P_2 be the slower jet. Let y m be the distance that P_2 is ahead of P_1, and x m be the distance between them.



Now $x^2 = y^2 + (12\,000)^2$ {Pythagoras}

$\therefore\ x = (y^2 + (12\,000)^2)^{\frac{1}{2}}$ $\{x \geqslant 0\}$

Now, $\dfrac{dx}{dt} = \dfrac{dx}{dy} \times \dfrac{dy}{dt}$ {chain rule}

$= [\frac{1}{2}(y^2 + (12\,000)^2)^{-\frac{1}{2}} \times 2y] \times \dfrac{dy}{dt}$ {chain rule, product rule}

$= \dfrac{y}{\sqrt{y^2 + (12\,000)^2}}\,\dfrac{dy}{dt}$

Particular case:

As P_1 is behind P_2, it is catching up at a rate of $50\ \text{m s}^{-1}$.

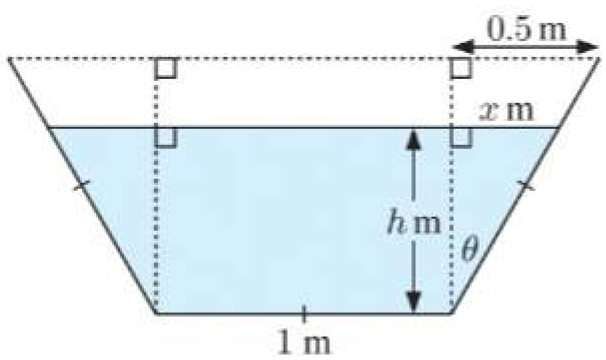
$\therefore\ \dfrac{dy}{dt} = -50\ \text{m s}^{-1}$

When $y = 5000$ and $\dfrac{dy}{dt} = -50$, $\dfrac{dx}{dt} = \dfrac{5000}{\sqrt{(5000)^2 + (12\,000)^2}} \times (-50)$

$= -\dfrac{250\,000}{13\,000} = -\frac{250}{13}$

$\therefore$ the distance between the jets is decreasing at $\frac{250}{13} \approx 19.2\ \text{m s}^{-1}$.

9 **a**



$\sin\theta = \dfrac{0.5}{1} = 0.5$

$\therefore\ \theta = 30^\circ$

Let the height of the water be h m.

$\therefore\ \tan 30^\circ = \dfrac{x}{h}$

$\therefore\ x = h\tan 30^\circ$

$\therefore\ x = \dfrac{h}{\sqrt{3}}$

$\therefore$ the water in the trough has volume $V = \dfrac{h}{2}[1 + (1 + 2x)] \times 6$

$\therefore\ V = \dfrac{h}{2}\left(2 + \dfrac{2h}{\sqrt{3}}\right) \times 6$

$\therefore\ V = 6h + 2\sqrt{3}h^2$ ✓

b $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$ {chain rule}

$= (6 + 4\sqrt{3}h)\,\dfrac{dh}{dt}$

Particular case:

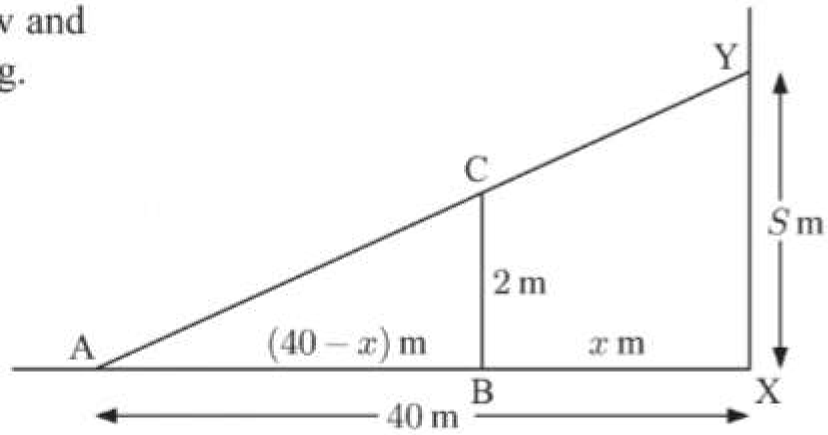
When $h = 0.2$ and $\dfrac{dV}{dt} = -0.1\ \text{m}^3$ per minute,

$-0.1 = (6 + 4\sqrt{3}(0.2))\,\dfrac{dh}{dt}$

$\therefore\ \dfrac{dh}{dt} = \dfrac{-0.1}{6 + 0.8\sqrt{3}} \approx -0.0135$ m per minute

$\therefore$ the water level is falling at about 1.35 cm per minute when the water is 20 cm deep.

10 Let S m be the height of the person's shadow and x m be the person's distance from the building.



Now triangles ABC and AXY are similar.

$$\therefore \quad \frac{AB}{AX} = \frac{BC}{XY}$$

$$\therefore \quad \frac{40-x}{40} = \frac{2}{S}$$

$$\therefore \quad S = \frac{80}{40-x} = 80(40-x)^{-1}$$

$$\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt} \qquad \{\text{chain rule}\}$$

$$= [-80(40-x)^{-2}(-1)] \times \frac{dx}{dt}$$

$$= \frac{80}{(40-x)^2}\,\frac{dx}{dt}$$

But $\dfrac{dx}{dt} = -1\ \text{m s}^{-1}$ for all values of t, so $\dfrac{dS}{dt} = -\dfrac{80}{(40-x)^2}$.

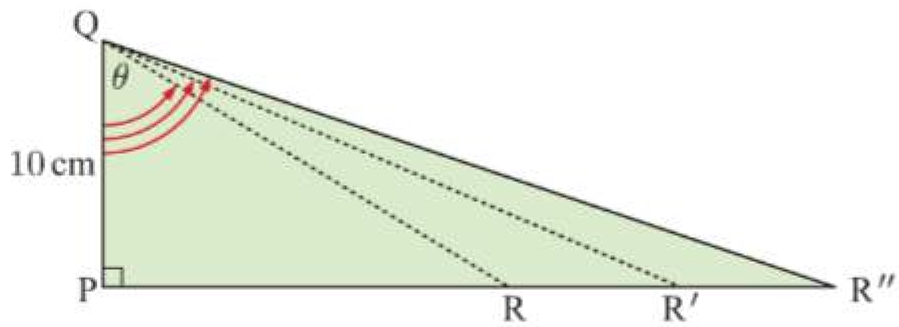
a When $x = 20$, $\dfrac{dS}{dt} = -\dfrac{80}{(40-20)^2} = -\frac{80}{400} = -0.2\ \text{m s}^{-1}$

$\therefore$ the person's shadow is shortening at $0.2\ \text{m s}^{-1}$.

b When $x = 10$, $\dfrac{dS}{dt} = -\dfrac{80}{(40-10)^2} = -\frac{80}{900} = -\frac{4}{45}\ \text{m s}^{-1}$

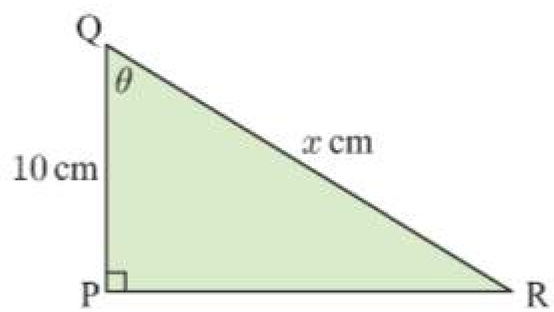
$\therefore$ the person's shadow is shortening at $\frac{4}{45}\ \text{m s}^{-1}$.

11 **a**



Since triangle PQR is right angled and QP remains constant at 10 cm, QR will increase as $\widehat{PQR}$ increases.

b Let QR $= x$ cm and $\widehat{PQR} = \theta$.



Now $\quad \cos\theta = \dfrac{10}{x}$

$$\therefore \quad x = \frac{10}{\cos\theta} = 10(\cos\theta)^{-1}$$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} \qquad \{\text{chain rule}\}$$

$$= [-10(\cos\theta)^{-2}(-\sin\theta)] \times \frac{d\theta}{dt}$$

$$= \frac{10\tan\theta}{\cos\theta}\,\frac{d\theta}{dt}$$

Particular case:

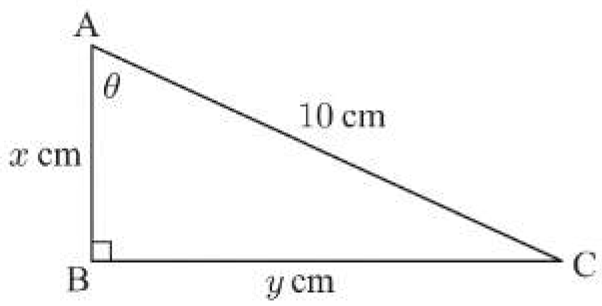
Now $\dfrac{d\theta}{dt} = 2°\ \text{min}^{-1} = \frac{\pi}{90}$ radians min^{-1},

so when $\theta = 60° = \frac{\pi}{3}$, $\dfrac{dx}{dt} = \dfrac{10\tan\frac{\pi}{3}}{\cos\frac{\pi}{3}} \times \dfrac{\pi}{90}$

$$= \frac{10\sqrt{3}}{\frac{1}{2}} \times \frac{\pi}{90}$$

$$= \frac{2\pi}{3\sqrt{3}} \approx 1.21 \text{ cm per minute}$$

$\therefore$ QR is increasing at approximately 1.21 cm per minute when $\widehat{PQR} = 60°$.

12 

Let AB $= x$ cm and BC $= y$ cm.

Now $\cos\theta = \dfrac{x}{10}$

$\therefore\ x = 10\cos\theta$

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$ {chain rule}

$= (-10\sin\theta)\,\dfrac{d\theta}{dt}$

If AB increases at $0.1\ \text{cm s}^{-1}$, $\dfrac{dx}{dt} = 0.1\ \text{cm s}^{-1}$

Particular case:

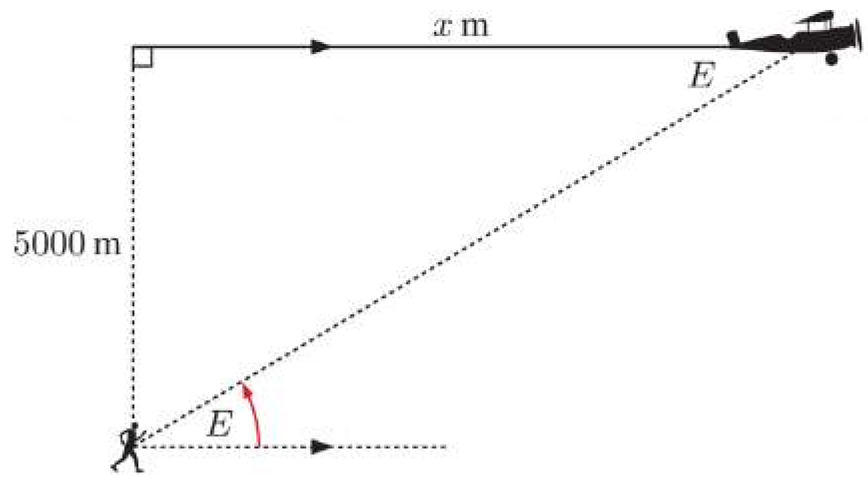
When triangle ABC is isosceles, $\theta = \frac{\pi}{4}$

$\therefore\ 0.1 = (-10\sin\frac{\pi}{4})\,\dfrac{d\theta}{dt}$

$\therefore\ \dfrac{d\theta}{dt} = \dfrac{0.1}{-10 \times \frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{100}$ radians per second

$\therefore$ $\widehat{CAB}$ is decreasing at $\frac{\sqrt{2}}{100}$ radians per second.

13 **a** Let the angle of elevation be E, and the horizontal distance between the observer and aeroplane be x m.



b $\tan E = \dfrac{5000}{x}$

$x = \dfrac{5000}{\tan E}$ $\{E \neq 0 \quad \therefore \quad \tan E \neq 0\}$

$= 5000(\tan E)^{-1}$

$\dfrac{dx}{dt} = \dfrac{dx}{dE} \times \dfrac{dE}{dt}$ {chain rule}

$= \left[-5000(\tan E)^{-2}\left(\dfrac{1}{\cos^2 E}\right)\right] \times \dfrac{dE}{dt}$

$= \left[-5000\left(\dfrac{\cos^2 E}{\sin^2 E}\right)\left(\dfrac{1}{\cos^2 E}\right)\right] \times \dfrac{dE}{dt}$

$= -\dfrac{5000}{\sin^2 E}\,\dfrac{dE}{dt}$

Now $\dfrac{dx}{dt} = 200 \text{ m s}^{-1}$

$\therefore \quad 200 = -\dfrac{5000}{\sin^2 E}\,\dfrac{dE}{dt}$

$\therefore \quad \dfrac{dE}{dt} = -\dfrac{\sin^2 E}{25}$

i When $E = 60°$,

$\dfrac{dE}{dt} = -\dfrac{\sin^2 60°}{25}$

$= -\dfrac{\left(\frac{\sqrt{3}}{2}\right)^2}{25}$

$= -\frac{3}{100}$ radians per second

$\therefore$ the angle of elevation is decreasing at $\frac{3}{100}$ radians per second.

ii When $E = 30°$,

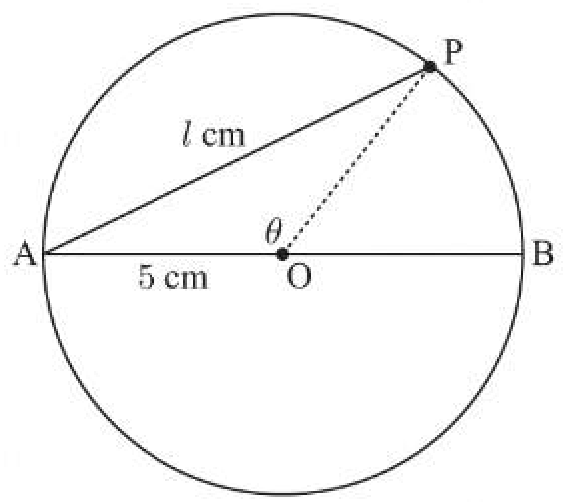
$\dfrac{dE}{dt} = -\dfrac{\sin^2 30°}{25}$

$= -\dfrac{(\frac{1}{2})^2}{25}$

$= -\frac{1}{100}$ radians per second

$\therefore$ the angle of elevation is decreasing at $\frac{1}{100}$ radians per second.

14



Let $\text{AP} = l$ cm and let $\text{A}\widehat{\text{O}}\text{P} = \theta$

$\therefore \quad l^2 = 5^2 + 5^2 - 2 \times 5 \times 5\cos\theta$ {cosine rule}

$\therefore \quad l^2 = 50 - 50\cos\theta$

$\therefore \quad l = \sqrt{50 - 50\cos\theta}$ $\{l \geqslant 0\}$

$= 5(2 - 2\cos\theta)^{\frac{1}{2}}$

$\dfrac{dl}{dt} = \dfrac{dl}{d\theta} \times \dfrac{d\theta}{dt}$ {chain rule}

$= [\frac{5}{2}(2 - 2\cos\theta)^{-\frac{1}{2}}(2\sin\theta)] \times \dfrac{d\theta}{dt}$

$= \dfrac{5\sin\theta}{\sqrt{2 - 2\cos\theta}}\,\dfrac{d\theta}{dt}$

Now the point moves at one revolution every 10 seconds.

$\therefore \quad \dfrac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ radians per second

$\therefore \quad \dfrac{dl}{dt} = \dfrac{\sin\theta}{\sqrt{2 - 2\cos\theta}}\,\pi$

a If $\text{AP} = l = 5$ cm, $\dfrac{dl}{dt} > 0$,

then $\theta = \frac{\pi}{3}$ {$\triangle$APO is equilateral}

$$\therefore \quad \frac{dl}{dt} = \frac{\sin\frac{\pi}{3}}{\sqrt{2 - 2\cos\frac{\pi}{3}}}\pi$$

$$= \frac{\frac{\sqrt{3}}{2}}{\sqrt{2 - 2(\frac{1}{2})}}\pi$$

$$= \tfrac{\sqrt{3}}{2}\pi \text{ cm s}^{-1}$$

$\therefore$ the distance AP is increasing at a rate of $\frac{\sqrt{3}}{2}\pi$ cm s^{-1}.

b If P is at B, then $l = 10$ cm and $\theta = \pi$

$$\therefore \quad \frac{dl}{dt} = \frac{\sin\pi}{\sqrt{2 - 2\cos\pi}}\pi$$

$$= 0 \text{ cm s}^{-1}$$

$\therefore$ the distance AP is not changing.

REVIEW SET 20A

1 $H(t) = 60 + 40\ln(2t + 1)$ cm, $t \geqslant 0$

a $H(0) = 60 + 40\ln(2(0) + 1)$

$\quad = 60$

$\therefore$ the tree was 60 cm tall when it was planted.

b **i** To find when $H = 150$, we graph $Y_1 = 60 + 40\ln(2x + 1)$ and $Y_2 = 150$ on the same set of axes, and find their point of intersection.

So, the tree reaches 150 cm after about 4.24 years, or about 4 years, 3 months.

ii To find when $H = 300$, we graph $Y_1 = 60 + 40\ln(2x + 1)$ and $Y_2 = 300$ on the same set of axes, and find their point of intersection.

So, the tree reaches 300 cm after about 201 years.

c $H(t) = 60 + 40\ln(2t + 1)$ cm, $t \geqslant 0$

$$\therefore \quad H'(t) = 40\left(\frac{2}{2t + 1}\right)$$

$$= \frac{80}{2t + 1} \text{ cm per year}$$

i $H'(2) = \dfrac{80}{2(2) + 1} = 16$

$\therefore$ after 2 years, the tree's height is increasing at a rate of 16 cm per year.

ii $H'(20) = \dfrac{80}{2(20) + 1}$

$\quad = \frac{80}{41} \approx 1.95$

$\therefore$ after 20 years, the tree's height is increasing at a rate of about 1.95 cm per year.

2 **a**

$$V = 20\,000e^{-0.4t} \text{ pounds}$$

When $t = 0$, $V = 20\,000e^{-0.4(0)}$
$= 20\,000$

$\therefore$ the purchase price of the car is £20 000.

b

$$V = 20\,000e^{-0.4t} \text{ pounds}$$

$$\therefore \frac{dV}{dt} = 20\,000e^{-0.4t}(-0.4)$$
$$= -8000e^{-0.4t} \text{ pounds per year}$$

When $t = 10$, $\dfrac{dV}{dt} = -8000e^{-0.4(10)}$
$= -8000e^{-4}$
≈ -146.53

$\therefore$ after 10 years, the value of the car is decreasing at about £146.53 per year.

3 **a** $C(v) = \dfrac{v^2}{20} + \dfrac{50\,000}{v}$ dollars

$$C(64) = \frac{64^2}{20} + \frac{50\,000}{64}$$
$$= 204.8 + 781.25$$
$$= 986.05$$

$\therefore$ the cost of running the train for 1 hour at 64 km h^{-1} is \$986.05

$\therefore$ the cost of running the train for 5 hours at 64 km h^{-1} is \$4930.25 .

b

$$C(v) = \frac{v^2}{20} + \frac{50\,000}{v} \text{ euros}, \quad v > 0$$
$$= \tfrac{1}{20}v^2 + 50\,000v^{-1}$$
$$\therefore C'(v) = \tfrac{1}{10}v - 50\,000v^{-2}$$
$$= \tfrac{1}{10}v - \frac{50\,000}{v^2}$$

i $C'(75) = \tfrac{1}{10}(75) - \dfrac{50\,000}{75^2}$
$= \tfrac{75}{10} - \tfrac{80}{9}$
$= -\tfrac{25}{18}$
≈ -1.39

$\therefore$ if the average speed is 75 km h^{-1}, the rate of change in the cost of running the train is decreasing at about \$1.39 per km h^{-1}.

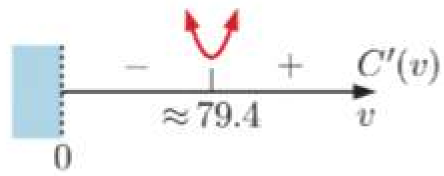
ii $C'(90) = \tfrac{1}{10}(90) - \dfrac{50\,000}{90^2}$
$= 9 - \tfrac{500}{81}$
$= \tfrac{229}{81}$
≈ 2.83

$\therefore$ if the average speed is 90 km h^{-1}, the rate of change in the cost of running the train is increasing at about \$2.83 per km h^{-1}.

c $C(v)$ is a minimum when $C'(v) = 0$

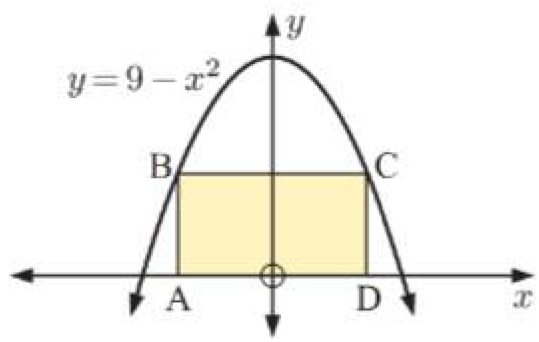
$$\therefore\ \tfrac{1}{10}v - \frac{50\,000}{v^2} = 0$$
$$\therefore\ \tfrac{1}{10}v^3 - 50\,000 = 0$$
$$\therefore\ \tfrac{1}{10}v^3 = 50\,000$$
$$\therefore\ v^3 = 500\,000$$
$$\therefore\ v \approx 79.4\ \text{km h}^{-1}$$

$C'(v)$ has sign diagram:



$\therefore$ the cost of running the train is a minimum when the average speed of the train is about $79.4\ \text{km h}^{-1}$.

4 **a** Let $\text{OD} = x$, so C has coordinates $(x,\ 9 - x^2)$.



Area of rectangle ABCD = length × width

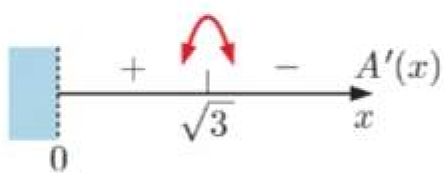
$$\therefore\ A(x) = 2x \times (9 - x^2)$$
$$= 18x - 2x^3$$

b $$A(x) = 18x - 2x^3$$
$$\therefore\ A'(x) = 18 - 6x^2$$

$A'(x) = 0$ when $18 - 6x^2 = 0$

$$\therefore\ 6x^2 = 18$$
$$\therefore\ x^2 = 3$$
$$\therefore\ x = \sqrt{3}\quad \{x > 0\}$$

which has sign diagram:



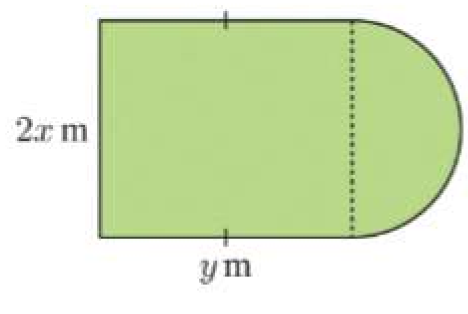
So, the area is a maximum when $x = \sqrt{3}$.

When $x = \sqrt{3}$, $y = 9 - (\sqrt{3})^2 = 6$

So, C has coordinates $(\sqrt{3}, 6)$.

5 **a** perimeter $= 2x + 2y + \pi x$

$$\therefore\ 200 = 2x + 2y + \pi x$$
$$\therefore\ 2y = 200 - 2x - \pi x$$
$$\therefore\ y = 100 - x - \tfrac{\pi}{2}x$$



b area of lawn A = area of rectangle + area of semi-circle

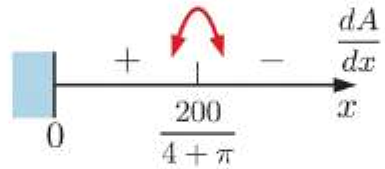
$$= 2x \times y + \tfrac{1}{2}\pi x^2$$
$$= 2x(100 - x - \tfrac{\pi}{2}x) + \tfrac{\pi}{2}x^2 \quad \{\text{using } \mathbf{a}\}$$
$$= 200x - 2x^2 - \pi x^2 + \tfrac{\pi}{2}x^2$$
$$\therefore\ A = 200x - 2x^2 - \tfrac{\pi}{2}x^2\ \ \text{m}^2$$

c $\dfrac{dA}{dx} = 200 - 4x - \pi x$

Now $\dfrac{dA}{dx} = 0$ when

$$\begin{aligned} 200 - 4x - \pi x &= 0 \\ \therefore\ 4x + \pi x &= 200 \\ \therefore\ x(4+\pi) &= 200 \\ \therefore\ x &= \frac{200}{4+\pi} \\ \therefore\ x &\approx 28.0 \end{aligned}$$

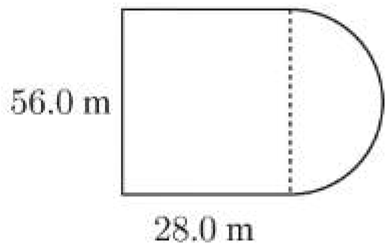
$\dfrac{dA}{dx}$ has sign diagram:



The area of the lawn is maximised when $x = \dfrac{200}{4+\pi} \approx 28.0$

and $y = 100 - \dfrac{200}{4+\pi} - \frac{\pi}{2}\left(\dfrac{200}{4+\pi}\right)$
≈ 28.0

The dimensions of the lawn of maximum area are:

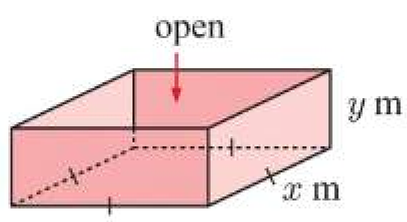


6 **a** capacity $= 1$ kL $\equiv 1\ \text{m}^3$

volume of box = area of base × height

$\therefore\ 1 = x^2 y$

$\therefore\ y = \dfrac{1}{x^2},\quad x > 0$



b area of steel needed = area of base + area of 4 sides

$$\begin{aligned} &= x^2 + 4xy \\ &= x^2 + 4x\left(\frac{1}{x^2}\right) \qquad \{\text{from } \mathbf{a}\} \\ &= x^2 + \frac{4}{x} \end{aligned}$$

Steel costs £2 per m^2, so total cost of steel $= \left(x^2 + \dfrac{4}{x}\right) \times 2$

$\therefore\ C(x) = 2x^2 + \dfrac{8}{x}$ pounds

c $$C(x) = 2x^2 + \frac{8}{x} = 2x^2 + 8x^{-1}$$

$$\therefore \; C'(x) = 4x - 8x^{-2} = 4x - \frac{8}{x^2}$$

$C'(x) = 0$ when $4x - \dfrac{8}{x^2} = 0$

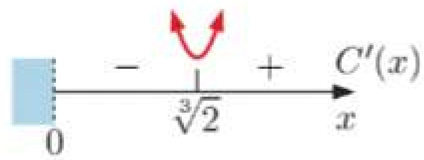
$$\therefore \; 4x = \frac{8}{x^2}$$

$$\therefore \; 4x^3 = 8$$

$$\therefore \; x^3 = 2$$

$$\therefore \; x = \sqrt[3]{2}$$

$C'(x)$ has sign diagram:



So, the cost is a minimum when $x = \sqrt[3]{2} \approx 1.26$

When $x = \sqrt[3]{2}$, $y = \dfrac{1}{(\sqrt[3]{2})^2} \approx 0.630$

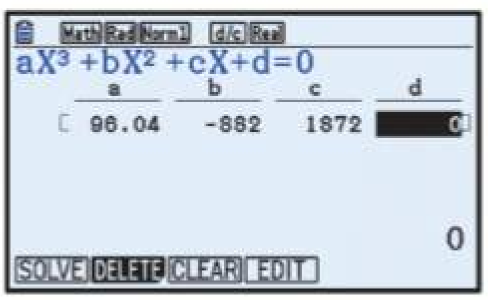
So, the box which would cost the least to make would have square base with sides about 1.26 m and height about 0.630 m.

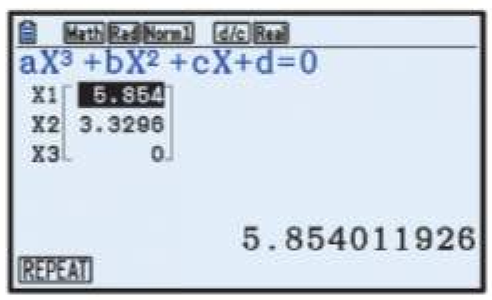
7 $D(t) = \sqrt{24.01t^4 - 294t^3 + 936t^2}$ metres

a The stone flies for 5.9 seconds before landing, so the domain of $D(t)$ is $\{t \mid 0 \leqslant t \leqslant 5.9\}$.

b $$D^2 = 24.01t^4 - 294t^3 + 936t^2$$

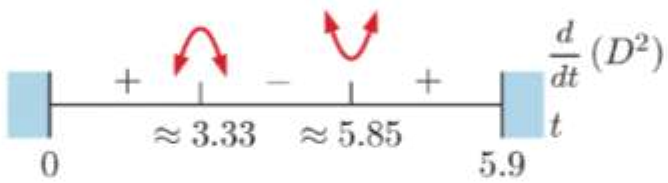
$$\therefore \; \frac{d}{dt}(D^2) = 96.04t^3 - 882t^2 + 1872t$$





Using technology, $\dfrac{d}{dt}(D^2) = 0$ when $t = 0$, 3.33, or 5.85.

$\therefore \; \dfrac{d}{dt}(D^2)$ has sign diagram:



So, D^2 is maximised when $t \approx 3.33$ seconds.

c D^2 is maximised when $t \approx 3.33$ seconds, so D is also maximised when $t \approx 3.33$ seconds.

$$D(3.33) \approx \sqrt{24.01(3.33)^4 - 294(3.33)^3 + 936(3.33)^2}$$
$$\approx 49.8$$

So, the maximum distance between the stone and Max is about 49.8 m.

d The stone lands when $t = 5.9$ seconds.

$$D(5.9) = \sqrt{24.01(5.9)^4 - 294(5.9)^3 + 936(5.9)^2}$$
$$\approx 36.0$$

So, the stone lands about 36.0 m from Max.

8 $h(t) = at^2 + bt + c$ metres

a The ball is thrown from a height of 1.6 m above the ground, so $h(0) = 1.6$

$$\therefore \ c = 1.6$$

b The ball initially gains height at 16.4 m s^{-1}, so $h'(0) = 16.4$

Now $h'(t) = 2at + b$

$$\therefore \ b = 16.4$$

c After 2 seconds the ball is falling at 3.2 m s^{-1}, so $h'(2) = -3.2$

$$\therefore \ 2a(2) + 16.4 = -3.2$$
$$\therefore \ 4a = -19.6$$
$$\therefore \ a = -4.9$$

d $h'(t) = -9.8t + 16.4 = 0$ when $-9.8t = -16.4$

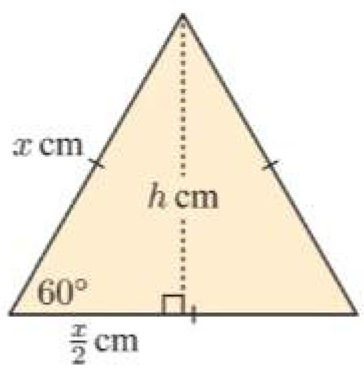
$$\therefore \ t = \frac{-16.4}{-9.8}$$
$$\therefore \ t \approx 1.67$$

$h'(t)$ has sign diagram:

$$h(t) = -4.9t^2 + 16.4t + 1.6$$
$$\therefore \ h(1.67) \approx -4.9(1.67)^2 + 16.4(1.67) + 1.6$$
$$\approx 15.3$$

So, the maximum height reached by the ball was about 15.3 m, which occurred about 1.67 seconds after the ball was thrown.

9 **a**



$$\sin 60° = \frac{h}{x}$$
$$\therefore \ h = x \sin 60° = \tfrac{\sqrt{3}}{2}x$$

Now $A = \frac{1}{2}xh$

$$\therefore \ A = \tfrac{1}{2}x\left(\tfrac{\sqrt{3}}{2}x\right) = \tfrac{\sqrt{3}}{4}x^2 \text{ cm}^2$$

b $\dfrac{dA}{dt} = \dfrac{dA}{dx} \times \dfrac{dx}{dt}$ {chain rule}

$$= \tfrac{\sqrt{3}}{2}x \frac{dx}{dt}$$

Particular case:

When $x = 15$ cm and $\dfrac{dx}{dt} = 3$ cm per minute,

$\dfrac{dA}{dt} = \tfrac{\sqrt{3}}{2} \times 15 \times 3 = \tfrac{45\sqrt{3}}{2}$ cm^2 per minute.

$\therefore$ the area is increasing at $\tfrac{45\sqrt{3}}{2}$ cm^2 per minute when the side length is 15 cm.

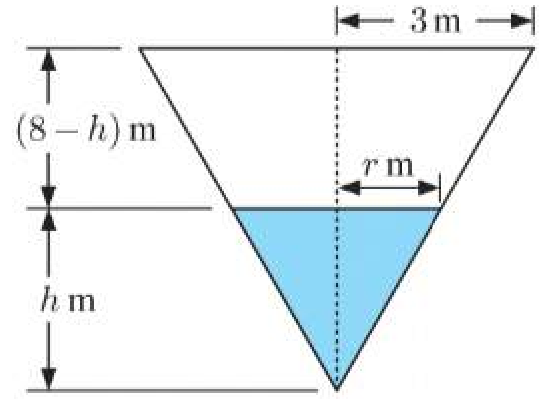
10 **a** Volume $V = \frac{1}{3}\pi r^2 h$

Using similar triangles, $\frac{h}{r} = \frac{8}{3}$

$\therefore\ h = \frac{8r}{3}$

$\therefore\ V = \frac{1}{3}\pi r^2\left(\frac{8r}{3}\right)$

$\therefore\ V(r) = \frac{8}{9}\pi r^3\ \text{ m}^3$



b $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ {chain rule}

$\therefore\ \frac{dV}{dt} = \frac{8}{3}\pi r^2\ \frac{dr}{dt}$

Particular case:

When $h = 5$, $r = \frac{3h}{8} = \frac{15}{8}$ and $\frac{dV}{dt} = -0.2 = -\frac{1}{5}$ m^3 per minute

$-\frac{1}{5} = \frac{8}{3}\pi\left(\frac{15}{8}\right)^2\ \frac{dr}{dt}$

$\therefore\ -\frac{1}{5} = \frac{225}{24}\pi\ \frac{dr}{dt}$

$\therefore\ \frac{dr}{dt} = -\frac{8}{375\pi} \approx -0.006\,79$ m per minute

$\therefore$ the radius is decreasing at approximately 0.006 79 m per minute.

REVIEW SET 20B

1 **a** $C(x) = 850 + 3.3x^{0.85} + 2.8x^{0.5}$ euros

$\therefore\ C'(x) = 3.3(0.85x^{-0.15}) + 2.8(0.5x^{-0.5})$

$= 2.805x^{-0.15} + 1.4x^{-0.5}$ euros per item

b $C'(1000) = 2.805(1000)^{-0.15} + 1.4(1000)^{-0.5}$

$\approx$ €1.04

This estimates the cost of making the 1001st item each day.

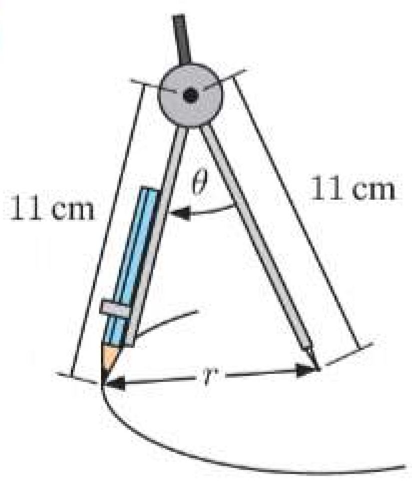
c $C(1001) - C(1000)$

$= 850 + 3.3(1001)^{0.85} + 2.8(1001)^{0.5} - \left(850 + 3.3(1000)^{0.85} + 2.8(1000)^{0.5}\right)$

$\approx$ €1.04

This is the actual cost of making the 1001st item each day. The answer in **b** is a very good estimate.

2 **a**



Using the cosine rule, $r^2 = 11^2 + 11^2 - 2 \times 11 \times 11 \times \cos\theta$

$= 121 + 121 - 242\cos\theta$

$= 242 - 242\cos\theta$

$= 242(1 - \cos\theta)$

Now, the area of the circle $A = \pi r^2$

$= \pi \times 242(1 - \cos\theta)$

$= 242\pi(1 - \cos\theta)\ \text{ cm}^2$

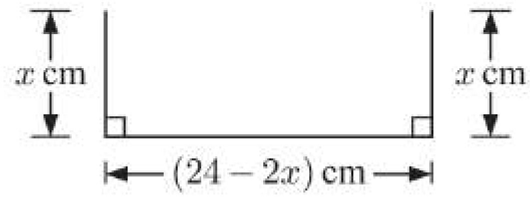
b $\quad A = 242\pi(1-\cos\theta)\ \text{cm}^2$

$$\therefore\ \frac{dA}{d\theta} = 242\pi(\sin\theta)$$
$$= 242\pi\sin\theta\ \ \text{cm}^2 \text{ per radian}$$

When $\theta = \frac{\pi}{4}$, $\quad \frac{dA}{d\theta} = 242\pi\sin\frac{\pi}{4}$
$$= 242\pi \times \frac{1}{\sqrt{2}}$$
$$= 121\sqrt{2}\pi \approx 538\ \text{cm}^2 \text{ per radian}$$

$\therefore$ the area of the circle is changing at a rate of $121\sqrt{2}\pi \approx 538\ \ \text{cm}^2$ per radian when $\theta = \frac{\pi}{4}$.

3 Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of the cross-section.



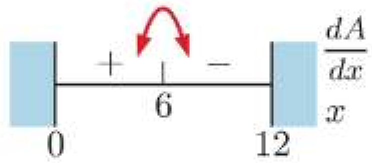
$$A = x(24-2x), \quad 0 \leqslant x \leqslant 12$$
$$= 24x - 2x^2$$

$$\therefore\ \frac{dA}{dx} = 24 - 4x$$

So, $\frac{dA}{dx} = 0$ when $24 - 4x = 0$

$\therefore\ x = 6$

$\frac{dA}{dx}$ has sign diagram:

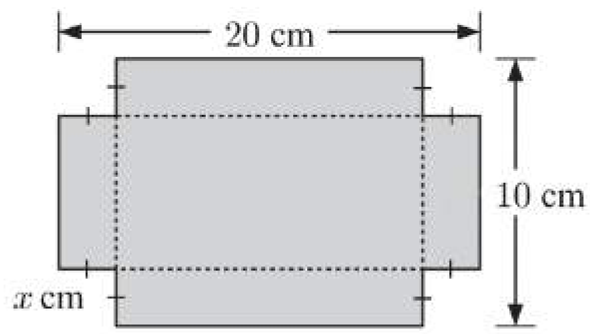


The maximum water is held when $x = 6$

$\therefore$ the bends must be made 6 cm from each end.

4 4 squares with sides x cm are cut from the corners.

$\therefore$ the remaining sides have length $(20-2x)$ cm and $(10-2x)$ cm.



Now, volume $V = \text{length} \times \text{width} \times \text{depth}$
$$= (20-2x)(10-2x)x$$
$$= (200 - 40x - 20x + 4x^2)x$$
$$= (200 - 60x + 4x^2)x$$
$$= 200x - 60x^2 + 4x^3\ \ \text{cm}^3$$

Since the side lengths must be positive, $x > 0$ and $10 - 2x > 0$

$\therefore\ 2x < 10$

$\therefore\ 0 < x < 5$

$$V = 4x^3 - 60x^2 + 200x$$

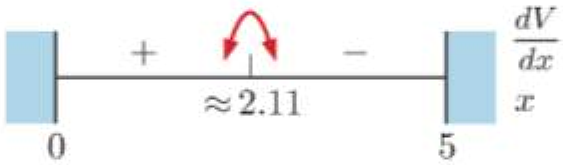
$$\therefore\ \frac{dV}{dx} = 12x^2 - 120x + 200$$

So, $\frac{dV}{dx} = 0$ when $12x^2 - 120x + 200 = 0$

$\therefore\ 3x^2 - 30x + 50 = 0$

$\therefore\ x \approx 2.11 \quad$ {using technology, $0 < x < 5$}

$\dfrac{dV}{dx}$ has sign diagram:



$\therefore$ the capacity of the container is maximised when $x \approx 2.11$.

5 **a** $\qquad D = 9.3 + 6.8\cos(0.507t)$ m

$$\therefore \ \frac{dD}{dt} = 6.8(-\sin(0.507t))(0.507)$$
$$= -3.4476\sin(0.507t)$$

This tells us the rate at which the depth of water is increasing or decreasing t hours after midnight.

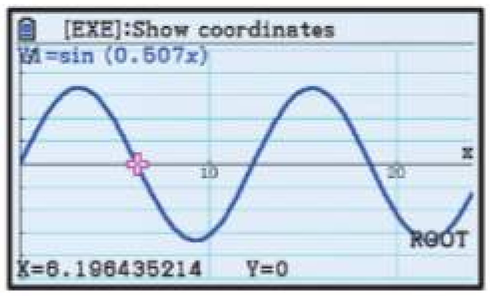
b When $t = 8$, $\quad D = 9.3 + 6.8\cos(0.507 \times 8)$
$$\approx 5.15$$

$\therefore$ the depth of the water at 8 am is about 5.15 m.

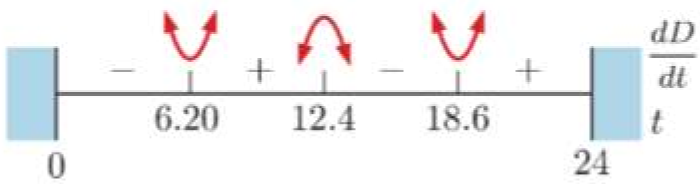
c When $t = 8$, $\quad \dfrac{dD}{dt} = -3.4476\sin(0.507 \times 8)$
$$\approx 2.73 > 0$$

$\therefore$ the tide is rising at 8 am at a rate of about 2.73 m per hour.

d $\dfrac{dD}{dt} = 0$ when $\quad -3.4476\sin(0.507t) = 0$
$$\therefore \ \sin(0.507t) = 0$$
$$\therefore \ t = 0,$$
$$\text{or} \approx 6.20,\ 12.4,\ 18.6$$
$$\{\text{on } 0 \leqslant t \leqslant 24\}$$



$\dfrac{dD}{dt}$ has sign diagram:



So, the tide is highest 0 hours and about 12.4 hours after midnight.

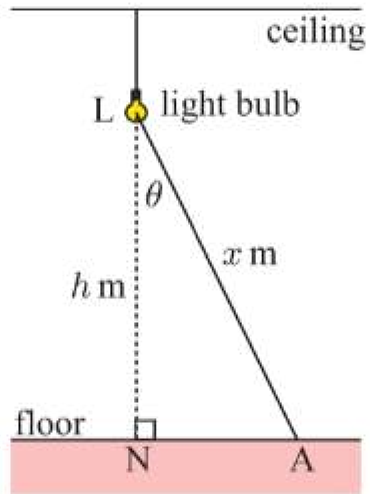
$$0.4 \text{ hours} = 0.4 \times 60$$
$$= 24 \text{ minutes}$$

$\therefore$ the tide will be highest at midnight and at about 12:24 pm.

When $t = 0$, $\quad D = 9.3 + 6.8\cos 0$
$$= 16.1 \text{ m}$$

$\therefore$ the maximum depth is 16.1 m.

6 $I = \dfrac{\sqrt{8}\cos\theta}{x^2}$ units

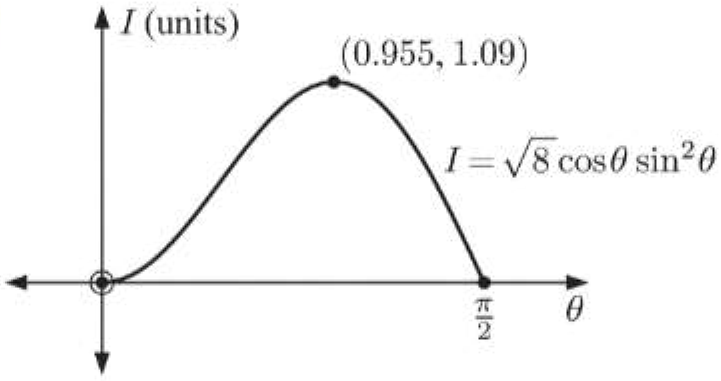


a If NA = 1 metre, $\sin\theta = \dfrac{\text{NA}}{x} = \dfrac{1}{x}$

$$\therefore \quad \frac{1}{x^2} = \sin^2\theta$$

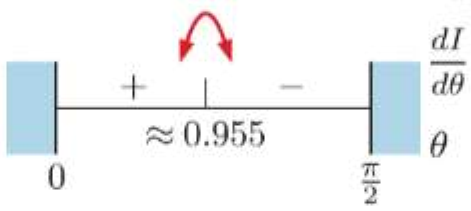
$\therefore$ at A, $I = \dfrac{\sqrt{8}\cos\theta}{x^2} = \sqrt{8}\cos\theta\sin^2\theta$ units

b **i**



ii From **i**, we see that $\dfrac{dI}{d\theta} \geqslant 0$ for $0 \leqslant \theta \leqslant 0.955$, and $\dfrac{dI}{d\theta} \leqslant 0$ for $0.955 \leqslant \theta \leqslant \frac{\pi}{2}$.

$\therefore$ $\dfrac{dI}{d\theta}$ has sign diagram:



iii The maximum value of I occurs when $\theta \approx 0.955\,31$.

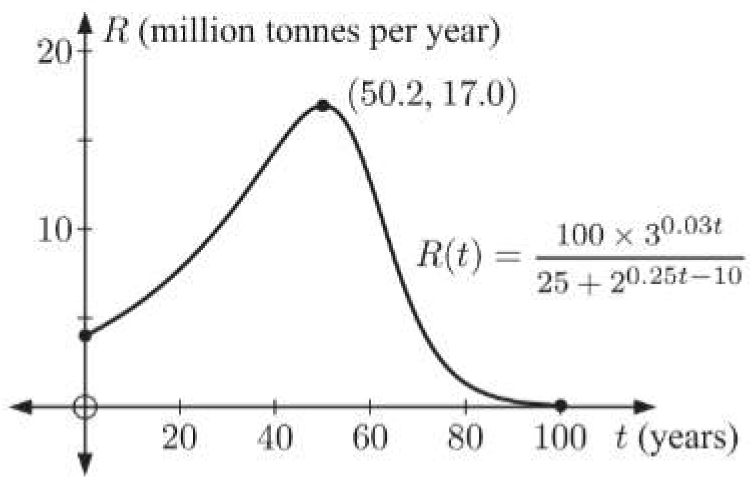
Now, $\tan\theta = \dfrac{1}{h}$

$$\therefore \quad h \approx \frac{1}{\tan(0.955\,31)}$$
$$\approx 0.707$$

The bulb should be about 0.707 m above the floor to provide the greatest illumination at A.

7 $R(t) = \dfrac{100 \times 3^{0.03t}}{25 + 2^{0.25t-10}}$ million tonnes per year, $t \geqslant 0$

a



b From **a**, we see that $R'(t) \geqslant 0$ for $0 \leqslant t \leqslant 50.2$, and $R'(t) \leqslant 0$ for $50.2 \leqslant t \leqslant 100$.

$\therefore$ $R'(t)$ has sign diagram:

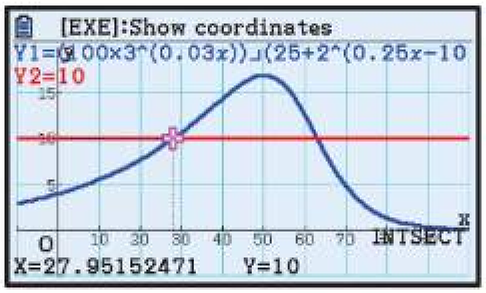
c $R(20) = \dfrac{100 \times 3^{0.03(20)}}{25 + 2^{0.25(20)-10}}$

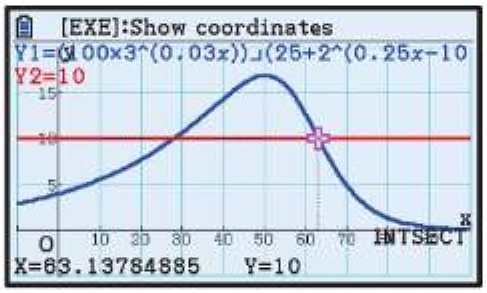
$= \dfrac{100 \times 3^{0.6}}{25 + 2^{-5}}$

≈ 7.72

So, the ore will be mined at a rate of about 7.72 million tonnes per year after 20 years.

d The rate of mining will be 10 million tonnes per year when $\dfrac{100 \times 3^{0.03t}}{25 + 2^{0.25t-10}} = 10.$





Using technology, $t \approx 28.0$ or 63.1

So, the rate of mining will be 10 million tonnes per year about 28.0 years and about 63.1 years after mining begins.

e From the graph in **a**, there is a local maximum at about $(50.2, 17.0)$.

So, the maximum rate of mining will be about 17.0 million tonnes per year, and will occur after about 50.2 years.

8 $S(t) = at^3 + bt^2 + ct + d$ dollars, $0 \leqslant t \leqslant 6.5$

a The share price was initially \$22.81, so $S(0) = 22.81$

$\therefore$ $d = 22.81$

b The share price immediately began to fall at a rate of 16 cents per hour,

so $S'(0) = -0.16$

Now $S'(t) = 3at^2 + 2bt + c$

$\therefore$ $c = -0.16$

c At 12:30 pm, when $t = 3$, the share price was \$22.49, so $S(3) = 22.49$

$\therefore$ $a(3)^3 + b(3)^2 - 0.16(3) + 22.81 = 22.49$

$\therefore$ $27a + 9b - 0.48 + 22.81 = 22.49$

$\therefore$ $27a + 9b = 0.16$ .... (1)

At 2:30 pm, when $t = 5$, the share price was \$22.60, so $S(5) = 22.6$

$\therefore$ $a(5)^3 + b(5)^2 - 0.16(5) + 22.81 = 22.6$

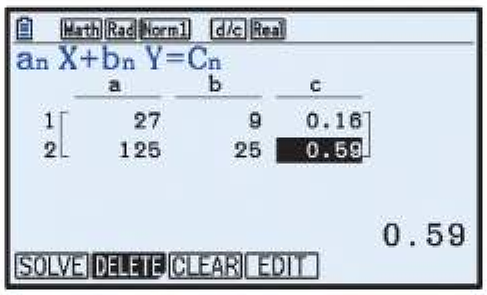
$\therefore$ $125a + 25b - 0.8 + 22.81 = 22.6$

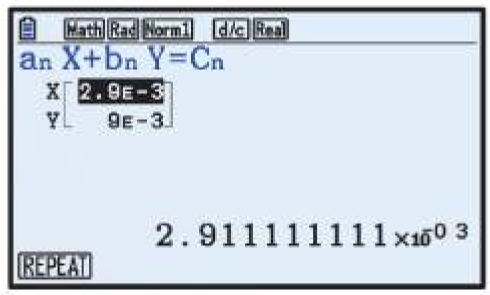
$\therefore$ $125a + 25b = 0.59$ .... (2)

We solve (1) and (2) simultaneously using technology.

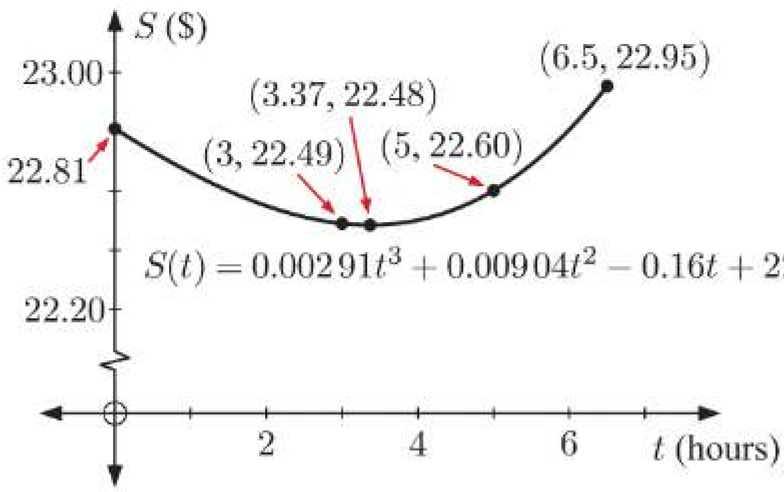
$\therefore \quad a \approx 0.002\,91$

and $\quad b \approx 0.009\,04$





d



e At 4 pm, $t = 6.5$

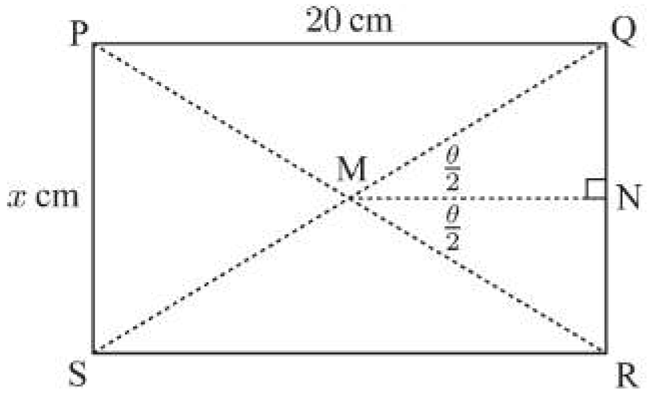
$S(6.5) \approx 0.00291(6.5)^3 + 0.009\,04(6.5)^2 - 0.16(6.5) + 22.81$

≈ 22.95

So, the share price at 4 pm was about \$22.95 .

f Using the graph in **d**, the share price is at a minimum value of about \$22.48 after about 3.37 hours, or around 12:52 pm, and at a maximum value of about \$22.95 when the market closed at 4 pm.

9



Let N be the midpoint of [QR] in isosceles triangle QMR.

$\therefore \quad \text{MN} = 10 \text{ cm}$

Let $\text{QR} = x$ cm and let $\widehat{\text{QMR}} = \theta$.

In triangle MNQ, $\tan \frac{\theta}{2} = \dfrac{\frac{x}{2}}{10} = \dfrac{x}{20}$

$\therefore \quad x = 20 \tan \frac{\theta}{2}$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} \qquad \{\text{chain rule}\}$$

$$= \tfrac{1}{2} \times 20 \times \frac{1}{\cos^2 \frac{\theta}{2}} \times \frac{d\theta}{dt}$$

$$= \frac{10}{\cos^2 \frac{\theta}{2}} \, \frac{d\theta}{dt}$$

Particular case:

When [QR] $= x = 15$ cm, $\tan\frac{\theta}{2} = \frac{15}{20} = \frac{3}{4}$

Also, $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$

$$\therefore \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} + \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{1}{\cos^2\frac{\theta}{2}}$$

$$\therefore \frac{1}{\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2} + 1$$

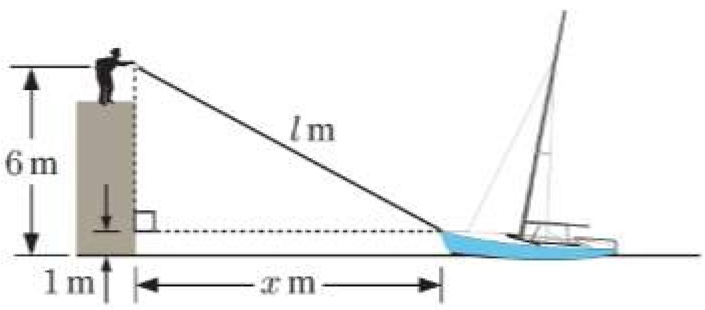
$$= \left(\tfrac{3}{4}\right)^2 + 1 = \tfrac{25}{16}$$

Now, $\frac{dx}{dt} = 2\text{ cm s}^{-1}$, so $2 = 10\left(\frac{25}{16}\right)\frac{d\theta}{dt}$

$$\therefore \frac{d\theta}{dt} = 0.128 \text{ radians per second}$$

$\therefore$ the acute angle between the diagonals is increasing at a rate of 0.128 radians per second when [QR] is 15 cm long.

10 Let l m be the length of rope and x m be the distance of the boat from the jetty.



Then $x^2 + 5^2 = l^2$ {Pythagoras}

$\therefore l = (x^2+25)^{\frac{1}{2}}$ $\{l \geqslant 0\}$

$$\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt} \quad \{\text{chain rule}\}$$

$$= \tfrac{1}{2}(x^2+25)^{-\frac{1}{2}}(2x) \times \frac{dx}{dt}$$

$$= \frac{x}{\sqrt{x^2+25}}\frac{dx}{dt}$$

Particular case:

When $x = 15$ m, $\frac{dl}{dt} = -20$ m per minute

$$\therefore -20 = \frac{15}{\sqrt{15^2+25}}\frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = -\frac{20\sqrt{10}}{3} \approx -21.1 \text{ m per minute}$$

$\therefore$ the boat is approaching the jetty at approximately 21.1 metres per minute.

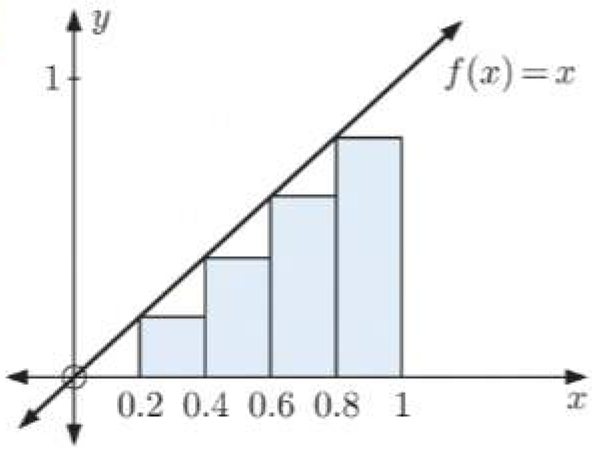
Chapter 21

INTRODUCTION TO INTEGRATION

EXERCISE 21A.1

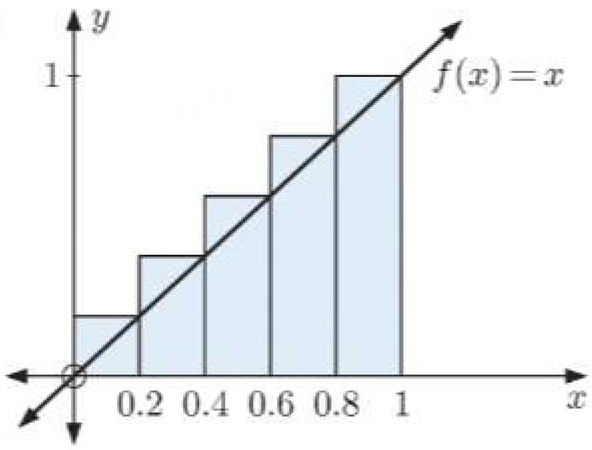
1 **a** The rectangles are $\frac{1}{5} = 0.2$ units wide.

i



$$\begin{aligned} A_L &= 0.2 \times f(0) + 0.2 \times f(0.2) + 0.2 \times f(0.4) \\ &\quad + 0.2 \times f(0.6) + 0.2 \times f(0.8) \\ &= (0.2 \times 0) + (0.2 \times 0.2) + (0.2 \times 0.4) \\ &\quad + (0.2 \times 0.6) + (0.2 \times 0.8) \\ &= 0.4 \text{ units}^2 \end{aligned}$$

ii



$$\begin{aligned} A_U &= 0.2 \times f(0.2) + 0.2 \times f(0.4) + 0.2 \times f(0.6) \\ &\quad + 0.2 \times f(0.8) + 0.2 \times f(1) \\ &= (0.2 \times 0.2) + (0.2 \times 0.4) + (0.2 \times 0.6) \\ &\quad + (0.2 \times 0.8) + (0.2 \times 1) \\ &= 0.6 \text{ units}^2 \end{aligned}$$

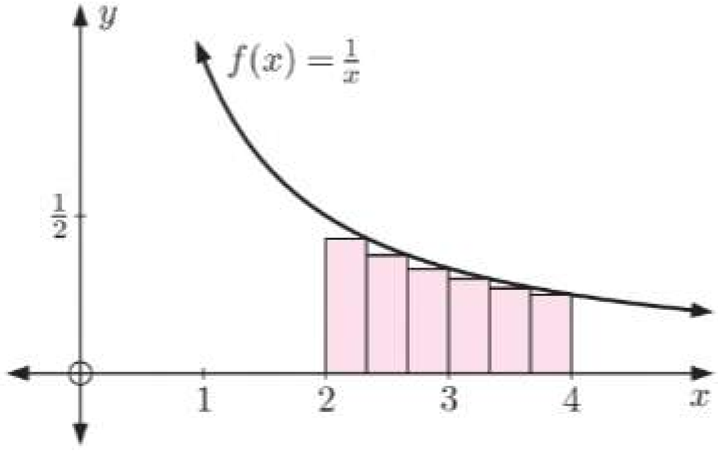
b The area between $y = x$ and the x-axis from $x = 0$ to $x = 1$ is a triangle.

$$\begin{aligned} \therefore \quad \text{area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times 1 \times 1 \\ &= 0.5 \text{ units}^2 \end{aligned}$$

$\therefore$ $A_L < \text{area} < A_U$, and both A_L and A_U are within 0.1 units^2, or 20%, of the actual area.

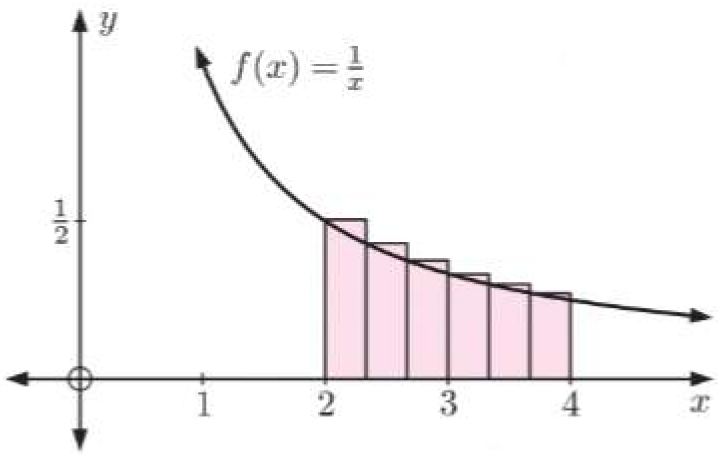
2 The rectangles are $\frac{2}{6} = \frac{1}{3}$ units wide.

a



$$\begin{aligned} A_L &= \tfrac{1}{3} \times f(\tfrac{7}{3}) + \tfrac{1}{3} \times f(\tfrac{8}{3}) + \tfrac{1}{3} \times f(3) + \tfrac{1}{3} \times f(\tfrac{10}{3}) + \tfrac{1}{3} \times f(\tfrac{11}{3}) + \tfrac{1}{3} \times f(4) \\ &= (\tfrac{1}{3} \times \tfrac{3}{7}) + (\tfrac{1}{3} \times \tfrac{3}{8}) + (\tfrac{1}{3} \times \tfrac{1}{3}) + (\tfrac{1}{3} \times \tfrac{3}{10}) + (\tfrac{1}{3} \times \tfrac{3}{11}) + (\tfrac{1}{3} \times \tfrac{1}{4}) \\ &\approx 0.653 \text{ units}^2 \end{aligned}$$

b



$$A_U = \tfrac{1}{3} \times f(2) + \tfrac{1}{3} \times f(\tfrac{7}{3}) + \tfrac{1}{3} \times f(\tfrac{8}{3}) + \tfrac{1}{3} \times f(3) + \tfrac{1}{3} \times f(\tfrac{10}{3}) + \tfrac{1}{3} \times f(\tfrac{11}{3})$$
$$= (\tfrac{1}{3} \times \tfrac{1}{2}) + (\tfrac{1}{3} \times \tfrac{3}{7}) + (\tfrac{1}{3} \times \tfrac{3}{8}) + (\tfrac{1}{3} \times \tfrac{1}{3}) + (\tfrac{1}{3} \times \tfrac{3}{10}) + (\tfrac{1}{3} \times \tfrac{3}{11})$$
$$\approx 0.737 \text{ units}^2$$

3 Using provided software,

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3} = 2.\overline{3}$

4 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** A_L and A_U converge to $0.25 = \dfrac{1}{4} = \dfrac{1}{3+1}$

ii A_L and A_U converge to $0.5 = \dfrac{1}{2} = \dfrac{1}{1+1}$

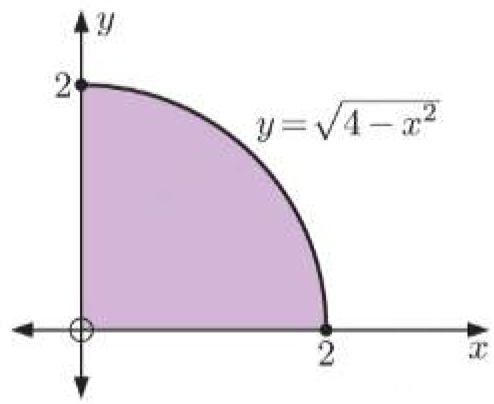
iii A_L and A_U converge to $0.\overline{6} = \dfrac{2}{3} = \dfrac{1}{\frac{1}{2}+1}$

iv A_L and A_U converge to $0.75 = \dfrac{3}{4} = \dfrac{1}{\frac{1}{3}+1}$

c From **b**, it appears that the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ and any number $a > 0$ is $\dfrac{1}{a+1}$.

5 **a**

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$



b $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is approximately $3.1408 < \pi < 3.1429$

This is a better approximation than our estimates in **a** using $n = 10$, 50, 100, 200, or 1000 rectangles. Only $n = 10\,000$ gives us a better estimate than that of Archimedes.

INVESTIGATION 1 THE AREA UNDER $f(x) = x^2$

1 **a** $f(x) = x^2$, $0 \leqslant x \leqslant 1$ is divided into n subintervals.

Each lower rectangle has width $\dfrac{1}{n}$.

Since $f(x)$ is increasing on $0 \leqslant x \leqslant 1$, the ith lower rectangle has height $f\left(\dfrac{i-1}{n}\right)$.

$$\therefore \text{ the total area of lower rectangles } A_L = \sum_{i=1}^{n}\left(\frac{1}{n} \times f\left(\frac{i-1}{n}\right)\right) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)$$

b

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right) &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{i^2-2i+1}{n^2} \\
&= \frac{1}{n^3}\sum_{i=1}^{n}(i^2-2i+1) \\
&= \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) \\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2 \times \frac{n(n+1)}{2} + n\right) \quad \{\text{using provided formulae}\} \\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n(n+1) + n\right) \\
&= \frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6} - n^2 - n + n\right) \\
&= \frac{1}{n^3}\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n - n^2\right) \\
&= \frac{1}{n^3}\left(\tfrac{1}{3}n^3 - \tfrac{1}{2}n^2 + \tfrac{1}{6}n\right)
\end{aligned}$$

$$\therefore A_L = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

c As $n \to \infty$, $\frac{1}{2n} \to 0$ and $\frac{1}{6n^2} \to 0$

$\therefore$ as $n \to \infty$, $A_L \to \frac{1}{3}$.

2 **a** Each upper rectangle has width $\frac{1}{n}$.

Since $f(x)$ is increasing on $0 \leqslant x \leqslant 1$, the ith upper rectangle has height $f\left(\frac{i}{n}\right)$.

$\therefore$ the total area of upper rectangles $A_U = \sum_{i=1}^{n}\left(\frac{1}{n} \times f\left(\frac{i}{n}\right)\right)$

$$= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$

b

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \qquad \left\{\text{using } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right\} \\
&= \frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right) \\
&= \frac{1}{n^3}\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n\right)
\end{aligned}$$

$\therefore$ $A_U = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$

c As $n \to \infty$, $\frac{1}{2n} \to 0$ and $\frac{1}{6n^2} \to 0$

$\therefore$ as $n \to \infty$, $A_U \to \frac{1}{3}$.

3 We know that $A_L < A < A_U$, and as $n \to \infty$, A_L and A_U both converge to the value $\frac{1}{3}$.

$\therefore$ $A = \frac{1}{3}$ units^2.

EXERCISE 21A.2

1 **a** $n = 4$, $a = 2$, $b = 4$, $f(x) = \dfrac{2}{\sqrt{x}}$

$h = \dfrac{b-a}{n} = \dfrac{1}{2}$

$x_i = 2 + \frac{1}{2}i$

i	x_i	$f(x_i)$
0	2	1.414 214
1	$2\frac{1}{2}$	1.264 911
2	3	1.154 701
3	$3\frac{1}{2}$	1.069 045
4	4	1

Using the trapezoidal rule, the area $\approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$

≈ 2.3479 units^2

b $n = 4,\ a = 1,\ b = 3,\ f(x) = -x^2 + 6x - 4$

$h = \frac{b-a}{n} = \frac{1}{2}$

$x_i = 1 + \frac{1}{2}i$

i	x_i	$f(x_i)$
0	1	1
1	$1\frac{1}{2}$	2.75
2	2	4
3	$2\frac{1}{2}$	4.75
4	3	5

Using the trapezoidal rule, the area $\approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$

≈ 7.25 units^2

2 **a** $n = 6,\ a = 0,\ b = 3,\ f(x) = 3 - x$

$h = \frac{b-a}{n} = \frac{1}{2}$

$x_i = 0 + \frac{1}{2}i$

i	x_i	$f(x_i)$
0	0	3
1	$\frac{1}{2}$	$2\frac{1}{2}$
2	1	2
3	$1\frac{1}{2}$	$1\frac{1}{2}$
4	2	1
5	$2\frac{1}{2}$	$\frac{1}{2}$
6	3	0

Using the trapezoidal rule, the area $= \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_5) + f(x_6)\right)$

$= 4\frac{1}{2}$ units^2

b Since $y = 3 - x$ is a straight line, the area of the ith trapezium is the same as the area between the x-axis and $y = f(x)$ from $x = x_{i-1}$ to $x = x_i$.
So, the calculation in **a** is exact for this function.

3 **a** $n = 8,\ a = 0,\ b = 4,\ f(x) = \sqrt{x}$

$h = \frac{b-a}{n} = \frac{1}{2}$

$x_i = 0 + \frac{1}{2}i$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{2}$	0.707 107
2	1	1
3	$1\frac{1}{2}$	1.224 745
4	2	1.414 214
5	$2\frac{1}{2}$	1.581 139
6	3	1.732 051
7	$3\frac{1}{2}$	1.870 829
8	4	2

Using the trapezoidal rule, the area $\approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_7) + f(x_8)\right)$

≈ 5.2650 units^2

b $n = 8, \ a = 0, \ b = 1, \ f(x) = \sqrt{x}e^{-\pi x}$

$h = \frac{b-a}{n} = \frac{1}{8}$

$x_i = 0 + \frac{1}{8}i$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{8}$	0.238 731
2	$\frac{1}{4}$	0.227 969
3	$\frac{3}{8}$	0.188 527
4	$\frac{1}{2}$	0.146 993
5	$\frac{5}{8}$	0.110 970
6	$\frac{3}{4}$	0.082 082
7	$\frac{7}{8}$	0.059 865
8	1	0.043 214

Using the trapezoidal rule, the area $\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_7) + f(x_8))$

≈ 0.1346 units^2

c $n = 8, \ a = -0.6, \ b = 1, \ f(x) = x^3 - 2x^2 + 1$

$h = \frac{b-a}{n} = 0.2$

$x_i = -0.6 + 0.2i$

i	x_i	$f(x_i)$
0	-0.6	0.064
1	-0.4	0.616
2	-0.2	0.912
3	0	1
4	0.2	0.928
5	0.4	0.744
6	0.6	0.496
7	0.8	0.232
8	1	0

Using the trapezoidal rule, the area $\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_7) + f(x_8))$

≈ 0.992 units^2

4 **a** Using the software provided:

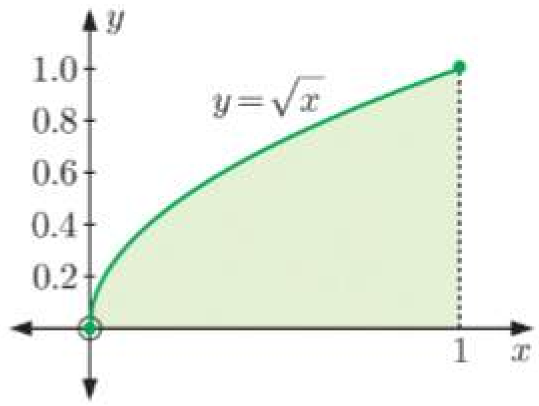
n	*Area estimate*
8	3.0898
40	3.1369
100	3.1404
1000	3.1416

b From **Exercise 21A.1** question **5 b**, Archimedes' approximation is about $3.1408 < \pi < 3.1428$.

Only $n = 1000$ gives us a better estimate than that of Archimedes.

EXERCISE 21B

1 **a**



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\displaystyle\int_0^1 \sqrt{x}\,dx \approx 0.67$

d $n = 8,\ a = 0,\ b = 1,\ f(x) = \sqrt{x}$

$h = \dfrac{b-a}{n} = \frac{1}{8}$

$x_i = 0 + \frac{1}{8}i$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{8}$	0.353 553
2	$\frac{1}{4}$	0.5
3	$\frac{3}{8}$	0.612 372
4	$\frac{1}{2}$	0.707 107
5	$\frac{5}{8}$	0.790 569
6	$\frac{3}{4}$	0.866 025
7	$\frac{7}{8}$	0.935 414
8	1	1

Using the trapezoidal rule, the area $\approx \dfrac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_7) + f(x_8)\right)$

≈ 0.6581 units^2

$\therefore\ \displaystyle\int_0^1 \sqrt{x}\,dx \approx 0.6581$

With just 8 subintervals, the trapezoidal rule is more accurate than lower and upper rectangles were with 50 subintervals.

2 **a** The rectangles will have width $\frac{2-0}{n} = \frac{2}{n}$.

Let $x_i = \frac{2i}{n}$ for $i = 0, ..., n$.

Since $y = \sqrt{1+x^3}$ is increasing on $0 \leqslant x \leqslant 2$, the ith lower rectangle has height $\sqrt{1+x_{i-1}^{\ 3}}$ and the ith upper rectangle has height $\sqrt{1+x_i^{\ 3}}$.

The lower rectangle sum will be

$$\begin{aligned} A_L &= \frac{2}{n} \times \sqrt{1+x_0^{\ 3}} + \frac{2}{n} \times \sqrt{1+x_1^{\ 3}} + ... + \frac{2}{n} \times \sqrt{1+x_{n-1}^{\ 3}} \\ &= \frac{2}{n} \sum_{i=0}^{n-1} \sqrt{1+x_i^{\ 3}} \end{aligned}$$

and the upper rectangle sum will be

$$\begin{aligned} A_U &= \frac{2}{n}\sqrt{1+x_1^{\ 3}} + \frac{2}{n}\sqrt{1+x_2^{\ 3}} + ... + \frac{2}{n}\sqrt{1+x_n^{\ 3}} \\ &= \frac{2}{n} \sum_{i=1}^{n} \sqrt{1+x_i^{\ 3}} \end{aligned}$$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

c $\int_0^2 \sqrt{1+x^3}\,dx \approx 3.24$

d $n = 10,\ a = 0,\ b = 2,\ f(x) = \sqrt{1+x^3}$

$h = \frac{b-a}{n} = 0.2$

$x_i = 0 + 0.2i$

i	x_i	$f(x_i)$
0	0	1
1	0.2	1.003 992
2	0.4	1.031 504
3	0.6	1.102 724
4	0.8	1.229 634
5	1	1.414 214
6	1.2	1.651 666
7	1.4	1.934 942
8	1.6	2.257 432
9	1.8	2.613 809
10	2	3

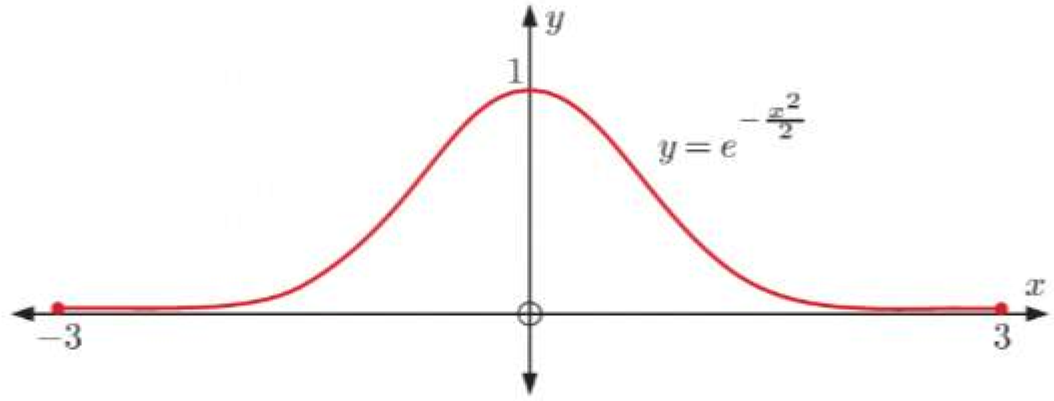
Using the trapezoidal rule, the area $\approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10})\right)$

≈ 3.2480 units^2

$\therefore \int_0^2 \sqrt{1+x^3}\,dx \approx 3.2480$

With just 10 subintervals, the trapezoidal rule is more accurate than lower and upper rectangles were with 100 subintervals.

3 **a**



b Using provided software with the following settings:

From: 0

To: 3

Method: Upper/lower rectangles

Partitions: 2250

we find that $A_L \approx 1.2493$ and $A_U \approx 1.2506$

c Since $y = e^{-\frac{x^2}{2}}$ is symmetrical about the y-axis, the lower and upper rectangle sums for $-3 \leqslant x \leqslant 0$ with $n = 2250$ is equivalent to the lower and upper rectangle sums for $0 \leqslant x \leqslant 3$ with $n = 2250$.

$\therefore$ $A_L \approx 1.2493$ and $A_U \approx 1.2506$

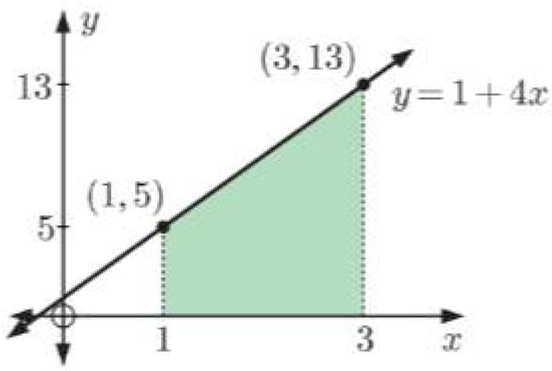
So, $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx = 2\int_0^3 e^{-\frac{x^2}{2}}\,dx \approx 2\left(\frac{A_L + A_U}{2}\right) \approx 2.4999$.

Using technology, $\sqrt{2\pi} \approx 2.5066$, which is close to our estimate.

d Using the trapezoidal rule with $n = 6$ subintervals, the area ≈ 1.2493, which does not lie between the values for A_L and A_U calculated in **b**.

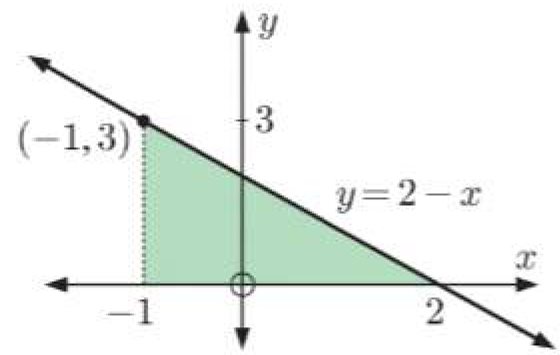
Using the trapezoidal rule with $n = 7$ subintervals, the area ≈ 1.2494, which lies between $A_L \approx 1.2493$ and $A_U \approx 1.2506$, so 7 subintervals are necessary to get a more accurate estimate than lower and upper rectangles with 2250 subintervals.

4 **a**



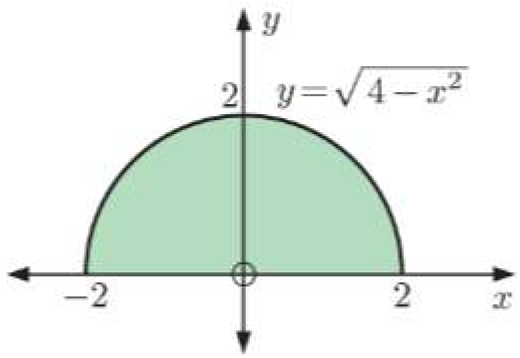
$$\int_1^3 (1+4x)\,dx = \text{shaded area}$$
$$= \left(\frac{5+13}{2}\right) \times 2$$
$$= 18$$

b



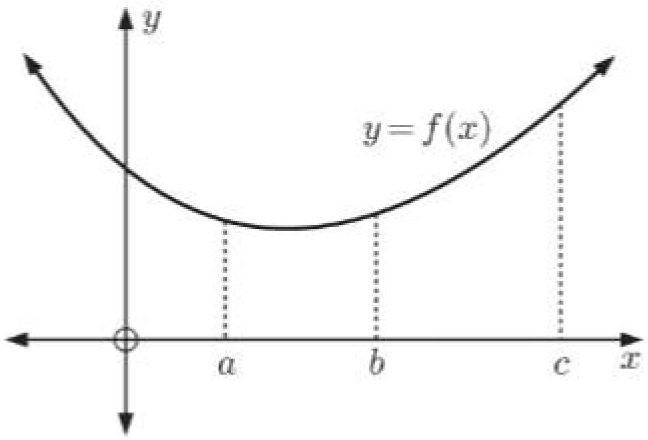
$$\int_{-1}^2 (2-x)\,dx = \text{shaded area}$$
$$= \tfrac{1}{2}(3 \times 3)$$
$$= 4.5$$

c



$$\int_{-2}^{2} \sqrt{4-x^2}\, dx = \text{shaded area}$$
$$= \tfrac{1}{2}(\pi \times 2^2)$$
$$= 2\pi$$

5 **a**



i From the diagram, we can see that there is no area under the curve for $a \leqslant x \leqslant a$.

$\therefore \displaystyle\int_a^a f(x)\, dx = 0$ for any positive function $f(x)$.

ii From the diagram, we can see that the area under the curve for $a \leqslant x \leqslant c$ is made up of the area under the curve for $a \leqslant x \leqslant b$ and $b \leqslant x \leqslant c$.

$\therefore \displaystyle\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$ for any positive function $f(x)$,

provided that $a \leqslant b \leqslant c$.

b **i** $\displaystyle\int_5^5 f(x)\, dx = 0 \qquad \left\{\int_a^a f(x)\, dx = 0\right\}$

ii $\displaystyle\int_2^9 f(x)\, dx$

$$= \int_2^5 f(x)\, dx + \int_5^9 f(x)\, dx \qquad \left\{\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx\right\}$$
$$= 10 + 12$$
$$= 22$$

EXERCISE 21C

1 **a** **i** $\dfrac{d}{dx}(x^2) = 2x$

$\therefore \dfrac{d}{dx}\left(\tfrac{1}{2}x^2\right) = x$

$\therefore$ the antiderivative of x is $\tfrac{1}{2}x^2$ or $\dfrac{x^2}{2}$.

ii $\dfrac{d}{dx}(x^3) = 3x^2$

$\therefore \dfrac{d}{dx}\left(\tfrac{1}{3}x^3\right) = x^2$

$\therefore$ the antiderivative of x^2 is $\tfrac{1}{3}x^3$ or $\dfrac{x^3}{3}$.

iii $\frac{d}{dx}(x^6) = 6x^5$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$

$\therefore$ the antiderivative of x^5 is $\frac{1}{6}x^6$ or $\frac{x^6}{6}$.

iv $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$\therefore$ $\frac{d}{dx}(-x^{-1}) = x^{-2}$

$\therefore$ the antiderivative of x^{-2} is $-x^{-1}$ or $-\frac{1}{x}$.

v $\frac{d}{dx}(x^{-3}) = -3x^{-4}$

$\therefore$ $\frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$

$\therefore$ the antiderivative of x^{-4} is $-\frac{1}{3}x^{-3} = -\frac{1}{3x^3}$.

vi $\frac{d}{dx}(x^{\frac{4}{3}}) = \frac{4}{3}x^{\frac{1}{3}}$

$\therefore$ $\frac{d}{dx}(\frac{3}{4}x^{\frac{4}{3}}) = x^{\frac{1}{3}}$

$\therefore$ the antiderivative of $x^{\frac{1}{3}}$ is $\frac{3}{4}x^{\frac{4}{3}}$.

vii $\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{-\frac{1}{3}}$

$\therefore$ $\frac{d}{dx}(\frac{3}{2}x^{\frac{2}{3}}) = x^{-\frac{1}{3}}$

$\therefore$ the antiderivative of $x^{-\frac{1}{3}}$ is $\frac{3}{2}x^{\frac{2}{3}}$.

viii $\frac{d}{dx}(x^{\frac{5}{3}}) = \frac{5}{3}x^{\frac{2}{3}}$

$\therefore$ $\frac{d}{dx}(\frac{3}{5}x^{\frac{5}{3}}) = x^{\frac{2}{3}}$

$\therefore$ the antiderivative of $x^{\frac{2}{3}}$ is $\frac{3}{5}x^{\frac{5}{3}}$.

b The antiderivative of x^n is $\frac{x^{n+1}}{n+1}$, for $n \neq -1$.

2 **a** **i** $\frac{d}{dx}(e^{3x}) = 3e^{3x}$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$

$\therefore$ the antiderivative of e^{3x} is $\frac{1}{3}e^{3x}$.

ii $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$

$\therefore$ the antiderivative of e^{5x} is $\frac{1}{5}e^{5x}$.

iii $\frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$

$\therefore$ $\frac{d}{dx}(2e^{\frac{1}{2}x}) = e^{\frac{1}{2}x}$

$\therefore$ the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$.

iv $\frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$

$\therefore$ $\frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$

$\therefore$ the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$.

v $\frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$

$\therefore$ $\frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$

$\therefore$ the antiderivative of $e^{\pi x}$ is $\frac{1}{\pi}e^{\pi x}$.

vi $\frac{d}{dx}(e^{\frac{x}{3}}) = \frac{1}{3}e^{\frac{x}{3}}$

$\therefore$ $\frac{d}{dx}(3e^{\frac{x}{3}}) = e^{\frac{x}{3}}$

$\therefore$ the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$.

b The antiderivative of e^{kx} is $\frac{1}{k}e^{kx}$, where $k \neq 0$ is a constant.

3 **a** $\frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$

$\therefore$ $\frac{d}{dx}(2x^3 + 2x^2) = 6x^2 + 4x$

$\therefore$ the antiderivative of $6x^2 + 4x$ is $2x^3 + 2x^2$.

b $\frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

$\therefore$ $\frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$

$\therefore$ the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$.

c $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}$

$$= -\frac{1}{2x\sqrt{x}}$$

$$\therefore \frac{d}{dx}\left(-\frac{2}{\sqrt{x}}\right) = \frac{1}{x\sqrt{x}}$$

$\therefore$ the antiderivative of $\dfrac{1}{x\sqrt{x}}$ is $-\dfrac{2}{\sqrt{x}}$.

INVESTIGATION 2 THE AREA FUNCTION

1 $F(t) = 5t$

$\therefore$ $F'(t) = 5 = f(t)$

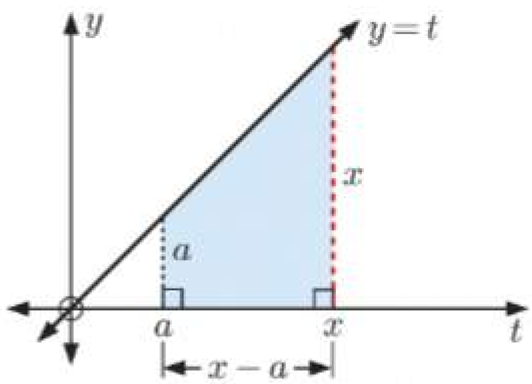
$\therefore$ $F(t)$ is the antiderivative of $f(t)$.

2 **a** $A(x) = \left(\dfrac{x+a}{2}\right)(x-a)$

$$= \frac{x^2-a^2}{2}$$

$$= \frac{x^2}{2} - \frac{a^2}{2}$$

$$= F(x) - F(a) \quad \text{where} \quad F(t) = \frac{t^2}{2}$$



b $F(t) = \dfrac{t^2}{2}$

$\therefore$ $F'(t) = t = f(t)$

$\therefore$ $F(t)$ is the antiderivative of $f(t)$.

3 **a** $A(x) = \left(\dfrac{2x+3+2a+3}{2}\right)(x-a)$

$$= \left(\frac{2(x+a)+6}{2}\right)(x-a)$$

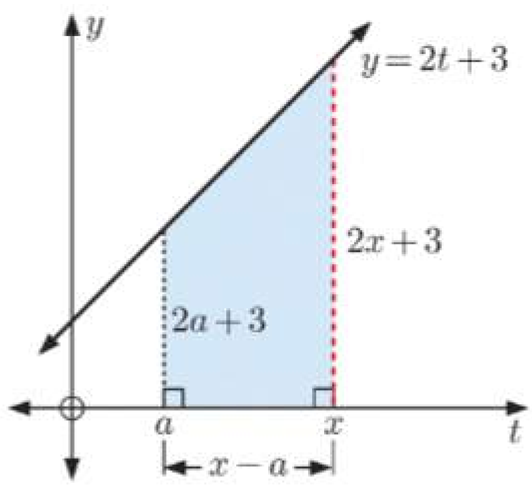
$$= (x+a+3)(x-a)$$

$$= (x+a)(x-a) + 3(x-a)$$

$$= x^2 - a^2 + 3x - 3a$$

$$= x^2 + 3x - (a^2 + 3a)$$

$$= F(x) - F(a) \quad \text{where} \quad F(t) = t^2 + 3t$$



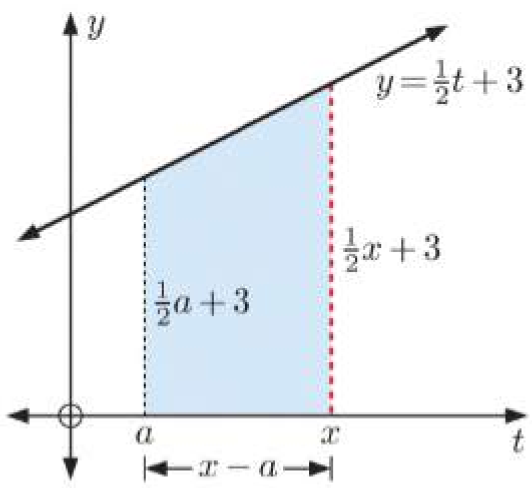
b $F(t) = t^2 + 3t$

$\therefore$ $F'(t) = 2t + 3 = f(t)$

$\therefore$ $F(t)$ is the antiderivative of $f(t)$.

4 **a** Consider $f(t) = \frac{1}{2}t + 3$.

The corresponding area function is



$$\begin{aligned}
A(x) &= \int_a^x \left(\tfrac{1}{2}t + 3\right) dt \\
&= \text{shaded area} \\
&= \left(\frac{\frac{1}{2}x + 3 + \frac{1}{2}a + 3}{2}\right)(x-a) \\
&= \left(\tfrac{1}{4}x + \tfrac{3}{2} + \tfrac{1}{4}a + \tfrac{3}{2}\right)(x-a) \\
&= \tfrac{1}{4}x^2 - \tfrac{1}{4}ax + \tfrac{3}{2}x - \tfrac{3}{2}a + \tfrac{1}{4}ax - \tfrac{1}{4}a^2 + \tfrac{3}{2}x - \tfrac{3}{2}a \\
&= \tfrac{1}{4}x^2 + 3x - \tfrac{1}{4}a^2 - 3a \\
&= \tfrac{1}{4}x^2 + 3x - \left(\tfrac{1}{4}a^2 + 3a\right) \\
&= F(x) - F(a) \quad \text{where } F(t) = \tfrac{1}{4}t^2 + 3t
\end{aligned}$$

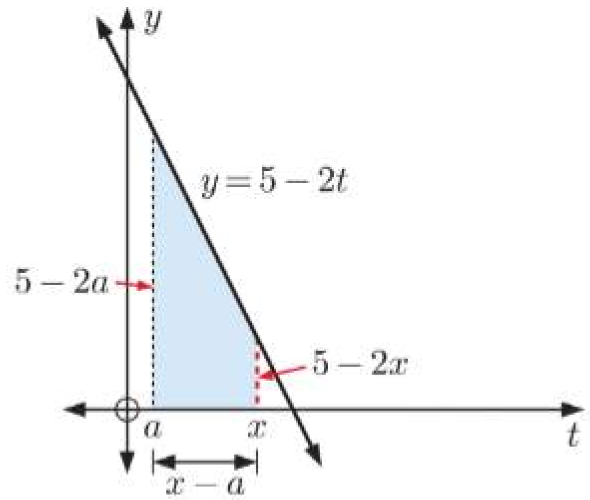
Now $F(t) = \frac{1}{4}t^2 + 3t$

$\therefore$ $F'(t) = \frac{1}{2}t + 3 = f(t)$

$\therefore$ $F(t)$ is the antiderivative of $f(t)$.

b Consider $f(t) = 5 - 2t$.

The corresponding area function is



$$\begin{aligned}
A(x) &= \int_a^x (5 - 2t)\, dt \\
&= \text{shaded area} \\
&= \left(\frac{5 - 2x + 5 - 2a}{2}\right)(x-a) \\
&= \left(\tfrac{5}{2} - x + \tfrac{5}{2} - a\right)(x-a) \\
&= \tfrac{5}{2}x - \tfrac{5}{2}a - x^2 + ax + \tfrac{5}{2}x - \tfrac{5}{2}a - ax + a^2 \\
&= 5x - x^2 - 5a + a^2 \\
&= 5x - x^2 - (5a - a^2) \\
&= F(x) - F(a) \quad \text{where } F(t) = 5t - t^2
\end{aligned}$$

Now $F(t) = 5t - t^2$

$\therefore$ $F'(t) = 5 - 2t = f(t)$

$\therefore$ $F(t)$ is the antiderivative of $f(t)$.

5 $f(t) = 3t^2 + 4t + 5$

We predict that $F(t)$ is the antiderivative of $f(t)$.

$\frac{d}{dt}(t^3 + 2t^2 + 5t) = 3t^2 + 4t + 5 = f(t)$

$\therefore$ $F(t) = t^3 + 2t^2 + 5t$

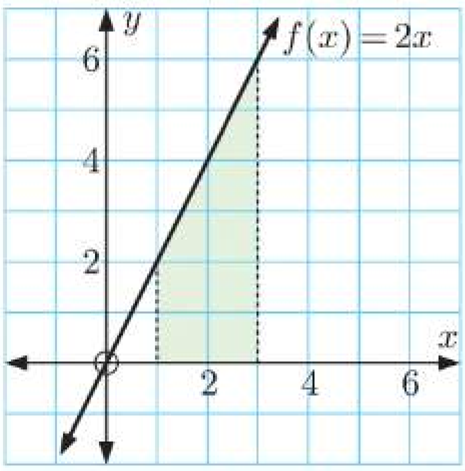
EXERCISE 21D

1 **a** $\dfrac{d}{dx}(x^2) = 2x$

$\therefore$ the antiderivative of $f(x) = 2x$ is $F(x) = x^2$.

b $$\int_1^3 2x\,dx = F(3) - F(1) = 3^2 - 1^2 = 8 \text{ units}^2$$

c



$$\int_1^3 2x\,dx = \text{shaded area} = \left(\frac{2+6}{2}\right) \times 2 = 8 \text{ units}^2$$

2 **a** $$\frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \tfrac{3}{2}x^{\frac{1}{2}} = \tfrac{3}{2}\sqrt{x}$$

$$\therefore \frac{d}{dx}(\tfrac{2}{3}x^{\frac{3}{2}}) = x^{\frac{1}{2}} = \sqrt{x}$$

$\therefore$ the antiderivative of $f(x) = \sqrt{x}$ is $F(x) = \frac{2}{3}x^{\frac{3}{2}}$.

b $$\int_0^1 \sqrt{x}\,dx = F(1) - F(0) = \tfrac{2}{3} - 0 = \tfrac{2}{3} \text{ units}^2$$

c $\frac{2}{3} \approx 0.67$ to 2 significant figures

$\therefore$ the answer is the same as **Exercise 21B** question **1**.

3 **a** $$\frac{d}{dx}(x^4) = 4x^3$$

$$\therefore \frac{d}{dx}\left(\tfrac{1}{4}x^4\right) = x^3$$

$\therefore$ the antiderivative of $f(x) = x^3$ is $F(x) = \frac{1}{4}x^4$.

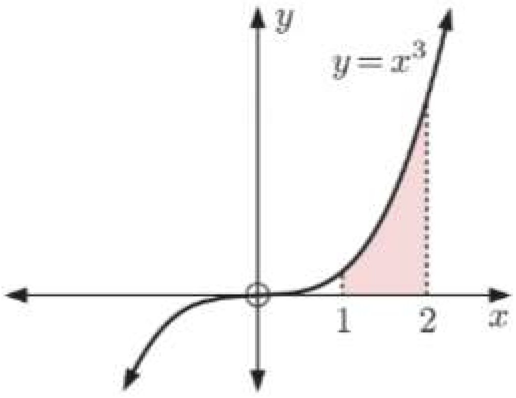
i $$\int_0^2 x^3\,dx = F(2) - F(0) = 4 - 0 = 4 \text{ units}^2$$

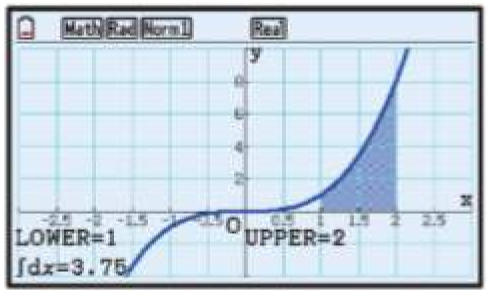
ii $$\int_2^3 x^3\,dx = F(3) - F(2) = \tfrac{81}{4} - 4 = 16\tfrac{1}{4} \text{ units}^2$$

iii $$\int_0^3 x^3\,dx = F(3) - F(0) = \tfrac{81}{4} - 0 = 20\tfrac{1}{4} \text{ units}^2$$

b $$\int_0^3 x^3\,dx = \int_0^2 x^3\,dx + \int_2^3 x^3\,dx$$

4 a





$f(x) = x^3$ has antiderivative $F(x) = \frac{x^4}{4}$

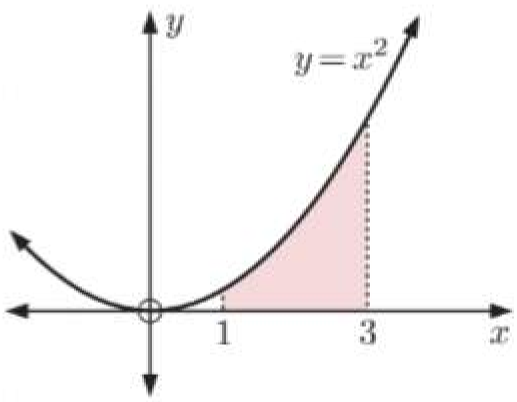
$\therefore$ shaded area $= \int_1^2 x^3\,dx$

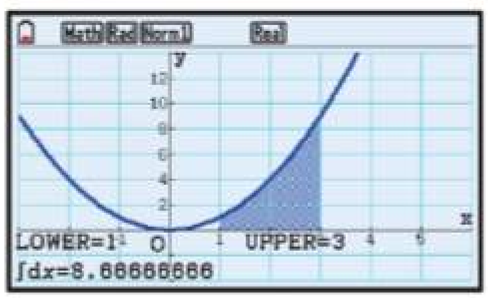
$= F(2) - F(1)$

$= 4 - \frac{1}{4}$

$= 3\frac{3}{4}$ units^2

b





$f(x) = x^2$ has antiderivative $F(x) = \frac{x^3}{3}$

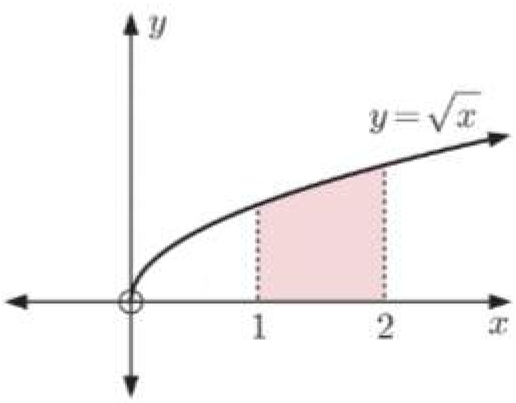
$\therefore$ shaded area $= \int_1^3 x^2\,dx$

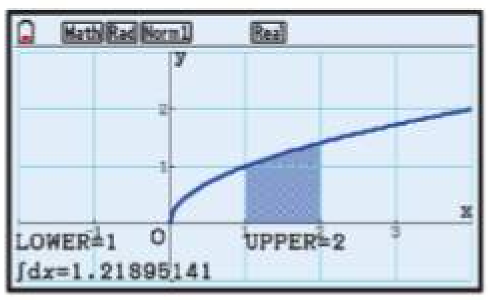
$= F(3) - F(1)$

$= 9 - \frac{1}{3}$

$= 8\frac{2}{3}$ units^2

c





$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

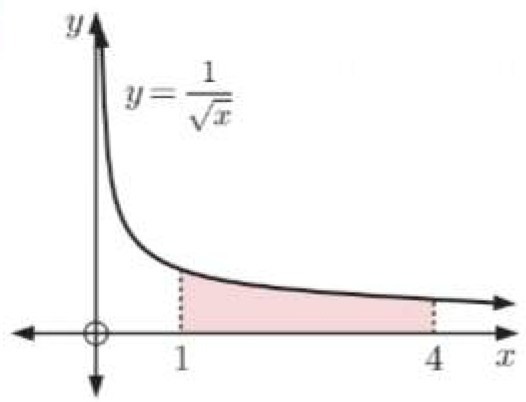
$\therefore$ shaded area $= \int_1^2 \sqrt{x}\,dx$

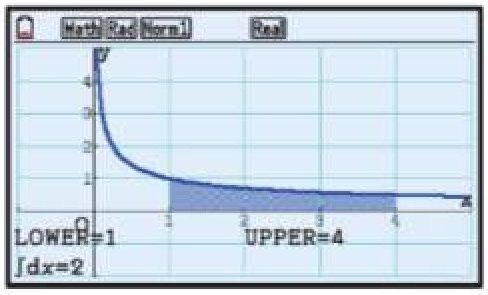
$= F(2) - F(1)$

$= \frac{2}{3}(2\sqrt{2}) - \frac{2}{3}(1\sqrt{1})$

$= \frac{4\sqrt{2}}{3} - \frac{2}{3}$

$= \frac{4\sqrt{2} - 2}{3}$ units^2

d 



$f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$\begin{aligned}\therefore \text{ shaded area} &= \int_1^4 \frac{1}{\sqrt{x}}\,dx \\ &= F(4) - F(1) \\ &= 2\sqrt{4} - 2\sqrt{1} \\ &= 2 \text{ units}^2\end{aligned}$$

5 Let $F(x)$ be the antiderivative of $f(x)$ and $G(x)$ be the antiderivative of $g(x)$.

a $\int_a^a f(x)\,dx = F(a) - F(a) = 0$

$\int_a^a f(x)\,dx =$ area of the region under the curve $y = f(x)$ between $x = a$ and $x = a$.

This region has 0 width, so its area $= 0$.

b The antiderivative of $f(x) = k$ is $F(x) = kx$.

$$\begin{aligned}\therefore \int_a^b k\,dx &= F(b) - F(a) \\ &= kb - ka \\ &= k(b-a)\end{aligned}$$

c
$$\begin{aligned}\int_b^a f(x)\,dx &= F(a) - F(b) \\ &= -[F(b) - F(a)] \\ &= -\int_a^b f(x)\,dx\end{aligned}$$

d
$$\frac{d}{dx} F(x) = f(x)$$

$\therefore \frac{d}{dx}(k\,F(x)) = k\,f(x)$

$\therefore$ $k\,F(x)$ is the antiderivative of $k\,f(x)$.

$$\begin{aligned}\text{So, } \int_a^b k\,f(x)\,dx &= k\,F(b) - k\,F(a) \\ &= k[F(b) - F(a)] \\ &= k\int_a^b f(x)\,dx\end{aligned}$$

e $\dfrac{d}{dx}F(x) = f(x)$ and $\dfrac{d}{dx}G(x) = g(x)$

$\therefore \quad \dfrac{d}{dx}[F(x) + G(x)] = f(x) + g(x)$

$\therefore \quad F(x) + G(x)$ is the antiderivative of $f(x) + g(x)$.

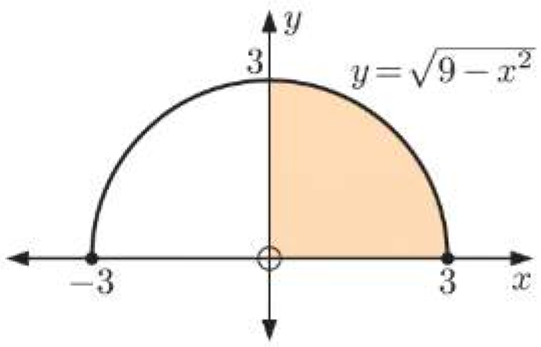
So, $\displaystyle\int_a^b [f(x) + g(x)]\,dx$

$= [F(b) + G(b)] - [F(a) + G(a)]$

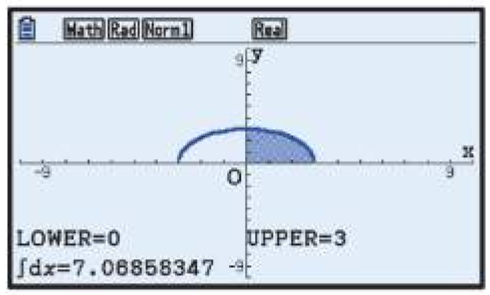
$= [F(b) - F(a)] + [G(b) - G(a)]$

$= \displaystyle\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

6



Using technology, area $= \displaystyle\int_0^3 \sqrt{9 - x^2}\,dx \approx 7.07$ units^2



Check: The area is a quarter circle with radius 3 units.

$\therefore$ area $= \frac{1}{4}\pi r^2$

$= \frac{1}{4} \times \pi \times 3^2$

$= \frac{9\pi}{4}$

≈ 7.07 units^2 ✓

7 **a** If $\dfrac{d}{dx}F(x) = f(x)$ then $\dfrac{d}{dx}(-F(x)) = -f(x)$

$\therefore \quad \displaystyle\int_a^b (-f(x))\,dx = -F(b) - (-F(a))$

$= -(F(b) - F(a))$

$= -\displaystyle\int_a^b f(x)\,dx$

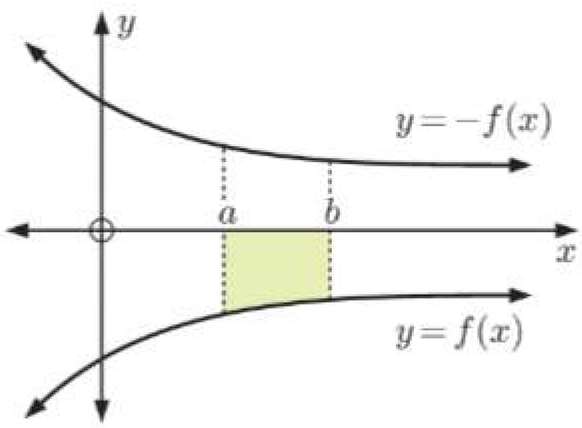
b Since $y=-f(x)$ is a reflection of $y=f(x)$ in the x-axis, then

shaded area

$=$ area between the x-axis and $y=-f(x)$ from $x=a$ to $x=b$

$$= \int_a^b (-f(x))\,dx$$

$$= -\int_a^b f(x)\,dx \quad \{\text{using } \mathbf{a}\}$$

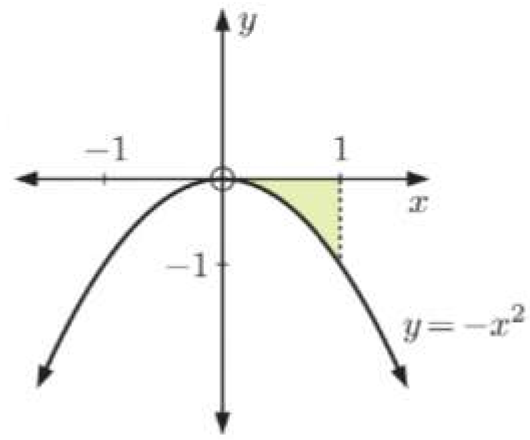


c **i** $\displaystyle\int_0^1 (-x^2)\,dx = -\int_0^1 x^2\,dx$

Now $f(x)=x^2$ has antiderivative $F(x)=\frac{1}{3}x^3$

$$\therefore \int_0^1 (-x^2)\,dx = -(F(1)-F(0))$$
$$= -\left(\tfrac{1}{3}-0\right)$$
$$= -\tfrac{1}{3}$$

The shaded region has area $\frac{1}{3}$ units^2.



ii $\displaystyle\int_0^1 (x^2-x)\,dx = -\int_0^1 (x-x^2)\,dx$

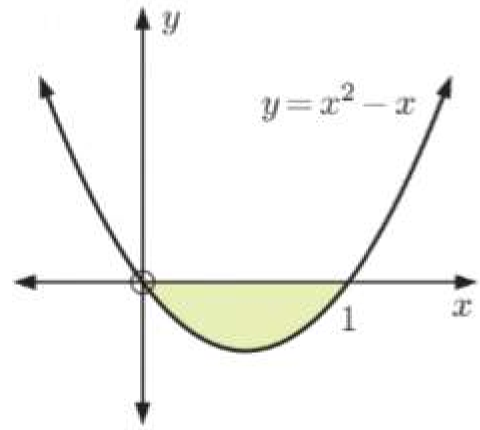
$\{x^2-x \leqslant 0$ for all $0 \leqslant x \leqslant 1\}$

Now $f(x)=x-x^2$ has antiderivative

$F(x)=\frac{1}{2}x^2-\frac{1}{3}x^3$

$$\therefore \int_0^1 (x^2-x)\,dx = -(F(1)-F(0))$$
$$= -\left(\tfrac{1}{2}-\tfrac{1}{3}-(0-0)\right)$$
$$= -\tfrac{1}{6}$$

The shaded region has area $\frac{1}{6}$ units^2.



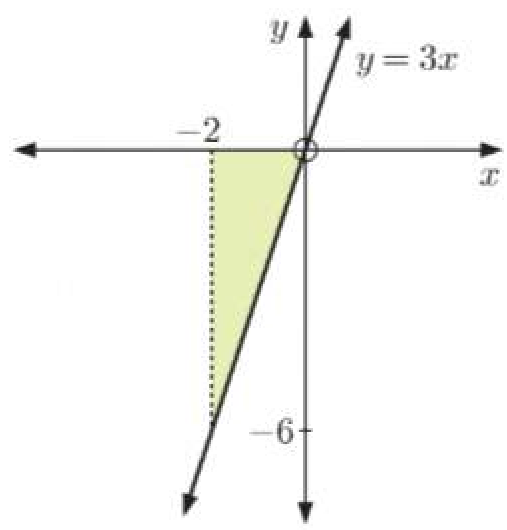
iii $\displaystyle\int_{-2}^0 3x\,dx = -\int_{-2}^0 -3x\,dx$

Now $f(x)=-3x$ has antiderivative

$F(x)=-\frac{3}{2}x^2$

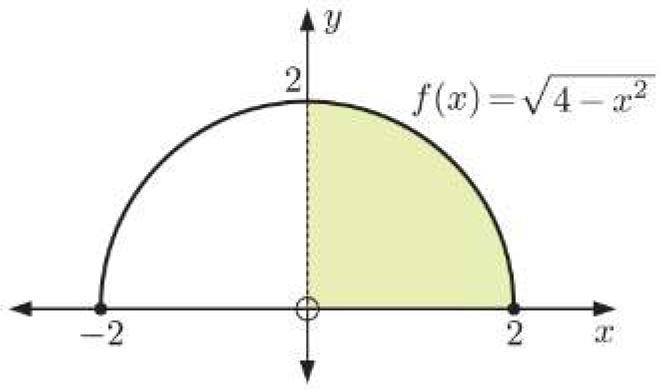
$$\therefore \int_{-2}^0 3x\,dx = -(F(0)-F(-2))$$
$$= -(0-(-6))$$
$$= -6$$

The shaded region has area 6 units^2.



d $\displaystyle\int_0^2 \left(-\sqrt{4-x^2}\right) dx = -\int_0^2 \sqrt{4-x^2}\, dx$

Now $f(x) = \sqrt{4-x^2}$ is the top half of a circle with radius 2 units and centre $(0, 0)$.

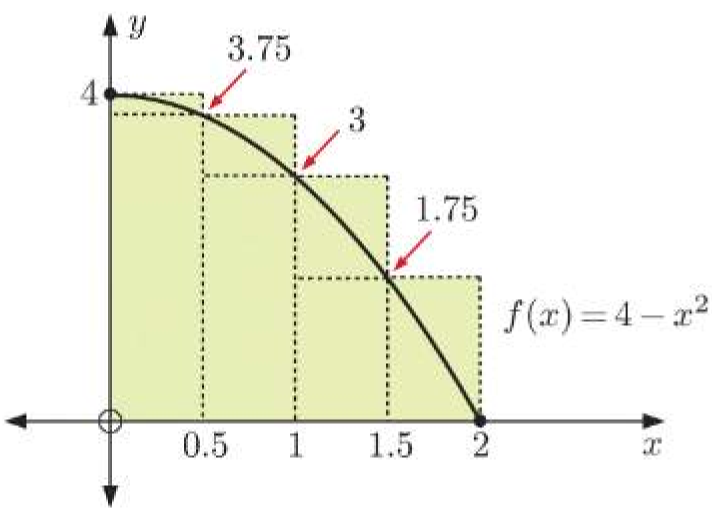


$$\therefore \int_0^2 \left(-\sqrt{4-x^2}\right) dx = -\int_0^2 \sqrt{4-x^2}\, dx$$
$$= -(\text{shaded area})$$
$$= -\tfrac{1}{4} \times \pi \times 2^2$$
$$= -\pi$$

REVIEW SET 21A

1 The rectangles are $\frac{2}{4} = \frac{1}{2}$ units wide.

a



$$A_L = \tfrac{1}{2}\left[f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right]$$
$$= \tfrac{1}{2}(\tfrac{15}{4} + 3 + \tfrac{7}{4} + 0)$$
$$= \tfrac{17}{4}$$
$$A_U = \tfrac{1}{2}\left[f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2})\right]$$
$$= \tfrac{1}{2}(4 + \tfrac{15}{4} + 3 + \tfrac{7}{4})$$
$$= \tfrac{25}{4}$$
$$\therefore \tfrac{17}{4} < \int_0^2 (4-x^2)\, dx < \tfrac{25}{4}$$
$$\therefore A = \tfrac{17}{4}, \quad B = \tfrac{25}{4}$$

b $n = 8, \ a = 0, \ b = 2, \ f(x) = 4 - x^2$

$h = \dfrac{b-a}{n} = \dfrac{1}{4}$

$x_i = 0 + \frac{1}{4}i$

i	x_i	$f(x_i)$
0	0	4
1	$\frac{1}{4}$	3.9375
2	$\frac{1}{2}$	3.75
3	$\frac{3}{4}$	3.4375
4	1	3
5	$1\frac{1}{4}$	2.4375
6	$1\frac{1}{2}$	1.75
7	$1\frac{3}{4}$	0.9375
8	2	0

Using the trapezoidal rule, the area $\approx \dfrac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_7) + f(x_8)\right)$

$$\approx 5.3125 \text{ units}^2$$

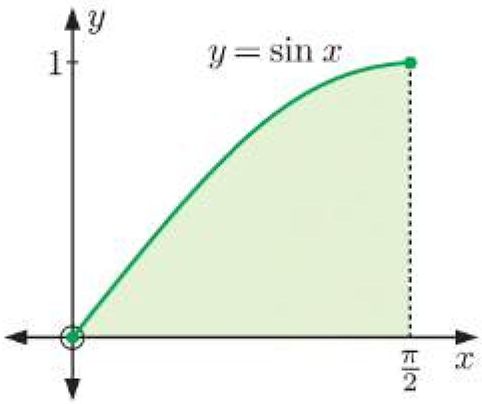
$$\therefore \int_0^2 (4-x^2)\, dx \approx 5.3125$$

c $\displaystyle\int_0^2 (4-x^2)\,dx = \left[4x - \tfrac{1}{3}x^3\right]_0^2$

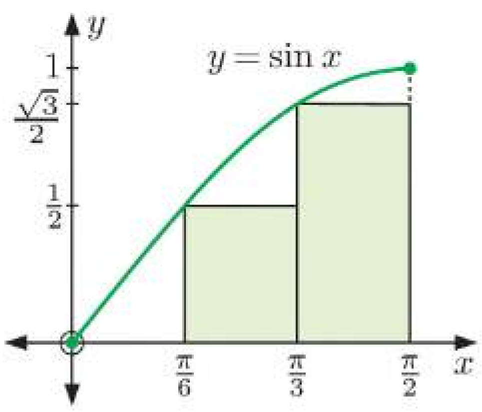
$$= \left(8 - \tfrac{8}{3}\right) - 0$$
$$= \tfrac{16}{3}$$
$$= 5\tfrac{1}{3}$$

The estimate from **a**, $\approx \dfrac{A+B}{2} \approx 5.25$, and the estimate from **b**, ≈ 5.3125, both provide good approximations of the integral.

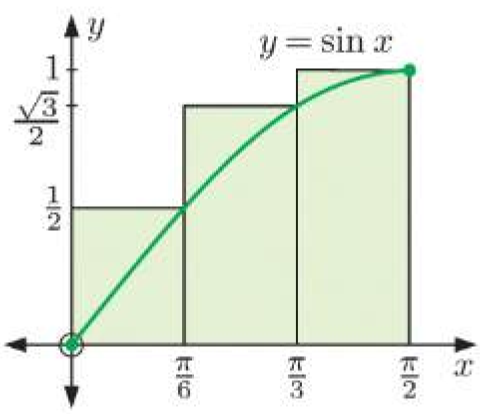
2 **a**



b The rectangles are $\dfrac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$ units wide.



$$A_L = \tfrac{\pi}{6}\left(\sin 0 + \sin\tfrac{\pi}{6} + \sin\tfrac{\pi}{3}\right)$$
$$= \tfrac{\pi}{6}\left(0 + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi(1+\sqrt{3})}{12}$$



$$A_U = \tfrac{\pi}{6}\left(\sin\tfrac{\pi}{6} + \sin\tfrac{\pi}{3} + \sin\tfrac{\pi}{2}\right)$$
$$= \tfrac{\pi}{6}\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} + 1\right)$$
$$= \tfrac{\pi}{6}\left(\frac{1+\sqrt{3}+2}{2}\right)$$
$$= \tfrac{\pi}{6}\left(\frac{3+\sqrt{3}}{2}\right)$$
$$= \frac{\pi(3+\sqrt{3})}{12}$$

$$\frac{\pi(1+\sqrt{3})}{12} < \int_0^{\frac{\pi}{2}} \sin x\,dx < \frac{\pi(3+\sqrt{3})}{12}$$

or $\quad 0.715 < \displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx < 1.24$

3 **a** $\dfrac{d}{dx}(x^5) = 5x^4$

$\therefore \dfrac{d}{dx}\left(\frac{1}{5}x^5\right) = x^4$

$\therefore$ the antiderivative of x^4 is $\frac{1}{5}x^5$ or $\dfrac{x^5}{5}$.

b $\dfrac{1}{2x^2} = \frac{1}{2}x^{-2}$

Now, $\dfrac{d}{dx}(x^{-1}) = -x^{-2}$

$\therefore \dfrac{d}{dx}\left(-\frac{1}{2}x^{-1}\right) = \frac{1}{2}x^{-2}$

$\therefore$ the antiderivative of $\dfrac{1}{2x^2}$ is $-\frac{1}{2}x^{-1} = -\dfrac{1}{2x}$.

c $\dfrac{d}{dx}(e^{-\frac{1}{2}x}) = -\frac{1}{2}e^{-\frac{1}{2}x}$

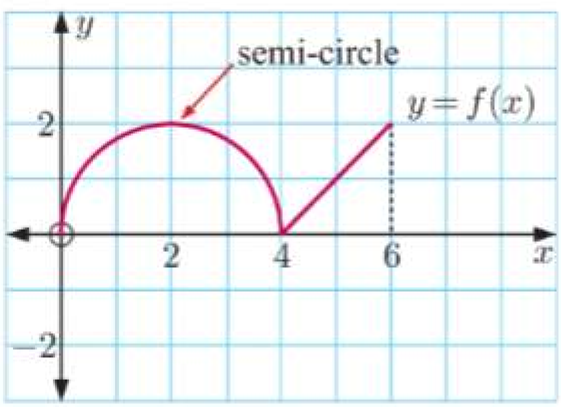
$\therefore \dfrac{d}{dx}(-2e^{-\frac{1}{2}x}) = e^{-\frac{1}{2}x}$

$\therefore$ the antiderivative of $e^{-\frac{1}{2}x}$ is $-2e^{-\frac{1}{2}x}$.

d $\dfrac{d}{dx}(\sin x) = \cos x$

$\therefore$ the antiderivative of $\cos x$ is $\sin x$.

4



a $\displaystyle\int_0^4 f(x)\,dx$ = area of semi-circle with radius 2

$= \frac{1}{2} \times \pi(2)^2$

$= 2\pi$

b $\displaystyle\int_4^6 f(x)\,dx$ = area of triangle

$= \frac{1}{2} \times 2 \times 2$

$= 2$

5 **a** $\frac{d}{dx}(x^3) = 3x^2$

$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$

$\therefore$ the antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$.

i $\int_0^1 x^2\,dx$
$= F(1) - F(0)$
$= \frac{1}{3} - 0$
$= \frac{1}{3}$ units^2

ii $\int_1^2 x^2\,dx$
$= F(2) - F(1)$
$= \frac{8}{3} - \frac{1}{3}$
$= \frac{7}{3} = 2\frac{1}{3}$ units^2

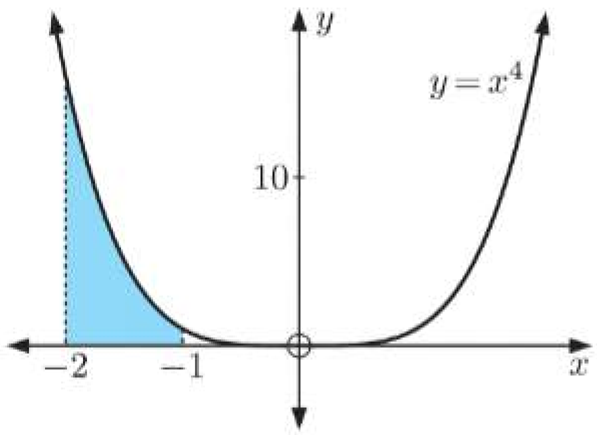
iii $\int_0^2 x^2\,dx$
$= F(2) - F(0)$
$= \frac{8}{3} - 0$
$= \frac{8}{3} = 2\frac{2}{3}$ units^2

b $\int_0^2 x^2\,dx = \int_0^1 x^2\,dx + \int_1^2 x^2\,dx$

6 **a** $n = 8,\ a = -2,\ b = -1,\ f(x) = x^4$

$h = \frac{b-a}{n} = \frac{1}{8}$

$x_i = -2 + \frac{1}{8}i$



i	x_i	$f(x_i)$
0	-2	16
1	$-1\frac{7}{8}$	12.359 62
2	$-1\frac{3}{4}$	9.378 91
3	$-1\frac{5}{8}$	6.972 90
4	$-1\frac{1}{2}$	5.0625
5	$-1\frac{3}{8}$	3.574 46
6	$-1\frac{1}{4}$	2.441 41
7	$-1\frac{1}{8}$	1.601 81
8	-1	1

Using the trapezoidal rule, the area $\approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_7) + f(x_8)\right)$

≈ 6.2365 units^2

$\therefore \int_{-2}^{-1} x^4\,dx \approx 6.2365$

b The estimate in **a** is an over estimate. Joining the interval endpoints with straight line segments would give a larger area than the shaded area.

c $\int_{-2}^{-1} x^4\,dx = \left[\frac{1}{5}x^5\right]_{-2}^{-1}$

$= \frac{1}{5}(-1)^5 - \frac{1}{5}(-2)^5$

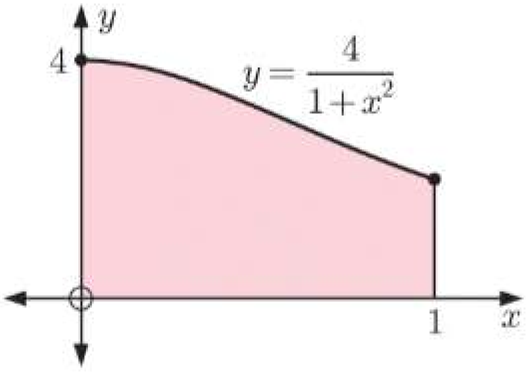
$= -\frac{1}{5} + \frac{32}{5}$

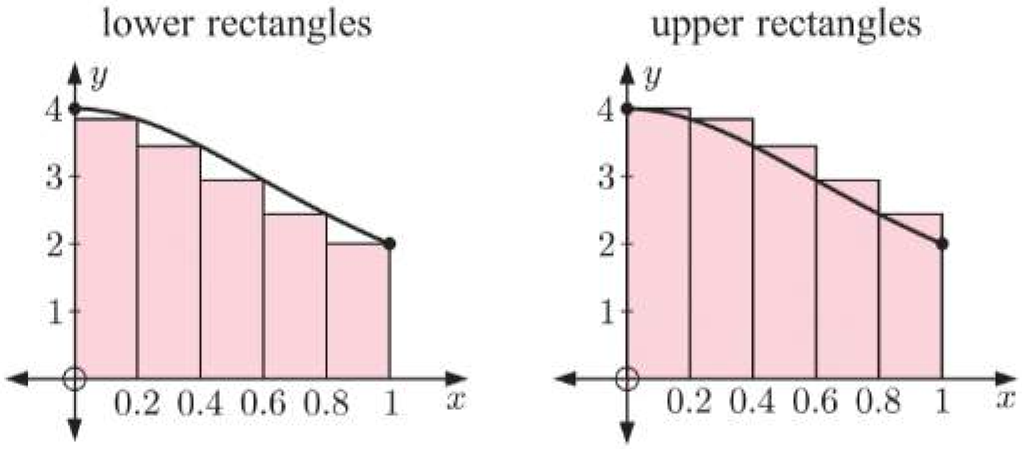
$= \frac{31}{5}$

$= 6.2 < 6.2365$ ✓

REVIEW SET 21B

1 **a**





b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c Using $n = 500$ rectangles,

$$\int_0^1 \frac{4}{1+x^2}\,dx \approx \frac{A_L + A_U}{2}$$
$$\approx \frac{3.1396 + 3.1436}{2}$$
$$\approx 3.1416$$

d $n = 10,\ a = 0,\ b = 1,\ f(x) = \frac{4}{1+x^2}$

$h = \frac{b-a}{n} = 0.1$

$x_i = 0 + 0.1i$

i	x_i	$f(x_i)$
0	0	4
1	0.1	3.9604
2	0.2	3.8462
3	0.3	3.6697
4	0.4	3.4483
5	0.5	3.2
6	0.6	2.9412
7	0.7	2.6846
8	0.8	2.4390
9	0.9	2.2099
10	1	2

Using the trapezoidal rule, the area $\approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10})\right)$

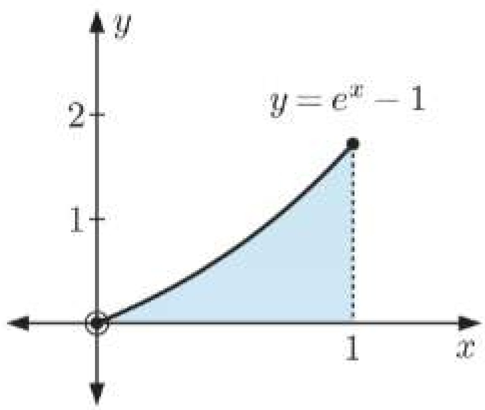
≈ 3.1399 units^2

$$\therefore \int_0^1 \frac{4}{1+x^2}\,dx \approx 3.1399$$

e $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \pi \approx 3.1416$

Both answers in **c** and **d** are good approximations of the integral, but the trapezoidal method required fewer subintervals and is hence more efficient.

2 **a**



b $n = 10,\ \ a = 0,\ \ b = 1,\ \ f(x) = e^x - 1$

$h = \dfrac{b-a}{n} = 0.1$

$x_i = a + hi = 0.1i$

i	x_i	$f(x_i)$
0	0	0
1	0.1	0.105 17
2	0.2	0.221 40
3	0.3	0.349 86
4	0.4	0.491 83
5	0.5	0.648 72
6	0.6	0.822 12
7	0.7	1.013 75
8	0.8	1.225 54
9	0.9	1.459 60
10	1	1.718 28

i $A_L = h(f(x_0) + f(x_1) + ... + f(x_9))$
≈ 0.6338 units^2

$\therefore\ \displaystyle\int_0^1 (e^x - 1)\,dx \approx 0.6338$

ii $A_U = h(f(x_1) + f(x_2) + ... + f(x_{10}))$
≈ 0.8056 units^2

$\therefore\ \displaystyle\int_0^1 (e^x - 1)\,dx \approx 0.8056$

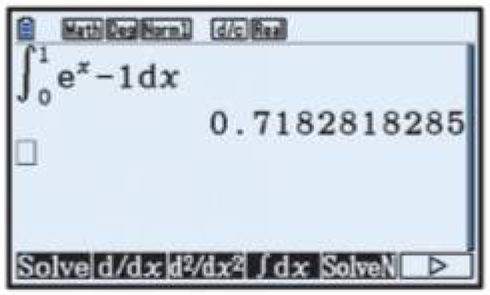
iii Using the trapezoidal rule, the area

$\approx \dfrac{h}{2}(f(x_0) + 2\,f(x_1) + 2\,f(x_2) + ... + 2\,f(x_9) + f(x_{10}))$

≈ 0.7197 units^2

$\therefore\ \displaystyle\int_0^1 (e^x - 1)\,dx \approx 0.7197$

c Using technology, $\int_0^1 (e^x - 1)\,dx \approx 0.7183$.

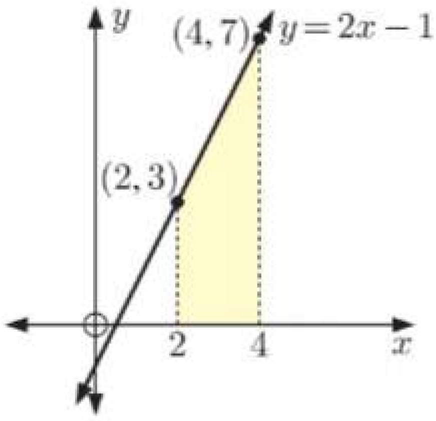


Using the upper and lower rectangle sums in **b i** and **b ii**:

$$\int_0^1 (e^x - 1)\,dx \approx \frac{0.6338 + 0.8056}{2}$$
$$\approx 0.7197$$

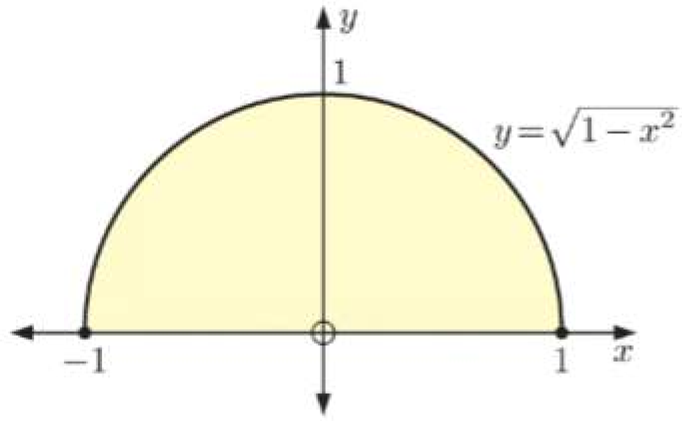
Our approximations of the integral using the rectangle and trapezoidal methods with 10 subintervals are very accurate.

3 **a**



$$\int_2^4 (2x-1)\,dx = \text{shaded area}$$
$$= \left(\frac{3+7}{2}\right) \times 2$$
$$= 10$$

b



$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \text{shaded area}$$
$$= \tfrac{1}{2}\pi r^2$$
$$= \tfrac{1}{2}(\pi \times 1^2)$$
$$= \tfrac{\pi}{2}$$

4 **a** $\frac{d}{dx}(x^3 - 2x) = 3x^2 - 2$

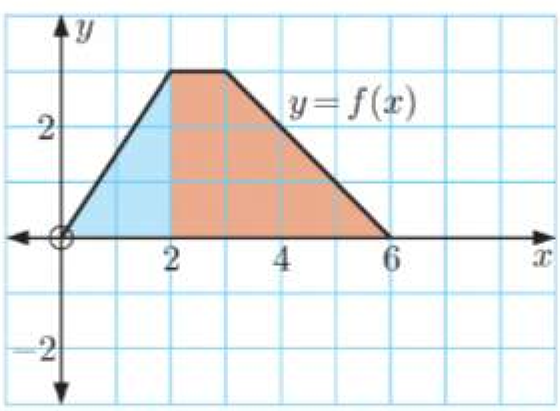
$\therefore$ the antiderivative of $3x^2 - 2$ is $x^3 - 2x$.

b $\frac{d}{dx}(x^{\frac{4}{3}}) = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x}$

$\therefore \frac{d}{dx}(\frac{3}{4}x^{\frac{4}{3}}) = \sqrt[3]{x}$

$\therefore$ the antiderivative of $\sqrt[3]{x}$ is $\frac{3}{4}x^{\frac{4}{3}}$.

5



a $\int_0^2 f(x)\,dx$

$= \text{area of blue triangle}$

$= \frac{1}{2} \times 2 \times 3$

$= 3$

b $\int_2^6 f(x)\,dx$

$= \text{area of red trapezium}$

$= \left(\frac{1+4}{2}\right) \times 3$

$= \frac{15}{2}$

6 **a** $\dfrac{d}{dx}(x^2) = 2x$

$\therefore \dfrac{d}{dx}(2x^2) = 4x$

$\therefore$ the antiderivative of $f(x) = 4x$ is $F(x) = 2x^2$.

$$\int_0^3 4x\,dx = F(3) - F(0) = 2(3)^2 - 2(0)^2 = 18 \text{ units}^2$$

b $\dfrac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$\therefore \dfrac{d}{dx}(\frac{2}{3}x^{\frac{3}{2}}) = \sqrt{x}$

$\therefore$ the antiderivative of $f(x) = \sqrt{x}$ is

$F(x) = \frac{2}{3}x^{\frac{3}{2}}$.

$$\int_0^9 \sqrt{x}\,dx = F(9) - F(0) = \tfrac{2}{3}(9)^{\frac{3}{2}} - \tfrac{2}{3}(0)^{\frac{3}{2}} = \tfrac{2}{3} \times 9 \times 3 = 18 \text{ units}^2$$

Chapter 22

TECHNIQUES FOR INTEGRATION

EXERCISE 22A

1 **a** $$\frac{d}{dx}(x^7) = 7x^6$$

$$\therefore \int 7x^6\,dx = x^7 + c$$

$$\therefore 7\int x^6\,dx = x^7 + c$$

$$\therefore \int x^6\,dx = \tfrac{1}{7}x^7 + c$$

b $$\frac{d}{dx}(x^{\frac{3}{2}}) = \tfrac{3}{2}x^{\frac{1}{2}} = \tfrac{3}{2}\sqrt{x}$$

$$\therefore \int \tfrac{3}{2}\sqrt{x}\,dx = x^{\frac{3}{2}} + c$$

$$\therefore \tfrac{3}{2}\int \sqrt{x}\,dx = x^{\frac{3}{2}} + c$$

$$\therefore \int \sqrt{x}\,dx = \tfrac{2}{3}x^{\frac{3}{2}} + c$$

c $$\frac{d}{dx}(x^{-\frac{1}{2}}) = -\tfrac{1}{2}x^{-\frac{3}{2}}$$

$$\therefore \int -\tfrac{1}{2}x^{-\frac{3}{2}}\,dx = x^{-\frac{1}{2}} + c$$

$$\therefore -\tfrac{1}{2}\int x^{-\frac{3}{2}}\,dx = x^{-\frac{1}{2}} + c$$

$$\therefore \int x^{-\frac{3}{2}}\,dx = -2x^{-\frac{1}{2}} + c$$

d $$\frac{d}{dx}(x^{n+1}) = (n+1)x^n, \quad n \neq -1$$

$$\therefore \int (n+1)x^n\,dx = x^{n+1} + c, \quad n \neq -1$$

$$\therefore (n+1)\int x^n\,dx = x^{n+1} + c, \quad n \neq -1$$

$$\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

2 **a** $$\frac{d}{dx}(e^{4x}) = 4e^{4x}$$

$$\therefore \int 4e^{4x}\,dx = e^{4x} + c$$

$$\therefore 4\int e^{4x}\,dx = e^{4x} + c$$

$$\therefore \int e^{4x}\,dx = \tfrac{1}{4}e^{4x} + c$$

b $$\frac{d}{dx}(e^{-\frac{x}{2}}) = -\tfrac{1}{2}e^{-\frac{x}{2}}$$

$$\therefore \int -\tfrac{1}{2}e^{-\frac{x}{2}}\,dx = e^{-\frac{x}{2}} + c$$

$$\therefore -\tfrac{1}{2}\int e^{-\frac{x}{2}}\,dx = e^{-\frac{x}{2}} + c$$

$$\therefore \int e^{-\frac{x}{2}}\,dx = -2e^{-\frac{x}{2}} + c$$

c $$\frac{d}{dx}(e^{kx}) = ke^{kx}, \quad k \neq 0$$

$$\therefore \int ke^{kx}\,dx = e^{kx} + c, \quad k \neq 0$$

$$\therefore k\int e^{kx}\,dx = e^{kx} + c, \quad k \neq 0$$

$$\therefore \int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

3 **a**
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\therefore \int \cos x \; dx = \sin x + c$$

b
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\therefore \int (-\sin x)\, dx = \cos x + c$$
$$\therefore -\int \sin x \; dx = \cos x + c$$
$$\therefore \int \sin x \; dx = -\cos x + c$$

c
$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$
$$\therefore \int \frac{1}{\cos^2 x} \, dx = \tan x + c$$

4
$$\frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$
$$\therefore \int (3x^2 + 2x)\, dx = x^3 + x^2 + c$$

5
$$\frac{d}{dx}(3x^4 - 2x^2) = 12x^3 - 4x$$
$$\therefore \int (12x^3 - 4x)\, dx = 3x^4 - 2x^2 + c$$
$$\therefore 4\int (3x^3 - x)\, dx = 3x^4 - 2x^2 + c$$
$$\therefore \int (3x^3 - x)\, dx = \tfrac{3}{4}x^4 - \tfrac{1}{2}x^2 + c$$

6 **a**
$$\begin{aligned}\frac{d}{dx}[F(x) + G(x)] &= F'(x) + G'(x) \\ &= f(x) + g(x)\end{aligned}$$

b Using **a**,
$$\begin{aligned}\int [f(x) + g(x)]\, dx &= F(x) + G(x) + c \\ &= \int f(x) \; dx + \int g(x) \; dx\end{aligned}$$

7 **a**
$$\begin{aligned}\frac{d}{dx}(\sin 3x) &= \cos 3x \times 3 \\ &= 3\cos 3x\end{aligned}$$
$$\therefore \int 3\cos 3x \; dx = \sin 3x + c$$
$$\therefore 3\int \cos 3x \; dx = \sin 3x + c$$
$$\therefore \int \cos 3x \; dx = \tfrac{1}{3}\sin 3x + c$$

b
$$\begin{aligned}\frac{d}{dx}\left(\cos\left(\tfrac{\pi}{3} - x\right)\right) &= -\sin\left(\tfrac{\pi}{3} - x\right) \times (-1) \\ &= \sin\left(\tfrac{\pi}{3} - x\right)\end{aligned}$$
$$\therefore \int \sin\left(\tfrac{\pi}{3} - x\right) dx = \cos\left(\tfrac{\pi}{3} - x\right) + c$$

c $\frac{d}{dx}\left(e^{3x+1}\right) = 3e^{3x+1}$

$\therefore \int 3e^{3x+1}\,dx = e^{3x+1} + c$

$\therefore 3\int e^{3x+1}\,dx = e^{3x+1} + c$

$\therefore \int e^{3x+1}\,dx = \frac{1}{3}e^{3x+1} + c$

d $\frac{d}{dx}\left(\sqrt{5x-1}\right) = \frac{d}{dx}\left((5x-1)^{\frac{1}{2}}\right)$

$= \frac{1}{2}(5x-1)^{-\frac{1}{2}} \times 5$

$= \frac{5}{2\sqrt{5x-1}}$

$\therefore \int \frac{5}{2\sqrt{5x-1}}\,dx = \sqrt{5x-1} + c$

$\therefore \frac{5}{2}\int \frac{1}{\sqrt{5x-1}}\,dx = \sqrt{5x-1} + c$

$\therefore \int \frac{1}{\sqrt{5x-1}}\,dx = \frac{2}{5}\sqrt{5x-1} + c$

e $\frac{d}{dx}\left((2x+1)^4\right) = 4(2x+1)^3 \times 2$

$= 8(2x+1)^3$

$\therefore \int 8(2x+1)^3\,dx = (2x+1)^4 + c$

$\therefore 8\int (2x+1)^3\,dx = (2x+1)^4 + c$

$\therefore \int (2x+1)^3\,dx = \frac{1}{8}(2x+1)^4 + c$

f $\frac{d}{dx}\left((x^2-x)^3\right) = 3(x^2-x)^2 \times (2x-1)$

$= 3(2x-1)(x^2-x)^2$

$\therefore \int 3(2x-1)(x^2-x)^2\,dx = (x^2-x)^3 + c$

$\therefore 3\int (2x-1)(x^2-x)^2\,dx = (x^2-x)^3 + c$

$\therefore \int (2x-1)(x^2-x)^2\,dx = \frac{1}{3}(x^2-x)^3 + c$

8 **a** For $x > 0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

b For $x < 0$, $\frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$

c $\int \frac{1}{x}\,dx = \begin{cases} \ln x + c & \text{if } x > 0 \\ \ln(-x) + c & \text{if } x < 0 \end{cases}$

$\therefore \int \frac{1}{x}\,dx = \ln|x| + c, \quad x \neq 0$

EXERCISE 22B

1 **a** $\int (x^2 + 3x - 2)\,dx$

$= \frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x + c$

b $\int (2x^2 - 3x + 1)\,dx$

$= \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + c$

c $\int(-x^3+4x^2-3)\,dx$
$= -\frac{1}{4}x^4+\frac{4}{3}x^3-3x+c$

d $\int(\frac{1}{2}x+x^2+x^3)\,dx$
$= \frac{1}{4}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+c$

e $\int(x^4-x^2-x+2)\,dx$
$= \frac{1}{5}x^5-\frac{1}{3}x^3-\frac{1}{2}x^2+2x+c$

f $\int\left(4x^2+\frac{1}{x}\right)dx$
$= \frac{4}{3}x^3+\ln|\,x\,|+c$

g $\int\left(2\sqrt{x}-\frac{3}{\sqrt{x}}\right)dx$
$= \int(2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})\,dx$
$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{3x^{\frac{1}{2}}}{\frac{1}{2}}+c$
$= \frac{4}{3}x^{\frac{3}{2}}-6x^{\frac{1}{2}}+c$

h $\int\left(\frac{1}{3x}-\frac{2}{x^2}\right)dx$
$= \int\left(\frac{1}{3}x^{-1}-2x^{-2}\right)dx$
$= \frac{1}{3}\ln|\,x\,|-\frac{2x^{-1}}{(-1)}+c$
$= \frac{1}{3}\ln|\,x\,|+\frac{2}{x}+c$

i $\int(x\sqrt{x}-9)\,dx = \int(x^{\frac{3}{2}}-9)\,dx$
$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}}-9x+c$
$= \frac{2}{5}x^{\frac{5}{2}}-9x+c$

j $\int(3x^{-\frac{3}{2}}+x^{\frac{1}{4}})\,dx$
$= \frac{3x^{-\frac{1}{2}}}{(-\frac{1}{2})}+\frac{x^{\frac{5}{4}}}{\frac{5}{4}}+c$
$= -6x^{-\frac{1}{2}}+\frac{4}{5}x^{\frac{5}{4}}+c$

2 **a** $\int(2e^x-3x)\,dx$
$= 2e^x-\frac{3}{2}x^2+c$

b $\int\left(\frac{4}{x}+x^2-e^x\right)dx$
$= 4\ln|\,x\,|+\frac{1}{3}x^3-e^x+c$

c $\int\left(5e^x+\frac{1}{2}x^2\right)dx$
$= 5e^x+\frac{1}{6}x^3+c$

d $\int(3\sin x-2)\,dx$
$= -3\cos x-2x+c$

e $\int(4x-2\cos x)\,dx$
$= 2x^2-2\sin x+c$

f $\int(\sin x-2\cos x+e^x)\,dx$
$= -\cos x-2\sin x+e^x+c$

g $\int(x^2\sqrt{x}-10\sin x)\,dx$
$= \int(x^{\frac{5}{2}}-10\sin x)\,dx$
$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}}+10\cos x+c$
$= \frac{2}{7}x^{\frac{7}{2}}+10\cos x+c$

h $\int\left(\frac{x(x-1)}{3}+\frac{1}{\cos^2 x}\right)dx$
$= \int\left(\frac{1}{3}x^2-\frac{1}{3}x+\frac{1}{\cos^2 x}\right)dx$
$= \frac{1}{9}x^3-\frac{1}{6}x^2+\tan x+c$

i $\displaystyle\int(-\sin x + 2\sqrt{x})\,dx$

$\displaystyle= \int(-\sin x + 2x^{\frac{1}{2}})\,dx$

$\displaystyle= \cos x + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \cos x + \frac{4}{3}x^{\frac{3}{2}} + c$

3 **a** $\dfrac{dy}{dx} = 6$

$\therefore\ y = \displaystyle\int 6\,dx$

$= 6x + c$

b $\dfrac{dy}{dx} = 4x^2$

$\therefore\ y = \displaystyle\int 4x^2\,dx$

$= \frac{4}{3}x^3 + c$

c $\dfrac{dy}{dx} = \dfrac{1}{x^2} = x^{-2}$

$\therefore\ y = \displaystyle\int x^{-2}\,dx$

$= \dfrac{x^{-1}}{(-1)} + c$

$= -\dfrac{1}{x} + c$

d $\dfrac{dy}{dx} = \dfrac{2}{\sqrt[3]{x}} = 2x^{-\frac{1}{3}}$

$\therefore\ y = \displaystyle\int 2x^{-\frac{1}{3}}\,dx$

$= \dfrac{2x^{\frac{2}{3}}}{\frac{2}{3}} + c$

$= 3x^{\frac{2}{3}} + c$

e $\dfrac{dy}{dx} = 2x^3 - 4$

$\therefore\ y = \displaystyle\int(2x^3 - 4)\,dx$

$= \frac{1}{2}x^4 - 4x + c$

f $\dfrac{dy}{dx} = 4x^3 + 3x^2$

$\therefore\ y = \displaystyle\int(4x^3 + 3x^2)\,dx$

$= x^4 + x^3 + c$

g $\dfrac{dy}{dx} = 2 - \dfrac{1}{x}$

$\therefore\ y = \displaystyle\int\left(2 - \frac{1}{x}\right)dx$

$= 2x - \ln|x| + c$

h $\dfrac{dy}{dx} = \sin x + 2\cos x$

$\therefore\ y = \displaystyle\int(\sin x + 2\cos x)\,dx$

$= -\cos x + 2\sin x + c$

i $\dfrac{dy}{dx} = 2e^x - 5 + x$

$\therefore\ y = \displaystyle\int(2e^x - 5 + x)\,dx$

$= 2e^x - 5x + \frac{1}{2}x^2 + c$

4 **a** $\displaystyle\int(2e^t - 4\sin t)\,dt$

$= 2e^t + 4\cos t + c$

b $\displaystyle\int\left(3\cos t - \frac{1}{t}\right)dt$

$= 3\sin t - \ln|t| + c$

c
$$\int (5\sin t - \sqrt{t})\,dt$$
$$= \int (5\sin t - t^{\frac{1}{2}})\,dt$$
$$= -5\cos t - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= -5\cos t - \tfrac{2}{3}t\sqrt{t} + c$$

d
$$\int \left(\frac{1}{\cos^2 x} + 2\sin x\right)\,dx$$
$$= \tan x - 2\cos x + c$$

e
$$\int (\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}\theta^2 + \cos\theta + c$$

f
$$\int \left(\frac{2}{\theta} - \frac{1}{\cos^2\theta}\right)\,d\theta$$
$$= 2\ln|\,\theta\,| - \tan\theta + c$$

5 **a**
$$\int (2x+1)^2\,dx$$
$$= \int (4x^2 + 4x + 1)\,dx$$
$$= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c$$
$$= \tfrac{4}{3}x^3 + 2x^2 + x + c$$

b
$$\int \left(x + \frac{1}{x}\right)^2\,dx$$
$$= \int \left(x^2 + 2 + \frac{1}{x^2}\right)\,dx$$
$$= \int (x^2 + 2 + x^{-2})\,dx$$
$$= \frac{x^3}{3} + 2x + \frac{x^{-1}}{(-1)} + c$$
$$= \tfrac{1}{3}x^3 + 2x - \frac{1}{x} + c$$

c
$$\int \left(\frac{1-4x}{x\sqrt{x}}\right)\,dx$$
$$= \int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}}\right)\,dx$$
$$= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}})\,dx$$
$$= \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= -2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c$$
$$= -\frac{2}{\sqrt{x}} - 8\sqrt{x} + c$$

d
$$\int \frac{2x-1}{\sqrt{x}}\,dx$$
$$= \int \left(2\sqrt{x} - \frac{1}{\sqrt{x}}\right)\,dx$$
$$= \int (2x^{\frac{1}{2}} - x^{-\frac{1}{2}})\,dx$$
$$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$
$$= \tfrac{4}{3}x\sqrt{x} - 2\sqrt{x} + c$$

e
$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2\,dx$$
$$= \int \left(x - 2 + \frac{1}{x}\right)\,dx$$
$$= \tfrac{1}{2}x^2 - 2x + \ln|\,x\,| + c$$

f
$$\int \left(\frac{1-x^2}{x}\right)\,dx$$
$$= \int \left(\frac{1}{x} - x\right)\,dx$$
$$= \ln|\,x\,| - \tfrac{1}{2}x^2 + c$$

g
$$\begin{aligned}
&\int \left(\frac{2}{x}+1\right)^2 dx \\
&= \int \left(\frac{4}{x^2}+\frac{4}{x}+1\right) dx \\
&= \int (4x^{-2}+4x^{-1}+1)\, dx \\
&= \frac{4x^{-1}}{(-1)} + 4\ln|x| + x + c \\
&= -\frac{4}{x} + 4\ln|x| + x + c
\end{aligned}$$

h
$$\begin{aligned}
&\int \frac{x^2-4x+10}{x}\, dx \\
&= \int \left(x-4+\frac{10}{x}\right) dx \\
&= \tfrac{1}{2}x^2 - 4x + 10\ln|x| + c
\end{aligned}$$

i
$$\begin{aligned}
&\int \frac{3x^3-2x^2+5}{x^2}\, dx \\
&= \int \left(3x-2+\frac{5}{x^2}\right) dx \\
&= \int (3x-2+5x^{-2})\, dx \\
&= \frac{3x^2}{2} - 2x + \frac{5x^{-1}}{(-1)} + c \\
&= \tfrac{3}{2}x^2 - 2x - \frac{5}{x} + c
\end{aligned}$$

6 **a**
$$\begin{aligned}
f'(x) &= (1-2x)^2 \\
\therefore\ f(x) &= \int (1-2x)^2\, dx \\
&= \int (1-4x+4x^2)\, dx \\
&= x - \frac{4x^2}{2} + \frac{4x^3}{3} + c \\
&= x - 2x^2 + \tfrac{4}{3}x^3 + c
\end{aligned}$$

b
$$\begin{aligned}
f'(x) &= \sqrt{x} - \frac{2}{\sqrt{x}} \\
&= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\
\therefore\ f(x) &= \int (x^{\frac{1}{2}} - 2x^{-\frac{1}{2}})\, dx \\
&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= \tfrac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c \\
&= \tfrac{2}{3}x\sqrt{x} - 4\sqrt{x} + c
\end{aligned}$$

c
$$\begin{aligned}
f'(x) &= \frac{x^2-5}{x^2} \\
&= 1 - 5x^{-2} \\
\therefore\ f(x) &= \int (1-5x^{-2})\, dx \\
&= x - \frac{5x^{-1}}{(-1)} + c \\
&= x + \frac{5}{x} + c
\end{aligned}$$

7 **a**
$$\int (\sqrt{x} + \tfrac{1}{2}\cos x)\,dx$$
$$= \int (x^{\frac{1}{2}} + \tfrac{1}{2}\cos x)\,dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \tfrac{1}{2}\sin x + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} + \tfrac{1}{2}\sin x + c$$

b
$$\int (2e^x - 4\sin x)\,dx$$
$$= 2e^x + 4\cos x + c$$

c
$$\int (3\cos x - \sin x)\,dx$$
$$= 3\sin x + \cos x + c$$

8 **a**
$$\int \left(x^2 - \frac{1}{x}\right)^2 dx$$
$$= \int \left(x^4 - 2x + \frac{1}{x^2}\right) dx$$
$$= \int (x^4 - 2x + x^{-2})\,dx$$
$$= \frac{x^5}{5} - \frac{2x^2}{2} + \frac{x^{-1}}{(-1)} + c$$
$$= \tfrac{1}{5}x^5 - x^2 - \frac{1}{x} + c$$

b
$$\int \frac{x^2 - 4x + 2}{\sqrt{x}}\,dx$$
$$= \int \frac{x^2 - 4x + 2}{x^{\frac{1}{2}}}\,dx$$
$$= \int (x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2x^{-\frac{1}{2}})\,dx$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{2}{5}x^{\frac{5}{2}} - \tfrac{8}{3}x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c$$

c
$$\int \sqrt{x}(3x-1)^2\,dx$$
$$= \int x^{\frac{1}{2}} \times (9x^2 - 6x + 1)\,dx$$
$$= \int (9x^{\frac{5}{2}} - 6x^{\frac{3}{2}} + x^{\frac{1}{2}})\,dx$$
$$= \frac{9x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{6x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \tfrac{18}{7}x^{\frac{7}{2}} - \tfrac{12}{5}x^{\frac{5}{2}} + \tfrac{2}{3}x^{\frac{3}{2}} + c$$

EXERCISE 22C

1 **a**
$$f'(x) = 2x - 1$$
$$\therefore\ f(x) = \int (2x-1)\,dx$$
$$= \frac{2x^2}{2} - x + c$$
$$= x^2 - x + c$$
But $f(0) = 3$, so $0 - 0 + c = 3$
$$\therefore\ c = 3$$
$$\therefore\ f(x) = x^2 - x + 3$$

b
$$f'(x) = 3x^2 + 2x$$
$$\therefore\ f(x) = \int (3x^2 + 2x)\,dx$$
$$= \frac{3x^3}{3} + \frac{2x^2}{2} + c$$
$$= x^3 + x^2 + c$$
But $f(2) = 5$, so $8 + 4 + c = 5$
$$\therefore\ c = -7$$
$$\therefore\ f(x) = x^3 + x^2 - 7$$

c $f'(x) = 2 + \dfrac{1}{\sqrt{x}} = 2 + x^{-\frac{1}{2}}$

$\therefore \; f(x) = \displaystyle\int (2 + x^{-\frac{1}{2}})\,dx$

$= 2x + \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2x + 2x^{\frac{1}{2}} + c$

But $f(1) = 1$, so $2 + 2 + c = 1$

$\therefore \; c = -3$

$\therefore \; f(x) = 2x + 2\sqrt{x} - 3$

d $f'(x) = x - \dfrac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}}$

$\therefore \; f(x) = \displaystyle\int (x - 2x^{-\frac{1}{2}})\,dx$

$= \dfrac{x^2}{2} - \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{1}{2}x^2 - 4\sqrt{x} + c$

But $f(1) = 2$, so $\frac{1}{2} - 4 + c = 2$

$\therefore \; c = \frac{11}{2}$

$\therefore \; f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

e $f'(x) = \sqrt{x} - 2 = x^{\frac{1}{2}} - 2$

$\therefore \; f(x) = \displaystyle\int (x^{\frac{1}{2}} - 2)\,dx$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2x + c$

$= \frac{2}{3}x^{\frac{3}{2}} - 2x + c$

But $f(4) = 0$, so $\frac{2}{3}(4)^{\frac{3}{2}} - 2(4) + c = 0$

$\therefore \; \frac{16}{3} - 8 + c = 0$

$\therefore \; c = \frac{8}{3}$

$\therefore \; f(x) = \frac{2}{3}x^{\frac{3}{2}} - 2x + \frac{8}{3}$

f $f'(x) = \dfrac{1}{x}$

$\therefore \; f(x) = \displaystyle\int \frac{1}{x}\,dx$

$= \ln|x| + c$

But $f(e) = 2$, so $\ln e + c = 2$

$\therefore \; c = 1$

$\therefore \; f(x) = \ln|x| + 1$

2 $\dfrac{dy}{dx} = x - 2x^2$

$\therefore \; y = \displaystyle\int (x - 2x^2)\,dx$

$= \frac{1}{2}x^2 - \frac{2}{3}x^3 + c$

But the curve passes through $(2, 4)$, so when $x = 2$, $y = 4$.

$\therefore \; 4 = \frac{1}{2}(2)^2 - \frac{2}{3}(2)^3 + c$

$\therefore \; 4 = 2 - \frac{16}{3} + c$

$\therefore \; c = \frac{22}{3}$

$\therefore \; y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{22}{3}$

3 $\dfrac{dy}{dx} = 1 - e^x$

$\therefore \; y = \displaystyle\int (1 - e^x)\,dx$

$= x - e^x + c$

But the curve passes through $(3, e^3)$, so when $x = 3$, $y = e^3$.

$\therefore \; e^3 = 3 - e^3 + c$

$\therefore \; c = 2e^3 - 3$

$\therefore \; y = x - e^x + 2e^3 - 3$

4 **a**
$$f'(x) = x^2 - 4\cos x$$
$$\therefore\ f(x) = \int (x^2 - 4\cos x)\,dx$$
$$= \tfrac{1}{3}x^3 - 4\sin x + c$$
But $f(0) = 3$, so $c = 3$
$$\therefore\ f(x) = \tfrac{1}{3}x^3 - 4\sin x + 3$$

b
$$f'(x) = 2\cos x - 3\sin x$$
$$\therefore\ f(x) = \int (2\cos x - 3\sin x)\,dx$$
$$= 2\sin x + 3\cos x + c$$
But $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$,

so $2\sin\frac{\pi}{4} + 3\cos\frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$
$$\therefore\ \tfrac{2}{\sqrt{2}} + \tfrac{3}{\sqrt{2}} + c = \tfrac{1}{\sqrt{2}}$$
$$\therefore\ c = -\tfrac{4}{\sqrt{2}}$$
$$\therefore\ c = -2\sqrt{2}$$
$$\therefore\ f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$$

c
$$f'(x) = \sqrt{x} - \frac{2}{\cos^2 x} = x^{\frac{1}{2}} - \frac{2}{\cos^2 x}$$
$$\therefore\ f(x) = \int \left(x^{\frac{1}{2}} - \frac{2}{\cos^2 x}\right) dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2\tan x + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x + c$$
But $f(\pi) = 0$,

so $\frac{2}{3}\pi^{\frac{3}{2}} - 2\tan\pi + c = 0$
$$\therefore\ c = -\tfrac{2}{3}\pi^{\frac{3}{2}}$$
$$f(x) = \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x - \tfrac{2}{3}\pi^{\frac{3}{2}}$$
$$= \tfrac{2}{3}x\sqrt{x} - 2\tan x - \tfrac{2}{3}\pi\sqrt{\pi}$$

d
$$f'(x) = e^x + 3\cos x$$
$$\therefore\ f(x) = \int (e^x + 3\cos x)\,dx$$
$$= e^x + 3\sin x + c$$
But $f(\pi) = 0$, so $e^\pi + 3\sin\pi + c = 0$
$$\therefore\ e^\pi + c = 0$$
$$\therefore\ c = -e^\pi$$
$$\therefore\ f(x) = e^x + 3\sin x - e^\pi$$

5
$$f'(x) = ax + 1$$
$$\therefore\ f(x) = \int (ax + 1)\,dx$$
$$= \frac{ax^2}{2} + x + c$$
Now $f(0) = 3$, so $c = 3$
$$\therefore\ f(x) = \tfrac{1}{2}ax^2 + x + 3$$
and $f(3) = -3$
$$\therefore\ \tfrac{1}{2}a(3)^2 + 3 + 3 = -3$$
$$\therefore\ \tfrac{9}{2}a + 6 = -3$$
$$\therefore\ \tfrac{9}{2}a = -9$$
$$\therefore\ a = -2$$
$$\therefore\ f(x) = \tfrac{1}{2}(-2)x^2 + x + 3$$
$$= -x^2 + x + 3$$

6 $f'(x) = ax^2 + bx$

$$\therefore\ f(x) = \int (ax^2 + bx)\,dx = \frac{ax^3}{3} + \frac{bx^2}{2} + c$$

Now $f(0) = 1$, so $c = 1$

$\therefore\ f(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + 1,$

$f(-1) = -2$ and $f(1) = 4$

$\therefore\ \frac{1}{3}a(-1)^3 + \frac{1}{2}b(-1)^2 + 1 = -2$

$\therefore\ -\frac{1}{3}a + \frac{1}{2}b + 1 = -2$

$\therefore\ -\frac{1}{3}a + \frac{1}{2}b = -3$ .... (1)

$\therefore\ \frac{1}{3}a + \frac{1}{2}b + 1 = 4$

$\therefore\ \frac{1}{3}a + \frac{1}{2}b = 3$ .... (2)

Adding (1) and (2) together gives: $b = 0$

Substituting $b = 0$ into (1) gives: $-\frac{1}{3}a = -3$

$\therefore\ a = 9$

$$\therefore\ f(x) = \tfrac{1}{3}(9)x^3 + \tfrac{1}{2}(0)x^2 + 1 = 3x^3 + 1$$

7 **a** $f''(x) = 2x + 1$

$$\therefore\ f'(x) = \int (2x+1)\,dx = x^2 + x + c$$

But $f'(1) = 3$, so $1 + 1 + c = 3$

$\therefore\ c = 1$

$\therefore\ f'(x) = x^2 + x + 1$

$$\therefore\ f(x) = \int (x^2 + x + 1)\,dx = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x + d$$

But $f(2) = 7$,

so $\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2 + d = 7$

$\therefore\ \frac{8}{3} + 2 + 2 + d = 7$

$\therefore\ d = \frac{1}{3}$

$\therefore\ f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}} = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\therefore\ f'(x) = \int (15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})\,dx = \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$$

But $f'(1) = 12$, so $10 + 6 + c = 12$

$\therefore\ c = -4$

$\therefore\ f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$

$$\therefore\ f(x) = \int (10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4)\,dx = \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + d = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + d$$

But $f(0) = 5$, so $d = 5$

$\therefore\ f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c
$$f''(x) = \cos x$$
$$\therefore \quad f'(x) = \int \cos x \, dx$$
$$= \sin x + c$$
But $f'(\frac{\pi}{2}) = 0$, so $\sin \frac{\pi}{2} + c = 0$
$$\therefore \quad 1 + c = 0$$
$$\therefore \quad c = -1$$
$$\therefore \quad f'(x) = \sin x - 1$$
$$\therefore \quad f(x) = \int (\sin x - 1) \, dx$$
$$= -\cos x - x + d$$
But $f(0) = 3$, so $-\cos 0 + d = 3$
$$\therefore \quad -1 + d = 3$$
$$\therefore \quad d = 4$$
$$\therefore \quad f(x) = -\cos x - x + 4$$

d
$$f''(x) = 2x$$
$$\therefore \quad f'(x) = \int 2x \, dx$$
$$= x^2 + c$$
$$\therefore \quad f(x) = \int (x^2 + c) \, dx$$
$$= \tfrac{1}{3}x^3 + cx + d$$
But $(1, 0)$ and $(0, 5)$ lie on the curve $y = f(x)$
$$\therefore \quad f(1) = 0$$
$$\therefore \quad \tfrac{1}{3} + c + d = 0$$
$$\therefore \quad c + d = -\tfrac{1}{3} \quad .... \ (*)$$
and $f(0) = 5$
$$\therefore \quad d = 5$$
Subsituting $d = 5$ into $(*)$ gives:
$$c + 5 = -\tfrac{1}{3}$$
$$\therefore \quad c = -\tfrac{16}{3}$$
$$\therefore \quad f(x) = \tfrac{1}{3}x^3 - \tfrac{16}{3}x + 5$$

8
$$f''(x) = 3e^{-x}$$
$$\therefore \quad f'(x) = \int 3e^{-x} \, dx$$
$$= -3e^{-x} + c$$
$$\therefore \quad f(x) = \int (-3e^{-x} + c) \, dx$$
$$= 3e^{-x} + cx + d$$

But $f(1) = \dfrac{3}{e}$ and $f(3) = \dfrac{3}{e^3} - 2$

$$\therefore \quad 3e^{-1} + c + d = \frac{3}{e}$$
$$\therefore \quad \frac{3}{e} + c + d = \frac{3}{e}$$
$$\therefore \quad c + d = 0$$
$$\therefore \quad d = -c \quad .... \ (1)$$

$$\therefore \quad 3e^{-3} + 3c + d = \frac{3}{e^3} - 2$$
$$\therefore \quad \frac{3}{e^3} + 3c + d = \frac{3}{e^3} - 2$$
$$\therefore \quad 3c + d = -2 \quad .... \ (2)$$

Subsituting (1) into (2) gives: $3c + (-c) = -2$
$$\therefore \quad 2c = -2$$
$$\therefore \quad c = -1$$
Subsituting $c = -1$ into (1) gives $d = -(-1)$
$$= 1$$
$$\therefore \quad f(x) = 3e^{-x} - x + 1$$

EXERCISE 22D

1 **a** $\displaystyle\int (2x+5)^3\,dx$

$= \frac{1}{2} \times \dfrac{(2x+5)^4}{4} + c$

$= \frac{1}{8}(2x+5)^4 + c$

b $\displaystyle\int (4x-3)^7\,dx$

$= \frac{1}{4} \times \dfrac{(4x-3)^8}{8} + c$

$= \frac{1}{32}(4x-3)^8 + c$

c $\displaystyle\int 3(1-x)^4\,dx$

$= 3\displaystyle\int (1-x)^4\,dx$

$= 3 \times \frac{1}{-1} \times \dfrac{(1-x)^5}{5} + c$

$= -\frac{3}{5}(1-x)^5 + c$

d $\displaystyle\int \frac{1}{(3-2x)^2}\,dx$

$= \displaystyle\int (3-2x)^{-2}\,dx$

$= \frac{1}{-2} \times \dfrac{(3-2x)^{-1}}{-1} + c$

$= \dfrac{1}{2(3-2x)} + c$

e $\displaystyle\int \frac{4}{(2x-1)^4}\,dx$

$= 4\displaystyle\int (2x-1)^{-4}\,dx$

$= 4 \times \frac{1}{2} \times \dfrac{(2x-1)^{-3}}{-3} + c$

$= -\dfrac{2}{3(2x-1)^3} + c$

f $\displaystyle\int \sqrt{3x-4}\,dx$

$= \displaystyle\int (3x-4)^{\frac{1}{2}}\,dx$

$= \frac{1}{3} \times \dfrac{(3x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + c$

g $\displaystyle\int \frac{10}{\sqrt{1-5x}}\,dx$

$= 10\displaystyle\int (1-5x)^{-\frac{1}{2}}\,dx$

$= 10 \times \frac{1}{-5} \times \dfrac{(1-5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -4\sqrt{1-5x} + c$

h $\displaystyle\int \frac{4}{\sqrt{3-4x}}\,dx$

$= 4\displaystyle\int (3-4x)^{-\frac{1}{2}}\,dx$

$= 4 \times \frac{1}{-4} \times \dfrac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -2\sqrt{3-4x} + c$

i $\displaystyle\int \frac{5}{(3x-2)^3}\,dx$

$= 5\displaystyle\int (3x-2)^{-3}\,dx$

$= 5 \times \frac{1}{3} \times \dfrac{(3x-2)^{-2}}{-2} + c$

$= -\dfrac{5}{6(3x-2)^2} + c$

2 $\dfrac{dy}{dx} = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$

$$\therefore \; y = \int (2x-7)^{\frac{1}{2}}\, dx$$

$$= \tfrac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore \; y = f(x) = \tfrac{1}{3}(2x-7)^{\frac{3}{2}} + c$$

But $f(8) = 11$, so $\tfrac{1}{3}(2(8)-7)^{\frac{3}{2}} + c = 11$

$$\therefore \; \tfrac{1}{3}(9)^{\frac{3}{2}} + c = 11$$

$$\therefore \; \tfrac{1}{3}(27) + c = 11$$

$$\therefore \; 9 + c = 11$$

$$\therefore \; c = 2$$

$$\therefore \; y = \tfrac{1}{3}(2x-7)^{\frac{3}{2}} + 2$$

3 $f'(x) = \dfrac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$

$$\therefore \; f(x) = \int 4(1-x)^{-\frac{1}{2}}\, dx$$

$$= 4\int (1-x)^{-\frac{1}{2}}\, dx$$

$$= 4 \times \tfrac{1}{-1} \times \frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -8\sqrt{1-x} + c$$

But $y = f(x)$ passes through $(-3, -11)$, so $-8\sqrt{1-(-3)} + c = -11$

$$\therefore \; -8\sqrt{4} + c = -11$$

$$\therefore \; -16 + c = -11$$

$$\therefore \; c = 5$$

$$\therefore \; f(x) = 5 - 8\sqrt{1-x}$$

Now $f(-8) = 5 - 8\sqrt{1-(-8)}$

$$= 5 - 8(3)$$

$$= -19$$

So, the point on the graph of $y = f(x)$ with x-coordinate -8 is $(-8, -19)$.

4 **a** $\displaystyle\int 3(2x-1)^2\, dx$

$$= 3\int (2x-1)^2\, dx$$

$$= 3 \times \tfrac{1}{2} \times \frac{(2x-1)^3}{3} + c$$

$$= \tfrac{1}{2}(2x-1)^3 + c$$

b $\displaystyle\int (4x-5)^2\, dx$

$$= \tfrac{1}{4} \times \frac{(4x-5)^3}{3} + c$$

$$= \tfrac{1}{12}(4x-5)^3 + c$$

c $\int (1-3x)^3\, dx$
$= \frac{1}{-3} \times \frac{(1-3x)^4}{4} + c$
$= -\frac{1}{12}(1-3x)^4 + c$

d $\int (2-5x)^2\, dx$
$= \frac{1}{-5} \times \frac{(2-5x)^3}{3} + c$
$= -\frac{1}{15}(2-5x)^3 + c$

e $\int 4\sqrt{5-x}\, dx$
$= 4\int (5-x)^{\frac{1}{2}}\, dx$
$= 4 \times \frac{1}{-1} \times \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c$

f $\int (7x+1)^4\, dx$
$= \frac{1}{7} \times \frac{(7x+1)^5}{5} + c$
$= \frac{1}{35}(7x+1)^5 + c$

5 $\frac{dy}{dx} = x - \frac{5}{(1-x)^2}$
$= x - 5(1-x)^{-2}$
$\therefore\ y = \int (x - 5(1-x)^{-2})\, dx$
$= \frac{1}{2}x^2 - 5 \times \frac{1}{-1} \times \frac{(1-x)^{-1}}{-1} + c$
$= \frac{1}{2}x^2 - \frac{5}{1-x} + c$

But when $x = 2,\ y = 0$
$\therefore\ 0 = \frac{1}{2}(2)^2 - \frac{5}{1-2} + c$
$\therefore\ 0 = \frac{1}{2} \times 4 - \frac{5}{-1} + c$
$\therefore\ 0 = 2 + 5 + c$
$\therefore\ c = -7$

$\therefore\ y = \frac{1}{2}x^2 - \frac{5}{1-x} - 7$

6 **a** $\int \sin 3x\, dx$
$= -\frac{1}{3}\cos 3x + c$

b $\int (2\cos(-4x) + 1)\, dx$
$= 2 \times (\frac{1}{-4})\sin(-4x) + x + c$
$= -\frac{1}{2}\sin(-4x) + x + c$

c $\int \frac{1}{\cos^2 2x}\, dx$
$= \frac{1}{2}\tan 2x + c$

d $\int 3\cos\frac{x}{2}\, dx$
$= 6\sin\frac{x}{2} + c$

e $\int (3\sin 2x - e^{-x})\, dx$

$= -\frac{3}{2}\cos 2x + e^{-x} + c$

f $\int 2\sin\left(2x + \frac{\pi}{6}\right) dx$

$= 2 \times \left(\frac{1}{2}\right)\left(-\cos\left(2x + \frac{\pi}{6}\right)\right) + c$

$= -\cos\left(2x + \frac{\pi}{6}\right) + c$

g $\int -3\cos\left(\frac{\pi}{4} - x\right) dx$

$= -3 \times \left(\frac{1}{-1}\right)\sin\left(\frac{\pi}{4} - x\right) + c$

$= 3\sin\left(\frac{\pi}{4} - x\right) + c$

h $\int \left(e^{2x} - \frac{2}{\cos^2\left(\frac{x}{2}\right)}\right) dx$

$= \frac{1}{2}e^{2x} - 2\left(\frac{1}{\frac{1}{2}}\right)\tan\frac{x}{2} + c$

$= \frac{1}{2}e^{2x} - 4\tan\frac{x}{2} + c$

i $\int \frac{4}{\cos^2\left(\frac{\pi}{3} - 2x\right)}\, dx$

$= 4 \times \left(\frac{1}{-2}\right)\tan\left(\frac{\pi}{3} - 2x\right) + c$

$= -2\tan\left(\frac{\pi}{3} - 2x\right) + c$

j $\int (\cos 2x + \sin 2x)\, dx$

$= \frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x + c$

k $\int (2\sin 3x + 5\cos 4x)\, dx$

$= -\frac{2}{3}\cos 3x + \frac{5}{4}\sin 4x + c$

l $\int \left(\frac{1}{2}\cos 8x - 3\sin x\right) dx$

$= \frac{1}{2} \times \frac{1}{8}\sin 8x + 3\cos x + c$

$= \frac{1}{16}\sin 8x + 3\cos x + c$

7 **a** $\int (2e^{x} + 5e^{2x})\, dx$

$= 2e^{x} + 5\left(\frac{1}{2}\right)e^{2x} + c$

$= 2e^{x} + \frac{5}{2}e^{2x} + c$

b $\int (3e^{5x-2})\, dx$

$= 3\left(\frac{1}{5}\right)e^{5x-2} + c$

$= \frac{3}{5}e^{5x-2} + c$

c $\int (e^{7-3x})\, dx$

$= \left(\frac{1}{-3}\right)e^{7-3x} + c$

$= -\frac{1}{3}e^{7-3x} + c$

d $\int (e^{x} + e^{-x})^2\, dx$

$= \int (e^{2x} + 2 + e^{-2x})\, dx$

$= \frac{1}{2}e^{2x} + 2x + \left(\frac{1}{-2}\right)e^{-2x} + c$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

e $\int (e^{-x} + 2)^2\, dx$

$= \int (e^{-2x} + 4e^{-x} + 4)\, dx$

$= \left(\frac{1}{-2}\right)e^{-2x} + 4\left(\frac{1}{-1}\right)e^{-x} + 4x + c$

$= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

f $\int \frac{(e^{2x} - 5)^2}{e^{x}}\, dx$

$= \int \frac{e^{4x} - 10e^{2x} + 25}{e^{x}}\, dx$

$= \int (e^{3x} - 10e^{x} + 25e^{-x})\, dx$

$= \frac{1}{3}e^{3x} - 10e^{x} - 25e^{-x} + c$

8 $\dfrac{dy}{dx} = (1 - e^x)^2 = 1 - 2e^x + e^{2x}$

$\therefore \; y = \displaystyle\int (1 - 2e^x + e^{2x})\, dx$

$= x - 2e^x + \frac{1}{2}e^{2x} + c$

When $x = 0, \; y = 4$

$\therefore \; 0 - 2 + \frac{1}{2} + c = 4$

$\therefore \; c = \frac{11}{2}$

$\therefore \; y = x - 2e^x + \frac{1}{2}e^{2x} + \frac{11}{2}$

9 $f'(x) = 2e^{-2x}$

$\therefore \; f(x) = 2(\frac{1}{-2})e^{-2x} + c$

$= -e^{-2x} + c$

But $f(0) = 3$, so $-e^0 + c = 3$

$\therefore \; -1 + c = 3$

$\therefore \; c = 4$

$\therefore \; f(x) = -e^{-2x} + 4$

10 $f'(x) = p \sin \frac{x}{2}$

$\therefore \; f(x) = \displaystyle\int p \sin \tfrac{x}{2}\, dx$

$= p \displaystyle\int \sin \tfrac{x}{2}\, dx$

$= p \times 2\left(-\cos \frac{x}{2}\right) + c$

$= -2p \cos \frac{x}{2} + c$

But $f(0) = 1$, so $-2p + c = 1$ .... (1)

and $f(2\pi) = 0$, so $2p + c = 0$ .... (2)

(2) − (1) gives: $2p + c - (-2p + c) = 0 - 1$

$\therefore \; 4p = -1$

$\therefore \; p = -\frac{1}{4}$ and $\therefore \; c = -2p$ {using (2)}

$= -2(-\frac{1}{4})$

$= \frac{1}{2}$

$\therefore \; f(x) = \frac{1}{2} \cos \frac{x}{2} + \frac{1}{2}$

11 $g''(x) = -\sin 2x$

$\therefore \; g'(x) = \displaystyle\int -\sin 2x\, dx$

$= \frac{1}{2} \cos 2x + c$

Now $g'(\pi) = \frac{1}{2} \cos 2\pi + c$

$= \frac{1}{2} + c$

and $g'(-\pi) = \frac{1}{2} \cos(-2\pi) + c$

$= \frac{1}{2} + c$

$= g'(\pi)$

$\therefore$ the gradients of the tangents to $y = g(x)$ at $x = \pi$ and $x = -\pi$ are equal.

12 $\frac{dy}{dx} = \sqrt{x} + \frac{1}{2}e^{-4x} = x^{\frac{1}{2}} + \frac{1}{2}e^{-4x}$

$$\therefore \ y = \int (x^{\frac{1}{2}} + \frac{1}{2}e^{-4x})\,dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}(\frac{1}{-4})e^{-4x} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + c$$

But $y = 0$ when $x = 1$ so $\frac{2}{3} - \frac{1}{8}e^{-4} + c = 0$

$$\therefore \ c = \frac{1}{8}e^{-4} - \frac{2}{3}$$

$\therefore \ y = \frac{2}{3}x\sqrt{x} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

13 **a** $\int \frac{6}{x+4}\,dx = 6\ln|x+4| + c$

b $\int \frac{1}{2x-1}\,dx = \frac{1}{2}\ln|2x-1| + c$

c $\int \frac{5}{1-3x}\,dx$

$= 5\int \frac{1}{1-3x}\,dx$

$= 5(\frac{1}{-3})\ln|1-3x| + c$

$= -\frac{5}{3}\ln|1-3x| + c$

d $\int \left(4 + \frac{1}{5x-2}\right)dx$

$= 4x + \frac{1}{5}\ln|5x-2| + c$

e $\int \left(1 - 2x + \frac{4}{x-3}\right)dx$

$= x - x^2 + 4\ln|x-3| + c$

f $\int \left(\sin 2x - \frac{3}{1-2x}\right)dx$

$= -\frac{1}{2}\cos 2x - 3(\frac{1}{-2})\ln|1-2x| + c$

$= -\frac{1}{2}\cos 2x + \frac{3}{2}\ln|1-2x| + c$

14 Differentiating Tracy's answer: $\frac{d}{dx}\left(\frac{1}{4}\ln 4x + c\right) = \frac{1}{4}\left(\frac{4}{4x}\right) + 0, \ x > 0$

$$= \frac{1}{4x}, \ x > 0$$

Differentiating Nadine's answer: $\frac{d}{dx}\left(\frac{1}{4}\ln x + c\right) = \frac{1}{4}\left(\frac{1}{x}\right) + 0, \ x > 0$

$$= \frac{1}{4x}, \ x > 0$$

Both answers give the correct derivative and both are correct. This result occurs because $\ln 4x = \ln 4 + \ln x$. So, their answers differ by a constant which is accounted for by c.

15
$$\frac{3x-1}{x+2} = \frac{3(x+2)-6-1}{x+2}$$
$$= \frac{3(x+2)-7}{x+2}$$
$$= 3 - \frac{7}{x+2}$$
$$\therefore \int \frac{3x-1}{x+2}\,dx = \int \left(3 - \frac{7}{x+2}\right) dx$$
$$= 3x - 7\ln|x+2| + c$$

16
$$f'(x) = 2x - \frac{2}{1-x}$$
$$\therefore f(x) = \int \left(2x - \frac{2}{1-x}\right) dx$$
$$= x^2 - 2(\tfrac{1}{-1})\ln|1-x| + c$$
$$= x^2 + 2\ln|1-x| + c$$
But $f(-1) = 3$
$$\therefore (-1)^2 + 2\ln|1-(-1)| + c = 3$$
$$\therefore 1 + 2\ln 2 + c = 3$$
$$\therefore c = 2 - 2\ln 2$$
$$\therefore f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$$

EXERCISE 22E

1
$$\frac{d}{dx}\left((2x^2-5x)^3\right) = 3(4x-5)(2x^2-5x)^2$$
$$\therefore \int 3(4x-5)(2x^2-5x)^2\,dx = (2x^2-5x)^3 + c$$
$$\therefore 3\int (4x-5)(2x^2-5x)^2\,dx = (2x^2-5x)^3 + c$$
$$\therefore \int (4x-5)(2x^2-5x)^2\,dx = \tfrac{1}{3}(2x^2-5x)^3 + c$$

2
$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \times 2x$$
$$= 2x\cos(x^2)$$
$$\therefore \int 2x\cos(x^2)\,dx = \sin(x^2) + c$$
$$\therefore 2\int x\cos(x^2)\,dx = \sin(x^2) + c$$
$$\therefore \int x\cos(x^2)\,dx = \tfrac{1}{2}\sin(x^2) + c$$

3
$$\frac{d}{dx}\left(\ln(5-3x+x^2)\right) = \frac{-3+2x}{5-3x+x^2}$$
$$\therefore \int \frac{-3+2x}{5-3x+x^2}\,dx = \ln\left|5-3x+x^2\right| + c$$
$$\therefore 2\int \frac{-3+2x}{5-3x+x^2}\,dx = 2\ln\left|5-3x+x^2\right| + c$$
$$\therefore \int \frac{4x-6}{5-3x+x^2}\,dx = 2\ln\left|5-3x+x^2\right| + c$$

4 **a** $u = x^3 + 1, \quad \dfrac{du}{dx} = 3x^2$

$$\therefore \int 3x^2(x^3+1)^4\,dx = \int u^4 \frac{du}{dx}\,dx$$
$$= \int u^4\,du$$
$$= \tfrac{1}{5}u^5 + c$$
$$= \tfrac{1}{5}(x^3+1)^5 + c$$

b $u = x^3 + 1, \quad \dfrac{du}{dx} = 3x^2$

$$\therefore \int x^2 e^{x^3+1}\,dx = \tfrac{1}{3}\int (3x^2)e^{x^3+1}\,dx$$
$$= \tfrac{1}{3}\int e^u \frac{du}{dx}\,dx$$
$$= \tfrac{1}{3}\int e^u\,du$$
$$= \tfrac{1}{3}e^u + c$$
$$= \tfrac{1}{3}e^{x^3+1} + c$$

c $u = \sin x, \quad \dfrac{du}{dx} = \cos x$

$$\therefore \int \sin^4 x\cos x\,dx = \int u^4 \frac{du}{dx}\,dx$$
$$= \int u^4\,du$$
$$= \frac{u^5}{5} + c$$
$$= \tfrac{1}{5}\sin^5 x + c$$

d $u = x^2 - 3, \quad \dfrac{du}{dx} = 2x$

$$\therefore \int 2x\cos(x^2-3)\,dx$$
$$= \int \cos u \frac{du}{dx}\,dx$$
$$= \int \cos u\,du$$
$$= \sin u + c$$
$$= \sin(x^2-3) + c$$

5 **a** $$\int 4x^3(2+x^4)^3\,dx = \int u^3 \frac{du}{dx}\,dx \qquad \left\{u = 2 + x^4, \quad \frac{du}{dx} = 4x^3\right\}$$
$$= \int u^3\,du$$
$$= \frac{u^4}{4} + c$$
$$= \tfrac{1}{4}(2+x^4)^4 + c$$

b $$\int \frac{2x}{\sqrt{x^2+3}}\,dx$$
$$= \int 2x(x^2+3)^{-\frac{1}{2}}\,dx$$
$$= \int u^{-\frac{1}{2}} \frac{du}{dx}\,dx$$
$$\left\{u = x^2 + 3, \quad \frac{du}{dx} = 2x\right\}$$
$$= \int u^{-\frac{1}{2}}\,du$$
$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 2\sqrt{u} + c$$
$$= 2\sqrt{x^2+3} + c$$

c $$\int \frac{6x^2}{(2x^3-1)^4}\,dx$$
$$= \int 6x^2(2x^3-1)^{-4}\,dx$$
$$= \int u^{-4}\frac{du}{dx}\,dx$$
$$\left\{u = 2x^3 - 1, \quad \frac{du}{dx} = 6x^2\right\}$$
$$= \int u^{-4}\,du$$
$$= \frac{u^{-3}}{-3} + c$$
$$= -\frac{1}{3u^3} + c$$
$$= -\frac{1}{3(2x^3-1)^3} + c$$

d
$$\int (x^3+2x+1)^4(3x^2+2)\,dx$$
$$= \int u^4 \frac{du}{dx}\,dx \quad \{u = x^3+2x+1, \quad \frac{du}{dx} = 3x^2+2\}$$
$$= \int u^4\,du$$
$$= \frac{u^5}{5} + c$$
$$= \tfrac{1}{5}(x^3+2x+1)^5 + c$$

e
$$\int \frac{x}{(1-x^2)^5}\,dx$$
$$= -\tfrac{1}{2}\int (1-x^2)^{-5} \times (-2x)\,dx$$
$$= -\tfrac{1}{2}\int u^{-5} \frac{du}{dx}\,dx \quad \{u = 1-x^2, \quad \frac{du}{dx} = -2x\}$$
$$= -\tfrac{1}{2}\int u^{-5}\,du$$
$$= -\tfrac{1}{2}\,\frac{u^{-4}}{-4} + c$$
$$= \frac{1}{8(1-x^2)^4} + c$$

f
$$\int \frac{x+2}{(x^2+4x-3)^2}\,dx = \tfrac{1}{2}\int (x^2+4x-3)^{-2}(2x+4)\,dx$$
$$= \tfrac{1}{2}\int u^{-2}\frac{du}{dx}\,dx \qquad \{u = x^2+4x-3, \quad \frac{du}{dx} = 2x+4\}$$
$$= \tfrac{1}{2}\int u^{-2}\,du$$
$$= \tfrac{1}{2}\,\frac{u^{-1}}{-1} + c$$
$$= -\frac{1}{2(x^2+4x-3)} + c$$

6 **a**
$$\int -2e^{1-2x}\,dx$$
$$= \int e^u \frac{du}{dx}\,dx \quad \{u = 1-2x, \quad \frac{du}{dx} = -2\}$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{1-2x} + c$$

b
$$\int 2xe^{x^2}\,dx$$
$$= \int e^u \frac{du}{dx}\,dx \quad \{u = x^2, \quad \frac{du}{dx} = 2x\}$$
$$= \int e^u\,du$$
$$= e^u + c$$
$$= e^{x^2} + c$$

c
$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$$
$$= 2\int e^u \frac{du}{dx}\,dx \qquad \{u = \sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}\}$$
$$= 2\int e^u\,du$$
$$= 2e^u + c$$
$$= 2e^{\sqrt{x}} + c$$

7 **a**
$$\int \frac{2x}{x^2+1}\,dx$$
$$= \int \frac{1}{x^2+1}\,(2x)\,dx$$
$$= \int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\{u = x^2+1, \quad \frac{du}{dx} = 2x\}$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|\,u\,| + c$$
$$= \ln\left|\,x^2+1\,\right| + c$$

b
$$\int \frac{x}{2-x^2}\,dx$$
$$= -\tfrac{1}{2}\int \frac{1}{2-x^2}\,(-2x)\,dx$$
$$= -\tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\{u = 2-x^2, \quad \frac{du}{dx} = -2x\}$$
$$= -\tfrac{1}{2}\int \frac{1}{u}\,du$$
$$= -\tfrac{1}{2}\ln|\,u\,| + c$$
$$= -\tfrac{1}{2}\ln\left|\,2-x^2\,\right| + c$$

c
$$\int \frac{2x-3}{x^2-3x}\,dx$$
$$= \int \frac{1}{x^2-3x}\,(2x-3)\,dx$$
$$= \int \frac{1}{u}\,\frac{du}{dx}\,dx \qquad \{u = x^2-3x, \quad \frac{du}{dx} = 2x-3\}$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|\,u\,| + c$$
$$= \ln\left|\,x^2-3x\,\right| + c$$

8 **a**
$$\int x^2(3-x^3)^2\,dx$$
$$= -\tfrac{1}{3}\int (-3x^2)(3-x^3)^2\,dx$$
$$= -\tfrac{1}{3}\int u^2\,\frac{du}{dx}\,dx$$
$$\{u = 3-x^3, \quad \frac{du}{dx} = -3x^2\}$$
$$= -\tfrac{1}{3}\int u^2\,du$$
$$= -\tfrac{1}{3}\times\frac{u^3}{3} + c$$
$$= -\tfrac{1}{9}(3-x^3)^3 + c$$

b
$$\int x\sqrt{1-x^2}\,dx$$
$$= -\tfrac{1}{2}\int (-2x)(1-x^2)^{\frac{1}{2}}\,dx$$
$$= -\tfrac{1}{2}\int u^{\frac{1}{2}}\,\frac{du}{dx}\,dx$$
$$\{u = 1-x^2, \quad \frac{du}{dx} = -2x\}$$
$$= -\tfrac{1}{2}\int u^{\frac{1}{2}}\,du$$
$$= -\tfrac{1}{2}\times\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= -\tfrac{1}{3}u^{\frac{3}{2}} + c$$
$$= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

c
$$\int xe^{1-x^2}\,dx$$
$$= -\tfrac{1}{2}\int (-2x)e^{1-x^2}\,dx$$
$$= -\tfrac{1}{2}\int e^u\,\frac{du}{dx}\,dx$$
$$\{u = 1 - x^2, \quad \frac{du}{dx} = -2x\}$$
$$= -\tfrac{1}{2}\int e^u\,du$$
$$= -\tfrac{1}{2}e^u + c$$
$$= -\tfrac{1}{2}e^{1-x^2} + c$$

d
$$\int \frac{(\ln x)^3}{x}\,dx$$
$$= \int u^3\,\frac{du}{dx}\,dx$$
$$\{u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}\}$$
$$= \int u^3\,du$$
$$= \frac{u^4}{4} + c$$
$$= \tfrac{1}{4}(\ln x)^4 + c$$

e
$$\int (2x-1)e^{x-x^2}\,dx$$
$$= -\int (1-2x)e^{x-x^2}\,dx$$
$$= -\int e^u\,\frac{du}{dx}\,dx$$
$$\{u = x - x^2, \quad \frac{du}{dx} = 1 - 2x\}$$
$$= -\int e^u\,du$$
$$= -e^u + c$$
$$= -e^{x-x^2} + c$$

f
$$\int \frac{1-x^2}{x^3-3x}\,dx$$
$$= -\tfrac{1}{3}\int \frac{1}{x^3-3x}\,(3x^2-3)\,dx$$
$$= -\tfrac{1}{3}\int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\{u = x^3 - 3x, \quad \frac{du}{dx} = 3x^2 - 3\}$$
$$= -\tfrac{1}{3}\int \frac{1}{u}\,du$$
$$= -\tfrac{1}{3}\ln|\,u\,| + c$$
$$= -\tfrac{1}{3}\ln\left|\,x^3 - 3x\,\right| + c$$

9 **a**
$$\int \sin^7 x\cos x\,dx$$
$$= \int u^7\,\frac{du}{dx}\,dx$$
$$\{u = \sin x, \quad \frac{du}{dx} = \cos x\}$$
$$= \int u^7\,du$$
$$= \frac{u^8}{8} + c$$
$$= \tfrac{1}{8}\sin^8 x + c$$

b
$$\int \cos^5 x\sin x\,dx$$
$$= -\int \cos^5 x(-\sin x)\,dx$$
$$= -\int u^5\,\frac{du}{dx}\,dx$$
$$\{u = \cos x, \quad \frac{du}{dx} = -\sin x\}$$
$$= -\int u^5\,du$$
$$= -\tfrac{1}{6}u^6 + c$$
$$= -\tfrac{1}{6}\cos^6 x + c$$

c

$$\int \frac{\sin x}{\sqrt{\cos x}}\,dx$$
$$= -\int \frac{-\sin x}{\sqrt{\cos x}}\,dx$$
$$= -\int u^{-\frac{1}{2}}\,\frac{du}{dx}\,dx$$
$$\left\{u = \cos x,\quad \frac{du}{dx} = -\sin x\right\}$$
$$= -\int u^{-\frac{1}{2}}\,du$$
$$= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= -2(\cos x)^{\frac{1}{2}} + c$$
$$= -2\sqrt{\cos x} + c$$

d

$$\int \tan x\,dx$$
$$= \int \frac{\sin x}{\cos x}\,dx$$
$$= -\int \frac{-\sin x}{\cos x}\,dx$$
$$= -\int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\left\{u = \cos x,\quad \frac{du}{dx} = -\sin x\right\}$$
$$= -\int \frac{1}{u}\,du$$
$$= -\ln|\,u\,| + c$$
$$= -\ln|\,\cos x\,| + c$$

e

$$\int \sqrt{\sin x}\cos x\,dx$$
$$= \int u^{\frac{1}{2}}\,\frac{du}{dx}\,dx$$
$$\left\{u = \sin x,\quad \frac{du}{dx} = \cos x\right\}$$
$$= \int u^{\frac{1}{2}}\,du$$
$$= \tfrac{2}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{2}{3}(\sin x)^{\frac{3}{2}} + c$$

f

$$\int \frac{\cos x}{(2+\sin x)^2}\,dx$$
$$= \int u^{-2}\,\frac{du}{dx}\,dx$$
$$\left\{u = 2+\sin x,\quad \frac{du}{dx} = \cos x\right\}$$
$$= \int u^{-2}\,du$$
$$= -u^{-1} + c$$
$$= -(2+\sin x)^{-1} + c$$
$$= -\frac{1}{2+\sin x} + c$$

g

$$\int \frac{\sin x}{1-\cos x}\,dx$$
$$= \int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\left\{u = 1-\cos x,\quad \frac{du}{dx} = \sin x\right\}$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|\,u\,| + c$$
$$= \ln|\,1-\cos x\,| + c$$

h

$$\int \frac{\cos 2x}{\sin 2x - 3}\,dx$$
$$= \tfrac{1}{2}\int \frac{2\cos 2x}{\sin 2x - 3}\,dx$$
$$= \tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\left\{u = \sin 2x - 3,\quad \frac{du}{dx} = 2\cos 2x\right\}$$
$$= \tfrac{1}{2}\int \frac{1}{u}\,du$$
$$= \tfrac{1}{2}\ln|\,u\,| + c$$
$$= \tfrac{1}{2}\ln|\,\sin 2x - 3\,| + c$$

i
$$\begin{aligned}\int x\sin(x^2)\,dx &= \tfrac{1}{2}\int (2x)\sin(x^2)\,dx\\ &= \tfrac{1}{2}\int \sin u\,\frac{du}{dx}\,dx \qquad \{u = x^2,\ \frac{du}{dx} = 2x\}\\ &= \tfrac{1}{2}\int \sin u\,du\\ &= \tfrac{1}{2}(-\cos u) + c\\ &= -\tfrac{1}{2}\cos(x^2) + c\end{aligned}$$

10 **a**
$$f'(x) = \frac{e^{\tan x}}{\cos^2 x}$$
$$\begin{aligned}\therefore\ f(x) &= \int \frac{e^{\tan x}}{\cos^2 x}\,dx\\ &= \int e^{\tan x}\frac{1}{\cos^2 x}\,dx\\ &= \int e^u\,\frac{du}{dx}\,dx\\ &\qquad \{u = \tan x,\ \frac{du}{dx} = \frac{1}{\cos^2 x}\}\\ &= \int e^u\,du\\ &= e^u + c\\ &= e^{\tan x} + c\end{aligned}$$

b
$$\begin{aligned}&\int \cos^3 x\,dx\\ &= \int \cos^2 x\cos x\,dx\\ &= \int (1-\sin^2 x)\cos x\,dx\\ &= \int (1-u^2)\,\frac{du}{dx}\,dx\\ &\qquad \{u = \sin x,\ \frac{du}{dx} = \cos x\}\\ &= \int (1-u^2)\,du\\ &= u - \frac{u^3}{3} + c\\ &= \sin x - \frac{\sin^3 x}{3} + c\\ &= \sin x - \tfrac{1}{3}\sin^3 x + c\end{aligned}$$

c
$$\begin{aligned}&\int \sin^3 2x\cos 2x\,dx\\ &= \tfrac{1}{2}\int \sin^3 2x(2\cos 2x)\,dx\\ &= \tfrac{1}{2}\int u^3\,\frac{du}{dx}\,dx\\ &\qquad \{u = \sin 2x,\ \frac{du}{dx} = 2\cos 2x\}\\ &= \tfrac{1}{2}\int u^3\,du\\ &= \tfrac{1}{2}\times\frac{u^4}{4} + c\\ &= \tfrac{1}{8}\sin^4 2x + c\end{aligned}$$

REVIEW SET 22A

1
$$\frac{d}{dx}(x^4 - x^2) = 4x^3 - 2x$$
$$\therefore \int (4x^3 - 2x)\,dx = x^4 - x^2 + c$$
$$\therefore 2\int (2x^3 - x)\,dx = x^4 - x^2 + c$$
$$\therefore \int (2x^3 - x)\,dx = \tfrac{1}{2}x^4 - \tfrac{1}{2}x^2 + c$$

2
$$\frac{d}{dx}\left(\sin\left(\tfrac{\pi}{3} - 2x\right)\right) = \cos\left(\tfrac{\pi}{3} - 2x\right) \times (-2)$$
$$= -2\cos\left(\tfrac{\pi}{3} - 2x\right)$$
$$\therefore \int -2\cos\left(\tfrac{\pi}{3} - 2x\right)dx = \sin\left(\tfrac{\pi}{3} - 2x\right) + c$$
$$\therefore -2\int \cos\left(\tfrac{\pi}{3} - 2x\right)dx = \sin\left(\tfrac{\pi}{3} - 2x\right) + c$$
$$\therefore \int \cos\left(\tfrac{\pi}{3} - 2x\right)dx = -\tfrac{1}{2}\sin\left(\tfrac{\pi}{3} - 2x\right) + c$$

3 **a**
$$\int \left(\sqrt{x} - \frac{2}{x^2}\right)dx$$
$$= \int (x^{\frac{1}{2}} - 2x^{-2})\,dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2 \times \frac{x^{-1}}{-1} + c$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} + 2x^{-1} + c$$
$$= \tfrac{2}{3}x\sqrt{x} + \frac{2}{x} + c$$

b
$$\int \left(2x - \frac{3}{\sqrt[3]{x}}\right)dx$$
$$= \int (2x - 3x^{-\frac{1}{3}})\,dx$$
$$= x^2 - \frac{3x^{\frac{2}{3}}}{\frac{2}{3}} + c$$
$$= x^2 - \tfrac{9}{2}x^{\frac{2}{3}} + c$$

c
$$\int \frac{6x + 5}{\sqrt{x}}\,dx$$
$$= \int (6x^{\frac{1}{2}} + 5x^{-\frac{1}{2}})\,dx$$
$$= 6 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 4x^{\frac{3}{2}} + 10x^{\frac{1}{2}} + c$$
$$= 4x\sqrt{x} + 10\sqrt{x} + c$$

4 **a**
$$\int \frac{4}{\sqrt{x}}\,dx$$
$$= 4\int x^{-\frac{1}{2}}\,dx$$
$$= 4 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 8\sqrt{x} + c$$

b
$$\int \left(\tfrac{1}{3}x^3 + 2x\right)dx$$
$$= \tfrac{1}{3} \times \frac{x^4}{4} + x^2 + c$$
$$= \tfrac{1}{12}x^4 + x^2 + c$$

c
$$\int \frac{1 - 2x}{x^3}\,dx$$
$$= \int \left(\frac{1}{x^3} - \frac{2}{x^2}\right)dx$$
$$= \int (x^{-3} - 2x^{-2})\,dx$$
$$= \frac{x^{-2}}{-2} - \frac{2x^{-1}}{-1} + c$$
$$= -\tfrac{1}{2}x^{-2} + 2x^{-1} + c$$
$$= -\frac{1}{2x^2} + \frac{2}{x} + c$$

5 **a** $\int(-3x^4+6x^2)\,dx$
$= -\frac{3x^5}{5}+\frac{6x^3}{3}+c$
$= -\frac{3}{5}x^5+2x^3+c$

b $\int \frac{3x^3-x^2-1}{x^2}\,dx$
$= \int\left(3x-1-\frac{1}{x^2}\right)dx$
$= \int(3x-1-x^{-2})\,dx$
$= \frac{3x^2}{2}-x-\frac{x^{-1}}{-1}+c$
$= \frac{3}{2}x^2-x+\frac{1}{x}+c$

c $\int(2x-\sqrt{x})^2\,dx$
$= \int(4x^2-4x\sqrt{x}+x)\,dx$
$= \int(4x^2-4x^{\frac{3}{2}}+x)\,dx$
$= \frac{4x^3}{3}-\frac{4x^{\frac{5}{2}}}{\frac{5}{2}}+\frac{x^2}{2}+c$
$= \frac{4}{3}x^3-\frac{8}{5}x^{\frac{5}{2}}+\frac{1}{2}x^2+c$

d $\int\left(4e^x-\frac{3}{x}\right)dx$
$= 4e^x-3\ln|\,x\,|+c$

e $\int \sin(4x-5)\,dx$
$= -\frac{1}{4}\cos(4x-5)+c$

f $\int e^{4-3x}\,dx$
$= (\frac{1}{-3})e^{4-3x}+c$
$= -\frac{1}{3}e^{4-3x}+c$

6 **a** $\frac{dy}{dx}=3e^{-x}-2\sin\left(\frac{\pi}{2}-x\right)$
$\therefore\ y=\int\left(3e^{-x}-2\sin\left(\frac{\pi}{2}-x\right)\right)dx$
$= 3\left(\frac{1}{-1}\right)e^{-x}-2\left(\frac{1}{-1}\right)\left(-\cos\left(\frac{\pi}{2}-x\right)\right)+c$
$= -3e^{-x}-2\cos\left(\frac{\pi}{2}-x\right)+c$

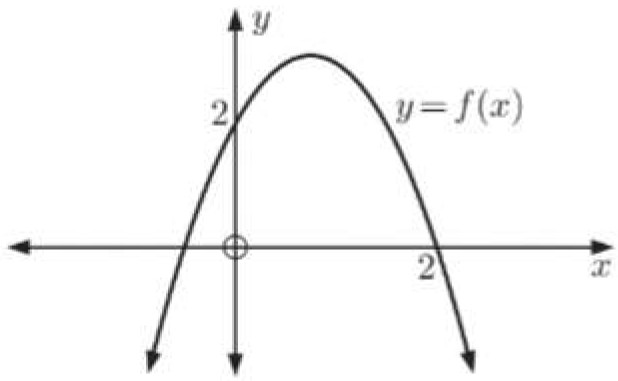
b $\frac{dy}{dx}=\cos 4x-\frac{1}{2}x^2$
$\therefore\ y=\int\left(\cos 4x-\frac{1}{2}x^2\right)dx$
$= \frac{1}{4}\sin 4x-\frac{1}{2}\times\frac{x^3}{3}+c$
$= \frac{1}{4}\sin 4x-\frac{1}{6}x^3+c$

7 $f'(x)=3x^2-4x+1$
$\therefore\ f(x)=\int(3x^2-4x+1)\,dx$
$= \frac{3x^3}{3}-\frac{4x^2}{2}+x+c$
$= x^3-2x^2+x+c$
But $f(0)=2$, so $c=2$
$\therefore\ f(x)=x^3-2x^2+x+2$

8 $f'(x) = ax + 3$

$\therefore\ f(x) = \int (ax+3)\,dx$

$= \dfrac{ax^2}{2} + 3x + c$



But the y-intercept is 2, so $f(0) = 2$

$\therefore\ c = 2$

$\therefore\ f(x) = \dfrac{ax^2}{2} + 3x + 2$

and the x-intercept is 2, so $f(2) = 0$

$\therefore\ \dfrac{a(2)^2}{2} + 3(2) + 2 = 0$

$\therefore\ 2a + 6 + 2 = 0$

$\therefore\ 2a = -8$

$\therefore\ a = -4$

$\therefore$ the equation of the curve is $y = f(x) = \dfrac{(-4)x^2}{2} + 3x + 2$

$= -2x^2 + 3x + 2$

9 $f'(x) = 3e^{2x}$

$\therefore\ f(x) = \int 3e^{2x}\,dx$

$= \frac{3}{2}e^{2x} + c$

But $f(0) = 2$, so $\frac{3}{2} + c = 2$

$\therefore\ c = \frac{1}{2}$

$\therefore\ f(x) = \frac{3}{2}e^{2x} + \frac{1}{2}$

10 **a** $\int \dfrac{x^2-7}{x}\,dx$

$= \int \left(x - \dfrac{7}{x}\right) dx$

$= \frac{1}{2}x^2 - 7\ln|x| + c$

b $\int \left(e^{2x-3} - \dfrac{2}{3x-1}\right) dx$

$= \frac{1}{2}e^{2x-3} - \frac{2}{3}\ln|3x-1| + c$

c $\int \left((4-3x)^3 + \sin(-2x)\right) dx$

$= \left(\frac{1}{-3}\right)\dfrac{(4-3x)^4}{4} + \left(\frac{1}{-2}\right)(-\cos(-2x)) + c$

$= -\frac{1}{12}(4-3x)^4 + \frac{1}{2}\cos(-2x) + c$

11
$$\begin{aligned}
f'(x) &= a\cos 3x \\
\therefore\ f(x) &= \int a\cos 3x\,dx \\
&= \frac{a}{3}\sin 3x + c
\end{aligned}$$

But $f(0)=-1$, so $c=-1$

$$\begin{aligned}
\therefore\ f(x) &= \frac{a}{3}\sin 3x - 1 \\
\text{and}\quad f(\tfrac{\pi}{4}) &= 1 \\
\therefore\ \frac{a}{3}\sin\tfrac{3\pi}{4} - 1 &= 1 \\
\therefore\ \frac{a}{3}\times\frac{1}{\sqrt{2}} &= 2 \\
\therefore\ \frac{a}{3\sqrt{2}} &= 2 \\
\therefore\ a &= 6\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\therefore\ f(x) &= \frac{6\sqrt{2}}{3}\sin 3x - 1 \\
&= 2\sqrt{2}\sin 3x - 1
\end{aligned}$$

12
$$\begin{aligned}
\sqrt{x^2-4} &= (x^2-4)^{\frac{1}{2}} \\
\therefore\ \frac{d}{dx}\left(\sqrt{x^2-4}\right) &= \tfrac{1}{2}(x^2-4)^{-\frac{1}{2}}\times 2x \\
&= \frac{x}{\sqrt{x^2-4}} \\
\therefore\ \int\frac{x}{\sqrt{x^2-4}}\,dx &= \sqrt{x^2-4}+c
\end{aligned}$$

13 $u = x^2+\frac{\pi}{3}$, $\frac{du}{dx}=2x$

$$\begin{aligned}
\therefore\ \int x\sin\left(x^2+\tfrac{\pi}{3}\right)dx &= \tfrac{1}{2}\int 2x\sin\left(x^2+\tfrac{\pi}{3}\right)dx \\
&= \tfrac{1}{2}\int \sin u\,\frac{du}{dx}\,dx \\
&= \tfrac{1}{2}\int \sin u\,du \\
&= \tfrac{1}{2}(-\cos u)+c \\
&= -\tfrac{1}{2}\cos\left(x^2+\tfrac{\pi}{3}\right)+c
\end{aligned}$$

14 **a**
$$\begin{aligned}
&\int \frac{x+2}{x^2+4x}\,dx\\
&= \tfrac{1}{2}\int \frac{2x+4}{x^2+4x}\,dx\\
&= \tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx\\
&\qquad \left\{u = x^2+4x,\quad \frac{du}{dx} = 2x+4\right\}\\
&= \tfrac{1}{2}\int \frac{1}{u}\,du\\
&= \tfrac{1}{2}\ln|u|+c\\
&= \tfrac{1}{2}\ln\left|x^2+4x\right|+c
\end{aligned}$$

b
$$\begin{aligned}
&\int 2xe^{x^2-1}\,dx\\
&= \int e^u\,\frac{du}{dx}\,dx\\
&\qquad \left\{u = x^2-1,\quad \frac{du}{dx} = 2x\right\}\\
&= \int e^u\,du\\
&= e^u+c\\
&= e^{x^2-1}+c
\end{aligned}$$

c
$$\begin{aligned}
&\int \sin^9 x\cos x\,dx\\
&= \int u^9\,\frac{du}{dx}\,dx\\
&\qquad \left\{u = \sin x,\quad \frac{du}{dx} = \cos x\right\}\\
&= \int u^9\,du\\
&= \tfrac{1}{10}u^{10}+c\\
&= \tfrac{1}{10}\sin^{10}x+c
\end{aligned}$$

d
$$\begin{aligned}
&\int \tan 2x\,dx\\
&= \int \frac{\sin 2x}{\cos 2x}\,dx\\
&= -\tfrac{1}{2}\int \frac{-2\sin 2x}{\cos 2x}\,dx\\
&= -\tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx\\
&\qquad \left\{u = \cos 2x,\quad \frac{du}{dx} = -2\sin 2x\right\}\\
&= -\tfrac{1}{2}\int \frac{1}{u}\,du\\
&= -\tfrac{1}{2}\ln|u|+c\\
&= -\tfrac{1}{2}\ln|\cos 2x|+c
\end{aligned}$$

REVIEW SET 22B

1
$$\begin{aligned}
\frac{d}{dx}(6e^{-2x}) &= 6e^{-2x}\times(-2)\\
&= -12e^{-2x}\\
\therefore\ \int -12e^{-2x}\,dx &= 6e^{-2x}+c\\
\therefore\ -12\int e^{-2x}\,dx &= 6e^{-2x}+c\\
\therefore\ \int e^{-2x}\,dx &= -\tfrac{1}{2}e^{-2x}+c
\end{aligned}$$

2
$$\begin{aligned}
\frac{d}{dx}(\ln(2x+1)) &= \frac{2}{2x+1}\\
\therefore\ \int \frac{2}{2x+1}\,dx &= \ln|2x+1|+c\\
\therefore\ 2\int \frac{1}{2x+1}\,dx &= \ln|2x+1|+c\\
\therefore\ \int \frac{1}{2x+1}\,dx &= \tfrac{1}{2}\ln|2x+1|+c
\end{aligned}$$

3 **a** $\int \frac{x^2-2}{x^2}\,dx$

$= \int \left(1 - \frac{2}{x^2}\right) dx$

$= \int (1 - 2x^{-2})\,dx$

$= x - \frac{2x^{-1}}{-1} + c$

$= x + \frac{2}{x} + c$

b $\int (3x-4)^2\,dx$

$= \int (9x^2 - 24x + 16)\,dx$

$= \frac{9x^3}{3} - \frac{24x^2}{2} + 16x + c$

$= 3x^3 - 12x^2 + 16x + c$

c $\int (4 - 2x^2)\,dx$

$= 4x - \frac{2}{3}x^3 + c$

4 **a** $\int (x^{\frac{1}{3}} + 3)\,dx$

$= \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 3x + c$

$= \frac{3}{4}x^{\frac{4}{3}} + 3x + c$

b $\int (3x^2 - 2)\,dx$

$= \frac{3x^3}{3} - 2x + c$

$= x^3 - 2x + c$

c $\int (3+2x)^2\,dx$

$= \int (9 + 12x + 4x^2)\,dx$

$= 9x + \frac{12x^2}{2} + \frac{4x^3}{3} + c$

$= 9x + 6x^2 + \frac{4}{3}x^3 + c$

5 $f'(x) = x^2 - 3x + 2$

$\therefore\ f(x) = \int (x^2 - 3x + 2)\,dx$

$= \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$

$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c$

But $f(1) = 3$, so $\frac{1}{3} - \frac{3}{2} + 2 + c = 3$

$\therefore\ c = \frac{13}{6} = 2\frac{1}{6}$

$\therefore\ f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 2\frac{1}{6}$

6 **a** $\frac{dy}{dx} = (x^2-1)^2 = x^4 - 2x^2 + 1$

$\therefore\ y = \int (x^4 - 2x^2 + 1)\,dx$

$= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c$

b $\frac{dy}{dx} = 400 - 20x^{-\frac{1}{2}}$

$\therefore\ y = \int (400 - 20x^{-\frac{1}{2}})\,dx$

$= 400x - \frac{20x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 400x - 40x^{\frac{1}{2}} + c$

7 **a** $\int (2x^3 - 5x + 7)\,dx$

$= \frac{2x^4}{4} - \frac{5x^2}{2} + 7x + c$

$= \frac{1}{2}x^4 - \frac{5}{2}x^2 + 7x + c$

b $\int \left(3x - \frac{1}{x}\right) dx$

$= \frac{3}{2}x^2 - \ln|x| + c$

c $\displaystyle\int (2e^{-x}+3)\,dx$
$= 2(\frac{1}{-1})e^{-x}+3x+c$
$= -2e^{-x}+3x+c$

d $\displaystyle\int 4\cos 2x\,dx$
$= 4\times\frac{1}{2}\sin 2x+c$
$= 2\sin 2x+c$

e $\displaystyle\int \left(3+e^{2x-1}\right)^2\,dx$
$= \displaystyle\int \left(9+6e^{2x-1}+(e^{2x-1})^2\right)dx$
$= \displaystyle\int \left(9+6e^{2x-1}+e^{4x-2}\right)dx$
$= 9x+6\left(\frac{1}{2}\right)e^{2x-1}+\frac{1}{4}e^{4x-2}+c$
$= 9x+3e^{2x-1}+\frac{1}{4}e^{4x-2}+c$

f $\displaystyle\int \frac{1}{\cos^2\left(\frac{\pi}{4}-x\right)}\,dx$
$= \left(\frac{1}{-1}\right)\tan\left(\frac{\pi}{4}-x\right)+c$
$= -\tan\left(\frac{\pi}{4}-x\right)+c$

8 $f'(x)=\dfrac{2}{x}-1$

$\therefore\ f(x)=\displaystyle\int\left(\frac{2}{x}-1\right)dx$
$= 2\ln|x|-x+c$

But $f(2)=e$, so $2\ln 2-2+c=e$
$\therefore\ c=e+2-2\ln 2$

$\therefore\ f(x)=2\ln|x|-x+e+2-2\ln 2$

9 $\dfrac{dy}{dx}=ax^2+b\sqrt{x-1}=ax^2+b(x-1)^{\frac{1}{2}}$

$\therefore\ y=\displaystyle\int\left(ax^2+b(x-1)^{\frac{1}{2}}\right)dx$
$= \dfrac{ax^3}{3}+\dfrac{b(x-1)^{\frac{3}{2}}}{\frac{3}{2}}+c$
$= \frac{1}{3}ax^3+\frac{2}{3}b(x-1)^{\frac{3}{2}}+c$

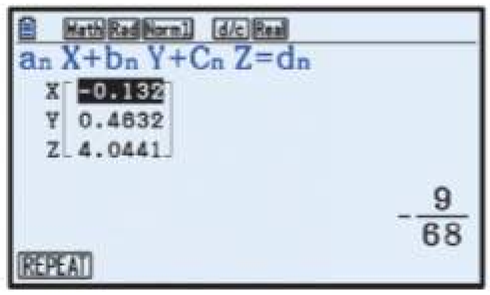
But the curve passes through $(1,4)$, $(2,4)$, and $(5,1)$

$\therefore\ \frac{1}{3}a(1)^3+\frac{2}{3}b(1-1)^{\frac{3}{2}}+c=4$
$\therefore\ \frac{1}{3}a+c=4$ .... (1)

and $\frac{1}{3}a(2)^3+\frac{2}{3}b(2-1)^{\frac{3}{2}}+c=4$
$\therefore\ \frac{8}{3}a+\frac{2}{3}b+c=4$ .... (2)

and $\frac{1}{3}a(5)^3+\frac{2}{3}b(5-1)^{\frac{3}{2}}+c=1$
$\therefore\ \frac{125}{3}a+\frac{2}{3}b(4)^{\frac{3}{2}}+c=1$
$\therefore\ \frac{125}{3}a+\frac{16}{3}b+c=1$ .... (3)

We solve (1), (2), and (3) simultaneously using technology.



$\therefore \quad a = -\frac{9}{68}, \quad b = \frac{63}{136}, \quad c = \frac{275}{68}$

$\therefore \quad y = \frac{1}{3}\left(-\frac{9}{68}\right)x^3 + \frac{2}{3}\left(\frac{63}{136}\right)(x-1)^{\frac{3}{2}} + \frac{275}{68}$

$\therefore \quad y = -\frac{3}{68}x^3 + \frac{21}{68}(x-1)^{\frac{3}{2}} + \frac{275}{68}$

10 $\quad f'(x) = \dfrac{3}{\sqrt{4-3x}} = 3(4-3x)^{-\frac{1}{2}}$

$$\begin{aligned} \therefore \quad f(x) &= \int 3(4-3x)^{-\frac{1}{2}}\,dx \\ &= 3\left(\tfrac{1}{-3}\right)\frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -2\sqrt{4-3x} + c \end{aligned}$$

But $f(-4) = 0$, so

$$\begin{aligned} -2\sqrt{4-3(-4)} + c &= 0 \\ \therefore \quad -2\sqrt{16} + c &= 0 \\ \therefore \quad -8 + c &= 0 \\ \therefore \quad c &= 8 \end{aligned}$$

$\therefore \quad f(x) = -2\sqrt{4-3x} + 8$

11 **a** $$\begin{aligned} \int \frac{1}{3-2x}\,dx &= \tfrac{1}{-2}\ln|3-2x| + c \\ &= -\tfrac{1}{2}\ln|3-2x| + c \end{aligned}$$

b $$\begin{aligned} \int \frac{4}{5x+1}\,dx &= 4\int \frac{1}{5x+1}\,dx \\ &= \tfrac{4}{5}\ln|5x+1| + c \end{aligned}$$

12 $$\frac{d}{dx}\left((3x^2+x)^3\right) = 3(3x^2+x)^2(6x+1)$$

$\therefore \quad \int 3(3x^2+x)^2(6x+1)\,dx = (3x^2+x)^3 + c$

$\therefore \quad 3\int (3x^2+x)^2(6x+1)\,dx = (3x^2+x)^3 + c$

$\therefore \quad \int (3x^2+x)^2(6x+1)\,dx = \frac{1}{3}(3x^2+x)^3 + c$

13 **a**

$$\int \frac{2x}{\sqrt{x^2-5}}\,dx$$
$$= \int 2x(x^2-5)^{-\frac{1}{2}}\,dx$$
$$= \int u^{-\frac{1}{2}}\frac{du}{dx}\,dx$$
$$\left\{u = x^2-5, \quad \frac{du}{dx} = 2x\right\}$$
$$= \int u^{-\frac{1}{2}}\,du$$
$$= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 2\sqrt{u} + c$$
$$= 2\sqrt{x^2-5} + c$$

b

$$\int \frac{\sin x}{\cos^4 x}\,dx$$
$$= -\int \frac{-\sin x}{\cos^4 x}\,dx$$
$$= -\int u^{-4}\frac{du}{dx}\,dx$$
$$\left\{u = \cos x, \quad \frac{du}{dx} = -\sin x\right\}$$
$$= -\int u^{-4}\,du$$
$$= -\frac{u^{-3}}{-3} + c$$
$$= \frac{1}{3\cos^3 x} + c$$

c

$$\int 4xe^{-x^2}\,dx$$
$$= -2\int -2xe^{-x^2}\,dx$$
$$= -2\int e^u\frac{du}{dx}\,dx$$
$$\left\{u = -x^2, \quad \frac{du}{dx} = -2x\right\}$$
$$= -2\int e^u\,du$$
$$= -2e^u + c$$
$$= -2e^{-x^2} + c$$

d

$$\int \sin^3 x\,dx$$
$$= \int \sin^2 x \sin x\,dx$$
$$= \int (1-\cos^2 x)\sin x\,dx$$
$$= -\int (1-\cos^2 x)(-\sin x)\,dx$$
$$= -\int (1-u^2)\frac{du}{dx}\,dx$$
$$\left\{u = \cos x, \quad \frac{du}{dx} = -\sin x\right\}$$
$$= -\int (1-u^2)\,du$$
$$= -(u - \tfrac{1}{3}u^3 + c)$$
$$= -u + \tfrac{1}{3}u^3 + c$$
$$= -\cos x + \tfrac{1}{3}\cos^3 x + c$$

14 **a**

$$\int \frac{x}{x^2-9}\,dx$$
$$= \tfrac{1}{2}\int \frac{2x}{x^2-9}\,dx$$
$$= \tfrac{1}{2}\int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$\{u = x^2-9, \quad \frac{du}{dx} = 2x\}$$
$$= \tfrac{1}{2}\int \frac{1}{u}\,du$$
$$= \tfrac{1}{2}\ln|\,u\,| + c$$
$$= \tfrac{1}{2}\ln\left|\,x^2-9\,\right| + c$$

b

$$\int \frac{x}{x^2-9}\,dx$$

If $x = 3\sin t$, then $\dfrac{dx}{dt} = 3\cos t$

$$\therefore \quad \int \frac{x}{x^2-9}\,dx$$
$$= \int \frac{3\sin t}{(3\sin t)^2-9}\,\frac{dx}{dt}\,dt$$
$$= \int \frac{3\sin t}{9\sin^2 t-9}\,(3\cos t)\,dt$$
$$= \int \frac{9\sin t\cos t}{-9(1-\sin^2 t)}\,dt$$
$$= \int \frac{\sin t\cos t}{-\cos^2 t}\,dt$$
$$= \int -\frac{\sin t}{\cos t}\,dt$$
$$= \int \frac{1}{u}\,\frac{du}{dt}\,dt \qquad \{u = \cos t, \quad \frac{du}{dt} = -\sin t\}$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|\,u\,| + c$$
$$= \ln|\cos t\,| + c$$

Now, $x = 3\sin t$

$$\therefore \quad \frac{x}{3} = \sin t$$
$$\therefore \quad t = \sin^{-1}\left(\tfrac{x}{3}\right)$$
$$\therefore \quad \int \frac{x}{x^2-9}\,dx = \ln\left|\cos\left(\sin^{-1}\left(\tfrac{x}{3}\right)\right)\right| + c$$

Chapter 23

DEFINITE INTEGRALS

EXERCISE 23A

1 **a** $\displaystyle\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{\frac{1}{2}}\,dx$
$= \left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$
$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$
$= \frac{2}{3}(8) - \frac{2}{3}(1)$
$= \frac{14}{3}$

$\displaystyle\int_1^4 (-\sqrt{x})\,dx = \int_1^4 (-x^{\frac{1}{2}})\,dx$
$= \left[-\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$
$= \left[-\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$
$= -\frac{2}{3}(8) - (-\frac{2}{3}(1))$
$= -\frac{14}{3}$

b $\displaystyle\int_0^1 x^7\,dx = \left[\frac{x^8}{8}\right]_0^1$
$= \frac{1}{8} - 0$
$= \frac{1}{8}$

$\displaystyle\int_0^1 (-x^7)\,dx = \left[-\frac{x^8}{8}\right]_0^1$
$= -\frac{1}{8} - 0$
$= -\frac{1}{8}$

Property: $\displaystyle\int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$

2 **a** $\displaystyle\int_0^1 x^2\,dx$
$= \left[\frac{x^3}{3}\right]_0^1$
$= \frac{1}{3} - 0$
$= \frac{1}{3}$

b $\displaystyle\int_1^2 x^2\,dx$
$= \left[\frac{x^3}{3}\right]_1^2$
$= \frac{8}{3} - \frac{1}{3}$
$= \frac{7}{3}$

c $\displaystyle\int_0^2 x^2\,dx$
$= \left[\frac{x^3}{3}\right]_0^2$
$= \frac{8}{3} - 0$
$= \frac{8}{3}$

d $\displaystyle\int_0^1 3x^2\,dx$
$= \left[x^3\right]_0^1$
$= 1 - 0$
$= 1$

Properties: $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where c is a constant

3 **a**
$$\begin{aligned}&\int_0^2 (x^3-4x)\,dx\\&=\left[\frac{x^4}{4}-2x^2\right]_0^2\\&=\left(\tfrac{16}{4}-2(4)\right)-(0-0)\\&=-4\end{aligned}$$

b
$$\begin{aligned}&\int_2^3 (x^3-4x)\,dx\\&=\left[\frac{x^4}{4}-2x^2\right]_2^3\\&=\left(\tfrac{81}{4}-2(9)\right)-\left(\tfrac{16}{4}-2(4)\right)\\&=\tfrac{25}{4}\\&=6\tfrac{1}{4}\end{aligned}$$

c
$$\begin{aligned}\int_0^3 (x^3-4x)\,dx&=\left[\frac{x^4}{4}-2x^2\right]_0^3\\&=\left(\tfrac{81}{4}-2(9)\right)-(0-0)\\&=\tfrac{9}{4}\\&=2\tfrac{1}{4}\end{aligned}$$

Property: $\displaystyle\int_a^b f(x)\,dx+\int_b^c f(x)\,dx=\int_a^c f(x)\,dx$

4 **a**
$$\begin{aligned}\int_0^1 2x^2\,dx&=\left[\tfrac{2}{3}x^3\right]_0^1\\&=\tfrac{2}{3}(1)-0\\&=\tfrac{2}{3}\end{aligned}$$

b
$$\begin{aligned}\int_0^1 \sqrt{x}\,dx&=\int_0^1 x^{\frac{1}{2}}\,dx\\&=\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1\\&=\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^1\\&=\tfrac{2}{3}(1)-0\\&=\tfrac{2}{3}\end{aligned}$$

c
$$\begin{aligned}&\int_0^1 (2x^2+\sqrt{x})\,dx\\&=\int_0^1 (2x^2+x^{\frac{1}{2}})\,dx\\&=\left[\tfrac{2}{3}x^3+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1\\&=\left[\tfrac{2}{3}x^3+\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^1\\&=\left(\tfrac{2}{3}(1)+\tfrac{2}{3}(1)\right)-(0+0)\\&=\tfrac{4}{3}\end{aligned}$$

Property: $\displaystyle\int_a^b f(x)\,dx+\int_a^b g(x)\,dx=\int_a^b [f(x)+g(x)]\,dx$

5 **a**
$$\begin{aligned}\int_0^1 x^3\,dx&=\left[\frac{x^4}{4}\right]_0^1\\&=\tfrac{1}{4}-0\\&=\tfrac{1}{4}\end{aligned}$$

b
$$\begin{aligned}\int_0^2 (x^2-x)\,dx&=\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_0^2\\&=(\tfrac{8}{3}-2)-(0-0)\\&=\tfrac{2}{3}\end{aligned}$$

c $\displaystyle\int_0^2 (3x^2 - x + 6)\,dx$

$= \left[\dfrac{3x^3}{3} - \dfrac{x^2}{2} + 6x\right]_0^2$

$= \left[x^3 - \dfrac{x^2}{2} + 6x\right]_0^2$

$= (8 - 2 + 12) - 0$

$= 18$

d $\displaystyle\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx$

$= \displaystyle\int_1^4 (x - 3x^{-\frac{1}{2}})\,dx$

$= \left[\dfrac{x^2}{2} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$

$= \left[\dfrac{x^2}{2} - 6\sqrt{x}\right]_1^4$

$= \left(\frac{16}{2} - 12\right) - \left(\frac{1}{2} - 6\right)$

$= \frac{3}{2}$

e $\displaystyle\int_1^4 \left(x + 2\sqrt{x}\right) dx$

$= \displaystyle\int_1^4 (x + 2x^{\frac{1}{2}})\,dx$

$= \left[\dfrac{x^2}{2} + \dfrac{2x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$

$= \left[\dfrac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}}\right]_1^4$

$= \left(\frac{16}{2} + \frac{4}{3}(8)\right) - \left(\frac{1}{2} + \frac{4}{3}\right)$

$= \frac{101}{6}$

f $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

$= \displaystyle\int_4^9 (x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})\,dx$

$= \left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - 6\sqrt{x}\right]_4^9$

$= \left(\frac{2}{3}(27) - 6(3)\right) - \left(\frac{2}{3}(8) - 6(2)\right)$

$= (18 - 18) - \left(\frac{16}{3} - 12\right)$

$= \frac{20}{3}$

g $\displaystyle\int_1^3 \frac{1}{x^2}\,dx = \int_1^3 x^{-2}\,dx$

$= \left[\dfrac{x^{-1}}{-1}\right]_1^3$

$= \left[-\dfrac{1}{x}\right]_1^3$

$= -\frac{1}{3} - (-1)$

$= \frac{2}{3}$

h $\displaystyle\int_1^2 (x+3)^2\,dx$

$= \displaystyle\int_1^2 (x^2 + 6x + 9)\,dx$

$= \left[\dfrac{x^3}{3} + \dfrac{6x^2}{2} + 9x\right]_1^2$

$= \left[\dfrac{x^3}{3} + 3x^2 + 9x\right]_1^2$

$= \left(\frac{8}{3} + 12 + 18\right) - \left(\frac{1}{3} + 3 + 9\right)$

$= \frac{61}{3}$

i $\displaystyle\int_1^4 \left(x^2 + \frac{1}{x}\right) dx = \left[\frac{x^3}{3} + \ln|x|\right]_1^4$

$= \left(\frac{64}{3} + \ln 4\right) - \left(\frac{1}{3} + \ln 1\right)$

$= 21 + \ln 4$

6
$$\int_m^{2m} (2x-1)\,dx = 4$$
$$\therefore\ \left[x^2 - x\right]_m^{2m} = 4$$
$$\therefore\ (4m^2 - 2m) - (m^2 - m) = 4$$
$$\therefore\ 3m^2 - m - 4 = 0$$
$$\therefore\ (3m-4)(m+1) = 0$$
$$\therefore\ m = -1 \text{ or } \tfrac{4}{3}$$

7 **a**
$$\int_0^1 (3x+1)^4\,dx$$
$$= \left[\frac{1}{3}\,\frac{(3x+1)^5}{5}\right]_0^1$$
$$= \left[\frac{(3x+1)^5}{15}\right]_0^1$$
$$= \frac{4^5}{15} - \frac{1^5}{15}$$
$$= \tfrac{1024}{15} - \tfrac{1}{15}$$
$$= \tfrac{341}{5} = 68\tfrac{1}{5}$$

b
$$\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$$
$$= \int_2^6 (2x-3)^{-\frac{1}{2}}\,dx$$
$$= \left[\frac{1}{2}\,\frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^6$$
$$= \left[\sqrt{2x-3}\right]_2^6$$
$$= \sqrt{9} - \sqrt{1}$$
$$= 2$$

c
$$\int_{-3}^0 \sqrt{1-x}\,dx = \int_{-3}^0 (1-x)^{\frac{1}{2}}\,dx$$
$$= \left[\left(\frac{1}{-1}\right)\frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-3}^0$$
$$= \left[-\tfrac{2}{3}(1-x)^{\frac{3}{2}}\right]_{-3}^0$$
$$= (-\tfrac{2}{3}(1)^{\frac{3}{2}}) - (-\tfrac{2}{3}(4)^{\frac{3}{2}})$$
$$= -\tfrac{2}{3} - \left(-\tfrac{16}{3}\right)$$
$$= \tfrac{14}{3} = 4\tfrac{2}{3}$$

8 **a**
$$\int_0^1 e^x\,dx = \left[e^x\right]_0^1$$
$$= e^1 - e^0$$
$$= e - 1$$

b
$$\int_0^3 (2e^x - 3)\,dx = \left[2e^x - 3x\right]_0^3$$
$$= (2e^3 - 9) - (2 - 0)$$
$$= 2e^3 - 11$$

c
$$\int_0^2 e^{3x}\,dx = \left[\tfrac{1}{3}e^{3x}\right]_0^2$$
$$= \tfrac{1}{3}e^6 - \tfrac{1}{3}e^0$$
$$= \tfrac{1}{3}e^6 - \tfrac{1}{3}$$
$$= \tfrac{1}{3}(e^6 - 1)$$

d
$$\int_0^1 e^{1-x}\,dx = \left[(\tfrac{1}{-1})e^{1-x}\right]_0^1$$
$$= \left[-e^{1-x}\right]_0^1$$
$$= -e^0 - (-e^1)$$
$$= e - 1$$

e $\displaystyle\int_0^{\ln 4} e^x(e^x - 2)\,dx = \int_0^{\ln 4} (e^{2x} - 2e^x)\,dx$

$$\begin{aligned}
&= \left[\tfrac{1}{2}e^{2x} - 2e^x\right]_0^{\ln 4} \\
&= \left(\tfrac{1}{2}e^{2\ln 4} - 2e^{\ln 4}\right) - \left(\tfrac{1}{2}e^0 - 2e^0\right) \\
&= \left(\tfrac{1}{2}e^{\ln 16} - 2e^{\ln 4}\right) - \left(\tfrac{1}{2} - 2\right) \\
&= \left(\tfrac{1}{2}(16) - 2(4)\right) - \left(-\tfrac{3}{2}\right) \\
&= \tfrac{3}{2}
\end{aligned}$$

f $\displaystyle\int_1^2 (e^{-x} + 1)^2\,dx = \int_1^2 (e^{-2x} + 2e^{-x} + 1)\,dx$

$$\begin{aligned}
&= \left[(\tfrac{1}{-2})e^{-2x} + 2(\tfrac{1}{-1})e^{-x} + x\right]_1^2 \\
&= \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x\right]_1^2 \\
&= \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2\right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1\right) \\
&= -\frac{1}{2e^4} - \frac{3}{2e^2} + \frac{2}{e} + 1
\end{aligned}$$

9 **a** $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\,dx = \Big[\sin x\Big]_0^{\frac{\pi}{6}}$

$$\begin{aligned}
&= \sin\tfrac{\pi}{6} - \sin 0 \\
&= \tfrac{1}{2} - 0 \\
&= \tfrac{1}{2}
\end{aligned}$$

b $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx = \Big[-\cos x\Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$$\begin{aligned}
&= \left(-\cos\tfrac{\pi}{2}\right) - \left(-\cos\tfrac{\pi}{3}\right) \\
&= 0 - \left(-\tfrac{1}{2}\right) \\
&= \tfrac{1}{2}
\end{aligned}$$

c $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx = \Big[\tan x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$$\begin{aligned}
&= \tan\tfrac{\pi}{3} - \tan\tfrac{\pi}{4} \\
&= \sqrt{3} - 1
\end{aligned}$$

d $\displaystyle\int_0^{\frac{\pi}{6}} \sin 3x\,dx = \Big[-\tfrac{1}{3}\cos 3x\Big]_0^{\frac{\pi}{6}}$

$$\begin{aligned}
&= \left(-\tfrac{1}{3}\cos\tfrac{\pi}{2}\right) - \left(-\tfrac{1}{3}\cos 0\right) \\
&= 0 - \left(-\tfrac{1}{3}\right) \\
&= \tfrac{1}{3}
\end{aligned}$$

e $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\left(x - \tfrac{\pi}{3}\right)dx$

$$\begin{aligned}
&= \Big[\sin\left(x - \tfrac{\pi}{3}\right)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \sin\left(\tfrac{\pi}{2} - \tfrac{\pi}{3}\right) - \sin\left(\tfrac{\pi}{6} - \tfrac{\pi}{3}\right) \\
&= \sin\tfrac{\pi}{6} - \sin\left(-\tfrac{\pi}{6}\right) \\
&= \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) \\
&= 1
\end{aligned}$$

f $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(2x - \tfrac{\pi}{4}\right)dx$

$$\begin{aligned}
&= \Big[-\tfrac{1}{2}\cos\left(2x - \tfrac{\pi}{4}\right)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \left(-\tfrac{1}{2}\cos\tfrac{3\pi}{4}\right) - \left(-\tfrac{1}{2}\cos\tfrac{\pi}{4}\right) \\
&= \tfrac{1}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}} \\
&= \tfrac{1}{\sqrt{2}}
\end{aligned}$$

10 $\displaystyle\int_0^{\frac{\pi}{6}} (\sin 3x - \cos x)\,dx = \left[-\tfrac{1}{3}\cos 3x - \sin x\right]_0^{\frac{\pi}{6}}$

$= \left(-\tfrac{1}{3}(0) - \tfrac{1}{2}\right) - \left(-\tfrac{1}{3}(1) - 0\right)$

$= -\tfrac{1}{6}$

11 $\displaystyle\int_3^{12} \frac{1}{x}\,dx = \left[\ln|x|\right]_3^{12}$

$= \ln 12 - \ln 3$

$= \ln\left(\tfrac{12}{3}\right) \qquad \left\{\ln a - \ln b = \ln\left(\frac{a}{b}\right)\right\}$

$= \ln 4$

12 **a** $\displaystyle\int_{-6}^{-2} \frac{1}{x}\,dx$

$= \left[\ln|x|\right]_{-6}^{-2}$

$= \ln 2 - \ln 6$

$= \ln\left(\tfrac{2}{6}\right) \qquad \left\{\ln a - \ln b = \ln\left(\frac{a}{b}\right)\right\}$

$= \ln\left(\tfrac{1}{3}\right)$

$= \ln(3^{-1})$

$= -\ln 3$

b $\displaystyle\int_{-1}^{5} \frac{1}{x+4}\,dx$

$= \left[\ln|x+4|\right]_{-1}^{5}$

$= \ln 9 - \ln 3$

$= \ln\left(\tfrac{9}{3}\right) \qquad \left\{\ln a - \ln b = \ln\left(\frac{a}{b}\right)\right\}$

$= \ln 3$

c $\displaystyle\int_1^8 \frac{2}{3x+4}\,dx$

$= \left[2\left(\tfrac{1}{3}\right)\ln|3x+4|\right]_1^8$

$= \left[\tfrac{2}{3}\ln|3x+4|\right]_1^8$

$= \tfrac{2}{3}\ln 28 - \tfrac{2}{3}\ln 7$

$= \tfrac{2}{3}(\ln 28 - \ln 7)$

$= \tfrac{2}{3}\ln\left(\tfrac{28}{7}\right) \qquad \left\{\ln a - \ln b = \ln\left(\frac{a}{b}\right)\right\}$

$= \tfrac{2}{3}\ln 4$

$= \tfrac{4}{3}\ln 2$

d $\displaystyle\int_{-4}^{0} \frac{4}{5-2x}\,dx$

$= \left[4\left(\tfrac{1}{-2}\right)\ln|5-2x|\right]_{-4}^{0}$

$= \left[-2\ln|5-2x|\right]_{-4}^{0}$

$= -2\ln 5 - (-2\ln 13)$

$= -2\ln 5 + 2\ln 13$

$= 2(\ln 13 - \ln 5)$

$= 2\ln\left(\tfrac{13}{5}\right) \qquad \left\{\ln a - \ln b = \ln\left(\frac{a}{b}\right)\right\}$

13 $\dfrac{4x+1}{x-1} = \dfrac{4x-4+1+4}{x-1}$

$= \dfrac{4(x-1)+5}{x-1}$

$= \dfrac{4(x-1)}{x-1} + \dfrac{5}{x-1}$

$= 4 + \dfrac{5}{x-1}$ as required

$\therefore \displaystyle\int_3^5 \frac{4x+1}{x-1}\,dx$

$= \displaystyle\int_3^5 \left(4 + \frac{5}{x-1}\right)dx$

$= \left[4x + 5\ln|x-1|\right]_3^5$

$= (4(5) + 5\ln|5-1|) - (4(3) + 5\ln|3-1|)$

$= 20 + 5\ln 4 - 12 - 5\ln 2$

$= 20 + 5\ln(2^2) - 12 - 5\ln 2$

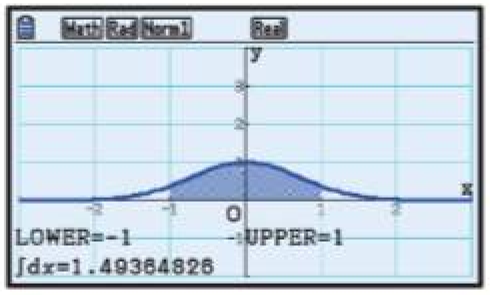
$= 8 + 10\ln 2 - 5\ln 2$

$= 8 + 5\ln 2$

14 **a**

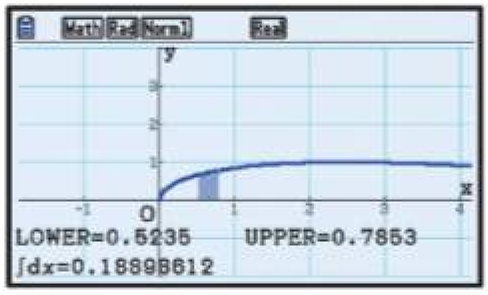
$$\therefore \quad \int_1^3 \ln x \, dx \approx 1.30$$

b



$$\therefore \quad \int_{-1}^1 e^{-x^2} \, dx \approx 1.49$$

c



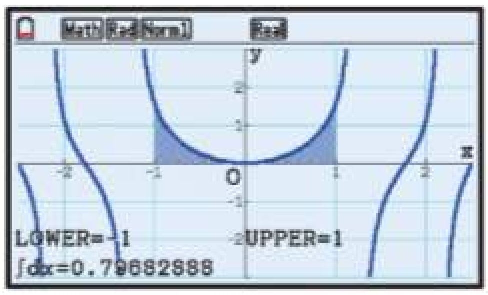
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin(\sqrt{x}) \, dx \approx 0.189$$

$$\therefore \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x}) \, dx \approx -0.189$$

d

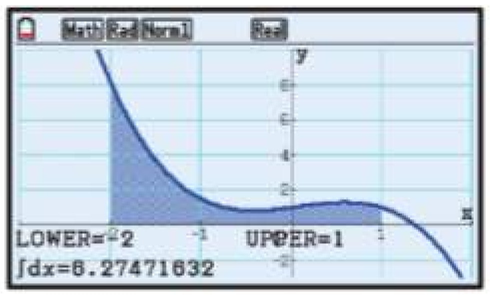
$$\therefore \quad \int_2^3 \frac{1+\sqrt{x}}{x^2+1} \, dx \approx 0.363$$

e



$$\therefore \quad \int_{-1}^1 \tan(x^2) \, dx \approx 0.797$$

f



$$\therefore \quad \int_{-2}^1 (2^x - x^3) \, dx \approx 6.27$$

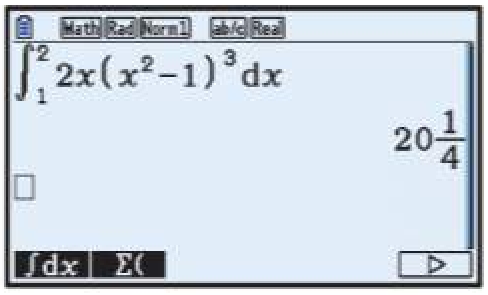
EXERCISE 23B

1 **a** Let $u = x^2 - 1 \quad \therefore \quad \dfrac{du}{dx} = 2x$

When $x = 1, \; u = 0$

When $x = 2, \; u = 3$

$$\begin{aligned}\therefore \int_1^2 2x(x^2-1)^3\,dx &= \int_1^2 u^3 \frac{du}{dx}\,dx \\ &= \int_0^3 u^3\,du \\ &= \left[\frac{u^4}{4}\right]_0^3 \\ &= \left(\tfrac{81}{4} - 0\right) \\ &= \tfrac{81}{4} = 20\tfrac{1}{4}\end{aligned}$$

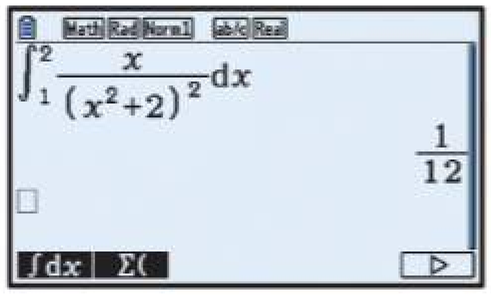


b Let $u = x^2 + 2 \quad \therefore \quad \dfrac{du}{dx} = 2x$

When $x = 1, \; u = 3$

When $x = 2, \; u = 6$

$$\begin{aligned}\therefore \int_1^2 \frac{x}{(x^2+2)^2}\,dx &= \int_1^2 u^{-2}\left(\tfrac{1}{2}\,\frac{du}{dx}\right)dx \\ &= \tfrac{1}{2}\int_3^6 u^{-2}\,du \\ &= \tfrac{1}{2}\left[\frac{u^{-1}}{-1}\right]_3^6 \\ &= \tfrac{1}{2}\left[-\tfrac{1}{6} - \left(-\tfrac{1}{3}\right)\right] \\ &= \tfrac{1}{12}\end{aligned}$$

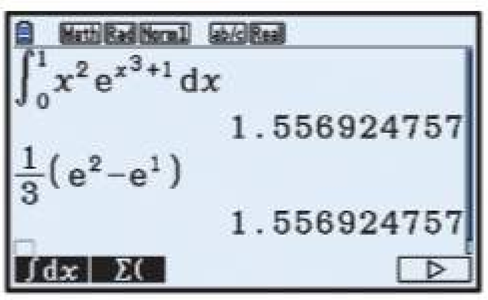


c Let $u = x^3 + 1 \quad \therefore \quad \dfrac{du}{dx} = 3x^2$

When $x = 0, \; u = 1$

When $x = 1, \; u = 2$

$$\begin{aligned}\therefore \int_0^1 x^2 e^{x^3+1}\,dx &= \int_0^1 e^u\left(\tfrac{1}{3}\,\frac{du}{dx}\right)dx \\ &= \tfrac{1}{3}\int_1^2 e^u\,du \\ &= \tfrac{1}{3}\left[e^u\right]_1^2 \\ &= \tfrac{1}{3}(e^2 - e) \\ &\approx 1.56\end{aligned}$$

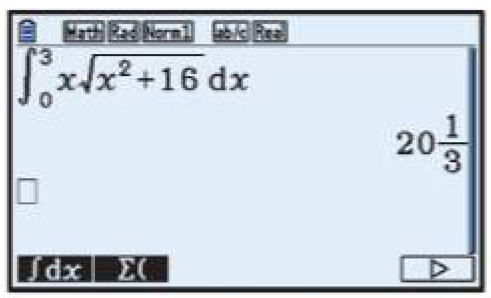


d Let $u = x^2 + 16 \quad \therefore \quad \dfrac{du}{dx} = 2x$

When $x = 0, \; u = 16$

When $x = 3, \; u = 25$

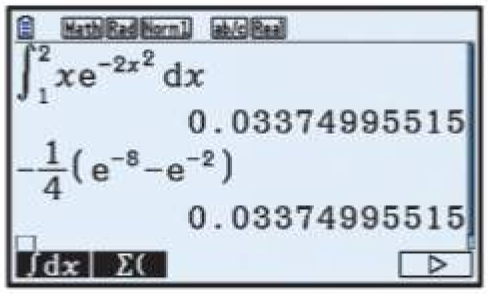
$$\begin{aligned}\therefore \int_0^3 x\sqrt{x^2+16}\,dx &= \int_0^3 u^{\frac{1}{2}}\left(\tfrac{1}{2}\,\frac{du}{dx}\right)dx \\ &= \tfrac{1}{2}\int_{16}^{25} u^{\frac{1}{2}}\,du \\ &= \tfrac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{16}^{25} \\ &= \tfrac{1}{2}\times\tfrac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{25} \\ &= \tfrac{1}{3}(125 - 64) \\ &= \tfrac{61}{3} = 20\tfrac{1}{3}\end{aligned}$$



e Let $u = -2x^2 \quad \therefore \quad \dfrac{du}{dx} = -4x$

When $x = 1, \ u = -2$
When $x = 2, \ u = -8$

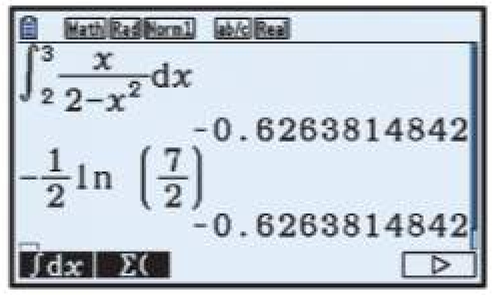
$$\therefore \int_1^2 xe^{-2x^2}\,dx = \int_1^2 e^u\left(-\tfrac{1}{4}\,\frac{du}{dx}\right)dx$$
$$= -\tfrac{1}{4}\int_{-2}^{-8} e^u\,du$$
$$= -\tfrac{1}{4}\left[e^u\right]_{-2}^{-8}$$
$$= -\tfrac{1}{4}(e^{-8} - e^{-2})$$
$$\approx 0.0337$$



f Let $u = 2 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

When $x = 2, \ u = -2$
When $x = 3, \ u = -7$

$$\therefore \int_2^3 \frac{x}{2 - x^2}\,dx = \int_2^3 \frac{1}{u}\left(-\tfrac{1}{2}\,\frac{du}{dx}\right)dx$$
$$= -\tfrac{1}{2}\int_{-2}^{-7} \frac{1}{u}\,du$$
$$= -\tfrac{1}{2}\left[\ln|u|\right]_{-2}^{-7}$$
$$= -\tfrac{1}{2}(\ln 7 - \ln 2)$$
$$= -\tfrac{1}{2}\ln\left(\tfrac{7}{2}\right)$$
$$\approx -0.626$$



2 **a** Let $u = \cos x \quad \therefore \quad \dfrac{du}{dx} = -\sin x$

When $x = 0, \quad u = \cos 0 = 1$
When $x = \frac{\pi}{3}, \quad u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx = \int_0^{\frac{\pi}{3}} u^{-\frac{1}{2}}\left(-\frac{du}{dx}\right)dx$$
$$= -\int_1^{\frac{1}{2}} u^{-\frac{1}{2}}\,du$$
$$= -\left[2u^{\frac{1}{2}}\right]_1^{\frac{1}{2}}$$
$$= -\left(2\sqrt{\tfrac{1}{2}} - 2\sqrt{1}\right)$$
$$= 2 - \sqrt{2}$$

b Let $u = \sin x \quad \therefore \quad \dfrac{du}{dx} = \cos x$

When $x = 0, \quad u = \sin 0 = 0$
When $x = \frac{\pi}{6}, \quad u = \sin\frac{\pi}{6} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x\cos x\,dx = \int_0^{\frac{\pi}{6}} u^2\,\frac{du}{dx}\,dx$$
$$= \int_0^{\frac{1}{2}} u^2\,du$$
$$= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}}$$
$$= \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3$$
$$= \tfrac{1}{24}$$

c Let $u = \cos x \quad \therefore \quad \dfrac{du}{dx} = -\sin x$

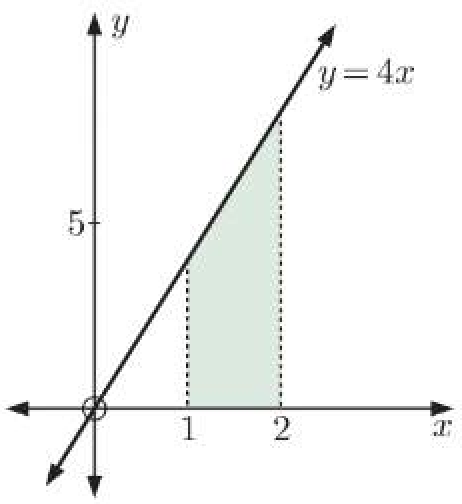
When $x = -\frac{\pi}{6}, \quad u = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

When $x = \frac{\pi}{3}, \quad u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned}
\therefore \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, dx &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx \\
&= -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u} \, du \\
&= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u} \, du \\
&= \Big[\ln|u|\Big]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \\
&= \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right) \\
&= \ln\left(\frac{\sqrt{3}}{2}\right) + \ln 2 \\
&= \ln\sqrt{3} \\
&= \tfrac{1}{2}\ln 3
\end{aligned}$$

EXERCISE 23C

1 **a**



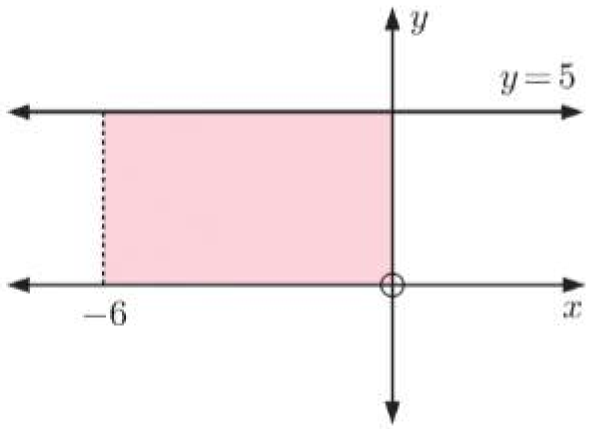
When $x = 1, \quad y = 4(1) = 4$

When $x = 2, \quad y = 4(2) = 8$

$$\begin{aligned}
\text{Area} &= \text{area of trapezium} \\
&= \left(\frac{4+8}{2}\right) \times 1 \\
&= 6 \text{ units}^2
\end{aligned}$$

b
$$\begin{aligned}
\text{Area} &= \int_1^2 4x \, dx \\
&= \left[2x^2\right]_1^2 \\
&= 2(2)^2 - 2(1)^2 \\
&= 6 \text{ units}^2
\end{aligned}$$

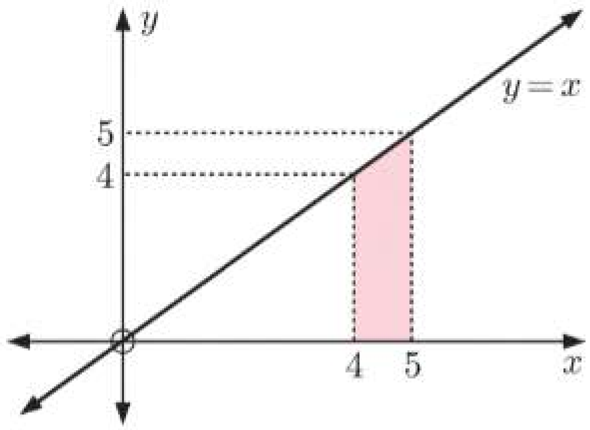
2 **a**



i Area $= 6 \times 5$
$= 30 \text{ units}^2$

ii Area $= \displaystyle\int_{-6}^{0} 5\,dx$
$= \big[5x\big]_{-6}^{0}$
$= 5(0) - 5(-6)$
$= 30 \text{ units}^2$

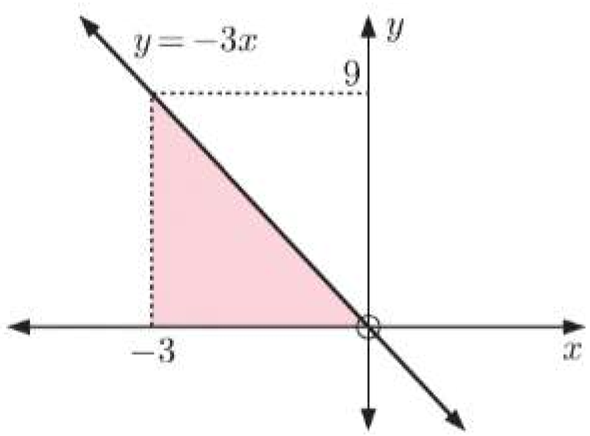
b



i Area = area of trapezium
$= \left(\frac{4+5}{2}\right) \times 1$
$= \frac{9}{2}$
$= 4\frac{1}{2} \text{ units}^2$

ii Area $= \displaystyle\int_{4}^{5} x\,dx$
$= \left[\frac{1}{2}x^2\right]_{4}^{5}$
$= \frac{1}{2}(5)^2 - \frac{1}{2}(4)^2$
$= \frac{25}{2} - \frac{16}{2}$
$= 4\frac{1}{2} \text{ units}^2$

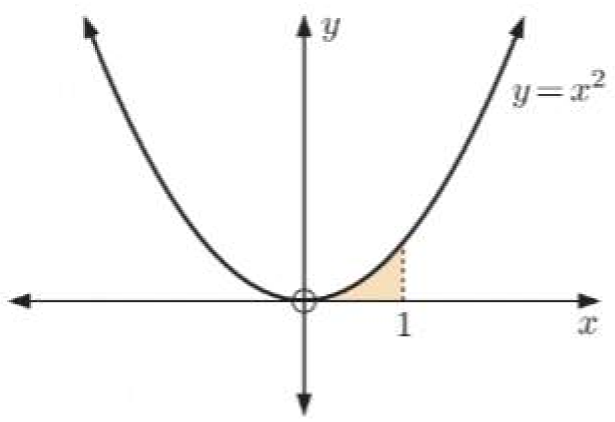
c



i Area $= \frac{1}{2} \times 3 \times 9$
$= 13\frac{1}{2} \text{ units}^2$

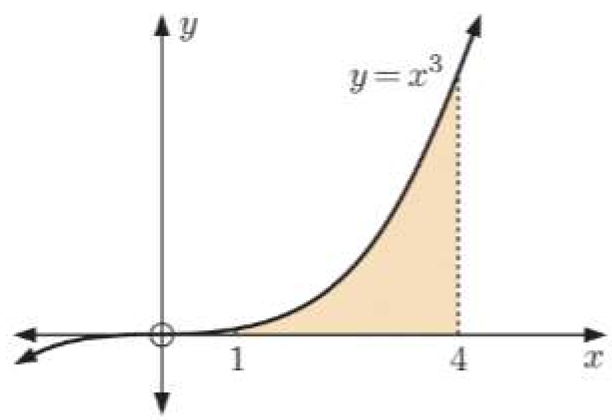
ii Area $= \displaystyle\int_{-3}^{0} -3x\,dx$
$= \left[-\frac{3}{2}x^2\right]_{-3}^{0}$
$= 0 - \left(-\frac{3}{2}(9)\right)$
$= 13\frac{1}{2} \text{ units}^2$

3 **a**



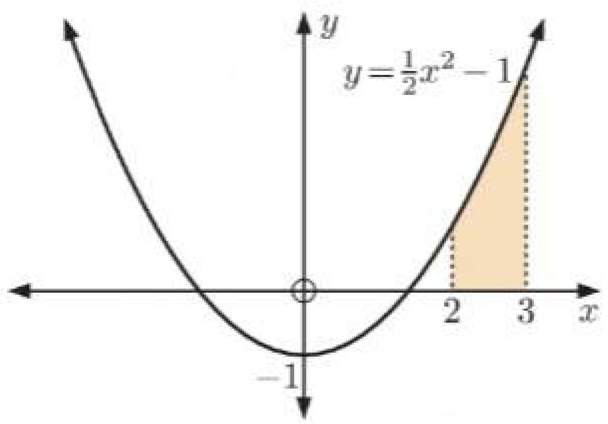
$$\text{Area} = \int_0^1 x^2\,dx$$
$$= \left[\tfrac{1}{3}x^3\right]_0^1$$
$$= \tfrac{1}{3} - 0$$
$$= \tfrac{1}{3}\ \text{units}^2$$

b



$$\text{Area} = \int_1^4 x^3\,dx$$
$$= \left[\tfrac{1}{4}x^4\right]_1^4$$
$$= 64 - \tfrac{1}{4}$$
$$= 63\tfrac{3}{4}\ \text{units}^2$$

c



$$\text{Area} = \int_2^3 \left(\tfrac{1}{2}x^2 - 1\right)dx$$
$$= \left[\tfrac{1}{6}x^3 - x\right]_2^3$$
$$= \left(\tfrac{27}{6} - 3\right) - \left(\tfrac{8}{6} - 2\right)$$
$$= 2\tfrac{1}{6}\ \text{units}^2$$

4 **a** A and B are the x-intercepts of $y = -x^2 + x + 6$.

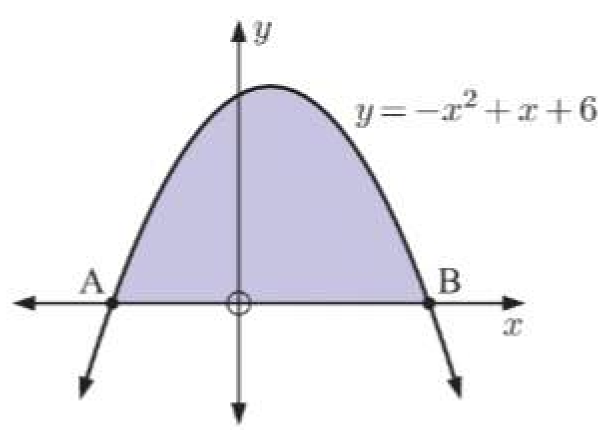
When $y = 0$, $-x^2 + x + 6 = 0$

$\therefore\ x^2 - x - 6 = 0$

$\therefore\ (x+2)(x-3) = 0$

$\therefore\ x = -2$ or 3

$\therefore$ A is $(-2, 0)$ and B is $(3, 0)$.



b $$\text{Area} = \int_{-2}^{3} (-x^2 + x + 6)\,dx$$
$$= \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 6x\right]_{-2}^{3}$$
$$= \left(-9 + \tfrac{9}{2} + 18\right) - \left(\tfrac{8}{3} + 2 - 12\right)$$
$$= 13\tfrac{1}{2} - \left(-7\tfrac{1}{3}\right)$$
$$= 20\tfrac{5}{6}\ \text{units}^2$$

5 **a** $y = -x^2 + 7x - 10$

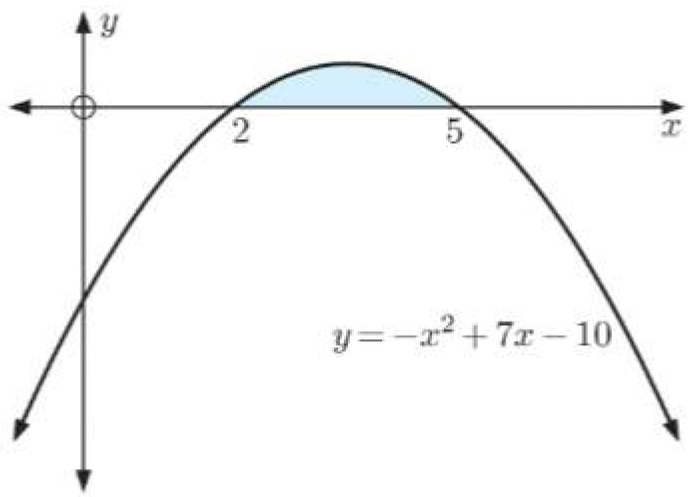
When $y = 0$, $-x^2 + 7x - 10 = 0$

$\therefore\ x^2 - 7x + 10 = 0$

$\therefore\ (x-2)(x-5) = 0$

$\therefore\ x = 2$ or 5

$\therefore$ the x-intercepts are 2 and 5.



$$\therefore \text{ enclosed area} = \int_2^5 (-x^2 + 7x - 10)\, dx$$
$$= \left[-\tfrac{1}{3}x^3 + \tfrac{7}{2}x^2 - 10x\right]_2^5$$
$$= \left(-\tfrac{125}{3} + \tfrac{175}{2} - 50\right) - \left(-\tfrac{8}{3} + 14 - 20\right)$$
$$= 4\tfrac{1}{2} \text{ units}^2$$

b $y = -2x^2 + 2x + 4$

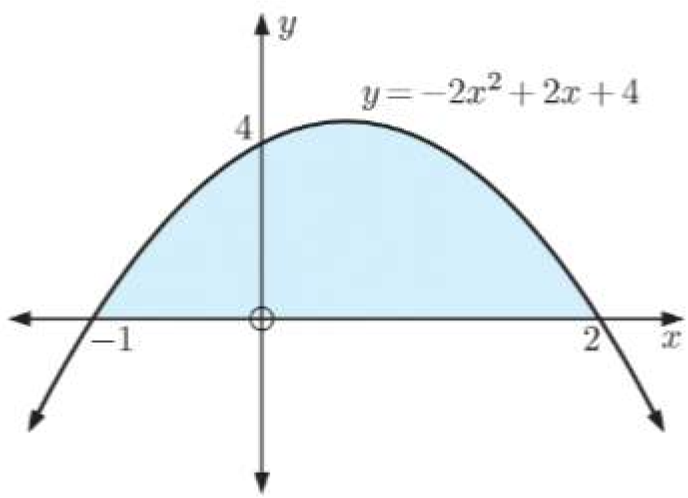
When $y = 0$, $-2x^2 + 2x + 4 = 0$

$\therefore\ -2(x^2 - x - 2) = 0$

$\therefore\ -2(x+1)(x-2) = 0$

$\therefore\ x = -1$ or 2

$\therefore$ the x-intercepts are -1 and 2.



$$\therefore \text{ enclosed area} = \int_{-1}^{2} (-2x^2 + 2x + 4)\, dx$$
$$= \left[-\tfrac{2}{3}x^3 + x^2 + 4x\right]_{-1}^{2}$$
$$= \left(-\tfrac{16}{3} + 4 + 8\right) - \left(\tfrac{2}{3} + 1 - 4\right)$$
$$= 9 \text{ units}^2$$

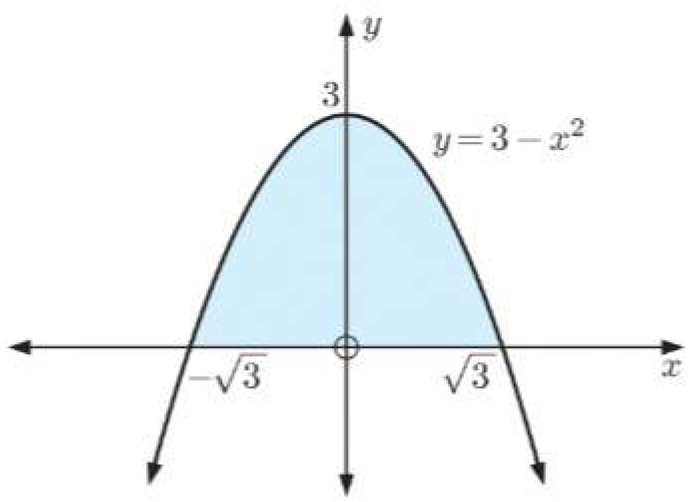
c $y = 3 - x^2$

When $y = 0$, $3 - x^2 = 0$

$\therefore\ x^2 = 3$

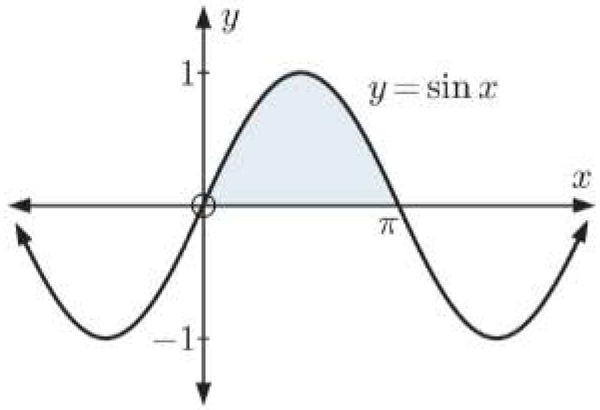
$\therefore\ x = \pm\sqrt{3}$

$\therefore$ the x-intercepts are $\sqrt{3}$ and $-\sqrt{3}$.



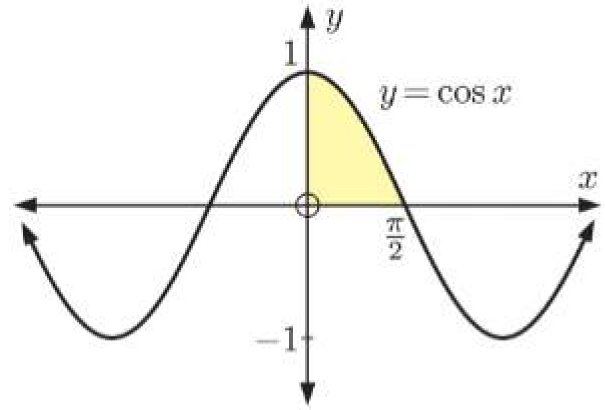
$$\therefore \text{ enclosed area} = \int_{-\sqrt{3}}^{\sqrt{3}} (3 - x^2)\, dx$$
$$= \left[3x - \tfrac{1}{3}x^3\right]_{-\sqrt{3}}^{\sqrt{3}}$$
$$= \left(3\sqrt{3} - \sqrt{3}\right) - \left(-3\sqrt{3} + \sqrt{3}\right)$$
$$= 4\sqrt{3} \text{ units}^2$$

6



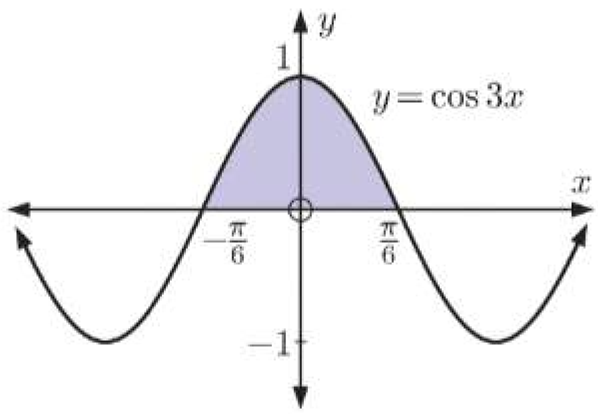
$$\begin{aligned}\text{Area} &= \int_0^{\pi} \sin x \, dx \\ &= \left[-\cos x\right]_0^{\pi} \\ &= [-\cos\pi + \cos 0] \\ &= -(-1)+1 \\ &= 2 \text{ units}^2\end{aligned}$$

7



$$\begin{aligned}\text{Area} &= \int_0^{\frac{\pi}{2}} \cos x \, dx \\ &= \left[\sin x\right]_0^{\frac{\pi}{2}} \\ &= \sin\tfrac{\pi}{2} - \sin 0 \\ &= 1 \text{ unit}^2\end{aligned}$$

8



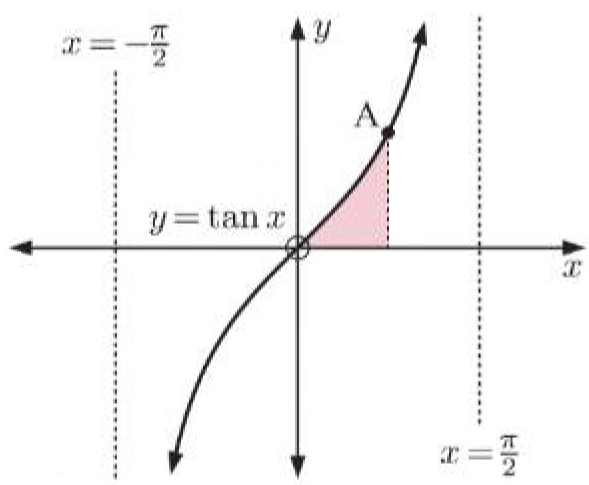
The period of $y = \cos 3x$ is $\frac{2\pi}{3}$, so the x-intercepts under one arch are $-\frac{\pi}{6}$ and $\frac{\pi}{6}$.

$$\begin{aligned}\text{The required area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x \, dx \\ &= \left[\tfrac{1}{3}\sin 3x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\ &= \tfrac{1}{3}\left(\sin\tfrac{\pi}{2} - \sin\left(-\tfrac{\pi}{2}\right)\right) \\ &= \tfrac{1}{3}(1-(-1)) \\ &= \tfrac{2}{3} \text{ units}^2\end{aligned}$$

9 **a** $y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y = 1$ when $x = \frac{\pi}{4}$

$\therefore$ A has x-coordinate $\frac{\pi}{4}$.



b
$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= -\int \frac{-\sin x}{\cos x}\, dx \\ &= -\int \frac{1}{u}\frac{du}{dx}\, dx \\ &\qquad \left\{u = \cos x,\quad \frac{du}{dx} = -\sin x\right\} \\ &= -\int \frac{1}{u}\, du \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c\end{aligned}$$

c Shaded area $= \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \Big[-\ln|\cos x|\Big]_0^{\frac{\pi}{4}}$

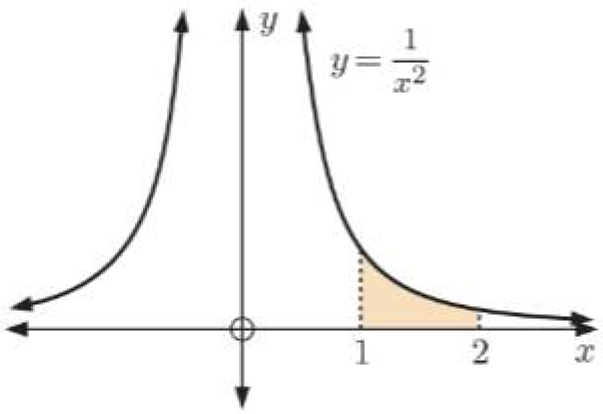
$= -\ln\left(\frac{1}{\sqrt{2}}\right) - (-\ln 1)$

$= -\ln(2^{-\frac{1}{2}})$

$= \ln(2^{\frac{1}{2}})$

$= \ln\sqrt{2}$ units^2

10 **a**



Area $= \int_1^2 \frac{1}{x^2} \, dx$

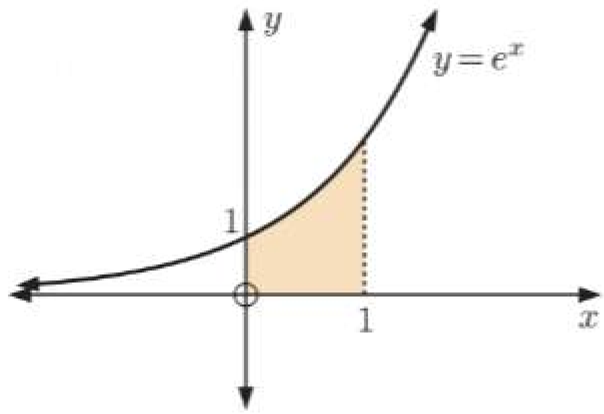
$= \int_1^2 x^{-2} \, dx$

$= \left[-\frac{1}{x}\right]_1^2$

$= -\frac{1}{2} - (-1)$

$= \frac{1}{2}$ units^2

b



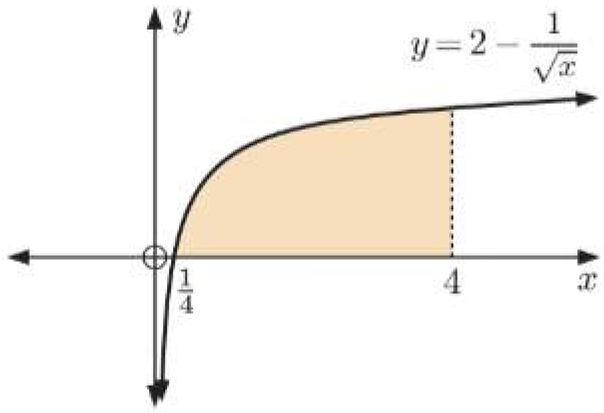
Area $= \int_0^1 e^x \, dx$

$= \left[e^x\right]_0^1$

$= e^1 - e^0$

$= (e-1)$ units^2

c



When $y = 0$, $2 - \dfrac{1}{\sqrt{x}} = 0$

$\therefore\ 2 = \dfrac{1}{\sqrt{x}}$

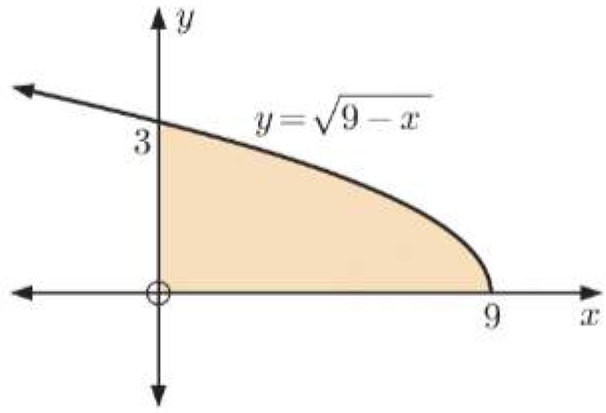
$\therefore\ \sqrt{x} = \frac{1}{2}$

$\therefore\ x = \frac{1}{4}\quad \{x > 0\}$

$\therefore$ the x-intercept is $\frac{1}{4}$.

$$\begin{aligned}\therefore\ \text{area} &= \int_{\frac{1}{4}}^{4} \left(2 - \frac{1}{\sqrt{x}}\right) dx \\ &= \int_{\frac{1}{4}}^{4} \left(2 - x^{-\frac{1}{2}}\right) dx \\ &= \left[2x - 2x^{\frac{1}{2}}\right]_{\frac{1}{4}}^{4} \\ &= (8 - 4) - \left(\tfrac{1}{2} - 1\right) \\ &= 4\tfrac{1}{2} \text{ units}^2\end{aligned}$$

d



When $y = 0$, $\sqrt{9 - x} = 0$

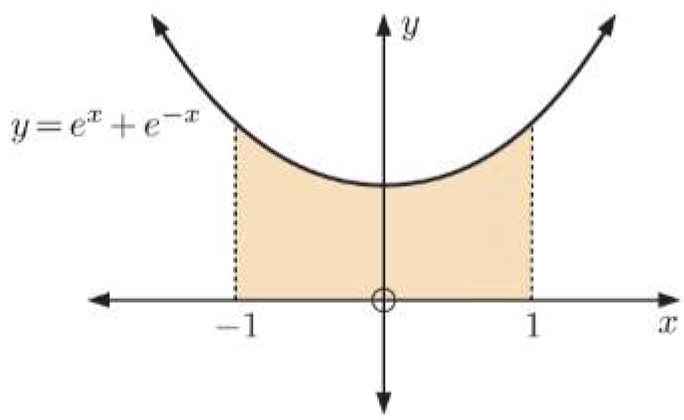
$\therefore\ 9 - x = 0$

$\therefore\ x = 9$

$\therefore$ the x-intercept is 9.

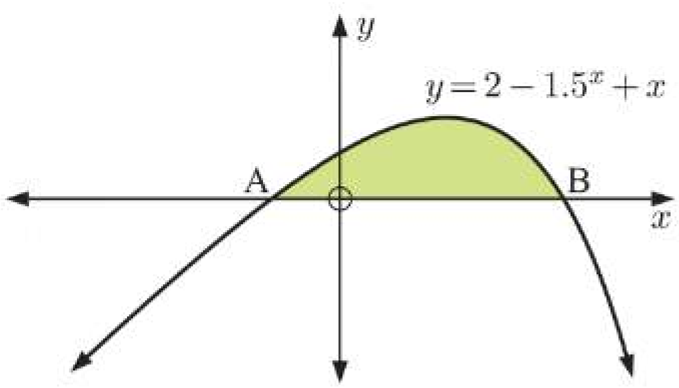
$$\begin{aligned}\therefore\ \text{area} &= \int_0^9 \sqrt{9 - x}\, dx \\ &= \int_0^9 (9 - x)^{\frac{1}{2}}\, dx \\ &= \left[-\tfrac{2}{3}(9 - x)^{\frac{3}{2}}\right]_0^9 \\ &= 0 - \left(-\tfrac{2}{3}(27)\right) \\ &= 18 \text{ units}^2\end{aligned}$$

e

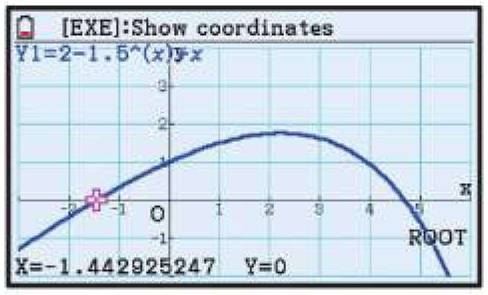


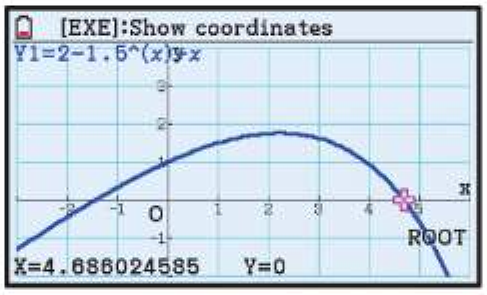
$$\begin{aligned}\text{Area} &= \int_{-1}^{1} (e^x + e^{-x})\, dx \\ &= \left[e^x - e^{-x}\right]_{-1}^{1} \\ &= \left(e - \frac{1}{e}\right) - \left(\frac{1}{e} - e\right) \\ &= \left(2e - \frac{2}{e}\right) \text{ units}^2\end{aligned}$$

11



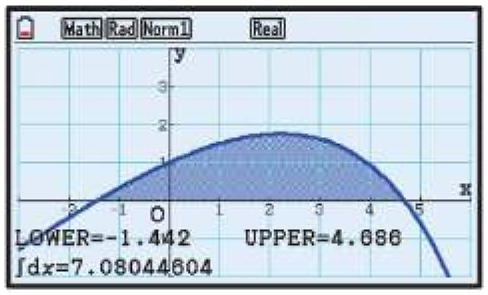
a





$\therefore$ the x-coordinate of A ≈ -1.44,
and the x-coordinate of B ≈ 4.69.

b



$\therefore$ the shaded area is about 7.08 units^2.

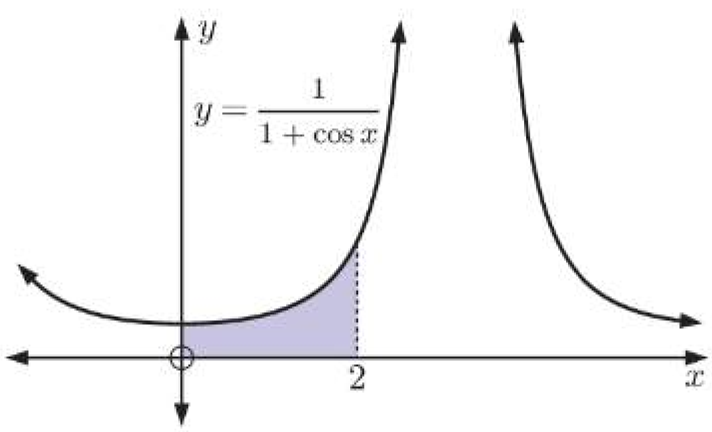
12 **a**

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) &= \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}\\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}\\ &= \frac{1+\cos x}{(1+\cos x)^2}\\ &= \frac{1}{1+\cos x}\end{aligned}$$

b

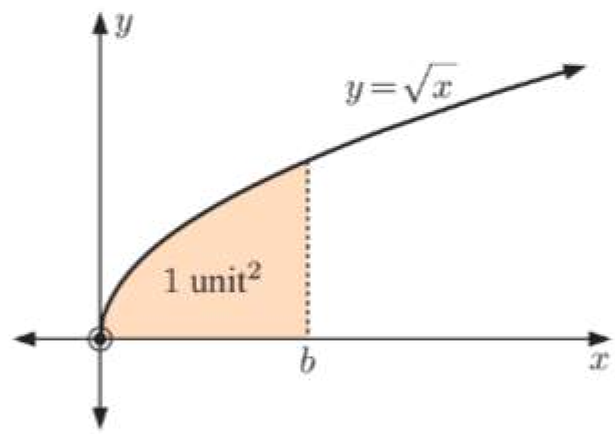
$$\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) = \frac{1}{1+\cos x}$$

$$\therefore \int \frac{1}{1+\cos x}\,dx = \frac{\sin x}{1+\cos x} + c$$



$$\begin{aligned}\therefore \text{ shaded area} &= \int_0^2 \frac{1}{1+\cos x}\,dx\\ &= \left[\frac{\sin x}{1+\cos x}\right]_0^2\\ &= \frac{\sin 2}{1+\cos 2} - \frac{\sin 0}{1+\cos 0} \text{ units}^2\\ &= \frac{\sin 2}{1+\cos 2} \text{ units}^2 \quad (\approx 1.56 \text{ units}^2)\end{aligned}$$

13 **a** Area $= \int_0^b \sqrt{x}\,dx$



$\therefore\ 1 = \int_0^b x^{\frac{1}{2}}\,dx$

$\therefore\ 1 = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^b$

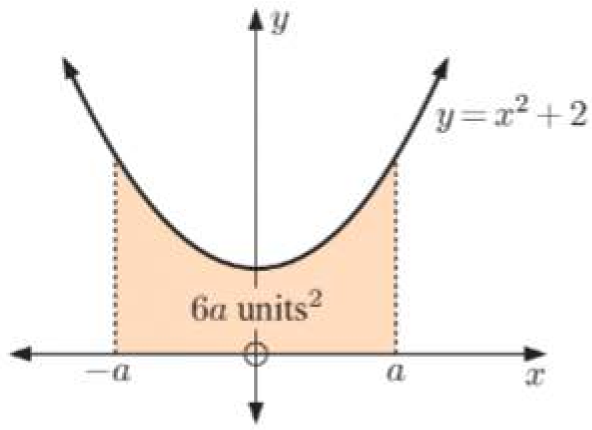
$\therefore\ 1 = \frac{2}{3}b^{\frac{3}{2}} - 0$

$\therefore\ \frac{3}{2} = b^{\frac{3}{2}}$

$\therefore\ b = \left(\frac{3}{2}\right)^{\frac{2}{3}}$

$\therefore\ b \approx 1.3104$

b Area $= \int_{-a}^{a} (x^2+2)\,dx$



$\therefore\ 6a = \left[\frac{1}{3}x^3 + 2x\right]_{-a}^{a}$

$= \left(\frac{1}{3}a^3 + 2a\right) - \left(-\frac{1}{3}a^3 - 2a\right)$

$= \frac{2}{3}a^3 + 4a$

$\therefore\ \frac{2}{3}a^3 - 2a = 0$

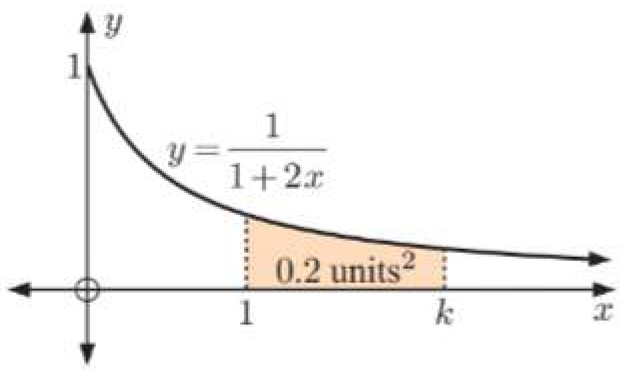
$\therefore\ a^3 - 3a = 0$

$\therefore\ a(a^2-3) = 0$

$\therefore\ a = 0,\ \pm\sqrt{3}$

but $a > 0$, $\therefore\ a = \sqrt{3}$

c Area $= \int_1^k \frac{1}{1+2x}\,dx$



$\therefore\ 0.2 = \left[\frac{1}{2}\ln|1+2x|\right]_1^k$

$= \frac{1}{2}\ln(1+2k) - \frac{1}{2}\ln 3 \quad \{k > 1\}$

$= \frac{1}{2}[\ln(1+2k) - \ln 3]$

$\therefore\ 0.4 = \ln\left(\frac{1+2k}{3}\right)$

$\therefore\ \frac{1+2k}{3} = e^{0.4}$

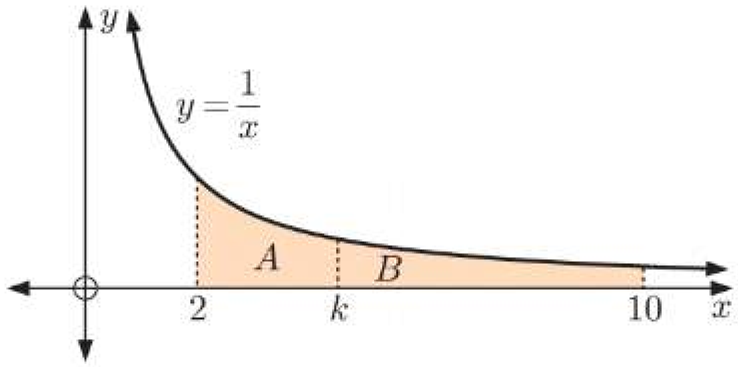
$\therefore\ 1+2k = 3e^{0.4}$

$\therefore\ 2k = 3e^{0.4} - 1$

$\therefore\ k = \frac{3e^{0.4}-1}{2} \approx 1.7377$

d Area of $A =$ Area of B

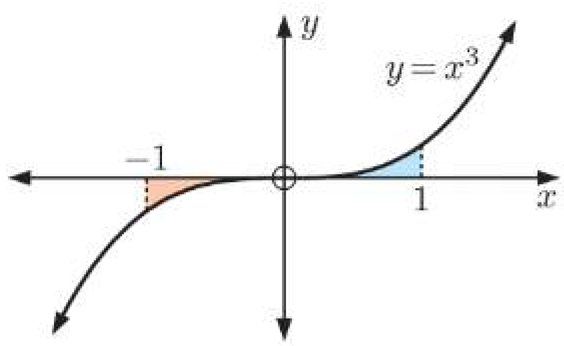
$$\therefore \int_2^k \frac{1}{x}\,dx = \int_k^{10} \frac{1}{x}\,dx$$
$$\therefore \big[\ln|x|\big]_2^k = \big[\ln|x|\big]_k^{10}$$
$$\therefore \ln k - \ln 2 = \ln 10 - \ln k \quad \{2 < k < 10\}$$
$$\therefore 2\ln k = \ln 10 + \ln 2$$
$$\therefore \ln k^2 = \ln 20$$
$$\therefore k^2 = 20$$
$$\therefore k = 2\sqrt{5} \quad \{2 < k < 10\}$$



INVESTIGATION $\int_a^b f(x)\,dx$ AND AREAS

1 **a** $\displaystyle\int_0^1 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{4} - 0 = \tfrac{1}{4}$ and $\displaystyle\int_{-1}^1 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{-1}^1 = \tfrac{1}{4} - \tfrac{1}{4}(-1)^4 = \tfrac{1}{4} - \tfrac{1}{4} = 0$

b Since the curve lies on or above the x-axis for $0 \leqslant x \leqslant 1$, the first integral in **a** is the area bounded by $y = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.
The second integral in **a** does *not* give an area as the curve lies on or below the x-axis for $-1 \leqslant x \leqslant 0$.



c $\displaystyle\int_{-1}^0 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{-1}^0 = 0 - \tfrac{1}{4}(-1)^4 = -\tfrac{1}{4}$

The answer is negative since the curve lies on or below the x-axis for $-1 \leqslant x \leqslant 0$.

d $\displaystyle\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = -\tfrac{1}{4} + \tfrac{1}{4}$ {using **a** and **c**}
$\displaystyle= 0 = \int_{-1}^1 x^3\,dx$

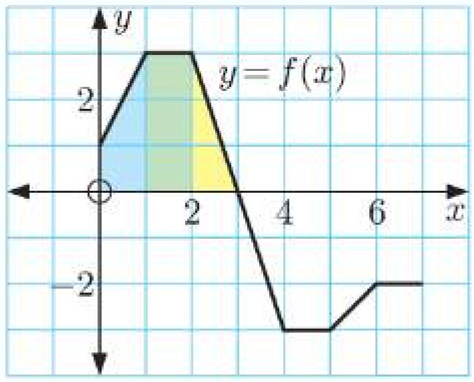
e $\displaystyle\int_0^{-1} x^3\,dx = -\int_{-1}^0 x^3\,dx = -\left(-\tfrac{1}{4}\right)$ {from **c**} $= \tfrac{1}{4}$

The area between the curve and the x-axis between $x = -1$ and $x = 0$ is $\frac{1}{4}$ units^2.

2 Area $= -\int_a^b f(x)\,dx$

3 **a** $\int_0^3 f(x)\,dx =$ area of blue trapezium + area of green rectangle + area of yellow triangle

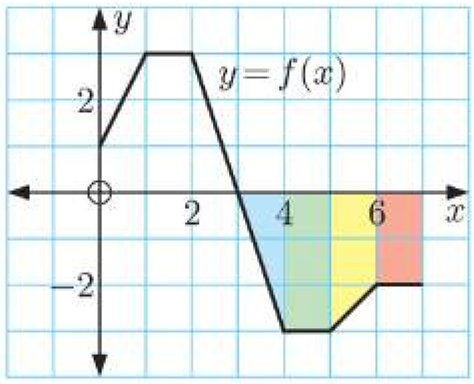
$$= \left(\frac{1+3}{2}\right)\times 1 + (1\times 3) + \left(\tfrac{1}{2}\times 1\times 3\right)$$
$$= 2+3+\tfrac{3}{2}$$
$$= \tfrac{13}{2}$$



b $\int_3^7 f(x)\,dx$

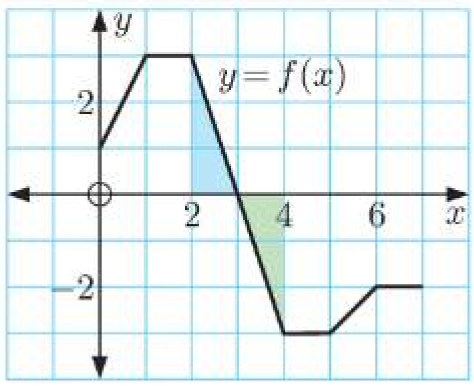
$= -($area of blue triangle + area of green rectangle + area of yellow trapezium + area of red rectangle$)$

$$= -\left[\left(\tfrac{1}{2}\times 1\times 3\right) + (1\times 3) + \left(\frac{2+3}{2}\right)\times 1 + (1\times 2)\right]$$
$$= -\left(\tfrac{3}{2}+3+\tfrac{5}{2}+2\right)$$
$$= -9$$



c $\int_2^4 f(x)\,dx =$ area of blue triangle − area of green triangle

$$= \left(\tfrac{1}{2}\times 1\times 3\right) - \left(\tfrac{1}{2}\times 1\times 3\right)$$
$$= 0$$



d $\int_0^7 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx$

$= \tfrac{13}{2} + (-9)$ {using **a** and **b**}

$= -\tfrac{5}{2}$

4 **a** $\int_0^4 f(x)\,dx =$ area of blue semi-circle

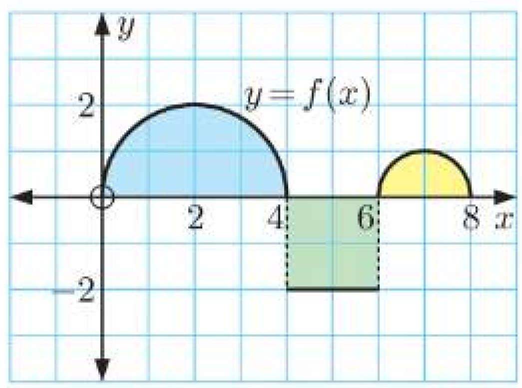
$$= \tfrac{1}{2}\times\pi\times 2^2$$
$$= 2\pi$$

b $\int_4^6 f(x)\,dx = -$area of green square

$$= -(2\times 2)$$
$$= -4$$

c $\int_6^8 f(x)\,dx =$ area of yellow semi-circle

$$= \tfrac{1}{2}\times\pi\times 1^2$$
$$= \tfrac{\pi}{2}$$



d $\int_0^8 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx + \int_6^8 f(x)\,dx$

$= 2\pi + (-4) + \frac{\pi}{2}$ {using **a**, **b**, and **c**}

$= \frac{5\pi}{2} - 4$

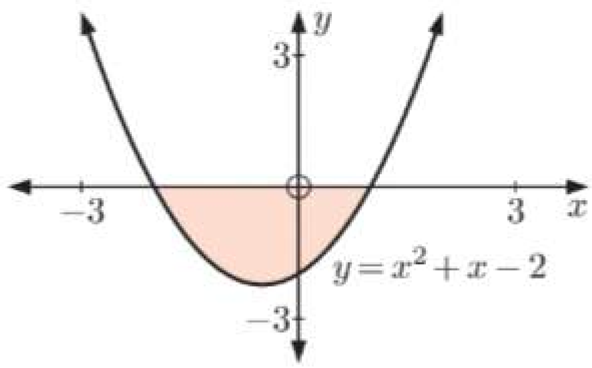
EXERCISE 23D

1 **a** The curve cuts the x-axis when $y = 0$

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x+2)(x-1) = 0$

$\therefore\ x = -2$ or 1

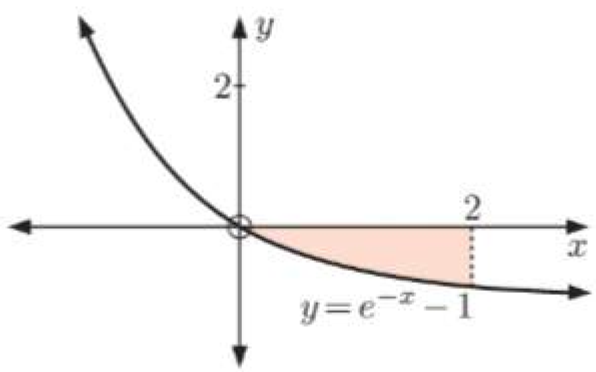


$\therefore$ the x-intercepts are -2 and 1.

$$\begin{aligned}\text{Area} &= -\int_{-2}^{1} (x^2 + x - 2)\,dx\\ &= -\left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 2x\right]_{-2}^{1}\\ &= -\left[\left(\tfrac{1}{3} + \tfrac{1}{2} - 2\right) - \left(-\tfrac{8}{3} + 2 + 4\right)\right]\\ &= -\left[-\tfrac{7}{6} - \tfrac{10}{3}\right]\\ &= 4\tfrac{1}{2}\ \text{units}^2\end{aligned}$$

b

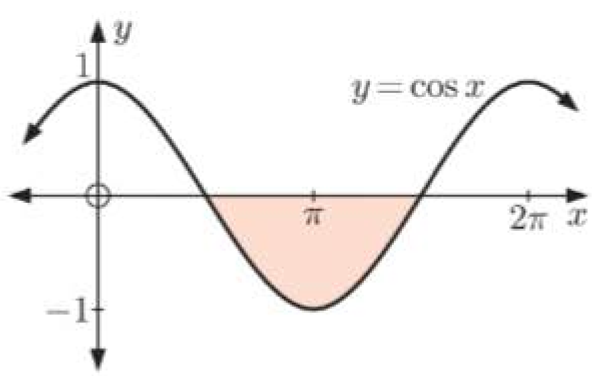
$$\begin{aligned}\text{Area} &= -\int_0^2 (e^{-x} - 1)\,dx\\ &= -\left[-e^{-x} - x\right]_0^2\\ &= -\left[(-e^{-2} - 2) - (-e^0 - 0)\right]\\ &= -(-e^{-2} - 1)\\ &= (1 + e^{-2})\ \text{units}^2\end{aligned}$$



c The curve cuts the x-axis when $y = 0$

$\therefore\ \cos x = 0$

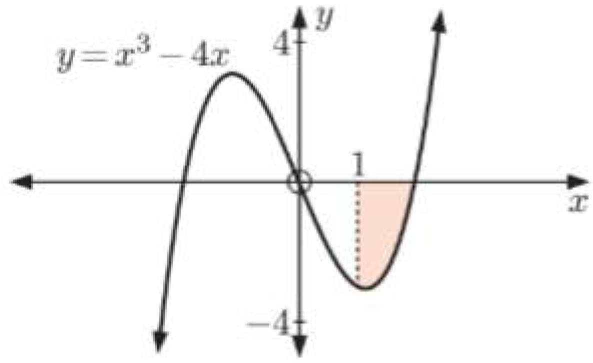
$\therefore\ x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$



$$\begin{aligned}\text{Area} &= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx\\ &= -\Big[\sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\\ &= -[(-1) - 1]\\ &= 2\ \text{units}^2\end{aligned}$$

d The curve cuts the x-axis when $y = 0$

$$\therefore \; x^3 - 4x = 0$$
$$\therefore \; x(x^2 - 4) = 0$$
$$\therefore \; x(x+2)(x-2) = 0$$



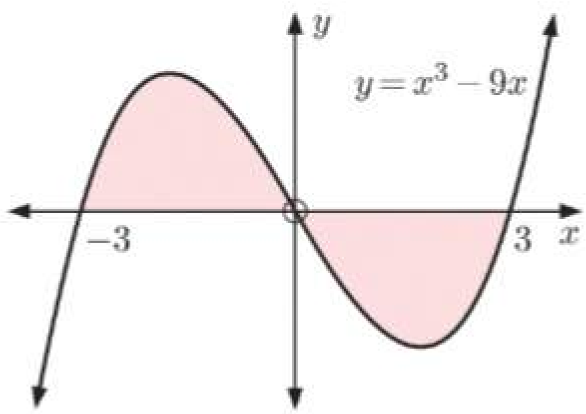
$\therefore$ the x-intercepts are -2, 0, and 2.

$$\begin{aligned}\text{Area} &= -\int_1^2 (x^3 - 4x)\,dx \\ &= -\left[\tfrac{1}{4}x^4 - 2x^2\right]_1^2 \\ &= -\left[(4-8) - \left(\tfrac{1}{4} - 2\right)\right] \\ &= -\left[-4 - \left(-\tfrac{7}{4}\right)\right] \\ &= 2\tfrac{1}{4} \text{ units}^2\end{aligned}$$

2 **a** $f(x) = x^3 - 9x$

$$\begin{aligned} &= x(x^2 - 9) \\ &= x(x+3)(x-3)\end{aligned}$$

$\therefore$ $y = f(x)$ cuts the x-axis at -3, 0, and 3.

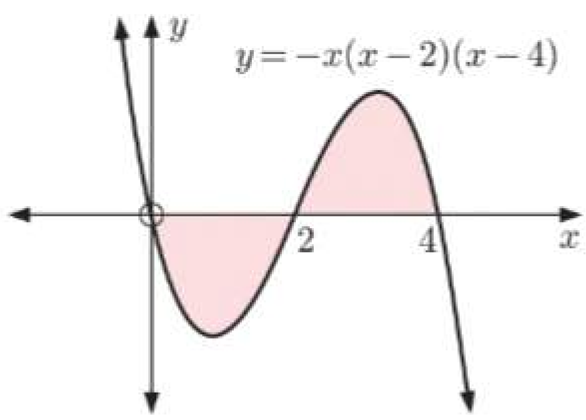


$$\begin{aligned}\text{Total area} &= \int_{-3}^0 (x^3 - 9x)\,dx - \int_0^3 (x^3 - 9x)\,dx \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right]_{-3}^0 - \left[\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right]_0^3 \\ &= \left(0 - \left(\tfrac{81}{4} - \tfrac{81}{2}\right)\right) - \left(\left(\tfrac{81}{4} - \tfrac{81}{2}\right) - 0\right) \\ &= 40\tfrac{1}{2} \text{ units}^2\end{aligned}$$

b $f(x) = -x(x-2)(x-4)$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 2, and 4.

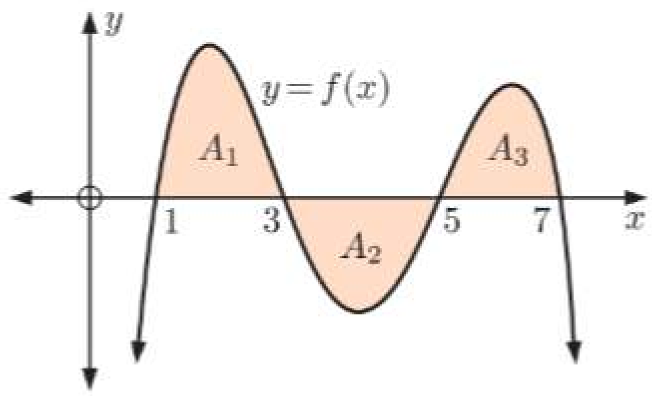
$$\begin{aligned} f(x) &= -x(x-2)(x-4) \\ &= -x(x^2 - 6x + 8) \\ &= -x^3 + 6x^2 - 8x\end{aligned}$$



$$\begin{aligned}\text{Total area} &= -\int_0^2 (-x^3 + 6x^2 - 8x)\,dx + \int_2^4 (-x^3 + 6x^2 - 8x)\,dx \\ &= -\left[-\tfrac{1}{4}x^4 + 2x^3 - 4x^2\right]_0^2 + \left[-\tfrac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4 \\ &= -((-4 + 16 - 16) - 0) + ((-64 + 128 - 64) - (-4 + 16 - 16)) \\ &= 8 \text{ units}^2\end{aligned}$$

3 **a** $\int_1^7 f(x)\,dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leqslant x \leqslant 7$.

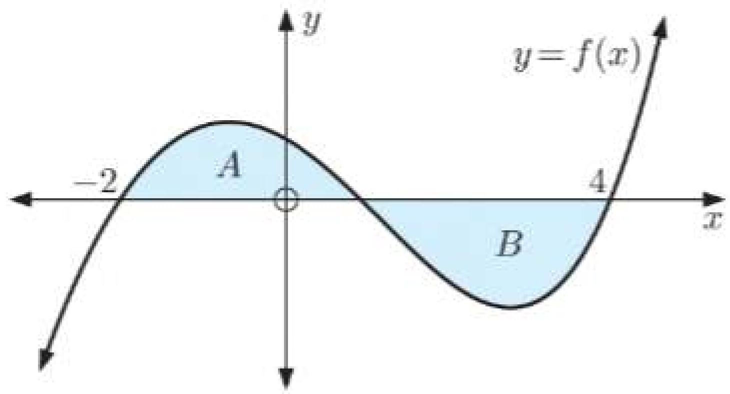
But $f(x)$ is not positive for $3 \leqslant x \leqslant 5$, so $\int_1^7 f(x)\,dx = A_1 - A_2 + A_3$ which is *not* the shaded area.



b Total shaded area $= \int_1^3 f(x)\,dx + \int_3^5 (-f(x))\,dx + \int_5^7 f(x)\,dx$

$$= \int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$$

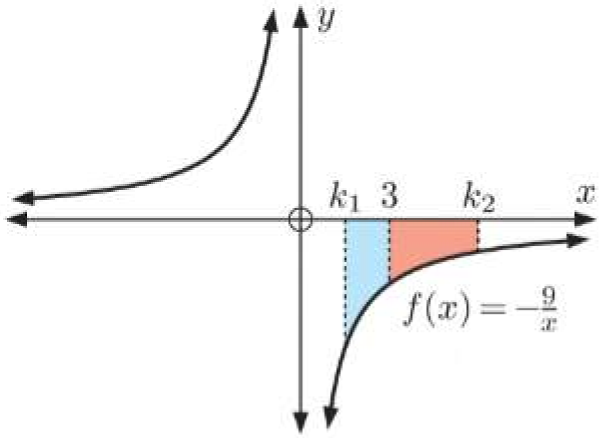
4



$$\int_{-2}^4 f(x)\,dx = \text{area of region } A + (-\text{area of region } B) \qquad \{\text{since region } B \text{ is below the } x\text{-axis}\}$$
$$= \text{area of region } A - \text{area of region } B$$
$$= -6$$

$\therefore$ area of region $B >$ area of region A

5



Blue area = red area = $9\ln 2$ units^2

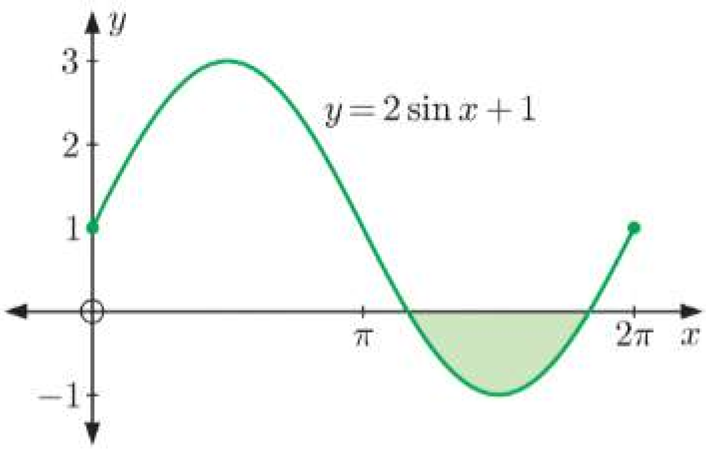
$$\therefore\ -\int_{k_1}^{3} -\frac{9}{x}\,dx = 9\ln 2$$
$$\therefore\ \int_{k_1}^{3} \frac{9}{x}\,dx = 9\ln 2$$
$$\therefore\ \left[9\ln|x|\right]_{k_1}^{3} = 9\ln 2$$
$$\therefore\ 9\ln 3 - 9\ln k_1 = 9\ln 2 \quad \{k_1 > 0\}$$
$$\therefore\ \ln 3 - \ln k_1 = \ln 2$$
$$\therefore\ \ln k_1 = \ln 3 - \ln 2$$
$$\therefore\ \ln k_1 = \ln\left(\tfrac{3}{2}\right)$$
$$\therefore\ k_1 = \tfrac{3}{2}$$

or

$$-\int_{3}^{k_2} -\frac{9}{x}\,dx = 9\ln 2$$
$$\therefore\ \int_{3}^{k_2} \frac{9}{x}\,dx = 9\ln 2$$
$$\therefore\ \left[9\ln|x|\right]_{3}^{k_2} = 9\ln 2$$
$$\therefore\ 9\ln k_2 - 9\ln 3 = 9\ln 2 \quad \{k_2 > 0\}$$
$$\therefore\ \ln k_2 - \ln 3 = \ln 2$$
$$\therefore\ \ln k_2 = \ln 2 + \ln 3$$
$$\therefore\ \ln k_2 = \ln 6$$
$$\therefore\ k_2 = 6$$

So, $k = \frac{3}{2}$ or 6

6 **a**



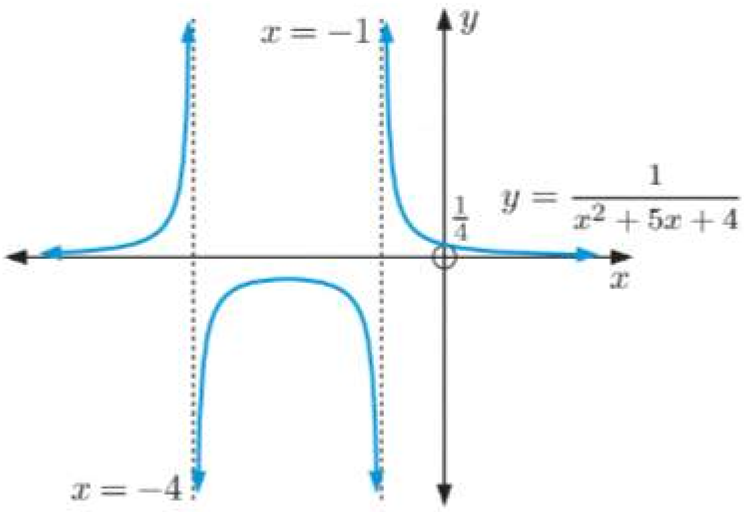
b The curve cuts the x-axis when $y = 0$

$$\therefore\ 2\sin x + 1 = 0$$
$$\therefore\ 2\sin x = -1$$
$$\therefore\ \sin x = -\tfrac{1}{2}$$
$$\therefore\ x = -\tfrac{\pi}{6},\ \tfrac{7\pi}{6},\ \tfrac{11\pi}{6},\ ....$$

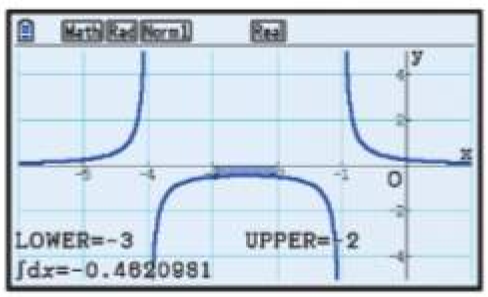
$\therefore$ the x-intercepts are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$$\begin{aligned}
\text{Area} &= -\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (2\sin x + 1)\, dx \\
&= -\Big[-2\cos x + x\Big]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\
&= -\left[\left(-2\cos\tfrac{11\pi}{6} + \tfrac{11\pi}{6}\right) - \left(-2\cos\tfrac{7\pi}{6} + \tfrac{7\pi}{6}\right)\right] \\
&= -\left[\left(-\sqrt{3} + \tfrac{11\pi}{6}\right) - \left(\sqrt{3} + \tfrac{7\pi}{6}\right)\right] \\
&= \left(2\sqrt{3} - \tfrac{2\pi}{3}\right) \text{ units}^2 \\
&\approx 1.37 \text{ units}^2
\end{aligned}$$

7 **a**



b

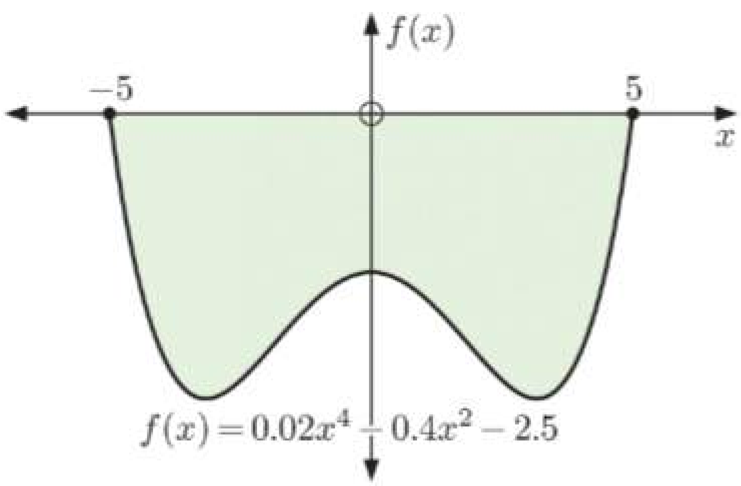


$$\therefore \int_{-3}^{-2} \frac{1}{x^2 + 5x + 4}\, dx \approx -0.462$$

Now, $y = f(x)$ is below the x-axis on this interval, so the area bounded by $y = f(x)$, the x-axis, $x = -3$, and $x = -2$ is about 0.462 units^2.

8 **a** Cross-sectional area of gutter

$$\begin{aligned}
&= -\int_{-5}^{5} (0.02x^4 - 0.4x^2 - 2.5)\, dx \\
&= -\left[0.004x^5 - \tfrac{0.4}{3}x^3 - 2.5x\right]_{-5}^{5} \\
&= -\left[-16\tfrac{2}{3} - 16\tfrac{2}{3}\right] \\
&= 33\tfrac{1}{3} \text{ cm}^2
\end{aligned}$$



b Volume of gutter

$= \text{area of cross-section} \times \text{length of gutter}$

$= 33\frac{1}{3}\ \text{cm}^2 \times 20\ \text{m}$

$= 33\frac{1}{3}\ \text{cm}^2 \times 2000\ \text{cm}$

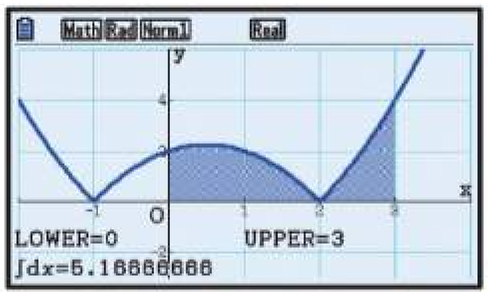
$= 66\,666\frac{2}{3}\ \text{cm}^3$

$\therefore$ the gutter can hold $66\,666\frac{2}{3}$ mL $\{1\ \text{cm}^3 \equiv 1\ \text{mL}\}$

$= 66\frac{2}{3}$ L of water in total.

ACTIVITY 1 CALCULATING AREAS USING TECHNOLOGY

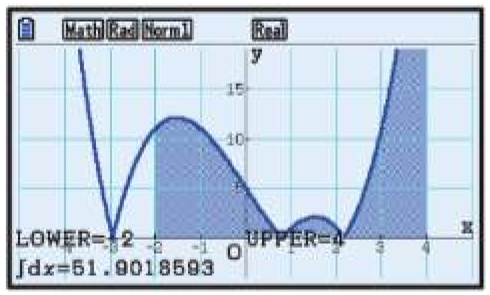
1 a



$$\therefore \int_1^3 |\,x^2 - x - 2\,|\,dx \approx 5.17$$

So, the area is about 5.17 units^2.

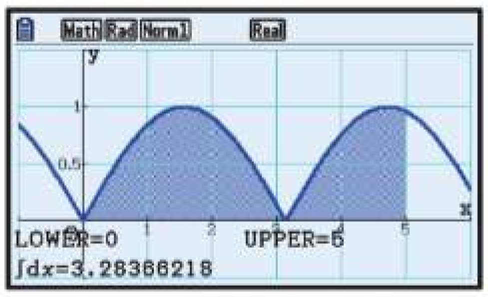
b



$$\therefore \int_{-2}^4 |\,x^3 - 7x + 5\,|\,dx \approx 51.9$$

So, the area is about 51.9 units^2.

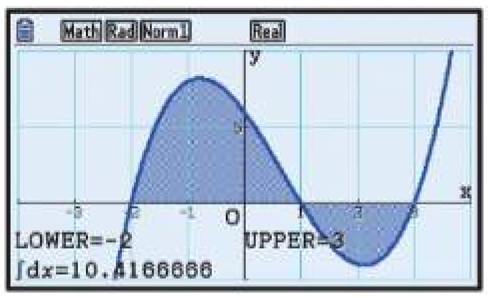
c



$$\therefore \int_0^5 |\,\sin x\,|\,dx \approx 3.28$$

So, the area is about 3.28 units^2.

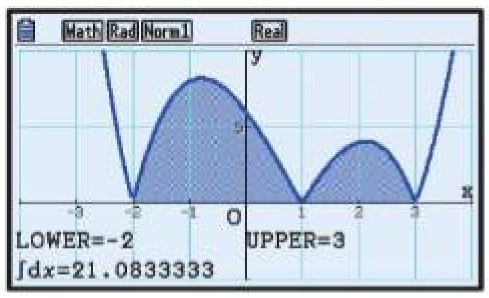
2 a i



$$\therefore \int_{-2}^3 (x^3 - 2x^2 - 5x + 6)\,dx$$

≈ 10.4

ii

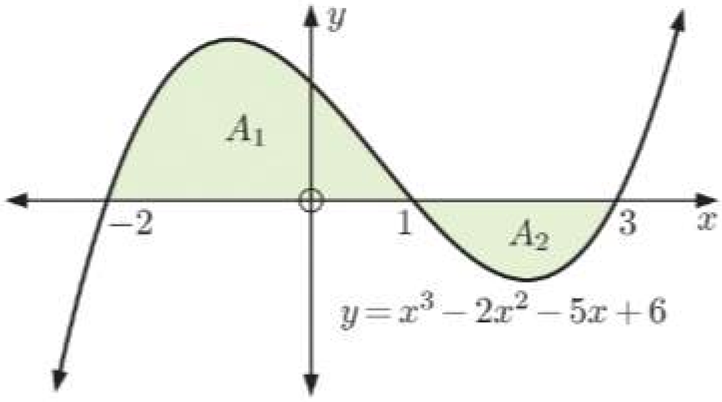


$$\therefore \int_{-2}^3 |\,x^3 - 2x^2 - 5x + 6\,|\,dx$$

≈ 21.1

b $\displaystyle\int_{-2}^{3} (x^3 - 2x^2 - 5x + 6)\,dx = A_1 - A_2$

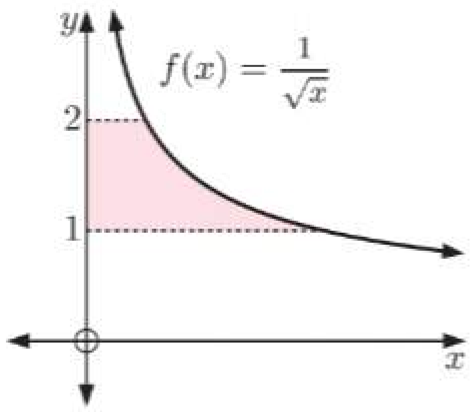
$\displaystyle\int_{-2}^{3} \left| x^3 - 2x^2 - 5x + 6 \right| dx = A_1 + A_2$



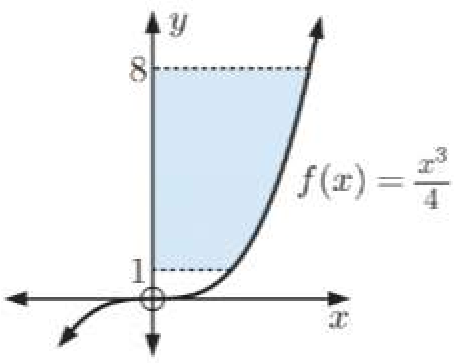
EXERCISE 23E

1 **a**
$$y = \frac{1}{\sqrt{x}}$$
$$\therefore\ y^2 = \frac{1}{x}$$
$$\therefore\ x = \frac{1}{y^2}$$
$$\therefore\ f^{-1}(y) = \frac{1}{y^2}$$

b
$$\text{Area} = \int_1^2 \frac{1}{y^2}\,dy$$
$$= \left[-y^{-1}\right]_1^2$$
$$= -\tfrac{1}{2} - (-1)$$
$$= \tfrac{1}{2} \text{ units}^2$$

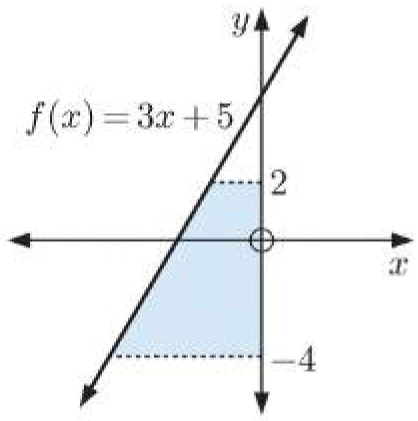


2 **a**
$$y = \frac{x^3}{4}$$
$$\therefore\ x^3 = 4y$$
$$\therefore\ x = \sqrt[3]{4y}$$
$$\therefore\ f^{-1}(y) = \sqrt[3]{4y}$$

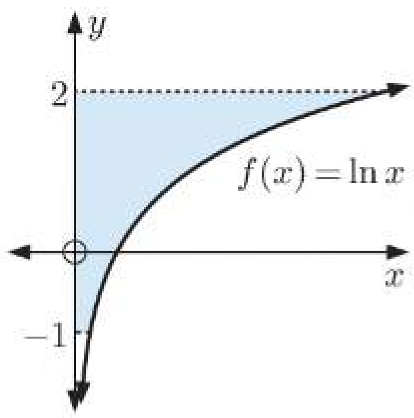


$$\therefore\ \text{area} = \int_1^8 (4y)^{\frac{1}{3}}\,dy$$
$$= 4^{\frac{1}{3}} \int_1^8 y^{\frac{1}{3}}\,dy$$
$$= 4^{\frac{1}{3}} \left[\tfrac{3}{4} y^{\frac{4}{3}}\right]_1^8$$
$$= 4^{\frac{1}{3}} \left(\tfrac{3}{4}(8)^{\frac{4}{3}} - \tfrac{3}{4}\right)$$
$$= 2^{\frac{2}{3}} \times \tfrac{45}{4}$$
$$= 2^{\frac{2}{3}} \times \frac{45}{2^2}$$
$$= \frac{45}{2^{\frac{4}{3}}}$$
$$= \frac{45}{2 \times \sqrt[3]{2}} \text{ units}^2$$

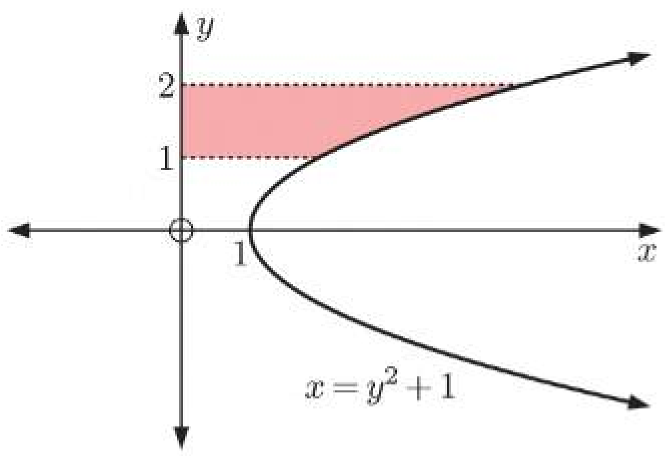
b
$$y = 3x + 5$$
$$\therefore\ 3x = y - 5$$
$$\therefore\ x = \frac{y-5}{3}$$
$$\therefore\ f^{-1}(y) = \tfrac{1}{3}y - \tfrac{5}{3}$$
$$\therefore\ \text{area} = -\int_{-4}^{2} \left(\tfrac{1}{3}y - \tfrac{5}{3}\right) dy$$
$$= -\left[\tfrac{1}{6}y^2 - \tfrac{5}{3}y\right]_{-4}^{2}$$
$$= -\left[\left(\tfrac{1}{6}(2)^2 - \tfrac{5}{3}(2)\right) - \left(\tfrac{1}{6}(-4)^2 - \tfrac{5}{3}(-4)\right)\right]$$
$$= -[-\tfrac{8}{3} - (\tfrac{28}{3})]$$
$$= 12 \text{ units}^2$$



c
$$y = \ln x$$
$$\therefore\ x = e^y$$
$$\therefore\ f^{-1}(y) = e^y$$
$$\therefore\ \text{area} = \int_{-1}^{2} e^y\, dy$$
$$= \left[e^y\right]_{-1}^{2}$$
$$= e^2 - e^{-1}$$
$$= e^2 - \frac{1}{e}$$
$$= \frac{e^3 - 1}{e} \text{ units}^2$$



3 **a**
$$x = y^2 + 1$$
$$\therefore\ f^{-1}(y) = y^2 + 1$$
$$\therefore\ \text{area} = \int_{1}^{2} (y^2 + 1)\, dy$$
$$= \left[\tfrac{1}{3}y^3 + y\right]_{1}^{2}$$
$$= (\tfrac{8}{3} + 2) - (\tfrac{1}{3} + 1)$$
$$= \tfrac{7}{3} + 1$$
$$= 3\tfrac{1}{3} \text{ units}^2$$



b $x = \sqrt{y+5}$

$\therefore \ f^{-1}(y) = (y+5)^{\frac{1}{2}}$

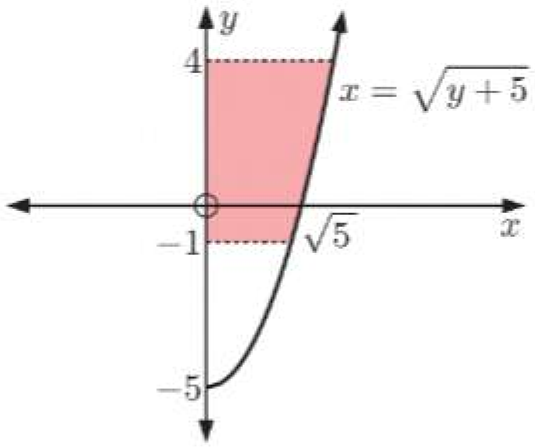
$\therefore$ area $= \int_{-1}^{4} (y+5)^{\frac{1}{2}}\, dy$

$= \left[\frac{2}{3}(y+5)^{\frac{3}{2}}\right]_{-1}^{4}$

$= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(4)^{\frac{3}{2}}$

$= \frac{38}{3}$

$= 12\frac{2}{3}$ units^2



4 a $y = \sqrt{x}$

$\therefore \ x = y^2$

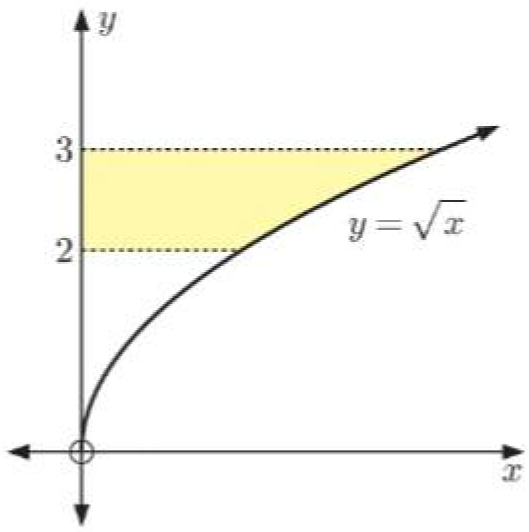
$\therefore \ f^{-1}(y) = y^2$

$\therefore$ area $= \int_{2}^{3} y^2\, dy$

$= \left[\frac{1}{3}y^3\right]_2^3$

$= \frac{1}{3} \times 27 - \frac{1}{3} \times 8$

$= \frac{19}{3} = 6\frac{1}{3}$ units^2



b $y = \frac{2}{x}$

$\therefore \ x = \frac{2}{y}$

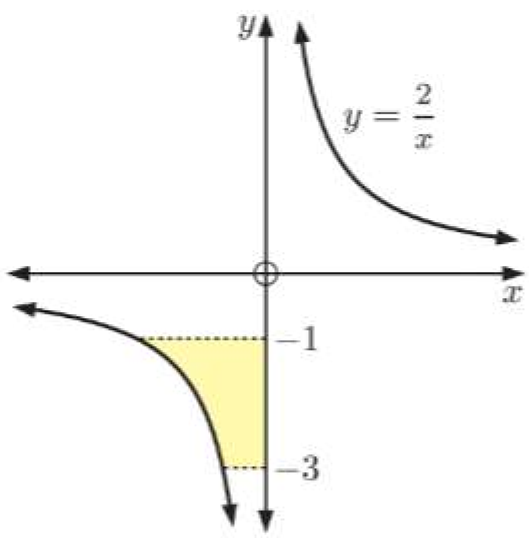
$\therefore \ f^{-1}(y) = \frac{2}{y}$

$\therefore$ area $= -\int_{-3}^{-1} \frac{2}{y}\, dy$

$= -\left[2\ln|y|\right]_{-3}^{-1}$

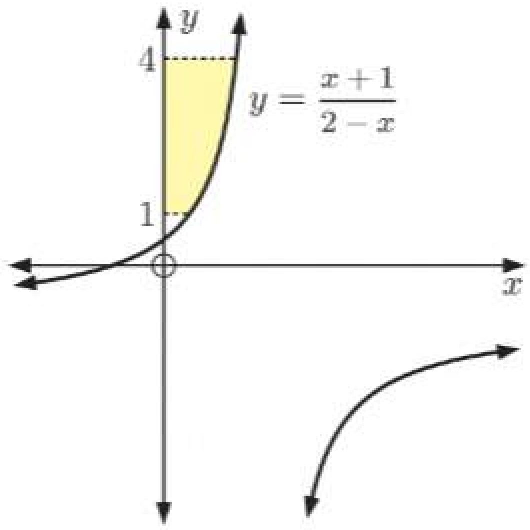
$= -(2\ln 1 - 2\ln 3)$

$= 2\ln 3$ units^2



c

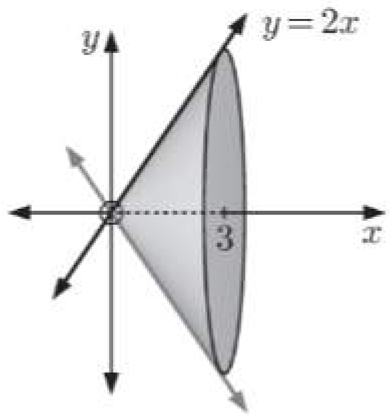
$$y = \frac{x+1}{2-x}$$
$$\therefore\ 2y - xy = x + 1$$
$$\therefore\ xy + x = 2y - 1$$
$$\therefore\ x(y+1) = 2y - 1$$
$$\therefore\ x = \frac{2y-1}{y+1}$$
$$\therefore\ x = \frac{2(y+1)-3}{y+1}$$
$$\therefore\ f^{-1}(y) = 2 - \frac{3}{y+1}$$



$$\therefore\ \text{area} = \int_1^4 \left(2 - \frac{3}{y+1}\right) dy$$
$$= \left[2y - 3\ln|y+1|\right]_1^4$$
$$= 8 - 3\ln 5 - (2 - 3\ln 2)$$
$$= 6 - 3\ln\tfrac{5}{2} \text{ units}^2$$

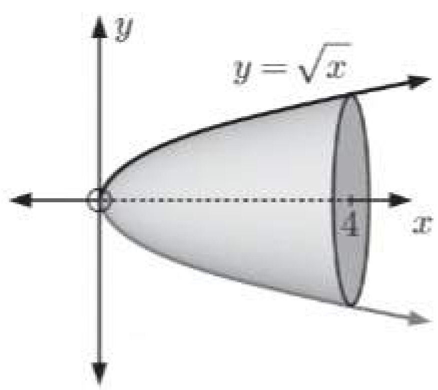
EXERCISE 23F.1

1 **a**



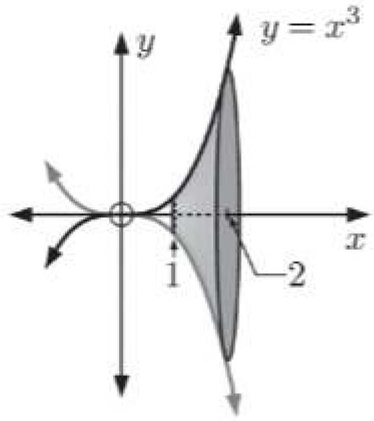
$$\text{Volume} = \pi\int_0^3 y^2\,dx$$
$$= \pi\int_0^3 (2x)^2\,dx$$
$$= 4\pi\int_0^3 x^2\,dx$$
$$= 4\pi\left[\tfrac{1}{3}x^3\right]_0^3$$
$$= 4\pi(9-0)$$
$$= 36\pi \text{ units}^3$$

b



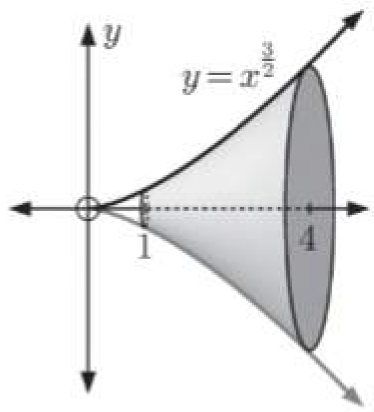
$$\text{Volume} = \pi\int_0^4 y^2\,dx$$
$$= \pi\int_0^4 (\sqrt{x})^2\,dx$$
$$= \pi\int_0^4 x\,dx$$
$$= \pi\left[\tfrac{1}{2}x^2\right]_0^4$$
$$= \pi(8-0)$$
$$= 8\pi \text{ units}^3$$

c



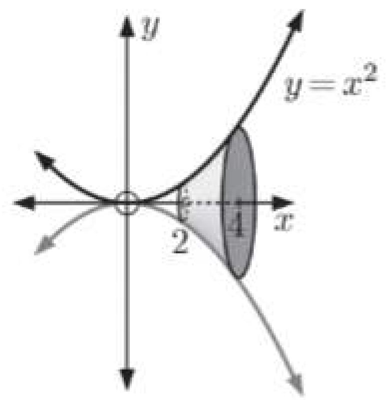
$$\text{Volume} = \pi\int_1^2 y^2\,dx$$
$$= \pi\int_1^2 (x^3)^2\,dx$$
$$= \pi\int_1^2 x^6\,dx$$
$$= \pi\left[\tfrac{1}{7}x^7\right]_1^2$$
$$= \pi\left(\tfrac{128}{7} - \tfrac{1}{7}\right)$$
$$= \tfrac{127\pi}{7} \text{ units}^3$$

d



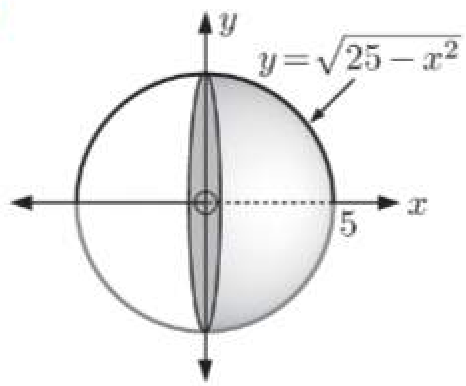
$$\begin{aligned}\text{Volume} &= \pi\int_1^4 y^2\,dx\\ &= \pi\int_1^4 (x^{\frac{3}{2}})^2\,dx\\ &= \pi\int_1^4 x^3\,dx\\ &= \pi\left[\tfrac{1}{4}x^4\right]_1^4\\ &= \pi\left(\tfrac{256}{4}-\tfrac{1}{4}\right)\\ &= \tfrac{255\pi}{4}\text{ units}^3\end{aligned}$$

e



$$\begin{aligned}\text{Volume} &= \pi\int_2^4 y^2\,dx\\ &= \pi\int_2^4 (x^2)^2\,dx\\ &= \pi\int_2^4 x^4\,dx\\ &= \pi\left[\tfrac{1}{5}x^5\right]_2^4\\ &= \pi\left(\tfrac{1024}{5}-\tfrac{32}{5}\right)\\ &= \tfrac{992\pi}{5}\text{ units}^3\end{aligned}$$

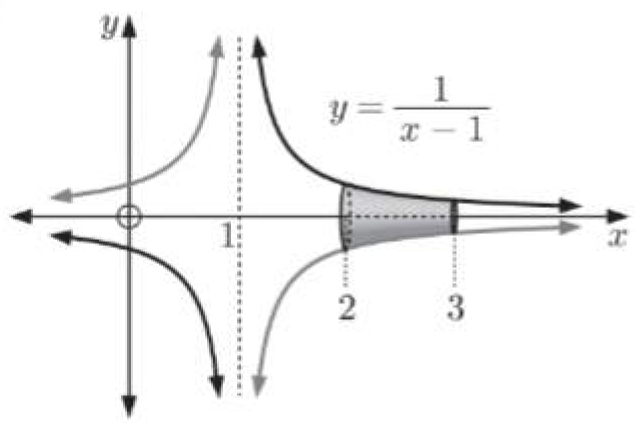
f



Volume

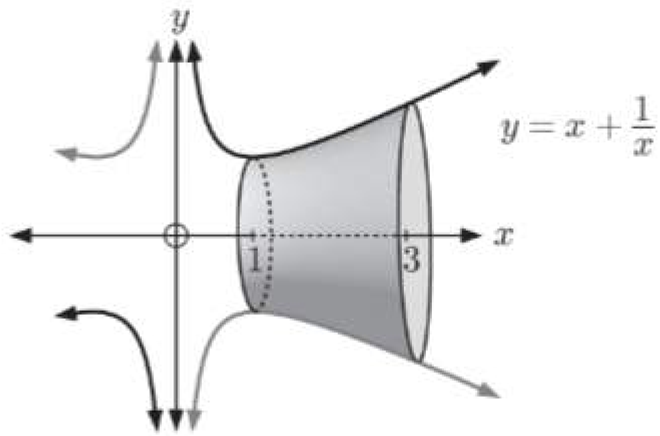
$$\begin{aligned}&= \pi\int_0^5 y^2\,dx\\ &= \pi\int_0^5 (25-x^2)\,dx\\ &= \pi\left[25x-\frac{x^3}{3}\right]_0^5\\ &= \pi\left(125-\tfrac{125}{3}\right)\\ &= \pi\left(\tfrac{2}{3}\right)125\\ &= \tfrac{250\pi}{3}\text{ units}^3\end{aligned}$$

g



$$\begin{aligned}\text{Volume} &= \pi\int_2^3 y^2\,dx\\ &= \pi\int_2^3 \left(\frac{1}{x-1}\right)^2 dx\\ &= \pi\int_2^3 (x-1)^{-2}\,dx\\ &= \pi\left[-\frac{1}{x-1}\right]_2^3\\ &= \pi(-\tfrac{1}{2}+1)\\ &= \tfrac{\pi}{2}\text{ units}^3\end{aligned}$$

h



$$\begin{aligned}\text{Volume} &= \pi\int_1^3 y^2\,dx\\ &= \pi\int_1^3 \left(x+\frac{1}{x}\right)^2 dx\\ &= \pi\int_1^3 (x^2+2+x^{-2})\,dx\\ &= \pi\left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_1^3\\ &= \pi[(9+6-\tfrac{1}{3})-(\tfrac{1}{3}+2-1)]\\ &= \tfrac{40\pi}{3}\text{ units}^3\end{aligned}$$

2 **a** Volume $= \pi \int_0^6 y^2\, dx$

$= \pi \int_0^6 \left(\frac{x}{2}+4\right)^2 dx$

$= \pi \int_0^6 \left(\frac{1}{4}x^2+4x+16\right) dx$

$= \pi \left[\frac{x^3}{12}+\frac{4x^2}{2}+16x\right]_0^6$

$= \pi[(18+72+96)-0]$

$= 186\pi$ units^3

b Volume

$= \pi \int_1^2 y^2\, dx$

$= \pi \int_1^2 (x^2+3)^2\, dx$

$= \pi \int_1^2 (x^4+6x^2+9)\, dx$

$= \pi \left[\frac{x^5}{5}+\frac{6x^3}{3}+9x\right]_1^2$

$= \pi\left[\left(\frac{32}{5}+16+18\right)-\left(\frac{1}{5}+2+9\right)\right]$

$= \pi\left(\frac{146}{5}\right)$

$= \frac{146\pi}{5}$ units^3

c Volume $= \pi \int_0^4 y^2\, dx$

$= \pi \int_0^4 (e^x)^2\, dx$

$= \pi \int_0^4 e^{2x}\, dx$

$= \pi\left[\frac{1}{2}e^{2x}\right]_0^4$

$= \pi\left(\frac{1}{2}e^8-\frac{1}{2}\right)$

$= \frac{\pi}{2}(e^8-1)$ units^3

d Volume

$= \pi \int_0^3 y^2\, dx$

$= \pi \int_0^3 (e^{-x}+1)^2\, dx$

$= \pi \int_0^3 (e^{-2x}+2e^{-x}+1)\, dx$

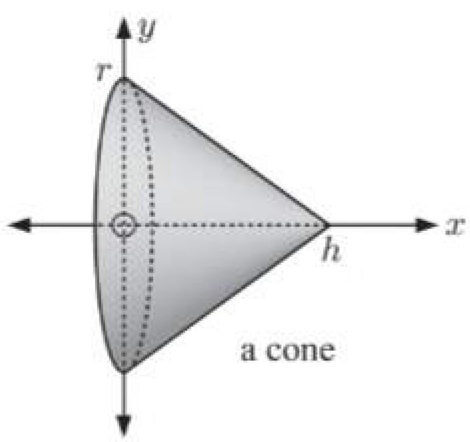
$= \pi\left[-\frac{1}{2}e^{-2x}-2e^{-x}+x\right]_0^3$

$= \pi\left[-\frac{1}{2}e^{-6}-2e^{-3}+3-\left(-\frac{1}{2}-2+0\right)\right]$

$= \pi\left(-\frac{1}{2}e^{-6}-2e^{-3}+5\frac{1}{2}\right)$

≈ 17.0 units^3

3 **a** a cone of base radius r and height h



b (AB) has gradient $= \frac{r-0}{0-h} = -\frac{r}{h}$

$\therefore$ its equation is $\quad y = -\left(\frac{r}{h}\right)x + r$

c $V = \pi \int_0^h y^2\, dx$

$= \pi \int_0^h \left(\frac{-r}{h}x+r\right)^2 dx$

$= \pi r^2 \int_0^h \left(-\frac{x}{h}+1\right)^2 dx$

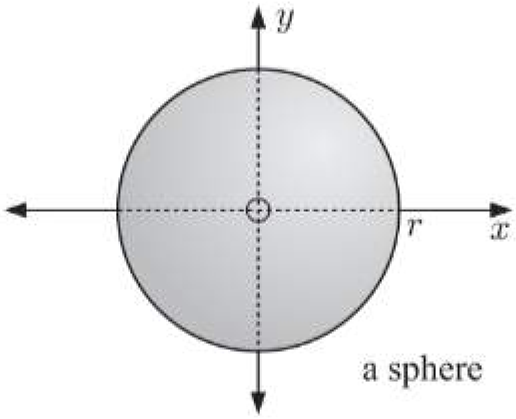
$= \pi r^2 \int_0^h \left(\frac{x^2}{h^2}-\frac{2x}{h}+1\right) dx$

$= \pi r^2 \left[\frac{x^3}{3h^2}-\frac{x^2}{h}+x\right]_0^h$

$= \pi r^2\left[\left(\frac{h}{3}-h+h\right)-0\right]$

$= \frac{1}{3}\pi r^2 h$ units^3

4 **a** a sphere of radius r



b $V = \pi \int_{-r}^{r} y^2 \, dx = 2\pi \int_0^r (r^2 - x^2) \, dx$

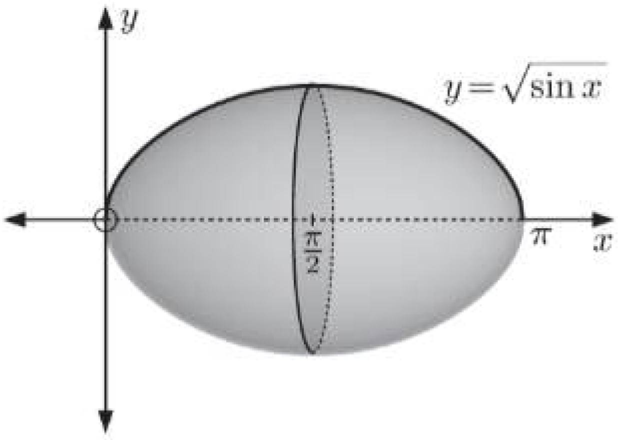
$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$

$= 2\pi \left[\left(r^3 - \frac{r^3}{3} \right) - 0 \right]$

$= 2\pi \times \frac{2}{3} r^3$

$= \frac{4}{3}\pi r^3$ units^3

5 **a**



Volume $= \pi \int_0^{\pi} y^2 \, dx$

$= \pi \int_0^{\pi} \sin x \, dx$

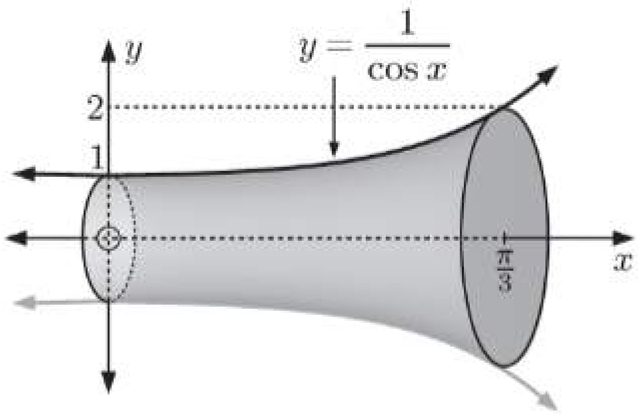
$= \pi [-\cos x]_0^{\pi}$

$= \pi [-\cos \pi - (-\cos 0)]$

$= \pi(2)$

$= 2\pi$ units^3

b



Volume $= \pi \int_0^{\frac{\pi}{3}} y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx$

$= \pi \left[\tan x \right]_0^{\frac{\pi}{3}}$

$= \pi \left(\tan \frac{\pi}{3} - \tan 0 \right)$

$= \pi(\sqrt{3} - 0)$

$= \pi\sqrt{3}$ units^3

6 **a** Volume $= \pi \int_1^5 y^2 \, dx$

$= \pi \int_1^5 (\ln x)^2 \, dx$

$\approx 4.857\pi$ {using technology}

≈ 15.3 units^3

b Volume $= \pi \int_0^{\frac{\pi}{4}} y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{4}} (4 \sin 2x)^2 \, dx$

$= 2\pi^2$ {using technology}

≈ 19.7 units^3

c Volume $= \pi \int_1^3 y^2 \, dx$

$= \pi \int_1^3 \left(\frac{x^3}{x^2 + 1} \right)^2 dx$

$\approx 5.926\pi$ {using technology}

≈ 18.6 units^3

d Volume $= \pi \int_0^2 y^2 \, dx$

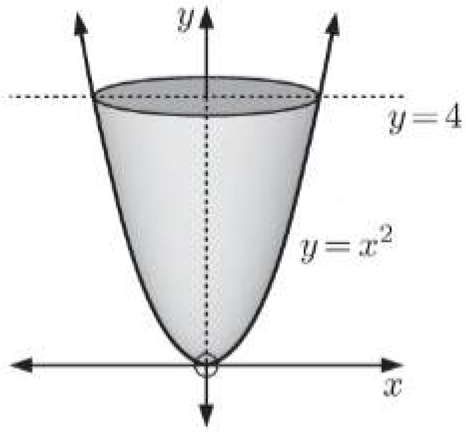
$= \pi \int_0^2 \left(e^{\sin x} \right)^2 \, dx$

$\approx 9.613\pi$ {using technology}

≈ 30.2 units^3

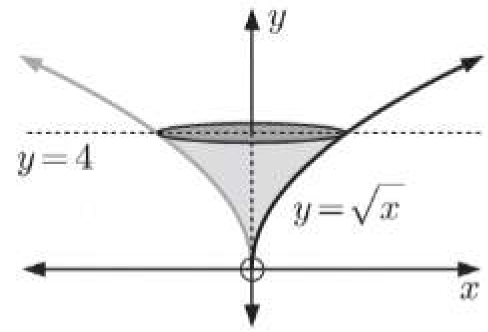
EXERCISE 23F.2

1 **a**



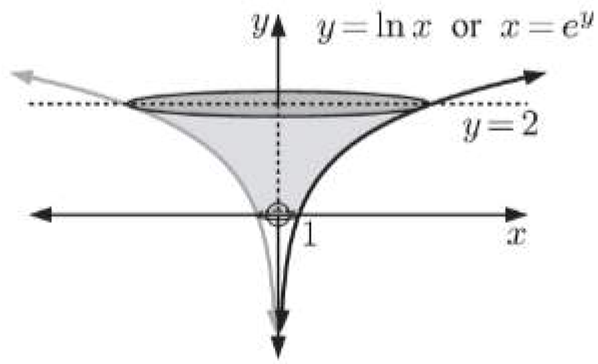
$$\text{Volume} = \pi \int_0^4 x^2\, dy$$
$$= \pi \int_0^4 y\, dy$$
$$= \pi \left[\frac{y^2}{2}\right]_0^4$$
$$= \pi(8 - 0)$$
$$= 8\pi \text{ units}^3$$

b



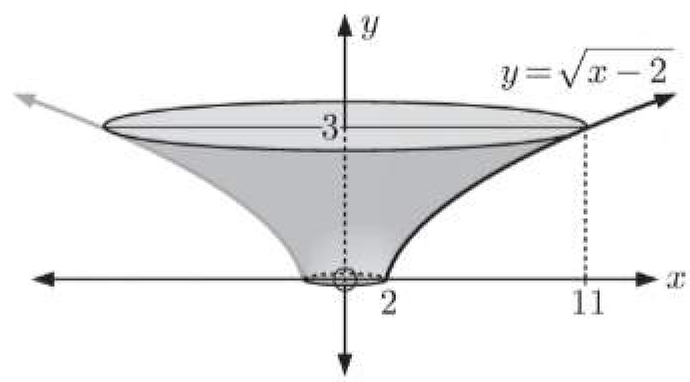
$$\text{Volume} = \pi \int_1^4 x^2\, dy$$
$$= \pi \int_1^4 y^4\, dy$$
$$= \pi \left[\frac{y^5}{5}\right]_1^4$$
$$= \pi\left(\frac{4^5}{5} - \frac{1}{5}\right)$$
$$= \frac{1023\pi}{5} \text{ units}^3$$

c



$$\text{Volume} = \pi \int_0^2 x^2\, dy$$
$$= \pi \int_0^2 (e^y)^2\, dy$$
$$= \pi \int_0^2 e^{2y}\, dy$$
$$= \pi\left[\tfrac{1}{2}e^{2y}\right]_0^2$$
$$= \pi\left(\tfrac{1}{2}e^4 - \tfrac{1}{2}\right)$$
$$= \tfrac{\pi}{2}(e^4 - 1) \text{ units}^3$$

d



When $x = 2, \quad y = 0$

When $x = 11, \quad y = 3$

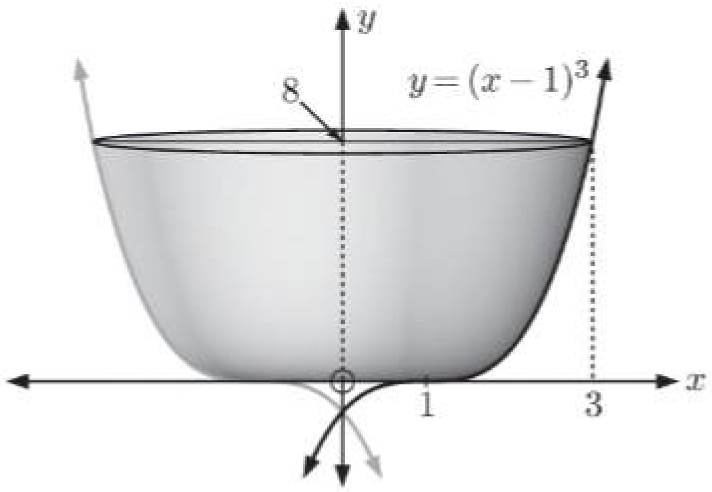
Now $y = \sqrt{x-2}$

$\therefore\ y^2 = x - 2$

$\therefore\ x = y^2 + 2$

$$\therefore\ \text{volume} = \pi \int_0^3 x^2\, dy$$
$$= \pi \int_0^3 (y^2+2)^2\, dy$$
$$= \pi \int_0^3 (y^4 + 4y^2 + 4)\, dy$$
$$= \pi\left[\tfrac{1}{5}y^5 + \tfrac{4}{3}y^3 + 4y\right]_0^3$$
$$= \pi\left(\tfrac{243}{5} + 36 + 12 - 0\right)$$
$$= \tfrac{483\pi}{5} \text{ units}^3$$

e



When $x=1, \quad y=0$

When $x=3, \quad y=8$

Now $y=(x-1)^3$

$\therefore \quad x-1=y^{\frac{1}{3}}$

$\therefore \quad x=y^{\frac{1}{3}}+1$

$\therefore$ volume

$$= \pi \int_0^8 x^2 \, dy$$

$$= \pi \int_0^8 (y^{\frac{1}{3}}+1)^2 \, dy$$

$$= \pi \int_0^8 \left(y^{\frac{2}{3}}+2y^{\frac{1}{3}}+1\right) dy$$

$$= \pi \left[\tfrac{3}{5}y^{\frac{5}{3}}+\tfrac{3}{2}y^{\frac{4}{3}}+y\right]_0^8$$

$$= \pi\left(\tfrac{3}{5}\times 32+\tfrac{3}{2}\times 16+8-0\right)$$

$$= \tfrac{256\pi}{5} \text{ units}^3$$

2 **a** Volume $= \pi \displaystyle\int_3^5 x^2 \, dy$

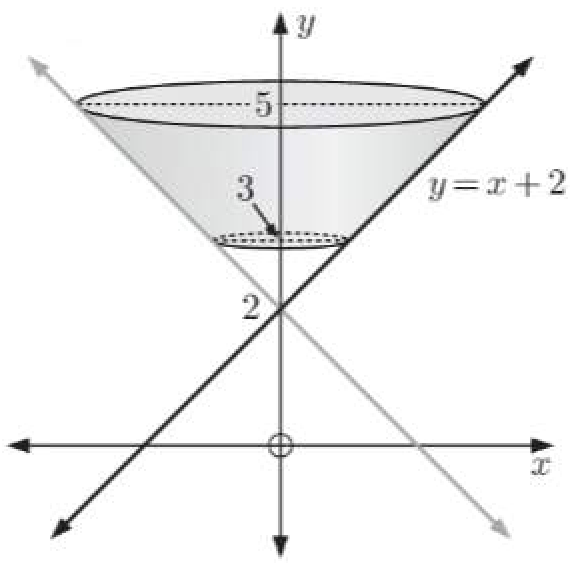
$$= \pi \int_3^5 (y-2)^2 \, dy$$

$$= \pi \int_3^5 (y^2-4y+4) \, dy$$

$$= \pi \left[\tfrac{1}{3}y^3-2y^2+4y\right]_3^5$$

$$= \pi \left[\left(\tfrac{125}{3}-50+20\right)-(9-18+12)\right]$$

$$= \tfrac{26}{3}\pi \text{ units}^3$$



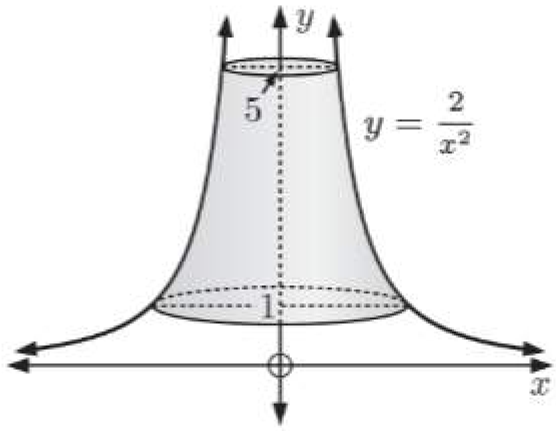
b $\text{Volume} = \pi \int_1^5 x^2\,dy$

$= \pi \int_1^5 \frac{2}{y}\,dy$

$= \pi \left[2\ln|y|\right]_1^5$

$= 2\pi \ln 5 - 2\pi \ln 1$

$= 2\pi \ln 5$ units^3



c $\text{Volume} = \pi \int_2^6 x^2\,dy$

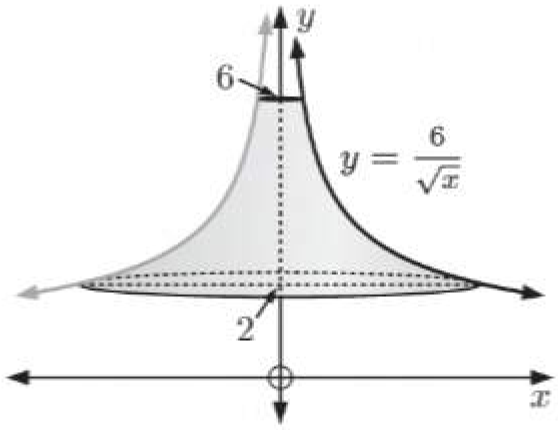
$= \pi \int_2^6 \left(\frac{6}{y}\right)^4 dy$

$= \pi \int_2^6 6^4 y^{-4}\,dy$

$= \pi \left[\frac{6^4}{-3} y^{-3}\right]_2^6$

$= \pi(-2 + 54)$

$= 52\pi$ units^3



3 $\frac{x^2}{9} + \frac{y^2}{16} = 1, \quad x \geqslant 0 \qquad \therefore \quad x^2 = 9\left(1 - \frac{y^2}{16}\right)$

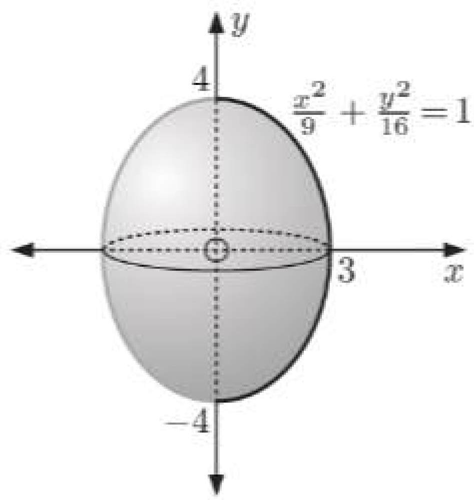
$\therefore \quad \text{volume} = \pi \int_{-4}^4 x^2\,dy$

$= \pi \int_{-4}^4 \left(9 - \frac{9}{16}y^2\right) dy$

$= \pi \left[9y - \frac{3}{16}y^3\right]_{-4}^4$

$= \pi[(36 - 12) - (-36 + 12)]$

$= 48\pi$ units^3



4 $y = \frac{1}{4}x^2$

$\therefore \quad x^2 = 4y$

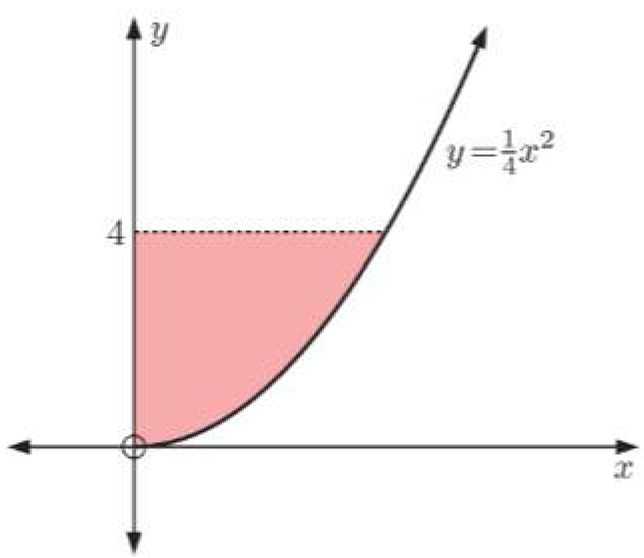
$\therefore \quad \text{volume} = \pi \int_0^4 x^2\,dy$

$= \pi \int_0^4 4y\,dy$

$= \pi \left[2y^2\right]_0^4$

$= \pi(2(4^2) - 0)$

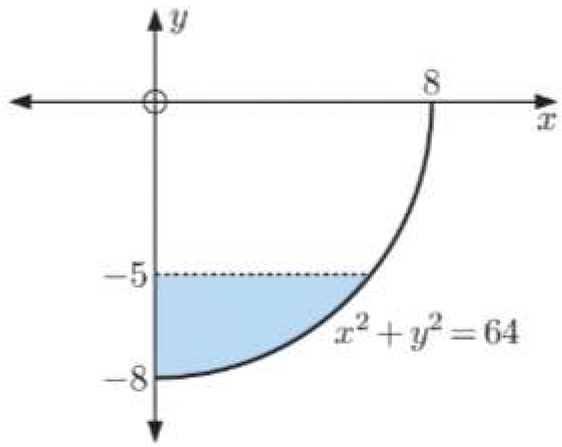
$= 32\pi$ units^3



$\therefore$ the capacity of the bowl is 32π units^3.

5 **a** $x^2 + y^2 = 64$

$\therefore \; x^2 = 64 - y^2$



$$\begin{aligned}
\therefore \text{ volume} &= \pi \int_{-8}^{-5} x^2 \, dy \\
&= \pi \int_{-8}^{-5} (64 - y^2) \, dy \\
&= \pi \left[64y - \tfrac{1}{3}y^3\right]_{-8}^{-5} \\
&= \pi \left[\left(64(-5) - \tfrac{1}{3}(-5)^3\right) - \left(64(-8) - \tfrac{1}{3}(-8)^3\right)\right] \\
&= \pi \left[-\tfrac{835}{3} + \tfrac{1024}{3}\right] \\
&= 63\pi \text{ units}^3
\end{aligned}$$

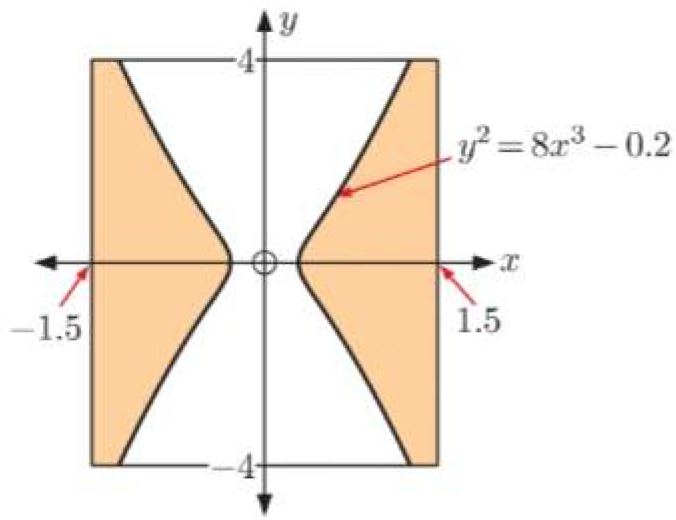
b The volume of a hemispherical bowl of radius 8 cm which contains water to a depth of 3 cm is described by the integral in part **a**. The units are cm.

$\therefore$ the volume of water in the bowl is $63\pi \approx 198 \text{ cm}^3$.

6 $y^2 = 8x^3 - 0.2$

$\therefore \; x^3 = \dfrac{y^2 + 0.2}{8}$

$\therefore \; x = \frac{1}{2}\sqrt[3]{y^2 + 0.2}$

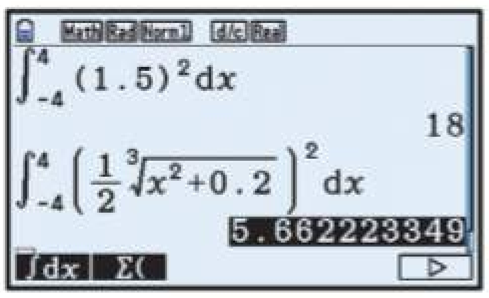


Now, the volume of the cylinder $= \pi \displaystyle\int_{-4}^{4} (1.5)^2 \, dy$, and

the volume of the inner casing $= \pi \displaystyle\int_{-4}^{4} \left(\tfrac{1}{2}\sqrt[3]{y^2 + 0.2}\right)^2 dy$

$\therefore$ the volume of air between the inner and outer casings

$$\begin{aligned}
&= \pi \int_{-4}^{4} (1.5)^2 \, dy - \pi \int_{-4}^{4} \left(\tfrac{1}{2}\sqrt[3]{y^2 + 0.2}\right)^2 dy \\
&\approx 18\pi - 5.662\pi \quad \{\text{using technology}\} \\
&\approx 38.8 \text{ cm}^3
\end{aligned}$$



EXERCISE 23G

1 **a** 8:20 am is 20 minutes after 8 am.

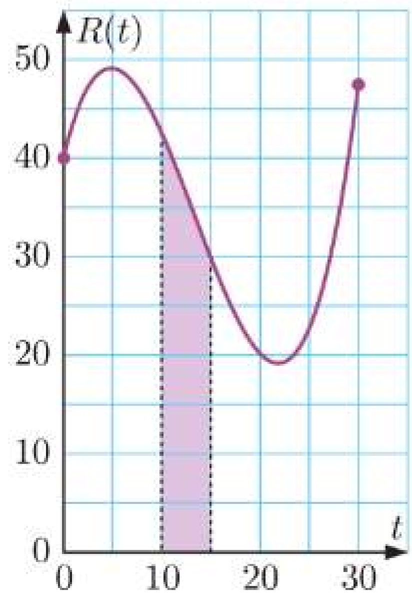
$$R(t) = \frac{t^3}{80} - \frac{t^2}{2} + 4t + 40$$

$$\therefore \quad R(20) = \frac{20^3}{80} - \frac{20^2}{2} + 4(20) + 40$$
$$= 20$$

So, the rate of traffic flow at 8:20 am was 20 cars per minute.

b The traffic flow was greatest when $R(t)$ was a maximum. Looking at the graph, the maximum value of $R(t)$ occurred when $t \approx 5$.

$\therefore$ the traffic flow was greatest at about 5 minutes after 8 am, that is, about 8:05 am.

c 

$\int_{10}^{15} R(t)\,dt$ represents the total number of cars going past the pedestrian crossing from 8:10 am to 8:15 am.

d 8 am is 0 minutes after 8 am and 8:30 am is 30 minutes after 8 am.

Total number of cars which passed the crossing between 8 am and 8:30 am

$$= \int_0^{30} \left(\frac{t^3}{80} - \frac{t^2}{2} + 4t + 40\right) dt$$
$$= \left[\tfrac{1}{320}t^4 - \tfrac{1}{6}t^3 + 2t^2 + 40t\right]_0^{30}$$
$$= 1031.25 \approx 1031 \text{ cars}$$

2 **a** $R_1(t) = 5 - 5e^{-0.2t}$, $\quad R_2(t) = 6 - 6e^{-0.1t}$

i $R_1(2) = 5 - 5e^{-0.2(2)}$
≈ 1.65 litres per minute

ii $R_2(2) = 6 - 6e^{-0.1(2)}$
≈ 1.09 litres per minute

b The rate of water leaking into the kayak is greater than the rate of water being bailed from the kayak after 2 minutes. So, the amount of water in the kayak is increasing after 2 minutes.

c **i**
$$\begin{aligned}\int_0^3 R_1(t)\,dt &= \int_0^3 (5 - 5e^{-0.2t})\,dt \\ &= \left[5t + 25e^{-0.2t}\right]_0^3 \\ &= (15 + 25e^{-0.6}) - (0 + 25) \\ &\approx 3.72\end{aligned}$$

About 3.72 litres of water have leaked into the kayak in the first 3 minutes.

ii
$$\begin{aligned}\int_2^5 R_2(t)\,dt &= \int_2^5 (6 - 6e^{-0.1t})\,dt \\ &= \left[6t + 60e^{-0.1t}\right]_2^5 \\ &= (30 + 60e^{-0.5}) - (12 + 60e^{-0.2}) \\ &\approx 5.27\end{aligned}$$

About 5.27 litres of water have been bailed out of the kayak from $t = 2$ minutes to $t = 5$ minutes.

iii
$$\begin{aligned}\int_0^8 [R_1(t) - R_2(t)]\,dt &= \int_0^8 (5 - 5e^{-0.2t} - (6 - 6e^{-0.1t}))\,dt \\ &= \int_0^8 (-1 - 5e^{-0.2t} + 6e^{-0.1t})\,dt \\ &= \left[-t + 25e^{-0.2t} - 60e^{-0.1t}\right]_0^8 \\ &= (-8 + 25e^{-1.6} - 60e^{-0.8}) - (0 + 25 - 60) \\ &\approx 5.09\end{aligned}$$

There are about 5.09 litres of water in the kayak 8 minutes after striking the rock.

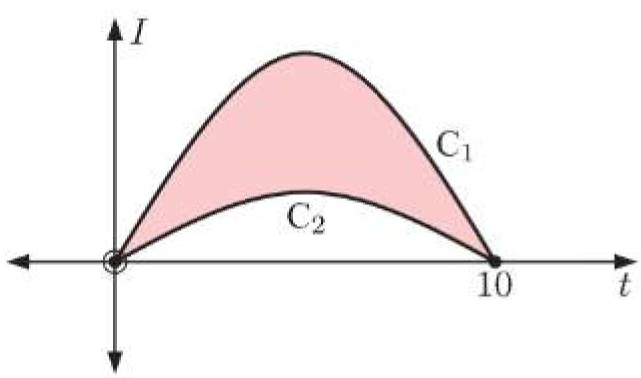
d
$$\begin{aligned}\int_0^{10} [R_1(t) - R_2(t)]\,dt &= \int_0^{10} (5 - 5e^{-0.2t} - (6 - 6e^{-0.1t}))\,dt \\ &= \int_0^{10} (-1 - 5e^{-0.2t} + 6e^{-0.1t})\,dt \\ &= \left[-t + 25e^{-0.2t} - 60e^{-0.1t}\right]_0^{10} \\ &= (-10 + 25e^{-2} - 60e^{-1}) - (0 + 25 - 60) \\ &\approx 6.31\end{aligned}$$

There are about 6.31 litres of water in the kayak 10 minutes after striking the rock.

3 **a** $y = 3\sin\frac{\pi t}{10}$ has amplitude 3, which is larger than the amplitude of $y = \sin\frac{\pi t}{10}$ which has amplitude 1.

$\therefore$ C_1 is $y = 3\sin\frac{\pi t}{10}$ and

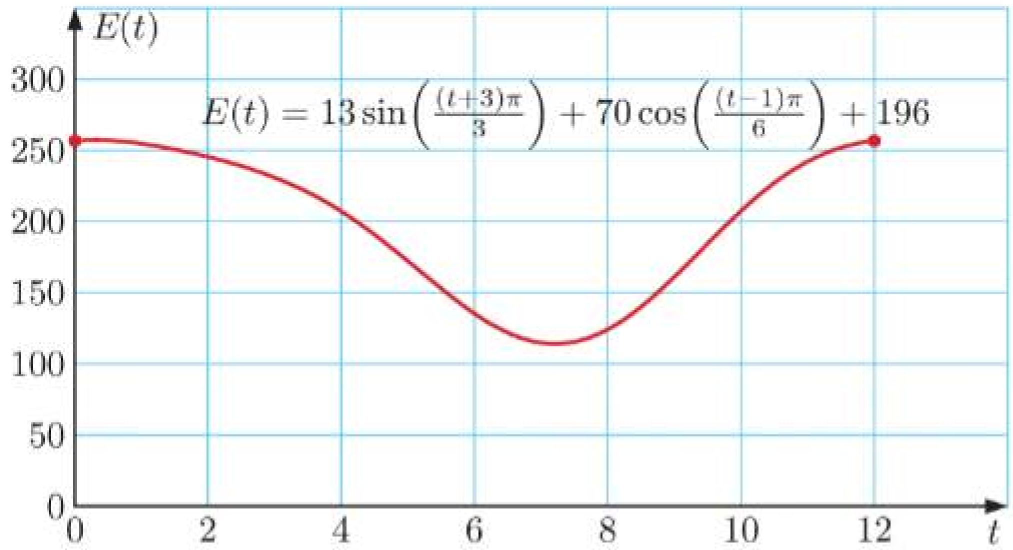
C_2 is $y = \sin\frac{\pi t}{10}$



b $$\begin{aligned}\text{Area} &= \int_0^{10} \left(3\sin\tfrac{\pi t}{10} - \sin\tfrac{\pi t}{10}\right) dt\\ &= \int_0^{10} 2\sin\tfrac{\pi t}{10}\, dt\\ &= \left[-\tfrac{20}{\pi}\cos\tfrac{\pi t}{10}\right]_0^{10}\\ &= -\tfrac{20}{\pi}\cos\pi + \tfrac{20}{\pi}\cos 0\\ &= \tfrac{20}{\pi} + \tfrac{20}{\pi}\\ &= \tfrac{40}{\pi}\text{ units}\end{aligned}$$

c The area in **b** represents the total amount of energy that enters the greenhouse in the first 10 hours.

4 **a**



b $$E(t) = 13\sin\left(\tfrac{(t+3)\pi}{3}\right) + 70\cos\left(\tfrac{(t-1)\pi}{6}\right) + 196 \quad \text{TWh per month}$$

$$\begin{aligned}\therefore \int E(t)\,dt &= \int \left(13\sin\left(\tfrac{(t+3)\pi}{3}\right) + 70\cos\left(\tfrac{(t-1)\pi}{6}\right) + 196\right) dt\\ &= 13\left(-\cos\left(\tfrac{(t+3)\pi}{3}\right)\right)\left(\tfrac{3}{\pi}\right) + 70\sin\left(\tfrac{(t-1)\pi}{6}\right)\left(\tfrac{6}{\pi}\right) + 196t + c\\ &= -\tfrac{39}{\pi}\cos\left(\tfrac{(t+3)\pi}{3}\right) + \tfrac{420}{\pi}\sin\left(\tfrac{(t-1)\pi}{6}\right) + 196t + c\end{aligned}$$

i $$\begin{aligned}\int_3^4 E(t)\,dt &= \left(-\tfrac{39}{\pi}\cos\tfrac{7\pi}{3} + \tfrac{420}{\pi}\sin\tfrac{\pi}{2} + 784\right) - \left(-\tfrac{39}{\pi}\cos 2\pi + \tfrac{420}{\pi}\sin\tfrac{\pi}{3} + 588\right)\\ &\approx 220.12 \text{ TWh}\end{aligned}$$

The power consumption of the United Kingdom in April is about 220.12 TWh.

ii $$\begin{aligned}\int_5^8 E(t)\,dt &= \left(-\tfrac{39}{\pi}\cos\tfrac{11\pi}{3} + \tfrac{420}{\pi}\sin\tfrac{7\pi}{6} + 1568\right) - \left(-\tfrac{39}{\pi}\cos\tfrac{8\pi}{3} + \tfrac{420}{\pi}\sin\tfrac{2\pi}{3} + 980\right)\\ &\approx 392.96 \text{ TWh}\end{aligned}$$

The power consumption of the United Kingdom for June 1st to September 1st is about 392.96 TWh.

iii $$\begin{aligned}\int_0^{12} E(t)\,dt &= \left(\cancel{-\tfrac{39}{\pi}\cos 5\pi} + \cancel{\tfrac{420}{\pi}\sin\tfrac{11\pi}{6}} + 2352\right) - \left(\cancel{-\tfrac{39}{\pi}\cos\pi} + \cancel{\tfrac{420}{\pi}\sin\left(-\tfrac{\pi}{6}\right)}\right)\\ &= 2352 \text{ TWh}\end{aligned}$$

The yearly power consumption of the United Kingdom is 2352 TWh.

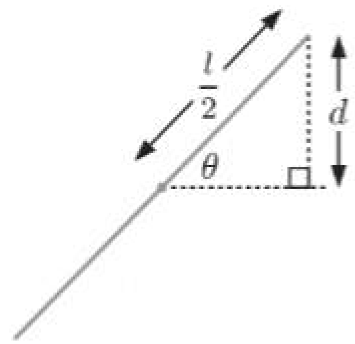
ACTIVITY 2 BUFFON'S NEEDLE PROBLEM

CASE 1: THE SHORT NEEDLE

1 **a** $0 \leqslant \theta \leqslant \pi$ **b** $0 \leqslant D \leqslant \frac{w}{2}$

2 Assuming that the needle toss is "random", that is, it is tossed vertically rather than being cast on a particular orientation, it is reasonable to assume that θ and D will take values in their ranges with equal probability.

3



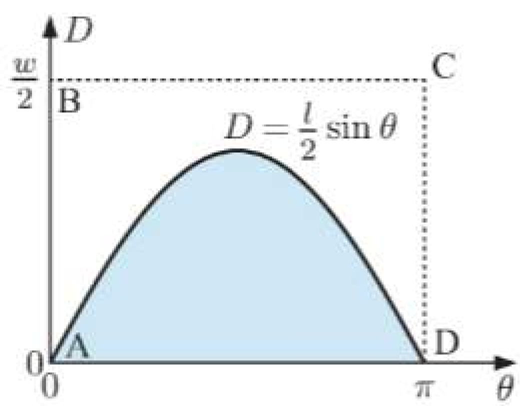
$$\sin\theta = \frac{d}{\left(\frac{l}{2}\right)}$$

$$\therefore \quad d = \frac{l}{2}\sin\theta$$

The needle will lie on a line if $D \leqslant d$

$$\therefore \quad D \leqslant \frac{l}{2}\sin\theta$$

4 θ is on the horizontal axis. The length of the rectangle, AD, covers the range of values for θ found in **1 a**.
D is on the vertical axis. The height of the rectangle, AB, covers the range of values for D found in **1 b**.
So, any toss of the needle can be described by a unique point (θ, D) which lies in the rectangle. Under the assumption in **2**, each possible outcome (θ, D) is equally likely.
Assuming $l \leqslant w$, the shaded area describes the set of points for which $D \leqslant \frac{l}{2}\sin\theta$. From **3**, this is the set of points for which the needle will lie on a line.



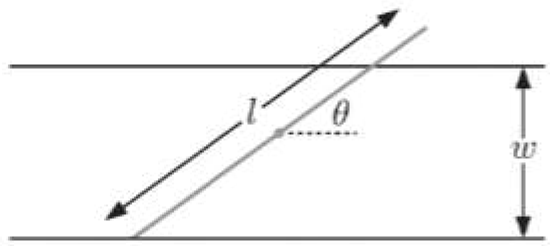
$$\therefore \quad \text{P(needle lies on a line)} = \frac{\text{shaded area}}{\text{area of rectangle ABCD}}$$

5
$$\begin{aligned}
\text{Shaded area} &= \int_0^{\pi} \frac{l}{2}\sin\theta \, d\theta \\
&= \frac{l}{2}\int_0^{\pi} \sin\theta \, d\theta \\
&= \frac{l}{2}\Big[-\cos\theta\Big]_0^{\pi} \\
&= \frac{l}{2}(-\cos\pi + \cos 0) \\
&= l
\end{aligned}$$

6 Using **4** and **5**,
$$\text{P(needle lies on a line)} = \frac{l}{\pi\left(\frac{w}{2}\right)} = \frac{2l}{w\pi}$$

CASE 2: THE LONG NEEDLE

1 For a needle with length $l > w$, the curve $D = \frac{l}{2}\sin\theta$ leaves the rectangle.

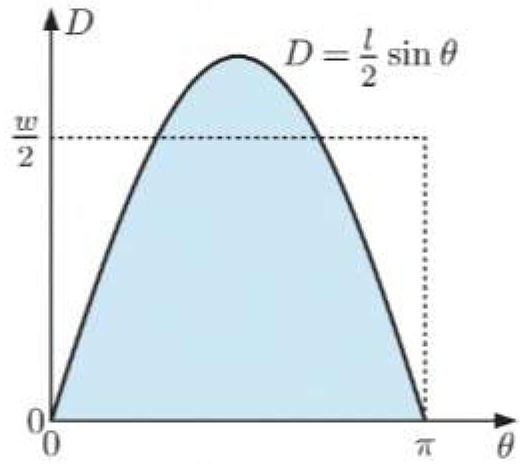


Physically, this means that there is a range of angles $\theta^* < \theta < \pi - \theta^*$ for which the needle is *certain* to lie on a line.

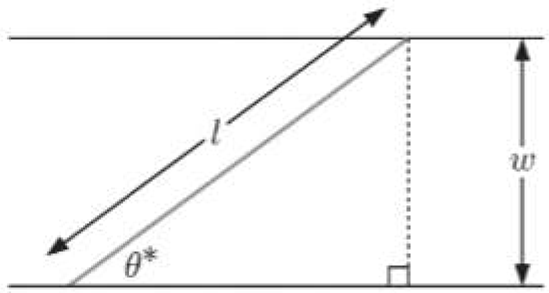
Since any toss of the needle is still described by the set of points in the rectangle, the part of the shaded area *outside* the rectangle is impossible.

So, for $l > w$,

$$\text{P(needle lies on a line)} = \frac{\text{shaded area } \textit{within} \text{ rectangle}}{\text{area of rectangle}}.$$



2

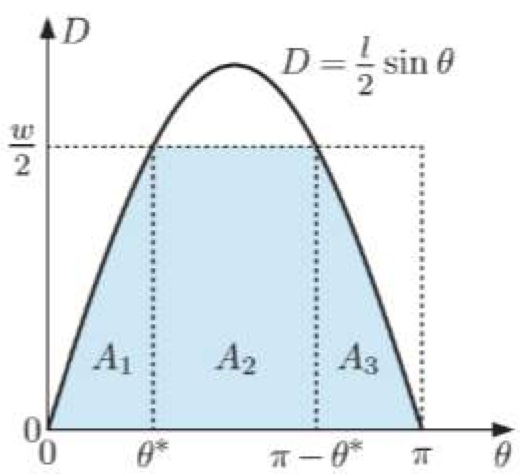


θ^* is the critical value of θ for which the needle *exactly* fits between two lines.

$$\sin\theta^* = \frac{w}{l}$$

$$\therefore \quad \theta^* = \sin^{-1}\left(\tfrac{w}{l}\right)$$

3 **a**



$$A_1 = \int_0^{\theta^*} \frac{l}{2}\sin\theta \, d\theta$$

$$A_2 = (\pi - \theta^* - \theta^*)\,\frac{w}{2}$$

$$= (\pi - 2\theta^*)\,\frac{w}{2}$$

$$A_3 = A_1 \qquad \{\text{by symmetry}\}$$

$\therefore$ P(needle lies on a line)

$$= \frac{A_1 + A_2 + A_3}{\pi\left(\frac{w}{2}\right)}$$

$$= \frac{(\pi - 2\theta^*)\frac{w}{2} + 2\displaystyle\int_0^{\theta^*}\left(\frac{l}{2}\sin\theta\right)d\theta}{\frac{w\pi}{2}}$$

b Continuing from **a**, P(needle lies on a line)

$$= \frac{\pi - 2\theta^*}{\pi} + \frac{4}{w\pi}\,\frac{l}{2}\int_0^{\theta^*} \sin\theta \, d\theta$$

$$= 1 - \tfrac{2}{\pi}\theta^* + \frac{2l}{w\pi}\left[-\cos\theta\right]_0^{\theta^*}$$

$$= 1 - \tfrac{2}{\pi}\theta^* + \frac{2l}{w\pi}(-\cos\theta^* + \cos 0)$$

$$= 1 - \tfrac{2}{\pi}\theta^* + \frac{2l}{w\pi}(1 - \cos\theta^*)$$

$$= 1 - \tfrac{2}{\pi}\theta^* + \frac{2l}{w\pi}\left(1 - \sqrt{1 - \sin^2\theta^*}\right) \qquad \{\text{since } 0 < \theta^* \leqslant \tfrac{\pi}{2}\}$$

$$= 1 - \tfrac{2}{\pi}\sin^{-1}\left(\tfrac{w}{l}\right) + \frac{2l}{w\pi}\left(1 - \sqrt{1 - \left(\sin\left(\sin^{-1}\left(\tfrac{w}{l}\right)\right)^2\right)}\right) \qquad \{\text{using } \mathbf{2}\}$$

$$= 1 - \tfrac{2}{\pi}\sin^{-1}\left(\tfrac{w}{l}\right) + \frac{2l}{w\pi}\left(1 - \sqrt{1 - \frac{w^2}{l^2}}\right)$$

$$= 1 - \tfrac{2}{\pi}\sin^{-1}\left(\tfrac{w}{l}\right) + \frac{2l}{w\pi}\left(1 - \frac{\sqrt{l^2 - w^2}}{l}\right)$$

4 For the boundary case $l = w$, the formula in **3 b** gives

$$\text{P(needle lies on a line)} = 1 - \tfrac{2}{\pi}\sin^{-1}(1) + \frac{2l}{w\pi}\left(1 - \frac{\sqrt{0}}{l}\right)$$

$$= 1 - \tfrac{2}{\pi} \times \tfrac{\pi}{2} + \frac{2l}{w\pi}$$

$$= \frac{2l}{w\pi}$$

which agrees with the formula for the short needle.

REVIEW SET 23A

1 **a**
$$\int_{-2}^{0}(1 - 3x)\,dx = \left[x - \tfrac{3}{2}x^2\right]_{-2}^{0}$$
$$= 0 - \left(-2 - \tfrac{3}{2}(4)\right)$$
$$= 0 - (-8)$$
$$= 8$$

b
$$\int_0^{\frac{1}{2}}(x - \sqrt{x})\,dx = \int_0^{\frac{1}{2}}\left(x - x^{\frac{1}{2}}\right)dx$$
$$= \left[\tfrac{1}{2}x^2 - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$
$$= \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}x^{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$
$$= \tfrac{1}{2}\left(\tfrac{1}{4}\right) - \tfrac{2}{3}\left(\tfrac{1}{2\sqrt{2}}\right) - 0$$
$$= \tfrac{1}{8} - \tfrac{1}{3\sqrt{2}}$$

c
$$\int_1^2 (x^2 + 1)^2\,dx$$
$$= \int_1^2 (x^4 + 2x^2 + 1)\,dx$$
$$= \left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\right]_1^2$$
$$= \left(\tfrac{32}{5} + \tfrac{16}{3} + 2\right) - \left(\tfrac{1}{5} + \tfrac{2}{3} + 1\right)$$
$$= \tfrac{178}{15}$$

2 **a** $\displaystyle\int_{-5}^{-1} \sqrt{1-3x}\,dx$

$\displaystyle= \int_{-5}^{-1} (1-3x)^{\frac{1}{2}}\,dx$

$\displaystyle= \left[\left(\tfrac{1}{-3}\right)\frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{-1}$

$\displaystyle= \left[-\tfrac{2}{9}(1-3x)^{\frac{3}{2}}\right]_{-5}^{-1}$

$= -\frac{2}{9}(4)^{\frac{3}{2}} - (-\frac{2}{9}(16)^{\frac{3}{2}})$

$= -\frac{16}{9} + \frac{128}{9}$

$= \frac{112}{9}$

b $\displaystyle\int_0^{\frac{\pi}{2}} \cos\tfrac{x}{2}\,dx = \left[2\sin\tfrac{x}{2}\right]_0^{\frac{\pi}{2}}$

$= 2\sin\frac{\pi}{4} - 2\sin 0$

$= \sqrt{2}$

c $\displaystyle\int_2^6 \frac{2}{x}\,dx = \left[2\ln|\,x\,|\right]_2^6$

$= 2\ln 6 - 2\ln 2$

$= 2(\ln 6 - \ln 2)$

$= 2\ln\left(\frac{6}{2}\right)$

$= 2\ln 3$

3 $\displaystyle\frac{d}{dx}(e^{-2x}\sin x) = -2e^{-2x}\sin x + e^{-2x}(\cos x)$

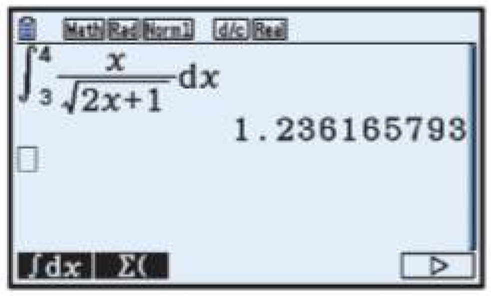
$= e^{-2x}(\cos x - 2\sin x)$

$\displaystyle\therefore \int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right]dx = \left[e^{-2x}\sin x\right]_0^{\frac{\pi}{2}}$

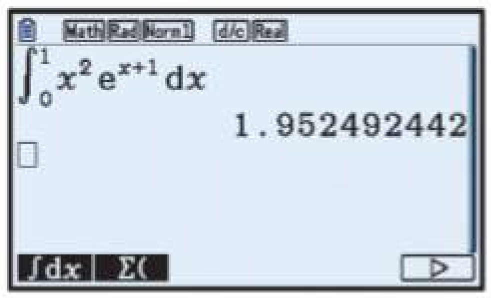
$= \left(e^{-\pi}\sin\frac{\pi}{2}\right) - 0$

$= e^{-\pi}$

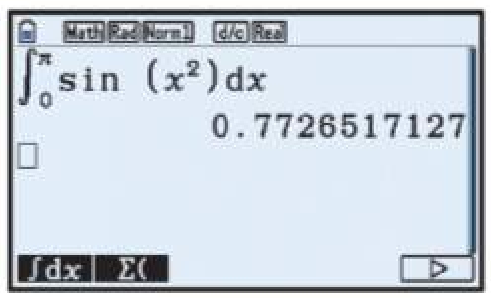
4 **a** Using technology, $\displaystyle\int_3^4 \frac{x}{\sqrt{2x+1}}\,dx \approx 1.236\,17$



b Using technology, $\displaystyle\int_0^1 x^2 e^{x+1}\,dx \approx 1.952\,49$



c Using technology, $\displaystyle\int_0^{\pi} \sin(x^2)\,dx \approx 0.772\,652$



5 **a** $\displaystyle\int 2x(x^2+1)^3\,dx = \int u^3 \frac{du}{dx}\,dx \qquad \{u = x^2+1,\ \ \frac{du}{dx} = 2x\}$

$$= \int u^3\,du$$
$$= \tfrac{1}{4}u^4 + c$$
$$= \tfrac{1}{4}(x^2+1)^4 + c$$

b **i** $\displaystyle\int_0^1 2x(x^2+1)^3\,dx = \left[\tfrac{1}{4}(x^2+1)^4\right]_0^1$

$$= \tfrac{1}{4}(16) - \tfrac{1}{4}(1)$$
$$= \tfrac{15}{4}$$

ii $\displaystyle\int_{-1}^2 -x(1+x)^3\,dx$

$$= -\tfrac{1}{2}\int_{-1}^2 2x(x^2+1)^3\,dx$$
$$= -\tfrac{1}{2}\left[\tfrac{1}{4}(x^2+1)^4\right]_1^2$$
$$= -\tfrac{1}{2}\left[\tfrac{1}{4}(5)^4 - \tfrac{1}{4}(2)^4\right]$$
$$= -\tfrac{1}{2}\left(\tfrac{625}{4} - 4\right)$$
$$= -\tfrac{609}{8}$$

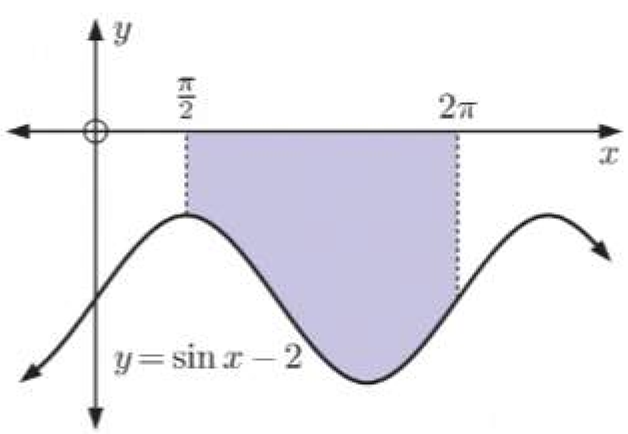
6 Let $u = 1 - x^3 \quad \therefore \quad \dfrac{du}{dx} = -3x^2$

When $x = 0,\ u = 1$
When $x = 1,\ u = 0$

$$\therefore \int_0^1 x^2 e^{1-x^3}\,dx = -\tfrac{1}{3}\int_0^1 -3x^2 e^{1-x^3}\,dx$$
$$= -\tfrac{1}{3}\int_0^1 e^u \frac{du}{dx}\,dx$$
$$= -\tfrac{1}{3}\int_1^0 e^u\,du$$
$$= -\tfrac{1}{3}\left[e^u\right]_1^0$$
$$= -\tfrac{1}{3}(1-e)$$
$$= \frac{e-1}{3}$$

7 **a** Area $\displaystyle = -\int_{\frac{\pi}{2}}^{2\pi} (\sin x - 2)\,dx$

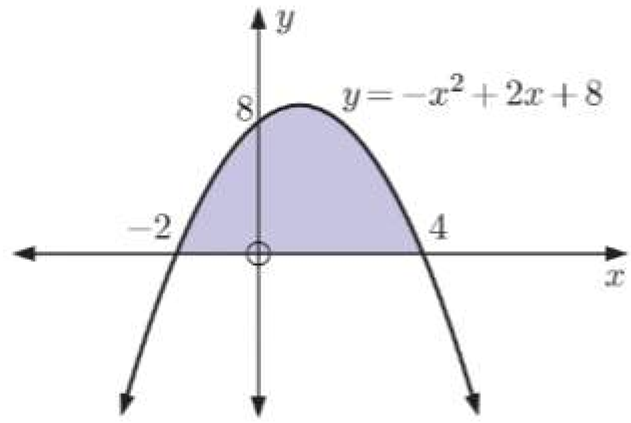
$$= -\left[-\cos x - 2x\right]_{\frac{\pi}{2}}^{2\pi}$$
$$= -[(-1-4\pi) - (0-\pi)]$$
$$= (3\pi + 1) \text{ units}^2$$



b The curve cuts the x-axis when $y = 0$

$$\therefore \ -x^2 + 2x + 8 = 0$$
$$\therefore \ x^2 - 2x - 8 = 0$$
$$\therefore \ (x+2)(x-4) = 0$$
$$\therefore \ x = -2 \text{ or } 4$$

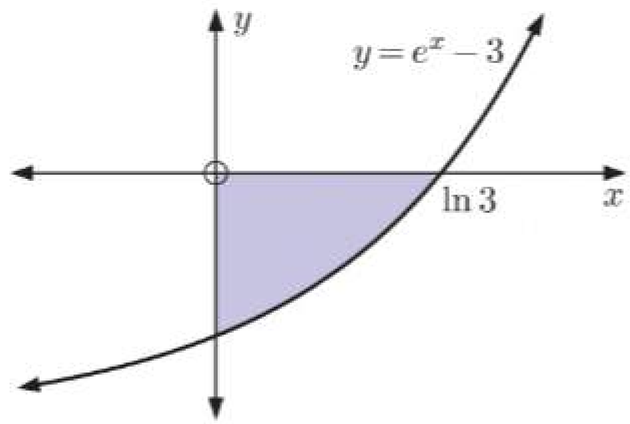
$\therefore$ the x-intercepts are -2 and 4.



$$\begin{aligned}\text{Area} &= \int_{-2}^{4} (-x^2 + 2x + 8)\, dx \\ &= \left[-\tfrac{1}{3}x^3 + x^2 + 8x\right]_{-2}^{4} \\ &= \left(-\tfrac{64}{3} + 16 + 32\right) - \left(\tfrac{8}{3} + 4 - 16\right) \\ &= \tfrac{80}{3} - \left(-\tfrac{28}{3}\right) \\ &= 36 \text{ units}^2\end{aligned}$$

c The curve cuts the x-axis when $y = 0$

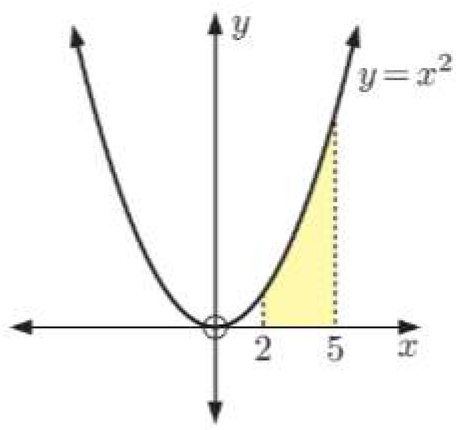
$$\therefore \ e^x - 3 = 0$$
$$\therefore \ e^x = 3$$
$$\therefore \ x = \ln 3$$



$$\begin{aligned}\text{Area} &= -\int_{0}^{\ln 3} (e^x - 3)\, dx \\ &= -\left[e^x - 3x\right]_{0}^{\ln 3} \\ &= -[(3 - 3\ln 3) - 1] \\ &= (3\ln 3 - 2) \text{ units}^2\end{aligned}$$

8 **a**

$$\begin{aligned}\text{Area} &= \int_{2}^{5} x^2\, dx \\ &= \left[\tfrac{1}{3}x^3\right]_{2}^{5} \\ &= \tfrac{125}{3} - \tfrac{8}{3} \\ &= 39 \text{ units}^2\end{aligned}$$



b Area $= \int_1^2 \sqrt{5-x}\,dx$

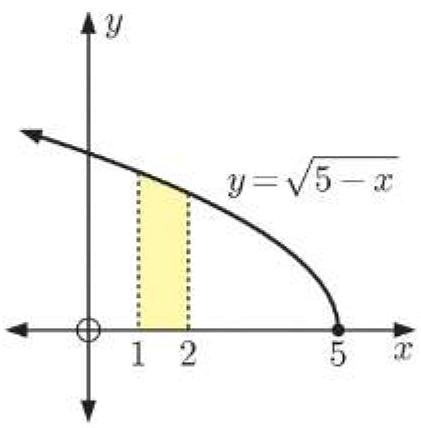
$= \int_1^2 (5-x)^{\frac{1}{2}}\,dx$

$= \left[\left(\frac{1}{-1}\right)\frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2$

$= \left[-\frac{2}{3}(5-x)^{\frac{3}{2}}\right]_1^2$

$= -\frac{2}{3}(3)^{\frac{3}{2}} - (-\frac{2}{3}(4)^{\frac{3}{2}})$

$= \left(\frac{16}{3} - 2\sqrt{3}\right)$ units^2



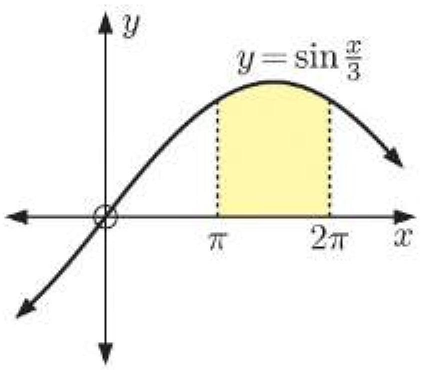
c Area $= \int_\pi^{2\pi} \sin\frac{x}{3}\,dx$

$= \left[-3\cos\frac{x}{3}\right]_\pi^{2\pi}$

$= -3\cos\frac{2\pi}{3} - \left(-3\cos\frac{\pi}{3}\right)$

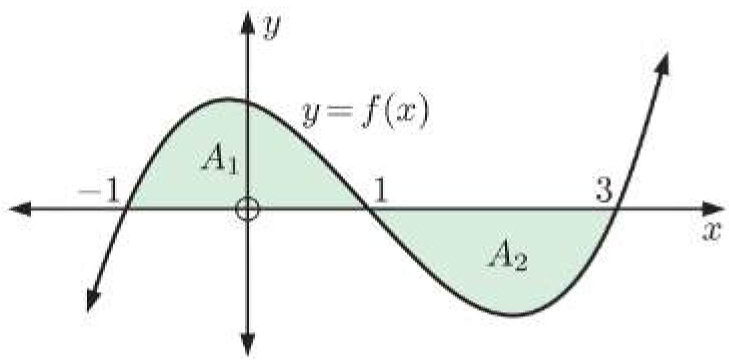
$= \frac{3}{2} - \left(-\frac{3}{2}\right)$

$= 3$ units^2



9 $\int_{-1}^3 f(x)\,dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $-1 \leqslant x \leqslant 3$.

But $f(x)$ is not positive for $1 \leqslant x \leqslant 3$, so $\int_{-1}^3 f(x)\,dx = A_1 - A_2$ which is *not* the shaded area.



The area of the shaded region $= \int_{-1}^1 f(x)\,dx - \int_1^3 f(x)\,dx$

10 $y = x^2$ meets $y = k$ where $x^2 = k$

$$\therefore \; x = \pm\sqrt{k}$$

Now, the area $= 5\frac{1}{3}$

$$\therefore \int_{-\sqrt{k}}^{\sqrt{k}} (k - x^2)\, dx = 5\tfrac{1}{3}$$

$$\therefore \left[kx - \frac{x^3}{3}\right]_{-\sqrt{k}}^{\sqrt{k}} = 5\tfrac{1}{3}$$

$$\therefore \left(k\sqrt{k} - \frac{k\sqrt{k}}{3}\right) - \left(-k\sqrt{k} - \left(-\frac{k\sqrt{k}}{3}\right)\right) = \tfrac{16}{3}$$

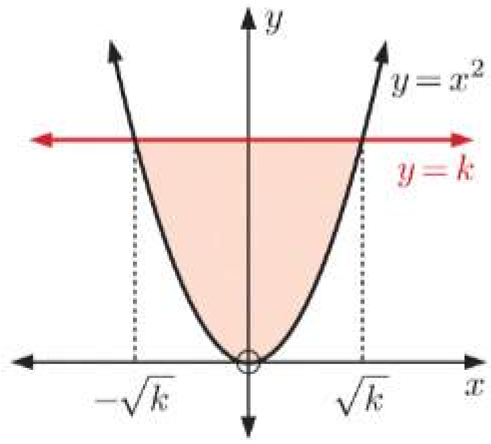
$$\therefore \; 2k\sqrt{k} - \frac{2k\sqrt{k}}{3} = \tfrac{16}{3}$$

$$\therefore \; \tfrac{4}{3}k^{\frac{3}{2}} = \tfrac{16}{3}$$

$$\therefore \; k^{\frac{3}{2}} = 4$$

$$\therefore \; k^3 = 16$$

$$\therefore \; k = \sqrt[3]{16}$$



11 **a**

$$y = \sqrt{x+3}$$

$$\therefore \; y^2 = x + 3$$

$$\therefore \; x = y^2 - 3$$

$$\therefore \; f^{-1}(y) = y^2 - 3$$

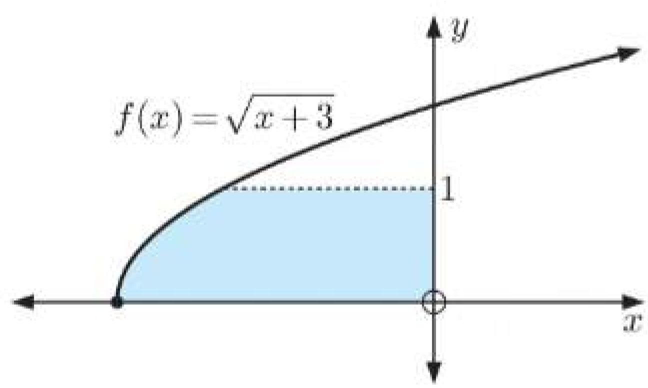
$$\therefore \; \text{area} = -\int_0^1 (y^2 - 3)\, dy$$

$$= -\left[\tfrac{1}{3}y^3 - 3y\right]_0^1$$

$$= -\left[\left(\tfrac{1}{3} - 3\right) - 0\right]$$

$$= \tfrac{8}{3}$$

$$= 2\tfrac{2}{3} \text{ units}^2$$



b

$$y = 3\ln(5 - x)$$

$$\therefore \; 5 - x = e^{\frac{y}{3}}$$

$$\therefore \; x = 5 - e^{\frac{y}{3}}$$

$$\therefore \; f^{-1}(y) = 5 - e^{\frac{y}{3}}$$

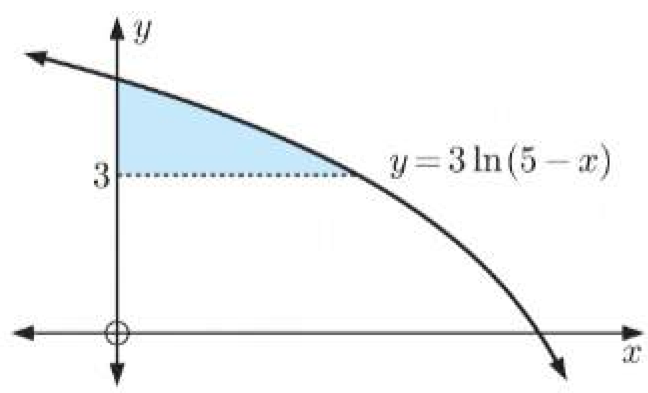
Now, when $x = 0$, $y = 3\ln 5$

$$\therefore \; \text{area} = \int_3^{3\ln 5} \left(5 - e^{\frac{y}{3}}\right) dy$$

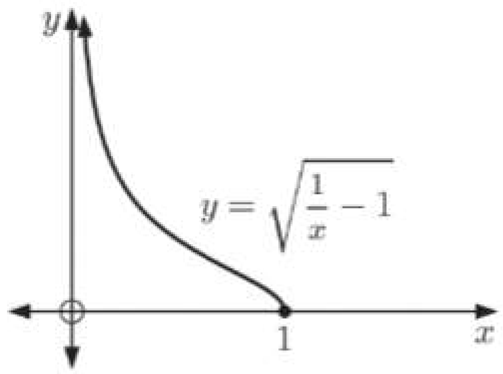
$$= \left[5y - 3e^{\frac{y}{3}}\right]_3^{3\ln 5}$$

$$= 15\ln 5 - 3e^{\ln 5} - (15 - 3e)$$

$$= (15\ln 5 - 30 + 3e) \text{ units}^2$$



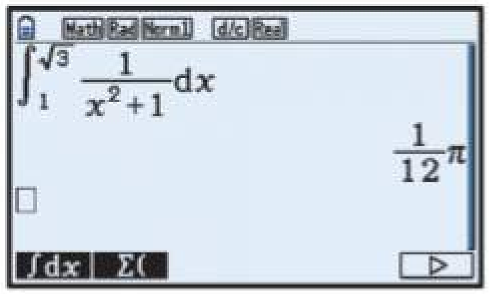
12 **a**



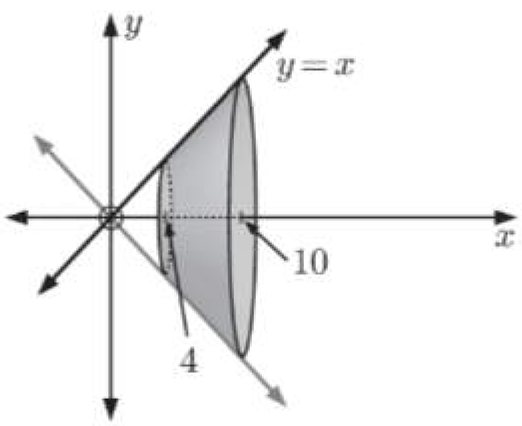
b

$$\begin{aligned} y &= \sqrt{\frac{1}{x}-1} \\ \therefore\ y^2 &= \frac{1}{x}-1 \\ \therefore\ xy^2 &= 1-x \\ \therefore\ xy^2+x &= 1 \\ \therefore\ x(y^2+1) &= 1 \\ \therefore\ x &= \frac{1}{y^2+1} \\ \therefore\ f^{-1}(y) &= \frac{1}{y^2+1} \end{aligned}$$

$$\begin{aligned} \therefore\ \text{area} &= \int_1^{\sqrt{3}} \frac{1}{y^2+1}\,dy \\ &= \tfrac{\pi}{12}\text{ units}^2 \qquad \{\text{using technology}\} \\ &\approx 0.262\text{ units}^2 \end{aligned}$$

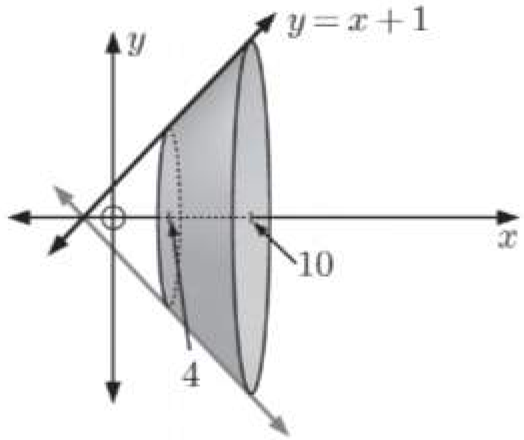


13 **a**



$$\begin{aligned} \text{Volume} &= \pi\int_4^{10} y^2\,dx \\ &= \pi\int_4^{10} x^2\,dx \\ &= \pi\left[\frac{x^3}{3}\right]_4^{10} \\ &= \pi\left(\tfrac{1000}{3}-\tfrac{64}{3}\right) \\ &= \tfrac{936\pi}{3} \\ &= 312\pi\text{ units}^3 \end{aligned}$$

b



$$\begin{aligned} \text{Volume} &= \pi\int_4^{10} y^2\,dx \\ &= \pi\int_4^{10} (x+1)^2\,dx \\ &= \pi\left[\frac{(x+1)^3}{3}\right]_4^{10} \\ &= \pi\left(\tfrac{11^3}{3}-\tfrac{5^3}{3}\right) \\ &= \tfrac{1206\pi}{3} \\ &= 402\pi\text{ units}^3 \end{aligned}$$

c Volume $= \pi \int_0^{\pi} y^2 \, dx$

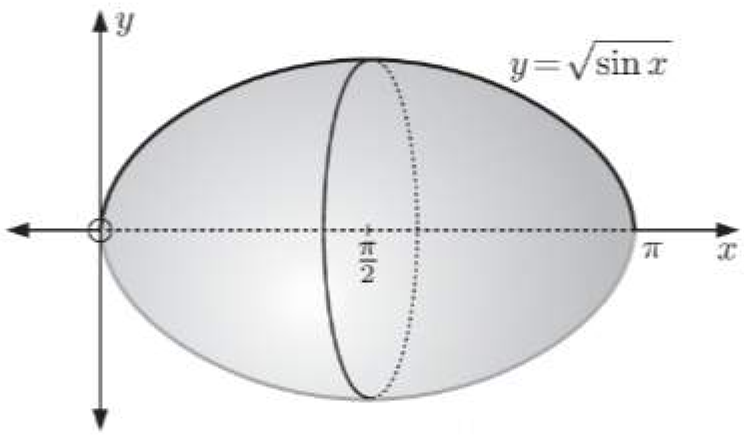
$= \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2 \, dx$

$= \pi \int_0^{\pi} \sin x \, dx$

$= \pi \left[-\cos x\right]_0^{\pi}$

$= \pi(1 - (-1))$

$= 2\pi$ units^3



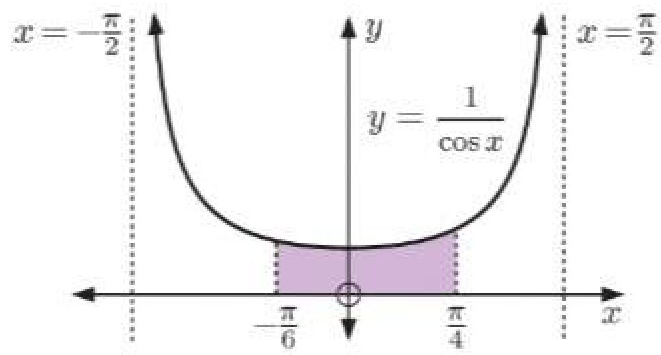
14 **a** Volume $= \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} y^2 \, dx$

$= \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1}{\cos x}\right)^2 \, dx$

$= \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos^2 x} \, dx$

$= \pi \left[\tan x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \pi \left(1 + \frac{1}{\sqrt{3}}\right)$ units^3



b Volume $= \pi \int_0^2 y^2 \, dx$

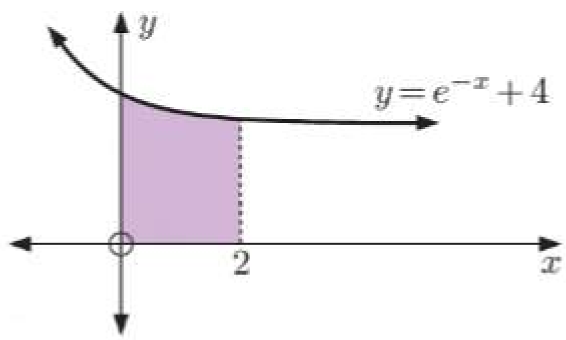
$= \pi \int_0^2 (e^{-x} + 4)^2 \, dx$

$= \pi \int_0^2 (e^{-2x} + 8e^{-x} + 16) \, dx$

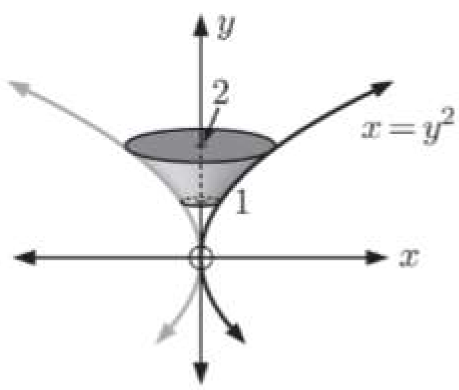
$= \pi \left[\frac{1}{-2} e^{-2x} + \frac{8}{-1} e^{-x} + 16x\right]_0^2$

$= \pi \left[\left(-\frac{1}{2} e^{-4} - 8e^{-2} + 32\right) - \left(-\frac{1}{2} - 8\right)\right]$

$= \pi \left(\frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2}\right)$ units$^3 \approx 124$ units^3



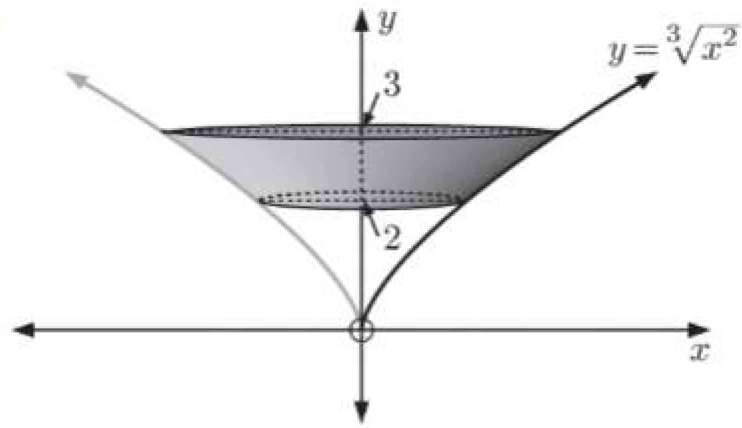
15 **a**



$y = \sqrt{x} \quad \therefore \quad x^2 = y^4$

$$\begin{aligned} \text{Volume} &= \pi \int_1^2 x^2 \, dy \\ &= \pi \int_1^2 y^4 \, dy \\ &= \pi \left[\frac{y^5}{5}\right]_1^2 \\ &= \pi \left(\tfrac{32}{5} - \tfrac{1}{5}\right) \\ &= \tfrac{31\pi}{5} \text{ units}^3 \end{aligned}$$

b



$y = \sqrt[3]{x^2} \quad \therefore \quad x^2 = y^3$

$$\begin{aligned} \text{Volume} &= \pi \int_2^3 x^2 \, dy \\ &= \pi \int_2^3 y^3 \, dy \\ &= \pi \left[\frac{y^4}{4}\right]_2^3 \\ &= \pi \left(\tfrac{81}{4} - \tfrac{16}{4}\right) \\ &= \tfrac{65\pi}{4} \text{ units}^3 \end{aligned}$$

16 $R_1(t) = 6.4, \quad R_2(t) = 2.5 - 1.25e^{-0.2t}$

a **i**

$$\begin{aligned} \int_0^{\frac{1}{2}} R_2(t) \, dt &= \int_0^{\frac{1}{2}} (2.5 - 1.25e^{-0.2t}) \, dt \\ &= \left[2.5t - 1.25\left(\tfrac{1}{-0.2}\right) e^{-0.2t}\right]_0^{\frac{1}{2}} \\ &= \left[2.5t + 6.25e^{-0.2t}\right]_0^{\frac{1}{2}} \\ &= (2.5(\tfrac{1}{2}) + 6.25e^{-0.2(\frac{1}{2})}) - (0 + 6.25) \\ &\approx 0.655 \end{aligned}$$

About 655 millilitres of water leak from the watering can in the first 30 seconds.

ii

$$\begin{aligned} \int_0^1 [R_1(t) - R_2(t)] \, dt &= \int_0^1 \left[6.4 - \left(2.5 - 1.25e^{-0.2t}\right)\right] dt \\ &= \int_0^1 \left(3.9 + 1.25e^{-0.2t}\right) dt \\ &= \left[3.9t + 1.25\left(\tfrac{1}{-0.2}\right) e^{-0.2t}\right]_0^1 \\ &= \left[3.9t - 6.25e^{-0.2t}\right]_0^1 \\ &= (3.9 - 6.25e^{-0.2}) - (0 - 6.25) \\ &\approx 5.03 \end{aligned}$$

There are about 5.03 litres of water in the watering can after 1 minute.

b Suppose it takes x seconds for the watering can to be full.

We find x such that $\int_0^x [R_1(t) - R_2(t)]\, dt = 16$

$$\therefore \left[3.9t - 6.25e^{-0.2t}\right]_0^x = 16 \qquad \{\text{using } \textbf{a ii}\}$$

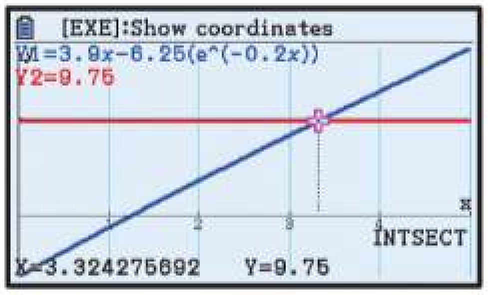
$$\therefore (3.9x - 6.25e^{-0.2x}) - (0 - 6.25) = 16$$

$$\therefore 3.9x - 6.25e^{-0.2x} + 6.25 = 16$$

$$\therefore 3.9x - 6.25e^{-0.2x} = 9.75$$

$$\therefore x \approx 3.324$$

{using technology}



$$3.324 \text{ minutes} \approx 3.324 \times 60 \text{ seconds}$$
$$\approx 199 \text{ seconds}$$

$\therefore$ it will take about 199 seconds for the watering can to be full.

REVIEW SET 23B

1 **a** $$\int_2^3 \frac{1}{\sqrt{3x}}\, dx = \int_2^3 (3x)^{-\frac{1}{2}}\, dx$$
$$= \left[\left(\tfrac{1}{3}\right)\frac{(3x)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3$$
$$= \left[\tfrac{2}{3}\sqrt{3x}\right]_2^3$$
$$= \tfrac{2}{3}\sqrt{9} - \tfrac{2}{3}\sqrt{6}$$
$$= 2 - \tfrac{2}{3}\sqrt{2}\sqrt{3}$$
$$= 2 - \tfrac{2\sqrt{2}}{\sqrt{3}}$$

b $$\int_1^4 \left(x - \tfrac{1}{2}x^2\right) dx$$
$$= \left[\tfrac{1}{2}x^2 - \tfrac{1}{6}x^3\right]_1^4$$
$$= \left(\tfrac{1}{2}(16) - \tfrac{1}{6}(64)\right) - \left(\tfrac{1}{2} - \tfrac{1}{6}\right)$$
$$= -\tfrac{8}{3} - \tfrac{1}{3}$$
$$= -3$$

c $$\int_0^1 \left(x^2 + \tfrac{1}{3}\right)^2 dx$$
$$= \int_0^1 \left(x^4 + \tfrac{2}{3}x^2 + \tfrac{1}{9}\right) dx$$
$$= \left[\tfrac{1}{5}x^5 + \tfrac{2}{9}x^3 + \tfrac{1}{9}x\right]_0^1$$
$$= \left(\tfrac{1}{5} + \tfrac{2}{9} + \tfrac{1}{9}\right) - 0$$
$$= \tfrac{8}{15}$$

2 **a** $$\int_2^3 \frac{1}{\sqrt{3x-4}}\, dx = \int_2^3 (3x-4)^{-\frac{1}{2}}\, dx$$
$$= \left[\left(\tfrac{1}{3}\right)\frac{(3x-4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3$$
$$= \left[\tfrac{2}{3}\sqrt{3x-4}\right]_2^3$$
$$= \tfrac{2}{3}\sqrt{5} - \tfrac{2}{3}\sqrt{2}$$
$$= \tfrac{2}{3}(\sqrt{5} - \sqrt{2})$$

b $$\int_{2e}^{3e} \frac{4}{x+e}\, dx = \left[4\ln|x+e|\right]_{2e}^{3e}$$
$$= 4\ln 4e - 4\ln 3e$$
$$= 4(\ln 4e - \ln 3e)$$
$$= 4\ln\left(\frac{4e}{3e}\right)$$
$$= 4\ln\left(\tfrac{4}{3}\right)$$

c $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin x + 1)\, dx = \left[-2\cos x + x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$$= \left(-2\cos\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) - \left(-2\cos\tfrac{\pi}{4} + \tfrac{\pi}{4}\right)$$

$$= \tfrac{\pi}{2} - \left(-\sqrt{2} + \tfrac{\pi}{4}\right)$$

$$= \tfrac{\pi}{2} + \sqrt{2} - \tfrac{\pi}{4}$$

$$= \tfrac{\pi}{4} + \sqrt{2}$$

3 $\displaystyle\int_0^b \cos x\, dx = \tfrac{1}{\sqrt{2}}, \quad 0 < b < \pi$

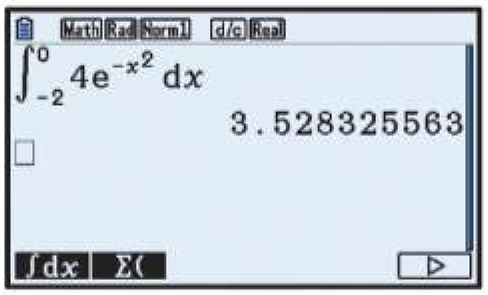
$\therefore\ \left[\sin x\right]_0^b = \tfrac{1}{\sqrt{2}}$

$\therefore\ \sin b - 0 = \tfrac{1}{\sqrt{2}}$

$\therefore\ \sin b = \tfrac{1}{\sqrt{2}}$

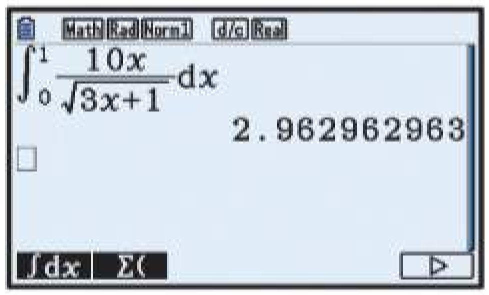
$\therefore\ b = \tfrac{\pi}{4}$ or $\tfrac{3\pi}{4}$

4 **a**



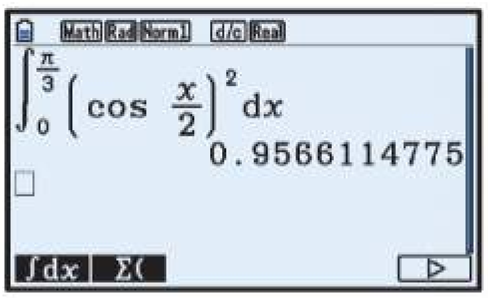
$\therefore\ \displaystyle\int_{-2}^{0} 4e^{-x^2}\, dx \approx 3.528$

b



$\therefore\ \displaystyle\int_0^1 \frac{10x}{\sqrt{3x+1}}\, dx \approx 2.963$

c



$\therefore\ \displaystyle\int_0^{\frac{\pi}{3}} \cos^2\left(\tfrac{x}{2}\right) dx \approx 0.9566$

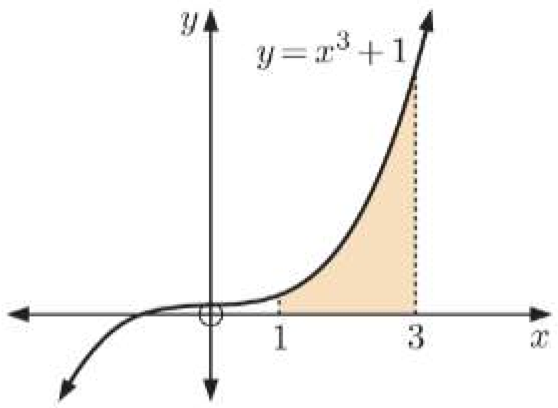
5 Let $u = 1 + \sin x \quad \therefore \quad \dfrac{du}{dx} = \cos x$

When $x = 0, \quad u = 1$

When $x = \frac{\pi}{6}, \quad u = 1 + \frac{1}{2} = \frac{3}{2}$

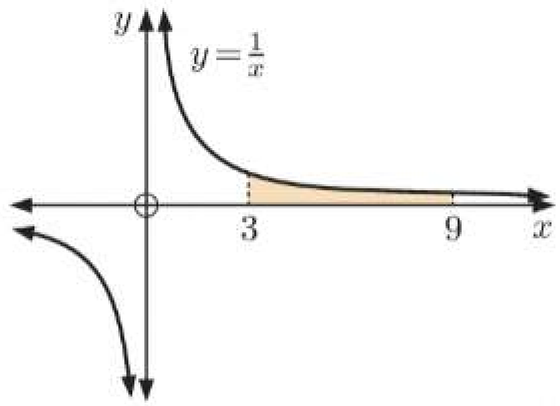
$$\therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{(1+\sin x)^3}\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{u^3}\frac{du}{dx}\,dx$$
$$= \int_1^{\frac{3}{2}} u^{-3}\,du$$
$$= \left[\frac{u^{-2}}{-2}\right]_1^{\frac{3}{2}}$$
$$= -\tfrac{1}{2}\left(\tfrac{2}{3}\right)^2 + \tfrac{1}{2}$$
$$= \tfrac{5}{18}$$

6 **a**



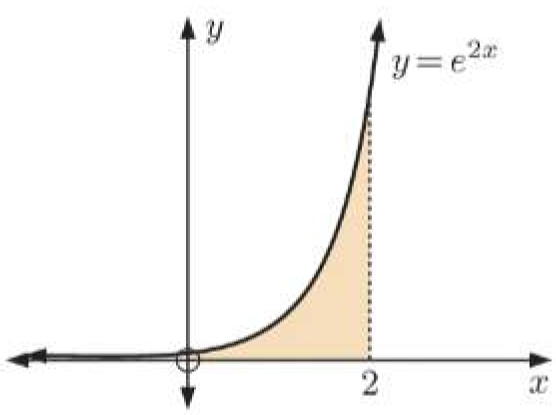
$$\text{Area} = \int_1^3 (x^3+1)\,dx$$
$$= \left[\tfrac{1}{4}x^4 + x\right]_1^3$$
$$= \left(\tfrac{81}{4}+3\right) - \left(\tfrac{1}{4}+1\right)$$
$$= 22 \text{ units}^2$$

b



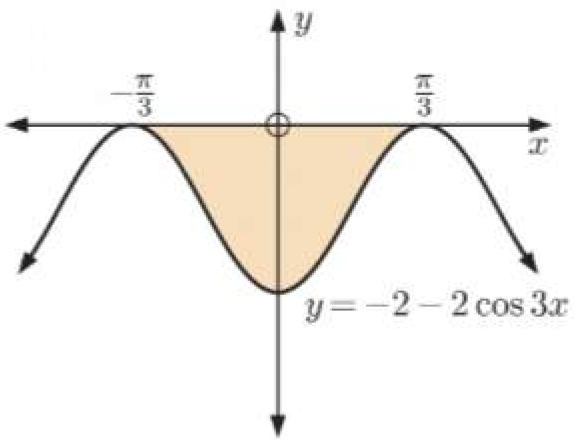
$$\text{Area} = \int_3^9 \frac{1}{x}\,dx$$
$$= \left[\ln|x|\right]_3^9$$
$$= \ln 9 - \ln 3$$
$$= \ln\left(\tfrac{9}{3}\right)$$
$$= \ln 3 \text{ units}^2$$

c



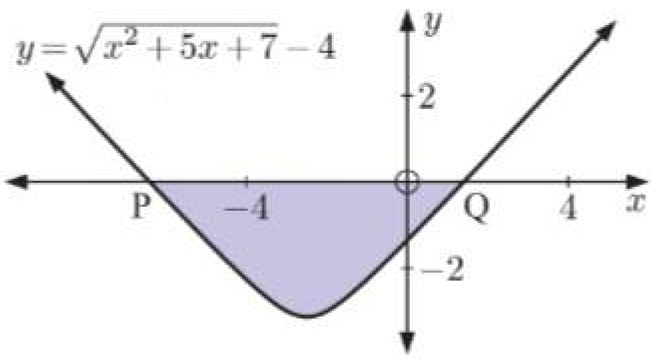
$$\text{Area} = \int_0^2 e^{2x}\,dx$$
$$= \left[\tfrac{1}{2}e^{2x}\right]_0^2$$
$$= \tfrac{1}{2}e^4 - \tfrac{1}{2}$$
$$= \frac{e^4-1}{2} \text{ units}^2$$

d

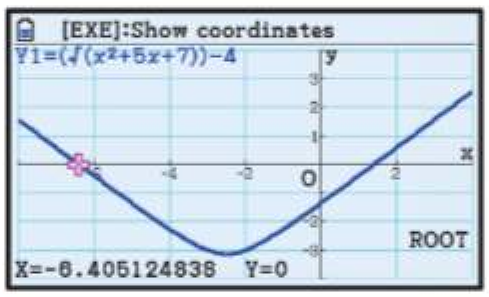


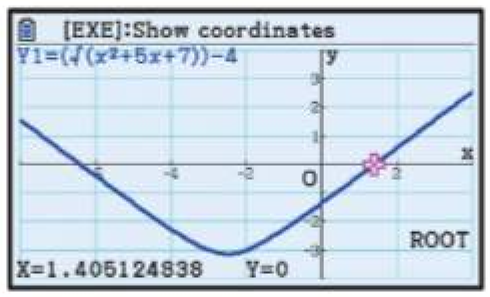
$$\begin{aligned}\text{Area} &= -\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (-2-2\cos 3x)\,dx \\ &= -\left[-2x-\tfrac{2}{3}\sin 3x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= -\left[\left(-\tfrac{2\pi}{3}-\tfrac{2}{3}\sin \pi\right)-\left(\tfrac{2\pi}{3}-\tfrac{2}{3}\sin(-\pi)\right)\right] \\ &= -\left(-\tfrac{2\pi}{3}-\tfrac{2\pi}{3}\right) \\ &= \tfrac{4\pi}{3} \text{ units}^2\end{aligned}$$

7

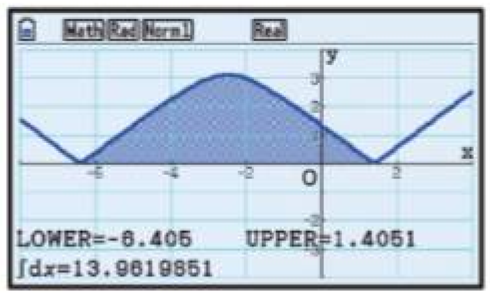


a Using technology, the x-coordinate of P ≈ -6.41, and the x-coordinate of Q ≈ 1.41.





b Using technology, the shaded area ≈ 14.0 units^2.

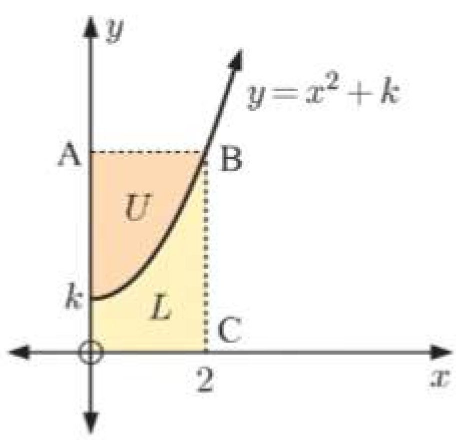


8 B has coordinates $(2,\ 2^2+k)$, or $(2,\ 4+k)$.

$\therefore$ the horizontal line from A to B is $y=4+k$.

Now, upper area U = lower area L

$$\begin{aligned}\therefore \int_0^2 [(4+k)-(x^2+k)]\,dx &= \int_0^2 (x^2+k)\,dx \\ \therefore \int_0^2 (-x^2+4)\,dx &= \int_0^2 (x^2+k)\,dx \\ \therefore \left[-\tfrac{1}{3}x^3+4x\right]_0^2 &= \left[\tfrac{1}{3}x^3+kx\right]_0^2 \\ \therefore \left(-\tfrac{8}{3}+8\right)-0 &= \left(\tfrac{8}{3}+2k\right)-0 \\ \therefore \tfrac{16}{3} &= \tfrac{8}{3}+2k \\ \therefore \tfrac{8}{3} &= 2k \\ \therefore k &= \tfrac{4}{3}\end{aligned}$$



9 Area $= \int_0^m \sin x \, dx = \frac{1}{2}$

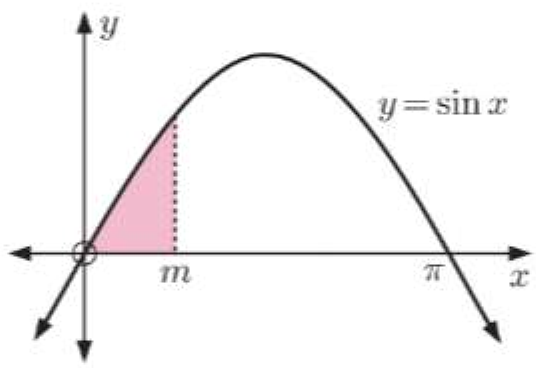
$\therefore \; \left[-\cos x\right]_0^m = \frac{1}{2}$

$\therefore \; -\cos m - (-1) = \frac{1}{2}$

$\therefore \; -\cos m = -\frac{1}{2}$

$\therefore \; \cos m = \frac{1}{2}$

$\therefore \; m = \frac{\pi}{3}$



10 **a** $y = \frac{1}{x}$

$\therefore \; x = \frac{1}{y}$

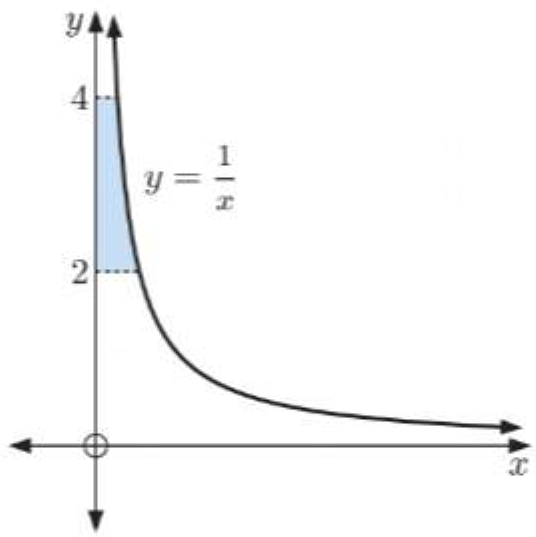
$\therefore \; f^{-1}(y) = \frac{1}{y}$

$\therefore$ area $= \int_2^4 \frac{1}{y} \, dy$

$= \left[\ln|\,y\,|\right]_2^4$

$= \ln 4 - \ln 2$

$= \ln 2$ units^2



b $y = x^3 + 2$

$\therefore \; x^3 = y - 2$

$\therefore \; x = \sqrt[3]{y-2}$

$\therefore \; f^{-1}(y) = \sqrt[3]{y-2}$

$\therefore$ area $= \int_3^6 (y-2)^{\frac{1}{3}} \, dy$

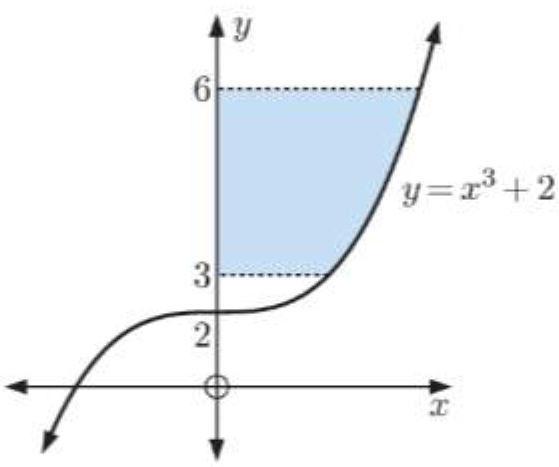
$= \left[\dfrac{(y-2)^{\frac{4}{3}}}{\frac{4}{3}}\right]_3^6$

$= \left[\frac{3}{4}(y-2)^{\frac{4}{3}}\right]_3^6$

$= \frac{3}{4}(4)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}}$

$= \frac{3}{4} \times 4 \times \sqrt[3]{4} - \frac{3}{4}$

$= \left(3\sqrt[3]{4} - \frac{3}{4}\right)$ units^2

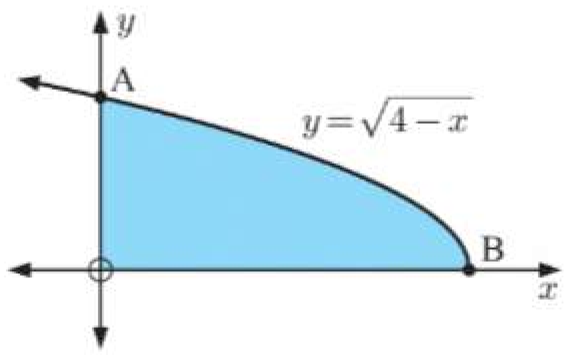


11 **a** When $x = 0$, $y = \sqrt{4-0}$
$= 2$

$\therefore$ A has coordinates $(0, 2)$.

When $y = 0$, $\sqrt{4-x} = 0$
$\therefore\ 4 - x = 0$
$\therefore\ x = 4$

$\therefore$ B has coordinates $(4, 0)$.



b **i** $y = \sqrt{4-x}$

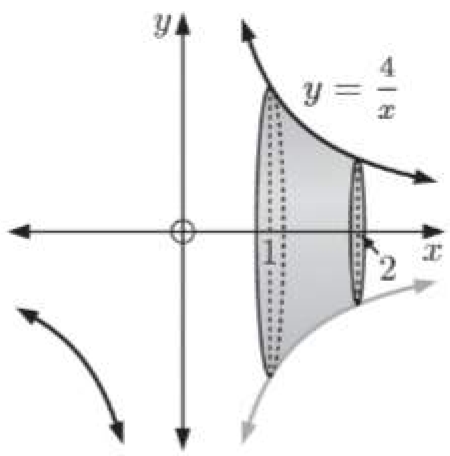
$$\therefore\ \text{area} = \int_0^4 (4-x)^{\frac{1}{2}}\,dx$$
$$= \left[-\tfrac{2}{3}(4-x)^{\frac{3}{2}}\right]_0^4$$
$$= -\tfrac{2}{3}(0)^{\frac{3}{2}} - \left(-\tfrac{2}{3}(4)^{\frac{3}{2}}\right)$$
$$= \tfrac{16}{3} = 5\tfrac{1}{3}\text{ units}^2$$

ii $y = \sqrt{4-x}$
$\therefore\ y^2 = 4 - x$
$\therefore\ x = 4 - y^2$

$$\therefore\ \text{area} = \int_0^2 (4 - y^2)\,dy$$
$$= \left[4y - \tfrac{1}{3}y^3\right]_0^2$$
$$= 8 - \tfrac{8}{3} - (0 - 0)$$
$$= \tfrac{16}{3} = 5\tfrac{1}{3}\text{ units}^2$$

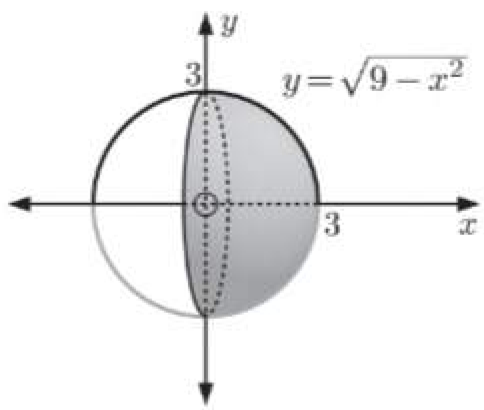
The area is the same whichever axis is being considered.

12 **a**



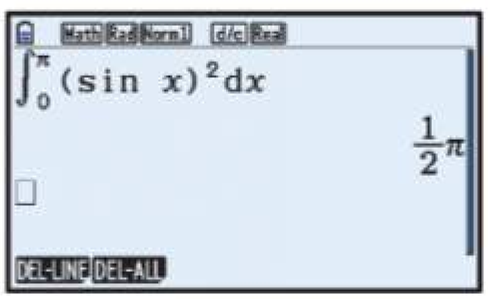
$$\text{Volume} = \pi\int_1^2 \frac{16}{x^2}\,dx$$
$$= \pi\left[-\frac{16}{x}\right]_1^2$$
$$= \pi(-8 + 16)$$
$$= 8\pi\text{ units}^3$$

b



$$\text{Volume} = \pi\int_0^3 (9 - x^2)\,dx$$
$$= \pi\left[9x - \frac{x^3}{3}\right]_0^3$$
$$= \pi\left(27 - \tfrac{27}{3} - 0\right)$$
$$= 18\pi\text{ units}^3$$

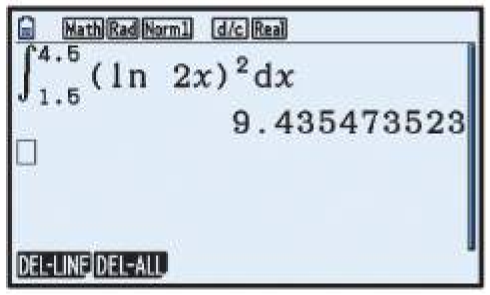
13 **a** Using technology, $\text{volume} = \pi\displaystyle\int_0^\pi \sin^2 x\,dx$
$= \pi \times \frac{1}{2}\pi$
$= \dfrac{\pi^2}{2}$
$\approx 4.93\text{ units}^3$



b Using technology, volume $= \pi \displaystyle\int_{1.5}^{4.5} [\ln(2x)]^2 \, dx$

$\approx 9.4355\pi$

≈ 29.6 units^3



14 **a** $\dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{-\sin x \sin x - \cos x \cos x}{\sin^2 x}$ {quotient rule}

$= \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= -\dfrac{1}{\sin^2 x}$

b Volume $= \pi \displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} y^2 \, dx$

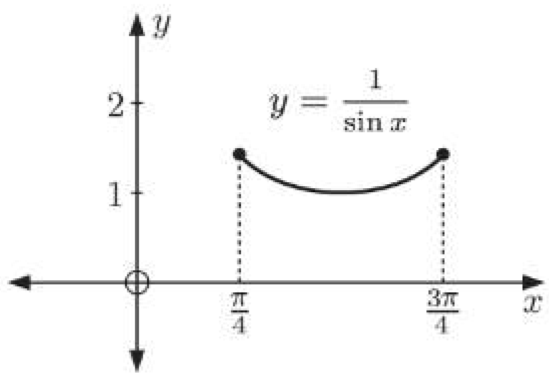
$= \pi \displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{\sin^2 x} \, dx$

$= \pi \left[-\dfrac{\cos x}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$ {using **a**}

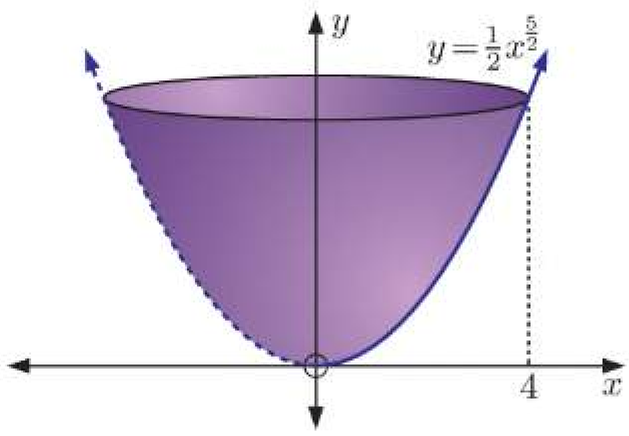
$= \pi \left(-\dfrac{\cos \frac{3\pi}{4}}{\sin \frac{3\pi}{4}} + \dfrac{\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}}\right)$

$= \pi(-(-1) + 1)$

$= 2\pi$ units^3



15 **a** The height of the vase is $y = \frac{1}{2}(4)^{\frac{5}{2}} = 16$ cm.



b $y = \frac{1}{2}x^{\frac{5}{2}} \quad \therefore \quad x^2 = (2y)^{\frac{4}{5}}$

$$\begin{aligned}\text{Volume} &= \pi \int_0^{16} x^2 \, dy \\ &= \pi \int_0^{16} (2y)^{\frac{4}{5}} \, dy \\ &= 2^{\frac{4}{5}} \pi \left[\tfrac{5}{9} y^{\frac{9}{5}} \right]_0^{16} \\ &= 2^{\frac{4}{5}} \pi \left(\tfrac{5}{9} \times 2^{\frac{36}{5}} - 0 \right) \\ &= 2^8 \pi \times \tfrac{5}{9} \\ &= \tfrac{1280\pi}{9} \text{ cm}^3\end{aligned}$$

c

$$\begin{aligned}\text{Volume} &= \pi \int_0^{10} (2y)^{\frac{4}{5}} \, dy \\ &= 2^{\frac{4}{5}} \pi \left[\tfrac{5}{9} y^{\frac{9}{5}} \right]_0^{10} \\ &= 2^{\frac{4}{5}} \pi \left(\tfrac{5}{9} \times 10^{\frac{9}{5}} - 0 \right) \\ &\approx 192\end{aligned}$$

$\therefore$ the vase contains about 192 mL of water.

16 $E(t) = 2\sin\left(\frac{t-5}{5}\right) + \frac{1}{2}\sin\left(\frac{t-5}{4}\right)$ kW

$$\begin{aligned}\therefore \int E(t)\, dt &= \int \left(2\sin\left(\tfrac{t-5}{5}\right) + \tfrac{1}{2}\sin\left(\tfrac{t-5}{4}\right) \right) dt \\ &= 2\left(-\cos\left(\tfrac{t-5}{5}\right)(5) \right) + \tfrac{1}{2}\left(-\cos\left(\tfrac{t-5}{4}\right)(4) \right) + c \\ &= -10\cos\left(\tfrac{t-5}{5}\right) - 2\cos\left(\tfrac{t-5}{4}\right) + c\end{aligned}$$

a

$$\begin{aligned}\int_5^{12} E(t)\, dt &= \left[-10\cos\left(\tfrac{t-5}{5}\right) - 2\cos\left(\tfrac{t-5}{4}\right) \right]_5^{12} \\ &= \left(-10\cos\tfrac{7}{5} - 2\cos\tfrac{7}{4} \right) - \left(-10\cos 0 - 2\cos 0 \right) \\ &\approx 10.66\end{aligned}$$

The solar energy transferred into Callum's solar panels from 5 am to 12 pm is about 10.7 kWh.

b

$$\begin{aligned}\int_{12}^{20} E(t)\, dt &= \left[-10\cos\left(\tfrac{t-5}{5}\right) - 2\cos\left(\tfrac{t-5}{4}\right) \right]_{12}^{20} \\ &= \left(-10\cos 3 - 2\cos\tfrac{15}{4} \right) - \left(-10\cos\tfrac{7}{5} - 2\cos\tfrac{7}{4} \right) \\ &\approx 12.88\end{aligned}$$

The solar energy transferred into Callum's solar panels from 12 pm to 8 pm is about 12.9 kWh.

c

$$\begin{aligned}\int_5^{20} E(t)\, dt &= \int_5^{12} E(t)\, dt + \int_{12}^{20} E(t)\, dt \\ &\approx 10.66 + 12.88 \qquad \{\text{using } \mathbf{a} \text{ and } \mathbf{b}\} \\ &\approx 23.54\end{aligned}$$

The solar energy transferred into Callum's solar panels from 5 am to 8 pm is about 23.5 kWh.

Chapter 24

KINEMATICS

EXERCISE 24A

1 $s(t) = 5 - t$ cm, $0 \leqslant t \leqslant 10$ s

a $s(0) = 5 - 0 = 5$ cm

$\therefore$ the initial displacement of the object is 5 cm to the right of the origin.

b **i** $s(3) = 5 - 3 = 2$ cm

At time $t = 3$ seconds, the object is 2 cm to the right of the origin.

ii $s(10) = 5 - 10 = -5$ cm

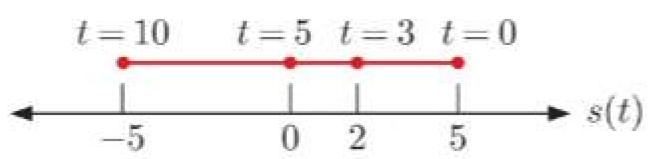
At time $t = 10$ seconds, the object is 5 cm to the left of the origin.

c $s(t) = 5 - t = 0$ when $t = 5$

$\therefore$ the object reaches the origin at time $t = 5$ seconds.

d No, the displacement function $s(t)$ is linear, so it has no turning points.

e

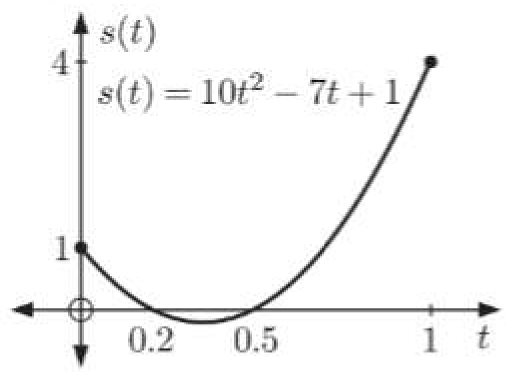


2 $s(t) = 10t^2 - 7t + 1$ m, $0 \leqslant t \leqslant 1$ s

a $s(0) = 1$ m

$\therefore$ the initial displacement of the object is 1 m to the right of the origin.

b $s(t) = 10t^2 - 7t + 1$
$= (5t - 1)(2t - 1)$



c The object changes direction at the turning point of $s(t)$.

This occurs when $t = \dfrac{-(-7)}{2(10)}$

$= \frac{7}{20} = 0.35$ s

$s(0.35) = 10(0.35)^2 - 7(0.35) + 1$

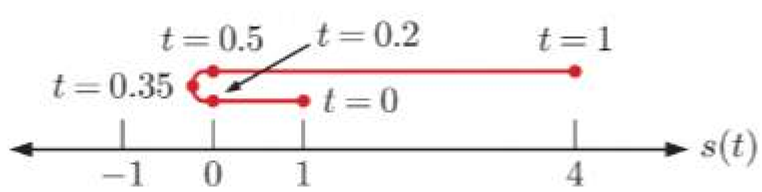
$= -0.225$

$\therefore$ the object changes direction after 0.35 seconds, when it is 0.225 m to the left of the origin.

d The object is to the right of the origin when $s(t) > 0$.

This occurs for $0 \leqslant t < 0.2$ s and 0.5 s $< t \leqslant 1$ s.

e



3 $x(t) = 8\sin 8\pi t + 6$ cm, $0 \leqslant t \leqslant 0.25$ s

a $x(t) = 6$

$\therefore\ 8\sin 8\pi t + 6 = 6$

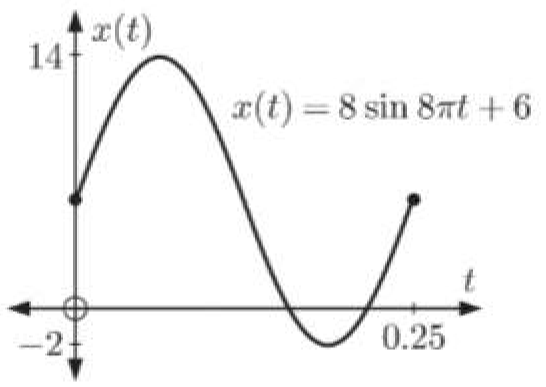
$\therefore\ 8\sin 8\pi t = 0$

$\therefore\ \sin 8\pi t = 0$

$\therefore\ 8\pi t = 0,\ \pi,\ 2\pi,\ ....$ $\{\text{since } t \geqslant 0\}$

$\therefore\ t = 0$ s, 0.125 s, or 0.25 s $\{0 \leqslant t \leqslant 0.25 \text{ s}\}$

b



c The mass changes direction at the turning points of $x(t)$.

This occurs when $\sin 8\pi t = 1$ or $\sin 8\pi t = -1$, $0 \leqslant t \leqslant 0.25$

$\therefore\ 8\pi t = \frac{\pi}{2}$ $\therefore\ \sin 8\pi t = \frac{3\pi}{2}$

$\therefore\ t = \frac{1}{16}$ $\therefore\ t = \frac{3}{16}$

$= 0.0625$ $= 0.1875$

When $t = 0.0625$, $\sin 8\pi t = 1$

$\therefore\ 8\sin 8\pi t + 6 = 8(1) + 6$

$= 14$

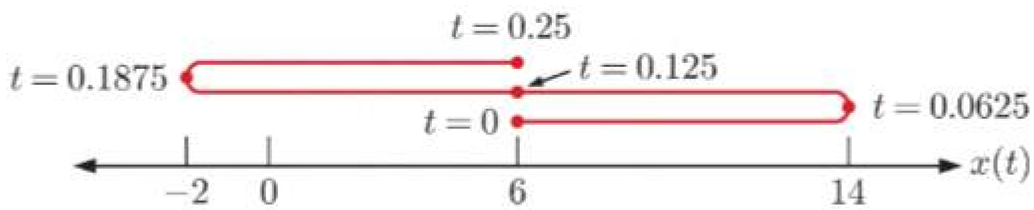
When $t = 0.1875$, $\sin 8\pi t = -1$

$\therefore\ 8\sin 8\pi t + 6 = 8(-1) + 6$

$= -2$

$\therefore$ the mass changes direction 14 cm to the right of the origin, at $t = 0.0625$ seconds, and 2 cm to the left of the origin, at $t = 0.1875$ seconds.

d



EXERCISE 24B.1

1 **a** **i** At $t = 2$ seconds, the displacement is 3 m.

ii At $t = 8$ seconds, the displacement is -3 m.

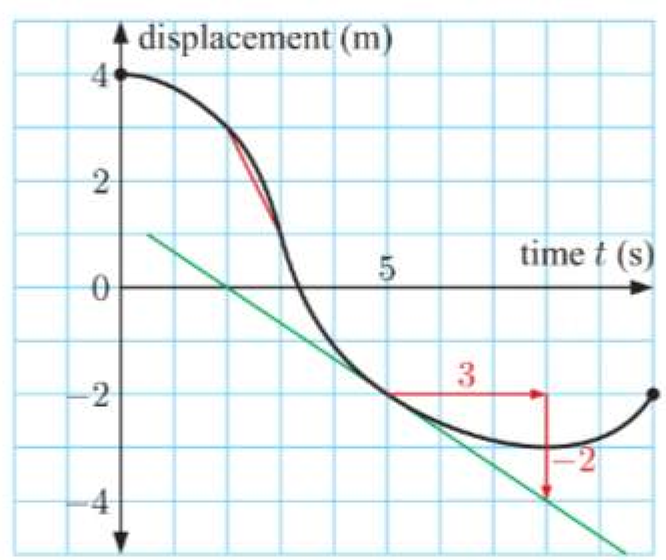
b average velocity $= \dfrac{s(3) - s(2)}{3 - 2}$

$= \dfrac{1 - 3}{1}$

$= -2 \text{ m s}^{-1}$

c The gradient of the tangent at $t = 5$ seconds is $-\frac{2}{3}$.

$\therefore$ the instantaneous velocity at $t = 5$ seconds is $-\frac{2}{3} \text{ m s}^{-1}$.



2 $s(t) = t^2 - 6t + 1$ m, $t \geqslant 0$ s

a $s(1) = (1)^2 - 6(1) + 1$
$\quad = 1 - 6 + 1$
$\quad = -4$ m

$s(3) = (3)^2 - 6(3) + 1$
$\quad = 9 - 18 + 1$
$\quad = -8$ m

$$\begin{aligned}\text{average velocity} &= \frac{s(3) - s(1)}{3 - 1}\\ &= \frac{-8 - (-4)}{2}\\ &= \frac{-4}{2}\\ &= -2 \text{ m s}^{-1}\end{aligned}$$

b $v(t) = s'(t) = 2t - 6 \text{ m s}^{-1}$

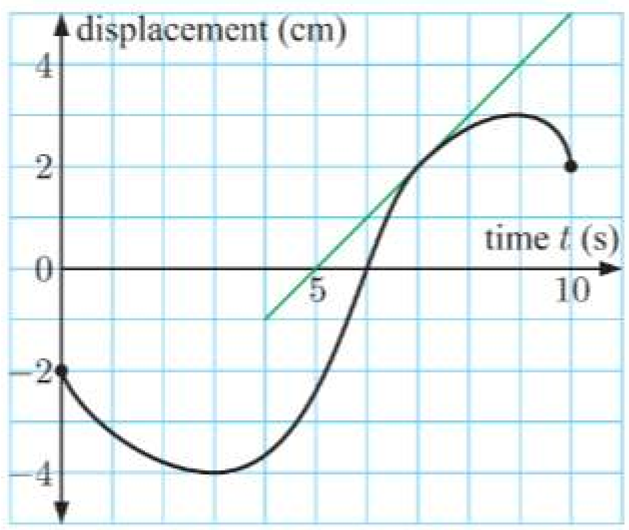
c **i** $v(1) = 2(1) - 6$
$\quad = 2 - 6$
$\quad = -4 \text{ m s}^{-1}$

$\therefore$ the instantaneous velocity at $t = 1$ second is -4 m s^{-1}.

ii $v(5) = 2(5) - 6$
$\quad = 10 - 6$
$\quad = 4 \text{ m s}^{-1}$

$\therefore$ the instantaneous velocity at $t = 5$ seconds is 4 m s^{-1}.

3 **a** At time $t = 0$ seconds, the displacement of the object is -2 cm.
$\therefore$ the object is initially 2 cm to the left of the origin.

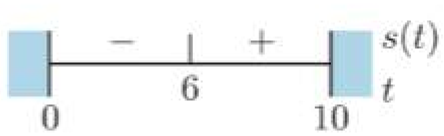


b The displacement of the object is 0 cm at time $t = 6$ seconds.
$\therefore$ the object is at the origin when $t = 6$ seconds.

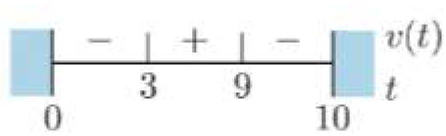
c At time $t = 5$ seconds, the object has negative displacement, but this value is increasing.
$\therefore$ the object is moving to the right when $t = 5$ seconds.

d The object changes direction at the turning points of the displacment graph.
$\therefore$ the object changes direction at times $t = 3$ seconds and $t = 9$ seconds.

e **i**



ii



f The gradient of the tangent at $t = 7$ is 1.
$\therefore$ the instantaneous velocity at $t = 7$ seconds is 1 cm s^{-1}.

4 $s(t) = 2\sqrt{t} + 3$ cm, $t \geqslant 0$ s

a $s(1) = 2\sqrt{1} + 3 = 2 + 3 = 5$ cm $\qquad s(4) = 2\sqrt{4} + 3 = 4 + 3 = 7$ cm

$$\text{average velocity} = \frac{s(4) - s(1)}{4 - 1} = \frac{7-5}{3} = \tfrac{2}{3} \text{ cm s}^{-1}$$

b $s(0) = 2\sqrt{0} + 3 = 3$ cm

$\therefore$ the initial position of the object is 3 cm to the right of the origin.

c $s(t) = 2t^{\frac{1}{2}} + 3$

$\therefore \ v(t) = s'(t) = t^{-\frac{1}{2}} = \dfrac{1}{\sqrt{t}}$ cm s^{-1}

d **i** $v(4) = \dfrac{1}{\sqrt{4}} = \frac{1}{2}$ cm s^{-1}

$\therefore$ the instantaneous velocity at $t = 4$ seconds is $\frac{1}{2}$ cm s^{-1}.

ii $v(16) = \dfrac{1}{\sqrt{16}} = \frac{1}{4}$ cm s^{-1}

$\therefore$ the instantaneous velocity at $t = 16$ seconds is $\frac{1}{4}$ cm s^{-1}.

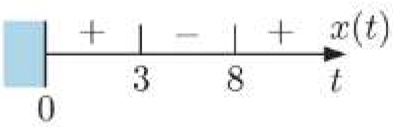
5 $x(t) = t^3 - 11t^2 + 24t$ m, $t \geqslant 0$ s

a $\dot{x}(t) = x'(t) = 3t^2 - 22t + 24$ m s^{-1}

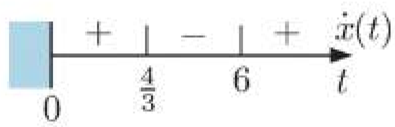
b $x(0) = 0$ m, $\dot{x}(0) = 24$ m s^{-1}

$\therefore$ the object is initially at the origin, moving to the right at 24 m s^{-1}.

c $x(t) = t(t-3)(t-8)$ has sign diagram:



$\dot{x}(t) = (3t-4)(t-6)$ has sign diagram:



d $x(t) = 0$ when $t = 0, 3,$ or 8 {from **c**}

$\therefore$ the object is at O at $t = 0$ seconds, 3 seconds, and 8 seconds.

e The object changes direction when $\dot{x}(t)$ changes sign.

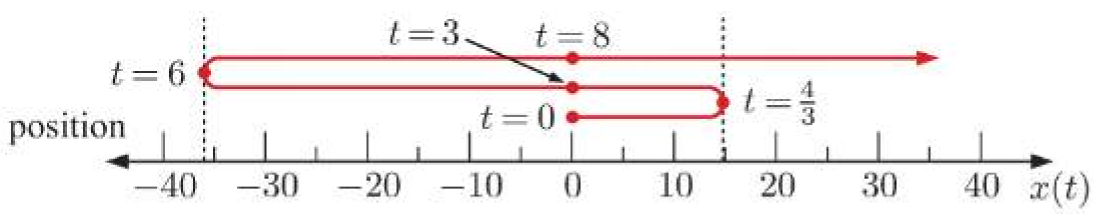
This occurs when $t = \frac{4}{3}$ and $t = 6$ {from **c**}

$x\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + 24\left(\frac{4}{3}\right) \approx 14.8$ m $\qquad x(6) = (6)^3 - 11(6)^2 + 24(6) = -36$ m

$\therefore$ the object changes direction at $t = \frac{4}{3}$ seconds when it is about 14.8 m to the right of the origin, and at $t = 6$ seconds, when it is 36 m to the left of the origin.

f The object starts at O, and moves towards the right at 24 m s^{-1}. Its velocity is decreasing. After $\frac{4}{3}$ seconds, when it is about 14.8 m to the right of O, it changes direction and moves to the left, passing O after 3 seconds. After 6 seconds, when it is 36 m to the left of O, it changes direction again and moves towards the right, passing O once more after 8 seconds.

g



6 $s(t) = bt - 4.9t^2$ m, $t \geqslant 0$ s

a $v(t) = s'(t) = b - 9.8t$

$\therefore\ v(0) = b - 9.8(0)$
$= b\ \text{m}\,\text{s}^{-1}$

$\therefore$ the initial velocity is $b\ \text{m}\,\text{s}^{-1}$ upwards.

b **i** The shell reaches its maximum height after 7.1 seconds.

$\therefore$ the velocity of the shell at $t = 7.1$ seconds is zero.

$\therefore\ v(7.1) = 0$

$\therefore\ b - 9.8(7.1) = 0$

$\therefore\ b - 69.58 = 0$

$\therefore\ b = 69.58$

$\therefore$ the initial velocity of the shell is $69.58\ \text{m}\,\text{s}^{-1}$ upwards.

ii $s(t) = 69.58t - 4.9t^2$

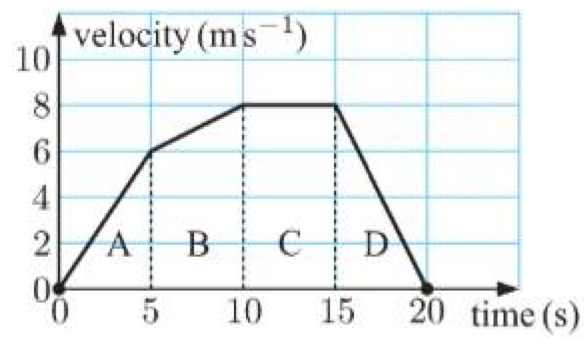
The shell reaches its maximum height after 7.1 seconds.

$s(7.1) = 69.58(7.1) - 4.9(7.1)^2$
≈ 247 m

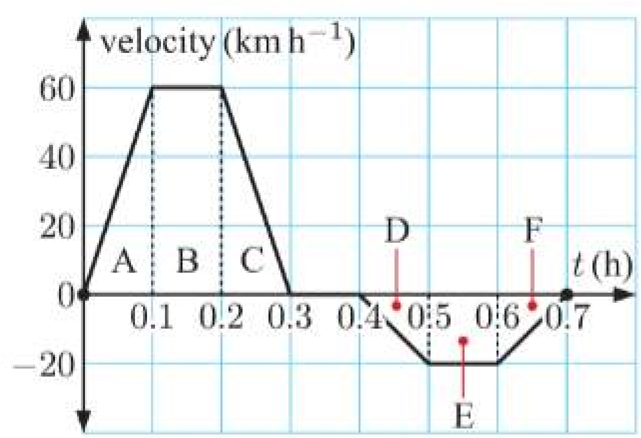
$\therefore$ the shell reached a maximum height of about 247 m.

EXERCISE 24B.2

1 Total distance travelled
= total area under the graph
= area A + area B + area C + area D
$= \frac{1}{2}(5)(6) + \left(\frac{6+8}{2}\right)(5) + (5)(8) + \frac{1}{2}(5)(8)$
$= 15 + 35 + 40 + 20$
$= 110$ m



2



a **i** When the graph is above the t-axis, the car is travelling forwards.

ii When the graph is below the t-axis, the car is travelling backwards (in the opposite direction).

b Total distance travelled
= total area between the graph and the t-axis
= area A + area B + area C + area D + area E + area F
$= \frac{1}{2}(0.1)(60) + (0.1)(60) + \frac{1}{2}(0.1)(60) + \frac{1}{2}(0.1)(20) + (0.1)(20) + \frac{1}{2}(0.1)(20)$
$= 3 + 6 + 3 + 1 + 2 + 1$
$= 16$ km

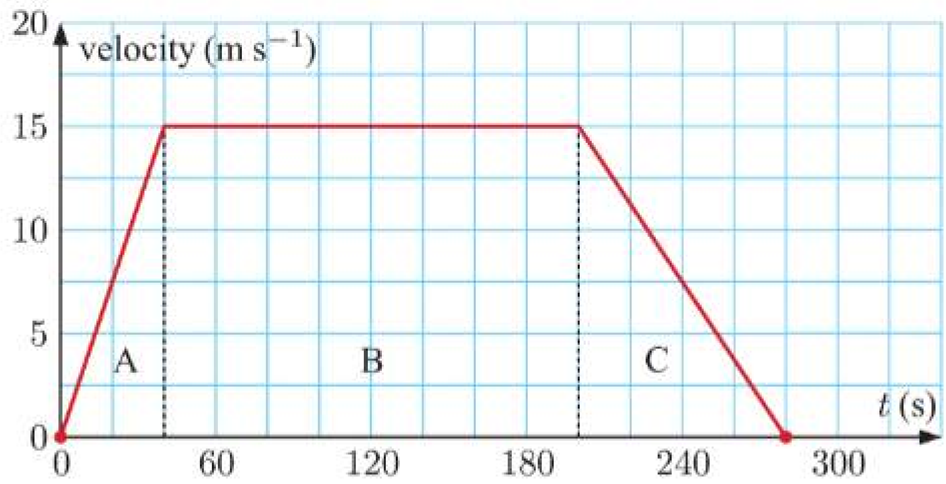
c Displacement = forward distance travelled − backward distance travelled

$= \text{area A} + \text{area B} + \text{area C} - \text{area D} - \text{area E} - \text{area F}$

$= 3 + 6 + 3 - 1 - 2 - 1$

$= 8$ km from the starting point (on positive side)

3 **a**



b Total distance travelled = total area under graph

$= \text{area A} + \text{area B} + \text{area C}$

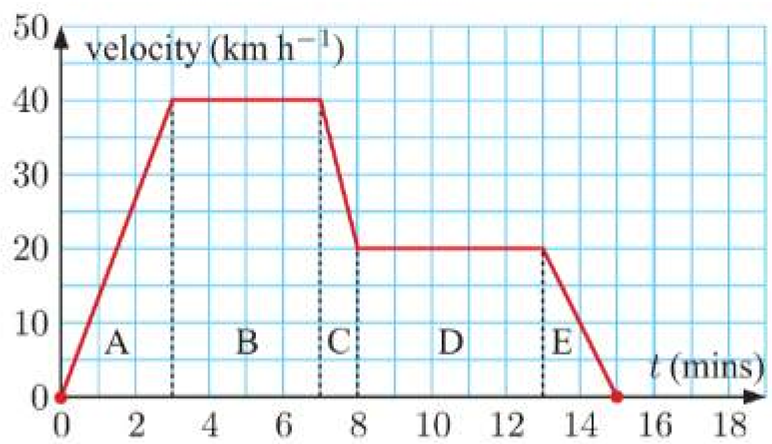
$= \frac{1}{2}(40)(15) + (160)(15) + \frac{1}{2}(80)(15)$

$= 300 + 2400 + 600$

$= 3300$ m

$= 3.3$ km

4 **a**



b Total distance travelled

= total area under graph

$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$

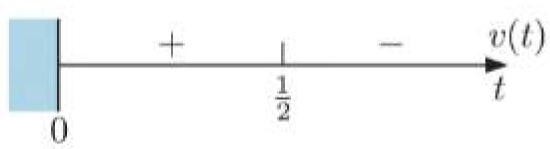
$= \frac{1}{2}\left(\frac{3}{60}\right)(40) + \left(\frac{4}{60}\right)(40) + \left(\frac{40+20}{2}\right)\left(\frac{1}{60}\right) + \left(\frac{5}{60}\right)(20) + \frac{1}{2}\left(\frac{2}{60}\right)(20)$ $\quad \{t \text{ min} = \frac{t}{60} \text{ hours}\}$

$= 1 + \frac{8}{3} + \frac{1}{2} + \frac{5}{3} + \frac{1}{3}$

$= 6\frac{1}{6}$ km

5 **a** $v(t) = s'(t) = 1 - 2t$

$\therefore$ the sign diagram of v is:

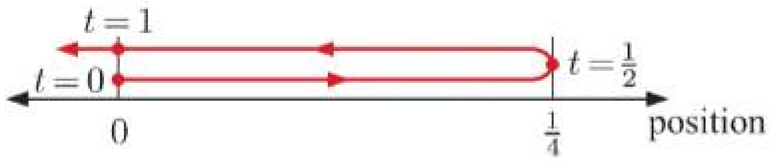


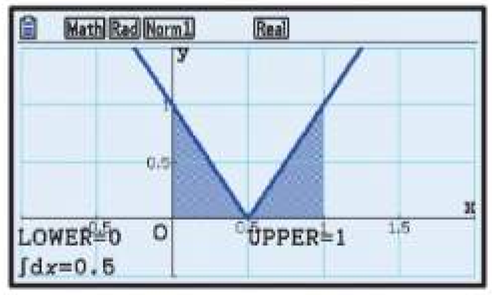
Since the sign changes, the particle changes direction at $t = \frac{1}{2}$ second.

b $s(t) = \int (1 - 2t)\, dt$
$\quad = t - t^2 + c$

We choose the initial displacement to be zero,
so $\quad s(0) = c = 0$
$\therefore \quad s\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$
$\quad s(1) = 1 - 1 = 0$

Motion diagram:





$\therefore \quad$ total distance travelled $= \frac{1}{4} + \frac{1}{4}$
$\quad = \frac{1}{2}$ cm

c Displacement = final position − original position
$\quad = s(1) - s(0)$
$\quad = 0 - 0$
$\quad = 0$ cm

So, the particle returned to its original position after one second.

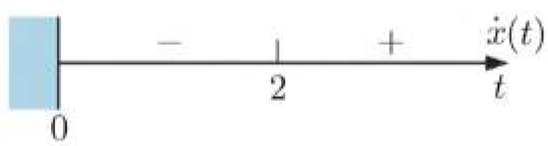
6 **a** $\dot{x}(t) = x'(t) = t^2 - t - 2$

Now $\quad x(t) = \int (t^2 - t - 2)\, dt$
$\quad = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t + c$

The particle is initially at the origin.
$\therefore \quad x(0) = 0$
$\quad \therefore \quad c = 0$
$\therefore \quad x(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t$ cm

b $\dot{x}(t) = t^2 - t - 2$
$\quad = (t + 1)(t - 2)$

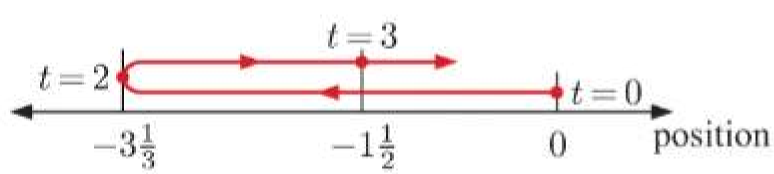
$\therefore \quad$ the sign diagram of $\dot{x}$ is:



Since the sign changes, the particle changes direction at $t = 2$ seconds.

Now $\quad x(0) = 0 \qquad x(2) = \frac{8}{3} - 2 - 4 \qquad x(3) = 9 - \frac{9}{2} - 6$
$\qquad\qquad\qquad = -3\frac{1}{3} \qquad\qquad = -1\frac{1}{2}$

Motion diagram:



$\therefore \quad$ total distance travelled $= \left(0 - \left(-3\frac{1}{3}\right)\right) + \left(-1\frac{1}{2} - \left(-3\frac{1}{3}\right)\right)$
$\quad = 5\frac{1}{6}$ cm

c Displacement = final position − original position

$$= x(3) - x(0)$$
$$= -1\tfrac{1}{2} - 0$$
$$= -1\tfrac{1}{2}$$

So, the particle's displacement is $1\frac{1}{2}$ cm left of its starting position.

7 **a** $v(t) = s'(t) = 29.4 - 9.8t$

$$\therefore \; s(t) = \int (29.4 - 9.8t)\, dt$$
$$= 29.4t - 4.9t^2 + c$$

The ball is initially 1 metre above ground level.

$\therefore \; s(0) = 1$

$\therefore \; c = 1$

$\therefore \; s(t) = 29.4t - 4.9t^2 + 1$ m

b The maximum height reached by the ball occurs when its velocity equals 0.

$\therefore \; 29.4 - 9.8t = 0$

$\therefore \; 9.8t = 29.4$

$\therefore \; t = 3$

So, the maximum height is reached at $t = 3$ seconds.

$s(3) = 29.4(3) - 4.9(3)^2 + 1$

$= 45.1$ m

$\therefore$ the maximum height reached by the ball is 45.1 m.

8 **a** $v(t) = s'(t) = 32 + 4t$

$$\therefore \; s(t) = \int (32 + 4t)\, dt$$
$$= 32t + 2t^2 + c$$

Now $s(0) = 16$

$\therefore \; c = 16$

$\therefore \; s(t) = 32t + 2t^2 + 16$ m

b The moving object changes direction when $v(t) = 0$

$\therefore \; 32 + 4t = 0$

$\therefore \; t = -8$

But $t \geqslant 0$, so there is no change of direction.

$\therefore$ displacement = total distance travelled = $s(\tau) - s(0)$

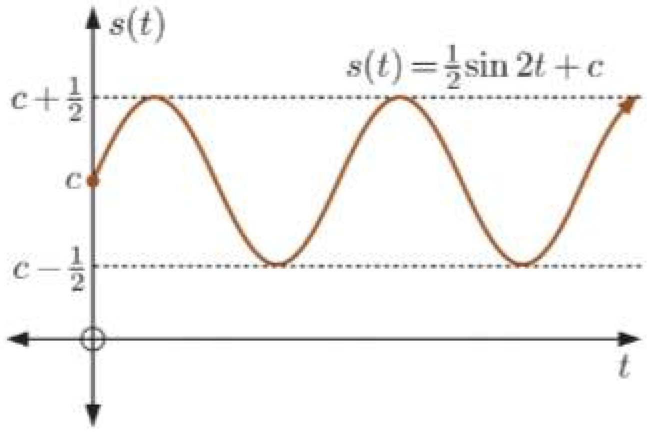
$$= \int_0^{\tau} (32 + 4t)\, dt$$

c Total distance travelled in first 4 seconds $= \int_0^4 (32+4t)\,dt$

$$= \left[32t + 2t^2\right]_0^4$$
$$= (128+32) - 0$$
$$= 160 \text{ m}$$

9 **a** $v(t) = s'(t) = \cos 2t$

$$\therefore \quad s(t) = \int \cos 2t \; dt$$
$$= \tfrac{1}{2}\sin 2t + c$$



The graph shows that the particle oscillates between positions $c+\frac{1}{2}$ and $c-\frac{1}{2}$.

Distance $= (c+\frac{1}{2}) - (c-\frac{1}{2})$

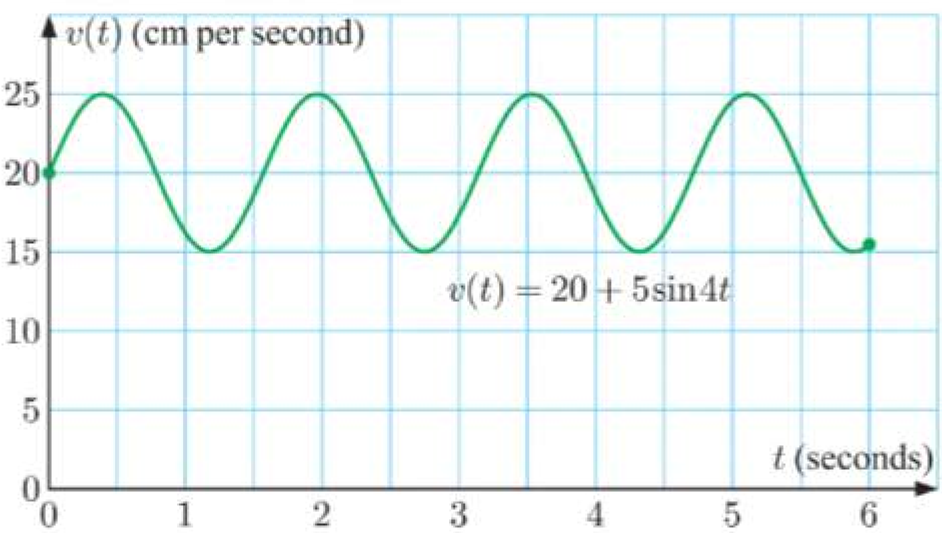
$= 1$ m

b $s\left(\frac{\pi}{4}\right) = 1, \quad \therefore \quad \frac{1}{2}\sin\frac{\pi}{2} + c = 1$

$$\therefore \quad \tfrac{1}{2}(1) + c = 1$$
$$\therefore \quad c = \tfrac{1}{2}$$

$$\therefore \quad s(t) = \tfrac{1}{2}\sin 2t + \tfrac{1}{2}$$
$$\therefore \quad s\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}\sin\tfrac{2\pi}{3} + \tfrac{1}{2}$$
$$= \tfrac{\sqrt{3}}{4} + \tfrac{1}{2}$$
$$= \frac{\sqrt{3}+2}{4} \text{ m}$$

10 **a**



b $v(4.5) = 20 + 5\sin(4 \times 4.5)$

$$= 20 + 5\sin 18$$
$$\approx 16.2 \text{ cm s}^{-1}$$

$\therefore$ the pendulum's velocity after 4.5 seconds is about 16.2 cm s^{-1}.

c Distance travelled by the tip of the pendulum in the first 2 seconds

$$= \int_0^2 |\, v(t) \,| \, dt$$
$$= \int_0^2 v(t) \, dt$$
$$= \int_0^2 (20 + 5\sin 4t) \, dt$$
$$= \left[20t - \tfrac{5}{4}\cos 4t\right]_0^2$$
$$= \left(40 - \tfrac{5}{4}\cos 8\right) - \left(0 - \tfrac{5}{4}\right)$$
$$\approx 41.4 \text{ cm}$$

11 **a**
$$\dot{x}(t) = x'(t) = -4 + t^{\frac{1}{2}}$$
$$\therefore \; x(t) = \int (-4 + t^{\frac{1}{2}}) \, dt$$
$$= -4t + \tfrac{2}{3}t^{\frac{3}{2}} + c$$

$x(0) = 0, \quad \therefore \; c = 0$

$$\therefore \; x(t) = -4t + \tfrac{2}{3}t^{\frac{3}{2}} \text{ m}$$

b The object changes direction when $\dot{x}(t) = 0$

$$\therefore \; -4 + \sqrt{t} = 0$$
$$\therefore \; \sqrt{t} = 4$$
$$\therefore \; t = 16$$

$\therefore$ the object changes direction at $t = 16$ seconds.

c Change in displacement $= x(30) - x(0)$

$$= -4(30) + \tfrac{2}{3}(30)^{\frac{3}{2}} - 0$$
$$\approx -10.5 \text{ m}$$

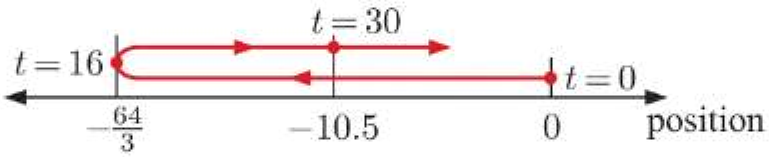
$\therefore$ after the first 30 seconds the particle is about 10.5 m to the left of the origin.

d $x(0) = 0$

$$x(16) = -4(16) + \tfrac{2}{3}(16)^{\frac{3}{2}}$$
$$= -\tfrac{64}{3}$$

$x(30) \approx -10.5$

Motion diagram:



$\therefore$ total distance travelled $\approx \left(0 - \left(-\tfrac{64}{3}\right)\right) + \left(-10.5 - \left(-\tfrac{64}{3}\right)\right)$

$$\approx 32.2 \text{ m}$$

12 **a** $v(t) = 10\sqrt{t}\ \text{m s}^{-1}$

i $v(1) = 10\sqrt{1}$
$\quad = 10$

So the motorcyclist's velocity after 1 second is $10\ \text{m s}^{-1}$.

ii $v(2) = 10\sqrt{2}$

So the motorcyclist's velocity after 2 seconds is $10\sqrt{2}\ \text{m s}^{-1}$.

b
$$\begin{aligned} s(t) &= \int v(t)\, dt \\ &= \int 10\sqrt{t}\, dt \\ &= \int 10t^{\frac{1}{2}}\, dt \\ &= \tfrac{20}{3}t^{\frac{3}{2}} + c \ \text{ m} \end{aligned}$$

We assume that $s(0) = 0,\ c = 0$

$\therefore\ s(t) = \frac{20}{3}t^{\frac{3}{2}}$ m

c
$$\begin{aligned} \int_0^2 v(t)\, dt &= \left[\tfrac{20}{3}t^{\frac{3}{2}}\right]_0^2 \\ &= \tfrac{20}{3}(2^{\frac{3}{2}}) - 0 \\ &\approx 18.9 \end{aligned}$$

The motorcyclist travels about 18.9 m in the first 2 seconds.

d **i** $v(t) = 20$ when
$$\begin{aligned} 10\sqrt{t} &= 20 \\ \therefore\ \sqrt{t} &= 2 \\ \therefore\ t &= 4 \end{aligned}$$

It will take 4 seconds for the motorcyclist to reach a speed of $20\ \text{m s}^{-1}$.

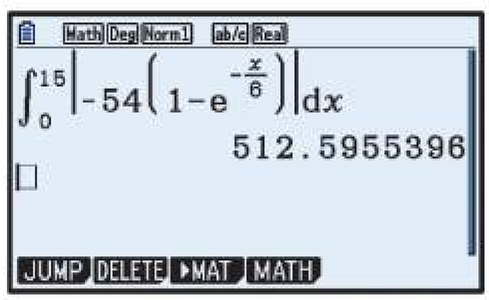
ii
$$\begin{aligned} \text{Distance travelled in first 4 seconds} &= \int_0^4 v(t)\, dt \\ &= \left[\tfrac{20}{3}t^{\frac{3}{2}}\right]_0^4 \\ &= \tfrac{20}{3}(4^{\frac{3}{2}}) - 0 \\ &= 53\tfrac{1}{3}\ \text{m} \end{aligned}$$

$\therefore$ yes, the motorcyclist has given himself enough distance as he only needs $53\frac{1}{3}$ m to reach the required speed.

13 **a** $v(t) = -54(1 - e^{-\frac{t}{6}})\ \ \text{m s}^{-1}$

$$\begin{aligned} \int_0^{15} |\, v(t)\,|\, dt &= \int_0^{15} \left| -54(1 - e^{-\frac{t}{6}}) \right| dt \\ &\approx 513 \qquad \{\text{using technology}\} \end{aligned}$$

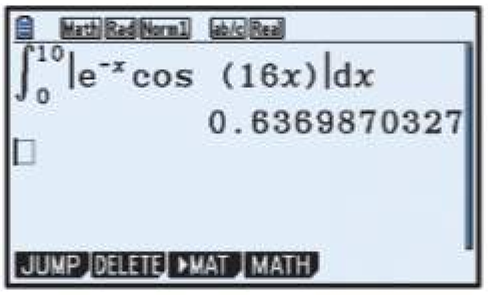
$\therefore$ the skydiver travels a total distance of about 513 m in the first 15 seconds.



b $v(t) = e^{-t}\cos 16t \quad \text{cm s}^{-1}$

$$\int_0^{10} |\, v(t)\,|\, dt = \int_0^{10} \left| e^{-t}\cos 16t \right| dt$$
$$\approx 0.637$$

$\therefore$ the mass on the spring travels a total distance of about 0.637 cm in the first 10 seconds.



EXERCISE 24C

1 **a** $v(t) = 10t - t^2 \quad \text{cm s}^{-1}, \quad t \geqslant 0 \text{ s}$

$$\begin{aligned} v(2) &= 10(2) - (2)^2 \\ &= 20 - 4 \\ &= 16 \text{ cm s}^{-1} \end{aligned}$$

$\therefore$ the velocity of the pariticle at $t = 2$ seconds is 16 cm s^{-1}.

b
$$\begin{aligned} \text{average acceleration} &= \frac{v(3) - v(1)}{3 - 1} \\ &= \frac{\left(10(3) - (3)^2\right) - \left(10(1) - (1)^2\right)}{3 - 1} \\ &= \frac{(30 - 9) - (10 - 1)}{2} \\ &= \frac{21 - 9}{2} \\ &= 6 \text{ cm s}^{-2} \end{aligned}$$

$\therefore$ the average acceleration from $t = 1$ to $t = 3$ seconds is 6 cm s^{-2}.

c $a(t) = v'(t) = 10 - 2t \text{ cm s}^{-2}$

d
$$\begin{aligned} a(3) &= 10 - 2(3) \\ &= 4 \text{ cm s}^{-2} \end{aligned}$$

$\therefore$ the instantaneous acceleration of the particle at $t = 3$ seconds is 4 cm s^{-2}.

2 **a** $x(t) = t^3 - t^2 - 5 \text{ m}, \quad t \geqslant 0 \text{ s}$

$\therefore \dot{x}(t) = x'(t) = 3t^2 - 2t \text{ m s}^{-1}$

$\therefore \ddot{x}(t) = \dot{x}'(t) = 6t - 2 \text{ m s}^{-2}$

$$\therefore \begin{aligned} x(2) &= 2^3 - 2^2 - 5 \\ &= 8 - 4 - 5 \\ &= -1 \text{ m} \end{aligned} \qquad \begin{aligned} \dot{x}(2) &= 3(2)^2 - 2(2) \\ &= 12 - 4 \\ &= 8 \text{ m s}^{-1} \end{aligned} \qquad \begin{aligned} \ddot{x}(2) &= 6(2) - 2 \\ &= 12 - 2 \\ &= 10 \text{ m s}^{-2} \end{aligned}$$

At $t = 2$ seconds, the object has displacement -1 m, velocity 8 m s^{-1}, and acceleration 10 m s^{-2}.

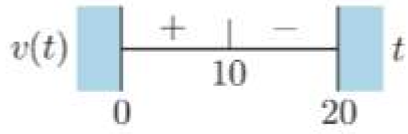
b $\ddot{x}(t) = 0$ when $6t - 2 = 0$

$$\therefore 6t = 2$$
$$\therefore t = \tfrac{1}{3}$$

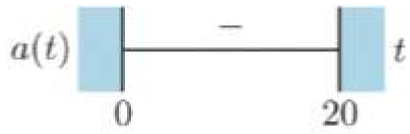
$\therefore$ the object has zero acceleration at $t = \frac{1}{3}$ seconds.

3 **a** $s(t) = 98t - 4.9t^2$ m

$\therefore\ v(t) = s'(t) = 98 - 9.8t\ \text{m s}^{-1}$ which has sign diagram:



$\therefore\ a(t) = v'(t) = -9.8\ \text{m s}^{-2}$ which has sign diagram:



b $s(0) = 0$ m $\qquad v(0) = 98\ \text{m s}^{-1}$

The stone is initially 0 m above the ground, moving upward with velocity $98\ \text{m s}^{-1}$.

c $s(5) = 98(5) - 4.9(5)^2 = 367.5$ m $\qquad v(5) = 98 - 9.8(5) = 49\ \text{m s}^{-1}$ $\qquad a(5) = -9.8\ \text{m s}^{-2}$

At $t = 5$ seconds, the stone is 367.5 m above the ground and moving upward at $49\ \text{m s}^{-1}$. It has acceleration $-9.8\ \text{m s}^{-2}$.

$s(12) = 98(12) - 4.9(12)^2 = 470.4$ m $\qquad v(12) = 98 - 9.8(12) = -19.6\ \text{m s}^{-1}$ $\qquad a(12) = -9.8\ \text{m s}^{-2}$

At $t = 12$ seconds, the stone is 470.4 m above the ground and moving downward at $19.6\ \text{m s}^{-1}$. It has acceleration $-9.8\ \text{m s}^{-2}$.

d The maximum height reached by the stone occurs when its upward velocity equals 0.

$$\therefore\ 98 - 9.8t = 0$$
$$\therefore\ 9.8t = 98$$
$$\therefore\ t = 10$$

So, the maximum height is reached at $t = 10$ seconds.

$$s(10) = 98(10) - 4.9(10)^2 = 490 \text{ m}$$

$\therefore$ the maximum height reached by the stone is 490 m.

e The stone is on the ground when its displacement equals 0.

$$\therefore\ 98t - 4.9t^2 = 0$$
$$\therefore\ t(98 - 4.9t) = 0$$
$$\therefore\ t = 0 \text{ or } 98 - 4.9t = 0$$
$$4.9t = 98$$
$$t = 20$$

After it is fired from the catapult, it takes 20 seconds for the stone to hit the ground.

4 **a** $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm

$$\therefore\ v(t) = 100 + 200\left(-\tfrac{1}{5}\right)e^{-\frac{t}{5}} \qquad \{v(t) = s'(t)\}$$
$$= 100 - 40e^{-\frac{t}{5}}\ \text{cm s}^{-1}$$

and
$$a(t) = -40\left(-\tfrac{1}{5}\right)e^{-\frac{t}{5}} \qquad \{a(t) = v'(t)\}$$
$$= 8e^{-\frac{t}{5}}\ \text{cm s}^{-2}$$

b When $t = 0$, $s(0) = 200$ cm
$v(0) = 60\ \text{cm s}^{-1}$
$a(0) = 8\ \text{cm s}^{-2}$

$\therefore$ the particle is initially 200 cm to the right of the origin, moving to the right at $60\ \text{cm s}^{-1}$, and has acceleration $8\ \text{cm s}^{-2}$.

c As $t \to \infty$, $v(t) \to 100^-$

$\therefore$ the velocity of P approaches $100\ \text{cm s}^{-1}$ from below as $t \to \infty$.

d As $t \to \infty$, $a(t) \to 0^+$

$\therefore$ the acceleration of P approaches $0\ \text{cm s}^{-2}$ from above as $t \to \infty$.

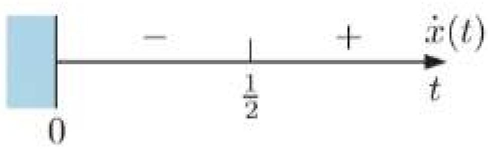
5 **a** $x(t) = t - \ln(2t+1)$ cm

$\therefore$ $x(0) = 0 - \ln(2(0)+1)$
$= -\ln(1)$
$= 0$

$\therefore$ the object is initially at the origin. ✓

b $\dot{x}(t) = 1 - \dfrac{2}{2t+1}\ \text{cm s}^{-1}$ $\{\dot{x}(t) = x'(t)\}$

c $\dot{x}(t) = 1 - \dfrac{2}{2t+1}$
$= \dfrac{2t+1-2}{2t+1}$
$= \dfrac{2t-1}{2t+1}$ which has sign diagram:



i The object is moving to the right when the velocity is positive.

$\therefore$ the object is moving to the right for $t > \frac{1}{2}$ second.

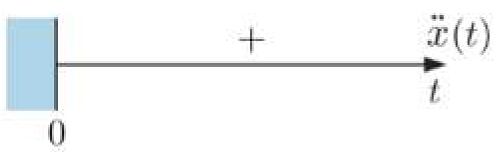
ii The object is moving to the left when the velocity is negative.

$\therefore$ the object is moving to the left for $0 \leqslant t < \frac{1}{2}$ second.

d $\dot{x}(t) = 1 - 2(2t+1)^{-1}\ \text{cm s}^{-1}$

$\therefore$ $\ddot{x}(t) = 2(2t+1)^{-2}(2)$ {chain rule}
$= \dfrac{4}{(2t+1)^2}\ \text{cm s}^{-2}$

e $\ddot{x}(t)$ has sign diagram:



$\therefore$ the object's acceleration is positive for all $t \geqslant 0$.

f $\ddot{x}(2) = \dfrac{4}{(2(2)+1)^2}$
$= \frac{4}{25}\ \text{cm s}^{-2}$

$\therefore$ the acceleration of the object after 2 seconds is $\frac{4}{25}\ \text{cm s}^{-2}$.

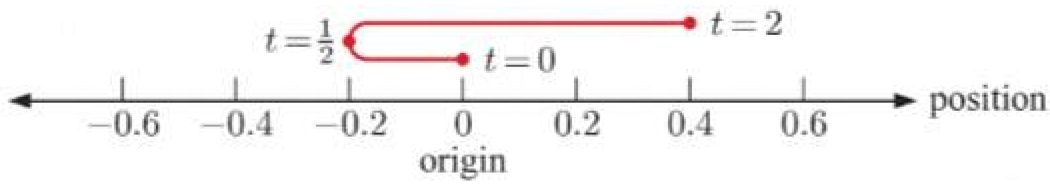
g $\dot{x}(t)$ changes sign when $t = \frac{1}{2}$, so this is where the object changes direction.

$x\left(\frac{1}{2}\right) = \frac{1}{2} - \ln\left(2\left(\frac{1}{2}\right) + 1\right)$
$\quad = \frac{1}{2} - \ln 2$ cm

and

$x(2) = 2 - \ln(2(2) + 1)$
$\quad = 2 - \ln 5$ cm

The motion diagram of P is:



$\therefore$ the total distance travelled by the object in the first 2 seconds

$= \left(0 - \left(\frac{1}{2} - \ln 2\right)\right) + \left(2 - \ln 5 - \left(\frac{1}{2} - \ln 2\right)\right)$
$= -\frac{1}{2} + \ln 2 + 2 - \ln 5 - \frac{1}{2} + \ln 2$
$= 1 + 2\ln 2 - \ln 5$
$= 1 + \ln 4 - \ln 5$
$= 1 + \ln\left(\frac{4}{5}\right) \approx 0.777$ cm

6 $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}

a $v(0) = 50 - 10e^{0}$
$\quad = 40$

So, the initial velocity is 40 m s^{-1}.

b $v(3) = 50 - 10e^{-0.5(3)}$
$\quad \approx 47.8$

So, the velocity after 3 seconds is about 47.8 m s^{-1}.

c $v(t) = 45$ when $50 - 10e^{-0.5t} = 45$

$\therefore -10e^{-0.5t} = -5$
$\therefore e^{-0.5t} = \frac{1}{2}$
$\therefore -0.5t = \ln\left(\frac{1}{2}\right)$
$\therefore t = 2\ln 2 \approx 1.39$

So, it will take about 1.39 seconds for the particle's velocity to reach 45 m s^{-1}.

d As $t \to \infty$, $10e^{-0.5t} \to 0$ from above, thus $v(t) \to 50$ m s^{-1} from below.

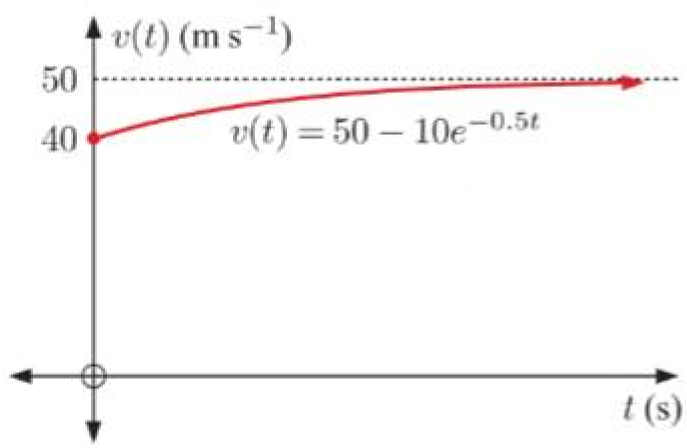
e $a(t) = v'(t)$
$\quad = 5e^{-0.5t}$

And as $e^{x} > 0$ for all x,
then $a(t) = 5e^{-0.5t} > 0$ for all t. ✓

f $a(t) = 2$ when $5e^{-0.5t} = 2$

$\therefore e^{-0.5t} = \frac{2}{5}$
$\therefore -0.5t = \ln\left(\frac{2}{5}\right)$
$\therefore -0.5t = -\ln\left(\frac{5}{2}\right)$
$\therefore t = 2\ln\left(\frac{5}{2}\right)$ s

g



h The particle does not change direction.

$\therefore$ total distance travelled in first 3 seconds $= \int_0^3 v(t)\,dt$

$$= \int_0^3 (50 - 10e^{-0.5t})\,dt$$
$$= \left[50t + 20e^{-0.5t}\right]_0^3$$
$$= 150 + 20e^{-1.5} - (20)$$
$$\approx 134 \text{ m}$$

7 **a**

$$v(t) = \int a(t)\,dt$$
$$= \int \left(\frac{t}{10} - 3\right) dt$$
$$= \frac{t^2}{20} - 3t + c$$

Now $v(0) = 45$

$\therefore\ c = 45$

$\therefore\ v(t) = \frac{t^2}{20} - 3t + 45\ \text{m s}^{-1}$

b

$$\int_0^{60} v(t)\,dt$$
$$= \int_0^{60} \left(\frac{t^2}{20} - 3t + 45\right) dt$$
$$= \left[\frac{t^3}{60} - \tfrac{3}{2}t^2 + 45t\right]_0^{60}$$
$$= (3600 - 5400 + 2700) - 0$$
$$= 900$$

The train travels a total of 900 m in the first 60 seconds (the train does not change direction).

8 **a**

$$\dot{x}(t) = \int \ddot{x}(t)\,dt$$
$$= \int 4e^{-\frac{t}{20}}\,dt$$
$$= -80e^{-\frac{t}{20}} + c$$

Now $\dot{x}(0) = 20$

$\therefore\ -80 + c = 20$

$\therefore\ c = 100$

$\therefore\ \dot{x}(t) = 100 - 80e^{-\frac{t}{20}}\ \text{m s}^{-1}$

As $t \to \infty$, $80e^{-\frac{t}{20}} \to 0$,

and thus $\dot{x}(t) \to 100\ \text{m s}^{-1}$.

b The object does not change direction.

$\therefore$ total distance travelled in first 10 seconds

$$= \int_0^{10} \dot{x}(t)\,dt$$
$$= \int_0^{10} (100 - 80e^{-\frac{t}{20}})\,dt$$
$$= \left[100t + 1600e^{-\frac{t}{20}}\right]_0^{10}$$
$$= (1000 + 1600e^{-\frac{1}{2}}) - 1600$$
$$\approx 370 \text{ m}$$

9 **a** $v(t) = \int a(t)\,dt$

$= \int 2(t+1)^{-3}\,dt$

$= -(t+1)^{-2} + c$

At $t = 0$ the particle is stationary

$\therefore\ v(0) = 0$

$\therefore\ -1^{-2} + c = 0$

$\therefore\ c = 1$

$\therefore\ v(t) = -\dfrac{1}{(t+1)^2} + 1\ \text{m s}^{-1}$

b $s(t) = \int v(t)\,dt$

$= \int (-(t+1)^{-2} + 1)\,dt$

$= (t+1)^{-1} + t + c$

At $t = 0$ the particle is at the origin

$\therefore\ s(0) = 0$

$\therefore\ 1^{-1} + 0 + c = 0$

$\therefore\ c = -1$

$\therefore\ s(t) = \dfrac{1}{t+1} + t - 1\ \text{m}$

c $a(2) = \dfrac{2}{(2+1)^3}$

$= \frac{2}{27}\ \text{m s}^{-2}$

$v(2) = -\dfrac{1}{(2+1)^2} + 1$

$= -\frac{1}{9} + 1$

$= \frac{8}{9}\ \text{m s}^{-1}$

$s(2) = \dfrac{1}{2+1} + 2 - 1$

$= \frac{1}{3} + 1$

$= \frac{4}{3}\ \text{m}$

$\therefore$ at $t = 2$, the particle is $\frac{4}{3}$ metres to the right of the origin, moving to the right at $\frac{8}{9}\ \text{m s}^{-1}$, and accelerating at $\frac{2}{27}\ \text{m s}^{-2}$.

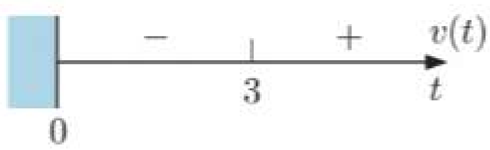
EXERCISE 24D

1 **a** $s(t) = t^2 - 6t + 7\ \text{m}$

$\therefore\ v(t) = 2t - 6$ $\quad\{v(t) = s'(t)\}$

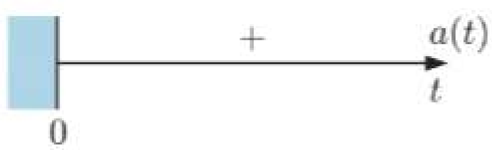
$= 2(t-3)\ \text{m s}^{-1}$

which has sign diagram:



and $a(t) = 2\ \text{m s}^{-2}$ $\quad\{a(t) = v'(t)\}$

which has sign diagram:



b **i** When $t = 1$, speed $= |v(1)|$

$= |2(1) - 6|$

$= |-4|$

$= 4\ \text{m s}^{-1}$

ii When $t = 4$, speed $= |v(4)|$

$= |2(4) - 6|$

$= |2|$

$= 2\ \text{m s}^{-1}$

c When $t = 0$, $s(0) = 7\ \text{m}$

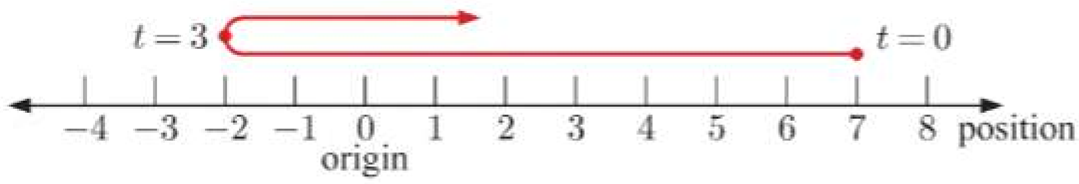
$v(0) = -6\ \text{m s}^{-1}$

$a(0) = 2\ \text{m s}^{-2}$

$\therefore$ the object is initially 7 m to the right of O, moving to the left at $6\ \text{m s}^{-1}$, with acceleration $2\ \text{m s}^{-2}$.

d $v(t)$ changes sign when $t = 3$, so this is when the object changes direction.
$s(3) = 9 - 18 + 7 = -2$
So, the object changes direction when it is 2 m to the left of O.

e



f The object's speed is decreasing when $v(t)$ and $a(t)$ have opposite signs. This occurs for $0 \leqslant t \leqslant 3$.

2 $s(t) = 1.2 + 28.1t - 4.9t^2$ m

a When the ball was first released, $t = 0$.
$s(0) = 1.2$
So, the ball was released 1.2 m above ground level.

b $s'(t) = 28.1 - 9.8t$ m s^{-1} which is the instantaneous velocity of the ball t seconds after being released.

c $s'(t) = 0$ when $28.1 - 9.8t = 0$

$$\therefore \ 28.1 = 9.8t$$
$$\therefore \ t = \frac{28.1}{9.8}$$

So, the ball has reached its maximum height after $\frac{28.1}{9.8}$ seconds.

$$s\left(\frac{28.1}{9.8}\right) = 1.2 + 28.1\left(\frac{28.1}{9.8}\right) - 4.9\left(\frac{28.1}{9.8}\right)^2 \approx 41.5$$

So, the maximum height reached by the ball is about 41.5 m.

d $s'(t) = 28.1 - 9.8t$ m s^{-1}

i $s'(0) = 28.1$
The ball's speed when released is 28.1 m s^{-1}.

ii $s'(2) = 28.1 - 9.8(2) = 8.5$
The ball's speed at $t = 2$ seconds is 8.5 m s^{-1}.

iii $s'(5) = 28.1 - 9.8(5) = -20.9$
The ball's speed at $t = 5$ seconds is 20.9 m s^{-1}.

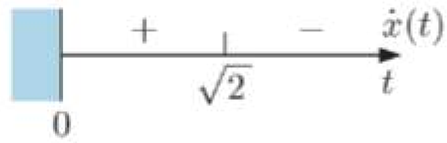
3 **a**
$$x(t) = 12t - 2t^3 - 1 \text{ cm}$$
$$\therefore \ \dot{x}(t) = 12 - 6t^2 \text{ cm s}^{-1} \qquad \{\dot{x}(t) = x'(t)\}$$
$$\therefore \ \ddot{x}(t) = -12t \text{ cm s}^{-2} \qquad \{\ddot{x}(t) = \dot{x}'(t)\}$$

b $x(0) = -1$ cm, $\dot{x}(0) = 12$ cm s^{-1}, $\ddot{x}(0) = 0$ cm s^{-2}
The particle is initially 1 cm to the left of the origin, travelling to the right at a constant speed of 12 cm s^{-1}.

c $\dot{x}(t) = 12 - 6t^2$

$= 6(2 - t^2)$

$= 6(\sqrt{2} + t)(\sqrt{2} - t)$ cm s^{-1} which has sign diagram:



$\dot{x}(t)$ changes sign when $t = \sqrt{2}$ seconds, so this is when the particle changes direction.

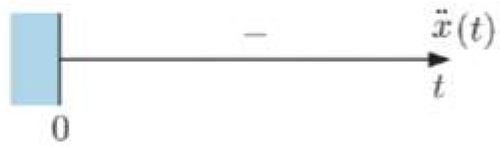
$x(\sqrt{2}) = 12(\sqrt{2}) - 2(\sqrt{2})^3 - 1$

$= 12\sqrt{2} - 4\sqrt{2} - 1$

$= 8\sqrt{2} - 1$

So, the particle changes direction when it is $(8\sqrt{2} - 1)$ cm to the right of O.

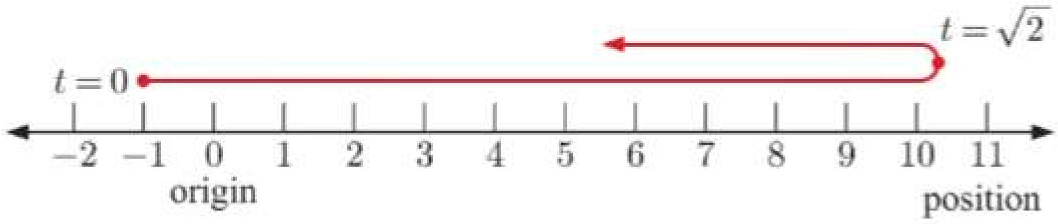
d $\ddot{x}(t) = -12t$ has sign diagram:



i The particle's speed is increasing when $\dot{x}(t)$ and $\ddot{x}(t)$ have the same sign. This occurs for $t \geqslant \sqrt{2}$.

ii The particle's velocity is never increasing.

e

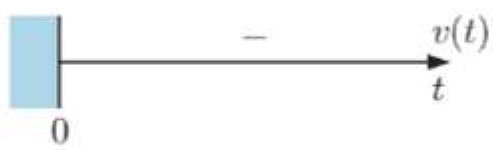


4 **a** $s(t) = 4 - \sqrt{t+1} = 4 - (t+1)^{\frac{1}{2}}$ m

$\therefore\ v(t) = -\frac{1}{2}(t+1)^{-\frac{1}{2}}$ $\quad \{v(t) = s'(t)\}$

$= -\dfrac{1}{2\sqrt{t+1}}$ m s^{-1}

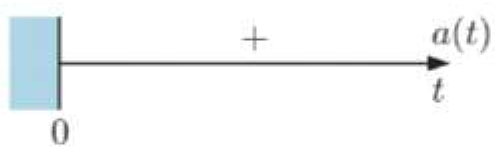
which has sign diagram:



and $a(t) = \frac{1}{4}(t+1)^{-\frac{3}{2}}$ $\quad \{a(t) = v'(t)\}$

$= \dfrac{1}{4(t+1)^{\frac{3}{2}}}$ m s^{-2}

which has sign diagram:



b $s(0) = 3$ m, $\quad v(0) = -\frac{1}{2}$ m s^{-1}, $\quad a(0) = \frac{1}{4}$ m s^{-2}

Initially, the particle is 3 m to the right of O, moving to the left at $\frac{1}{2}$ m s^{-1} with acceleration $\frac{1}{4}$ m s^{-2}.

c $s(3) = 4 - \sqrt{4} = 2$ m, $v(3) = -\dfrac{1}{2\sqrt{4}} = -\frac{1}{4}$ m s^{-1}, $a(3) = \dfrac{1}{4(4)^{\frac{3}{2}}} = \frac{1}{32}$ m s^{-2}

After 3 seconds, the particle is 2 m to the right of O, moving to the left at $\frac{1}{4}$ m s^{-1}, with acceleration $\frac{1}{32}$ m s^{-2}.

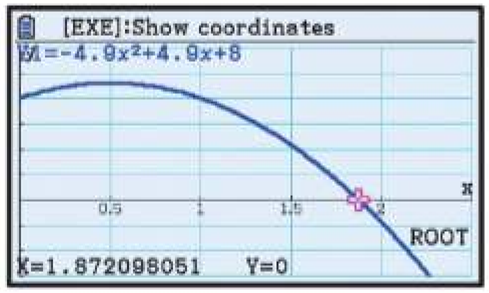
d The particle's speed is continuously decreasing.

5 **a** When the device reaches the water,

$$s(t) = 0$$

$$\therefore\ -4.9t^2 + 4.9t + 8 = 0$$

$$\therefore\ t \approx 1.87 \quad \{\text{using technology, } t > 0\}$$



So, the device takes approximately 1.87 seconds to reach the water.

$\therefore\ k \approx 1.87$

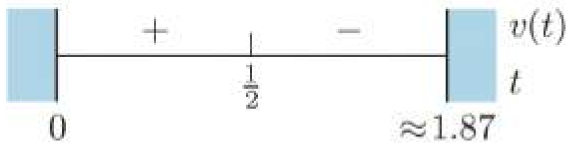
b

$$s(t) = -4.9t^2 + 4.9t + 8 \text{ m}$$

$$\therefore\ v(t) = -9.8t + 4.9 \text{ m s}^{-1} \qquad \{v(t) = s'(t)\}$$

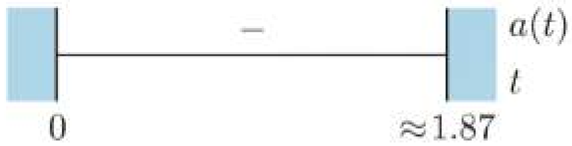
$$= 4.9(1 - 2t)$$

which has sign diagram:



and $a(t) = -9.8$ m s^{-2} $\qquad \{a(t) = v'(t)\}$

which has sign diagram:



c **i** Looking at the sign diagrams for $v(t)$ and $a(t)$, $v(0.2) > 0$ and $a(0.2) < 0$

$\therefore$ $v(0.2)$ and $a(0.2)$ have opposite sign.

$\therefore$ the speed of the device is decreasing after 0.2 seconds.

ii $v(1) < 0$ and $a(1) < 0$

$\therefore$ $v(1)$ and $a(1)$ have the same sign.

$\therefore$ the speed of the device is increasing after 1 second.

6 **a**

$$x(t) = 1 - 2\cos t \text{ cm}$$

$$\therefore\ \dot{x}(t) = 2\sin t \text{ cm s}^{-1} \qquad \{\dot{x}(t) = x'(t)\}$$

$$\therefore\ \ddot{x}(t) = 2\cos t \text{ cm s}^{-2} \qquad \{\ddot{x}(t) = \dot{x}'(t)\}$$

When $t = 0$, $x(0) = 1 - 2\cos 0 = -1$ cm

$\dot{x}(0) = 2\sin 0 = 0$ cm s^{-1}

$\ddot{x}(0) = 2\cos 0 = 2$ cm s^{-2}

$\therefore$ P is initially 1 cm to the left of the origin, instantaneously at rest, and accelerating at 2 cm s^{-2}.

b $x\left(\frac{\pi}{4}\right) = 1 - 2\cos\frac{\pi}{4} = 1 - \sqrt{2} = -(\sqrt{2} - 1)$ cm, $\dot{x}\left(\frac{\pi}{4}\right) = 2\sin\frac{\pi}{4} = \sqrt{2}$ cm s^{-1},

$\ddot{x}\left(\frac{\pi}{4}\right) = 2\cos\frac{\pi}{4} = \sqrt{2}$ cm s^{-2}

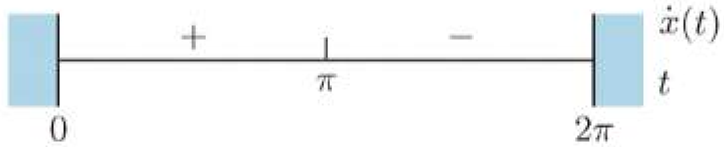
$\therefore$ at $t = \frac{\pi}{4}$ seconds, the particle is $(\sqrt{2} - 1)$ cm left of O, moving to the right at $\sqrt{2}$ cm s^{-1}, with acceleration $\sqrt{2}$ cm s^{-2}.

c The particle reverses direction when $\dot{x}(t) = 0$

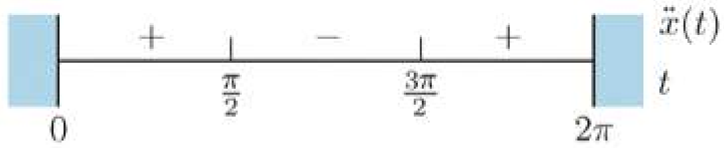
$$\therefore\ 2\sin t = 0$$
$$\therefore\ t = \pi \qquad \{0 < t < 2\pi\}$$
$$x(\pi) = 1 - 2\cos\pi = 3 \text{ cm}$$

$\therefore$ the particle reverses direction at $t = \pi$ seconds, 3 cm to the right of the origin.

d $\dot{x}(t) = 2\sin t$ has sign diagram:



$\ddot{x}(t) = 2\cos t$ has sign diagram:



The particle's speed is increasing when $\dot{x}(t)$ and $\ddot{x}(t)$ have the same sign.

This occurs for $0 \leqslant t \leqslant \frac{\pi}{2}$ seconds and $\pi \leqslant t \leqslant \frac{3\pi}{2}$ seconds.

7 **a** $x(t) = 8\sin\frac{t}{2}$ m

i $x(3) = 8\sin\frac{3}{2}$
$\approx 7.98 > 0$

$\therefore$ after 3 seconds, the dog is to the right of its kennel.

ii $x(7) = 8\sin\frac{7}{2}$
$\approx -2.81 < 0$

$\therefore$ after 7 seconds, the dog is to the left of its kennel.

b $x(t) = 8\sin\frac{t}{2}$ m

$$\therefore\ \dot{x}(t) = 8\left(\tfrac{1}{2}\right)\cos\tfrac{t}{2} \qquad \{\dot{x}(t) = x'(t)\}$$
$$= 4\cos\tfrac{t}{2} \text{ m s}^{-1}$$

c **i** $\dot{x}(4) = 4\cos 2$
$\approx -1.66 < 0$

$\therefore$ after 4 seconds, the dog is moving to the left.

ii $\dot{x}(10) = 4\cos 5$
$\approx 1.13 > 0$

$\therefore$ after 10 seconds, the dog is moving to the right.

d $\ddot{x}(t) = 4\left(-\frac{1}{2}\right)\sin\frac{t}{2} \qquad \{\ddot{x}(t) = \dot{x}'(t)\}$
$= -2\sin\frac{t}{2}$ m s^{-2}

e $\dot{x}(2) = 4\cos 1$
≈ 2.16 m s^{-1}

$\ddot{x}(2) = -2\sin 1$
≈ -1.68 m s^{-2}

$\dot{x}(2) > 0$ and $\ddot{x}(2) < 0$

$\therefore$ $\dot{x}(2)$ and $\ddot{x}(2)$ have opposite sign.

$\therefore$ the dog's speed is decreasing after 2 seconds.

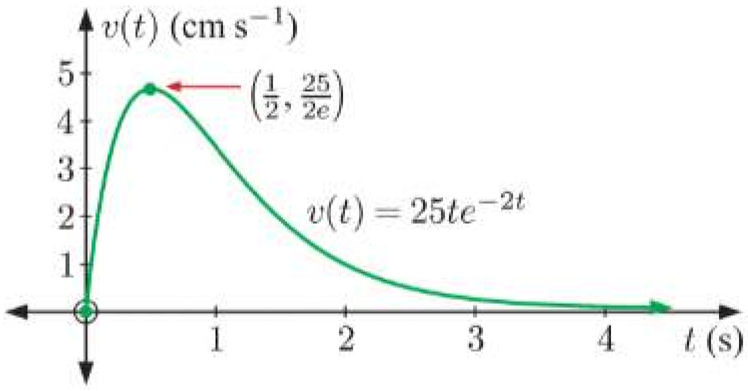
f $\left|\dot{x}(t)\right|$ is a maximum when $\ddot{x}(t) = 0$

$$\therefore\ -2\sin\tfrac{t}{2} = 0$$
$$\therefore\ \frac{t}{2} = k\pi \qquad \{k \in \mathbb{Z}\}$$
$$\therefore\ t = 2k\pi$$

Now $x(2k\pi) = 8\sin k\pi$
$= 0 \qquad \{k \in \mathbb{Z}\}$

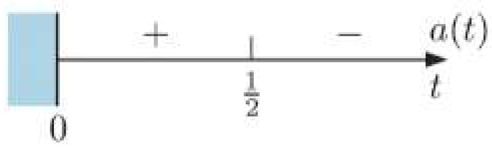
$\therefore$ the dog's speed is maximised when it is moving past its kennel.

8 **a**



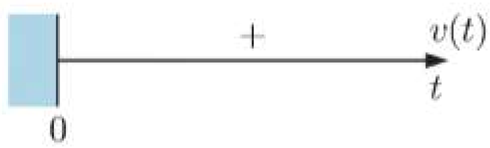
b
$$v(t) = 25te^{-2t} \text{ cm s}^{-1}, \quad t \geqslant 0$$
$$\therefore \quad a(t) = v'(t) = 25e^{-2t} + 25t(-2)e^{-2t} \qquad \{\text{product rule}\}$$
$$= 25e^{-2t} - 50te^{-2t}$$
$$= 25(1-2t)e^{-2t} \text{ cm s}^{-2}, \quad t \geqslant 0 \quad ✓$$

c $a(t) = 25(1-2t)e^{-2t}$ has sign diagram:



The acceleration is positive and hence the velocity is increasing for $0 \leqslant t \leqslant \frac{1}{2}$ second.

d $v(t) = 25te^{-2t}$ has sign diagram:

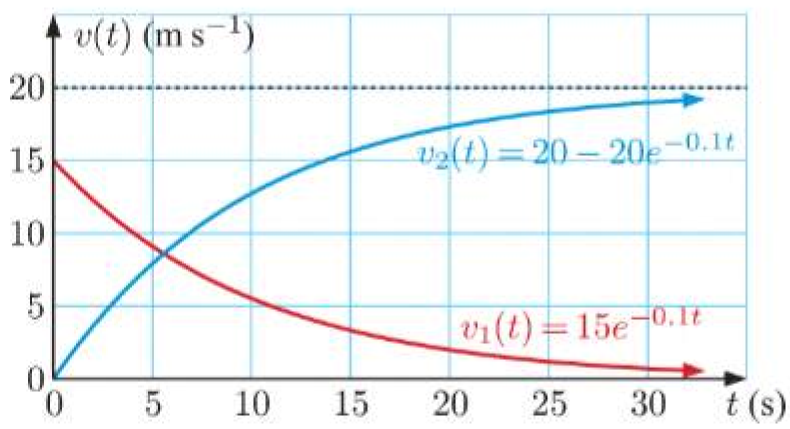


The speed of the object is decreasing when $v(t)$ and $a(t)$ have opposite signs.
This occurs when $t \geqslant \frac{1}{2}$ second.

9 Lion: $v_1(t) = 15e^{-0.1t}$ m s^{-1}
Zebra: $v_2(t) = 20 - 20e^{-0.1t}$ m s^{-1}

a After 1 second, speed of lion $= v_1(1) = 15e^{-0.1(1)}$
≈ 13.6 m s^{-1}
speed of zebra $= v_2(1) = 20 - 20e^{-0.1(1)}$
≈ 1.90 m s^{-1}

b



As shown by the graph, the lion's speed $v_1(t)$ decreases over time whereas the zebra's speed $v_2(t)$ increases over time.

c $$\begin{aligned}\int_0^3 v_1(t)\,dt &= \int_0^3 15e^{-0.1t}\,dt \\ &= \left[-150e^{-0.1t}\right]_0^3 \\ &= -150e^{-0.1(3)} - (-150) \\ &= 150 - 150e^{-0.3} \\ &\approx 38.9\end{aligned}$$

The lion has travelled a total distance of about 38.9 metres in the first 3 seconds.

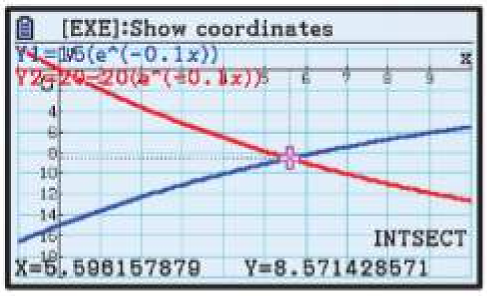
d $$\begin{aligned}\int_0^3 [v_1(t) - v_2(t)]\,dt &= \int_0^3 \left[15e^{-0.1t} - (20 - 20e^{-0.1t})\right] dt \\ &= \int_0^3 (35e^{-0.1t} - 20)\,dt \\ &= \left[-350e^{-0.1t} - 20t\right]_0^3 \\ &= (-350e^{-0.3} - 60) - (-350) \\ &= 290 - 350e^{-0.3} \\ &\approx 30.7\end{aligned}$$

In the first 3 seconds, the lion has gained about 30.7 metres on the zebra.

e At the time when $v_1(t) = v_2(t)$, the lion and the zebra will be moving at the same speed. Since the lion's speed decreases over time and the zebra's speed increases over time, the zebra will be faster than the lion after that time. So, they will be closest at the point when their speeds are equal.

f $v_1(t) = v_2(t)$ when $15e^{-0.1t} = 20 - 20e^{-0.1t}$

$\therefore\ t \approx 5.60$ {technology}



So, $v_1(t) = v_2(t)$ after about 5.60 seconds.

g From **e** and **f**, the lion is closest to the zebra after about 5.60 seconds.

$$\begin{aligned}\int_0^{5.60} [v_1(t) - v_2(t)]\,dt &\approx \left[-350e^{-0.1t} - 20t\right]_0^{5.60} \\ &\approx (-350e^{-0.560} - 20 \times 5.60) - (-350) \\ &\approx 38.08\end{aligned}$$

So, after about 5.60 seconds, the lion has travelled about 38.08 metres more than the zebra has travelled. The zebra was however initially 40 metres ahead of the lion, so at their closest point, the zebra will still be about $40\text{ m} - 38.08\text{ m} \approx 1.92\text{ m}$ ahead of the lion.

So, the lion did not catch the zebra but was about 1.92 m from the zebra at their closest point.

EXERCISE 24E

1 $a = v\,\dfrac{dv}{ds}$
$= 4s^3 \times 12s^2$
$= 48s^5 \text{ m s}^{-2}$

2 **a** $a = v\,\dfrac{dv}{ds}$
$= -3s^2 \times -6s$
$= 18s^3 \text{ m s}^{-2}$

b When $s = 4$, $v = -3(4)^2 = -48 \text{ m s}^{-1}$
and $a = 18(4)^3 = 1152 \text{ m s}^{-2}$

3 **a** $a = v\,\dfrac{dv}{ds}$
$= \dfrac{6}{s} \times -\dfrac{6}{s^2}$
$= -\dfrac{36}{s^3} \text{ m s}^{-2}$

b When $v = 4$, $\dfrac{6}{s} = 4$
$\therefore\ s = \frac{3}{2}$
$\therefore\ a = -\dfrac{36}{(\frac{3}{2})^3}$
$= -\dfrac{36 \times 8}{27}$
$= -\frac{32}{3} \text{ m s}^{-2}$

4 $v \propto \sqrt{s}$
$\therefore\ v = k\sqrt{s}$ for some constant k

If $v = ks^{\frac{1}{2}}$, $\dfrac{dv}{ds} = \frac{1}{2}ks^{-\frac{1}{2}} = \dfrac{k}{2\sqrt{s}}$

So, $a = v\,\dfrac{dv}{ds}$
$= k\sqrt{s} \times \dfrac{k}{2\sqrt{s}}$
$= \dfrac{k^2}{2}$, where k is a constant

$\therefore$ the acceleration is constant.

5 **a** $a = v\,\dfrac{dv}{ds}$
$= (3s - s^2)(3 - 2s)$
$= s(3 - s)(3 - 2s)$
$= s(2s^2 - 9s + 9) \text{ m s}^{-2}$

b **i** Initially, $s = 5$
$\therefore\ v = 3(5) - (5)^2 = -10 \text{ m s}^{-1}$
and $a = 5(2(5)^2 - 9(5) + 9)$
$= 5 \times 14$
$= 70 \text{ m s}^{-2}$

ii The particle initially moves to the left, and while $3 < s < 5$, $v < 0$ and $a > 0$. As time progresses, the particle will approach $s = 3$, and v and a will approach 0.

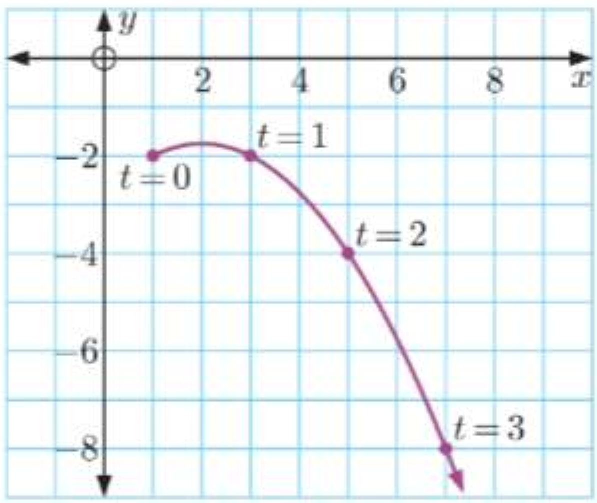
EXERCISE 24F

1 **a** $x(0) = 1, \quad y(0) = -2$

$\therefore$ the initial position is $(1, -2)$.

b

t	$x(t)$	$y(t)$
0	1	-2
1	3	-2
2	5	-4
3	7	-8



c $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 2 \\ -2t+1 \end{pmatrix}$.

d **i** When $t = 0$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ speed $= \sqrt{2^2 + 1^2}$
$= \sqrt{5}\ \text{m s}^{-1}$

ii When $t = 2$, $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\therefore$ speed $= \sqrt{2^2 + (-3)^2}$
$= \sqrt{13}\ \text{m s}^{-1}$

e **i** P is moving parallel to $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ when $\begin{pmatrix} 2 \\ -2t+1 \end{pmatrix} = k\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ for some constant k.

$\therefore$ $2 = k$ and $-2t + 1 = -5k$
$\therefore$ $-2t + 1 = -10$
$\therefore$ $-2t = -11$
$\therefore$ $t = 5.5$ s

ii P is moving perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ when $\begin{pmatrix} 2 \\ -2t+1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0$

$\therefore$ $10 - 4t + 2 = 0$
$\therefore$ $-4t = -12$
$\therefore$ $t = 3$ s

2 **a** $x(0) = 6\sqrt{1} + 5 = 11, \quad y(0) = -10$

$\therefore$ the truck is initially at $(11, -10)$.

b $x(t) = 6(t+1)^{\frac{1}{2}} + 5 \qquad \therefore\ x'(t) = 3(t+1)^{-\frac{1}{2}}$

$y(t) = 4t - 10 \qquad \therefore\ y'(t) = 4$

$\therefore$ $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \dfrac{3}{\sqrt{t+1}} \\ 4 \end{pmatrix}$

c When $t = 0$, $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\therefore$ speed $= \sqrt{3^2 + 4^2}$
$= 5 \text{ m s}^{-1}$

d As $t \to \infty$, $\dfrac{3}{\sqrt{t+1}} \to 0$, so $\mathbf{v} \to \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$\therefore$ over time, the speed of the truck decreases to 4 m s^{-1}.

e The truck was travelling at 4.5 m s^{-1} when $\sqrt{\left(\dfrac{3}{\sqrt{t+1}}\right)^2 + 4^2} = \frac{9}{2}$

$$\therefore \frac{9}{t+1} + 16 = \tfrac{81}{4}$$
$$\therefore \frac{9}{t+1} = \tfrac{17}{4}$$
$$\therefore t + 1 = \tfrac{36}{17}$$
$$\therefore t = \tfrac{19}{17} \text{ s}$$

3 **a** $x(0) = 4e^0 - 10 = -6$, $y(0) = 3e^0 + 4 = 7$
$\therefore$ the cat is initially at $(-6, 7)$.

b $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -4e^{-t} \\ -3e^{-t} \end{pmatrix}$

c **i** The cat is moving in a straight line, as it is always moving parallel to $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{-3e^{-t}}{-4e^{-t}} = \frac{3}{4}$ which is constant.

ii The velocity vector changes over time, so the cat is not moving with constant velocity.

d The speed of the cat $= \sqrt{(-4e^{-t})^2 + (-3e^{-t})^2}$
$= \sqrt{16e^{-2t} + 9e^{-2t}}$
$= 5e^{-t}$ which decreases as t increases.

So, the cat moves in a straight line from $(-6, 7)$ to $(-10, 4)$ with decreasing speed.

e τ is the time at which the cat is 10 cm from her toy.

$$\therefore \sqrt{(4e^{-\tau} - 10 + 10)^2 + (3e^{-\tau} + 4 - 4)^2} = 0.1$$
$$\therefore \sqrt{16e^{-2\tau} + 9e^{-2\tau}} = 0.1$$
$$\therefore 5e^{-\tau} = 0.1$$
$$\therefore \tau \approx 3.91 \quad \{\text{technology}\}$$

4 **a** $x_A(0) = 2$, $y_A(0) = -6$
$\therefore$ S is $(2, -6)$.

b $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 \\ t - 2 \end{pmatrix}$

c When $t = 1$, $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\therefore \text{ speed} = \sqrt{1^2 + (-1)^2} = \sqrt{2}\ \text{m s}^{-1}$$

d **i** At time t, particle B has been travelling for $(t - k)$ seconds.

$$\therefore \ x_B(t) = x_A(t-k) = (t-k) + 2$$
$$y_B(t) = y_A(t-k) = \tfrac{1}{2}(t-k)^2 - 2(t-k) - 6, \quad t \geqslant k$$

ii
$$y_B(10) = 0$$
$$\therefore \ \tfrac{1}{2}(10-k)^2 - 2(10-k) - 6 = 0$$
$$\therefore \ \tfrac{1}{2}(100 - 20k + k^2) - 20 + 2k - 6 = 0$$
$$\therefore \ \tfrac{1}{2}k^2 - 8k + 24 = 0$$
$$\therefore \ k^2 - 16k + 48 = 0$$
$$\therefore \ (k-4)(k-12) = 0$$
$$\therefore \ k = 4 \qquad \{k \leqslant 10\}$$

iii $x_A(6) = 8$, $y_A(6) = \frac{1}{2}(6)^2 - 2(6) - 6 = 0$

When $t = 6$, A is at $(8, 0)$.

$x_B(6) = (6-4) + 2 = 4$, $y_B(6) = \frac{1}{2}(6-4)^2 - 2(6-4) - 6 = -8$

When $t = 6$, B is at $(4, -8)$.

$$\therefore \text{ the distance between the particles} = \sqrt{(8-4)^2 + (0-(-8))^2} = \sqrt{80} = 4\sqrt{5}\ \text{m}$$

iv At time t, particle A has velocity vector $\begin{pmatrix} 1 \\ t-2 \end{pmatrix}$ and particle B has velocity vector $\begin{pmatrix} 1 \\ (t-4)-2 \end{pmatrix} = \begin{pmatrix} 1 \\ t-6 \end{pmatrix}$.

$\therefore$ the particles are moving perpendicular to each other when

$$\begin{pmatrix} 1 \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ t-6 \end{pmatrix} = 0$$
$$\therefore \ 1 + t^2 - 8t + 12 = 0$$
$$\therefore \ t^2 - 8t + 13 = 0$$
$$\therefore \ t = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(13)}}{2}$$
$$\therefore \ t = \frac{8 \pm \sqrt{12}}{2}$$
$$\therefore \ t = 4 + \sqrt{3}\ \text{s} \qquad \{t \geqslant 4\}$$

5 **a** $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2t-2 \\ 3t^2-3 \end{pmatrix}$

$\therefore \quad x(t) = \int (2t-2)\, dt = t^2 - 2t + c$

Now $x(0) = 2$, so $c = 2$.

$\therefore \quad x(t) = t^2 - 2t + 2, \quad t \geqslant 0$

$y(t) = \int (3t^2 - 3)\, dt = t^3 - 3t + c$

Now $y(0) = 5$, so $c = 5$.

$\therefore \quad y(t) = t^3 - 3t + 5, \quad t \geqslant 0$

b $x(2) = 2^2 - 2(2) + 2 = 2$

and $y(2) = 2^3 - 3(2) + 5 = 7$

After 2 seconds, the particle is at $(2, 7)$.

c When $t = 3$, $\mathbf{v} = \begin{pmatrix} 4 \\ 24 \end{pmatrix}$

$\therefore$ speed $= \sqrt{4^2 + 24^2}$

$= \sqrt{592} = 4\sqrt{37}\ \text{m s}^{-1}$

6 **a** When $t = 0$, $\mathbf{v} = \begin{pmatrix} 6 + 2\sin 0 \\ 8 - 2\sin 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

$\therefore$ speed $= \sqrt{6^2 + 8^2}$

$= 10\ \text{m s}^{-1}$

b $S = \sqrt{(6 + 2\sin t)^2 + (8 - 2\sin t)^2}$

$\therefore \quad S^2 = 36 + 24\sin t + 4\sin^2 t + 64 - 32\sin t + 4\sin^2 t$

$\therefore \quad S^2 = 100 - 8\sin t + 8\sin^2 t$

c Using technology, S^2 has maximum value 116 and minimum value 98.

$\therefore$ the highest speed $= \sqrt{116} \approx 10.8\ \text{m s}^{-1}$

and the lowest speed $= \sqrt{98} \approx 9.90\ \text{m s}^{-1}$

d $x(t) = \int (6 + 2\sin t)\, dt = 6t - 2\cos t + c$

Now $x(0) = 0$, so $-2 + c = 0$

$\therefore \quad c = 2$

$\therefore \quad x(t) = 6t - 2\cos t + 2, \quad t \geqslant 0$

$y(t) = \int (8 - 2\sin t)\, dt = 8t + 2\cos t + c$

Now $y(0) = 0$, so $2 + c = 0$

$\therefore \quad c = -2$

$\therefore \quad y(t) = 8t + 2\cos t - 2, \quad t \geqslant 0$

e $x(10) = 62 - 2\cos 10, \quad y(10) = 78 + 2\cos 10$

$\therefore$ distance between the boat and $(0, 0)$ after 10 seconds

$= \sqrt{(62 - 2\cos 10)^2 + (78 + 2\cos 10)^2}$

≈ 99.4 m

7 **a** $x(0) = 2\cos 0 = 2, \quad y(0) = 2\sin 0 = 0$

$\therefore$ P is initially at $(2, 0)$.

b $x\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} = 0, \quad y\left(\frac{\pi}{2}\right) = 2\sin\frac{\pi}{2} = 2$

$\therefore$ after $\frac{\pi}{2}$ seconds, P is at $(0, 2)$.

c
$$\begin{aligned} x^2 + y^2 &= (2\cos t)^2 + (2\sin t)^2 \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \end{aligned}$$

P moves in a circle centred at $(0, 0)$ with radius 2 m.

d $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix}$

e
$$\begin{aligned} \text{Speed} &= |\,\mathbf{v}\,| \\ &= \sqrt{(-2\sin t)^2 + (2\cos t)^2} \\ &= 2\sqrt{\sin^2 t + \cos^2 t} \\ &= 2\ \text{m s}^{-1} \end{aligned}$$

8 **a** **i** $x(0) = 5\cos 0 = 5, \quad y(0) = 5\sin 0 = 0$

$\therefore$ P is initially at $(5, 0)$.

ii $x\left(\frac{\pi}{3}\right) = 5\cos\left(\frac{2\pi}{3}\right) = -\frac{5}{2}, \quad y\left(\frac{\pi}{3}\right) = 5\sin\left(\frac{2\pi}{3}\right) = \frac{5\sqrt{3}}{2}$

$\therefore$ after $\frac{\pi}{3}$ seconds, P is at $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.

b $x\left(\frac{3\pi}{4}\right) = 5\cos\left(\frac{3\pi}{2}\right) = 0, \quad y\left(\frac{3\pi}{4}\right) = 5\sin\left(\frac{3\pi}{2}\right) = -5$

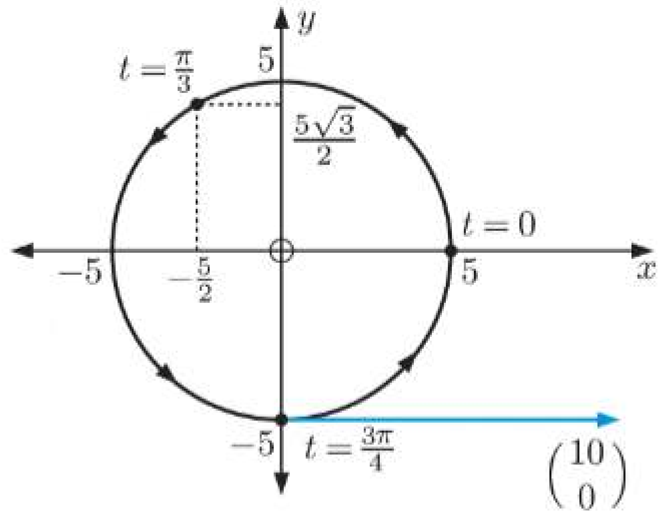
$\therefore$ after $\frac{3\pi}{4}$ seconds, P is at $(0, -5)$.

P has velocity vector $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -10\sin 2t \\ 10\cos 2t \end{pmatrix}$.

When $t = \frac{3\pi}{4}$, $\mathbf{v} = \begin{pmatrix} -10\sin\left(\frac{3\pi}{2}\right) \\ 10\cos\left(\frac{3\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$.

c $x^2 + y^2 = (5\cos 2t)^2 + (5\sin 2t)^2$
$= 25(\cos^2 2t + \sin^2 2t)$
$= 25 \quad \{\cos^2\theta + \sin^2\theta = 1\}$

Starting from $(5, 0)$, P moves anticlockwise in a circle centred at $(0, 0)$ with radius 5 m.



d Speed $= |\mathbf{v}|$
$= \sqrt{(-10\sin 2t)^2 + (10\cos 2t)^2}$
$= 10\sqrt{\sin^2 2t + \cos^2 2t}$
$= 10 \text{ m s}^{-1}$

9 **a** $x^2 + y^2 = \left(30\cos\frac{t}{6}\right)^2 + \left(30\sin\frac{t}{6}\right)^2$
$= 30^2\left(\cos^2\frac{t}{6} + \sin^2\frac{t}{6}\right)$
$= 30^2 \quad \{\cos^2\theta + \sin^2\theta = 1\}$

$\therefore$ the radius of the track is 30 cm.

b The train returns to its starting point when $\frac{t}{6} = 2\pi$
$\therefore\ t = 12\pi$

$\therefore$ the train takes 12π seconds to complete a lap of the track.

c $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -5\sin\frac{t}{6} \\ 5\cos\frac{t}{6} \end{pmatrix}$

d Speed $= |\mathbf{v}|$
$= \sqrt{\left(-5\sin\frac{t}{6}\right)^2 + \left(5\cos\frac{t}{6}\right)^2}$
$= 5\sqrt{\sin^2\frac{t}{6} + \cos^2\frac{t}{6}}$
$= 5 \text{ cm s}^{-1}$

e **i** At time t, the second train is in the same position the first train was at time $(t-4)$, $t \geqslant 4$.
$\therefore$ the position equations for the second train are
$x(t) = 30\cos\left(\frac{t-4}{6}\right),\quad y(t) = 30\sin\left(\frac{t-4}{6}\right),\quad t \geqslant 4.$

ii For the first train, $x(12) = 30\cos 2,\quad y(12) = 30\sin 2$
For the second train, $x(12) = 30\cos\frac{8}{6},\quad y(12) = 30\sin\frac{8}{6}$

After 12 seconds, the first train is at $(30\cos 2, 30\sin 2)$, and the second train is at $(30\cos\frac{4}{3}, 30\sin\frac{4}{3})$.

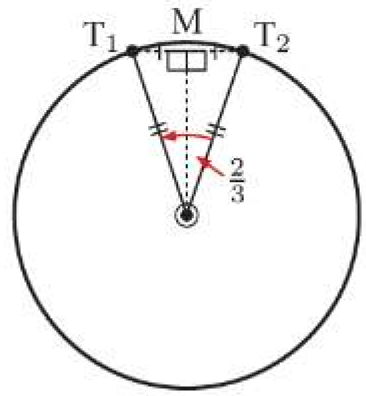
iii In 4 seconds, the trains travel through an angle of $\frac{4}{6}=\frac{2}{3}$.

In right angled triangle OMT_1,

$$M\widehat{O}T_1=\frac{1}{3} \quad \text{and} \quad \sin\frac{1}{3}=\frac{T_1M}{30}$$

$$\therefore \quad T_1M = 30\sin\frac{1}{3}$$

$$\therefore \quad T_1T_2 = 60\sin\frac{1}{3} \approx 19.6 \text{ cm}$$

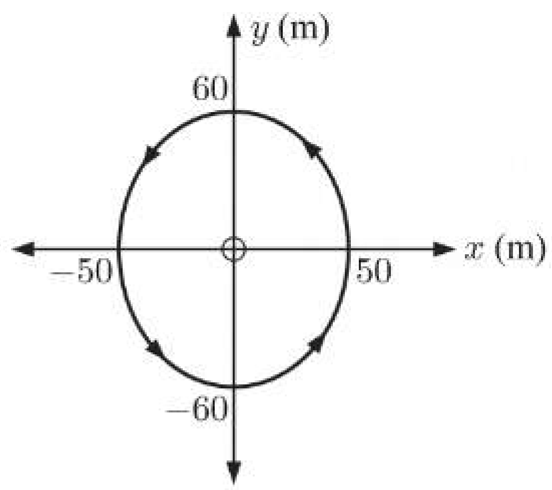


10 **a** $x(0)=50\cos 0=50, \quad y(0)=60\sin 0 = 0$

$\therefore$ Kristen's starting position is $(50, 0)$.

b $$\frac{x^2}{50^2}+\frac{y^2}{60^2}=\frac{\left[50\cos\left(\frac{\pi t}{40}\right)\right]^2}{50^2}+\frac{\left[60\sin\left(\frac{\pi t}{40}\right)\right]^2}{60^2}$$
$$=\cos^2\left(\tfrac{\pi t}{40}\right)+\sin^2\left(\tfrac{\pi t}{40}\right)$$
$$=1 \qquad \{\cos^2\theta+\sin^2\theta=1\}$$

c



The path Kristen runs along is an ellipse with width 100 m and height 120 m.

d Kristen will return to her starting position when $\frac{\pi t}{40}=2\pi$

$$\therefore \quad t = 80$$

$\therefore$ Kristen takes 80 seconds to complete one lap of the lake.

e $$\mathbf{v}=\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -\frac{5}{4}\pi\sin\left(\frac{\pi t}{40}\right) \\ \frac{3}{2}\pi\cos\left(\frac{\pi t}{40}\right) \end{pmatrix}$$

When $t=0$, $\mathbf{v}=\begin{pmatrix} -\frac{5}{4}\pi\sin 0 \\ \frac{3}{2}\pi\cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2}\pi \end{pmatrix}$

$\therefore$ Kristin's initial speed is $\frac{3}{2}\pi \approx 4.71 \text{ m s}^{-1}$.

ACTIVITY 1 ORBITS

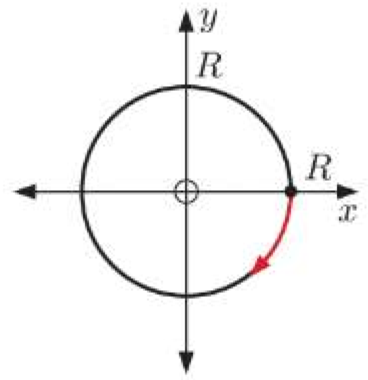
1 **a** P moves in a circular orbit with radius R units and centre $(0, 0)$. P starts at $(R, 0)$ and moves anticlockwise. The time it takes to perform a complete revolution is $\frac{2\pi}{\omega}$.

b **i** We specify $R > 0$ so that we can describe the radius of the orbit as R rather than $|R|$.

ii We specify $\omega > 0$ so that we can describe the time taken to perform a complete revolution as $\frac{2\pi}{\omega}$ rather than $\frac{2\pi}{|\omega|}$.

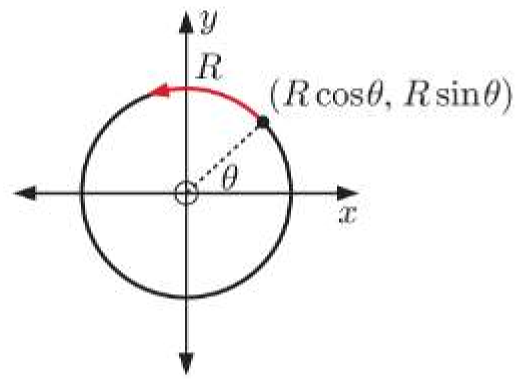
2 **a** $x(t) = R\cos(-\omega t),\quad y(t) = R\sin(-\omega t),\quad t \geqslant 0,\ R > 0,\ \omega > 0$

The object moves in a circular orbit with radius R units and centre $(0, 0)$. The object starts at $(R, 0)$ and moves *clockwise*.



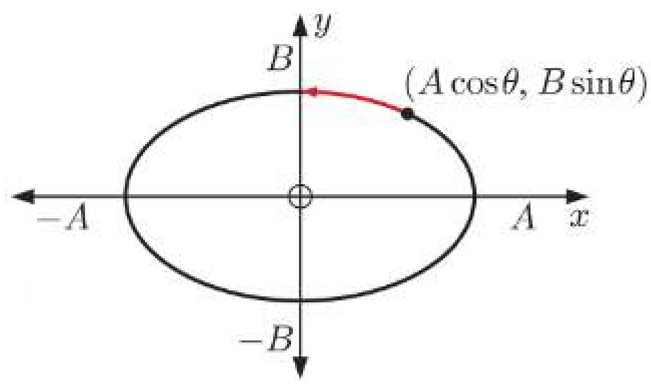
b $x(t) = R\cos(\omega t + \theta),\quad y(t) = R\sin(\omega t + \theta),\quad t \geqslant 0,\ R > 0,\ \omega > 0$

The object moves in a circular orbit with radius R units and centre $(0, 0)$. The object starts at $(R\cos\theta, R\sin\theta)$ and moves anticlockwise.



c $x(t) = A\cos(\omega t + \theta),\quad y(t) = B\sin(\omega t + \theta),\quad t \geqslant 0,\ A > 0,\ B > 0$

The object moves in an elliptical orbit with width $2A$, height $2B$, and centre $(0, 0)$. The object starts at $(A\cos\theta, B\sin\theta)$ and moves anticlockwise.



ACTIVITY 2 BÉZIER CURVES

1 **a** $x_0 = -3,\quad x_1 = 0,\quad x_2 = 2,\quad x_3 = 3$

$y_0 = -1,\quad y_1 = 3,\quad y_2 = 4,\quad y_3 = 2$

$$a_x = 3 - 3(2) + 3(0) - (-3) = 0 \qquad a_y = 2 - 3(4) + 3(3) - (-1) = 0$$

$$b_x = 3(2) - 6(0) + 3(-3) = -3 \qquad b_y = 3(4) - 6(3) + 3(-1) = -9$$

$$c_x = 3(0) - 3(-3) = 9 \qquad c_y = 3(3) - 3(-1) = 12$$

$$d_x = -3 \qquad d_y = -1$$

$$\therefore\ x(t) = -3t^2 + 9t - 3,\quad 0 \leqslant t \leqslant 1 \qquad \therefore\ y(t) = -9t^2 + 12t - 1,\quad 0 \leqslant t \leqslant 1$$

b $\underbrace{x(0) = -3,\quad y(0) = -1}_{\text{starting point}}$ ✓ $\qquad \underbrace{x(1) = 3,\quad y(1) = 2}_{\text{ending point}}$ ✓

c $x\left(\frac{1}{2}\right) = -3\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 3 = \frac{3}{4}$

$y\left(\frac{1}{2}\right) = -9\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) - 1 = 2\frac{3}{4}$

$\therefore$ Q is $\left(\frac{3}{4}, 2\frac{3}{4}\right)$.

d The highest point R occurs when $y'(t) = 0$ (since R is not at either endpoint).

$y'(t) = -18t + 12$

$\therefore$ $y'(t) = 0$ when $-18t + 12 = 0$

$\therefore$ $18t = 12$

$\therefore$ $t = \frac{12}{18} = \frac{2}{3}$

$x\left(\frac{2}{3}\right) = -3\left(\frac{2}{3}\right)^2 + 9\left(\frac{2}{3}\right) - 3 = 1\frac{2}{3}$

$y\left(\frac{2}{3}\right) = -9\left(\frac{2}{3}\right)^2 + 12\left(\frac{2}{3}\right) - 1 = 3$

$\therefore$ R is $\left(1\frac{2}{3}, 3\right)$.

2 **a** $x_0 = 4,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 3$

$y_0 = 1,\quad y_1 = 3,\quad y_2 = 0,\quad y_3 = -2$

$a_x = 3 - 3(2) + 3(1) - 4 = -4$

$b_x = 3(2) - 6(1) + 3(4) = 12$

$c_x = 3(1) - 3(4) = -9$

$d_x = 4$

$a_y = -2 - 3(0) + 3(3) - 1 = 6$

$b_y = 3(0) - 6(3) + 3(1) = -15$

$c_y = 3(3) - 3(1) = 6$

$d_y = 1$

$\therefore$ $x(t) = -4t^3 + 12t^2 - 9t + 4,\ 0 \leqslant t \leqslant 1$

$\therefore$ $y(t) = 6t^3 - 15t^2 + 6t + 1,\ 0 \leqslant t \leqslant 1$

b $x_0 = 2,\quad x_1 = -1,\quad x_2 = -2,\quad x_3 = 3$

$y_0 = 4,\quad y_1 = -1,\quad y_2 = 5,\quad y_3 = 6$

$a_x = 3 - 3(-2) + 3(-1) - 2 = 4$

$b_x = 3(-2) - 6(-1) + 3(2) = 6$

$c_x = 3(-1) - 3(2) = -9$

$d_x = 2$

$a_y = 6 - 3(5) + 3(-1) - 4 = -16$

$b_y = 3(5) - 6(-1) + 3(4) = 33$

$c_y = 3(-1) - 3(4) = -15$

$d_y = 4$

$\therefore$ $x(t) = 4t^3 + 6t^2 - 9t + 2,\ 0 \leqslant t \leqslant 1$

$\therefore$ $y(t) = -16t^3 + 33t^2 - 15t + 4,\ 0 \leqslant t \leqslant 1$

c $x_0 = -1,\quad x_1 = 0,\quad x_2 = 2,\quad x_3 = -3$

$y_0 = 2,\quad y_1 = 0,\quad y_2 = 3,\quad y_3 = 8$

$a_x = -3 - 3(2) + 3(0) - (-1) = -8$

$b_x = 3(2) - 6(0) + 3(-1) = 3$

$c_x = 3(0) - 3(-1) = 3$

$d_x = -1$

$a_y = 8 - 3(3) + 3(0) - 2 = -3$

$b_y = 3(3) - 6(0) + 3(2) = 15$

$c_y = 3(0) - 3(2) = -6$

$d_y = 2$

$\therefore$ $x(t) = -8t^3 + 3t^2 + 3t - 1,\ 0 \leqslant t \leqslant 1$

$\therefore$ $y(t) = -3t^3 + 15t^2 - 6t + 2,\ 0 \leqslant t \leqslant 1$

3 **a** $x_0 = 1,\quad x_1 = 0,\quad x_2 = -4,\quad x_3 = -2$
$y_0 = 2,\quad y_1 = 4,\quad y_2 = -2,\quad y_3 = 4$

$a_x = -2 - 3(-4) + 3(0) - 1 = 9$
$b_x = 3(-4) - 6(0) + 3(1) = -9$
$c_x = 3(0) - 3(1) = -3$
$d_x = 1$

$a_y = 4 - 3(-2) + 3(4) - 2 = 20$
$b_y = 3(-2) - 6(4) + 3(2) = -24$
$c_y = 3(4) - 3(2) = 6$
$d_y = 2$

$\therefore\ x(t) = 9t^3 - 9t^2 - 3t + 1,\ 0 \leqslant t \leqslant 1$
$\therefore\ y(t) = 20t^3 - 24t^2 + 6t + 2,\ 0 \leqslant t \leqslant 1$

b **i** The highest and lowest points occur when $y'(t) = 0$ or when $t = 0$ or 1.

Now $y(t) = 20t^3 - 24t^2 + 6t + 2$

$$\therefore\ y'(t) = 60t^2 - 48t + 6 = 6(10t^2 - 8t + 1)$$

Using technology, $y'(t) = 0$ when $t \approx 0.645$ or 0.155.

Checking these points and the boundary points:

t	$x(t)$	$y(t)$
0	1	2
0.645	-2.26	1.25
0.155	0.352	2.43
1	-2	4

$\therefore$ the highest point is $(-2, 4)$ and the lowest point is $\approx (-2.26, 1.25)$.

ii The left-most and right-most points occur when $x'(t) = 0$ or when $t = 0$ or 1.

Now $x(t) = 9t^3 - 9t^2 - 3t + 1$

$$\therefore\ x'(t) = 27t^2 - 18t - 3 = 3(9t^2 - 6t - 1)$$

Using technology, $x'(t) = 0$ when $t \approx 0.805$ or -1.38 {excluded as $t > 0$}

Checking $t \approx 0.805$ and the boundary points:

t	$x(t)$	$y(t)$
0	1	2
0.805	-2.55	1.71
1	-2	4

$\therefore$ the left-most point is $\approx (-2.55, 1.71)$ and the right-most point is $(1, 2)$.

4 Defining a curve mathematically allows the curve to be replicated exactly by anyone who knows the parametric equations of the curve.

INVESTIGATION — PROJECTILE MOTION

1 $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

By integrating each component of $\mathbf{a}$, $\mathbf{v} = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$ for constants c_1, c_2.

Now when $t = 0$, $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore\ c_1 = a$ and $c_2 = b$, so $\mathbf{v} = \begin{pmatrix} a \\ b - gt \end{pmatrix}$.

2 By integrating each component of $\mathbf{v}$, the position equations are

$x(t) = at + c_1, \quad y(t) = bt - \frac{1}{2}gt^2 + c_2, \quad$ for constants c_1, c_2.

Now $x(0) = x_0$ and $y(0) = y_0$

$\therefore \; c_1 = x_0$ and $c_2 = y_0$, so $x(t) = x_0 + at$

$y(t) = y_0 + bt - \frac{1}{2}gt^2$

EXERCISE 24G

1 **a** $x(0) = -0.6, \quad y(0) = 1.8$

$\therefore$ the initial position is $(-0.6, 1.8)$.

b $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 20.2 \\ 22.2 - 9.8t \end{pmatrix}$

When $t = 0$, $\mathbf{v} = \begin{pmatrix} 20.2 \\ 22.2 \end{pmatrix}$

$\therefore$ speed $= \sqrt{20.2^2 + 22.2^2}$

$\approx 30.0 \text{ m s}^{-1}$

c Distance from line $= |\,0 - (-0.6)\,|$

$= 0.6$ m

d The javelin lands when $y(t) = 0$

$\therefore \; 1.8 + 22.2t - 4.9t^2 = 0$

$\therefore \; t \approx 4.61$ {technology, $t > 0$}

The total flight time was approximately 4.61 seconds.

e The maximum height was reached when $y'(t) = 0$

$\therefore \; 22.2 - 9.8t = 0$

$\therefore \; t = \frac{22.2}{9.8}$

Now $y\left(\frac{22.8}{9.8}\right) = 1.8 + 22.2\left(\frac{22.8}{9.8}\right) - 4.9\left(\frac{22.8}{9.8}\right)^2$

≈ 26.9 m

So, the maximum height reached was approximately 26.9 m.

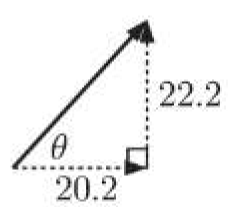
f $x(4.61) = -0.6 + 20.2(4.61)$

≈ 92.522

The javelin travelled $\approx 0.6 + 92.522$

≈ 93.1 m horizontally.

g When $t = 0$, $\mathbf{v} = \begin{pmatrix} 20.2 \\ 22.2 \end{pmatrix}$

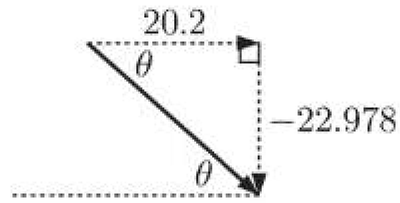


$\therefore \; \tan\theta = \frac{22.2}{20.2}$

$\therefore \; \theta \approx 47.7°$

The javelin was released at approximately 47.7° above the horizontal.

h When $t \approx 4.61$, $\mathbf{v} \approx \begin{pmatrix} 20.2 \\ 22.2 - 9.8(4.61) \end{pmatrix} \approx \begin{pmatrix} 20.2 \\ -22.978 \end{pmatrix}$



$\therefore \tan\theta \approx \dfrac{22.978}{20.2}$

$\therefore \theta \approx 48.7°$

The javelin strikes the ground at an angle of approximately 48.7°.

2 **a** **i** $x(0) = 3$, $y(0) = 1.3$

$\therefore$ the initial position is $(3, 1.3)$.

ii $x(2) = 3 + 12.7(2) = 28.4$, $y(2) = 1.3 + 13.2(2) - 4.9(2)^2 = 8.1$

$\therefore$ after 2 seconds, the position is $(28.4, 8.1)$.

b The stone hits the ground when $y(t) = 0$

$$\therefore 1.3 + 13.2t - 4.9t^2 = 0$$
$$\therefore t \approx 2.79 \quad \{\text{technology, } t \geqslant 0\}$$

The stone took approximately 2.79 seconds to hit the ground.

c The stone travelled a horizontal distance of $x(2.79) - x(0) \approx (3 + 12.7(2.79)) - 3$

$$\approx 12.7 \times 2.79$$
$$\approx 35.4 \text{ m}$$

d The maximum height is reached when $y'(t) = 0$

$$\therefore 13.2 - 9.8t = 0$$
$$\therefore t = \frac{13.2}{9.8}$$

$\therefore$ the maximum height reached by the stone $= y\left(\frac{13.2}{9.8}\right)$

$$= 1.3 + 13.2\left(\tfrac{13.2}{9.8}\right) - 4.9\left(\tfrac{13.2}{9.8}\right)^2$$
$$\approx 10.2 \text{ m}$$

e The stone has velocity vector $\mathbf{v} = \begin{pmatrix} 12.7 \\ 13.2 - 9.8t \end{pmatrix}$.

When $t = 2$, $\mathbf{v} = \begin{pmatrix} 12.7 \\ -6.4 \end{pmatrix}$.

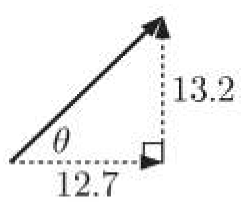
$\therefore$ speed $= \sqrt{12.7^2 + (-6.4)^2}$

$\approx 14.2 \text{ m s}^{-1}$

f When $t = 0$, $\mathbf{v} = \begin{pmatrix} 12.7 \\ 13.2 \end{pmatrix}$

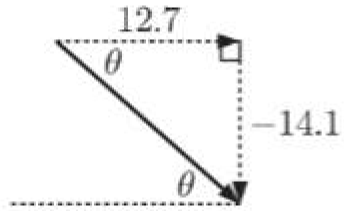
$\therefore \tan\theta = \dfrac{13.2}{12.7}$

$\therefore \theta \approx 46.1°$



The angle of release is approximately 46.1°.

g The stone strikes the ground when $t \approx 2.79$ seconds.



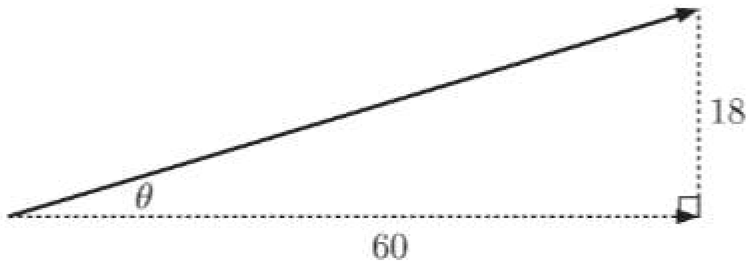
When $t \approx 2.79$, $\mathbf{v} \approx \begin{pmatrix} 12.7 \\ -14.1 \end{pmatrix}$

$\therefore \tan\theta \approx \dfrac{14.1}{12.7}$

$\therefore \theta \approx 48.1°$

The angle at which the stone strikes the ground is approximately $48.1°$.

3 **a** When $t = 0$, $\mathbf{v} = \begin{pmatrix} 60 \\ 18 \end{pmatrix}$



$\therefore$ speed $= \sqrt{60^2 + 18^2}$
$\approx 62.6 \text{ m s}^{-1}$

$\therefore \tan\theta = \dfrac{18}{60}$

$\therefore \theta \approx 16.7°$

The initial angle of trajectory is approximately $16.7°$.

b The maximum height was reached when $18 - gt = 0$

$\therefore t = \dfrac{18}{g}$

$\therefore t \approx \dfrac{18}{9.8} \approx 1.84$ s

c $y(t) = \int (18 - 9.8t)\, dt$
$= 18t - 4.9t^2 + c$

But $y(0) = 0$, so $c = 0$

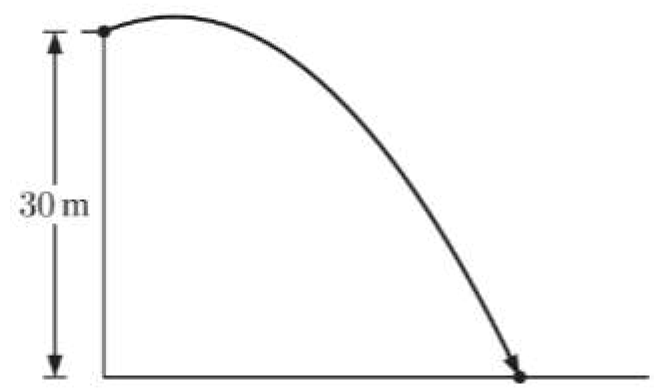
$\therefore y(t) = 18t - 4.9t^2$

$\therefore y(1.84) \approx 18(1.84) - 4.9(1.84)^2 \approx 16.5$

$\therefore$ the maximum height is approximately 16.5 m.

d No, the golf ball would be slowed by air resistance and would have a lower maximum height.

4 **a** **i** When $t = 0$, $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$



$\therefore$ speed $= \sqrt{5^2 + 5^2}$
$= 5\sqrt{2}$
$\approx 7.07 \text{ m s}^{-1}$

ii When $t = 0.5$, $\mathbf{v} \approx \begin{pmatrix} 5 \\ 0.1 \end{pmatrix}$ $\{g \approx 9.8\}$

$\therefore$ speed $\approx \sqrt{5^2 + 0.1^2}$
$\approx 5.00 \text{ m s}^{-1}$

b $$y(t) = \int (5 - 9.8t)\, dt$$
$$= 5t - 4.9t^2 + c$$

But $y(0) = 30$, so $c = 30$

$\therefore\ y(t) = 5t - 4.9t^2 + 30$

Now $y(t) = 0$ when $5t - 4.9t^2 + 30 = 0$

$\therefore\ t \approx 3.04$ s $\quad$ {technology, $t > 0$}

c $$x(t) = \int 5\, dt$$
$$= 5t + c$$

But $x(0) = 0$, so $c = 0$

$\therefore\ x(t) = 5t$

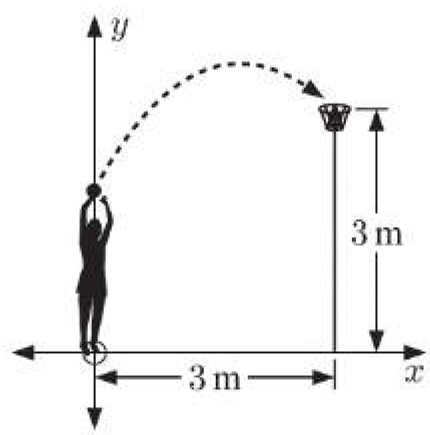
$\therefore\ x(3.04) \approx 5(3.04) \approx 15.2$

The rock landed approximately 15.2 m from the castle.

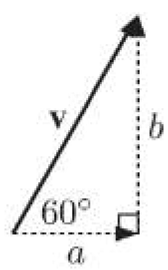
5 **a** The initial position of the ball is $(0, 2)$,

and the initial velocity vector is $\begin{pmatrix} a \\ b \end{pmatrix}$.

So, $x(t) = at$, $\quad y(t) = 2 + bt - 4.9t^2$, $\quad t \geqslant 0$



b



$$\tan 60° = \frac{b}{a}$$
$$\therefore\ \sqrt{3} = \frac{b}{a}$$
$$\therefore\ b = a\sqrt{3}$$

c The goal is at the point $(3, 3)$.

Now when $x(t) = 3$, $\quad at = 3$

$$\therefore\ t = \frac{3}{a}$$

We require a such that $y\left(\frac{3}{a}\right) = 3$

$$\therefore\ 2 + b\left(\frac{3}{a}\right) - 4.9\left(\frac{3}{a}\right)^2 = 3$$
$$\therefore\ 2 + a\sqrt{3}\left(\frac{3}{a}\right) - 4.9\left(\frac{9}{a^2}\right) = 3$$
$$\therefore\ 3\sqrt{3} - 1 = \frac{44.1}{a^2}$$
$$\therefore\ a^2 = \frac{44.1}{3\sqrt{3} - 1}$$
$$\therefore\ a = \sqrt{\frac{44.1}{3\sqrt{3} - 1}} \quad \{a > 0\}$$
$$\approx 3.24$$

d $\text{Speed} = \sqrt{a^2+b^2}$

$$\begin{aligned} &= \sqrt{a^2+(a\sqrt{3})^2} \\ &= \sqrt{a^2+3a^2} \\ &= 2a \\ &\approx 6.48\ \text{m s}^{-1} \end{aligned}$$

e The maximum height is reached when $y'(t) = 0$

$$\begin{aligned} \therefore\ b - 9.8t &= 0 \\ \therefore\ a\sqrt{3} - 9.8t &= 0 \\ \therefore\ t &= \frac{a\sqrt{3}}{9.8} \approx 0.573 \end{aligned}$$

$\therefore$ the ball reaches the maximum height at $t \approx 0.573$ seconds.

Now $y(0.573) \approx 2 + a\sqrt{3}(0.573) - 4.9(0.573)^2$
≈ 3.61

The maximum height reached by the ball is approximately 3.61 m.

f The ball has velocity vector $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a \\ a\sqrt{3} - 9.8t \end{pmatrix}$.

The ball passes through the net when $t = \dfrac{3}{a}$.

At this time, $\mathbf{v} = \begin{pmatrix} a \\ a\sqrt{3} - 9.8\left(\frac{3}{a}\right) \end{pmatrix} \approx \begin{pmatrix} 3.24 \\ -3.45 \end{pmatrix}$

and speed $\approx \sqrt{3.24^2 + (-3.45)^2} \approx 4.74\ \text{m s}^{-1}$

6 The projectile has position equations $x(t) = at$, $y(t) = bt - \frac{1}{2}gt^2$, and velocity vector $\mathbf{v} = \begin{pmatrix} a \\ b - gt \end{pmatrix}$, $t \geqslant 0$.

a **i** The maximum height is reached when $b - gt = 0$

$$\therefore\ t = \frac{b}{g}$$

ii
$$\begin{aligned} y\left(\frac{b}{g}\right) &= b\left(\frac{b}{g}\right) - \tfrac{1}{2}g\left(\frac{b}{g}\right)^2 \\ &= \frac{b^2}{g} - \tfrac{1}{2}\,\frac{b^2}{g} \\ &= \frac{b^2}{2g} \end{aligned}$$

$\therefore$ the maximum height is $\dfrac{b^2}{2g}$.

iii $y(t) = 0$ when $bt - \frac{1}{2}gt^2 = 0$

$$\begin{aligned} \therefore\ t(b - \tfrac{1}{2}gt) &= 0 \\ \therefore\ t = 0 \text{ or } t &= \frac{2b}{g} \end{aligned}$$

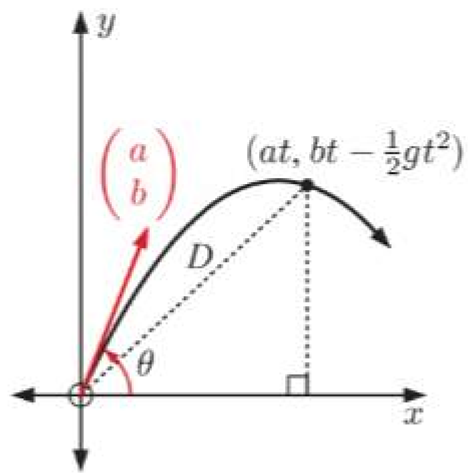
$\therefore$ the flight time until the projectile returns to the ground is $\dfrac{2b}{g}$.

iv $x\left(\frac{2b}{g}\right) = a\left(\frac{2b}{g}\right) = \frac{2ab}{g}$

$\therefore$ the horizontal distance travelled is $\frac{2ab}{g}$.

b **i** $D^2 = (at)^2 + (bt - \frac{1}{2}gt^2)^2$

$\therefore\ D = \sqrt{(at)^2 + (bt - \frac{1}{2}gt^2)^2}$ $\{D > 0\}$



ii $D^2 = (at)^2 + (bt)^2 - 2(bt)\left(\frac{1}{2}gt^2\right) + \left(\frac{1}{2}gt^2\right)^2$

$\therefore\ D^2 = a^2t^2 + b^2t^2 - bgt^3 + \frac{1}{4}g^2t^4$

$\therefore\ \frac{d}{dt}(D^2) = 2a^2t + 2b^2t - 3bgt^2 + g^2t^3$

iii We first note that, for $a, b > 0$, $\tan\theta = \frac{b}{a}$.

Now D is maximised when D^2 is maximised.

$$\frac{d}{dt}(D^2) = 0 \quad \text{when}$$

$2a^2t + 2b^2t - 3bgt^2 + g^2t^3 = 0$

$\therefore\ t(2a^2 + 2b^2 - 3bgt + g^2t^2) = 0$

$\therefore\ g^2t^2 - 3bgt + 2a^2 + 2b^2 = 0$ $\{t = 0$ corresponds to a local minimum for $D^2\}$

$$\therefore\ t = \frac{3bg \pm \sqrt{(-3bg)^2 - 4(g^2)(2a^2 + 2b^2)}}{2g^2}$$

$$= \frac{3bg \pm \sqrt{9b^2g^2 - 8a^2g^2 - 8b^2g^2}}{2g^2}$$

$$= \frac{3bg \pm g\sqrt{b^2 - 8a^2}}{2g^2}$$

$$= \frac{3b \pm \sqrt{b^2 - 8a^2}}{2g}$$

$\therefore$ a local maximum will exist if $b^2 - 8a^2 > 0$

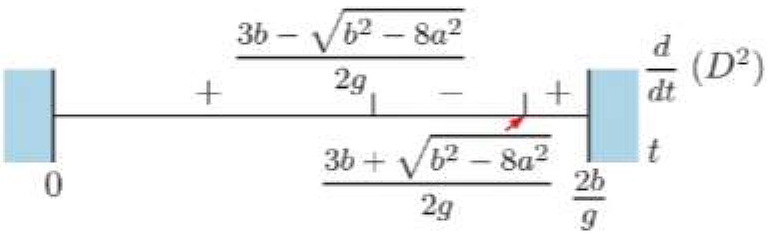
$$\therefore\ b^2 > 8a^2$$

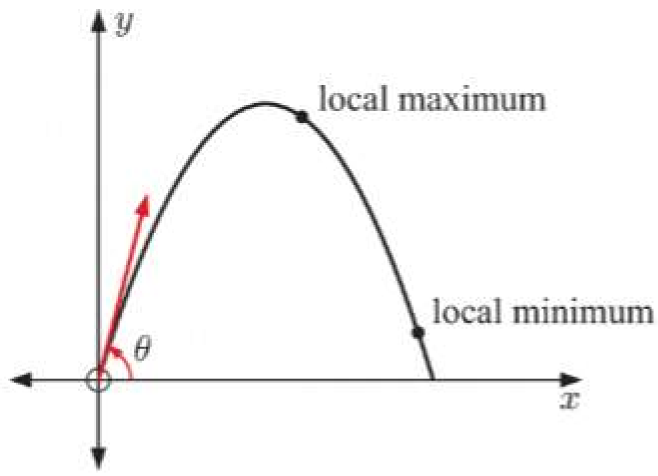
$$\therefore\ \frac{b^2}{a^2} > 8$$

$$\therefore\ \frac{b}{a} > 2\sqrt{2} \qquad \{a, b > 0\}$$

But $\tan\theta = \frac{b}{a}$, so a local maximum exists if $\tan\theta > 2\sqrt{2}$.

In this case, $\dfrac{d}{dt}(D^2)$ has sign diagram





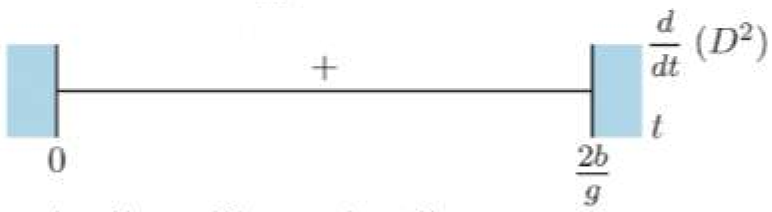
So, D will reach a local maximum at

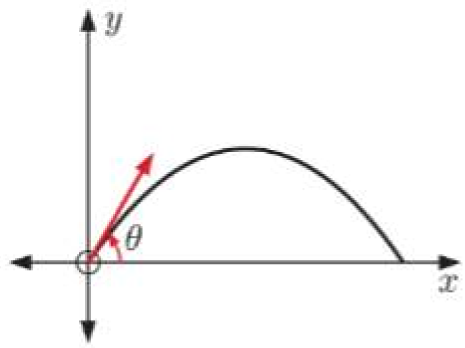
$t = \dfrac{3b-\sqrt{b^2-8a^2}}{2g}$, and a local minimum

at $t = \dfrac{3b+\sqrt{b^2-8a^2}}{2g}$.

$\left(\text{Note that } \sqrt{b^2-8a^2} < b, \text{ so } \dfrac{3b+\sqrt{b^2-8a^2}}{2g} < \dfrac{2b}{g}.\right)$

If $0 \leqslant \tan\theta \leqslant 2\sqrt{2}$, $\dfrac{d}{dt}(D^2)$ has sign diagram





So, the projectile will continually move further from the origin while in flight.

REVIEW SET 24A

1 **a** $s(t) = 12 - 2t$ m, $\quad 0 \leqslant t \leqslant 10$ s

$s(0) = 12 - 2(0)$
$\quad = 12$ m

$\therefore$ the initial displacement of the object is 12 m to the right of the origin.

b **i** $s(1) = 12 - 2(1)$
$\quad = 12 - 2$
$\quad = 10$ m

$\therefore$ the displacement of the object at $t = 1$ second is 10 m to the right of the origin.

ii $s(3) = 12 - 2(3)$
$\quad = 12 - 6$
$\quad = 6$ m

$\therefore$ the displacement of the object at $t = 3$ seconds is 6 m to the right of the origin.

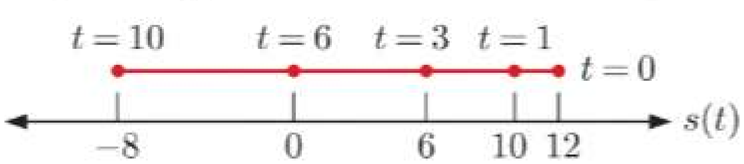
c The object is at the origin when $s(t) = 0$

$\therefore 12 - 2t = 0$
$\therefore 2t = 12$
$\therefore t = 6$

$\therefore$ the object reaches the origin at $t = 6$ seconds.

d No, the displacement function is linear, so it has no turning points.

e

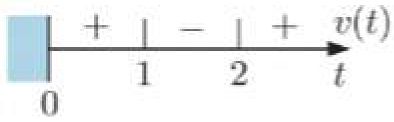


2 **a** $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, $t \geqslant 0$ s

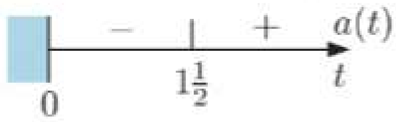
$\therefore$ $v(t) = 6t^2 - 18t + 12$ cm s^{-1} $\{v(t) = s'(t)\}$

$\therefore$ $a(t) = 12t - 18$ cm s^{-2} $\{a(t) = v'(t)\}$

$v(t) = 6(t-1)(t-2)$ has sign diagram:



$a(t) = 6(2t-3)$ has sign diagram:



b $s(0) = -5$ cm

$v(0) = 12$ cm s^{-1}

$a(0) = -18$ cm s^{-2}

The particle P is initially 5 cm to the left of the origin, moving to the right at 12 cm s^{-1}, with acceleration -18 cm s^{-2} (decreasing speed).

c $s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 5$
$= 16 - 36 + 24 - 5$
$= -1$ cm

$v(2) = 6(2)^2 - 18(2) + 12$
$= 24 - 36 + 12$
$= 0$ cm s^{-1}

$a(2) = 12(2) - 18$
$= 24 - 18$
$= 6$ cm s^{-2}

At $t = 2$, the particle is 1 cm to the left of the origin, is instantaneously stationary, and is beginning to accelerate.

d The particle changes direction when $v(t)$ changes sign.

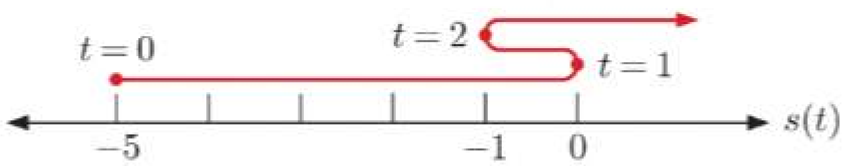
From the sign diagram in **a**, this occurs at $t = 1$ second and $t = 2$ seconds.

$s(1) = 2 - 9 + 12 - 5$
$= 0$

$s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 5$
$= 16 - 36 + 24 - 5$
$= -1$

$\therefore$ the particle changes direction at $t = 1$ second when it is at the origin, and at $t = 2$ seconds when it is 1 cm to the left of the origin.

e



f The particle's speed is increasing when $v(t)$ and $a(t)$ have the same sign.

From the sign diagrams in **a**, this occurs for $1 \leqslant t \leqslant 1\frac{1}{2}$ and $t \geqslant 2$ seconds.

3 **a** $\dot{x}(t) = t^2 - 6t + 8$
$= (t-2)(t-4)$

$\therefore$ the sign diagram of $\dot{x}(t)$ is:

b Since the signs change, the particle reverses direction at $t = 2$ and $t = 4$ seconds.

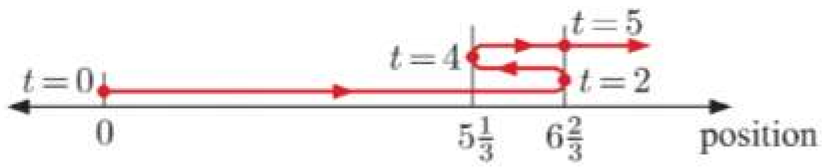
$$\begin{aligned} \text{Now} \quad x(t) &= \int \dot{x}(t)\, dt \\ &= \int (t^2 - 6t + 8)\, dt \\ &= \tfrac{1}{3}t^3 - 3t^2 + 8t + c \end{aligned}$$

We choose the initial displacement to be zero,

so $x(0) = c = 0$

$$\begin{aligned} \therefore \quad x(2) &= \tfrac{1}{3}(2)^3 - 3(2)^2 + 8(2) = \tfrac{8}{3} - 12 + 16 = 6\tfrac{2}{3} \\ x(4) &= \tfrac{1}{3}(4)^3 - 3(4)^2 + 8(4) = \tfrac{64}{3} - 48 + 32 = 5\tfrac{1}{3} \\ x(5) &= \tfrac{1}{3}(5)^3 - 3(5)^2 + 8(5) = \tfrac{125}{3} - 75 + 40 = 6\tfrac{2}{3} \end{aligned}$$

Motion diagram:



The particle initially moves in the positive direction, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$, it is $6\frac{2}{3}$ m from its starting point again.

c After 5 seconds, the particle is $6\frac{2}{3}$ metres from its original position.

d
$$\begin{aligned} \text{Total distance travelled} &= 6\tfrac{2}{3} + \left(6\tfrac{2}{3} - 5\tfrac{1}{3}\right) + \left(6\tfrac{2}{3} - 5\tfrac{1}{3}\right) \\ &= 6\tfrac{2}{3} + 1\tfrac{1}{3} + 1\tfrac{1}{3} \\ &= 9\tfrac{1}{3} \text{ metres} \end{aligned}$$

4 **a** $v(t) = 2.75 - t + 0.5t^{1.2}$ m s^{-1}, $0 \leqslant t \leqslant 6$ s

i $v(0) = 2.75 \text{ m s}^{-1}$

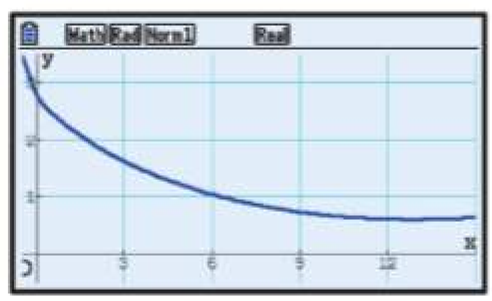
$\therefore$ the velocity of the kayak after the kayaker stops paddling is 2.75 m s^{-1}.

ii $v(3) = 2.75 - 3 + 0.5(3)^{1.2}$

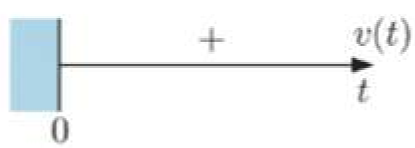
≈ 1.62

$\therefore$ the velocity of the kayak after 3 seconds is about 1.62 m s^{-1}.

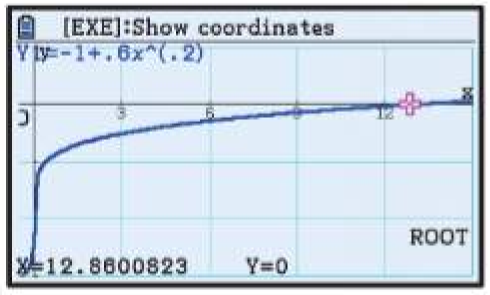
b $v(t) > 0$ for all t {using technology}



So, $v(t)$ has sign diagram:



$$a(t) = -1 + 0.5(1.2)t^{0.2} \qquad \{a(t) = v'(t)\}$$
$$= -1 + 0.6t^{0.2} \text{ m s}^{-2}$$



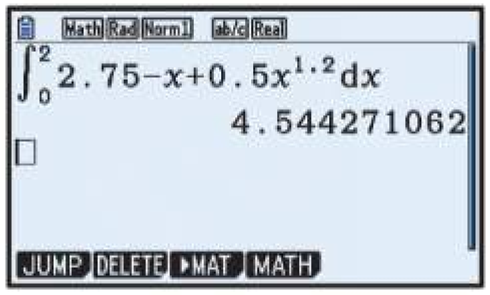
$a(t) = 0$ when $t \approx 12.9$ {using technology}

So, $a(t)$ has sign diagram:

Now, our model only considers the interval $0 \leqslant t \leqslant 6$ seconds, where $v(t)$ and $a(t)$ have opposite sign.

$\therefore$ the kayak's speed is decreasing during the 6 second period.

c $$\int_0^2 v(t)\,dt = \int_0^2 (2.75 - t + 0.5t^{1.2})\,dt$$
$$\approx 4.54 \qquad \{\text{using technology}\}$$



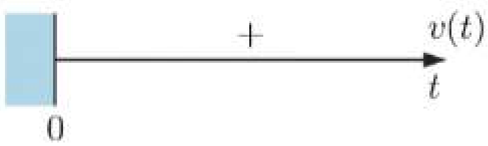
The kayak travels approximately 4.54 m in the first 2 seconds after the kayaker stops paddling.

5 **a** $$s(t) = 15t - \frac{60}{(t+1)^2} \text{ cm}, \quad t \geqslant 0 \text{ s}$$
$$= 15t - 60(t+1)^{-2}$$
$$\therefore\ v(t) = 15 + 120(t+1)^{-3} \qquad \{v(t) = s'(t)\}$$
$$= 15 + \frac{120}{(t+1)^3} \text{ cm s}^{-1}$$
$$\therefore\ a(t) = -360(t+1)^{-4} \qquad \{a(t) = v'(t)\}$$
$$= -\frac{360}{(t+1)^4} \text{ cm s}^{-2}$$

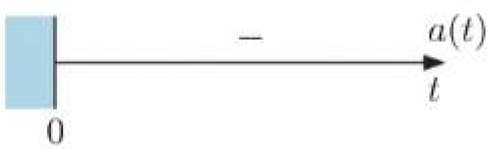
b $$s(3) = 15(3) - \frac{60}{(3+1)^2} = 41.25 \text{ cm}$$
$$v(3) = 15 + \frac{120}{(3+1)^3} = 16.875 \text{ cm s}^{-1}$$
$$a(3) = -\frac{360}{(3+1)^4} \approx -1.41 \text{ cm s}^{-2}$$

So, at time $t = 3$ seconds, the particle is 41.25 cm to the right of O, moving to the right at 16.875 cm s^{-1}, with decreasing speed $(a(3) \approx -1.41 \text{ cm s}^{-2})$.

c $v(t)$ has sign diagram:



$a(t)$ has sign diagram:



The signs of $v(t)$ and $a(t)$ are never the same.

$\therefore$ the particle's speed is never increasing.

6 **a** $x(t) = 3 + 2\sin \pi t$ m, $t \geqslant 0$ s

$\therefore\ \dot{x}(t) = 2\pi \cos \pi t \qquad \{\dot{x}(t) = x'(t)\}$

$\therefore\ \ddot{x}(t) = -2\pi^2 \sin \pi t \qquad \{\ddot{x}(t) = \dot{x}'(t)\}$

$x(0) = 3 + 2\sin 0 = 3$ m $\qquad \dot{x}(0) = 2\pi \cos 0 = 2\pi$ m s^{-1} $\qquad \ddot{x}(0) = -2\pi^2 \sin 0 = 0$ m s^{-2}

The object is initially 3 m to the right of the origin, moving to the right at 2π m s^{-1}, and has acceleration 0 m s^{-2}.

b $\dot{x}(t) = 0$ when $2\pi \cos \pi t = 0$

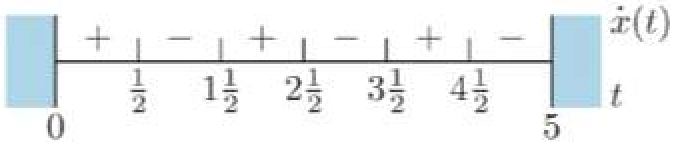
$\therefore\ \cos \pi t = 0$

$\therefore\ \pi t = \left(k + \frac{1}{2}\right)\pi, \quad k \in \mathbb{Z}$

$\therefore\ t = k + \frac{1}{2}, \quad k \in \mathbb{Z}$

$\therefore\ t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}$ seconds $\qquad \{0 \leqslant t \leqslant 5\}$

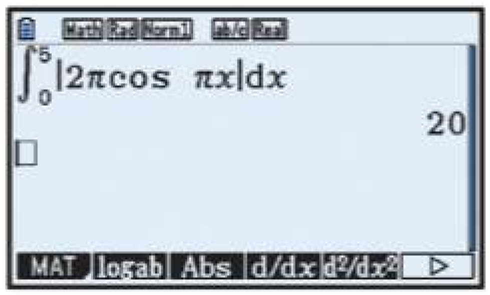
$\dot{x}(t)$ has sign diagram:



So, the spotlight changes direction at $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, and $4\frac{1}{2}$ seconds during the first 5 seconds.

c $\displaystyle\int_0^5 |\,\dot{x}(t)\,|\ dt = \int_0^5 |\,2\pi\cos \pi t\,|\ dt$

$= 20 \qquad$ {using technology}

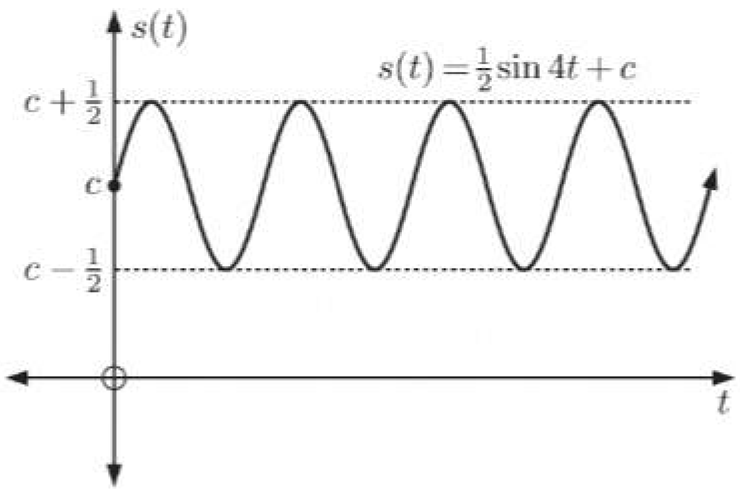


$\therefore$ the total distance travelled by the spotlight in the first 5 seconds is 20 m.

7 **a** $v(t) = s'(t) = 2\cos 4t$ m s^{-1}

$\therefore\ s(t) = \displaystyle\int 2\cos 4t\ dt$

$= \frac{1}{2}\sin 4t + c$ m



The graph shows that the particle oscillates between positions $c + \frac{1}{2}$ and $c - \frac{1}{2}$.

Distance $= (c + \frac{1}{2}) - (c - \frac{1}{2})$

$= 1$ m

b $s(t) = \frac{1}{2}\sin 4t + c$

$s\left(\frac{\pi}{12}\right) = 6$

$\therefore\ \frac{1}{2}\sin \frac{\pi}{3} + c = 6$

$\therefore\ \frac{\sqrt{3}}{4} + c = 6$

$\therefore\ c = 6 - \frac{\sqrt{3}}{4}$

$s\left(\frac{\pi}{6}\right) = \frac{1}{2}\sin \frac{2\pi}{3} + 6 - \frac{\sqrt{3}}{4}$

$= \frac{\sqrt{3}}{4} + 6 - \frac{\sqrt{3}}{4}$

$= 6$ m

c $\displaystyle\int_0^{\pi} |\,v(t)\,|\,dt = \int_0^{\pi} |\,2\cos 4t\,|\,dt$

$= 4$ {using technology}

$\therefore$ the total distance travelled by the particle in the first π seconds is 4 m.

8 **a** $a(t) = -2\ \text{m s}^{-2}$

$\therefore\ v(t) = \displaystyle\int a(t)\,dt$

$= \displaystyle\int -2\,dt$

$= -2t + c\ \text{m s}^{-1}$

But $v(0) = 65$

$\therefore\ c = 65$

$\therefore\ v(t) = -2t + 65\ \text{m s}^{-1}$

b $s(t) = \displaystyle\int v(t)\,dt$

$= \displaystyle\int (-2t + 65)\,dt$

$= -t^2 + 65t + c$

But $s(0) = 0$

$\therefore\ c = 0$

$\therefore\ s(t) = -t^2 + 65t$ m

c **i** $v(t) = 3$ when $-2t + 65 = 3$

$\therefore\ 2t = 62$

$\therefore\ t = 31$

$\therefore$ it will take 31 seconds for the plane to reduce its speed to 3 m s^{-1}.

ii $s(31) = -31^2 + 65(31)$

$= 1054$

$\therefore$ the plane will have travelled 1054 m along the runway after 31 seconds.

9 $v(t) = \dfrac{100}{(t+2)^2}\ \text{m s}^{-1}$

a $v(0) = \dfrac{100}{2^2} = \dfrac{100}{4} = 25$

The initial velocity of the boat was 25 m s^{-1}.

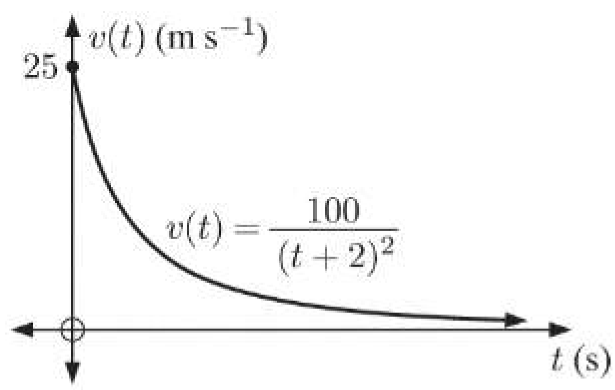
$v(3) = \dfrac{100}{5^2} = \dfrac{100}{25} = 4$

The velocity of the boat after 3 seconds was 4 m s^{-1}.

b As $t \to \infty$, $(t+2)^2 \to \infty$

$\therefore\ v(t) \to 0$ from above

c



d $\displaystyle\int_0^2 v(t)\,dt = \int_0^2 100(t+2)^{-2}\,dt$

$= \left[-100(t+2)^{-1}\right]_0^2$

$= -\frac{100}{4} - \left(-\frac{100}{2}\right)$

$= 25$

The boat travels a total distance of 25 m in the first 2 seconds after its engine is turned off.

e Suppose the boat travels 30 m after T seconds, then

$$\begin{aligned}\int_0^T (100(t+2)^{-2})\,dt &= 30\\ \therefore\ \left[-100(t+2)^{-1}\right]_0^T &= 30\\ \therefore\ \left(-\frac{100}{T+2}\right)-\left(-\frac{100}{2}\right) &= 30\\ \therefore\ -\frac{100}{T+2} &= -20\\ \therefore\ 20T+40 &= 100\\ \therefore\ 20T &= 60\\ \therefore\ T &= 3\end{aligned}$$

So it will take 3 seconds for the boat to travel 30 metres.

10 **a**

$$\begin{aligned} v &= s^2\\ \therefore\ a &= v\,\frac{dv}{ds}\\ &= s^2\times 2s\\ &= 2s^3\ \mathrm{m\,s^{-2}}\end{aligned}$$

b When $s=-2$, $v=(-2)^2=4$ and $a=2(-2)^3=-16$

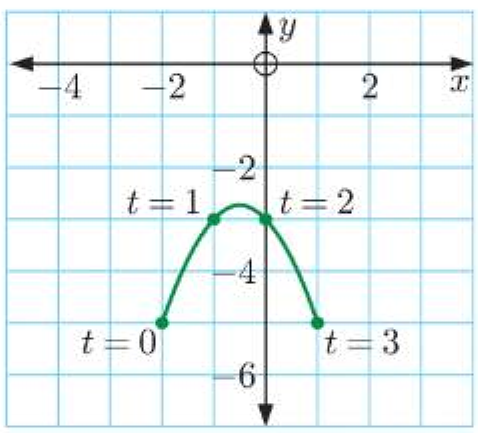
$\therefore$ initial velocity $= 4\ \mathrm{m\,s^{-1}}$ and initial acceleration $= -16\ \mathrm{m\,s^{-2}}$.

11 **a** $x(0)=-2$, $y(0)=-5$

$\therefore$ the initial position of P is $(-2, -5)$.

b

t	$\mathrm{P}(x(t), y(t))$
0	$(-2, -5)$
1	$(-1, -3)$
2	$(0, -3)$
3	$(1, -5)$



c P has velocity vector $\mathbf{v}=\begin{pmatrix}x'(t)\\ y'(t)\end{pmatrix}=\begin{pmatrix}1\\ -2t+3\end{pmatrix}$

d **i** When $t=0$, $\mathbf{v}=\begin{pmatrix}1\\ 3\end{pmatrix}$

$\therefore$ speed $=\sqrt{1^2+3^2}$
$=\sqrt{10}\ \mathrm{m\,s^{-1}}$

ii When $t=4$, $\mathbf{v}=\begin{pmatrix}1\\ -5\end{pmatrix}$

$\therefore$ speed $=\sqrt{1^2+(-5)^2}$
$=\sqrt{26}\ \mathrm{m\,s^{-1}}$

12 **a** **i** $x(0)=10\cos 0=10$
$y(0)=10\sin 0=0$
$\therefore$ the seal is initially at $(10, 0)$.

ii $x(\pi)=10\cos\frac{\pi}{2}=0$
$y(\pi)=10\sin\frac{\pi}{2}=10$
$\therefore$ after π seconds, the seal is at $(0, 10)$.

b The seal is swimming anticlockwise.

c The seal will return to its starting point when $\frac{t}{2} = 2\pi$

$$\therefore \quad t = 4\pi$$

The seal will take 4π seconds to complete a lap of the pool.

d The velocity vector $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -5\sin\frac{t}{2} \\ 5\cos\frac{t}{2} \end{pmatrix}$

e $$\begin{aligned}\text{Speed} &= \sqrt{\left(-5\sin\tfrac{t}{2}\right)^2 + \left(5\cos\tfrac{t}{2}\right)^2} \\ &= 5\sqrt{\sin^2\tfrac{t}{2} + \cos^2\tfrac{t}{2}} \\ &= 5 \text{ m s}^{-1} \qquad \{\cos^2\theta + \sin^2\theta = 1\}\end{aligned}$$

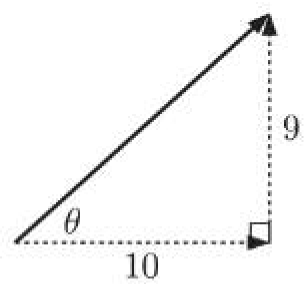
13 **a** When $t = 0$, $\mathbf{v} = \begin{pmatrix} 10 \\ 9 \end{pmatrix}$

$$\begin{aligned}\therefore \quad \text{speed} &= \sqrt{10^2 + 9^2} \\ &= \sqrt{181} \\ &\approx 13.5 \text{ m s}^{-1}\end{aligned}$$

b $\tan\theta = \frac{9}{10}$

$\therefore \quad \theta \approx 42.0°$

The shot was released at an angle of approximately $42.0°$.



c The maximum height is reached when $9 - gt = 0$

$$\therefore \quad t = \frac{9}{g} \approx \frac{9}{9.8}$$

Now $$\begin{aligned}y(t) &= \int (9 - 9.8t)\, dt \\ &= 9t - 4.9t^2 + c\end{aligned}$$

But $y(0) = 1.5$,

$\therefore \quad c = 1.5$

So, $y(t) = 9t - 4.9t^2 + 1.5$

$$\begin{aligned}\therefore \quad y\left(\tfrac{9}{9.8}\right) &= 9\left(\tfrac{9}{9.8}\right) - 4.9\left(\tfrac{9}{9.8}\right)^2 + 1.5 \\ &\approx 5.63\end{aligned}$$

The maximum height reached by the shot was approximately 5.63 m.

d The shot hits the ground when $y(t) = 0$

$$\therefore \quad 9t - 4.9t^2 + 1.5 = 0$$

$$\therefore \quad t \approx 1.99 \qquad \{\text{technology, } t \geqslant 0\}$$

$\therefore$ the shot was in the air for approximately 1.99 seconds.

e $x(t) = \int 10\,dt$
$= 10t + c$

But $x(0) = 0$
$\therefore\ c = 0$
$\therefore\ x(t) = 10t$
$\therefore\ x(1.99) \approx 19.9$

The shot travelled approximately 19.9 m horizontally before hitting the ground.

REVIEW SET 24B

1 **a** $s(t) = t^2 + 4t + 1$ m, $t \geqslant 0$ s
$s(0) = 1$ m
$\therefore$ the object is initially 1 m to the right of the origin.

b average velocity $= \dfrac{s(3) - s(1)}{3 - 1}$
$= \dfrac{3^2 + 4(3) + 1 - (1^2 + 4(1) + 1)}{2}$
$= \dfrac{22 - 6}{2}$
$= 8\ \text{m s}^{-1}$

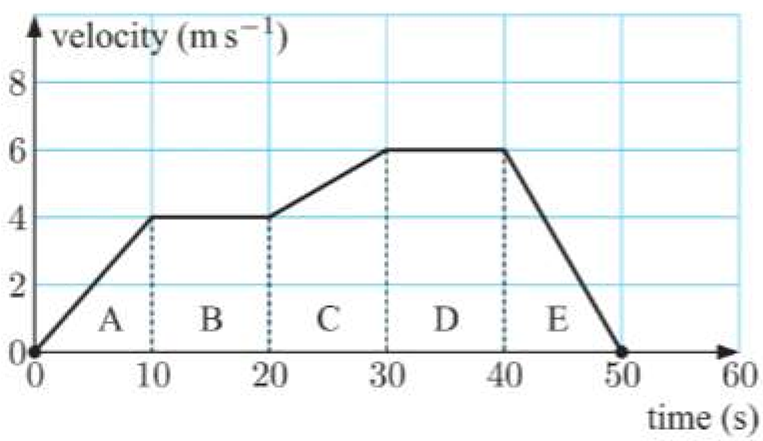
$\therefore$ the average velocity from $t = 1$ to $t = 3$ seconds is $8\ \text{m s}^{-1}$.

c $v(t) = s'(t) = 2t + 4\ \text{m s}^{-1}$

d $v(1) = 2(1) + 4$
$= 6\ \text{m s}^{-1}$

$\therefore$ the instantaneous velocity at $t = 1$ second is $6\ \text{m s}^{-1}$.

2



Total distance travelled = total area under graph
$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$
$= \frac{1}{2}(10)(4) + (10)(4) + \left(\frac{6+4}{2}\right)(10) + (10)(6) + \frac{1}{2}(10)(6)$
$= 20 + 40 + 50 + 60 + 30$
$= 200$ metres

3 **a** $v(t) = \int a(t)\,dt$
$= \int (6t - 30)\,dt$
$= 3t^2 - 30t + c$

Now $v(0) = 27$
$\therefore\ c = 27$
$\therefore\ v(t) = 3t^2 - 30t + 27\ \text{cm s}^{-1}$

b $s(t) = \int v(t)\,dt$
$= \int (3t^2 - 30t + 27)\,dt$
$= t^3 - 15t^2 + 27t + c$

Now $s(0) = 0$
$\therefore\ c = 0$
$\therefore\ s(t) = t^3 - 15t^2 + 27t\ \text{cm}$
$\therefore\ s(6) = 6^3 - 15(6)^2 + 27(6)$
$= 216 - 540 + 162$
$= -162$

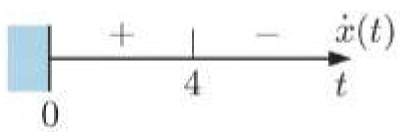
$\therefore$ the particle is 162 cm to the left of the origin after 6 seconds.

4 **a** $x(t) = 3t - t\sqrt{t}\ \text{cm},\ \ t \geqslant 0\ \text{s}$
$= 3t - t^{\frac{3}{2}}$

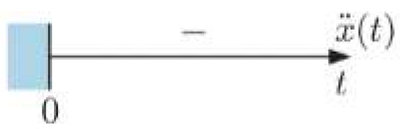
$\therefore\ \dot{x}(t) = 3 - \frac{3}{2}t^{\frac{1}{2}}\ \text{cm s}^{-1}$ $\quad \{\dot{x}(t) = x'(t)\}$
$= 3 - \frac{3}{2}\sqrt{t}\ \text{cm s}^{-1}$

$\therefore\ \ddot{x}(t) = -\frac{3}{4}t^{-\frac{1}{2}}\ \text{cm s}^{-2}$ $\quad \{\ddot{x}(t) = \dot{x}'(t)\}$
$= -\dfrac{3}{4\sqrt{t}}\ \text{cm s}^{-2}$

$\dot{x}(t)$ has sign diagram:



$\ddot{x}(t)$ has sign diagram:



b $x(0) = 0\ \text{cm}$ $\qquad \dot{x}(0) = 3\ \text{cm s}^{-1}$
The particle is initially at the origin, moving to the right at $3\ \text{cm s}^{-1}$.

c $x(2) = 3(2) - 2\sqrt{2}$
$= 6 - 2\sqrt{2}$
$\approx 3.17\ \text{cm}$

$\dot{x}(2) = 3 - \frac{3}{2}\sqrt{2}$
$\approx 0.879\ \text{cm s}^{-1}$

$\ddot{x}(2) = -\dfrac{3}{4\sqrt{2}}$
$\approx -0.530\ \text{cm s}^{-2}$

So, at time $t = 2$ seconds, the particle is about 3.17 cm to the right of the origin, travelling to the right at about $0.879\ \text{cm s}^{-1}$, with decreasing speed $(\ddot{x}(2) \approx -0.530\ \text{cm s}^{-2})$.

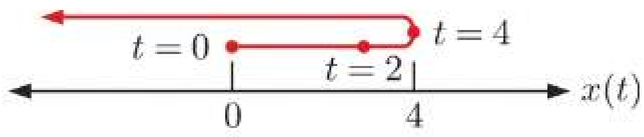
d The particle reverses direction when the sign of $\dot{x}(t)$ changes.
This occurs at $t = 4$ seconds.

$x(4) = 3(4) - 4\sqrt{4}$
$= 12 - 8$
$= 4\ \text{cm}$

$\therefore$ the particle reverses direction at $t = 4$ seconds, when it is 4 cm to the right of the origin.

e The particle's speed is decreasing when $\dot{x}(t)$ and $\ddot{x}(t)$ have opposite sign.
From the sign diagrams in **a**, this occurs when $0 \leqslant t \leqslant 4$ seconds.

f



g $x(6) = 3(6) - 6\sqrt{6}$
$= 18 - 6\sqrt{6}$
≈ 3.30 cm

Total distance travelled in first 6 seconds
$\approx 4 + (4 - 3.30)$
≈ 4.70 cm

5 **a** $v(t) = 4.8t^2 - 0.8t^3 \text{ m s}^{-1}, \quad 0 \leqslant t \leqslant 6$ s
$a(t) = 9.6t - 2.4t^2 \text{ m s}^{-2} \qquad \{a(t) = v'(t)\}$

i $a(1) = 9.6 - 2.4$
$= 7.2 \text{ m s}^{-2}$
$\therefore$ the acceleration of the human cannonball after 1 second is 7.2 m s^{-2}.

ii $a(2) = 9.6(2) - 2.4(2)^2$
$= 19.2 - 9.6$
$= 9.6 \text{ m s}^{-2}$
$\therefore$ the acceleration of the human cannonball after 2 seconds is 9.6 m s^{-2}.

iii $a(4) = 9.6(4) - 2.4(4)^2$
$= 38.4 - 38.4$
$= 0 \text{ m s}^{-2}$
$\therefore$ the acceleration of the human cannonball after 4 seconds is 0 m s^{-2}.

iv $a(5) = 9.6(5) - 2.4(5)^2$
$= 48 - 60$
$= -12 \text{ m s}^{-2}$
$\therefore$ the acceleration of the human cannonball after 5 seconds is -12 m s^{-2}.

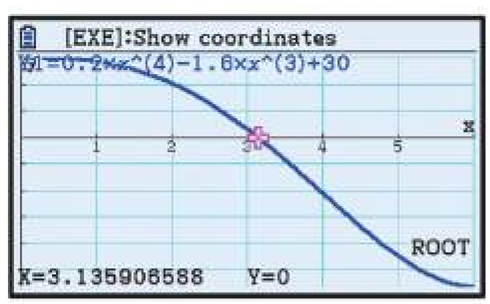
b $$\int_0^3 v(t)\,dt = \int_0^3 \left(4.8t^2 - 0.8t^3\right) dt$$
$$= \left[1.6t^3 - 0.2t^4\right]_0^3$$
$$= \left(1.6(3)^3 - 0.2(3)^4\right) - 0$$
$$= 27$$

The human cannonball travels 27 m in the first 3 seconds.

c Suppose the human cannonball has travelled 30 m after T seconds, then

$$\int_0^T v(t)\,dt = 30$$
$$\therefore \int_0^T \left(4.8t^2 - 0.8t^3\right) dt = 30$$
$$\therefore \left[1.6t^3 - 0.2t^4\right]_0^T = 30$$
$$\therefore 1.6T^3 - 0.2T^4 = 30$$
$$\therefore 0.2T^4 - 1.6T^3 + 30 = 0$$
$$\therefore T \approx 3.14 \quad \{0 \leqslant T \leqslant 6\}$$
{using technology}

$\therefore$ it takes about 3.14 seconds for the human cannonball to travel 30 m.



6 **a** $s(t) = 80e^{-\frac{t}{10}} - 40t$ m, $t \geqslant 0$ s

$\therefore\ v(t) = -8e^{-\frac{t}{10}} - 40\ \text{m s}^{-1}$ $\{v(t) = s'(t)\}$

$\therefore\ a(t) = \frac{8}{10}e^{-\frac{t}{10}} = \frac{4}{5}e^{-\frac{t}{10}}\ \text{m s}^{-2}$ $\{a(t) = v'(t)\}$

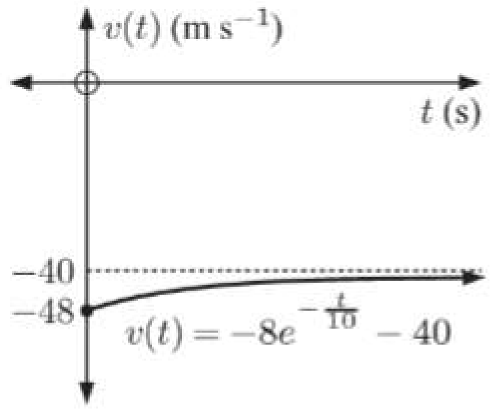
b When $t = 0$, $s(0) = 80$ m

$v(0) = -8 - 40 = -48\ \text{m s}^{-1}$

$a(0) = \frac{4}{5}\ \text{m s}^{-2}$

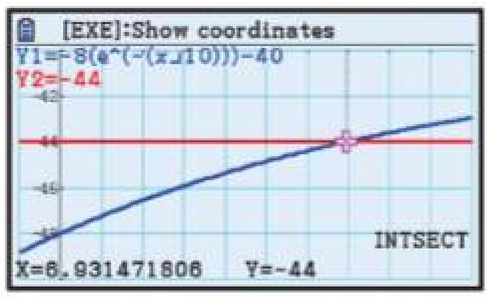
$\therefore$ the particle is initially 80 m to the right of the origin, moving to the left at 48 m s^{-1} with acceleration 0.8 m s^{-2}.

c



d When $v = -44$, $-8e^{-\frac{t}{10}} - 40 = -44$

$\therefore\ t \approx 6.93$ {technology}



$\therefore$ the particle P has velocity $-44\ \text{m s}^{-1}$ at $t \approx 6.93$ seconds.

7 **a** $s(t) = 30 + \cos \pi t$ cm, $t \geqslant 0$ s

$\therefore\ v(t) = -\pi \sin \pi t\ \ \text{cm s}^{-1}$ $\{v(t) = s'(t)\}$

$v(0) = -\pi \sin 0 = 0\ \text{cm s}^{-1}$,

$v(\frac{1}{2}) = -\pi \sin \frac{\pi}{2} = -\pi\ \text{cm s}^{-1}$,

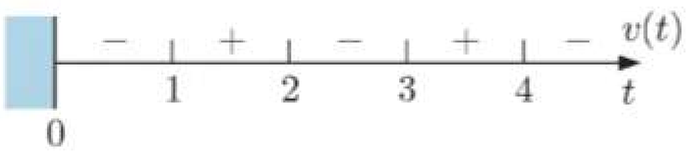
$v(1) = -\pi \sin \pi = 0\ \text{cm s}^{-1}$,

$v(1\frac{1}{2}) = -\pi \sin \frac{3\pi}{2} = \pi\ \text{cm s}^{-1}$

$v(2) = -\pi \sin 2\pi = 0\ \text{cm s}^{-1}$

b The cork is falling when its velocity is negative.

$v(t)$ has sign diagram:



$v(t)$ is negative when $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, and so on.

So, the cork is falling when $2n \leqslant t \leqslant 2n + 1$, $n \in \{0, 1, 2, 3, ...\}$

8 **a** $v(t) = \dfrac{(t^{1.1} + 3t)^{1.5}}{10}\ \text{m s}^{-1}$

$v(4) = \dfrac{(4^{1.1} + 3(4))^{1.5}}{10}$

$\approx 6.76\ \text{m s}^{-1}$

b $a(t) = v'(t)$

$= \frac{1}{10} \times 1.5(t^{1.1} + 3t)^{0.5}(1.1t^{0.1} + 3)$ {chain rule}

$= 0.15(t^{1.1} + 3t)^{0.5}(1.1t^{0.1} + 3)\ \ \text{m s}^{-2}$

c $a(2) = 0.15(2^{1.1} + 3(2))^{0.5}(1.1(2)^{0.1} + 3)$

$\approx 1.79\ \text{m s}^{-2}$

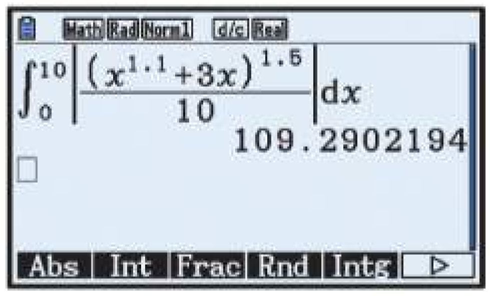
$\therefore$ the acceleration of the skier after 2 seconds is about $1.79\ \text{m s}^{-2}$.

d Total distance travelled in first 10 seconds

$= \displaystyle\int_0^{10} |\, v(t)\, |\ dt$

$= \displaystyle\int_0^{10} \left| \frac{(t^{1.1} + 3t)^{1.5}}{10} \right| dt$

≈ 109 m {using technology}



9 **a** $\dot{x}(t) = -\frac{1}{24}t^3 - \frac{1}{12}t\ \ \text{m s}^{-1}$

$x(t) = \displaystyle\int \dot{x}(t)\ dt$

$= \displaystyle\int \left(-\tfrac{1}{24}t^3 - \tfrac{1}{12}t\right) dt$

$= -\frac{1}{96}t^4 - \frac{1}{24}t^2 + c$ m

But $x(0) = 2$

$\therefore\ c = 2$

$\therefore\ x(t) = -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2$ m

b The feather is on the ground when

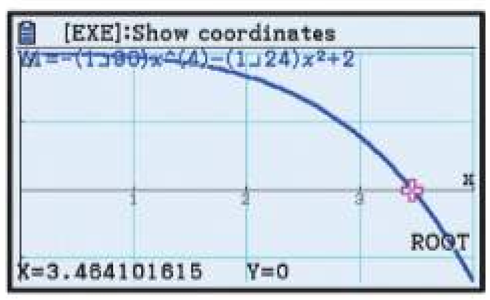
$x(t) = 0$

$\therefore\ -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2 = 0$

$\therefore\ t \approx 3.46$ or -3.46 {using technology}

$\therefore\ t \approx 3.46$ $\{t \geqslant 0\}$

$\therefore$ it takes about 3.46 seconds for the feather to reach the ground.



10 **a** After 2 seconds, $v_1(2) = 10(1 - e^{-1.25(2)})$ $\approx 9.18\ \text{m s}^{-1}$ $\quad v_2(2) = 10.5(1 - e^{-2})$ $\approx 9.08\ \text{m s}^{-1}$

$\therefore$ Tyson is running faster after 2 seconds.

b $\displaystyle\int_0^5 v_1(t)\,dt = \int_0^5 10(1-e^{-1.25t})\,dt$

$\displaystyle = \int_0^5 (10 - 10e^{-1.25t})\,dt$

$= \left[10t + 8e^{-1.25t}\right]_0^5$

$\approx (50 + 0.015) - (8)$

≈ 42.0

Tyson has travelled about 42.0 m in the first 5 seconds of the race.

c $\displaystyle s_1(t) = \int v_1(t)\,dt$

$\displaystyle = \int 10(1-e^{-1.25t})\,dt$

$\displaystyle = \int \left(10 - 10e^{-1.25t}\right)\,dt$

$= 10t + 8e^{-1.25t} + c$

Now $s_1(0) = 0$

$\therefore\ 0 + 8e^0 + c = 0$

$\therefore\ 8 + c = 0$

$\therefore\ c = -8$

$\therefore\ s_1(t) = 10t + 8e^{-1.25t} - 8$ m

$\displaystyle s_2(t) = \int v_2(t)\,dt$

$\displaystyle = \int 10.5(1-e^{-t})\,dt$

$\displaystyle = \int \left(10.5 - 10.5e^{-t}\right)\,dt$

$= 10.5t + 10.5e^{-t} + c$

Now $s_2(0) = 0$

$\therefore\ 0 + 10.5e^0 + c = 0$

$\therefore\ 10.5 + c = 0$

$\therefore\ c = -10.5$

$\therefore\ s_2(t) = 10.5t + 10.5e^{-t} - 10.5$ m

d After 3 seconds, $s_1(3) = 10(3) + 8e^{-1.25(3)} - 8 \approx 22.2$ m

$s_2(3) = 10.5(3) + 10.5e^{-3} - 10.5 \approx 21.5$ m

$\therefore$ Tyson is winning the race after 3 seconds.

e $s_1(10.8) = 10(10.8) + 8e^{-1.25(10.8)} - 8$

≈ 100 m

$\therefore$ Tyson completes the race in approximately 10.8 seconds.

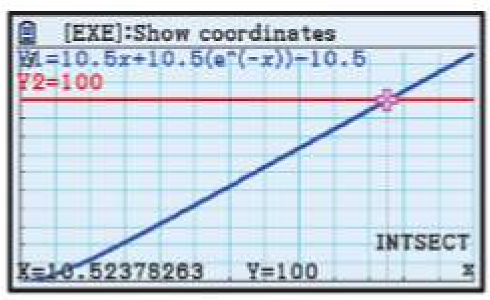
f When Maurice completes the race,

$s_2(t) = 100$

$\therefore\ 10.5t + 10.5e^{-t} - 10.5 = 100$

$\therefore\ t \approx 10.5$ {using technology}

$\therefore$ Maurice completes the race in approximately 10.5 seconds, so Maurice wins the race.



11 $\displaystyle v \propto \frac{1}{\sqrt{s}}$, so $\displaystyle v = \frac{k}{\sqrt{s}} = ks^{-\frac{1}{2}}$ for some constant k.

$\displaystyle\therefore\ a = v\,\frac{dv}{ds}$

$= ks^{-\frac{1}{2}} \times -\frac{1}{2}ks^{-\frac{3}{2}}$

$\displaystyle = -\frac{k^2}{2s^2}$, which is always negative.

12 **a** $\dfrac{d}{dt}\left(\dfrac{\sin\frac{t}{5}}{e^{\frac{t}{5}}}\right) = \dfrac{\frac{1}{5}\cos\frac{t}{5}\times e^{\frac{t}{5}} - \sin\frac{t}{5}\times\frac{1}{5}e^{\frac{t}{5}}}{(e^{\frac{t}{5}})^2}$ {quotient rule}

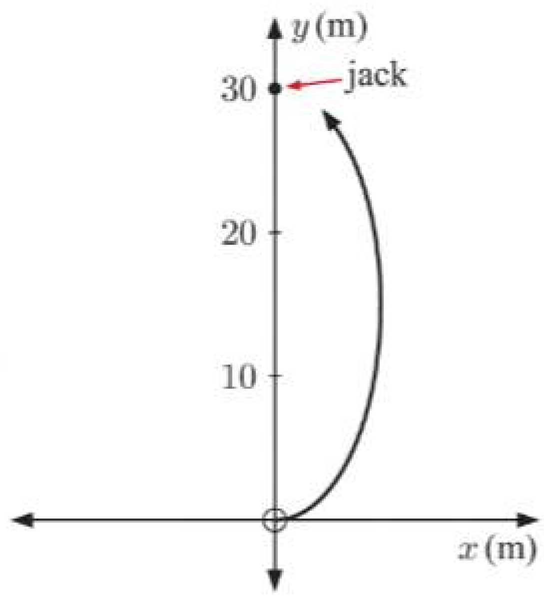
$$= \frac{\cos\frac{t}{5} - \sin\frac{t}{5}}{5e^{\frac{t}{5}}}$$

b **i** When $t = 0$, $v_x = \frac{6}{5}e^0(\cos 0 - \sin 0) = \frac{6}{5}$

$v_y = 5e^0 = 5$

$\therefore$ speed $= \sqrt{\left(\frac{6}{5}\right)^2 + 5^2}$

$\approx 5.14\ \text{m s}^{-1}$



ii $x(t) = \displaystyle\int \left(\tfrac{6}{5}e^{-\frac{t}{5}}\left(\cos\tfrac{t}{5} - \sin\tfrac{t}{5}\right)\right) dt$

$= 6\left(\dfrac{\sin\frac{t}{5}}{e^{\frac{t}{5}}}\right) + c$ {using **a**}

But $x(0) = 0$

$\therefore$ $c = 0$

$\therefore$ $x(t) = \dfrac{6\sin\frac{t}{5}}{e^{\frac{t}{5}}}$

$y(t) = \displaystyle\int 5e^{-\frac{t}{8}}\,dt$

$= -40e^{-\frac{t}{8}} + c$

But $y(0) = 0$

$\therefore$ $-40 + c = 0$

$\therefore$ $c = 40$

$\therefore$ $y(t) = -40e^{-\frac{t}{8}} + 40$

iii $x(10) = \dfrac{6\sin 2}{e^2} \approx 0.738$

$y(10) = -40e^{-\frac{5}{4}} + 40 \approx 28.540$

Distance between $(0.738, 28.540)$ and $(0, 30) \approx \sqrt{(0-0.738)^2 + (30-28.540)^2}$

≈ 1.64 m

iv When $t = 10$, $v_x = \frac{6}{5}e^{-2}(\cos 2 - \sin 2) \approx -0.2153$

$v_y = 5e^{-\frac{5}{4}} \approx 1.4325$

$\therefore$ speed $\approx \sqrt{(-0.2153)^2 + 1.4325^2}$

$\approx 1.45\ \text{m s}^{-1}$

Chapter 25

DIFFERENTIAL EQUATIONS

EXERCISE 25A

1 $\dfrac{dP}{dt}$ is proportional to P.

$\therefore \quad \dfrac{dP}{dt} = kP, \ k > 0 \quad$ {the population is increasing, so $\dfrac{dP}{dt} > 0$}

2 $\dfrac{dG}{dt}$ is proportional to $\sqrt{G}$.

$\therefore \quad \dfrac{dG}{dt} = k\sqrt{G}, \ k > 0 \quad$ {the algae is growing, so $\dfrac{dG}{dt} > 0$}

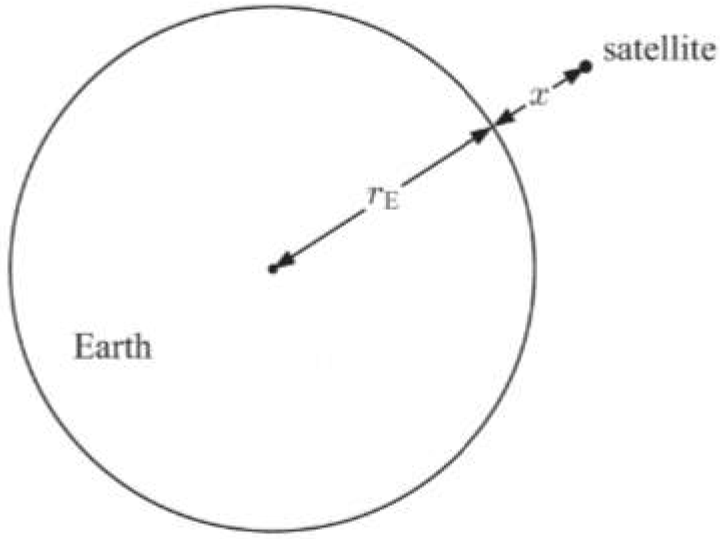
3 $\dfrac{dT}{dt}$ is proportional to $T - \tau \quad (T > \tau)$

$\therefore \quad \dfrac{dT}{dt} = -k(T - \tau), \ k > 0 \quad$ {the temperature is falling, so $\dfrac{dT}{dt} < 0$}

4 **a** **i** mg **ii** $-kv^2$

b The air resistance is proportional to the square of the velocity. The air resistance *reduces* the resultant force of the parachutist, so its coefficient is negative.

5 **a** The distance between the centre of the Earth and the satellite is $r = x + r_E$.



By Newton's law of universal gravitation, the force of attraction between the Earth and the satellite is $F = \dfrac{Gm_E m_S}{(x + r_E)^2}$ .... (1)

By Newton's second law of motion, the acceleration exerted on the satellite due to this force is $a = \dfrac{F}{m_S}$ .... (2)

Substituting (1) into (2) gives $\quad a = \dfrac{Gm_E \cancel{m_S}}{(x + r_E)^2 \cancel{m_S}}$

$$\text{or} \quad \ddot{x} = \frac{Gm_E}{(x + r_E)^2}$$

b For an object at low altitude, x is negligible compared to r_E, so we can ignore it.

So, the acceleration due to gravity $\quad g \approx \dfrac{Gm_E}{r_E^2}$

$$\approx \frac{6.7743 \times 10^{-11} \times 5.9722 \times 10^{24}}{(6.378 \times 10^6)^2}$$

$$\approx 9.95 \text{ m s}^{-2}$$

EXERCISE 25B

1 **a** If $y = x^4$, then

$$\frac{dy}{dx} = 4x^3 \quad \text{as required.}$$

b If $y = 5e^{2x}$, then

$$\begin{aligned}\frac{dy}{dx} &= 10e^{2x}\\ &= 2(5e^{2x})\\ &= 2y \quad \text{as required.}\end{aligned}$$

c If $y = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$, then

$$\begin{aligned}\frac{dy}{dx} &= \tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\\ &= \frac{x}{\sqrt{x^2+1}}\\ &= \frac{x}{y} \quad \text{as required.}\end{aligned}$$

d If $y = -\frac{1}{x} = -x^{-1}$, then

$$\begin{aligned}\frac{dy}{dx} &= x^{-2}\\ &= (-x^{-1})^2\\ &= y^2 \quad \text{as required.}\end{aligned}$$

e If $y = 3e^{\frac{x^2}{2}+x}$, then

$$\begin{aligned}\frac{dy}{dx} &= 3\left(\tfrac{1}{2}(2x)+1\right)e^{\frac{x^2}{2}+x}\\ &= (x+1)3e^{\frac{x^2}{2}+x}\\ &= (x+1)y\\ &= xy + y\end{aligned}$$

$$\therefore \quad \frac{dy}{dx} - y = xy \quad \text{as required.}$$

2 If $y = \sqrt[3]{x^2+1} = (x^2+1)^{\frac{1}{3}}$, then

$$\begin{aligned}\frac{dy}{dx} &= \tfrac{1}{3}(x^2+1)^{-\frac{2}{3}} \times 2x\\ &= \frac{2x}{3\left(\sqrt[3]{x^2+1}\right)^2}\\ &= \frac{2x}{3y^2}\end{aligned}$$

If $y = \sqrt{x+3} = (x+3)^{\frac{1}{2}}$, then

$$\begin{aligned}\frac{dy}{dx} &= \tfrac{1}{2}(x+3)^{-\frac{1}{2}}\\ &= \frac{1}{2\sqrt{x+3}}\\ &= \frac{1}{2y}\end{aligned}$$

If $y = \frac{1}{x^2} = x^{-2}$, then

$$\begin{aligned}\frac{dy}{dx} &= -2x^{-3}\\ &= -\frac{2}{x^3}\\ &= -\frac{2}{x} \times \frac{1}{x^2}\\ &= -\frac{2y}{x}\end{aligned}$$

$\therefore$ **a** **B** **b** **C** **c** **A**

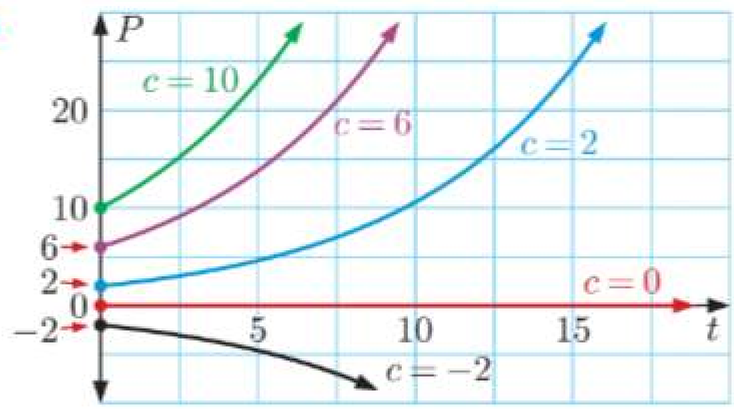
3 **a** If $y = x^3 + c$, then
$\frac{dy}{dx} = 3x^2$ as required.

b If $y = ce^{-x}$, then
$\frac{dy}{dx} = -ce^{-x}$
$= -y$ as required.

c If $y = -\frac{2}{x^2+c} = -2(x^2+c)^{-1}$, then
$\frac{dy}{dx} = 2(x^2+c)^{-2}(2x)$
$= \frac{4x}{(x^2+c)^2}$
$= x\left(-\frac{2}{x^2+c}\right)^2 = xy^2$ as required.

4 **a** If $P = ce^{\frac{t}{6}}$, then $\frac{dP}{dt} = \frac{c}{6}e^{\frac{t}{6}}$
$\therefore \frac{dP}{dt} = \frac{P}{6}$
$\therefore \frac{dP}{dt} - \frac{P}{6} = 0$ as required.

b



c c must be positive so that the population of guinea pigs is positive.

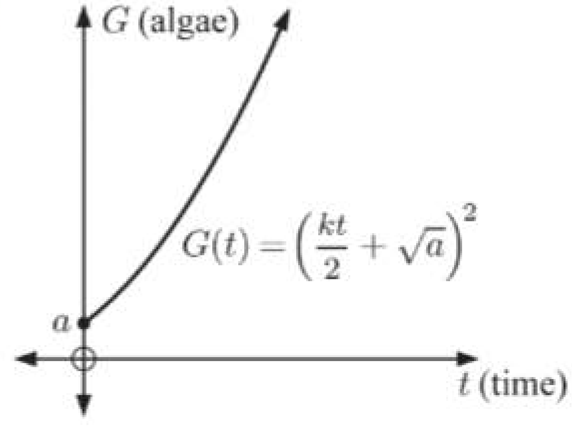
d When $t = 0$, $P = 30$, so $30 = ce^{\frac{0}{6}}$
$\therefore c = 30$
$\therefore$ the particular solution is $P = 30e^{\frac{t}{6}}$.

5 **a** If $G(t) = \left(\frac{kt}{2} + \sqrt{a}\right)^2$, $G'(t) = 2\left(\frac{kt}{2} + \sqrt{a}\right) \times \left(\frac{k}{2}\right)$ {a, k are constant}
$= k\left(\frac{kt}{2} + \sqrt{a}\right)$
$= k\sqrt{G}$ as required {$k, t, a \geqslant 0$}

b $G(0) = (\sqrt{a})^2$
$= a$
$\therefore$ a is the initial amount of algae.

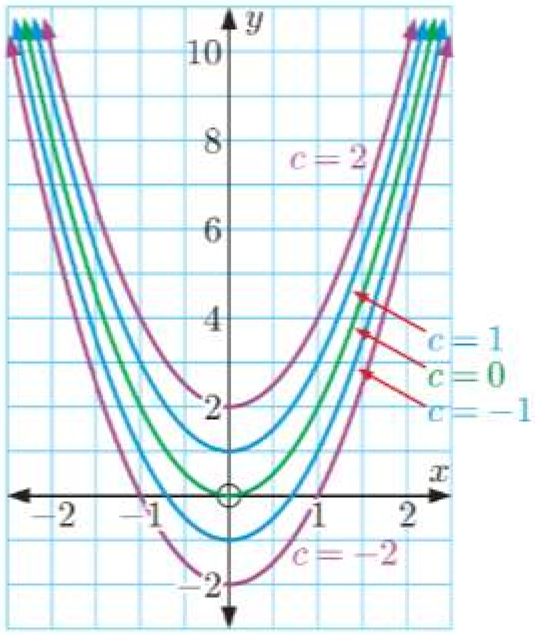
c $G(t) = \left(\frac{kt}{2}\right)^2 + 2\left(\frac{kt}{2}\right)\left(\sqrt{a}\right) + \left(\sqrt{a}\right)^2$
$= \frac{k^2}{4}t^2 + k\sqrt{a}t + a$
which is a quadratic function with vertex
at $t = -\frac{k\sqrt{a}}{2\left(\frac{k^2}{4}\right)}$
$= -\frac{2\sqrt{a}}{k}$
Since $a, k > 0$, the vertex of $G(t)$ occurs for some $t < 0$.

d



6 **a** If $y = 2x^2 + c$, then $\dfrac{dy}{dx} = 4x$ for any constant c as required.

b



c From **a**, $y = 2x^2 + c$ is a general solution to the differential equation.

The particular solution passes through $(1, \frac{1}{2})$, so

$$\tfrac{1}{2} = 2(1)^2 + c$$

$$\therefore \quad c = \tfrac{1}{2} - 2 = -\tfrac{3}{2}$$

$\therefore$ the particular solution is $y = 2x^2 - \frac{3}{2}$

d $\dfrac{dy}{dx} = 4x$

$\therefore$ at the point $(1, \frac{1}{2})$, $\dfrac{dy}{dx} = 4(1) = 4$

$\therefore$ the gradient of the tangent to the particular solution $y = 2x^2 - \frac{3}{2}$ at $(1, \frac{1}{2})$, is 4.

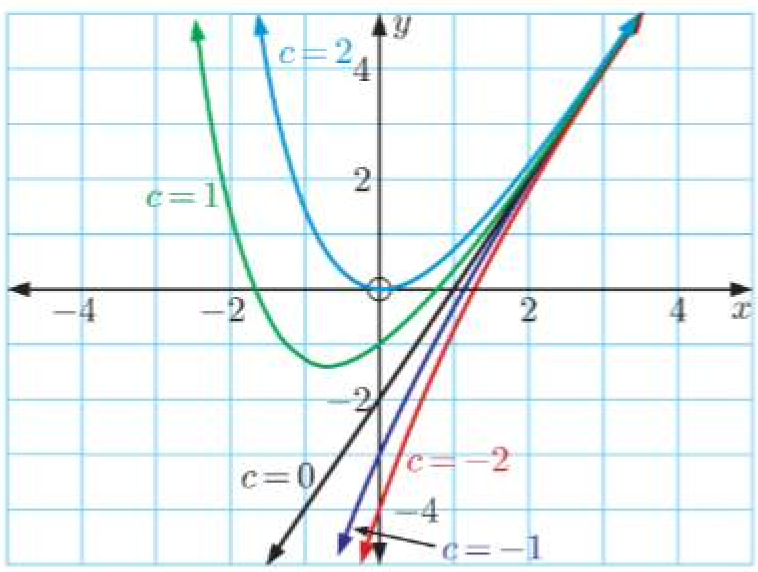
$\therefore$ the equation of the tangent is $y = 4(x-1) + \frac{1}{2}$

$$\therefore \quad y = 4x - \tfrac{7}{2}$$

7 **a** If $y = 2x - 2 + ce^{-x}$, then

$$\begin{aligned} \frac{dy}{dx} &= 2 - ce^{-x} \\ &= 2x - (2x - 2 + ce^{-x}) \\ &= 2x - y \quad \text{for any constant } c \text{ as required.} \end{aligned}$$

b



c From **a**, $y = 2x - 2 + ce^{-x}$ is a general solution to the differential equation.

The particular solution passes through $(0, 1)$, so $1 = -2 + c$

$$\therefore \quad c = 3$$

$\therefore$ the particular solution is $y = 2x - 2 + 3e^{-x}$

d $\dfrac{dy}{dx} = 2x - y$

$\therefore$ at the point $(0, 1)$, $\dfrac{dy}{dx} = 0 - 1 = -1$

$\therefore$ the gradient of the tangent to the particular solution $y = 2x - 2 + 3e^{-x}$ at $(0, 1)$, is -1.

$\therefore$ the equation of the tangent is $y = -(x - 0) + 1$

$$\therefore \quad y = -x + 1$$

8 **a** **i** If $x = \sin\omega t$, then $\dfrac{dx}{dt} = \omega\cos\omega t$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2\sin\omega t = -\omega^2 x$$

$$\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{as required.}$$

ii If $x = \sin(\omega t + b)$, then $\dfrac{dx}{dt} = \omega\cos(\omega t + b)$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2\sin(\omega t + b) = -\omega^2 x$$

$$\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{as required.}$$

iii If $x = \cos(\omega t + b)$, $\dfrac{dx}{dt} = -\omega\sin(\omega t + b)$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2\cos(\omega t + b) = -\omega^2 x$$

$$\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{as required.}$$

b If $x = e^{\lambda t}$, then $\dfrac{dx}{dt} = \lambda e^{\lambda t}$

$$\therefore \quad \frac{d^2x}{dt^2} = \lambda^2 e^{\lambda t}$$

$$\begin{aligned}\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x &= \lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t}\\ &= e^{\lambda t}(\lambda^2 + \omega^2)\end{aligned}$$

So, $x = e^{\lambda t}$ is a solution to $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ if $\lambda^2 + \omega^2 = 0$

$$\therefore \quad \lambda^2 = -\omega^2$$

$$\therefore \quad \lambda = \pm i\omega$$

In each case, λ is purely imaginary.

c Suppose $x_1(t)$ and $x_2(t)$ are solutions to the SHM equation.

If $x = Ax_1(t) + Bx_2(t)$ then $\dfrac{dx}{dt} = Ax_1'(t) + Bx_2'(t)$

$$\begin{aligned}\therefore \quad \frac{d^2x}{dt^2} &= Ax_1''(t) + Bx_2''(t)\\ &= A(-\omega^2 x_1(t)) + B(-\omega^2 x_2(t))\\ &\qquad \{x_1(t) \text{ and } x_2(t) \text{ are solutions}\}\\ &= -\omega^2(Ax_1(t) + Bx_2(t))\\ &= -\omega^2 x\end{aligned}$$

$$\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{as required.}$$

EXERCISE 25C

1 **a** $\dfrac{dy}{dx} = 4x^3$

$\therefore\ y = \displaystyle\int 4x^3\, dx$

$\therefore\ y = x^4 + c$

b $\dfrac{dy}{dx} = x^2 + 6x$

$\therefore\ y = \displaystyle\int (x^2 + 6x)\, dx$

$\therefore\ y = \frac{1}{3}x^3 + 3x^2 + c$

c $\dfrac{dy}{dx} = e^{3x} + 4$

$\therefore\ y = \displaystyle\int (e^{3x} + 4)\, dx$

$\therefore\ y = \frac{1}{3}e^{3x} + 4x + c$

d $\dfrac{dy}{dx} = \cos x + \sin 2x$

$\therefore\ y = \displaystyle\int (\cos x + \sin 2x)\, dx$

$\therefore\ y = \sin x - \frac{1}{2}\cos 2x + c$

e $\dfrac{dy}{dx} = \dfrac{1}{x+4}$

$\therefore\ y = \displaystyle\int \frac{1}{x+4}\, dx$

$\therefore\ y = \ln|x+4| + c$

f $\dfrac{dy}{dx} + \dfrac{2}{x} = \sqrt{x}$

$\therefore\ \dfrac{dy}{dx} = \sqrt{x} - \dfrac{2}{x}$

$\therefore\ y = \displaystyle\int \left(x^{\frac{1}{2}} - \frac{2}{x}\right) dx$

$\therefore\ y = \frac{2}{3}x^{\frac{3}{2}} - 2\ln|x| + c$

2 **a** $\dfrac{dM}{dt} = \dfrac{3t^2}{t^3 - 4}$

$\therefore\ M = \displaystyle\int \frac{3t^2}{t^3-4}\, dt$

$\therefore\ M = \ln\left|t^3 - 4\right| + c$

b $\dfrac{dy}{dt} = \dfrac{t}{\sqrt{25-t^2}}$

$\therefore\ y = \displaystyle\int \frac{t}{\sqrt{25-t^2}}\, dt$

$\therefore\ y = \displaystyle\int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,\frac{du}{dt}\right) dt$

$\{u = 25 - t^2,\quad \dfrac{du}{dt} = -2t\}$

$\therefore\ y = -\frac{1}{2}\displaystyle\int u^{-\frac{1}{2}}\, du$

$\therefore\ y = -\frac{1}{2}(2u^{\frac{1}{2}} + c)$

$\therefore\ y = -\sqrt{25 - t^2} + c$

c $f'(t) = te^{-t^2+1} + 2$

$\therefore\ f(t) = \displaystyle\int \left(te^{-t^2+1} + 2\right) dt$

$\therefore\ f(t) = \displaystyle\int te^{-t^2+1}\, dt + \int 2\, dt$

$\therefore\ f(t) = \displaystyle\int e^u\left(-\frac{1}{2}\,\frac{du}{dt}\right) dt + \int 2\, dt$

$\{u = -t^2 + 1,\quad \dfrac{du}{dt} = -2t\}$

$\therefore\ f(t) = -\frac{1}{2}\displaystyle\int e^u\, du + \int 2\, dt$

$\therefore\ f(t) = -\frac{1}{2}e^u + 2t + c$

$\therefore\ f(t) = -\frac{1}{2}e^{-t^2+1} + 2t + c$

3 **a** $\dfrac{dy}{dx} = 3x - 2$

$\therefore\ y = \displaystyle\int (3x - 2)\, dx$

$\therefore\ y = \frac{3}{2}x^2 - 2x + c$

Now $y(0) = 5$

$\therefore\ 5 = 0 - 0 + c$

$\therefore\ c = 5$

So, the solution is $y = \frac{3}{2}x^2 - 2x + 5$.

b $\dfrac{dy}{dx} = e^{3x} + 1$

$\therefore\ y = \displaystyle\int (e^{3x} + 1)\, dx$

$\therefore\ y = \frac{1}{3}e^{3x} + x + c$

Now $y(0) = 0$

$\therefore\ 0 = \frac{1}{3}e^0 + 0 + c$

$\therefore\ c = -\frac{1}{3}$

So, the solution is $y = \frac{1}{3}e^{3x} + x - \frac{1}{3}$.

c $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore\ y = \displaystyle\int \frac{1}{x}\, dx$

$\therefore\ y = \ln|\,x\,| + c$

Now $y(2) = \ln 12$

$\therefore\ \ln 12 = \ln|\,2\,| + c$

$\therefore\ c = \ln 12 - \ln 2 = \ln 6$

So, the solution is $y = \ln|\,x\,| + \ln 6$.

4 **a** $\dfrac{dy}{dt} = 2e^{2t} - e^{-t}$

$\therefore\ y = \displaystyle\int (2e^{2t} - e^{-t})\, dt$

$\therefore\ y = e^{2t} + e^{-t} + c$

Now $y(\ln 2) = 2.5$

$\therefore\ 2.5 = e^{2\ln 2} + e^{-\ln 2} + c$

$\therefore\ 2.5 = 4 + 0.5 + c$

$\therefore\ c = -2$

So, the solution is $y = e^{2t} + e^{-t} - 2$.

b $\dfrac{dM}{d\alpha} = \cos 2\alpha - 3\sin\alpha$

$\therefore\ M = \displaystyle\int (\cos 2\alpha - 3\sin\alpha)\, d\alpha$

$\therefore\ M = \frac{1}{2}\sin 2\alpha + 3\cos\alpha + c$

Now $M\left(\frac{\pi}{2}\right) = 2$

$\therefore\ 2 = \frac{1}{2}\sin\pi + 3\cos\frac{\pi}{2} + c$

$\therefore\ c = 2$

So, the solution is

$M = \frac{1}{2}\sin 2\alpha + 3\cos\alpha + 2$.

5 $f'(x) = 2x - 5$

$\therefore\ f(x) = \displaystyle\int (2x - 5)\, dx$

$= x^2 - 5x + c$

Now $f(2) = -18$

$\therefore\ -18 = 2^2 - 5(2) + c$

$\therefore\ c = -12$

$\therefore\ f(x) = x^2 - 5x - 12$

$\therefore\ f(-2) = (-2)^2 - 5(-2) - 12$

$= 4 + 10 - 12$

$= 2$

6 **a** $\dfrac{dy}{dx} = e^x - e^{-x}$

$\therefore \; y = \displaystyle\int (e^x - e^{-x})\, dx$

$\therefore \; y = e^x + e^{-x} + c$

Now $y(0) = 1$

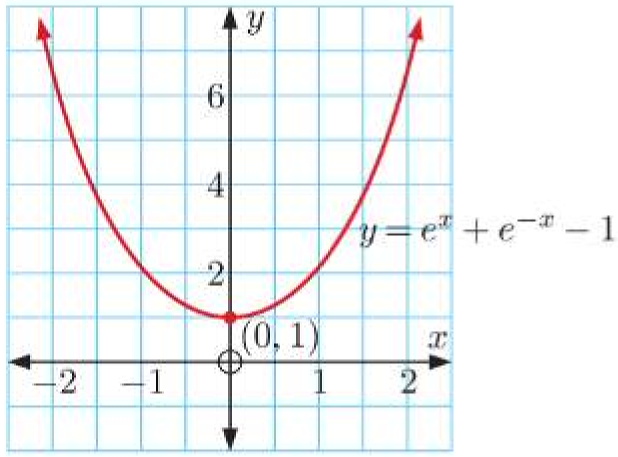
$\therefore \; 1 = e^0 + e^0 + c$

$\therefore \; 1 = 2 + c$

$\therefore \; c = -1$

So, the solution is $y = e^x + e^{-x} - 1$.

b



c When $x = \ln 2$, $\dfrac{dy}{dx} = e^{\ln 2} - e^{-\ln 2}$ and $y = e^{\ln 2} + e^{-\ln 2} - 1$

$= 2 - \frac{1}{2}$ $\qquad = 2 + \frac{1}{2} - 1$

$= \frac{3}{2}$ $\qquad = \frac{3}{2}$

$\therefore$ the equation of the tangent is $y = \frac{3}{2}(x - \ln 2) + \frac{3}{2}$

$\therefore \; 2y = 3x - 3\ln 2 + 3$

$\therefore \; 3x - 2y = 3\ln 2 - 3$

7 $f'(x) = ax + bx^{-2}$

$\therefore \; f(x) = \displaystyle\int (ax + bx^{-2})\, dx$

$= \dfrac{a}{2}x^2 - bx^{-1} + c$

Now $f(-1) = -2$ and $f(1) = 0$

$\therefore \; \dfrac{a}{2}(-1)^2 - b(-1)^{-1} + c = -2$

$\therefore \; \dfrac{a}{2} + b + c = -2 \quad .... \, (1)$

and $\dfrac{a}{2}(1)^2 - b(1)^{-1} + c = 0$

$\therefore \; \dfrac{a}{2} - b + c = 0 \quad .... \, (2)$

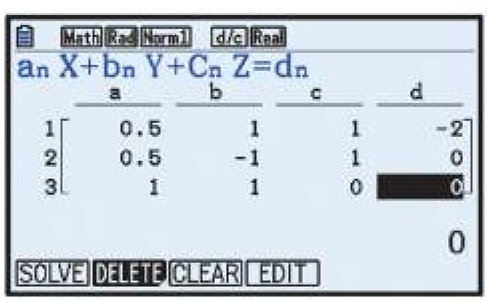
Also, $f'(1) = 0$

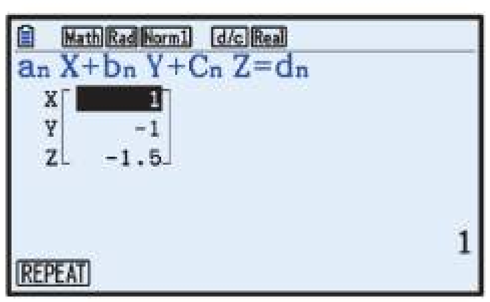
$\therefore \; a(1) + b(1)^{-2} = 0$

$\therefore \; a + b = 0 \quad .... \, (3)$

Solving (1), (2), and (3) simultaneously we find that $a = 1$, $b = -1$, and $c = -\frac{3}{2}$.

$\therefore \; f(x) = \frac{1}{2}x^2 + \dfrac{1}{x} - \frac{3}{2}$





8 **a** $f''(x) = 6x - 4$

$\therefore\ f'(x) = \int (6x - 4)\, dx = 3x^2 - 4x + c$

$\therefore\ f(x) = \int (3x^2 - 4x + c)\, dx = x^3 - 2x^2 + cx + d$

Now, $f'(1) = 3$ and $f(2) = 7$

$\therefore\ 3 = 3(1)^2 - 4(1) + c$ $\therefore\ 7 = 2^3 - 2(2)^2 + 4(2) + d$

$\therefore\ c = 4$ $\therefore\ d = -1$

So, the solution is $f(x) = x^3 - 2x^2 + 4x - 1$.

b $\dfrac{d^2y}{dx^2} = \sin 2x$

$\therefore\ \dfrac{dy}{dx} = \int \sin 2x\, dx = -\frac{1}{2}\cos 2x + c$

$\therefore\ y = \int \left(-\frac{1}{2}\cos 2x + c\right) dx = -\frac{1}{4}\sin 2x + cx + d$

Now, $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 2\pi$

$\therefore\ 0 = -\frac{1}{4}\sin 0 + c(0) + d$ $\therefore\ 2\pi = -\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right) + c\left(\frac{\pi}{2}\right)$

$\therefore\ d = 0$ $\therefore\ c = 4$

So, the solution is $y = -\frac{1}{4}\sin 2x + 4x$.

9 The marginal cost is $C'(x) = 3.15 + 0.004x$ pounds per gadget

$$\begin{aligned}
\therefore\ C(x) &= \int (3.15 + 0.004x)\, dx \\
&= 3.15x + 0.002x^2 + c \ \text{ pounds}
\end{aligned}$$

But $C(0) = 450$ pounds, so $c = 450$

$$\begin{aligned}
\therefore\ C(x) &= 3.15x + 0.002x^2 + 450 \ \text{ pounds} \\
\therefore\ C(800) &= 3.15(800) + 0.002(800)^2 + 450 \ \text{ pounds} \\
&= 4250 \text{ pounds}
\end{aligned}$$

$\therefore$ the total cost is £4250.

10 **a** The marginal profit is $P'(x) = 15 - 0.03x$ pounds per plate

$$\begin{aligned}
\therefore\ P(x) &= \int (15 - 0.03x)\, dx \\
&= 15x - 0.015x^2 + c \ \text{ pounds}
\end{aligned}$$

But $P(0) = -650$ pounds, so $c = -650$

$\therefore\ P(x) = 15x - 0.015x^2 - 650$ pounds

b $P''(x) = -0.03 < 0$

$\therefore$ the maximum profit occurs when $P'(x) = 0$

$$\therefore\ 0 = 15 - 0.03x$$
$$\therefore\ 0.03x = 15$$
$$\therefore\ x = 500$$

Now $P(500) = 15(500) - 0.015(500)^2 - 650 = 3100$

$\therefore$ the maximum profit per week is £3100 when 500 plates are made.

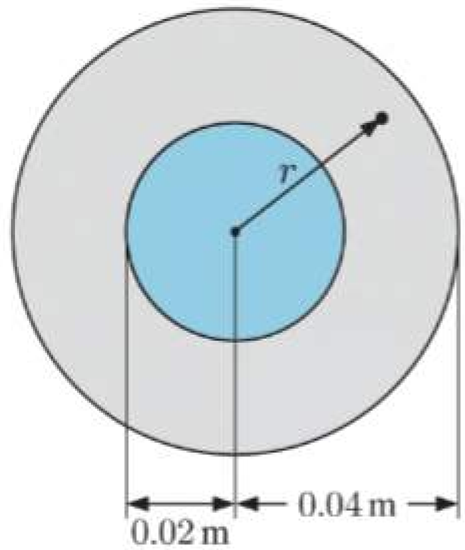
c In order for a profit to be made, $P(x)$ must be greater than 0

$\therefore$ $15x - 0.015x^2 - 650 > 0$

Using technology, the x-intercepts of $P(x)$ are $x_1 \approx 45.39$ and $x_2 \approx 954.6$.

Since we cannot produce part of a plate, a profit is made when between 46 and 954 plates inclusive are made per week.

11 Since $\dfrac{dT}{dr} = -\dfrac{q}{2\pi kr}$, $T = -\dfrac{q}{2\pi k}\displaystyle\int \frac{1}{r}\,dr$ $\{q, k \text{ constant}\}$



$$\therefore\ T = -\frac{680}{2\pi(0.2)}\ln r + c \quad \{r > 0\}$$
$$\therefore\ T = -\tfrac{1700}{\pi}\ln r + c$$

But when $r = 0.02$, $T = 600$

$$\therefore\ 600 = -\tfrac{1700}{\pi}\ln(0.02) + c$$
$$\therefore\ 600 = \tfrac{1700}{\pi}\ln 50 + c$$
$$\therefore\ c = 600 - \tfrac{1700}{\pi}\ln 50$$

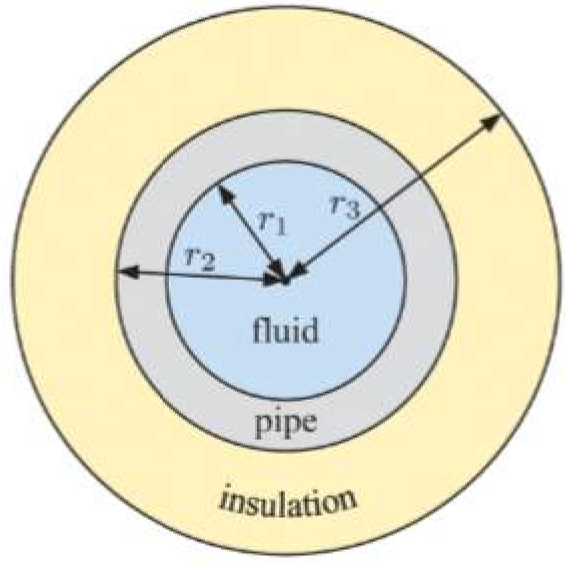
Thus $T = -\tfrac{1700}{\pi}\ln r + 600 - \tfrac{1700}{\pi}\ln 50$

$\therefore$ $T = 600 - \tfrac{1700}{\pi}\ln 50r$

When $r = 0.04$, $T = 600 - \tfrac{1700}{\pi}\ln 2 \approx 225$

The external temperature of the tube is about 225°C.

12 Since $\dfrac{dT}{dr} = -\dfrac{q}{2\pi kr}$, $T(r) = -\dfrac{q}{2\pi k}\displaystyle\int \frac{1}{r}\,dr$ $\{q, k \text{ constant}\}$



$$\therefore\ T(r) = -\frac{60}{2\pi k}\ln r + c \quad \{r > 0\}$$
$$\therefore\ T(r) = -\frac{30}{\pi k}\ln r + c$$

a For the pipe, $k = 19$.

$T(r_1) = T(0.14) = 400$

$$\therefore\ 400 = -\tfrac{30}{19\pi}\ln(0.14) + c$$
$$\therefore\ c = 400 + \tfrac{30}{19\pi}\ln(0.14)$$

Thus for the pipe, $T(r) = -\tfrac{30}{19\pi}\ln r + 400 + \tfrac{30}{19\pi}\ln(0.14)$

$$= 400 + \tfrac{30}{19\pi}\ln\left(\frac{0.14}{r}\right)$$

$T(r_2) = T(0.20) = 400 + \tfrac{30}{19\pi}\ln(0.7) \approx 400$

The temperature on the outer surface of the pipe is about 400°C.

b For the insulation, $k = 0.018$.

$$T(r_2) = T(0.20) = 400 + \tfrac{30}{19\pi}\ln(0.7) \qquad \{\text{from } \mathbf{a}\}$$

$$\therefore\ 400 + \tfrac{30}{19\pi}\ln(0.7) = -\frac{30}{\pi(0.018)}\ln(0.20) + c$$

$$\therefore\ c = 400 + \tfrac{30}{19\pi}\ln(0.7) + \tfrac{5000}{3\pi}\ln(0.20)$$

Thus for the insulation, $T(r) = -\tfrac{5000}{3\pi}\ln r + 400 + \tfrac{30}{19\pi}\ln(0.7) + \tfrac{5000}{3\pi}\ln(0.20)$

$$\therefore\ \tfrac{5000}{3\pi}\ln r = 400 + \tfrac{30}{19\pi}\ln(0.7) + \tfrac{5000}{3\pi}\ln(0.20) - T$$

$$\therefore\ \ln r = \tfrac{6\pi}{25} + \tfrac{9}{9500}\ln(0.7) + \ln(0.20) - \tfrac{3\pi}{5000}T$$

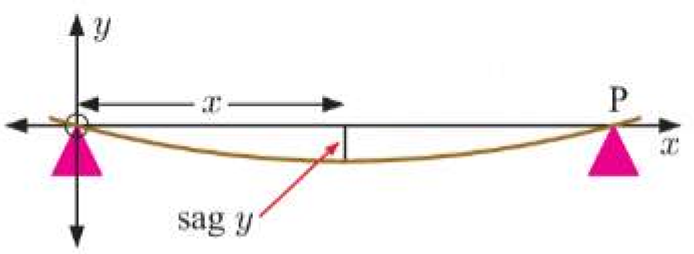
$$T(r_3) = 50 \quad \therefore\ \ln r_3 = \tfrac{6\pi}{25} + \tfrac{9}{9500}\ln(0.7) + \tfrac{3\pi}{5000}\ln(0.20) - \tfrac{3\pi}{100}$$

$$\therefore\ r_3 \approx 0.387$$

The outer radius of the insulation is $r_3 \approx 0.387$ m.

c Thickness of insulation needed $= r_3 - r_2 \approx 0.387 - 0.20 \approx 0.187$ m.

13



a Since $\dfrac{d^2y}{dx} = 0.01\left(2x - \dfrac{x^2}{2}\right)$, $\dfrac{dy}{dx} = 0.01\displaystyle\int\left(2x - \frac{x^2}{2}\right)dx$

$$= \tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 + c$$

and so, $y(x) = \displaystyle\int\left(\tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 + c\right)dx$

$$= \tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 + cx + d$$

But $y(0) = 0$

$\therefore\ 0 = 0 - 0 + 0 + d$

$\therefore\ d = 0$

and $y(4) = 0$

$\therefore\ 0 = \tfrac{1}{300}(4)^3 - \tfrac{1}{2400}(4)^4 + 4c$

$\therefore\ 4c = \tfrac{1}{2400}(4)^4 - \tfrac{1}{300}(4)^3$

$\therefore\ c = \tfrac{1}{2400}(4)^3 - \tfrac{1}{300}(4)^2$

$\therefore\ c = -\tfrac{2}{75}$

So, the sag $y(x) = \left(\tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 - \tfrac{2}{75}x\right)$ metres.

b $\dfrac{dy}{dx} = \tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 - \tfrac{2}{75}$ {from **a**}

The maximum sag occurs when $\dfrac{dy}{dx} = 0$

$$\therefore\ \tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 - \tfrac{2}{75} = 0$$

$$\therefore\ 6x^2 - x^3 - 16 = 0$$

Using technology, the three solutions are $x \approx -1.464$, 2, and ≈ 5.464.

But the maximum lies between 0 and 4, so it must occur when $x = 2$.

When $x = 2$, $y = \tfrac{1}{300}(2)^3 - \tfrac{1}{2400}(2)^4 - \tfrac{2}{75}(2) \approx -0.0333$

$\therefore$ the maximum sag from the horizontal is about 0.0333 m which is about 3.33 cm.

Yes, it seems reasonable that the maximum sag occurs when $x = 2$, as it is the middle point of the plank.

c At the point 1 m from P, $x = 3$.

$y(3) = \frac{1}{300}(3)^3 - \frac{1}{2400}(3)^4 - \frac{2}{75}(3) = -0.023\,75$

$\therefore$ the sag 1 m away from P is 0.023 75 m which is 2.375 cm.

d When $x = 3$, $\frac{dy}{dx} = \frac{1}{100}(3)^2 - \frac{1}{600}(3)^3 - \frac{2}{75} = \frac{11}{600}$

$\therefore$ the angle θ that the plank makes with the horizontal satisfies $\tan\theta = \frac{11}{600}$

$$\therefore\ \theta = \tan^{-1}\left(\tfrac{11}{600}\right) \approx 1.05^\circ$$

EXERCISE 25D

1 a

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y^2} \\ \therefore\ y^2\frac{dy}{dx} &= x \\ \therefore\ \int y^2\frac{dy}{dx}\,dx &= \int x\,dx \\ \therefore\ \int y^2\,dy &= \int x\,dx \\ \therefore\ \tfrac{1}{3}y^3 &= \tfrac{1}{2}x^2 + c \\ \therefore\ y^3 &= \tfrac{3}{2}x^2 + c \\ \therefore\ y &= \sqrt[3]{\tfrac{3}{2}x^2 + c} \end{aligned}$$

b

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{e^y} \\ \therefore\ e^y\frac{dy}{dx} &= 2x \\ \therefore\ \int e^y\frac{dy}{dx}\,dx &= \int 2x\,dx \\ \therefore\ \int e^y\,dy &= \int 2x\,dx \\ \therefore\ e^y &= x^2 + c \\ \therefore\ y &= \ln(x^2 + c) \end{aligned}$$

c

$$\begin{aligned} \frac{dy}{dx} &= 3xy \\ \therefore\ \frac{1}{y}\frac{dy}{dx} &= 3x \\ \therefore\ \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int 3x\,dx \\ \therefore\ \int \frac{1}{y}\,dy &= \int 3x\,dx \\ \therefore\ \ln|y| &= \tfrac{3}{2}x^2 + c \\ \therefore\ y &= \pm e^{\frac{3}{2}x^2 + c} \\ \therefore\ y &= Ae^{\frac{3}{2}x^2} \\ &\{A = \pm e^c\} \end{aligned}$$

d

$$\begin{aligned} \frac{dy}{dx} &= 2x\sqrt{y} \\ \therefore\ \frac{1}{\sqrt{y}}\frac{dy}{dx} &= 2x \\ \therefore\ \int \frac{1}{\sqrt{y}}\frac{dy}{dx}\,dx &= \int 2x\,dx \\ \therefore\ \int y^{-\frac{1}{2}}\,dy &= \int 2x\,dx \\ \therefore\ 2y^{\frac{1}{2}} &= x^2 + c \\ \therefore\ \sqrt{y} &= \tfrac{1}{2}x^2 + c \\ \therefore\ y &= \left(\frac{x^2}{2} + c\right)^2 \end{aligned}$$

e

$$\begin{aligned} \frac{dy}{dx} &= y\sin x \\ \therefore\ \frac{1}{y}\frac{dy}{dx} &= \sin x \\ \therefore\ \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int \sin x\,dx \\ \therefore\ \int \frac{1}{y}\,dy &= \int \sin x\,dx \\ \therefore\ \ln|y| &= -\cos x + c \\ \therefore\ y &= \pm e^{-\cos x + c} \\ \therefore\ y &= Ae^{-\cos x} \quad \{A = \pm e^c\} \end{aligned}$$

f

$$\frac{dy}{dx} = -x\sqrt{y+1}$$
$$\therefore \frac{1}{\sqrt{y+1}}\frac{dy}{dx} = -x$$
$$\therefore \int \frac{1}{\sqrt{y+1}}\frac{dy}{dx}\,dx = \int -x\,dx$$
$$\therefore \int (y+1)^{-\frac{1}{2}}\,dy = \int -x\,dx$$
$$\therefore 2(y+1)^{\frac{1}{2}} = -\tfrac{1}{2}x^2 + c$$
$$\therefore \sqrt{y+1} = -\tfrac{1}{4}x^2 + c$$
$$\therefore y+1 = \left(-\tfrac{1}{4}x^2 + c\right)^2$$
$$\therefore y = \left(-\tfrac{1}{4}x^2 + c\right)^2 - 1$$

g

$$\frac{dy}{dx} = \frac{y}{x}$$
$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$$
$$\therefore \int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$$
$$\therefore \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$
$$\therefore \ln|y| = \ln|x| + c$$
$$\therefore \ln|y| - \ln|x| = c$$
$$\therefore \ln\left|\frac{y}{x}\right| = c$$
$$\therefore \frac{y}{x} = \pm e^c$$
$$\therefore y = Ax \qquad \{A = \pm e^c\}$$

h

$$\frac{dy}{dx} = 3x^2e^y$$
$$\therefore e^{-y}\frac{dy}{dx} = 3x^2$$
$$\therefore \int e^{-y}\frac{dy}{dx}\,dx = \int 3x^2\,dx$$
$$\therefore \int e^{-y}\,dy = \int 3x^2\,dx$$
$$\therefore -e^{-y} = x^3 + c$$
$$\therefore e^{-y} = -x^3 + c$$
$$\therefore -y = \ln(c - x^3)$$
$$\therefore y = -\ln(c - x^3)$$

i

$$\frac{dy}{dx} = \frac{y+2}{x-1}$$
$$\therefore \frac{1}{y+2}\frac{dy}{dx} = \frac{1}{x-1}$$
$$\therefore \int \frac{1}{y+2}\frac{dy}{dx}\,dx = \int \frac{1}{x-1}\,dx$$
$$\therefore \int \frac{1}{y+2}\,dy = \int \frac{1}{x-1}\,dx$$
$$\therefore \ln|y+2| = \ln|x-1| + c$$
$$\therefore \ln|y+2| - \ln|x-1| = c$$
$$\therefore \ln\left|\frac{y+2}{x-1}\right| = c$$
$$\therefore \frac{y+2}{x-1} = \pm e^c$$
$$\therefore y+2 = A(x-1) \qquad \{A = \pm e^c\}$$
$$\therefore y = A(x-1) - 2$$

2 **a**

$$\frac{dy}{dx} = y$$
$$\therefore \frac{1}{y}\frac{dy}{dx} = 1$$
$$\therefore \int \frac{1}{y}\frac{dy}{dx}\,dx = \int 1\,dx$$
$$\therefore \int \frac{1}{y}\,dy = \int 1\,dx$$
$$\therefore \ln|y| = x + c$$
$$\therefore y = \pm e^{x+c}$$
$$\therefore y = Ae^x \qquad \{A = \pm e^c\}$$

b

$$\frac{dy}{dx} = \frac{1}{y}$$
$$\therefore y\frac{dy}{dx} = 1$$
$$\therefore \int y\frac{dy}{dx}\,dx = \int 1\,dx$$
$$\therefore \int y\,dy = \int 1\,dx$$
$$\therefore \tfrac{1}{2}y^2 = x + c$$
$$\therefore y^2 = 2x + c$$
$$\therefore y = \pm\sqrt{2x+c}$$

c
$$\frac{dy}{dt} = y - 4$$
$$\therefore \quad \frac{1}{y-4}\,\frac{dy}{dt} = 1$$
$$\therefore \quad \int \frac{1}{y-4}\,\frac{dy}{dt}\,dt = \int 1\,dt$$
$$\therefore \quad \int \frac{1}{y-4}\,dy = \int 1\,dt$$
$$\therefore \quad \ln|\,y-4\,| = t + c$$
$$\therefore \quad y - 4 = \pm e^{t+c}$$
$$\therefore \quad y = Ae^t + 4 \quad \{A = \pm e^c\}$$

d
$$\frac{dP}{dt} = 3\sqrt{P}$$
$$\therefore \quad \frac{1}{\sqrt{P}}\,\frac{dP}{dt} = 3$$
$$\therefore \quad \int \frac{1}{\sqrt{P}}\,\frac{dP}{dt}\,dt = \int 3\,dt$$
$$\therefore \quad \int P^{-\frac{1}{2}}\,dP = \int 3\,dt$$
$$\therefore \quad 2P^{\frac{1}{2}} = 3t + c$$
$$\therefore \quad \sqrt{P} = \tfrac{3}{2}t + c$$
$$\therefore \quad P = \left(\tfrac{3}{2}t + c\right)^2$$

e
$$\frac{dQ}{dt} = 2Q + 3$$
$$\therefore \quad \frac{1}{2Q+3}\,\frac{dQ}{dt} = 1$$
$$\therefore \quad \int \frac{1}{2Q+3}\,\frac{dQ}{dt}\,dt = \int 1\,dt$$
$$\therefore \quad \int \frac{1}{2Q+3}\,dQ = \int 1\,dt$$
$$\therefore \quad \tfrac{1}{2}\ln|\,2Q+3\,| = t + c$$
$$\therefore \quad \ln|\,2Q+3\,| = 2t + c$$
$$\therefore \quad 2Q + 3 = \pm e^{2t+c}$$
$$\therefore \quad 2Q = Ae^{2t} - 3 \quad \{A = \pm e^c\}$$
$$\therefore \quad Q = Ae^{2t} - \tfrac{3}{2}$$

f
$$\frac{dQ}{dt} = \frac{1}{2Q+3}$$
$$\therefore \quad (2Q+3)\,\frac{dQ}{dt} = 1$$
$$\therefore \quad \int (2Q+3)\,\frac{dQ}{dt}\,dt = \int 1\,dt$$
$$\therefore \quad \int (2Q+3)\,dQ = \int 1\,dt$$
$$\therefore \quad Q^2 + 3Q = t + c$$
$$\therefore \quad Q^2 + 3Q + \left(\tfrac{3}{2}\right)^2 = t + c$$
$$\therefore \quad \left(Q + \tfrac{3}{2}\right)^2 = t + c$$
$$\therefore \quad Q + \tfrac{3}{2} = \pm\sqrt{t+c}$$
$$\therefore \quad Q = -\tfrac{3}{2} \pm \sqrt{t+c}$$

3
$$\frac{dy}{dx} = y^2$$
$$\therefore \quad \frac{1}{y^2}\,\frac{dy}{dx} = 1$$
$$\therefore \quad \int \frac{1}{y^2}\,\frac{dy}{dx}\,dx = \int 1\,dx$$
$$\therefore \quad \int y^{-2}\,dy = \int 1\,dx$$
$$\therefore \quad -y^{-1} = x + c$$
$$\therefore \quad y = \frac{1}{c-x} \quad \text{which is defined for } x \neq c.$$

4 **a**

$$\frac{dy}{dx} = \frac{3x}{y^2}$$
$$\therefore\ y^2\,\frac{dy}{dx} = 3x$$
$$\therefore\ \int y^2\,\frac{dy}{dx}\,dx = \int 3x\,dx$$
$$\therefore\ \int y^2\,dy = \int 3x\,dx$$
$$\therefore\ \tfrac{1}{3}y^3 = \tfrac{3}{2}x^2 + c$$
$$\therefore\ y^3 = \tfrac{9}{2}x^2 + c$$
$$\therefore\ y = \sqrt[3]{\tfrac{9}{2}x^2 + c}$$

But $y(0) = 1$, so $1 = \sqrt[3]{c}$

$$\therefore\ c = 1$$

The particular solution is

$y = \sqrt[3]{\tfrac{9}{2}x^2 + 1}.$

b

$$\frac{dy}{dx} = \frac{\sqrt{y}}{3}$$
$$\therefore\ \frac{1}{\sqrt{y}}\,\frac{dy}{dx} = \frac{1}{3}$$
$$\therefore\ \int \frac{1}{\sqrt{y}}\,\frac{dy}{dx}\,dx = \int \tfrac{1}{3}\,dx$$
$$\therefore\ \int y^{-\frac{1}{2}}\,dy = \int \tfrac{1}{3}\,dx$$
$$\therefore\ 2y^{\frac{1}{2}} = \tfrac{1}{3}x + c$$
$$\therefore\ y^{\frac{1}{2}} = \tfrac{1}{6}x + c$$
$$\therefore\ \sqrt{y} = \tfrac{1}{6}x + c$$

But $y(44) = 9$, so $\sqrt{9} = \frac{44}{6} + c$

$$\therefore\ 3 = \tfrac{22}{3} + c$$
$$\therefore\ c = -\tfrac{13}{3}$$

So, $\sqrt{y} = \frac{1}{6}x - \frac{13}{3}$

The particular solution is

$$y = \left(\tfrac{1}{6}x - \tfrac{13}{3}\right)^2 = \left(\tfrac{1}{6}x - \tfrac{26}{6}\right)^2 = \tfrac{1}{36}(x-26)^2$$

c

$$\frac{dy}{dx} = y + yx^2 = y(1+x^2)$$
$$\therefore\ \frac{1}{y}\,\frac{dy}{dx} = 1 + x^2$$
$$\therefore\ \int \frac{1}{y}\,\frac{dy}{dx}\,dx = \int (1+x^2)\,dx$$
$$\therefore\ \int \frac{1}{y}\,dy = \int (1+x^2)\,dx$$
$$\therefore\ \ln|y| = x + \tfrac{1}{3}x^3 + c$$
$$\therefore\ y = \pm e^{x+\frac{1}{3}x^3+c}$$
$$\therefore\ y = Ae^{x+\frac{1}{3}x^3} \quad \{A = \pm e^c\}$$

But $y(0) = 1$, so $1 = Ae^0$

$$\therefore\ A = 1$$

The particular solution is $y = e^{x+\frac{1}{3}x^3}$.

d

$$\frac{dy}{dx} = \frac{3x}{\cos y}$$
$$\therefore\ \cos y\,\frac{dy}{dx} = 3x$$
$$\therefore\ \int \cos y\,\frac{dy}{dx}\,dx = \int 3x\,dx$$
$$\therefore\ \int \cos y\,dy = \int 3x\,dx$$
$$\therefore\ \sin y = \tfrac{3}{2}x^2 + c$$

But $y(1) = 0$, so $\sin 0 = \frac{3}{2} + c$

$$\therefore\ 0 = \tfrac{3}{2} + c$$
$$\therefore\ c = -\tfrac{3}{2}$$

So, $\sin y = \frac{3}{2}x^2 - \frac{3}{2}$

The particular solution is

$y = \sin^{-1}\left(\frac{3}{2}x^2 - \frac{3}{2}\right).$

e

$$\frac{dy}{dx} = \frac{6\cos 2x}{\sqrt{y}}$$

$$\therefore \quad \sqrt{y}\,\frac{dy}{dx} = 6\cos 2x$$

$$\therefore \quad \int \sqrt{y}\,\frac{dy}{dx}\,dx = \int 6\cos 2x\,dx$$

$$\therefore \quad \int y^{\frac{1}{2}}\,dy = \int 6\cos 2x\,dx$$

$$\therefore \quad \tfrac{2}{3}y^{\frac{3}{2}} = 3\sin 2x + c$$

$$\therefore \quad y^{\frac{3}{2}} = \tfrac{9}{2}\sin 2x + c$$

But $y(0) = 3$, so $3^{\frac{3}{2}} = c$

$$\therefore \quad c = 3\sqrt{3}$$

So, $y^{\frac{3}{2}} = \frac{9}{2}\sin 2x + 3\sqrt{3}$

The particular solution is $y = \left(\frac{9}{2}\sin 2x + 3\sqrt{3}\right)^{\frac{2}{3}}$.

5

$$\frac{dy}{dx} = 2\sqrt{y}$$

$$\therefore \quad \frac{1}{\sqrt{y}}\,\frac{dy}{dx} = 2$$

$$\therefore \quad \int \frac{1}{\sqrt{y}}\,\frac{dy}{dx}\,dx = \int 2\,dx$$

$$\therefore \quad \int y^{-\frac{1}{2}}\,dy = \int 2\,dx$$

$$\therefore \quad 2y^{\frac{1}{2}} = 2x + c$$

$$\therefore \quad y^{\frac{1}{2}} = x + c$$

$$\therefore \quad \sqrt{y} = x + c$$

But $y(0) = 9$, so $\sqrt{9} = c$

$$\therefore \quad c = 3$$

So, $\sqrt{y} = x + 3$ which is defined for $x \geqslant -3$.

The particular solution is $y = (x+3)^2$, defined for $x \geqslant -3$.

6 **a**

$$\frac{dP}{dt} = \frac{1}{2}P$$

$$\therefore \frac{1}{P}\frac{dP}{dt} = \frac{1}{2}$$

$$\therefore \int \frac{1}{P}\frac{dP}{dt}\,dt = \int \frac{1}{2}\,dt$$

$$\therefore \int \frac{1}{P}\,dP = \int \frac{1}{2}\,dt$$

$$\therefore \ln|P| = \tfrac{1}{2}t + c$$

$$\therefore P = e^{\frac{t}{2}+c} \qquad \{\text{since } P \geqslant 0\}$$

$$\therefore P = Ae^{\frac{t}{2}} \qquad \{A = e^c\}$$

When $t = 0$, $P = 40$

$$\therefore 40 = Ae^0 \quad \text{and so} \quad A = 40$$

$$\therefore P = 40e^{\frac{t}{2}}$$

b When $t = 6$, $P = 40e^3$

$$\therefore P \approx 803$$

After 6 months, there were approximately 803 rabbits.

7 **a**

$$\frac{dI}{dt} = -0.4I$$

$$\therefore \frac{1}{I}\frac{dI}{dt} = -0.4$$

$$\therefore \int \frac{1}{I}\frac{dI}{dt}\,dt = \int -0.4\,dt$$

$$\therefore \int \frac{1}{I}\,dI = \int -0.4\,dt$$

$$\therefore \ln|I| = -0.4t + c$$

$$\therefore I = \pm e^{-0.4t+c}$$

$$\therefore I = Ae^{-0.4t} \qquad \{A = \pm e^c\}$$

When $t = 0$, $I = 350$

$$\therefore 350 = Ae^0 \quad \text{and so} \quad A = 350$$

$$\therefore I = 350e^{-0.4t}$$

b When $t = 5$, $I = 350e^{-2}$

$$\therefore I \approx 47.4$$

After 5 milliseconds, the current is approximately 47.4 milliamps.

c When $I = 20$, we have $20 = 350e^{-0.4t}$

$$\therefore t \approx 7.16 \qquad \{\text{technology}\}$$

It takes approximately 7.16 milliseconds for the current to fall to 20 milliamps.

8
$$\frac{dC_A}{dt} = -kC_A$$
$$\therefore \frac{1}{C_A}\frac{dC_A}{dt} = -k$$
$$\therefore \int \frac{1}{C_A}\frac{dC_A}{dt}\,dt = -\int k\,dt$$
$$\therefore \int \frac{1}{C_A}\,dC_A = -\int k\,dt$$
$$\therefore \ln|C_A| = -kt + c$$
$$\therefore C_A = e^{-kt+c} \quad \{\text{since } C_A \geqslant 0\}$$
$$\therefore C_A = Ae^{-0.31t} \quad \{A = e^c\}$$

When $t = 0$, $C_A = 1$
$$\therefore 1 = Ae^0 \text{ and so } A = 1$$
$$\therefore C_A = e^{-0.31t}$$

When $C_A = 1 - 0.8 = 0.2$, $0.2 = e^{-0.31t}$
$$\therefore t \approx 5.19 \quad \{\text{technology}\}$$

It will take approximately 5.19 minutes for 80% of the ethylene oxide to be used up.

9 The rate of change in the weight w of raw sugar is directly proportional to weight w.

So, $\frac{dw}{dt} = kw$ for some constant $k \neq 0$
$$\therefore \frac{1}{w}\frac{dw}{dt} = k$$
$$\therefore \int \frac{1}{w}\frac{dw}{dt}\,dt = \int k\,dt$$
$$\therefore \int \frac{1}{w}\,dw = \int k\,dt$$
$$\therefore \ln|w| = kt + c$$
$$\therefore w = e^{kt+c} \quad \{\text{since } w \geqslant 0\}$$
$$\therefore w = Ae^{kt} \quad \{A = e^c\}$$

Let w_0 be the initial weight of the sugar.

When $t = 0$, $w = w_0$
$$\therefore w_0 = Ae^0 \text{ and so } A = w_0$$
$$\therefore w = w_0e^{kt}$$

When $t = 10$, $w = (1 - 0.8)w_0$
$$\therefore 0.2w_0 = w_0e^{10k}$$
$$\therefore 0.2 = e^{10k}$$
$$\therefore e^k = (0.2)^{\frac{1}{10}}$$
$$\therefore w = w_0(0.2)^{\frac{t}{10}}$$

When $t = 30$,
$$w = w_0(0.2)^{\frac{30}{10}} = w_0(0.2)^3 = 0.008w_0$$

$\therefore$ 0.8% of raw sugar remains after 30 hours.

10 Let $A(t)$ be the size of the algae, in m^2, after t days.

$$\frac{dA}{dt} = k\sqrt{A}, \quad k > 0$$

$$\therefore \quad \frac{1}{\sqrt{A}} \frac{dA}{dt} = k$$

$$\therefore \quad \int \frac{1}{\sqrt{A}} \frac{dA}{dt} \, dt = \int k \, dt$$

$$\therefore \quad \int A^{-\frac{1}{2}} \, dA = \int k \, dt$$

$$\therefore \quad 2A^{\frac{1}{2}} = kt + c$$

$$\therefore \quad \sqrt{A} = \tfrac{1}{2}kt + c$$

When $t = 0$, $A = 16$

$$\therefore \quad 4 = c$$

When $t = 3$, $A = 32$

$$\therefore \quad \sqrt{32} = \tfrac{3}{2}k + 4$$

$$\therefore \quad 8\sqrt{2} - 8 = 3k$$

$$\therefore \quad k = \frac{8\sqrt{2} - 8}{3}$$

So, $\sqrt{A} = \dfrac{4\sqrt{2} - 4}{3} t + 4$

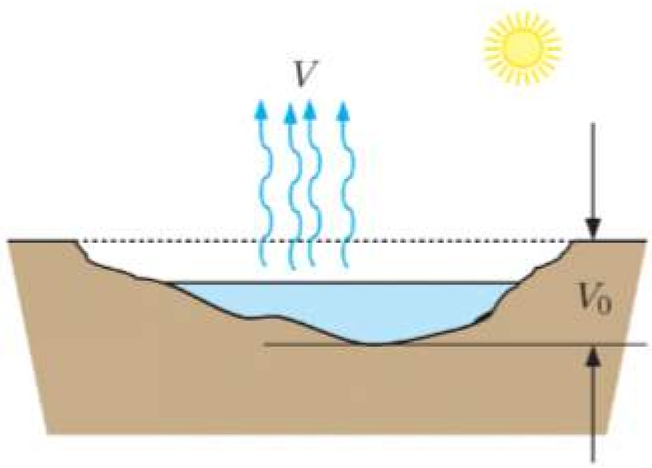
$$\therefore \quad \sqrt{A} = \tfrac{4}{3}\left((\sqrt{2} - 1)t + 3\right)$$

$$\therefore \quad A(t) = \tfrac{16}{9}\left((\sqrt{2} - 1)t + 3\right)^2$$

11 **a** The amount of water remaining in the lake is $(V_0 - V)$.

The water evaporates at a rate proportional to the volume of water remaining.

$$\therefore \quad \frac{dV}{dt} = k(V_0 - V) \quad \text{for some constant } k.$$



b

$$\frac{dV}{dt} = k(V_0 - V)$$

$$\therefore \quad \frac{1}{V_0 - V} \frac{dV}{dt} = k$$

$$\therefore \quad \int \frac{1}{V_0 - V} \frac{dV}{dt} \, dt = \int k \, dt$$

$$\therefore \quad \int \frac{1}{V_0 - V} \, dV = \int k \, dt$$

$$\therefore \quad -\ln|V_0 - V| = kt + c$$

$$\therefore \quad \ln|V_0 - V| = -kt + c$$

$$\therefore \quad V_0 - V = e^{-kt+c} \qquad \{\text{since } V_0 - V \geqslant 0\}$$

$$\therefore \quad V = V_0 - Ae^{-kt} \qquad \{A = e^c\}$$

When $t = 0$, $V = 0$

$\therefore\ 0 = V_0 - Ae^0$ and so $A = V_0$

$\therefore\ V = V_0 - V_0 e^{-kt}$

When $t = 20$, $V = \frac{1}{2}V_0$

$\therefore\ \frac{1}{2}V_0 = V_0 - V_0 e^{-20k}$

$\therefore\ \frac{1}{2} = 1 - e^{-20k}$

$\therefore\ e^{-20k} = \frac{1}{2}$

$\therefore\ e^{-k} = \left(\frac{1}{2}\right)^{\frac{1}{20}}$

$\therefore\ e^{-kt} = \left(\frac{1}{2}\right)^{\frac{t}{20}}$

$\therefore\ V = V_0 - V_0\left(\frac{1}{2}\right)^{\frac{t}{20}}$

When $t = 50$, $V = V_0 - V_0\left(\frac{1}{2}\right)^{\frac{50}{20}}$

$= V_0 - 2^{-\frac{5}{2}}V_0$

$= V_0(1 - 2^{-\frac{5}{2}})$ is the amount of water that has *evaporated.*

$\therefore$ the amount of water *remaining* $= V_0 - V_0(1 - 2^{-\frac{5}{2}})$

$= 2^{-\frac{5}{2}}V_0$

$\approx 0.177V_0$

$\therefore$ approximately 17.7% of the original water remains after 50 days without rain.

12

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\therefore\ \frac{dT}{dt} = k(T - T_m) \quad \text{for some constant } k.$$

$$\therefore\ \frac{1}{T - T_m}\frac{dT}{dt} = k$$

$$\therefore\ \int \frac{1}{T - T_m}\frac{dT}{dt}\,dt = \int k\,dt$$

$$\therefore\ \int \frac{1}{T - T_m}\,dT = \int k\,dt$$

$$\therefore\ \ln|T - T_m| = kt + c$$

$$\therefore\ T - T_m = \pm e^{kt+c}$$

$$\therefore\ T = T_m + Ae^{kt} \qquad \{A = \pm e^c\}$$

a $T_m = 5$

When $t = 0$, $T = 100$

$\therefore\ 100 = 5 + Ae^0$ and so $A = 95$

$\therefore\ T = 5 + 95e^{kt}$

When $t = 1$, $T = 80$

$\therefore\ 80 = 5 + 95e^k$

$\therefore\ 75 = 95e^k$

$\therefore\ e^k = \frac{75}{95} = \frac{15}{19}$

$\therefore\ T = 5 + 95\left(\frac{15}{19}\right)^t$

When $T = 10$, $10 = 5 + 95\left(\frac{15}{19}\right)^t$

$\therefore\ t \approx 12.5$ {technology}

It will take about 12.5 minutes for the temperature of the object to drop to 10°C.

b Let t be the time in hours since 6 am, and $T_m = 5$.

When $t = 0$, $T = 13$

$\therefore\ 13 = 5 + Ae^0$ and so $A = 8$

$\therefore\ T = 5 + 8e^{kt}$

When $t = 3$, $T = 9$

$$\therefore\ 9 = 5 + 8e^{3k}$$
$$\therefore\ e^{3k} = \tfrac{4}{8} = \tfrac{1}{2}$$
$$\therefore\ e^k = \left(\tfrac{1}{2}\right)^{\frac{1}{3}}$$
$$\therefore\ T = 5 + 8\left(\tfrac{1}{2}\right)^{\frac{t}{3}}$$

When $T = 37$, $37 = 5 + 8\left(\tfrac{1}{2}\right)^{\frac{t}{3}}$

$\therefore\ t = -6$ {technology}

$\therefore$ the person died 6 hours before 6 am.

$\therefore$ the time of death was 12 am which is midnight.

13 **a**

$$L\,\frac{dI}{dt} + RI = E$$
$$\therefore\ 0.3\,\frac{dI}{dt} + 10I = 20$$
$$\therefore\ \tfrac{3}{10}\,\frac{dI}{dt} = 20 - 10I$$
$$\therefore\ \frac{dI}{dt} = \tfrac{10}{3}(-10)(I-2)$$
$$\therefore\ \int \frac{1}{I-2}\,\frac{dI}{dt} = -\tfrac{100}{3}$$
$$\therefore\ \int \frac{1}{I-2}\,\frac{dI}{dt}\,dt = \int -\tfrac{100}{3}\,dt$$
$$\therefore\ \int \frac{1}{I-2}\,dI = \int -\tfrac{100}{3}\,dt$$
$$\therefore\ \ln|I-2| = -\tfrac{100}{3}t + c$$
$$\therefore\ I - 2 = \pm e^{-\frac{100}{3}t + c}$$
$$\therefore\ I(t) = Ae^{-\frac{100}{3}t} + 2 \quad \{A = \pm e^c\}$$

b

$$I(0) = 0$$
$$\therefore\ Ae^0 + 2 = 0$$
$$\therefore\ A = -2$$

The particular solution is

$I(t) = 2 - 2e^{-\frac{100}{3}t}$

c As $t \to \infty$, $e^{-\frac{100}{3}t} \to 0$

$\therefore\ I(t) \to 2$

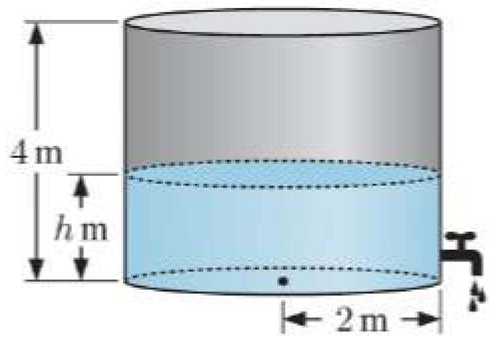
The limiting current is 2 amps.

d When $I = 0.99 \times 2 = 1.98$, $1.98 = 2 - 2e^{-\frac{100}{3}t}$

$\therefore\ t \approx 0.138$ {technology}

It will take about 0.138 seconds for the current to reach 99% of its limiting value.

14 We are given that



$$\frac{dV}{dt} \propto \sqrt{h} \quad \text{where } h \text{ is the depth of the water.}$$

$$\therefore \frac{dV}{dt} = k\sqrt{h} \quad \text{where } k \text{ is a constant.}$$

$$\therefore \frac{dV}{dh}\frac{dh}{dt} = k\sqrt{h} \quad \{\text{chain rule}\}$$

$$\therefore 4\pi\frac{dh}{dt} = k\sqrt{h} \quad \left\{V = \pi r^2 h = 4\pi h \quad \therefore \frac{dV}{dh} = 4\pi\right\}$$

$$\therefore \frac{4\pi}{\sqrt{h}}\frac{dh}{dt} = k$$

$$\therefore \int 4\pi h^{-\frac{1}{2}}\frac{dh}{dt}\,dt = \int k\,dt$$

$$\therefore 4\pi\int h^{-\frac{1}{2}}\,dh = \int k\,dt$$

$$\therefore 4\pi\frac{h^{\frac{1}{2}}}{\frac{1}{2}} = kt + c$$

$$\therefore 8\pi\sqrt{h} = kt + c$$

Now when $t = 0$, $h = 4$

$$\therefore 8\pi\sqrt{4} = c$$

$$\therefore c = 16\pi$$

$$\therefore 8\pi\sqrt{h} = kt + 16\pi$$

Also, when $t = 2$, $h = 1$

$$\therefore 8\pi\sqrt{1} = 2k + 16\pi$$

$$\therefore 2k = -8\pi$$

$$\therefore k = -4\pi$$

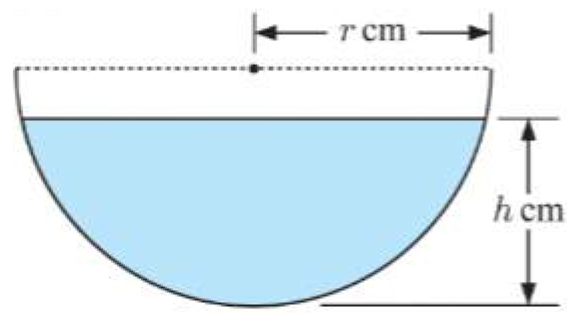
So, the equation connecting the depth of the water and the time t is $8\pi\sqrt{h} = -4\pi t + 16\pi$.

The tank is empty when $h = 0$. This occurs when $4\pi t = 16\pi$

$$\therefore t = 4$$

The tank empties in 4 hours.

15 **a** $V = \frac{1}{3}\pi h^2(3r - h) = \pi r h^2 - \frac{1}{3}\pi h^3$



$$\therefore \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$$

$$= (2\pi r h - \pi h^2)\frac{dh}{dt}$$

Now $\frac{dV}{dt} = -r^2$, so $-r^2 = (2\pi r h - \pi h^2)\frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{-r^2}{2\pi r h - \pi h^2}$$

$$= \frac{r^2}{\pi h^2 - 2\pi r h}$$

b **i** $(\pi h^2 - 2\pi r h)\dfrac{dh}{dt} = r^2$ {using **a**}

$$\therefore \int (\pi h^2 - 2\pi r h)\frac{dh}{dt}\,dt = \int r^2\,dt$$

$$\therefore \int (\pi h^2 - 2\pi r h)\,dh = \int r^2\,dt$$

$$\therefore \tfrac{1}{3}\pi h^3 - \pi r h^2 = r^2 t + c$$

$$\therefore \tfrac{1}{3}\pi h^3 - 10\pi h^2 = 100t + c \quad \{r = 10\}$$

When $t = 0$, $h = 10$

$$\therefore \tfrac{1000}{3}\pi - 1000\pi = c$$

$$\therefore c = -\tfrac{2000}{3}\pi$$

$$\therefore 100t = \tfrac{1}{3}\pi h^3 - 10\pi h^2 + \tfrac{2000}{3}\pi$$

$$\therefore t = \tfrac{\pi}{300}h^3 - \tfrac{\pi}{10}h^2 + \tfrac{2000}{300}\pi$$

$$\therefore t = \tfrac{\pi}{300}(h^3 - 30h^2 + 2000)$$

ii When $h = 5$, $t = \frac{1375\pi}{300}$

$$\therefore t \approx 14.4$$

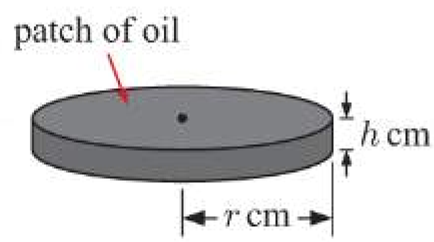
It will take about 14.4 hours for the depth of the water to fall to 5 cm.

iii When $h = 0$, $t = \frac{20\pi}{3}$

$$\therefore t \approx 20.9$$

It will take about 20.9 hours for the bowl to empty.

16 **a** We are given that



$\dfrac{dr}{dt} \propto h$ where h is the thickness of the patch.

$\therefore \dfrac{dr}{dt} = kh$ where k is a constant.

$\therefore \dfrac{dr}{dt} = \dfrac{kV}{\pi r^2}$ $\{V = \pi r^2 h \quad \therefore h = \dfrac{V}{\pi r^2}\}$

$$\therefore r^2\frac{dr}{dt} = \frac{kV}{\pi}$$

$$\therefore \int r^2\frac{dr}{dt}\,dt = \int \frac{kV}{\pi}\,dt$$

$$\therefore \int r^2\,dr = \int \frac{kV}{\pi}\,dt$$

$$\therefore \tfrac{1}{3}r^3 = \frac{kV}{\pi}t + c$$

$$\therefore r^3 = \frac{3kV}{\pi}t + c$$

$$\therefore r = \sqrt[3]{\frac{3kV}{\pi}t + c}$$

b Since $V = 1 \text{ L} = 1000 \text{ cm}^3$, $r = \sqrt[3]{\frac{3000k}{\pi}t + c}$.

When $t = 0$, $r = 20$

$\therefore\ 20 = \sqrt[3]{c}$

$\therefore\ c = 8000$

$\therefore\ r = \sqrt[3]{\frac{3000k}{\pi}t + 8000}$

When $t = 2$, $r = 50$

$\therefore\ 50 = \sqrt[3]{\frac{6000k}{\pi} + 8000}$

$\therefore\ 125\,000 = \frac{6000k}{\pi} + 8000$

$\therefore\ \frac{6000k}{\pi} = 117\,000$

$\therefore\ \frac{3000k}{\pi} = 58\,500$

$\therefore\ r = \sqrt[3]{58\,500t + 8000}$

When $r = 500$, $500 = \sqrt[3]{58\,500t + 8000}$

$\therefore\ 125\,000\,000 = 58\,500t + 8000$

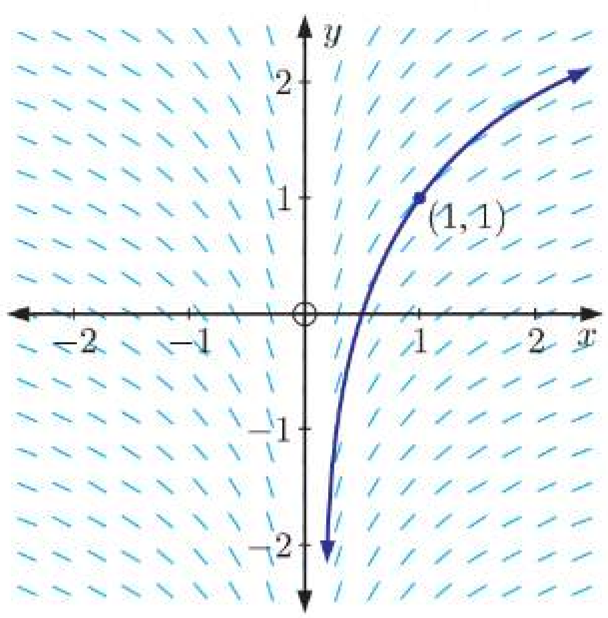
$\therefore\ 58\,500t = 124\,992\,000$

$\therefore\ t \approx 2136.6$

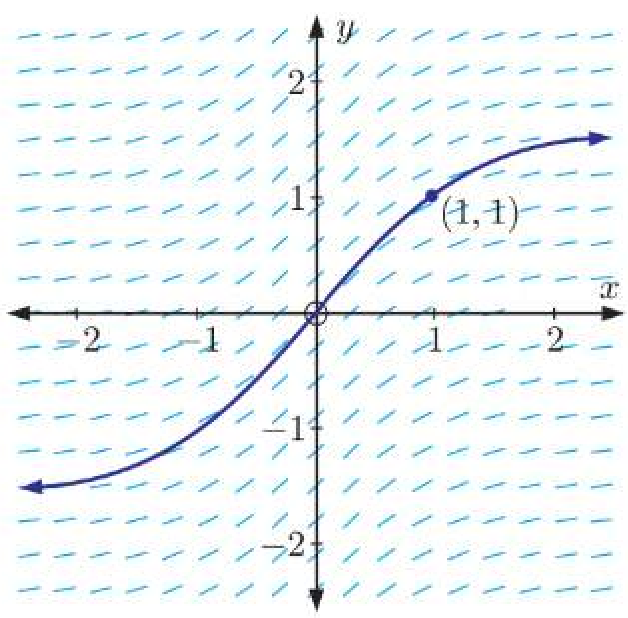
It will take about 2136.6 seconds or 35.6 minutes for the spill radius to reach 5 m.

EXERCISE 25E

1 **a**



b

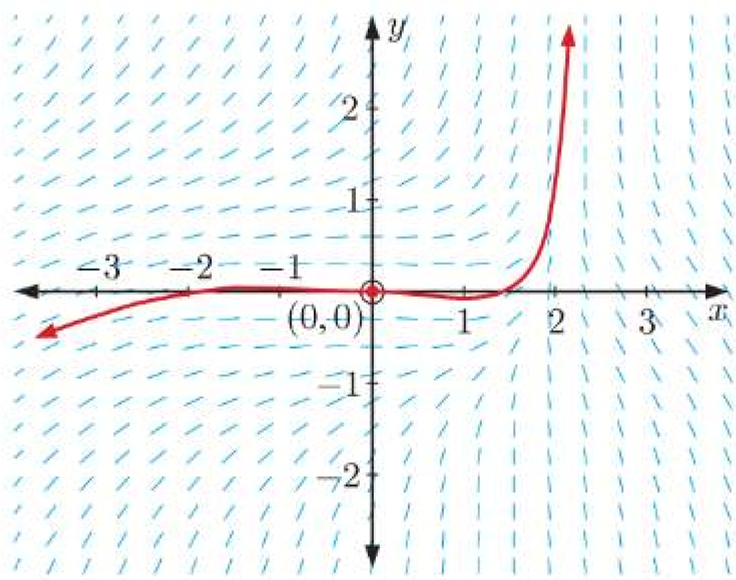


2 **a** At the point $(0, 0)$,

$$\frac{dy}{dx} = \frac{-1+0+0}{0-0+10} = -\frac{1}{10}$$

$\therefore$ the gradient of the tangent to the solution curve at the origin is $-\frac{1}{10}$.

b



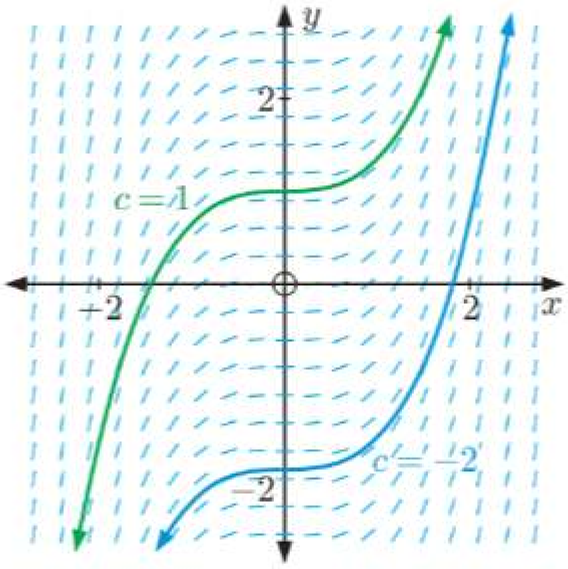
3 **a** $\frac{dy}{dx} = x^2$

$\therefore \; y = \int x^2 \, dx$

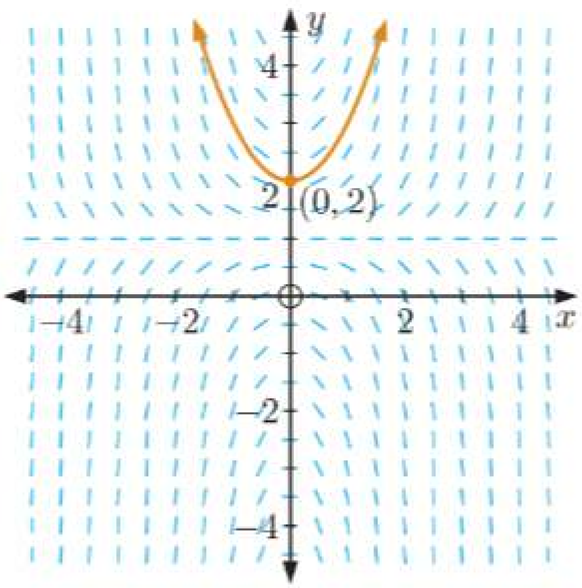
$\therefore \; y = \frac{1}{3}x^3 + c$

b **i** When $c = 1$, the general solution is $y = \frac{1}{3}x^3 + 1$.

ii When $c = -2$, the general solution is $y = \frac{1}{3}x^3 - 2$.



4 **a**



b
$$\frac{dy}{dx} = x(y-1)$$
$$\therefore \; \frac{1}{y-1}\frac{dy}{dx} = x$$
$$\therefore \; \int \frac{1}{y-1}\frac{dy}{dx}\,dx = \int x\,dx$$
$$\therefore \; \int \frac{1}{y-1}\,dy = \int x\,dx$$
$$\therefore \; \ln|y-1| = \frac{1}{2}x^2 + c$$
$$\therefore \; y - 1 = \pm e^{\frac{1}{2}x^2 + c}$$
$$\therefore \; y = Ae^{\frac{1}{2}x^2} + 1 \quad \{A = \pm e^c\}$$

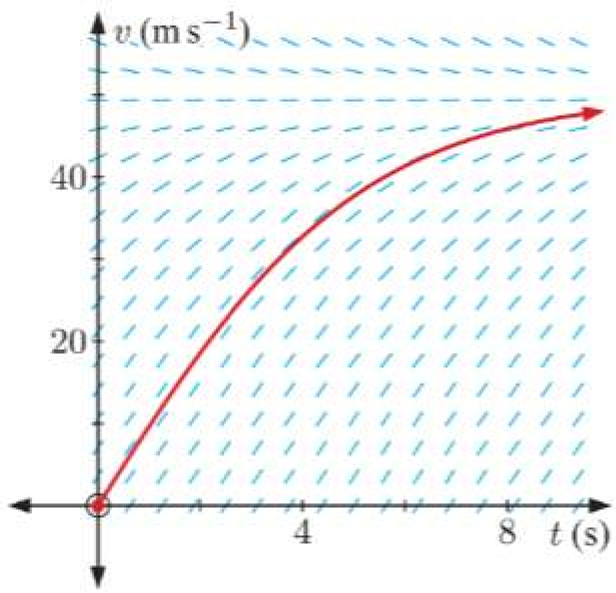
The particular solution passes through $(0, 2)$.

$\therefore \; 2 = Ae^0 + 1$

$\therefore \; A = 1$

Hence $(0, 2)$ lies on the particular solution curve $y = e^{\frac{1}{2}x^2} + 1$.

5 **a**



b The skydiver approaches a terminal velocity of approximately 50 m s^{-1}.

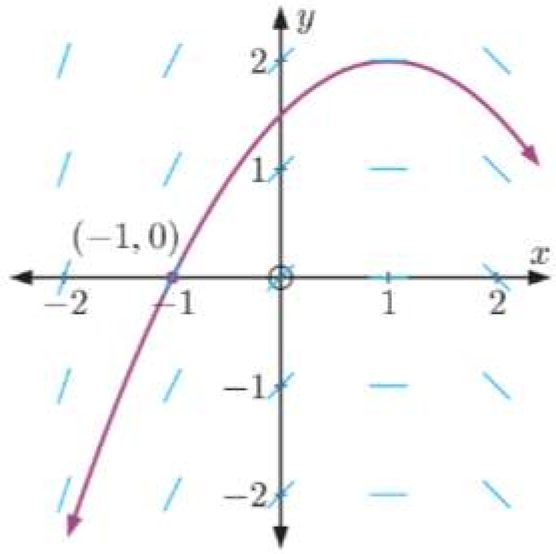
6 **a** The particular solution passes through $(-1, 0)$ $\therefore\ 0 = -1 - \frac{1}{2}(-1)^2 + c$

$\therefore\ c = \frac{3}{2}$

So, the particular solution is $y = x - \frac{1}{2}x^2 + \frac{3}{2}$.

b We calculate $\dfrac{dy}{dx}$ for each grid point:

y \ x	-2	-1	0	1	2
-2	3	2	1	0	-1
-1	3	2	1	0	-1
0	3	2	1	0	-1
1	3	2	1	0	-1
2	3	2	1	0	-1



7 **a**

$$\frac{dy}{dx} = \frac{1}{2}xy$$

$$\therefore\ \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}x$$

$$\therefore\ \int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{2}x\,dx$$

$$\therefore\ \int \frac{1}{y}\,dy = \int \frac{1}{2}x\,dx$$

$$\therefore\ \ln|y| = \tfrac{1}{4}x^2 + c$$

$$\therefore\ y = \pm e^{\frac{1}{4}x^2 + c}$$

$$\therefore\ y = Ae^{\frac{1}{4}x^2} \qquad \{A = \pm e^c\}$$

The particular solution passes through $(1, -1)$.

$\therefore\ -1 = Ae^{\frac{1}{4}}$

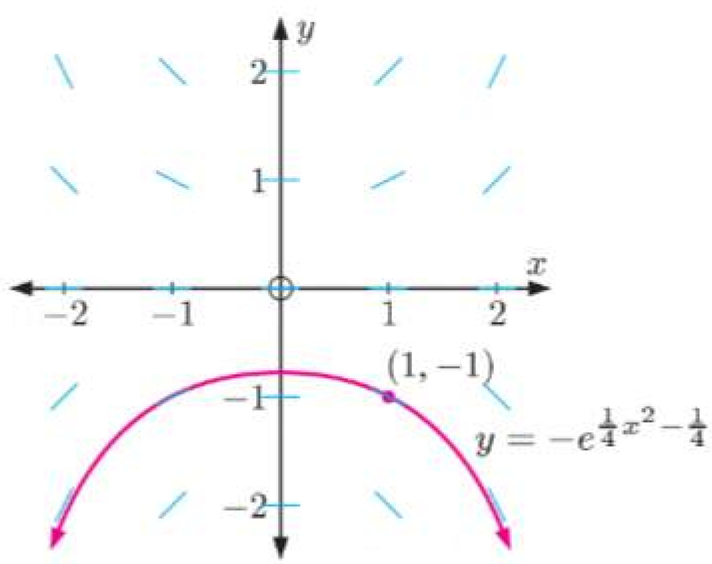
$\therefore\ A = -e^{-\frac{1}{4}}$

Hence $(1, -1)$ lies on the particular solution curve $y = -e^{-\frac{1}{4}} \times e^{\frac{1}{4}x^2}$

$\therefore\ y = -e^{\frac{1}{4}x^2 - \frac{1}{4}}$

b We calculate $\dfrac{dy}{dx}$ for each grid point:

y \ x	-2	-1	0	1	2
-2	2	1	0	-1	-2
-1	1	0.5	0	-0.5	-1
0	0	0	0	0	0
1	-1	-0.5	0	0.5	1
2	-2	-1	0	1	2



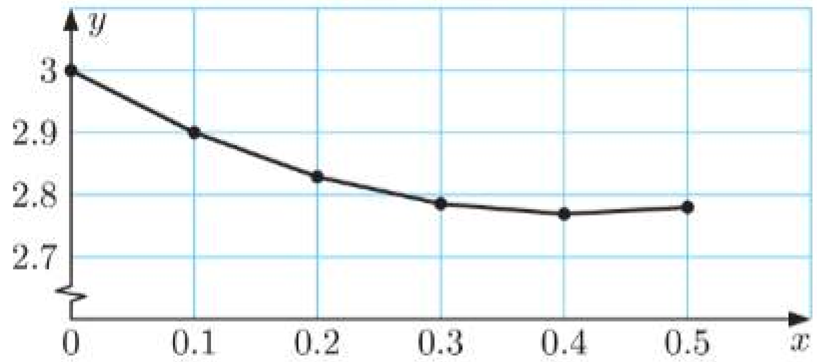
EXERCISE 25F

1 $\dfrac{dy}{dx} = xy - 1$ with initial point $(0, 3)$.

a

Iteration	x_{i-1}	y_{i-1}	$\dfrac{dy}{dx}$	x_i	y_i
1	0	3	-1	0.1	2.9
2	0.1	2.9	-0.71	0.2	2.829
3	0.2	2.829	-0.4342	0.3	2.7856
4	0.3	2.7856	-0.1643	0.4	2.7691
5	0.4	2.7691	0.1077	0.5	2.7799

b



2 $\dfrac{dy}{dx} = 3e^{2x} - 1, \quad y(0) = 2$

$y(0) = 2$ gives us $x_0 = 0$ and $y_0 = 2$.

a **i**

Iteration	x_{i-1}	y_{i-1}	$\dfrac{dy}{dx}$	x_i	y_i
1	0	2	2	0.25	2.5
2	0.25	2.5	3.9462	0.5	3.4865

$\therefore \quad y(0.5) \approx 3.4865$

ii

Iteration	x_{i-1}	y_{i-1}	$\dfrac{dy}{dx}$	x_i	y_i
1	0	2	2	0.1	2.2
2	0.1	2.2	2.6642	0.2	2.4664
3	0.2	2.4664	3.4755	0.3	2.8140
4	0.3	2.8140	4.4664	0.4	3.2606
5	0.4	3.2606	5.6766	0.5	3.8283

$\therefore \quad y(0.5) \approx 3.8283$

b $y(0.5) = y(0) + \int_0^{0.5} \frac{dy}{dx}\, dx$ {Fundamental Theorem of Calculus}

$$\begin{aligned} &= 2 + \int_0^{0.5} (3e^{2x} - 1)\, dx \\ &= 2 + \left[\tfrac{3}{2}e^{2x} - x\right]_0^{0.5} \\ &= 2 + \left(\tfrac{3}{2}e - 0.5\right) - \left(\tfrac{3}{2} - 0\right) \\ &= \tfrac{3}{2}e \\ &\approx 4.0774 \end{aligned}$$

The accuracy of Euler's method was improved by decreasing the step size.

3 $\frac{dy}{dx} = 1 + 2x - 3y, \quad y(0) = 1$

$y(0) = 1$ gives us $x_0 = 0$ and $y_0 = 1$.

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	1	-2	0.2	0.6
2	0.2	0.6	-0.4	0.4	0.52
3	0.4	0.52	0.24	0.6	0.568
4	0.6	0.568	0.496	0.8	0.6672
5	0.8	0.6672	0.5984	1	0.7869

$\therefore \quad y(1) \approx 0.7869$

4 $\frac{dy}{dx} = -\cos x, \quad y(0) = 0$

$y(0) = 0$ gives us $x_0 = 0$ and $y_0 = 0$.

a

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	0	-1	0.1	-0.1
2	0.1	-0.1	-0.9950	0.2	-0.1995
3	0.2	-0.1995	-0.9801	0.3	-0.2975
4	0.3	-0.2975	-0.9553	0.4	-0.3930
5	0.4	-0.3930	-0.9211	0.5	-0.4851

$\therefore \quad y(0.5) \approx -0.4851$

b Using technology to estimate $y(0.5)$ using Euler's method with step size 0.001 for 500 steps, we get $y(0.5) \approx -0.4795$.

c $y(0.5) = y(0) + \int_0^{0.5} \frac{dy}{dx}\,dx$ {Fundamental Theorem of Calculus}

$$= 0 + \int_0^{0.5} (-\cos x)\,dx$$

$$= \left[-\sin x\right]_0^{0.5}$$

$$= -\sin(0.5) + 0$$

$$\approx -0.4794$$

The accuracy of Euler's method was improved by decreasing the step size.

5 $\frac{dy}{dx} = x\cos y$ with initial point $(0, 0)$

a

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	0	0	1	0
2	1	0	1	2	1

$\therefore\ y(2) \approx 1$

b Using technology to estimate $y(2)$ using Euler's method with step size 0.1 for 20 steps, we get $y(2) \approx 1.3021$.

c Using technology to estimate $y(2)$ using Euler's method with step size 0.01 for 200 steps, we get $y(2) \approx 1.3018$.

ACTIVITY 1 LOGISTIC GROWTH

1 a i In $\frac{dA}{dt} = 0.1A\left(1 - \frac{A}{500}\right)$, we have $r = 0.1$ and $K = 500$.

$N_0 = 20$

$$\therefore\ A = \frac{500}{1 + \left(\frac{500}{20} - 1\right)e^{-0.1t}}$$

$$\therefore\ A = \frac{500}{1 + 24e^{-0.1t}}$$

ii As $t \to \infty$, $e^{-0.1t} \to 0$

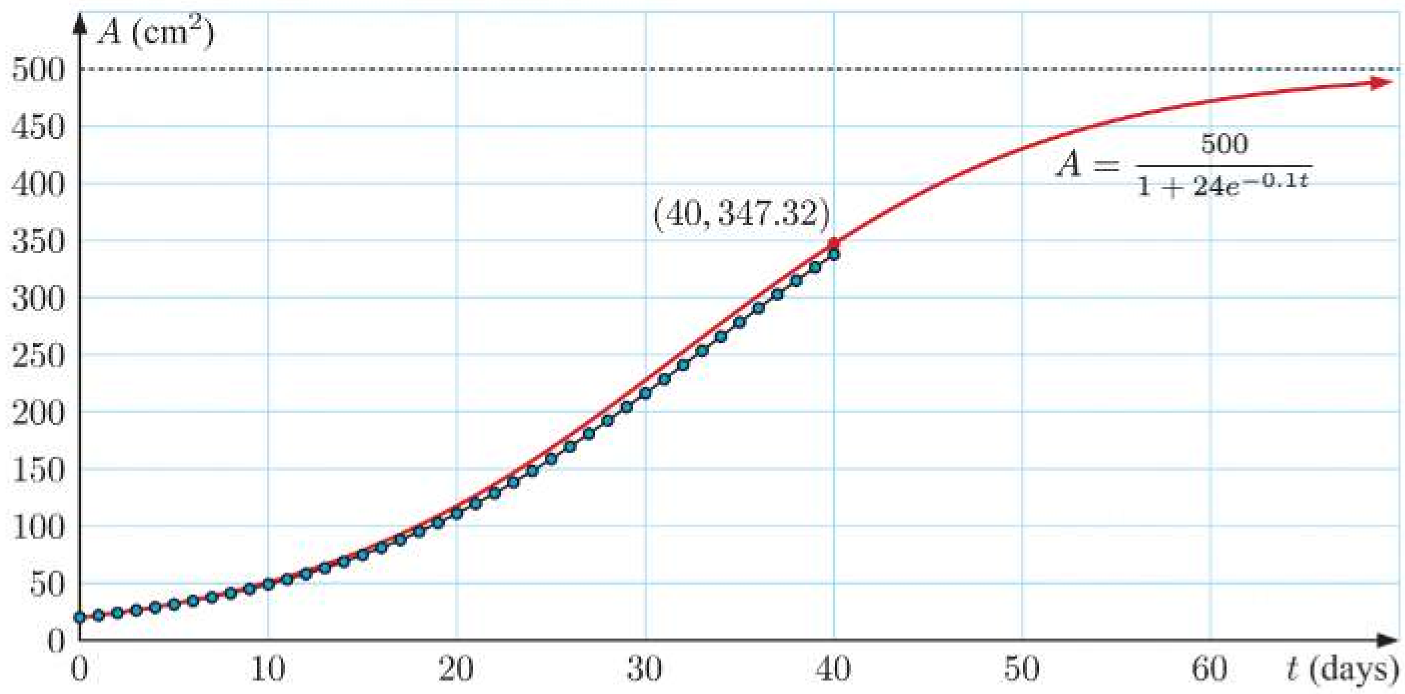
$$\therefore\ A \to \frac{500}{1+0} = 500$$

The limiting size of the lily is 500 cm^2.

b Using the spreadsheet with step size 1:

Time (t days)	*Estimate* (cm^2)
0	20.0
1	21.92
2	24.0159
3	26.3021
4	28.794
5	31.5076
6	34.4598
7	37.6683
8	41.1513
9	44.9278
10	49.0168
11	53.438
12	58.2107
13	63.354
14	68.8867
15	74.8263
16	81.1891
17	87.9897
18	95.2402
19	102.9501
20	111.1254

Time (t days)	*Estimate* (cm^2)
21	119.7681
22	128.8761
23	138.4419
24	148.4528
25	158.8905
26	169.7303
27	180.9416
28	192.4878
29	204.3263
30	216.4091
31	228.6834
32	241.0925
33	253.5767
34	266.0741
35	278.5224
36	290.8597
37	303.0258
38	314.9635
39	326.6194
40	337.9453



c The approximation in **b** is reasonably accurate. The approximation underestimates the day 40 area by about 9.38 cm^2.

Reducing the step size to 0.5 reduces this underestimate to about 4.61 cm^2.

2 **a**

Iteration	t_{i-1}	N_{i-1}	$\frac{dN}{dt}$	t_i	N_i
1	0	2.0	1.5947	1	3.5947
2	1	3.5947	2.8585	2	6.4532
3	2	6.4532	5.107	3	11.5602
4	3	11.5602	9.07	4	20.6301
5	4	20.6301	15.9366	5	36.5668
6	5	36.5668	27.4706	6	64.0374
7	6	64.0374	45.7622	7	109.7996
8	7	109.7996	71.7651	8	181.5646
9	8	181.5646	101.2974	9	282.8621
10	9	282.8621	119.6084	10	402.4705

Reducing the step size will improve the accuracy of the model.

b We have $r = 0.8$, $K = 600$, $N_0 = 2$.

$$\therefore \quad N = \frac{600}{1 - \left(\frac{600}{2} - 1\right) e^{-0.8t}}$$

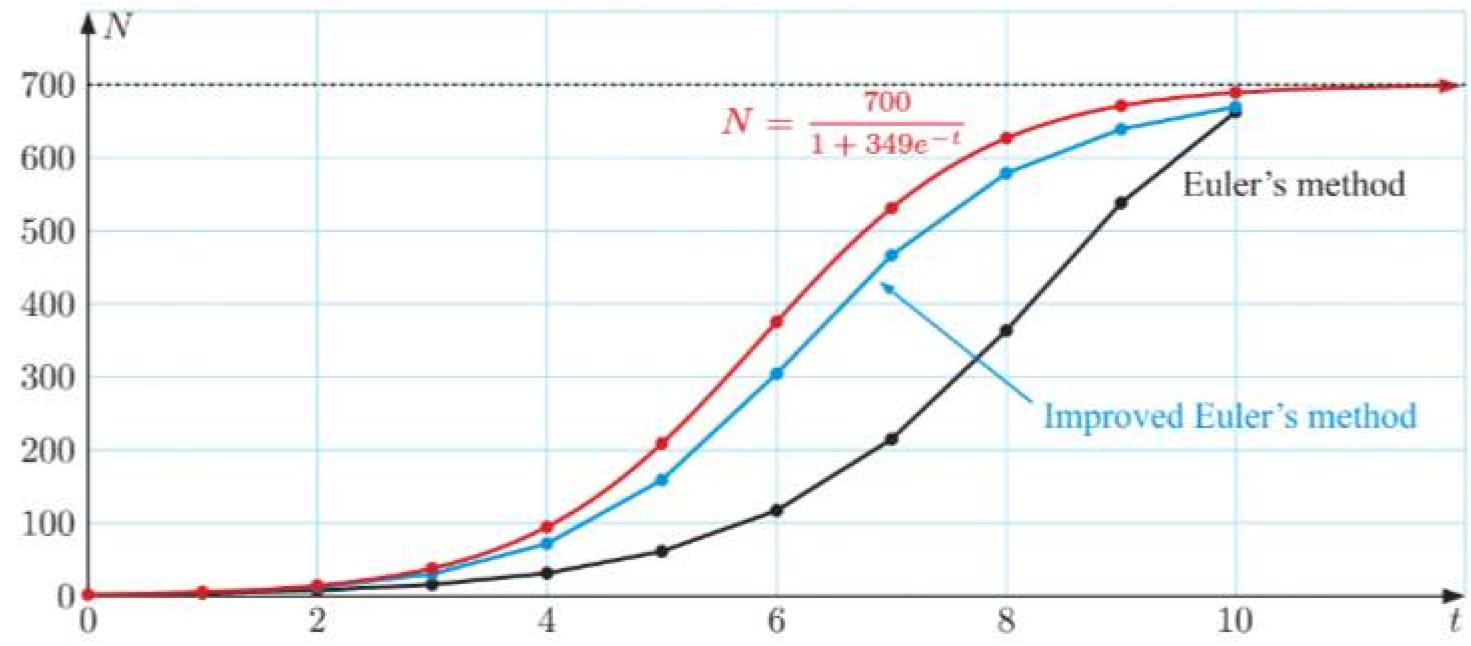
$$\therefore \quad N = \frac{600}{1 + 299e^{-0.8t}}$$

INVESTIGATION IMPROVING EULER'S METHOD

2 **a, b**

t	*Euler's method*	*Improved Euler's method*	*Analytic solution*
0	2	2	2
1	3.9943	4.9829	5.41
2	7.9658	12.3513	14.5132
3	15.8409	30.2321	38.0938
4	31.3233	71.7749	94.6949
5	61.245	158.8293	208.859
6	117.1316	304.3849	375.3181
7	214.6634	466.4843	531.0083
8	363.4977	578.9077	626.6357
9	538.2375	639.1404	671.0959
10	662.6184	669.5502	689.0818

c



d The improved Euler's method is more accurate than Euler's method, especially when the graph is steep.

3 **a**

t	*Euler's method*	*Improved Euler's method*
0	0	0
1	6.0	5.82
2	11.64	10.9915
3	16.2851	15.1419
4	19.6331	18.2191
5	21.7785	20.3754
6	23.0355	21.8301
7	23.7291	22.7875
8	24.0984	23.4076
9	24.2911	23.8051
10	24.3905	24.0584
11	24.4415	24.2191
12	24.4677	24.3208
13	24.481	24.3851
14	24.4878	24.4257
15	24.4913	24.4513

b The results obtained in Euler's method are greater than those of the improved Euler's method, due to the velocity increasing steeply at first.

As the iteration progresses and the parachutist approaches terminal velocity, the difference between the two methods decreases.

ACTIVITY 2 THE SPRUCE BUDWORM

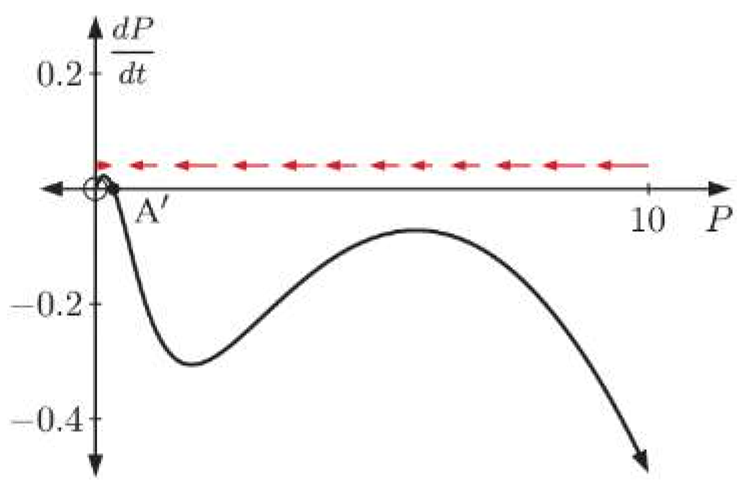
1 Between O and A, the effect of predators is small, and the population is not inhibited by the carrying capacity, so the population increases.

Between A and B, the predators start to have a greater influence, and the rate of reproduction at this population level is insufficient to counteract the predators, so the population decreases.

Between B and C, the higher population levels mean that the budworms are reproducing sufficiently to counteract the predators, so the population increases.

At population levels greater than C, the population is approaching its carrying capacity, which inhibits the reproduction of the budworms. The effect of the predators now exceeds the reproduction, so the population decreases.

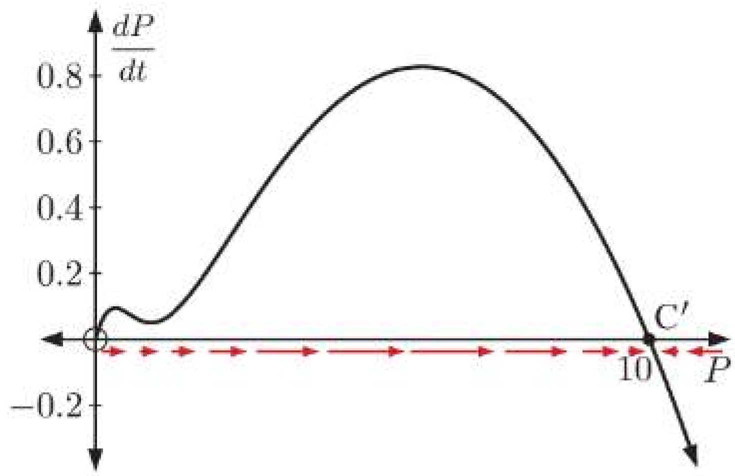
2 **a** **i**



ii O is an unstable equilibrium point, and A′ is a stable equilibrium point.

iii The lower reproduction rate means that, even at higher population levels, the reproduction is unable to counteract the effect of the predators. So, the population decreases to A′.

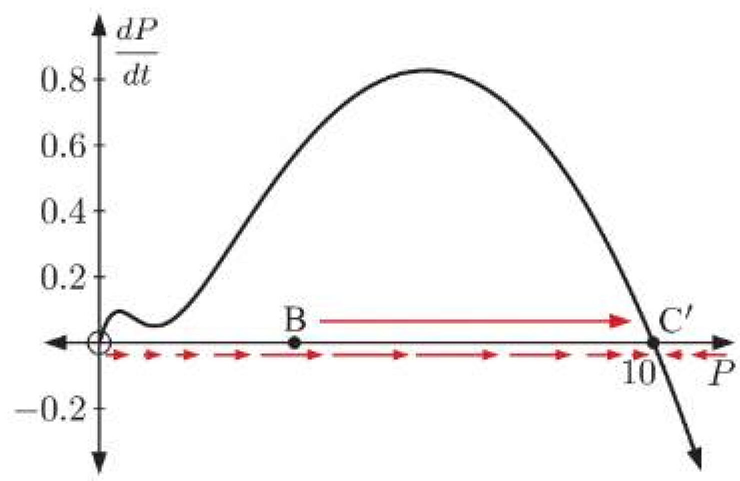
b **i**



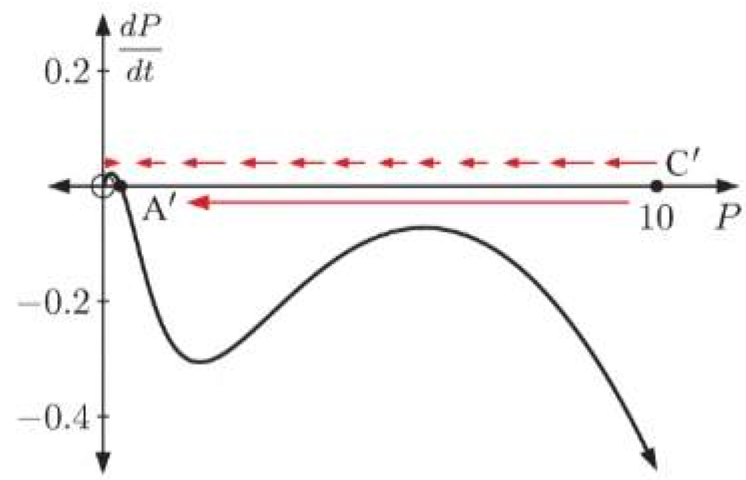
ii O is an unstable equilibrium point, and C′ is a stable equilibrium point.

iii The higher reproduction rate means that, even at lower population levels, the reproduction is able to counteract the effect of the predators. So, the population increases to C′.

3 **a** From the original equilibrium point B, if the reproduction rate increases to $r = 0.6$, the population will increase to the equilibrium point C′.



b From equilibrium point C′, if the reproduction rate falls to $r = 0.3$, the population will decrease to the equilibrium point A′.



REVIEW SET 25A

1 $\dfrac{dM}{dt}$ is inversely proportional to M^2

$\therefore \dfrac{dM}{dt} = -\dfrac{k}{M^2}$, $k > 0$ {the mass is decreasing, so $\dfrac{dM}{dt} < 0$}

2 **a** If $y = 3e^{-\frac{1}{x}}$, then
$$\frac{dy}{dx} = 3e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right)$$
$$= \frac{y}{x^2} \quad \text{as required.}$$

b If $y = 2 + ce^{-x^3}$ then
$$\frac{dy}{dx} + 3x^2y = -3x^2(ce^{-x^3}) + 3x^2(2 + ce^{-x^3})$$
$$= -3cx^2e^{-x^3} + 6x^2 + 3cx^2e^{-x^3}$$
$$= 6x^2 \quad \text{as required.}$$

3 **a**
$$\frac{dy}{dx} = \cos 2x - \sin x$$
$$\therefore \ y = \int (\cos 2x - \sin x)\, dx$$
$$\therefore \ y = \tfrac{1}{2}\sin 2x + \cos x + c$$

b
$$\frac{dy}{dx} = 3 - e^{-2x}$$
$$\therefore \ y = \int (3 - e^{-2x})\, dx$$
$$= 3x + \tfrac{1}{2}e^{-2x} + c$$

c
$$\frac{dy}{dx} = \frac{1}{2x+1}$$
$$\therefore \ y = \int \frac{1}{2x+1}\, dx$$
$$= \tfrac{1}{2}\ln|2x+1| + c$$

Now $y(0) = 2$

$\therefore \ 2 = \frac{1}{2}\ln 1 + c$

$\therefore \ c = 2$

So, the solution is $y = \frac{1}{2}\ln|2x+1| + 2$.

d $\dfrac{dy}{dt} - t = te^{t^2}$

$\therefore\ \dfrac{dy}{dt} = t + te^{t^2}$

$\therefore\ y = \displaystyle\int t\left(1 + e^{t^2}\right) dt$

$\therefore\ y = \displaystyle\int \tfrac{1}{2}(1 + e^u)\,\dfrac{du}{dt}\,dt \qquad \{u = t^2,\ \ \dfrac{du}{dt} = 2t\}$

$\therefore\ y = \tfrac{1}{2}\displaystyle\int (1 + e^u)\,du$

$\therefore\ y = \tfrac{1}{2}(u + e^u) + c$

$\therefore\ y = \tfrac{1}{2}\left(t^2 + e^{t^2}\right) + c$

Now $y(1) = 2e$

$\therefore\ 2e = \tfrac{1}{2}(1 + e) + c$

$\therefore\ c = \tfrac{3}{2}e - \tfrac{1}{2}$

So, the solution is $y = \tfrac{1}{2}\left(t^2 + e^{t^2}\right) + \tfrac{3}{2}e - \tfrac{1}{2}$

$\therefore\ y = \tfrac{1}{2}(t^2 + e^{t^2} + 3e - 1)$

4 **a** $S'(t) = \dfrac{4000e^{-0.05t}}{(1 + 4e^{-0.05t})^2}$

i At noon, $t = 0$, so $S'(0) = \dfrac{4000e^0}{(1 + 4e^0)^2}$

$= 160$

At noon, spectators were entering the stadium at a rate of 160 spectators per minute.

ii At 12:30 pm, $t = 30$, so $S'(30) = \dfrac{4000e^{-1.5}}{(1 + 4e^{-1.5})^2}$

≈ 249

At 12:30 pm, spectators were entering the stadium at a rate of about 249 spectators per minute.

b $\dfrac{d}{dt}\left(\dfrac{1}{1 + 4e^{-0.05t}}\right) = \dfrac{d}{dt}\left((1 + 4e^{-0.05t})^{-1}\right)$

$= -(1 + 4e^{-0.05t})^{-2}(-0.2e^{-0.05t}) \qquad$ {chain rule}

$= \dfrac{0.2e^{-0.05t}}{(1 + 4e^{-0.05t})^2}$

c $$S(t) = \int S'(t)\,dt$$
$$= \int \frac{4000e^{-0.05t}}{(1+4e^{-0.05t})^2}\,dt$$
$$= 20\,000 \int \frac{0.2e^{-0.05t}}{(1+4e^{-0.05t})^2}\,dt$$
$$= \frac{20\,000}{1+4e^{-0.05t}} + c \qquad \{\text{from } \textbf{b}\}$$

Now $S(0) = 4000$

$$\therefore\ \frac{20\,000}{1+4e^{0}} + c = 4000$$
$$\therefore\ 4000 + c = 4000$$
$$\therefore\ c = 0$$
$$\therefore\ S(t) = \frac{20\,000}{1+4e^{-0.05t}}$$

d At 1:40 pm, $t = 100$, so $S(100) = \dfrac{20\,000}{1+4e^{-0.05\times 100}} \approx 19\,500$

There were about 19 500 spectators in the stadium at 1:40 pm.

5 **a**
$$\frac{dy}{dx} = 5x^2y$$
$$\therefore\ \frac{1}{y}\frac{dy}{dx} = 5x^2$$
$$\therefore\ \int \frac{1}{y}\frac{dy}{dx}\,dx = \int 5x^2\,dx$$
$$\therefore\ \int \frac{1}{y}\,dy = \int 5x^2\,dx$$
$$\therefore\ \ln|y| = \tfrac{5}{3}x^3 + c$$
$$\therefore\ y = \pm e^{\frac{5}{3}x^3 + c}$$
$$\therefore\ y = Ae^{\frac{5}{3}x^3} \qquad \{A = \pm e^c\}$$

b
$$\frac{dy}{dx} = 2xy^2 - y^2 = y^2(2x-1)$$
$$\therefore\ \frac{1}{y^2}\frac{dy}{dx} = 2x - 1$$
$$\therefore\ \int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int (2x-1)\,dx$$
$$\therefore\ \int y^{-2}\,dy = \int (2x-1)\,dx$$
$$\therefore\ -y^{-1} = x^2 - x + c$$
$$\therefore\ y = -\frac{1}{x^2 - x + c}$$

6 **a** We are given that $\dfrac{dM}{dt} \propto M$

$$\therefore\ \frac{dM}{dt} = -kM, \quad k > 0 \qquad \{\text{the mass is decreasing, so } \frac{dM}{dt} < 0\}$$

The initial mass is M_0, so $M(0) = M_0$.

b $$\frac{dM}{dt} = -kM$$

$$\therefore \int \frac{1}{M}\,dM = \int -k\,dt$$

$$\therefore \ln|M| = -kt + c$$

$$\therefore M = \pm e^{-kt+c}$$

$$\therefore M = Ae^{-kt} \qquad \{A = \pm e^c\}$$

When $t = 0$, $M = M_0$

$\therefore M_0 = Ae^0$ and so $A = M_0$

$\therefore M = M_0 e^{-kt}$

When $t = 30$, $M = \frac{4}{5}M_0$

$\therefore \frac{4}{5}M_0 = M_0 e^{-30k}$

$\therefore \frac{4}{5} = e^{-30k}$

$\therefore e^{-k} = \left(\frac{4}{5}\right)^{\frac{1}{30}}$

$\therefore M = M_0\left(\frac{4}{5}\right)^{\frac{t}{30}}$

When $M = \frac{1}{2}M_0$, $\frac{1}{2}M_0 = M_0\left(\frac{4}{5}\right)^{\frac{t}{30}}$

$$\therefore \left(\frac{4}{5}\right)^{\frac{t}{30}} = \frac{1}{2}$$

$$\therefore t \approx 93.2 \qquad \{\text{technology}\}$$

It will take about 93.2 days for the substance to decay to half its original mass.

7 $$\frac{dy}{dx} = -\frac{3}{e^y}$$

$$\therefore \int e^y\,dy = \int -3\,dx$$

$$\therefore e^y = -3x + c$$

$$\therefore y = \ln(-3x + c)$$

But $y(0) = 1$, so $1 = \ln c$

$\therefore c = e$

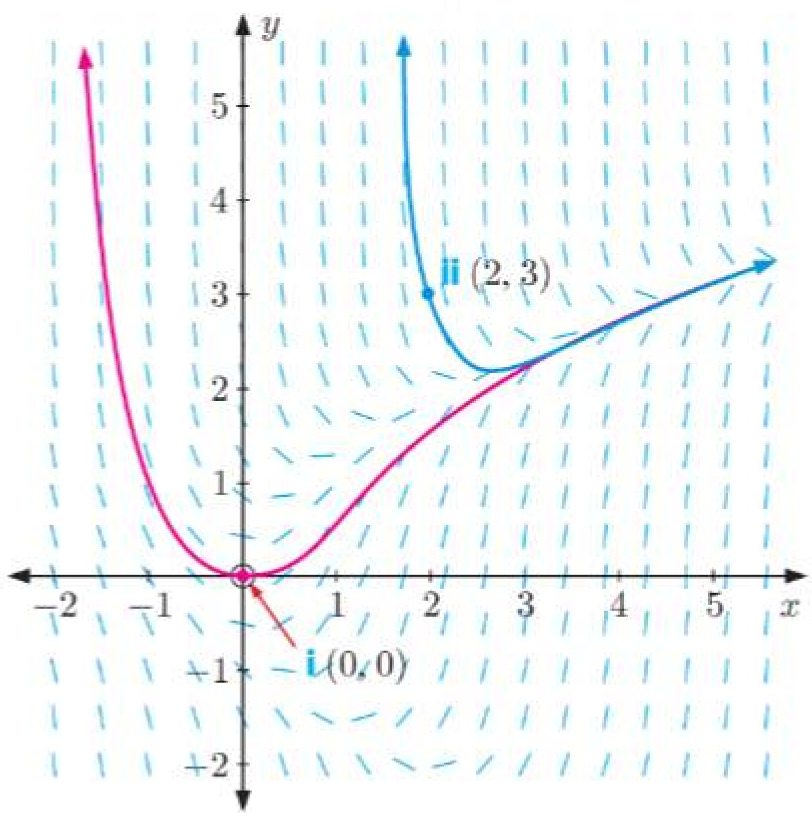
The particular solution is $y = \ln(e - 3x)$.

8 **a** At the point $(1, -1)$,

$$\frac{dy}{dx} = 2(1) - (-1)^2 = 1$$

So the gradient of the tangent to the solution curve at $(1, -1)$ is 1.

b



9 **a** $$\frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore \int y\,dy = \int -x\,dx$$

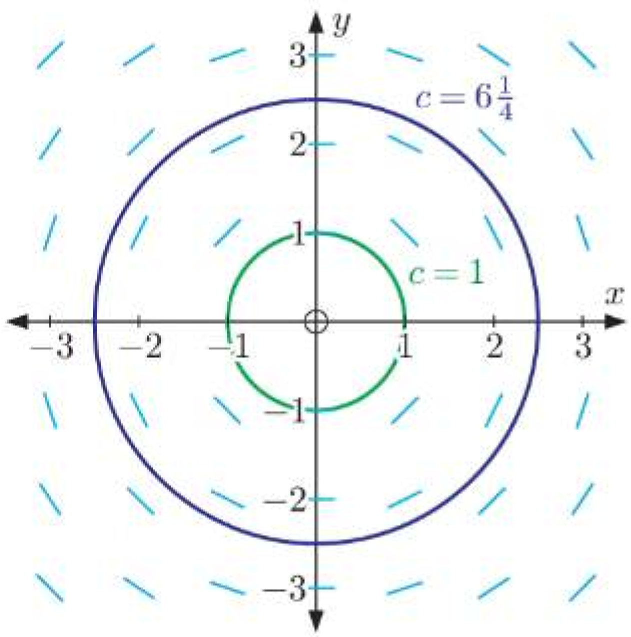
$$\therefore \tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + c$$

$$\therefore \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 = c$$

$$\therefore x^2 + y^2 = c$$

b We calculate $\frac{dy}{dx}$ for each grid point:

y \ x	-3	-2	-1	0	1	2	3
-3	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
-1	-3	-2	-1	0	1	2	3
0	-	-	-	-	-	-	-
1	3	2	1	0	-1	-2	-3
2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$
3	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1



10 $\frac{dy}{dx} = x - 2y$ with initial point $(1, 2)$

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	1	2	-3	1.1	1.7
2	1.1	1.7	-2.3	1.2	1.47
3	1.2	1.47	-1.74	1.3	1.296
4	1.3	1.296	-1.292	1.4	1.1668
5	1.4	1.1668	-0.9336	1.5	1.073 44
6	1.5	1.073 44	$-0.646 88$	1.6	1.008 752

$\therefore \; y(1.6) \approx 1.0088$

REVIEW SET 25B

1 $\frac{dD}{dt}$ is proportional to $\sqrt{D}$.

$\therefore \; \frac{dD}{dt} = k\sqrt{D}, \;\; k > 0 \quad \{$the distance is increasing, so $\frac{dD}{dt} > 0\}$

2 If $y = 3\sin x - 2\cos x$, then $\dfrac{dy}{dx} = 3\cos x + 2\sin x$

$$\therefore\ 3\frac{dy}{dx} = 9\cos x + 6\sin x$$

$$\therefore\ 3\frac{dy}{dx} - 2y = 9\cos x + 6\sin x - 2(3\sin x - 2\cos x)$$

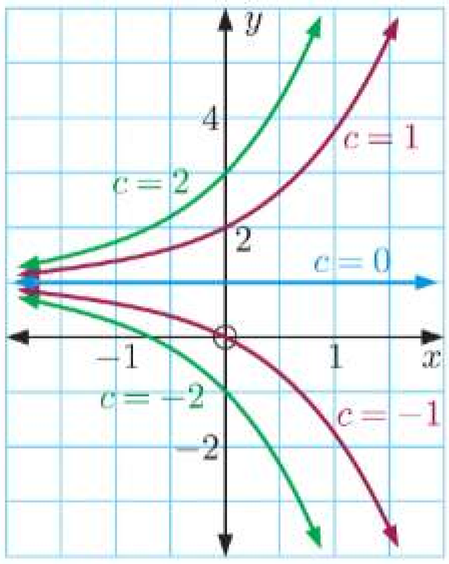
$$\therefore\ 3\frac{dy}{dx} - 2y = 13\cos x \quad \text{as required.}$$

3 **a** If $y = ce^x + 1$, then $\dfrac{dy}{dx} = ce^x$

$$= (ce^x + 1) - 1$$

$$= y - 1 \quad \text{for any constant } c \text{ as required.}$$

b



c From **a**, $y = ce^x + 1$ is a general solution to the differential equation.

The particular solution passes through $(0, 4)$, so

$$4 = ce^0 + 1$$

$$\therefore\ c = 3$$

$\therefore$ the particular solution is $y = 3e^x + 1$.

d $\dfrac{dy}{dx} = y - 1$

$\therefore$ at the point $(0, 4)$, $\dfrac{dy}{dx} = 4 - 1 = 3$

$\therefore$ the gradient of the tangent to the particular solution $y = 3e^x + 1$ at $(0, 4)$, is 3.

$\therefore$ the equation of the tangent is $y = 3(x - 0) + 4$

$$\therefore\ y = 3x + 4$$

4 **a** $\dfrac{dy}{dx} = \dfrac{e^x}{e^x - 2}$

$$\therefore\ y = \int \frac{e^x}{e^x - 2}\,dx$$

$$\therefore\ y = \ln|\,e^x - 2\,| + c$$

b $\dfrac{dy}{dx} = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x\right)$

$$\therefore\ y = \int \tfrac{1}{2}\cos\left(\tfrac{\pi}{3} - 2x\right)dx$$

$$= -\tfrac{1}{4}\sin\left(\tfrac{\pi}{3} - 2x\right) + c$$

Now $y\left(\frac{\pi}{2}\right) = 0$

$$\therefore\ 0 = -\tfrac{1}{4}\sin\left(\tfrac{\pi}{3} - \pi\right) + c$$

$$\therefore\ c = \tfrac{1}{4}\sin\left(-\tfrac{2\pi}{3}\right)$$

$$= -\tfrac{1}{4}\sin\tfrac{2\pi}{3}$$

$$= -\tfrac{1}{4}\left(\tfrac{\sqrt{3}}{2}\right)$$

$$= -\tfrac{\sqrt{3}}{8}$$

So, the solution is $y = -\frac{1}{4}\sin\left(\frac{\pi}{3} - 2x\right) - \frac{\sqrt{3}}{8}$.

5 **a**
$$\frac{dy}{dx} = 2y^4$$
$$\therefore \frac{1}{y^4}\frac{dy}{dx} = 2$$
$$\therefore \int \frac{1}{y^4}\frac{dy}{dx}\,dx = \int 2\,dx$$
$$\therefore \int y^{-4}\,dy = \int 2\,dx$$
$$\therefore -\tfrac{1}{3}y^{-3} = 2x + c$$
$$\therefore y^{-3} = c - 6x$$
$$\therefore y^3 = \frac{1}{c-6x}$$
$$\therefore y = \frac{1}{\sqrt[3]{c-6x}}$$

b
$$(t^2+1)\frac{dP}{dt} = Pt$$
$$\therefore \frac{1}{P}\frac{dP}{dt} = \frac{t}{t^2+1}$$
$$\therefore \int \frac{1}{P}\,dP = \tfrac{1}{2}\int \frac{2t}{t^2+1}\,dt$$
$$\therefore \ln|P| = \tfrac{1}{2}\ln\left|t^2+1\right| + c$$
$$\therefore \ln|P| = \tfrac{1}{2}\ln(t^2+1) + c \qquad \{t^2+1>0\}$$
$$\therefore P = \pm e^c\sqrt{t^2+1}$$
$$\therefore P = A\sqrt{t^2+1} \qquad \{A = \pm e^c\}$$

6 **a**
$$\frac{dy}{dx} = \sqrt{y}$$
$$\therefore \frac{1}{\sqrt{y}}\frac{dy}{dx} = 1$$
$$\therefore \int y^{-\frac{1}{2}}\,dy = \int 1\,dx$$
$$\therefore 2y^{\frac{1}{2}} = x + c$$
$$\therefore \sqrt{y} = \tfrac{1}{2}x + c$$

But $y(0) = 4$, so $\sqrt{4} = \tfrac{1}{2}(0) + c$

$$\therefore c = 2$$

So, $\sqrt{y} = \tfrac{1}{2}x + 2$.

The particular solution is $y = \left(\tfrac{1}{2}x + 2\right)^2$.

b
$$\frac{dy}{dx} = y\cos x$$
$$\therefore \frac{1}{y}\frac{dy}{dx} = \cos x$$
$$\therefore \int \frac{1}{y}\,dy = \int \cos x\,dx$$
$$\therefore \ln|y| = \sin x + c$$
$$\therefore y = \pm e^{\sin x + c}$$
$$\therefore y = Ae^{\sin x} \qquad \{A = \pm e^c\}$$

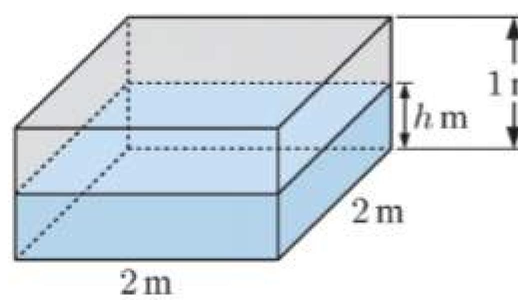
But $y\left(\frac{\pi}{2}\right) = \frac{1}{e^2}$, so $\frac{1}{e^2} = Ae^{\sin\frac{\pi}{2}}$

$$\therefore e^{-2} = Ae$$
$$\therefore A = e^{-3}$$

The particular solution is $y = e^{\sin x - 3}$.

7 **a** We are given that $\frac{dV}{dt} \propto \sqrt{h}$ where h is the depth of the water, and V is the volume of water in the tank.

$\therefore \frac{dV}{dt} = -k\sqrt{h}, \quad k > 0 \quad \{V \text{ is decreasing, so } \frac{dV}{dt} < 0\}$



b $V = l \times w \times h = 2 \times 2 \times h = 4h \text{ m}^3$

$$\therefore \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \qquad \{\text{chain rule}\}$$
$$= 4\frac{dh}{dt}$$

From **a**, $\frac{dV}{dt} = -k\sqrt{h}$, so $-k\sqrt{h} = 4\frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = -\frac{k}{4}\sqrt{h}, \quad k > 0$$

c
$$\frac{dh}{dt} = -\frac{k}{4}\sqrt{h}$$
$$\therefore\ \frac{1}{\sqrt{h}}\frac{dh}{dt} = -\frac{k}{4}$$
$$\therefore\ \int \frac{1}{\sqrt{h}}\frac{dh}{dt}\,dt = \int -\frac{k}{4}\,dt$$
$$\therefore\ \int h^{-\frac{1}{2}}\,dh = \int -\frac{k}{4}\,dt$$
$$\therefore\ 2h^{\frac{1}{2}} = -\frac{k}{4}t + c$$
$$\therefore\ \sqrt{h} = -\frac{k}{8}t + c$$

Now when $t = 0$, $h = 1$
$$\therefore\ \sqrt{1} = c$$
$$\therefore\ c = 1$$
$$\therefore\ \sqrt{h} = -\frac{k}{8}t + 1$$

Also, when $t = 2$, $h = 1 - 0.19 = 0.81$
$$\therefore\ \sqrt{0.81} = -\frac{k}{8}(2) + 1$$
$$\therefore\ 0.9 = -\frac{k}{4} + 1$$
$$\therefore\ -\frac{k}{4} = -\frac{1}{10}$$
$$\therefore\ k = \tfrac{2}{5}$$

So, the equation connecting the depth of the water and the time t is $\sqrt{h} = 1 - \frac{1}{20}t$.

The tank is empty when $h = 0$.

This occurs when $\frac{t}{20} = 1$
$$\therefore\ t = 20$$

The tank empties in 20 minutes.

8 **a** $\frac{dy}{dx} = y + 1$

So, $\frac{dy}{dx} = 0$ when $y = -1$.

This corresponds to the slope field **B**, which has horizontal line segments along $y = -1$.

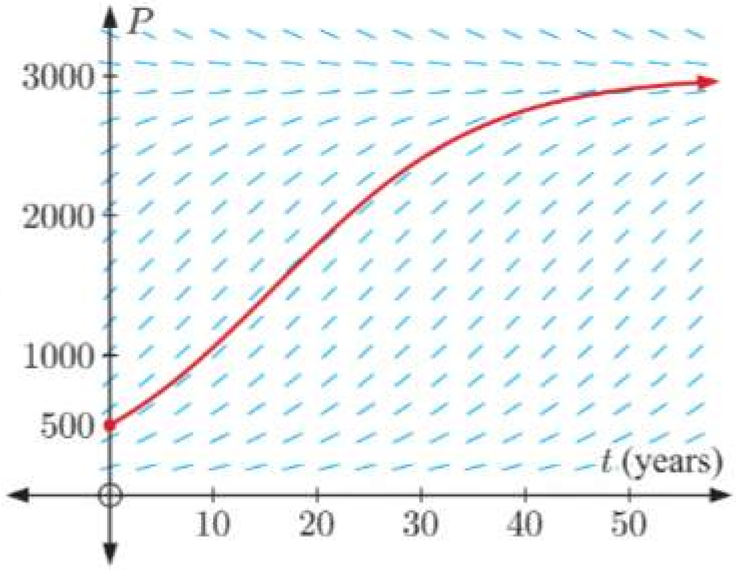
b $\frac{dy}{dx} = x - y$

So, $\frac{dy}{dx} = 0$ when $x = y$.

This corresponds to the slope field **C**, which has horizontal line segments along $y = x$.

c The remaining slope field is **A**.

9 **a**



b The population of rodents approaches the limiting population of 3000.

10 $\frac{dy}{dx} = \sin(x+y), \quad y(0) = 0.5$

$y(0) = 0.5$ gives us $x_0 = 0$ and $y_0 = 0.5$.

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	0.5	0.4794	0.1	0.5479
2	0.1	0.5479	0.6035	0.2	0.6083
3	0.2	0.6083	0.7231	0.3	0.6806
4	0.3	0.6806	0.8308	0.4	0.7637
5	0.4	0.7637	0.9183	0.5	0.8555

$\therefore \quad y(0.5) \approx 0.8555$

Chapter 26

COUPLED DIFFERENTIAL EQUATIONS

EXERCISE 26A

1 $\begin{cases} \dfrac{dx}{dt} = e^y \\ \dfrac{dy}{dt} = x \end{cases}$

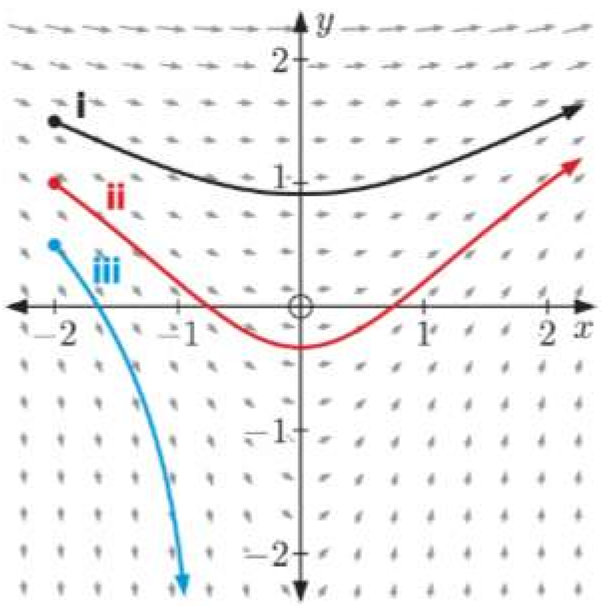
a **i** At $(1, 0)$, $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} e^0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

On the phase portrait, the vector at $(1, 0)$ has the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

ii At $(-1, 1)$, $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} e^1 \\ -1 \end{pmatrix} = \begin{pmatrix} e \\ -1 \end{pmatrix}$

On the phase portrait, the vector at $(-1, 1)$ has the direction $\begin{pmatrix} e \\ -1 \end{pmatrix}$.

b



2 $\begin{cases} \dot{x} = \cos \pi y \\ \dot{y} = x \end{cases}$

a **i** At $(0, \frac{1}{2})$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos \frac{\pi}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ $(0, \frac{1}{2})$ is an equilibrium point.

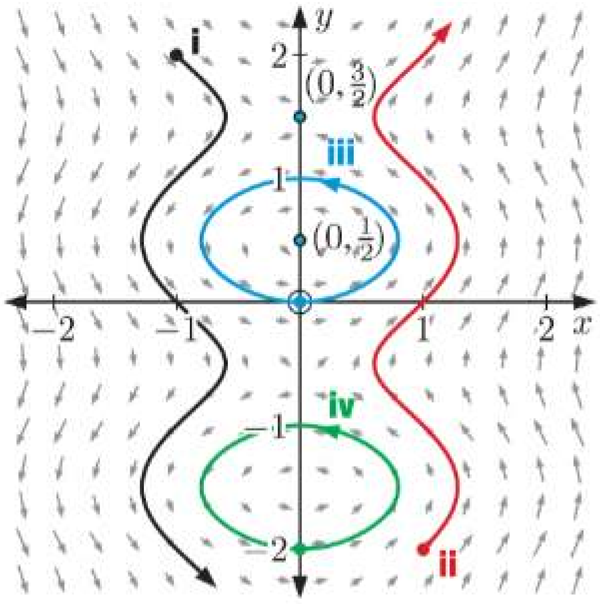
From the phase portrait, we see that it is a centre.

ii At $(0, \frac{3}{2})$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos \frac{3\pi}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

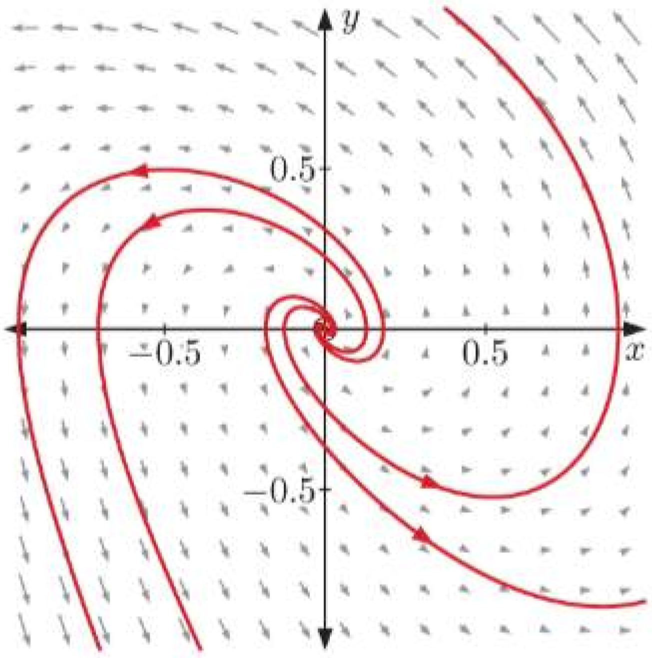
$\therefore$ $(0, \frac{3}{2})$ is an equilibrium point.

From the phase portrait, we see that it is a saddle point.

b



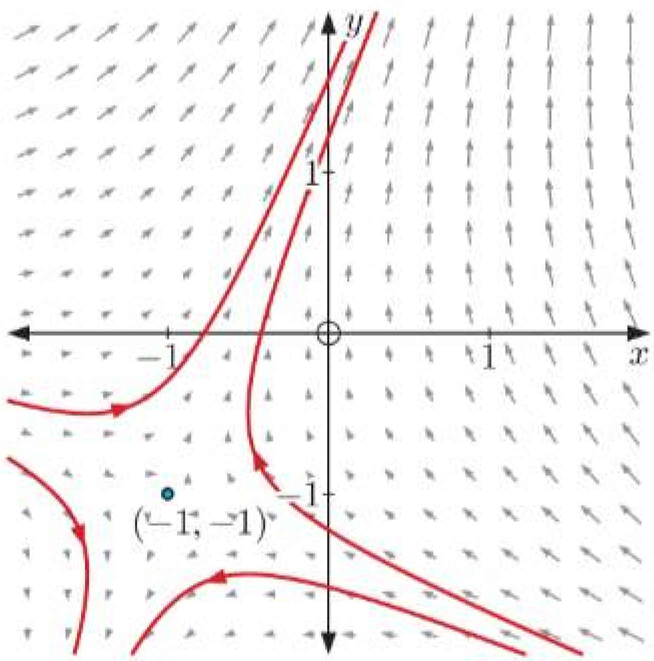
3 **a**



We draw some trajectories on the phase portrait.

It appears that $(0, 0)$ is an equilibrium point. It is an unstable spiral.

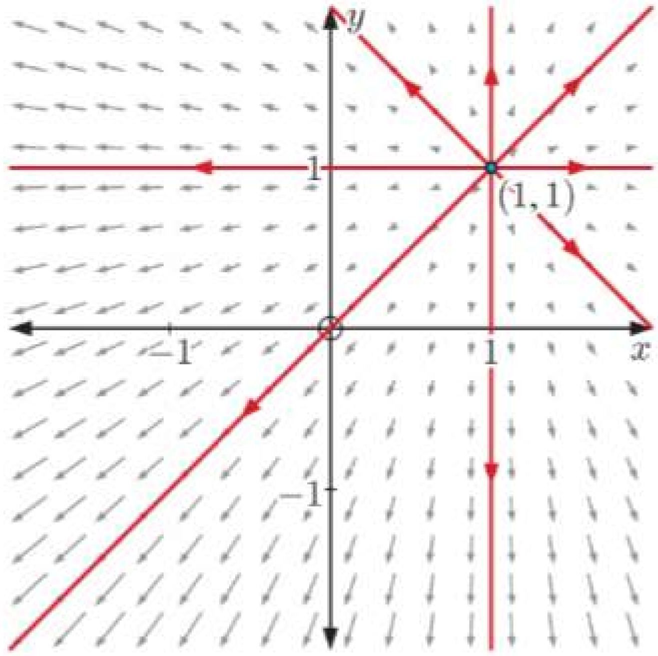
b



We draw some trajectories on the phase portrait.

It appears that $(-1, -1)$ is an equilibrium point. It is a saddle point.

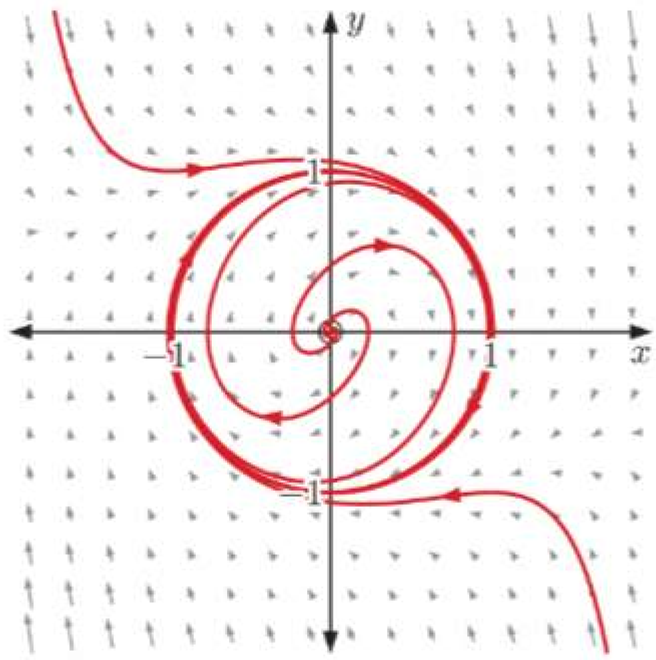
c



We draw some trajectories on the phase portrait.

It appears that $(1,\ 1)$ is an equilibrium point. It is an unstable fixed point.

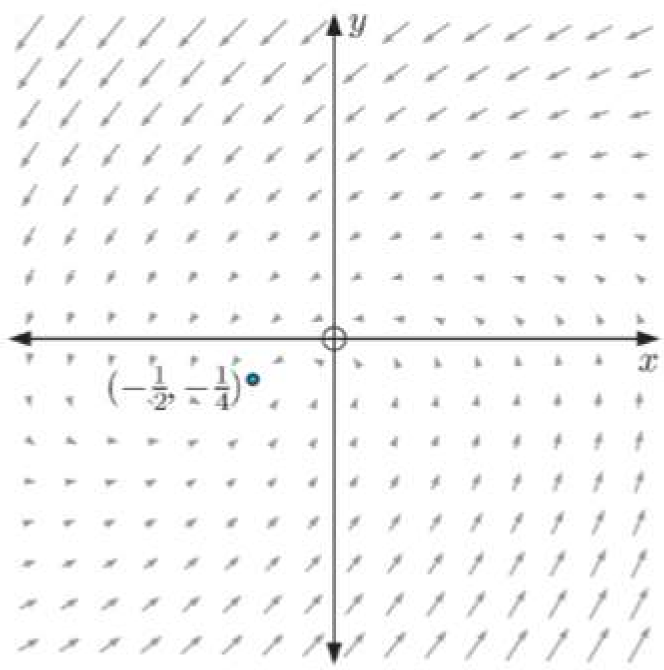
d



We draw some trajectories on the phase portrait.

It appears that $(0,\ 0)$ is an equilibrium point. It is a centre.

4 $\begin{cases} \dfrac{dx}{dt} = -2y - \frac{1}{2} \\ \dfrac{dy}{dt} = x - 2y \end{cases}$



Equilibrium points occur where

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0$$

$$\therefore \quad -2y - \tfrac{1}{2} = 0 \quad \text{and} \quad x - 2y = 0$$

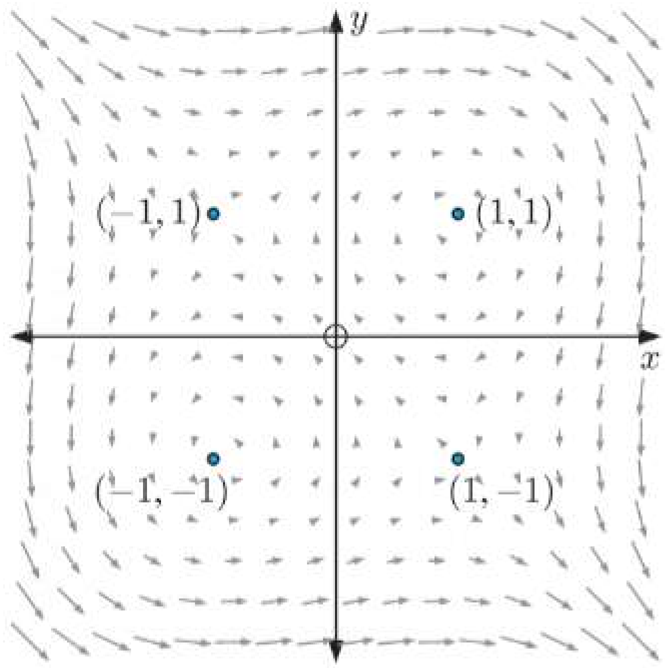
$$\therefore \quad y = -\tfrac{1}{4} \quad \text{and} \quad x = 2y$$

$$\therefore \quad y = -\tfrac{1}{4} \quad \text{and} \quad x = -\tfrac{1}{2}$$

So, $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ is an equilibrium point.

From the phase portrait, we see that it is a stable spiral.

5 $\begin{cases} \dfrac{dx}{dt} = y^2 - 1 \\ \dfrac{dy}{dt} = 1 - x^2 \end{cases}$



Equilibrium points occur where

$$\begin{aligned} \frac{dx}{dt} &= 0 &\quad\text{and}\quad& \frac{dy}{dt} = 0 \\ \therefore\ y^2 - 1 &= 0 &\quad\text{and}\quad& 1 - x^2 = 0 \\ \therefore\ y^2 &= 1 &\quad\text{and}\quad& x^2 = 1 \\ \therefore\ y &= \pm 1 &\quad\text{and}\quad& x = \pm 1 \end{aligned}$$

So, $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ are equilibrium points.

From the phase portrait, we see that $(1, 1)$ and $(-1, -1)$ are centres, and $(1, -1)$ and $(-1, 1)$ are saddle points.

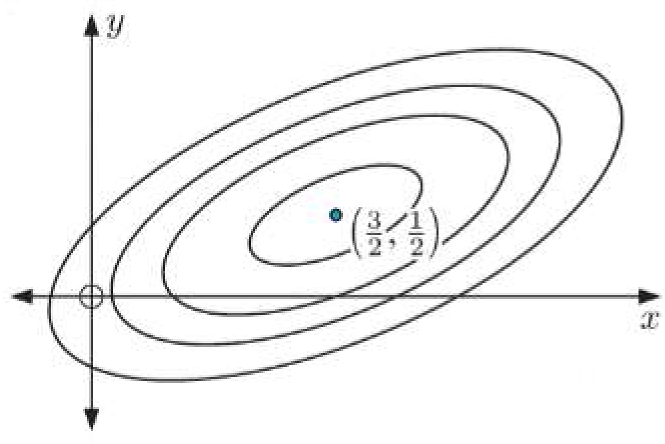
6 $\begin{cases} \dot{x} = x - 3y \\ \dot{y} = x - y - 1 \end{cases}$

a Equilibrium points occur where

$$\begin{aligned} \dot{x} &= 0 &\quad\text{and}\quad& \dot{y} = 0 \\ \therefore\ x - 3y &= 0 &\quad\text{and}\quad& x - y - 1 = 0 \\ \therefore\ x &= 3y &\quad\text{and}\quad& y = x - 1 \\ &&& \therefore\ y = 3y - 1 \\ &&& \therefore\ 2y = 1 \\ &&& \therefore\ y = \tfrac{1}{2} \text{ and } x = \tfrac{3}{2} \end{aligned}$$

So, $\left(\frac{3}{2}, \frac{1}{2}\right)$ is an equilibrium point.

From the phase portrait, we see that $\left(\frac{3}{2}, \frac{1}{2}\right)$ is a centre.



b At $(0, 0)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 - 0 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

On the phase portrait, the vector at $(0, 0)$ has the direction $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

So, the solution curves rotate anticlockwise.

ACTIVITY 1 POPULATION MODELS

1

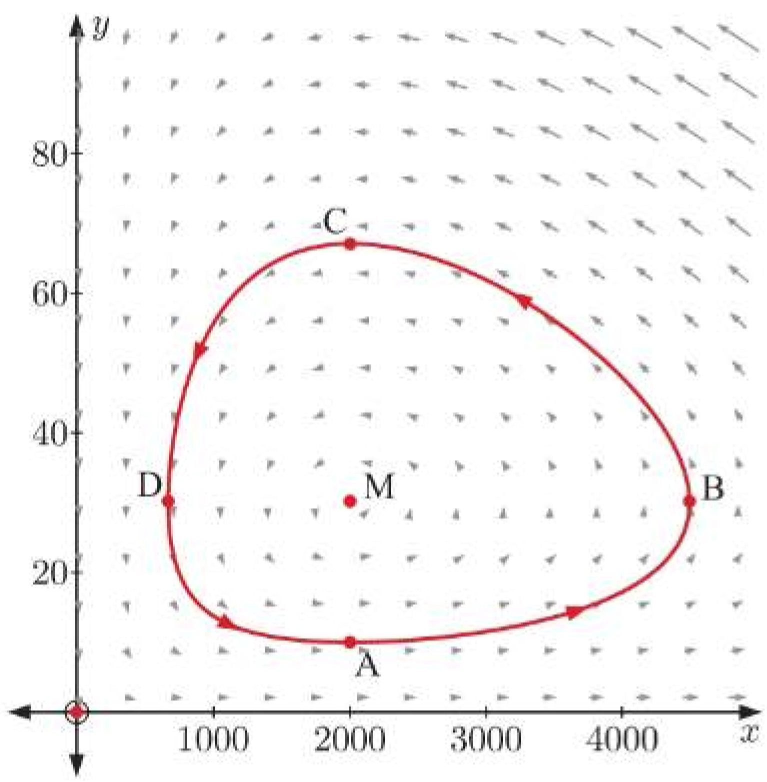
$$\begin{cases} \dfrac{dx}{dt} = \overbrace{ax}^{\text{exponential growth}} \; \overbrace{- \, bxy}^{\text{loss due to predators}} \\[2ex] \dfrac{dy}{dt} = \underbrace{cxy}_{\text{growth with available prey}} \; \underbrace{- \, dy}_{\text{natural death}} \end{cases}$$

where x is the prey,
y is the predator,
and $a, b, c, d > 0$.

a **i** The predator feeds on the prey to survive.
So, the growth term of the predator is affected by how much prey is available, whereas the growth term of the prey is unaffected by the number of predators.
Also, both the prey and the predator die due to natural causes, but the prey also die due to the predator and so their death is affected by the number of predators.

ii If x is constant, then $\dfrac{dy}{dt} = (cx - d)y$ which is an exponential growth model for y.

b $$\begin{cases} \dfrac{dx}{dt} = 30x - xy \\[1ex] \dfrac{dy}{dt} = 0.015xy - 30y \end{cases}$$



i At $(0, 0)$, $\begin{pmatrix} \dfrac{dx}{dt} \\[1ex] \dfrac{dy}{dt} \end{pmatrix}$

$$= \begin{pmatrix} 30(0) - 0(0) \\ 0.015(0)(0) - 30(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ $(0, 0)$ is an equilibrium point.

From the phase portrait, we see that it is a saddle point.

ii The equilibrium point $M(x, y)$ has $x \neq 0$ and $y \neq 0$.

$$\begin{aligned} &\text{Now } \frac{dx}{dt} = 0 && \text{and} && \frac{dy}{dt} = 0 \\ \therefore\ & 30x - xy = 0 && \text{and} && 0.015xy - 30y = 0 \\ \therefore\ & (30 - y)x = 0 && \text{and} && (0.015x - 30)y = 0 \\ &\therefore\ y = 30 && \text{and} && x = 2000 \end{aligned}$$

$\therefore$ the equilibrium point M has coordinates $(2000, 30)$.

From the phase portrait, we see that M is a centre.

When there are 2000 seals and 30 orcas, the populations are at equilibrium. This means the populations of seals and orcas do not change over time.

iii At point B, the population of seals reaches its maximum. With plentiful prey, the number of orcas increase rapidly which in turn decreases the number of seals.

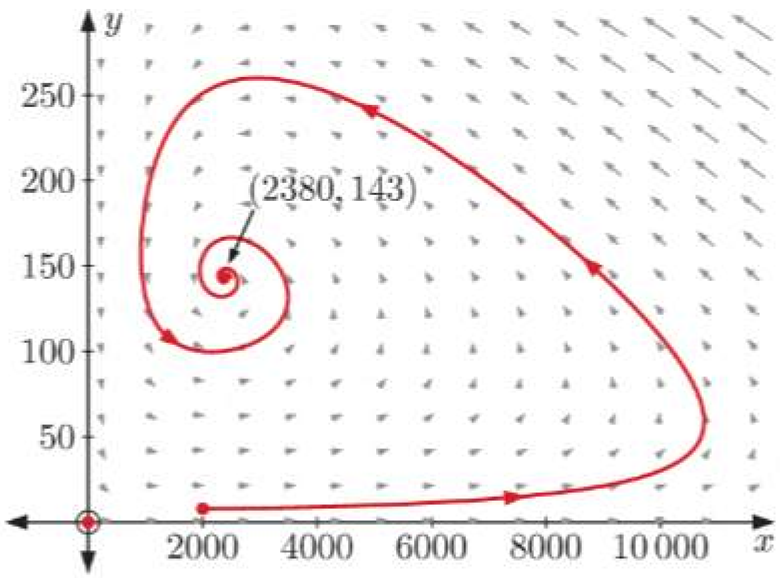
At point C, the population of orcas reaches its maximum. There is not enough prey to sustain all of the predators, so the number of orcas starts to decrease.

At point D, the population of seals reaches its minimum. With so little prey, the number of orcas decreases very rapidly, eventually allowing the population of seals to recover.

2 $\begin{cases} \dfrac{dx}{dt} = ax - bx^2 - cxy \\ \dfrac{dy}{dt} = dxy - ey - fy^2 \end{cases}$

a The new terms $-bx^2$ and $-fy^2$ represent loss from self-competition for resources in the prey and predator, respectively.

b $\begin{cases} \dfrac{dx}{dt} = 5x - 0.0003x^2 - 0.03xy \\ \dfrac{dy}{dt} = 0.0006xy - y - 0.003y^2 \end{cases}$



The equilibrium point (x, y) which is not $(0, 0)$ satisfies $x \neq 0$ and $y \neq 0$.

$$\text{Now} \quad \frac{dx}{dt} = 0$$

$$\therefore \quad 5x - 0.0003x^2 - 0.03xy = 0$$

$$\therefore \quad (5 - 0.0003x - 0.03y)x = 0$$

$$\therefore \quad 5 - 0.0003x - 0.03y = 0 \qquad \{x \neq 0\}$$

$$\therefore \quad x = \frac{50\,000}{3} - 100y \qquad \text{.... } (*)$$

$$\text{Also} \quad \frac{dy}{dt} = 0$$

$$\therefore \quad 0.0006xy - y - 0.003y^2 = 0$$

$$\therefore \quad (0.0006x - 1 - 0.003y)y = 0$$

$$\therefore \quad 0.0006x - 1 - 0.003y = 0 \qquad \{y \neq 0\}$$

$$\therefore \quad 0.0006\left(\frac{50\,000}{3} - 100y\right) - 1 - 0.003y = 0 \qquad \{\text{using } (*)\}$$

$$\therefore \quad 10 - 0.06y - 1 - 0.003y = 0$$

$$\therefore \quad 0.063y = 9$$

$$\therefore \quad y = \frac{1000}{7} \approx 143$$

Substituting into $(*)$, $x = \dfrac{50\,000}{3} - 100\left(\dfrac{1000}{7}\right)$

$$= \frac{50\,000}{21} \approx 2380$$

$\therefore$ the equilibrium point is $\left(\frac{50\,000}{21}, \frac{1000}{7}\right)$ which is about $(2380, 143)$.

From the phase portrait, we see that $(2380, 143)$ is a stable spiral.

Using the trajectory from $(2000, 10)$, we see that the population of the prey and the predator will fluctuate until eventually there are about 2380 prey and 143 predators. At this point the populations of prey and predator do not change over time.

3 $\begin{cases} \dfrac{dx}{dt} = ax - bx^2 - cxy \\ \dfrac{dy}{dt} = dy - ey^2 - fxy \end{cases}$

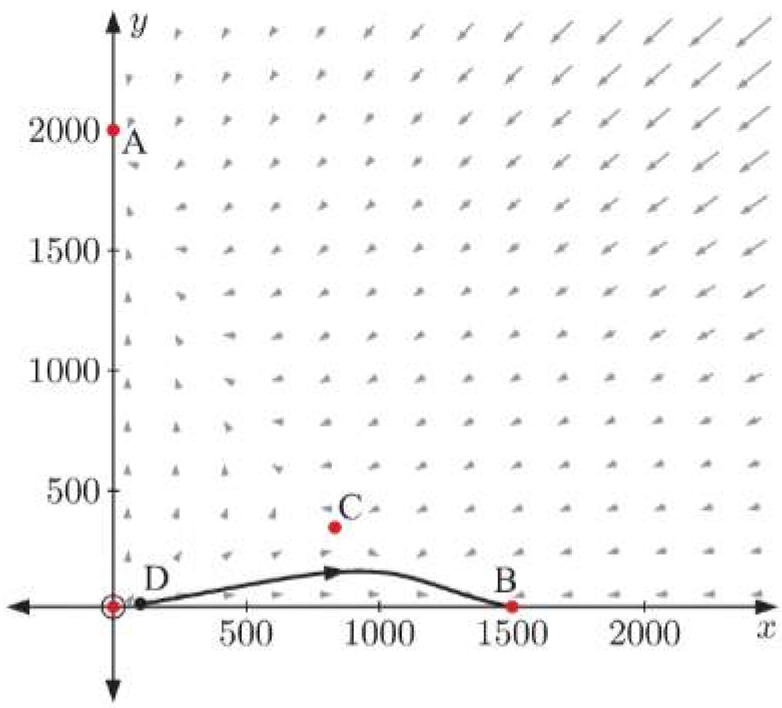
a **i** The terms ax and dy model exponential growth.

ii The terms $-bx^2$ and $-ey^2$ model self-competition.

iii The terms $-cxy$ and $-fxy$ model inter-species competition.

b $\begin{cases} \dfrac{dx}{dt} = 1.5x - 0.001x^2 - 0.002xy \\ \dfrac{dy}{dt} = 2y - 0.001y^2 - 0.002xy \end{cases}$

i Equilibrium points occur where $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0.$



Now $\dfrac{dx}{dt} = 0$ when $1.5x - 0.001x^2 - 0.002xy = 0$

$\therefore \ (1.5 - 0.001x - 0.002y)x = 0$

$\therefore \ x = 0$ or $x = 1500 - 2y$

Also $\dfrac{dy}{dt} = 0$ when $2y - 0.001y^2 - 0.002xy = 0$

$\therefore \ (2 - 0.001y - 0.002x)y = 0$

$\therefore \ y = 0$ or $y = 2000 - 2x$

$\therefore$ O(0, 0) is an equilibrium point. $(x = 0, \ y = 0)$

At O, both species are extinct.

If $x = 0$ and $y = 2000 - 2x$

$\therefore \ y = 2000$

$\therefore$ A(0, 2000) is an equilibrium point.

At A, species x is extinct and species y has a population of 2000.

If $x = 1500 - 2y$ and $y = 0$

$\therefore \ x = 1500$

$\therefore$ B(1500, 0) is an equilibrium point.

At B, species y is extinct and species x has a population of 1500.

If $x = 1500 - 2y$ and $y = 2000 - 2x$

$$\therefore\ y = 2000 - 2(1500 - 2y)$$
$$\therefore\ y = 4y - 1000$$
$$\therefore\ 3y = 1000$$
$$\therefore\ y = \frac{1000}{3} \approx 333$$
$$\text{and}\quad x = 1500 - 2\left(\frac{1000}{3}\right)$$
$$= \frac{2500}{3} \approx 833$$

$\therefore$ $C\left(\frac{2500}{3}, \frac{1000}{3}\right)$ which is about $(833, 333)$ is an equilibrium point.

At C, both species coexist with about 833 from species x and about 333 from species y.

ii From the phase portrait, C is a saddle point.

iii At point D, the population of species x is greater than the population of species y. Thus, species x initially grows faster than species y. As species x grows larger, there are not enough members of species y to compete for resources, so the population of species y declines to extinction. Eventually equilibrium point A is reached with species y extinct and species x with a population of 1500.

iv The only equilibrium point with the population of species x and species y *both* non-zero is C.

From **ii**, C is a saddle point, so there are *two* trajectories which approach C in the long term. Every other trajectory close to C will eventually approach either A or B.

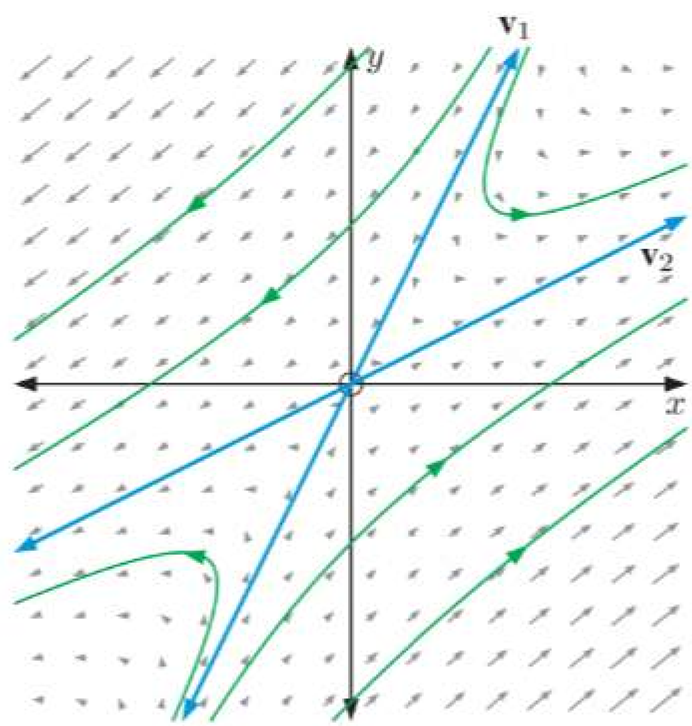
So, in theory it is possible for both species to coexist, but in practice it is almost impossible for this to happen.

INVESTIGATION 1 COUPLED LINEAR DIFFERENTIAL EQUATIONS

1 At $(0, 0)$, $\dot{\mathbf{x}} = \mathbf{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ $(0, 0)$ is the equilibrium point for any system of coupled linear differential equations.

2 $\begin{cases} \dfrac{dx}{dt} = 3x - 2y \\[2mm] \dfrac{dy}{dt} = 2x - 2y \end{cases}$



a The system has matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

where $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

b From **1**, the equilibrium point is $(0, 0)$.
From the phase portrait, we see that it is a saddle point.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{vmatrix} = 0$

$$\therefore \ (\lambda - 3)(\lambda + 2) + 4 = 0$$
$$\therefore \ \lambda^2 - \lambda - 6 + 4 = 0$$
$$\therefore \ \lambda^2 - \lambda - 2 = 0$$
$$\therefore \ (\lambda + 1)(\lambda - 2) = 0$$
$$\therefore \ \lambda = -1 \text{ or } 2$$

The eigenvalues are -1 and 2.

For $\lambda = -1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \ \begin{pmatrix} -4 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \ -2a + b = 0$$

Letting $a = t$, $t \neq 0$ then $b = 2t$

$$\therefore \ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue -1.

For $\lambda = 2$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \ \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \ -a + 2b = 0$$

Letting $b = t$, $t \neq 0$ then $a = 2t$

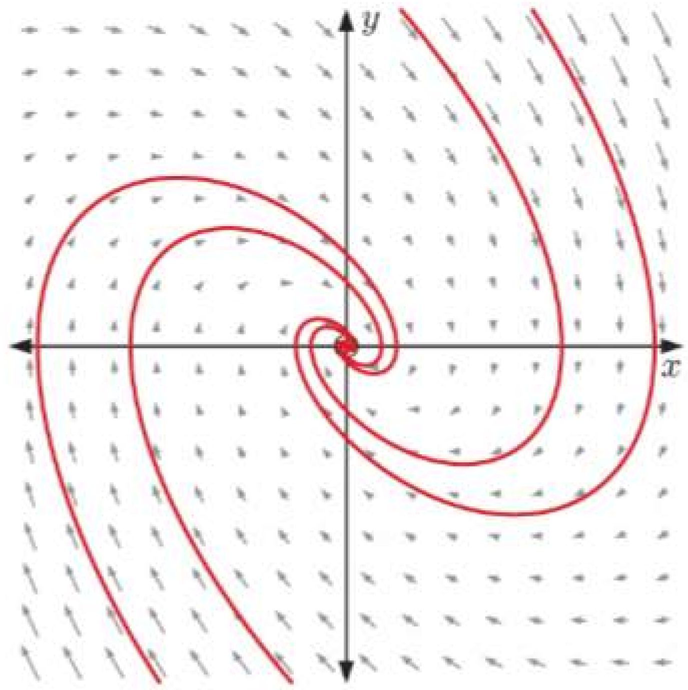
$$\therefore \ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ choosing $t = 1$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 2.

d The trajectories which are straight lines are along the lines $k\mathbf{v}_1$, $k \in \mathbb{R}$ and $k\mathbf{v}_2$, $k \in \mathbb{R}$.

e It appears that no trajectories cross the eigenvectors of $\mathbf{A}$.

3 $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - y \end{cases}$



a The system has matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

b From **1**, the equilibrium point is $(0, 0)$.
Drawing some trajectories near $(0, 0)$, we see that it is a stable spiral.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & -1 \\ 1 & \lambda + 1 \end{vmatrix} = 0$

$$\therefore \lambda(\lambda + 1) + 1 = 0$$

$$\therefore \lambda^2 + \lambda + 1 = 0$$

$$\therefore \lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

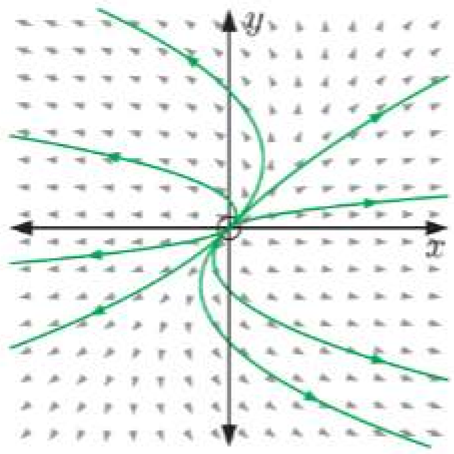
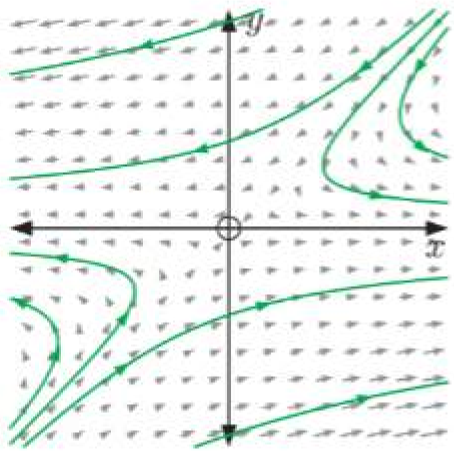
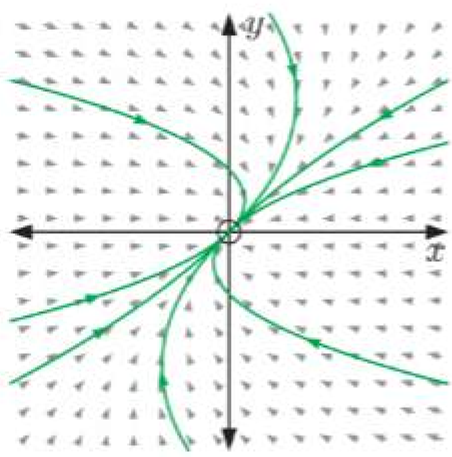
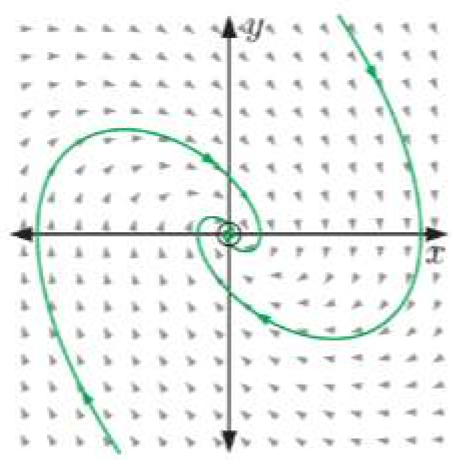
$$= -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i$$

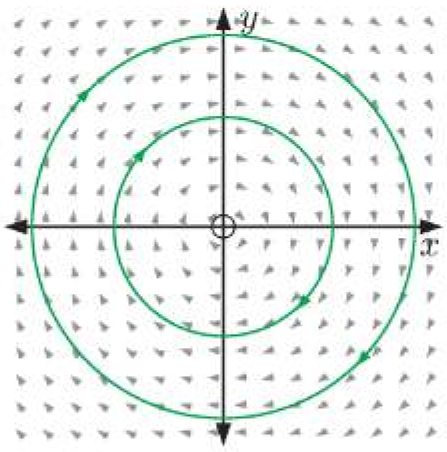
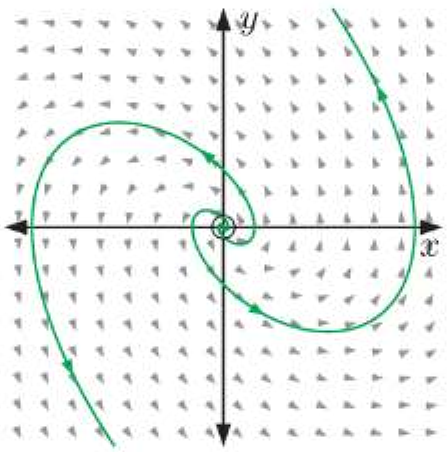
The eigenvalues are $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

d It appears that there are no trajectories which are straight lines.

INVESTIGATION 2 TRAJECTORIES

1

Matrix $\mathbf{A}$	*Eigenvalues*	*Phase portrait*	*Type of equilibrium point*
$\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$	2 and 1		unstable fixed
$\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$	2 and -1		saddle point
$\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$	-1 and -2		stable fixed
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$		stable spiral

Matrix **A**	*Eigenvalues*	*Phase portrait*	*Type of equilibrium point*
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\pm i$		centre
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$		unstable spiral

2 Let λ_1 and λ_2 be the eigenvalues of **A**. The equilibrium point is:

a a stable fixed point if $0 > \lambda_1 > \lambda_2$

b a saddle point if $\lambda_1 > 0 > \lambda_2$

c an unstable fixed point if $\lambda_1 > \lambda_2 > 0$

d a stable spiral if λ_1 and λ_2 are complex with negative real part

e a centre if λ_1 and λ_2 are purely imaginary

f an unstable spiral if λ_1 and λ_2 are complex with positive real part.

3 λ_1, λ_2 are real.

a If $\mathbf{x}$ is a trajectory along $\mathbf{v}$, then $\mathbf{x} = Ae^{\lambda t}\mathbf{v}$.

i If $\lambda > 0$, then $e^{\lambda t} \to \infty$ as $t \to \infty$

$\therefore$ the trajectory $\mathbf{x}$ moves away from O.

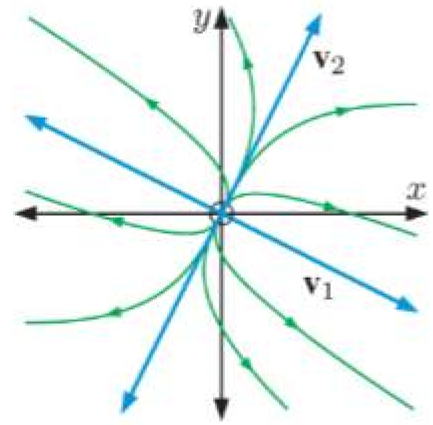
ii If $\lambda < 0$, then $e^{\lambda t} \to 0$ as $t \to \infty$

$\therefore$ the trajectory $\mathbf{x}$ moves towards O.

b $\lambda_1 > \lambda_2 > 0$

i From **2 c**, the equilibrium point is an unstable fixed point.

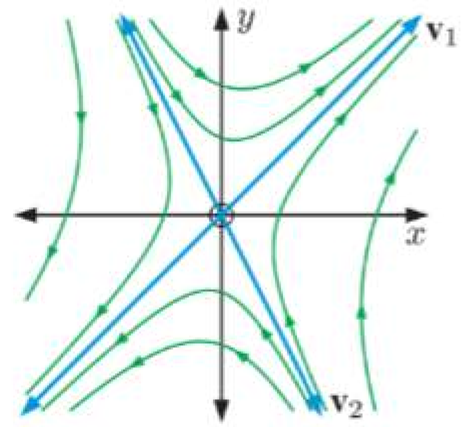
ii As $\lambda_1 > \lambda_2 > 0$, $e^{\lambda_1 t} > e^{\lambda_2 t}$ for large t.
The trajectories become parallel to $\mathbf{v}_1$ as $t \to \infty$.



c $\lambda_1 > 0 > \lambda_2$

i From **2 b**, the equilibrium point is a saddle point.

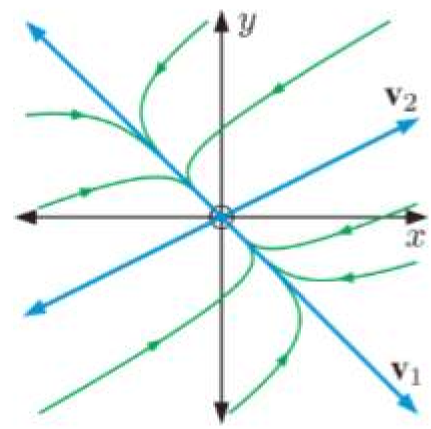
ii As $t \to \infty$, $e^{\lambda_1 t} \to \infty$ and $e^{\lambda_2 t} \to 0$.
The trajectories approach $k\mathbf{v}_1$, $k \in \mathbb{R}$ as $t \to \infty$.



d $0 > \lambda_1 > \lambda_2$

i From **2 a**, the equilibrium point is a stable fixed point.

ii As $0 > \lambda_1 > \lambda_2$, $e^{\lambda_1 t} > e^{\lambda_2 t}$ for large t.
$\therefore$ $e^{\lambda_2 t}$ tends to 0 faster.
The trajectories approach O along $k\mathbf{v}_1$, $k \in \mathbb{R}$.



4 $\lambda_1, \lambda_2 = \alpha \pm i\theta$

The general solution is $\mathbf{x} = e^{\alpha t}[Ae^{i\theta t}\,\mathbf{v}_1 + Be^{-i\theta t}\,\mathbf{v}_2]$.

a If $\alpha < 0$ then $0 < e^{\alpha} < 1$
$\therefore$ as $t \to \infty$, $e^{\alpha t} \to 0$
So, the trajectories spiral inwards.

b If $\alpha = 0$, then $e^{\alpha t} = 1$ for all t
$\therefore$ $\mathbf{x} = Ae^{i\theta t}\,\mathbf{v}_1 + Be^{-i\theta t}\,\mathbf{v}_2$ for all t.
So, the trajectories are circular.

c If $\alpha > 0$, then $e^{\alpha} > 1$
$\therefore$ as $t \to \infty$, $e^{\alpha t} \to \infty$
So, the trajectories spiral outwards.

EXERCISE 26B

1 **a** Using the given eigenvalues and eigenvectors, the general solution to the system is

$$\mathbf{x} = Ae^{-t}\begin{pmatrix}2\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

b **i** When $t=0$, $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

$$\therefore\ \dot{\mathbf{x}} = \begin{pmatrix}0 & -2\\-1 & 1\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}-6\\0\end{pmatrix}$$

ii When $t=0$, $\mathbf{x} = \begin{pmatrix}3\\3\end{pmatrix}$

$$\therefore\ A\begin{pmatrix}2\\1\end{pmatrix} + B\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}^{-1}\begin{pmatrix}3\\3\end{pmatrix}$$

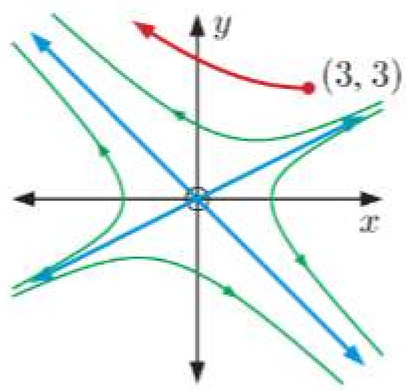
$$\therefore\ \begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

The particular solution is $\mathbf{x} = 2e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + e^{2t}\begin{pmatrix}-1\\1\end{pmatrix}$.

c The eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$ are real and satisfy $\lambda_2 > 0 > \lambda_1$.

$\therefore$ the equilibrium point at O is a saddle point.

d



e As $t \to \infty$, $e^{-t} \to 0$

$$\therefore\ \mathbf{x} \to e^{2t}\begin{pmatrix}-1\\1\end{pmatrix}$$

$\therefore$ the line $k\begin{pmatrix}-1\\1\end{pmatrix}$ will be an asymptote as $t \to \infty$.

2 **a** The eigenvalues $\pm 3i$ are purely imaginary.

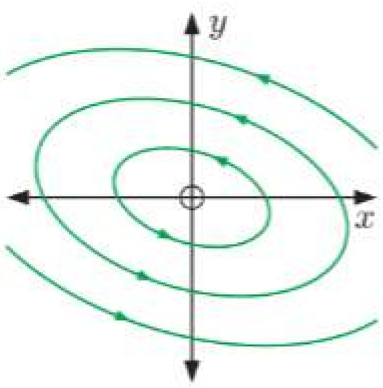
$\therefore$ the equilibrium point at O is a centre.

b **i** At $(1, 0)$, $\begin{pmatrix}\dot{x}\\\dot{y}\end{pmatrix} = \begin{pmatrix}-1-5(0)\\2(1)+0\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

$\therefore$ the trajectory at the point $(1, 0)$ is $\begin{pmatrix}-1\\2\end{pmatrix}$.

ii Using **i**, the trajectories rotate anticlockwise.

c



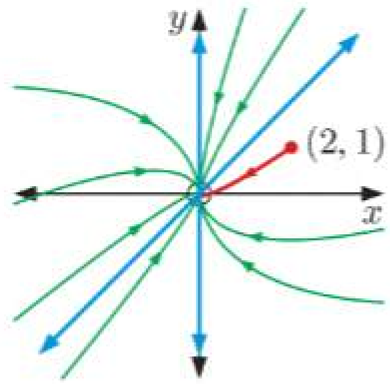
3 **a** The eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -2$ are real and satisfy $0 > \lambda_2 > \lambda_1$.

$\therefore$ the equilibrium point at O is a stable fixed point.

When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ the initial trajectory is

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & 0 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -4 \end{pmatrix}.$$



b The eigenvalues are complex with positive real part.

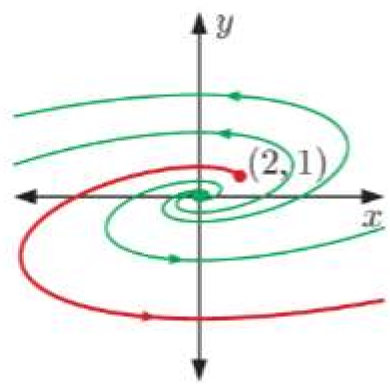
$\therefore$ the equilibrium point at O is an unstable spiral.

When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ the initial trajectory is

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & -6 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

So the rotation is anticlockwise.



4 $\begin{cases} \dfrac{dx}{dt} = -5x + 7y \\ \dfrac{dy}{dt} = -2x + y \end{cases}$ can be written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} -5 & 7 \\ -2 & 1 \end{pmatrix}$.

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda + 5 & -7 \\ 2 & \lambda - 1 \end{vmatrix} = 0$

$$\therefore\ (\lambda + 5)(\lambda - 1) + 14 = 0$$
$$\therefore\ \lambda^2 + 4\lambda - 5 + 14 = 0$$
$$\therefore\ \lambda^2 + 4\lambda + 9 = 0$$
$$\therefore\ \lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(9)}}{2} = \frac{-4 \pm \sqrt{-20}}{2} = \frac{-4 \pm 2i\sqrt{5}}{2} = -2 \pm i\sqrt{5}$$

The eigenvalues are $-2 \pm i\sqrt{5}$.

b The eigenvalues are complex with negative real part.

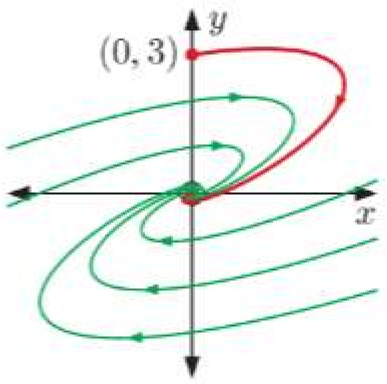
$\therefore$ the equilibrium point at O is a stable spiral.

c When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\therefore$ the initial trajectory is $\dot{\mathbf{x}} = \begin{pmatrix} -5 & 7 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \end{pmatrix}$

So the rotation is clockwise.

d



5 $\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = x - 3y \end{cases}$ can be written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} -3 & 4 \\ 1 & -3 \end{pmatrix}$.

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda+3 & -4 \\ -1 & \lambda+3 \end{vmatrix} = 0$

$$\therefore\ (\lambda+3)^2 - 4 = 0$$
$$\therefore\ (\lambda+3)^2 = 4$$
$$\therefore\ \lambda + 3 = \pm 2$$
$$\therefore\ \lambda = -1 \text{ or } -5$$

The eigenvalues are -1 and -5.

For $\lambda_1 = -1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$ where $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -a + 2b = 0$$

Letting $b = t$, $t \neq 0$ then $a = 2t$

$\therefore$ $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$.

For $\lambda_2 = -5$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$ where $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore\ \begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ -a - 2b = 0$$

Letting $b = t,\ t \neq 0$ then $a = -2t$

$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} t, \quad t \neq 0$

$\therefore$ choosing $t = 1$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -5$.

b The eigenvalues λ_1 and λ_2 are real and satisfy $0 > \lambda_1 > \lambda_2$.

$\therefore$ the equilibrium point at O is a stable fixed point.

c **i** When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore \dot{\mathbf{x}} = \begin{pmatrix} -3 & 4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 18 \\ -11 \end{pmatrix}$

ii Using the eigenvalues and eigenvectors in **a**, the general solution to the system is

$\mathbf{x} = Ae^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + Be^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

When $t = 0$, we know $\mathbf{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore A\begin{pmatrix} 2 \\ 1 \end{pmatrix} + B\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\therefore \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

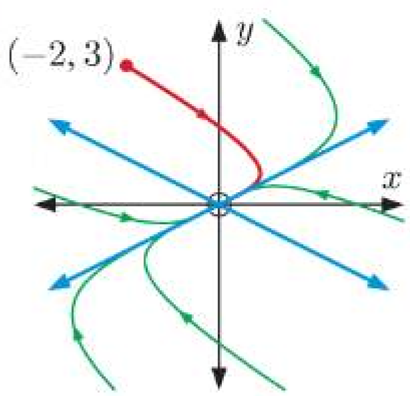
The particular solution is $\mathbf{x} = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

d When $t = 1$, $\mathbf{x} = e^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2e^{-5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2e^{-1} - 4e^{-5} \\ e^{-1} + 2e^{-5} \end{pmatrix} \approx \begin{pmatrix} 0.709 \\ 0.381 \end{pmatrix}$

$\therefore$ the particular solution is at about $(0.709, 0.381)$ when $t = 1$.

e



f $0 > \lambda_1 > \lambda_2$

$\therefore$ the trajectory approaches O along $k\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $k \in \mathbb{R}$, as $t \to \infty$.

6 **a** **i** V is proportional to $\frac{dI}{dt}$, and acts in the opposite direction.

$\therefore \quad V = -L\,\frac{dI}{dt}$ for some "inductance" L.

ii I is proportional to $\frac{dV}{dt}$.

$\therefore \quad I = C\,\frac{dV}{dt}$ for some "capacitance" C.

b $L = 4 \times 10^{-2}$ H and $C = 10^{-5}$ F

i From **a**, $\begin{cases} \frac{dI}{dt} = -25V \\ \frac{dV}{dt} = 10^5 I \end{cases}$

$$\therefore \quad \begin{pmatrix} \frac{dI}{dt} \\ \frac{dV}{dt} \end{pmatrix} = \mathbf{A}\begin{pmatrix} I \\ V \end{pmatrix} \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 0 & -25 \\ 10^5 & 0 \end{pmatrix}.$$

ii If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & 25 \\ -10^5 & \lambda \end{vmatrix} = 0$

$$\therefore \quad \lambda^2 + 2\,500\,000 = 0$$
$$\therefore \quad \lambda^2 = -2\,500\,000$$
$$\therefore \quad \lambda = \pm 500i\sqrt{10}$$

The eigenvalues are $\pm 500i\sqrt{10}$.

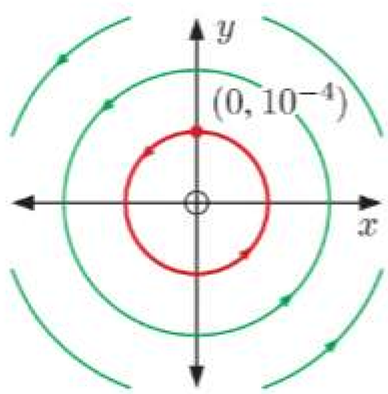
c **i** The eigenvalues $\pm 500i\sqrt{10}$ are purely imaginary.

$\therefore$ the equilibrium point at O is a centre.

When $t = 0$, $\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 10^{-4} \end{pmatrix}$

$$\therefore \quad \text{the initial trajectory is} \quad \begin{pmatrix} \frac{dI}{dt} \\ \frac{dV}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -25 \\ 10^5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 10^{-4} \end{pmatrix} = \begin{pmatrix} -0.0025 \\ 0 \end{pmatrix}$$

So the rotation is anticlockwise.



ii Over time the voltage and current oscillate, such that when one is zero, the other is at its maximum or minimum value.

ACTIVITY 2 EQUILIBRIUM REACTIONS

$2\,SO_2 + O_2 \rightleftharpoons 2\,SO_3$

1 In the forward direction, the amount of SO_3 formed from SO_2 is $0.04D$.
In the reverse direction, the amount of SO_3 broken down into SO_2 is $0.01T$.

$\therefore$ the *change* in the amount of SO_3 is $\dfrac{dT}{dt} = 0.04D - 0.01T$.

Now, as SO_3 is *formed*, SO_2 is *lost*, and as SO_3 is *lost*, SO_2 is *formed*.

$$\therefore \quad \frac{dD}{dt} = -\frac{dT}{dt} = -0.04D + 0.01T$$

Thus, the system can be written as $\begin{cases} \dfrac{dD}{dt} = -0.04D + 0.01T \\ \dfrac{dT}{dt} = 0.04D - 0.01T. \end{cases}$

2 The system has matrix form $\begin{pmatrix} \dot{D} \\ \dot{T} \end{pmatrix} = \mathbf{A}\begin{pmatrix} D \\ T \end{pmatrix}$ where $\mathbf{A} = \begin{pmatrix} -0.04 & 0.01 \\ 0.04 & -0.01 \end{pmatrix}$.

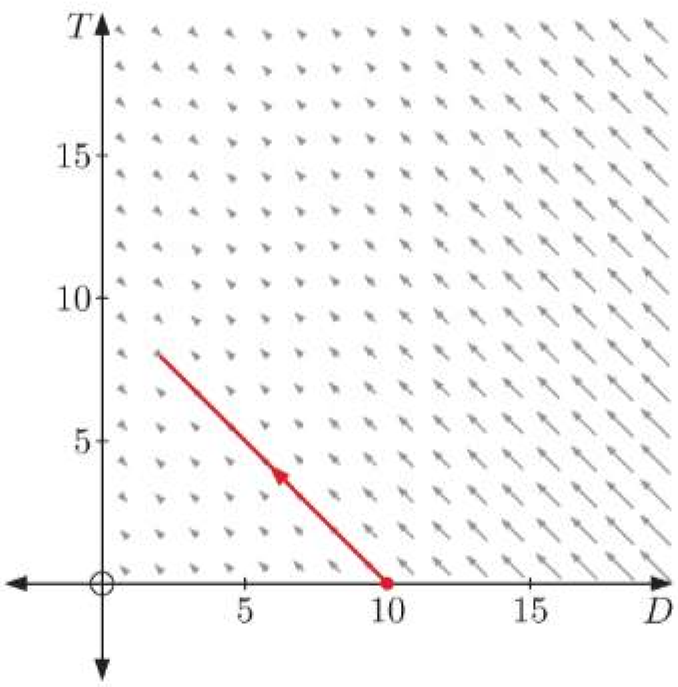
3 If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda + 0.04 & -0.01 \\ -0.04 & \lambda + 0.01 \end{vmatrix} = 0$

$$\therefore \quad (\lambda + 0.04)(\lambda + 0.01) - 0.0004 = 0$$
$$\therefore \quad \lambda^2 + 0.05\lambda + 0.0004 - 0.0004 = 0$$
$$\therefore \quad \lambda(\lambda + 0.05) = 0$$
$$\therefore \quad \lambda = 0 \text{ or } -0.05$$

The eigenvalues are 0 and -0.05.

4 When $t = 0$, $\begin{pmatrix} D \\ T \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

$$\therefore \quad \begin{pmatrix} \dot{D} \\ \dot{T} \end{pmatrix} = \begin{pmatrix} -0.04 & 0.01 \\ 0.04 & -0.01 \end{pmatrix}\begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.4 \\ 0.4 \end{pmatrix}$$



5 **a** From the reaction equation, one molecule of SO_2 is used to create one molecule of SO_3 and vice versa.
So, the total amount of molecules in the system must remain constant.
$\therefore$ $D+T=c$, where c is a constant, which is the equation of a straight line.
Thus, all trajectories on the diagram must be straight lines of the form $D+T=c$.

b The forward direction occurs 4 times faster than the reverse direction.
So, in the long term, the amount of SO_3 will be 4 times the amount of SO_2.
Thus, the system will always tend to an equilibrium point of the form $(d, 4d)$.
We can also observe in the system of coupled differential equations that when $T=4D$, $\dfrac{dD}{dt}=\dfrac{dT}{dt}=0$.

6 From **5 a** and **5 b**, the equilibrium point must be the intersection of the lines $D+T=c$ and $T=4D$.
So, the initial amount of molecules in the system determines the equilibrium point to which the system tends.

EXERCISE 26C

1 $\dfrac{d^2x}{dt^2}+\omega^2x=0$

a Let $y=\dfrac{dx}{dt}$

$\therefore \dfrac{dy}{dt}=\dfrac{d^2x}{dt^2}$

$\therefore$ the system is $\begin{cases}\dfrac{dx}{dt}=y\\ \dfrac{dy}{dt}=-\omega^2x.\end{cases}$

b The system has matrix form $\dot{\mathbf{x}}=\begin{pmatrix}0 & 1\\ -\omega^2 & 0\end{pmatrix}\mathbf{x}$.

2 $\dfrac{d^2x}{dt^2}+k\,\dfrac{dx}{dt}+\omega^2x=0, \quad k\geqslant 0$

a Let $y=\dfrac{dx}{dt}$

$\therefore \dfrac{dy}{dt}=\dfrac{d^2x}{dt^2}$

$\therefore \dfrac{dy}{dt}+ky+\omega^2x=0$

$\therefore$ the system is $\begin{cases}\dfrac{dx}{dt}=y\\ \dfrac{dy}{dt}=-\omega^2x-ky.\end{cases}$

b The system has matrix form $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{pmatrix}0 & 1\\ -\omega^2 & -k\end{pmatrix}$.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & -1 \\ \omega^2 & \lambda + k \end{vmatrix} = 0$

$$\therefore\ \lambda(\lambda + k) + \omega^2 = 0$$

$$\therefore\ \lambda^2 + k\lambda + \omega^2 = 0$$

$$\therefore\ \lambda = \frac{-k \pm \sqrt{k^2 - 4(1)(\omega^2)}}{2} = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$$

The eigenvalues are $\dfrac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$.

d If $k = 0$, then the eigenvalues are $\pm\dfrac{\sqrt{-4\omega^2}}{2} = \pm i\omega$, which are purely imaginary.

$\therefore$ the equilibrium point at O is a centre.

So, the spring oscillates indefinitely.

e If $0 < k < 2\omega$, then $k^2 - 4\omega^2 < 0$, and the eigenvalues $\dfrac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$ are complex with negative real part.

$\therefore$ the equilibrium point at O is a stable spiral.

So, the spring eventually comes to rest.

3 $m\ddot{x} = -mg - k(\dot{x})^2$, $g \approx 9.8$, $k > 0$

Let $v = \dot{x}$

$\therefore\ \dot{v} = \ddot{x}$

$\therefore\ m\dot{v} = -mg - kv^2$

$\therefore$ the system is $\begin{cases} \dot{x} = v \\ \dot{v} = -g - \dfrac{k}{m}v^2. \end{cases}$

4 $50\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 0$

a Let $y = \dfrac{dx}{dt}$

$\therefore\ \dfrac{dy}{dt} = \dfrac{d^2x}{dt^2}$

$\therefore$ the system is $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\frac{1}{50}y. \end{cases}$

b
$$\frac{dy}{dt} = -\tfrac{1}{50}y$$

$$\therefore\ \frac{1}{y}\frac{dy}{dt} = -\tfrac{1}{50}$$

$$\therefore\ \int \frac{1}{y}\,dy = \int -\tfrac{1}{50}\,dt$$

$$\therefore\ \ln|y| = -\tfrac{1}{50}t + c$$

$$\therefore\ y = \pm e^{-\frac{t}{50}+c}$$

$$\therefore\ y(t) = Ae^{-\frac{t}{50}} \qquad \{A = \pm e^c\}$$

c $\dfrac{dx}{dt} = y = Ae^{-\frac{t}{50}}$ {using **b**}

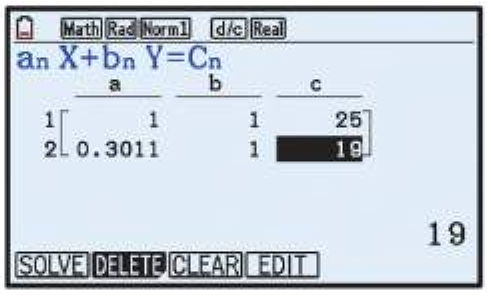
$\therefore\ x(t) = -50Ae^{-\frac{t}{50}} + c,$ for constants A, c

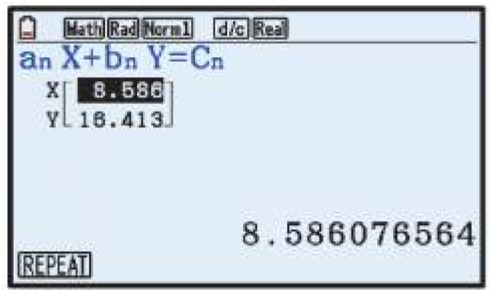
d **i** Let $B = -50A$ $\therefore\ x(t) = Be^{-\frac{t}{50}} + c$

Now $x(0) = 25$ and $x(60) = 19$

$\therefore\ B + c = 25$ .... (1) $\therefore\ Be^{-\frac{6}{5}} + c = 19$ .... (2)

Solving (1) and (2) simultaneously using technology, we find that $B \approx 8.59$ and $c \approx 16.4$





The particular solution is $x(t) \approx 8.59e^{-\frac{t}{50}} + 16.4$.

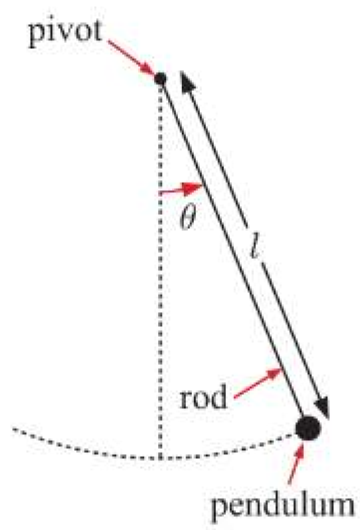
ii As $t \to \infty,\ e^{-\frac{t}{50}} \to 0$

$\therefore\ x(t) \to 16.4$

In the long term, the temperature of the drink will approach about 16.4°C.

ACTIVITY 3 THE SIMPLE RIGID PENDULUM

1 $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{l}\sin\theta$ and $v = \dfrac{d\theta}{dt}$



a **i** If $v < 0$, the angle is *decreasing*.

$\therefore$ the pendulum is moving clockwise.

ii If $v = 0$, the angle is not changing.

$\therefore$ the pendulum is at rest.

iii If $v > 0$, the angle is *increasing*.

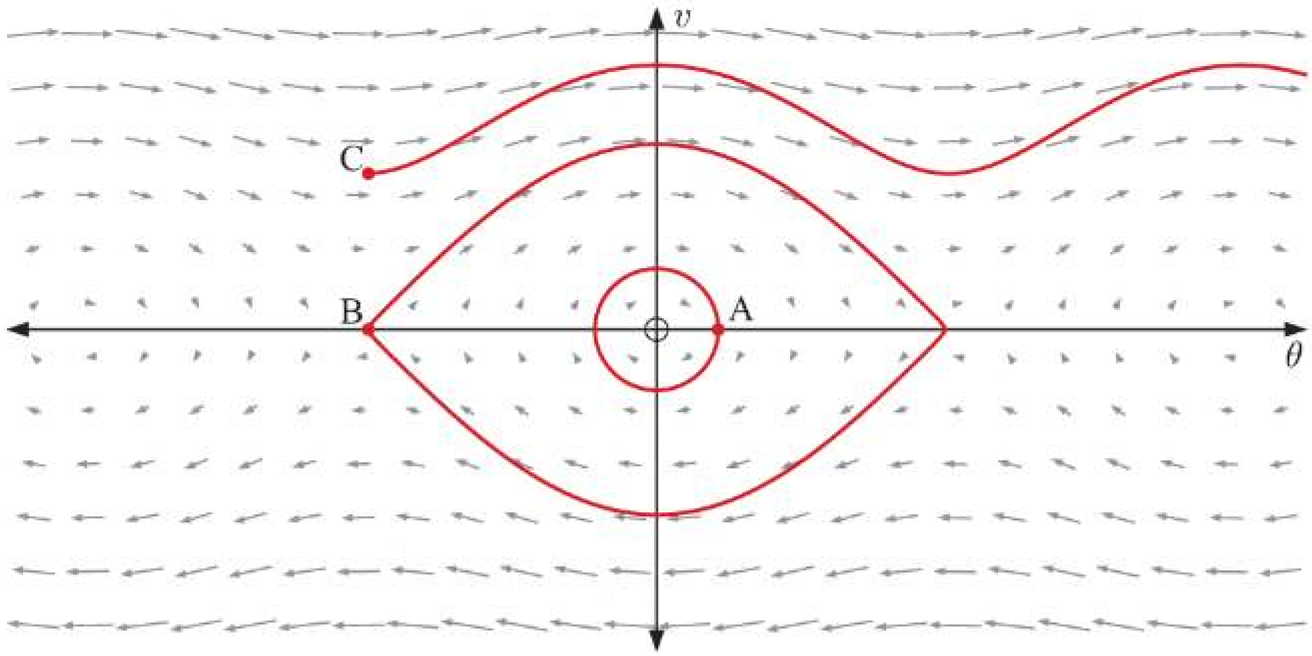
$\therefore$ the pendulum is moving anticlockwise.

b $v = \dfrac{d\theta}{dt}$

$\therefore\ \dfrac{dv}{dt} = \dfrac{d^2\theta}{dt^2}$

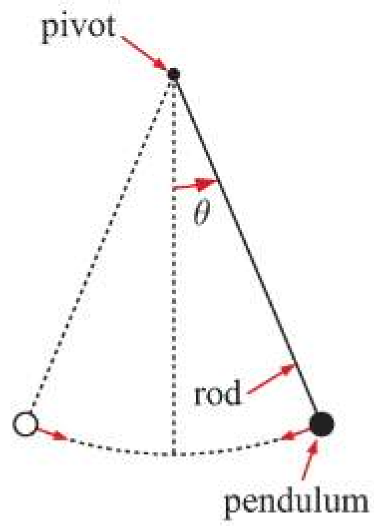
$\therefore$ the system is $\begin{cases} \dfrac{d\theta}{dt} = v \\ \dfrac{dv}{dt} = -\dfrac{g}{l}\sin\theta. \end{cases}$

c



i From A, the pendulum first moves clockwise until it comes to rest, then moves anticlockwise until it comes to rest at the initial position A.

It continues to oscillate in this manner indefinitely.

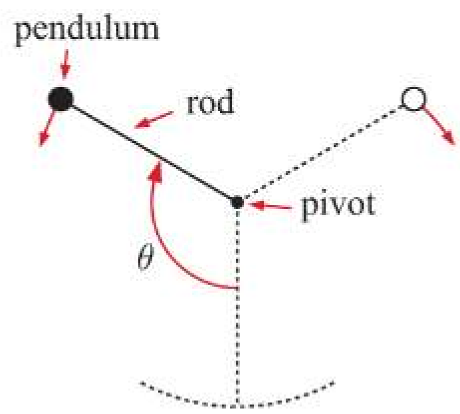


ii At initial point B, the pendulum is released from rest from a position with a large negative angle.

From B, the pendulum first moves anticlockwise until it comes to rest, then moves clockwise until it comes to rest at the initial position B.

It continues to oscillate in this manner indefinitely.

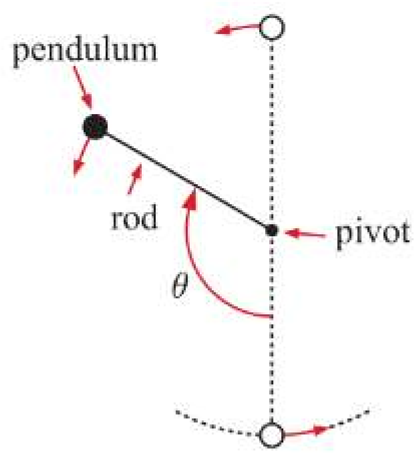
The maximum angular velocity obtained is greater than in **i**.



iii At initial point C, the pendulum is released with large positive angular velocity from a position with a large negative angle.

From C, the pendulum first moves anticlockwise reaching its maximum angular velocity as it passes through its resting position. It continues moving anticlockwise, reaching its minimum angular velocity as it moves through the point directly above the pivot.

It then continues to move anticlockwise around the pivot, and behaves in this manner indefinitely.



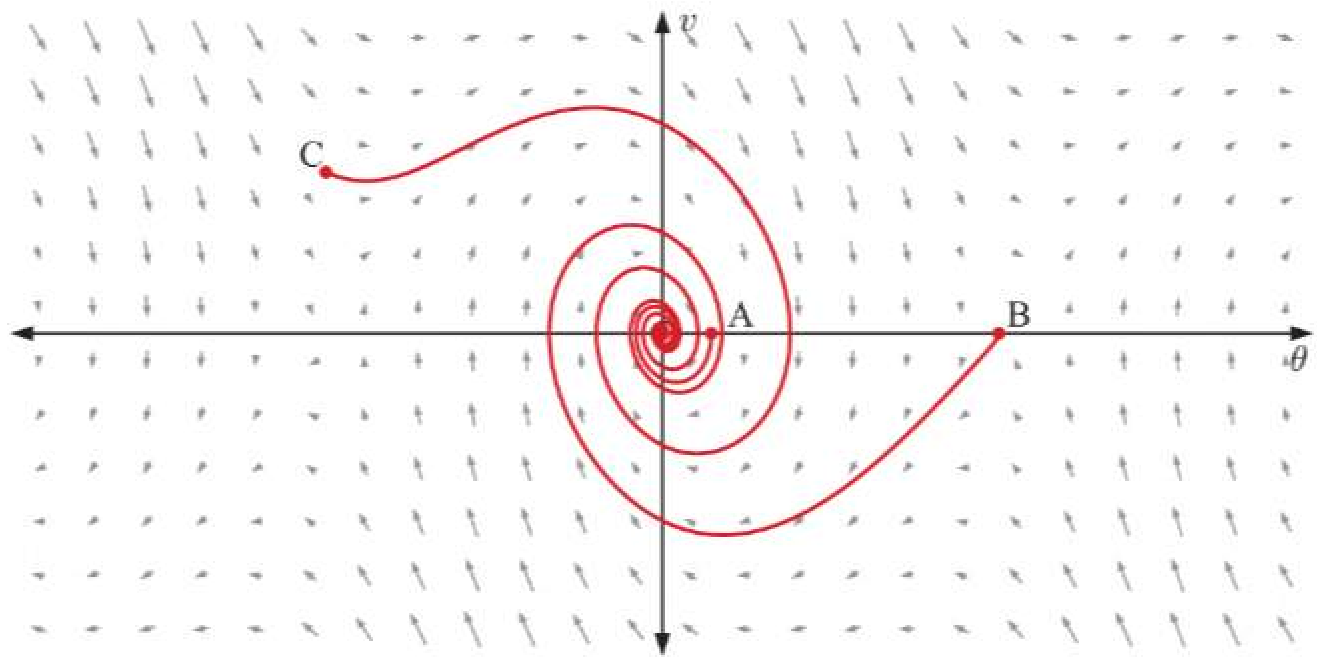
2 $\dfrac{d^2\theta}{dt^2} + k\,\dfrac{d\theta}{dt} = -\dfrac{g}{l}\,\sin\theta$

a Let $v = \dfrac{d\theta}{dt}$

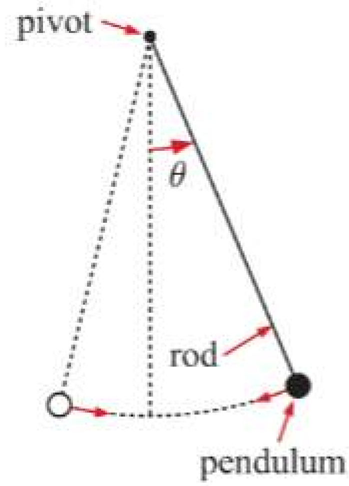
$\therefore\ \dfrac{dv}{dt} = \dfrac{d^2\theta}{dt^2}$

$\therefore$ the system is $\begin{cases} \dfrac{d\theta}{dt} = v \\ \dfrac{dv}{dt} = -\dfrac{g}{l}\,\sin\theta - kv. \end{cases}$

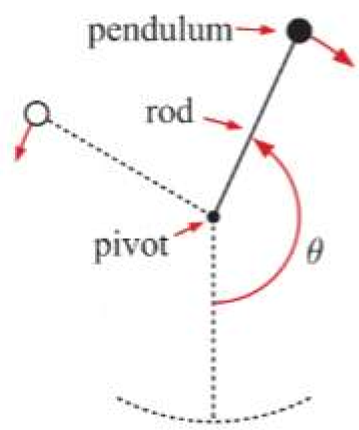
b



At initial position A, the pendulum is released from rest from a position with a small positive angle. It oscillates back and forth as in **1 c i**, but the angle is reduced each oscillation until eventually it comes to rest.

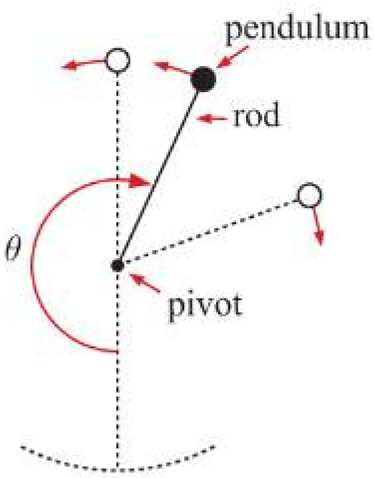


At initial point B, the pendulum is released from rest from a position with a large positive angle. It oscillates back and forth, as above, until it eventually comes to rest.



At initial point C, the pendulum is released with large positive angular velocity from a position with a large negative angle.

From C, the pendulum first moves anticlockwise, slowing down as it continues through the point directly above the pivot. It then oscillates back and forth, until it eventually comes to rest.



EXERCISE 26D

1 $\dfrac{d^2x}{dt^2} + 9x = 0$

a If $x(t) = 2\cos 3t$, then $\dfrac{dx}{dt} = -6\sin 3t$

$$\therefore \quad \frac{d^2x}{dt^2} = -18\cos 3t$$

$$\therefore \quad \frac{d^2x}{dt^2} = -9(2\cos 3t)$$

$$\therefore \quad \frac{d^2x}{dt^2} = -9x$$

$$\therefore \quad \frac{d^2x}{dt^2} + 9x = 0 \quad \checkmark$$

Also $x(0) = 2\cos 0 = 2$ ✓

and at $t = 0$, $\dfrac{dx}{dt} = -6\sin 0 = 0$ ✓

$\therefore$ $x(t) = 2\cos 3t$ is a particular solution which satisfies the initial conditions.

b Let $\dfrac{dx}{dt} = y$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{dy}{dt}$$

$\therefore$ the system is $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -9x. \end{cases}$

c Using step size h, $t_i = t_{i-1} + h$,

$$x_i = x_{i-1} + hy_{i-1},$$

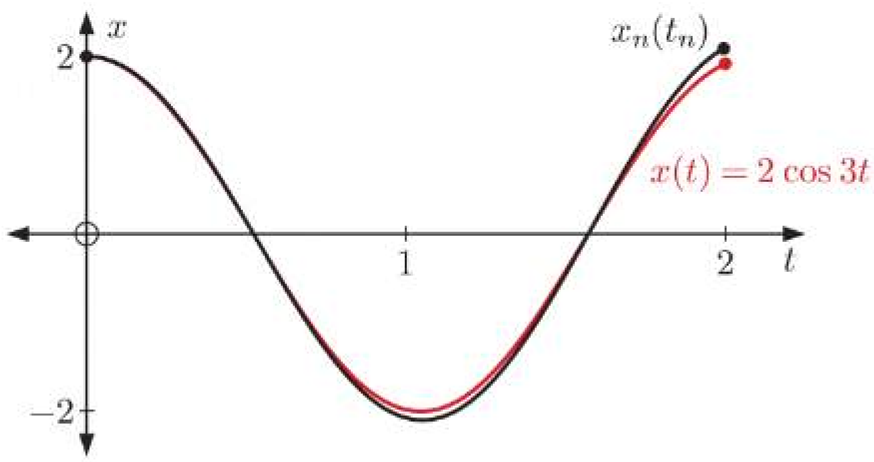
and $y_i = y_{i-1} + h(-9x_{i-1}) = y_{i-1} - 9hx_{i-1}$.

d At time $t_0 = 0$, $x_0 = 2$ and $y_0 = 0$.

Using **c** with step size $h = 0.01$, $x_1 = 2 + 0.01(0) = 2$

and $y_1 = 0 - 9(0.01)(2) = -0.18$.

e



With $h = 0.01$, the points (t_i, x_i) approximate a smooth curve similar to the analytic solution $x(t) = 2\cos 3t$. Euler's method is less accurate at times when the curve changes gradient faster.

2 $\begin{cases} \dfrac{dx}{dt} = 2x - xy \\ \dfrac{dy}{dt} = xy - y \end{cases}$

a Using step size h, $t_i = t_{i-1} + h$,

$x_i = x_{i-1} + h(2x_{i-1} - x_{i-1}\,y_{i-1})$,

and $y_i = y_{i-1} + h(x_{i-1}\,y_{i-1} - y_{i-1})$

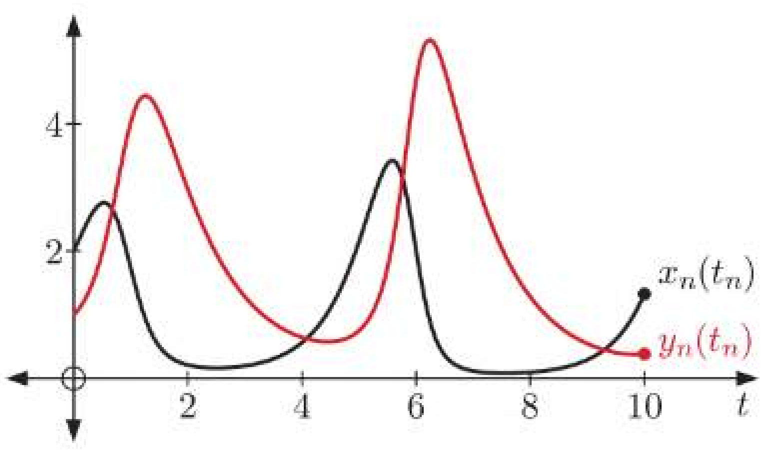
b At time $t_0 = 0$, $x_0 = 2$ and $y_0 = 1$

Using **a** with $h = 0.05$, $t_i = t_{i-1} + 0.05$,

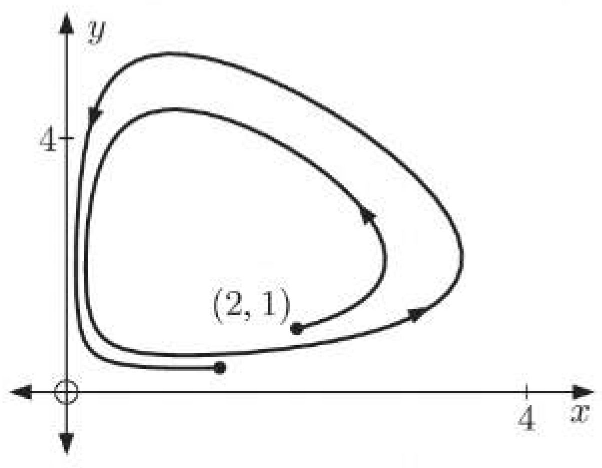
$x_i = x_{i-1} + 0.05(2x_{i-1} - x_{i-1}\,y_{i-1})$,

and $y_i = y_{i-1} + 0.05(x_{i-1}\,y_{i-1} - y_{i-1})$.

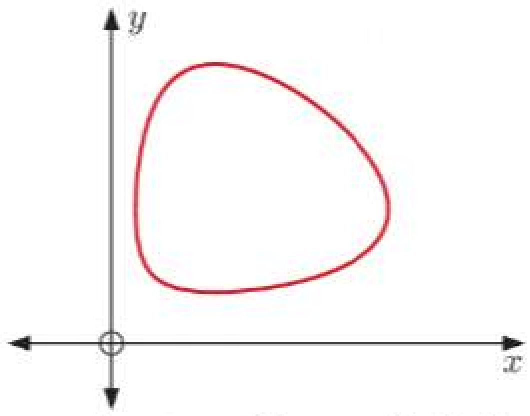
i



ii



iii The exact solution indicates that the system repeats itself in a closed loop, whereas the numerical solution moves in an unstable spiral. The numerical solution appears to suggest one or both of the populations will become extinct in the long term, which is not the case.



$x - \ln x + y - 2\ln y = 3 - \ln 2$

3 $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 1.5(1-x^2)y - x \end{cases}$

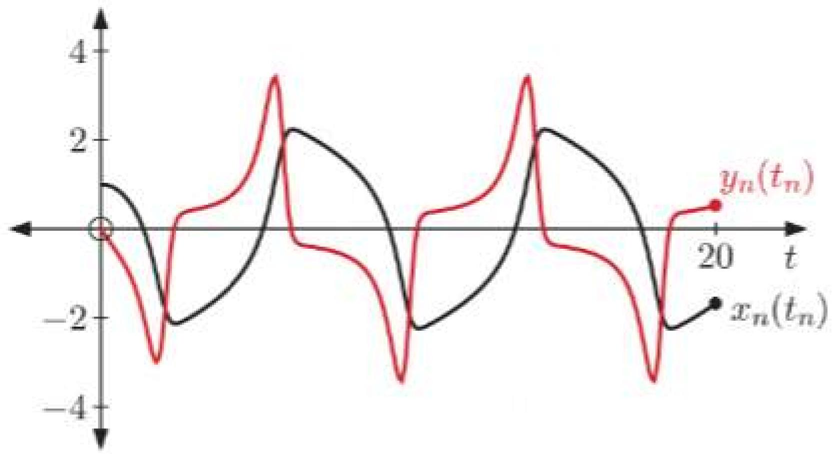
At time $t_0 = 0$, $x_0 = 1$ and $y_0 = 0$.

Using step size $h = 0.1$, $t_i = t_{i-1} + 0.1$,

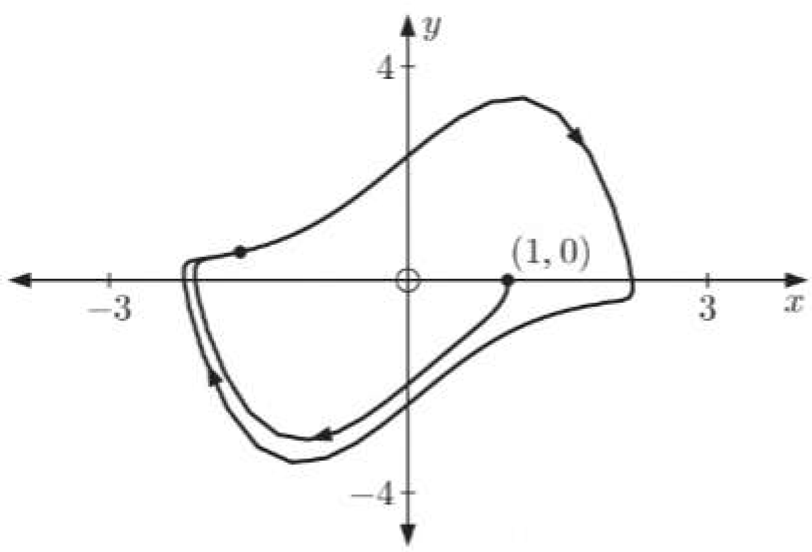
$$x_i = x_{i-1} + 0.1y_{i-1},$$

and $y_i = y_{i-1} + 0.1[1.5(1 - x_{i-1}^2)y_{i-1} - x_{i-1}]$.

a



b



4 $\dfrac{d^2x}{dt^2} = 9.81 - 0.375\left(\dfrac{dx}{dt}\right)^2 \;\; \text{m s}^{-2}$

a Let $y = \dfrac{dx}{dt}$

$\therefore \dfrac{dy}{dt} = \dfrac{d^2x}{dt^2}$

$\therefore$ the system is $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 9.81 - 0.375y^2. \end{cases}$

b At time $t_0 = 0$, $x_0 = 0$ and $y_0 = 0$.

Using step size h, $t_i = t_{i-1} + h$,

$x_i = x_{i-1} + hy_{i-1}$,

and $y_i = y_{i-1} + h(9.81 - 0.375y_{i-1}^2)$.

Using technology:

h	*Steps*	$x(5)$
0.5	10	≈ 22.927
0.1	50	≈ 23.616
0.05	100	≈ 23.673
0.01	500	≈ 23.715

So, in the first 5 seconds, the anchor falls about 23.7 m.

c Using Euler's method with step size $h = 0.01$ for 500 steps, we see that $y_i \approx 5.1147$ for large i.

$\therefore$ the terminal velocity of the anchor is about 5.1147 m s^{-1}.

d The anchor has no acceleration when $\dfrac{d^2x}{dt^2} = 0$

$$\therefore\; 9.81 - 0.375\left(\frac{dx}{dt}\right)^2 = 0$$

$$\therefore\; \left(\frac{dx}{dt}\right)^2 = \frac{9.81}{0.375}$$

$$\therefore\; \frac{dx}{dt} = \sqrt{\frac{9.81}{0.375}} \approx 5.1147$$

So, the terminal velocity of the anchor is about 5.1147 m s^{-1}, which is consistent with the answer to **c**.

5 $\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{600}(1200 - x - 2y) \\ \dfrac{dy}{dt} = \dfrac{y}{600}\left(1500 - \frac{3}{2}x - y\right) \end{cases}$

a The number of hermit crabs cannot be negative.

$\therefore\; x, y \geqslant 0$

b Equilibrium points occur where $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

Now $\frac{dx}{dt} = 0$ when $\frac{x}{600}(1200 - x - 2y) = 0$

$\therefore \ x = 0$ or $x = 1200 - 2y$

Also $\frac{dy}{dt} = 0$ when $\frac{y}{600}\left(1500 - \frac{3}{2}x - y\right) = 0$

$\therefore \ y = 0$ or $y = 1500 - \frac{3}{2}x$

$\therefore$ $(0, 0)$ is an equilibrium point. $(x = 0, \ y = 0)$

If $x = 0$ and $y = 1500 - \frac{3}{2}x$

$\therefore \ y = 1500$

$\therefore$ $(0, 1500)$ is an equilibrium point.

If $x = 1200 - 2y$ and $y = 0$

$\therefore \ x = 1200$

$\therefore$ $(1200, 0)$ is an equilibrium point.

If $x = 1200 - 2y$ and $y = 1500 - \frac{3}{2}x$

$$\therefore \ y = 1500 - \tfrac{3}{2}(1200 - 2y)$$
$$\therefore \ y = 3y - 300$$
$$\therefore \ 2y = 300$$
$$\therefore \ y = 150$$
and $x = 1200 - 2(150) = 900$

$\therefore$ $(900, 150)$ is an equilibrium point.

c **i** At time $t_0 = 0$, $x_0 = 175$ and $y_0 = 25$.

Using step size $h = \frac{1}{4}$, $t_i = t_{i-1} + \frac{1}{4}$,

$$x_i = x_{i-1} + \frac{x_{i-1}}{2400}\left(1200 - x_{i-1} - 2y_{i-1}\right),$$

and $$y_i = y_{i-1} + \frac{y_{i-1}}{2400}\left(1500 - \tfrac{3}{2}x_{i-1} - y_{i-1}\right).$$

Using technology, $x_8 \approx 820$ and $y_8 \approx 170$.

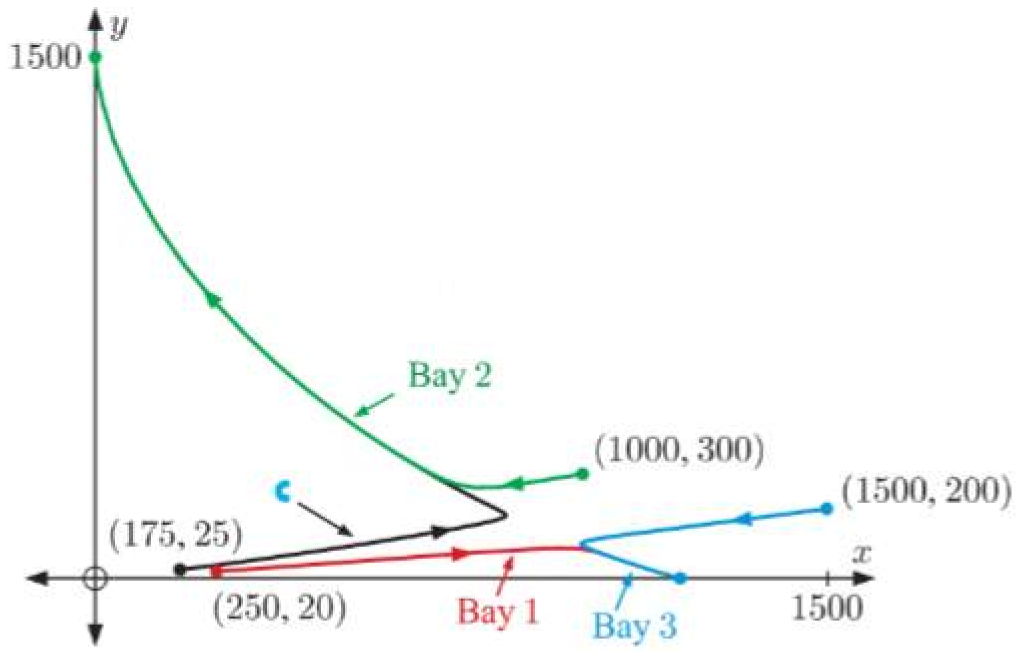
So, after 2 years, there are about 820 common hermit crabs and about 170 yellow-footed hermit crabs.

ii Using step size $h = \frac{1}{4}$, $x_i \approx 0$ and $y_i \approx 1500$ for large i.

So, in the long term, there will be no common hermit crabs and 1500 yellow-footed hermit crabs.

d We calculate the trajectory for each bay using the formulas in **c i** with step size $h = \frac{1}{4}$.

Bay	x_0	y_0
1	250	20
2	1000	300
3	1500	200

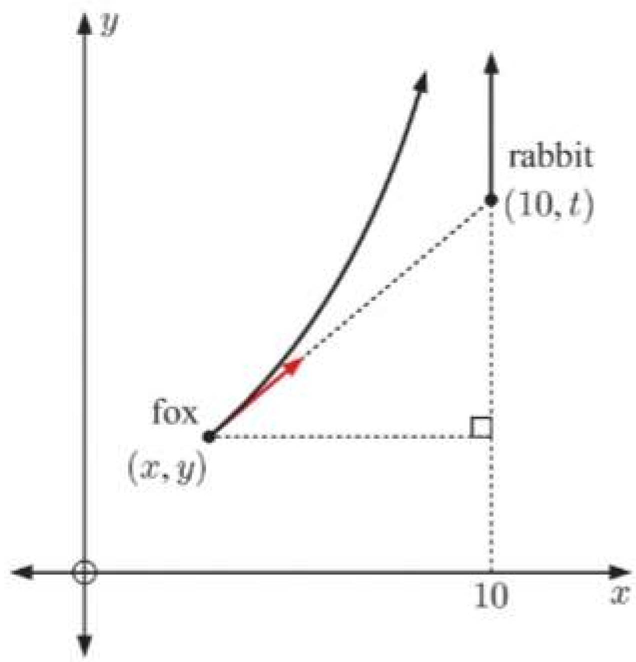


e The only stable equilibrium points are $(0, 1500)$ and $(1200, 0)$. Unless the populations are initially $x_0 = 900$, $y_0 = 150$, one of the species will become extinct in the long term.

6 **a** The position of the rabbit has vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t, \quad t \geqslant 0$$

$\therefore$ at time t, the rabbit is at $(10, t)$.



b The fox is always running directly toward the rabbit.

$\therefore$ at time t, the fox is moving in the direction

$$\mathbf{b} = \begin{pmatrix} 10 - x \\ t - y \end{pmatrix}.$$

The fox runs at a constant speed of F units per second.

$\therefore$ the fox has velocity vector

$$\mathbf{v} = \frac{F}{|\mathbf{b}|}\mathbf{b} = \frac{F}{\sqrt{(10-x)^2 + (t-y)^2}} \begin{pmatrix} 10 - x \\ t - y \end{pmatrix}$$

So, the position (x, y) of the fox satisfies

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{10 - x}{\sqrt{(10-x)^2 + (t-y)^2}} F \\[2ex] \dfrac{dy}{dt} = \dfrac{t - y}{\sqrt{(10-x)^2 + (t-y)^2}} F. \end{cases}$$

c If $F = 1$, $\begin{cases} \dfrac{dx}{dt} = \dfrac{10 - x}{\sqrt{(10-x)^2 + (t-y)^2}} \\[2ex] \dfrac{dy}{dt} = \dfrac{t - y}{\sqrt{(10-x)^2 + (t-y)^2}} \end{cases}$

At time $t_0 = 0$, $x_0 = 0$ and $y_0 = 0$.

i Using step size $h = 0.1$, $t_i = t_{i-1} + 0.1$,

$$x_i = x_{i-1} + 0.1\left(\frac{10 - x_{i-1}}{\sqrt{(10 - x_{i-1})^2 + (t_{i-1} - y_{i-1})^2}}\right),$$

and $$y_i = y_{i-1} + 0.1\left(\frac{t - y_{i-1}}{\sqrt{(10 - x_{i-1})^2 + (t_{i-1} - y_{i-1})^2}}\right).$$

Using technology,

t (seconds)	*Position*	t (seconds)	*Position*
1	(0.998, 0.047)	11	(8.213, 6.135)
2	(1.986, 0.202)	12	(8.531, 7.083)
3	(2.947, 0.474)	13	(8.794, 8.048)
4	(3.867, 0.865)	14	(9.011, 9.024)
5	(4.729, 1.371)	15	(9.190, 10.008)
6	(5.518, 1.984)	16	(9.337, 10.997)
7	(6.224, 2.691)	17	(9.457, 11.990)
8	(6.844, 3.475)	18	(9.556, 12.985)
9	(7.378, 4.320)	19	(9.637, 13.982)
10	(7.832, 5.211)	20	(9.703, 14.979)

ii Using step size $h = 0.1$, $\sqrt{(10 - x_i)^2 - (t_i - y_i)^2} \approx 5.03$ for large i.

$\therefore$ the fox does not appear to catch the rabbit, and the distance between them appears to converge to about 5.03 units.

d If $F = 2$, $$\begin{cases} \dfrac{dx}{dt} = \dfrac{20 - 2x}{\sqrt{(10 - x)^2 + (t - y)^2}} \\[2ex] \dfrac{dy}{dt} = \dfrac{2t - 2y}{\sqrt{(10 - x)^2 + (t - y)^2}} \end{cases}$$

i Using step size $h = 0.05$, $t_i = t_{i-1} + 0.05$,

$$x_i = x_{i-1} + 0.05\left(\frac{20 - 2x_{i-1}}{\sqrt{(10 - x_{i-1})^2 + (t_{i-1} - y_{i-1})^2}}\right),$$

and $$y_i = y_{i-1} + 0.05\left(\frac{2t - 2y_{i-1}}{\sqrt{(10 - x_{i-1})^2 + (t_{i-1} - y_{i-1})^2}}\right).$$

Using technology,

t (seconds)	*Position*	t (seconds)	*Position*
1	(1.996, 0.102)	6	(9.828, 5.332)
2	(3.964, 0.451)	7	(10.000, 6.917)
3	(5.852, 1.102)	8	(10.000, 7.917)
4	(7.571, 2.117)	9	(10.000, 8.917)
5	(8.967, 3.538)	10	(10.000, 9.917)

ii Using technology, the y-coordinate of the fox first exceeds that of the rabbit when $t = 6.7$. At this time, the x-coordinate of the fox ≈ 10.000, which is the same x-coordinate as the rabbit.

So, the fox appears to catch the rabbit after approximately 6.7 seconds.

ACTIVITY 4 THE SIR MODEL FOR INFECTIOUS DISEASES

1 $$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta IS}{N} \\ \dfrac{dI}{dt} = \dfrac{\beta IS}{N} - \gamma I \\ \dfrac{dR}{dt} = \gamma I \end{cases}$$

a The term $\dfrac{\beta IS}{N}$ represents how many susceptible individuals become infectious when in contact with already infectious individuals each day. Notice that R is not present as recovered individuals are immune to the disease.

The term γI represents how many infectious individuals recover from the disease each day.

b **i** The sign of $\dfrac{dS}{dt}$ is negative. This means that the number of susceptible individuals decreases over time.

ii If $\dfrac{\beta IS}{N} > \gamma I$, $\dfrac{dI}{dt}$ is positive. More people are becoming infected than are recovering, so the number of infected people increases.

If $\gamma I > \dfrac{\beta IS}{N}$, $\dfrac{dI}{dt}$ is negative. More people are recovering than becoming infected, so the number of infected people decreases.

iii The sign of $\dfrac{dR}{dt}$ is positive. This means that the number of recovered individuals increases over time.

d γ is the proportion of infected individuals who recover each day.

$\therefore$ the expected duration of an individual infection is $\dfrac{1}{\gamma}$ days.

e Equilibrium occurs where $\dfrac{dS}{dt} = \dfrac{dI}{dt} = \dfrac{dR}{dt} = 0$.

$\therefore\ -\dfrac{\beta IS}{N} = 0, \quad \dfrac{\beta IS}{N} - \gamma I = 0, \quad \text{and} \quad \gamma I = 0.$

This occurs when $I = 0$ (that is, there are no infectious individuals).

f **i** An outbreak occurs if $\dfrac{dI}{dt} > 0$

$$\therefore\ \frac{\beta IS}{N} - \gamma I > 0$$

$$\therefore\ I\left(\frac{\beta S}{N} - \gamma\right) > 0$$

$$\therefore\ \frac{\beta S}{N} - \gamma > 0 \qquad \{\text{as } I \geqslant 0\}$$

$$\therefore\ \frac{S}{N} > \frac{\gamma}{\beta}$$

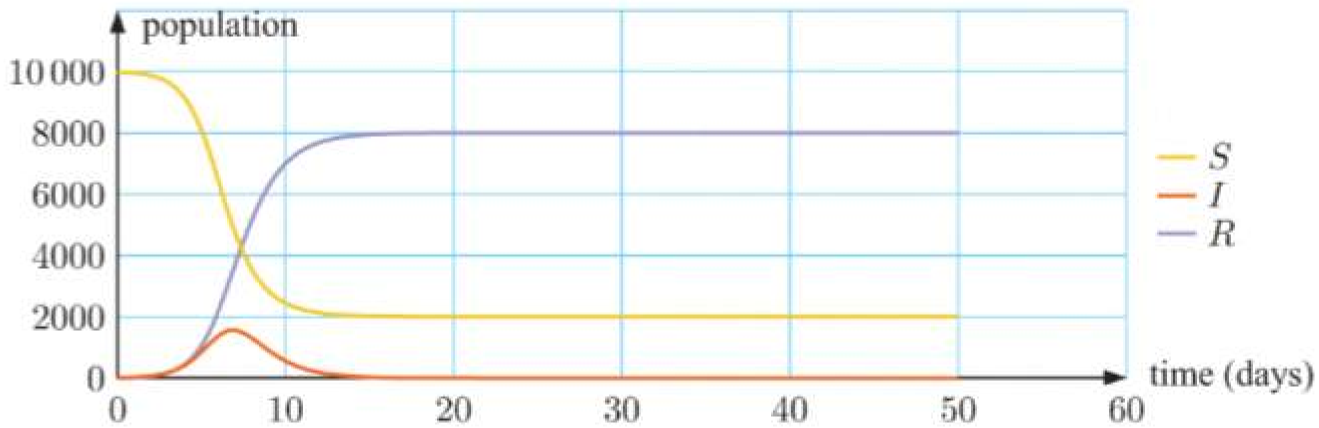
ii From **i**, an outbreak occurs if $\dfrac{S}{N} > \dfrac{\gamma}{\beta}$

$\therefore$ an outbreak is avoided if $\dfrac{S}{N} \leqslant \dfrac{\gamma}{\beta}$

$$\therefore \quad S \leqslant \frac{\gamma}{\beta} N$$

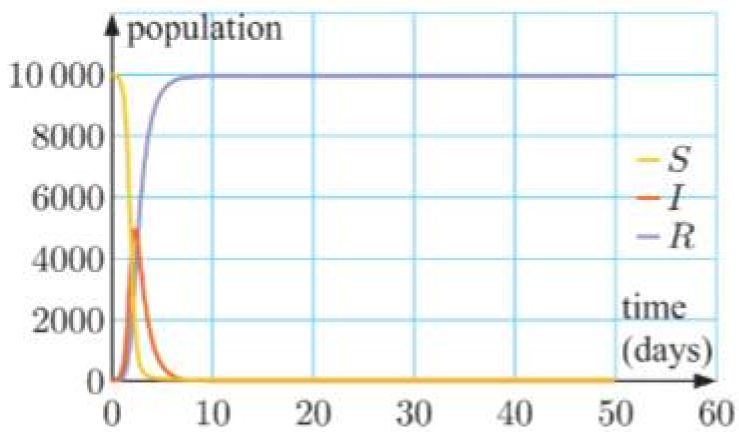
So, to avoid an outbreak, at least $\dfrac{\gamma}{\beta}$ of the population must be vaccinated.

g $\beta = 2, \quad \gamma = 1$

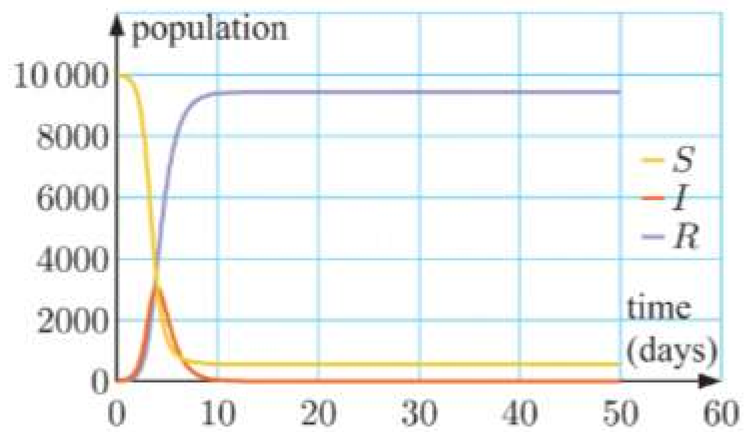


i Initially, the number of susceptible individuals decreases dramatically as they become infected. So, the number of infected individuals increases, but those infected recover quite quickly.
After about 7 days, the number of infectious individuals starts to decline as more people recover, and there are fewer susceptible people to catch the disease.
In the long term, there are about 8000 recovered individuals, and about 2000 susceptible individuals. The number of infectious individuals decreases to 0 which means the disease is eventually eradicated.
In this case, about 80% of the population will catch the disease in the long term.

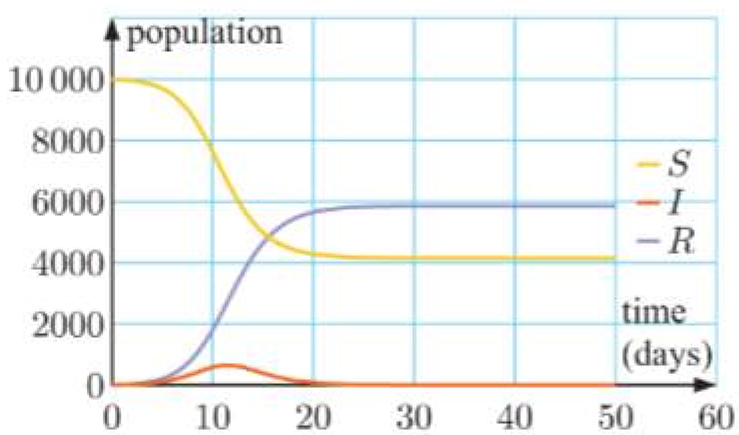
ii $\beta = 5, \quad \gamma = 1$



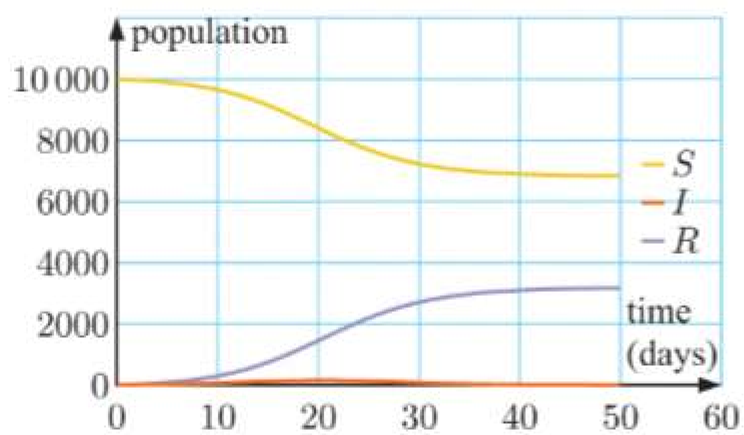
$\beta = 3, \quad \gamma = 1$



$\beta = 1.5, \quad \gamma = 1$



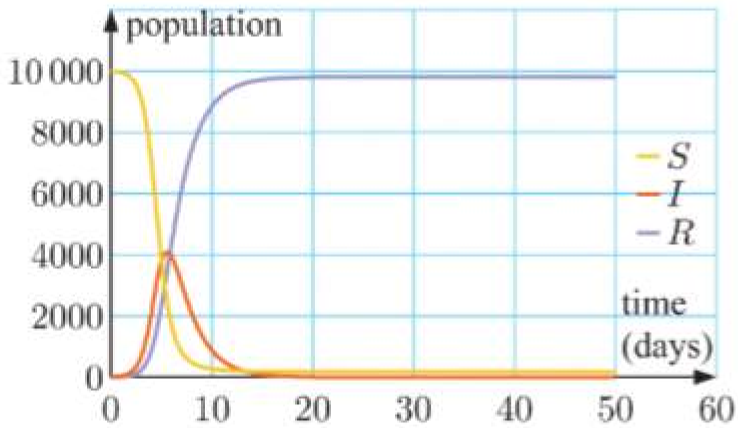
$\beta = 1.2, \quad \gamma = 1$



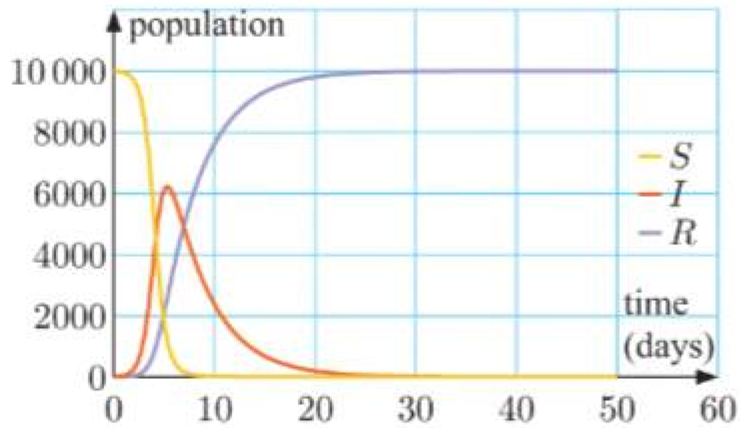
Decreasing the value of β appears to decrease the proportion of the population who will catch the disease in the long term.

So, limiting contact between infectious and susceptible individuals appears to be an effective means of controlling an outbreak and limiting its impact.

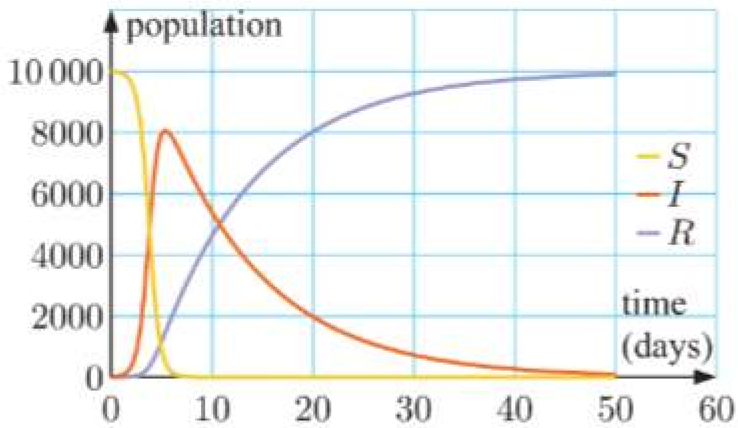
iii $\beta = 2, \quad \gamma = \frac{1}{2}$



$\beta = 2, \quad \gamma = \frac{1}{4}$

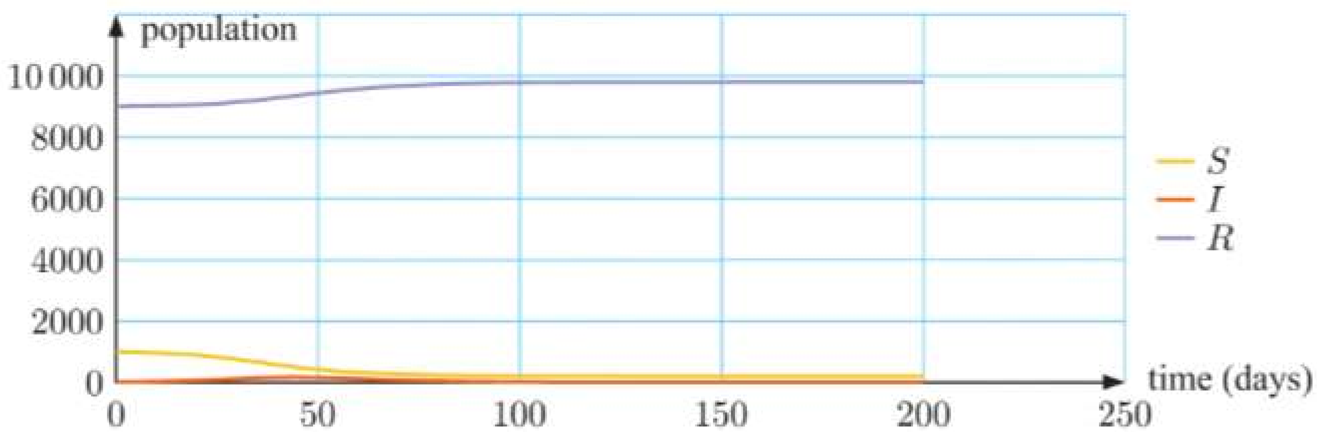


$\beta = 2, \quad \gamma = \frac{1}{10}$



Decreasing the value of γ increases the proportion of the population who will catch the disease in the long term, and increases the time it takes for the disease to become eradicated. So, a lack of medical care and longer recovery times means there is more time for an infectious individual to infect a susceptible individual. Hence a greater proportion of the population will catch the disease in the long term.

iv $\beta = 2, \quad \gamma = \frac{1}{10}$



Even with a slow recovery rate, vaccinating 90% of the population significantly reduces the rate of infection. Only about 8% of the population became infected, and it appears that the disease has been eliminated. So, vaccination is very effective in limiting outbreaks and hence the load on a hospital system.

h From **f**, an outbreak occurs if $\dfrac{dI}{dt} > 0$, and from **f ii**, this occurs when $\dfrac{S}{N} > \dfrac{\gamma}{\beta}$.

So, if the number of recovered or vaccinated individuals who are susceptible to the new strain forces the susceptible proportion of the population above $\dfrac{\gamma}{\beta}$, then an outbreak of the new strain can occur.

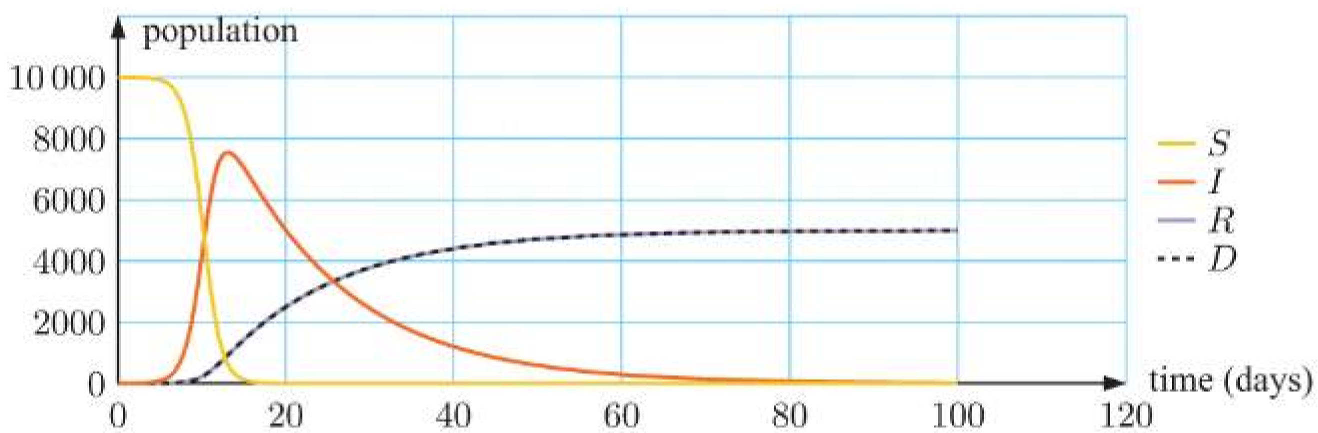
2 $$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta IS}{N} \\ \dfrac{dI}{dt} = \dfrac{\beta IS}{N} - \gamma I - \mu I \\ \dfrac{dR}{dt} = \gamma I \\ \dfrac{dD}{dt} = \mu I \end{cases}$$

a μ is the proportion of infected individuals who die each day.

γ is the proportion of infected individuals who recover each day.

$\therefore$ $\dfrac{\mu}{\gamma + \mu}$ is the death rate, that is, the proportion of infected individuals who will die from the disease.

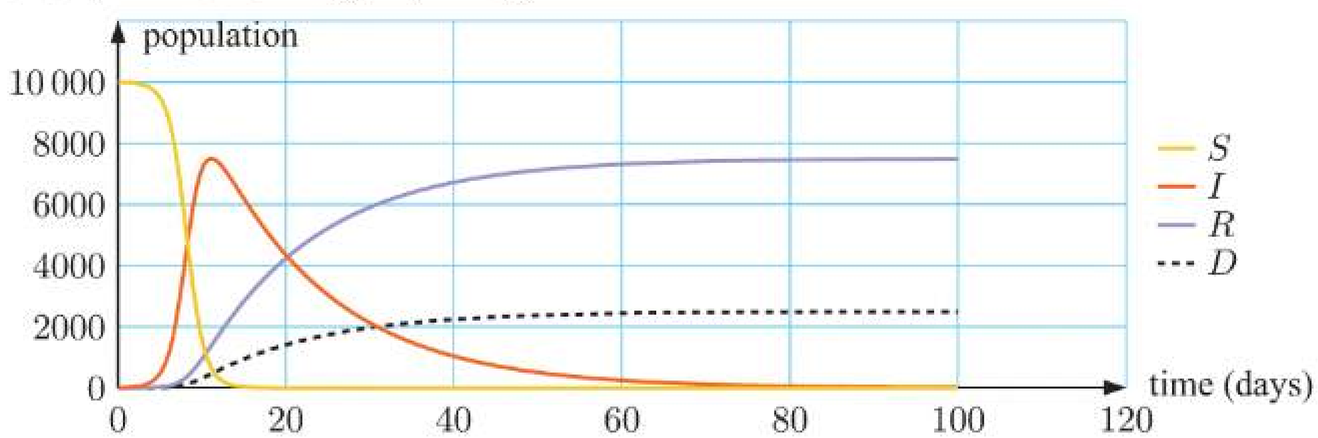
c **i** $\beta = 1$, $\gamma = \frac{1}{28}$, $\mu = \frac{1}{28}$



With only one infected individual, the number of susceptible individuals who become infected is initially very low. After about 5 days, the number of infected individuals increases dramatically.
After about 15 days, almost everyone has been infected with the disease.
In the long term, only half the original population is expected to survive.

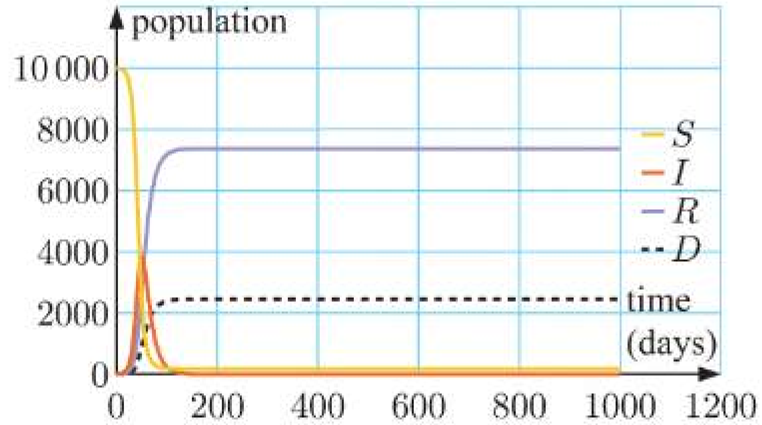
ii From the spreadsheet, rounding to the nearest individual, $S(2) \approx 9994$, $I(2) \approx 6$, $R(2) \approx 0$, and $D(2) \approx 0$.

(1) $\beta = 1$, $\gamma = \frac{3}{56}$, $\mu = \frac{1}{56}$

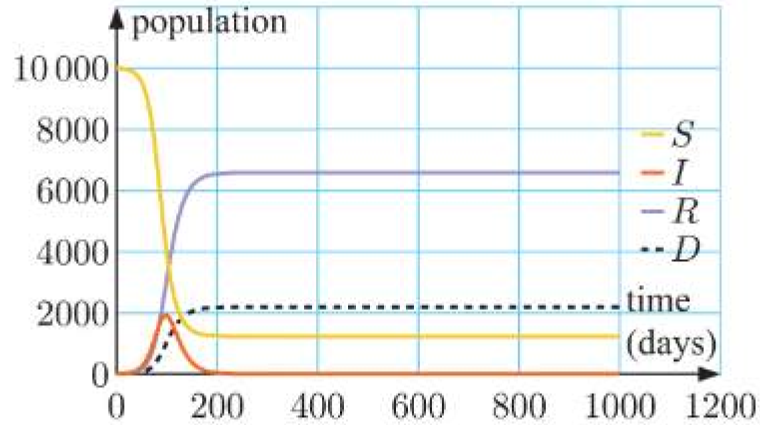


The disease continues to spread quite fast and eventually infects the whole population. However, many more individuals are expected to recover in the long term, with about 75% of the original population expected to survive.

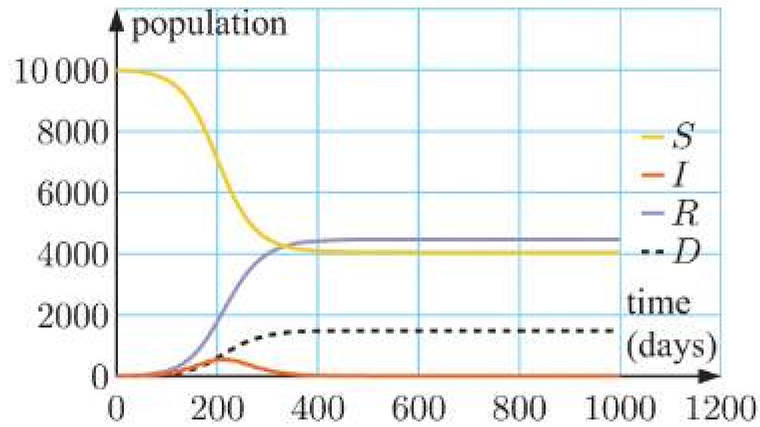
(2) $\beta = 0.25, \quad \gamma = \frac{3}{56}, \quad \mu = \frac{1}{56}$



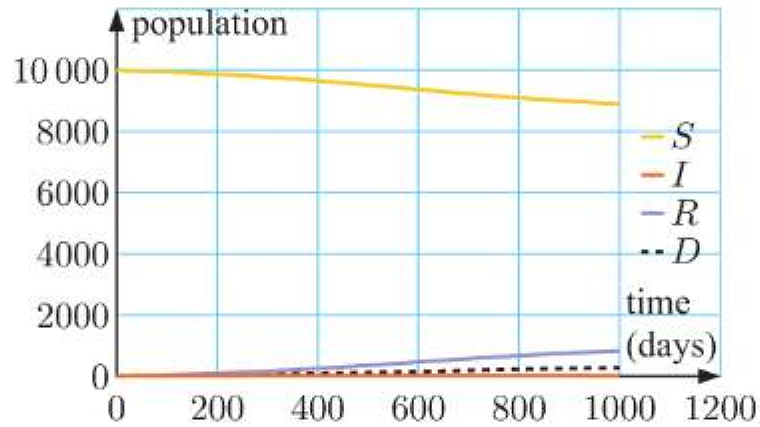
$\beta = 0.15, \quad \gamma = \frac{3}{56}, \quad \mu = \frac{1}{56}$



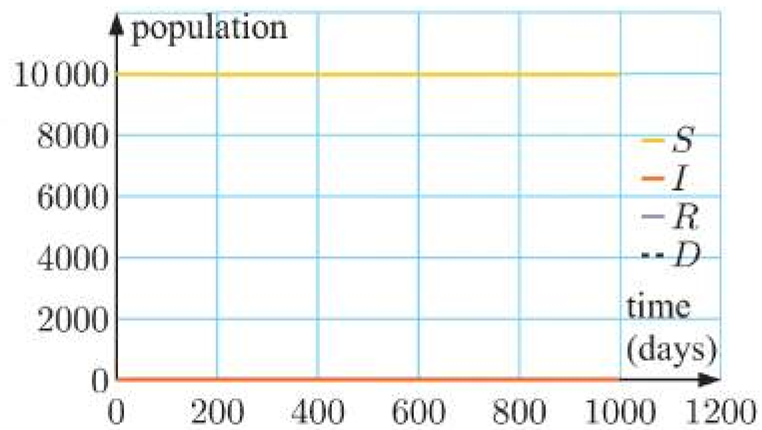
$\beta = 0.1, \quad \gamma = \frac{3}{56}, \quad \mu = \frac{1}{56}$



$\beta = 0.075, \quad \gamma = \frac{3}{56}, \quad \mu = \frac{1}{56}$



$\beta = 0.05, \quad \gamma = \frac{3}{56}, \quad \mu = \frac{1}{56}$



With more and more efficient containment zones and biohazard equipment, a larger proportion of the original population will not get infected at all, and more people will survive in the long term. So, slowing the spread of disease is an important factor in controlling an Ebola outbreak.

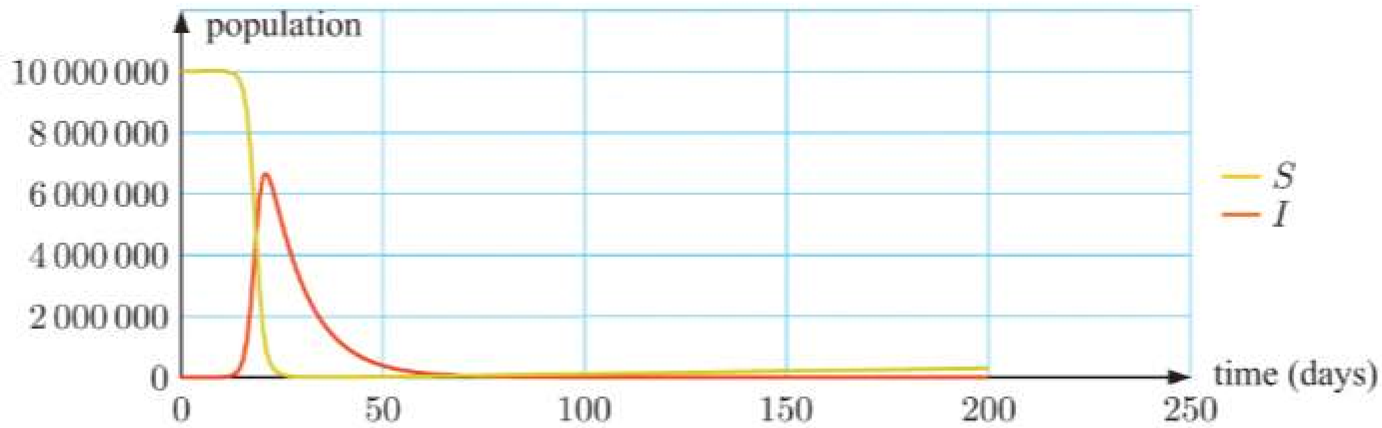
3 $$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta IS}{N} + \alpha(1-\rho)N - \delta S \\ \dfrac{dI}{dt} = \dfrac{\beta IS}{N} - \gamma I - \mu I - \delta I \\ \dfrac{dR}{dt} = \gamma I + \alpha\rho N - \delta R \end{cases}$$

a The term $\alpha(1-\rho)N$ is the number of children who have not been vaccinated against the disease.

The terms δS, δI, and δR are the number of susceptible, infectious, and recovered individuals, respectively, who die due to other causes.

The term $\alpha\rho N$ is the number of children who have been vaccinated against the disease.

b $\beta = 1$, $\gamma = 0.1$, $\mu = 0.005$, $\alpha = 0.0002$, $\delta = 0.000\,05$, $\rho = 0$

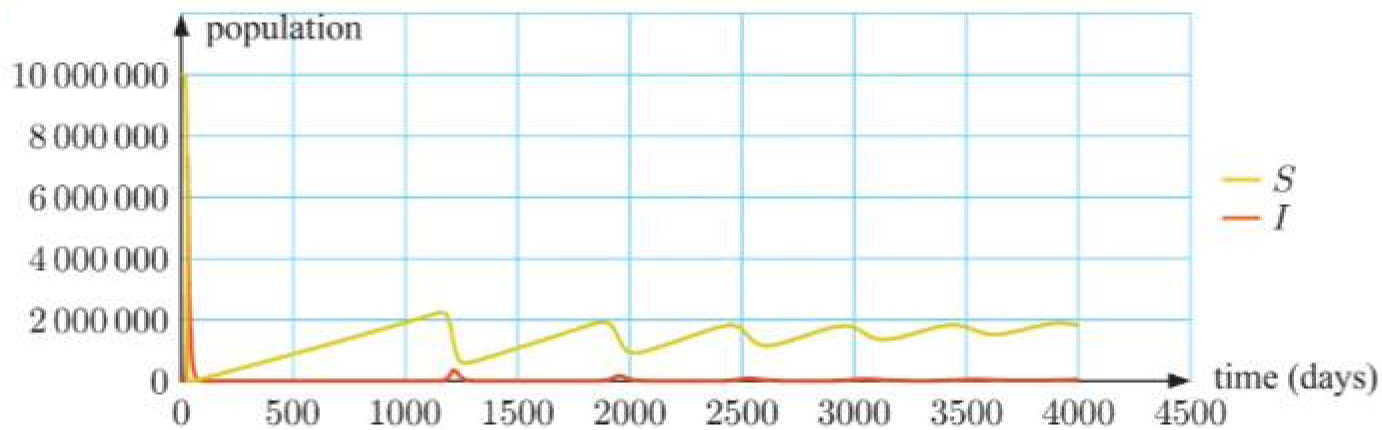


With only one infected individual, the number of susceptible individuals who become infected is initially very low. After about 10 days, the number of infected individuals increases dramatically.

After about 20 days, almost all susceptible individuals are infected.

After about 100 days, the number of infectious individuals appears to have decreased to 0. It appears that the disease has been eliminated.

c



It appears that a number of outbreaks occur, indicating that the disease has not been fully eliminated.

Each outbreak occurs when $\dfrac{dI}{dt} > 0$

$$\therefore \quad \frac{\beta IS}{N} - \gamma I - \mu I - \delta I > 0$$

$$\therefore \quad I\left(\frac{\beta S}{N} - \gamma - \mu - \delta\right) > 0$$

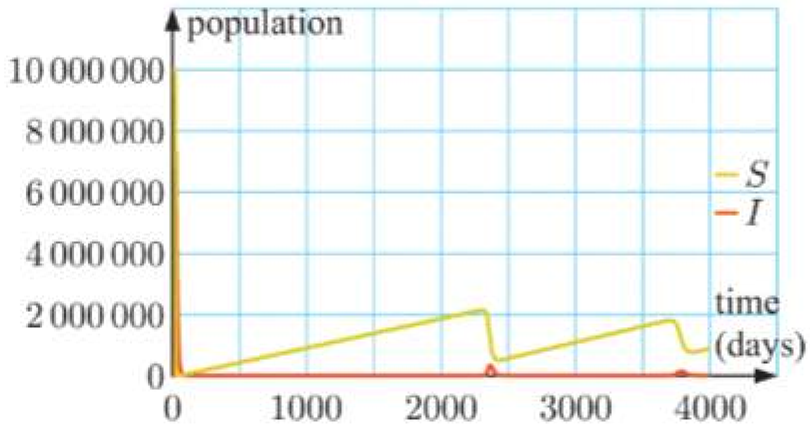
$$\therefore \quad \frac{\beta S}{N} - \gamma - \mu - \delta > 0 \qquad \{\text{as } I \geqslant 0\}$$

$$\therefore \quad \frac{S}{N} > \frac{\gamma + \mu + \delta}{\beta}$$

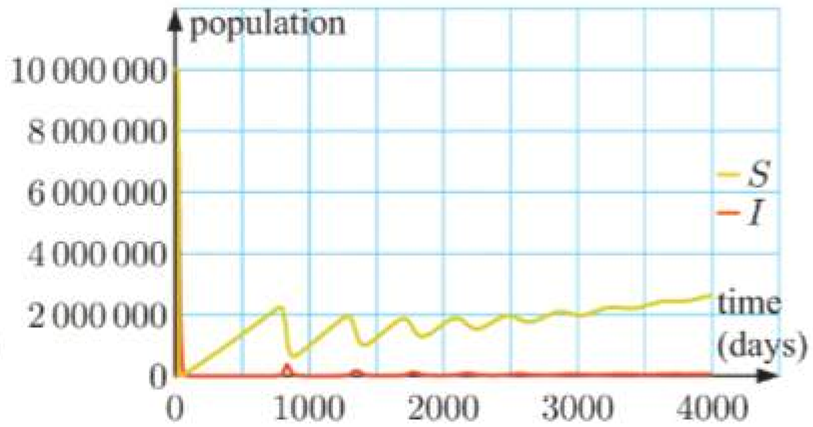
So, as the number of susceptible individuals increases due to births, and hence the susceptible proportion of the population increases beyond $\dfrac{\gamma+\mu+\delta}{\beta}$, outbreaks are likely to occur.

This is consistent with the pattern in the graph.

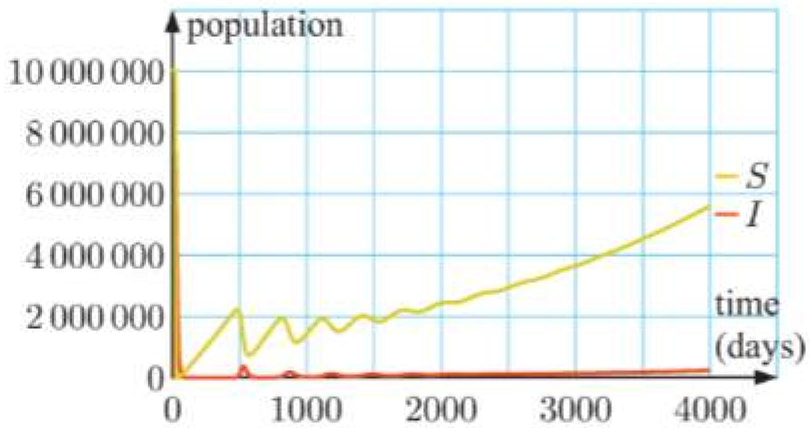
d $\alpha = 0.0001$



$\alpha = 0.0003$



$\alpha = 0.0005$

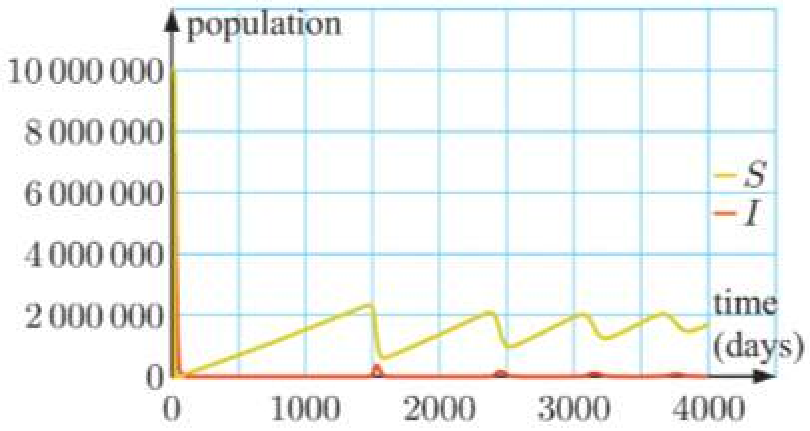


i It appears that the frequency of outbreaks increases as the birth rate increases. This is because the number of susceptible individuals increases faster with a higher birth rate, and hence there are more individuals who are able to catch the disease and cause an outbreak.

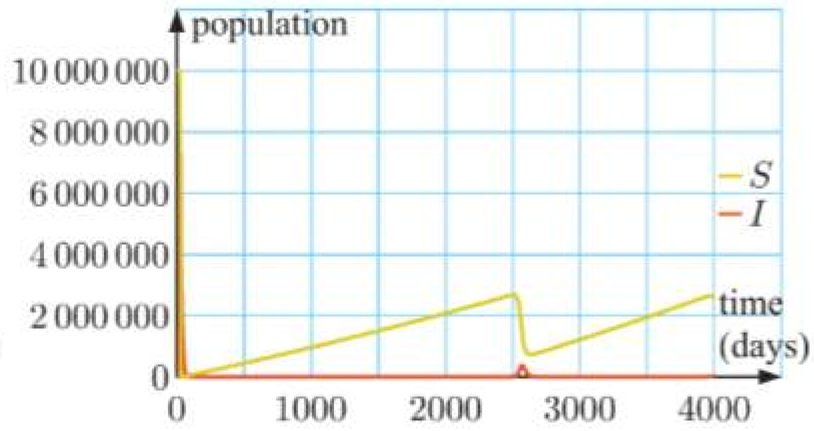
ii As people are born, the susceptible proportion $\dfrac{S}{N}$ increases as the infectious proportion $\dfrac{I}{N}$ decreases.

For sufficiently high birth rates, $\dfrac{S}{N}$ will reach $\dfrac{\gamma+N+\delta}{\beta}$ before $\dfrac{I}{N}$ becomes negligible, resulting in another outbreak.

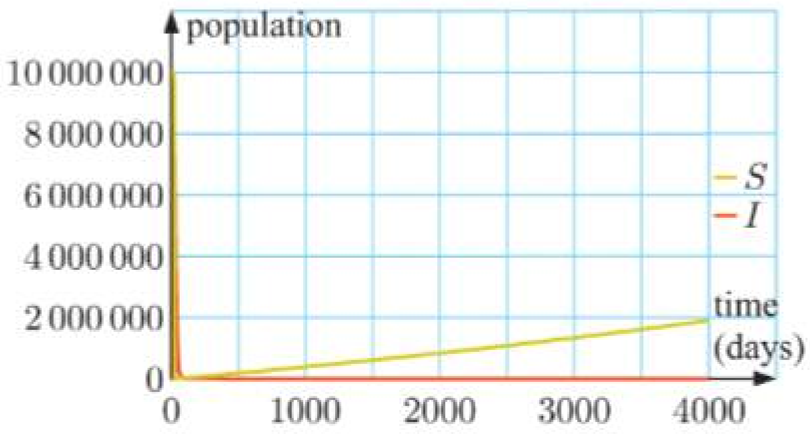
e $\rho = 0.2$



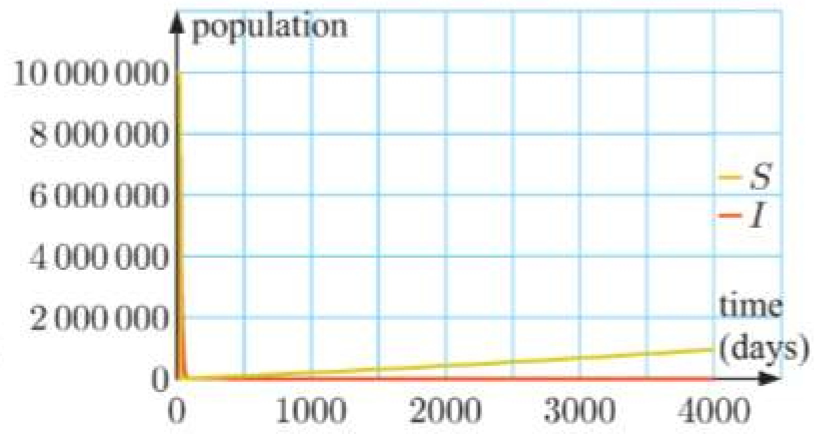
$\rho = 0.5$



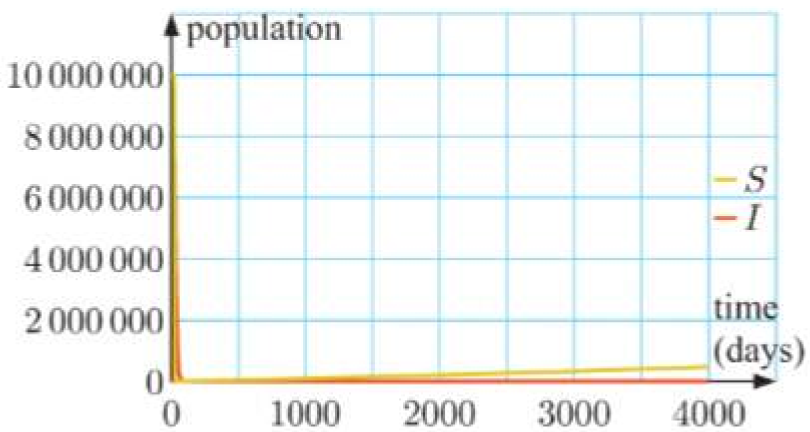
$\rho = 0.8$



$\rho = 0.9$



$\rho = 0.95$



It appears that immunising a greater proportion will significantly delay the occurrence of future outbreaks.

With at least 80% of children vaccinated it appears that any future outbreaks have been prevented.

4 Let σ be the proportion of recovered individuals who become immunocompromised each day.

$\therefore$ σR individuals move from the recovered category to the susceptible category.

So, the system becomes
$$\begin{cases} \dfrac{dS}{dt} = -\dfrac{\beta IS}{N} + \alpha(1-\rho)N - \delta S + \sigma R \\ \dfrac{dI}{dt} = \dfrac{\beta IS}{N} - \gamma I - \mu I - \delta I \\ \dfrac{dR}{dt} = \gamma I + \alpha\rho N - \delta R - \sigma R. \end{cases}$$

REVIEW SET 26A

1 $\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = 3x - y + 1 \end{cases}$

Equilibrium points occur where

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0$$

$$\therefore\ 2x - y = 0 \quad \text{and} \quad 3x - y + 1 = 0$$

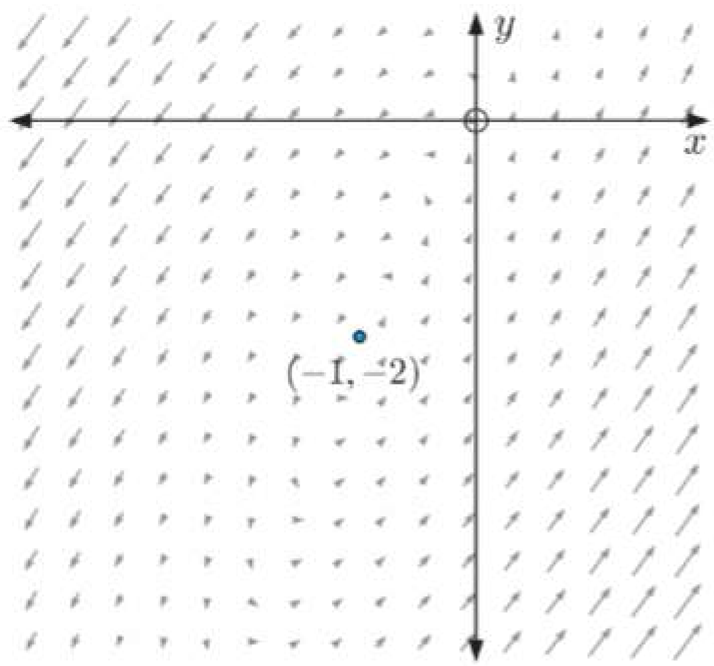
$$\therefore\ y = 2x \quad \text{and} \quad 3x = y - 1$$

$$\therefore\ 3x = 2x - 1$$

$$\therefore\ x = -1 \text{ and } y = -2$$

So, $(-1, -2)$ is an equilibrium point.

From the phase portrait, we see that it is an unstable spiral.



2 $\begin{cases} \dot{x} = xy - x \\ \dot{y} = y^2 - 4 \end{cases}$

a At $(-1, 3)$, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} (-1)(3) - (-1) \\ 3^2 - 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

b Equilibrium points occur where $\dot{x} = \dot{y} = 0$

$$\dot{x} = 0 \quad \text{and} \quad \dot{y} = 0$$

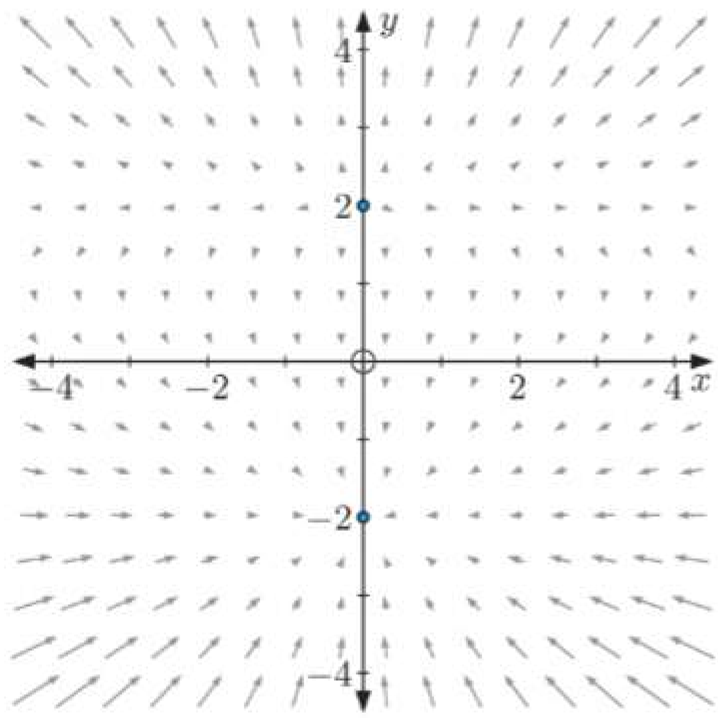
$$\therefore\ xy - x = 0 \quad \text{and} \quad y^2 - 4 = 0$$

$$\therefore\ x(y-1) = 0 \quad \text{and} \quad y^2 = 4$$

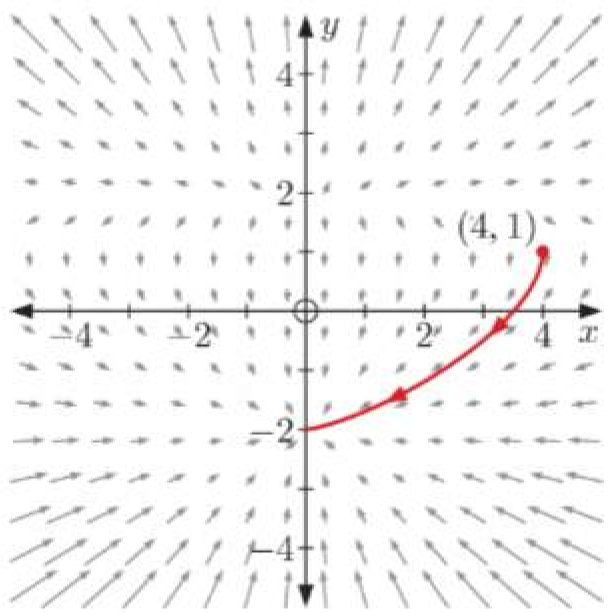
$$\therefore\ x = 0 \quad \text{and} \quad y = \pm 2$$

So, $(0, 2)$ and $(0, -2)$ are equilibrium points.

From the phase portrait, we see that $(0, 2)$ is an unstable fixed point, and $(0, -2)$ is a stable fixed point.



c



3 $\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 5x - 3y \end{cases}$ can be written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 5 & -3 \end{pmatrix}$.

a If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - 4 & 3 \\ -5 & \lambda + 3 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 4)(\lambda + 3) + 15 = 0$$
$$\therefore\ \lambda^2 - \lambda - 12 + 15 = 0$$
$$\therefore\ \lambda^2 - \lambda + 3 = 0$$
$$\therefore\ \lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(3)}}{2}$$
$$= \frac{1 \pm \sqrt{-11}}{2}$$
$$= \tfrac{1}{2} \pm \tfrac{\sqrt{11}}{2}\, i$$

The eigenvalues are $\frac{1}{2} \pm \frac{\sqrt{11}}{2}\, i$.

b The eigenvalues are complex with positive real part.

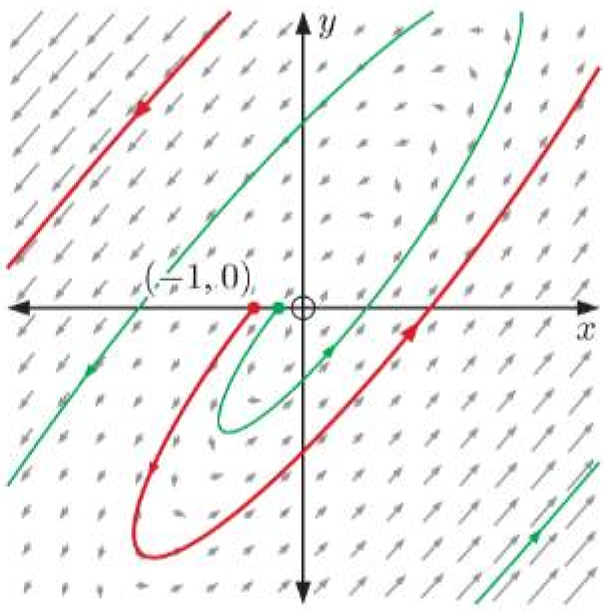
$\therefore$ the equilibrium point at O is an unstable spiral.

c **i** When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\therefore$ the initial trajectory is $\dot{\mathbf{x}} = \begin{pmatrix} 4 & -3 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$.

ii Using **i**, the spirals rotate anticlockwise.

d



4 $\dfrac{d^2P}{dt^2} + 8\,\dfrac{dP}{dt} + 30P = 200$

Let $S = \dfrac{dP}{dt}$

$\therefore \dfrac{dS}{dt} = \dfrac{d^2P}{dt^2}$

$\therefore \dfrac{dS}{dt} + 8S + 30P = 200$

$\therefore$ the system is $\begin{cases} \dfrac{dP}{dt} = S \\ \dfrac{dS}{dt} = 200 - 30P - 8S. \end{cases}$

5 **a** The eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$ satisfy $\lambda_2 > 0 > \lambda_1$.

$\therefore$ the equilibrium point at O is a saddle point.

b Using the given eigenvalues and eigenvectors, the general solution to the system is

$\mathbf{x} = Ae^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + Be^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

c When $t=0$, $\mathbf{x}=\begin{pmatrix}3\\-3\end{pmatrix}$

$$\therefore\ A\begin{pmatrix}-1\\1\end{pmatrix}+B\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\-3\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}-1&1\\1&1\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}3\\-3\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}-1&1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}3\\-3\end{pmatrix}$$

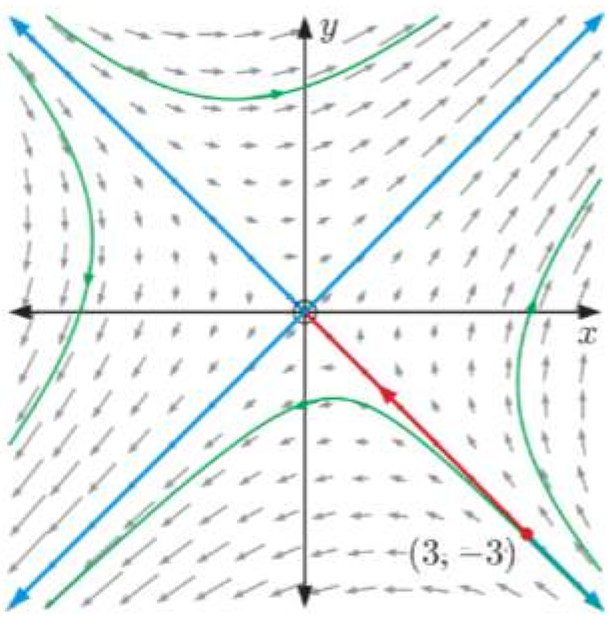
$$\therefore\ \begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}$$

The particular solution is $\mathbf{x}=-3e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}$.

d As $t\to\infty$, $e^{-2t}\to 0$

$\therefore$ the trajectory approaches O along $k\begin{pmatrix}-1\\1\end{pmatrix}$ as $t\to\infty$.

e



6 $\begin{cases}\dfrac{dx}{dt}=\dfrac{x}{50}(30-0.5y-0.003x)\\[2mm]\dfrac{dy}{dt}=\dfrac{y}{50}(-2+0.02x-0.9y)\end{cases}$

a Equilibrium points occur where $\dfrac{dx}{dt}=\dfrac{dy}{dt}=0$.

Now $\dfrac{dx}{dt}=0$ where $\dfrac{x}{50}(30-0.5y-0.003x)=0$

$\therefore\ x=0$ or $x=10\,000-\frac{500}{3}y$

Also $\dfrac{dy}{dt}=0$ where $\dfrac{y}{50}(-2+0.02x-0.9y)=0$

$\therefore\ y=0$ or $y=\frac{1}{45}x-\frac{20}{9}$

$\therefore$ (0, 0) is an equilibrium point. $(x=0,\ y=0)$

At (0, 0), both populations have died out.

If $x = 0$ and $y = \frac{1}{45}x - \frac{20}{9}$

$\therefore\ y = -\frac{20}{9}$

$\therefore\ \left(0, -\frac{20}{9}\right)$ is an equilibrium point.

However, this point is not valid as we cannot have a negative number of lions.

If $x = 10\,000 - \frac{500}{3}y$ and $y = 0$

$\therefore\ x = 10\,000$

$\therefore\ (10\,000, 0)$ is an equilibrium point.

At $(10\,000, 0)$, the lions have died out.

If $x = 10\,000 - \frac{500}{3}y$ and $y = \frac{1}{45}x - \frac{20}{9}$

$$\therefore\ y = \frac{1}{45}\left(10\,000 - \frac{500}{3}y\right) - \frac{20}{9}$$

$$\therefore\ y = 220 - \frac{100}{27}y$$

$$\therefore\ \frac{127}{27}y = 220$$

$$\therefore\ y = \frac{5940}{127} \approx 46.8$$

$$\text{and}\quad x = 10\,000 - \frac{500}{3}\left(\frac{5940}{127}\right) = \frac{840\,000}{381} \approx 2200$$

$\therefore\ \left(\frac{840\,000}{381}, \frac{5940}{127}\right)$ which is about $(2200, 46.8)$ is an equilibrium point.

At about $(2200, 46.8)$, the zebras and the lions coexist in equilibrium.

b **i** At time $t_0 = 0$, $x_0 = 800$ and $y_0 = 35$

Using step size $h = 0.1$, $t_i = t_{i-1} + 0.1$,

$$x_i = x_{i-1} + \frac{x_{i-1}}{500}(30 - 0.5y_{i-1} - 0.003x_{i-1}),$$

$$\text{and}\quad y_i = y_{i-1} + \frac{y_{i-1}}{500}(-2 + 0.02x_{i-1} - 0.9y_{i-1}).$$

Using technology, $x_{10} \approx 1010$ and $y_{10} \approx 27.4$

So, after 1 year, there are about 1010 zebras and about 27 lions.

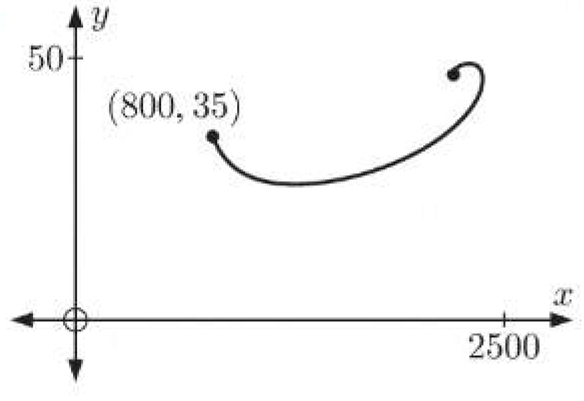
ii Using step size $h = 0.1$, we see that $x_i \approx 2200$ and $y_i \approx 46.8$ for large i.

So in the long term we predict that there will be about 2200 zebras and about 47 lions.

iii Using step size $h = 0.1$, we see that $y_i \approx 26$ for $i = 20$.

$\therefore$ the lowest population of lions is about 26 which occurs after about 2 years.

iv



REVIEW SET 26B

1 $\begin{cases} \dot{x} = e^y - 1 \\ \dot{y} = x^2 - 3x \end{cases}$

a **i** At $(0, 0)$,
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} e^0 - 1 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ $(0, 0)$ is an equilibrium point.

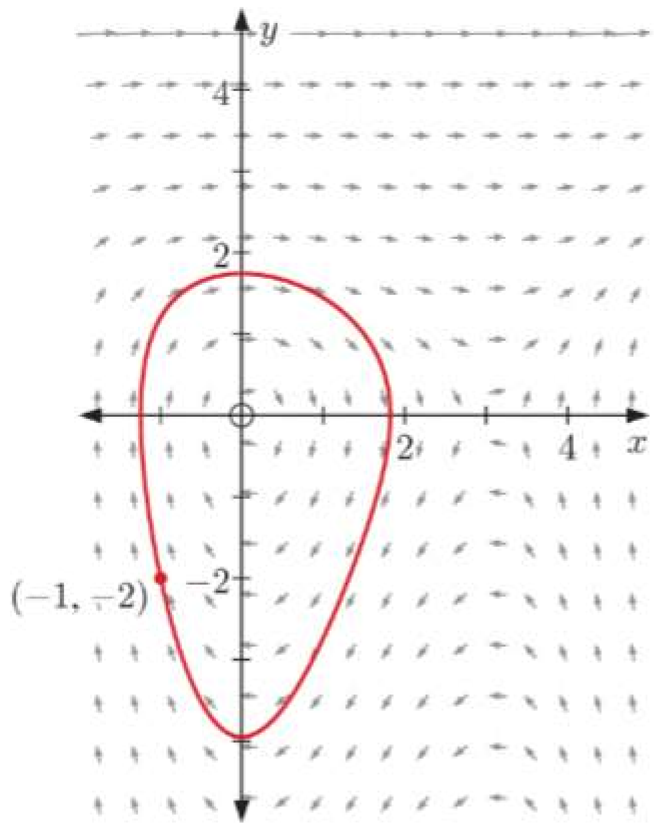
From the phase portrait, we see that it is a centre.

ii At $(3, 0)$,
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} e^0 - 1 \\ 3^2 - 3(3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ $(3, 0)$ is an equilibrium point.

From the phase portrait, we see that it is a saddle point.

b



2 $\begin{cases} \dfrac{dx}{dt} = \cos \pi y \\ \dfrac{dy}{dt} = 3x - 2y \end{cases}$

a Equilibrium points occur where

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0$$
$$\therefore \cos \pi y = 0 \quad \text{and} \quad 3x - 2y = 0$$
$$\therefore \pi y = \tfrac{\pi}{2} + k\pi, \ k \in \mathbb{Z} \quad \text{and} \quad 3x = 2y$$
$$\therefore y = \tfrac{1}{2} + k, \ k \in \mathbb{Z} \quad \text{and} \quad x = \tfrac{2}{3}y$$

So, the equilibrium points are $\left(\frac{1}{3}+\frac{2}{3}k,\ \frac{1}{2}+k\right)$ where $k \in \mathbb{Z}$.

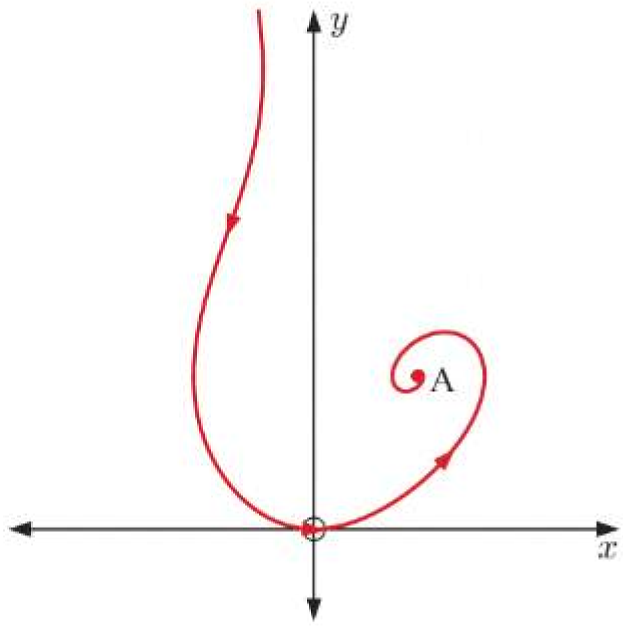
Equilibrium point A appears to be the first such point with positive x and y-coordinates.

$\therefore$ equilibrium point A is at $\left(\frac{1}{3}+\frac{2}{3}(0),\ \frac{1}{2}+0\right)$ which is $\left(\frac{1}{3},\ \frac{1}{2}\right)$.

b The solution curve passes through the origin.

At $(0, 0)$, $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \cos 0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So, the solution curve rotates inwards.



3 **a** Using the given eigenvalues and eigenvectors, the general solution to the system is

$$\mathbf{x} = Ae^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + Be^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

b **i** When $t=0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$$\therefore \ \mathbf{x} = \begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -21 \\ 27 \end{pmatrix}$$

ii When $t=0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$$\therefore \ A\begin{pmatrix} -3 \\ 1 \end{pmatrix} + B\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

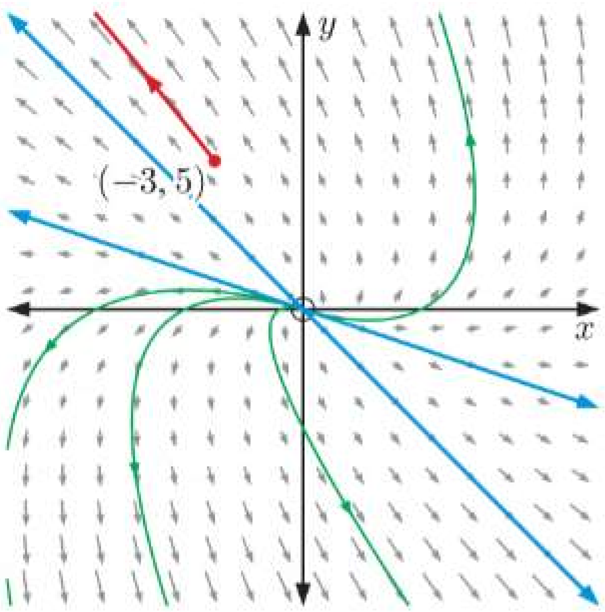
$$\therefore \ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

The particular solution is $\mathbf{x} = -e^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + 6e^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

c The eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 5$ satisfy $\lambda_2 > \lambda_1 > 0$.

$\therefore$ the equilibrium point at O is an unstable fixed point.

d



e $\lambda_2 > \lambda_1 > 0$

$\therefore$ the trajectory becomes parallel to $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as $t \to \infty$.

4 $\dfrac{d^2x}{dt^2} + 4x = 0$

a If $x(t) = 3\sin 2t$, then $\dfrac{dx}{dt} = 6\cos 2t$

$$\therefore \frac{d^2x}{dt^2} = -12\sin 2t$$

$$\therefore \frac{d^2x}{dt^2} = -4(3\sin 2t)$$

$$\therefore \frac{d^2x}{dt^2} = -4x$$

$$\therefore \frac{d^2x}{dt^2} + 4x = 0 \quad \checkmark$$

Also, $x(0) = 3\sin 0 = 0$ ✓

and at $t = 0$, $\dfrac{dx}{dt} = 6\cos 0 = 6$ ✓

$\therefore$ $x(t) = 3\sin 2t$ is a particular solution which satisfies the initial conditions.

b Let $y = \dfrac{dx}{dt}$

$\therefore$ $\dfrac{dy}{dt} = \dfrac{d^2x}{dt^2}$

$\therefore$ the system is $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -4x. \end{cases}$

c Using step size h, $t_i = t_{i-1} + h$,

$x_i = x_{i-1} + hy_{i-1}$,

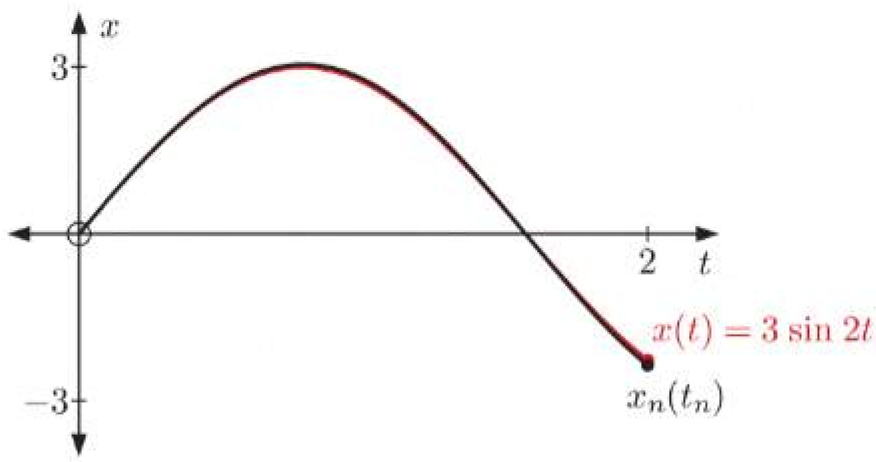
and $y_i = y_{i-1} + h(-4x_{i-1}) = y_{i-1} - 4hx_{i-1}$.

d At time $t_0 = 0$, $x_0 = 0$ and $y_0 = 6$

Using **c** with step size $h = 0.01$, $x_1 = 0 + 0.01(6) = 0.06$

and $y_1 = 6 - 4(0.01)(0) = 6$.

e



The points (t_i, x_i) approximate a smooth curve similar to the analytic solution $x(t) = 3\sin 2t$. Euler's method is less accurate at times when the curve changes gradient faster.

5 $\dfrac{d^2I}{dt^2} + 4\,\dfrac{dI}{dt} + 16I = 0$

a Let $J = \dfrac{dI}{dt}$

$\therefore\ \dfrac{dJ}{dt} = \dfrac{d^2I}{dt^2}$

$\therefore\ \dfrac{dJ}{dt} + 4J + 16I = 0$

$\therefore$ the system is $\begin{cases} \dfrac{dI}{dt} = J \\ \dfrac{dJ}{dt} = -16I - 4J. \end{cases}$

b The system has matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -16 & -4 \end{pmatrix}$.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & -1 \\ 16 & \lambda+4 \end{vmatrix} = 0$

$$\therefore\ \lambda(\lambda+4) + 16 = 0$$

$$\therefore\ \lambda^2 + 4\lambda + 16 = 0$$

$$\therefore\ \lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(16)}}{2} = \frac{-4 \pm \sqrt{-48}}{2} = \frac{-4 \pm 4i\sqrt{3}}{2} = -2 \pm 2i\sqrt{3}$$

The eigenvalues are $-2 \pm 2i\sqrt{3}$ which are complex with negative real part.

$\therefore$ the equilibrium point at O is a stable spiral.

d **i** When $t = 0$, $\begin{pmatrix} I \\ J \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\therefore$ the initial trajectory is $\begin{pmatrix} \dfrac{dI}{dt} \\ \dfrac{dJ}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16 & -4 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -84 \end{pmatrix}$.

ii At time $t_0 = 0$, $I_0 = 4$ and $J_0 = 5$

Using step size $h = 0.05$, $t_i = t_{i-1} + 0.05$,

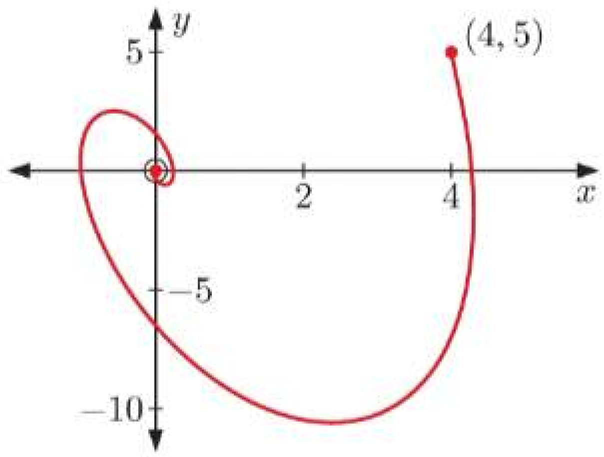
$$I_i = I_{i-1} + 0.05J_{i-1},$$

and $J_i = J_{i-1} + 0.05(-16I_{i-1} - 4J_{i-1})$.

Using technology, we see that $x_{12} \approx 0.0825$ and $x_{13} \approx -0.259$.

So, the first time the current is zero is after about $12 \times 0.05 = 0.6$ ms.

iii



iv In the long term, the current converges to zero.

6 $\begin{cases} x(t) = R\cos(\omega t + \phi) \\ y(t) = R\sin(\omega t + \phi) \end{cases}$

a $\dot{x} = -R\omega \sin(\omega t + \phi)$

$\dot{y} = R\omega \cos(\omega t + \phi)$

b From **a**, $\dot{x} = -\omega y$

and $\dot{y} = \omega x$.

$\therefore$ the system has the form $\dot{\mathbf{x}} = \mathbf{Ax}$ where $\mathbf{A} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & \omega \\ -\omega & \lambda \end{vmatrix} = 0$

$$\therefore \lambda^2 + \omega^2 = 0$$
$$\therefore \lambda^2 = -\omega^2$$
$$\therefore \lambda = \pm i\omega$$

The eigenvalues are $\pm i\omega$ which are purely imaginary.

$\therefore$ the equilibrium point at O is a centre.

d At $t = 0$, $\dot{\mathbf{x}} = \begin{pmatrix} -R\omega\sin(0+\phi) \\ R\omega\cos(0+\phi) \end{pmatrix} = \begin{pmatrix} -R\omega\sin\phi \\ R\omega\cos\phi \end{pmatrix}$.

e **i** If $\omega > 0$, the angle (measured anticlockwise) increases, so the rotation is anticlockwise.

If $\omega < 0$, the rotation is clockwise.

ii If $\mathbf{A}$ has the form $\begin{pmatrix} 0 & - \\ + & 0 \end{pmatrix}$:

- the x-direction has the opposite sign to the y-coordinate of the object (the object moves left above the x-axis, and right below the x-axis)
- the y-direction has the same sign as the x-coordinate of the object (the object moves up to the right of the y-axis, and down to the left of the y-axis).

So, the rotation is anticlockwise.

Similarly, if $\mathbf{A}$ has the form $\begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$, the rotation is clockwise.

f $\dot{\mathbf{x}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}$

i Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0$

$$\therefore\ (\lambda - a)(\lambda - d) - bc = 0$$

$$\therefore\ \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\begin{aligned} \therefore\ \lambda &= \frac{a + d \pm \sqrt{(a+d)^2 - 4(1)(ad - bc)}}{2} \\ &= \frac{a+d}{2} \pm \tfrac{1}{2}\sqrt{a^2 + 2ad + d^2 - 4ad + 4bc} \\ &= \frac{a+d}{2} \pm \tfrac{1}{2}\sqrt{a^2 - 2ad + d^2 + 4bc} \\ &= \frac{a+d}{2} \pm \tfrac{1}{2}\sqrt{(a-d)^2 + 4bc} \end{aligned}$$

The eigenvalues are $\frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.

ii For the eigenvalues to be complex, we require

$(a-d)^2 + 4bc < 0$

$\therefore\ (a-d)^2 < -4bc$

$\therefore\ -4bc > 0 \quad \{\text{as } (a-d)^2 \geqslant 0\}$

$\therefore\ bc < 0$

So, bc must be negative.

iii If $b < 0$, $c > 0$, the matrix has the form $\begin{pmatrix} a & - \\ + & d \end{pmatrix}$, so the rotation is anticlockwise.

If $b > 0$, $c < 0$, the matrix has the form $\begin{pmatrix} a & + \\ - & d \end{pmatrix}$, so the rotation is clockwise.

Chapter 27

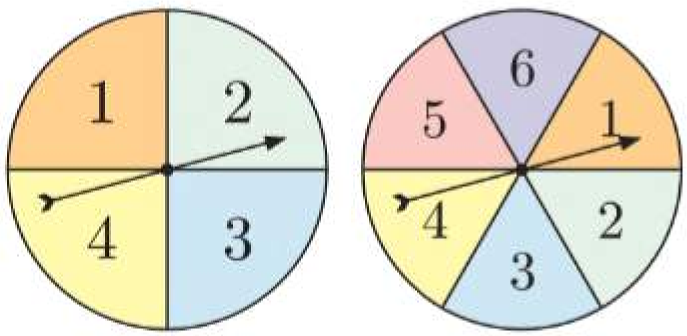
DISCRETE RANDOM VARIABLES

EXERCISE 27A

1 **a** The quantity of fat in a sausage is a continuous random variable.
b The mark out of 50 for a geography test is a discrete random variable.
c The weight of a Year 12 student is a continuous random variable.
d The volume of water in a cup of coffee is a continuous random variable.
e The number of trout in a lake is a discrete random variable.
f The number of the hairs on a cat is a discrete random variable.
g The length of a horse's mane is a continuous random variable.
h The height of a skyscraper is a continuous random variable.

2 **a** **i** The random variable X is the height of water in the rain gauge.
ii The variable is a continuous random variable. **iii** $0 \leqslant X \leqslant 400$ mm
b **i** The random variable X is the stopping distance.
ii The variable is a continuous random variable. **iii** $0 \leqslant X \leqslant 50$ m
c **i** The random variable X is the number of times that the switch is turned off and on before it fails.
ii The variable is a discrete random variable. **iii** X can be any integer $\geqslant 1$

3 **a** X is the sum of a number from one spinner and a number from the other spinner. So X is a discrete random variable because X has a set of distinct possible values.



b $X = 2, 3, 4, 5, 6, 7, 8, 9,$ or 10

4 **a** The two teams play against each other until one team wins 4 games (best out of 7).
$\therefore \quad X = 4, 5, 6,$ or 7
b **i** $X = 5$ **ii** $X = 6$ or 7

5 **a** There are four weighing devices and X is the number which are accurate.
$\therefore \quad X = 0, 1, 2, 3,$ or 4
b

A B C D	A B C D	A B C D	A B C D	A B C D
✓✓✓✓	✓✓✓✗	✓✓✗✗	✗✗✗✓	✗✗✗✗
	✓✓✗✓	✓✗✓✗	✗✗✓✗	
	✓✗✓✓	✓✗✗✓	✗✓✗✗	
	✗✓✓✓	✗✗✓✓	✓✗✗✗	
		✗✓✗✓		
		✗✓✓✗		
$(X = 4)$	$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

c **i** If exactly two devices are accurate, then $X = 2$.

ii If at least two devices are accurate, then 2, 3, or 4 are accurate $\therefore$ $X = 2, 3,$ or 4.

6 **a** If 3 coins are tossed then the number of heads X can be 0, 1, 2, or 3.

b Let H represent heads, and T represent tails.

HHH	HHT	TTH	TTT
	HTH	THT	
	THH	HTT	
$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

c $\text{P}(X = 0) = \frac{1}{8}$, $\text{P}(X = 1) = \frac{3}{8}$, $\text{P}(X = 2) = \frac{3}{8}$, $\text{P}(X = 3) = \frac{1}{8}$

Since $\text{P}(X = 2) \neq \text{P}(X = 3)$, the possible values of X are not equally likely to occur.

EXERCISE 27B

1 **a** **i**

x	1	2	3	4
$\text{P}(X = x)$	0.2	0.4	0.15	0.25

$$\sum_{x=1}^{4} \text{P}(X = x) = 0.2 + 0.4 + 0.15 + 0.25 = 1$$

Since $\sum_{x=1}^{4} \text{P}(X = x) = 1$ and $0 \leqslant \text{P}(X = x) \leqslant 1$ for all x, it is a valid probability distribution.

ii

x	0	1	2	3
$\text{P}(X = x)$	0.2	0.3	0.4	0.2

$$\sum_{x=0}^{3} \text{P}(X = x) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$$

Since $\sum_{x=0}^{3} \text{P}(X = x) > 1$, it is not a valid probability distribution.

iii

x	0	1	2	3	4
$\text{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

$$\sum_{x=0}^{4} \text{P}(X = x) = 0.2 + 0.2 + 0.2 + 0.2 + 0.2 = 1$$

Since $\sum_{x=0}^{4} \text{P}(X = x) = 1$ and $0 \leqslant \text{P}(X = x) \leqslant 1$ for all x, it is a valid probability distribution.

iv

x	2	3	4	5
$\text{P}(X = x)$	0.3	0.4	0.5	-0.2

Since $\text{P}(X = 5) = -0.2 < 0$, it is not a valid probability distribution.

b X is a uniform random variable for the probability distribution in **a iii**, since $p_i = 0.2$ for each value of i.

2 **a**

x	0	1	2
$\mathrm{P}(X=x)$	0.3	k	0.5

$$\sum_{x=0}^{2} \mathrm{P}(X=x)=1$$

$$\therefore\ 0.3+k+0.5=1$$

$$\therefore\ k=0.2$$

b

x	0	1	2	3
$\mathrm{P}(X=x)$	k	$2k$	$3k$	k

$$\sum_{x=0}^{3} \mathrm{P}(X=x)=1$$

$$\therefore\ k+2k+3k+k=1$$

$$\therefore\ 7k=1$$

$$\therefore\ k=\tfrac{1}{7}$$

3 **a**

$$\sum_{x=0}^{3} \mathrm{P}(X=x)=1$$

$$\therefore\ 0.1+0.25+0.45+a=1$$

$$\therefore\ a=0.2$$

x	0	1	2	3
$\mathrm{P}(X=x)$	0.1	0.25	0.45	a

b Since $\mathrm{P}(X=0) \neq \mathrm{P}(X=1)$, the probabilities of each outcome are not all equal, so X is not a uniform discrete random variable.

c Since $\mathrm{P}(X=2)$ is the greatest probability, 2 is the mode of the distribution.

d $\mathrm{P}(X \geqslant 2)=\mathrm{P}(X=2)+\mathrm{P}(X=3)$
$=0.45+0.2$
$=0.65$

4

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a From the table, $P(2)=0.1088$.

b Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i)=1$

$$\therefore\ a+0.3333+0.1088+0.0084+0.0007+0.0000=1$$

$$\therefore\ a+0.4512=1$$

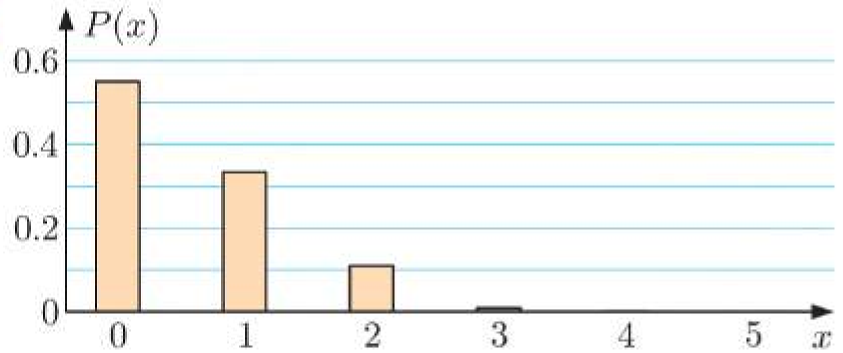
$$\therefore\ a=0.5488$$

This is the probability that Jason does not hit a home run in a game.

c $P(1)+P(2)+P(3)+P(4)+P(5)=0.3333+0.1088+0.0084+0.0007+0.0000$
$=0.4512$

This is the probability that Jason will hit one or more home runs in a game.

d



e Jason is most likely to score 0 home runs, so this is the mode of the distribution.
Using **b**, $P(0)=0.5488 \geqslant 0.5$, so the median is 0 home runs.

5

x	0	1	2	3	4
$P(X = x)$	0.68	0.2	0.06	k	0.02

a $$\sum_{x=0}^{4} P(X = x) = 1$$

$\therefore\ 0.68 + 0.2 + 0.06 + k + 0.02 = 1$

$\therefore\ k = 0.04$

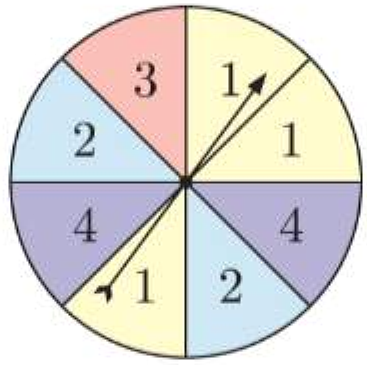
b It is most likely that the number of tyres which needed replacing is 0, so the mode of the distribution is 0 tyres.

c $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4)$
$= 0.06 + 0.04 + 0.02$
$= 0.12$

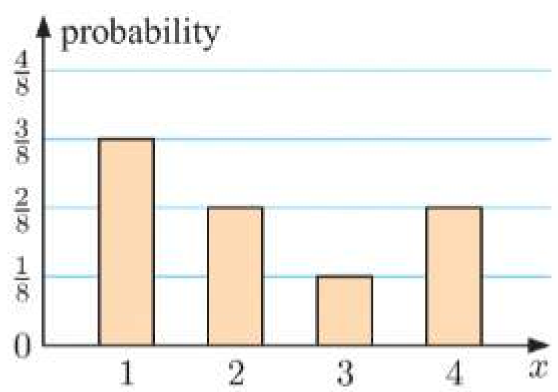
This is the probability that more than 1 tyre will need replacing on a car being inspected.

6 **a**

x	1	2	3	4
$P(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$



b



c The spinner is most likely to land on 1, so this is the mode of the distribution.

$p_1 = \frac{3}{8} = 0.375$

$p_1 + p_2 = \frac{3}{8} + \frac{2}{8} = 0.625$

Since $p_1 + p_2 \geqslant 0.5$, the median is 2.

d $P(X \leqslant 3) = P(X = 1) + P(X = 2) + P(X = 3)$
$= \frac{3}{8} + \frac{2}{8} + \frac{1}{8}$
$= \frac{6}{8}$
$= \frac{3}{4}$

7 **a** $X = 1, 2, 3,$ or 4

b $P(X = 1) = \frac{24}{100} = 0.24$

$P(X = 2) = \frac{35}{100} = 0.35$

$P(X = 3) = \frac{27}{100} = 0.27$

$P(X = 4) = \frac{14}{100} = 0.14$

$\therefore$ the probability table for X is

x	1	2	3	4
$P(X = x)$	0.24	0.35	0.27	0.14

c It is most likely for a randomly selected person to have 2 bedrooms in their house, so this is the mode of the distribution.

$$p_1 = 0.24$$
$$p_1 + p_2 = 0.24 + 0.35 = 0.59$$

Since $p_1 + p_2 \geqslant 0.5$, the median is 2 bedrooms.

8 **a** $X = 1$, 2, 3, or 4

b $\text{P}(X = 1) = \frac{12}{25} = 0.48$

$\text{P}(X = 2) = \frac{7}{25} = 0.28$

$\text{P}(X = 3) = \frac{2}{25} = 0.08$

$\text{P}(X = 4) = \dfrac{25 - (12 + 7 + 2)}{25} = 0.16$

$\therefore$ the probability table for X is

x	1	2	3	4
$\text{P}(X = x)$	0.48	0.28	0.08	0.16

c It is most likely for a randomly selected player to only need 1 shot to score a goal, so this is the mode of the distribution.

$$p_1 = 0.48$$
$$p_1 + p_2 = 0.48 + 0.28 = 0.76$$

Since $p_1 + p_2 \geqslant 0.5$, the median is 2 shots.

9 **a** $P(x) = \dfrac{x+1}{10}$, $x = 0, 1, 2, 3$

$\therefore$ $P(0) = \frac{1}{10}$, $P(1) = \frac{2}{10}$, $P(2) = \frac{3}{10}$, $P(3) = \frac{4}{10}$

$0 \leqslant P(x_i) \leqslant 1$ in each case, and $\sum_{i=1}^{n} P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

b $P(x) = \dfrac{6}{11x}$, $x = 1, 2, 3$

$\therefore$ $P(1) = \frac{6}{11}$, $P(2) = \frac{6}{22} = \frac{3}{11}$, $P(3) = \frac{6}{33} = \frac{2}{11}$

$0 \leqslant P(x_i) \leqslant 1$ in each case, and $\sum_{i=1}^{n} P(x_i) = \frac{6}{11} + \frac{3}{11} + \frac{2}{11} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

10 **a** $P(x) = k(x+2)$, $x = 1, 2, 3$

$\therefore$ $P(1) = 3k$, $P(2) = 4k$, $P(3) = 5k$

Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i) = 1$

$$\therefore\ 3k + 4k + 5k = 1$$
$$\therefore\ 12k = 1$$
$$\therefore\ k = \tfrac{1}{12}$$

b $P(x) = \dfrac{k}{x+1}, \quad x = 0, 1, 2, 3$

$\therefore \quad P(0) = k, \quad P(1) = \frac{k}{2}, \quad P(2) = \frac{k}{3}, \quad P(3) = \frac{k}{4}$

Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i) = 1$

$$\therefore \quad k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$$

$$\therefore \quad \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \quad \frac{25k}{12} = 1$$

$$\therefore \quad k = \tfrac{12}{25}$$

11 **a** $P(x) = \dfrac{4x - x^2}{a}, \quad x = 0, 1, 2, 3$

$\therefore \quad P(0) = 0, \quad P(1) = \frac{3}{a}, \quad P(2) = \frac{4}{a}, \quad P(3) = \frac{3}{a}$

Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i) = 1$

$$\therefore \quad 0 + \frac{3}{a} + \frac{4}{a} + \frac{3}{a} = 1$$

$$\therefore \quad \frac{10}{a} = 1$$

$$\therefore \quad a = 10$$

b $\mathrm{P}(X = 1) = P(1) = \dfrac{3}{a} = \dfrac{3}{10}$

c Since $\mathrm{P}(X = 2) = P(2) = \frac{4}{10}$ is the greatest probability, the mode of the distribution is 2.

EXERCISE 27C.1

1 **a**

x_i	1	2	3
p_i	0.4	0.5	0.1

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 1(0.4) + 2(0.5) + 3(0.1) \\ &= 1.7\end{aligned}$$

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 0(0.1) + 1(0.2) + 2(0.15) + 3(0.2) + 4(0.35) \\ &= 2.5\end{aligned}$$

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 0(0.2) + 2(0.35) + 5(0.27) + 10(0.18) \\ &= 3.85\end{aligned}$$

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= 10\left(\tfrac{1}{4}\right) + 15\left(\tfrac{1}{3}\right) + 30\left(\tfrac{1}{12}\right) + 60\left(\tfrac{1}{3}\right) \\ &= 30\end{aligned}$$

2 **a** Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i) = 1$

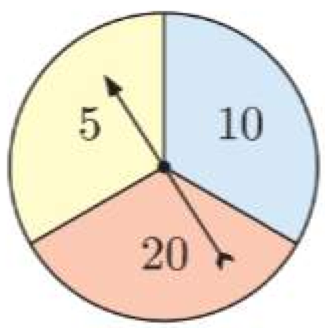
x	1	3	5
$\mathrm{P}(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

$$\therefore \quad \tfrac{2}{5} + a + \tfrac{1}{10} = 1$$
$$\therefore \quad a = \tfrac{1}{2}$$

b Since $\mathrm{P}(X = 3)$ is the greatest probability, 3 is the mode of the distribution.

c $$\begin{aligned}\mu = \mathrm{E}(X) &= 1\left(\tfrac{2}{5}\right) + 3\left(\tfrac{1}{2}\right) + 5\left(\tfrac{1}{10}\right) \\ &= \tfrac{2}{5} + \tfrac{3}{2} + \tfrac{5}{10} \\ &= \tfrac{4}{10} + \tfrac{15}{10} + \tfrac{5}{10} \\ &= 2\tfrac{2}{5}\end{aligned}$$

3 Each coloured region on the spinner has the same area.
The probability table is:



Number of points	5	10	20
Probability	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= \left(5 \times \tfrac{1}{3}\right) + \left(10 \times \tfrac{1}{3}\right) + \left(20 \times \tfrac{1}{3}\right) \\ &= \tfrac{35}{3} \\ &\approx 11.7 \text{ points}\end{aligned}$$

In the long term, we can expect to be awarded an average of about 11.7 points per spin.

4

Number of fish	0	1	2	3
Probability	0.17	0.28	0.36	0.19

$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= (0 \times 0.17) + (1 \times 0.28) + (2 \times 0.36) + (3 \times 0.19) \\ &= 0.28 + 0.72 + 0.57 \\ &= 1.57 \text{ fish} \end{aligned}$$

On average, you would expect Ernie to catch 1.57 fish per trip.

5

Number of books	1	2	3	4	5
Probability	0.16	0.15	a	0.28	0.16

a Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i) = 1$

$$\therefore \quad 0.16 + 0.15 + a + 0.28 + 0.16 = 1$$
$$\therefore \quad a = 0.25$$

b Pam is most likely to borrow 4 books when she visits the library, so this is the mode of the distribution.

c $$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= (1 \times 0.16) + (2 \times 0.15) + (3 \times 0.25) + (4 \times 0.28) + (5 \times 0.16) \\ &= 0.16 + 0.30 + 0.75 + 1.12 + 0.80 \\ &= 3.13 \text{ books} \end{aligned}$$

On average, Pam borrows 3.13 books per visit.

6

Colour	*Number of lollies*
Red	4
Green	6
White	10

There are 5 red balls, 2 green balls, and 1 white ball, so in total there are $5 + 2 + 1 = 8$ balls.

Number of lollies	4	6	10
Probability	$\frac{5}{8} = 0.625$	$\frac{2}{8} = 0.25$	$\frac{1}{8} = 0.125$

$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= (4 \times 0.625) + (6 \times 0.25) + (10 \times 0.125) \\ &= 2.5 + 1.5 + 1.25 \\ &= 5.25 \text{ lollies} \end{aligned}$$

On average, Lachlan can expect to receive 5.25 lollies.

7 **a** P(all ten pins) $= 1 - \frac{1}{3} - \frac{2}{5}$

$= \frac{15}{15} - \frac{5}{15} - \frac{6}{15}$

$= \frac{4}{15}$

b

Number of pins knocked down	8	9	10
Probability	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{4}{15}$

$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{n} x_i p_i \\ &= \left(8 \times \tfrac{1}{3}\right) + \left(9 \times \tfrac{2}{5}\right) + \left(10 \times \tfrac{4}{15}\right) \\ &= \tfrac{40}{15} + \tfrac{54}{15} + \tfrac{40}{15} \\ &= \tfrac{134}{15} \\ &\approx 8.93 \text{ pins} \end{aligned}$$

On average, Jenna knocks down about 8.93 pins with her first bowl.

8 **a** When Brad's soccer team plays an offensive strategy, P(draw) $= 1 - 0.3 - 0.55 = 0.15$
When Brad's soccer team plays a defensive strategy, P(draw) $= 1 - 0.2 - 0.3 = 0.5$

b Let X be the number of points awarded per game when Brad's soccer team plays an offensive strategy.

$$\begin{aligned} \mathrm{E}(X) &= (3 \times 0.3) + (1 \times 0.15) + (0 \times 0.55) \\ &= 0.9 + 0.15 \\ &= 1.05 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	3	1	0
Probability	0.3	0.15	0.55

Let Y be the number of points awarded per game when Brad's soccer team plays a defensive strategy.

$$\begin{aligned} \mathrm{E}(Y) &= (3 \times 0.2) + (1 \times 0.5) + (0 \times 0.3) \\ &= 0.6 + 0.5 \\ &= 1.1 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	3	1	0
Probability	0.2	0.5	0.3

c It is better for the team to play a defensive strategy in the long run as the team is expected to gain more points per game.

d If 4 points are awarded instead of 3 points for a win:

$$\begin{aligned} \mathrm{E}(X) &= (4 \times 0.3) + (1 \times 0.15) + (0 \times 0.55) \\ &= 1.2 + 0.15 \\ &= 1.35 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	4	1	0
Probability	0.3	0.15	0.55

and
$$\begin{aligned} \mathrm{E}(Y) &= (4 \times 0.2) + (1 \times 0.5) + (0 \times 0.3) \\ &= 0.8 + 0.5 \\ &= 1.3 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	4	1	0
Probability	0.2	0.5	0.3

The team is expected to gain more points per game when they play an offensive strategy. The team should change their strategy.

9 Since this is a probability distribution,

x	1	2	3	4
$P(X = x)$	0.3	a	b	0.2

$$\sum_{i=1}^{n} P(x_i) = 1$$

$\therefore\ 0.3 + a + b + 0.2 = 1$

$\therefore\ b = 0.5 - a$ .... $(*)$

Now, $E(X) = 2.5$

$\therefore\ (1 \times 0.3) + (2 \times a) + (3 \times b) + (4 \times 0.2) = 2.5$

$\therefore\ 0.3 + 2a + 3(0.5 - a) + 0.8 = 2.5$ $\{$using $(*)\}$

$\therefore\ 2a + 1.5 - 3a = 1.4$

$\therefore\ a = 0.1$ and $b = 0.4$

10 **a** **i** car park B
ii car park A
iii car park B

Car park A

Time	*Cost*
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

Time	*Cost*
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

b Let \$$X$ be the amount Zoe pays for parking.

When Zoe parks her car at car park A:

$$\begin{aligned} E(X) &= (7 \times 0) + (12 \times 0.2) + (15 \times 0.7) + (19 \times 0.1) \\ &= 2.4 + 10.5 + 1.9 \\ &= \$14.80 \end{aligned}$$

When Zoe parks her car at car park B:

$$\begin{aligned} E(X) &= (6.5 \times 0) + (11 \times 0.2) + (16 \times 0.7) + (18.5 \times 0.1) \\ &= 2.2 + 11.2 + 1.85 \\ &= \$15.25 \end{aligned}$$

$\therefore$ Zoe should choose car park A as it has the lower expected cost.

11 The probability of the ring not being stolen or lost is $\begin{aligned} P(\text{ring is safe}) &= 1 - 0.0025 - 0.03 \\ &= 0.9675 \end{aligned}$

Let \$$X$ be the amount the insurance company pays the policy owner.

$$\begin{aligned} E(X) &= (0 \times 0.9675) + (20\,000 \times 0.0025) + (8000 \times 0.03) \\ &= 50 + 240 \\ &= \$290 \text{ per policy} \end{aligned}$$

$\therefore$ the insurance company should charge \$390 per policy to have an expected return of \$100.

EXERCISE 27C.2

1 Let X denote the return from one game.

Number	1	2	3	4	5	6
Winnings	\$3	\$1	\$3	\$1	\$3	\$1
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned}\mathrm{E}(X) &= 3\left(\tfrac{1}{6}\times 3\right)+3\left(\tfrac{1}{6}\times 1\right)\\ &= \tfrac{9}{6}+\tfrac{3}{6}\\ &= \tfrac{12}{6}\\ &= 2\end{aligned}$$

So, \$2 is the expected return.

Since the game costs \$2 to play, the expected gain $=$ expected return $-$ \$2
$= \$2 - \$2 = \$0$

Since the expected gain is zero, the game is fair.

2 **a** Let X denote the return from each roll.

$$\begin{aligned}\mathrm{E}(X) &= \left(\tfrac{1}{6}\times 1\right)+\left(\tfrac{1}{6}\times 2\right)+\left(\tfrac{1}{6}\times 3\right)+\left(\tfrac{1}{6}\times 4\right)+\left(\tfrac{1}{6}\times 5\right)+\left(\tfrac{1}{6}\times 6\right)\\ &= \tfrac{1}{6}\times 21\\ &= \$3.50\end{aligned}$$

b The expected gain is $\$3.50 - \$4 = -\$0.50$

c The player should not play many games, as on average he would expect to lose \$0.50 with each roll.

3 **a** Let X denote the return from each bet.

$$\begin{aligned}\mathrm{E}(X) &= \left(\tfrac{18}{37}\times 2\right)+\left(\tfrac{19}{37}\times -2\right)\\ &= \tfrac{36}{37}-\tfrac{38}{37}\\ &= -\tfrac{2}{37}\\ &\approx -\$0.05\end{aligned}$$

b $100 \times -\frac{2}{37} \approx -\$5.41$

From 100 bets, I would expect to lose about \$5.41 .

4

Result	*Win*
HH	\$10
HT or TH	\$3
TT	\$1

Let X be the gain from each game, and
Y be the return from each game.

$$\begin{aligned}\mathrm{E}(Y) &= \left(\tfrac{1}{4}\times 10\right)+\left(\tfrac{2}{4}\times 3\right)+\left(\tfrac{1}{4}\times 1\right)\\ &= \tfrac{10}{4}+\tfrac{6}{4}+\tfrac{1}{4}\\ &= \$4.25\end{aligned}$$

The expected return per game is \$4.25. It costs \$5.00 to play the game.

So, the expected gain $\mathrm{E}(X) = \mathrm{E}(Y) - \$5$
$= \$4.25 - \$5.00$
$= -\$0.75$

So we expect a loss of \$0.75 per game on average.

5 **a**

Disc colour	Black	Blue	Gold
Winnings	\$1	\$5	\$20
Probability	$\frac{10}{15}$	$\frac{4}{15}$	$\frac{1}{15}$

Let X be the return from each game.

$$\begin{aligned} E(X) &= \left(1 \times \tfrac{10}{15}\right) + \left(5 \times \tfrac{4}{15}\right) + \left(20 \times \tfrac{1}{15}\right) \\ &= \frac{10+20+20}{15} \\ &= \frac{50}{15} \approx \$3.33 \end{aligned}$$

The expected return per game is \$3.33. It costs \$4.00 to play the game.

So, the expected gain $\approx \$3.33 - \$4.00$

$\approx -\$0.67 \neq \$0$, so the game is not fair.

b Let the new prize money for selecting the gold disc be $\$x$.

Now, for the game to be fair, the expected return must be equal to the cost of each game.

$\therefore\ E(X) = \left(1 \times \frac{10}{15}\right) + \left(5 \times \frac{4}{15}\right) + \left(x \times \frac{1}{15}\right) = 4$ {the cost of the game is \$4}

$$\begin{aligned} \therefore\ \tfrac{10}{15} + \tfrac{20}{15} + \frac{x}{15} &= 4 \\ \therefore\ \frac{30+x}{15} &= 4 \\ \therefore\ 30 + x &= 60 \\ \therefore\ x &= 30 \end{aligned}$$

So, the new prize money for selecting the gold disc is \$30.

6 **a** **i** $P(\text{win 5 tokens}) = \frac{6}{20}$ {there are 6 multiples of 3 between 1 and 20}

$= \frac{3}{10}$

$= 0.3$

ii $P(\text{win 10 tokens}) = \frac{2}{20}$ {there are 2 multiples of 10 between 1 and 20}

$= \frac{1}{10}$

$= 0.1$

b
$$\begin{aligned} E(X) &= (0 \times 0.6) + (5 \times 0.3) + (10 \times 0.1) \\ &= 1.5 + 1 \\ &= 2.5 \text{ tokens} \end{aligned}$$

c It costs 3 tokens to play the game. So, the expected gain $= 2.5 - 3 = -0.5$ tokens.

We do not recommend playing the game many times as the player can expect to lose half a token on average per game.

7 $P(RRR) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$

$P(BBB) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{24}{1320}$

$P(GGG) = \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} = \frac{6}{1320}$

$P(RBG) = P(RGB) = P(BRG) = P(BGR) = P(GRB) = P(GBR) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$

$$\begin{aligned} P(\text{winning}) &= P(\text{all the same colour or one of each}) \\ &= P(RRR) + P(BBB) + P(GGG) + P(RBG) + P(RGB) \\ &\quad + P(BRG) + P(BGR) + P(GRB) + P(GBR) \\ &= \tfrac{60}{1320} + \tfrac{24}{1320} + \tfrac{6}{1320} + \tfrac{60}{1320} \times 6 \\ &= \tfrac{60+24+6+360}{1320} \\ &= \tfrac{450}{1320} = \tfrac{15}{44} \end{aligned}$$

The player expects to win $11 \times \frac{15}{44} = \$3.75$

The organiser makes \$1 when the player loses \$1.

Now, the expected gain for the player = expected win − cost to play

$\therefore\ -\$1.00 = \$3.75 - \text{cost to play}$

$\therefore\ \text{cost to play} = \$4.75$

EXERCISE 27D

1 **a**

x	1	2	3
$\text{P}(X = x)$	0.3	0.4	0.3

i $\mu = \sum x_i p_i$
$= 1(0.3) + 2(0.4) + 3(0.3)$
$= 2$

ii $\sigma^2 = \sum (x_i - \mu)^2 p_i$
$= (1-2)^2(0.3) + (2-2)^2(0.4) + (3-2)^2(0.3)$
$= 0.6$

iii $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{0.6}$
≈ 0.775

b

x	0	1	2	3
$\text{P}(X = x)$	0.2	0.4	0.1	0.3

i $\mu = \sum x_i p_i$
$= 0(0.2) + 1(0.4) + 2(0.1) + 3(0.3)$
$= 1.5$

ii $\sigma^2 = \sum (x_i - \mu)^2 p_i$
$= (0-1.5)^2(0.2) + (1-1.5)^2(0.4) + (2-1.5)^2(0.1) + (3-1.5)^2(0.3)$
$= 1.25$

iii $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{1.25}$
≈ 1.12

2

x_i	2	4	10	20
p_i	k	0.05	0.35	$3k$

a Since this is a probability distribution, $\sum_{i=1}^{n} p_i = 1$

$\therefore\ k + 0.05 + 0.35 + 3k = 1$

$\therefore\ 4k + 0.4 = 1$

$\therefore\ 4k = 0.6$

$\therefore\ k = 0.15$

b $3k = 0.45$

$\therefore$ the value 20 has the highest probability of occurring, so this is the mode of the distribution.

c $\mu = \sum x_i p_i$
$= 2(0.15) + 4(0.05) + 10(0.35) + 20(0.45)$
$= 13$

d $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{(2-13)^2(0.15) + (4-13)^2(0.05) + (10-13)^2(0.35) + (20-13)^2(0.45)}$
$= \sqrt{47.4}$
≈ 6.88

3 *Michelle*:

Number of aces	0	1	2	3	4
Probability	0.1	0.15	0.45	0.25	0.05

Amanda:

Number of aces	0	1	2	3	4
Probability	0.2	0.1	0.35	0.2	0.15

a For Michelle, $\mu = \sum x_i p_i$
$= 0(0.1) + 1(0.15) + 2(0.45) + 3(0.25) + 4(0.05)$
$= 2$ aces

For Amanda, $\mu = \sum x_i p_i$
$= 0(0.2) + 1(0.1) + 2(0.35) + 3(0.2) + 4(0.15)$
$= 2$ aces

$\therefore$ each player is expected to serve an average of 2 aces per set.

b For Michelle,

$\sigma^2 = \sum (x_i - \mu)^2 p_i$
$= (0-2)^2(0.1) + (1-2)^2(0.15) + (2-2)^2(0.45) + (3-2)^2(0.25) + (4-2)^2(0.05)$
$= 1 \text{ ace}^2$

$\therefore \sigma = \sqrt{1} = 1$ ace

For Amanda,

$\sigma^2 = \sum (x_i - \mu)^2 p_i$
$= (0-2)^2(0.2) + (1-2)^2(0.1) + (2-2)^2(0.35) + (3-2)^2(0.2) + (4-2)^2(0.15)$
$= 1.7 \text{ aces}^2$

$\therefore \sigma = \sqrt{1.7} \approx 1.30$ aces

c From **b**, Amanda has a higher variance and standard deviation.

$\therefore$ Amanda has the greater variation in the number of aces served.

4

x	0	1	2	3	4	5	>5
$\mathrm{P}(X=x)$	0.54	0.26	0.15	k	0.01	0.01	0.00

a Since this is a probability distribution, $\sum_{i=1}^{n} \mathrm{P}(x_i) = 1$

$$\therefore \quad 0.54 + 0.26 + 0.15 + k + 0.01 + 0.01 + 0.00 = 1$$
$$\therefore \quad k + 0.97 = 1$$
$$\therefore \quad k = 0.03$$

b $\mu = \sum x_i p_i$

$= 0(0.54) + 1(0.26) + 2(0.15) + 3(0.03) + 4(0.01) + 5(0.01)$

$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$

$= 0.74$

So, over a long period the mean number of deaths per dozen crayfish is 0.74.

c $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

$= \sqrt{(0-0.74)^2(0.54) + (1-0.74)^2(0.26) + (2-0.74)^2(0.15) + ... + (5-0.74)^2(0.01)}$

$= \sqrt{0.9924}$

≈ 0.996

5 **a**

x	1	2	3
$\mathrm{P}(X=x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

b $\mu = \sum x_i p_i$

$= 1(\frac{2}{6}) + 2(\frac{1}{6}) + 3(\frac{3}{6})$

$= \frac{2}{6} + \frac{2}{6} + \frac{9}{6}$

$= \frac{13}{6}$

≈ 2.17

c $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$

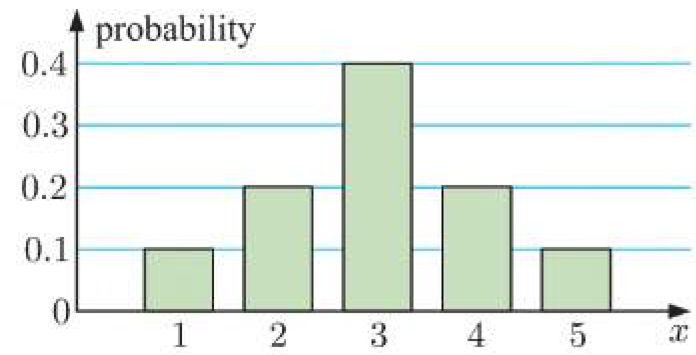
$= \sqrt{(1 - \frac{13}{6})^2(\frac{2}{6}) + (2 - \frac{13}{6})^2(\frac{1}{6}) + (3 - \frac{13}{6})^2(\frac{3}{6})}$

$= \sqrt{\frac{29}{36}}$

≈ 0.898

6 **a**

x_i	1	2	3	4	5
p_i	0.1	0.2	0.4	0.2	0.1



b $\mu = \sum x_i p_i$
$= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1)$
$= 3$

$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$
$= \sqrt{\sum x_i^{\,2} p_i - \mu^2}$
$= \sqrt{1^2(0.1) + 2^2(0.2) + 3^2(0.4) + 4^2(0.2) + 5^2(0.1) - 3^2}$
$= \sqrt{1.2}$
≈ 1.10

7 $\sigma^2 = \mathrm{E}[(X - \mu)^2]$
$= \sum (x_i - \mu)^2 p_i$
$= (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \ldots\ldots + (x_n - \mu)^2 p_n$
$= (x_1^{\,2} - 2x_1\mu + \mu^2)p_1 + (x_2^{\,2} - 2x_2\mu + \mu^2)p_2 + \ldots\ldots + (x_n^{\,2} - 2x_n\mu + \mu^2)p_n$
$= (x_1^{\,2}p_1 + x_2^{\,2}p_2 + x_3^{\,2}p_3 + \ldots\ldots + x_n^{\,2}p_n) - 2\mu(x_1p_1 + x_2p_2 + \ldots\ldots + x_np_n)$
$\quad + \mu^2(p_1 + p_2 + p_3 + \ldots\ldots + p_n)$
$= \sum x_i^{\,2}p_i - 2\mu(\sum x_ip_i) + \mu^2(1)$ $\quad \{p_1 + p_2 + \ldots\ldots + p_n = 1\}$
$= \sum x_i^{\,2}p_i - 2\mu(\mu) + \mu^2$ $\quad \{\text{since } \sum x_ip_i = \mu\}$
$= \sum x_i^{\,2}p_i - \mu^2$
$= \mathrm{E}(X^2) - (\mathrm{E}(X))^2$ as required.

8 **a** **i** Y would have the greater mean as, on average, we are more likely to obtain higher values from the maximum of two rolls than from a single roll.

ii X would have the greater standard deviation as the probabilities of obtaining each outcome are more spread out.

b

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

For the probability distribution of Y, we can see that:

$\mathrm{P}(Y = 1) = \mathrm{P}(\text{first roll} = 1) \times \mathrm{P}(\text{second roll} = 1)$
$= \frac{1}{4} \times \frac{1}{4}$
$= \frac{1}{16}$

$$\begin{aligned}
P(Y=2) &= P(\text{first roll}=1) \times P(\text{second roll}=2) \\
&\quad + P(\text{first roll}=2) \times P(\text{second roll}=1) \\
&\quad + P(\text{first roll}=2) \times P(\text{second roll}=2) \\
&= \tfrac{1}{4} \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{4} \\
&= \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} \\
&= \tfrac{3}{16}
\end{aligned}$$

$$\begin{aligned}
P(Y=3) &= P(\text{first roll} \leqslant 2) \times P(\text{second roll}=3) \\
&\quad + P(\text{first roll}=3) \times P(\text{second roll} \leqslant 2) \\
&\quad + P(\text{first roll}=3) \times P(\text{second roll}=3) \\
&= (\tfrac{1}{4} + \tfrac{1}{4}) \times \tfrac{1}{4} + \tfrac{1}{4} \times (\tfrac{1}{4} + \tfrac{1}{4}) + \tfrac{1}{4} \times \tfrac{1}{4} \\
&= \tfrac{2}{16} + \tfrac{2}{16} + \tfrac{1}{16} \\
&= \tfrac{5}{16}
\end{aligned}$$

$$\begin{aligned}
P(Y=4) &= P(\text{first roll} \leqslant 3) \times P(\text{second roll}=4) \\
&\quad + P(\text{first roll}=4) \times P(\text{second roll} \leqslant 3) \\
&\quad + P(\text{first roll}=4) \times P(\text{second roll}=4) \\
&= (\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}) \times \tfrac{1}{4} + \tfrac{1}{4} \times (\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}) + \tfrac{1}{4} \times \tfrac{1}{4} \\
&= \tfrac{3}{16} + \tfrac{3}{16} + \tfrac{1}{16} \\
&= \tfrac{7}{16}
\end{aligned}$$

y	1	2	3	4
$P(Y=y)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

c X:

$$\begin{aligned}
\mu &= \sum x_i p_i \\
&= 1(\tfrac{1}{4}) + 2(\tfrac{1}{4}) + 3(\tfrac{1}{4}) + 4(\tfrac{1}{4}) \\
&= \tfrac{1}{4} + \tfrac{2}{4} + \tfrac{3}{4} + \tfrac{4}{4} \\
&= \tfrac{10}{4} \\
&= 2.5
\end{aligned}$$

$$\begin{aligned}
\sigma &= \sqrt{\sum {x_i}^2 p_i - \mu^2} \\
&= \sqrt{1^2(\tfrac{1}{4}) + 2^2(\tfrac{1}{4}) + 3^2(\tfrac{1}{4}) + 4^2(\tfrac{1}{4}) - (\tfrac{10}{4})^2} \\
&= \sqrt{\tfrac{5}{4}} \\
&\approx 1.12
\end{aligned}$$

$$
\begin{aligned}
Y: \quad \mu &= \sum x_i p_i \\
&= 1(\tfrac{1}{16}) + 2(\tfrac{3}{16}) + 3(\tfrac{5}{16}) + 4(\tfrac{7}{16}) \\
&= \tfrac{1}{16} + \tfrac{6}{16} + \tfrac{15}{16} + \tfrac{28}{16} \\
&= \tfrac{50}{16} \\
&= 3.125
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= \sqrt{\sum x_i{}^2 p_i - \mu^2} \\
&= \sqrt{1^2(\tfrac{1}{16}) + 2^2(\tfrac{3}{16}) + 3^2(\tfrac{5}{16}) + 4^2(\tfrac{7}{16}) - (\tfrac{50}{16})^2} \\
&= \sqrt{\tfrac{55}{64}} \\
&\approx 0.927
\end{aligned}
$$

INVESTIGATION 1 PROPERTIES OF $aX + b$

1

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

a

$$
\begin{aligned}
\mathrm{E}(X) &= \sum x_i p_i \\
&= 1(0.2) + 2(0.2) + 3(0.2) + 4(0.2) + 5(0.2) \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X) &= \sum x_i{}^2 p_i - \mu^2 \\
&= 1^2(0.2) + 2^2(0.2) + 3^2(0.2) + 4^2(0.2) + 5^2(0.2) - 3^2 \\
&= 2
\end{aligned}
$$

b $Y = 2X + 3$

y	5	7	9	11	13
$\mathrm{P}(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$
\begin{aligned}
\mathrm{E}(2X + 3) &= \mathrm{E}(Y) \\
&= \sum y_i p_i \\
&= 5(0.2) + 7(0.2) + 9(0.2) + 11(0.2) + 13(0.2) \\
&= 9
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(2X + 3) &= \mathrm{Var}(Y) \\
&= \sum y_i{}^2 p_i - \mu^2 \\
&= 5^2(0.2) + 7^2(0.2) + 9^2(0.2) + 11^2(0.2) + 13^2(0.2) - 9^2 \\
&= 8
\end{aligned}
$$

c **i** $Y = 3X - 2$

y	1	4	7	10	13
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= 1(0.2) + 4(0.2) + 7(0.2) + 10(0.2) + 13(0.2) \\ &= 7 \end{aligned}$$

$$\begin{aligned} Var(Y) &= \sum {y_i}^2 p_i - \mu^2 \\ &= 1^2(0.2) + 4^2(0.2) + 7^2(0.2) + 10^2(0.2) + 13^2(0.2) - 7^2 \\ &= 18 \end{aligned}$$

ii $Y = -2X + 5$

y	3	1	-1	-3	-5
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= 3(0.2) + 1(0.2) + (-1)(0.2) + (-3)(0.2) + (-5)(0.2) \\ &= -1 \end{aligned}$$

$$\begin{aligned} Var(Y) &= \sum {y_i}^2 p_i - \mu^2 \\ &= 3^2(0.2) + 1^2(0.2) + (-1)^2(0.2) + (-3)^2(0.2) + (-5)^2(0.2) - (-1)^2 \\ &= 8 \end{aligned}$$

iii $Y = \dfrac{X+1}{2}$

y	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= 1(0.2) + \tfrac{3}{2}(0.2) + 2(0.2) + \tfrac{5}{2}(0.2) + 3(0.2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} Var(Y) &= \sum {y_i}^2 p_i - \mu^2 \\ &= 1^2(0.2) + (\tfrac{3}{2})^2(0.2) + 2^2(0.2) + (\tfrac{5}{2})^2(0.2) + 3^2(0.2) - 2^2 \\ &= \tfrac{1}{2} \end{aligned}$$

iv $Y = \dfrac{-X+2}{3}$

y	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1
$P(Y=y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned}E(Y) &= \sum y_i p_i \\ &= \tfrac{1}{3}(0.2) + 0(0.2) + (-\tfrac{1}{3})(0.2) + (-\tfrac{2}{3})(0.2) + (-1)(0.2) \\ &= -\tfrac{1}{3}\end{aligned}$$

$$\begin{aligned}\text{Var}(Y) &= \sum {y_i}^2 p_i - \mu^2 \\ &= (\tfrac{1}{3})^2(0.2) + 0^2(0.2) + (-\tfrac{1}{3})^2(0.2) + (-\tfrac{2}{3})^2(0.2) + (-1)^2(0.2) - (-\tfrac{1}{3})^2 \\ &= \tfrac{2}{9}\end{aligned}$$

3 **a** $E(aX+b) = a\,E(X) + b$

b $\text{Var}(aX+b) = a^2\,\text{Var}(X)$

4 $$\begin{aligned}\sigma(aX+b) &= \sqrt{\text{Var}(aX+b)} \\ &= \sqrt{a^2\,\text{Var}(X)} \qquad \{\text{using } \mathbf{3\ b}\} \\ &= \sqrt{a^2}\sqrt{\text{Var}(X)} \\ &= |\,a\,|\,\sigma(X)\end{aligned}$$

EXERCISE 27E

1 $$\begin{aligned}E(aX+b) &= E(aX) + E(b) && \{\text{using } E[g(X)+h(X)] = E[g(X)] + E[h(X)]\} \\ &= a\,E(X) + E(b) && \{\text{using } E(kX) = k\,E(X)\} \\ &= a\,E(X) + b && \{\text{using } E(k) = k,\ \ k \text{ a constant}\}\end{aligned}$$

2 **a** $$\begin{aligned}E(Y) &= E(3X+4) \\ &= 3\,E(X) + 4 \\ &= 3(3) + 4 \\ &= 13\end{aligned}$$

b $$\begin{aligned}E(Y) &= E(-2X+1) \\ &= -2\,E(X) + 1 \\ &= -2(3) + 1 \\ &= -5\end{aligned}$$

c $$\begin{aligned}E(Y) &= E\left(\frac{4X-2}{3}\right) \\ &= E\left(\tfrac{4}{3}X - \tfrac{2}{3}\right) \\ &= \tfrac{4}{3}\,E(X) - \tfrac{2}{3} \\ &= \tfrac{4}{3}(3) - \tfrac{2}{3} \\ &= 3\tfrac{1}{3}\end{aligned}$$

3 X has mean 6 and standard deviation 2.

$$\begin{aligned}E(Y) &= E(2X+5) \\ &= 2\,E(X) + 5 \\ &= 2(6) + 5 \\ &= 17\end{aligned}$$

$$\begin{aligned}\sigma(Y) &= \sigma(2X+5) \\ &= |\,2\,|\,\sigma(X) \\ &= 2(2) \\ &= 4\end{aligned}$$

For Y, the mean is 17 and the standard deviation is 4.

4 X has mean 5 and standard deviation 2.

a
$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(2X+3)\\ &= 2\,\mathrm{E}(X)+3\\ &= 2(5)+3\\ &= 13\end{aligned}$$
$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(2X+3)\\ &= 2^2\,\mathrm{Var}(X)\\ &= 4\times 2^2\\ &= 16\end{aligned}$$

b
$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(-5X+3)\\ &= -5\,\mathrm{E}(X)+3\\ &= -5(5)+3\\ &= -22\end{aligned}$$
$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(-5X+3)\\ &= (-5)^2\mathrm{Var}(X)\\ &= 25\times 2^2\\ &= 100\end{aligned}$$

c $Y = \dfrac{X-5}{2} = \frac{1}{2}X - \frac{5}{2}$
$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}\left(\tfrac{1}{2}X-\tfrac{5}{2}\right)\\ &= \tfrac{1}{2}\,\mathrm{E}(X)-\tfrac{5}{2}\\ &= \tfrac{1}{2}(5)-\tfrac{5}{2}\\ &= 0\end{aligned}$$
$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}\left(\tfrac{1}{2}X-\tfrac{5}{2}\right)\\ &= \left(\tfrac{1}{2}\right)^2\mathrm{Var}(X)\\ &= \tfrac{1}{4}\times 2^2\\ &= 1\end{aligned}$$

5

x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

a
$$\begin{aligned}\mathrm{E}(X) &= \textstyle\sum x_i p_i\\ &= 1(0.4)+2(0.3)+3(0.2)+4(0.1)\\ &= 2\end{aligned}$$

b
$$\begin{aligned}\mathrm{Var}(X) &= \textstyle\sum (x_i-\mu)^2 p_i\\ &= (1-2)^2(0.4)+(2-2)^2(0.3)+(3-2)^2(0.2)+(4-2)^2(0.1)\\ &= 1\end{aligned}$$

c
$$\begin{aligned}\sigma(X) &= \sqrt{\mathrm{Var}(X)}\\ &= \sqrt{1}\\ &= 1\end{aligned}$$

d
$$\begin{aligned}&\mathrm{E}(X+1)\\ &= \mathrm{E}(X)+1\\ &= 2+1\\ &= 3\end{aligned}$$

e
$$\begin{aligned}&\mathrm{Var}(3X+1)\\ &= 3^2\,\mathrm{Var}(X)\\ &= 9(1)\\ &= 9\end{aligned}$$

f
$$\begin{aligned}&\sigma(5-X)\\ &= |-1|\,\sigma(X)\\ &= 1(1)\\ &= 1\end{aligned}$$

g
$$\begin{aligned}&\mathrm{E}\left(\frac{2X+5}{3}\right)\\ &= \mathrm{E}\left(\tfrac{2}{3}X+\tfrac{5}{3}\right)\\ &= \tfrac{2}{3}\,\mathrm{E}(X)+\tfrac{5}{3}\\ &= \tfrac{2}{3}(2)+\tfrac{5}{3}\\ &= 3\end{aligned}$$

h
$$\begin{aligned}&\mathrm{Var}(20-4X)\\ &= (-4)^2\,\mathrm{Var}(X)\\ &= 16\times 1\\ &= 16\end{aligned}$$

6

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	a	$\frac{1}{6}$

a Since this is a probability distribution, $\sum_{i=1}^{n} P(x_i) = 1$

$$\therefore\ \tfrac{1}{6} + \tfrac{1}{3} + \tfrac{1}{12} + a + \tfrac{1}{6} = 1$$
$$\therefore\ a + \tfrac{9}{12} = 1$$
$$\therefore\ a = \tfrac{1}{4}$$

b **i**
$$\begin{aligned} \mathrm{E}(X) &= \sum x_i p_i \\ &= 1(\tfrac{1}{6}) + 2(\tfrac{1}{3}) + 3(\tfrac{1}{12}) + 4(\tfrac{1}{4}) + 5(\tfrac{1}{6}) \\ &= \tfrac{1}{6} + \tfrac{2}{3} + \tfrac{1}{4} + 1 + \tfrac{5}{6} \\ &= 2\tfrac{11}{12} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \sum {x_i}^2 p_i - \mu^2 \\ &= 1^2(\tfrac{1}{6}) + 2^2(\tfrac{1}{3}) + 3^2(\tfrac{1}{12}) + 4^2(\tfrac{1}{4}) + 5^2(\tfrac{1}{6}) - (2\tfrac{11}{12})^2 \\ &= \tfrac{1}{6} + \tfrac{4}{3} + \tfrac{3}{4} + 4 + \tfrac{25}{6} - \tfrac{1225}{144} \\ &= \tfrac{275}{144} \\ &\approx 1.91 \end{aligned}$$

$$\begin{aligned} \sigma(X) &= \sqrt{\mathrm{Var}(X)} \\ &= \sqrt{\tfrac{275}{144}} \\ &\approx 1.38 \end{aligned}$$

ii
$$\begin{aligned} \mathrm{E}(X+4) &= \mathrm{E}(X) + 4 \\ &= 2\tfrac{11}{12} + 4 \\ &= 6\tfrac{11}{12} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X+4) &= 1^2\,\mathrm{Var}(X) \\ &= \mathrm{Var}(X) \\ &\approx 1.91 \end{aligned}$$

$$\begin{aligned} \sigma(X+4) &= |\,1\,|\ \sigma(X) \\ &= \sigma(X) \\ &\approx 1.38 \end{aligned}$$

iii
$$\begin{aligned} \mathrm{E}(3X-1) &= 3\,\mathrm{E}(X) - 1 \\ &= 3(2\tfrac{11}{12}) - 1 \\ &= 7\tfrac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(3X-1) &= 3^2\,\mathrm{Var}(X) \\ &= 9 \times \tfrac{275}{144} \\ &\approx 17.2 \end{aligned}$$

$$\begin{aligned} \sigma(3X-1) &= |\,3\,|\ \sigma(X) \\ &= 3\,\sigma(X) \\ &= 3 \times \sqrt{\tfrac{275}{144}} \\ &\approx 4.15 \end{aligned}$$

7

x	0	1	2	3	4	5
$\mathrm{P}(X=x)$	0.17	0.25	0.3	0.15	0.1	0.03

a **i** $\begin{aligned}\mathrm{E}(X) &= \sum x_i p_i \\ &= 0(0.17)+1(0.25)+2(0.3)+3(0.15)+4(0.1)+5(0.03) \\ &= 0.25+0.6+0.45+0.4+0.15 \\ &= 1.85\end{aligned}$

ii $\begin{aligned}\mathrm{Var}(X) &= \sum {x_i}^2 p_i - \mu^2 \\ &= 0^2(0.17)+1^2(0.25)+2^2(0.3)+3^2(0.15)+4^2(0.1)+5^2(0.03)-(1.85)^2 \\ &= 1.7275 \\ &\approx 1.73\end{aligned}$

iii $\begin{aligned}\sigma(X) &= \sqrt{\mathrm{Var}(X)} \\ &= \sqrt{1.7275} \\ &\approx 1.31\end{aligned}$

b Dominic earns \$100 plus \$25 for each sale made.

$\therefore \quad Y = 25X + 100$

c **i** $\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(25X+100) \\ &= 25\,\mathrm{E}(X)+100 \\ &= 25(1.85)+100 \\ &= 146.25\end{aligned}$

ii $\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(25X+100) \\ &= 25^2\,\mathrm{Var}(X) \\ &= 625(1.7275) \\ &\approx 1080\end{aligned}$

iii $\begin{aligned}\sigma(Y) &= \mathrm{E}(25X+100) \\ &= |25|\,\sigma(X) \\ &= 25\times\sqrt{1.7275} \\ &\approx 32.9\end{aligned}$

8 $\begin{aligned}\mathrm{Var}(aX+b) &= \mathrm{E}\left[(aX+b-\mathrm{E}(aX+b))^2\right] \\ &= \mathrm{E}\left[(aX+\cancel{b}-a\,\mathrm{E}(X)-\cancel{b})^2\right] \\ &= \mathrm{E}\left[(a(X-\mathrm{E}(X))^2\right] \\ &= \mathrm{E}\left[a^2(X-\mu)^2\right] \\ &= a^2\,\mathrm{E}\left[(X-\mu)^2\right] \\ &= a^2\,\mathrm{Var}(X) \quad \text{as required.}\end{aligned}$

EXERCISE 27F

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (a head or a tail) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement, as we do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

2

$$\begin{array}{ccccccccccccccl} & & & & & 1 & & 1 & & & & & & & n=1 \\ & & & & 1 & & 2 & & 1 & & & & & & n=2 \\ & & & 1 & & 3 & & 3 & & 1 & & & & & n=3 \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & & & n=4 \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & n=5 \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & n=6 \end{array}$$

3 The number of trials is $n = 4$.
The probability of success with each toss is $p = \frac{1}{2}$.
Let X be the number of heads tossed.

$\therefore\ X \sim \mathrm{B}(4, \frac{1}{2})$

$$\begin{aligned}\therefore\ \mathrm{P}(X = x) &= \binom{4}{x}\left(\tfrac{1}{2}\right)^x\left(1-\tfrac{1}{2}\right)^{4-x} \\ &= \binom{4}{x}\left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{4-x}\end{aligned}$$

a P(4 heads)
$$\begin{aligned}&= \mathrm{P}(X=4) \\ &= \binom{4}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^{4-4} \\ &= \left(\tfrac{1}{2}\right)^4 \\ &= \tfrac{1}{16}\end{aligned}$$

b P(3 heads)
$$\begin{aligned}&= \mathrm{P}(X=3) \\ &= \binom{4}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^{4-3} \\ &= 4\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^1 \\ &= \tfrac{4}{16} \\ &= \tfrac{1}{4}\end{aligned}$$

c P(2 heads)
$$\begin{aligned}&= \mathrm{P}(X=2) \\ &= \binom{4}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{4-2} \\ &= 6\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2 \\ &= \tfrac{6}{16} \\ &= \tfrac{3}{8}\end{aligned}$$

4 The number of trials is $n = 5$.
The probability of success with each toss is $p = \frac{1}{2}$.
Let X be the number of heads from each toss.

$\therefore\ X \sim \mathrm{B}(5, \frac{1}{2})$

$$\begin{aligned}\therefore\ \mathrm{P}(X = x) &= \binom{5}{x}\left(\tfrac{1}{2}\right)^x\left(1-\tfrac{1}{2}\right)^{5-x} \\ &= \binom{5}{x}\left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{5-x}\end{aligned}$$

a P(4 heads)
$$\begin{aligned}&= \mathrm{P}(X=4) \\ &= \binom{5}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^{5-4} \\ &= 5\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right) \\ &= \tfrac{5}{32}\end{aligned}$$

b P(2 heads)
$$\begin{aligned}&= \mathrm{P}(X=2) \\ &= \binom{5}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{5-2} \\ &= 10\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^3 \\ &= \tfrac{10}{32} \\ &= \tfrac{5}{16}\end{aligned}$$

c P(4 heads *then* 1 tail)
$$\begin{aligned}&= \mathrm{P}(\mathrm{HHHHT}) \\ &= \left(\tfrac{1}{2}\right)^4 \times \left(\tfrac{1}{2}\right) \\ &= \tfrac{1}{32}\end{aligned}$$

5 The number of trials is $n = 4$.

The probability of success (getting a strawberry cream) is $p = \dfrac{2}{2+1} = \frac{2}{3}$.

Let X be the number of strawberry creams selected.

$\therefore \quad X \sim \mathrm{B}(4, \frac{2}{3})$

$$\therefore \quad \mathrm{P}(X = x) = \binom{4}{x}\left(\tfrac{2}{3}\right)^x\left(1-\tfrac{2}{3}\right)^{4-x} = \binom{4}{x}\left(\tfrac{2}{3}\right)^x\left(\tfrac{1}{3}\right)^{4-x}$$

a P(all strawberry creams)
$$= \mathrm{P}(X=4) = \binom{4}{4}\left(\tfrac{2}{3}\right)^4\left(\tfrac{1}{3}\right)^{4-4} = \left(\tfrac{2}{3}\right)^4 = \tfrac{16}{81}$$

b P(two of each type)
$$= \mathrm{P}(X=2) = \binom{4}{2}\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^{4-2} = 6\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^2 = \tfrac{8}{27}$$

c P(at least 2 strawberry creams)
$$= \mathrm{P}(X \geqslant 2)$$
$$= \mathrm{P}(X=2) + \mathrm{P}(X=3) + \mathrm{P}(X=4)$$
$$= \tfrac{8}{27} + \binom{4}{3}\left(\tfrac{2}{3}\right)^3\left(\tfrac{1}{3}\right)^1 + \tfrac{16}{81} \qquad \{\text{using } \mathbf{a} \text{ and } \mathbf{c}\}$$
$$= \tfrac{24}{81} + 4\left(\tfrac{8}{27}\right)\left(\tfrac{1}{3}\right) + \tfrac{16}{81}$$
$$= \tfrac{72}{81}$$
$$= \tfrac{8}{9}$$

6 The number of trials is $n = 6$.

The probability of success (selecting a "flat back") is $p = \dfrac{1}{3+1} = \frac{1}{4}$

Let X be the number of "flat backs" selected.

$\therefore \quad X \sim \mathrm{B}(6, \frac{1}{4})$

$$\therefore \quad \mathrm{P}(X = x) = \binom{6}{x}\left(\tfrac{1}{4}\right)^x\left(1-\tfrac{1}{4}\right)^{6-x} = \binom{6}{x}\left(\tfrac{1}{4}\right)^x\left(\tfrac{3}{4}\right)^{6-x}$$

a P(two "flat backs")
$$= \mathrm{P}(X=2) = \binom{6}{2}\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^{6-2} = 15\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^4 = \tfrac{1215}{4096}$$

b P(at least 3 "flat backs")
$$= \mathrm{P}(X \geqslant 3)$$
$$= \mathrm{P}(X=3) + \mathrm{P}(X=4) + \mathrm{P}(X=5) + \mathrm{P}(X=6)$$
$$= \binom{6}{3}\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^3 + \binom{6}{4}\left(\tfrac{1}{4}\right)^4\left(\tfrac{3}{4}\right)^2 + \binom{6}{5}\left(\tfrac{1}{4}\right)^5\left(\tfrac{3}{4}\right) + \binom{6}{6}\left(\tfrac{1}{4}\right)^6$$
$$= \frac{20\times 27}{4096} + \frac{15\times 9}{4096} + \frac{6\times 3}{4096} + \frac{1\times 1}{4096}$$
$$= \tfrac{694}{4096} = \tfrac{347}{2048}$$

c P(at most 3 "normal" kiwis) = P(at least 3 "flat backs")
$$= \tfrac{347}{2048} \qquad \{\text{from } \mathbf{b}\}$$

INVESTIGATION 2 THE GRAPH OF A BINOMIAL DISTRIBUTION

1 **a** $X \sim \text{B}(n,\ p)$

When $n = 25$, $p = 0.1$, the mode of X is 2.

b The distribution is positively skewed.

2 When $p = 0.5$, the distribution is symmetric.

When $p < 0.5$, the distribution is positively skewed.

When $p > 0.5$, the distribution is negatively skewed.

3 $p = 0.1$, and the value of n is free to change.

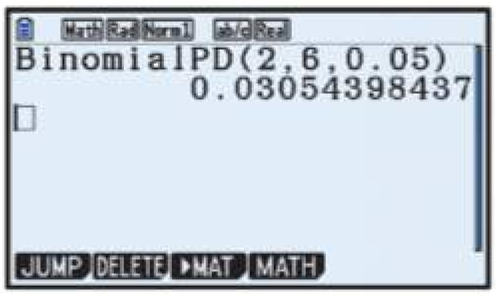
As n increases, the distribution becomes approximately symmetrical.

EXERCISE 27G

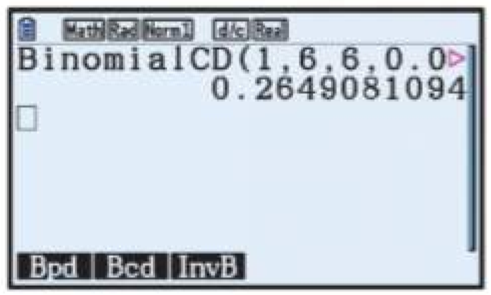
1 Let X be the number of defective light bulbs.

$n = 6$, so $X = 0, 1, 2, 3, 4, 5,$ or 6, and $p = 5\% = 0.05$

$\therefore\ X \sim \text{B}(6,\ 0.05)$

a 

$\text{P}(X = 2) \approx 0.0305$

b 

$\text{P}(X \geqslant 1) \approx 0.265$

2 Let X be the number of faulty items.

$n = 12$, so $X = 0, 1, 2, 3, \ldots,$ or 12, and $p = 6\% = 0.06$

$\therefore\ X \sim \text{B}(12,\ 0.06)$

a P(none will be faulty)

$= \text{P}(X = 0)$

$= \binom{12}{0}(0.06)^0(0.94)^{12}$

≈ 0.476

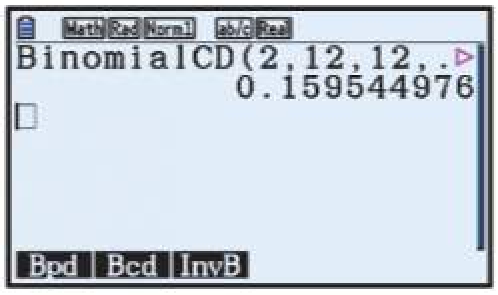
b P(at most one is faulty)

$= \text{P}(X \leqslant 1)$

$= \text{P}(X = 0) + \text{P}(X = 1)$

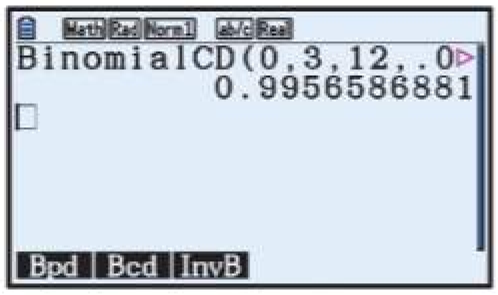
$\approx 0.476 + \binom{12}{1}(0.06)^1(0.94)^{11}$

≈ 0.840

c 

P(at least two are faulty) $= P(X \geqslant 2)$
≈ 0.160

or P(at least two are faulty)
$= 1 -$ P(at most one is faulty)
$\approx 1 - 0.840$ {from **b**}
≈ 0.160

d 

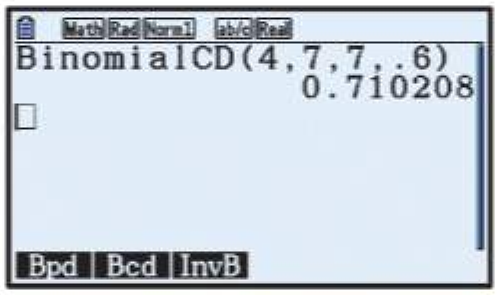
P(less than four are faulty) $= P(X < 4)$
$= P(X \leqslant 3)$
≈ 0.996

3 Let X be the number of times in a week that the bus is on time.
Since it is late 2 in every 5 days, then it is on time 3 in every 5 days, so $p = \frac{3}{5} = 0.6$.
$n = 7$, so $X = 0, 1, 2, 3, 4, 5, 6$, or 7, and $X \sim B(7, 0.6)$.

a $P(X = 7) = \binom{7}{7}(0.6)^7(0.4)^0$
≈ 0.0280

b P(on time only on Monday) $= 0.6 \times (0.4)^6$
$\approx 0.002\,46$

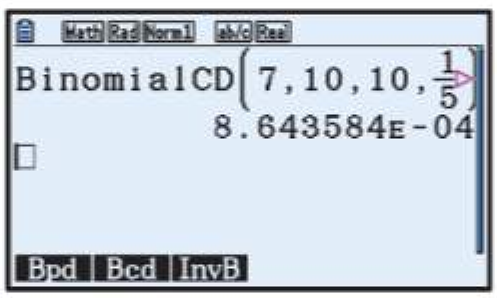
c $P(X = 6) = \binom{7}{6}(0.6)^6(0.4)$
≈ 0.131

d 

$P(X \geqslant 4) \approx 0.710$

4 Let X denote the number of questions Raj answers correctly.
$n = 10$, so $X = 0, 1, 2, ...,$ or 10, and $p = \frac{1}{5}$
$\therefore \ X \sim B(10, \frac{1}{5})$

P(Raj passes) $= P(X \geqslant 7)$
$\approx 0.000\,864$

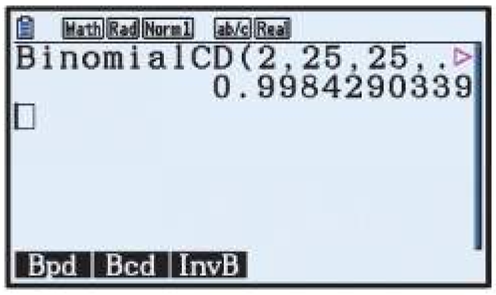


5 Let X be the number of students with the flu.

$n = 25$, so $X = 0, 1, 2, 3, \ldots, \text{or } 25$, and $p = 0.3$

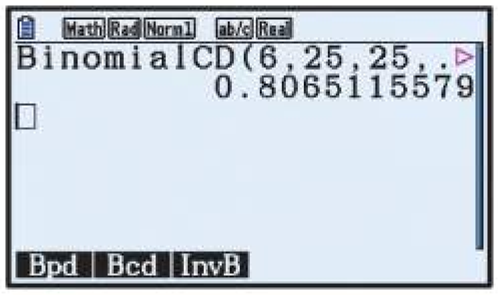
$\therefore \; X \sim B(25, 0.3)$

a



$P(X \geqslant 2) \approx 0.998$

b



20% of $25 = 5$

$\therefore \; P(\text{test cancelled}) = P(X \geqslant 6)$

≈ 0.807

6 Let X be the number of successful shots from the free throw line.

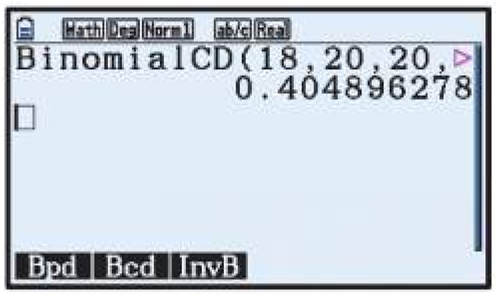
$n = 20$, so $X = 0, 1, 2, 3, \ldots, \text{or } 20$, and $p = 85\% = 0.85$

$\therefore \; X \sim B(20, 0.85)$

a $P(X = 20) = \binom{20}{20}(0.85)^{20}(0.15)^0$

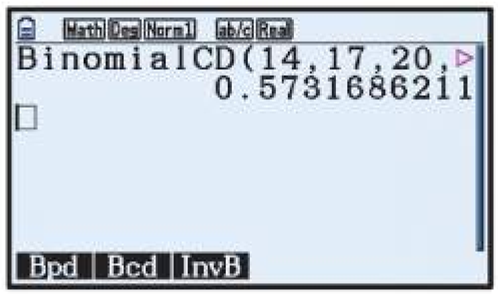
≈ 0.0388

b



$P(X \geqslant 18) \approx 0.405$

c



$P(14 \leqslant X \leqslant 17) \approx 0.573$

7 For Jelena to win a set of 6 games to 4, she must win 5 of the first 9 games, and then win the 10th game.

Let X be the number of games Jelena wins in the first 9 games.

$n = 9$, so $X = 0, 1, 2, 3, \ldots, \text{or } 9$

Now, Martina beats Jelena in 2 games out of 3, so the probability of Jelena winning a game is $p = 1 - \frac{2}{3} = \frac{1}{3}$.

$\therefore \; X \sim B(9, \frac{1}{3})$

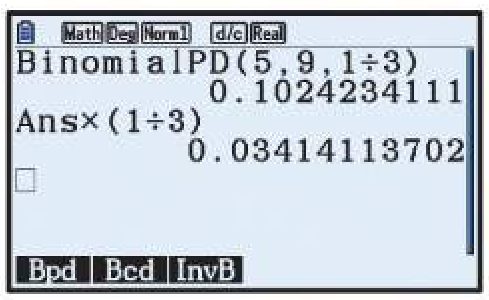
So, P(J wins 6 games to 4)

$= P(\text{J wins 5 of first 9 games}) \times P(\text{J wins 10th game})$

$= P(X = 5) \times \frac{1}{3}$

$\approx 0.1024 \times \frac{1}{3}$

≈ 0.0341



8 Let X be the number of heads.

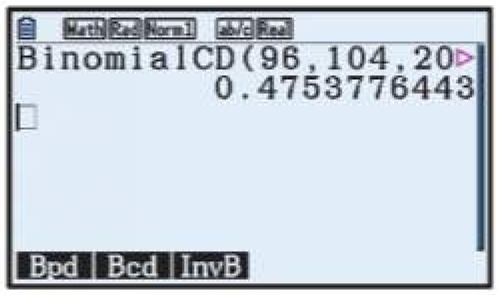
$n = 200$, so $X = 0, 1, 2, 3, ..., \text{or } 200$, and $p = \frac{1}{2}$

$\therefore \quad X \sim \text{B}(200, \frac{1}{2})$

a

$\text{P}(90 \leqslant X \leqslant 110) \approx 0.863$

b



$\text{P}(95 < X < 105) = \text{P}(96 \leqslant X \leqslant 104)$
≈ 0.475

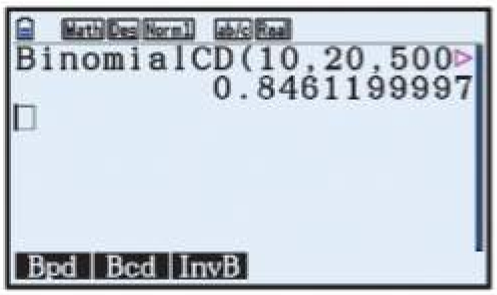
9 **a** $\text{P(rolling double sixes)} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

b Let X be the number of double sixes rolled.

$n = 500$, so $X = 0, 1, 2, 3, ..., \text{or } 500$, and $p = \frac{1}{36}$

$\therefore \quad X \sim \text{B}(500, \frac{1}{36})$

$\text{P}(10 \leqslant X \leqslant 20) \approx 0.846$

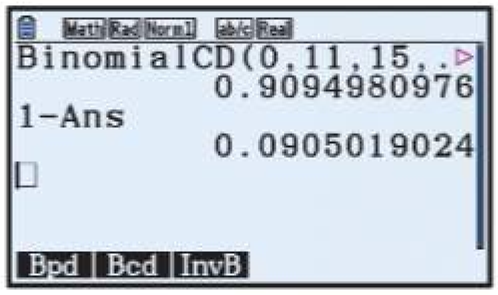


10 Let X be the number of traffic lights Shelley has stopped at.

$n = 15$, so $X = 0, 1, 2, 3, ..., \text{or } 15$, and $p = 0.6$

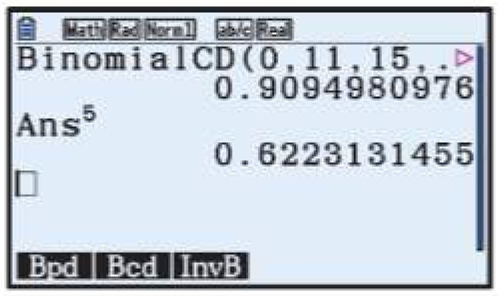
$\therefore \quad X \sim \text{B}(15, 0.6)$

a $\text{P(Shelley will be late)} = \text{P}(X > 11)$
$= 1 - \text{P}(X \leqslant 11)$
≈ 0.0905



b $\text{P(Shelley will be on time)} = \text{P}(X \leqslant 11)$
≈ 0.909

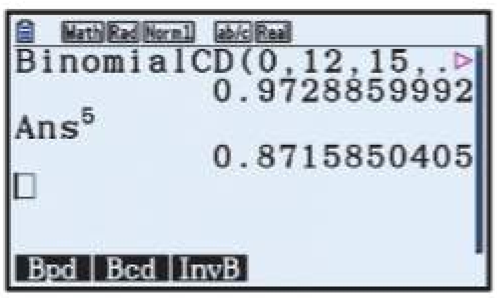
$\text{P(Shelley will be on time all 5 days)} = [\text{P}(X \leqslant 11)]^5$
≈ 0.622



c $\text{P(Shelley will be on time)} = \text{P}(X \leqslant 12)$
≈ 0.973

$\text{P(Shelley will be on time all 5 days)} = [\text{P}(X \leqslant 12)]^5$
≈ 0.872

$\therefore$ yes, the probability that Shelley is on time for work each day of a 5 day week is now about 87.2%.



11 Let X be the number of solar components which fail.

$n = 20$, so $X = 0, 1, 2, 3, ..., \text{or } 20$, and $p = 0.85$

$\therefore\ X \sim \text{B}(20, 0.85)$

a $\text{P(hot water unit fails within one year)} = \text{P(all 20 components fail)}$
$$= \text{P}(X = 20)$$
$$= (0.85)^{20}$$
$$\approx 0.0388$$

b P(hot water unit with n components fails within one year) $= (0.85)^n$

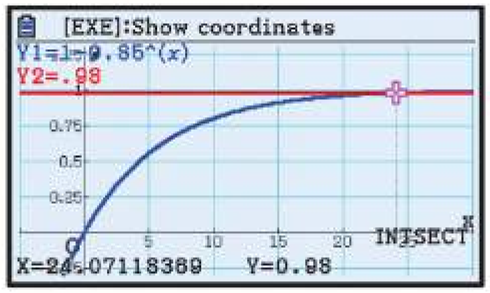
$\therefore$ P(hot water unit with n components is operating after one year) $= 1 - (0.85)^n$

$\therefore$ we need to find the smallest integer n such that $1 - (0.85)^n \geqslant 0.98$

Using technology, $1 - (0.85)^n = 0.98$ when $n \approx 24.1$ components.

$\therefore$ at least 25 solar components are needed.

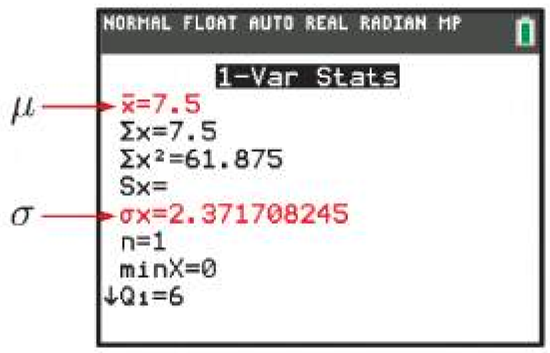
Check: When $n = 25$, the probability that at least one component will still work is $1 - (0.85)^{25} \approx 0.983 > 0.98$ ✓



INVESTIGATION 3 THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

1 $X \sim \text{B}(30, 0.25)$

Consult the graphics calculator instructions by clicking on the icon in the Investigation box if you need help obtaining the result shown.



2

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$\mu = 1$ $\sigma \approx 0.9487$	$\mu = 2.5$ $\sigma \approx 1.3693$	$\mu = 5$ $\sigma \approx 1.5811$	$\mu = 7$ $\sigma \approx 1.4491$
$n = 30$	$\mu = 3$ $\sigma \approx 1.6432$	$\mu = 7.5$ $\sigma \approx 2.3717$	$\mu = 15$ $\sigma \approx 2.7386$	$\mu = 21$ $\sigma \approx 2.5100$
$n = 50$	$\mu = 5$ $\sigma \approx 2.1213$	$\mu = 12.5$ $\sigma \approx 3.0619$	$\mu = 25$ $\sigma \approx 3.5355$	$\mu = 35$ $\sigma \approx 3.2404$

3

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$np = 1$ $\sqrt{np(1-p)}$ ≈ 0.9487	$np = 2.5$ $\sqrt{np(1-p)}$ ≈ 1.3693	$np = 5$ $\sqrt{np(1-p)}$ ≈ 1.5811	$np = 7$ $\sqrt{np(1-p)}$ ≈ 1.4491
$n = 30$	$np = 3$ $\sqrt{np(1-p)}$ ≈ 1.6432	$np = 7.5$ $\sqrt{np(1-p)}$ ≈ 2.3717	$np = 15$ $\sqrt{np(1-p)}$ ≈ 2.7386	$np = 21$ $\sqrt{np(1-p)}$ ≈ 2.5100
$n = 50$	$np = 5$ $\sqrt{np(1-p)}$ ≈ 2.1213	$np = 12.5$ $\sqrt{np(1-p)}$ ≈ 3.0619	$np = 25$ $\sqrt{np(1-p)}$ ≈ 3.5355	$np = 35$ $\sqrt{np(1-p)}$ ≈ 3.2404

Our results in **2** and **3** agree with the formulae $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

EXERCISE 27H

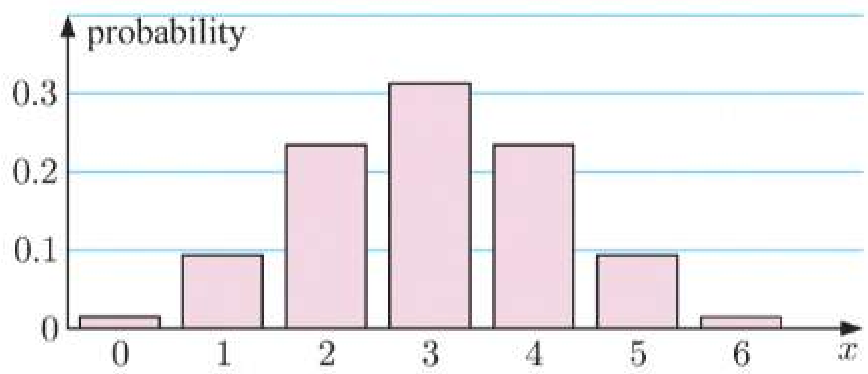
1 **a** $X \sim B(6,\ 0.5)$

i $\mu = np$
$= 6 \times 0.5$
$= 3$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{6 \times 0.5 \times 0.5}$
≈ 1.22

ii

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156



iii The distribution is symmetric.

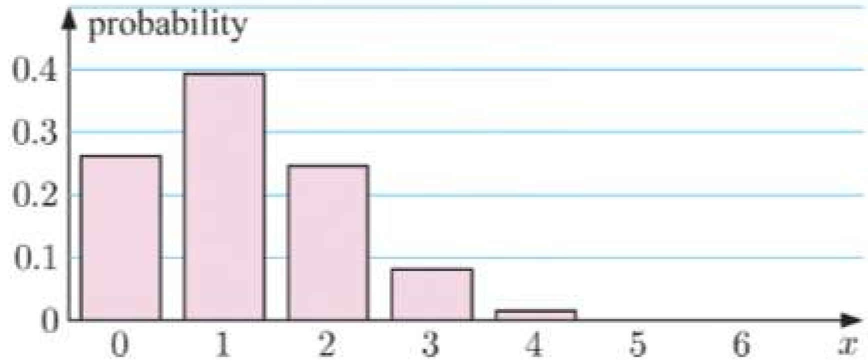
b $X \sim B(6,\ 0.2)$

i $\mu = np$
$= 6 \times 0.2$
$= 1.2$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{6 \times 0.2 \times 0.8}$
≈ 0.980

ii

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001



iii The distribution is positively skewed.

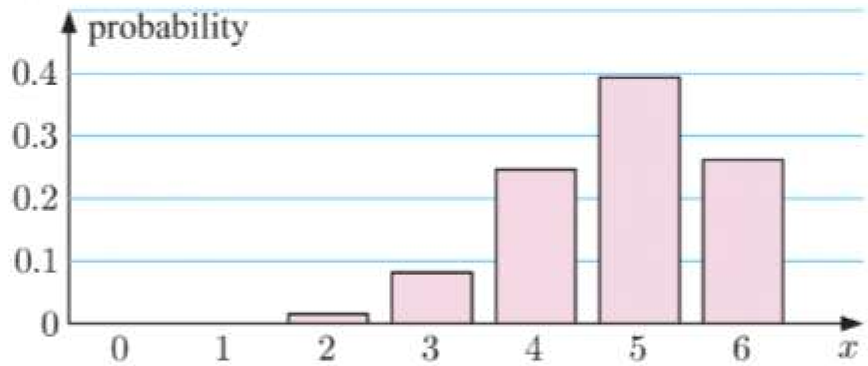
c $X \sim \text{B}(6, 0.8)$

i $\mu = np$
$= 6 \times 0.8$
$= 4.8$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{6 \times 0.8 \times 0.2}$
≈ 0.980

ii

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii The distribution is negatively skewed, and is the exact reflection of the distribution in **b**.

2 $X \sim \text{B}(10, 0.5)$ mean $\mu = np$
$= 10 \times \frac{1}{2}$
$= 5$

and variance $\sigma^2 = np(1-p)$
$= 10 \times \frac{1}{2} \times \frac{1}{2}$
$= 2.5$

3 **a** $X \sim \text{B}(30, 0.04)$

$\mu_X = np$
$= 30 \times 0.04$
$= 1.2$

$\sigma_X = \sqrt{np(1-p)}$
$= \sqrt{30 \times 0.04 \times 0.96}$
≈ 1.07

b $Y \sim \text{B}(30, 0.96)$

$\mu_Y = np$
$= 30 \times 0.96$
$= 28.8$

$\sigma_Y = \sqrt{np(1-p)}$
$= \sqrt{30 \times 0.96 \times 0.04}$
≈ 1.07

4 $X \sim B(30, 0.13)$

$$\begin{aligned}\mu &= np\\ &= 30 \times 0.13\\ &= 3.9\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{np(1-p)}\\ &= \sqrt{30 \times 0.13 \times 0.87}\\ &\approx 1.84\end{aligned}$$

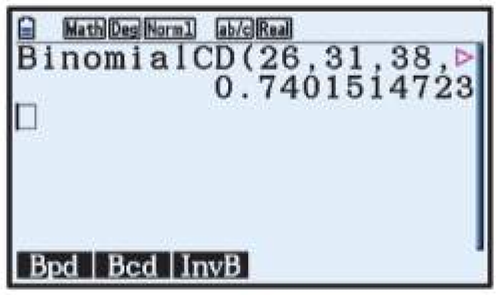
5 $X \sim B(38, 0.75)$

a
$$\begin{aligned}\mu &= np\\ &= 38 \times 0.75\\ &= 28.5\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{np(1-p)}\\ &= \sqrt{38 \times 0.75 \times 0.25}\\ &\approx 2.67\end{aligned}$$

b
$$\begin{aligned}\mu - \sigma &\approx 28.5 - 2.67\\ &\approx 25.8\end{aligned}$$

$$\begin{aligned}\mu + \sigma &\approx 28.5 + 2.67\\ &\approx 31.2\end{aligned}$$



$$\begin{aligned}\therefore\ P(\mu - \sigma < X < \mu + \sigma) &= P(26 \leqslant X \leqslant 31)\\ &\approx 0.740\end{aligned}$$

6 $X \sim B(100, \frac{1}{2})$, $Y \sim B(300, \frac{1}{6})$

a
$$\begin{aligned}\mu_X &= np\\ &= 100 \times \tfrac{1}{2}\\ &= 50\end{aligned}$$

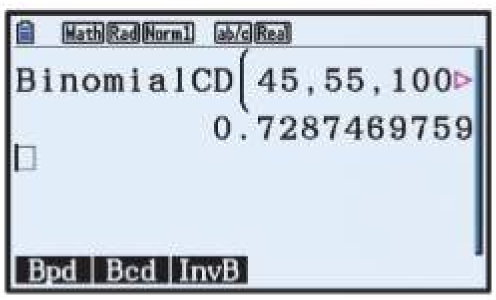
$$\begin{aligned}\mu_Y &= np\\ &= 300 \times \tfrac{1}{6}\\ &= 50\end{aligned}$$

b
$$\begin{aligned}\sigma_X &= \sqrt{np(1-p)}\\ &= \sqrt{100 \times \tfrac{1}{2} \times \tfrac{1}{2}}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

$$\begin{aligned}\sigma_Y &= \sqrt{np(1-p)}\\ &= \sqrt{300 \times \tfrac{1}{6} \times \tfrac{5}{6}}\\ &\approx 6.45\end{aligned}$$

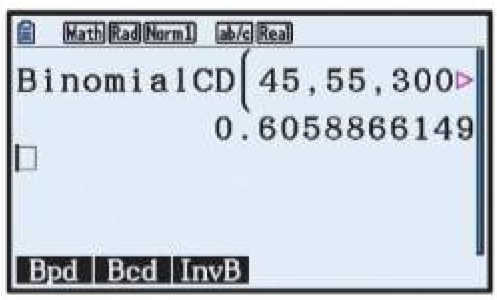
c X is more likely to lie between 45 and 55 inclusive because the standard deviation of X is lower than that of Y, which means there are more values of X which lie close to the mean.

d **i**



$P(45 \leqslant X \leqslant 55) \approx 0.729$

ii



$P(45 \leqslant Y \leqslant 55) \approx 0.606$

EXERCISE 27I

1 $X \sim Po(7.12)$

a $P(X = x) = \dfrac{(7.12)^x e^{-7.12}}{x!}$, $x = 0, 1, 2, \ldots.$

b **i** $E(X) = 7.12$

ii
$$\begin{aligned}Var(X) &= E(X)\\ &= 7.12\end{aligned}$$

iii
$$\begin{aligned}\sigma(X) &= \sqrt{Var(X)}\\ &= \sqrt{7.12}\\ &\approx 2.67\end{aligned}$$

c **i** $\mathrm{P}(X=2)=\dfrac{(7.12)^2e^{-7.12}}{2!}$

≈ 0.0205

ii $\mathrm{P}(X \leqslant 3)=\mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)$

$=\dfrac{(7.12)^0e^{-7.12}}{0!}+\dfrac{(7.12)^1e^{-7.12}}{1!}+\dfrac{(7.12)^2e^{-7.12}}{2!}+\dfrac{(7.12)^3e^{-7.12}}{3!}$

≈ 0.0757

iii $\mathrm{P}(X \geqslant 5)=\mathrm{P}(X>4)$

$=1-\mathrm{P}(X \leqslant 4)$

$\approx 1-0.162$

≈ 0.838

iv $\mathrm{P}(X \geqslant 3 \mid X \geqslant 1)=\dfrac{\mathrm{P}(X \geqslant 3 \;\cap\; X \geqslant 1)}{\mathrm{P}(X \geqslant 1)}$

$=\dfrac{\mathrm{P}(X \geqslant 3)}{\mathrm{P}(X \geqslant 1)}$

$=\dfrac{1-\mathrm{P}(X \leqslant 2)}{1-\mathrm{P}(X=0)}$

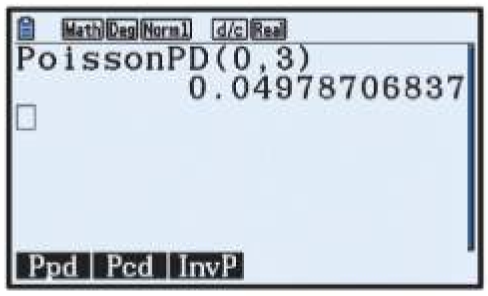
$\approx \dfrac{1-0.0271}{1-0.000\,809}$

≈ 0.974

2 Let X be the number of car rental requests that Top Cars receive on a particular day.

$\therefore \quad X \sim \mathrm{Po}(3)$

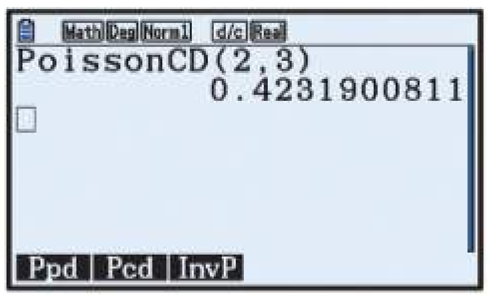
a $\mathrm{P}(X=0) \approx 0.0498$



b $\mathrm{P}(X \geqslant 3)=1-\mathrm{P}(X \leqslant 2)$

$\approx 1-0.423$

≈ 0.577

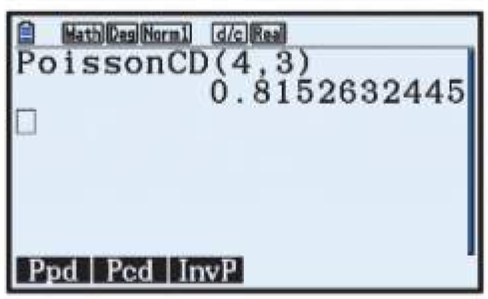


c $\mathrm{P}(\text{some requests are refused})=\mathrm{P}(X \geqslant 5)$

$=1-\mathrm{P}(X \leqslant 4)$

$\approx 1-0.815$

≈ 0.185



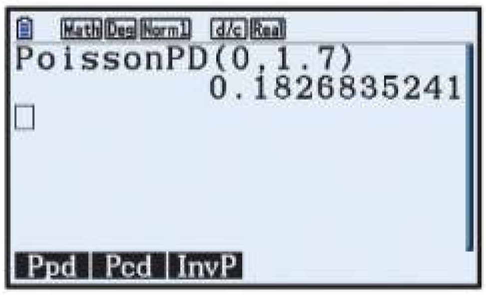
d $$\begin{aligned} P(X \geqslant 4 \mid X \geqslant 2) &= \frac{P(X \geqslant 4 \ \cap \ X \geqslant 2)}{P(X \geqslant 2)} \\ &= \frac{P(X \geqslant 4)}{P(X \geqslant 2)} \\ &= \frac{1 - P(X \leqslant 3)}{1 - P(X \leqslant 1)} \\ &\approx \frac{1 - 0.647\,23}{1 - 0.199\,14} \\ &\approx 0.440 \end{aligned}$$

3 Let X be the number of flaws in 1 metre of material.

$\therefore \quad X \sim \text{Po}(1.7)$

a $P(X = 3) \approx 0.150$

b $$\begin{aligned} &P(\text{at least one flaw in 2 metres}) \\ &= 1 - P(\text{no flaws in 2 metres}) \\ &= 1 - (P(X = 0))^2 \\ &\approx 1 - (0.1827)^2 \\ &\approx 0.967 \end{aligned}$$



c $$\begin{aligned} \mu &= E(X) \\ &= 1.7 \end{aligned} \qquad \begin{aligned} \sigma^2 &= \text{Var}(X) \\ &= E(X) \\ &= 1.7 \end{aligned} \qquad \begin{aligned} \sigma &= \sqrt{\text{Var}(X)} \\ &= \sqrt{1.7} \\ &\approx 1.30 \end{aligned}$$

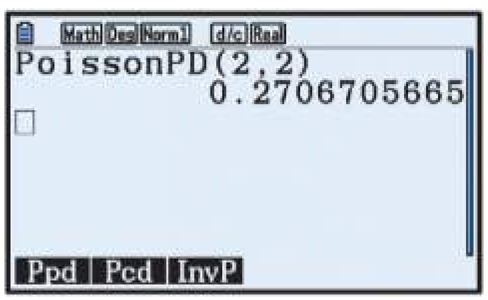
4 Let X be the number of aerofoils which disintegrate from a sample of 100.
Each aerofoil has a 2% chance of disintegrating,
so the rate at which aerofoils disintegrate $= 0.02 \times 100$
$= 2$ per sample of 100.

$\therefore \quad X \sim \text{Po}(2)$

a

$P(X = 1) \approx 0.271$

b



$P(X = 2) \approx 0.271$

c

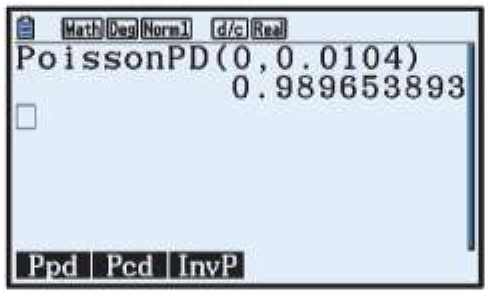
$P(X \leqslant 2) \approx 0.677$

5 **a** A person who drives 10 times per week will drive $10 \times 52 = 520$ times in one year.
Let X be the number of fatalities from driving 520 times.

The rate at which fatalities occur $= 0.000\,02 \times 520$
$= 0.0104$ fatalities per 520 times.

$\therefore \quad X \sim \text{Po}(0.0104)$

$$\begin{aligned} \text{P(surviving)} &= \text{P}(X = 0) \\ &\approx 0.989\,65 \\ &\approx 0.990 \end{aligned}$$

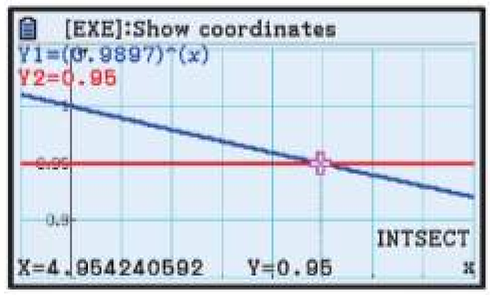


b Assuming that the probability of a fatal crash remains constant,
P(driving for n years and surviving) $\approx (0.989\,65)^n$

$\therefore$ we need to find n such that $(0.989\,65)^n = 0.95$

Using technology, $n \approx 4.95$

$\therefore$ you can drive for 4 years and still have a better than 95% chance of surviving.



6 $X \sim \text{Po}(\lambda)$

$$\therefore \quad \text{P}(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{where } x = 0, 1, 2, 3, 4, \ldots.$$

a If $\text{P}(X = 1) + \text{P}(X = 2) = \text{P}(X = 3)$,

$$\text{then} \quad \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{\lambda^3 e^{-\lambda}}{3!}$$

$$\therefore \quad \lambda + \frac{\lambda^2}{2} = \frac{\lambda^3}{6} \quad \{\div\, e^{-\lambda}\}$$

$$\therefore \quad 6\lambda + 3\lambda^2 = \lambda^3$$

$$\therefore \quad \lambda(\lambda^2 - 3\lambda - 6) = 0 \quad \text{where } \lambda \neq 0$$

$$\therefore \quad \lambda^2 - 3\lambda - 6 = 0$$

$$\therefore \quad \lambda = \frac{3 \pm \sqrt{9 - 4(1)(-6)}}{2} = \frac{3 \pm \sqrt{33}}{2}$$

But $\lambda > 0$, so $\lambda = \dfrac{3 + \sqrt{33}}{2} \approx 4.3723 \approx 4.37$

b **i** $\mathrm{E}(X) = \lambda$

$$= \frac{3+\sqrt{33}}{2}$$

$$\approx 4.37$$

ii $\sigma(X) = \sqrt{\lambda}$

$$\approx \sqrt{4.3723}$$

$$\approx 2.09$$

iii $\mathrm{P}(X \geqslant 3) = 1 - \mathrm{P}(X \leqslant 2)$

$$\approx 1 - 0.188$$

$$\approx 0.812$$

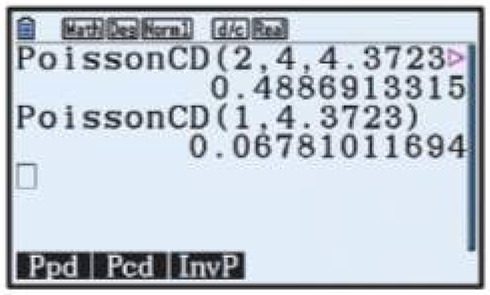
iv $\mathrm{P}(X \leqslant 4 \mid X \geqslant 2) = \dfrac{\mathrm{P}(X \leqslant 4 \ \cap \ X \geqslant 2)}{\mathrm{P}(X \geqslant 2)}$

$$= \frac{\mathrm{P}(2 \leqslant X \leqslant 4)}{\mathrm{P}(X \geqslant 2)}$$

$$= \frac{\mathrm{P}(2 \leqslant X \leqslant 4)}{1 - \mathrm{P}(X \leqslant 1)}$$

$$\approx \frac{0.4887}{1 - 0.067\,81}$$

$$\approx 0.524$$



7 **a** $Y \sim \mathrm{Po}(\lambda)$

$$\therefore \ \mathrm{P}(Y = y) = \frac{\lambda^y e^{-\lambda}}{y!}, \quad y = 0, 1, 2, 3, ...$$

$$\mathrm{P}(Y = 3) = \mathrm{P}(Y = 1) + 2\,\mathrm{P}(Y = 2)$$

$$\therefore \ \frac{\lambda^3 e^{-\lambda}}{3!} = \frac{\lambda^1 e^{-\lambda}}{1!} + 2\,\frac{\lambda^2 e^{-\lambda}}{2!}$$

$$\therefore \ \frac{\lambda^3}{6} = \lambda + \lambda^2 \qquad \{\times e^{\lambda}\}$$

$$\therefore \ \lambda^3 = 6\lambda + 6\lambda^2$$

$$\therefore \ \lambda(\lambda^2 - 6\lambda - 6) = 0 \quad \text{where} \quad \lambda \neq 0$$

$$\therefore \ \lambda^2 - 6\lambda - 6 = 0$$

$$\therefore \ \lambda = \frac{6 \pm \sqrt{36 - 4(1)(-6)}}{2}$$

$$= 3 \pm \sqrt{15}$$

But $\lambda > 0$, so $\lambda = 3 + \sqrt{15} \approx 6.8730$

b **i** $\mathrm{P}(1 < Y < 5)$

$$= \mathrm{P}(2 \leqslant Y \leqslant 4)$$

$$\approx 0.177$$

ii $$\begin{aligned}\mathrm{P}(2 \leqslant Y \leqslant 6 \mid Y \geqslant 4) &= \frac{\mathrm{P}(2 \leqslant Y \leqslant 6 \ \cap \ Y \geqslant 4)}{\mathrm{P}(Y \geqslant 4)} \\ &= \frac{\mathrm{P}(4 \leqslant Y \leqslant 6)}{\mathrm{P}(Y \geqslant 4)} \\ &= \frac{\mathrm{P}(4 \leqslant Y \leqslant 6)}{1 - \mathrm{P}(Y \leqslant 3)} \\ &\approx \frac{0.380}{1 - 0.0886} \\ &\approx 0.417\end{aligned}$$

8

Number per period	0	1	2	3	4	5	6	7
Frequency	91	156	132	75	33	9	3	1

a $$\begin{aligned}\text{mean} &= \frac{1 \times 156 + 2 \times 132 + 3 \times 75 + 4 \times 33 + 5 \times 9 + 6 \times 3 + 7 \times 1}{91 + 156 + 132 + 75 + 33 + 9 + 3 + 1} \\ &= \frac{847}{500} \\ &= 1.694\end{aligned}$$

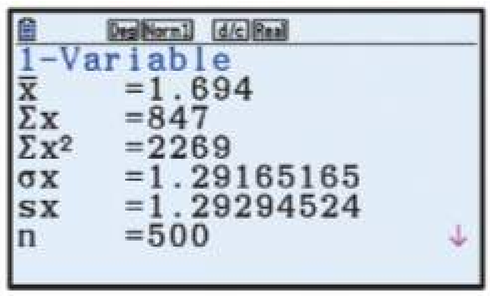
b $\lambda \approx \text{mean} = 1.694$

c Using technology, $\sigma \approx 1.292$

$$\therefore \ \sigma^2 \approx (1.292)^2 \approx 1.67$$

From **b**, the variance of the Poisson distribution is $\sigma^2 = \lambda \approx 1.694$.

The two values are very similar.



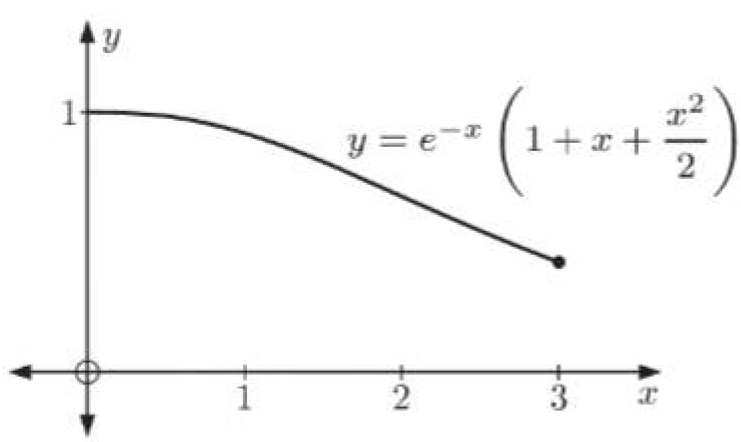
9 $U \sim \mathrm{Po}(x)$

$\therefore \ \mathrm{P}(U = u) = \dfrac{x^u e^{-x}}{u!}$ where $u = 0, 1, 2, 3, \ldots$

a $$\begin{aligned}y &= \mathrm{P}(U \leqslant 2) \\ &= \mathrm{P}(U = 0) + \mathrm{P}(U = 1) + \mathrm{P}(U = 2) \\ &= \frac{x^0 e^{-x}}{0!} + \frac{x^1 e^{-x}}{1!} + \frac{x^2 e^{-x}}{2!} \\ &= e^{-x} + x e^{-x} + \frac{x^2 e^{-x}}{2}\end{aligned}$$

$$\therefore \ y = e^{-x}\left(1 + x + \frac{x^2}{2}\right)$$

b



c $$y = e^{-x}\left(1 + x + \frac{x^2}{2}\right)$$

$$\therefore \quad \frac{dy}{dx} = -e^{-x}\left(1 + x + \frac{x^2}{2}\right) + e^{-x}(1 + x)$$

$$= -e^{-x} - xe^{-x} - \frac{x^2e^{-x}}{2} + e^{-x} + xe^{-x}$$

$$= -\frac{x^2e^{-x}}{2}$$

Since $x^2e^{-x} > 0$, $\frac{dy}{dx} < 0$ for all $x > 0$.

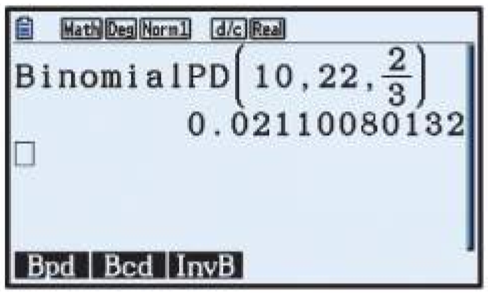
$\therefore$ as the mean x increases, $y = \mathrm{P}(U \leqslant 2)$ decreases.

10 $Y \sim \mathrm{Po}(20)$

a Y eggs are laid, and each egg has probability $\frac{2}{3}$ of hatching independently of one another.

b $(X \mid Y = 22) \sim \mathrm{B}(22, \frac{2}{3})$

$\therefore$ $\mathrm{P}(X = 10 \mid Y = 22) \approx 0.0211$



c $\mathrm{E}(X \mid Y) = \frac{2}{3}Y$

d $$\mathrm{E}(X) = \mathrm{E}\left[\mathrm{E}(X \mid Y)\right]$$
$$= \mathrm{E}\left(\tfrac{2}{3}Y\right)$$
$$= \tfrac{2}{3}\,\mathrm{E}(Y)$$
$$= \tfrac{2}{3} \times 20$$
$$= \tfrac{40}{3}$$

REVIEW SET 27A

1 **a** The number of attempts to pass a driving test is a discrete random variable.

b The length of time before a phone loses its battery charge is a continuous random variable.

c The number of phone calls made before a salesperson has sold 3 products is a discrete random variable.

2 **a** **i**

x	1	2	3
$\mathrm{P}(X = x)$	0.6	0.25	0.15

$$\sum_{x=1}^{3} \mathrm{P}(X = x) = 0.6 + 0.25 + 0.15 = 1$$

Since $\sum_{x=1}^{3} \mathrm{P}(X = x) = 1$ and $0 \leqslant \mathrm{P}(X = x) \leqslant 1$ for all x, it is a valid probability distribution.

ii

x	0	2	5	10
$P(X = x)$	0.3	0.5	0.1	0.2

$\sum p_i = 0.3 + 0.5 + 0.1 + 0.2 = 1.1$

Since $\sum p_i > 1$, it is not a valid probability distribution.

iii

x	0	1	2	3
$P(X = x)$	0.4	-0.2	0.35	0.45

Since $P(X = 1) = -0.2 < 0$, this is not a valid probability distribution.

iv

x	2	3	4	5
$P(X = x)$	0.25	0.25	0.25	0.25

$\sum_{x=2}^{5} P(X = x) = 0.25 + 0.25 + 0.25 + 0.25 = 1$

Since $\sum_{x=2}^{5} P(X = x) = 1$ and $0 \leqslant P(X = x) \leqslant 1$ for all x, it is a valid probability distribution.

v

x	2	3
$P(X = x)$	0.7	0.3

$\sum_{x=2}^{3} P(X = x) = 0.7 + 0.3 = 1$

Since $\sum_{x=2}^{3} P(X = x) = 1$ and $0 \leqslant P(X = x) \leqslant 1$ for all x, it is a valid probability distribution.

vi

x	0	1
$P(X = x)$	0.28	0.72

$\sum_{x=0}^{1} P(X = x) = 0.28 + 0.72 = 1$

Since $\sum_{x=0}^{1} P(X = x) = 1$ and $0 \leqslant P(X = x) \leqslant 1$ for all x, it is a valid probability distribution.

b The distribution in **a iv** is a uniform discrete random variable because $p_i = 0.25$ for each value of i.

3 **a** $\mathrm{P}(X=x)=\dfrac{a}{x^2+1}$ for $x=0,\ 1,\ 2,\ 3$

x	0	1	2	3
$\mathrm{P}(X=x)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

Since this is a probability mass function,

$$\sum_{x=0}^{3}\mathrm{P}(X=x)=1$$

$$\therefore\ a+\frac{a}{2}+\frac{a}{5}+\frac{a}{10}=1$$

$$\therefore\ 10a+5a+2a+a=10$$

$$\therefore\ 18a=10$$

$$\therefore\ a=\tfrac{5}{9}$$

b $\mathrm{P}(X\geqslant 1)=1-\mathrm{P}(X=0)$
$=1-\frac{5}{9}$
$=\frac{4}{9}$

4

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

a Since this is a probability distribution, $\sum_{i=1}^{n}P(x_i)=1$

$$\therefore\ 0.10+0.30+0.45+0.10+k=1$$

$$\therefore\ 0.95+k=1$$

$$\therefore\ k=0.05$$

b $\mathrm{P}(X\geqslant 3)=\mathrm{P}(X=3)+\mathrm{P}(X=4)$
$=0.10+0.05$
$=0.15$

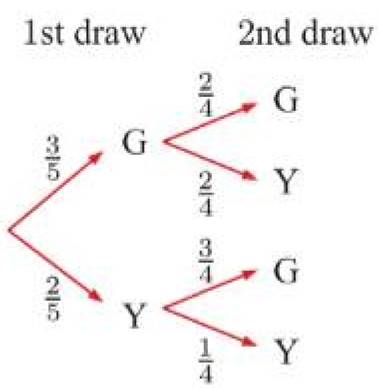
c Since $\mathrm{P}(X=2)$ is the greatest probability, 2 is the mode of this distribution.

d $\mathrm{E}(X)=\sum_{i=1}^{n}x_ip_i$
$=0(0.10)+1(0.30)+2(0.45)+3(0.10)+4(0.05)$
$=0+0.3+0.9+0.3+0.2$
$=1.7$

5 **a** X is a discrete random variable because it has a set of distinct possible values.

b $X=0$, 1, or 2

c



Outcome	X	Probability
GG	2	$\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$
GY	1	$\frac{3}{5}\times\frac{2}{4}=\frac{3}{10}$
YG	1	$\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$
YY	0	$\frac{2}{5}\times\frac{1}{4}=\frac{1}{10}$

x	0	1	2
$P(x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d $\text{E}(X) = \sum_{i=1}^{n} x_i p_i$
$= \left(0 \times \frac{1}{10}\right) + \left(1 \times \frac{3}{5}\right) + \left(2 \times \frac{3}{10}\right)$
$= \frac{6}{5}$
$= 1.2$ green balls

6 X has probability table:

x	1	3	4	6
$P(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

$\text{E}(X) = \sum_{i=1}^{n} x_i p_i$
$= 1\left(\frac{1}{6}\right) + 3\left(\frac{2}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{2}{6}\right)$
$= \frac{23}{6}$
≈ 3.83

7 **a** Let X denote the amount of money Lakshmi wins from one roll.
X has probability table:

x	2	4	6	8	10	12
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\text{E}(X) = \sum_{i=1}^{n} x_i p_i$
$= \left(2 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) + \left(8 \times \frac{1}{6}\right) + \left(10 \times \frac{1}{6}\right) + \left(12 \times \frac{1}{6}\right)$
$= 7$

$\therefore$ Lakshmi can expect to win \$7 from one roll of the die.

b Expected gain $=$ \$7 $-$ \$8 $= -$\$1.
So, Lakshmi should not play many games as she would lose \$1 per game in the long run.

8 **a** $\mu = \sum x_i p_i$
$= 1(0.15) + 2(0.1) + 3(0.35) + 4(0.4)$
$= 3$

x	1	2	3	4
$\text{P}(X = x)$	0.15	0.1	0.35	0.4

b $\sigma^2 = \sum {x_i}^2 p_i - \mu^2$
$= 1^2(0.15) + 2^2(0.1) + 3^2(0.35) + 4^2(0.4) - 3^2$
$= 1.1$

c $\sigma = \sqrt{\sum {x_i}^2 p_i - \mu^2}$
$= \sqrt{1.1}$
≈ 1.05

9 $P(x) = a(x^2 - 8x)$ where $x = 0, 1, 2, 3, ..., 8$

a X has probability table:

x	0	1	2	3	4	5	6	7	8
$P(x)$	0	$-7a$	$-12a$	$-15a$	$-16a$	$-15a$	$-12a$	$-7a$	0

If this is a probability distribution then $\sum_{i=1}^{n} P(x_i) = 1$

$$\therefore\ 0 + (-7a) + (-12a) + (-15a) + (-16a) + (-15a) + (-12a) + (-7a) + 0 = 1$$

$$\therefore\ -84a = 1$$

$$\therefore\ a = -\tfrac{1}{84}$$

b $\text{E}(X) = 0(0) + 1\left(\frac{7}{84}\right) + 2\left(\frac{12}{84}\right) + 3\left(\frac{15}{84}\right) + 4\left(\frac{16}{84}\right) + 5\left(\frac{15}{84}\right) + 6\left(\frac{12}{84}\right) + 7\left(\frac{7}{84}\right) + 8(0)$
$= 4$ marsupials

c $\sigma^2 = \sum {x_i}^2 p_i - \mu^2$
$= 0^2(0) + 1^2(\frac{7}{84}) + 2^2(\frac{12}{84}) + 3^2(\frac{15}{84}) + 4^2(\frac{16}{84}) + 5^2(\frac{15}{84}) + 6^2(\frac{12}{84}) + 7^2(\frac{7}{84}) + 8^2(0) - 4^2$
$= 3$

10 $Y = 4X + 3$

$\text{E}(Y) = \text{E}(4X + 3)$
$= 4\,\text{E}(X) + 3$
$= 4(6) + 3$
$= 27$

$\sigma(Y) = \sigma(4X + 3)$
$= |\,4\,|\ \sigma(X)$
$= 4(2)$
$= 8$

For Y, the mean is 27 and the standard deviation is 8.

11 The number of trials is $n = 5$.
The probability of success (kicking a goal) is $p = 80\% = 0.8$.
Let X be the number of goals scored, and G represent scoring a goal.
$\therefore\ X \sim \text{B}(5, 0.8)$

a P(3 goals *then* misses twice)
$= \text{P}(\text{GGGG}'\text{G}')$
$= (0.8)^3(1 - 0.8)^2$
$= 0.020\,48$

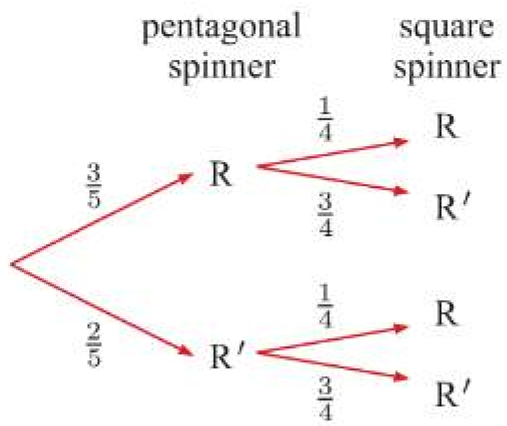
b P(3 goals and misses twice)
$= \text{P}(3 \text{ goals})$
$= \text{P}(X = 3)$
$= \binom{5}{3}(0.8)^3(0.2)^2$
$= 10 \times 0.020\,48$
$= 0.2048$

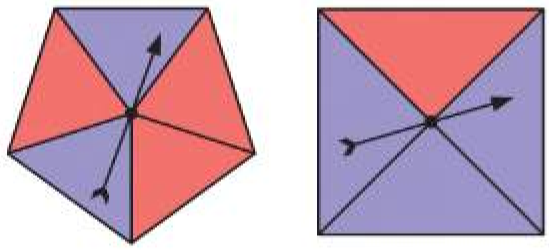
12 Let X denote the number of batteries that are defective.
$n = 20$, so $X = 0, 1, 2, 3, ..., \text{ or } 20$, and $p = \frac{3}{100}$
$\therefore\ X \sim \text{B}\left(20, \frac{3}{100}\right)$

a $\text{P}(X = 0) = \binom{20}{0}\left(\frac{3}{100}\right)^0\left(\frac{97}{100}\right)^{20}$
≈ 0.544

b $\text{P}(X \geqslant 1) = 1 - \text{P}(X = 0)$
$\approx 1 - 0.544$
≈ 0.456

13 **a**





b $$\begin{aligned} \text{P(exactly one red)} &= \text{P}(\text{RR}') + \text{P}(\text{R}'\text{R}) \\ &= \tfrac{3}{5} \times \tfrac{3}{4} + \tfrac{2}{5} \times \tfrac{1}{4} \\ &= \tfrac{9}{20} + \tfrac{1}{10} \\ &= \tfrac{11}{20} \end{aligned}$$

c **i** $X \sim \text{B}\left(10, \frac{11}{20}\right)$

ii $n = 10, \quad p = \frac{11}{20}$

$\text{P}(X = 1) = \binom{10}{1}\left(\frac{11}{20}\right)^1\left(\frac{9}{20}\right)^9 \approx 0.004\,16$

$\text{P}(X = 9) = \binom{10}{9}\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right)^1 \approx 0.0207$

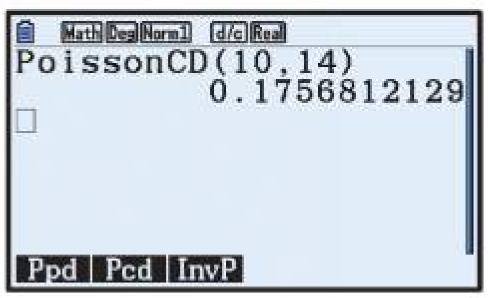
$\therefore$ it is more likely that exactly one red will occur 9 times.

iii $$\begin{aligned} \mu &= np \\ &= 10 \times \tfrac{11}{20} \\ &= 5.5 \end{aligned} \qquad \begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{10 \times \tfrac{11}{20} \times \tfrac{9}{20}} \\ &\approx 1.57 \end{aligned}$$

14 **a** Let X be the number of patients arriving between 8:00 am and 9:00 am.

$\therefore \quad X \sim \text{Po}(14)$

$$\begin{aligned} \text{P}(X > 10) &= 1 - \text{P}(X \leqslant 10) \\ &\approx 1 - 0.176 \\ &\approx 0.824 \end{aligned}$$



b Let Y be the number of patients arriving between 9:00 am and 9:45 am.

The rate at which patients arrive $= \frac{3}{4} \times 14$

$= 10.5$ patients per 45 minutes

$\therefore \quad Y \sim \text{Po}(10.5)$

$\text{P}(Y = 5) \approx 0.0293$

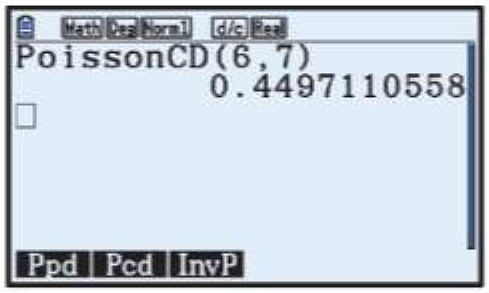
c Let W be the number of patients arriving between 10:00 am and 10:30 am.

The rate at which patients arrive $= \frac{1}{2} \times 14$

$= 7$ patients per 30 minutes

$\therefore \quad W \sim \text{Po}(7)$

$$\begin{aligned} \text{P}(W < 7) &= \text{P}(W \leqslant 6) \\ &\approx 0.450 \end{aligned}$$



REVIEW SET 27B

1

x	0	1	2	3	4	5
$\text{P}(X = x)$	0.07	0.14	k	0.46	0.08	0.02

a The random variable X represents the number of hits that Sally has in a softball match.
$X = 0, 1, 2, 3, 4,$ or 5

b **i** Since this is a probability distribution, $\sum\limits_{x=0}^{5} \text{P}(X = x) = 1$

$$\therefore \quad 0.07 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$$
$$\therefore \quad k + 0.77 = 1$$
$$\therefore \quad k = 0.23$$

ii $$\begin{aligned} \text{P}(X \geqslant 2) &= 1 - \text{P}(X \leqslant 1) \\ &= 1 - (0.07 + 0.14) \\ &= 0.79 \end{aligned}$$

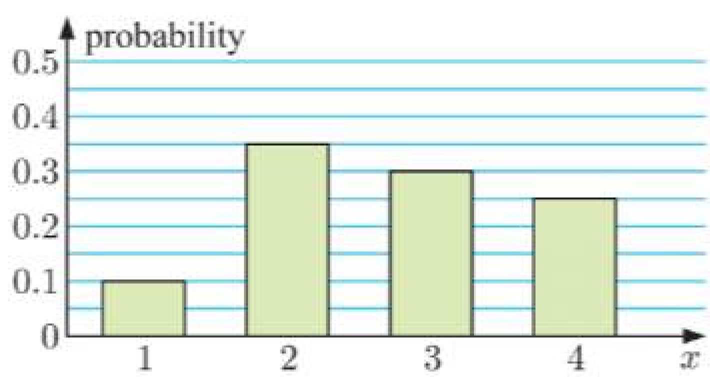
iii $$\begin{aligned} \text{P}(1 \leqslant X \leqslant 3) &= \text{P}(X = 1) + \text{P}(X = 2) + \text{P}(X = 3) \\ &= 0.14 + 0.23 + 0.46 \\ &= 0.83 \end{aligned}$$

c It is most likely that Sally will have 3 hits in one softball match, so 3 hits is the mode.

$$\begin{aligned} p_1 &= 0.07 \\ p_1 + p_2 &= 0.07 + 0.14 = 0.21 \\ p_1 + p_2 + p_3 &= 0.07 + 0.14 + 0.23 = 0.44 \\ p_1 + p_2 + p_3 + p_4 &= 0.07 + 0.14 + 0.23 + 0.46 = 0.9 \end{aligned}$$

Since $p_1 + p_2 + p_3 + p_4 \geqslant 0.5$, the median is 3 hits.

2 **a** 2 has the highest probability of occurring, so this is the mode of the distribution.



b $$\begin{aligned} p_1 &= 0.1 \\ p_1 + p_2 &= 0.1 + 0.35 = 0.45 \\ p_1 + p_2 + p_3 &= 0.1 + 0.35 + 0.3 = 0.75 \end{aligned}$$

Since $p_1 + p_2 + p_3 \geqslant 0.5$, the median is 3.

c $\text{E}(X) = (1 \times 0.1) + (2 \times 0.35) + (3 \times 0.3) + (4 \times 0.25)$
$= 2.7$

3 **a** $P(x) = \dfrac{e^x}{1+e}, \quad x = 0, 1$

$P(0) = \dfrac{1}{1+e}, \qquad P(1) = \dfrac{e}{1+e}$

Both of these obey $0 \leqslant P(x_i) \leqslant 1$, and $\sum\limits_{i=1}^{n} P(x_i) = \dfrac{1}{1+e} + \dfrac{e}{1+e} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

b $P(x) = \dfrac{x^2+x}{40}, \quad x = 1, 2, 3, 4$

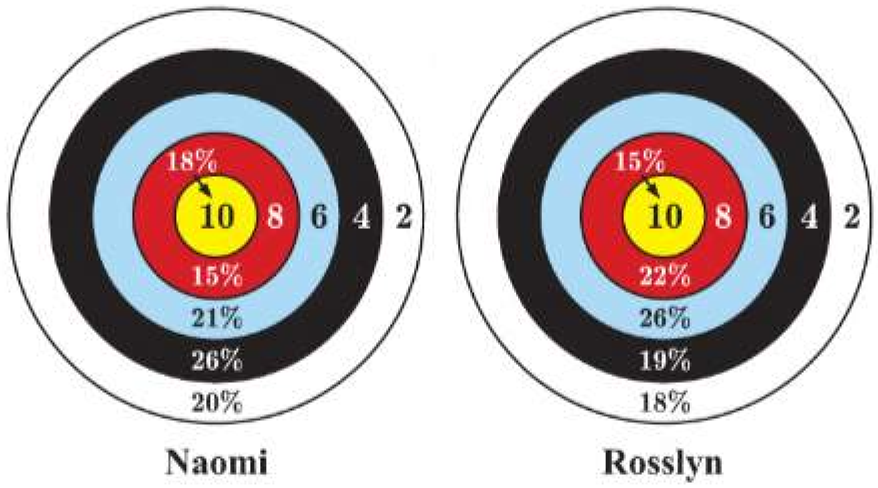
$P(1) = \dfrac{1+1}{40} = \dfrac{2}{40}, \quad P(2) = \dfrac{4+2}{40} = \dfrac{6}{40}, \quad P(3) = \dfrac{9+3}{40} = \dfrac{12}{40}, \quad P(4) = \dfrac{16+4}{40} = \dfrac{20}{40}$

All of these obey $0 \leqslant P(x_i) \leqslant 1$, and $\sum\limits_{i=1}^{n} P(x_i) = \frac{2}{40} + \frac{6}{40} + \frac{12}{40} + \frac{20}{40} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

4 Let N be the number of points Naomi scores per shot.
Let R be the number of points Rosslyn scores per shot.

a **i** $\text{P}(N = 10) = 0.18$
$\text{P}(R = 10) = 0.15$
$\therefore$ Naomi is more likely to score 10 points on a single shot.



ii $\text{P}(N \geqslant 6) = \text{P}(N = 6) + \text{P}(N = 8) + \text{P}(N = 10)$
$= 0.21 + 0.15 + 0.18$
$= 0.54$

$\text{P}(R \geqslant 6) = \text{P}(R = 6) + \text{P}(R = 8) + \text{P}(R = 10)$
$= 0.26 + 0.22 + 0.15$
$= 0.63$

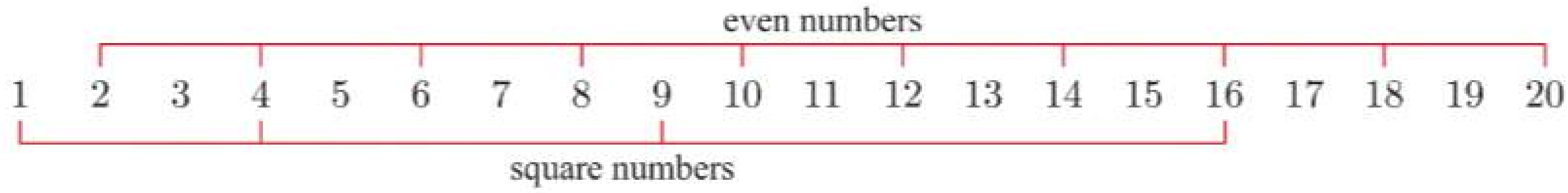
$\therefore$ Rosslyn is more likely to score at least 6 points.

b $\text{E}(N) = (2 \times 0.2) + (4 \times 0.26) + (6 \times 0.21) + (8 \times 0.15) + (10 \times 0.18)$
$= 5.7$ points

$\text{E}(R) = (2 \times 0.18) + (4 \times 0.19) + (6 \times 0.26) + (8 \times 0.22) + (10 \times 0.15)$
$= 5.94$ points

$\therefore$ in the long run, Rosslyn is expected to score more points per shot.

5



Let X denote the number written on the ticket drawn.

a **i** P(player wins \$3)
$= \text{P}(X \text{ is even but not square})$
$= \frac{8}{20}$
$= \frac{2}{5}$

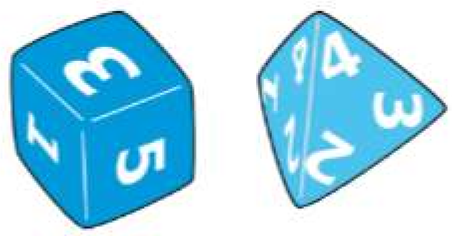
ii P(player wins \$6)
$= \text{P}(X \text{ is square but not even})$
$= \frac{2}{20}$
$= \frac{1}{10}$

iii P(player wins \$9)
$= \text{P}(X \text{ is even and square})$
$= \frac{2}{20}$
$= \frac{1}{10}$

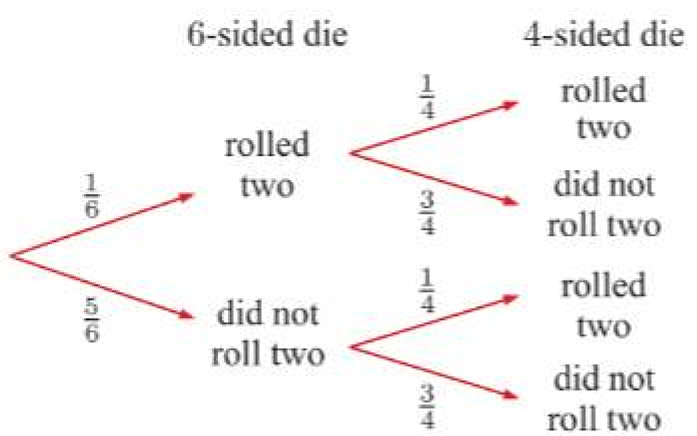
b The expected gain of one game is $\text{E}(X) = \left(0 \times \frac{8}{20}\right) + \left(3 \times \frac{2}{5}\right) + \left(6 \times \frac{1}{10}\right) + \left(9 \times \frac{1}{10}\right)$
$= \frac{27}{10}$
$= \$2.70$ per game

To make the game fair, the game must cost the same as the expected gain, so \$2.70 should be charged each game.

6 **a** X is not a binomial random variable because the probability of rolling a two is not the same for each die.



b



X	Probability
2	$\frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$
1	$\frac{1}{6} \times \frac{3}{4} = \frac{3}{24}$
1	$\frac{5}{6} \times \frac{1}{4} = \frac{5}{24}$
0	$\frac{5}{6} \times \frac{3}{4} = \frac{15}{24}$

$\frac{3}{24} + \frac{5}{24} = \frac{8}{24} = \frac{1}{3}$

x	0	1	2
$\text{P}(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

c $\text{E}(X) = 0\left(\frac{15}{24}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{24}\right) = \frac{5}{12}$

7 **a** Since this is a probability distribution, $\sum p_i = 1$

x_i	1	2	3	4
p_i	0.2	a	0.3	b

$$\therefore\ 0.2 + a + 0.3 + b = 1$$
$$\therefore\ b = 0.5 - a \quad .... (*)$$

$$\text{Now,}\quad \text{E}(X) = 2.8$$
$$\therefore\ (1 \times 0.2) + (2 \times a) + (3 \times 0.3) + (4 \times b) = 2.8$$
$$\therefore\ 0.2 + 2a + 0.9 + 4(0.5 - a) = 2.8 \qquad \{\text{using } (*)\}$$
$$\therefore\ 2a + 2 - 4a = 1.7$$
$$\therefore\ -2a = -0.3$$
$$\therefore\ a = 0.15 \ \text{ and } \ b = 0.35$$

b $\text{Var}(X) = \sum x_i^{\,2} p_i - \mu^2$
$= 1^2(0.2) + 2^2(0.15) + 3^2(0.3) + 4^2(0.35) - 2.8^2$
$= 1.26$

8

x	0	1	2	3	4
$\text{P}(X = x)$	0.1	0.2	0.3	0.3	0.1

a **i** $\text{E}(X) = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.3) + 4(0.1)$
$= 2.1$

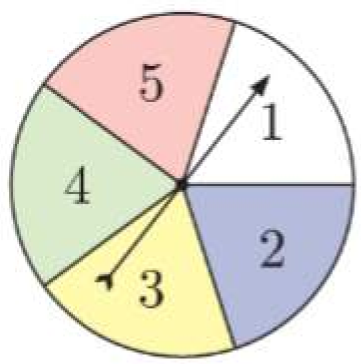
ii $\text{Var}(X) = \sum x_i^{\,2} p_i - \mu^2$
$= 0^2(0.1) + 1^2(0.2) + 2^2(0.3) + 3^2(0.3) + 4^2(0.1) - (2.1)^2$
$= 1.29$

b **i** $Y = 4 - X$

ii $\text{E}(Y) = \text{E}(4 - X)$
$= 4 - \text{E}(X)$
$= 4 - 2.1$
$= 1.9$

$\text{Var}(Y) = \text{Var}(4 - X)$
$= (-1)^2\,\text{Var}(X)$
$= \text{Var}(X)$
$= 1.29$

9 **a** The probability of success (spinning a 3) is the same for each spin, and the number of spins is fixed.



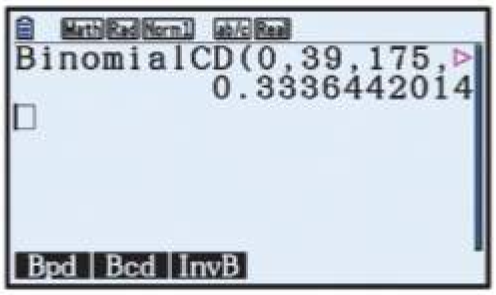
b $\mu = np$
$= 20 \times \frac{1}{5}$
$= 4$

$\sigma = \sqrt{np(1-p)}$
$= \sqrt{20 \times \frac{1}{5} \times \frac{4}{5}}$
$= \frac{4}{\sqrt{5}} \approx 1.79$

10 Let X be the number of visitors who make a voluntary donation upon entry.
$n = 175$, so $X = 0, 1, 2, 3, ...,$ or 175, and $p = 24\% = 0.24$
$\therefore\ X \sim \text{B}(175, 0.24)$

a $\text{E}(X) = \mu = np$
$= 175 \times 0.24$
$= 42$ donations

b **i**



$$\mathrm{P}(X < 40) = \mathrm{P}(X \leqslant 39) \approx 0.334$$

ii

$$\mathrm{P}(50 \leqslant X \leqslant 60) \approx 0.0931$$

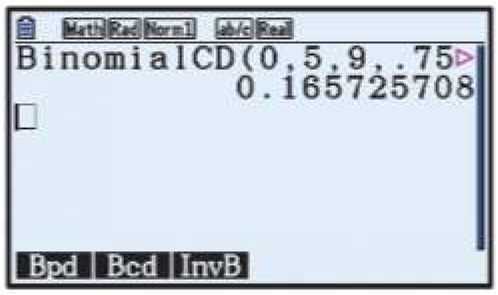
11 Let X denote the number of players who turn up to a game.

$n = 9$, so $X = 0, 1, 2, 3, ..., \text{ or } 9$, and $p = 75\% = 0.75$

$\therefore \ X \sim \mathrm{B}(9, 0.75)$

a **i** $\mathrm{P}(X = 9) = \binom{9}{9}(0.75)^9(0.25)^0 \approx 0.0751$

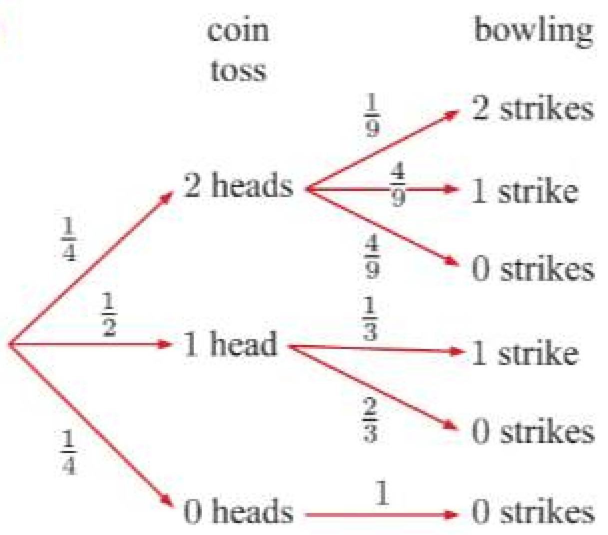
ii



$$\begin{aligned}\mathrm{P}(\text{forfeit}) &= \mathrm{P}(X < 6)\\ &= \mathrm{P}(X \leqslant 5)\\ &\approx 0.166\end{aligned}$$

b The team is expected to forfeit $30 \times 0.1657 \approx 4.97$ games throughout the season.

12 **a**



probability

$\frac{1}{4} \times \frac{1}{9} = \frac{1}{36}$

$\frac{1}{4} \times \frac{4}{9} = \frac{1}{9}$

$\frac{1}{4} \times \frac{4}{9} = \frac{1}{9}$

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

$\frac{1}{4} \times 1 = \frac{1}{4}$

b $\mathrm{P}(X = 0) = \frac{1}{9} + \frac{1}{3} + \frac{1}{4} = \frac{25}{36}$ $\quad \mathrm{P}(X = 1) = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$ $\quad \mathrm{P}(X = 2) = \frac{1}{36}$

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

c The expected return per game is $\mathrm{E}(X) = \left(0 \times \frac{25}{36}\right) + \left(10 \times \frac{5}{18}\right) + \left(20 \times \frac{1}{36}\right) = \frac{10}{3} \text{ dollars} \approx \$3.33$

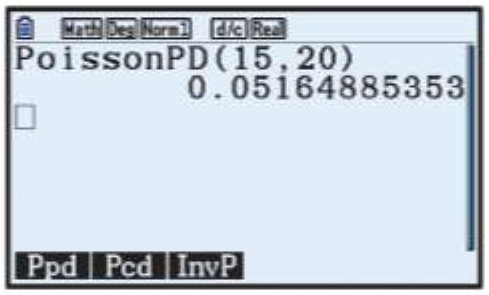
d Suvi's expected gain per game $\approx \$3.33 - \$5$
$\approx -\$1.67$

$\therefore$ Suvi should not play the game many times as she is expected to lose \$1.67 per game on average.

13 **a** Let X be the number of customers arriving at the shop in a 15 minute period.

$\therefore \;\; X \sim \text{Po}(20)$

$\text{P}(X = 15) \approx 0.0516$

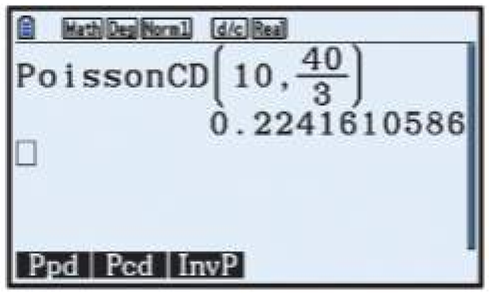


b Let Y be the number of customers arriving at the shop in a 10 minute period.

$$\begin{aligned}\text{The rate at which customers arrive} &= \tfrac{10}{15} \times 20\\ &= \tfrac{40}{3} \text{ customers per 10 minutes}\end{aligned}$$

$\therefore \;\; Y \sim \text{Po}\left(\frac{40}{3}\right)$

$$\begin{aligned}\text{P}(Y > 10) &= 1 - \text{P}(Y \leqslant 10)\\ &\approx 1 - 0.224\\ &\approx 0.776 < 0.8\end{aligned}$$



$\therefore$ the probability that more than 10 customers will arrive at the shop in a 10 minute period is *not* greater than 80%.

$\therefore$ the manager will not hire an extra shop assistant.

14 Suppose $X \sim \text{Po}(\lambda)$

a $$\text{P}(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, ...$$

Now $\text{P}(X = 1) = \text{P}(2 \leqslant x \leqslant 4)$

$$\therefore \;\; \text{P}(X = 1) = \text{P}(X = 2) + \text{P}(X = 3) + \text{P}(X = 4)$$

$$\therefore \;\; \frac{\lambda e^{-\lambda}}{1!} = \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \frac{\lambda^4 e^{-\lambda}}{4!}$$

$$\therefore \;\; \lambda = \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24} \qquad \{\times\, e^{\lambda}\}$$

$$\therefore \;\; 24\lambda = 12\lambda^2 + 4\lambda^3 + \lambda^4$$

$$\therefore \;\; \lambda(\lambda^3 + 4\lambda^2 + 12\lambda - 24) = 0 \quad \text{where } \lambda \neq 0$$

$$\therefore \;\; \lambda \approx 1.28 \quad \{\text{using technology}\}$$

mean of $X = \lambda \approx 1.28$

standard deviation $= \sqrt{\lambda} \approx 1.13$

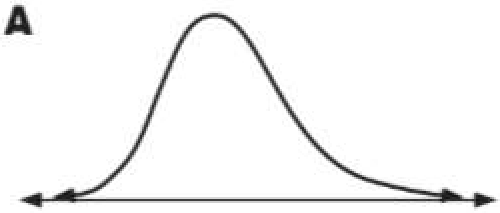
b $$\begin{aligned}\text{P}(X \geqslant 2) &= 1 - \text{P}(X \leqslant 1)\\ &\approx 1 - 0.634\\ &\approx 0.366\end{aligned}$$

Chapter 28

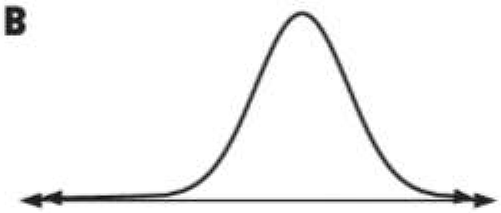
THE NORMAL DISTRIBUTION

EXERCISE 28A.1

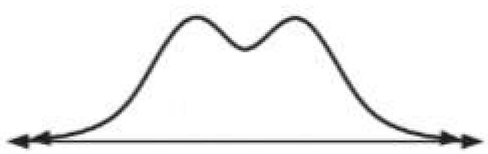
1 **A**



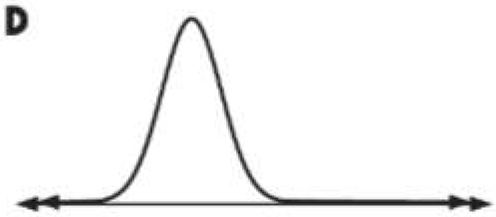
B



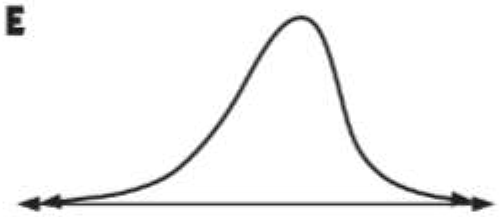
C



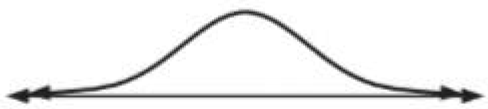
D



E



F



Distributions **B**, **D**, and **F** are symmetrical and bell-shaped.

$\therefore$ **B**, **D**, and **F** appear to be normally distributed.

2 Most measurements in each situation will be centred about the mean, with random variation about the mean explained by some of the factors listed below.

a The diameters may be affected by:

- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

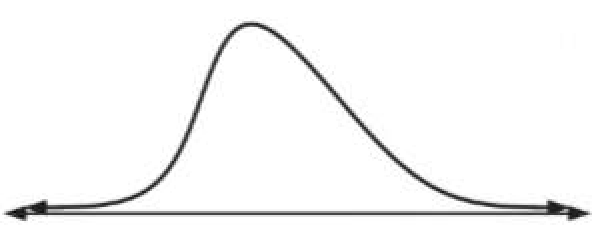
b The scores may be affected by:

- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

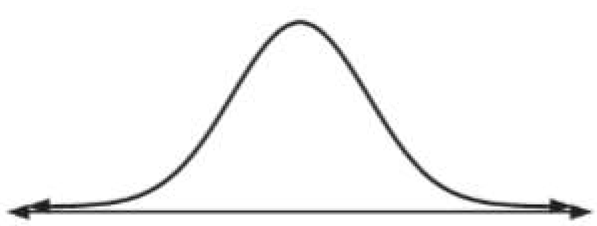
c The times may be affected by:

- weather conditions
- walking speed
- physical fitness
- traffic.

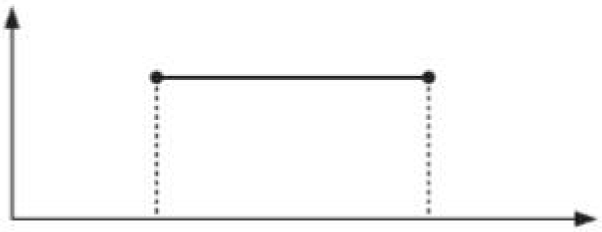
3 **a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



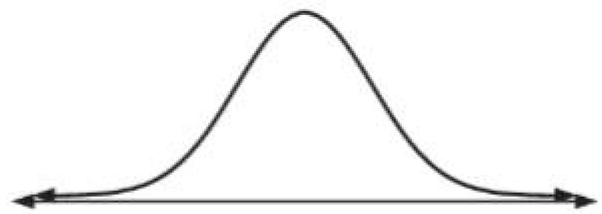
b The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



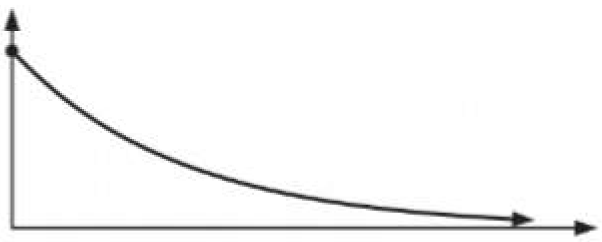
c The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



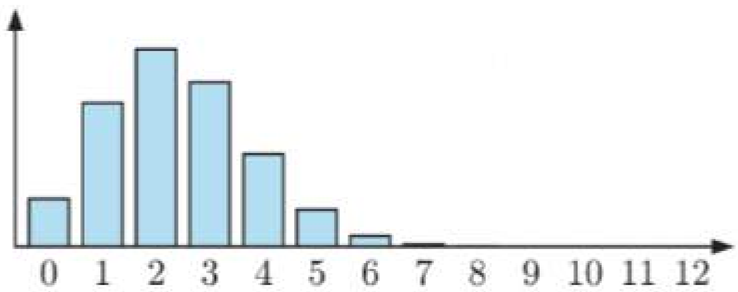
d The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



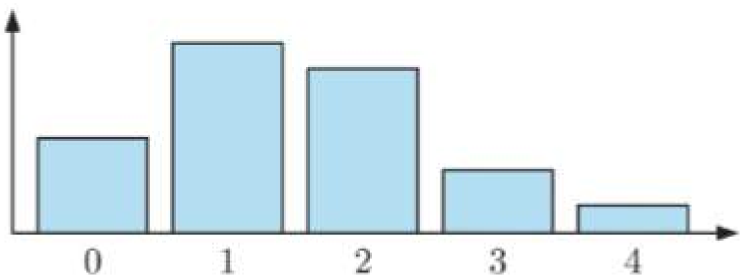
e The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



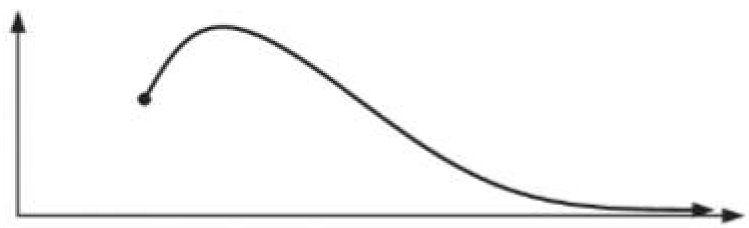
f The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



g The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



h The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



INVESTIGATION 1 PROPERTIES OF THE NORMAL CURVE

1 μ controls where the centre of the distribution is. As μ changes, the curve is translated horizontally, which is reasonable since μ is the mean and a measure of centre.

σ controls the shape of the curve. As σ increases, the curve becomes flatter and more spread out, which is reasonable since σ is the standard deviation and a measure of spread.

2 The curve has a vertical line of symmetry $x = \mu$.

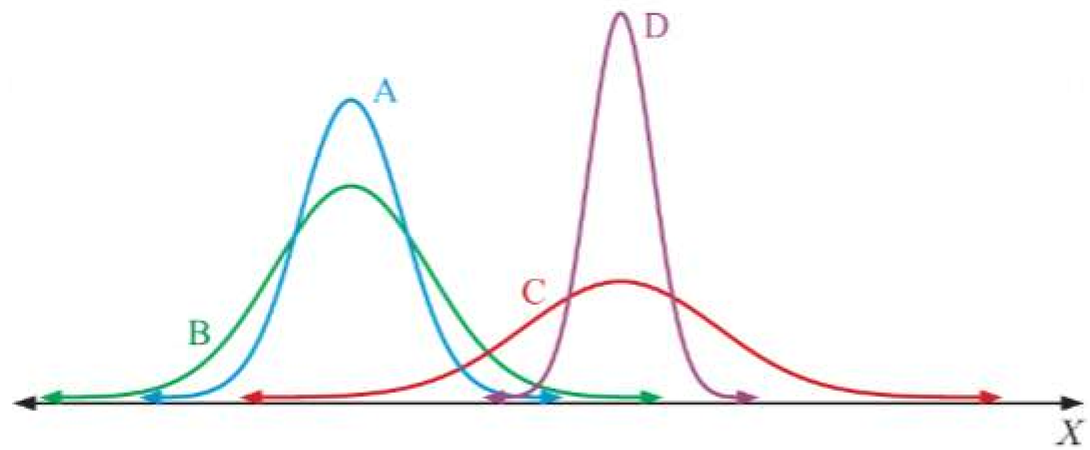
3 The function is never negative. This is important because a probability density function can never be negative.

4 As $x \to \pm\infty$, the curve approaches zero from above. The x-axis is a horizontal asymptote.

5 The area under the curve should remain constant as we change μ and σ, as the area under a probability density function must be 1.

EXERCISE 28A.2

1



A and B have the same mean, and C and D have the same mean.
The mean of A and B is lower than the mean of C and D.
B has a greater spread, and hence a larger standard deviation than A.
Similarly, C has a larger standard deviation than D.

a $\mu = 5,\ \sigma = 2$ corresponds to B

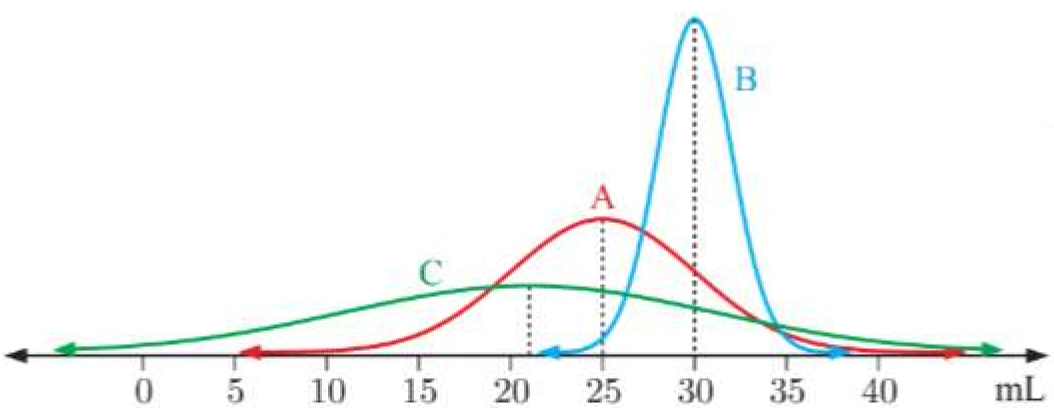
b $\mu = 15,\ \sigma = 0.5$ corresponds to D

c $\mu = 5,\ \sigma = 1$ corresponds to A

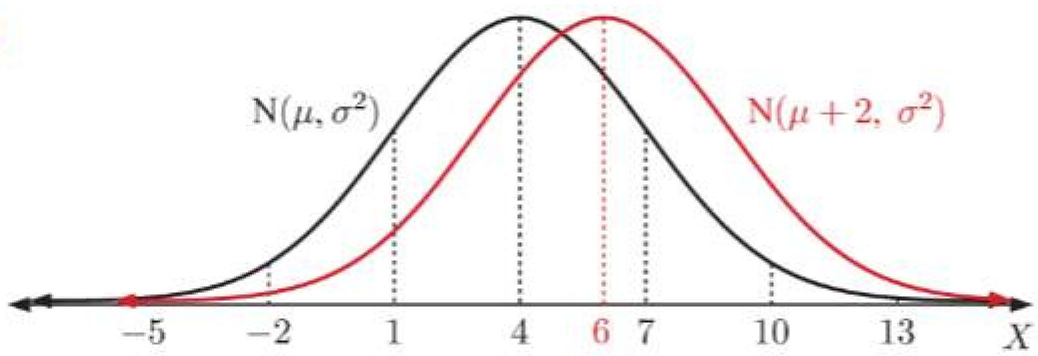
d $\mu = 15,\ \sigma = 3$ corresponds to C

2

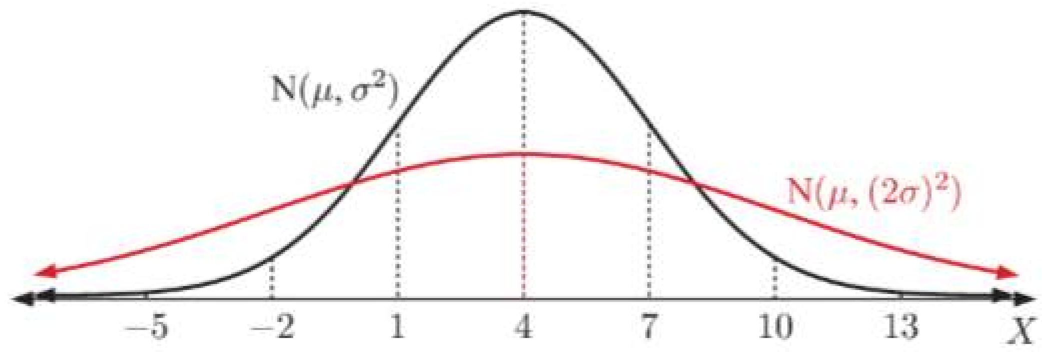
Distribution	*Mean* (mL)	*Standard deviation* (mL)
A	25	5
B	30	2
C	21	10



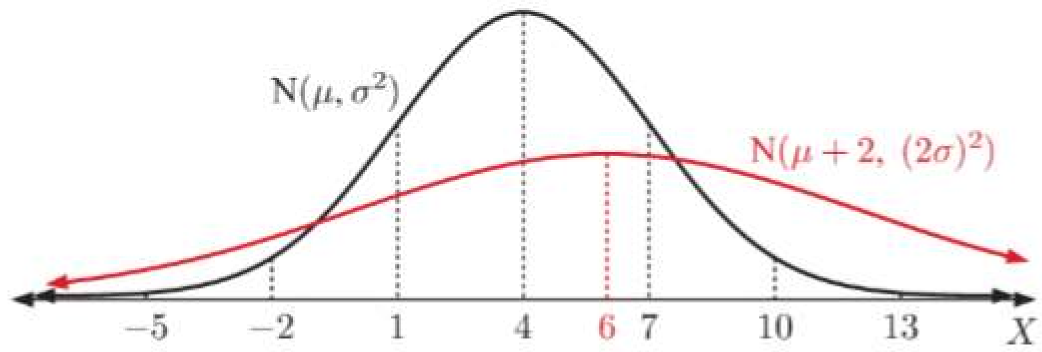
3 **a**



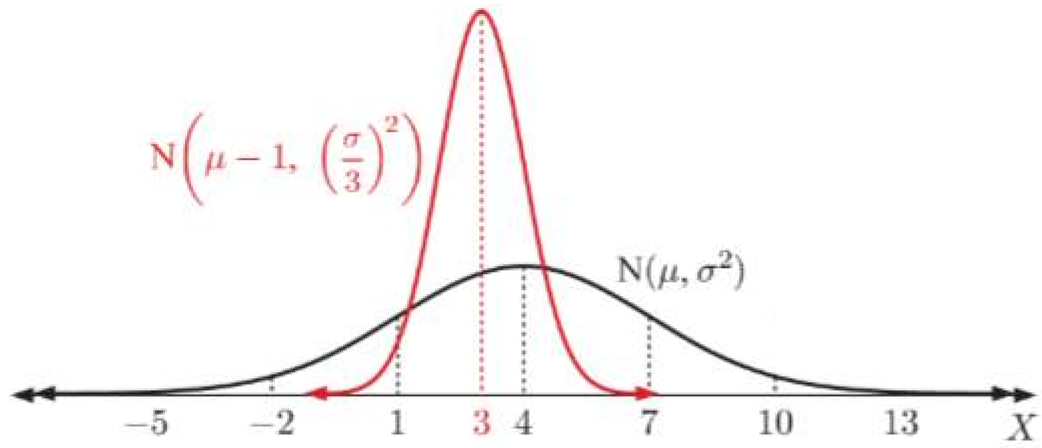
b



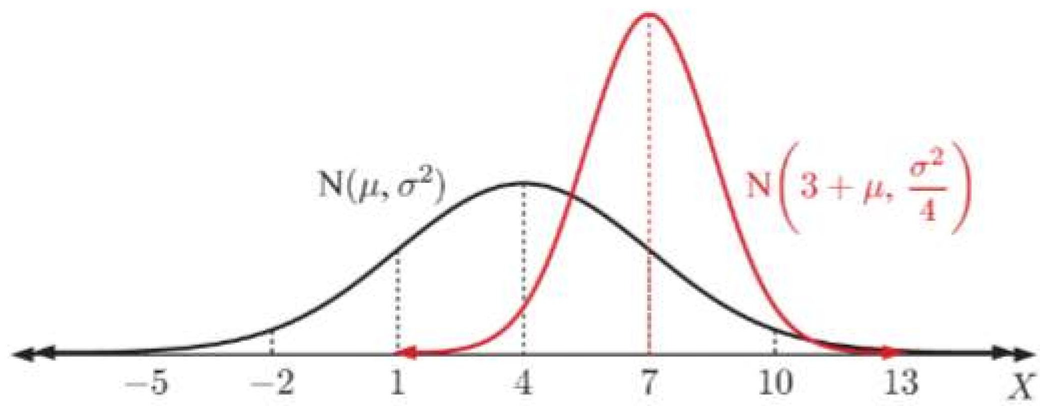
c



d



e



EXERCISE 28B.1

1 $X \sim \mathrm{N}(30,\ 5^2)$

a **i** The value which is 2 standard deviations above the mean $= 30 + 2 \times 5$
$= 40$

ii The value which is 1 standard deviation below the mean $= 30 - 5$
$= 25$

b **i** $35 = 30 + 5$

$\therefore$ 35 is 1 standard deviation above the mean.

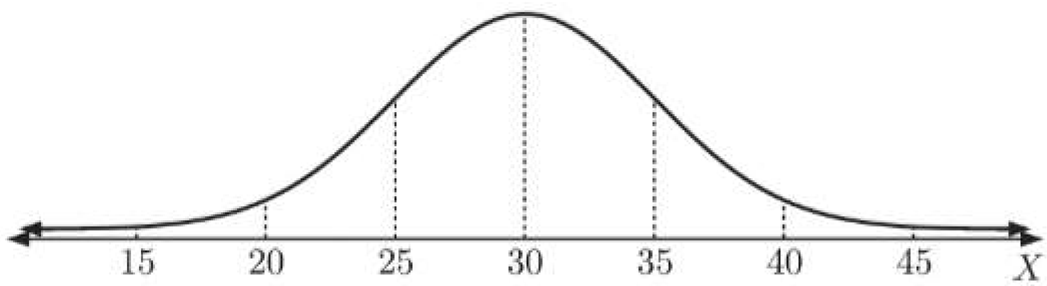
ii $20 = 30 - 2 \times 5$

$\therefore$ 20 is 2 standard deviations below the mean.

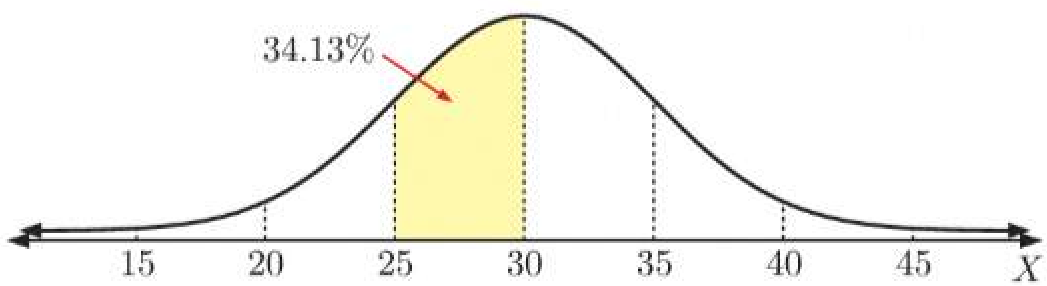
iii $45 = 30 + 3 \times 5$

$\therefore$ 45 is 3 standard deviations above the mean.

c

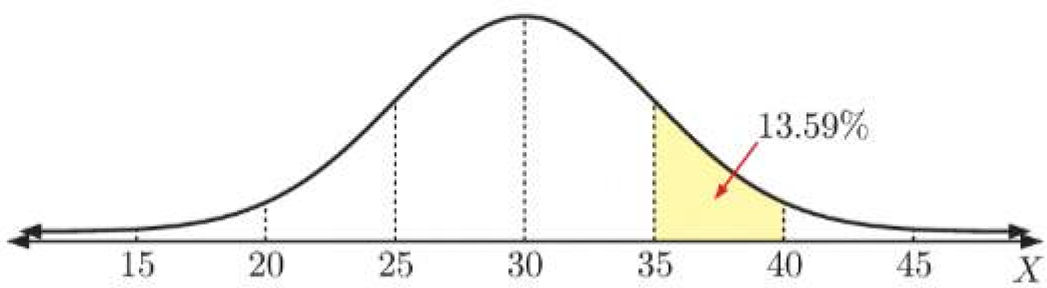


d



About 34.13% of the values of X are between 25 and 30.

e

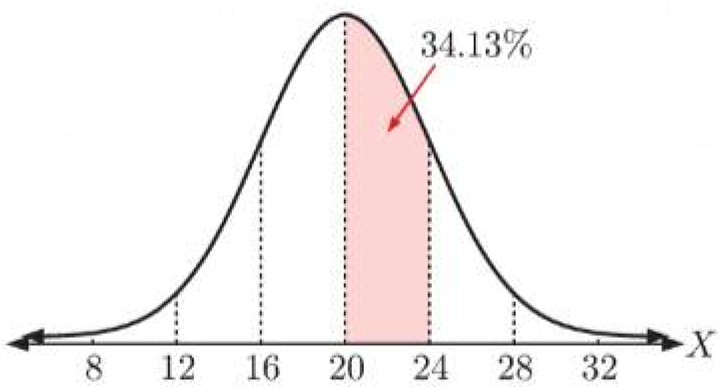


About 13.59% of the values of X are between 35 and 40.

$\therefore$ the probability that a randomly selected member of the population will measure between 35 and 40 is approximately 0.1359 .

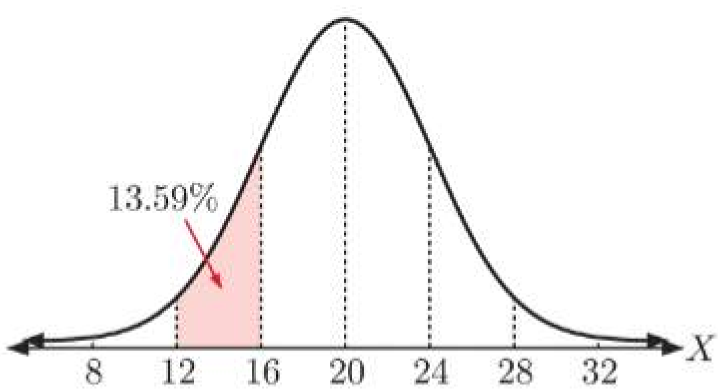
2 **a** $\mu = 20,\ \ \sigma = 4$

b **i**



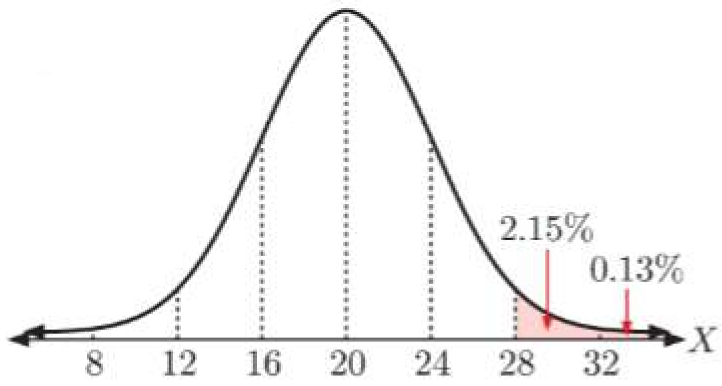
About 34.13% of X values are between 20 and 24.

ii



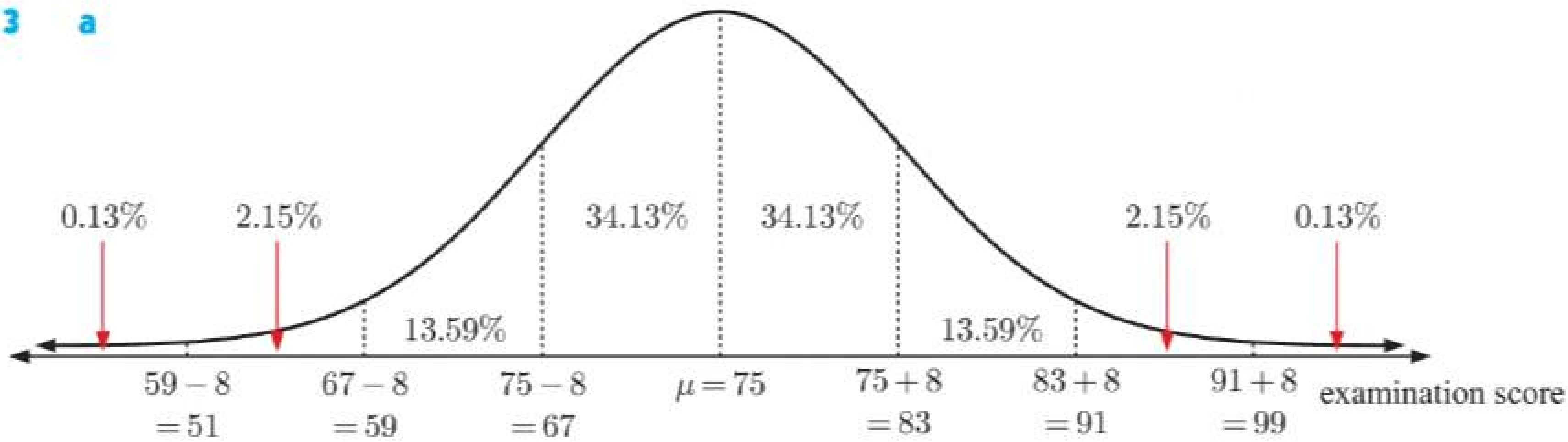
About 13.59% of X values are between 12 and 16.

iii

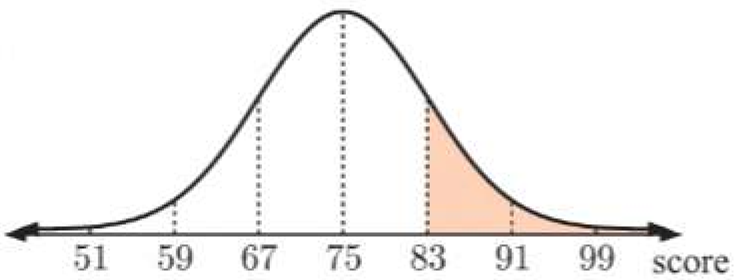


About $2.15\% + 0.13\% = 2.28\%$ of X values are greater than 28.

3 a

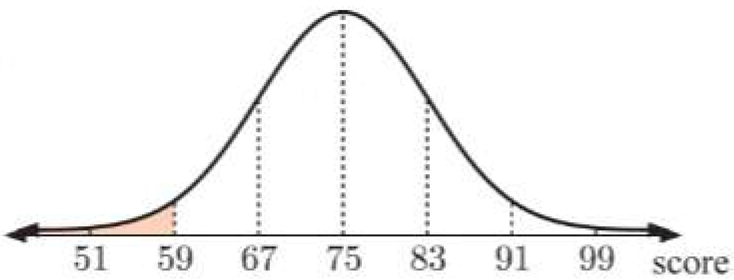


b i



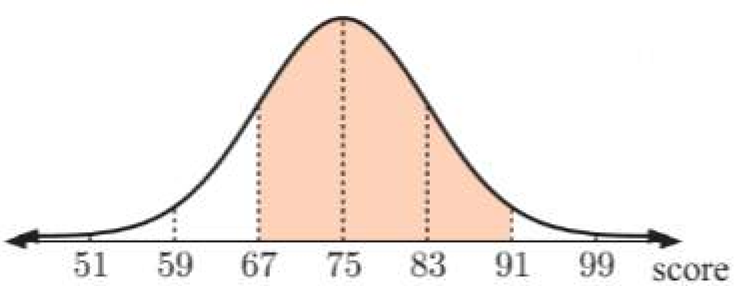
About $13.59\% + 2.15\% + 0.13\%$
$= 15.87\%$
of students would be expected to have scored more than 83.

ii



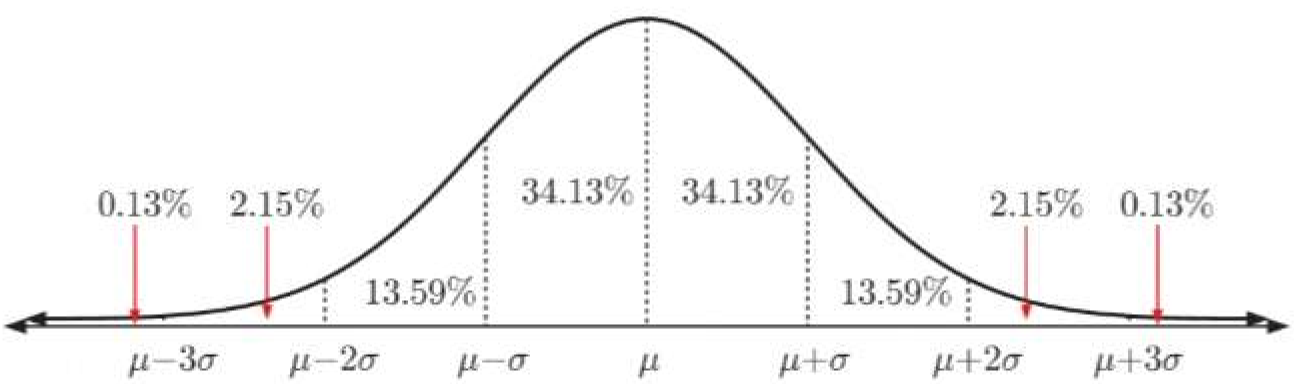
About $2.15\% + 0.13\% = 2.28\%$ of students would be expected to have scored less than 59.

iii



About $34.13\% + 34.13\% + 13.59\%$
$= 81.85\%$
of students would be expected to have scored between 67 and 91.

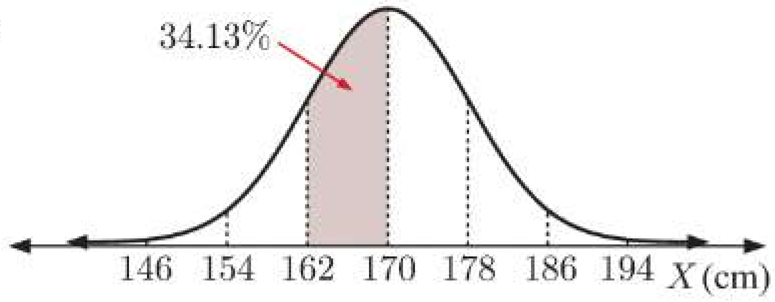
4



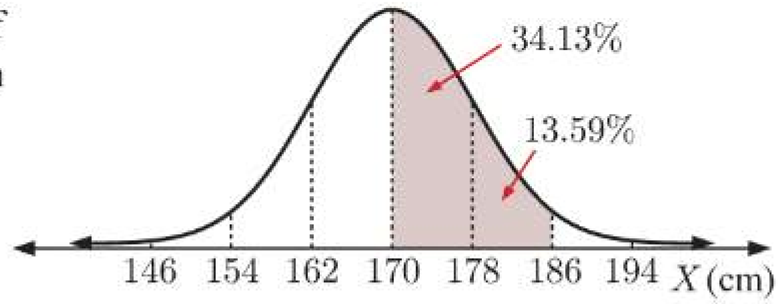
a P(value between $\mu - \sigma$ and $\mu + \sigma$)
$\approx 0.3413 + 0.3413$
≈ 0.6826

b P(value $> \mu + 2\sigma$)
$\approx 0.0215 + 0.0013$
≈ 0.0228

5 **a** **i** About 34.13% of female students have a height between 162 cm and 170 cm.



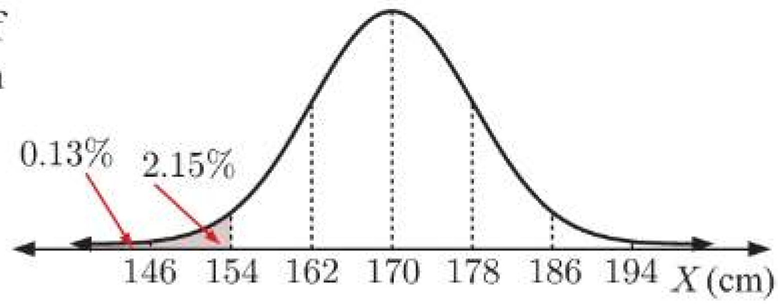
ii About $34.13\% + 13.59\% = 47.72\%$ of female students have a height between 170 cm and 186 cm.



b **i** About $2.15\% + 0.13\% = 2.28\%$ of female students have a height less than 154 cm.

$\therefore$ P(height is less than 154 cm)

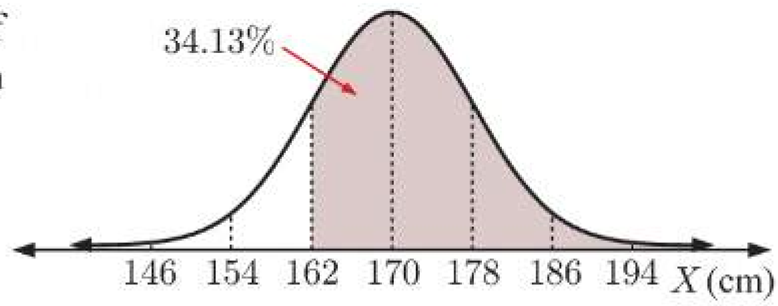
≈ 0.0228



ii About $34.13\% + 50\% = 84.13\%$ of female students have a height greater than 162 cm.

$\therefore$ P(height is greater than 162 cm)

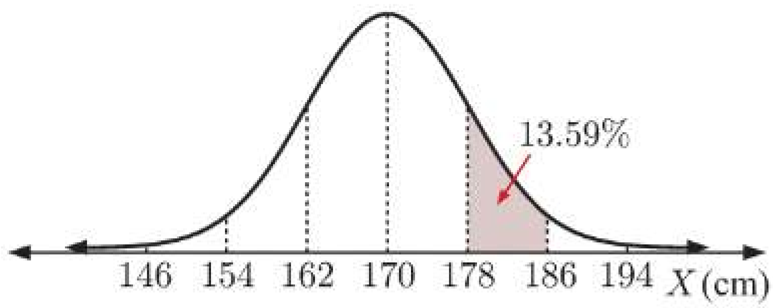
≈ 0.8413



c About 13.59% of the female students have a height between 178 cm and 186 cm.

So, we would expect about

13.59% of 500 $\approx$ 68 students to have a height between 178 cm and 186 cm.

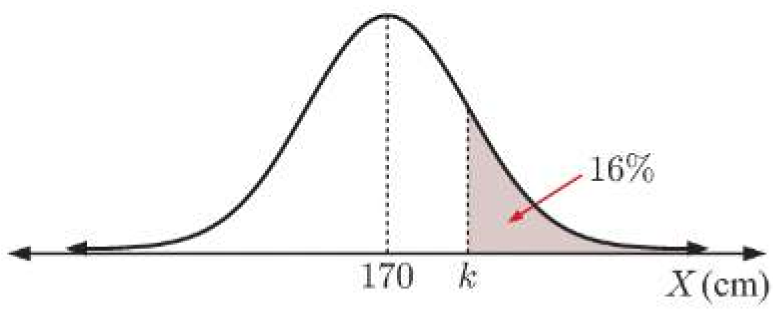


d Approximately 16% of data lies more than one standard deviation above the mean.

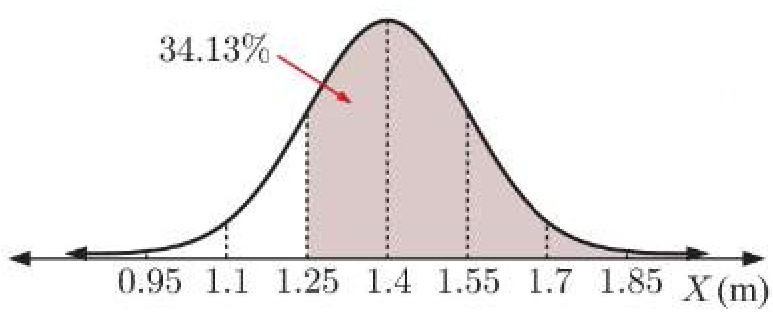
$\therefore$ k is about σ above the mean μ

$\therefore$ $k \approx 170 + 8$

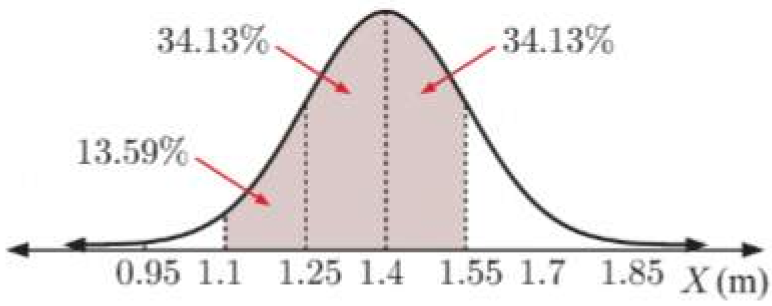
≈ 178



6 **a** About $34.13\% + 50\% = 84.13\%$ of adult female frilled sharks measure more than 1.25 m long.

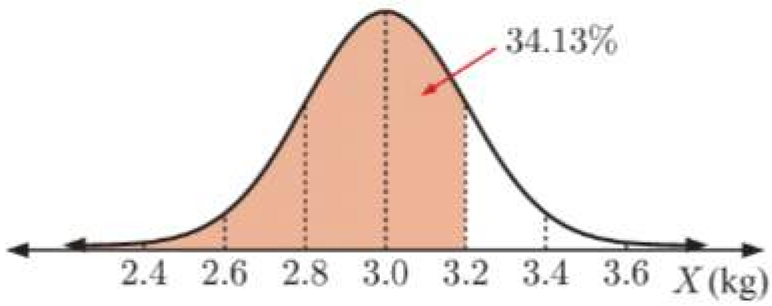


b About $13.59\% + 34.13\% + 34.13\%$
$= 81.85\%$ of adult female frilled sharks measure between 1.1 m and 1.55 m long.



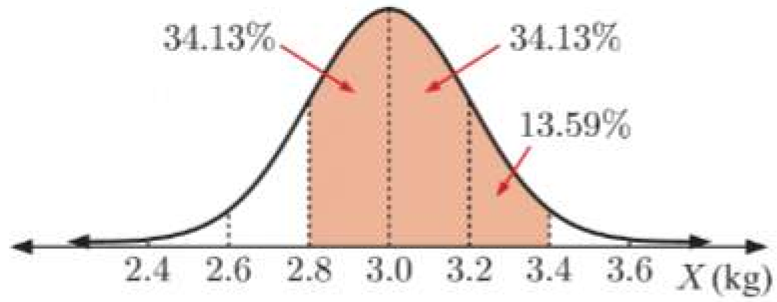
7 **a** About $50\% + 34.13\% = 84.13\%$ of babies born weighed less than 3.2 kg.

So, about $84.13\% \times 545 \approx 459$ babies born weighed less than 3.2 kg.

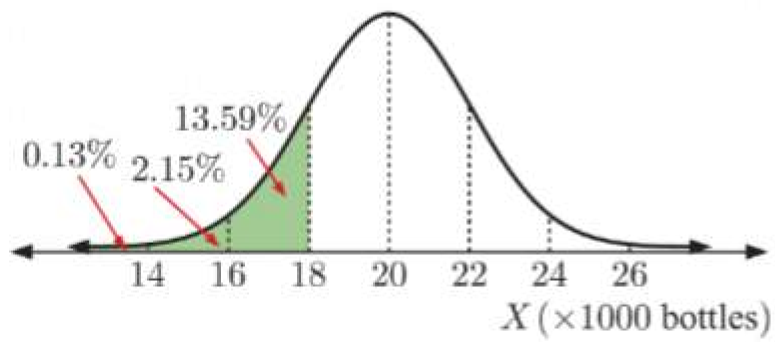


b About $34.13\% + 34.13\% + 13.59\%$
$= 81.85\%$ of babies born weighed between 2.8 kg and 3.4 kg.

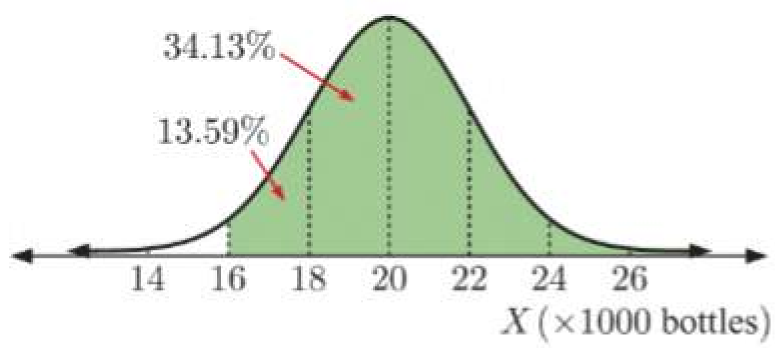
So, about $81.85\% \times 545 \approx 446$ babies born weighed between 2.8 kg and 3.4 kg.



8 **a** Under 18 000 bottles are filled on about $0.13\% + 2.15\% + 13.59\% = 15.87\%$ of days. So, under 18 000 bottles are filled on about $15.87\% \times 260 \approx 41$ days.

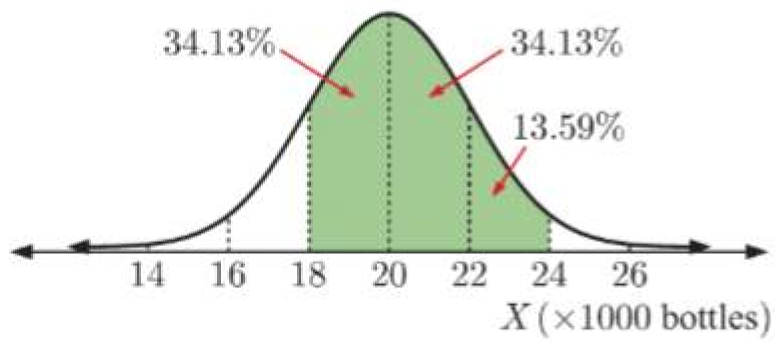


b Over 16 000 bottles are filled on about $13.59\% + 34.13\% + 50\% = 97.72\%$ of days. So, over 16 000 bottles are filled on about $97.72\% \times 260 \approx 254$ days.



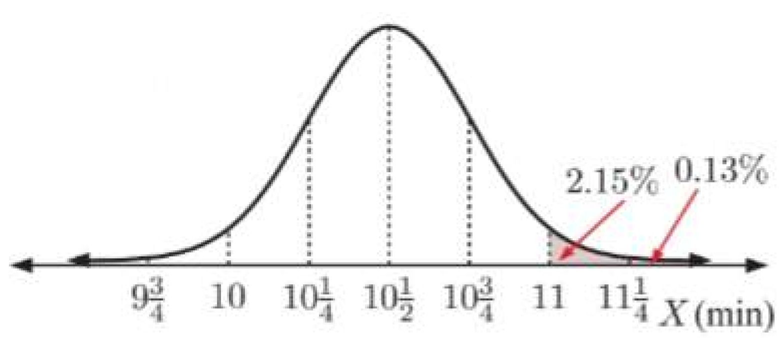
c Between 18 000 and 24 000 bottles are filled on about $34.13\% + 34.13\% + 13.59\%$
$= 81.85\%$ of days.

So, between 18 000 and 24 000 bottles are filled on about $81.85\% \times 260 \approx 213$ days.

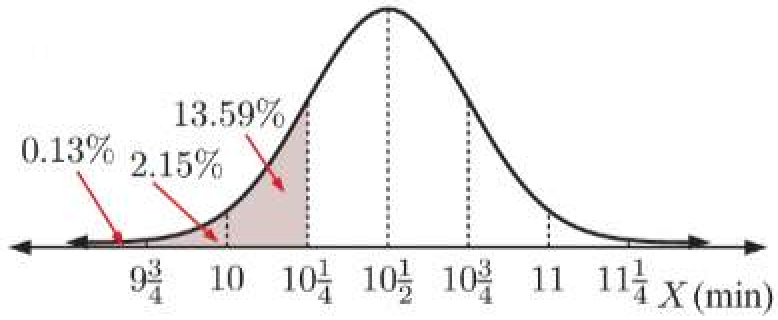


9 **a** About $2.15\% + 0.13\% = 2.28\%$ of competitors completed the race in a time longer than 11 minutes.

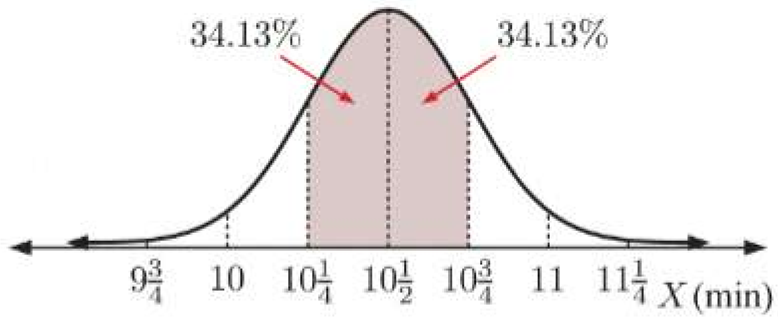
So, about $2.28\% \times 200 \approx 5$ competitors completed the race in a time longer than 11 minutes.



b About $0.13\% + 2.15\% + 13.59\% = 15.87\%$ of competitors completed the race in a time less than 10 minutes 15 seconds.
So, about $15.87\% \times 200 \approx 32$ competitors completed the race in a time less than 10 minutes 15 seconds.



c About $34.13\% + 34.13\% = 68.26\%$ of competitors completed the race in a time between 10 minutes 15 seconds and 10 minutes 45 seconds.
So, about $68.26\% \times 200 \approx 137$ competitors completed the race in a time between 10 minutes 15 seconds and 10 minutes 45 seconds.



10 **a** Approximately 84% of data is more than one standard deviation below the mean, and $34.13\% + 50\% \approx 84\%$.

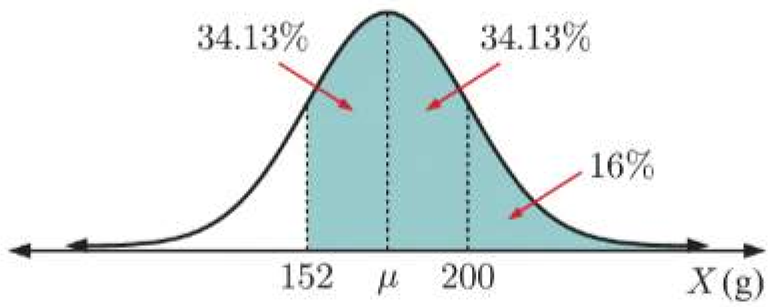
$\therefore$ 152 grams is about σ below the mean μ

$\therefore$ $\mu \approx 152 + \sigma$ .... (1)

Approximately 16% of data is more than one standard deviation above the mean, and $13.59\% + 2.15\% + 0.13\% \approx 16\%$.

$\therefore$ 200 grams is σ above the mean μ

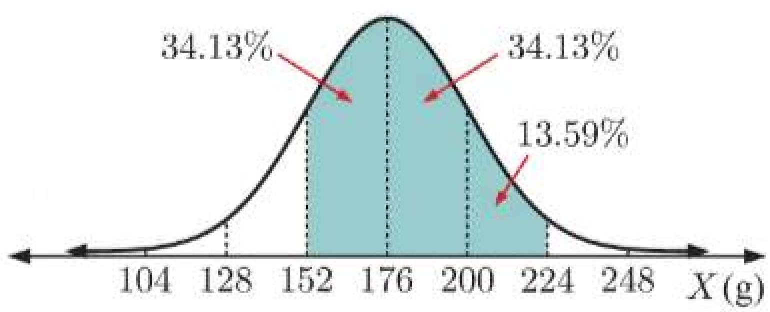
$\therefore$ $\mu \approx 200 - \sigma$ .... (2)



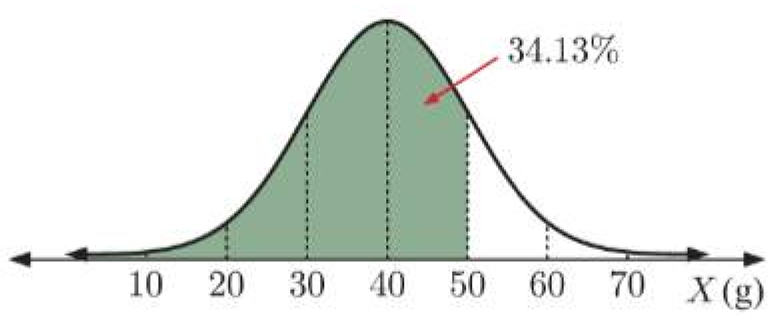
Equating (1) and (2) gives:
$$\begin{aligned} 152 + \sigma &\approx 200 - \sigma \\ \therefore\ 2\sigma &\approx 48 \\ \therefore\ \sigma &\approx 24 \\ \text{and}\quad \mu &\approx 200 - 24 \quad \{\text{using (2)}\} \\ &\approx 176 \end{aligned}$$

So, $\mu \approx 176$ grams and $\sigma \approx 24$ grams.

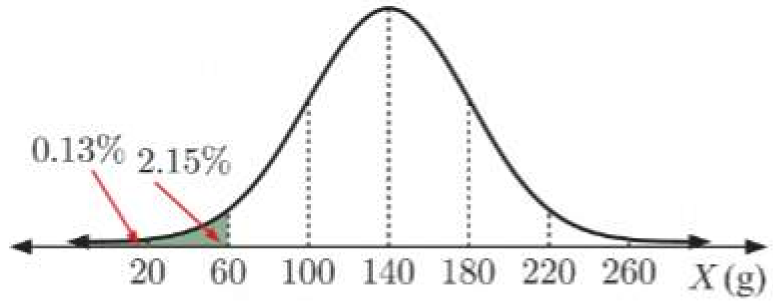
b About $34.13\% + 34.13\% + 13.59\%$
$= 81.85\%$ of the oranges weigh between 152 grams and 224 grams.



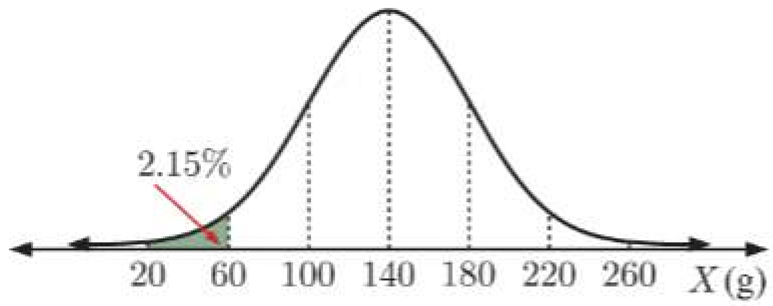
11 **a** **i** About $50\% + 34.13\% = 84.13\%$ of radishes grown without fertiliser will weigh less than 50 grams.



ii About $0.13\% + 2.15\% = 2.28\%$ of radishes grown with fertiliser will weigh less than 60 grams.

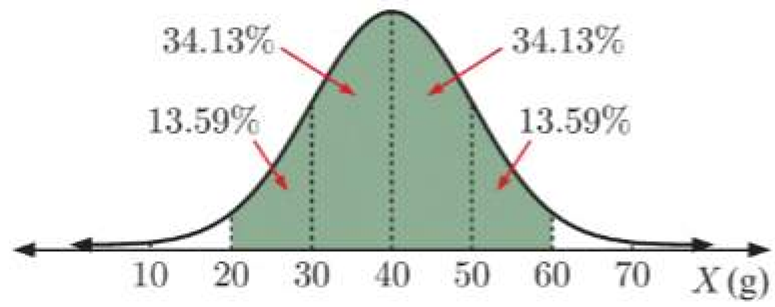


b **i** About 2.15% of radishes grown with fertiliser will weigh between 20 g and 60 g.



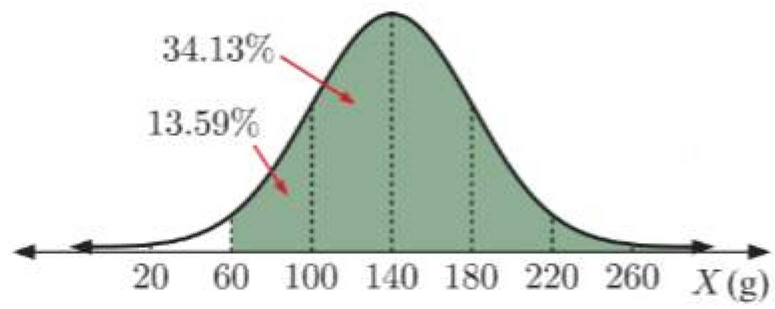
$\therefore$ P(radish grown with fertiliser weighs between 20 g and 60 g) ≈ 0.0215

ii About
$13.59\% + 34.13\% + 34.13\% + 13.59\%$
$= 95.44\%$
of radishes grown without fertiliser will weigh between 20 g and 60 g.



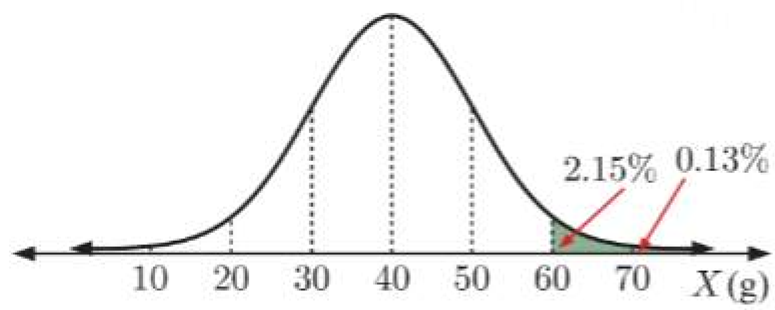
$\therefore$ P(radish grown without fertiliser weighs between 20 g and 60 g) ≈ 0.9544

c About $13.59\% + 34.13\% + 50\% = 97.72\%$ of radishes grown with fertiliser will weigh more than 60 g.



$\therefore$ P(radish grown with fertiliser weighs more than 60 g) ≈ 0.9772

About $2.15\% + 0.13\% = 2.28\%$ of radishes grown without fertiliser will weigh more than 60 g.



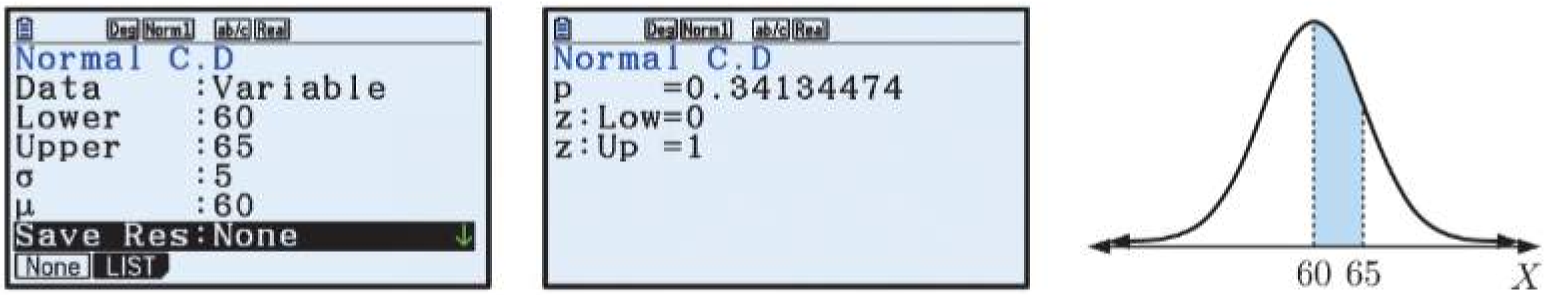
$\therefore$ P(radish grown without fertiliser weighs more than 60 g) ≈ 0.0228

P(*both* radishes weigh more than 60 g)
$=$ P(radish grown with fertiliser weighs more than 60 g)
$\times$ P(radish grown without fertiliser weighs more than 60 g)
$\approx 0.9772 \times 0.0228$
≈ 0.0223

EXERCISE 28B.2

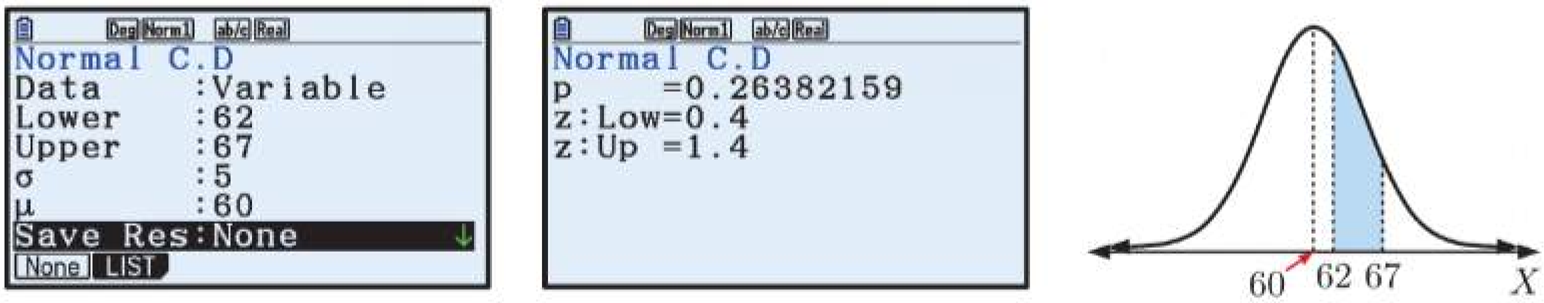
1 $X \sim \mathrm{N}(60, 5^2)$

a To find $\mathrm{P}(60 \leqslant X \leqslant 65)$, we set the lower bound to 60 and the upper bound to 65.



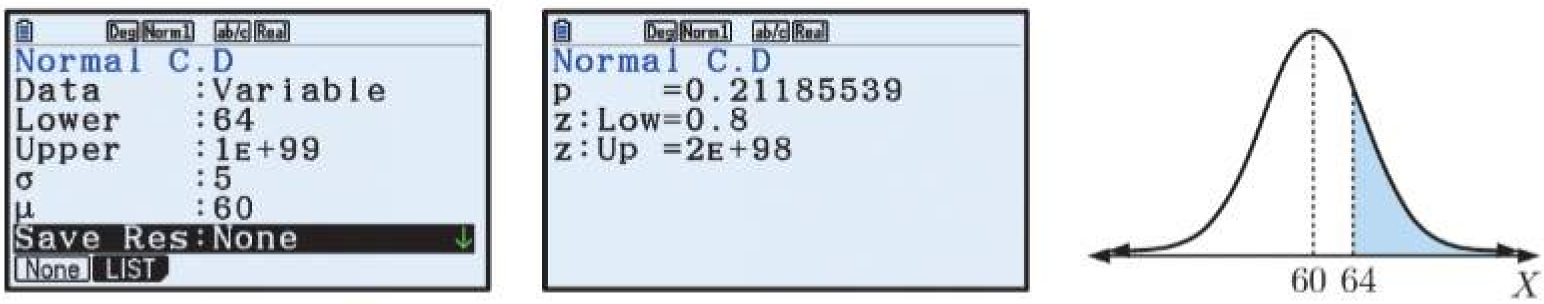
$\mathrm{P}(60 \leqslant X \leqslant 65) \approx 0.341$

b To find $\mathrm{P}(62 \leqslant X \leqslant 67)$, we set the lower bound to 62 and the upper bound to 67.



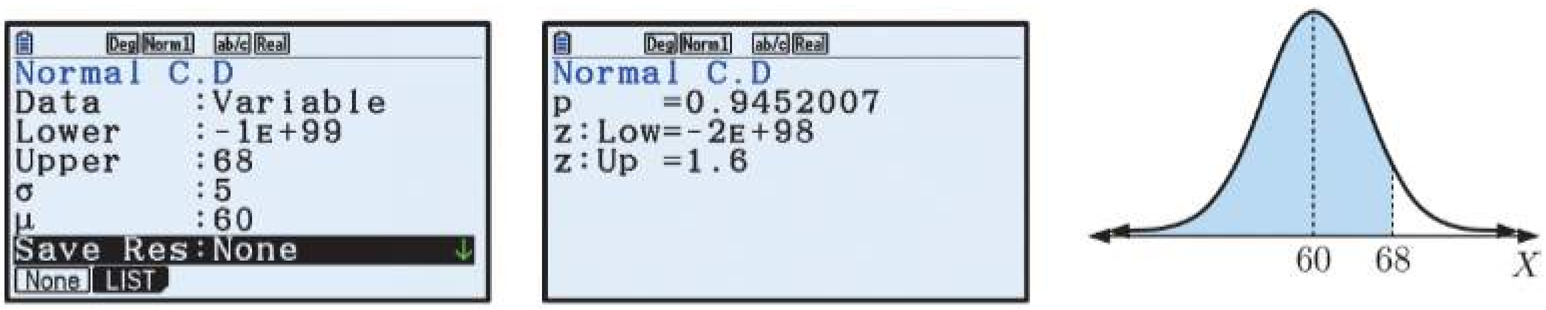
$\mathrm{P}(62 \leqslant X \leqslant 67) \approx 0.264$

c To find $\mathrm{P}(X \geqslant 64)$, we use a very high value such as 10^{99} to represent the upper bound.



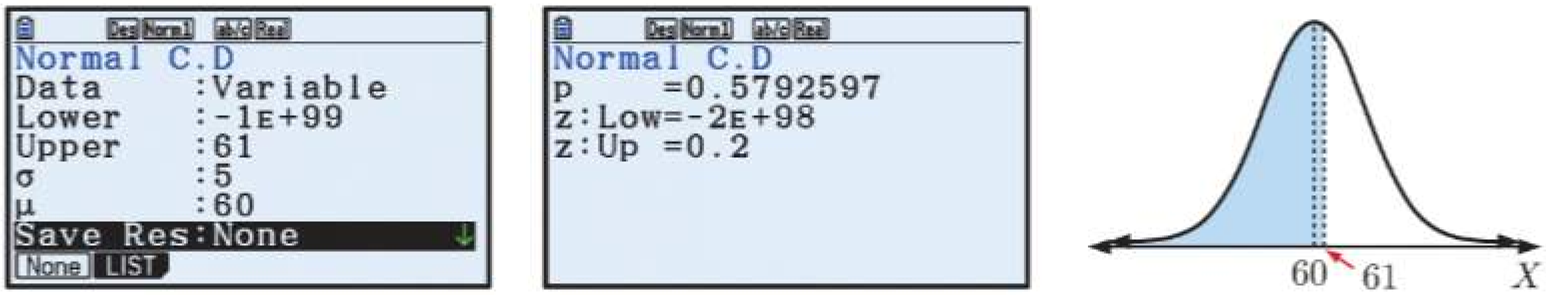
$\mathrm{P}(X \geqslant 64) \approx 0.212$

d To find $\mathrm{P}(X \leqslant 68)$, we use a very low value such as -10^{99} to represent the lower bound.



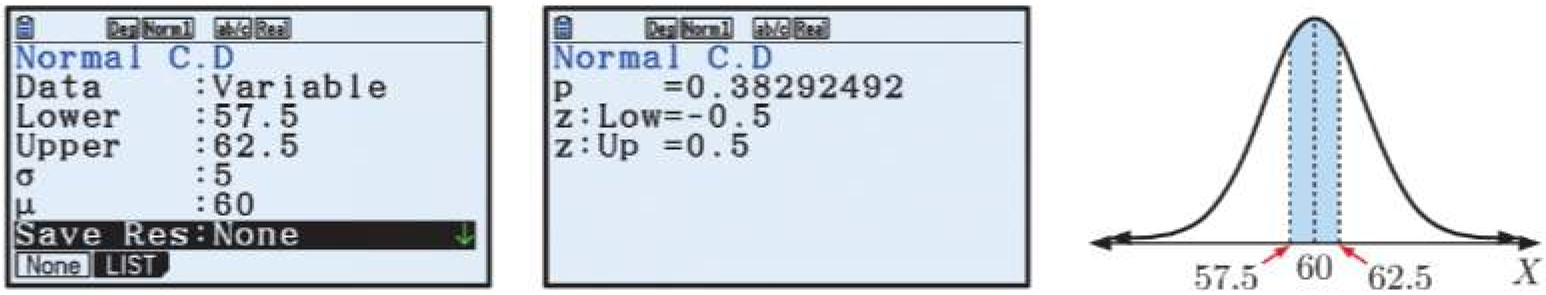
$\mathrm{P}(X \leqslant 68) \approx 0.945$

e To find $P(X \leqslant 61)$, we use a very low value such as -10^{99} to represent the lower bound.



$P(X \leqslant 61) \approx 0.579$

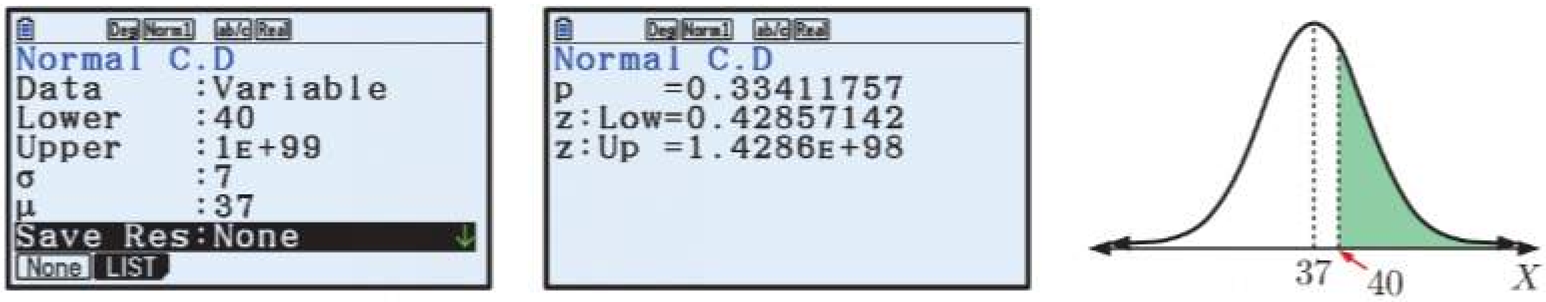
f To find $P(57.5 \leqslant X \leqslant 62.5)$, we set the lower bound to 57.5 and the upper bound to 62.5.



$P(57.5 \leqslant X \leqslant 62.5) \approx 0.383$

2 $X \sim N(37, 7^2)$

a To find $P(X > 40)$, we use a very high value such as 10^{99} to represent the upper bound.



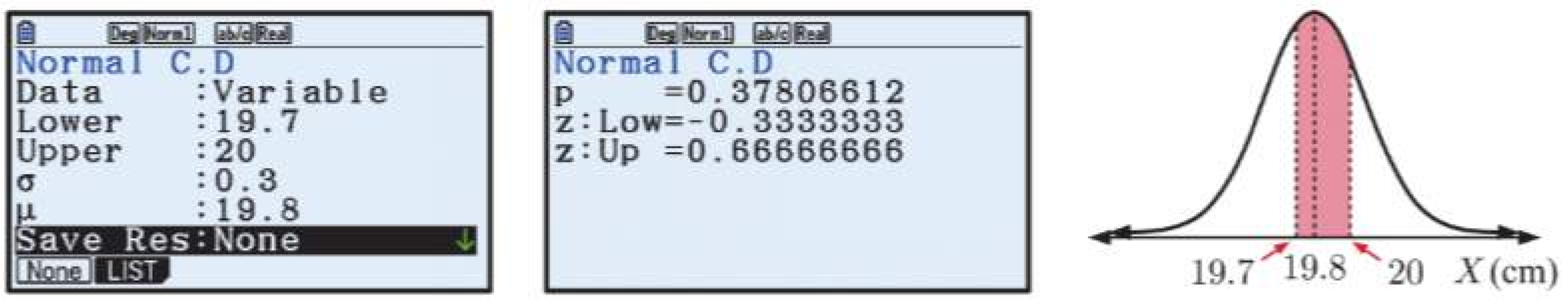
$P(X > 40) \approx 0.334$

b Since the mean of the distribution is 37, then $P(X > 37) = 0.5$

$$\begin{aligned} \therefore \quad P(37 \leqslant X \leqslant 40) &= P(X > 37) - P(X > 40) \\ &\approx 0.5 - 0.334 \\ &\approx 0.166 \end{aligned}$$

3 Let X cm be the length of a randomly selected bolt.

$\therefore \quad X \sim N(19.8, 0.3^2)$

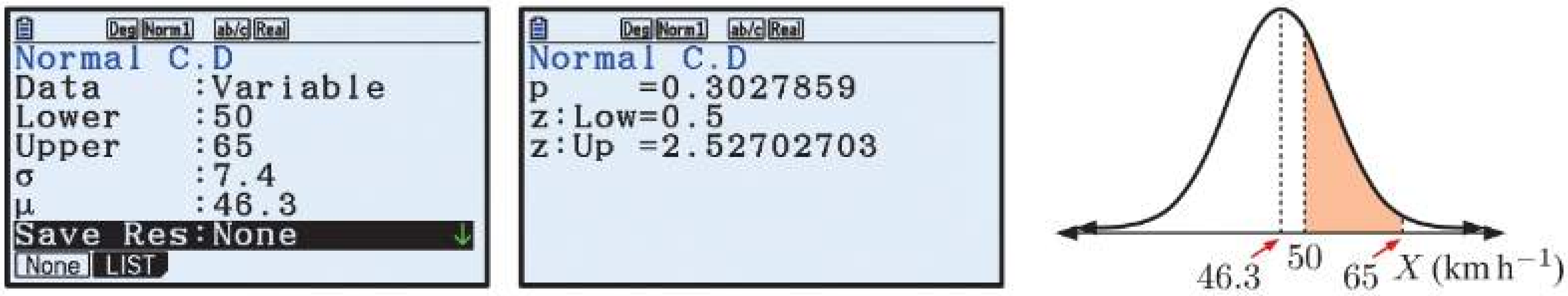


$P(19.7 < X < 20) \approx 0.378$

4 Let X km h^{-1} be the speed of a randomly selected car.

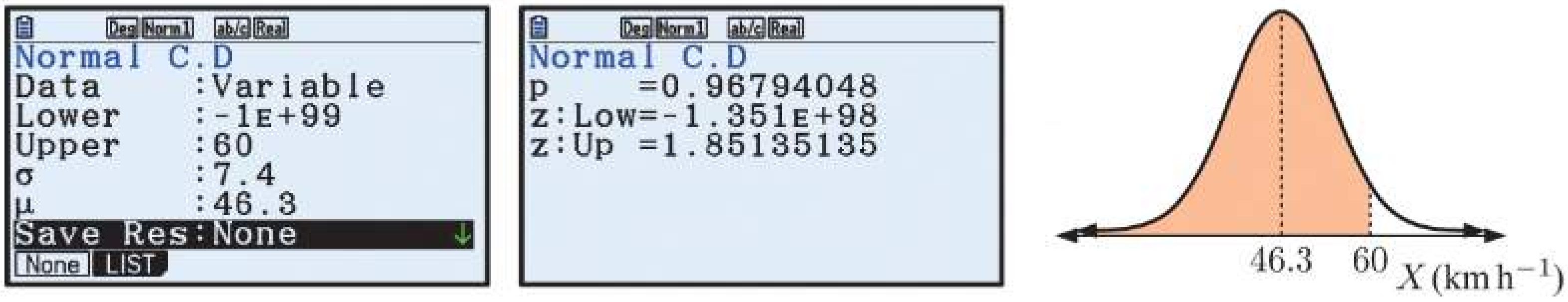
$\therefore \quad X \sim \text{N}(46.3,\ 7.4^2)$

a



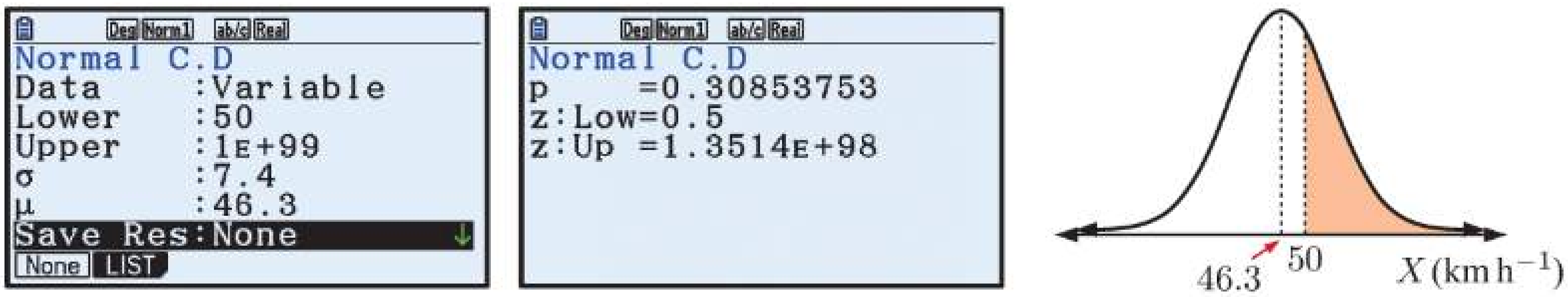
$\text{P}(50 < X < 65) \approx 0.303$

b



$\text{P}(X < 60) \approx 0.968$

c

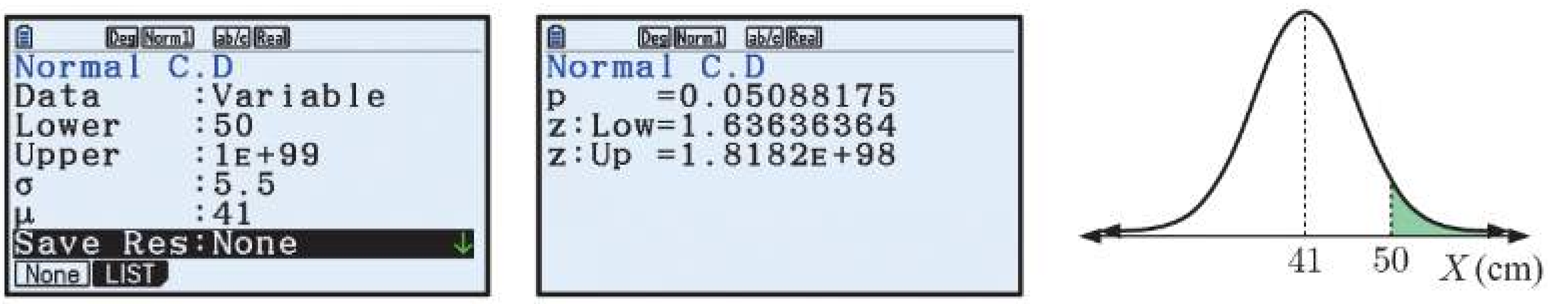


$\text{P}(X > 50) \approx 0.309$

5 Let X cm be the length of a randomly selected eel.

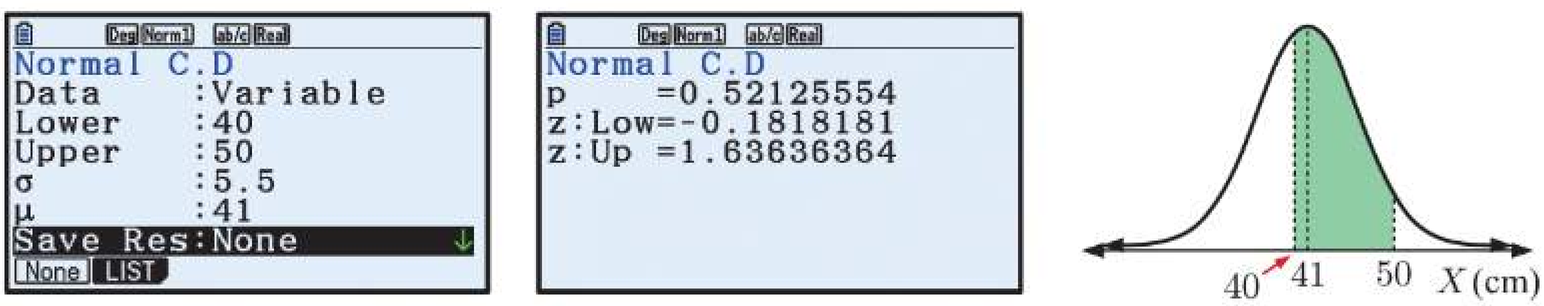
$\therefore \quad X \sim \text{N}(41,\ 5.5^2)$

a



$\text{P}(X \geqslant 50) \approx 0.0509$

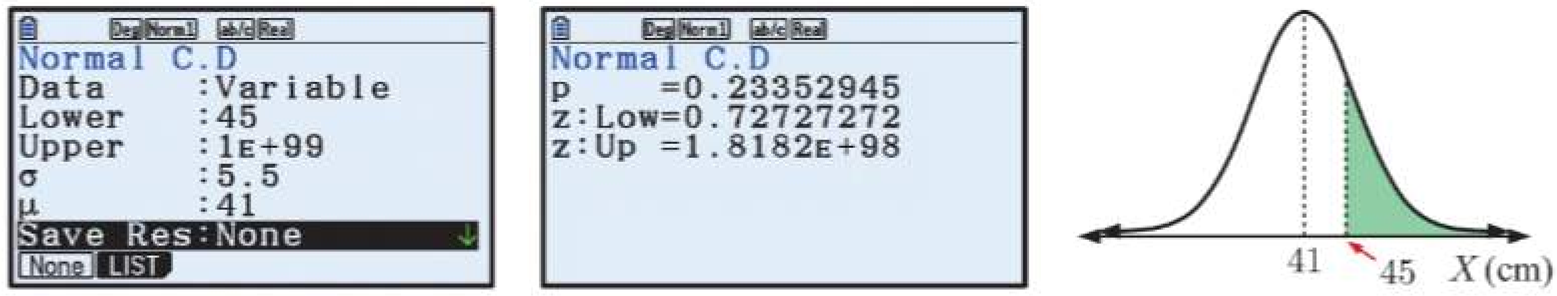
b



$\text{P}(40 < X < 50) \approx 0.521$

$\therefore$ about 52.1% of eels measure between 40 cm and 50 cm long.

c



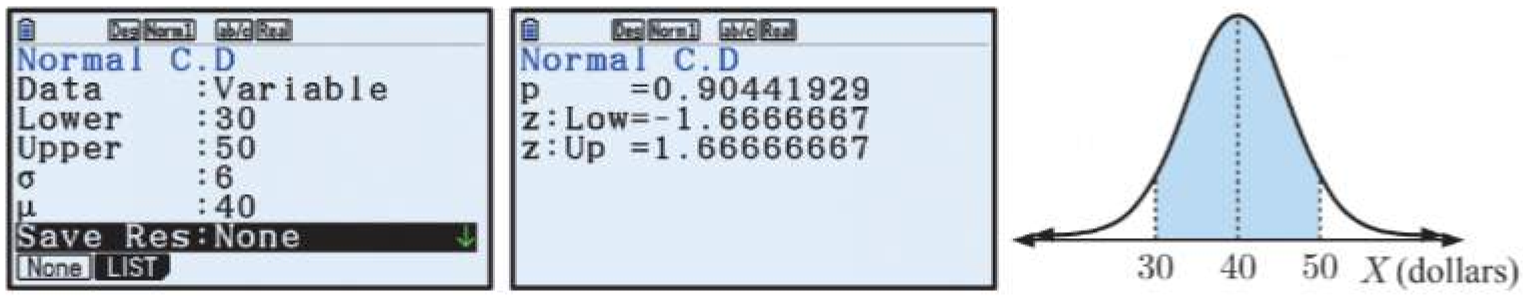
$P(X \geqslant 45) \approx 0.234$

$\therefore$ we would expect about $0.234 \times 200 \approx 47$ eels to measure at least 45 cm in length.

6 Let $\$X$ be the amount collected by Max in a randomly selected week.

$\therefore \quad X \sim N(40, 6^2)$

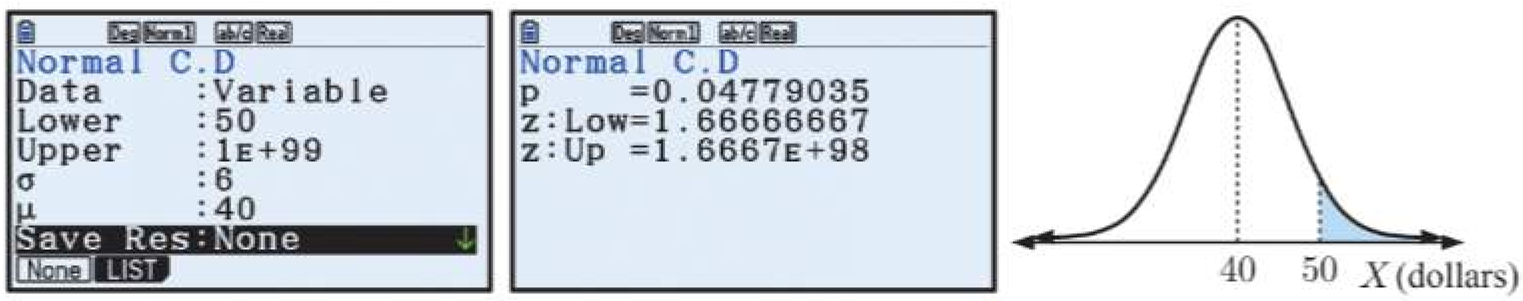
a **i**



$P(30 < X < 50) \approx 0.904$

$\therefore$ Max would expect to collect between \$30 and \$50 in about 90.4% of weeks.

ii



$P(X \geqslant 50) \approx 0.0478$

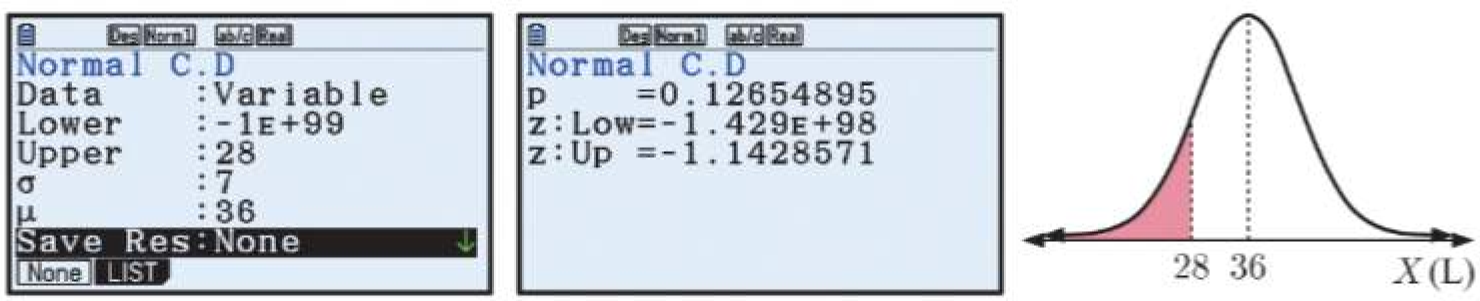
$\therefore$ Max would expect to collect at least \$50 in about 4.78% of weeks.

b There are about 52 weeks in a year, and the average weekly collection is \$40, so in 2 years we would expect Max to collect about $2 \times 52 \times \$40 = \$4160$.

7 Let X L be the amount of petrol bought by a randomly selected customer.

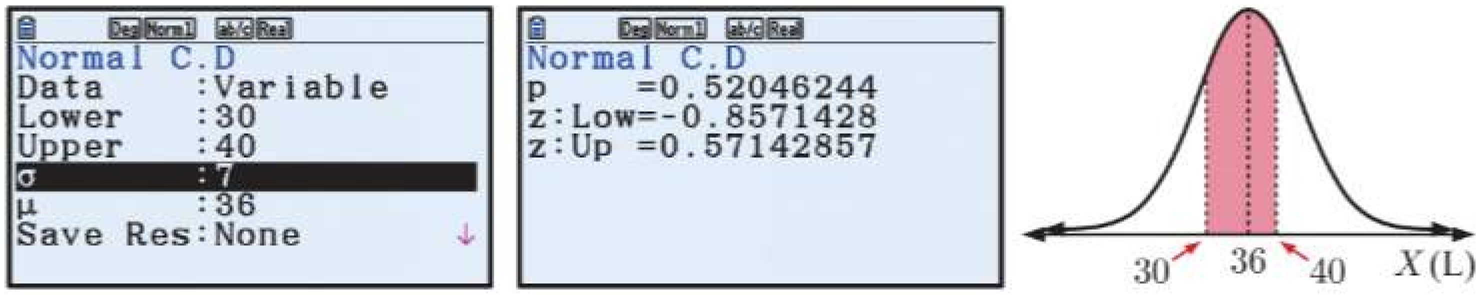
$\therefore \quad X \sim N(36, 7^2)$

a **i**



$P(X < 28) \approx 0.127$

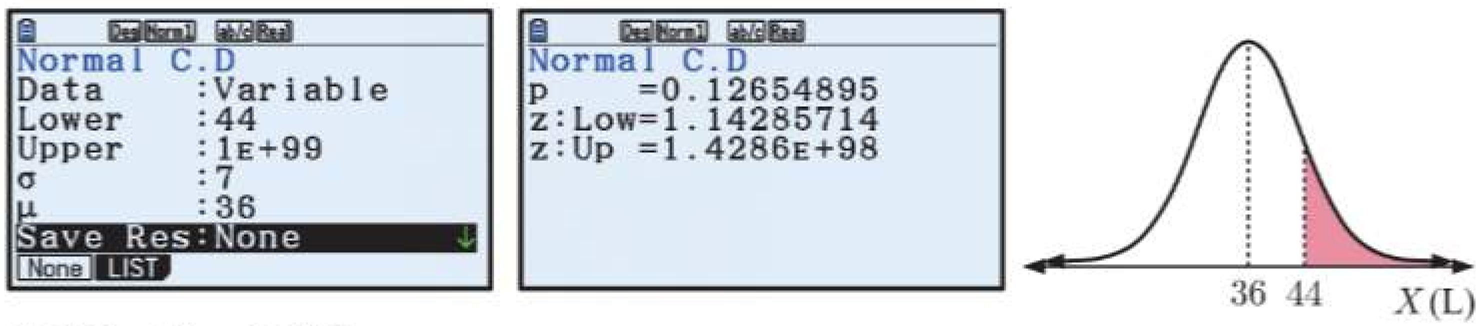
$\therefore$ about 12.7% of customers buy less than 28 L of petrol.

ii 

$P(30 < X < 40) \approx 0.520$

$\therefore$ about 52.0% of customers buy between 30 L and 40 L of petrol.

b **i** We would expect the petrol station to sell about $36 \text{ L} \times 600 = 21.6 \text{ kL}$ of petrol.

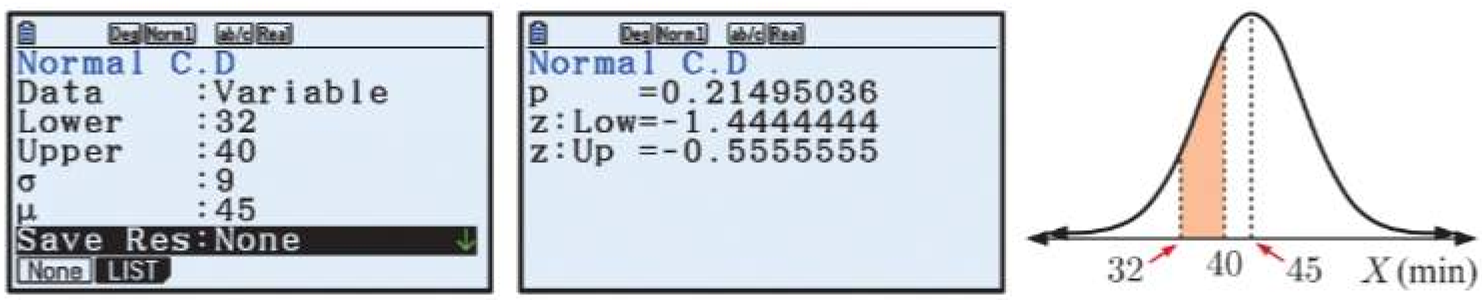
ii 

$P(X \geqslant 44) \approx 0.127$

$\therefore$ we would expect about $0.127 \times 600 \approx 76$ customers to buy at least 44 L of petrol.

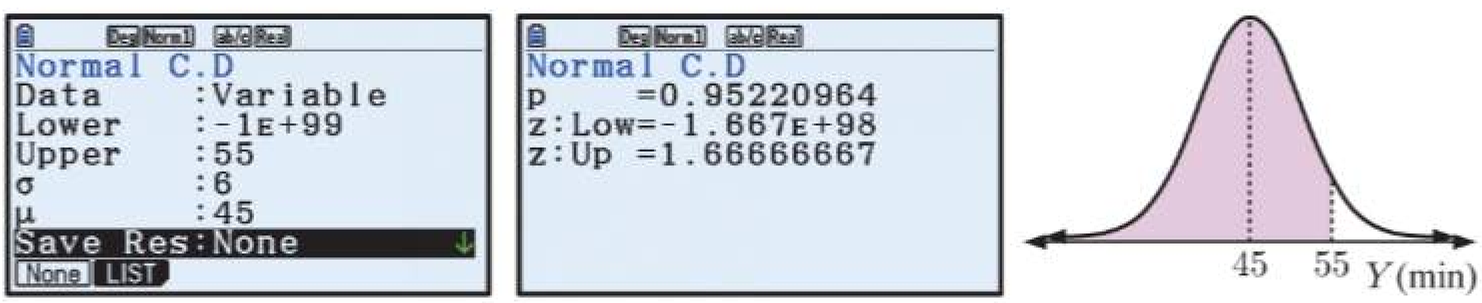
8 Let X minutes be the amount of time Enrique spends at the gym in a day, and Y minutes be the amount of time Damien spends at the gym in a day.

$\therefore \quad X \sim N(45, 9^2), \quad Y \sim N(45, 6^2)$

a **i** 

$P(32 < X < 40) \approx 0.215$

$\therefore$ Enrique will spend between 32 and 40 minutes at the gym on about 21.5% of days.

ii 

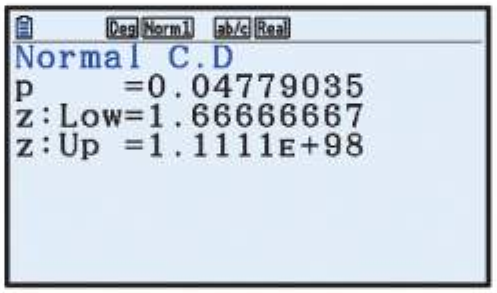
$P(Y < 55) \approx 0.952$

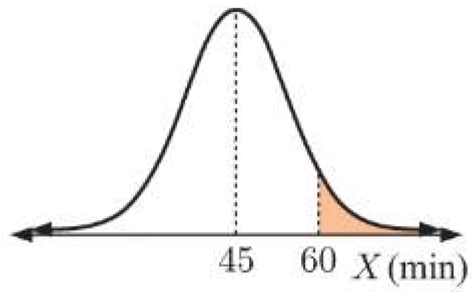
$\therefore$ Damien will spend less than 55 minutes at the gym on about 95.2% of days.

b **i** Enrique is more likely to spend at least 1 hour at the gym. The mean of both of their times is 45 minutes, but Enrique has a greater standard deviation, and so is more likely to exceed 1 hour.

ii Damien is more likely to spend between 40 and 50 minutes at the gym. Damien has the smaller standard deviation and is more likely to stay between 40 and 50 minutes, which is close to the mean of 45 minutes.

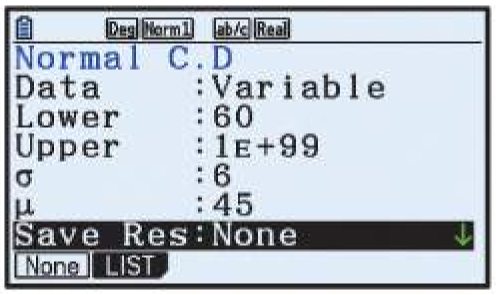
c **i** Enrique:

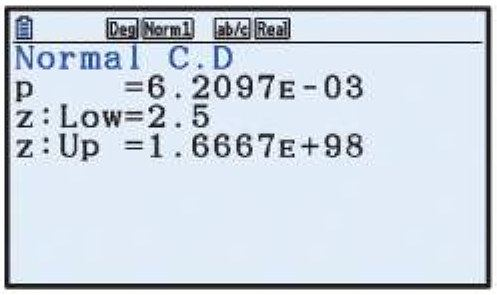


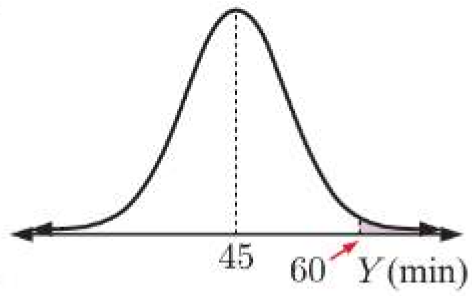


$P(X > 60) \approx 0.0478$

Damien:



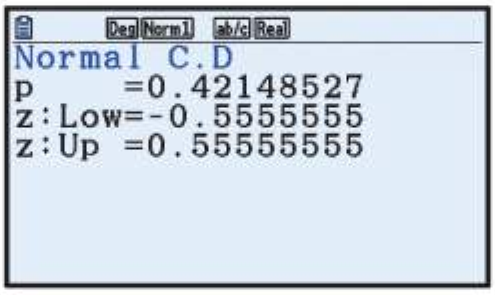


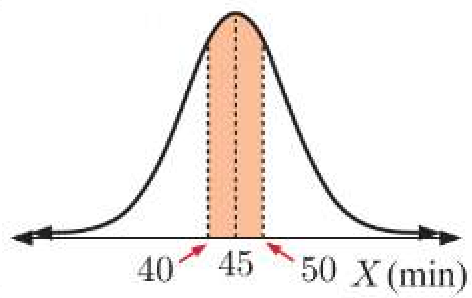


$P(Y > 60) \approx 0.006\,21$

$\therefore$ Enrique is more likely to spend at least 1 hour at the gym.

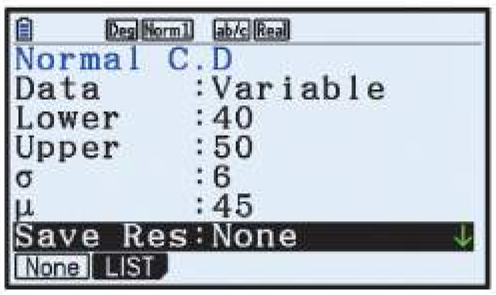
ii Enrique:

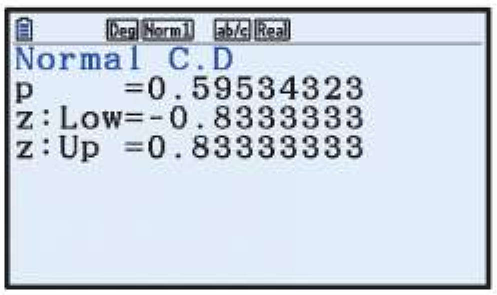


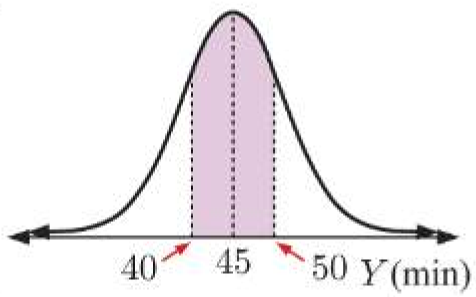


$P(40 < X < 50) \approx 0.421$

Damien:







$P(40 < Y < 50) \approx 0.595$

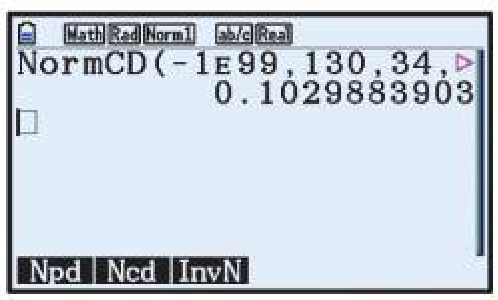
$\therefore$ Damien is more likely to spend between 40 and 50 minutes at the gym.

9 **a** Let X grams be the weight of a randomly selected apple.

$X \sim N(173, 34^2)$

$\therefore$ $P(X < 130) \approx 0.102\,99 \approx 0.103$

$\therefore$ about 10.3% of apples from the crop were too small to sell.

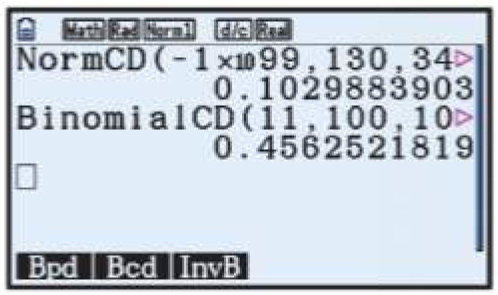


b Let Y be the number of apples which were too small to sell.

$Y \sim \text{B}(100, 0.102\,99)$

$\therefore\ \text{P}(Y > 10) = \text{P}(Y \geqslant 11)$

≈ 0.456



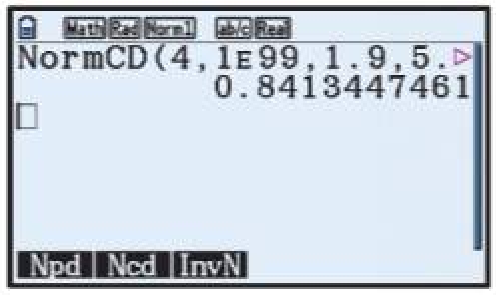
10 **a** Let X units be the drop in blood pressure of a randomly selected patient.

$X \sim \text{N}(5.9, 1.9^2)$

$\therefore\ \text{P}(X > 4) \approx 0.841\,34$

≈ 0.841

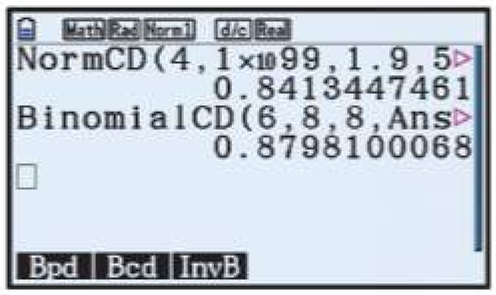
$\therefore$ about 84.1% of patients show a drop of more than 4 units.



b Let Y be the number of patients with a drop of more than 4 units.

$Y \sim \text{B}(8, 0.841\,34)$

$\therefore\ \text{P}(Y \geqslant 6) \approx 0.880$

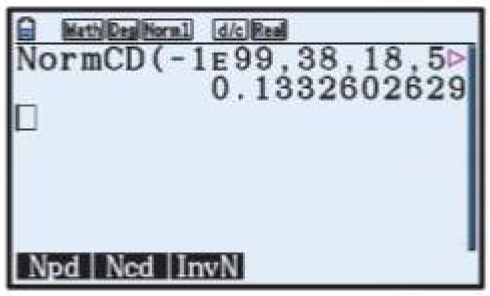


11 **a** Let X cm be the length of a randomly selected red snapper.

$X \sim \text{N}(58, 18^2)$

$\therefore\ \text{P}(X < 38) \approx 0.133\,26$

≈ 0.133

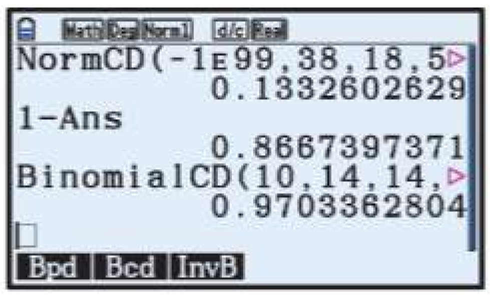


b Let Y be the number of red snapper that are long enough to keep.

From **a**, the probability that a single snapper can be *kept* is about $1 - 0.133\,26 \approx 0.866\,74$.

$Y \sim \text{B}(14, 0.866\,74)$

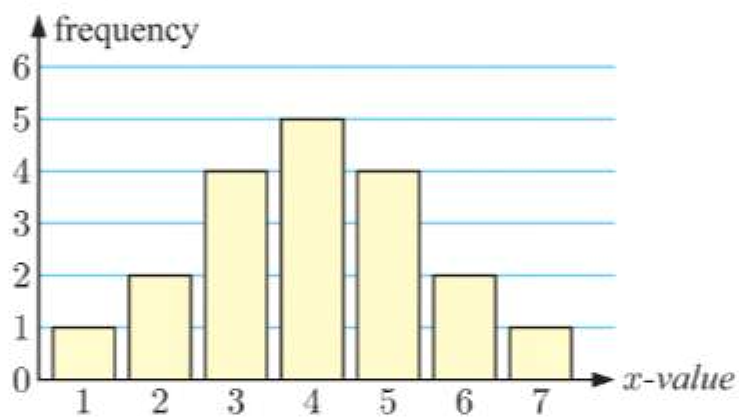
$\therefore\ \text{P}(Y \geqslant 10) \approx 0.970$



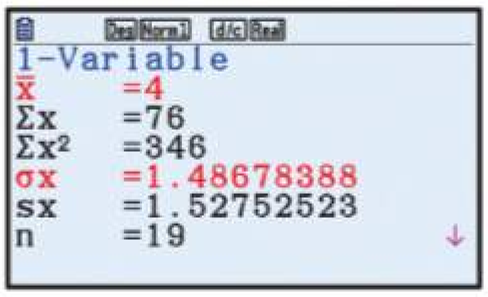
INVESTIGATION 3 z-SCORES

1 **a**

x-value	*Frequency*
1	1
2	2
3	4
4	5
5	4
6	2
7	1



b



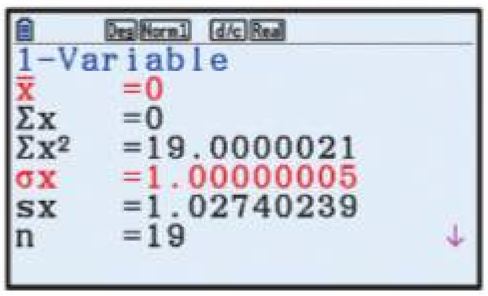
So, $\mu = 4$, $\sigma \approx 1.49$

c We calculate $z = \dfrac{x - \mu}{\sigma}$ to 3 significant figures.

x-value	1	2	3	4	5	6	7
z-score	-2.02	-1.35	-0.673	0.00	0.673	1.35	2.02

d

z-score	Frequency
-2.02	1
-1.35	2
-0.673	4
0.00	5
0.673	4
1.35	2
2.02	1



So, the z-scores have mean $\mu = 0$, and standard deviation $\sigma \approx 1$.

2 **c** Both histograms are approximately normally distributed. The histogram of the z-scores appears to be normally distributed with mean 0 and standard deviation 1 for any sample.

d If the original data is randomly generated from a normal distribution, the z-scores are also normally distributed with mean 0 and standard deviation 1.

If $X \sim N(\mu, \sigma^2)$ and $Z = \dfrac{X - \mu}{\sigma}$ then $Z \sim N(0, 1^2)$.

EXERCISE 28C.1

1 **a** For English, $z\text{-score} = \dfrac{48-40}{4.4} \approx 1.82$

For Mandarin, $z\text{-score} = \dfrac{81-60}{9} \approx 2.33$

For Geography, $z\text{-score} = \dfrac{84-55}{18} \approx 1.61$

For Biology, $z\text{-score} = \dfrac{68-50}{20} = 0.9$

For Mathematics, $z\text{-score} = \dfrac{84-50}{15} \approx 2.27$

Subject	*Emma's score*	μ	σ
English	48	40	4.4
Mandarin	81	60	9
Geography	84	55	18
Biology	68	50	20
Mathematics	84	50	15

b In order from best to worst: Mandarin, Mathematics, English, Geography, Biology.

c It is reasonable to compare Emma's performances using z-scores as the scores in each of Emma's classes are normally distributed.

2 **a**

Subject	*Sergio's score*	μ	σ
Physics	73%	78%	10.8%
Chemistry	77%	72%	11.6%
Mathematics	76%	74%	10.1%
German	91%	86%	9.6%
Biology	58%	62%	5.2%

For Physics, $z\text{-score} = \dfrac{73-78}{10.8} \approx -0.463$

For Chemistry, $z\text{-score} = \dfrac{77-72}{11.6} \approx 0.431$

For Mathematics, $z\text{-score} = \dfrac{76-74}{10.1} \approx 0.198$

For German, $z\text{-score} = \dfrac{91-86}{9.6} \approx 0.521$

For Biology, $z\text{-score} = \dfrac{58-62}{5.2} \approx -0.769$

b In order from best to worst: German, Chemistry, Mathematics, Physics, Biology.

3

Event	*Time* (seconds)	μ (seconds)	σ (seconds)
50 m freestyle	32.1	27.8	2.2
100 m backstroke	53.5	58.1	4.3
200 m breaststroke	140.0	143.7	6.4
100 m butterfly	59.6	57.7	5.5

a For 50 m freestyle, $z\text{-score} = \dfrac{32.1 - 27.8}{2.2} \approx 1.95$

For 100 m backstroke, $z\text{-score} = \dfrac{53.5 - 58.1}{4.3} \approx -1.07$

For 200 m breaststroke, $z\text{-score} = \dfrac{140.0 - 143.7}{6.4} \approx -0.578$

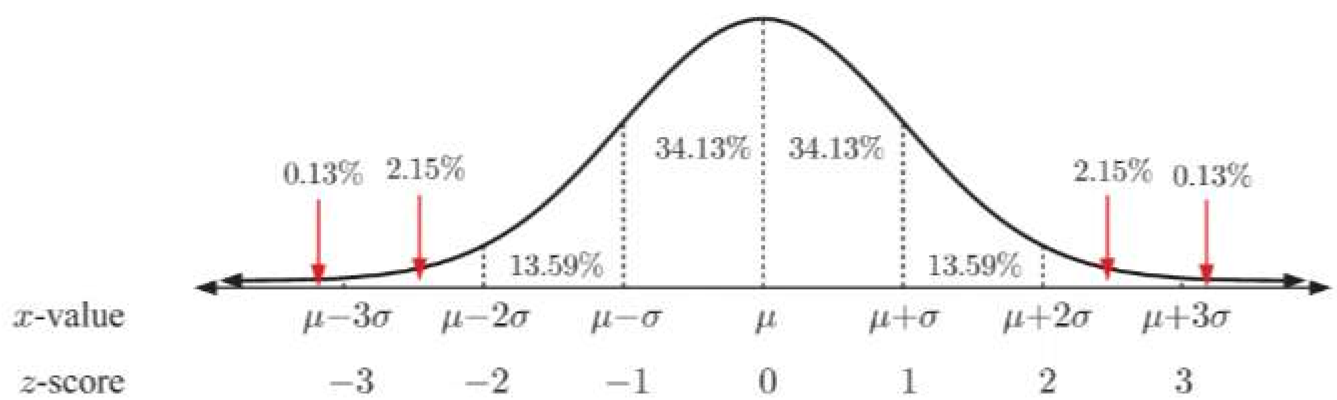
For 100 m butterfly, $z\text{-score} = \dfrac{59.6 - 57.7}{5.5} \approx 0.345$

b A lower z-score is better as it indicates that the time is lower, and hence that Frederick swam faster.

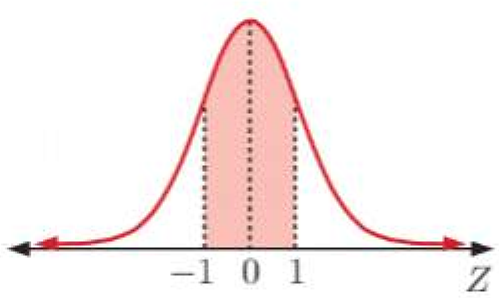
c In order from best to worst:
100 m backstroke, 200 m breaststroke, 100 m butterfly, 50 m freestyle.

EXERCISE 28C.2

1

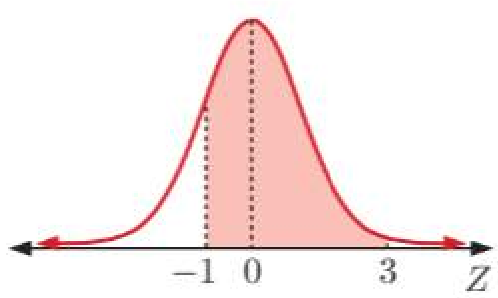


a



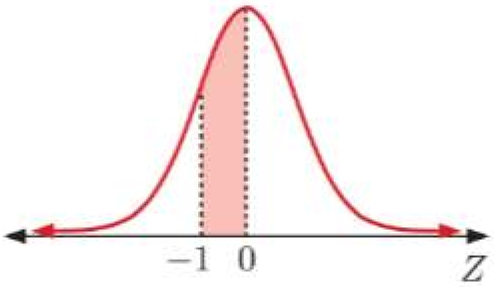
$$\begin{aligned} P(-1 < Z < 1) &\approx 34.13\% + 34.13\% \\ &\approx 68.26\% \\ &\approx 0.683 \end{aligned}$$

b



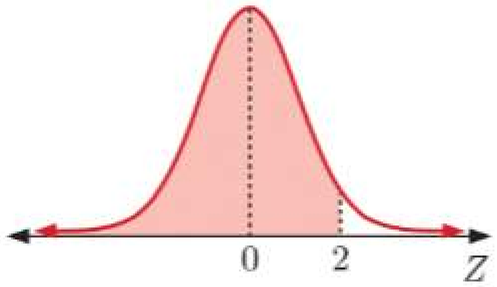
$P(-1 \leqslant Z \leqslant 3)$
$\approx 34.13\% + 34.13\% + 13.59\% + 2.15\%$
$\approx 84.00\%$
≈ 0.840

c



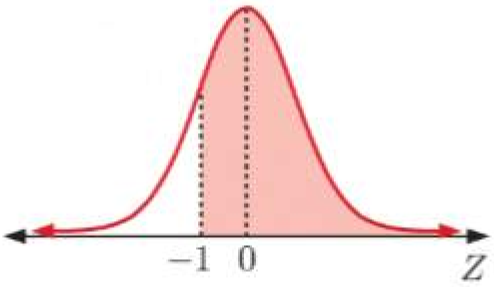
$$\mathrm{P}(-1 < Z < 0) \approx 34.13\%$$
$$\approx 0.341$$

d



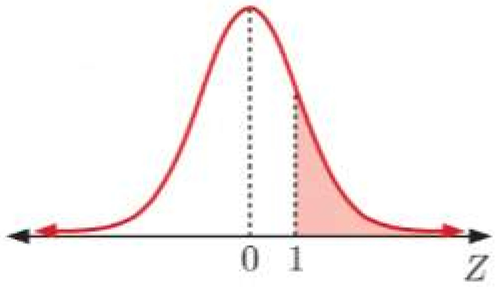
$$\mathrm{P}(Z < 2) \approx 50\% + 34.13\% + 13.59\%$$
$$\approx 97.72\%$$
$$\approx 0.977$$

e



$$\mathrm{P}(-1 < Z) = \mathrm{P}(Z > -1)$$
$$\approx 34.13\% + 50\%$$
$$\approx 84.13\%$$
$$\approx 0.841$$

f



$$\mathrm{P}(Z \geqslant 1) \approx 13.59\% + 2.15\% + 0.13\%$$
$$\approx 15.87\%$$
$$\approx 0.159$$

2 **a** If $\mathrm{P}(\mu - \sigma < X < \mu + 2\sigma) = \mathrm{P}(a < Z < b)$

then $a = \dfrac{(\mu - \sigma) - \mu}{\sigma}$ and $b = \dfrac{(\mu + 2\sigma) - \mu}{\sigma}$

$\therefore\ a = \dfrac{-\sigma}{\sigma} = -1$ $\qquad \therefore\ b = \dfrac{2\sigma}{\sigma} = 2$

$\therefore\ a = -1,\ b = 2$

b If $\mathrm{P}(\mu - 0.5\sigma < X < \mu) = \mathrm{P}(a < Z < b)$

then $a = \dfrac{(\mu - 0.5\sigma) - \mu}{\sigma}$ and $b = \dfrac{\mu - \mu}{\sigma}$

$\therefore\ a = \dfrac{-0.5\sigma}{\sigma} = -0.5$ $\qquad \therefore\ b = 0$

$\therefore\ a = -0.5,\ b = 0$

c If $\mathrm{P}(0 \leqslant Z \leqslant 3) = \mathrm{P}(\mu - a\sigma \leqslant X \leqslant \mu + b\sigma)$

then $\dfrac{(\mu - a\sigma) - \mu}{\sigma} = 0$ and $\dfrac{(\mu + b\sigma) - \mu}{\sigma} = 3$

$\therefore\ \mu - a\sigma - \mu = 0$ $\qquad \therefore\ \mu + b\sigma - \mu = 3\sigma$

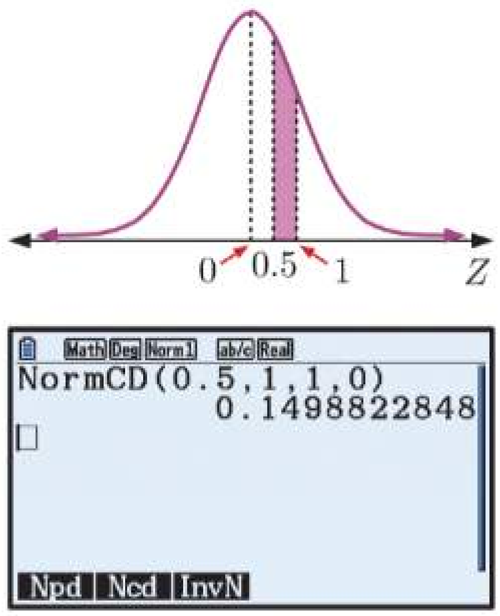
$\therefore\ -a\sigma = 0$ $\qquad \therefore\ b\sigma = 3\sigma$

$\therefore\ a = 0$ $\qquad \therefore\ b = 3$

$\therefore\ a = 0,\ b = 3$

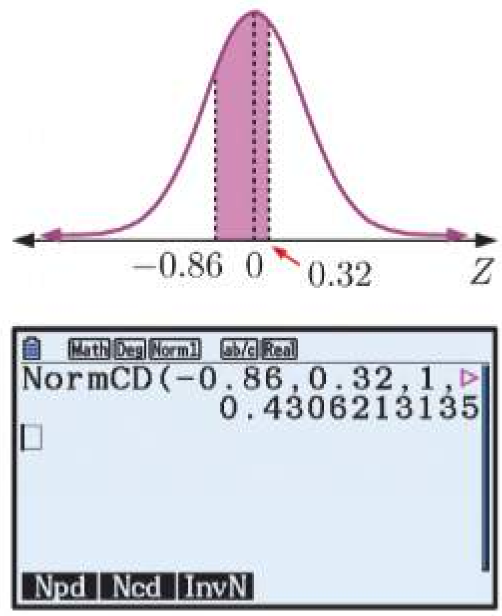
3 $Z \sim \text{N}(0, 1^2)$

a



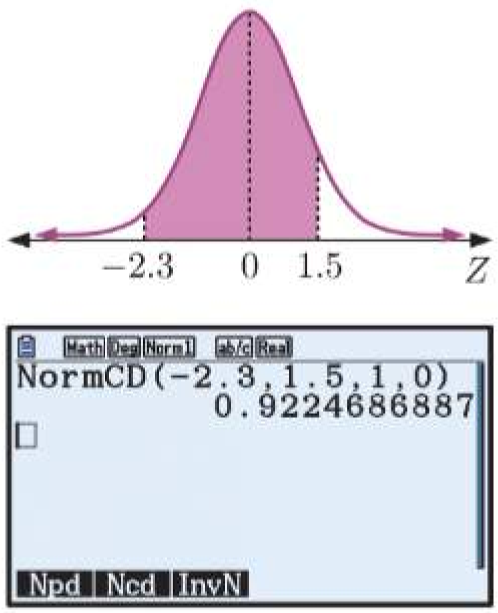
$\text{P}(0.5 \leqslant Z \leqslant 1) \approx 0.150$

b



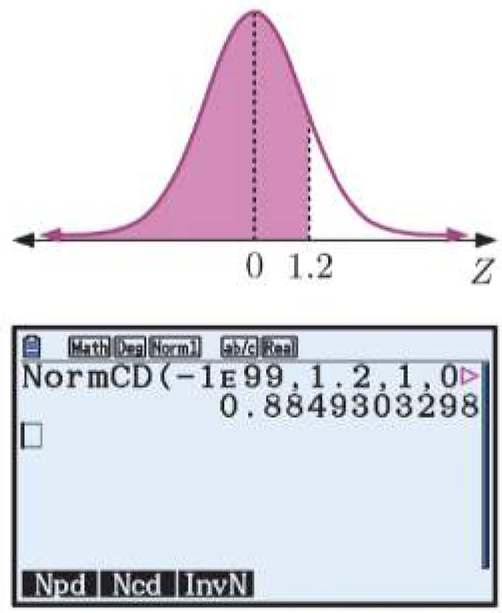
$\text{P}(-0.86 \leqslant Z \leqslant 0.32) \approx 0.431$

c



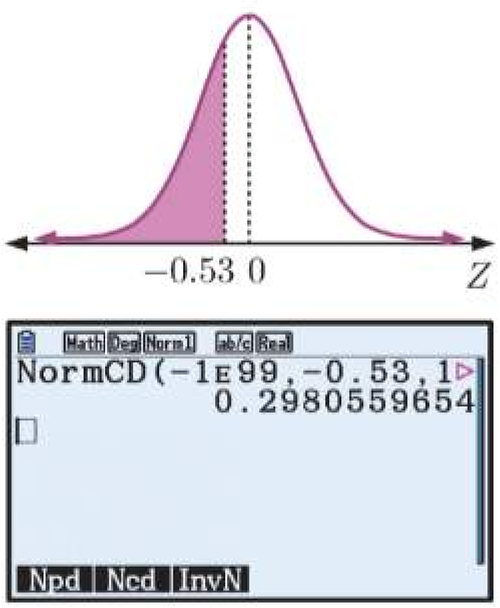
$\text{P}(-2.3 \leqslant Z \leqslant 1.5) \approx 0.922$

d



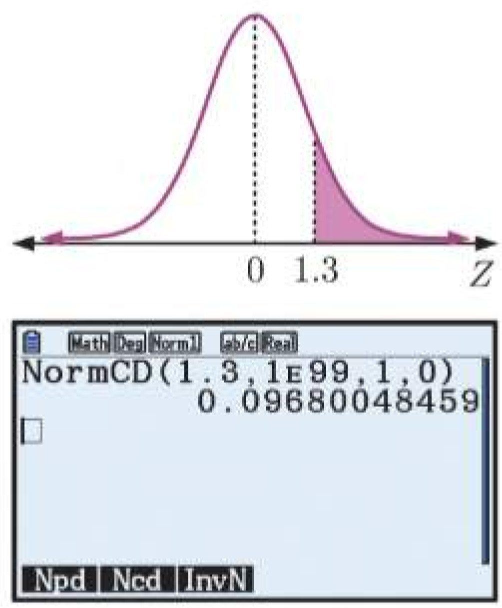
$\text{P}(Z \leqslant 1.2) \approx 0.885$

e



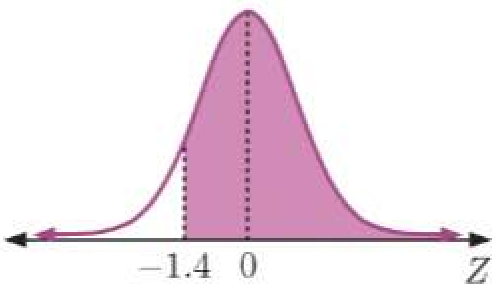
$\text{P}(Z \leqslant -0.53) \approx 0.298$

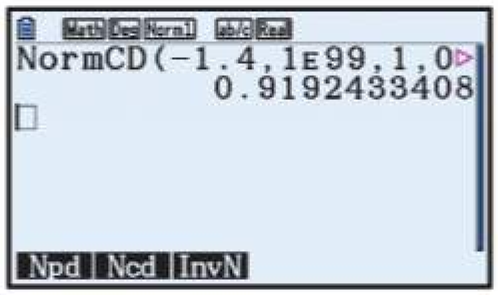
f



$\text{P}(Z \geqslant 1.3) \approx 0.0968$

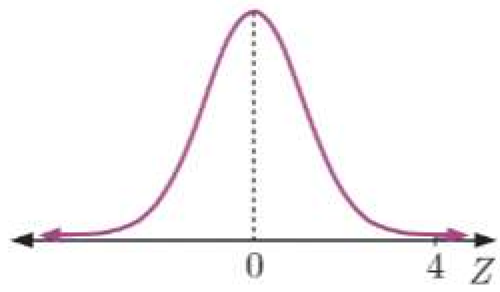
g

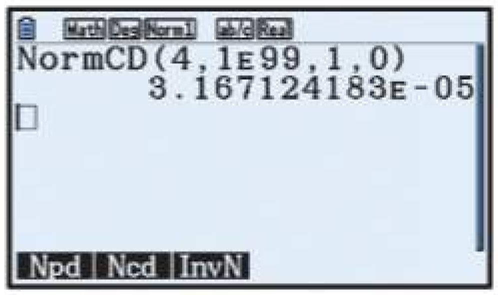




$P(Z \geqslant -1.4) \approx 0.919$

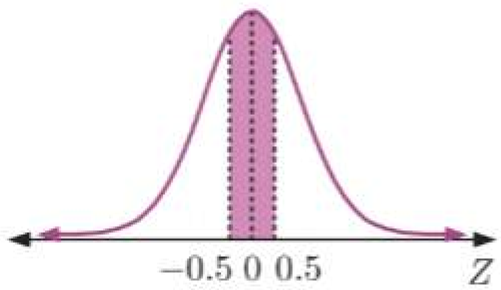
h

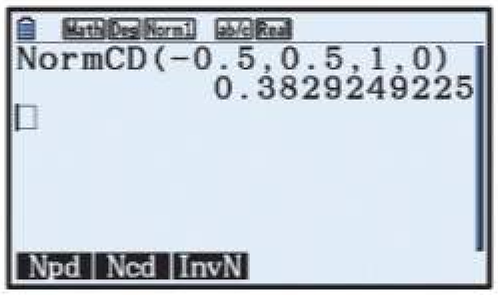




$P(Z > 4) \approx 0.000\,031\,7 \quad (3.17 \times 10^{-5})$

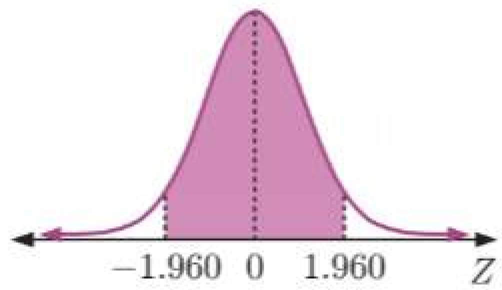
i

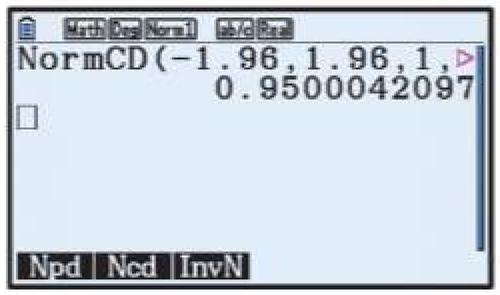




$P(-0.5 < Z < 0.5) \approx 0.383$

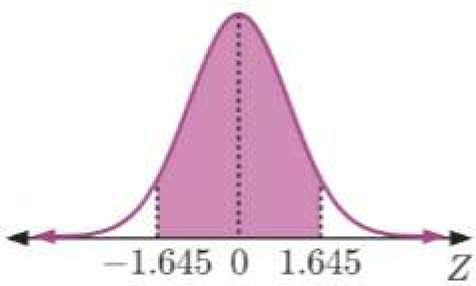
j

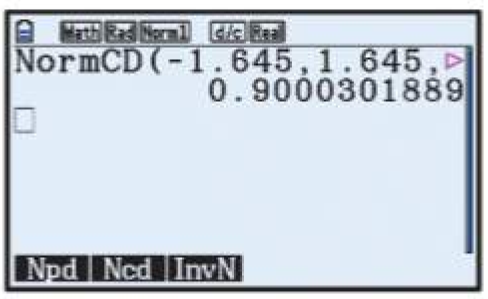




$P(-1.960 \leqslant Z \leqslant 1.960) \approx 0.950$

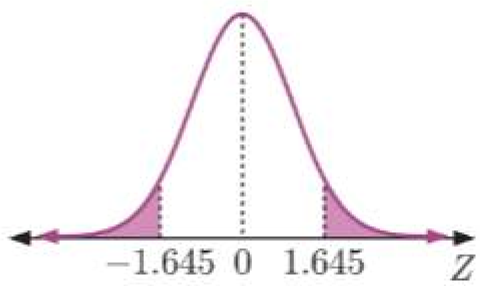
k

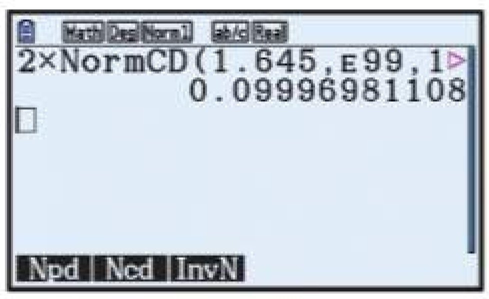




$P(-1.645 \leqslant Z \leqslant 1.645) \approx 0.900$

l



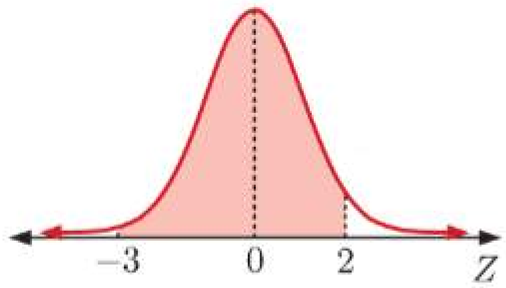


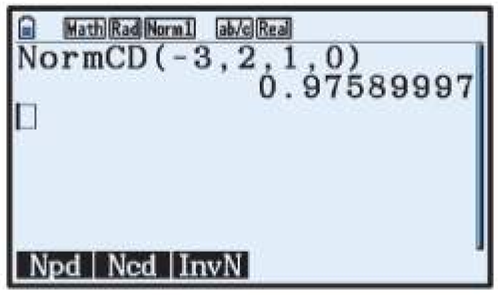
$$\begin{aligned} &P(|Z| > 1.645) \\ &= P(Z < -1.645) + P(Z > 1.645) \\ &= 2 \times P(Z > 1.645) \\ &\approx 0.100 \end{aligned}$$

4 **a** **i** Since $X \sim \mathrm{N}(\mu, \sigma^2)$, the Z-transformation of X is $Z = \dfrac{X-\mu}{\sigma}$.

Now
$$\begin{aligned}\mathrm{P}(\mu - 3\sigma < X < \mu + 2\sigma) &= \mathrm{P}(-3\sigma < X - \mu < 2\sigma)\\ &= \mathrm{P}\left(-3 < \frac{X-\mu}{\sigma} < 2\right)\\ &= \mathrm{P}(-3 < Z < 2)\end{aligned}$$

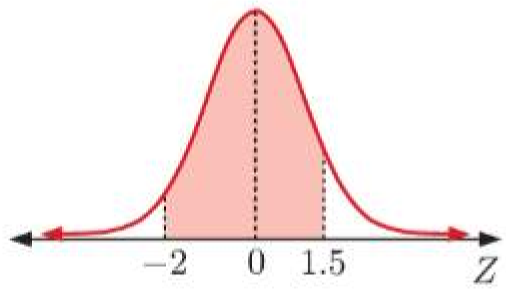
ii

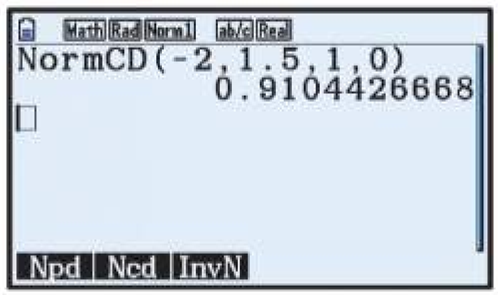




$$\begin{aligned}\mathrm{P}(\mu - 3\sigma < X < \mu + 2\sigma) &= \mathrm{P}(-3 < Z < 2)\\ &\approx 0.976\end{aligned}$$

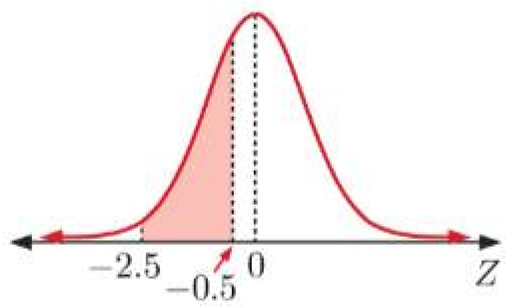
b **i**

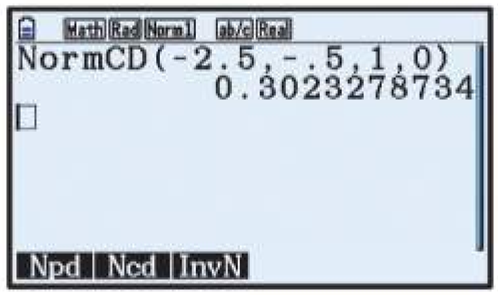




$$\begin{aligned}\mathrm{P}(\mu - 2\sigma < X < \mu + 1.5\sigma) &= \mathrm{P}(-2 < Z < 1.5)\\ &\approx 0.910\end{aligned}$$

ii





$$\begin{aligned}\mathrm{P}(\mu - 2.5\sigma < X < \mu - 0.5\sigma) &= \mathrm{P}(-2.5 < Z < -0.5)\\ &\approx 0.302\end{aligned}$$

5 $X \sim \mathrm{N}(58.3,\ 8.96^2)$

a **i** $z_1 = \dfrac{50.6 - 58.3}{8.96}$ $= -0.859\,375$ ≈ -0.859

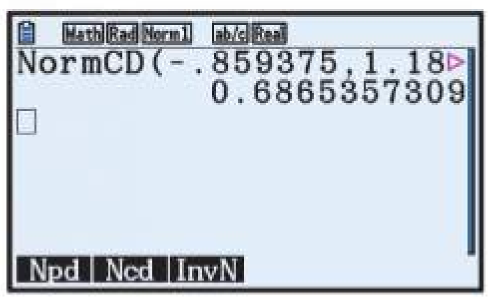
$z_2 = \dfrac{68.9 - 58.3}{8.96}$ $\approx 1.183\,036$ ≈ 1.18

ii $Z \sim \mathrm{N}(0,\ 1)$

$\quad \mathrm{P}(z_1 \leqslant Z \leqslant z_2)$

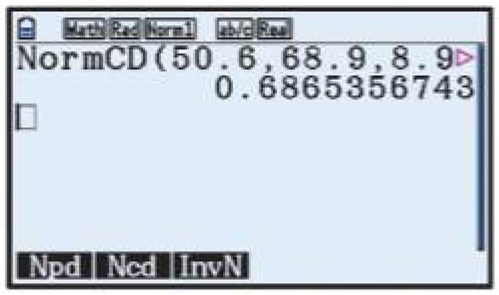
$= \mathrm{P}(-0.859\,375 \leqslant Z \leqslant 1.183\,036)$

≈ 0.687



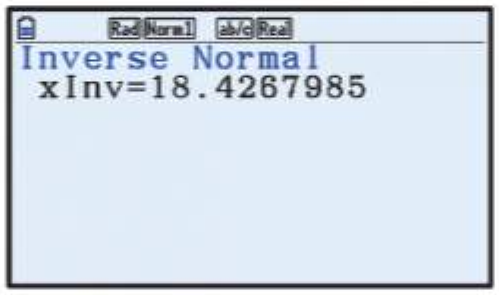
b Using technology,

$P(50.6 \leqslant X \leqslant 68.9) \approx 0.687$ ✓



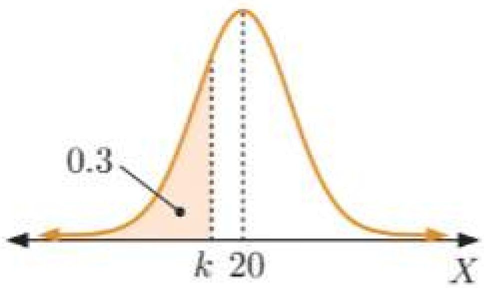
EXERCISE 28D

1 **a**

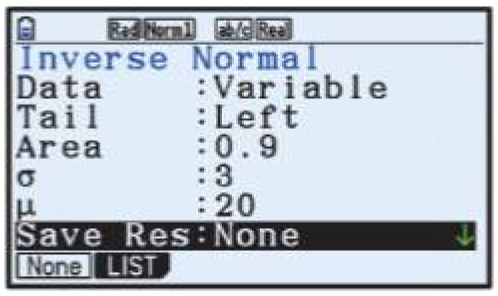


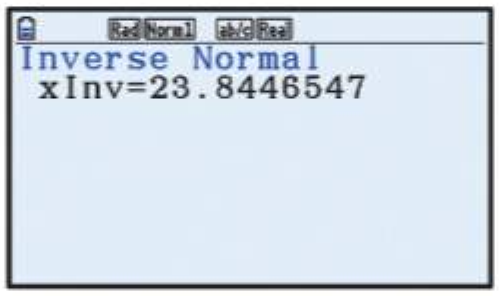
If $P(X \leqslant k) = 0.3$

then $k \approx 18.4$



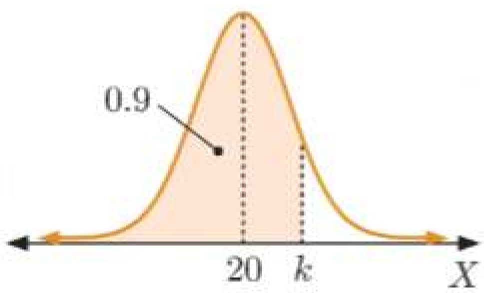
b



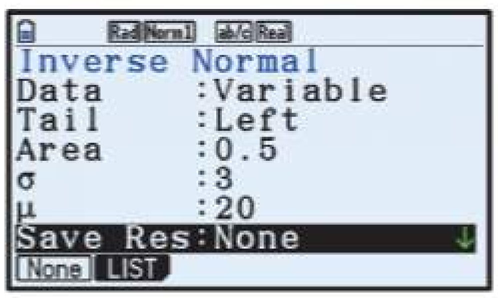


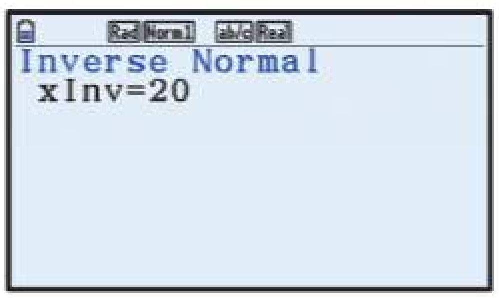
If $P(X \leqslant k) = 0.9$

then $k \approx 23.8$



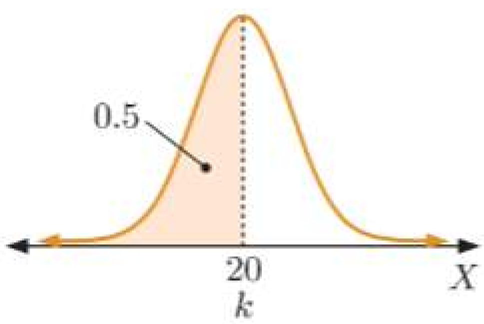
c



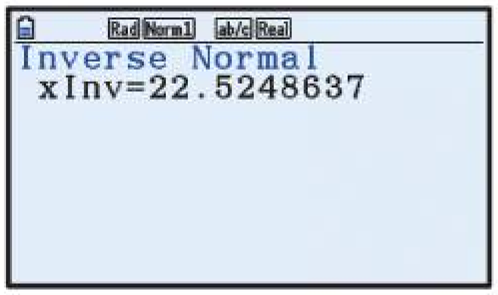


If $P(X \leqslant k) = 0.5$

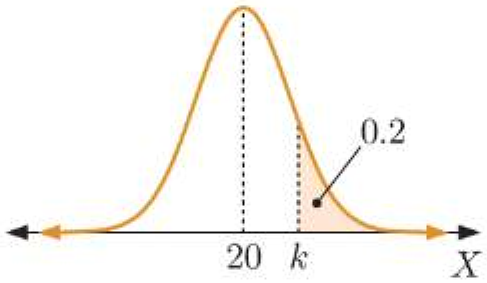
then $k = 20$



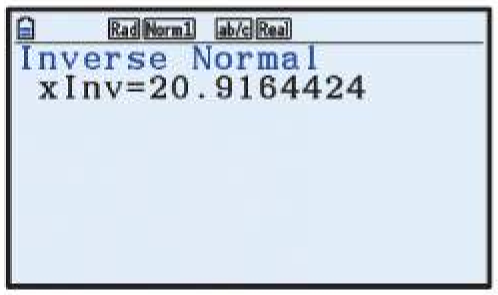
d



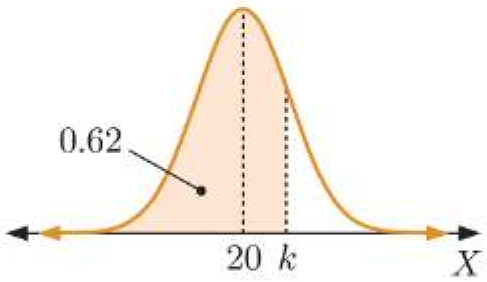
If $\mathrm{P}(X > k) = 0.2$
then $k \approx 22.5$



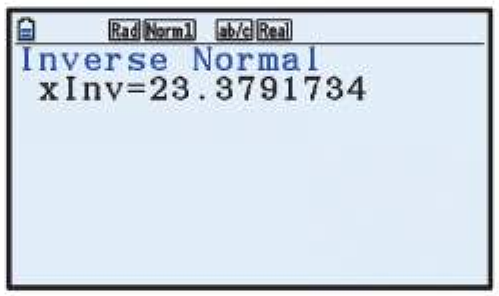
e



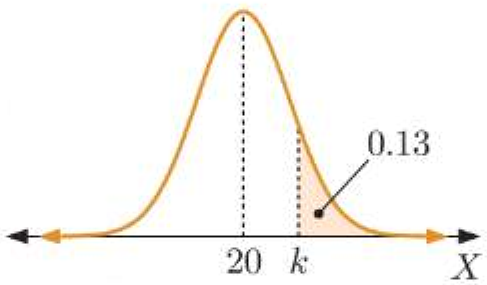
If $\mathrm{P}(X < k) = 0.62$
then $k \approx 20.9$



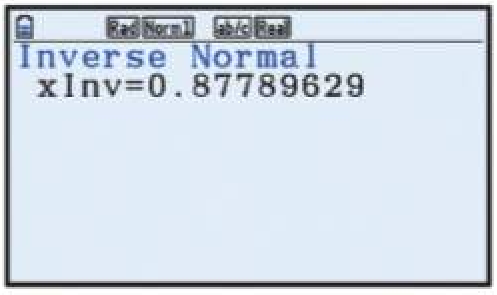
f



If $\mathrm{P}(X \geqslant k) = 0.13$
then $k \approx 23.4$

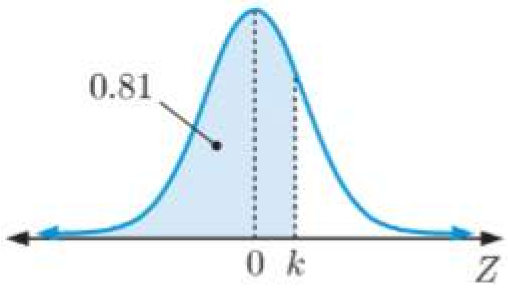


2 **a**

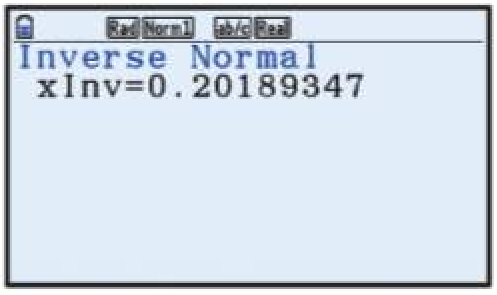


If $\text{P}(Z \leqslant k) = 0.81$

then $k \approx 0.878$

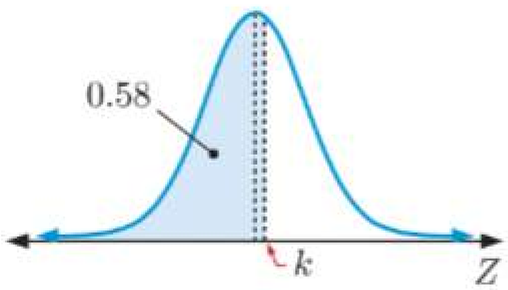


b

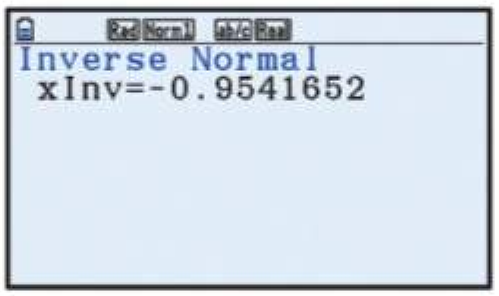


If $\text{P}(Z \leqslant k) = 0.58$

then $k \approx 0.202$

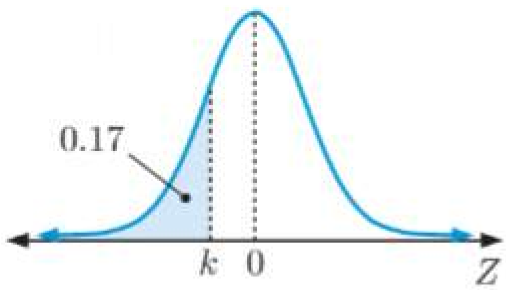


c

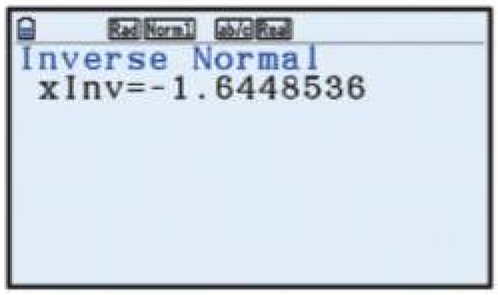


If $\text{P}(Z \leqslant k) = 0.17$

then $k \approx -0.954$

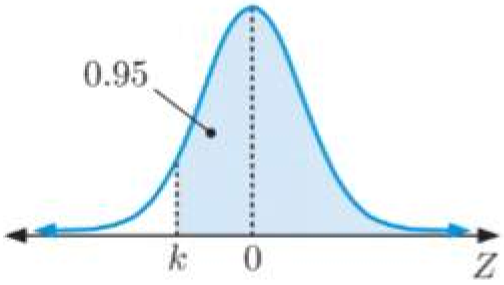


d

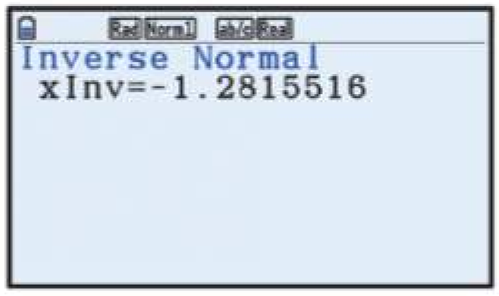


If $\mathrm{P}(Z \geqslant k) = 0.95$

$\therefore \quad k \approx -1.64$

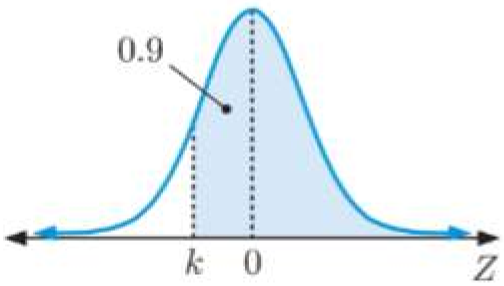


e

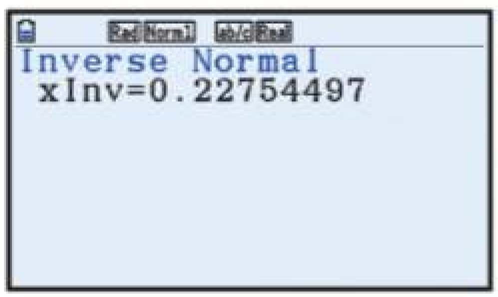


If $\mathrm{P}(Z \geqslant k) = 0.9$

$\therefore \quad k \approx -1.28$

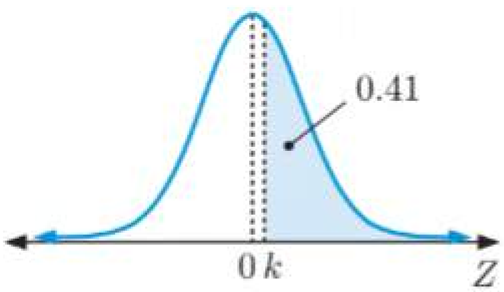


f

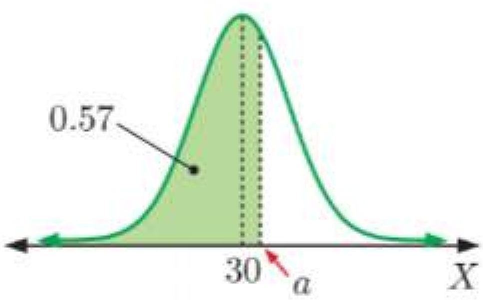


If $\mathrm{P}(Z \geqslant k) = 0.41$

$\therefore \quad k \approx 0.228$



3 **a** $X \sim \mathrm{N}(30, 5^2)$



$\therefore \quad a > 30$

b $P(X \leqslant a) = 0.57$

$\therefore \ a \approx 30.9$

c **i** $P(X \geqslant a) = 1 - P(X \leqslant a)$

$= 1 - 0.57$

$= 0.43$

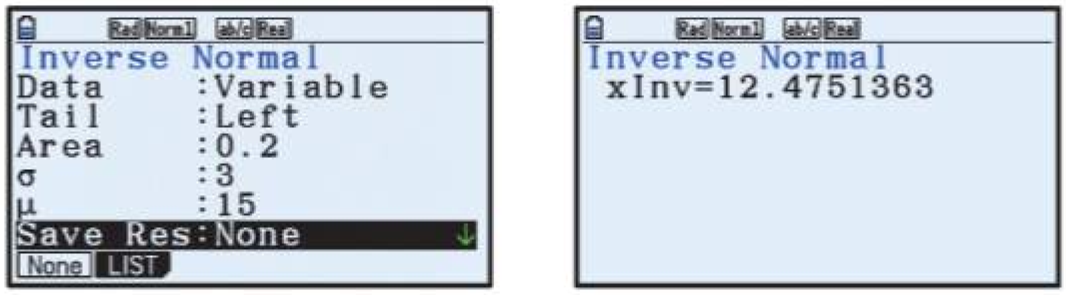
ii $P(30 \leqslant X \leqslant a) = P(X \leqslant a) - P(X \leqslant 30)$

$= 0.57 - 0.5$

$= 0.07$

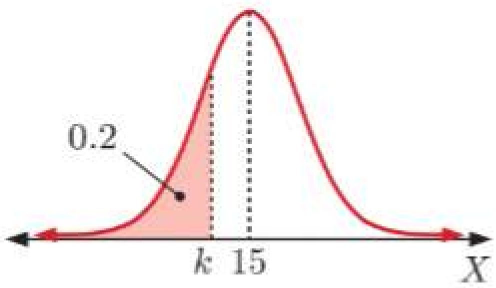
4 $X \sim N(15, 3^2)$

a

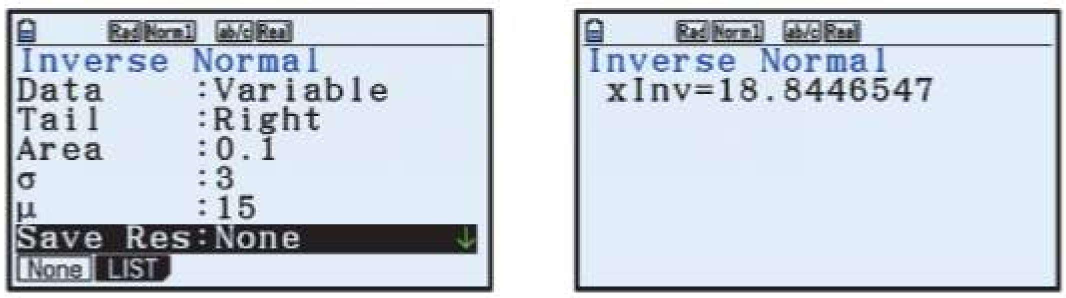


If $P(X < k) = 0.2$

then $k \approx 12.5$

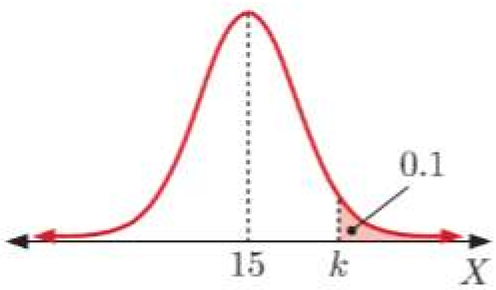


b

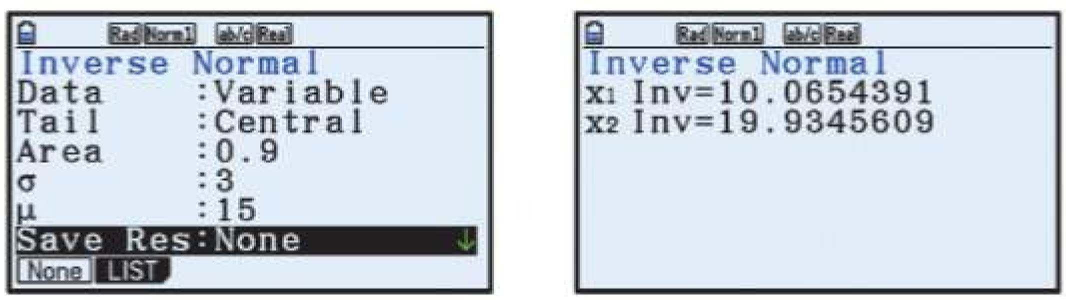


If $P(X > k) = 0.1$

then $k \approx 18.8$



c

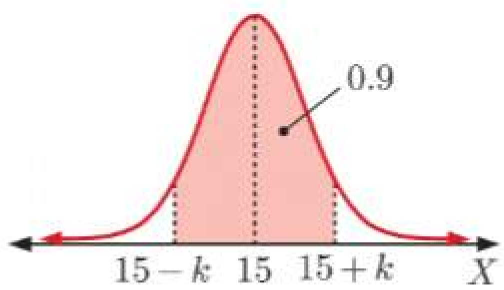


If $P(15 - k < X < 15 + k) = 0.9$

then $15 - k \approx 10.07$

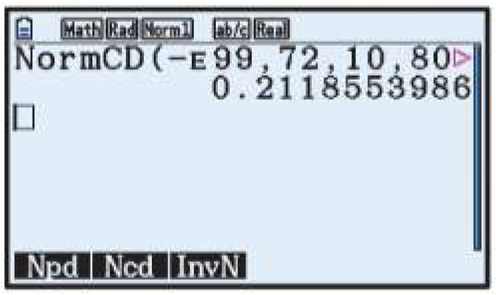
or $15 + k \approx 19.93$

$\therefore \ k \approx 4.93$



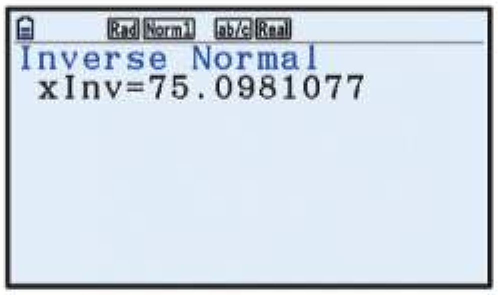
5 $X \sim \mathrm{N}(80, 10^2)$

a

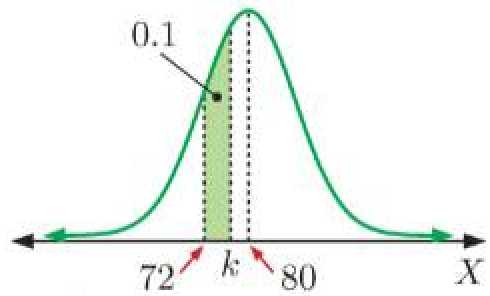


$\mathrm{P}(X \leqslant 72) \approx 0.212$

b

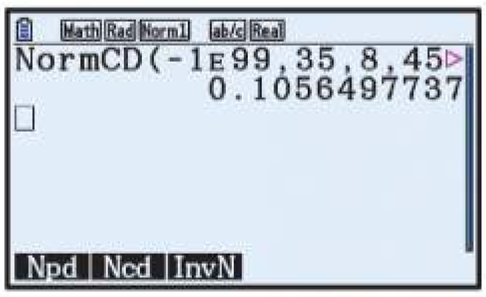


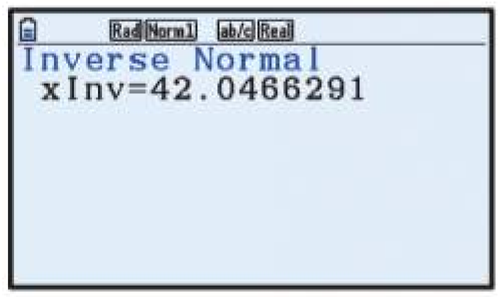
$$\mathrm{P}(72 \leqslant X \leqslant k) = 0.1$$
$$\therefore \mathrm{P}(X \leqslant k) - \mathrm{P}(X \leqslant 72) = 0.1$$
$$\therefore \mathrm{P}(X \leqslant k) - 0.212 \approx 0.1 \quad \{\text{using } \mathbf{a}\}$$
$$\therefore \mathrm{P}(X \leqslant k) \approx 0.312$$
$$\therefore k \approx 75.1$$



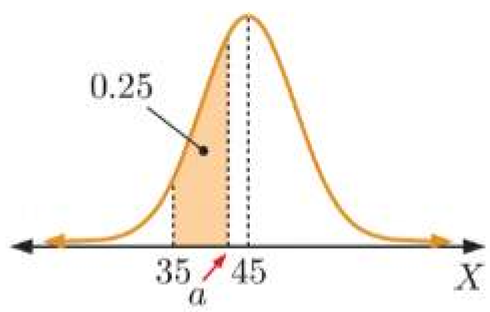
6 $X \sim \mathrm{N}(45, 8^2)$

a

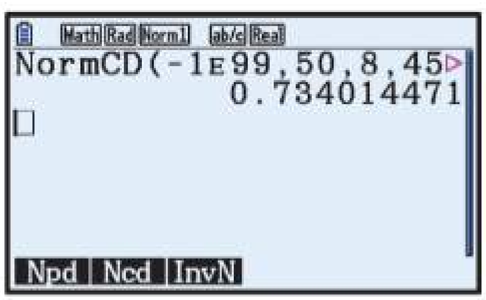


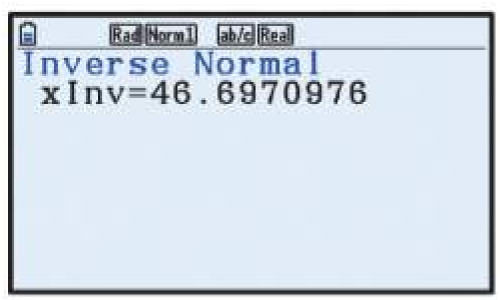


$$\mathrm{P}(35 \leqslant X \leqslant a) = 0.25$$
$$\therefore \mathrm{P}(X \leqslant a) - \mathrm{P}(X \leqslant 35) = 0.25$$
$$\therefore \mathrm{P}(X \leqslant a) - 0.106 \approx 0.25$$
$$\therefore \mathrm{P}(X \leqslant a) \approx 0.356$$
$$\therefore a \approx 42.0$$

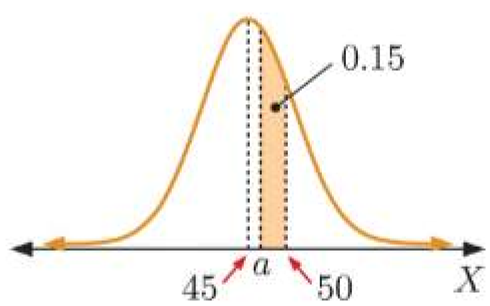


b



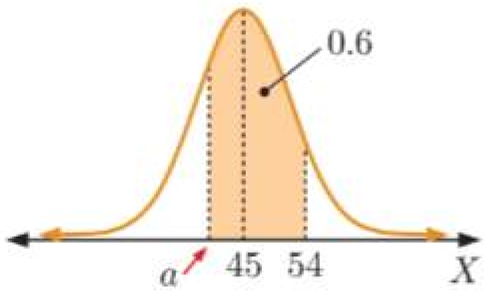


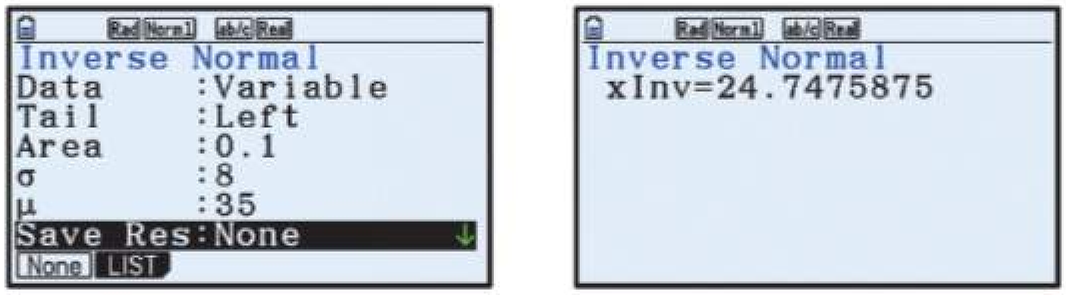
$$\mathrm{P}(a \leqslant X \leqslant 50) = 0.15$$
$$\therefore \mathrm{P}(X \leqslant 50) - \mathrm{P}(X \leqslant a) = 0.15$$
$$\therefore 0.734 - \mathrm{P}(X \leqslant a) \approx 0.15$$
$$\therefore \mathrm{P}(X \leqslant a) \approx 0.584$$
$$\therefore a \approx 46.7$$



c 

$$\begin{aligned} \mathrm{P}(a \leqslant X \leqslant 54) &= 0.6 \\ \therefore\ \mathrm{P}(X \leqslant 54) - \mathrm{P}(X \leqslant a) &= 0.6 \\ \therefore\ 0.870 - \mathrm{P}(X \leqslant a) &\approx 0.6 \\ \therefore\ \mathrm{P}(X \leqslant a) &\approx 0.27 \\ \therefore\ a &\approx 40.1 \end{aligned}$$



7 

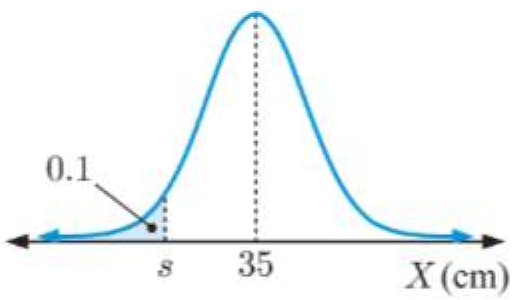
Let X cm be the length of a randomly selected fish, and s cm be the size of the smallest fish that can be harvested.

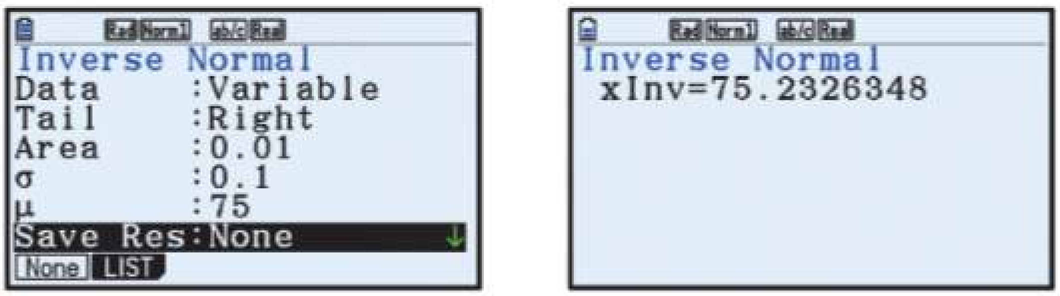
$X \sim \mathrm{N}(35, 8^2)$

$\mathrm{P}(X < s) = 0.1$

$\therefore\ s \approx 24.7$

$\therefore$ the size of the smallest fish that can be harvested is about 24.7 cm.



8 

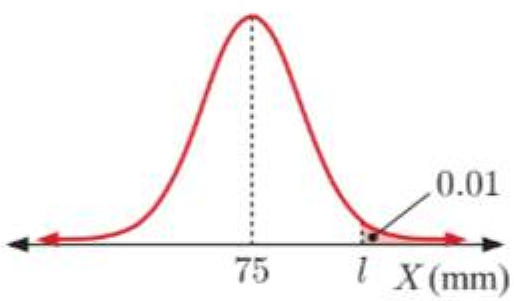
Let X mm be the length of a randomly selected screw, and l mm be the length of the smallest screw to be rejected.

$X \sim \mathrm{N}(75, 0.1^2)$

$\mathrm{P}(X \geqslant l) = 0.01$

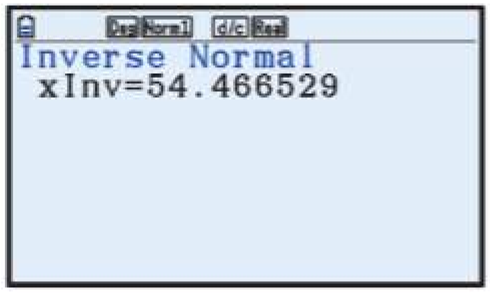
$\therefore\ l \approx 75.2$

$\therefore$ the length of the smallest screw to be rejected is about 75.2 mm.



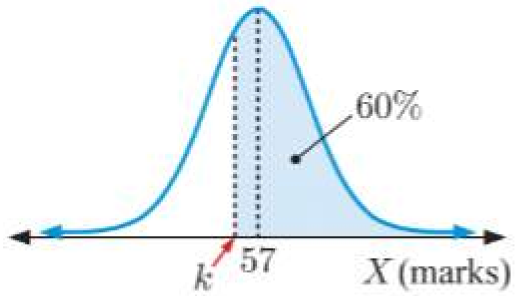
9 $X \sim \mathrm{N}(57, 10^2)$

a **i**

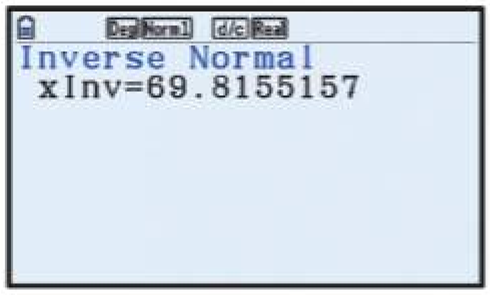


$\mathrm{P}(X \geqslant k) = 0.6$

$\therefore \quad k \approx 54.5$

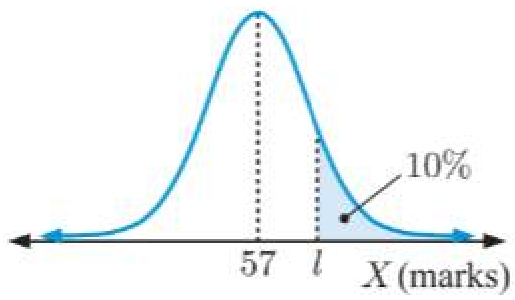


ii



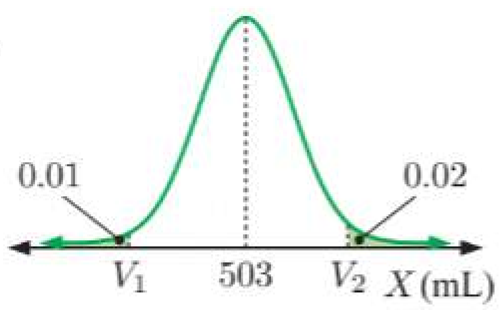
$\mathrm{P}(X \geqslant l) = 0.1$

$\therefore \quad l \approx 69.8$



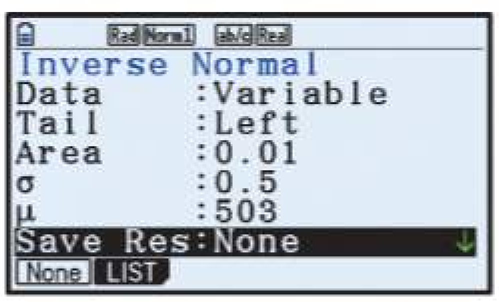
b
$$\begin{aligned} \mathrm{P}(k \leqslant X \leqslant l) &= \mathrm{P}(X \leqslant l) - \mathrm{P}(X < k) \\ &= (1 - \mathrm{P}(X > l)) - (1 - \mathrm{P}(X \geqslant k)) \\ &= (1 - 0.1) - (1 - 0.6) \\ &= 0.9 - 0.4 \\ &= 0.5 \quad ✓ \end{aligned}$$

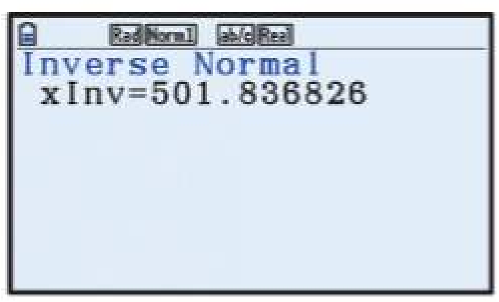
10 Let X mL be the volume in a randomly selected bottle.
Let V_1 mL be the volume of the smallest bottle kept, and V_2 mL be the volume of the largest bottle kept.
$X \sim \mathrm{N}(503, 0.5^2)$



$\mathrm{P}(X \leqslant V_1) = 0.01$

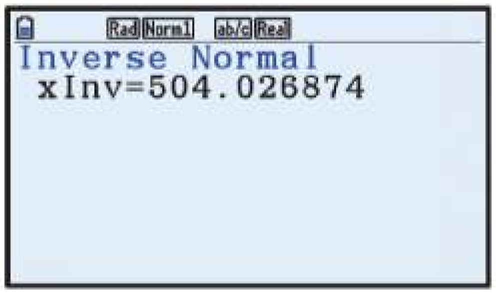
$\therefore \quad V_1 \approx 501.8$





$P(X \geqslant V_2) = 0.02$

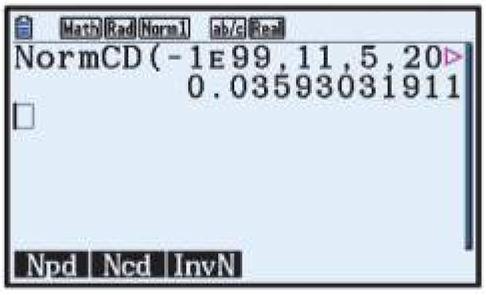
$\therefore \quad V_2 \approx 504.0$

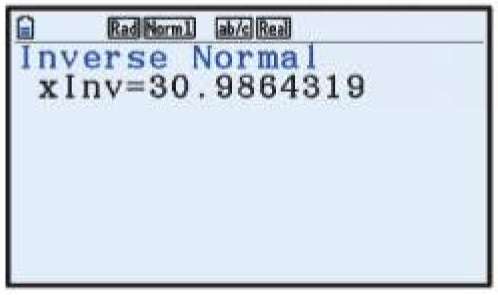


$\therefore$ the bottles which are kept have volumes ranging from about 501.8 mL to 504.0 mL.

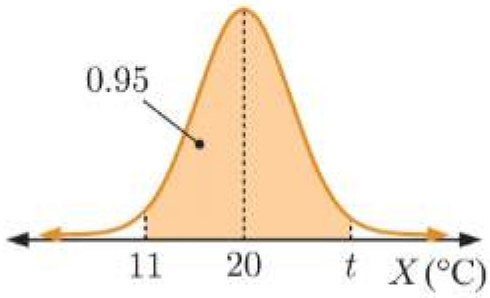
11 Let X°C be the temperature on a randomly selected morning, and t°C be the hottest temperature at which Abbey would still walk.

$X \sim \text{N}(20,\ 5^2)$





$$P(11 \leqslant X \leqslant t) = 0.95$$
$$\therefore \quad P(X \leqslant t) - P(X \leqslant 11) = 0.95$$
$$\therefore \quad P(X \leqslant t) - 0.0359 \approx 0.95$$
$$\therefore \quad P(X \leqslant t) \approx 0.986$$
$$\therefore \quad t \approx 31.0$$



$\therefore$ the upper limit of Abbey's walking temperatures is about 31.0°C.

INVESTIGATION 4 — THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

1 **a** As n increases, the distribution of X approaches that of the normal distribution.

b As n increases, the distribution of X approaches that of the normal distribution for all values of p used.

c It is reasonable to approximate the binomial distribution with a normal distribution as long as n is sufficiently large. The distribution should be symmetrical about the most commonly occurring value.

d We expect that $\mu = np$ and $\sigma = \sqrt{np(1-p)}$, like that of the binomial distribution.

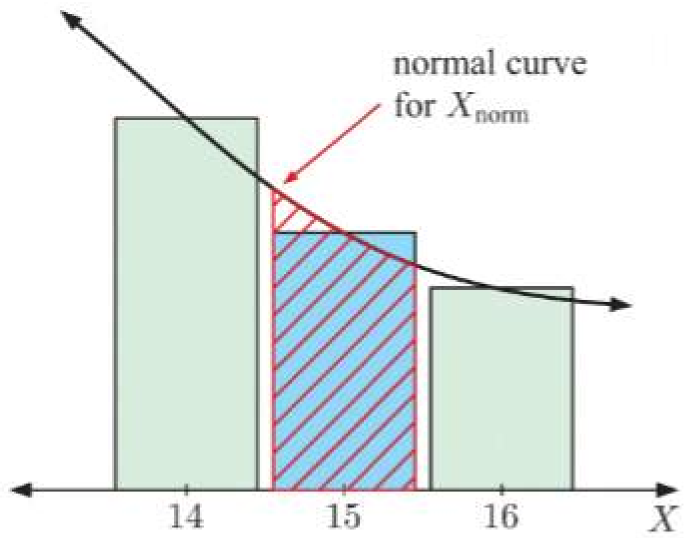
2 **a**

$$\begin{aligned} \mu &= np \\ &= 50 \times 0.2 \\ &= 10 \end{aligned} \qquad \begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{50 \times 0.2 \times 0.8} \\ &= \sqrt{8} \\ &\approx 2.83 \end{aligned}$$

b The blue shaded area represents $P(X = 15)$.
The red shaded area represents $P(14.5 \leqslant X_{norm} \leqslant 15.5)$.

The two areas are approximately equal.

$\therefore \quad P(X = 15) \approx P(14.5 \leqslant X_{norm} \leqslant 15.5)$.



c $X_{norm} \sim N(10, (\sqrt{8})^2)$

i $P(X \leqslant 10) \approx P(X_{norm} \leqslant 10.5)$

ii $P(X < 25) = P(X \leqslant 24) \approx P(X_{norm} \leqslant 24.5)$

iii $P(10 \leqslant X < 25) = P(10 \leqslant X \leqslant 24) \approx P(9.5 \leqslant X_{norm} \leqslant 24.5)$

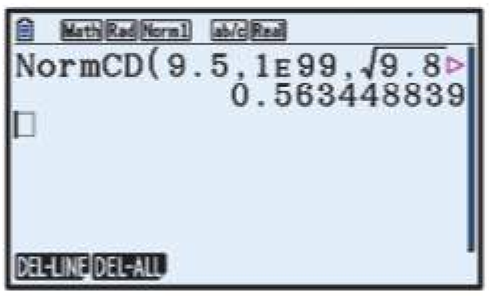
3 $X \sim B(500, 0.02)$

$$\mu = np = 500 \times 0.02 = 10$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{500 \times 0.02 \times 0.98} = \sqrt{9.8} \approx 3.13$$

$X_{norm} \sim N(10, (\sqrt{9.8})^2)$

$P(X \geqslant 10) \approx P(X_{norm} \geqslant 9.5)$
≈ 0.563



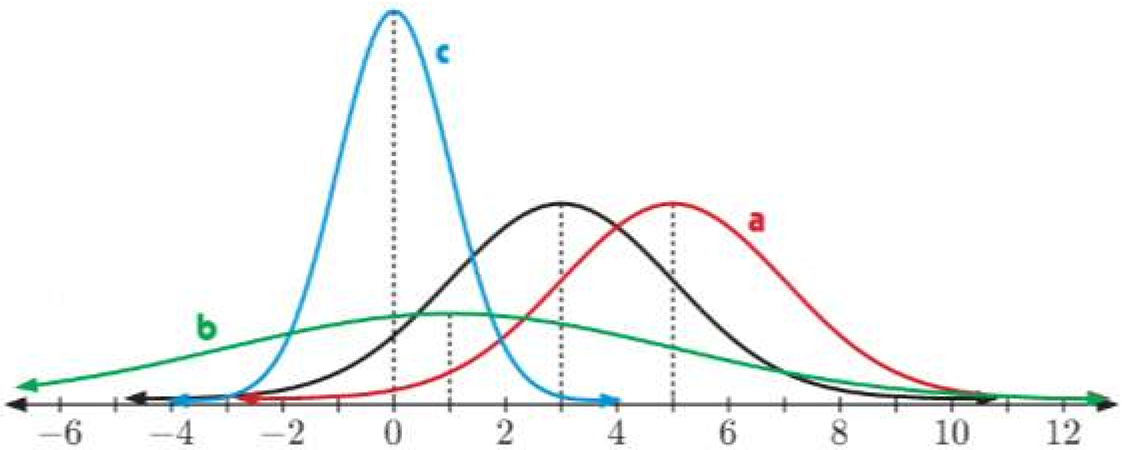
$\therefore$ we estimate that the probability that at least 10 tyres in the sample will be unfit for sale is approximately 0.563.

REVIEW SET 28A

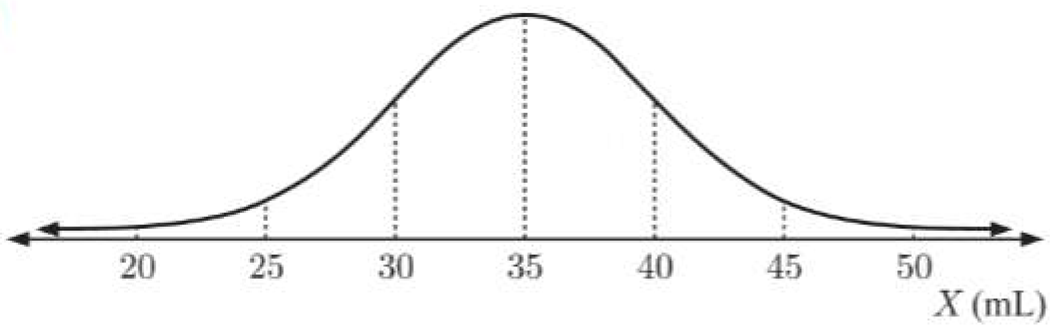
1 **a** The distribution of times taken for students to read a novel is likely to be positively skewed, and hence not normal.

b The mean amount spent on groceries at a supermarket is likely to occur most often with variations around the mean occurring symmetrically as a result of random variation in the prices of items bought and/or the quantities of items bought (for example weights of fruits and vegetables). So, the distribution is likely to be normal.

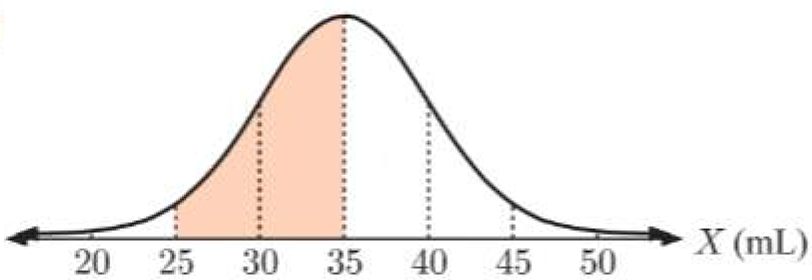
2



3 **a**

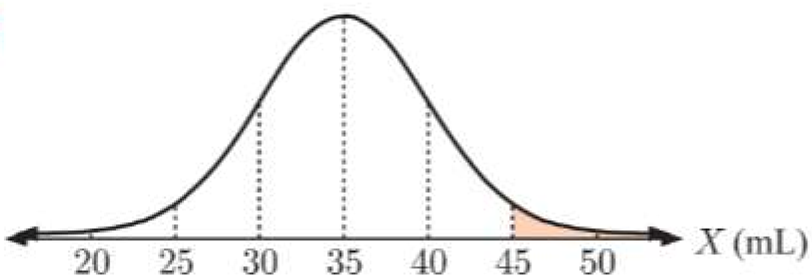


b **i**



About $13.59\% + 34.13\% = 47.72\%$ of Simon's lemons will produce between 25 mL and 35 mL of juice.

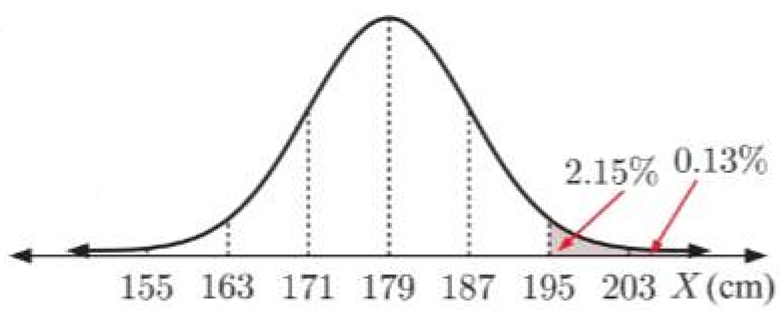
ii



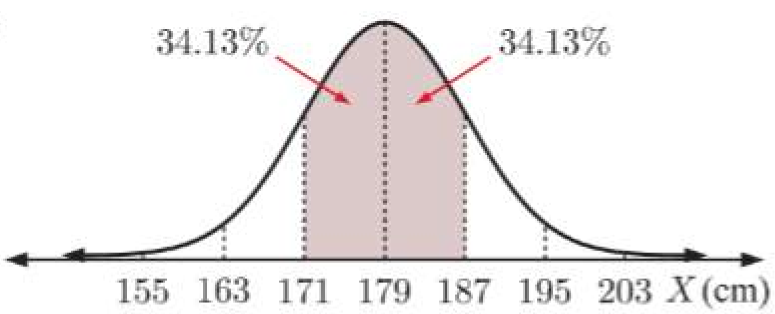
About $2.15\% + 0.13\% = 2.28\%$ of Simon's lemons will produce at least 45 mL of juice.

4 Let X cm be the height of a 17 year old boy.
$X \sim \text{N}(179,\ 8^2)$

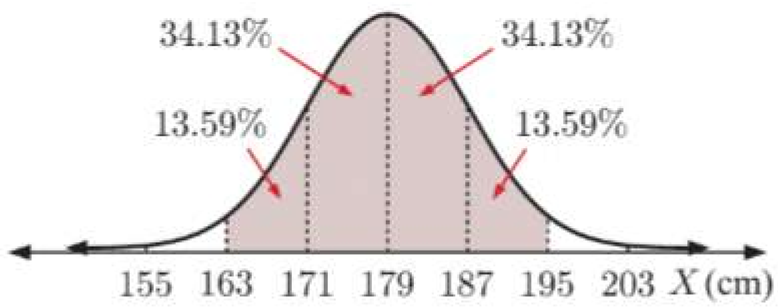
a About $2.15\% + 0.13\% = 2.28\%$ of 17 year old boys have a height more than 195 cm.



b About $34.13\% + 34.13\% = 68.26\%$ of 17 year old boys have a height between 171 cm and 187 cm.

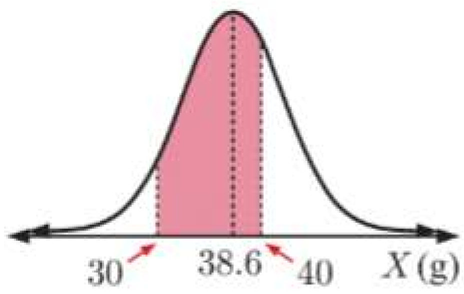


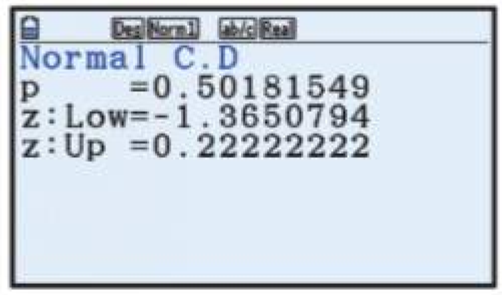
c About
$13.59\% + 34.13\% + 34.13\% + 13.59\%$
$= 95.44\%$ of 17 year old boys have a height between 163 cm and 195 cm.



5 Let X grams be the weight of the edible part of a randomly selected Coffin Bay oyster.

a

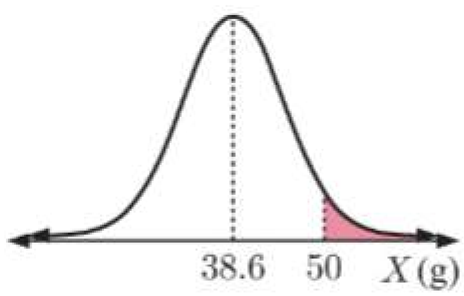


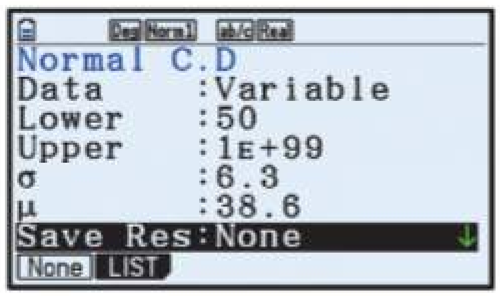


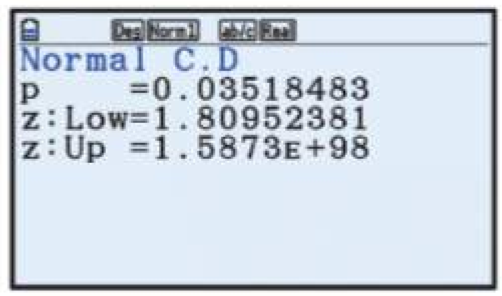
$P(30 < X < 40) \approx 0.502 \approx 50.2\%$

About 50.2% of oysters have an edible part that weighs between 30 g and 40 g.

b





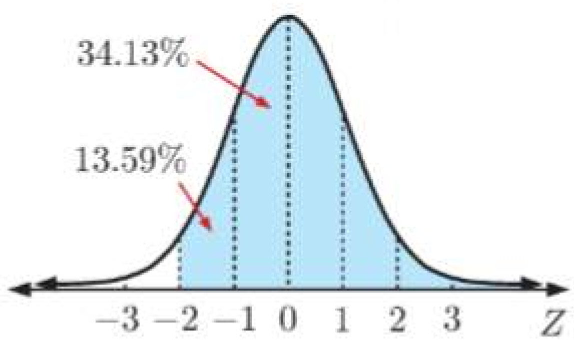


$P(X > 50) \approx 0.0352$

$\therefore$ we would expect about $0.0352 \times 200 \approx 7$ oysters to have an edible part that weighs more than 50 g.

6 **a** A z-score of -2 indicates that Harri's score is 2 standard deviations below the mean.

b About $13.59\% + 34.13\% + 50\% = 97.72\%$ of students obtained a better score than Harri.



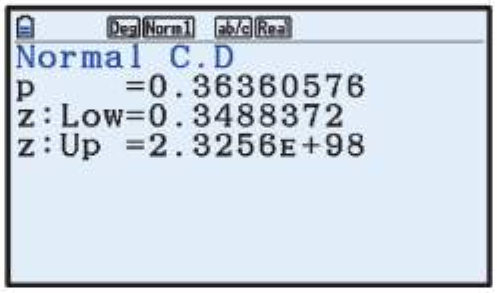
c $\mu = 61$ and $\mu - 2\sigma = 47$

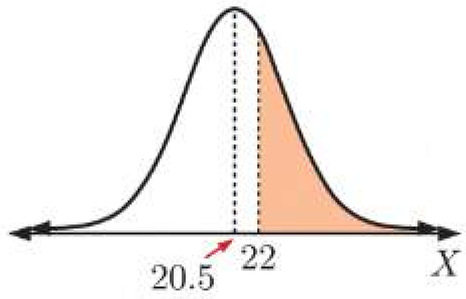
$$\therefore\ 61 - 2\sigma = 47$$
$$\therefore\ 2\sigma = 14$$
$$\therefore\ \sigma = 7$$

The standard deviation of the test scores was 7.

7 $X \sim \mathrm{N}(20.5,\ 4.3^2)$

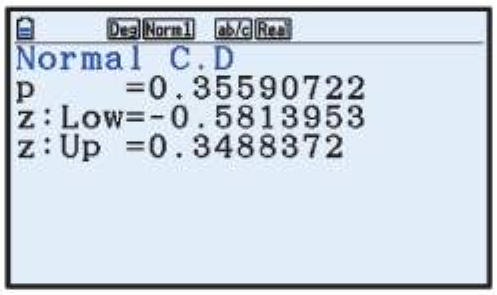
a

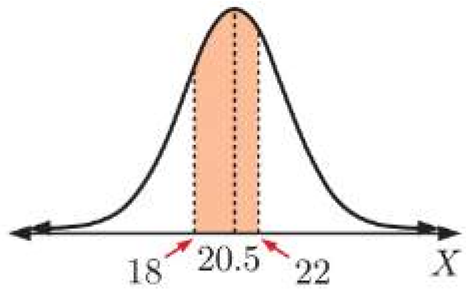




$\mathrm{P}(X \geqslant 22) \approx 0.364$

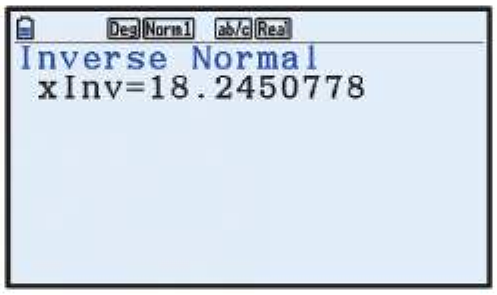
b

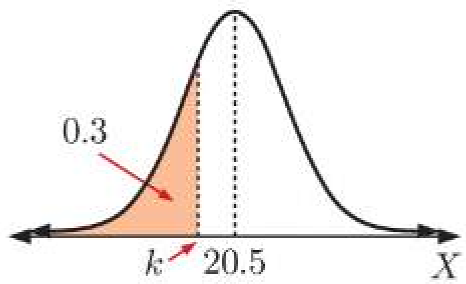




$\mathrm{P}(18 \leqslant X \leqslant 22) \approx 0.356$

c





If $\mathrm{P}(X \leqslant k) = 0.3$

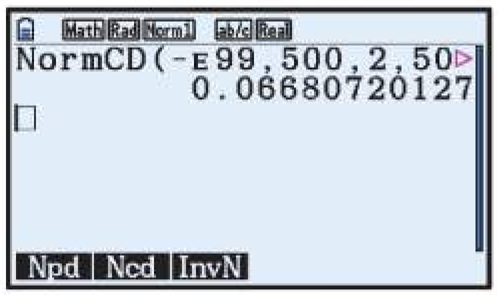
then $k \approx 18.2$

8 **a** $X \sim \mathrm{N}(503,\ 2^2)$

$\mathrm{P}(X < 500) \approx 0.066\,807$

≈ 0.0668

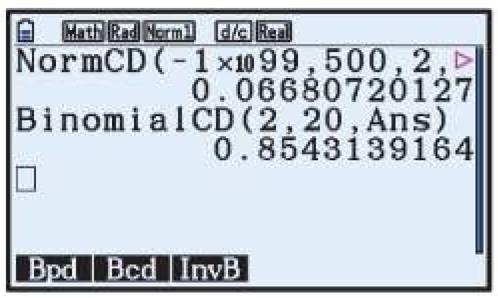
$\therefore$ approximately 6.68% of the bags are underweight.



b Let Y be the number of bags which are underweight.

$Y \sim \mathrm{B}(20,\ 0.066\,807)$

$\therefore\ \mathrm{P}(Y \leqslant 2) \approx 0.854$



9 Let X kJ be the daily energy intake of a randomly selected Canadian adult.

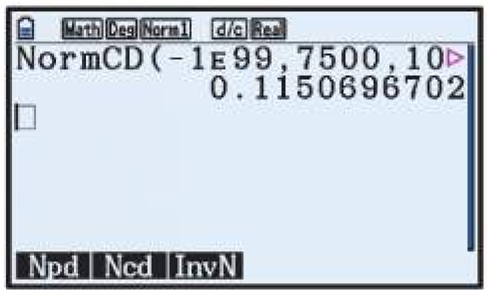
$X \sim \mathrm{N}(8700,\ 1000^2)$

a $\mathrm{P}(X > 8000) \approx 0.758$

$\therefore$ about 75.8% of Canadian adults have a daily energy intake of more than 8000 kJ.

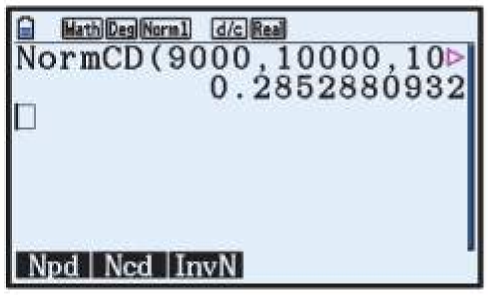
b $P(X < 7500) \approx 0.115$

$\therefore$ about 11.5% of Canadian adults have a daily energy intake of less than 7500 kJ.



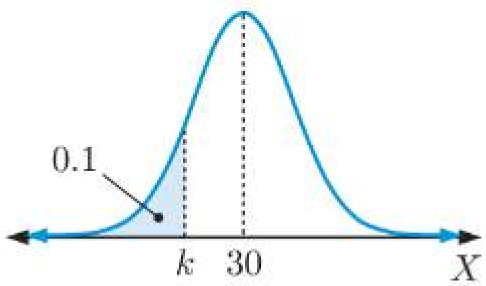
c $P(9000 < X < 10\,000) \approx 0.285$

$\therefore$ about 28.5% of Canadian adults have a daily energy intake between 9000 kJ and 10 000 kJ.



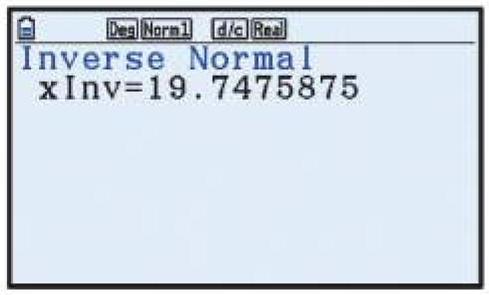
10 $X \sim N(30, 8^2)$

a

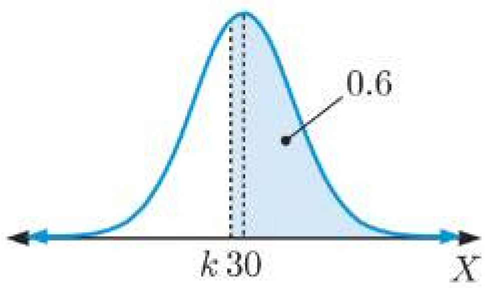


If $P(X \leqslant k) = 0.1$

then $k \approx 19.7$

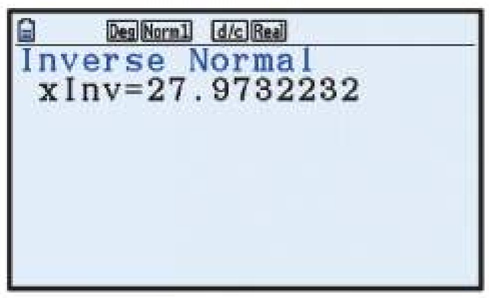


b



If $P(X \geqslant k) = 0.6$

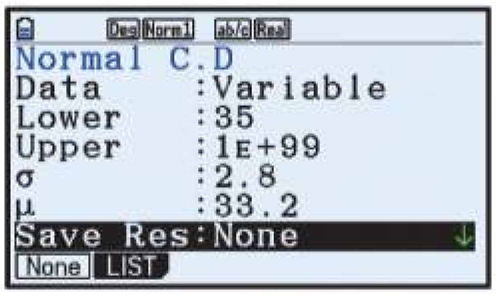
then $k \approx 28.0$

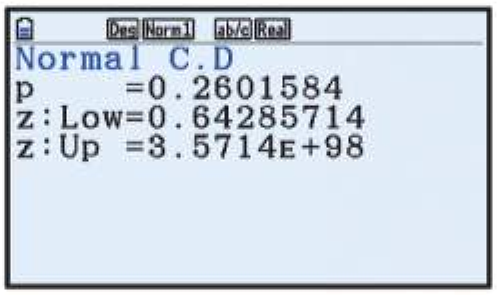


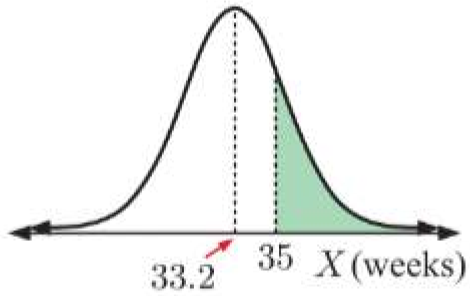
11 Let X be the life of a randomly selected battery in weeks.

$X \sim \text{N}(33.2,\ 2.8^2)$

a







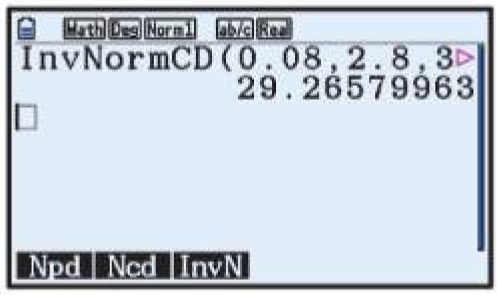
$\text{P}(X \geqslant 35) \approx 0.260$

b We need to find k such that

$\text{P}(X \leqslant k) = 0.08$

$\therefore\ k \approx 29.3$

So, the manufacturer can expect the batteries to last about 29.3 weeks before 8% of them fail.

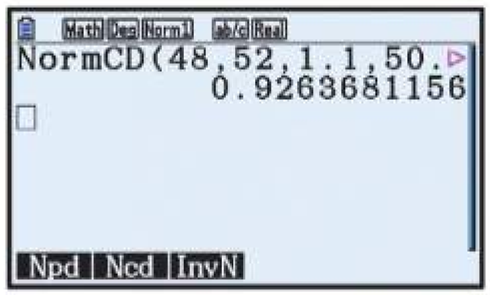


12 **a** Let X_A cm be the length of a randomly selected nail produced by machine A, and X_B cm be the length of a randomly selected nail produced by machine B.

i $X_A \sim \text{N}(50.2,\ 1.1^2)$

$\text{P}(48 \leqslant X_A \leqslant 52) \approx 0.9264$

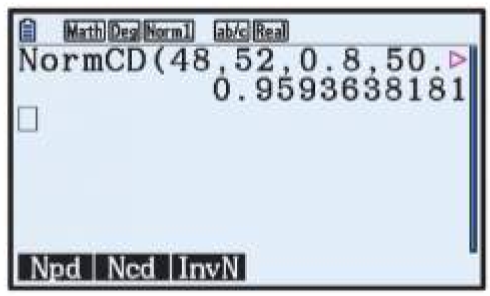
$\therefore$ the probability that a nail from machine A needs to be rejected is about $1 - 0.9264 \approx 0.0736$.



ii $X_B \sim \text{N}(50.6,\ 0.8^2)$

$\text{P}(48 \leqslant X_B \leqslant 52) \approx 0.9594$

$\therefore$ the probability that a nail from machine B needs to be rejected is about $1 - 0.9594 \approx 0.0406$.

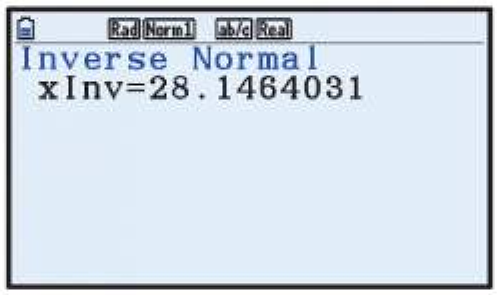


b
$$\begin{aligned}\text{P(made by machine A} \mid \text{rejected)} &= \frac{\text{P(made by machine A} \cap \text{rejected)}}{\text{P(rejected)}}\\ &\approx \frac{0.5 \times 0.0736}{0.5 \times 0.0736 + 0.5 \times 0.0406}\\ &\approx 0.644\end{aligned}$$

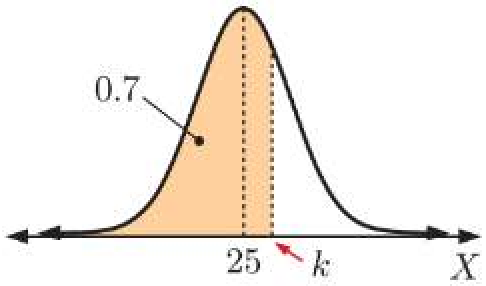
The probability that the nail was made by machine A *given* that it should be rejected is approximately 0.644.

13 $X \sim \mathrm{N}(25,\, 6^2)$

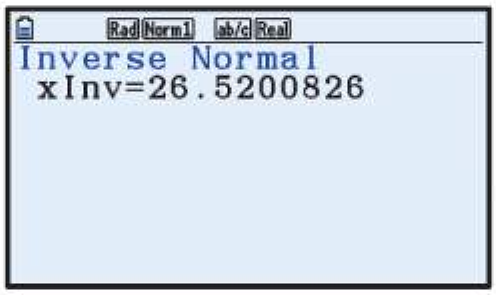
a



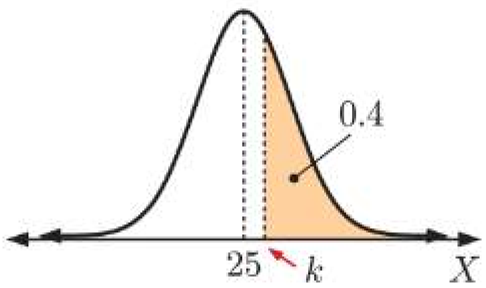
If $\mathrm{P}(X \leqslant k) = 0.7$
then $k \approx 28.1$



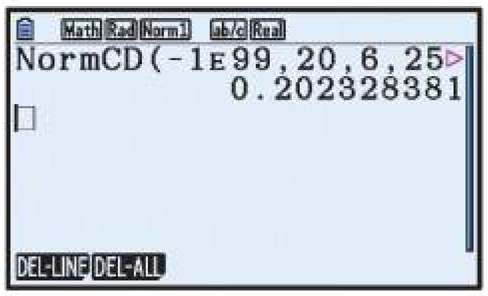
b

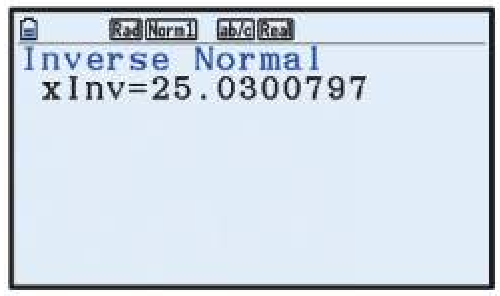


If $\mathrm{P}(X \geqslant k) = 0.4$
then $k \approx 26.5$

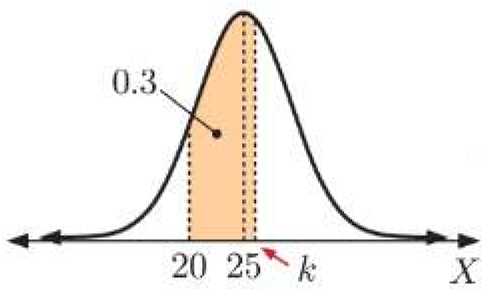


c





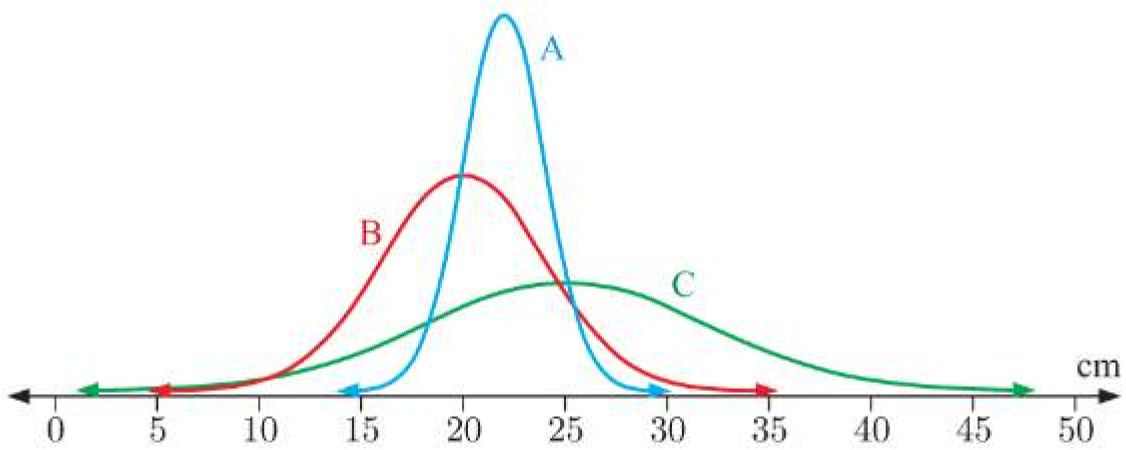
$$\mathrm{P}(20 \leqslant X \leqslant k) = 0.3$$
$$\therefore\ \mathrm{P}(X \leqslant k) - \mathrm{P}(X \leqslant 20) = 0.3$$
$$\therefore\ \mathrm{P}(X \leqslant k) - 0.202 \approx 0.3$$
$$\therefore\ \mathrm{P}(X \leqslant k) \approx 0.502$$
$$\therefore\ k \approx 25.0$$



REVIEW SET 28B

1

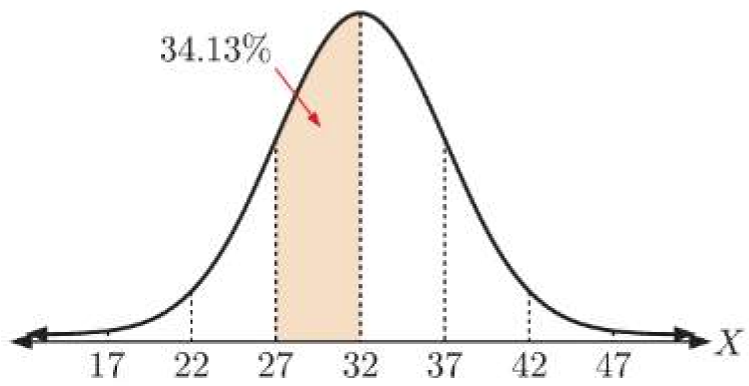
Distribution	*Mean* (cm)	*Standard deviation* (cm)
A	22	2
B	20	4
C	25	7



2 **a** The mean is $\mu = 32$, and the standard deviation is $\sigma = 5$.

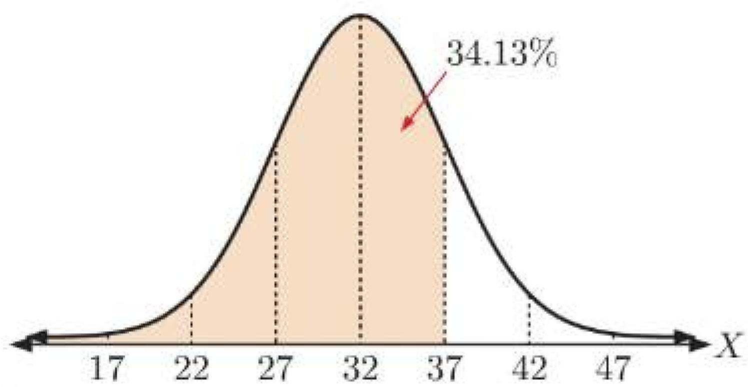
b $X \sim \text{N}(32,\ 5^2)$

i



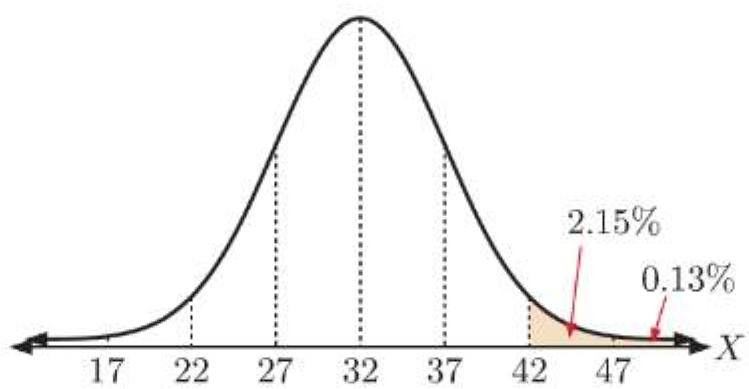
About 34.13% of values of X are between 27 and 32.

ii



About $50\% + 34.13\% = 84.13\%$ of values of X are less than 37.

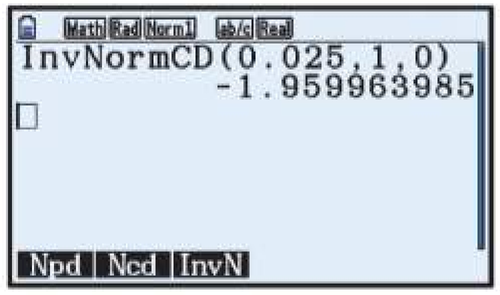
iii



About $2.15\% + 0.13\% = 2.28\%$ of values of X are greater than 42.

3

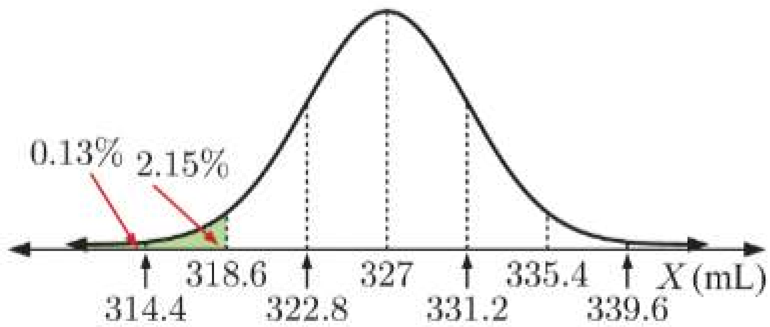
$$\text{If}\quad P(-k \leqslant Z \leqslant k) = 0.95$$
$$\therefore\quad 1 - P(Z \leqslant -k) - P(Z \geqslant k) = 0.95$$
$$\therefore\quad 1 - 2\,P(Z \leqslant -k) = 0.95 \qquad \{\text{symmetry of the normal distribution}\}$$
$$\therefore\quad 2\,P(Z \leqslant -k) = 0.05$$
$$\therefore\quad P(Z \leqslant -k) = 0.025$$
$$\therefore\quad k \approx 1.96$$



4 Let X mL be the contents of a can.

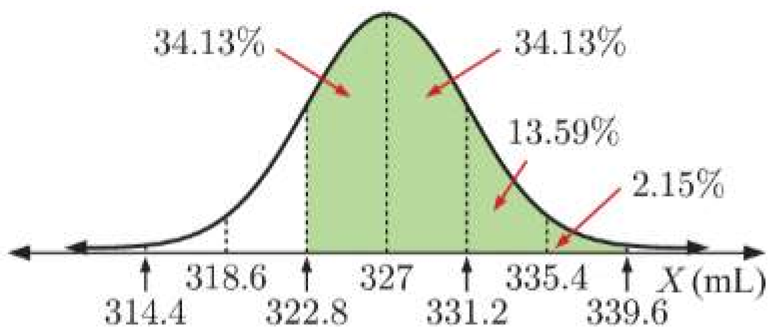
$X \sim N(327,\ 4.2^2)$

a **i** About $0.13\% + 2.15\% = 2.28\%$ of cans have contents less than 318.6 mL.



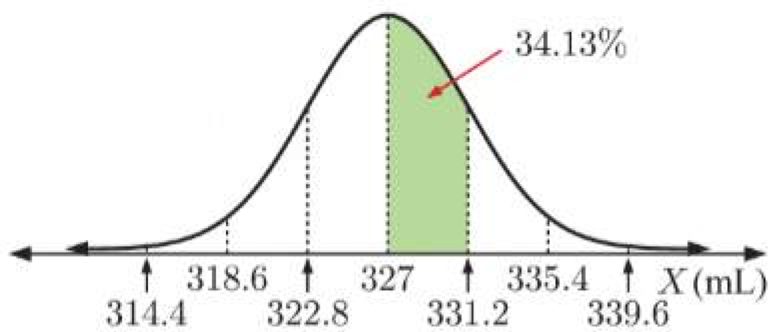
ii About

$34.13\% + 34.13\% + 13.59\% + 2.15\%$
$= 84.0\%$ of cans have contents between 322.8 mL and 339.6 mL.



b About 34.13% of cans have contents between 327 mL and 331.2 mL.

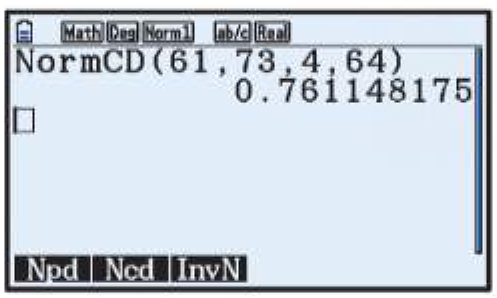
P(contents between 327 mL and 331.2 mL)
≈ 0.3413

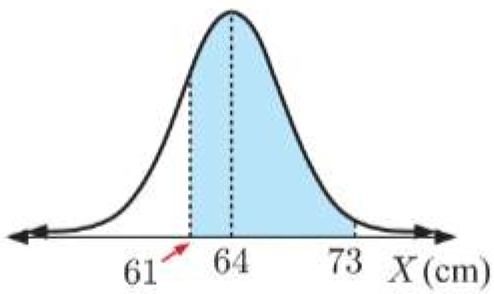


5 Let X cm be the arm length of a randomly selected 18 year old female.

$X \sim N(64,\ 4^2)$

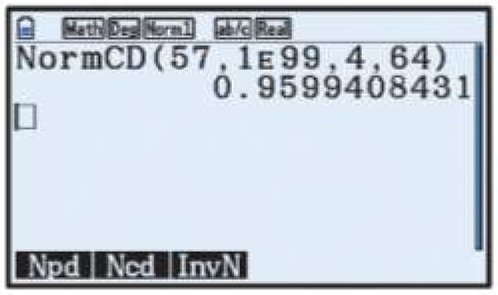
a **i**

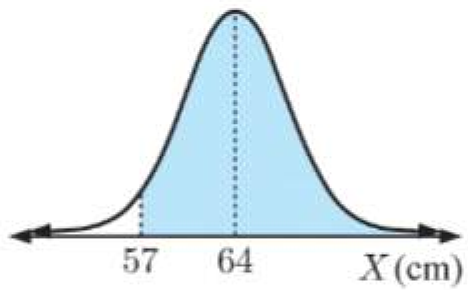




$P(61 < X < 73) \approx 0.761$

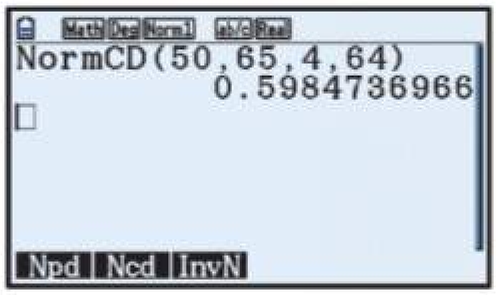
$\therefore$ approximately 76.1% of 18 year old females have an arm length between 61 cm and 73 cm.

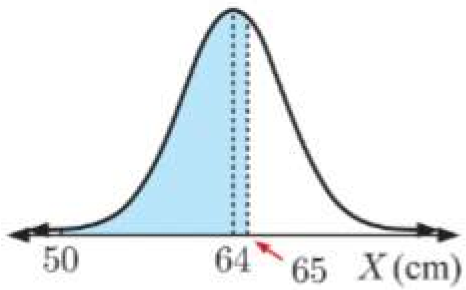
ii 



$P(X > 57) \approx 0.960$

$\therefore$ approximately 96.0% of 18 year old females have an arm length greater than 57 cm.

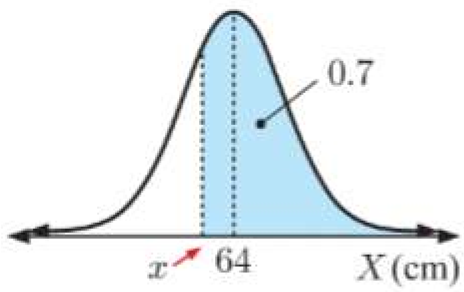
b 



$P(50 < X < 65) \approx 0.598$

$\therefore$ the probability that an 18 year old female has an arm length in the range 50 cm to 65 cm is approximately 0.598.

c 

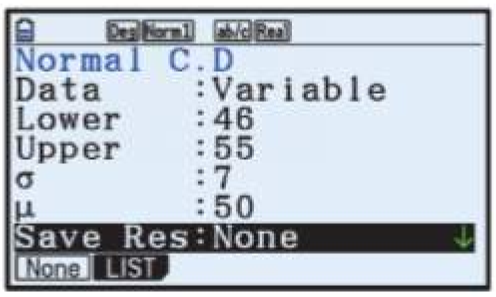


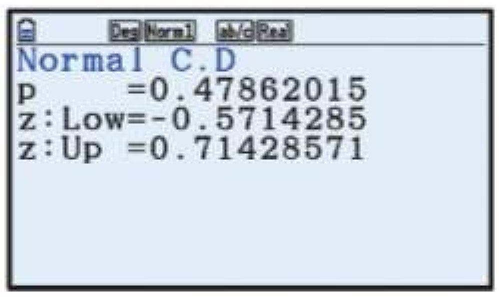
$P(X > x) = 0.7$

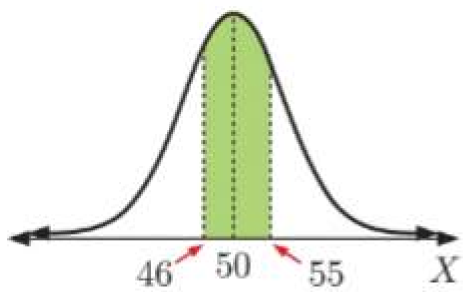
$\therefore P(X < x) = 0.3$

$\therefore x \approx 61.9$

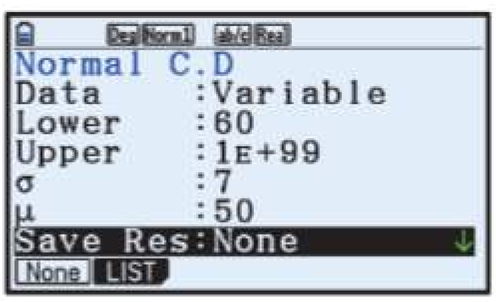
6 $X \sim N(50, 7^2)$

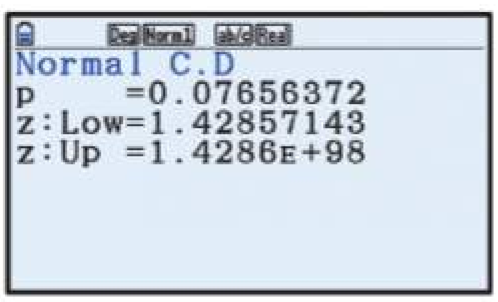
a 

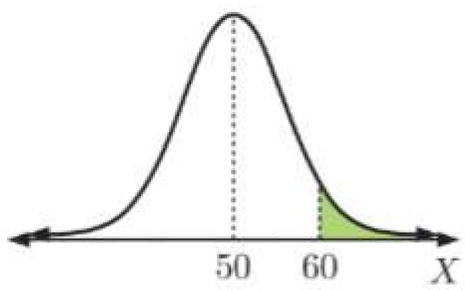




$P(46 \leqslant X \leqslant 55) \approx 0.479$

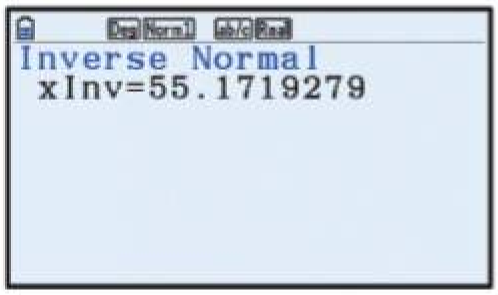
b 

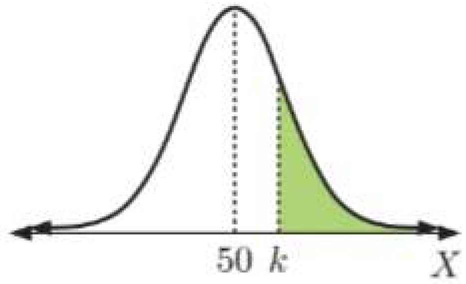




$P(X \geqslant 60) \approx 0.0766$

c





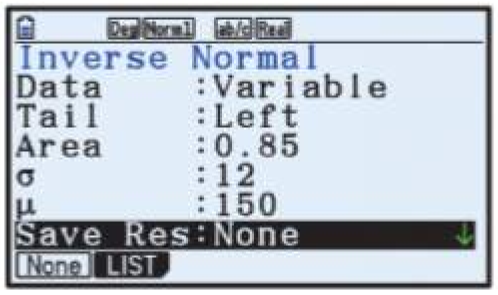
If $\quad P(X > k) = 0.23$

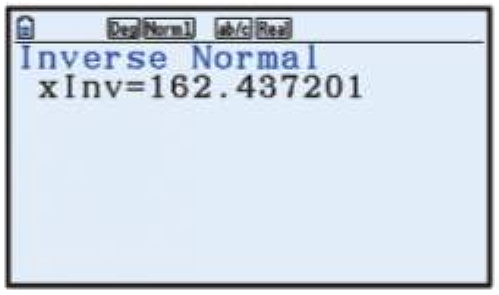
$\therefore \quad P(X < k) = 0.77$

$\therefore \quad k \approx 55.2$

7 Let X seconds be the time a contestant holds their breath.

$X \sim N(150, 12^2)$

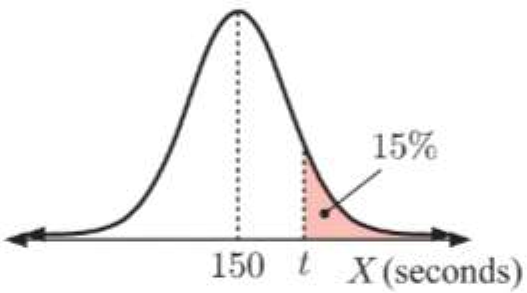




$P(X > t) = 0.15$

$\therefore \quad P(X < t) = 0.85$

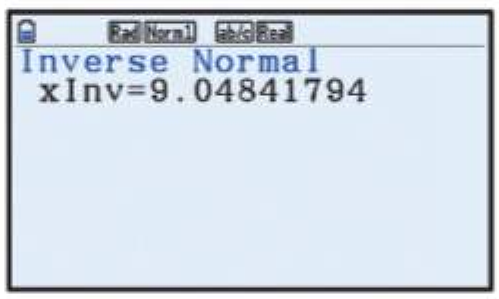
$\therefore \quad t \approx 162.4$



To advance to the final round, a contestant would need to hold their breath for about 162 seconds.

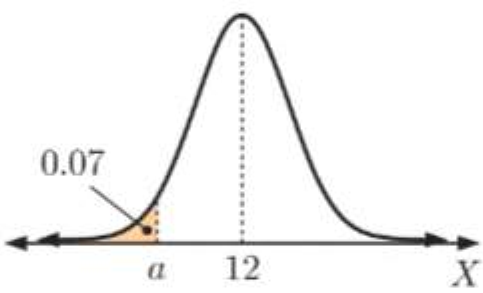
8 $X \sim N(12, 2^2)$

a

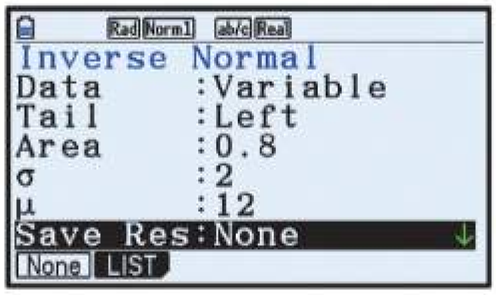


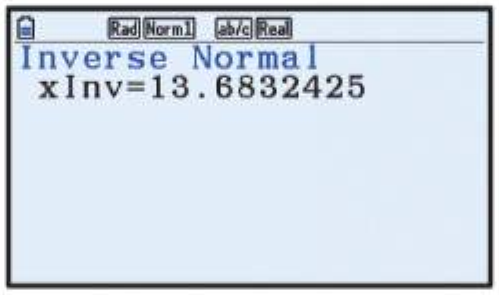
If $\quad P(X < a) = 0.07$

then $\quad a \approx 9.05$



b

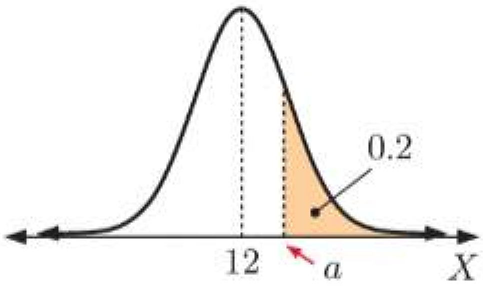




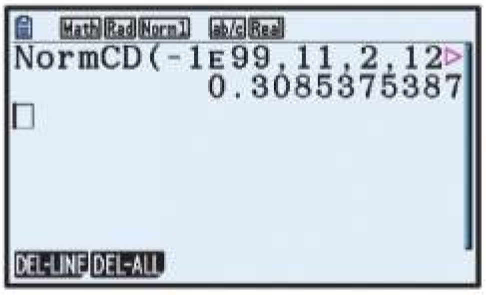
If $\quad P(X > a) = 0.2$

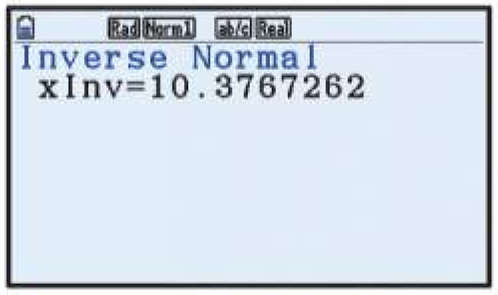
$\therefore \quad P(X < a) = 0.8$

$\therefore \quad a \approx 13.7$



c





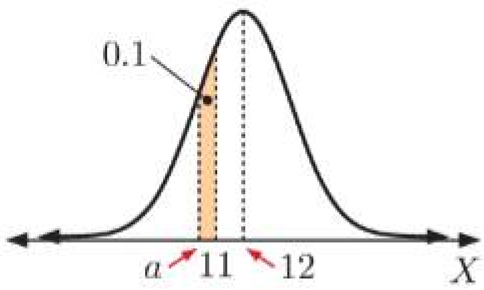
If $\quad P(a \leqslant X \leqslant 11) = 0.1$

$\therefore \quad P(X \leqslant 11) - P(X \leqslant a) = 0.1$

$\therefore \quad 0.3085 - P(X \leqslant a) \approx 0.1$

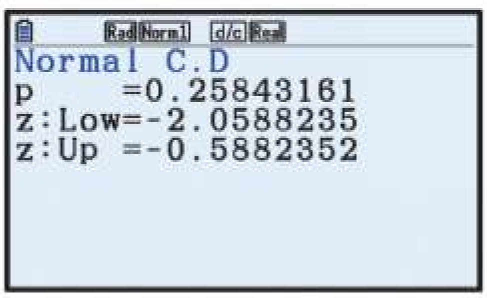
$\therefore \quad P(X \leqslant a) \approx 0.2085$

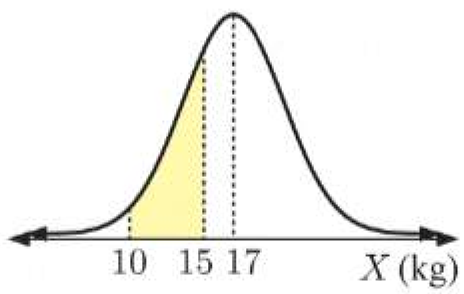
$\therefore \quad a \approx 10.4$



9 Let X kg be the weight of a suitcase at the airport. $\quad X \sim N(17, 3.4^2)$

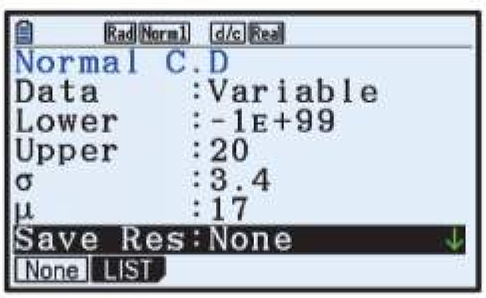
a

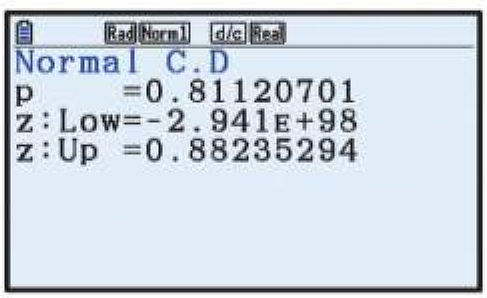


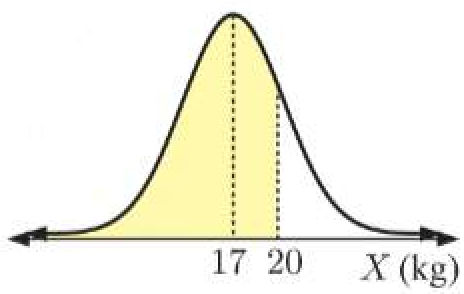


$P(10 < X < 15) \approx 0.258$

b



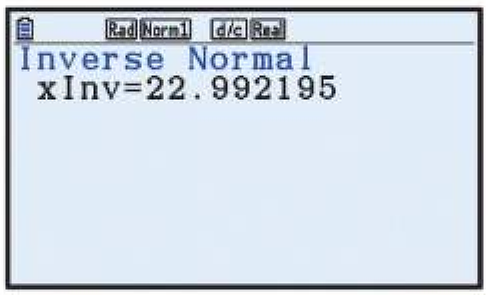


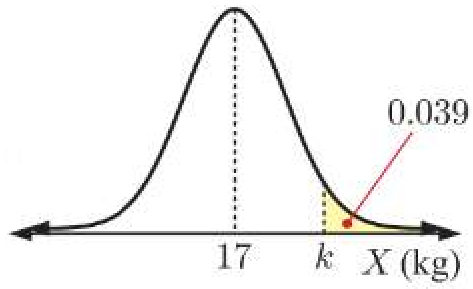


$P(X < 20) \approx 0.811$

$\therefore$ we would expect about $0.811 \times 300 \approx 243$ suitcases to be lighter than 20 kg.

c 





If $\quad P(X > k) = 0.039$

then $\quad k \approx 23.0$

$\therefore$ the maximum weight limit is ≈ 23.0 kg.

10 **a** The relative difficulty of each test is not known. We would need the mean mark and standard deviation for each test.

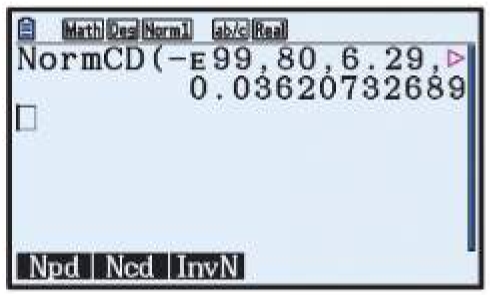
b Kerry's English z-score $= \dfrac{26-22}{4} = \dfrac{4}{4} = 1$ $\qquad$ Kerry's Chemistry z-score $= \dfrac{82-75}{7} = \dfrac{7}{7} = 1$

Since the z-scores are the same, Kerry's performance relative to the rest of the class is the same in both tests.

11 **a** Let X_F be the weight in kilograms of a female ostrich and X_M be the weight in kilograms of a male ostrich.

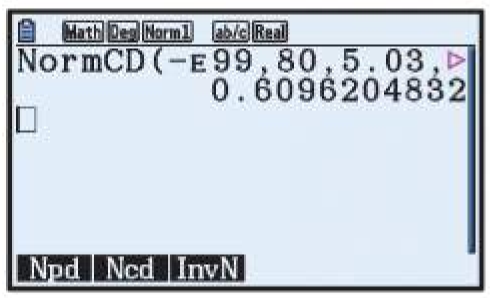
$X_F \sim N(78.6,\ 5.03^2) \quad$ and $\quad X_M \sim N(91.3,\ 6.29^2)$

i $P(X_M < 80) \approx 0.0362$



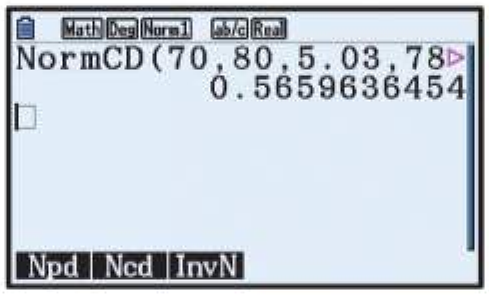
The probability that a randomly selected male ostrich will weigh less than 80 kg is about 0.0362.

ii $P(X_F < 80) \approx 0.610$



The probability that a randomly selected female ostrich will weigh less than 80 kg is about 0.610.

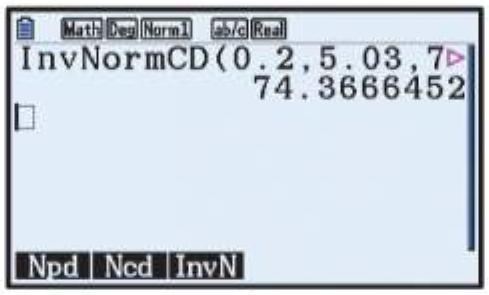
iii $P(70 < X_F < 80) \approx 0.566$



The probability that a randomly selected female ostrich will weigh between 70 and 80 kg is about 0.566.

b $P(X_F < k) = 0.2$

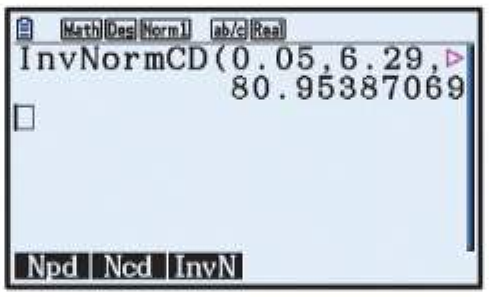
then $k \approx 74.4$



c We need to find a and b such that $P(X_M < a) = 0.05$ and $P(X_M > b) = 0.05$.

If $P(X_M < a) = 0.05$

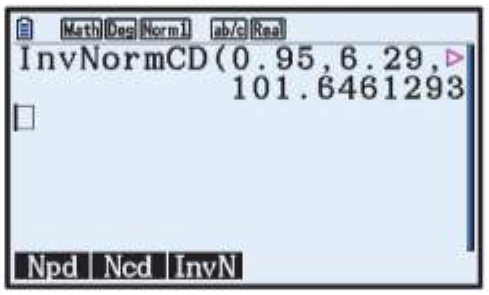
then $a \approx 81.0$



If $P(X_M > b) = 0.05$

then $P(X_M < b) = 0.95$

$\therefore\ b \approx 102$



d Probability that the ostrich weighs less than 80 kg

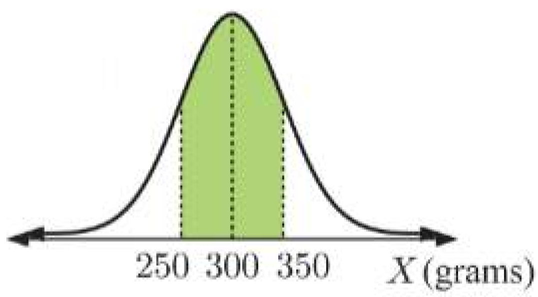
$= P(\text{ostrich is female} \cap \text{less than 80 kg}) + P(\text{ostrich is male} \cap \text{less than 80 kg})$

$\approx 0.82 \times 0.610 + 0.18 \times 0.0362$

≈ 0.506

12 **a** Let X grams be the weight of an apple.

$X \sim N(300, 50^2)$



$P(250 \leqslant X \leqslant 350) \approx 0.682\,689$

So, approximately 68.3% of apples are fit for sale.

b Let Y be the number of apples that are fit for sale.

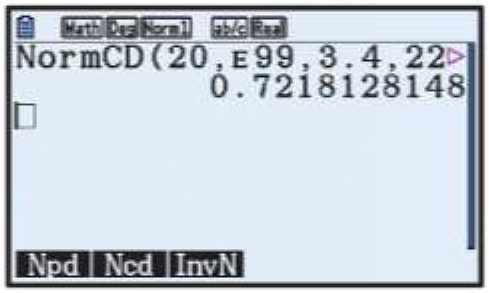
$Y \sim \mathrm{B}(100,\, 0.682\,689)$

$\mathrm{P}(Y \geqslant 75) \approx 0.0884$

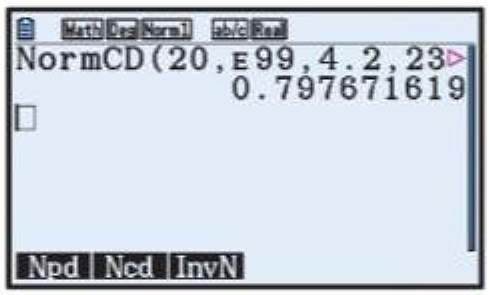
13 **a** Let X_G be the length in cm of a carrot from Giovanni's farm, and X_B be the length in cm of a carrot from Beppe's farm.

$X_G \sim \mathrm{N}(22,\, 3.4^2)$ and $X_B \sim \mathrm{N}(23.5,\, 4.2^2)$

i $\mathrm{P}(X_G > 20) \approx 0.722$



ii $\mathrm{P}(X_B > 20) \approx 0.798$



b Probability that neither carrot is longer than 20 cm $= \mathrm{P}(X_G < 20) \times \mathrm{P}(X_B < 20)$

$$\approx (1 - 0.722) \times (1 - 0.798)$$

$$\approx 0.0563$$

Chapter 29

ESTIMATION AND CONFIDENCE INTERVALS

INVESTIGATION 1 THE EXPECTATION AND VARIANCE OF $X+Y$

1 **a**

x	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

y	1	2	3	4
$\mathrm{P}(Y=y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b

$$\begin{aligned}&\mathrm{E}(X)\\&=1\left(\tfrac{1}{3}\right)+2\left(\tfrac{1}{3}\right)+3\left(\tfrac{1}{3}\right)\\&=\tfrac{1}{3}+\tfrac{2}{3}+1\\&=2\end{aligned}$$

$$\begin{aligned}&\mathrm{Var}(X)\\&=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2\\&=1^2\left(\tfrac{1}{3}\right)+2^2\left(\tfrac{1}{3}\right)+3^2\left(\tfrac{1}{3}\right)-2^2\\&=\tfrac{1}{3}+\tfrac{4}{3}+3-4\\&=\tfrac{2}{3}\end{aligned}$$

$$\begin{aligned}&\mathrm{E}(Y)\\&=1\left(\tfrac{1}{4}\right)+2\left(\tfrac{1}{4}\right)+3\left(\tfrac{1}{4}\right)+4\left(\tfrac{1}{4}\right)\\&=\tfrac{1}{4}+\tfrac{1}{2}+\tfrac{3}{4}+1\\&=\tfrac{5}{2}\\&=2\tfrac{1}{2}\end{aligned}$$

$$\begin{aligned}&\mathrm{Var}(Y)\\&=\mathrm{E}(Y^2)-[\mathrm{E}(Y)]^2\\&=1^2\left(\tfrac{1}{4}\right)+2^2\left(\tfrac{1}{4}\right)+3^2\left(\tfrac{1}{4}\right)+4^2\left(\tfrac{1}{4}\right)-\left(\tfrac{5}{2}\right)^2\\&=\tfrac{1}{4}+1+\tfrac{9}{4}+4-\tfrac{25}{4}\\&=\tfrac{5}{4}\\&=1\tfrac{1}{4}\end{aligned}$$

Variable	*Expectation*	*Variance*
X	2	$\frac{2}{3}$
Y	$2\frac{1}{2}$	$1\frac{1}{4}$

2

		Y			
		1	2	3	4
	1	2	3	4	5
X	2	3	4	5	6
	3	4	5	6	7

$\therefore$ the possible values of W are 2, 3, 4, 5, 6, 7.

3

$$\begin{aligned}\mathrm{P}(W=2)&=\mathrm{P}(X=1\ \cap\ Y=1)\\&=\mathrm{P}(X=1)\,\mathrm{P}(Y=1)\qquad\{X\text{ and }Y\text{ are independent}\}\\&=\tfrac{1}{3}\times\tfrac{1}{4}\\&=\tfrac{1}{12}\end{aligned}$$

$$\begin{aligned}\mathrm{P}(W=3)&=\mathrm{P}(X=1\ \cap\ Y=2)\ +\ \mathrm{P}(X=2\ \cap\ Y=1)\\&=\mathrm{P}(X=1)\,\mathrm{P}(Y=2)\ +\ \mathrm{P}(X=2)\,\mathrm{P}(Y=1)\qquad\{X\text{ and }Y\text{ are independent}\}\\&=\tfrac{1}{3}\times\tfrac{1}{4}\ +\ \tfrac{1}{3}\times\tfrac{1}{4}\\&=\tfrac{1}{6}\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(W=4) &= \mathrm{P}(X=1 \cap Y=3) + \mathrm{P}(X=2 \cap Y=2) + \mathrm{P}(X=3 \cap Y=3)\\
&= \mathrm{P}(X=1)\,\mathrm{P}(Y=3) + \mathrm{P}(X=2)\,\mathrm{P}(Y=2) + \mathrm{P}(X=3)\,\mathrm{P}(Y=3)\\
&\qquad\qquad \{X \text{ and } Y \text{ are independent}\}\\
&= \tfrac{1}{3}\times\tfrac{1}{4} + \tfrac{1}{3}\times\tfrac{1}{4} + \tfrac{1}{3}\times\tfrac{1}{4}\\
&= \tfrac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(W=5) &= \mathrm{P}(X=1 \cap Y=4) + \mathrm{P}(X=2 \cap Y=3) + \mathrm{P}(X=3 \cap Y=2)\\
&= \mathrm{P}(X=1)\,\mathrm{P}(Y=4) + \mathrm{P}(X=2)\,\mathrm{P}(Y=3) + \mathrm{P}(X=3)\,\mathrm{P}(Y=2)\\
&\qquad\qquad \{X \text{ and } Y \text{ are independent}\}\\
&= \tfrac{1}{3}\times\tfrac{1}{4} + \tfrac{1}{3}\times\tfrac{1}{4} + \tfrac{1}{3}\times\tfrac{1}{4}\\
&= \tfrac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(W=6) &= \mathrm{P}(X=2 \cap Y=4) + \mathrm{P}(X=3 \cap Y=3)\\
&= \mathrm{P}(X=2)\,\mathrm{P}(Y=4) + \mathrm{P}(X=3)\,\mathrm{P}(Y=3) \qquad \{X \text{ and } Y \text{ are independent}\}\\
&= \tfrac{1}{3}\times\tfrac{1}{4} + \tfrac{1}{3}\times\tfrac{1}{4}\\
&= \tfrac{1}{6}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(W=7) &= \mathrm{P}(X=3 \cap Y=4)\\
&= \mathrm{P}(X=3)\,\mathrm{P}(Y=4) \qquad \{X \text{ and } Y \text{ are independent}\}\\
&= \tfrac{1}{3}\times\tfrac{1}{4}\\
&= \tfrac{1}{12}
\end{aligned}$$

4 **a**
$$\begin{aligned}
\mathrm{E}(W) &= 2\left(\tfrac{1}{12}\right) + 3\left(\tfrac{1}{6}\right) + 4\left(\tfrac{1}{4}\right) + 5\left(\tfrac{1}{4}\right) + 6\left(\tfrac{1}{6}\right) + 7\left(\tfrac{1}{12}\right)\\
&= \tfrac{1}{6} + \tfrac{1}{2} + 1 + \tfrac{5}{4} + 1 + \tfrac{7}{12}\\
&= \tfrac{9}{2}\\
&= 4\tfrac{1}{2}
\end{aligned}$$

b
$$\begin{aligned}
\mathrm{Var}(W) &= \mathrm{E}(W^2) - [\mathrm{E}(W)]^2\\
&= 2^2\left(\tfrac{1}{12}\right) + 3^2\left(\tfrac{1}{6}\right) + 4^2\left(\tfrac{1}{4}\right) + 5^2\left(\tfrac{1}{4}\right) + 6^2\left(\tfrac{1}{6}\right) + 7^2\left(\tfrac{1}{12}\right) - \left(\tfrac{9}{2}\right)^2\\
&= \tfrac{1}{3} + \tfrac{3}{2} + 4 + \tfrac{25}{4} + 6 + \tfrac{49}{12} - \tfrac{81}{4}\\
&= \tfrac{23}{12}\\
&= 1\tfrac{11}{12}
\end{aligned}$$

5 **a**
$$\begin{aligned}
\mathrm{E}(X) + \mathrm{E}(Y) &= 2 + 2\tfrac{1}{2}\\
&= 4\tfrac{1}{2}\\
&= \mathrm{E}(W)
\end{aligned}$$

b
$$\begin{aligned}
\mathrm{Var}(X) + \mathrm{Var}(Y) &= \tfrac{2}{3} + 1\tfrac{1}{4}\\
&= 1\tfrac{11}{12}\\
&= \mathrm{Var}(W)
\end{aligned}$$

EXERCISE 29A

1 $\mathrm{E}(X) = 3$, $\mathrm{E}(Y) = 2$, $\mathrm{Var}(X) = \frac{3}{2}$, $\mathrm{Var}(Y) = \frac{5}{4}$

a $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
$= 3 + 2$
$= 5$

b $\mathrm{E}(2X - Y) = 2\,\mathrm{E}(X) - \mathrm{E}(Y)$
$= 2(3) - 2$
$= 4$

c $\mathrm{Var}(X+Y)$
$= \mathrm{Var}(X) + \mathrm{Var}(Y)$ {X and Y are independent}
$= \frac{3}{2} + \frac{5}{4}$
$= \frac{11}{4}$

d $\mathrm{Var}(2X - 3Y)$
$= 2^2\,\mathrm{Var}(X) + (-3)^2\,\mathrm{Var}(Y)$ {X and Y are independent}
$= 4\left(\frac{3}{2}\right) + 9\left(\frac{5}{4}\right)$
$= \frac{69}{4}$

2

	μ	σ
X	3.8	0.323
Y	5.7	1.02

a Mean $= \mathrm{E}(X + 2Y)$
$= \mathrm{E}(X) + 2\,\mathrm{E}(Y)$
$= 3.8 + 2(5.7)$
$= 15.2$

$\mathrm{Var}(X+2Y) = \mathrm{Var}(X) + 2^2\,\mathrm{Var}(Y)$ {independence}
$= (0.323)^2 + 4(1.02)^2$
$= 4.265\,929$

$\therefore$ standard deviation $= \sqrt{4.265\,929}$
≈ 2.07

b Mean $= \mathrm{E}(Y - X)$
$= \mathrm{E}(Y) - \mathrm{E}(X)$
$= 5.7 - 3.8$
$= 1.9$

$\mathrm{Var}(Y - X) = \mathrm{Var}(Y) + (-1)^2\,\mathrm{Var}(X)$ {independence}
$= (1.02)^2 + (0.323)^2$
$= 1.144\,729$

$\therefore$ standard deviation $= \sqrt{1.144\,729}$
≈ 1.07

c Mean $= \mathrm{E}(3X - 2Y)$
$= 3\,\mathrm{E}(X) - 2\,\mathrm{E}(Y)$
$= 3(3.8) - 2(5.7)$
$= 0$

$\mathrm{Var}(3X - 2Y) = 3^2\,\mathrm{Var}(X) + (-2)^2\,\mathrm{Var}(Y)$ {independence}
$= 9(0.323)^2 + 4(1.02)^2$
$= 5.100\,561$

$\therefore$ standard deviation $= \sqrt{5.100\,561}$
≈ 2.26

3 **a** **i**

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

ii

y	1	2	3	4	5	6
$\mathrm{P}(Y = y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b **i** $\mathrm{E}(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = 1$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
$= 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) - 1^2$
$= \frac{1}{2}$

ii $\mathrm{E}(Y) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$

$= \frac{7}{2}$

$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2$

$= 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 9\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 25\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) - \left(\frac{7}{2}\right)^2$

$= \frac{35}{12}$

iii $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$

$= 1 + \frac{7}{2}$

$= \frac{9}{2}$

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ {independence}

$= \frac{1}{2} + \frac{35}{12}$

$= \frac{41}{12}$

iv $\mathrm{E}(4X - 2Y) = 4\,\mathrm{E}(X) - 2\,\mathrm{E}(Y)$

$= 4(1) - 2\left(\frac{7}{2}\right)$

$= -3$

$\mathrm{Var}(4X - 2Y) = 4^2\,\mathrm{Var}(X) + (-2)^2\,\mathrm{Var}(Y)$ {independence}

$= 16\,\mathrm{Var}(X) + 4\,\mathrm{Var}(Y)$

$= 16\left(\frac{1}{2}\right) + 4\left(\frac{35}{12}\right)$

$= \frac{59}{3}$

4

x	0	1	3	6
$\mathrm{P}(X = x)$	0.25	0.25	0.25	0.25

y	0	2	8
$\mathrm{P}(Y = y)$	0.5	0.3	0.2

a $\sum_{i=1}^{n} \mathrm{P}(X = x_i) = 0.25 + 0.25 + 0.25 + 0.25$

$= 1$

and $\mathrm{P}(X = x_i) > 0$ for all i

$\therefore$ X has a well defined probability distribution.

$\sum_{i=1}^{n} \mathrm{P}(Y = y_i) = 0.5 + 0.3 + 0.2$

$= 1$

and $\mathrm{P}(Y = y_i) > 0$ for all i

$\therefore$ Y has a well defined probability distribution.

b **i** Mean $= \mathrm{E}(X)$

$= 0(0.25) + 1(0.25) + 3(0.25) + 6(0.25)$

$= 0 + 0.25 + 0.75 + 1.5$

$= 2.5$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

$= 0^2(0.25) + 1^2(0.25) + 3^2(0.25) + 6^2(0.25) - (2.5)^2$

$= 0 + 0.25 + 2.25 + 9 - 6.25$

$= 5.25$

$\therefore$ standard deviation $= \sqrt{5.25}$

≈ 2.29

ii Mean $= \mathrm{E}(Y)$
$$= 0(0.5) + 2(0.3) + 8(0.2)$$
$$= 0 + 0.6 + 1.6$$
$$= 2.2$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2$$
$$= 0^2(0.5) + 2^2(0.3) + 8^2(0.2) - (2.2)^2$$
$$= 0 + 1.2 + 12.8 - 4.84$$
$$= 9.16$$

Standard deviation $= \sqrt{9.16}$
$$\approx 3.03$$

c Expected gain $= \mathrm{E}(X+Y) - 5$
$$= \mathrm{E}(X) + \mathrm{E}(Y) - 5$$
$$= 2.5 + 2.2 - 5$$
$$= -0.3$$
$= -\$0.30$ or -30 cents

d It is not advisable to play this game many times because the expected gain from playing this game is negative.

5 $X \sim \mathrm{B}(1, p)$

a $\mathrm{E}(X) = 1 \times p = p$ and $\mathrm{Var}(X) = 1 \times p(1-p) = p(1-p)$

b **i** The outcome of each trial of the experiment is a Bernoulli random variable with probability of success p, $X_i \sim \mathrm{B}(1, p)$. So the total number of successes out of n independent trials is $\sum_{i=1}^{n} X_i$. That is $Y = \sum_{i=1}^{n} X_i$, a sum of n independent Bernoulli random variables.

ii
$$\mathrm{E}(Y) = \mathrm{E}\left(\sum_{i=1}^{n} X_i\right)$$
$$= \sum_{i=1}^{n} \mathrm{E}(X_i)$$
$$= \sum_{i=1}^{n} p \qquad \{X_i \sim \mathrm{B}(1, p) \text{ for all } i \text{ and using } \mathbf{a}\}$$
$$= np$$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$
$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) \qquad \{\text{independence}\}$$
$$= \sum_{i=1}^{n} p(1-p) \qquad \{\text{using } \mathbf{a}\}$$
$$= np(1-p)$$

EXERCISE 29B

1 $X \sim \mathrm{Po}(10)$ and $Y \sim \mathrm{Po}(17)$

a $T = X + Y$

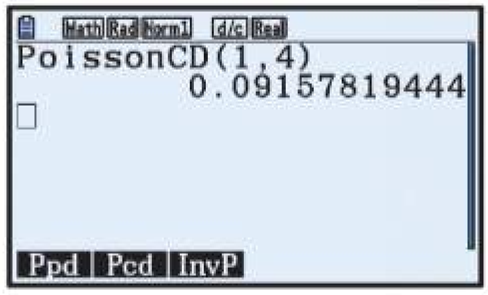
$\therefore \; T \sim \mathrm{Po}(27)$

b P(at least 30 people get on a bus at an interchange)
$= P(T \geqslant 30)$
$= 1 - P(T \leqslant 29)$
$\approx 1 - 0.693\,47$
≈ 0.307

2 $X \sim \text{Po}(3)$ and $(X+Y) \sim \text{Po}(7)$ where X and Y are independent.

a Since X and Y are independent, and $(X+Y) \sim \text{Po}(7)$ and $X \sim \text{Po}(3)$, Y follows a Poisson distribution with rate $7 - 3 = 4$.
That is, $Y \sim \text{Po}(4)$.

b $P(Y \geqslant 2) = 1 - P(Y \leqslant 1)$
$\approx 1 - 0.0916$
≈ 0.908



3 $X \sim \text{Po}(2)$ and $Y \sim \text{Po}(3)$ independently.
$\therefore\ (X+Y) \sim \text{Po}(5)$

a $P(X+Y=4) \approx 0.175$ {using technology}

b $P(Y \leqslant 1 \ \cap\ (X+Y)=4)$
$= P(Y=1 \ \cap\ X=3) + P(Y=0 \ \cap\ X=4)$
$= P(Y=1)\,P(X=3) + P(Y=0)\,P(X=4)$ {independence}

c $P(Y \leqslant 1 \ \cap\ (X+Y)=4)$
$\approx 0.149 \times 0.180 + 0.0498 \times 0.0902$ {using **b** and technology}
≈ 0.0314

d $P(Y \leqslant 1 \mid (X+Y)=4) = \dfrac{P(Y \leqslant 1 \ \cap\ (X+Y)=4)}{P(X+Y=4)}$
$\approx \dfrac{0.0314}{0.175}$ {using **c** and **a**}
≈ 0.179

4 Josie and Callie have 10 hours to collect signatures.
Let J be the number of signatures Josie collects in 10 hours,
and C be the number of signatures Callie collects in 10 hours.

The rate at which Josie collects signatures = 10 signatures per hour
= 100 signatures per 10 hours
$\therefore\ J \sim \text{Po}(100)$

The rate at which Callie collects signatures = 8 signatures per 40 minutes
= 12 signatures per hour
= 120 signatures per 10 hours
$\therefore\ C \sim \text{Po}(120)$

Assuming that J and C are independent, $(J+C) \sim \text{Po}(220)$.

Now, P(collect at least 250 signatures by 6 pm)
$= \text{P}(J+C \geqslant 250)$
$= 1 - \text{P}(J+C \leqslant 249)$
$\approx 1 - 0.9749$
≈ 0.0251

5 $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Po}(\lambda)$ independently.

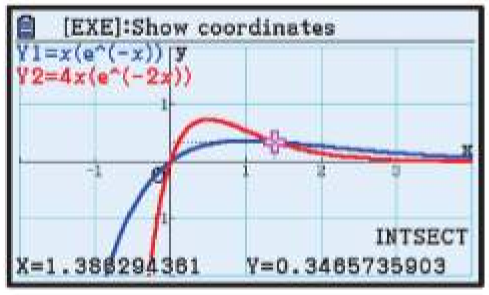
$\therefore \ (X+Y) \sim \text{Po}(2\lambda)$

Now $\text{P}(X=1) = 2\,\text{P}(X+Y=1)$

$$\therefore \ \frac{\lambda^1 e^{-\lambda}}{1!} = 2 \times \frac{(2\lambda)^1 e^{-2\lambda}}{1!}$$

$$\therefore \ \lambda e^{-\lambda} = 4\lambda e^{-2\lambda}$$

Using technology, $\lambda \approx 1.39$. $\{\lambda > 0\}$



6 **a** $X_1 \sim \text{Po}(\lambda_1)$, $X_2 \sim \text{Po}(\lambda_2)$, and $X_3 \sim \text{Po}(\lambda_3)$ independently.

i $(X_1 + X_2) \sim \text{Po}(\lambda_1 + \lambda_2)$

ii $(X_1 + X_2 + X_3)$
$= ([X_1 + X_2] + X_3)$
$\sim \text{Po}([\lambda_1 + \lambda_2] + \lambda_3)$ {using **i** and $X_1 + X_2$ is independent of X_3}
$\sim \text{Po}(\lambda_1 + \lambda_2 + \lambda_3)$

b From **a**, the sum of n independent Poisson random variables appears to be Poisson with rate given by the sum of the rates.

So, we predict that $Y = \sum\limits_{k=1}^{n} X_k \sim \text{Po}\left(\sum\limits_{k=1}^{n} \lambda_k\right)$.

$$\therefore \ Y \sim \text{Po}\left(\sum_{k=1}^{n} \lambda_k\right)$$

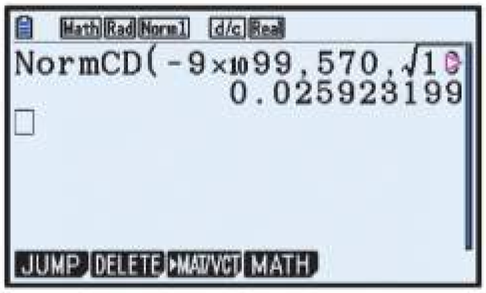
EXERCISE 29C

1 Let the weights of the pears be the independent random variables $X_1, X_2, \ldots, X_5$.

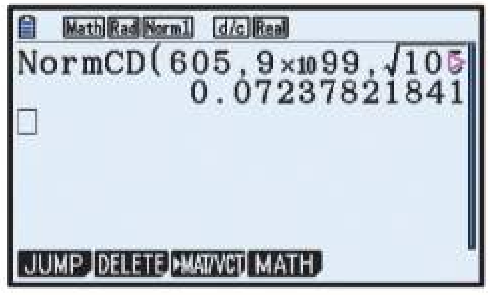
Let the sum of their weights be $Y_5 = \sum\limits_{k=1}^{5} X_k$, where $X_k \sim \text{N}(118, 4.6^2)$, $k = 1, 2, \ldots, 5$.

Now $\text{E}(Y_5) = \sum\limits_{k=1}^{5} \text{E}(X_k)$ and $\text{Var}(Y_5) = \sum\limits_{k=1}^{5} \text{Var}(X_k)$ {independence}
$= 5 \times 118$ $\qquad$ $= 5 \times (4.6)^2$
$= 590$ g $\qquad$ $= 105.8$ g^2

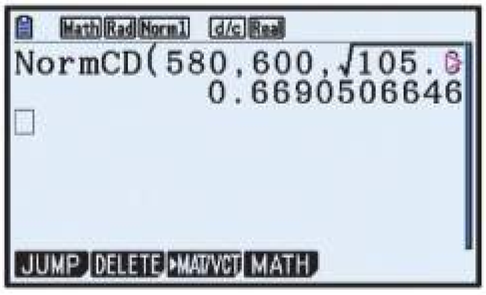
$\therefore \ Y_5 \sim \text{N}(590, 105.8)$

a 

$P(Y_5 < 570) \approx 0.0259$

b 

$P(Y_5 > 605) \approx 0.0724$

c 

$P(580 < Y_5 < 600) \approx 0.669$

2 Let the weights of the adults be the independent random variables A_1, A_2, A_3, A_4.

$\therefore \quad A_k \sim N(81, 11^2), \quad k = 1, 2, 3, 4$

Let the weights of the children be the independent random variables C_1, C_2, C_3.

$\therefore \quad C_j \sim N(48, 4^2), \quad j = 1, 2, 3$

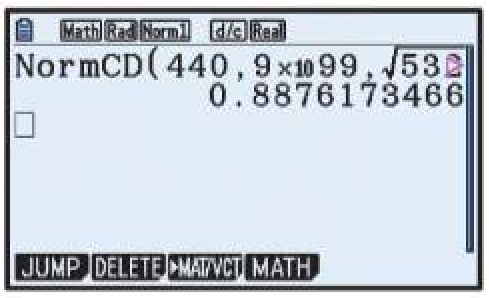
Let the sum of the weights be $S = A_1 + A_2 + A_3 + A_4 + C_1 + C_2 + C_3$

Now $E(S) = E(A_1) + E(A_2) + E(A_3) + E(A_4) + E(C_1) + E(C_2) + E(C_3)$
$= 4(81) + 3(48)$
$= 468$

and $\text{Var}(S) = \text{Var}(A_1) + \text{Var}(A_2) + ... + \text{Var}(C_3)$ {independence}
$= 4(11^2) + 3(4^2)$
$= 532$

$\therefore \quad S \sim N(468, 532)$

a $P(S > 440) \approx 0.888$



b We assume that the weights of each adult and each child are independent of one another.

3 Let C be the amount of black coffee dispensed and let F be the amount of froth.

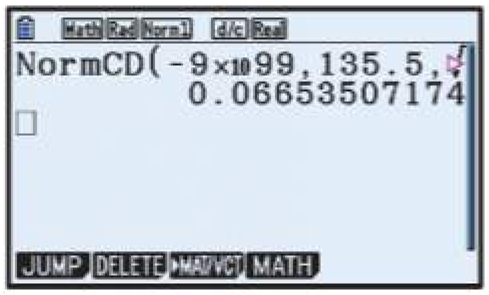
Then $C \sim N(120, 7^2)$ and $F \sim N(28, 4.5^2)$

Let the total volume of the cappuccino be $S = C + F$.

Now $E(S) = E(C) + E(F)$ and $\text{Var}(S) = \text{Var}(C) + \text{Var}(F)$ {independence}
$= 120 + 28$ $\qquad = 49 + 4.5^2$
$= 148$ mL $\qquad = 69.25$ mL^2

$P(S < 135.5) \approx 0.0665 \approx 6.65\%$ which is greater than 1%

$\therefore$ the manager needs to adjust the machine.



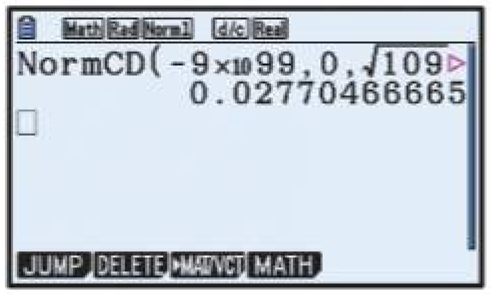
4 $X \sim N(-10, 1)$ and $Y \sim N(25, 25)$

a $E(U) = E(3X + 2Y)$
$= 3\,E(X) + 2\,E(Y)$
$= 3(-10) + 2(25)$
$= 20$

$\text{Var}(U) = 9\,\text{Var}(X) + 4\,\text{Var}(Y)$ {independence}
$= 9(1) + 4(25)$
$= 109$

$\therefore$ standard deviation $= \sqrt{109} \approx 10.4$

b $P(U < 0) \approx 0.0277$ {using technology}



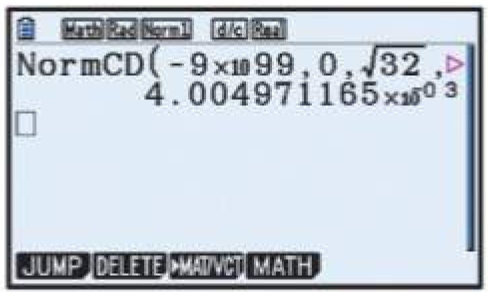
5 $S \sim N(280, 4)$ and $L \sim N(575, 16)$

a We need to find $P(L < 2S)$, which is $P(L - 2S < 0)$.

If $D = L - 2S$, $E(D) = E(L) - 2\,E(S)$ and $\text{Var}(D) = \text{Var}(L) + 4\,\text{Var}(S)$
$E(D) = 575 - 2 \times 280 = 15$
$\text{Var}(D) = 16 + 4 \times 4$ {independence} $= 32$

$\therefore$ $D \sim N(15, 32)$

Using technology, $P(D < 0) \approx 0.004\,00$



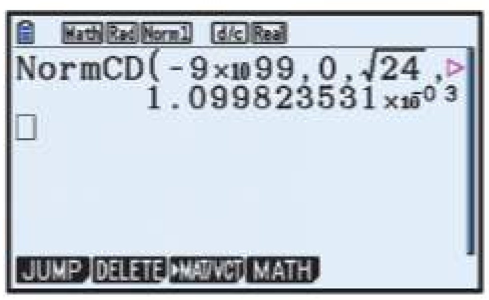
b We need to find $P(L < S_1 + S_2) = P(L - S_1 - S_2 < 0)$

Now $E(L - S_1 - S_2)$
$= E(L) - E(S_1) - E(S_2)$
$= 575 - 280 - 280$
$= 15$

and $\text{Var}(L - S_1 - S_2) = \text{Var}(L) + \text{Var}(S_1) + \text{Var}(S_2)$
$= 16 + 4 + 4$ {independence}
$= 24$

$\therefore$ $L - S_1 - S_2 \sim N(15, 24)$

Using technology, $P(L - S_1 - S_2 < 0) \approx 0.001\,10$



6 **a** $S \sim \text{N}(32, 5)$ and $L \sim \text{N}(92, 15)$

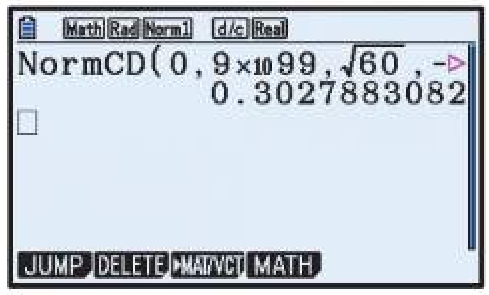
We need to find $\text{P}(L > 3S) = \text{P}(L - 3S > 0)$

Now $\text{E}(L - 3S) = \text{E}(L) - 3\,\text{E}(S)$
$= 92 - 3 \times 32$
$= -4$

and $\text{Var}(L - 3S)$
$= \text{Var}(L) + 9\,\text{Var}(S)$ {independence}
$= 15 + 9 \times 5$
$= 60$

$\therefore$ $(L - 3S) \sim \text{N}(-4, 60)$

Using technology, $\text{P}(L - 3S > 0) \approx 0.303$



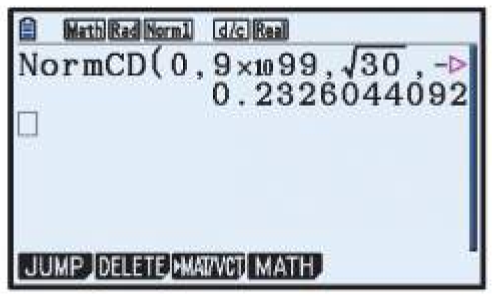
b We need to find $\text{P}(L > S_1 + S_2 + S_3) = \text{P}(L - S_1 - S_2 - S_3 > 0)$

Now $\text{E}(L - S_1 - S_2 - S_3) = \text{E}(L) - \text{E}(S_1) - \text{E}(S_2) - \text{E}(S_3)$
$= 92 - 32 - 32 - 32$
$= -4$

and $\text{Var}(L - S_1 - S_2 - S_3) = \text{Var}(L) + \text{Var}(S_1) + \text{Var}(S_2) + \text{Var}(S_3)$ {independence}
$= 15 + 3 \times 5$
$= 30$

$\therefore$ $(L - S_1 - S_2 - S_3) \sim \text{N}(-4, 30)$

Using technology, $\text{P}(L - S_1 - S_2 - S_3 > 0) \approx 0.233$



INVESTIGATION 2 THE DISTRIBUTION OF SAMPLE MEANS

1 The population X has mean μ and standard deviation σ.

For the random sample $\{X_1, ..., X_n\}$ from the population, $\text{E}(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$ for all $i = 1, ..., n$.

a **i** $\text{E}(\overline{X}_n) = \text{E}\left(\dfrac{1}{n}\sum_{i=1}^{n} X_i\right)$
$= \dfrac{1}{n}\sum_{i=1}^{n} \text{E}(X_i)$
$= \dfrac{1}{n}\sum_{i=1}^{n} \mu$
$= \dfrac{1}{n} \times n\mu$
$= \mu$

ii $\text{Var}(\overline{X}_n) = \text{Var}\left(\dfrac{1}{n}\sum_{i=1}^{n} X_i\right)$
$= \left(\dfrac{1}{n}\right)^2 \sum_{i=1}^{n} \text{Var}(X_i)$ {independence}
$= \dfrac{1}{n^2}\sum_{i=1}^{n} \sigma^2$
$= \dfrac{1}{n^2} \times n\sigma^2$
$= \dfrac{\sigma^2}{n}$

iii $\sigma(\overline{X}_n) = \sqrt{\text{Var}(\overline{X}_n)}$

$= \sqrt{\frac{\sigma^2}{n}}$ {using **ii**}

$= \frac{\sigma}{\sqrt{n}}$

b If the population is normally distributed, then each X_i is also normally distributed. $\overline{X}_n$ would therefore be a linear combination of normal random variables (each with coefficient $\frac{1}{n}$).

$\therefore$ $\overline{X}_n$ is itself normally distributed.

$\therefore$ $\overline{X}_n \sim \text{N}\left(\mu, \frac{\sigma^2}{n}\right)$.

2 **a** As the sample size n increases, the distribution becomes narrower and narrower, but remains bell-shaped and symmetrical about 0.

b **Normal distribution, $\mu = 0$, $\sigma = 1$**

n	*Mean of* $\overline{X}_n$	*Standard deviation of* $\overline{X}_n$	$\frac{\sigma}{\sqrt{n}}$
4	≈ 0.0453	≈ 0.4969	$\frac{1}{2} = 0.5$
9	≈ 0.0253	≈ 0.3141	$\frac{1}{3} \approx 0.333$
16	≈ 0.0058	≈ 0.2555	$\frac{1}{4} = 0.25$
25	≈ 0.001	≈ 0.1975	$\frac{1}{5} = 0.2$
100	≈ -0.0063	≈ 0.1015	$\frac{1}{10} = 0.1$

c The estimated mean of $\overline{X}_n \approx \mu = \text{E}(\overline{X}_n)$ and the estimated standard deviation of $\overline{X}_n \approx \frac{\sigma}{\sqrt{n}} = \sigma(\overline{X}_n)$.

The distribution of $\overline{X}_n$ stays bell-shaped and symmetrical, which suggests that $\overline{X}_n$ is normally distributed. This is consistent with the conclusion in **1 b**.

3 **a** **Negatively skewed, $\mu = 0$, $\sigma = 1$**

n	*Mean of* $\overline{X}_n$	*Standard deviation of* $\overline{X}_n$	$\frac{\sigma}{\sqrt{n}}$
4	≈ 0.0045	≈ 0.4973	$\frac{1}{2} = 0.5$
9	≈ -0.0068	≈ 0.3263	$\frac{1}{3} \approx 0.333$
16	≈ 0.0058	≈ 0.2518	$\frac{1}{4} = 0.25$
25	≈ 0.0014	≈ 0.2014	$\frac{1}{5} = 0.2$
100	≈ 0.0027	≈ 0.1026	$\frac{1}{10} = 0.1$

b For large values of n, the distribution of $\overline{X}_n$ is bell-shaped and symmetrical about 0, which does *not* resemble the negatively skewed population distribution.

c With the negatively skewed population we make similar observations regarding the mean and standard deviation of $\overline{X}_n$, and the shape of the distribution of $\overline{X}_n$ for large values of n, as for the normally distributed population.

However for the negatively skewed population, the distribution of $\overline{X}_n$ does not resemble the population. It becomes more bell-shaped as n increases.

EXERCISE 29D

1 X has mean $\mu = 70$ and standard deviation $\sigma = 12$.

a Mean of $\overline{X}_{64}$ = mean of X
$= \mu$
$= 70$

$$\text{Standard deviation of } \overline{X}_{64} = \frac{\sigma}{\sqrt{64}} = \frac{12}{8} = \tfrac{3}{2}$$

b Yes, 64 should be a sufficiently large sample for the Central Limit Theorem to apply. So we can assume that $\overline{X}_{64}$ is approximately normally distributed.

2 X is normally distributed with mean $\mu = 50$ and standard deviation $\sigma = 9$.

a **i** Mean of $\overline{X}_5$ = mean of X
$= \mu$
$= 50$

ii $$\text{Standard deviation of } \overline{X}_5 = \frac{\sigma}{\sqrt{5}} = \frac{9}{\sqrt{5}}$$

b Since X is normally distributed, $\overline{X}_5$ is also normally distributed.

3 X has mean $\mu = 80$ and standard deviation $\sigma = 9$.

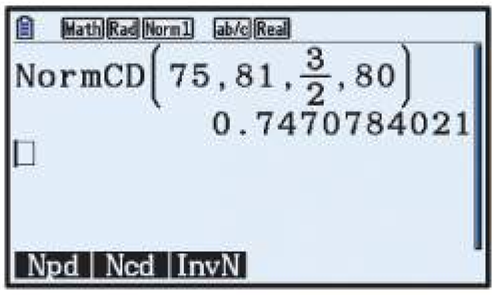
a Mean of $\overline{X}_{36}$ = mean of X
$= \mu$
$= 80$

$$\text{Standard deviation of } \overline{X}_{36} = \frac{\sigma}{\sqrt{36}} = \frac{9}{6} = \tfrac{3}{2}$$

b Yes, 36 should be a sufficiently large sample for the Central Limit Theorem to apply. So we can assume that $\overline{X}_{36}$ is approximately normally distributed.

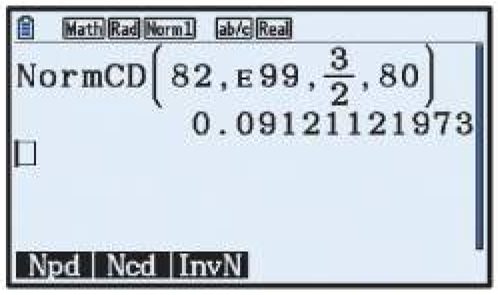
c Using technology:

i



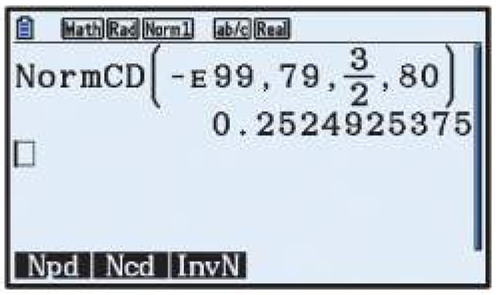
$P(75 \leqslant \overline{X}_{36} \leqslant 81) \approx 0.747$

ii



$P(\overline{X}_{36} > 82) \approx 0.0912$

iii



$P(\overline{X}_{36} < 79) \approx 0.252$

d No, we cannot find $P(70 < X < 85)$ because we do not know the distribution of X.

4 Y has mean $\mu = 3.6$ and standard deviation $\sigma = 0.7$.

a Mean of $\overline{Y}_{32}$ = mean of Y

$$= \mu$$
$$= 3.6$$

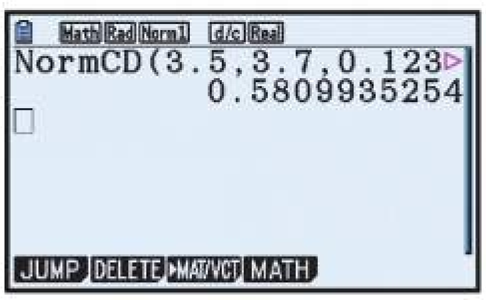
Standard deviation of $\overline{Y}_{32} = \frac{\sigma}{\sqrt{32}}$

$$= \frac{0.7}{\sqrt{32}}$$
$$\approx 0.124$$

b Yes, 32 should be a sufficiently large sample for the Central Limit Theorem to apply. So we can assume that $\overline{Y}_{32}$ is approximately normally distributed.

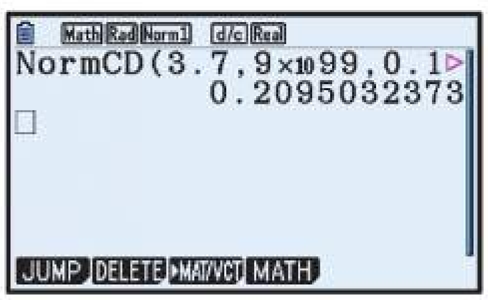
c Using technology:

i



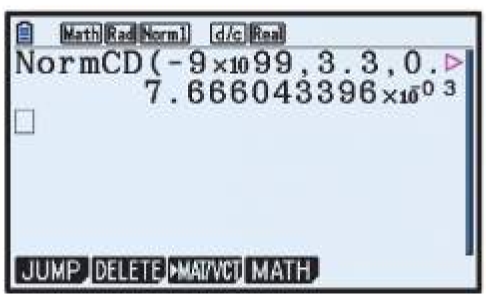
$P(3.5 < \overline{Y}_{32} < 3.7) \approx 0.581$

ii



$P(\overline{Y}_{32} > 3.7) \approx 0.210$

iii



$P(\overline{Y}_{32} < 3.3) \approx 0.007\,67$

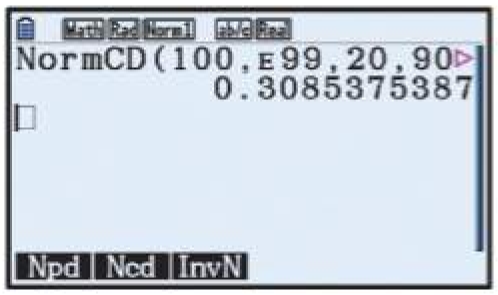
5 X is normally distributed with mean $\mu = 90$ and standard deviation $\sigma = 20$.

a Since X is normally distributed, $\overline{X}_5$ is also normally distributed.

b X has a larger standard deviation than $\overline{X}_5$, so X should have a greater proportion of scores over 100.

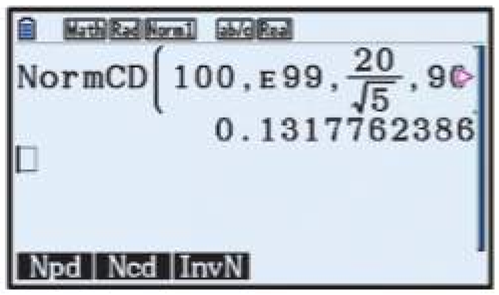
c $\overline{X}_5$ is normally distributed with mean 90 and standard deviation $\frac{20}{\sqrt{5}}$. Using technology:

i



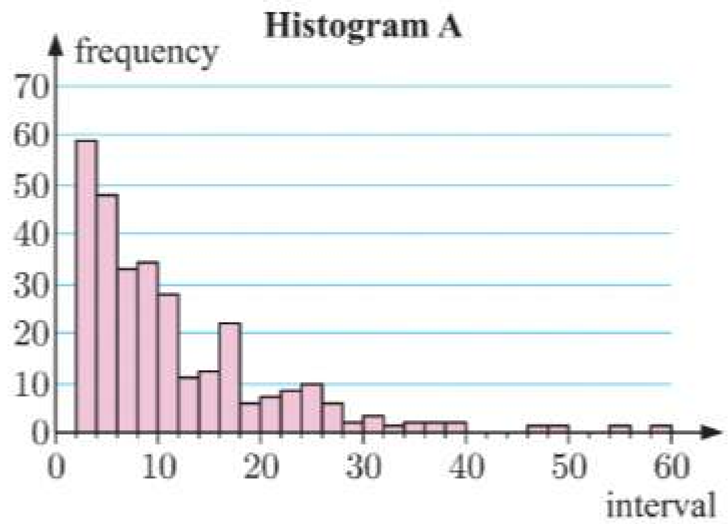
$\text{P}(X > 100) \approx 0.309$

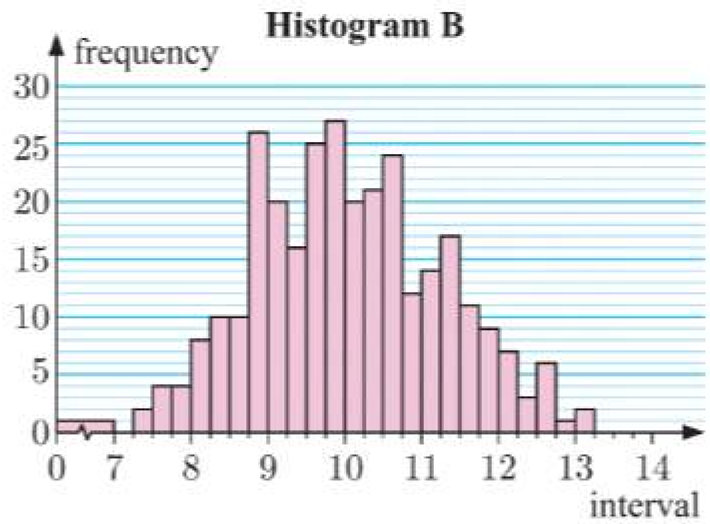
ii



$\text{P}(\overline{X}_5 > 100) \approx 0.132$

6





a Histogram B more closely resembles the normal distribution, and it has a much smaller spread than histogram A. So histogram B is from $\overline{X}_{64}$.

b Using histogram B,
$$\begin{aligned} \text{P}(\overline{X}_{64} < 9) &\approx \frac{1+0+2+4+4+8+10+10+26}{300} \\ &\approx \frac{65}{300} \\ &\approx 0.217 \end{aligned}$$

c X has mean $\mu = 10$ and standard deviation $\sigma = 10$.

$$\begin{aligned} \text{Mean of } \overline{X}_{64} &= \text{mean of } X \\ &= \mu \\ &= 10 \end{aligned}$$

$$\begin{aligned} \text{Standard deviation of } \overline{X}_{64} &= \frac{\sigma}{\sqrt{64}} \\ &= \frac{10}{8} \\ &= 1.25 \end{aligned}$$

Now, $\text{P}(\overline{X}_{64}$ is one standard deviation from the mean)
$$\begin{aligned} &= \text{P}(10 - 1.25 < \overline{X}_{64} < 10 + 1.25) \\ &= \text{P}(8.75 < \overline{X}_{64} < 11.25) \end{aligned}$$

Using histogram B, $\quad \text{P}(8.75 < \overline{X}_{64} < 11.25)$

$$\approx \frac{26+20+16+25+27+20+21+24+12+14}{300}$$

$$\approx \frac{205}{300}$$

$$\approx 0.683$$

Alternatively, since the sample size is $n = 64$, we can apply the Central Limit Theorem.

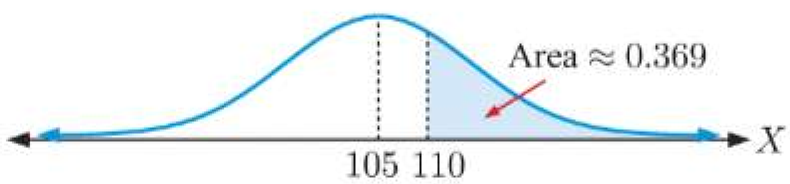
$\therefore$ $\overline{X}_{64}$ is approximately normally distributed.

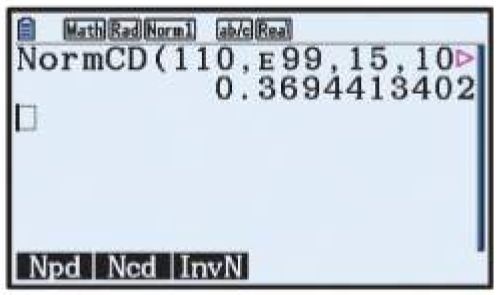
We know that for a normal distribution, the probability that a value is one standard deviation from the mean is ≈ 0.6826.

$\therefore$ our estimate using histogram B is very good.

7 **a** The IQ X of an individual 17 year old girl is normally distributed with mean 105 and standard deviation 15.

$\therefore$ $\text{P}(X > 110) \approx 0.369$

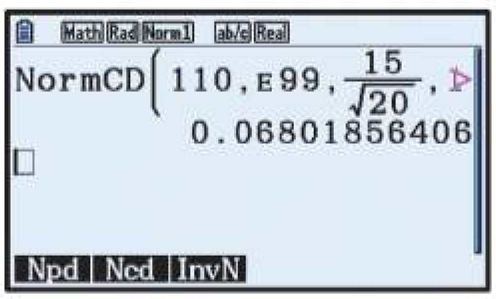


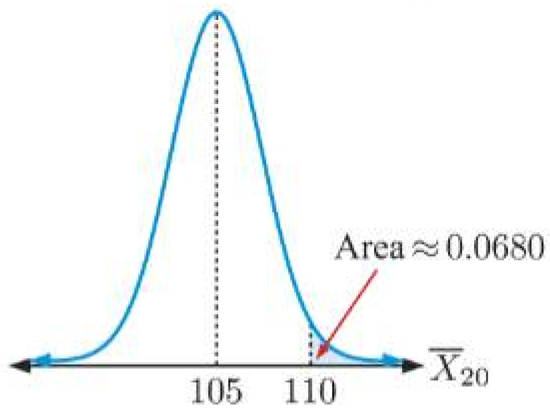


The probability that an individual 17 year old girl has an IQ of more than 110 is approximately 0.369.

b The average IQ $\overline{X}_{20}$ of a class of twenty 17 year old girls is normally distributed with mean 105 and standard deviation $\frac{15}{\sqrt{20}}$.

$\therefore$ $\text{P}(\overline{X}_{20} > 110) \approx 0.0680$

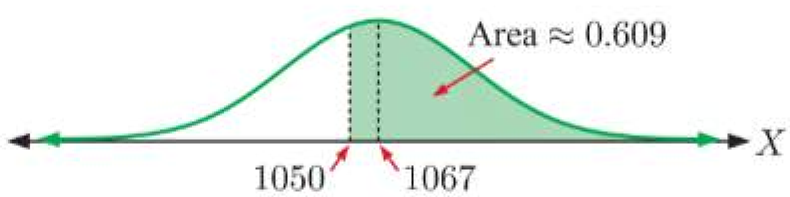


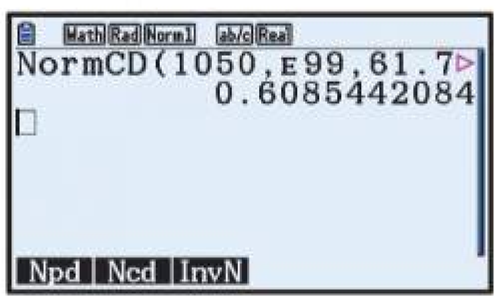


The probability that the mean IQ of a class of twenty 17 year old girls is greater than 110 is approximately 0.0680.

8 **a** The energy content X of an individual fruit bar is normally distributed with mean 1067 kJ and standard deviation 61.7 kJ.

$\therefore$ $\text{P}(X > 1050) \approx 0.609$

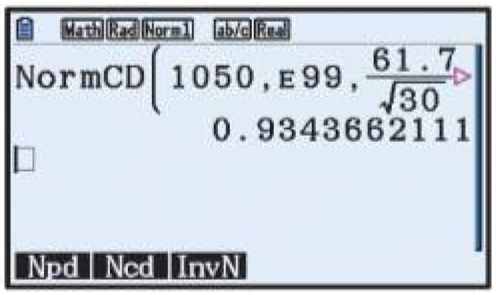


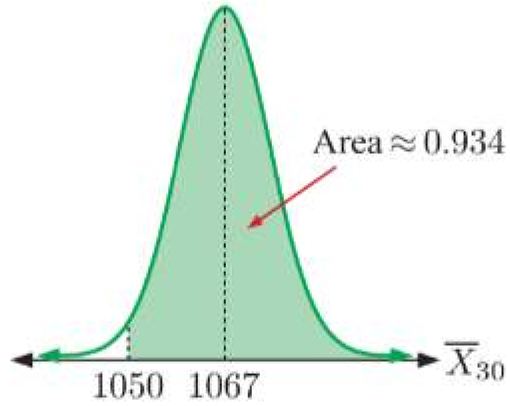


The probability that an individual fruit bar has more than 1050 kJ of energy is approximately 0.609.

b The average energy content $\overline{X}_{30}$ of a sample of 30 fruit bars is normally distributed with mean 1067 kJ and standard deviation $\frac{61.7}{\sqrt{30}}$ kJ.

$\therefore$ $P(\overline{X}_{30} > 1050) \approx 0.934$



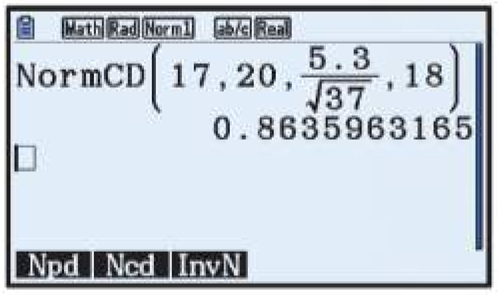


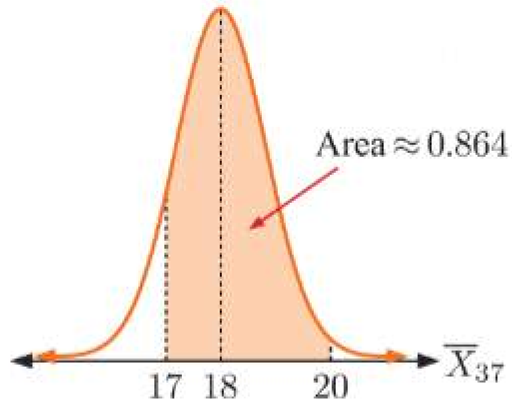
The probability that the mean energy content of a sample of 30 fruit bars is more than 1050 kJ is approximately 0.934.

9 Since the sample is of size $n = 37$, we apply the Central Limit Theorem.

$\therefore$ $\overline{X}_{37}$ is approximately normally distributed with mean 18 minutes and standard deviation $\frac{5.3}{\sqrt{37}}$ minutes.

$\therefore$ $P(17 < \overline{X}_{37} < 20) \approx 0.864$



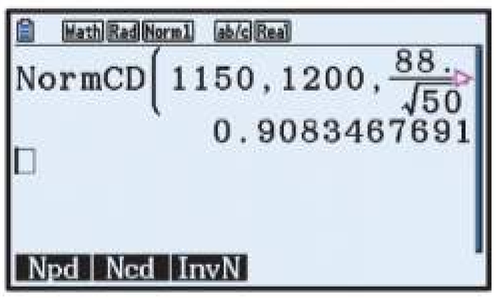


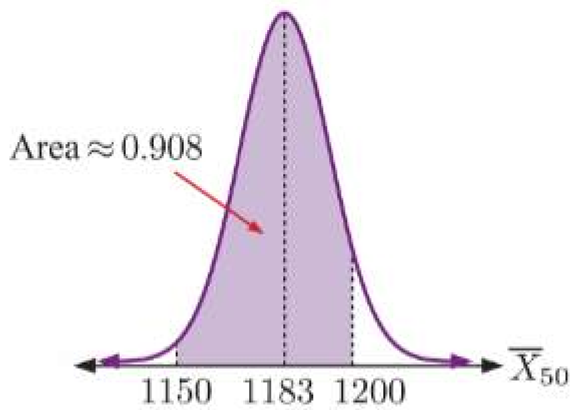
The probability that, in a sample of 37 customers, the mean stay in the shop is between 17 and 20 minutes is approximately 0.864.

10 Since the sample is of size $n = 50$, we apply the Central Limit Theorem.

$\therefore$ $\overline{X}_{50}$ is approximately normally distributed with mean 1183 mg and standard deviation $\frac{88.6}{\sqrt{50}}$ mg.

$\therefore$ $P(1150 < \overline{X}_{50} < 1200) \approx 0.908$

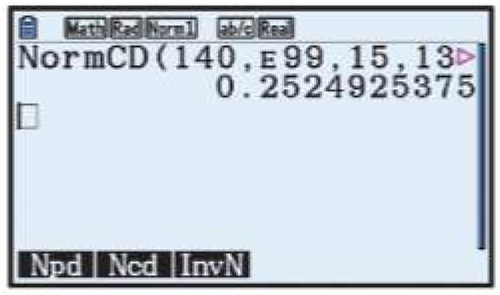


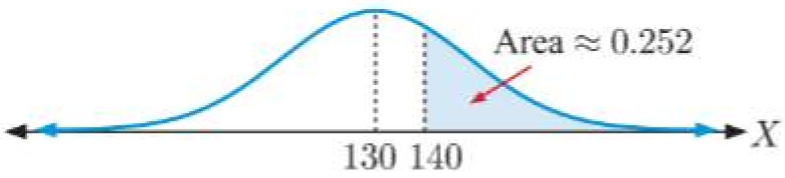


The probability that the mean sodium content per box for a sample of 50 boxes lies between 1150 mg and 1200 mg is approximately 0.908.

11 **a** The weight X of an individual orange is normally distributed with mean 130 g and standard deviation 15 g.

$\therefore$ $P(X > 140) \approx 0.252$

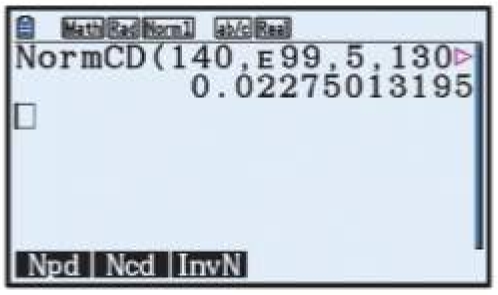


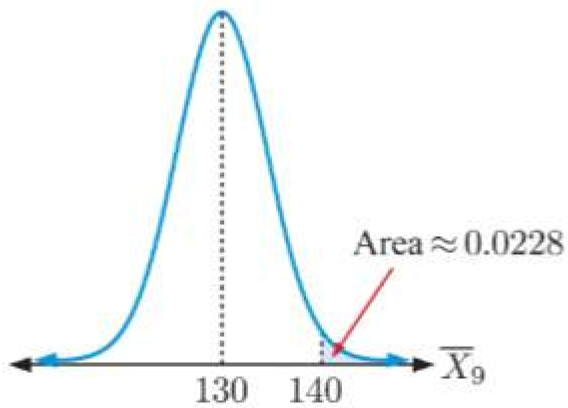


The probability that an individual orange weighs more than 140 g is approximately 0.252.

b The average weight $\overline{X}_9$ of a bag of 9 oranges is normally distributed with mean 130 g and standard deviation $\frac{15}{\sqrt{9}} = 5$ g.

$\therefore$ $P(\overline{X}_9 > 140) \approx 0.0228$

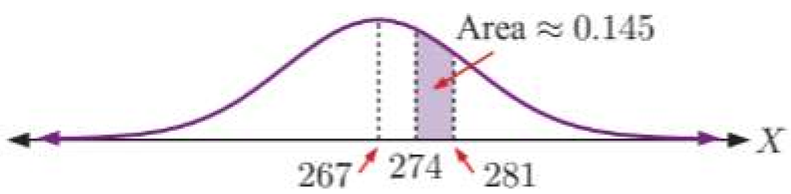


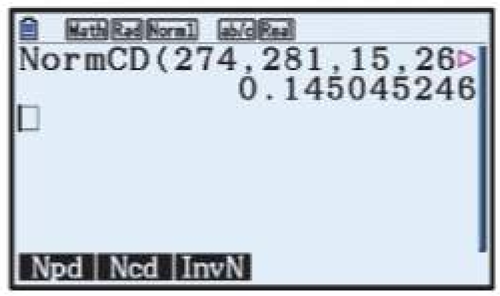


The probability that the average weight of oranges in a bag is more than 140 g is approximately 0.0228.

12 **a** The duration X of an individual pregnancy is normally distributed with mean 267 days and standard deviation 15 days.

$$\begin{aligned} \text{P(overdue by between 1 and 2 weeks)} &= \text{P(overdue by between 7 and 14 days)} \\ &= P(274 < X < 281) \\ &\approx 0.145 \end{aligned}$$



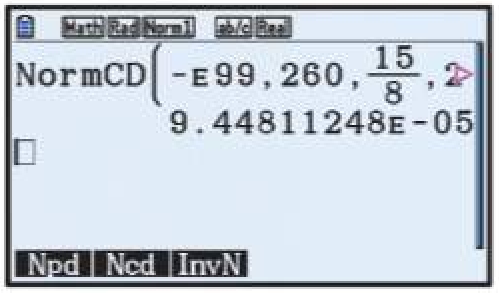


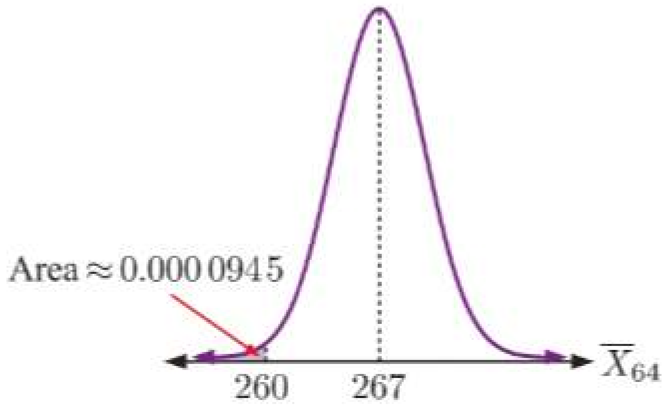
$\therefore$ about 14.5% of pregnancies will be overdue by between 1 and 2 weeks.

b **i** The average duration $\overline{X}_{64}$ of the group of 64 pregnant women is normally distributed with mean 267 days and standard deviation $\frac{15}{\sqrt{64}} = \frac{15}{8} = 1.875$ days.

ii A pregnancy is premature by at least one week if its duration is $\leqslant 260$ days.

$P(\overline{X}_{64} \leqslant 260) \approx 0.000\,094\,5$





The probability that the mean duration of these 64 patients' pregnancies will be premature by at least one week is approximately $0.000\,094\,5$.

c If X is not normally distributed, then the answer to **a** is not acceptable as the answer in **a** is calculated using a normal distribution. In part **b** however, we are considering the sample mean $\overline{X}_{64}$ which does follow an approximately normal distribution since the sample size of $n = 64$ is sufficiently large for the Central Limit Theorem to apply.

INVESTIGATION 3 APPLYING THE CENTRAL LIMIT THEOREM TO THE BINOMIAL DISTRIBUTION

$X \sim B(n, p)$ and $X = \sum_{i=1}^{n} X_i$ where $X_i \sim B(1, p)$ independently.

$\widehat{p} = \dfrac{X}{n}$

1 X is the sum of n independent and identically distributed random variables.

So $\widehat{p}$ has the form $\dfrac{1}{n}\sum_{i=1}^{n} X_i$, which is the same as the sample mean $\overline{X}_n$.

2
$$\begin{aligned} E(\widehat{p}) &= E\left(\frac{X}{n}\right) \\ &= \frac{1}{n}E(X) \\ &= \frac{1}{n} \times np \\ &= p \end{aligned} \qquad \begin{aligned} \text{Var}(\widehat{p}) &= \text{Var}\left(\frac{X}{n}\right) \\ &= \frac{1}{n^2}\text{Var}(X) \\ &= \frac{1}{n^2} \times np(1-p) \\ &= \frac{p(1-p)}{n} \end{aligned}$$

3 **a** For large values of n, by the Central Limit Theorem, $\widehat{p}$ is approximately normally distributed with mean p and standard deviation $\sqrt{\dfrac{p(1-p)}{n}}$.

$\therefore \quad \widehat{p} \sim N\left(p, \dfrac{p(1-p)}{n}\right)$

b From **a**, for large values of n $\widehat{p} \sim N\left(p, \dfrac{p(1-p)}{n}\right)$.

So, $X = n\widehat{p}$ is also approximately normally distributed with mean np and standard deviation $\sqrt{np(1-p)}$.

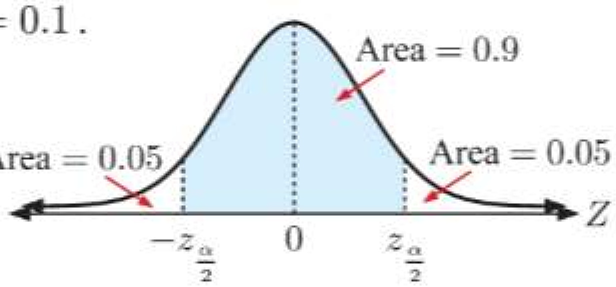
$\therefore \quad X \sim N(np, \ np(1-p))$

ACTIVITY PROPERTIES OF CONFIDENCE INTERVALS

1

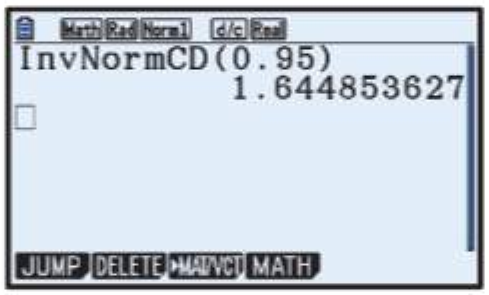
Confidence level	α	$z_{\frac{\alpha}{2}}$
90%	0.1	1.645
95%	0.05	1.960
98%	0.02	2.326
99%	0.01	2.576

For a 90% confidence level, $\alpha = 1 - 0.9 = 0.1$.

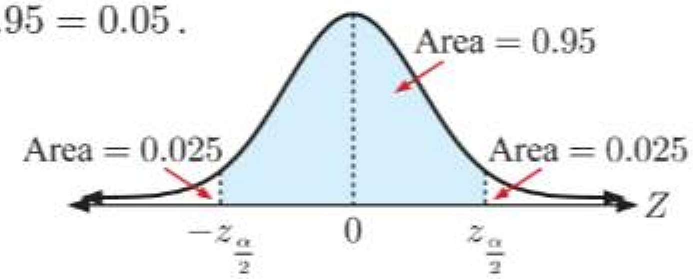


$\therefore\ \mathrm{P}(Z \leqslant z_{\frac{\alpha}{2}}) = 0.95$

$\therefore\ z_{\frac{\alpha}{2}} \approx 1.645$

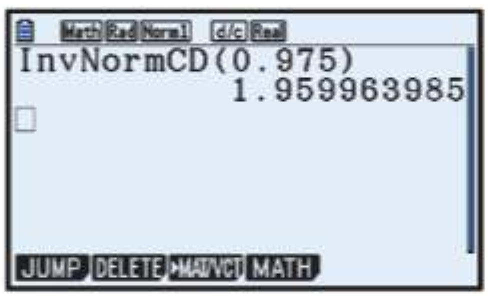


For a 95% confidence level, $\alpha = 1 - 0.95 = 0.05$.

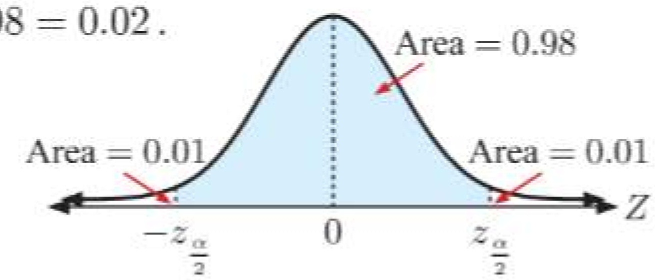


$\therefore\ \mathrm{P}(Z \leqslant z_{\frac{\alpha}{2}}) = 0.975$

$\therefore\ z_{\frac{\alpha}{2}} \approx 1.960$

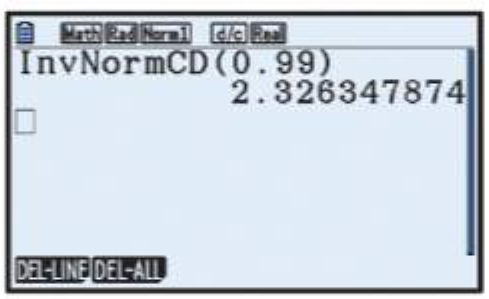


For a 98% confidence level, $\alpha = 1 - 0.98 = 0.02$.

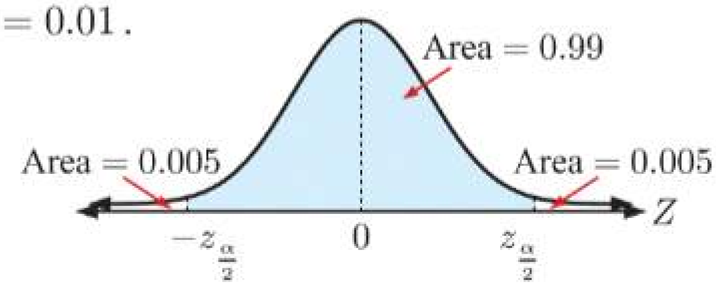


$\therefore\ \mathrm{P}(Z \leqslant z_{\frac{\alpha}{2}}) = 0.99$

$\therefore\ z_{\frac{\alpha}{2}} \approx 2.326$

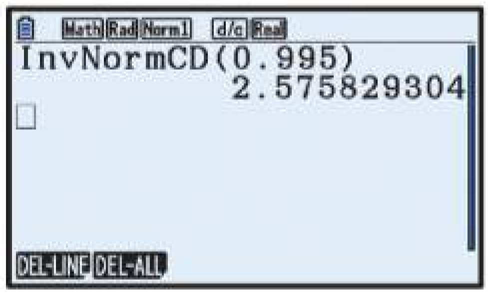


For a 99% confidence level, $\alpha = 1 - 0.99 = 0.01$.



$\therefore \ \mathrm{P}(Z \leqslant z_{\frac{\alpha}{2}}) = 0.995$

$\therefore \ z_{\frac{\alpha}{2}} \approx 2.576$



2 **a** For $n = 50$, the 95% confidence interval is

$$10 - 1.960 \times \frac{2}{\sqrt{50}} \leqslant \mu \leqslant 10 + 1.960 \times \frac{2}{\sqrt{50}}$$

$$\therefore \ 9.446 \leqslant \mu \leqslant 10.554$$

For $n = 100$, the 95% confidence interval is

$$10 - 1.960 \times \frac{2}{\sqrt{100}} \leqslant \mu \leqslant 10 + 1.960 \times \frac{2}{\sqrt{100}}$$

$$\therefore \ 9.608 \leqslant \mu \leqslant 10.392$$

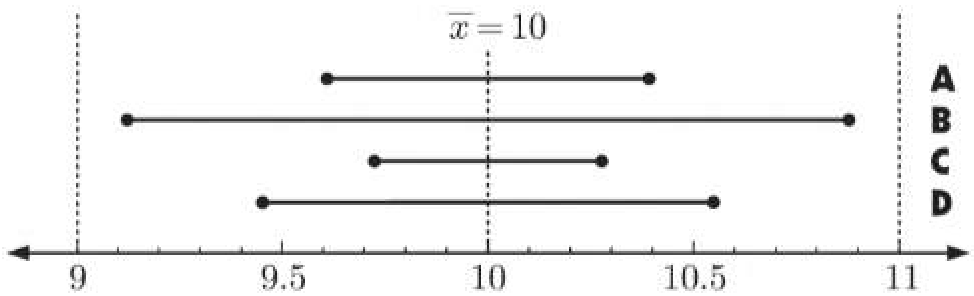
For $n = 200$, the 95% confidence interval is

$$10 - 1.960 \times \frac{2}{\sqrt{200}} \leqslant \mu \leqslant 10 + 1.960 \times \frac{2}{\sqrt{200}}$$

$$\therefore \ 9.723 \leqslant \mu \leqslant 10.277$$

n	*95% confidence interval*
20	$9.123 \leqslant \mu \leqslant 10.877$
50	$9.446 \leqslant \mu \leqslant 10.554$
100	$9.608 \leqslant \mu \leqslant 10.392$
200	$9.723 \leqslant \mu \leqslant 10.277$

b



A corresponds to $n = 100$.
B corresponds to $n = 20$.
C corresponds to $n = 200$.
D corresponds to $n = 50$.

c Increasing the sample size n produces confidence intervals of decreasing width.

EXERCISE 29E.1

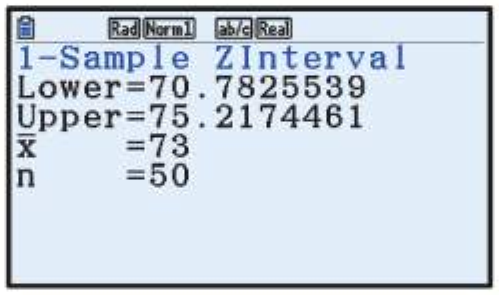
1 We are given that $\overline{x} = 73$, $\sigma = 8$, and $n = 50$.

The 95% confidence interval is $\overline{x} - 1.96\,\dfrac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\,\dfrac{\sigma}{\sqrt{n}}$

$$\therefore\ 73 - 1.96 \times \frac{8}{\sqrt{50}} \leqslant \mu \leqslant 73 + 1.96 \times \frac{8}{\sqrt{50}}$$

$$\therefore\ 70.8 \leqslant \mu \leqslant 75.2$$

Check using technology:

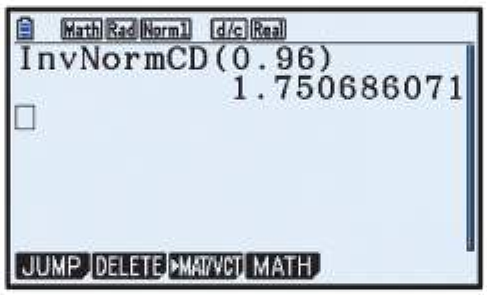


So, we are 95% confident that the population mean lies between 70.8 and 75.2.

2 **a** For a confidence level of 92%, we need to find a for which $\mathrm{P}(-a \leqslant Z \leqslant a) = 0.92$.
Using the symmetry of the normal distribution, the statement reduces to
$\mathrm{P}(Z \leqslant -a) = 0.04$ or $\mathrm{P}(Z \leqslant a) = 0.96$.

Using technology, we find that $a \approx 1.75$.

$\therefore\ z_{\frac{\alpha}{2}} \approx 1.75$



b We are given that $\overline{x} = 81.6$, $\sigma = 11$, and $n = 106$.

The 92% confidence interval is $\overline{x} - 1.75\,\dfrac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.75\,\dfrac{\sigma}{\sqrt{n}}$

$$\therefore\ 81.6 - 1.75 \times \frac{11}{\sqrt{106}} \leqslant \mu \leqslant 81.6 + 1.75 \times \frac{11}{\sqrt{106}}$$

$$\therefore\ 79.7 \leqslant \mu \leqslant 83.5$$

So, we are 92% confident that the population mean lies between 79.7 and 83.5.

3 We are given that $\overline{x} = 42.5$, $\sigma = 10$, and $n = 80$.

a **i** The 90% confidence interval is $\overline{x} - 1.645\,\dfrac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.645\,\dfrac{\sigma}{\sqrt{n}}$

$$\therefore\ 42.5 - 1.645 \times \frac{10}{\sqrt{80}} \leqslant \mu \leqslant 42.5 + 1.645 \times \frac{10}{\sqrt{80}}$$

$$\therefore\ 40.7 \leqslant \mu \leqslant 44.3$$

So, we are 90% confident that the population mean lies between 40.7 and 44.3.

ii The 95% confidence interval is $\overline{x} - 1.96\,\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\,\frac{\sigma}{\sqrt{n}}$

$$\therefore \quad 42.5 - 1.96 \times \frac{10}{\sqrt{80}} \leqslant \mu \leqslant 42.5 + 1.96 \times \frac{10}{\sqrt{80}}$$

$$\therefore \quad 40.3 \leqslant \mu \leqslant 44.7$$

So, we are 95% confident that the population mean lies between 40.3 and 44.7.

iii The 99% confidence interval is $\overline{x} - 2.576\,\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 2.576\,\frac{\sigma}{\sqrt{n}}$

$$\therefore \quad 42.5 - 2.576 \times \frac{10}{\sqrt{80}} \leqslant \mu \leqslant 42.5 + 2.576 \times \frac{10}{\sqrt{80}}$$

$$\therefore \quad 39.6 \leqslant \mu \leqslant 45.4$$

So, we are 99% confident that the population mean lies between 39.6 and 45.4.

b By increasing the confidence level, the margin of error is also increased.

4 **a** We are given that the 95% confidence interval for μ is $27.6 \leqslant \mu \leqslant 31.8$.

The sample mean $\overline{x}$ is the exact centre of the confidence interval.

Now, $\frac{27.6 + 31.8}{2} = 29.7$

$\therefore$ the sample mean $\overline{x} = 29.7$

b For the 95% confidence interval, the margin of error is

$$1.96\,\frac{\sigma}{\sqrt{n}} = 29.7 - 27.6 \quad (\text{or } 31.8 - 29.7)$$
$$= 2.1$$

$$\therefore \quad \frac{\sigma}{\sqrt{n}} = \frac{2.1}{1.96}$$

For a 99% confidence interval, the margin of error will be

$$2.576\,\frac{\sigma}{\sqrt{n}} = 2.576 \times \frac{2.1}{1.96}$$
$$= 2.76$$

So, the 99% confidence interval for μ is $\overline{x} - 2.576\,\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 2.576\,\frac{\sigma}{\sqrt{n}}$

$$\therefore \quad 29.7 - 2.76 \leqslant \mu \leqslant 29.7 + 2.76$$

$$\therefore \quad 26.94 \leqslant \mu \leqslant 32.46$$

So, we are 99% confident that the population mean lies between 26.9 and 32.5.

5 **a** We are given that $\overline{x} = 242.6$ mg, $\sigma = 7.3$ mg, and $n = 60$.

The 95% confidence interval is $\overline{x} - 1.96\,\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\,\frac{\sigma}{\sqrt{n}}$

$$\therefore \quad 242.6 - 1.96 \times \frac{7.3}{\sqrt{60}} \leqslant \mu \leqslant 242.6 + 1.96 \times \frac{7.3}{\sqrt{60}}$$

$$\therefore \quad 240.75 \leqslant \mu \leqslant 244.45 \text{ mg}$$

So, we are 95% confident that the mean amount of preservative added to the cans lies between 240.75 mg and 244.45 mg.

b Width of confidence interval $= 2 \times \text{margin of error}$

$$= 2 \times 1.96 \times \frac{7.3}{\sqrt{60}}$$

$$\approx 3.7 \text{ mg}$$

c The quality controller could obtain a 95% confidence interval with a smaller width by increasing the sample size. (The larger the sample size, the smaller the margin of error and thus the narrower the width of the confidence interval.)

EXERCISE 29E.2

1 **a** The required sample size $n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 34}{5}\right)^2$$

$$\approx 710.5$$

Rounding up, a sample size of 711 is required.

b The required sample size $n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 34}{1}\right)^2$$

$$\approx 17\,763.6$$

Rounding up to 3 significant figures, a sample size of 17 800 is required.

c The required sample size $n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 34}{0.1}\right)^2$$

$$\approx 1\,776\,355.8$$

Rounding up to 3 significant figures, a sample size of 1 780 000 is required.

2 The required sample size $n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 250.5}{140}\right)^2$$

$$\approx 49.2$$

Rounding up, a sample size of 50 crayfish is required.

3 **a** The required sample size $n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 17.8}{6}\right)^2$$

$$\approx 135.2$$

Rounding up, a sample size of 136 packets is required.

b The required sample size $n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 17.8}{4}\right)^2$$

$$\approx 304.3$$

Rounding up, a sample size of 305 packets is required.

4 **a** If a sample size n has a 95% confidence interval width $w = 2 \times 1.96\,\dfrac{\sigma}{\sqrt{n}}$,

then a sample size $2n$ has 95% confidence interval width $w' = 2 \times 1.96\,\dfrac{\sigma}{\sqrt{2n}}$

$$\therefore\ w' = 2 \times 1.96\,\frac{\sigma}{\sqrt{2}\sqrt{n}} = \frac{w}{\sqrt{2}}$$

Thus w is divided by $\sqrt{2}$.
So the width will decrease by a factor of $\sqrt{2}$.

b If a 95% confidence interval width w requires a sample size $n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2$,

then a 95% confidence interval width $\dfrac{w}{2}$ requires a sample size

$$n' = \left(\frac{2 \times 1.96\sigma}{\frac{w}{2}}\right)^2 = 4\left(\frac{2 \times 1.96\sigma}{w}\right)^2 = 4n$$

Thus the sample size n needs to be 4 times as large.

5 We are given that $\overline{x} = 1001$ mL, $\sigma = 2.3$ mL, and $n = 30$.

a The 95% confidence interval is $\overline{x} - 1.96\,\dfrac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\,\dfrac{\sigma}{\sqrt{n}}$

$$\therefore\ 1001 - 1.96 \times \frac{2.3}{\sqrt{30}} \leqslant \mu \leqslant 1001 + 1.96 \times \frac{2.3}{\sqrt{30}}$$

$$\therefore\ 1000.2 \leqslant \mu \leqslant 1001.8 \text{ mL}$$

So, we are 95% confident that the mean contents for all of the 1 litre bottles lies between 1000.2 mL and 1001.8 mL.

b The width of the confidence interval in **a** is $2 \times 1.96 \times \dfrac{2.3}{\sqrt{30}}$

$$\approx 1.6 \text{ mL}$$

c **i** The required sample size $n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2$

$$= \left(\frac{2 \times 1.96 \times 2.3}{0.5}\right)^2$$

$$\approx 325.2$$

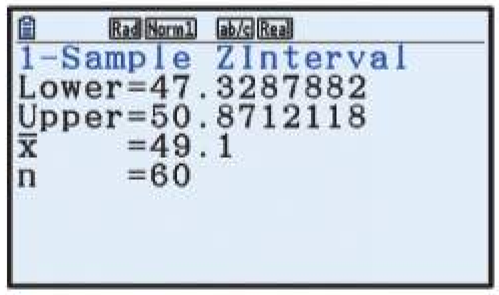
Rounding up, a sample size of at least 326 bottles is required.

ii It may take a long time and cost a lot of money to sample this many bottles.

EXERCISE 29E.3

1 We are given that $\overline{x} = 49.1$, $\sigma = 7$, and $n = 60$.

a Using technology:

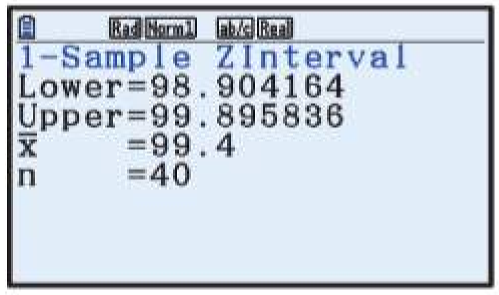


The 95% confidence interval for μ is $47.3 \leqslant \mu \leqslant 50.9$.

b 50 lies within the 95% confidence interval. Even though the sample mean $\overline{x} = 49.1$ is less than 50, we do not have enough evidence to reject the claim that $\mu = 50$.

2 We are given that $\overline{x} = 99.4$ g, $\sigma = 1.6$ g, and $n = 40$.

a Using technology:

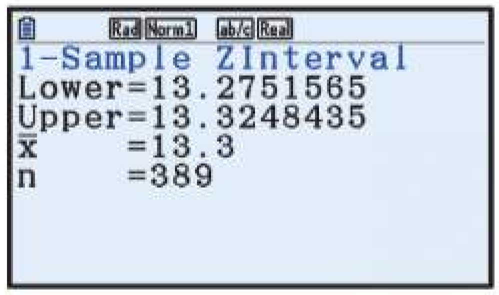


The 95% confidence interval for the population mean weight μ is $98.9 \leqslant \mu \leqslant 99.9$ g.

b 100 g lies outside of the confidence interval, so we reject the claim that $\mu = 100$ g.

3 We are given that $\overline{x} = 13.30$ dollars, $\sigma = 0.25$ dollars, and $n = 389$.

a Using technology:

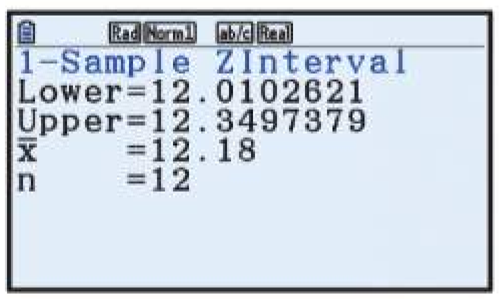


The 95% confidence interval for the mean price μ is $13.28 \leqslant \mu \leqslant 13.32$ dollars.

b The entire confidence interval lies below the old mean price of \$13.45. So we can reject the claim that $\mu = 13.45$ dollars. We therefore accept that the mean price has fallen.

4 We are given that $\overline{x} = 12.18$ seconds, $\sigma = 0.3$ seconds, and $n = 12$.

a Using technology:



The 95% confidence interval for Joan's mean running time μ is $12.01 \leqslant \mu \leqslant 12.35$ seconds.

b **i** The entire confidence interval lies below the old mean time of 12.46 seconds. So we can reject the claim that $\mu = 12.46$ seconds. We therefore accept the claim that Joan's running time has improved.

ii 12.26 seconds is within Joan's confidence interval, so we do not have enough evidence to accept the claim that Joan is now faster than Betty.

INVESTIGATION 4 THE DISTRIBUTION OF T

1 **a** The t-values appear to be approximately normally distributed. Changing the values of the mean and standard deviation does not appear to affect their distribution.

b As the value of n increases, the distribution of the t-values becomes less spread out, and the standard deviation gets slightly lower.

2 Estimating based on the frequency histogram with $n = 10$:

a about 60% of the t-values appear to lie between -1 and 1

b about 95% of the t-values appear to lie between -2 and 2

c about 99% of the t-values appear to lie between -3 and 3.

EXERCISE 29F

1 We are given that $\overline{x} = 8.7$ minutes, $s = 2.08$ minutes, and $n = 167$.

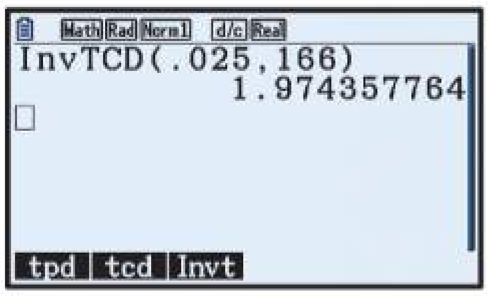
Since $n = 167$ is sufficiently large, $\overline{X}_n$ is approximately normally distributed.

{Central Limit Theorem}

σ is unknown and $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}} \sim t_{166}$.

For a 95% confidence interval, $\alpha = 0.05$. $\therefore \dfrac{\alpha}{2} = 0.025$.

Using technology, $t_{166,\,0.025} \approx 1.974$

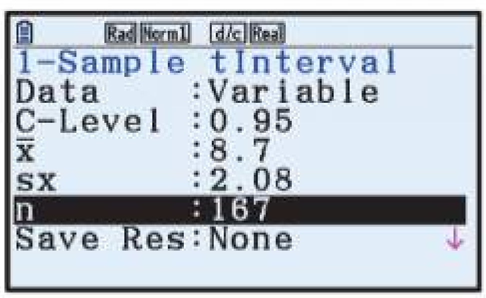


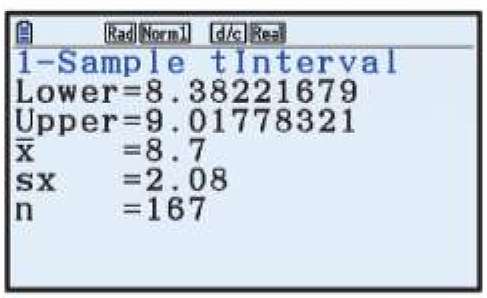
The 95% confidence interval for μ is $\overline{x} - t_{166,\,0.025}\dfrac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{166,\,0.025}\dfrac{s}{\sqrt{n}}$

$$\therefore\ 8.7 - 1.974 \times \frac{2.08}{\sqrt{167}} \leqslant \mu \leqslant 8.7 + 1.974 \times \frac{2.08}{\sqrt{167}}$$

$$\therefore\ 8.38 \leqslant \mu \leqslant 9.02$$

Check using technology:





So, we are 95% confident that the mean waiting time for all customer calls for software support lies between 8.38 minutes and 9.02 minutes.

2 We are given that $\overline{x} = 513.8$ g, $s = 14.9$ g, and $n = 75$.

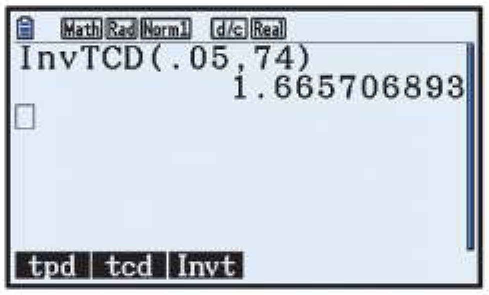
Since $n = 75$ is sufficiently large, $\overline{X}_n$ is approximately normally distributed.

{Central Limit Theorem}

σ is unknown and $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}} \sim t_{74}$.

For a 90% confidence interval, $\alpha = 0.1$. $\therefore \dfrac{\alpha}{2} = 0.05$.

Using technology, $t_{74,\,0.05} \approx 1.666$

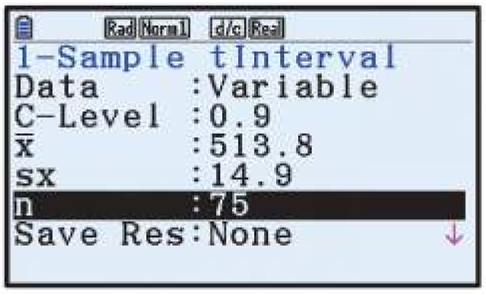


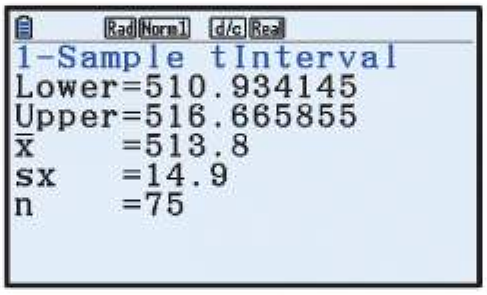
The 90% confidence interval for μ is $\overline{x} - t_{74,\,0.05}\dfrac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{74,\,0.05}\dfrac{s}{\sqrt{n}}$

$$\therefore\ 513.8 - 1.666 \times \frac{14.9}{\sqrt{75}} \leqslant \mu \leqslant 513.8 + 1.666 \times \frac{14.9}{\sqrt{75}}$$

$$\therefore\ 511 \leqslant \mu \leqslant 517$$

Check using technology:





So, we are 90% confident that the mean weight of all plastic packets of cereal lies between 511 grams and 517 grams.

3 We are given that $\overline{x} = 38.2$ days, $s = 4.7$ days, and $n = 42$.

Since $n = 42$ is sufficiently large, $\overline{X}_n$ is approximately normally distributed.

{Central Limit Theorem}

σ is unknown and $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}} \sim t_{41}$.

For a 90% confidence interval, $\alpha = 0.1$. $\therefore \dfrac{\alpha}{2} = 0.05$.

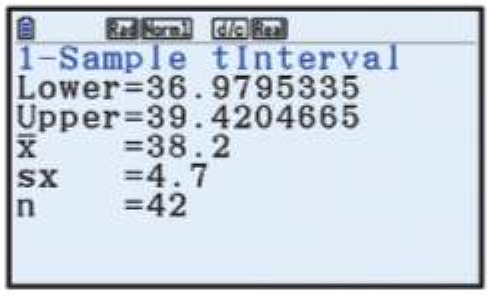
Using technology, $t_{41,\,0.05} \approx 1.683$

The 90% confidence interval for μ is $\overline{x} - t_{41,\,0.05}\dfrac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{41,\,0.05}\dfrac{s}{\sqrt{n}}$

$$\therefore\ 38.2 - 1.683 \times \frac{4.7}{\sqrt{42}} \leqslant \mu \leqslant 38.2 + 1.683 \times \frac{4.7}{\sqrt{42}}$$

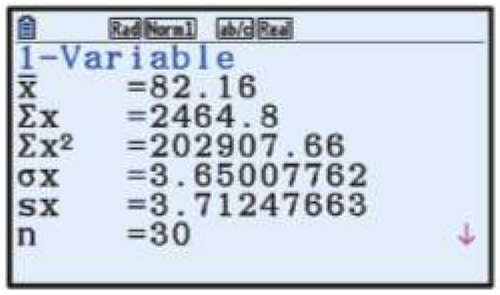
$$\therefore\ 37.0 \leqslant \mu \leqslant 39.4$$

Check using technology:



So, we are 90% confident that the average length of stay for all patients on the program lies between 37.0 days and 39.4 days.

4 **a** We use technology to find the mean and standard deviation of the data:
$\overline{x} = 82.16 \approx 82.2$ hours and
$s \approx 3.7125 \approx 3.71$ hours

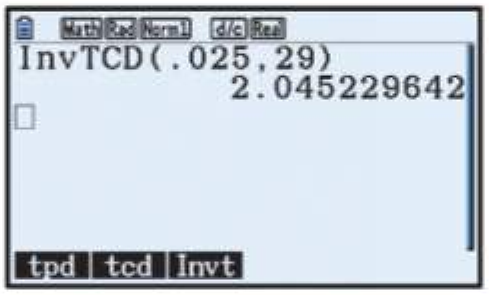


b Since $n = 30$ is sufficiently large, $\overline{X}_n$ is approximately normally distributed.
{Central Limit Theorem}

σ is unknown and $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}} \sim t_{29}$.

For a 95% confidence interval, $\alpha = 0.05$. $\therefore \dfrac{\alpha}{2} = 0.025$.

Using technology, $t_{29,\,0.025} \approx 2.045$

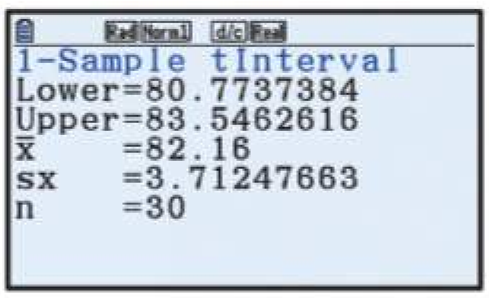


The 95% confidence interval is $\overline{x} - t_{29,\,0.025}\,\dfrac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{29,\,0.025}\,\dfrac{s}{\sqrt{n}}$

$$\therefore\ 82.16 - 2.045 \times \frac{3.7125}{\sqrt{30}} \leqslant \mu \leqslant 82.16 + 2.045 \times \frac{3.7125}{\sqrt{30}}$$

$$\therefore\ 80.8 \leqslant \mu \leqslant 83.5$$

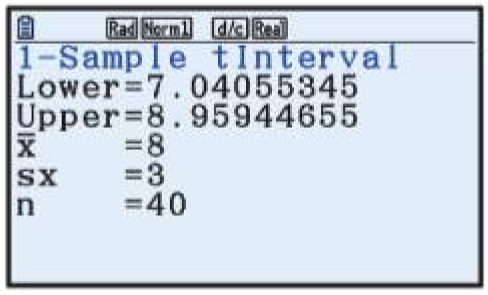
Check using technology:



So, we are 95% confident that the mean lifetime of the globes lies between 80.8 hours and 83.5 hours.

5 We are given that $\overline{x} = 8$ minutes, $s = 3$ minutes, and $n = 40$.

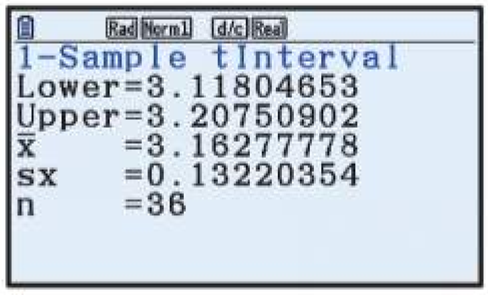
a Using technology:



The 95% confidence interval for the population mean waiting time μ is $7.04 \leqslant \mu \leqslant 8.96$ minutes.

b The entire confidence interval lies below the old mean time of 12 minutes. So we can reject the claim that $\mu = 12$ minutes. The confidence interval therefore supports the call centre's claim.

6 **a** Using technology:



The 95% confidence interval for the population mean stopping time μ is $3.12 \leqslant \mu \leqslant 3.21$ seconds.

b **i** The entire confidence interval lies below the old mean stopping time of 3.45 seconds. So we can reject the claim that $\mu = 3.45$ seconds. Therefore it appears that the new tyre tread does improve stopping times.

ii 3.1 seconds is below the confidence interval, so we reject the claim that $\mu = 3.1$ seconds. Therefore the production team's claim does not appear to be justified.

REVIEW SET 29A

1 **a**

$$\begin{aligned}\text{Mean} &= \mathrm{E}(X_1 + 2X_2 + 3X_3)\\ &= \mathrm{E}(X_1) + 2\,\mathrm{E}(X_2) + 3\,\mathrm{E}(X_3)\\ &= \mu + 2\mu + 3\mu\\ &= 6\mu\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X_1 + 2X_2 + 3X_3) &= \mathrm{Var}(X_1) + 2^2\,\mathrm{Var}(X_2) + 3^2\,\mathrm{Var}(X_3) \qquad \{\text{independence}\}\\ &= \sigma^2 + 4\sigma^2 + 9\sigma^2\\ &= 14\sigma^2\end{aligned}$$

$\therefore$ standard deviation $= \sigma\sqrt{14}$.

b $$\begin{aligned}\text{Mean} &= \mathrm{E}(2X_1 - 3X_2 + X_3)\\ &= 2\,\mathrm{E}(X_1) - 3\,\mathrm{E}(X_2) + \mathrm{E}(X_3)\\ &= 2\mu - 3\mu + \mu\\ &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(2X_1 - 3X_2 + X_3) &= 2^2\,\mathrm{Var}(X_1) + (-3)^2\,\mathrm{Var}(X_2) + \mathrm{Var}(X_3) \quad \{\text{independence}\}\\ &= 4\sigma^2 + 9\sigma^2 + \sigma^2\\ &= 14\sigma^2\end{aligned}$$

$\therefore$ standard deviation $= \sigma\sqrt{14}$.

2 Let S be the volume of a small bottle and let L be the volume of a large bottle.

$\therefore$ $S \sim \mathrm{N}(338,\, 3^2)$ and $L \sim \mathrm{N}(1010,\, 12^2)$

	μ (mL)	σ (mL)
small bottles	338	3
large bottles	1010	12

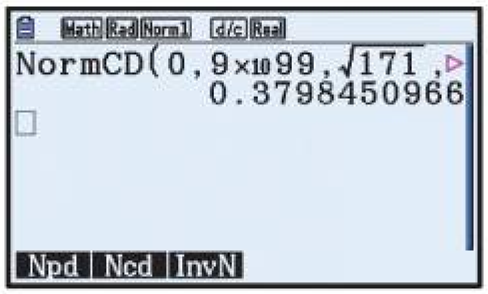
a Consider $$\begin{aligned}U &= L - (S_1 + S_2 + S_3)\\ &= L - S_1 - S_2 - S_3\end{aligned}$$

Now $$\begin{aligned}\mathrm{E}(U) &= \mathrm{E}(L) - \mathrm{E}(S_1) - \mathrm{E}(S_2) - \mathrm{E}(S_3)\\ &= 1010 - 3 \times 338\\ &= -4 \text{ mL}\end{aligned}$$

and $$\begin{aligned}\mathrm{Var}(U) &= \mathrm{Var}(L) + \mathrm{Var}(S_1) + \mathrm{Var}(S_2) + \mathrm{Var}(S_3) \quad \{\text{independence}\}\\ &= 12^2 + 3 \times 3^2\\ &= 171 \text{ mL}^2\end{aligned}$$

$\therefore$ $U \sim \mathrm{N}(-4,\, 171)$

$$\begin{aligned}\mathrm{P}(L > S_1 + S_2 + S_3) &= \mathrm{P}(L - S_1 - S_2 - S_3 > 0)\\ &= \mathrm{P}(U > 0)\\ &\approx 0.380\end{aligned}$$

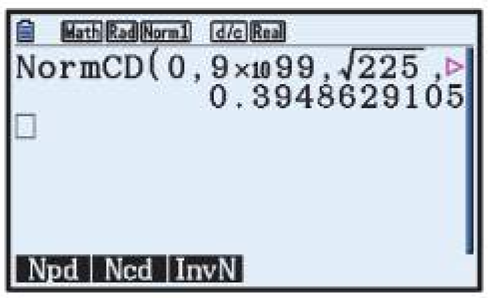


b Consider $V = L - 3S$

Now $$\begin{aligned}\mathrm{E}(V) &= \mathrm{E}(L) - 3\,\mathrm{E}(S)\\ &= 1010 - 3 \times 338\\ &= -4 \text{ mL}\end{aligned}$$ and $$\begin{aligned}\mathrm{Var}(V) &= \mathrm{Var}(L) + 9\,\mathrm{Var}(S) \quad \{\text{independence}\}\\ &= 12^2 + 9 \times 3^2\\ &= 225 \text{ mL}^2\end{aligned}$$

$\therefore$ $V \sim \mathrm{N}(-4,\, 225)$

$$\begin{aligned}\mathrm{P}(L > 3S) &= \mathrm{P}(L - 3S > 0)\\ &= \mathrm{P}(V > 0)\\ &\approx 0.395\end{aligned}$$



3 $X \sim \text{Po}(a)$ and $Y \sim \text{Po}(b)$ independently.

$\therefore\ (X+Y) \sim \text{Po}(a+b)$

$\therefore\ \text{E}(X+Y) = \text{Var}(X+Y) = a+b$ .... (1)

Now $\text{E}(X+Y) = a^2+b^2$ .... (2) and $\text{Var}(X+Y) = a^2-b^2+2$ .... (3)

$\therefore\ a^2+b^2 = a^2-b^2+2$ {equating (2) and (3)}

$\therefore\ 2b^2 = 2$

$\therefore\ b^2 = 1$

$\therefore\ b = 1$ $\{b > 0\}$

Substituting $b=1$ into (1) and (2) gives $a^2+1 = a+1$

$\therefore\ a^2 = a$

$\therefore\ a^2 - a = 0$

$\therefore\ a(a-1) = 0$

$\therefore\ a = 1$ $\{a > 0\}$

4 X has mean $\mu = 35$ and standard deviation $\sigma = 8$.

a Mean of $\overline{X}_4$ = mean of X

$= \mu$

$= 35$

Standard deviation of $\overline{X}_4 = \dfrac{\sigma}{\sqrt{4}}$

$= \dfrac{8}{2}$

$= 4$

b Mean of $\overline{X}_{30}$ = mean of X

$= \mu$

$= 35$

Standard deviation of $\overline{X}_{30} = \dfrac{\sigma}{\sqrt{30}}$

$= \dfrac{8}{\sqrt{30}}$

5 Let X be the height of a mature plant.

X has mean $\mu = 21$ cm and standard deviation $\sigma = \sqrt{90}$ cm.

Mean of $\overline{X}_{40}$ = mean of X

$= \mu$

$= 21$ cm

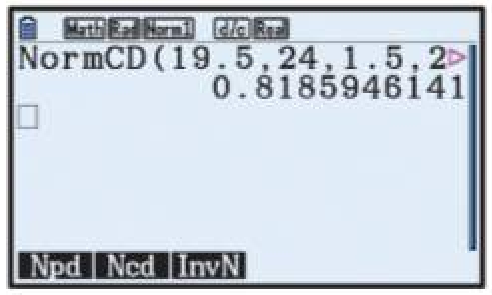
$$\begin{aligned}\text{Standard deviation of } \overline{X}_{40} &= \frac{\sigma}{\sqrt{40}}\\ &= \frac{\sqrt{90}}{\sqrt{40}}\\ &= \sqrt{\tfrac{90}{40}}\\ &= \sqrt{\tfrac{9}{4}}\\ &= \tfrac{\sqrt{9}}{\sqrt{4}}\\ &= \tfrac{3}{2} = 1.5 \text{ cm}\end{aligned}$$

Since $n = 40$ is sufficiently large, $\overline{X}_{40}$ is approximately normally distributed.

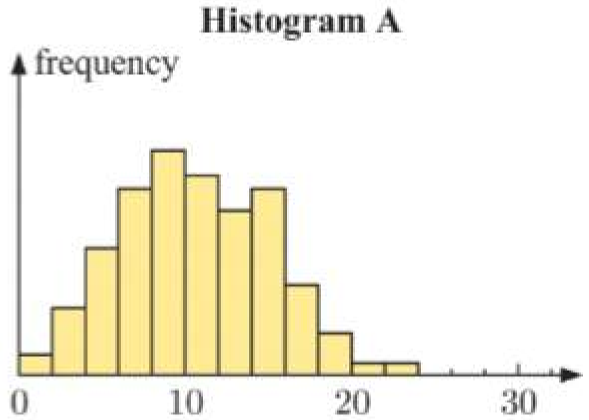
{Central Limit Theorem}

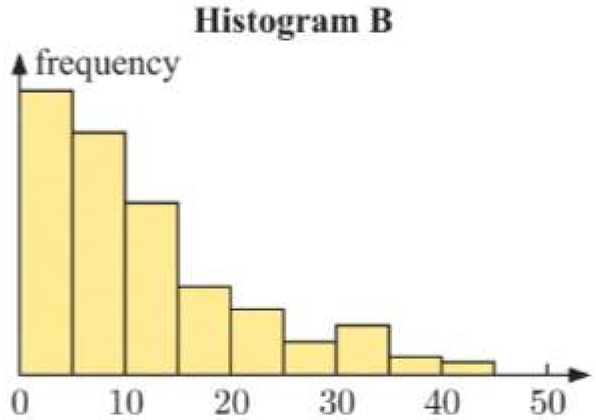
$\therefore\ \overline{X}_{40} \sim \text{N}(21,\ 1.5^2)$

$\therefore \quad P(19.5 \leqslant \overline{X}_{40} \leqslant 24) \approx 0.819$



6





a Histogram **B** is for X as it is not as close to a normal distribution, and histogram **A**'s standard deviation is much less than 9.21.

b Mean of $\overline{X}_9$ = mean of X

$$= \mu$$
$$= 11.4 \text{ g kg}^{-1}$$

Standard deviation of $\overline{X}_9 = \dfrac{\sigma}{\sqrt{9}}$

$$= \frac{9.21}{3}$$
$$= 3.07 \text{ g kg}^{-1}$$

7 We are given that $\overline{x} = 596.7$ g, $\sigma = 12.3$ g, and $n = 120$.

The 95% confidence interval for μ is $\overline{x} - 1.96\dfrac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\therefore \quad 596.7 - 1.96 \times \frac{12.3}{\sqrt{120}} \leqslant \mu \leqslant 596.7 + 1.96 \times \frac{12.3}{\sqrt{120}}$$

$$\therefore \quad 594.5 \leqslant \mu \leqslant 598.9 \text{ grams}$$

So, we are 95% confident that the mean weight of breakfast cereal lies between 594.5 g and 598.9 g.

8 **a** We are given that $\overline{x} = 828.2$ g, $\sigma = 16.3$ g, and $n = 65$.

The 95% confidence interval for μ is $\overline{x} - 1.96\dfrac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\dfrac{\sigma}{\sqrt{n}}$

$$\therefore \quad 828.2 - 1.96 \times \frac{16.3}{\sqrt{65}} \leqslant \mu \leqslant 828.2 + 1.96 \times \frac{16.3}{\sqrt{65}}$$

$$\therefore \quad 824.2 \leqslant \mu \leqslant 832.2 \text{ grams}$$

b The width of the confidence interval in **a** is $w = 2 \times 1.96 \times \dfrac{16.3}{\sqrt{65}}$

≈ 7.93 grams

$\therefore$ half of this width $\approx \frac{1}{2} \times 7.93 \approx 3.96$ grams.

The required sample size $n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2$

$$\approx \left(\frac{2 \times 1.96 \times 16.3}{3.96}\right)^2$$

$$\approx 260$$

A sample size of 260 cans is required.

9 $n = 15, \quad \displaystyle\sum_{i=1}^{15} (x_i - \overline{x})^2 = 230$

a $s^2 = \dfrac{1}{n-1} \displaystyle\sum_{i=1}^{n} (x_i - \overline{x})^2$

$= \frac{1}{14} \times 230$

≈ 16.4

b **i** A $k\%$ confidence interval for μ is $124.94 \leqslant \mu \leqslant 129.05$.

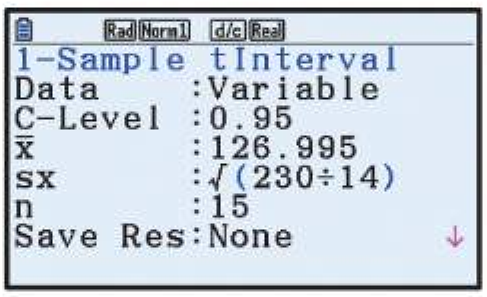
$\overline{x}$ is the midpoint of the confidence interval.

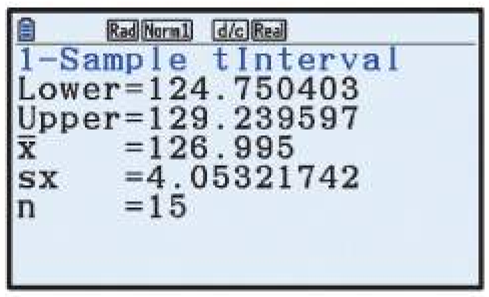
$$\therefore \quad \overline{x} = \frac{124.94 + 129.05}{2}$$

$$= 126.995$$

Since σ is unknown, we must use the t-distribution to calculate the 95% confidence interval.

Using technology:





The 95% confidence interval is $124.75 \leqslant \mu \leqslant 129.24$.

ii The general confidence interval for μ has the form

$$\overline{x} - t_{n-1,\,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{n-1,\,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

$$\therefore \quad 126.995 - t_{14,\,\frac{\alpha}{2}} \frac{\sqrt{\frac{230}{14}}}{\sqrt{15}} \leqslant \mu \leqslant 126.995 + t_{14,\,\frac{\alpha}{2}} \frac{\sqrt{\frac{230}{14}}}{\sqrt{15}}$$

$$\therefore \quad 126.995 - t_{14,\,\frac{\alpha}{2}} \sqrt{\tfrac{23}{21}} \leqslant \mu \leqslant 126.995 + t_{14,\,\frac{\alpha}{2}} \sqrt{\tfrac{23}{21}}$$

Equating the lower bounds of the confidence interval gives

$$126.995 - t_{14,\,\frac{\alpha}{2}}\sqrt{\tfrac{23}{21}} = 124.94$$

$$\therefore\ -t_{14,\,\frac{\alpha}{2}}\sqrt{\tfrac{23}{21}} = -2.055$$

$$\therefore\ t_{14,\,\frac{\alpha}{2}} = 2.055\sqrt{\tfrac{21}{23}} \approx 1.9636$$

Now $\frac{\alpha}{2} = \mathrm{P}(T \geqslant t_{14,\,\frac{\alpha}{2}})$

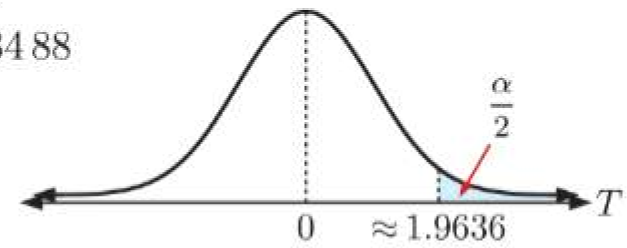
$\approx 0.034\,88$

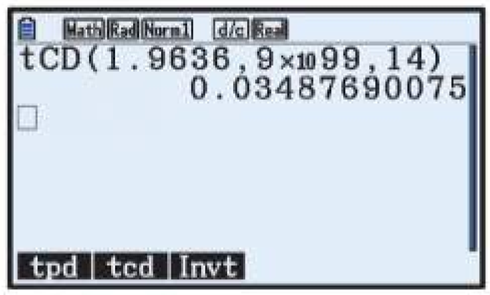
$\therefore\ \alpha \approx 2 \times 0.034\,88$

≈ 0.0698

$\therefore\ k \approx 100 - 6.98$

≈ 93.0





REVIEW SET 29B

1 **a** $Y = 2X_3 - 2X_2 - X_1$

i $\mathrm{E}(Y) = 2\,\mathrm{E}(X_3) - 2\,\mathrm{E}(X_2) - \mathrm{E}(X_1)$

$= 2a - 2(3) - (2)$

$= 2a - 8$

ii $\mathrm{Var}(Y) = 2^2\,\mathrm{Var}(X_3) + (-2)^2\,\mathrm{Var}(X_2) + (-1)^2\,\mathrm{Var}(X_1)$ {independence}

$= 4b + 4\left(\frac{1}{16}\right) + \frac{1}{8}$

$= 4b + \frac{3}{8}$

b If $\mathrm{E}(Y) = 0$, $2a - 8 = 0$

$\therefore\ a = 4$

If $\mathrm{Var}(Y) = 1$, $4b + \frac{3}{8} = 1$

$\therefore\ 4b = \frac{5}{8}$

$\therefore\ b = \frac{5}{32}$

c Y is a linear combination of normal random variables, so Y is also normally distributed.

So, $Y \sim \mathrm{N}(0,\, 1^2)$.

d $\mathrm{P}(Y \geqslant 8b) = \mathrm{P}(Y \geqslant 1.25)$

≈ 0.106

2 **a** Let A be the number of times Rowan's music is listened to on service A.
Let S be the number of times Rowan's music is listened to on service S.

$\therefore$ $A \sim \text{Po}(105)$ and $S \sim \text{Po}(250)$, independently.

Now $T = A + S$. $\therefore$ $T \sim \text{Po}(355)$

b $\text{P}(T > 400) = 1 - \text{P}(T \leqslant 400)$
$\approx 1 - 0.991\,21$
$\approx 0.008\,79$

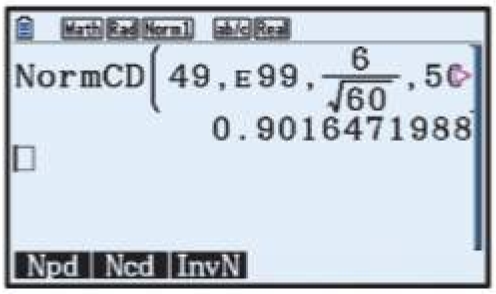
3 X has mean $\mu = 50$ and standard deviation $\sigma = 6$.

a Mean of $\overline{X}_{60}$ = mean of X
$= \mu$
$= 50$

Standard deviation of $\overline{X}_{60} = \dfrac{\sigma}{\sqrt{60}}$
$= \dfrac{6}{\sqrt{60}}$

b Yes, 60 should be a sufficiently large sample for the Central Limit Theorem to apply.
So we can assume that $\overline{X}_{60}$ is approximately normally distributed.

c Using technology:



$\text{P}(\overline{X}_{60} > 49) \approx 0.902$

4 X has mean $\mu = 500$ g and standard deviation $\sigma = 3.5$ g.

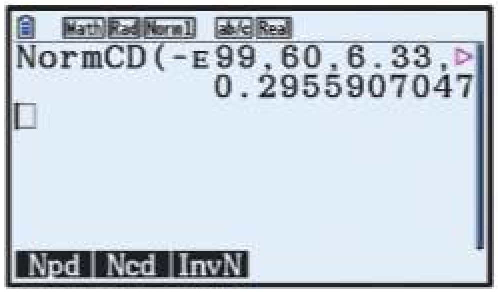
a Mean of $\overline{X}_{20}$ = mean of X
$= \mu$
$= 500$ g

Standard deviation of $\overline{X}_{20} = \dfrac{\sigma}{\sqrt{n}}$
$= \dfrac{3.5}{\sqrt{20}}$
≈ 0.783 g

b Since X is normally distributed, then $\overline{X}_{20}$ is also normally distributed.

5 **a** W is normally distributed with mean 63.4 g and standard deviation 6.33 g.

Using technology:



$\therefore \quad \text{P}(W \leqslant 60 \text{ g}) \approx 0.296$

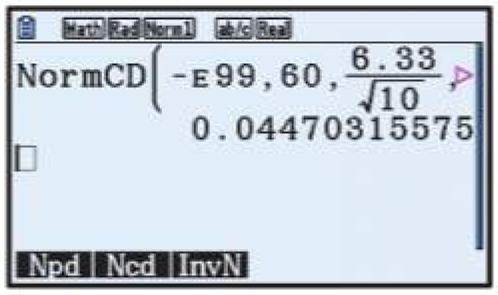
b Mean of $\overline{W}_{10}$ = mean of W

$= 63.4$ g

Standard deviation of $\overline{W}_{10} = \dfrac{6.33}{\sqrt{10}}$

≈ 2.00 g

c Using technology:

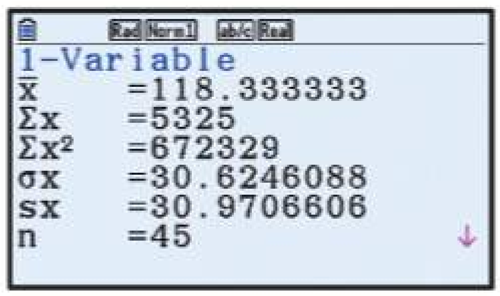


$\therefore \quad \text{P}(\overline{W} \leqslant 60 \text{ g}) \approx 0.0447$

The probability that the mean weight of a sample of 10 sausages will be less than or equal to 60 g is about 0.0447.

6 **a** We use technology to find the mean and standard deviation of the data:

$\overline{x} \approx 118$ points and $s \approx 31.0$ points



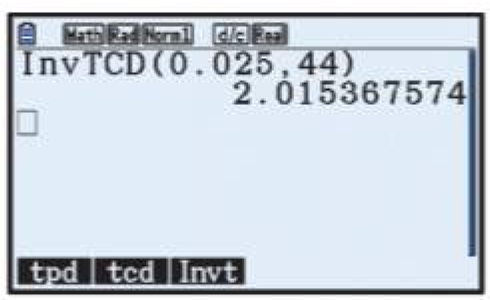
b Since $n = 45$ is sufficiently large, $\overline{X}_n$ is approximately normally distributed.

{Central Limit Theorem}

σ is unknown and $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}} \sim t_{44}$.

For a 95% confidence interval, $\alpha = 0.05$. $\quad \therefore \quad \dfrac{\alpha}{2} = 0.025$

Using technology, $t_{44,\, 0.025} \approx 2.015$

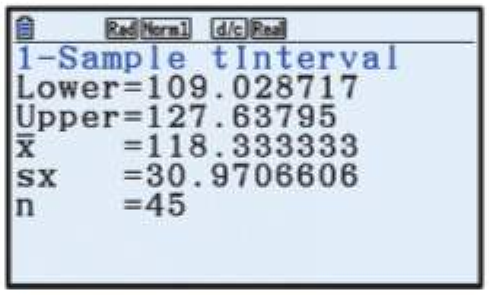


The 95% confidence interval for μ is $\overline{x} - t_{44,\,0.025}\,\dfrac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{44,\,0.025}\,\dfrac{s}{\sqrt{n}}$

$$\therefore\ 118 - 2.015 \times \frac{31.0}{\sqrt{45}} \leqslant \mu \leqslant 118 + 2.015 \times \frac{31.0}{\sqrt{45}}$$

$$\therefore\ 109 \leqslant \mu \leqslant 128$$

Check using technology:

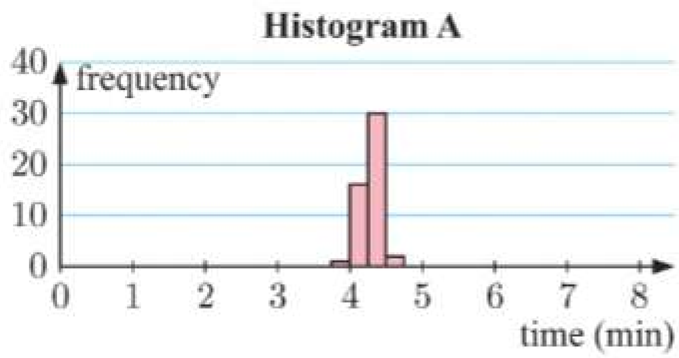


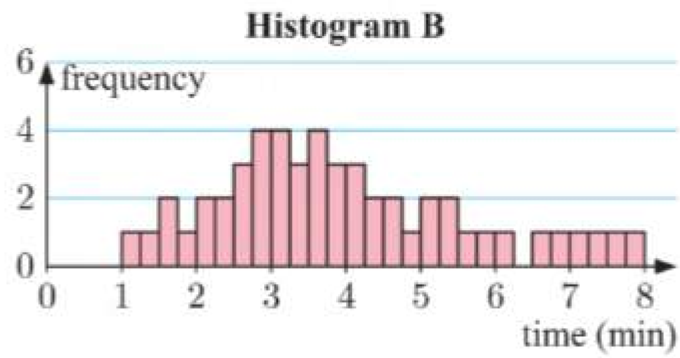
So, we are 95% confident that the mean score at the bowling centre lies between 109 points and 128 points.

c Width of confidence interval $\approx 128 - 109$

≈ 19 points

7 **a**





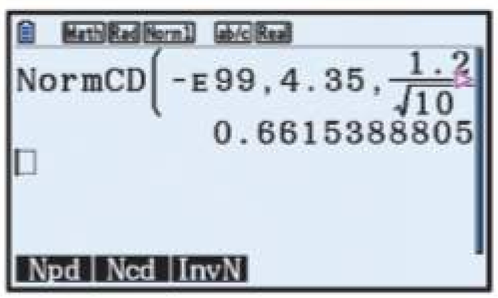
Histogram A represents a sample from $\overline{T}_{100}$, as histogram A is more symmetric and has a smaller spread than histogram B.

b **i** T has mean $\mu = 4.3$ minutes and standard deviation $\sigma = 1.2$ minutes.

Since the sample size $n = 100$ is sufficiently large, we apply the Central Limit Theorem.

$\therefore$ $\overline{T}_{100}$ is approximately normally distributed with mean 4.3 minutes and standard deviation $\dfrac{1.2}{\sqrt{100}} = \dfrac{1.2}{10} = 0.12$ minutes.

ii Using technology:



$\therefore$ $\mathrm{P}(\overline{T}_{100} \leqslant 4.35) \approx 0.662$

iii $7\frac{1}{4}$ hours $= 7.25 \times 60 = 435$ minutes

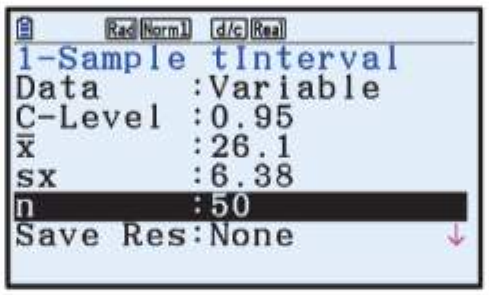
$\therefore$ to answer 100 calls over $7\frac{1}{4}$ hours means the average time to answer a single call is $\frac{435}{100} = 4.35$ minutes.

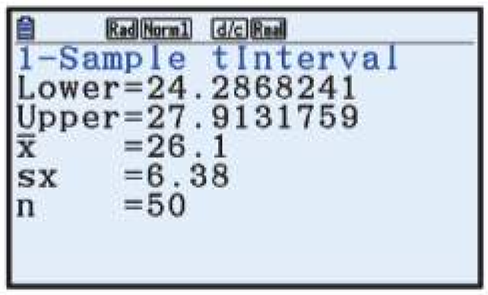
So, $\mathrm{P}(\text{operator can answer 100 calls in less than } 7\tfrac{1}{4} \text{ hours}) = \mathrm{P}(\overline{T}_{100} < 4.35 \text{ minutes})$

≈ 0.662

8 **a** Let X be the meat protein content of battery cage chickens on the feeding program.
We are given that $\overline{x} = 26.1$ units per kg, $s = 6.38$ units per kg, and $n = 50$.

Using technology:





The 95% confidence interval for the mean meat protein content is
$24.3 \leqslant \mu \leqslant 27.9$ units per kg.

b Free range chickens have mean meat protein content 24.9 units per kg. Since 24.9 units per kg lies within the 95% confidence interval, we do not have enough evidence to reject the claim that $\mu = 24.9$ units per kg. Thus, there is insufficient evidence to suggest that battery cage chickens on the feeding program have higher mean meat protein content than free range chickens.

9 Let X be the volume of a bottle in mL.

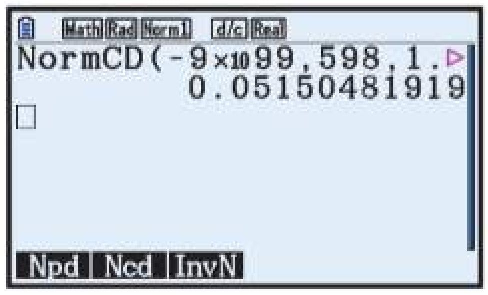
$X \sim \mathrm{N}(601, 1.84^2)$

$\overline{X}_{12}$ is the average volume of a pack of 12 bottles.

$\overline{X}_{12} \sim \mathrm{N}\left(601, \frac{1.84^2}{12}\right)$

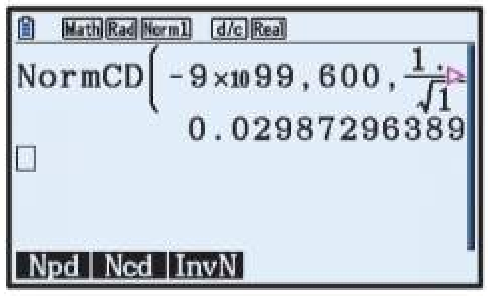
a $\mathrm{P}(X < 598) \approx 0.0515$

$\therefore$ the probability that an individual randomly selected bottle contains less than 598 mL is about 0.0515.



b $\mathrm{P}(\overline{X}_{12} < 600) \approx 0.0299$

$\therefore$ the probability that a randomly selected pack of 12 bottles has average volume less than 600 mL is about 0.0299.



c Let $X \sim \mathrm{N}(\mu, 1.84^2)$

We want $\mathrm{P}(\overline{X}_{12} < 600) = 0.01$

$$\therefore \mathrm{P}\left(\frac{\overline{X}_{12} - \mu}{\frac{1.84}{\sqrt{12}}} < \frac{600 - \mu}{\frac{1.84}{\sqrt{12}}}\right) = 0.01$$

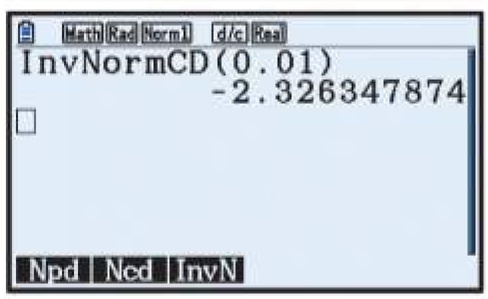
$$\therefore \mathrm{P}\left(Z < \frac{600 - \mu}{\frac{1.84}{\sqrt{12}}}\right) = 0.01$$

Thus $\dfrac{(600 - \mu)\sqrt{12}}{1.84} \approx -2.3263$

$\therefore 600 - \mu \approx -1.235\,67$

$\therefore \mu \approx 601.24$

So, the average volume dispensed needs to be 601.3 mL.



Chapter 30

HYPOTHESIS TESTING

EXERCISE 30A

1 **a** A Type I error involves rejecting a true null hypothesis.

b A Type II error involves accepting a false null hypothesis.

c The null hypothesis is a statement of *no difference*.

d The alternative hypothesis is a statement that there is a difference.

2 **a** A Type I error has been made. **b** A Type II error has been made.

3 **a** A Type II error has been made. **b** A Type I error has been made.

4 **a** The alternative hypothesis (H_1) would be that the person on trial is guilty.

b A Type I error has been made. **c** A Type II error has been made.

5 **a** A Type I error would result if X and Y are determined to have different effectiveness, when in fact they have the same.

b A Type II error would result if X and Y are determined to have the same effectiveness, when in fact they have different effectiveness.

6 **a** H_0: $\mu = 80$ {new globe lasts as long as old globe}
H_1: $\mu > 80$ {new globe lasts longer than old globe}

b H_0: $\mu = 80$ {new globe lasts as long as old globe}
H_1: $\mu < 80$ {new globe does not last as long as old globe}

7 H_0: $\mu = 26.3$ {new top speed is the same as old top speed}
H_1: $\mu > 26.3$ {new top speed is greater than old top speed}

8 H_0: $\mu = 80$ {mean weight of paper is 80 g per m^2}
H_1: $\mu \neq 80$ {mean weight of paper is *not* 80 g per m^2}

9 H_0: $\mu = 27$ {mean travel time is the same as before}
H_1: $\mu < 27$ {mean travel time is lower than before}

10 H_0: $\mu = 2.7$ {fat content of Brand B's muesli bars is 2.7 g}
H_1: $\mu > 2.7$ {fat content of Brand B's muesli bars is *greater* than 2.7 g}

11 Let p be the proportion of smokers in the British population.
H_0: $p = 0.21$, H_1: $p < 0.21$

12 Let p be the proportion of cancer patients who survive more than 5 years after diagnosis on the new treatment.

H_0: $p = 0.3$, H_1: $p > 0.3$

13 Let p be the proportion of Party A supporters.

a H_0: $p = 0.42$, H_1: $p > 0.42$ **b** H_0: $p = 0.42$, H_1: $p \neq 0.42$

EXERCISE 30B

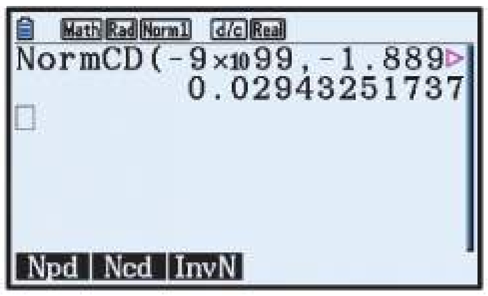
1 H_0: $\mu = 25$
H_1: $\mu < 25$

a **i** $\overline{x} = 23.75$, $\mu_0 = 25$, $\sigma = 3.97$, $n = 36$

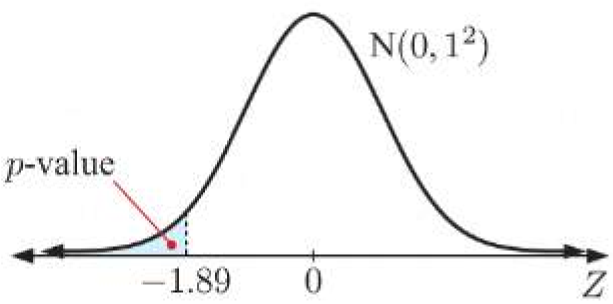
The value of the test statistic $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$

$$= \frac{23.75 - 25}{\frac{3.97}{\sqrt{36}}}$$

$$\approx -1.89$$

ii 

Since H_1: $\mu < 25$ and $z \approx -1.89$,
the p-value $= \mathrm{P}(Z \leqslant z)$ where $Z \sim \mathrm{N}(0,\ 1^2)$
$\approx \mathrm{P}(Z \leqslant -1.89)$
≈ 0.0294



b Since p-value < 0.05, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept that the population mean is less than 25.

2 **a** H_0: $\mu = 80$ {the population mean is 80}
H_1: $\mu > 80$ {the population mean is greater than 80}

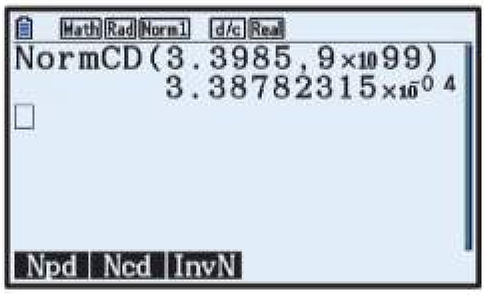
b $\overline{x} = 83.1$, $\mu_0 = 80$, $\sigma = 12.9$, $n = 200$

The value of the test statistic $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$

$$= \frac{83.1 - 80}{\frac{12.9}{\sqrt{200}}}$$

$$\approx 3.40$$

The null distribution is $\mathrm{N}(0,\ 1^2)$.

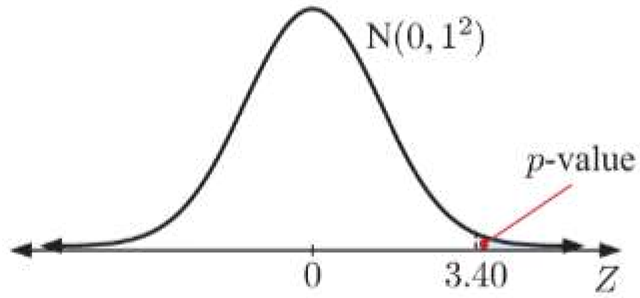
c 

Since H_1: $\mu > 80$ and $z \approx 3.40$,

the p-value $= \text{P}(Z \geqslant z)$ where $Z \sim \text{N}(0,\ 1^2)$

$\approx \text{P}(Z \geqslant 3.40)$

$\approx 0.000\,339$



d Since p-value $< 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level.

e We conclude that the population mean is greater than 80 at the 1% significance level.

3 *Step 1*: Let μ be the population mean weight of the bags. The factory wants to determine whether the weight has *decreased*, so the hypotheses to be considered are:

H_0: $\mu = 100$ {the mean weight is 100 g per bag}

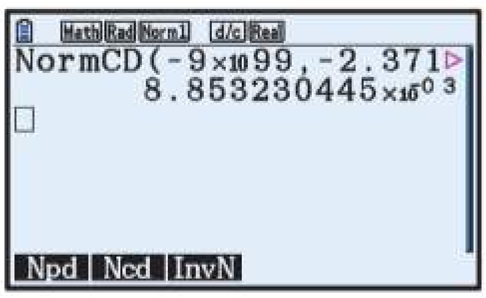
H_1: $\mu < 100$ {the mean weight is less than 100 g per bag}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 99.4$ g, $\mu_0 = 100$ g, $\sigma = 1.6$ g, $n = 40$ bags

The value of the test statistic $z = \dfrac{99.4 - 100}{\frac{1.6}{\sqrt{40}}}$

≈ -2.37

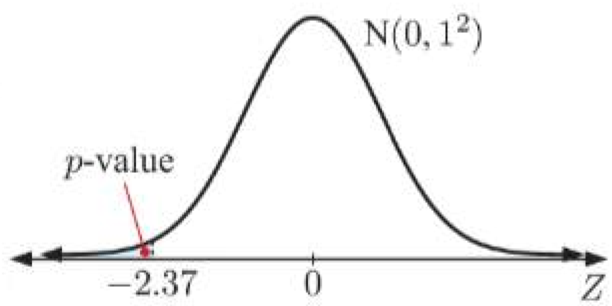
Step 4: 

Since H_1: $\mu < 100$,

the p-value $= \text{P}(Z \leqslant z)$ where $Z \sim \text{N}(0,\ 1^2)$

$\approx \text{P}(Z \leqslant -2.37)$

$\approx 0.008\,85$



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

Step 6: Since we have accepted H_1, we conclude on a 5% significance level that the mean weight is less than 100 g. The customer's claim is valid.

4 *Step 1*: Let μ be the population mean fineness of the herd's fleece. The breeder wants to determine whether the fineness has *changed*, so the hypotheses to be considered are:

H_0: $\mu = 20.3$ {the herd's mean fineness has not changed}

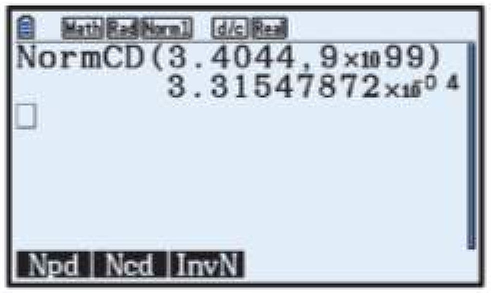
H_1: $\mu \neq 20.3$ {the herd's mean fineness has changed}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 19.2$ microns, $\mu_0 = 20.3$ microns, $\sigma = 2.89$ microns, $n = 80$ alpacas

$$\text{The value of the test statistic } z = \frac{19.2 - 20.3}{\frac{2.89}{\sqrt{80}}} \approx -3.40$$

Step 4:



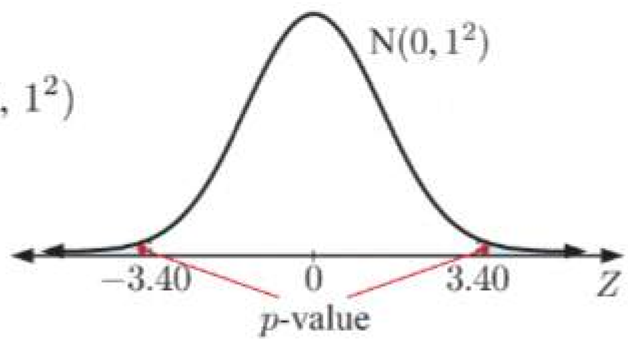
Since H_1: $\mu \neq 20.3$,

the p-value $= 2 \times \mathrm{P}(Z \geqslant |z|)$ where $Z \sim \mathrm{N}(0, 1^2)$

$\approx 2 \times \mathrm{P}(Z \geqslant 3.40)$

$\approx 2 \times 0.000\,331\,6$

$\approx 0.000\,663$



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

Step 6: Since we have rejected H_0, we conclude on a 5% significance level that the fineness of the herd's fleece has changed.

INVESTIGATION 1 MULTIPLE TESTING AND STATISTICAL FALLACY

1 **a** The value of the test statistic
$$z = \frac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{x} - 2}{\frac{\sigma}{\sqrt{10}}}$$

b

m	*Number of times H_0 is rejected*	*Proportion of samples where H_0 was rejected*
20	1	0.05
50	3	0.06
100	4	0.04
400	30	0.075
1000	52	0.052
10 000	523	0.0523

c **i**

m	*Number of times H_0 is rejected*	*Proportion of samples where H_0 was rejected*
20	3	0.15
50	4	0.08
100	9	0.09
400	46	0.115
1000	101	0.101
10 000	1013	0.1013

ii

m	*Number of times H_0 is rejected*	*Proportion of samples where H_0 was rejected*
20	1	0.05
50	1	0.02
100	4	0.04
400	12	0.03
1000	26	0.026
10 000	264	0.0264

iii

m	*Number of times H_0 is rejected*	*Proportion of samples where H_0 was rejected*
20	0	0
50	1	0.02
100	0	0
400	3	0.0075
1000	9	0.009
10 000	106	0.0106

In each table, including **b**, we see that the proportion of samples where H_0 was rejected is close to the particular value of α used. As the number of samples m increases, the proportion of samples rejected approaches α.

2 **a** H_0: $\mu = 2$ is true in every sample becuase each sample is generated from the distribution $N(2, 5^2)$, which has mean $\mu = 2$.

b P(incorrectly reject H_0) = P(reject H_0 | H_0 true)

$$= 2 \times P\left(Z \geqslant \mid z_{\frac{\alpha}{2}} \mid \,\middle|\, Z \sim N(0, 1^2)\right)$$
$$= 2 \times \frac{\alpha}{2}$$
$$= \alpha$$

c The number of times H_0 is rejected is a random variable with a fixed number of trials $n = m$, and the same probability of success (rejecting H_0) $p = \alpha$. It therefore has the distribution $B(m, \alpha)$, and so the *expected* number of samples where H_0 is incorrectly rejected is $m\alpha$.

3 **a** Sabeen is most likely to report on the effects of diet cola on females aged 15 to 19, as this result has the lowest p-value, and is therefore seen as the most significant result.

b From **2 c** we know that if we expect to see 1 result for which H_0 is incorrectly rejected, then $m\alpha = 1$.

Sabeen divided her data into 20 different groups, so we would expect to see H_0 rejected once on the significance level $\alpha = \frac{1}{m} = \frac{1}{20} = 0.05$.

This is exactly what we see in Sabeen's data, for the group of females aged 15 to 19, so it appears that this result is simply due to random variation.

Therefore, we would not expect Mysha to replicate Sabeen's finding that diet cola has a significant effect on females aged 15 to 19.

EXERCISE 30C

1 **a** In a Z-test, the null distribution is $Z \sim \mathrm{N}(0, 1^2)$, and does not depend on μ_0. For a given significance level α, the critical value, critical region, and acceptance region are the same for each set of hypotheses.

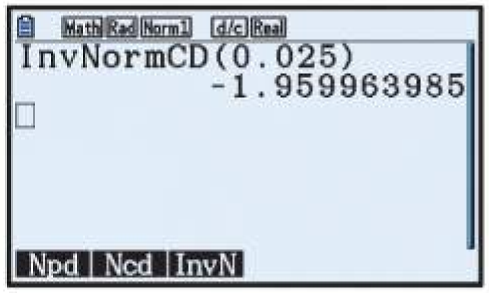
b Yes, the null distribution is still $Z \sim \mathrm{N}(0, 1^2)$. The critical value, critical region, and acceptance region will be the same.

2 **a** H_0: $\mu = 1.6$ {mean tread depth is 1.6 mm}

H_1: $\mu < 1.6$ {mean tread depth is less than 1.6 mm}

b **i** We require c such that $\mathrm{P}(Z \leqslant c) = 0.025$

$\therefore\ c = -z_{0.025} \approx -1.96$



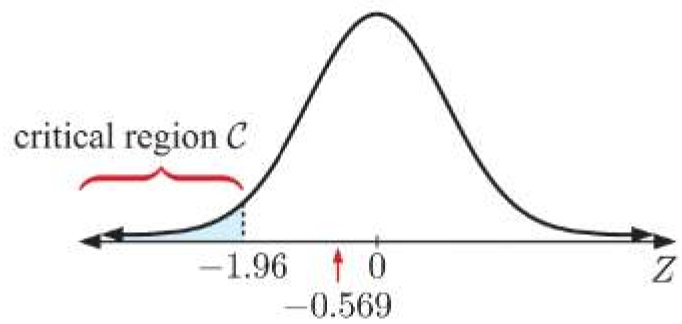
ii $\mathcal{C} = \{z \mid z \leqslant -1.96\}$

iii $\mathcal{A} = \{z \mid z > -1.96\}$

c $\overline{x} = 1.51$ mm, $\sigma = 0.5$ mm, $n = 10$

The observed value of the test statistic is

$$z = \frac{1.51 - 1.6}{\frac{0.5}{\sqrt{10}}} \approx -0.569$$



$z \notin \mathcal{C}$, so there is insufficient evidence to claim that the mean tread depth does not meet the legal requirement.

3 The hypotheses for the test are:

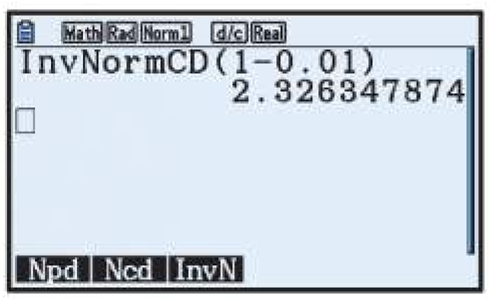
H_0: $\mu = 9210$ {mean energy intake has not changed}

H_1: $\mu \neq 9210$ {mean energy intake has changed}

a $\alpha = 0.02$

The critical value satisfies

$$\mathrm{P}(Z \leqslant -c \text{ or } Z \geqslant c) = 0.02$$
$$\therefore\ \mathrm{P}(Z \geqslant c) = 0.01$$
$$\therefore\ c = z_{0.01} \approx 2.33$$



i $\mathcal{C} = \{z \mid z \leqslant -2.33 \text{ or } z \geqslant 2.33\}$ **ii** $\mathcal{A} = \{z \mid -2.33 < z < 2.33\}$

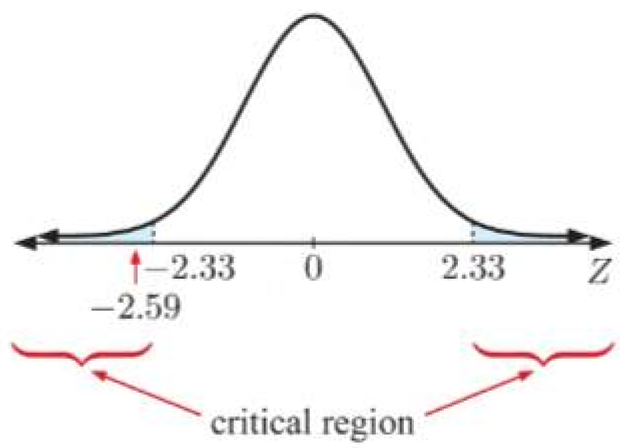
b $\overline{x} = 9100$ kJ, $\sigma = 300$ kJ, $n = 50$

The observed value of the test statistic is

$$z = \frac{9100 - 9210}{\frac{300}{\sqrt{50}}}$$

$$\approx -2.59$$

$z \in \mathcal{C}$, so there is sufficient evidence to claim that the mean energy intake has changed.



4 H_0: $\mu = -23$

H_1: $\mu \neq -23$

The observed test statistic is $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$

$$= \frac{\overline{x} - (-23)}{\frac{4}{\sqrt{100}}}$$

$$= \frac{\overline{x} + 23}{0.4}$$

The critical value c satisfies $\text{P}(Z \leqslant -c \text{ or } Z \geqslant c) = 0.05$

$\therefore \text{P}(Z \geqslant c) = 0.025$

$\therefore c = z_{0.025} \approx 1.96$ {from **1 b i**}

H_0 will be rejected if:

$z \leqslant -1.96$ or $z \geqslant 1.96$

$\therefore \dfrac{\overline{x} + 23}{0.4} \leqslant -1.96$ or $\dfrac{\overline{x} + 23}{0.4} \geqslant 1.96$

$\therefore \overline{x} \leqslant -23 - 1.96 \times 0.4$ or $\overline{x} \geqslant -23 + 1.96 \times 0.4$

$\therefore \overline{x} \leqslant -23.8$ or $\overline{x} \geqslant -22.2$

So, the null hypothesis will be rejected if $\overline{x} \leqslant -23.8$ or $\overline{x} \geqslant -22.2$.

5 The hypotheses to be tested are

H_0: $\mu = 504$ {the machine does not need to be adjusted}

H_1: $\mu \neq 504$ {the machine needs to be adjusted}

The observed test statistic is $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \dfrac{\overline{x} - 504}{\frac{3}{\sqrt{20}}}$.

The critical value c satisfies $\text{P}(Z \leqslant -c \text{ or } Z \geqslant c) = 0.02$

$\therefore \text{P}(Z \geqslant c) = 0.01$

$\therefore c = z_{0.01} \approx 2.33$ {from **3 a**}

H_0 will be accepted if:

$$-2.33 < z < 2.33$$

$$\therefore \quad -2.33 < \frac{\overline{x} - 504}{\frac{3}{\sqrt{20}}} < 2.33$$

$$\therefore \quad 504 - 2.33 \times \frac{3}{\sqrt{20}} < \overline{x} < 504 + 2.33 \times \frac{3}{\sqrt{20}}$$

$$\therefore \quad 502.4 < \overline{x} < 505.6$$

So, the quality controller should not adjust the machine for $502 \text{ mL} < \overline{x} < 506 \text{ mL}$.

EXERCISE 30D

1 $\overline{x} = 17.14, \quad s = 4.365, \quad n = 24$

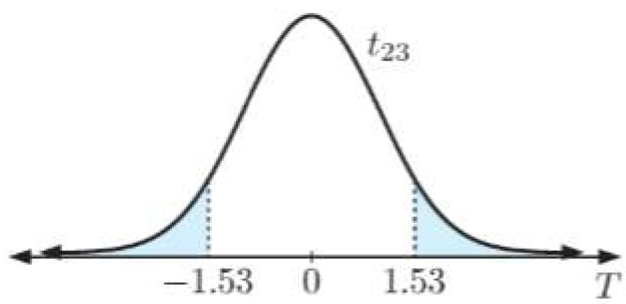
The hypotheses are H_0: $\mu = 18.5$
H_1: $\mu \neq 18.5$

a **i** $t = \dfrac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$

$= \dfrac{17.14 - 18.5}{\frac{4.365}{\sqrt{24}}}$

≈ -1.53

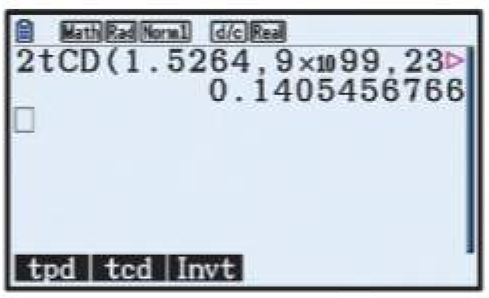
ii The null distribution is $T \sim t_{24-1}$ or $T \sim t_{23}$.



iii $p\text{-value} = 2 \times \text{P}(T \geqslant |t|)$ where $T \sim t_{23}$

$\approx 2 \times \text{P}(T \geqslant |-1.53|)$

≈ 0.141

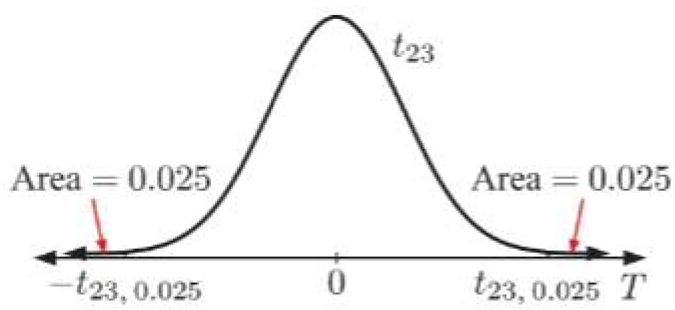


b $\alpha = 0.05$

i Since $p\text{-value} > 0.05 = \alpha$, we do not reject H_0 as there is insufficient evidence to do so. We accept at a 5% level that $\mu = 18.5$.

ii The critical region

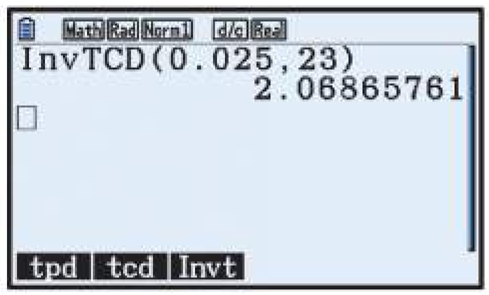
$\mathcal{C} = \{t \mid t \leqslant -t_{23,\, 0.025} \text{ or } t \geqslant t_{23,\, 0.025}\}$



Using technology, $t_{23,\,0.025} \approx 2.069$.

$\therefore \quad C = \{t \mid t \leqslant -2.069 \text{ or } t \geqslant 2.069\}$

Since $t \notin C$, we do not reject H_0 as there is insufficient evidence to do so. We accept at a 5% level that $\mu = 18.5$.



2 *Step 1*: Let μ be the population mean price for a 750 mL bottle.

The hypotheses to be considered are:

H_0: $\mu = 13.45$ {the mean price has not changed}

H_1: $\mu < 13.45$ {the mean price has fallen}

Step 2: The significance level is $\alpha = 0.02$.

Step 3: $\overline{x} = \$13.30$, $\mu_0 = \$13.45$, $s = \$0.25$, $n = 389$ bottles

The observed value of the test statistic is $t = \dfrac{13.30 - 13.45}{\frac{0.25}{\sqrt{389}}} \approx -11.8$

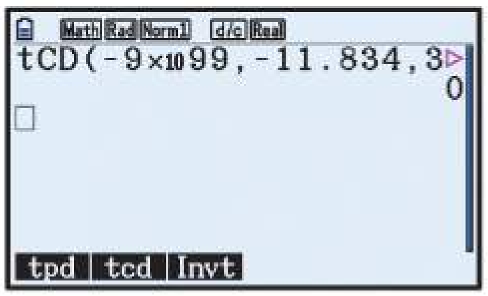
Step 4: Since $n = 389$ is sufficiently large, we can use the Central Limit Theorem to assume normality.

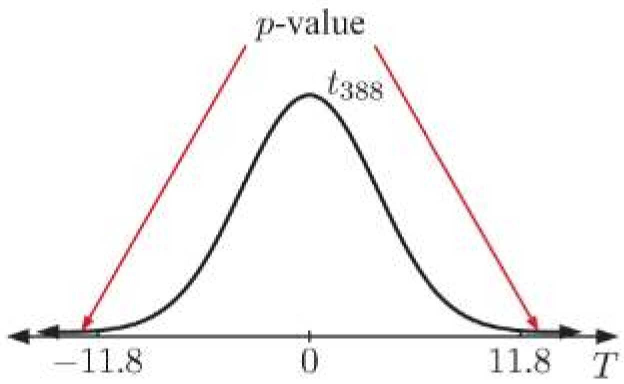
Since H_1: $\mu < 13.45$,

$p\text{-value} = \mathrm{P}(T \leqslant t)$ where $T \sim t_{388}$

$\approx \mathrm{P}(T \leqslant -11.8)$

≈ 0





Step 5: Since $p\text{-value} < 0.02 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 2% significance level.

Step 6: Since we have accepted H_1, we conclude that the mean price has fallen at a 2% level of significance.

3 *Step 1*: Let μ be the population mean volume of vinegar.

The hypotheses to be considered are:

H_0: $\mu = 500$ {the machine is filling the bottles correctly}

H_1: $\mu < 500$ {the machine is underfilling the bottles}

Step 2: The significance level is $\alpha = 0.01$.

Step 3: $\overline{x} = 499$ mL, $\mu_0 = 500$ mL, $s = 1.2$ mL, $n = 10$ bottles

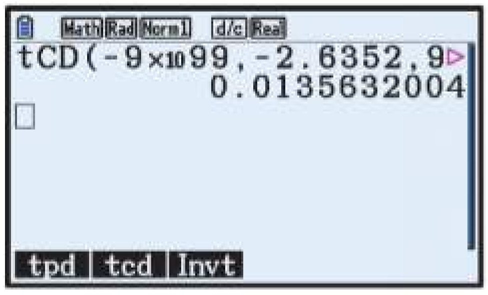
The observed value of the test statistic is $t = \dfrac{499 - 500}{\frac{1.2}{\sqrt{10}}} \approx -2.64$

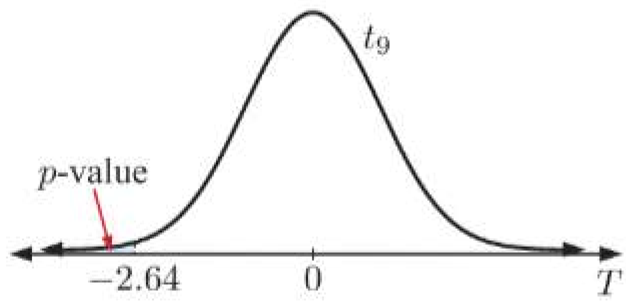
Step 4: Since H_1: $\mu < 500$,

p-value $= \mathrm{P}(T \leqslant t)$ where $T \sim t_9$

$\approx \mathrm{P}(T \leqslant -2.64)$

≈ 0.0136





Step 5: Since p-value $> 0.01 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the machine is not underfilling the bottles at a 1% level of significance.

4 *Step 1*: Let μ be the population mean screw length. The quality controller wants to determine whether the machine should be adjusted, so the hypotheses to be considered are:

H_0: $\mu = 2.00$ {the machine is producing screws of the correct length}

H_1: $\mu \neq 2.00$ {the machine is not producing screws of the correct length}

Step 2: The significance level is $\alpha = 0.02$.

Step 3: $\overline{x} = 2.04$ cm, $\mu_0 = 2.00$ cm, $s = 0.09$ cm, $n = 15$ screws

The observed value of the test statistic is

$$t = \frac{2.04 - 2.00}{\frac{0.09}{\sqrt{15}}} \approx 1.72$$

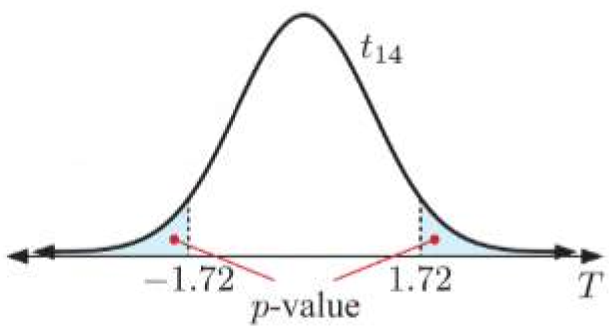
Step 4: Since H_1: $\mu \neq 2.00$,

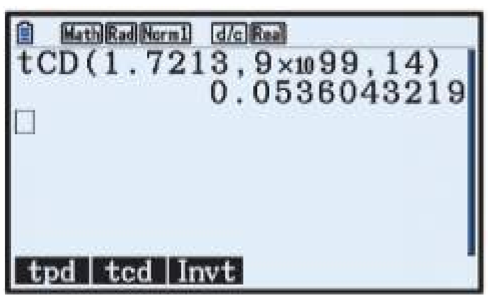
p-value $= 2 \times \mathrm{P}(T \geqslant |\,t\,|)$ where $T \sim t_{14}$

$\approx 2 \times \mathrm{P}(T \geqslant 1.72)$

$\approx 2 \times 0.0536$

≈ 0.107





Step 5: Since p-value $> 0.02 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 on a 2% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude on a 2% significance level that the mean length is significantly different from 2.00. Adjusting the machine is not justified.

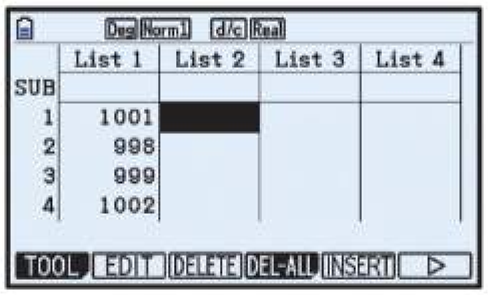
5 *Step 1*: Let μ be the population mean mass of the bags of sugar. We want to determine whether the machine is *under*-filling the bags, so the hypotheses to be considered are:

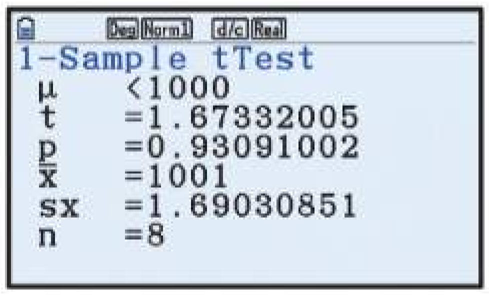
H_0: $\mu = 1000$ {the machine is filling the bags correctly}

H_1: $\mu < 1000$ {the machine is not filling the bags enough}

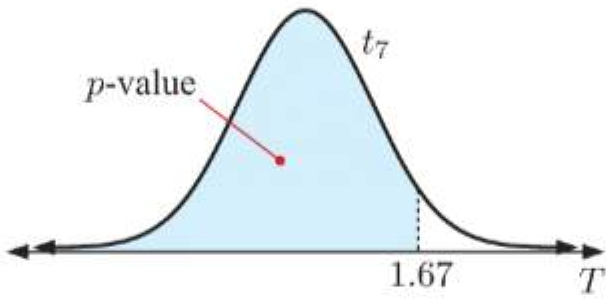
Step 2: The significance level is $\alpha = 0.01$.

Steps 3 and 4:





Using technology, the test statistic $t \approx 1.67$ and the p-value ≈ 0.931.



Step 5: Since p-value $> 0.01 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude on a 1% significance level that the bags have been under-filled.

6 **a** The distribution of the carrots' weights is likely to be centred around the mean, and will vary randomly due to factors such as amount of sunlight and soil conditions, so it is reasonable to assume that their weights are normally distributed.

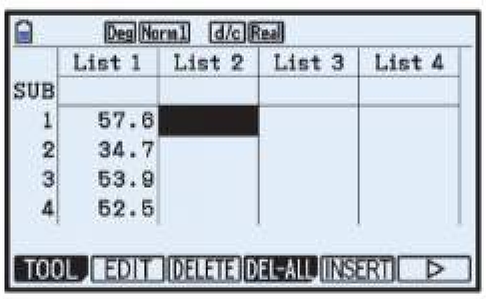
b *Step 1*: Let μ be the population mean weight of the carrots. The buyer wants to determine whether the mean is more than 50 g, so the hypotheses to be considered are:

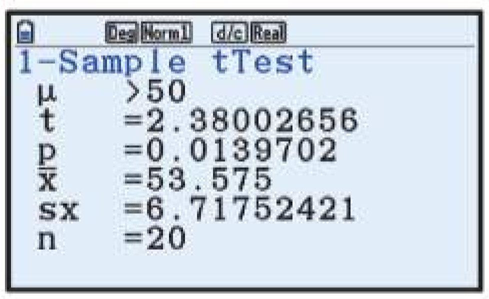
H_0: $\mu = 50$ {the carrots have a mean weight of 50 g}

H_1: $\mu > 50$ {the carrots have a mean weight of more than 50 g}

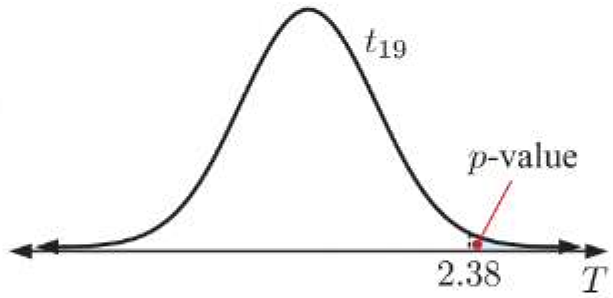
Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:





Using technology, the test statistic $t \approx 2.38$ and the p-value ≈ 0.0140.



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude on a 5% significance level that the mean weight is greater than 50 g. The buyer will therefore purchase the crop.

7 **a** *Step 1*: Let μ be the population mean income of the golf club members.
The hypotheses that should be tested are:

H_0: $\mu = 95\,000$ {the mean income is €95 000}

H_1: $\mu > 95\,000$ {the mean income is greater than €95 000}

Step 2: The significance level is $\alpha = 0.02$.

Step 3: $\overline{x} = €96\,318$, $\mu_0 = €95\,000$, $s = €14\,268$, $n = 113$

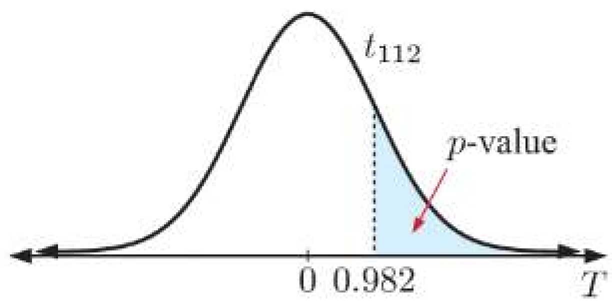
The observed value of the test statistic is

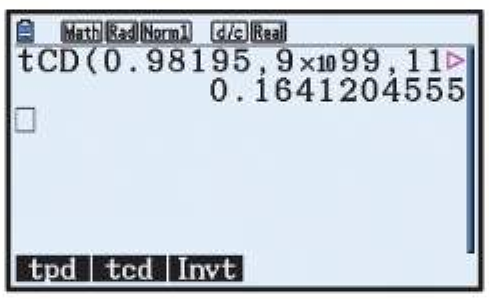
$$t = \frac{96\,318 - 95\,000}{\frac{14\,268}{\sqrt{113}}} \approx 0.982$$

Step 4: Since $n = 113$ is sufficiently large, we can use the Central Limit Theorem to assume normality.

Since H_1: $\mu > 95\,000$,

$$\begin{aligned} p\text{-value} &= \mathrm{P}(T \geqslant t) \quad \text{where } T \sim t_{112} \\ &\approx \mathrm{P}(T \geqslant 0.982) \\ &\approx 0.164 \end{aligned}$$





Step 5: Since p-value $> 0.02 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 0.02 significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the mean income is €95 000. There is insufficient evidence to support the management's claim.

b In **a**, H_0 was accepted. So if the conclusion made by the statistician is incorrect, then H_0 would be false.

$\therefore$ a Type II error would be made.

EXERCISE 30E

1 **a**

Athlete	A	B	C	D	E	F	G	H	I	J
Start (x_i)	10.3	10.5	10.6	10.4	10.8	11.1	9.9	10.6	10.6	10.8
End (y_i)	10.2	10.3	10.8	10.1	10.8	9.7	9.9	10.6	10.4	10.6
$d_i = y_i - x_i$	-0.1	-0.2	0.2	-0.3	0	-1.4	0	0	-0.2	-0.2

Athlete	K	L	M	N	O	P	Q	R	S	T
Start (x_i)	11.2	11.4	10.9	10.7	10.7	10.9	11.0	10.3	10.5	10.6
End (y_i)	10.8	11.2	11.0	10.5	10.7	11.0	11.1	10.5	10.3	10.2
$d_i = y_i - x_i$	-0.4	-0.2	0.1	-0.2	0	0.1	0.1	0.2	-0.2	-0.4

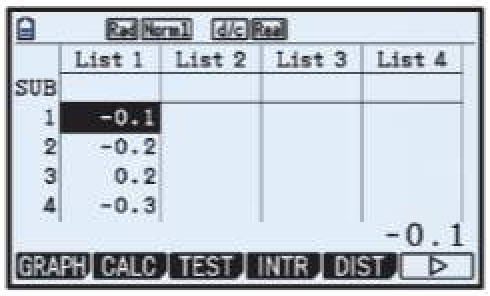
b **i** H_0: $\mu_D = 0$ {the athletes' performance has not changed}
H_1: $\mu_D < 0$ {the athletes' performance has improved}

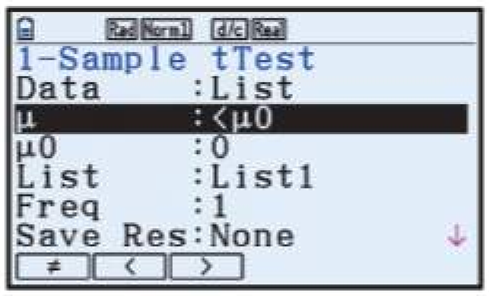
ii H_0: $\mu_D = 0$ {the athletes' performance has not changed}
H_1: $\mu_D \neq 0$ {the athletes' performance has changed}

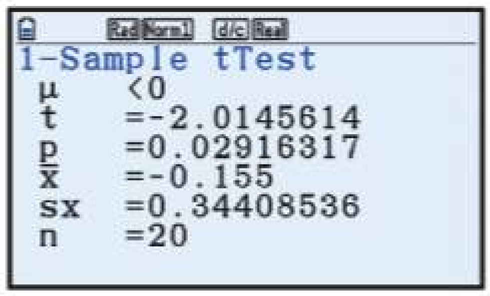
c *Step 1*: From **b i**, the hypotheses that should be considered are:
H_0: $\mu_D = 0$
H_1: $\mu_D < 0$

Step 2: The significance level is $\alpha = 0.05$.

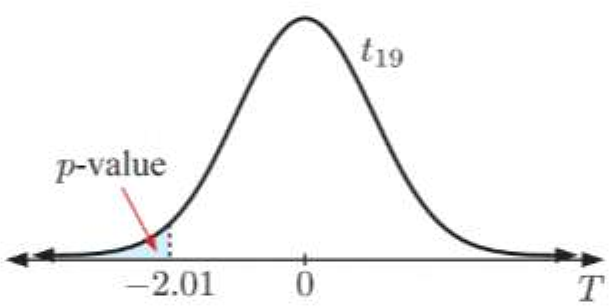
Steps 3 and 4:







Using technology, the test statistic $t \approx -2.01$ and the p-value ≈ 0.0292.



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude on a 5% significance level that the mean difference between the end and start time is less than zero. The sprint times of the athletes have improved.

2

Before tutoring (x_i)	15	17	25	11	28	20	23	34	27	14	26	26
After tutoring (y_i)	20	16	25	18	28	19	26	37	31	13	27	20
$d_i = y_i - x_i$	5	-1	0	7	0	-1	3	3	4	-1	1	-6

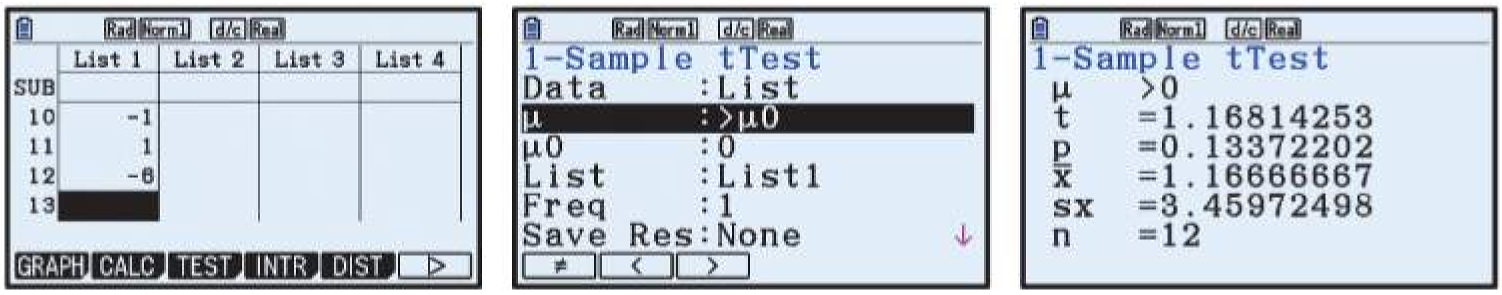
Step 1: The hypotheses to be considered are:

H_0: $\mu_D = 0$ {tutoring made no difference}

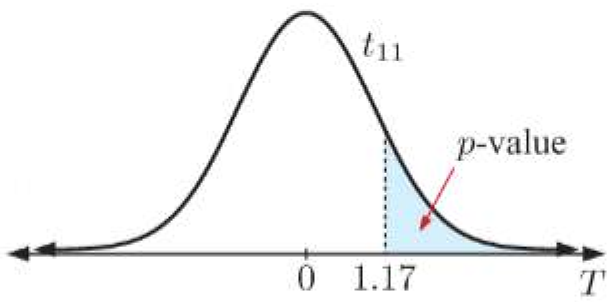
H_1: $\mu_D > 0$ {tutoring improved test results}

Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:



Using technology, the test statistic $t \approx 1.17$ and the p-value ≈ 0.134.



Step 5: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the mean difference between the test results is zero. There is insufficient evidence to support that there has been an improvement.

3 **a**

Child	A	B	C	D	E	F	G	H	I	J	K
Age 12 (x_i)	76	81	59	67	90	74	78	71	69	72	82
Age 13 (y_i)	79	82	66	72	93	76	77	82	75	77	86
$d_i = y_i - x_i$	3	1	7	5	3	2	-1	11	6	5	4

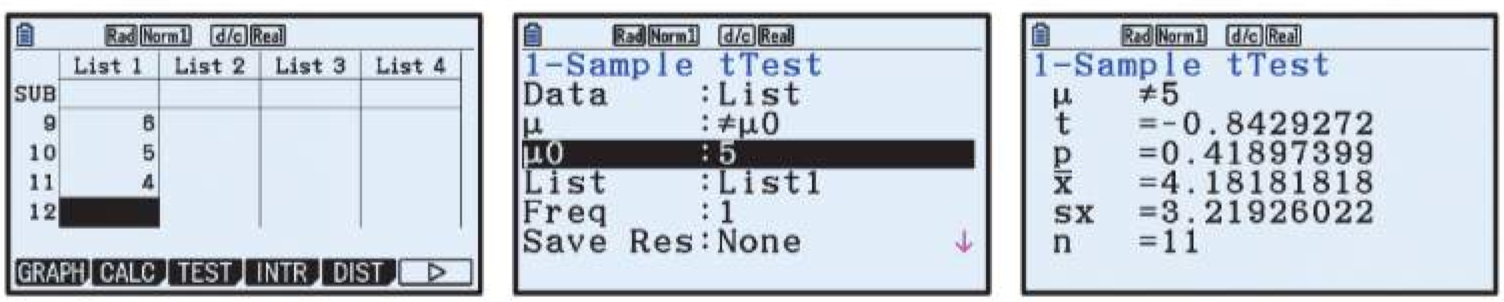
b *Step 1*: The hypotheses to be considered are:

H_0: $\mu_D = 5$ {the average throwing speed difference is 5 km h^{-1}}

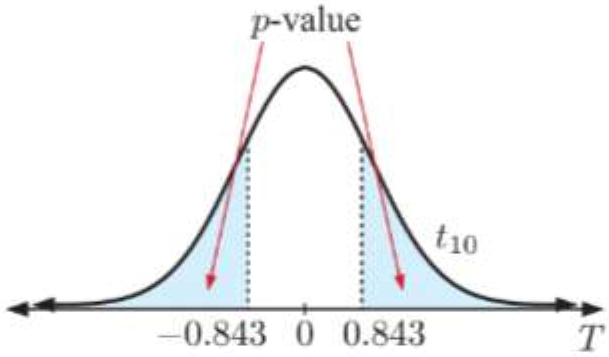
H_1: $\mu_D \neq 5$ {the average throwing speed difference is not 5 km h^{-1}}

Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:



Using technology, the test statistic $t \approx -0.843$ and the p-value ≈ 0.419.



Step 5: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the mean difference in throwing speeds is 5 km h^{-1}. There is insufficient evidence to reject the sports commission's claim.

EXERCISE 30F

1 **a** Let μ_1 be the population mean weight of tomatoes from last year's crop, and μ_2 be the population mean weight of tomatoes from this year's crop.

Frank wants to determine whether the new fertiliser increased the mean weight of the crop, so the hypotheses to be considered are:

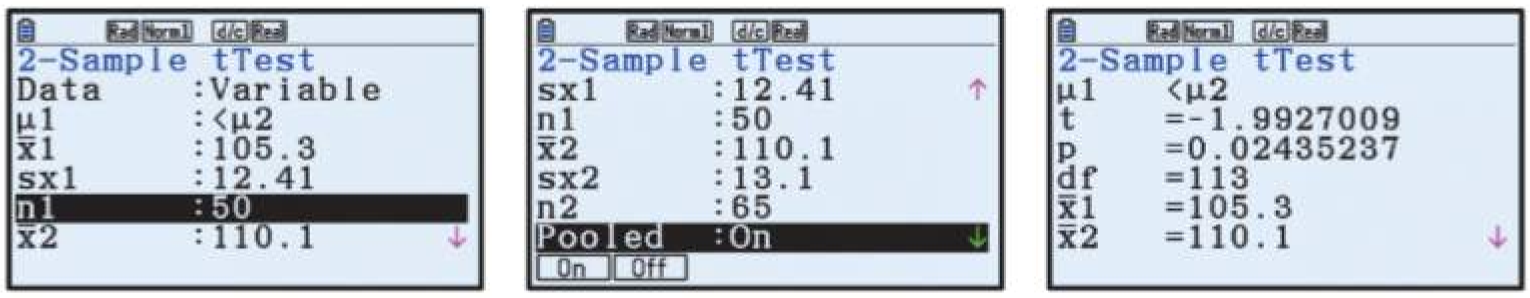
H_0: $\mu_1 = \mu_2$ {the mean weight of this year's crop is the same as last year's crop}

H_1: $\mu_1 < \mu_2$ {the mean weight of this year's crop is more than last year's crop}

b The significance level $\alpha = 0.01$.

For last year's sample, $\overline{x} = 105.3$, $s = 12.41$, and $n = 50$.

For this year's sample, $\overline{x} = 110.1$, $s = 13.1$, and $n = 65$.



Using technology, the test statistic $t \approx -1.99$ and the p-value ≈ 0.0244.

Since the p-value $> 0.01 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 1% significance level. Frank therefore cannot conclude that the new fertiliser was more effective.

2 *Step 1*: Let μ_1 be the population mean sleeping time for middle school students, and μ_2 be the population mean sleeping time for high school students.

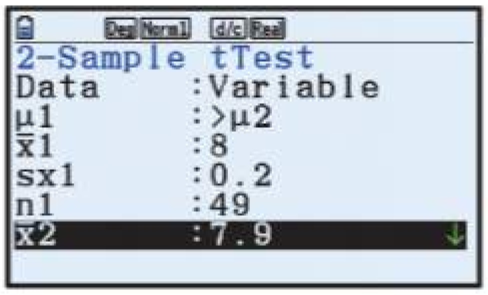
The hypotheses that should be considered are:

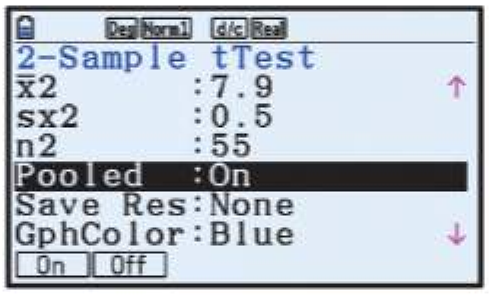
H_0: $\mu_1 = \mu_2$ {middle school and high school students sleep the same amount on average}

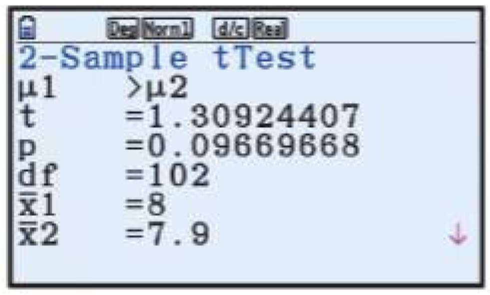
H_1: $\mu_1 > \mu_2$ {high school students sleep less on average than middle school students}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:







Using technology, the value of the test statistic $t \approx 1.31$.

Step 4: From the screenshots above, the p-value ≈ 0.0967.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude on a 5% significance level that high school students sleep less on average than middle school students. The researcher's claim is therefore invalid.

3 *Step 1*: Let μ_A be the population mean of the height of the seedlings using *Brand A*, and μ_B be the population mean of the height of the seedlings using *Brand B*.

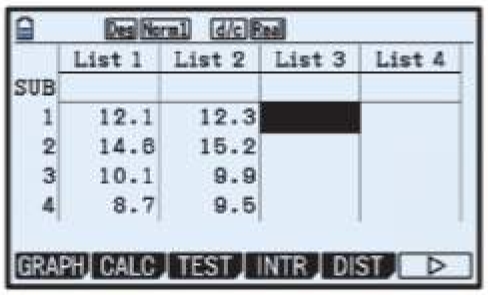
The hypotheses that should be considered are:

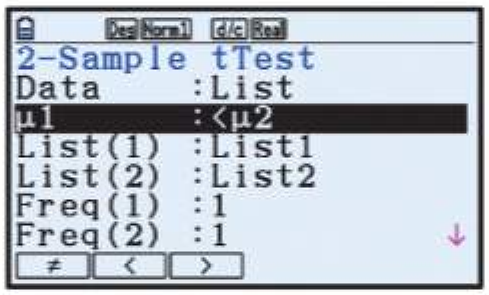
H_0: $\mu_A = \mu_B$ {the seedlings grow the same amount in both brands of soil}

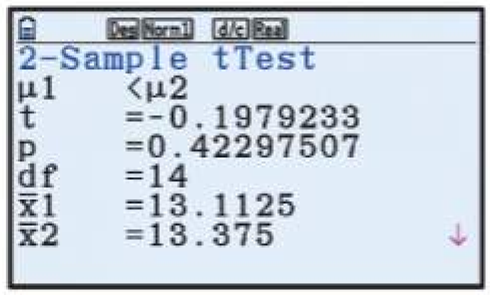
H_1: $\mu_A < \mu_B$ {the seedlings grow higher in *Brand B*'s soil than *Brand A*'s soil}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:







Using technology, the value of the test statistic $t \approx -0.198$.

Step 4: From the screenshots above, the p-value ≈ 0.423.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude on a 5% significance level that *Brand B*'s soil improves the growth of seedlings. *Brand B*'s guarantee is therefore invalid.

4 **a** *Step 1*: Let μ_1 be the population mean of Jesiah's 100 m sprint times, and μ_2 be the population mean of Billy's 100 m sprint times.

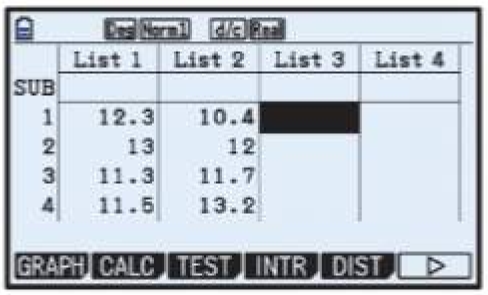
The hypotheses that should be considered are:

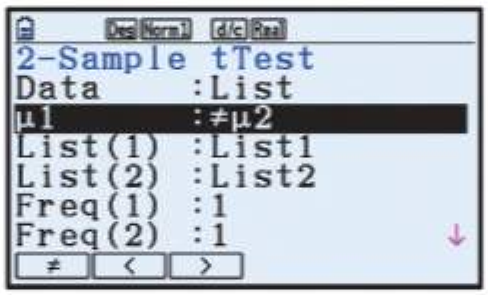
H_0: $\mu_1 = \mu_2$ {there is no difference between the runners' mean 100 m sprint times}

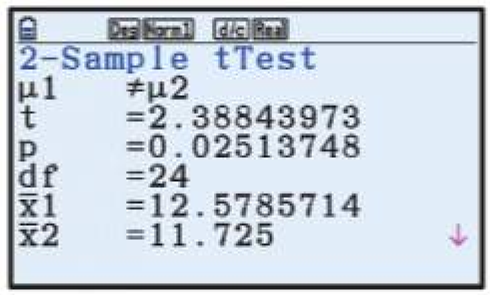
H_1: $\mu_1 \neq \mu_2$ {there is a significant difference between the runners' mean 100 m sprint times}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:







Using technology, the value of the test statistic $t \approx 2.39$.

Step 4: From the screenshots above, the p-value ≈ 0.0251.

Step 5: Since p-value $< 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that there is a significant difference between the runners' times on a 5% level of significance.

b The mean time for Jesiah was about 12.6 seconds and the mean time for Billy was about 11.7 seconds. So, Billy is faster.

EXERCISE 30G

1 The test statistic $T = \sum_{i=1}^{n} X_i$ is a Poisson random variable which can only take non-negative integer values, so the observed test statistic $t = \sum_{i=1}^{n} x_i$ must also be a non-negative integer.

2 **a** H_0: $\lambda = 2$ {the specimen is 5000 years old}
H_1: $\lambda < 2$ {the specimen is older than 5000 years}

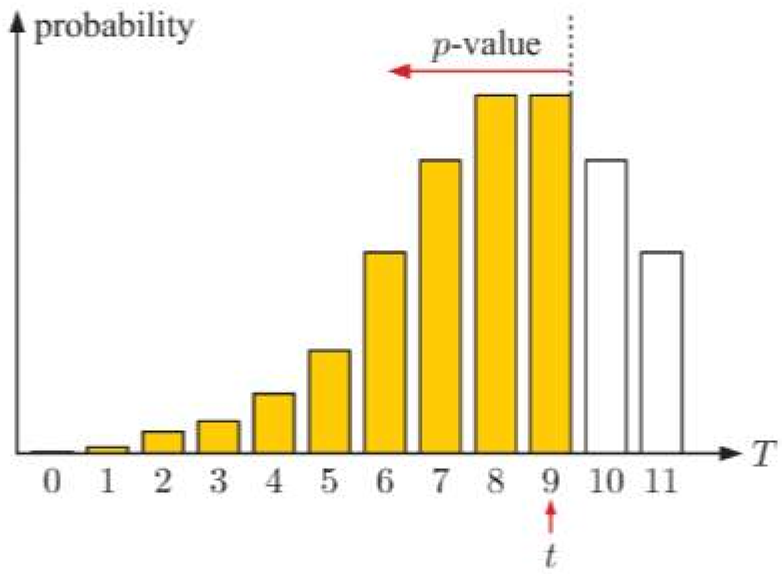
b The significance level is $\alpha = 0.05$.

The test statistic is $t = 9$.

$n \times \lambda_0 = 5 \times 2 = 10$

So the null distribution is $T \sim \text{Po}(10)$

$\therefore$ p-value $= \text{P}(T \leqslant t)$
$= \text{P}(T \leqslant 9)$
≈ 0.458



Since p-value $> 0.05 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

The scientist cannot conclude that the specimen is older than 5000 years.

3 H_0: $\lambda = 5.1$, $\alpha = 0.05$
H_1: $\lambda > 5.1$

a $t = n\overline{x}$
$= 16 \times 5.375$
$= 86$

b $n \times \lambda_0 = 16 \times 5.1 = 81.6$

So the null distribution is $T \sim \text{Po}(81.6)$

$\therefore$ $p\text{-value} = \text{P}(T \geqslant t)$
$= \text{P}(T \geqslant 86)$
≈ 0.328

Since p-value $> 0.05 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on an $\alpha = 0.05$ level of significance. So, H_0 should not be rejected.

4 **a** H_0: $\lambda = 30$ {30 cars pass through the intersection per minute}
H_1: $\lambda > 30$ {more than 30 cars pass through the intersection per minute}

b The significance level is $\alpha = 0.1$.

The test statistic is $t = 1100$.

$n \times \lambda_0 = 30 \times 30 = 900$

So the null distribution is $T \sim \text{Po}(900)$

$\therefore$ $p\text{-value} = \text{P}(T \geqslant t)$
$= \text{P}(T \geqslant 1100)$
$\approx 6.38 \times 10^{-11}$

Since p-value < 0.1, we have enough evidence to reject H_0 in favour of H_1 on a 10% significance level. We therefore conclude that the intersection should be upgraded.

c We assume that the number of cars passing through the intersection per minute during peak time has the same distribution every day and does not change during a particular peak time. Thus the 30 minute "sample" may not be entirely representative of the situation. If the data was collected on a different day, or earlier or later in this particular peak time, we may get very different results.

5 **a** Let λ be the mean number of siblings per student.

The hypotheses that Emily should consider are:

H_0: $\lambda = 1.5$ {mean number of siblings is 1.5}

H_1: $\lambda > 1.5$ {mean number of siblings is greater than 1.5}

b The significance level is $\alpha = 0.05$.

The test statistic is $t = \sum_{i=1}^{n} x_i = 42$.

$n \times \lambda_0 = 30 \times 1.5 = 45$

So the null distribution is $T \sim \text{Po}(45)$

$\therefore$ $p\text{-value} = \text{P}(T \geqslant t)$

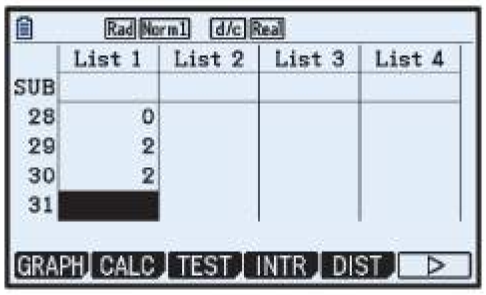
$= \text{P}(T \geqslant 42)$

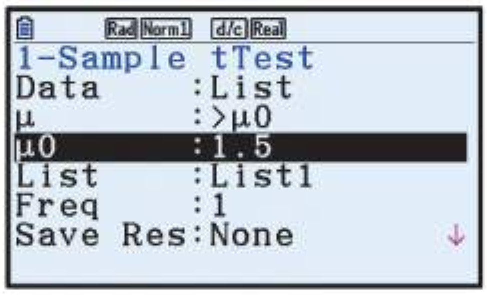
≈ 0.693

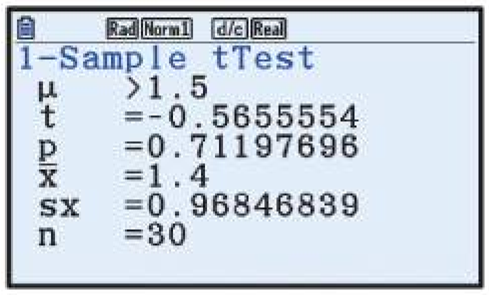
Since p-value $> 0.05 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Emily cannot conclude that the mean number of siblings per student is greater than 1.5.

c The significance level is $\alpha = 0.05$.







Using technology, the test statistic $t \approx -0.566$ and the p-value ≈ 0.712.

Since p-value $> 0.05 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Emily cannot conclude that the mean number of siblings per student is greater than 1.5.

d Both tests give similar results and the same conclusion. However, because the *number of siblings* is a discrete variable and the sample size may not be sufficiently large enough for the Central Limit Theorem to apply, the test in **b** is more appropriate for Emily.

6 **a** H_0: $\lambda = 2$ against H_1: $\lambda < 2$, $n = 3$, $\alpha = 0.05$

The null distribution is $T \sim \text{Po}(6)$.

t	$p\text{-value} = \text{P}(T \leqslant t)$	
0	$\approx 0.002\,48$	p-value $\leqslant \alpha$
1	≈ 0.0174	
2	≈ 0.0620	p-value $> \alpha$
3	≈ 0.151	
$\vdots$	$\vdots$	

i $\mathcal{C} = \{t \mid 0 \leqslant t \leqslant 1\} = \{0, 1\}$

ii $\mathcal{A} = \{t \mid t \geqslant 2\}$

iii The value in $\mathcal{C}$ with the largest p-value is 1.

$\therefore$ the critical value $c = 1$.

b H_0: $\lambda = 1.5$ against H_1: $\lambda > 1.5$, $n = 5$, $\alpha = 0.01$

The null distribution is $T \sim \text{Po}(7.5)$.

t	p-value $= \text{P}(T \geqslant t)$
0	1
1	≈ 0.999
2	≈ 0.995
3	≈ 0.980
4	≈ 0.941
5	≈ 0.868
6	≈ 0.759
7	≈ 0.622
8	≈ 0.475
9	≈ 0.338
10	≈ 0.224
11	≈ 0.138
12	≈ 0.0792
13	≈ 0.0427
14	≈ 0.0216
15	≈ 0.0103
16	$\approx 0.004\,61$
17	$\approx 0.001\,96$
$\vdots$	$\vdots$

$t = 0$ to 15: p-value $> \alpha$

$t \geqslant 16$: p-value $\leqslant \alpha$

i $\mathcal{C} = \{t \mid t \geqslant 16\}$

ii $\mathcal{A} = \{t \mid 0 \leqslant t \leqslant 15\}$
$= \{0, 1, ..., 15\}$

iii The value in $\mathcal{C}$ with the largest p-value is 16.
$\therefore$ the critical value $c = 16$.

c H_0: $\lambda = 0.95$ against H_1: $\lambda < 0.95$, $n = 10$, $\alpha = 0.1$

The null distribution is $T \sim \text{Po}(9.5)$.

t	p-value $= \text{P}(T \leqslant t)$
0	$\approx 0.000\,074\,9$
1	$\approx 0.000\,786$
2	$\approx 0.004\,16$
3	≈ 0.0149
4	≈ 0.0403
5	≈ 0.0885
6	≈ 0.165
7	≈ 0.269
8	≈ 0.392
$\vdots$	$\vdots$

$t = 0$ to 5: p-value $\leqslant \alpha$

$t \geqslant 6$: p-value $> \alpha$

i $\mathcal{C} = \{t \mid 0 \leqslant t \leqslant 5\}$
$= \{0, 1, 2, 3, 4, 5\}$

ii $\mathcal{A} = \{t \mid t \geqslant 6\}$

iii The value in $\mathcal{C}$ with the largest p-value is 5.
$\therefore$ the critical value $c = 5$.

7 H_0: $\lambda = 6$ against H_1: $\lambda > 6$, $n = 3$, $\alpha = 0.05$
The null distribution is $T \sim \text{Po}(18)$.

t	p-value $= \text{P}(T \geqslant t)$	
$\vdots$	$\vdots$	p-value $> \alpha$
19	≈ 0.438	
20	≈ 0.349	
21	≈ 0.269	
22	≈ 0.201	
23	≈ 0.145	
24	≈ 0.101	
25	≈ 0.0683	
26	≈ 0.0446	p-value $\leqslant \alpha$
27	≈ 0.0282	
$\vdots$	$\vdots$	

The critical region is $\mathcal{C} = \{t \mid t \geqslant 26\}$.

The test statistic $t = 20 \notin \mathcal{C}$, so we do not have sufficient evidence to reject H_0 in favour of H_1 on an $\alpha = 0.05$ level of significance.

So, we accept H_0 and conclude that $\lambda = 6$.

8 H_0: $\lambda = 4$ against H_1: $\lambda < 4$, $n = 2$
The null distribution is $T \sim \text{Po}(8)$.

t	p-value $= \text{P}(T \leqslant t)$	
0	$\approx 0.000\,335$	$\mathcal{C}$
1	$\approx 0.003\,02$	
2	≈ 0.0138	
3	≈ 0.0424	
4	≈ 0.0996	$\mathcal{A}$
$\vdots$	$\vdots$	

$t = 3$ is the value in $\mathcal{C}$ with the largest p-value ≈ 0.0424. So the significance level $\alpha \geqslant 0.0424$ because all of the values in $\mathcal{C}$ must result in rejecting H_0. Similarly, $t = 4$ is the value in the acceptance region $\mathcal{A}$ with the smallest p-value ≈ 0.0996. So, the significance level $\alpha < 0.0996$ because all of the values in $\mathcal{A}$ must result in retaining H_0. So, $0.0424 \leqslant \alpha < 0.0996$.

EXERCISE 30H

1 **a** Let p be the probability that the magician rolls a six with a fair die.
The hypotheses to be considered are:

H_0: $p = 0.9$ {the magician rolls a 6 nine times out of ten}
H_1: $p < 0.9$

b $n = 6$

The magician's claim will be accepted if he rolls at least four sixes.

$\therefore$ the magician's claim will be rejected if he rolls less than four sixes.

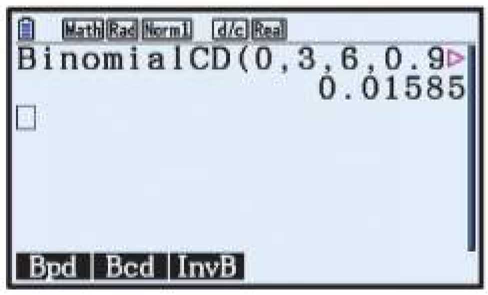
Let X be the number of sixes the magician rolls out of $n = 6$ rolls.

Under H_0, $X \sim B(6, 0.9)$

Now significance level $= P(\text{reject } H_0 \mid H_0 \text{ true})$

$= P(X < 4 \mid X \sim B(6, 0.9))$

≈ 0.0159



2 **a** Let p be the probability of rolling a four with the die.

The hypotheses to be considered are:

H_0: $p = \frac{1}{6}$ {the die is fair}

H_1: $p > \frac{1}{6}$ {the die is biased towards the number four}

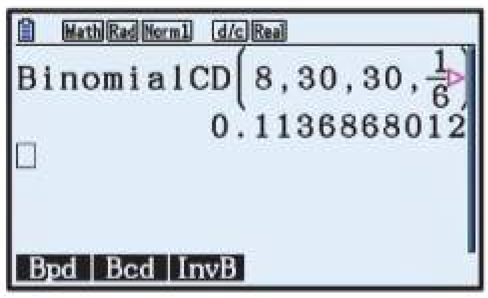
b The significance level is $\alpha = 0.05$.

c The test statistic X is the number of fours rolled in 30 rolls of the die.

The null distribution is $X \sim B\left(30, \frac{1}{6}\right)$.

d Since H_1: $p > \frac{1}{6}$, p-value $= P(X \geqslant 8)$

≈ 0.114



e Since p-value $> 0.05 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

The student does not have sufficient evidence to conclude that her die is biased towards the number 4.

3 **a** Let p be the probability of tossing heads with the coin.

The hypotheses to be considered are:

H_0: $p = \frac{1}{2}$ {the coin is unbiased}

H_1: $p < \frac{1}{2}$ {the coin is biased towards tails}

b The test statistic X is the number of heads in 80 tosses of the coin.

The null distribution is $X \sim B\left(80, \frac{1}{2}\right)$.

c Since H_1: $p < \frac{1}{2}$, p-value $= P(X \leqslant 37)$

≈ 0.288

d Since p-value $> 0.05 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.
There is not enough evidence to conclude that the coin is biased towards tails.

4 *Step 1*: Let p be the probability that a migraine sufferer's condition improves.
The hypotheses to be considered are:

H_0: $p = \frac{1}{2}$ {the new medication makes no difference}

H_1: $p > \frac{1}{2}$ {the new medication increases the probability of a participant's condition improving}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: The observed value of the test statistic is $x = 38$.

Step 4: The null distribution is $X \sim \text{B}\left(60, \frac{1}{2}\right)$.

Since H_1: $p > \frac{1}{2}$, p-value $= \text{P}(X \geqslant 38)$
≈ 0.0259

Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Step 6: Since we have accepted H_1, we conclude that the new medicine is better.

5 *Step 1*: Let p be the proportion of lottery tickets that win a prize.
The hypotheses to be considered are:

H_0: $p = 0.05$ {5% of tickets win a prize}

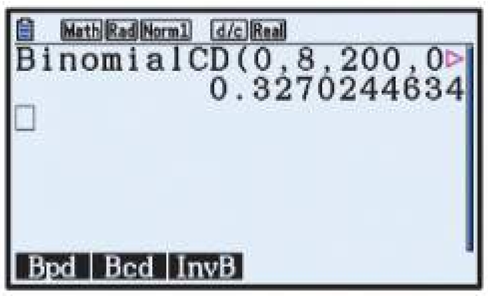
H_1: $p < 0.05$ {less than 5% of tickets win a prize}

Step 2: The significance level is $\alpha = 0.02$.

Step 3: The observed value of the test statistic is $x = 8$.

Step 4: The null distribution is $X \sim \text{B}(200, 0.05)$.

Since H_1: $p < 0.05$, p-value $= \text{P}(X \leqslant 8)$
≈ 0.327



Step 5: Since p-value $> 0.02 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 2% significance level.

Step 6: Since we have accepted H_0, we cannot conclude that the percentage of winning tickets is less than 5%.
The consumer group's suspicions are not supported.

6 *Step 1*: Let p be the proportion of adults who expect to be better off financially in 5 years' time. The hypotheses to be considered are:

H_0: $p = 0.206$ {people's feelings about their future financial situation have not changed}

H_1: $p > 0.206$ {people's feelings about their future financial situation have improved}

Step 2: The significance level is $\alpha = 0.01$.

Step 3: The observed value of the test statistic is $x = 66$.

Step 4: The null distribution is $X \sim \text{B}(300, 0.206)$.

Since H_1: $p > 0.206$, $p\text{-value} = \text{P}(X \geqslant 66)$
≈ 0.295

Step 5: Since $p\text{-value} > 0.01 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 1% significance level.

Step 6: We have accepted H_0 which does not support the hypothesis that people's feelings about their future financial situation have improved.

7 **a** H_0: $p = 0.6$ against H_1: $p < 0.6$, $n = 5$, $\alpha = 0.05$

The null distribution is $X \sim \text{B}(5, 0.6)$.

x	$p\text{-value} = \text{P}(X \leqslant x)$	
0	0.010 24	$p\text{-value} \leqslant \alpha$
1	0.087 04	
2	0.317 44	
3	0.663 04	$p\text{-value} > \alpha$
4	0.922 24	
5	1	

i $\mathcal{C} = \{0\}$

ii $\mathcal{A} = \{1, 2, 3, 4, 5\}$

iii The only value in $\mathcal{C}$ is 0.
$\therefore$ the critical value $c = 0$.

b H_0: $p = 0.35$ against H_1: $p > 0.35$, $n = 8$, $\alpha = 0.01$

The null distribution is $X \sim \text{B}(8, 0.35)$.

x	$p\text{-value} = \text{P}(X \geqslant x)$	
0	1	
1	≈ 0.968	
2	≈ 0.831	
3	≈ 0.572	$p\text{-value} > \alpha$
4	≈ 0.294	
5	≈ 0.106	
6	≈ 0.0253	
7	$\approx 0.003\,57$	$p\text{-value} \leqslant \alpha$
8	$\approx 0.000\,225$	

i $\mathcal{C} = \{7, 8\}$

ii $\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6\}$

iii The value in $\mathcal{C}$ with the largest p-value is 7.
$\therefore$ the critical value $c = 7$.

c H_0: $p = 0.74$ against H_1: $p < 0.74$, $n = 10$, $\alpha = 0.1$

The null distribution is $X \sim B(10, 0.74)$.

x	p-value $= P(X \leqslant x)$	
0	$\approx 1.41 \times 10^{-6}$	p-value $\leqslant \alpha$
1	$\approx 4.16 \times 10^{-5}$	
2	$\approx 0.000\,556$	
3	$\approx 0.004\,46$	
4	≈ 0.0239	
5	≈ 0.0904	
6	≈ 0.248	p-value $> \alpha$
7	≈ 0.504	
8	≈ 0.778	
9	≈ 0.951	
10	1	

i $\mathcal{C} = \{0, 1, 2, 3, 4, 5\}$

ii $\mathcal{A} = \{6, 7, 8, 9, 10\}$

iii The value in $\mathcal{C}$ with the largest p-value is 5.

$\therefore$ the critical value $c = 5$.

8 **a** H_0: $p = 0.5$ against H_1: $p > 0.5$, $n = 4$, $\alpha = 0.05$

The null distribution is $X \sim B(4, 0.5)$.

x	p-value $= P(X \geqslant x)$	
0	1	p-value $> \alpha$
1	0.9375	
2	0.6875	
3	0.3125	
4	0.0625	

i $\mathcal{C} = \varnothing$ (empty set)

ii $\mathcal{A} = \{0, 1, 2, 3, 4\}$

b No, the critical region is empty, so there is no value the test statistic can take which leads to rejecting H_0.

c To test his hypotheses more accurately, Judeau should toss the coin more times.

ACTIVITY — USING A Z-TEST TO TEST HYPOTHESES ABOUT A POPULATION PROPORTION

1 **a** $X \sim N(np_0,\ np_0(1-p_0))$

b $\widehat{p} \sim N\left(p_0,\ \dfrac{p_0(1-p_0)}{n}\right)$

2 **a** $\widehat{p} = \overline{X}_n$, and $\overline{X}_n \sim N\left(\mu_0,\ \left(\dfrac{\sigma}{\sqrt{n}}\right)^2\right)$ under the null hypothesis.

$$\therefore\ \widehat{p} \sim N\left(\mu_0,\ \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

So from **1 b**, $\mu_0 = p_0$ and $\left(\dfrac{\sigma}{\sqrt{n}}\right)^2 = \dfrac{p_0(1-p_0)}{n}$

$$\therefore \quad \sigma^2 = p_0(1-p_0)$$

$$\therefore \quad \sigma = \sqrt{p_0(1-p_0)}$$

i $\mu_0 = p_0$ **ii** $\sigma = \sqrt{p_0(1-p_0)}$

b $$Z = \frac{\overline{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{\widehat{p} - p_0}{\frac{\sqrt{p_0(1-p_0)}}{\sqrt{n}}} \quad \{\text{using } \textbf{a}\}$$

$$= \frac{\widehat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

3 Across questions **4** to **6** in **Exercise 30H**, we let:

$\widehat{p} = \dfrac{x}{n}$, where x is the observed value of the test statistic, and n is the sample size.

$p_0 = p$, where p is the probability under the null hypothesis.

We notice that in every question, the p-value calculated using a Z-test is smaller than the value calculated under the binomial distribution.

This means that the Z-test produces *more significant* results.

For example, in question **4**:

$x = 38$, $n = 60$, and $p = \frac{1}{2}$

$\therefore \quad \widehat{p} = \frac{38}{60}$ and $p_0 = \frac{1}{2}$

$$\therefore \quad z = \frac{\frac{38}{60} - \frac{1}{2}}{\sqrt{\frac{\frac{1}{2}(1-\frac{1}{2})}{60}}}$$

$$\approx 2.07$$

Now, the p-value $= \mathrm{P}(Z \geqslant z) \approx \mathrm{P}(Z \geqslant 2.07) \approx 0.0195$

This is smaller than the p-value of 0.0259 calculated using the binomial distribution.

INVESTIGATION 2 RANDOMISED TESTS

1 **a** Under H_0: $p = 0.5$, $X \sim \text{B}(10, 0.5)$.

b Using technology to calculate p-value $= \text{P}(X \leqslant x)$:

x	p-value
0	$\approx 0.000\,977$
1	≈ 0.0107
2	≈ 0.0547
3	≈ 0.172
4	≈ 0.377
5	≈ 0.623
6	≈ 0.828
7	≈ 0.945
8	≈ 0.989
9	≈ 0.999
10	1

c **i** p-value $\leqslant \alpha = 0.1$ when $x = 0$, 1, and 2.

$\therefore$ $\mathcal{C} = \{0, 1, 2\}$

ii p-value $\leqslant \alpha = 0.05$ when $x = 0$ and 1.

$\therefore$ $\mathcal{C} = \{0, 1\}$

iii p-value $\leqslant \alpha = 0.01$ when $x = 0$.

$\therefore$ $\mathcal{C} = \{0\}$

d For $\mathcal{C} = \{0, 1, 2\}$, P(incorrectly rejecting H_0) $= \text{P}(X \leqslant 2) \approx 0.0547 < 0.1$.

For $\mathcal{C} = \{0, 1\}$, P(incorrectly rejecting H_0) $= \text{P}(X \leqslant 1) \approx 0.0107 < 0.05$.

For $\mathcal{C} = \{0\}$, P(incorrectly rejecting H_0) $= \text{P}(X = 0) \approx 0.000\,977 < 0.01$.

e The *actual* significance of the test is the probability of incorrectly rejecting H_0, which is given by the maximum value of $\text{P}(X \leqslant x)$ for $x \in \mathcal{C}$.

Now x is only included in $\mathcal{C}$ if $\text{P}(X \leqslant x) \leqslant \alpha$, so the actual significance level must always be less than or equal to α.

2 **a** For $\alpha = 0.05$, the smallest value of X such that H_0 is not rejected is $a = 2$.

b **i** H_0 is rejected if $X < a$ *or* $X = a$ and heads is tossed.

So, P(incorrectly rejecting H_0) $= \text{P}(X < a) + \theta \times \text{P}(X = a)$

ii If P(incorrectly rejecting H_0) $= \alpha = 0.05$

$\therefore$ $\text{P}(X < a) + \theta \times \text{P}(X = a) = 0.05$

$$\therefore \theta = \frac{0.05 - \text{P}(X < 2)}{\text{P}(X = 2)} \qquad \{a = 2\}$$

$$\approx 0.893$$

3 When $\alpha = 0.1$, $a = 3$.

$\therefore$ P(incorrectly rejecting H_0) $= \mathrm{P}(X < 3) + \theta \times \mathrm{P}(X = 3) = 0.1$

$$\therefore \quad \theta = \frac{0.1 - \mathrm{P}(X < 3)}{\mathrm{P}(X = 3)}$$
$$\approx 0.387$$

When $\alpha = 0.01$, $a = 1$.

$\therefore$ P(incorrectly rejecting H_0) $= \mathrm{P}(X < 1) + \theta \times \mathrm{P}(X = 1) = 0.01$

$$\therefore \quad \theta = \frac{0.01 - \mathrm{P}(X = 0)}{\mathrm{P}(X = 1)}$$
$$= 0.924$$

It appears that the value of θ decreases as α increases.

4 For H_1: $p > 0.5$, we follow the randomised test outlined in **2**, but instead let a be the *largest* value of X such that H_0 is not rejected.
In this case, P(incorrectly rejecting H_0) $= \mathrm{P}(X > a) + \theta \times \mathrm{P}(X = a)$.

5 Under the null hypothesis, $T \sim \mathrm{Po}(n\lambda_0)$.
If H_1: $\lambda > \lambda_0$, let a be the largest value of T such that H_0 is not rejected.
In this case, if $T > a$ then H_0 is rejected, and if $T = a$ then H_0 is rejected with probability θ.
$\therefore$ P(incorrectly rejecting H_0) $= \mathrm{P}(T > a) + \theta \times \mathrm{P}(T = a)$
If H_1: $\lambda < \lambda_0$, let a be the smallest value of T such that H_0 is not rejected.
In this case, if $T < a$ then H_0 is rejected, and if $T = a$ then H_0 is rejected with probability θ.
$\therefore$ P(incorrectly rejecting H_0) $= \mathrm{P}(T < a) + \theta \times \mathrm{P}(T = a)$.

6 In general, $\mathcal{C}$ and $\mathcal{A}$ form the set of all outcomes of the test, but have no elements in common (they are disjoint).
However, in a randomised test, the value a could be a member of either $\mathcal{C}$ or $\mathcal{A}$, as observing a could lead to either rejecting or accepting H_0.
So, the critical and acceptance regions are not well defined.

7 Not "flipping a coin" means that the critical region is smaller than it would be with the coin. This means that H_0 is rejected less often.

- **a** Not flipping a coin means the results will be biased towards one party or another, so it *is reasonable* to use a coin.
- **b** We should be conservative when trialling new treatments, to avoid bad outcomes, so it is *not reasonable* to use a coin.
- **c** Similarly to **a**, we should be unbiased about which material performs the best, and so it *is reasonable* to use a coin.

EXERCISE 30I

1

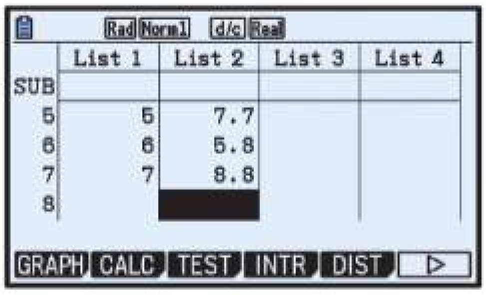
x	1	2	3	4	5	6	7
y	4.1	6.2	4.5	9.8	7.7	5.8	8.8

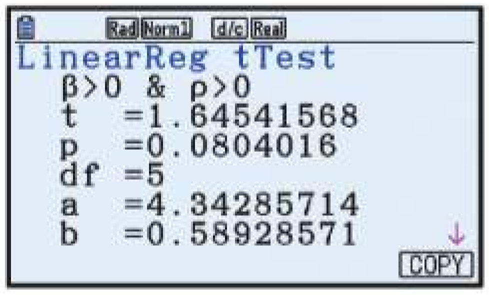
a Let ρ be the population product-moment correlation coefficient between the variables.
We use the hypotheses:

H_0: $\rho = 0$ {there is no correlation between the variables}

H_1: $\rho > 0$ {the variables are positively correlated}

b





The observed value of the test statistic ≈ 1.65, and the p-value ≈ 0.0804.

c Since p-value $> \alpha = 0.05$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.
We therefore accept H_0, and conclude that the variables are not positively correlated.

2

Mathematics mark (x)	13	10	8	14	6	11	10	5	12	13
Physics mark (y)	21	15	14	20	12	16	9	10	17	12

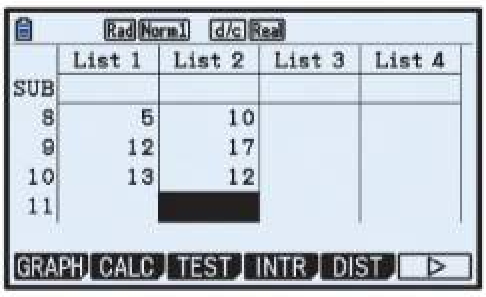
Step 1: Let ρ be the population product-moment correlation coefficient between *Mathematics mark* and *Physics mark*.
We use the hypotheses:

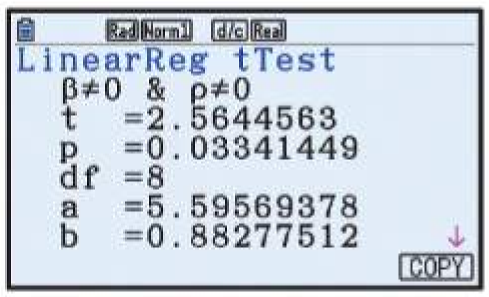
H_0: $\rho = 0$ {there is no association between the variables}

H_1: $\rho \neq 0$ {there is an association between the variables}

Step 2: The significance levels are: **a** $\alpha = 0.05$ **b** $\alpha = 0.01$.

Step 3:





The observed value of the test statistic ≈ 2.56.

Step 4: p-value ≈ 0.0334

a Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1, and conclude that there is an association between the variables.

b Since p-value $> 0.01 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_0, and conclude that there is no association between the variables.

3

Cholesterol level (x)	5.32	5.54	5.45	5.06	6.13	5.00	4.90	6.00	6.70	4.75
Resting heart rate (y)	55	48	55	53	74	44	49	68	78	51

Step 1: Let ρ be the population product-moment correlation coefficient between *cholesterol level* and *resting heart rate*.

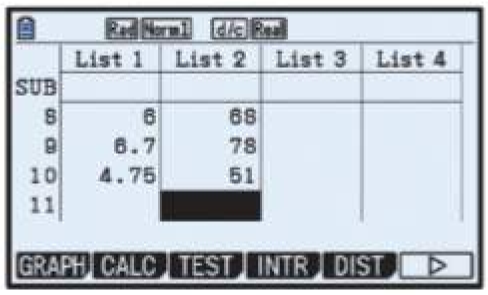
We use the hypotheses:

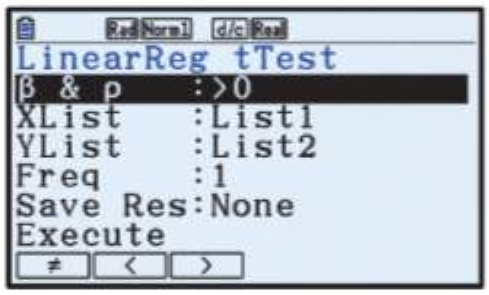
H_0: $\rho = 0$ {there is no correlation between the variables}

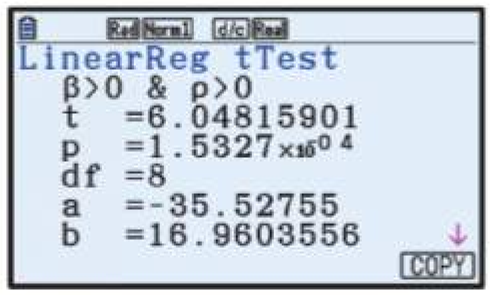
H_1: $\rho > 0$ {there is a positive correlation between the variables}

Step 2: The significance level is $\alpha = 0.01$.

Step 3:







The observed value of the test statistic ≈ 6.05.

Step 4: p-value $\approx 1.53 \times 10^{-4}$

Step 5: Since p-value $< \alpha = 0.01$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that there is a positive correlation between *cholesterol level* and *resting heart rate*. The researcher's claim is justified.

4

Number of Sudoku puzzles (x)	5	8	12	15	15	17	20	21	25	27
Number of logic puzzles (y)	3	11	9	6	15	13	25	15	13	20

Step 1: Let ρ be the population product-moment correlation coefficient between *number of Sudoku puzzles* and *number of logic puzzles*.

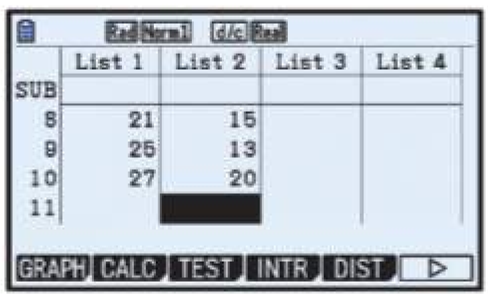
We use the hypotheses:

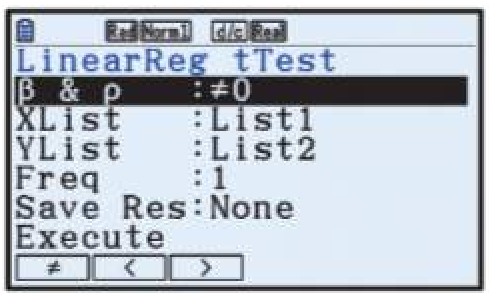
H_0: $\rho = 0$ {the variables are not linearly correlated}

H_1: $\rho \neq 0$ {the variables are linearly correlated}

Step 2: The significance level is $\alpha = 0.02$.

Step 3:





The observed value of the test statistic ≈ 2.74.

Step 4: p-value ≈ 0.0256

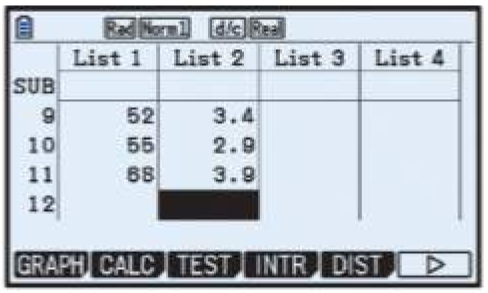
Step 5: Since p-value $> \alpha = 0.02$, we do not have enough evidence to reject H_0 in favour of H_1 on a 2% significance level. We therefore accept H_0.

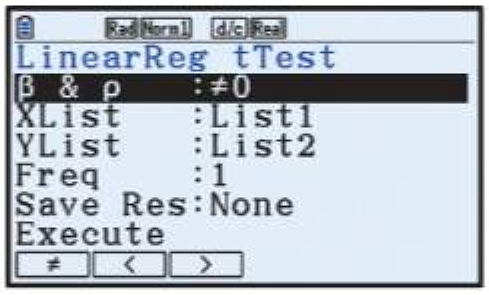
Step 6: Since we have accepted H_0, we conclude that the variables are not linearly correlated.

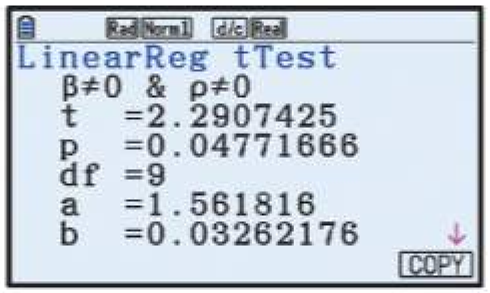
5

Weight of mother (x kg)	49	46	48	45	46	42	43	40	52	55	68
Birth weight of child (y kg)	3.8	3.1	2.5	3.0	3.2	2.8	3.1	2.9	3.4	2.9	3.9

a







The observed value of the test statistic ≈ 2.29.

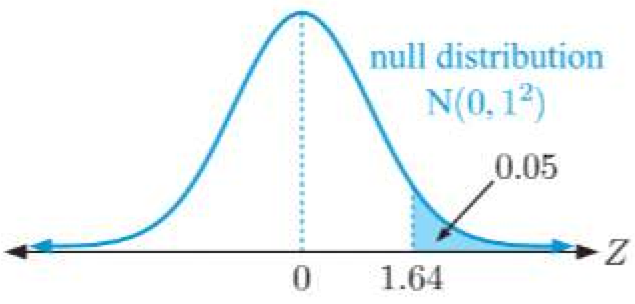
b $2.29 > 1.83$, so H_0 will be rejected for $\alpha = 0.1$.
$2.29 > 2.26$, so H_0 will be rejected for $\alpha = 0.05$.
$-2.82 < 2.29 < 2.82$, so H_0 will *not* be rejected for $\alpha = 0.02$.
$-3.25 < 2.29 < 3.25$, so H_0 will *not* be rejected for $\alpha = 0.01$.

c From the screenshots in **a**, the p-value ≈ 0.0477.
$0.0477 < 0.1$, so H_0 will be rejected for $\alpha = 0.1$.
$0.0477 < 0.05$, so H_0 will be rejected for $\alpha = 0.05$.
$0.0477 > 0.02$, so H_0 will *not* be rejected for $\alpha = 0.02$.
$0.0477 > 0.01$, so H_0 will *not* be rejected for $\alpha = 0.01$.
This is the same result as we found using critical regions.

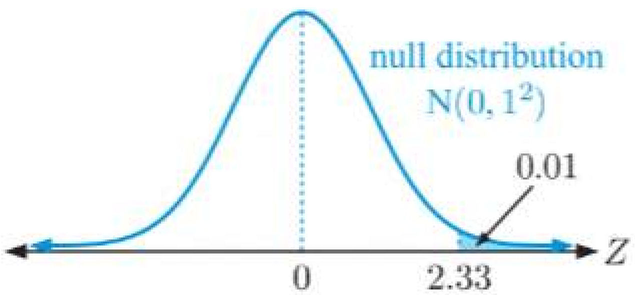
EXERCISE 30J.1

1 **a** H_0: $\mu = 27$ cm
H_1: $\mu > 27$ cm

i If $\alpha = 0.05$, then we retain H_0 if
$Z < z_{0.05} \approx 1.64$
$\therefore \quad \mathcal{C} = \{z \mid z \geqslant 1.64\}$



ii If $\alpha = 0.01$, then we retain H_0 if
$Z < z_{0.01} \approx 2.33$
$\therefore \quad \mathcal{C} = \{z \mid z \geqslant 2.33\}$



b P(Type I error) $= \alpha$

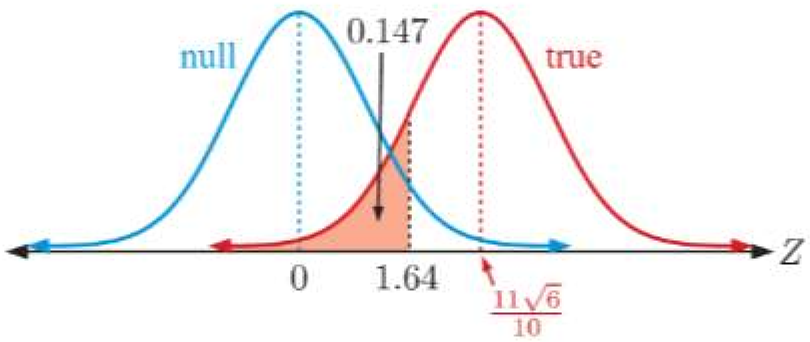
The true distribution of Z is $\text{N}\left(\dfrac{29.2-27}{\frac{\sqrt{6}}{\sqrt{9}}},\ 1^2\right) \equiv \text{N}\left(\frac{11\sqrt{6}}{10},\ 1^2\right)$.

i P(Type II error)

$= \text{P}(\text{Retain } H_0 \mid H_0 \text{ false})$

$\approx \text{P}\left(Z < 1.64 \;\middle|\; Z \sim \text{N}\left(\frac{11\sqrt{6}}{10},\ 1^2\right)\right)$ {using **a i**}

≈ 0.147

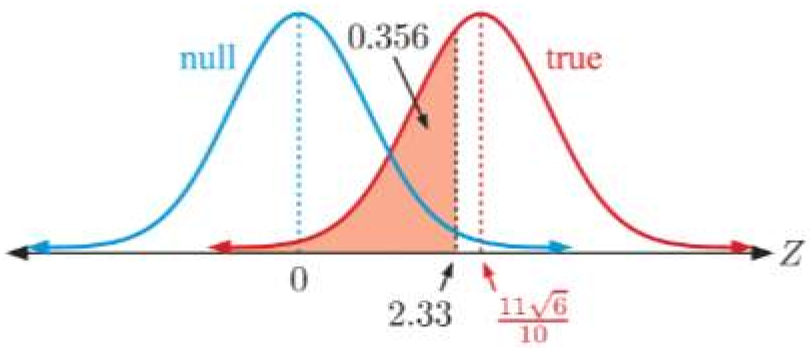


ii P(Type II error)

$= \text{P}(\text{Retain } H_0 \mid H_0 \text{ false})$

$\approx \text{P}\left(Z < 2.33 \;\middle|\; Z \sim \text{N}\left(\frac{11\sqrt{6}}{10},\ 1^2\right)\right)$ {using **a ii**}

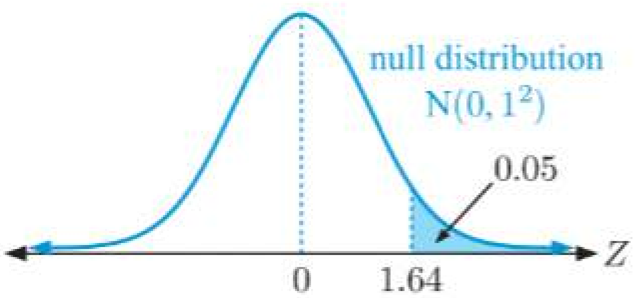
≈ 0.356



2 $\sigma = 0.7$ kg, $n = 15$ pumpkins, P(Type I error) $= \alpha = 0.05$, H_0: $\mu = 6$ kg

The true mean is $\mu = 6.4$ kg, so the true distribution of Z is $\text{N}\left(\dfrac{6.4-6}{\frac{0.7}{\sqrt{15}}},\ 1^2\right) \equiv \text{N}\left(\frac{4\sqrt{15}}{7},\ 1^2\right)$.

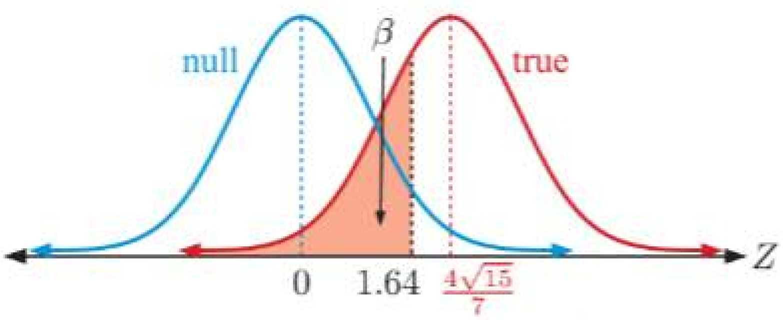
a H_1: $\mu > 6$ kg, so we retain H_0 if $Z < z_{0.05} \approx 1.64$.



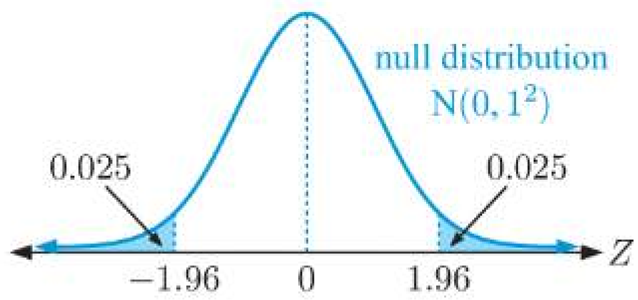
$\therefore\ \beta = \text{P}(\text{Retain } H_0 \mid H_0 \text{ false})$

$\approx \text{P}\left(Z < 1.64 \;\middle|\; Z \sim \text{N}\left(\frac{4\sqrt{15}}{7},\ 1^2\right)\right)$

≈ 0.285



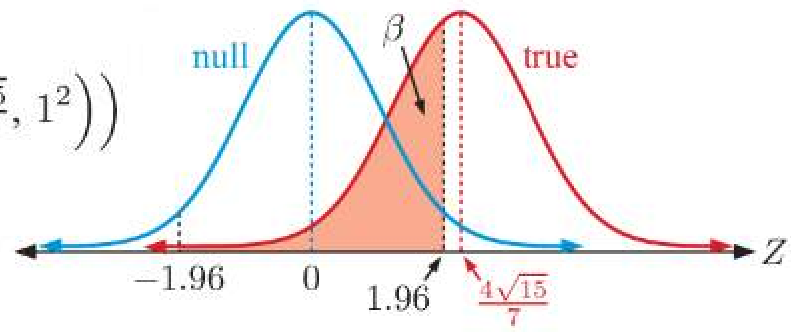
b H_1: $\mu \neq 6$ kg, so we retain H_0 if
$Z < z_{0.025} \approx 1.96$ and
$Z > -z_{0.025} \approx -1.96$.



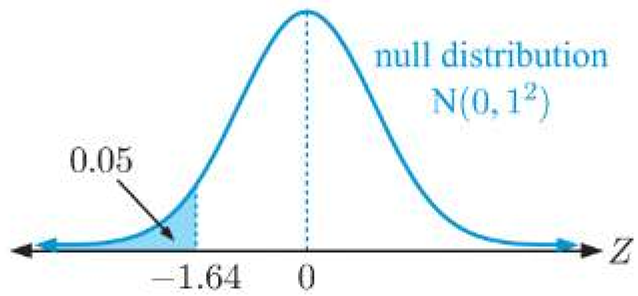
$\therefore$ $\beta = \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})$

$\approx \mathrm{P}\left(-1.96 < Z < 1.96 \;\middle|\; Z \sim \mathrm{N}\left(\frac{4\sqrt{15}}{7}, 1^2\right)\right)$

≈ 0.400



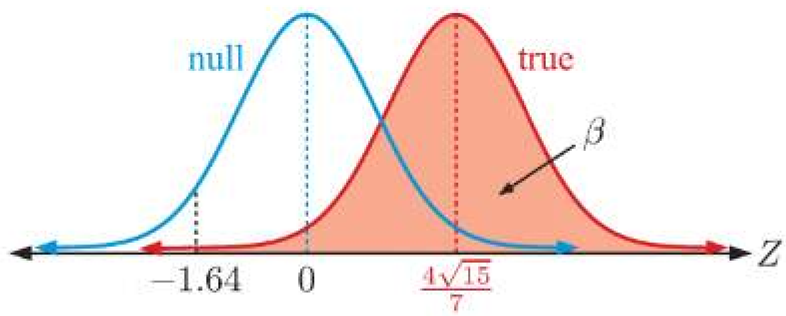
c H_1: $\mu < 6$ kg, so we retain H_0 if
$Z > -z_{0.05} \approx -1.64$.



$\therefore$ $\beta = \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})$

$\approx \mathrm{P}\left(Z > -1.64 \;\middle|\; Z \sim \mathrm{N}\left(\frac{4\sqrt{15}}{7}, 1^2\right)\right)$

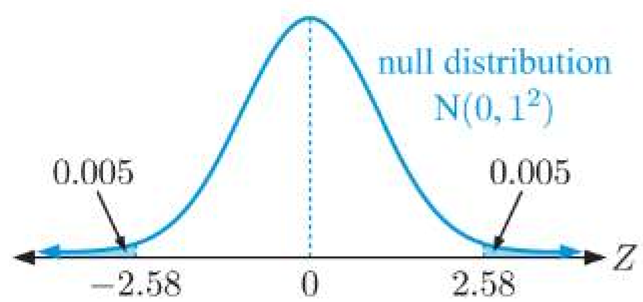
≈ 0.9999



3 $\sigma = 0.15$ m, $\mu_0 = 3.5$ m, $n = 20$ beams

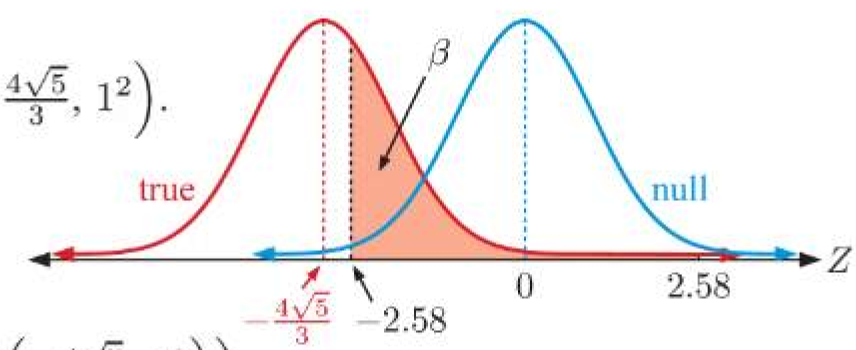
a H_0: $\mu = 3.5$ m
H_1: $\mu \neq 3.5$ m

b $\alpha = 0.01$, so we retain H_0 if
$Z < z_{0.005} \approx 2.58$ and $Z > -z_{0.005} \approx -2.58$.
$\therefore$ $\mathcal{C} = \{z \mid z \leqslant -2.58 \text{ or } z \geqslant 2.58\}$



c **i** The true mean is $\mu = 3.4$ m, so the true distribution

of Z is $\mathrm{N}\left(\dfrac{3.4 - 3.5}{\frac{0.15}{\sqrt{20}}}, 1^2\right) \equiv \mathrm{N}\left(-\frac{4\sqrt{5}}{3}, 1^2\right)$.



$\therefore$ P(Type II error)

$= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})$

$\approx \mathrm{P}\left(-2.58 < Z < 2.58 \;\middle|\; Z \sim \mathrm{N}\left(-\frac{4\sqrt{5}}{3}, 1^2\right)\right)$

≈ 0.343

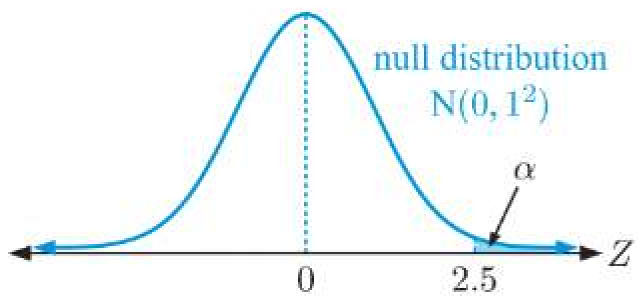
ii Power $= 1 - \text{P(Type II error)}$
≈ 0.657 {using **c i**}

4 **a** $\mu_0 = 150, \quad \sigma = \sqrt{64} = 8, \quad n = 16$

Under H_0, the z-score of 155 is $\dfrac{155 - 150}{\frac{8}{\sqrt{16}}} = 2.5$.

So, the decision rule can be written equivalently as: accept H_0 if $z < 2.5$
reject H_0 if $z \geqslant 2.5$.

$\therefore \ \alpha = \text{P(Type I error)}$
$= \text{P(Reject } H_0 \mid H_0 \text{ true)}$
$= \text{P}\left(Z \geqslant 2.5 \mid Z \sim \text{N}(0,\, 1^2)\right)$
$\approx 0.006\,21$

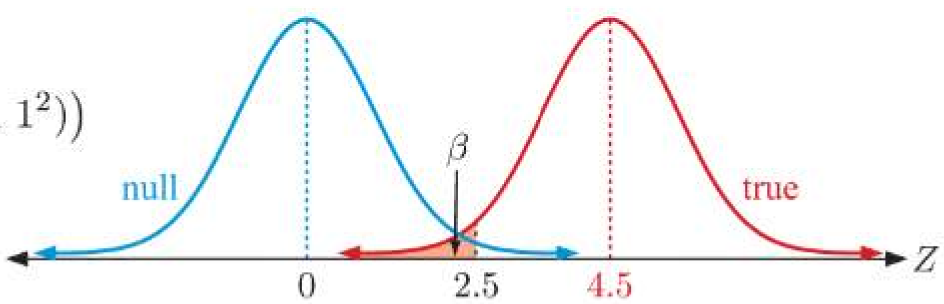


b **i** The true mean $\mu = 159$, so the true distribution

of Z is $\text{N}\left(\dfrac{159 - 150}{\frac{8}{\sqrt{16}}},\ 1^2\right) \equiv \text{N}(4.5,\, 1^2)$.

$\therefore \ \beta = \text{P(Type II error)}$
$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$
$= \text{P}\left(Z < 2.5 \mid Z \sim \text{N}(4.5,\, 1^2)\right)$
≈ 0.0228

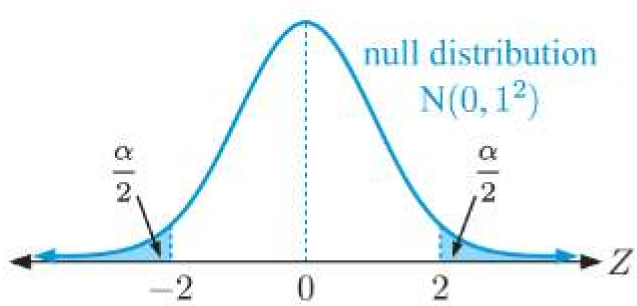
ii Power $= 1 - \beta$
≈ 0.977



5 **a** Under H_0, the z-score of 331 is $\dfrac{331 - 330}{\frac{2}{\sqrt{16}}} = 2$, and the z-score of 329 is $\dfrac{329 - 330}{\frac{2}{\sqrt{16}}} = -2$.

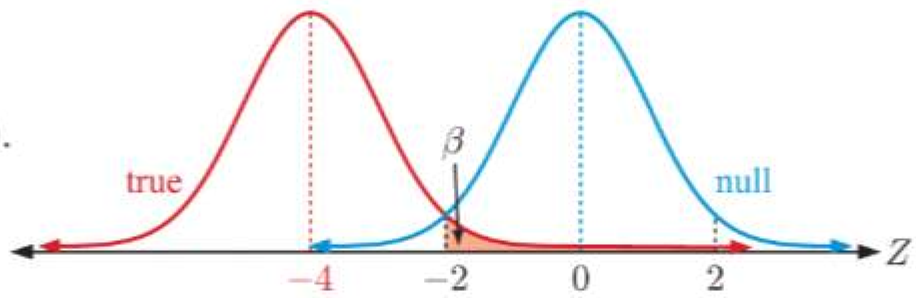
So, the decision rule can be equivalently written as: accept H_0 if $-2 < z < 2$
reject H_0 if $z \leqslant -2$ or $z \geqslant 2$.

$\therefore \ \alpha = \text{P(Type I error)}$
$= \text{P(Reject } H_0 \mid H_0 \text{ true)}$
$= \text{P}\left([Z \leqslant -2 \,\cup\, Z \geqslant 2] \mid Z \sim \text{N}(0,\, 1^2)\right)$
$= 1 - \text{P}\left(-2 < Z < 2 \mid Z \sim \text{N}(0,\, 1^2)\right)$
≈ 0.0455



b **i** The true mean $\mu = 328$ cm^3, so the true distribution of Z is

$$\mathrm{N}\left(\frac{328-330}{\frac{2}{\sqrt{16}}},\ 1^2\right) \equiv \mathrm{N}(-4,\ 1^2).$$



$$\begin{aligned}\therefore\ \beta &= \mathrm{P(Type\ II\ error)}\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}\left(-2 < Z < 2 \mid Z \sim \mathrm{N}(-4,\ 1^2)\right)\\ &\approx 0.0228\end{aligned}$$

ii $$\begin{aligned}\text{Power} &= 1-\beta\\ &\approx 0.977\end{aligned}$$

6 $H_0\colon\ \mu = 0,\quad H_1\colon\ \mu \neq 0$

$n = 4,\quad \sigma = 1$

a **i** If the true mean $\mu = 2$, then the true distribution of Z is $\mathrm{N}\left(\frac{2-0}{\frac{1}{\sqrt{4}}},\ 1^2\right) \equiv \mathrm{N}(4,\ 1^2)$.

For a given value of α, we retain H_0 if $-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}$.

$$\begin{aligned}\therefore\ \beta &= \mathrm{P(Type\ II\ error)}\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}\left(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}\ \middle|\ Z \sim \mathrm{N}(4,\ 1^2)\right)\end{aligned}$$

$\therefore$

α	0.1	0.05	0.01	0.001
β	$\approx 0.009\,26$	≈ 0.0207	≈ 0.0772	≈ 0.239

ii β increases as α decreases.

b **i** If the true mean $\mu = \pm a$, then the true distribution of Z is

$$\mathrm{N}\left(\frac{\pm a-0}{\frac{1}{\sqrt{4}}},\ 1^2\right) \equiv \mathrm{N}(\pm 2a,\ 1^2).$$

$\alpha = 0.05$, so we retain H_0 if $-z_{0.025} < Z < z_{0.025}$,
which is approximately $-1.96 < Z < 1.96$.

$$\begin{aligned}\therefore\ \beta &= \mathrm{P(Type\ II\ error)}\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}\left(-1.96 < Z < 1.96 \mid Z \sim \mathrm{N}(\pm 2a,\ 1^2)\right)\end{aligned}$$

Notice that the acceptance region $-1.96 < Z < 1.96$ is symmetric about 0, so
$\mathrm{P}(-1.96 < Z < 1.96 \mid Z \sim \mathrm{N}(2a,\ 1^2)) = \mathrm{P}(-1.96 < Z < 1.96 \mid Z \sim \mathrm{N}(-2a,\ 1^2))$.

$\therefore$

True value of μ	β
± 1	≈ 0.484
± 2	≈ 0.0207
± 5	≈ 0
± 10	≈ 0

ii The power of the test $1-\beta$, which is the probability of correctly rejecting H_0, measures how easy it is to detect a genuine difference.

As the true value of μ gets further and further away from 0, the value assumed under H_0, $\beta = \text{P(Type II error)} \to 0$.

So the power of the test $1 - \beta = \text{P(correctly rejecting } H_0) \to 1$.

That is, as μ gets further away from the null value, the more likely it is that a genuine difference will be detected.

EXERCISE 30J.2

1 **a** Let p be the probability of rolling a 4.

The hypotheses are:

H_0: $p = \frac{1}{4}$ {the die is not biased towards 4}

H_1: $p > \frac{1}{4}$ {the die is biased towards 4}

b **i** A Type I error is rejecting H_0 when it is actually true.

This means deciding the die is biased when it is actually fair.

ii The test statistic X is the number of 4s in a sample of 300 rolls.

$\therefore$ the null distribution is $X \sim B(300, 0.25)$.

$\therefore$ $\alpha = \text{P(Reject } H_0 \mid H_0 \text{ true)}$

$= P(X \geqslant 88 \mid X \sim B(300, 0.25))$

≈ 0.0496

The level of significance is about 4.96%.

c The true probability $p = 0.32$, so the true distribution of X is $B(300, 0.32)$.

$\therefore$ $\beta = \text{P(Type II error)}$

$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$

$= P(X \leqslant 87 \mid X \sim B(300, 0.32))$

≈ 0.146

2 **a** H_0: $p = 0.5$

H_1: $p > 0.5$

b **i** The region $X \geqslant 10$ is the critical region.

ii The null distribution is $X \sim \mathrm{B}(12, 0.5)$.

$$\begin{aligned}\alpha &= \mathrm{P}(\text{Type I error})\\ &= \mathrm{P}(\text{Reject } H_0 \mid H_0 \text{ true})\\ &= \mathrm{P}(X \geqslant 10 \mid X \sim \mathrm{B}(12, 0.5))\\ &\approx 0.0193\end{aligned}$$

The significance level is about 1.93%.

c **i** The true probability $p = 0.6$, so the true distribution of X is $\mathrm{B}(12, 0.6)$.

$$\begin{aligned}\therefore\ \beta &= \mathrm{P}(\text{Type II error})\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}(X \leqslant 9 \mid X \sim \mathrm{B}(12, 0.6))\\ &\approx 0.917\end{aligned}$$

$\therefore$ the power $1 - \beta \approx 0.0834$.

ii Eri is at risk of making a Type II error.

3 Each value x_i is independent and taken from $X \sim \mathrm{Po}(\lambda)$, so if $t = \sum_{i=1}^{9} x_i$ and t has distribution T, then $T \sim \mathrm{Po}(9\lambda)$.

a H_0: $\lambda = 3$, so the null distribution is $T \sim \mathrm{Po}(27)$.

$$\begin{aligned}\therefore\ \alpha &= \mathrm{P}(\text{Type I error})\\ &= \mathrm{P}(\text{Reject } H_0 \mid H_0 \text{ true})\\ &= \mathrm{P}(T \geqslant 38 \mid T \sim \mathrm{Po}(27))\\ &\approx 0.0263\end{aligned}$$

The significance level is about 2.63%.

b If the true mean is $\lambda = 4$, then the true distribution of T is $\mathrm{Po}(36)$.

$$\begin{aligned}\therefore\ \beta &= \mathrm{P}(\text{Type II error})\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}(T \leqslant 37 \mid T \sim \mathrm{Po}(36))\\ &\approx 0.609\end{aligned}$$

$\therefore$ the power $1 - \beta \approx 0.391$.

4 **a** The sum of n independent Poisson random variables with mean λ itself has distribution $\mathrm{Po}(n\lambda)$.

$\therefore$ the test statistic $T \sim \mathrm{Po}(5\lambda)$.

Now H_0: $\lambda = 2$, so the null distribution is $T \sim \mathrm{Po}(10)$.

$\mathrm{P}(T \geqslant 14) \approx 0.136 > \alpha = 0.1$ and $\mathrm{P}(T \geqslant 15) \approx 0.0836 < \alpha = 0.1$.

$\therefore\ \mathcal{C} = \{t \mid t \geqslant 15\}$

b The true population mean $\lambda = 2.5$, so the true distribution of T is $\mathrm{Po}(12.5)$.

$$\begin{aligned}\therefore\ \beta &= \mathrm{P}(\text{Type II error})\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}(T \leqslant 14 \mid T \sim \mathrm{Po}(12.5))\\ &\approx 0.725\end{aligned}$$

$\therefore$ the power $1 - \beta \approx 0.275$.

REVIEW SET 30A

1 Let μ be the population mean number of minutes after 9:45 am that the bus arrives.
The hypotheses that should be considered are:
H_0: $\mu = 0$ {the buses arrive on time}
H_1: $\mu > 0$ {the buses arrive late}

2 **a** There is a $\approx 7.94\%$ chance of observing this result if the null hypothesis is true.
b For a 10% significance level, we reject H_0 if there is less than a 10% chance of observing this result.
c As the p-value $< 0.1 = \alpha$, there is enough evidence to reject H_0 in favour of H_1.
d If H_0 is (incorrectly) accepted when it is actually false, a Type II error has been made.

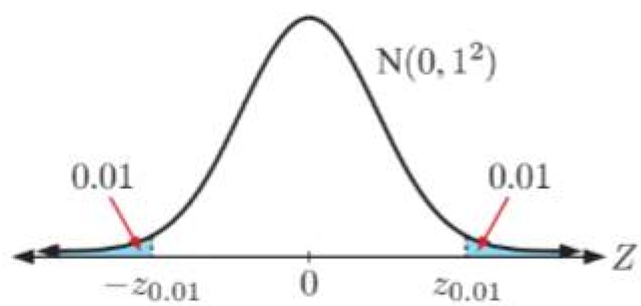
3 We are testing a claim about the mean of a population with known standard deviation, so we use a Z-test.

Step 1: Let μ be the population mean number of shaves per blade.
The hypotheses that should be considered are:
H_0: $\mu = 13$
H_1: $\mu \neq 13$

Step 2: The significance level is $\alpha = 0.02$.

Step 3: $\overline{x} = 12.8 \quad \sigma = 1.6 \quad n = 30$
The observed value of the test statistic is $z = \dfrac{12.8 - 13}{\frac{1.6}{\sqrt{30}}} \approx -0.685$

Step 4: We require a critical value c such that
$\text{P}(Z \leqslant -c \ \text{ or } \ Z \geqslant c) = 0.02$
$\therefore \ \text{P}(Z \geqslant c) = 0.01$
$\therefore \ c = z_{0.01} \approx 2.33$
So, the critical region
$\mathcal{C} = \{z \mid z \leqslant -2.33 \ \text{ or } \ z \geqslant 2.33\}$.

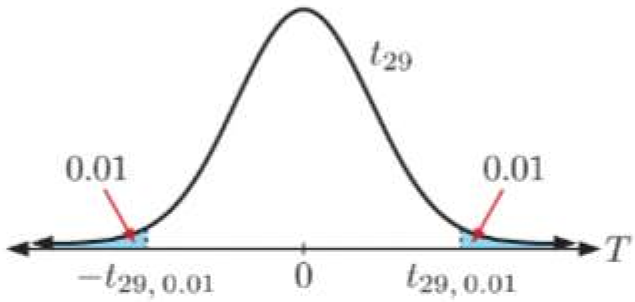


Step 5: The test statistic does not lie in the critical region, so we do not have enough evidence to reject H_0 in favour of H_1 at a 2% significance level.

Step 6: Since we have accepted H_0, we therefore accept the manufacturer's claim.

4 We are testing a claim about the mean of a population with unknown variance, so we use a one-sample t-test.
Let μ be the population mean house price in the town.
The hypotheses to be considered are:
H_0: $\mu = £438\,000$
H_1: $\mu \neq £438\,000$
The significance level is $\alpha = 0.02$.
The sample size $n = 30$, and the sample standard deviation $s = £23\,500$.

For a particular observed value of the sample mean $\overline{x}$, the observed value of the test statistic
$t = \dfrac{\overline{x} - 438\,000}{\frac{23\,500}{\sqrt{30}}}$.



The null distribution is $T \sim t_{29}$.

We require a critical value c such that

$$\mathrm{P}(T \leqslant -c \ \text{ or } \ T \geqslant c) = 0.02 = \alpha$$
$$\therefore \ \mathrm{P}(T \geqslant c) = 0.01$$
$$\therefore \ c = t_{29,\,0.01} \approx 2.46$$

So, the agent's claim would be supported at the 2% level of significance for any value of $\overline{x}$ such that $\quad -2.46 < t < 2.46$

$$\therefore \ -2.46 < \frac{\overline{x} - 438\,000}{\frac{23\,500}{\sqrt{30}}} < 2.46$$
$$\therefore \ 427\,436.71 < \overline{x} < 448\,563.29$$

$\therefore$ to the nearest £500, the agent's claim is supported if the sample mean house price $\overline{x}$ is between £428 000 and £448 000 inclusive.

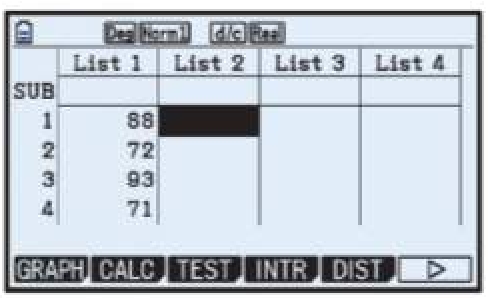
5 **a** Let μ be the population mean of Rosario's apricots this year. The hypotheses to be tested are:

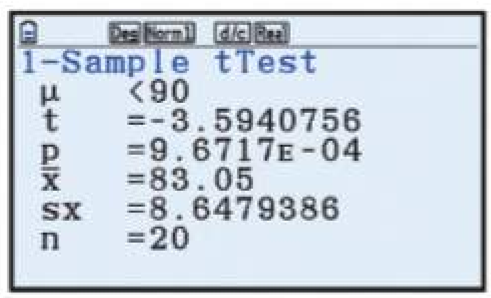
H_0: $\mu = 90$ {the mean weight of the harvest is the same as last year}

H_1: $\mu < 90$ {the mean weight of the harvest is less than it was last year}

b Rosario needs to test a hypothesis about the mean of a single population with unknown variance, so he should use Student's t-test for a population mean.

c The significance level is $\alpha = 0.01$.





Using technology, $t \approx -3.59$.

The null distribution is t_{19}.

$$\therefore \ \text{the critical region } \mathcal{C} = \{t \mid t \leqslant -t_{19,\,0.01}\} = \{t \mid t \leqslant -2.54\}$$

Since $-3.59 \in \mathcal{C}$, we have enough evidence to reject H_0 in favour of H_1 at the 1% significance level.

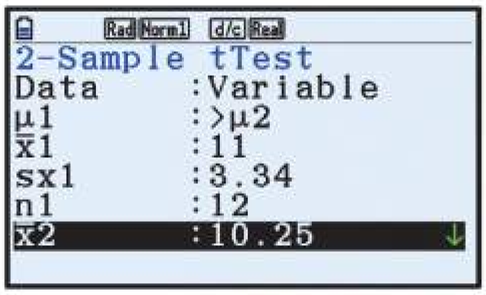
We therefore accept at the 1% significance level that the mean weight of Rosario's harvest is less than it was last year, and so his concerns are justified.

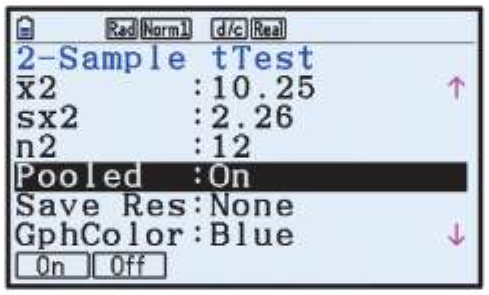
6 **a** Let μ_1 and μ_2 be the population mean numbers of fish caught per trip by Joe and Ruben respectively. The hypotheses that Joe should test are:

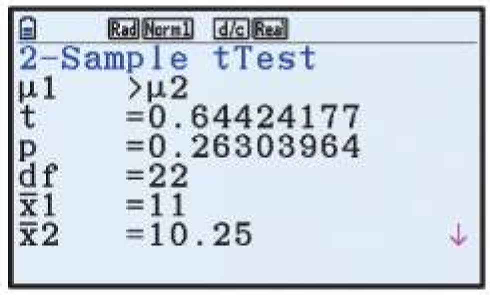
H_0: $\mu_1 = \mu_2$ {they catch the same number of fish}

H_1: $\mu_1 > \mu_2$ {Joe catches more fish than Ruben}

b The significance level is $\alpha = 0.05$.







Using technology, the value of the test statistic $t \approx 0.644$, and the p-value ≈ 0.263.

Since the p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at the 5% significance level.

We therefore conclude that the two friends catch the same number of fish on average. So Joe's claim is not justified.

7 **a** A two-sample t-test should be used, because:

- the data is not paired
- there are two different samples of customers.

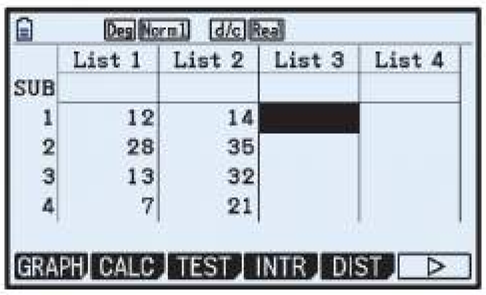
b *Step 1*: Let μ_A and μ_B be the population mean amount of time spent shopping in supermarkets A and B respectively. The hypotheses to be considered are:

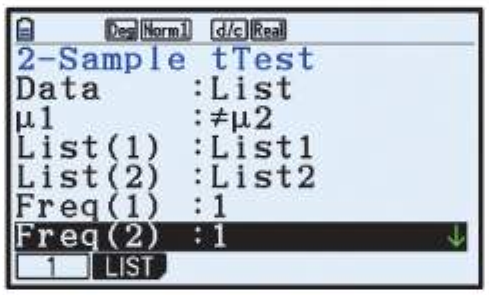
H_0: $\mu_A = \mu_B$ {customers spend the same amount of time at each supermarket}

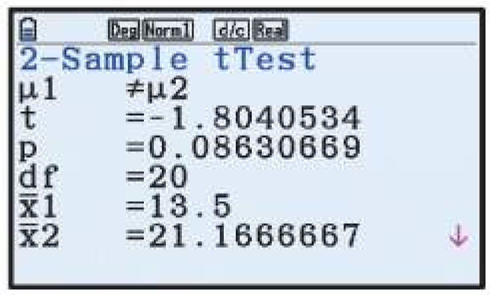
H_1: $\mu_A \neq \mu_B$ {customers spend different amounts of time at each supermarket}

Step 2: The significance level is $\alpha = 0.1$.

Step 3:







Using technology, the value of the test statistic is $t \approx -1.80$.

Step 4: From the screenshots above, the p-value ≈ 0.0863.

Step 5: Since the p-value $< 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 10% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1 at the 10% significance level, we therefore conclude that there is a significant difference between the mean amount of time spent at supermarket A and supermarket B.

8

E10 (x_i km L^{-1})	9.6	11.2	9.2	14.1	11.5	9.2	11.8	12.3	12.0	10.3
Regular unleaded (y_i km L^{-1})	10.7	11.8	9.3	13.4	12.4	9.5	12.2	13.1	12.8	11.0
$d_i = y_i - x_i$	1.1	0.6	0.1	−0.7	0.9	0.3	0.4	0.8	0.8	0.7

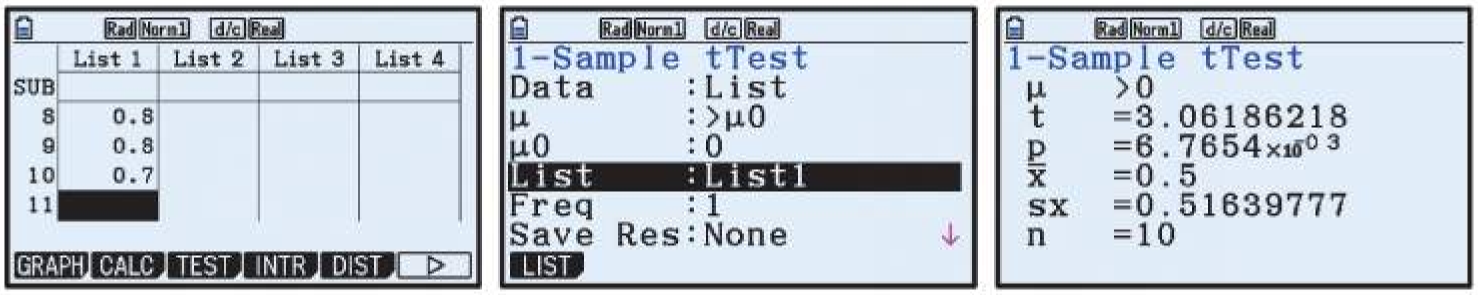
Step 1: The hypotheses to be considered are:

H_0: $\mu_D = 0$ {fuel economy of E10 is just as good as regular unleaded}

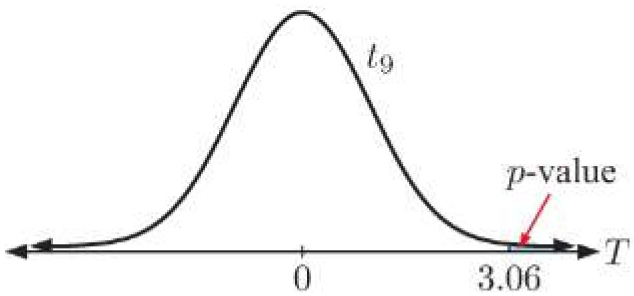
H_1: $\mu_D > 0$ {fuel economy of E10 is worse than regular unleaded}

Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:



Using technology, the test statistic $t \approx 3.06$ and the p-value $\approx 0.006\,77$.



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 with a 5% level of significance. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the fuel economy of E10 is worse than regular unleaded, and so the claim is not justified with a 5% level of significance.

9 *Step 1*: Let p be the probability that a randomly selected person who tries the new exercises will successfully learn to type accurately.

The hypotheses that should be considered are:

H_0: $p = 0.7$ {the new method has the same success rate as the traditional method}

H_1: $p > 0.7$ {the new method has a better success rate than the traditional method}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: The observed value of the test statistic is $x = 26$.

Step 4: The null distribution is $X \sim \text{B}(30, 0.7)$.

The alternative hypothesis is H_1: $p > 0.7$, so we use the upper tail of the null distribution.

$$p\text{-value} = \text{P}(X \geqslant 26) \approx 0.0302$$

Step 5: Since p-value $< \alpha = 0.05$, we have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the new method is indeed better than the traditional method. The claim is justified.

10 *Step 1*: Let λ be the population mean number of calls per 5 minute interval.

The hypotheses to be considered are:

H_0: $\lambda = 2$ {a new receptionist should not be hired}

H_1: $\lambda > 2$ {a new receptionist should be hired}

Step 2: The significance level is $\alpha = 0.01$.

Step 3: The test statistic is $t = 29$.

Step 4: There are 12 5 minute intervals in 1 hour, so $n \times \lambda_0 = 12 \times 2 = 24$.

So, the null distribution is $T \sim \text{Po}(24)$

$$\therefore \; p\text{-value} = \text{P}(T \geqslant t)$$
$$= \text{P}(T \geqslant 29)$$
$$\approx 0.177 \quad \{\text{using technology}\}$$

Step 5: Since p-value $> \alpha = 0.01$, we do not have enough evidence to reject H_0 in favour of H_1 with a 1% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that a new receptionist should not be hired.

11 *Step 1*: Let ρ be the population product-moment correlation coefficient between the men's *height* and *weight*.

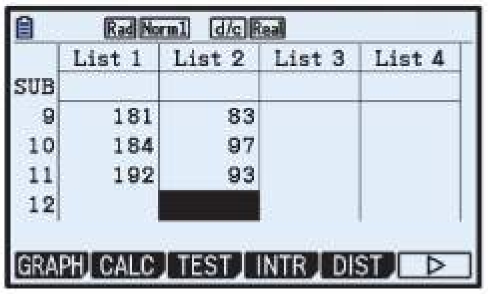
We use the hypotheses:

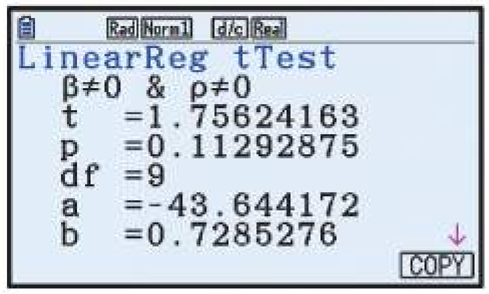
H_0: $\rho = 0$ {the variables are not linearly correlated}

H_1: $\rho \neq 0$ {the variables are linearly correlated}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:





The observed value of the test statistic ≈ 1.76.

Step 4: p-value ≈ 0.113

Step 5: Since p-value $> \alpha = 0.05$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that *height* and *weight* are not linearly correlated.

12 X is normally distributed with known population standard deviation, so our test statistic has a standard normal distribution.

a Under H_0, the z-score for 498 is $z = \dfrac{498 - 500}{\frac{3.71}{\sqrt{13}}} \approx -1.94$

and the z-score for 502 is $z = \dfrac{502 - 500}{\frac{3.71}{\sqrt{13}}} \approx 1.94$.

So the decision rule can be equivalently written as:

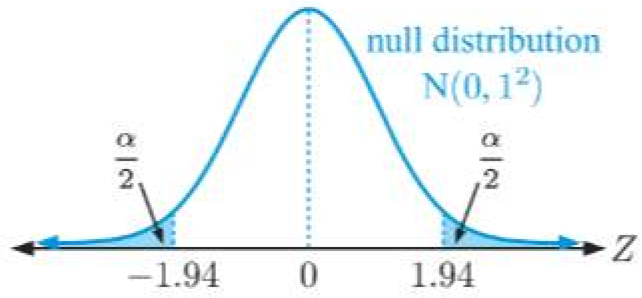
accept H_0 if $-1.94 < z < 1.94$

reject H_0 if $z \leqslant -1.94$ or $z \geqslant 1.94$

$\therefore$ the critical region $C = \{z \mid z \leqslant -1.94 \text{ or } z \geqslant 1.94\}$.

b The significance level

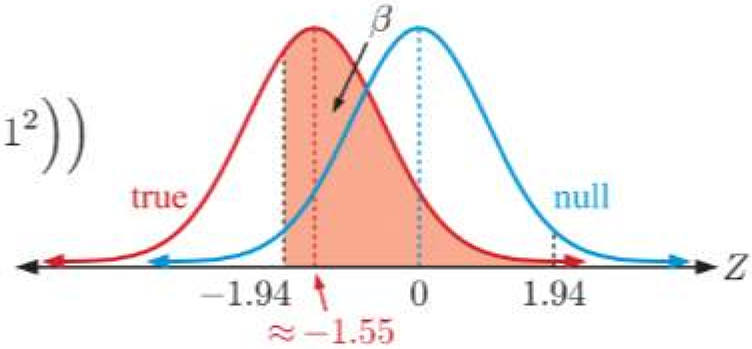
$\alpha = \text{P(Type I error)}$
$= \text{P(Reject } H_0 \mid H_0 \text{ true)}$
$\approx \text{P}(Z \leqslant -1.94 \text{ or } Z \geqslant 1.94 \mid Z \sim \text{N}(0,\, 1^2))$
$\approx 2\,\text{P}(Z \geqslant 1.94 \mid Z \sim \text{N}(0,\, 1^2))$
≈ 0.0519



c The true mean $\mu = 498.4$, so the true distribution of Z is

$$\text{N}\left(\frac{498.4 - 500}{\frac{3.71}{\sqrt{13}}},\ 1^2\right) \equiv \text{N}\left(-\tfrac{160\sqrt{13}}{371},\ 1^2\right), \quad \text{where} \quad -\tfrac{160\sqrt{13}}{371} \approx -1.55.$$

$\beta = \text{P(Type II error)}$
$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$
$= \text{P}\left(-1.94 < Z < 1.94 \,\middle|\, Z \sim \text{N}\left(-\tfrac{160\sqrt{13}}{371},\ 1^2\right)\right)$
≈ 0.651



13 Let X be the number of households out of a sample of 100 who use Ongodo washing powder.

a **i** $\mathcal{A} = \{x \mid x \geqslant 13\}$ **ii** $\mathcal{C} = \{x \mid x \leqslant 12\}$

b The null distribution is $X \sim \text{B}(100,\, 0.2)$.

$\alpha = \text{P(Type I error)}$
$= \text{P(Reject } H_0 \mid H_0 \text{ true)}$
$= \text{P}(X \leqslant 12 \mid X \sim \text{B}(100,\, 0.2))$
≈ 0.0253

The significance level of the test is about 2.53%.

c The true population proportion is $17.5\% = 0.175$, so the true distribution of X is $\text{B}(100,\, 0.175)$.

$\therefore\ \beta = \text{P(Type II error)}$
$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$
$= \text{P}(X \geqslant 13 \mid X \sim \text{B}(100,\, 0.175))$
≈ 0.910

$\therefore$ the power $1 - \beta \approx 0.0898$.

REVIEW SET 30B

1 **a** A Type I error would occur if we concluded that the seafood company is supplying undersize fish when they are in fact not.

b A Type II error would occur if we concluded that the seafood company is not supplying undersize fish when they actually are.

2 We are performing a test on the mean of a population with unknown standard deviation, so we use a one-sample t-test.

Step 1: Let μ be Yarni's population mean resting pulse rate.
The hypotheses to be considered are:
H_0: $\mu = 68$ {Yarni's pulse rate has not changed}
H_1: $\mu < 68$ {Yarni's pulse rate has decreased}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 65$, $s = 1.732$, $n = 42$
$\therefore$ the observed value of the test statistic $t = \dfrac{65 - 68}{\frac{1.732}{\sqrt{42}}} \approx -11.2$.

Step 4: $p\text{-value} = \text{P}(T \leqslant t)$ where $T \sim t_{41}$
$\approx \text{P}(T \leqslant -11.2)$
≈ 0

Step 5: Since p-value $< \alpha = 0.05$, we have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that Yarni's pulse rate has decreased.

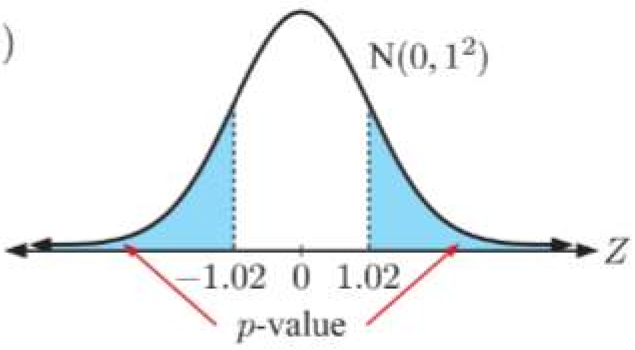
3 We are performing a test on the mean of a population with known standard deviation, so we use a Z-test.

Step 1: Let μ be the population mean colony weight.
The hypotheses to be considered are:
H_0: $\mu = 7.82$ {the mean weight has not changed}
H_1: $\mu \neq 7.82$ {the mean weight has changed}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 7.55$, $\sigma = 1.83$, $n = 48$
$\therefore$ the observed value of the test statistic $z = \dfrac{7.55 - 7.82}{\frac{1.83}{\sqrt{48}}} \approx -1.02$.

Step 4: $p\text{-value} = 2 \times \text{P}(Z \geqslant |\,z\,|)$ where $Z \sim \text{N}(0,\, 1^2)$
$\approx 2 \times \text{P}(Z \geqslant 1.02)$
$\approx 2 \times 0.153\,34$
≈ 0.307



Step 5: Since p-value $> \alpha = 0.05$, we do not have enough evidence to reject H_0 in favour of H_1 at a 5% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the colony's mean weight in 2012 does not differ from 2011.

4 We need to test a hypothesis about a single population mean, so we conduct Student's t-test for a population mean.

Step 1: Let μ be the population mean systolic blood pressure of the employees.

The hypotheses to be tested are:

H_0: $\mu = 140$ {the employees' average blood pressure is not too high}

H_1: $\mu > 140$ {the employee's average blood pressure is too high}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 143.7$ mm Hg, $\mu_0 = 140$ mm Hg, $s = 11.2$ mm Hg, $n = 35$ employees.

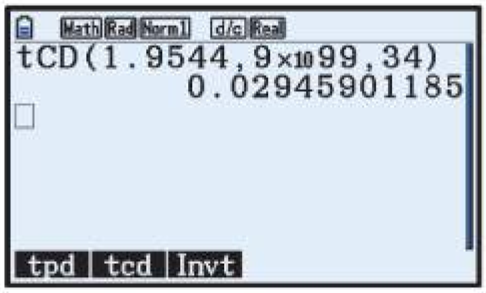
The value of the test statistic is $t = \dfrac{143.7 - 140}{\frac{11.2}{\sqrt{35}}} \approx 1.95$.

Step 4: Since H_1: $\mu > 140$ and $n = 35$,

the p-value $= \text{P}(T \geqslant t)$ where $T \sim t_{34}$

$\approx \text{P}(T \geqslant 1.95)$

≈ 0.0295



Step 5: Since p-value $< 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the company's concerns are justified at a 5% significance level.

5 We need to test hypotheses about a single population mean, so we conduct Student's t-test for a population mean.

Step 1: Let μ be the population mean distance that Arthur can hit a golf ball.

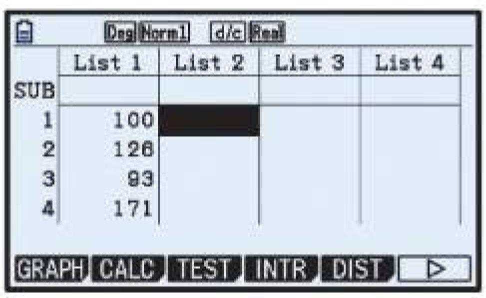
The hypotheses to be tested are:

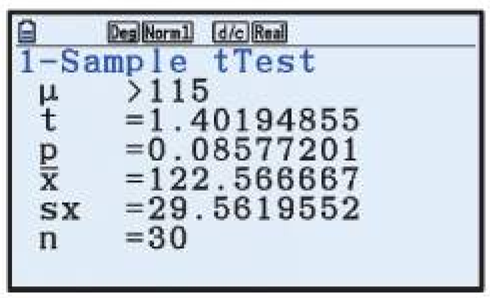
H_0: $\mu = 115$ {the professional did not help Arthur's drive distance}

H_1: $\mu > 115$ {the professional improved Arthur's drive distance}

Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:





Using technology, $t \approx 1.40$ and the p-value ≈ 0.0858.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that there is insufficient evidence at the 5% significance level to claim that Arthur has improved.

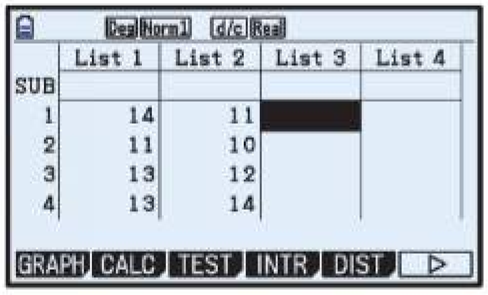
6 *Step 1*: Let μ_1 and μ_2 be the population mean points totals of people living in Maple Grove and Berkton respectively. The hypotheses to be tested are:

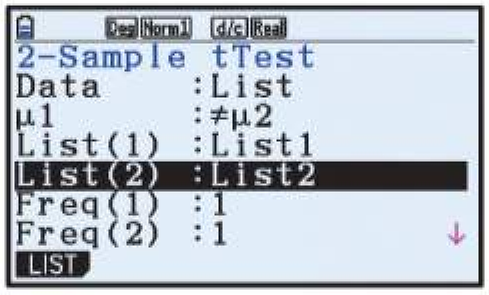
H_0: $\mu_1 = \mu_2$ {there is no difference in average points between the two suburbs}

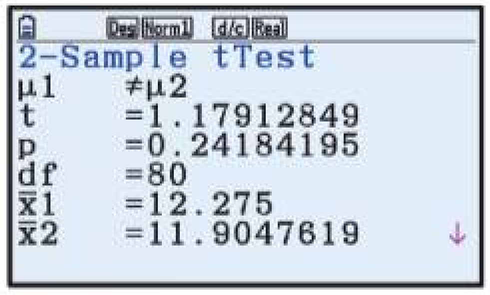
H_1: $\mu_1 \neq \mu_2$ {there is a difference in average points between the two suburbs}

Step 2: The significance level is $\alpha = 0.1$.

Steps 3 and 4:





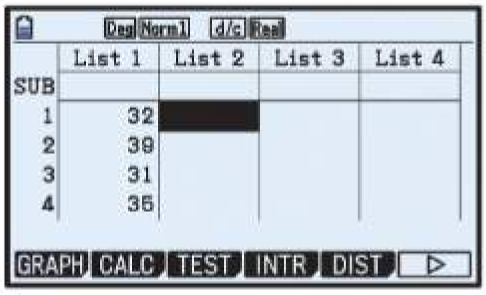


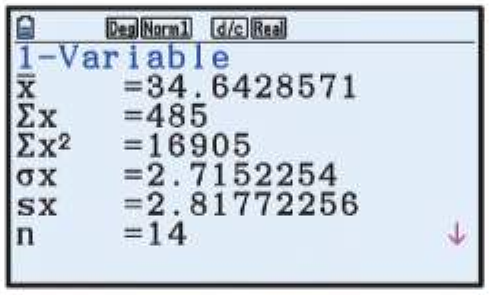
Using technology, $t \approx 1.18$ and the p-value ≈ 0.242.

Step 5: Since p-value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at the 10% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that there is not a significant difference between the points totals of the two suburbs.

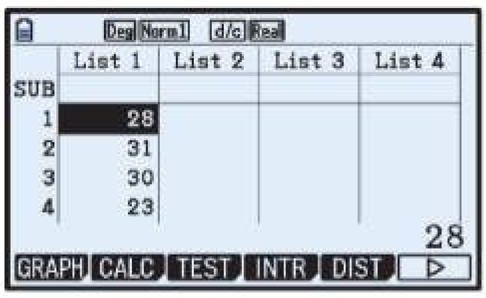
7 **a** *Revision course*:

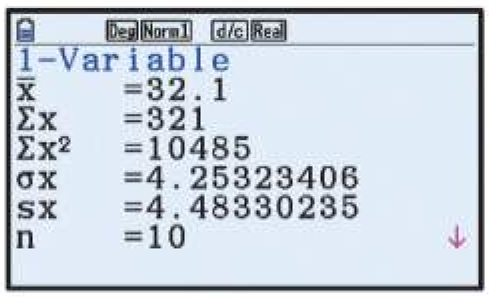




Using technology, the sample mean is $\overline{x}_1 \approx 34.6$, and $s_1 \approx 2.82$.

No revision course:





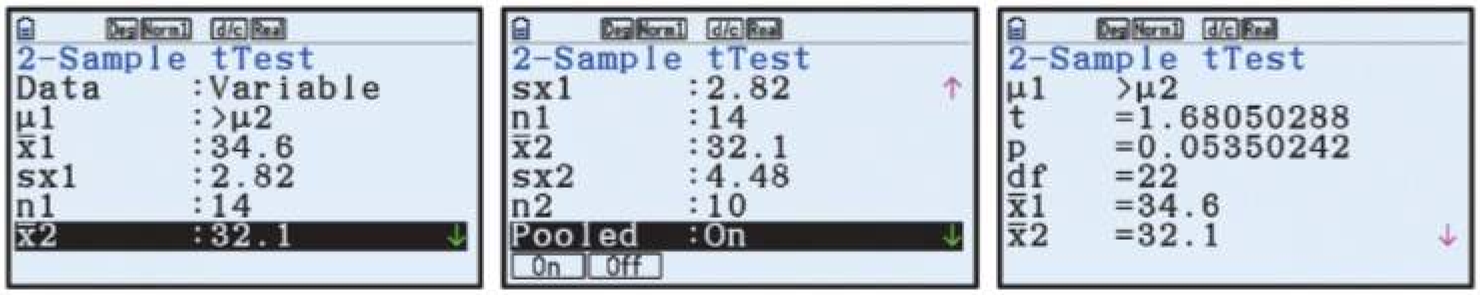
Using technology, the sample mean is $\overline{x}_2 \approx 32.1$, and $s_2 \approx 4.48$.

b *Step 1*: H_0: $\mu_1 = \mu_2$ {the revision course had no effect}

H_1: $\mu_1 > \mu_2$ {the revision course improved examination scores}

Step 2: The significance level is $\alpha = 0.1$.

Step 3:



Using technology, the value of the test statistic is $t \approx 1.68$.

Step 4: From the screenshots above, the p-value ≈ 0.0535.

Step 5: Since the p-value $< 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 10% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that there is sufficient evidence at the 10% significance level to say that the revision course was effective.

8 **a** Josie recorded 2 results from each member of the sample group, so she should use a paired t-test.

b

Silent (x_i seconds)	34	42	32	56	43	32	45	47	46	37	55	44	34	18	51
Yelling (y_i seconds)	39	47	42	65	51	42	54	52	41	45	60	48	32	21	58
$d_i = y_i - x_i$	5	5	10	9	8	10	9	5	-5	8	5	4	-2	3	7

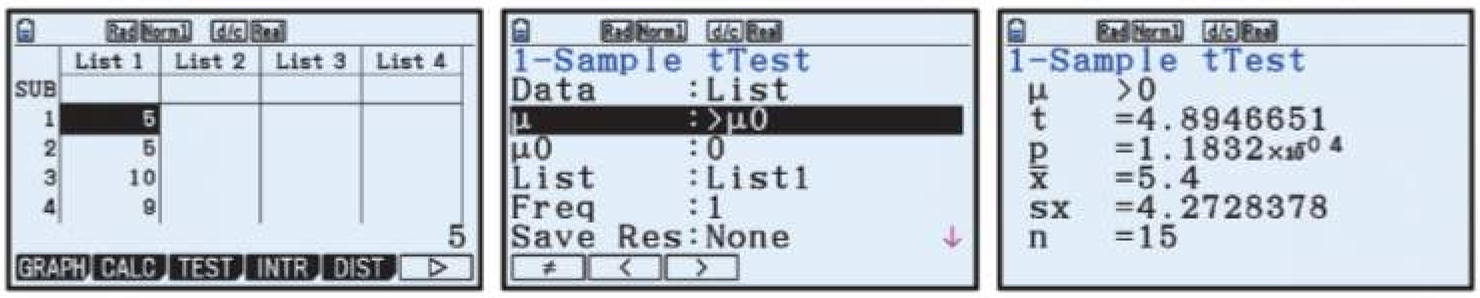
Step 1: The hypotheses to be considered are:

H_0: $\mu_D = 0$ {there is no difference between being silent and yelling}

H_1: $\mu_D > 0$ {yelling makes people withstand pain longer than being silent}

Step 2: The significance level is $\alpha = 0.1$.

Steps 3 and 4:



Using technology, the observed value of the test statistic $t \approx 4.89$, and the p-value $\approx 0.000\,118$.

Step 5: Since p-value $< 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 10% level of significance. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that yelling helps people withstand pain for longer than staying silent.

c We are assuming that:

- Each *Silent* measurement is independent of every other *Silent* measurement. Similarly for the *Yelling* measurements.
- The order in which the *Silent* and *Yelling* measurements were taken does not matter.
- The differences D are normally distributed.

9 We are testing a set of hypotheses about a population proportion.
The test statistic X is the number of sixes rolled in 25 rolls of the die.

Step 1: Let p be the probability of getting a six when the die is rolled.
The hypotheses to be considered are:
H_0: $p = \frac{1}{6}$ {the die is not biased towards six}
H_1: $p > \frac{1}{6}$ {the die is biased towards six}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: The observed value of the test statistic is $x = 8$.

Step 4: The null distribution is $X \sim \mathrm{B}\left(25, \frac{1}{6}\right)$.
H_1: $p > \frac{1}{6}$, so we use the upper tail of the null distribution.
p-value $= \mathrm{P}(X \geqslant 8)$
≈ 0.0447 {using technology}

Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 with a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the die is biased towards rolling a six.

10 **a** We are testing a set of hypotheses about a population proportion.
Let p be the population probability of a train being behind schedule.
The hypotheses to be considered are:
H_0: $p = 0.3$ {the probability of being behind schedule did not change}
H_1: $p < 0.3$ {the probability of being behind schedule improved}

b The test statistic X is the number of trains that were behind schedule from a sample of 10 trains.
$\therefore$ the null distribution is $X \sim \mathrm{B}(10, 0.3)$.

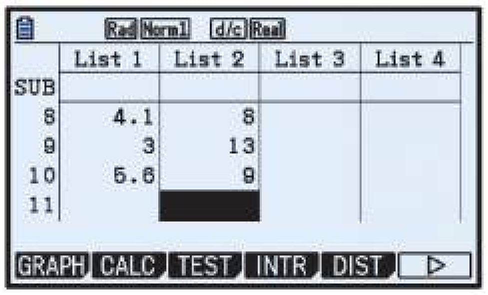
c The significance level is $\alpha = 0.05$.
The critical region $\mathcal{C}$ is the set of all values c such that $\mathrm{P}(X \leqslant c) \leqslant 0.05 = \alpha$.
Now using technology, $\mathrm{P}(X \leqslant 0) \approx 0.0282$ and $\mathrm{P}(X \leqslant 1) \approx 0.149$.
$\therefore$ $\mathcal{C} = \{0\}$

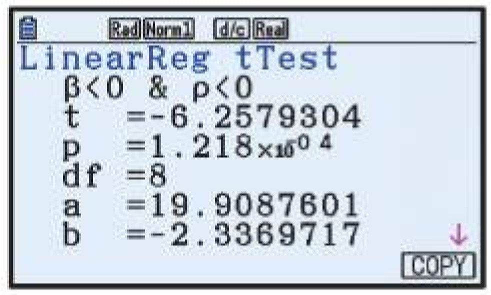
d The observed value of the test statistic is $x = 1$, which is not within the critical region, and so we do not have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0.
Since we have accepted H_0, we conclude that the new train did not change the probability that a train is behind schedule.

11 *Step 1*: Let ρ be the population product-moment correlation coefficient between x and y.
We use the hypotheses:
H_0: $\rho = 0$ {x and y are not correlated}
H_1: $\rho < 0$ {x and y are negatively correlated}

Step 2: The significance level is $\alpha = 0.01$.

Step 3:





The observed value of the test statistic is about -6.26.

Step 4: p-value $\approx 0.000\,122$

Step 5: Since p-value $< 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at a 1% level of significance. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that *distance from goal* and *successful shots* are negatively correlated.

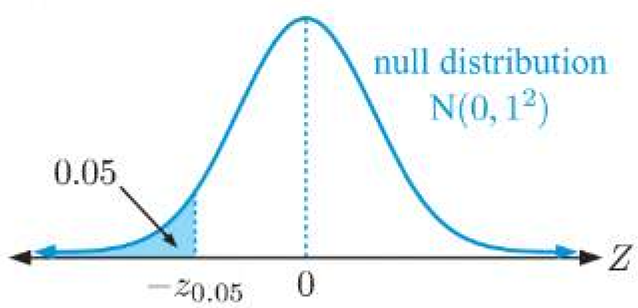
12 **a** We are testing a set of hypotheses about the mean of a population with known variance, so we use a Z-test.

The alternative hypothesis has the form H_1: $\mu < \mu_0$, with $\alpha = \text{P(Type I error)} = 0.05$, so the critical region $\mathcal{C}$ is the set of values of z such that $\text{P}(Z \leqslant z) \leqslant 0.05$.

Now $\text{P}(Z \leqslant -z_{0.05}) = 0.05$

$\therefore$ $\text{P}(Z < -1.64) \approx 0.05$ {using technology}

$\therefore$ $\mathcal{C} = \{z \mid z \leqslant -1.64\}$



b The true mean $\mu = 36$, so the true distribution of Z is

$$\text{N}\left(\frac{36-37}{\frac{\sqrt{7.5}}{\sqrt{30}}},\ 1^2\right) \equiv \text{N}(-2,\ 1^2).$$

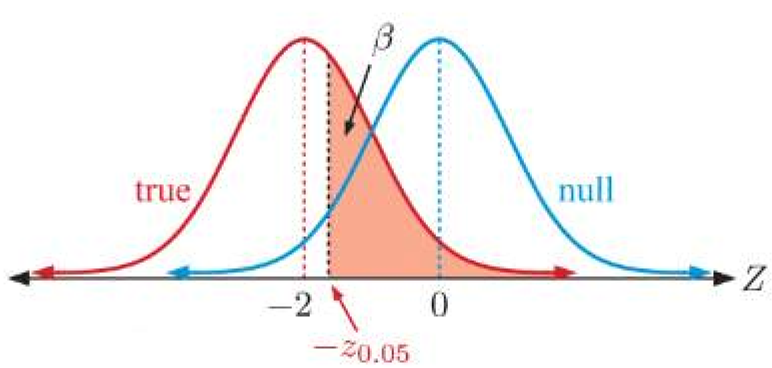
$\therefore$ $\beta = \text{P(Type II error)}$

$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$

$= \text{P}(Z > -1.64 \mid Z \sim \text{N}(-2,\ 1^2))$

$\approx 0.361\,24$

$\therefore$ the power $1 - \beta \approx 0.639$.



c $\alpha = \text{P(Type I error)} = 0.05$

Now, $\text{P(Type II error)} = \text{P(Retain } H_0 \mid H_0 \text{ false)}$

$$= \text{P}\left(Z \in \mathcal{A} \;\middle|\; Z \sim \text{N}\left(\frac{\mu - 37}{\frac{\sqrt{7.5}}{\sqrt{30}}},\ 1^2\right)\right)$$

$$= \text{P}\left(Z > -z_\alpha \mid Z \sim \text{N}(2(\mu - 37),\ 1^2)\right)$$

$$= \text{P}\left(Z > \frac{-z_{0.05} - 2(\mu - 37)}{1} \;\middle|\; Z \sim \text{N}(0,\, 1^2)\right)$$

$$= \text{P}(Z > 74 - 2\mu - z_{0.05} \mid Z \sim \text{N}(0,\, 1^2))$$

So, if $\text{P(Type II error)} = 0.1$,

then $\text{P}(Z > 74 - 2\mu - z_{0.05} \mid Z \sim \text{N}(0,\, 1^2)) = 0.1$

$$\therefore\ 74 - 2\mu - z_{0.05} = z_{0.1}$$

$$\therefore\ \mu = \frac{74 - z_{0.05} - z_{0.1}}{2}$$

$$\approx \frac{74 - 1.64 - 1.28}{2}$$

$$\approx 35.5$$

The true value of μ is about 35.5.

13 **a** The critical region $\mathcal{C}$ is the set of all values of the test statistic that result in rejecting H_0.

The observed value of the test statistic is $t = \sum_{i=1}^{15} x_i$, and we reject H_0 if $t \leqslant 33$.

$\therefore\ \mathcal{C} = \{t \mid t \leqslant 33\}$

b Each value x_i is independent and taken from $X \sim \text{Po}(\lambda)$, and $t = \sum_{i=1}^{15} x_i$, so if t has distribution T then $T \sim \text{Po}(15\lambda)$.

Now H_0: $\lambda = 3$, so the null distribution is $T \sim \text{Po}(45)$.

$\therefore\ \text{P(Type I error)} = \text{P(Reject } H_0 \mid H_0 \text{ true)}$

$$= \text{P}(T \leqslant 33 \mid T \sim \text{Po}(45))$$

$$\approx 0.0383 \quad \{\text{using technology}\}$$

c The true mean $\lambda = 2.5$, so the true distribution of T is $\text{Po}(15 \times 2.5) \equiv \text{Po}(37.5)$.

$\therefore\ \text{P(Type II error)} = \text{P(Retain } H_0 \mid H_0 \text{ false)}$

$$= \text{P}(T \geqslant 34 \mid T \sim \text{Po}(37.5))$$

$$\approx 0.738 \quad \{\text{using technology}\}$$

Chapter 31

χ^2 HYPOTHESIS TESTS

EXERCISE 31A

1 **a** Let p_1 and p_2 be the population proportions of "heads" and "tails" respectively.
The hypotheses that should be tested are:

H_0: $p_1 = 0.5$, $p_2 = 0.5$ {the coin is unbiased}

H_1: at least one of $p_1 \neq 0.5$ or $p_2 \neq 0.5$ {the coin is biased}

b There were 54 heads in 96 tosses, so there were $96 - 54 = 42$ tails.

c If the coin is fair, then $\text{P(head)} = \text{P(tail)} = 0.5$, so the expected frequencies are $96 \times 0.5 = 48$ heads and 48 tails.

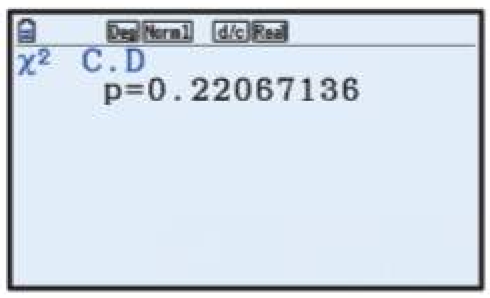
d

Side	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
heads	54	48	0.75
tails	42	48	0.75
		Total	1.5

So, $\chi^2_{\text{calc}} = 1.5$

e There are two categories, so $\text{df} = 2 - 1 = 1$.

f Using technology,
the p-value ≈ 0.221



g The significance level is $\alpha = 0.05$, and the p-value is > 0.05, so there is insufficient evidence to reject H_0 at the 5% significance level. There is therefore insufficient evidence to conclude that the coin is biased.

2 *Step 1*: Let p_{A}, p_{B}, and p_{C} be the population proportions of the votes received in the last election by parties A, B, and C respectively.
The hypotheses that should be tested are:

H_0: $p_{\text{A}} = 0.54$, $p_{\text{B}} = 0.3$, $p_{\text{C}} = 1 - 0.54 - 0.3 = 0.16$

H_1: at least one of $p_{\text{A}} \neq 0.54$, $p_{\text{B}} \neq 0.3$, or $p_{\text{C}} \neq 0.16$.

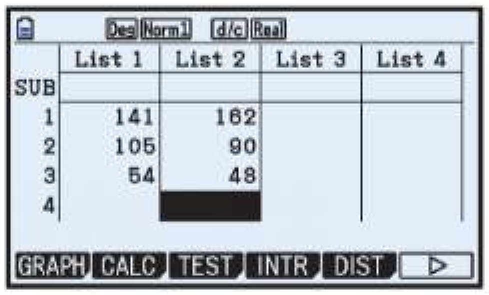
Step 2: The significance level is $\alpha = 0.01$.

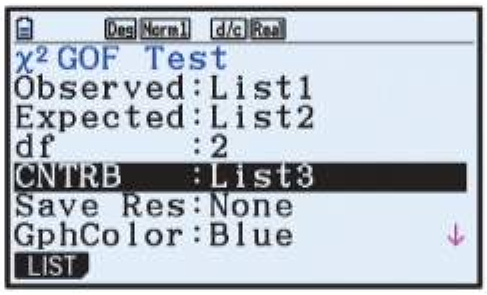
Step 3:

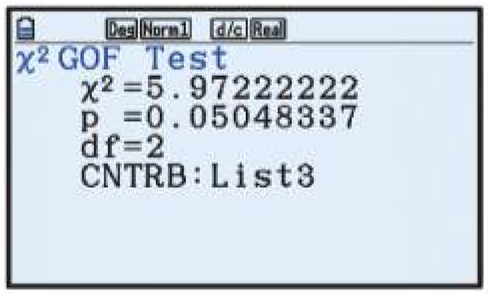
Party	f_{obs}	f_{exp}	$\dfrac{(f_{obs}-f_{exp})^2}{f_{exp}}$
A	141	$0.54 \times 300 = 162$	≈ 2.7222
B	105	$0.3 \times 300 = 90$	2.5
C	54	$0.16 \times 300 = 48$	0.75
		Total	≈ 5.9722

So, $\chi^2_{calc} \approx 5.97$

Step 4: $\text{df} = 3 - 1 = 2$







Using technology, the p-value ≈ 0.0505.

Step 5: Since p-value $> 0.01 = \alpha$, there is insufficient evidence to reject H_0 in favour of H_1 on a 1% significance level.

Step 6: Since we have not rejected H_0, there is insufficient evidence to conclude on a 1% significance level that the proportion of voters supporting each party since the last election has changed.

3 *Step 1*: Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of the ice cream flavours sold. The hypotheses that should be tested are:

H_0: $p_1 = \frac{1}{5}$, $p_2 = \frac{1}{5}$, $p_3 = \frac{1}{5}$, $p_4 = \frac{1}{5}$, $p_5 = \frac{1}{5}$

H_1: at least one of $p_1 \neq \frac{1}{5}$, $p_2 \neq \frac{1}{5}$, $p_3 \neq \frac{1}{5}$, $p_4 \neq \frac{1}{5}$, or $p_5 \neq \frac{1}{5}$.

Step 2: The significance level is $\alpha = 0.1$.

Step 3:

Flavour	f_{obs}	f_{exp}	$\dfrac{(f_{obs}-f_{exp})^2}{f_{exp}}$
chocolate	54	$205 \times \frac{1}{5} = 41$	≈ 4.1220
strawberry	48	41	≈ 1.1951
vanilla	35	41	≈ 0.8780
honeycomb	28	41	≈ 4.1220
choc-chip	40	41	≈ 0.0244
		Total	≈ 10.3415

So, $\chi^2_{calc} \approx 10.3$

Step 4: df $= 5 - 1 = 4$

Using technology, the p-value ≈ 0.0351.

Step 5: Since p-value $< 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 10% significance level.

Step 6: Since we have accepted H_1, we conclude that the proportions of each ice cream flavour sold are not all the same at a 10% significance level. Brian should therefore change the amounts of each ice cream flavour that he makes.

4 *Step 1*: Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of people living in London who identify as being White, Asian/Asian British, Black/Black British, Mixed, and Other respectively.

The hypotheses that should be tested are:

H_0: $p_1 = 0.712$, $p_2 = 0.121$, $p_3 = 0.109$, $p_4 = 0.032$, $p_5 = 0.026$

H_1: at least one of $p_1 \neq 0.712$, $p_2 \neq 0.121$, ..., or $p_5 \neq 0.026$.

Step 2: Since no significance level is specified, we assume that $\alpha = 0.05$.

Step 3:

Ethnic group	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
White	4 887 435	$8\,173\,941 \times 0.712 \approx 5\,819\,846$	$\approx 149\,384$
Asian/Asian British	1 511 546	$8\,173\,941 \times 0.121 \approx 989\,047$	$\approx 276\,029$
Black/Black British	1 088 640	$8\,173\,941 \times 0.109 \approx 890\,960$	$\approx 43\,860$
Mixed	405 279	$8\,173\,941 \times 0.032 \approx 261\,566$	$\approx 78\,961$
Other	281 041	$8\,173\,941 \times 0.026 \approx 212\,522$	$\approx 22\,091$
		Total	$\approx 570\,325$

So, $\chi^2_{\text{calc}} \approx 570\,000$

Step 4: df $= 5 - 1 = 4$

Using technology, the p-value ≈ 0.

Step 5: Since p-value $< 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude on a 5% significance level that there was a significant change in London's demographics between 2001 and 2011.

5 **a**

Band	*Expected frequency*
10	$150 \times 0.079 = 11.85$
9	$150 \times 0.167 = 25.05$
8	$150 \times 0.298 = 44.7$
7	$150 \times 0.297 = 44.55$
6	$150 \times 0.135 = 20.25$
5 and below	$150 \times 0.024 = 3.6$

b *Step 1*: Let p_{10}, p_9, p_8, p_7, p_6, and p_5 be the population proportions of Year 9 students at a particular school who are in NAPLAN bands 10, 9, 8, 7, 6, and 5 and below respectively.

The hypotheses that should be tested are:

H_0: $p_{10} = 0.079$, $p_9 = 0.167$, $p_8 = 0.298$, $p_7 = 0.297$, $p_6 = 0.135$, $p_5 = 0.024$

H_1: at least one of $p_{10} \neq 0.079$, $p_9 \neq 0.167$, ..., or $p_5 \neq 0.024$.

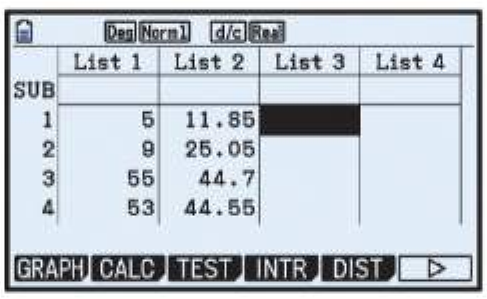
Step 2: The significance level is $\alpha = 0.01$.

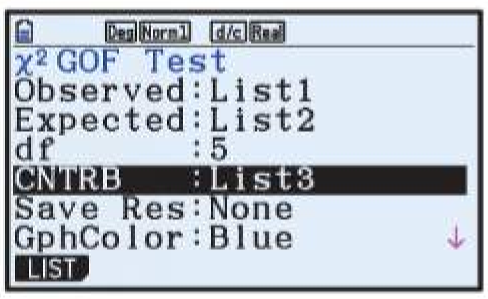
Step 3:

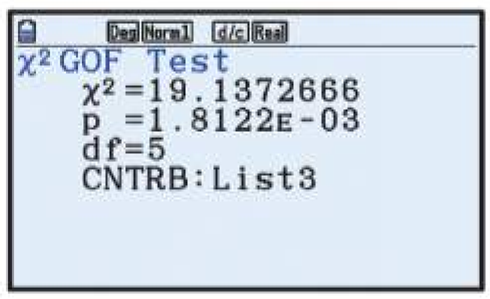
Band	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
10	5	11.85	≈ 3.9597
9	9	25.05	≈ 10.2835
8	55	44.7	≈ 2.3734
7	53	44.55	≈ 1.6027
6	23	20.25	≈ 0.3735
5 and below	5	3.6	≈ 0.5444
		Total	≈ 19.1372

So, $\chi^2_{\text{calc}} \approx 19.1$

Step 4: $\text{df} = 6 - 1 = 5$







Using technology, the p-value $\approx 0.001\,81$.

Step 5: Since p-value $< 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude on a 1% significance level that the NAPLAN results of Year 9s at this school are significantly different from the national results.

c Since the expected frequency for "Band 5 and below" is less than 5, the sample size is not large enough for χ^2 to be distributed appropriately. By combining "Band 6" and "Band 5 and below" we can obtain more reliable results.

d *Step 1*: Let p_{10}, p_9, p_8, p_7, and p_6 be the population proportions of Year 9 students at a particular school who are in NAPLAN bands 10, 9, 8, 7, and 6 and below respectively.

The hypotheses that should be tested are:

H_0: $p_{10} = 0.079$, $p_9 = 0.167$, $p_8 = 0.298$, $p_7 = 0.297$,
$p_6 = 0.135 + 0.024 = 0.159$

H_1: at least one of $p_{10} \neq 0.079$, $p_9 \neq 0.167$, ..., or $p_6 \neq 0.159$.

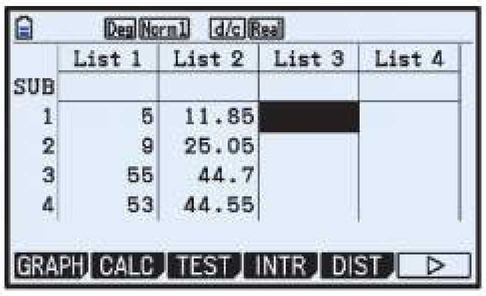
Step 2: The significance level is $\alpha = 0.01$.

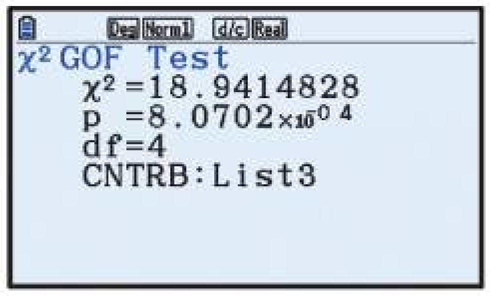
Step 3:

Band	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
10	5	11.85	≈ 3.9597
9	9	25.05	≈ 10.2835
8	55	44.7	≈ 2.3734
7	53	44.55	≈ 1.6027
6 and below	$23 + 5 = 28$	$20.25 + 3.6 = 23.85$	≈ 0.7221
		Total	≈ 18.9414

So, $\chi^2_{\text{calc}} \approx 18.9$

Step 4: $\text{df} = 5 - 1 = 4$





Using technology, the p-value $\approx 0.000\,807$.

Step 5: Since p-value $< 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, there is still sufficient evidence to support the claim that there is a substantial difference between the school's results and the rest of the nation at a 1% significance level.

Although we have obtained the same result both times, the result is more reliable now that each of the expected frequencies is sufficiently large.

6 **a**

- Each question can be answered either correctly (success) or incorrectly (failure). So, each trial has only two possible results.
- The probability of answering correctly (success) is the same for each question.
- There are 10 questions, and we assume the probability of answering each question correctly does not change based on the results from previous questions. So, there are a fixed number of independent trials.

So, the number of questions the person answers correctly X is a binomial random variable.

$X \sim \text{B}(10, \frac{1}{4})$

b It is unlikely that a person will have to guess the answer to *every* question. Some questions are more likely to be answered correctly than others. So the statistician's model is not completely appropriate.

c *Step 1*: The hypotheses are:

H_0: the data is from $\mathrm{B}\left(10, \frac{1}{4}\right)$

H_1: the data is not from $\mathrm{B}\left(10, \frac{1}{4}\right)$.

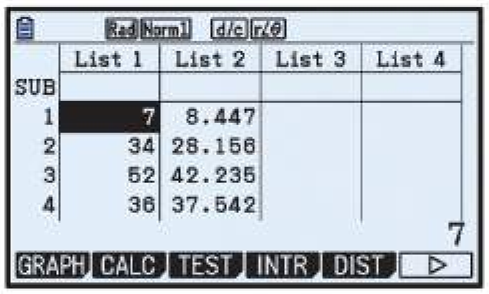
Step 2: The significance level is $\alpha = 0.1$.

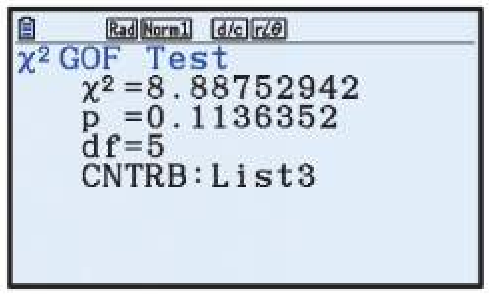
Step 3: $X \sim \mathrm{B}\left(10, \frac{1}{4}\right)$

x	$\mathrm{P}(X = x)$	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	$\approx 0.056\,31$	7	$\approx 0.056\,31 \times 150 \approx 8.45$	≈ 0.2479
1	$\approx 0.187\,71$	34	$\approx 0.187\,71 \times 150 \approx 28.16$	≈ 1.2126
2	$\approx 0.281\,57$	52	$\approx 0.281\,57 \times 150 \approx 42.24$	≈ 2.2577
3	$\approx 0.250\,28$	36	$\approx 0.250\,28 \times 150 \approx 37.54$	≈ 0.0634
4	$\approx 0.146\,00$	12	$\approx 0.146\,00 \times 150 \approx 21.90$	≈ 4.4751
5 or more	$\approx 0.078\,13$	9	$\approx 0.078\,13 \times 150 \approx 11.72$	≈ 0.6309
			Total	≈ 8.8875

So, $\chi^2_{\text{calc}} \approx 8.89$

Step 4: $\text{df} = 6 - 1 = 5$





Using technology, p-value ≈ 0.114.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is from the binomial distribution $\mathrm{B}\left(10, \frac{1}{4}\right)$. The statistician's model is therefore appropriate.

7 *Step 1*: The hypotheses are:

H_0: the data is from $\mathrm{Po}(5)$

H_1: the data is not from $\mathrm{Po}(5)$.

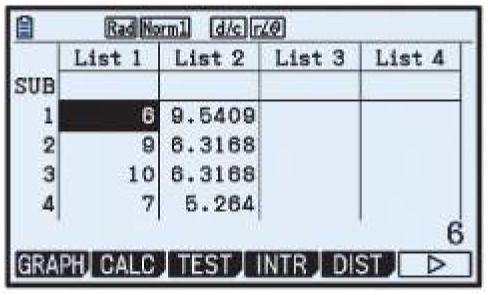
Step 2: The significance level is $\alpha = 0.01$.

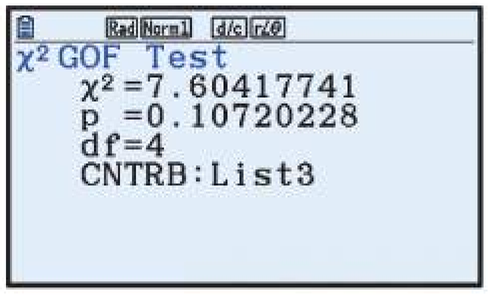
Step 3: $X \sim \mathrm{Po}(5)$

Score (x)	$\mathrm{P}(X = x)$	f_{obs}	f_{exp}	$\dfrac{(f_{obs} - f_{exp})^2}{f_{exp}}$
$\leqslant 3$	$\approx 0.265\,03$	6	$\approx 0.265\,03 \times 36 \approx 9.54$	≈ 1.3141
4	$\approx 0.175\,47$	9	$\approx 0.175\,47 \times 36 \approx 6.32$	≈ 1.1397
5	$\approx 0.175\,47$	10	$\approx 0.175\,47 \times 36 \approx 6.32$	≈ 2.1476
6	$\approx 0.146\,22$	7	$\approx 0.146\,22 \times 36 \approx 5.26$	≈ 0.5725
$\geqslant 7$	$\approx 0.237\,82$	4	$\approx 0.237\,82 \times 36 \approx 8.56$	≈ 2.4302
			Total	≈ 7.6042

So, $\chi^2_{calc} \approx 7.60$

Step 4: $\mathrm{df} = 5 - 1 = 4$





Using technology, p-value ≈ 0.107.

Step 5: Since p-value $> 0.01 = \alpha$, we do not have enough evidence to reject H_0 on a 1% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is from the Poisson distribution $\mathrm{Po}(5)$.

ACTIVITY MENDEL'S DATA

1

Type of pea	*Proportion*	*Expected frequency*
Yellow round seeds	$\dfrac{9}{9+3+3+1} = \dfrac{9}{16}$	$556 \times \frac{9}{16} = 312.75$
Green round seeds	$\dfrac{3}{9+3+3+1} = \dfrac{3}{16}$	$556 \times \frac{3}{16} = 104.25$
Yellow wrinkled seeds	$\dfrac{3}{9+3+3+1} = \dfrac{3}{16}$	$556 \times \frac{3}{16} = 104.25$
Green wrinkled seeds	$\dfrac{1}{9+3+3+1} = \dfrac{1}{16}$	$556 \times \frac{1}{16} = 34.75$

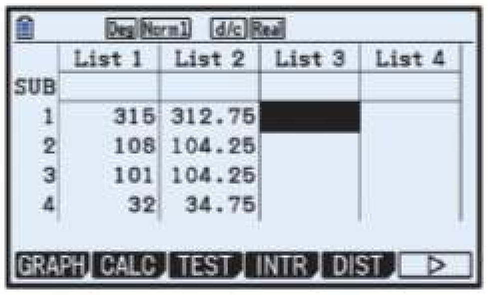
2 Let p_1, p_2, p_3, and p_4 be the population proportions of yellow round, green round, yellow wrinkled, and green wrinkled seeds respectively.

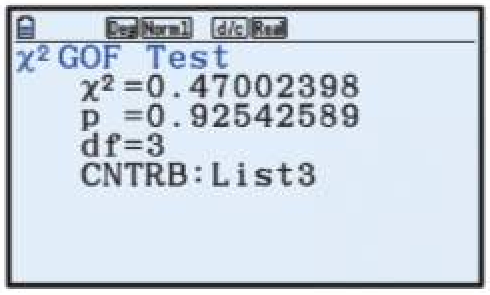
The hypotheses to be tested are:

H_0: $p_1 = \frac{9}{16}$, $p_2 = \frac{3}{16}$, $p_3 = \frac{3}{16}$, $p_4 = \frac{1}{16}$ {Mendel's model is true}

H_1: at least one of $p_1 \neq \frac{9}{16}$, $p_2 \neq \frac{3}{16}$, $p_3 \neq \frac{3}{16}$, or $p_4 \neq \frac{1}{16}$. {Mendel's model is false}

Type of pea	f_{obs}	f_{exp}	$\dfrac{(f_{obs}-f_{exp})^2}{f_{exp}}$
Yellow round seeds	315	312.75	≈ 0.0162
Green round seeds	108	104.25	≈ 0.1349
Yellow wrinkled seeds	101	104.25	≈ 0.1013
Green wrinkled seeds	32	34.75	≈ 0.2176
		Total	≈ 0.4700





So, $\chi^2_{calc} \approx 0.470$, and using technology, the p-value ≈ 0.925.

3 For common significance levels ($\alpha = 0.1$, 0.05, or 0.01 for example), we would retain H_0 since the p-value is quite large.
So, we would conclude that Mendel's model is true.

EXERCISE 31B

1

Number of days	0	1	2	3	4	5	6	7
Number of students	9	16	22	38	57	62	41	26

Step 1: H_0: the data is from a binomial distribution.
H_1: the data is *not* from a binomial distribution.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: In this case, $n = 7$.

$$\text{Now} \quad \overline{x} = \frac{0(9)+1(16)+....+7(26)}{9+16+....+26} = \frac{1140}{271} \approx 4.2066$$

$$\therefore \quad p \approx \frac{\overline{x}}{n} \approx \frac{4.2066}{7} \approx 0.6009$$

So the proposed distribution is $X \sim \mathrm{B}(7, 0.6009)$.

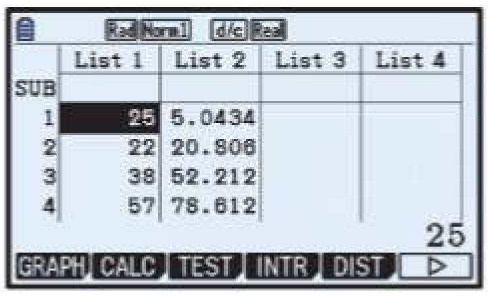
Number of days (x)	$P(X = x)$	f_{obs}	f_{exp}	
0	$\approx 0.001\,61$	9	$\approx 0.001\,61 \times 271 \approx 0.4371$	$\Big\} < 5$
1	$\approx 0.017\,00$	16	$\approx 0.0170 \times 271 \approx 4.6064$	
2	$\approx 0.076\,78$	22	$\approx 0.076\,78 \times 271 \approx 20.8066$	
3	$\approx 0.192\,66$	38	$\approx 0.192\,66 \times 271 \approx 52.2121$	
4	$\approx 0.290\,08$	57	$\approx 0.290\,08 \times 271 \approx 78.6125$	
5	$\approx 0.262\,06$	62	$\approx 0.262\,06 \times 271 \approx 71.0171$	
6	$\approx 0.131\,52$	41	$\approx 0.131\,52 \times 271 \approx 35.6420$	
7	$\approx 0.028\,29$	26	$\approx 0.028\,29 \times 271 \approx 7.6663$	

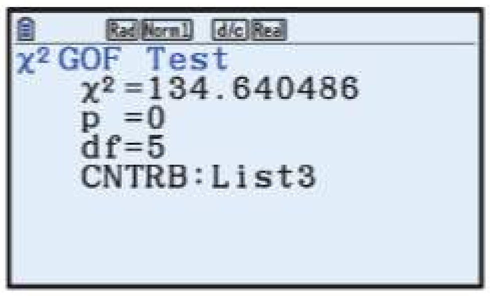
There are expected frequencies less than 5, so we combine "categories" appropriately:

Number of days (x)	f_{obs}	f_{exp}	$\dfrac{(f_{obs} - f_{exp})^2}{f_{exp}}$
$\leqslant 1$	25	≈ 5.0434	≈ 78.9668
2	22	≈ 20.8066	≈ 0.0684
3	38	≈ 52.2121	≈ 3.8685
4	57	≈ 78.6125	≈ 5.9418
5	62	≈ 71.0171	≈ 1.1449
6	41	≈ 35.6420	≈ 0.8055
7	26	≈ 7.6663	≈ 43.8446
		Total	≈ 134.6405

So, $\chi^2_{calc} \approx 135$

Step 4: $\text{df} = 7 - 1 - 1 = 5$





Using technology, p-value ≈ 0.

Step 5: Since p-value $< 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the data is not binomially distributed at the 5% level of significance.

2

Number of accidents	0	1	2	3
Number of weeks	52	22	20	10

a $\text{mean} = \dfrac{0(52) + 1(22) + 2(20) + 3(10)}{52 + 22 + 20 + 10}$

$= \dfrac{92}{104}$

≈ 0.885 accidents per week

b *Step 1*: H_0: the data is from a Poisson distribution.
H_1: the data is *not* from a Poisson distribution.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $X \sim \text{Po}(0.885)$

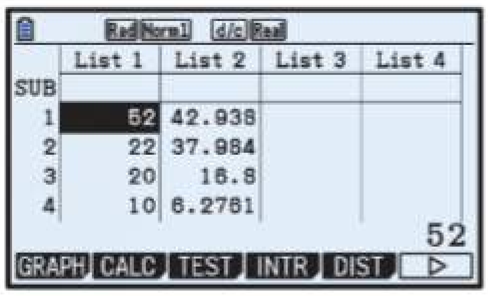
Number of accidents (x)	$\text{P}(X = x)$	f_{obs}	f_{exp}	
0	≈ 0.41287	52	$\approx 0.41287 \times 104 \approx 42.9388$	
1	≈ 0.36523	22	$\approx 0.36523 \times 104 \approx 37.9843$	
2	≈ 0.16155	20	$\approx 0.16155 \times 104 \approx 16.8008$	
3	≈ 0.04764	10	$\approx 0.04764 \times 104 \approx 4.9541$	$\} < 5$
$\geqslant 4$	≈ 0.01271	0	$\approx 0.01271 \times 104 \approx 1.3221$	

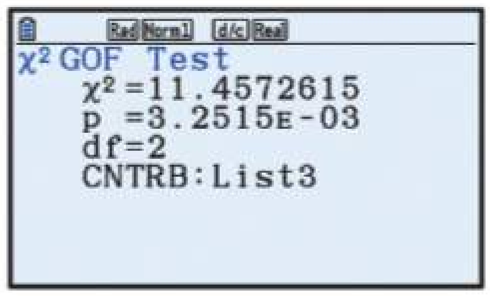
There are expected frequencies less than 5, so we combine "categories" appropriately:

Number of accidents (x)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	52	≈ 42.9388	≈ 1.9122
1	22	≈ 37.9843	≈ 6.7264
2	20	≈ 16.8008	≈ 0.6092
$\geqslant 3$	10	≈ 6.2762	≈ 2.2095
		Total	≈ 11.4573

So, $\chi^2_{\text{calc}} \approx 11.5$.

Step 4: $\text{df} = 4 - 1 - 1 = 2$





Using technology, p-value ≈ 0.00325.

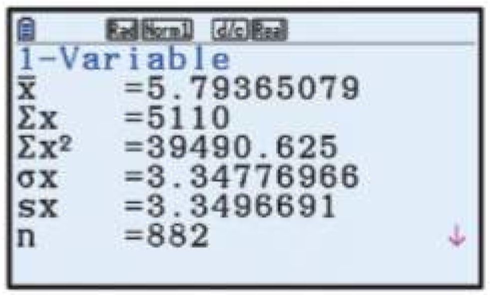
Step 5: Since p-value $< 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the data is not from a Poisson distribution on a 5% level of significance.

3 **a**

x	*Midpoint*	*Frequency*
$0 \leqslant x < 2.5$	1.25	159
$2.5 \leqslant x < 5$	3.75	245
$5 \leqslant x < 7.5$	6.25	210
$7.5 \leqslant x < 10$	8.75	164
$10 \leqslant x < 12.5$	11.25	74
$12.5 \leqslant x < 15$	13.75	30
	Total	882

Using technology, $\sigma \approx 3.35$.



b *Step 1*: H_0: the data is from a normal distribution with mean 5.

H_1: the data is *not* from a normal distribution with mean 5.

Step 2: The significance level is $\alpha = 0.05$.

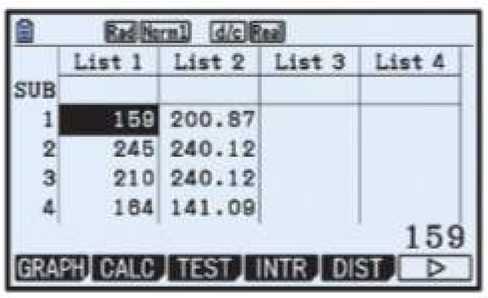
Step 3: $\mu = 5$ and from **a**, $\sigma \approx 3.35$

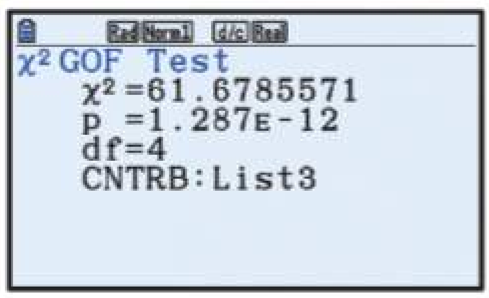
So, the proposed distribution is $X \sim N(5, 3.35^2)$.

x	*Probability*	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
$0 \leqslant x < 2.5$	$\approx 0.227\,75$	159	$\approx 0.227\,75 \times 882 \approx 200.8778$	≈ 8.7304
$2.5 \leqslant x < 5$	$\approx 0.272\,25$	245	$\approx 0.272\,25 \times 882 \approx 240.1222$	≈ 0.0991
$5 \leqslant x < 7.5$	$\approx 0.272\,25$	210	$\approx 0.272\,25 \times 882 \approx 240.1222$	≈ 3.7787
$7.5 \leqslant x < 10$	$\approx 0.159\,97$	164	$\approx 0.159\,97 \times 882 \approx 141.0965$	≈ 3.7178
$10 \leqslant x < 12.5$	$\approx 0.055\,19$	74	$\approx 0.055\,19 \times 882 \approx 48.6819$	≈ 13.1672
$12.5 \leqslant x < 15$	$\approx 0.012\,58$	30	$\approx 0.0125 \times 882 \approx 11.0993$	≈ 32.1853
			Total	≈ 61.6786

So, $\chi^2_{\text{calc}} \approx 61.7$.

Step 4: $\text{df} = 6 - 1 - 1 = 4$





Using technology, p-value $\approx 1.29 \times 10^{-12}$.

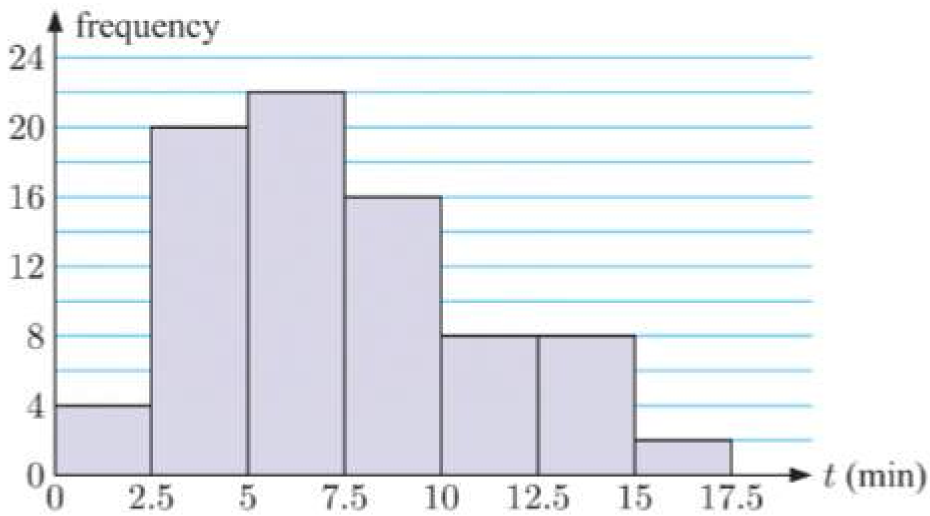
Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the data is *not* from a normal distribution with mean 5.

4 **a**

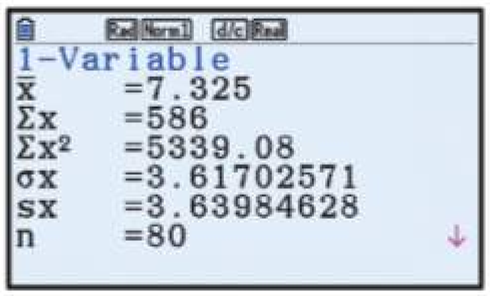
Time (t min)	*Tally*	*Frequency*
$0 \leqslant t < 2.5$	||||	4
$2.5 \leqslant t < 5$	卌 卌 卌 卌	20
$5 \leqslant t < 7.5$	卌 卌 卌 卌 ||	22
$7.5 \leqslant t < 10$	卌 卌 卌 |	16
$10 \leqslant t < 12.5$	卌 |||	8
$12.5 \leqslant t < 15$	卌 |||	8
$15 \leqslant t < 17.5$	||	2
	Total	80

b



c From the histogram, we can see that the data is slightly positively skewed.

d Using the original data,
$\mu \approx \overline{x} = 7.325$ minutes and
$\sigma \approx s \approx 3.64$ minutes.



e *Step 1*: H_0: the data is from a normal distribution.
H_1: the data is *not* from a normal distribution.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: From **d**, $\mu \approx 7.325$ and $\sigma \approx 3.64$.
So, the proposed distribution is $X \sim \text{N}(7.325, 3.64^2)$.

Time (t min)	*Probability*	f_{obs}	f_{exp}	
$0 \leqslant t < 2.5$	$\approx 0.092\,49$	4	$\approx 0.092\,49 \times 80 \approx 7.3996$	
$2.5 \leqslant t < 5$	$\approx 0.169\,00$	20	$\approx 0.169\,00 \times 80 \approx 13.5202$	
$5 \leqslant t < 7.5$	$\approx 0.257\,68$	22	$\approx 0.257\,68 \times 80 \approx 20.6140$	
$7.5 \leqslant t < 10$	$\approx 0.249\,62$	16	$\approx 0.249\,62 \times 80 \approx 19.1699$	
$10 \leqslant t < 12.5$	$\approx 0.153\,65$	8	$\approx 0.153\,65 \times 80 \approx 12.2918$	
$12.5 \leqslant t < 15$	$\approx 0.060\,06$	8	$\approx 0.060\,06 \times 80 \approx 4.8050$	< 5
$15 \leqslant t < 17.5$	$\approx 0.017\,49$	2	$\approx 0.017\,49 \times 80 \approx 1.3995$	

There are expected frequencies less than 5, so we combine "categories" appropriately:

Time (t min)	f_{obs}	f_{exp}	$\dfrac{(f_{obs}-f_{exp})^2}{f_{exp}}$
$0 \leqslant t < 2.5$	4	≈ 7.3996	≈ 1.5619
$2.5 \leqslant t < 5$	20	≈ 13.5202	≈ 3.1055
$5 \leqslant t < 7.5$	22	≈ 20.6140	≈ 0.0932
$7.5 \leqslant t < 10$	16	≈ 19.9699	≈ 0.7892
$10 \leqslant t < 12.5$	8	≈ 12.2918	≈ 1.4985
$12.5 \leqslant t < 17.5$	10	≈ 6.2045	≈ 2.3218
		Total	≈ 9.3701

So, $\chi^2_{calc} \approx 9.37$.

Step 4: $\text{df} = 6 - 2 - 1 = 3$

Using technology, p-value ≈ 0.0248.

Step 5: Since p-value $> 0.01 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is from a normal distribution at a 1% level of significance.

EXERCISE 31C

1 **a** There are 5 categories so $\text{df} = 5 - 1 = 4$, and the significance level is $\alpha = 0.05$.
So, from the table, $\chi^2_{crit} = 9.49$.

b $\chi^2_{calc} = 10.3 > \chi^2_{crit}$, so we reject H_0 on a 5% significance level.

2 **a** Let p_1, p_2, p_3, and p_4 be the population proportions of the lollies made by Chewy Chews, corresponding to the colours red, yellow, green, and blue respectively.
Chewy Chews claims to make the same proportions of each colour, so the hypotheses that should be tested are:

H_0: $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{4}$, $p_4 = \frac{1}{4}$

H_1: at least one of $p_1, p_2, p_3, p_4 \neq \frac{1}{4}$.

b

Colour	f_{obs}	f_{exp}	$\dfrac{(f_{obs}-f_{exp})^2}{f_{exp}}$
red	12	$65 \times 0.25 = 16.25$	≈ 1.112
yellow	17	16.25	≈ 0.035
green	20	16.25	≈ 0.865
blue	16	16.25	≈ 0.004
		Total	≈ 2.016

So, $\chi^2_{calc} \approx 2.02$

c There are 4 categories, so $df = 4 - 1 = 3$, and the significance level $\alpha = 0.1$.
So, from the table, $\chi^2_{crit} = 6.25$.

d $\chi^2_{calc} < \chi^2_{crit}$, so H_0 should not be rejected at a 10% significance level. Lucy therefore does not have enough evidence to reject the manufacturer's claim.

e

Using technology, the p-value ≈ 0.569

p-value $> 0.1 = \alpha$, so we again conclude that H_0 should not be rejected at a 10% significance level.

3 *Step 1*: Let p_1, p_2, p_3, p_4, and p_5 be the proportions of responses from the initial survey, corresponding to "very satisfied", "satisfied", "neutral", "dissatisfied", and "very satisfied" respectively.

The hypotheses that should be tested are:

H_0: $p_1 = 0.05$, $p_2 = 0.25$, $p_3 = 0.41$, $p_4 = 0.2$, $p_5 = 0.09$

H_1: at least one of $p_1 \neq 0.05$, $p_2 \neq 0.25$, ..., or $p_5 \neq 0.09$.

Step 2: The significance level is $\alpha = 0.01$.

Step 3:

Response	f_{obs}	f_{exp}	$\dfrac{(f_{obs}-f_{exp})^2}{f_{exp}}$
very satisfied	25	$233 \times 0.05 = 11.65$	≈ 15.298
satisfied	78	$233 \times 0.25 = 58.25$	≈ 6.696
neutral	77	$233 \times 0.41 = 95.53$	≈ 3.594
dissatisfied	36	$233 \times 0.2 = 46.6$	≈ 2.411
very dissatisfied	17	$233 \times 0.09 = 20.97$	≈ 0.752
		Total	≈ 28.751

So, $\chi^2_{calc} \approx 28.8$

Step 4: $df = 5 - 1 = 4$, so from the table, $\chi^2_{crit} = 13.28$.

Step 5: $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, so we have sufficient evidence to reject H_0 in favour of H_1 at the 1% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude at the 1% significance level that the ISP's changes had a significant impact. The expected numbers of people who were either "dissatisfied" or "very dissatisfied" based on the initial survey were both higher than the results observed in the latter survey, so we conclude that the changes were effective.

4 *Step 1*: H_0: the data is from Poisson distribution with rate $\lambda = 2.2$.
H_1: the data is *not* from a Poisson distribution with rate $\lambda = 2.2$.

Step 2: The significance level is $\alpha = 0.1$.

Step 3: $X \sim \text{Po}(2.2)$

Number of people served (x)	$\text{P}(X = x)$	f_{obs}	f_{exp}	
0	$\approx 0.110\,80$	14	$\approx 0.110\,80 \times 160 \approx 17.7285$	
1	$\approx 0.243\,77$	38	$\approx 0.243\,77 \times 160 \approx 39.0027$	
2	$\approx 0.268\,14$	49	$\approx 0.268\,14 \times 160 \approx 42.9030$	
3	$\approx 0.196\,64$	36	$\approx 0.196\,64 \times 160 \approx 31.4622$	
4	$\approx 0.108\,15$	14	$\approx 0.108\,15 \times 160 \approx 17.3042$	
5	$\approx 0.047\,59$	4	$\approx 0.047\,59 \times 160 \approx 7.6138$	
6	$\approx 0.017\,45$	4	$\approx 0.017\,45 \times 160 \approx 2.7917$	$\} < 5$
$\geqslant 7$	$\approx 0.007\,46$	1	$\approx 0.007\,46 \times 160 \approx 1.1938$	

There are expected frequencies less than 5, so we combine "categories" appropriately:

Number of people served (x)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	14	≈ 17.7285	≈ 0.7841
1	38	≈ 39.0027	≈ 0.0258
2	49	≈ 42.9030	≈ 0.8665
3	36	≈ 31.4622	≈ 0.6545
4	14	≈ 17.3042	≈ 0.6309
$\geqslant 5$	9	≈ 11.5994	≈ 0.5825
		Total	≈ 3.5443

So, $\chi^2_{\text{calc}} \approx 3.54$.

Step 4: $\text{df} = 6 - 1 = 5$, so from the table, $\chi^2_{\text{crit}} = 9.24$.

Step 5: $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, so we do not have enough evidence to reject H_0 in favour of H_1 on the 10% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data comes from a Poisson distribution with rate $\lambda = 2.2$.

5 *Step 1*: H_0: the data is binomially distributed.
H_1: the data is *not* binomially distributed.

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

Number of tyres	*Frequency*
0	307
1	180
2	54
3	9
4	0

In this case, $n = 4$.

$$\text{Now} \quad \overline{x} = \frac{0(307) + 1(180) + 2(54) + 3(9) + 4(0)}{307 + 180 + 54 + 9 + 0} = \frac{315}{550} \approx 0.5727$$

$$\therefore \quad p \approx \frac{\overline{x}}{n} \approx \frac{0.5727}{4} \approx 0.1432$$

So the proposed distribution is $X \sim \text{B}(4, 0.1432)$.

Number of tyres (x)	$\text{P}(X = x)$	f_{obs}	f_{exp}	
0	$\approx 0.538\,96$	307	$\approx 0.538\,96 \times 550 \approx 296.4267$	
1	$\approx 0.360\,26$	180	$\approx 0.360\,26 \times 550 \approx 198.1420$	
2	$\approx 0.090\,30$	54	$\approx 0.090\,30 \times 550 \approx 49.6669$	
3	$\approx 0.010\,06$	9	$\approx 0.010\,06 \times 550 \approx 5.5332$	
4	$\approx 0.000\,42$	0	$\approx 0.000\,42 \times 550 \approx 0.2312$	$\leftarrow < 5$

There are expected frequencies less than 5, so we combine "categories" appropriately:

Number of tyres (x)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	307	≈ 296.4267	≈ 0.3771
1	180	≈ 198.1420	≈ 1.6611
2	54	≈ 49.6669	≈ 0.3780
$\geqslant 3$	9	≈ 5.7643	≈ 1.8162
		Total	≈ 4.2325

So, $\chi^2_{\text{calc}} \approx 4.23$.

Step 4: $\text{df} = 4 - 1 - 1 = 2$, so from the table, $\chi^2_{\text{crit}} = 5.99$.

Step 5: $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, so we do not have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is binomially distributed.

EXERCISE 31D.1

Note: Expected frequency tables can be found either by hand, or using the χ^2 test functionality on your graphics calculator.

1 **a**

	Junior school	*Middle school*	*High school*	*Sum*
Plays sport	$\frac{165 \times 58}{250} = 38.28$	$\frac{165 \times 86}{250} = 56.76$	$\frac{165 \times 106}{250} = 69.96$	165
Does not play sport	$\frac{85 \times 58}{250} = 19.72$	$\frac{85 \times 86}{250} = 29.24$	$\frac{85 \times 106}{250} = 36.04$	85
Sum	58	86	106	250

b

	Drove to work	*Cycled to work*	*Public transport*	*Sum*
Male	$\frac{44 \times 46}{80} = 25.3$	$\frac{44 \times 14}{80} = 7.7$	$\frac{44 \times 20}{80} = 11$	44
Female	$\frac{36 \times 46}{80} = 20.7$	$\frac{36 \times 14}{80} = 6.3$	$\frac{36 \times 20}{80} = 9$	36
Sum	46	14	20	80

c The 2×3 contingency table is:

	Wore hat and sunscreen	*Wore hat or sunscreen*	*Wore neither*	*Sum*
Sunburnt	3	5	13	$3+5+13=21$
Not sunburnt	36	17	1	$36+17+1=54$
Sum	$3+36=39$	$5+17=22$	$13+1=14$	$21+54=75$

The expected frequency table is:

	Wore hat and sunscreen	*Wore hat or sunscreen*	*Wore neither*
Sunburnt	$\frac{21 \times 39}{75} = 10.92$	$\frac{21 \times 22}{75} = 6.16$	$\frac{21 \times 14}{75} = 3.92$
Not sunburnt	$\frac{54 \times 39}{75} = 28.08$	$\frac{54 \times 22}{75} = 15.84$	$\frac{54 \times 14}{75} = 10.08$

2 **a**

	Pass Maths test	*Fail Maths test*	*Sum*
Male	$\frac{50 \times 60}{100} = 30$	$\frac{50 \times 40}{100} = 20$	50
Female	$\frac{50 \times 60}{100} = 30$	$\frac{50 \times 40}{100} = 20$	50
Sum	60	40	100

b In a sample of 100 students, we would expect 30 to be male and pass the Maths test.

c

f_{obs}	f_{exp}	$f_{obs} - f_{exp}$	$(f_{obs} - f_{exp})^2$	$\frac{(f_{obs} - f_{exp})^2}{f_{exp}}$
24	30	$24 - 30 = -6$	$(-6)^2 = 36$	$\frac{36}{30} = 1.2$
26	20	$26 - 20 = 6$	$6^2 = 36$	$\frac{36}{20} = 1.8$
36	30	$36 - 30 = 6$	$6^2 = 36$	$\frac{36}{30} = 1.2$
14	20	$14 - 20 = -6$	$(-6)^2 = 36$	$\frac{36}{20} = 1.8$
			Total	6

So, $\chi^2_{calc} = 6$.

3 *Step 1*: H_0: *weight* and *diabetes* are independent
H_1: *weight* and *diabetes* are dependent.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\text{df} = (2-1)(3-1) = 2$

Step 4: The 2×3 contingency table is:

	Weight		
	light	*medium*	*heavy*
Diabetic	11	19	26
Non-diabetic	79	68	69

Using technology, $\chi^2_{\text{calc}} \approx 6.61$.

Step 5: From the screenshots above, the p-value ≈ 0.0368.

Step 6: Since the p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 5% significance level. We therefore accept H_1.

Step 7: We conclude that *weight* and *diabetes* are dependent.

4 **a** $\text{df} = (2-1)(3-1) = 2$, and the significance level $\alpha = 0.1$, so from the table, $\chi^2_{\text{crit}} = 4.61$.

b *Step 1*: H_0: *age of a voter* and *party they intend to vote for* are independent
H_1: *age of a voter* and *party they intend to vote for* are dependent.

Steps 2 and 3: From **a**, $\alpha = 0.1$ and $\text{df} = 2$.

Step 4: The 2×3 contingency table is:

	Age of voter		
	18 to 35	36 to 59	60+
Party A	85	95	131
Party B	168	197	173

Using technology, $\chi^2_{\text{calc}} \approx 8.58$.

Step 5: From **a**, $\chi^2_{\text{crit}} = 4.61$, so $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$.

Step 6: Since $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, we have sufficient evidence to reject H_0 in favour of H_1 at a 10% significance level. We therefore accept H_1.

Step 7: Since we have accepted H_1, we conclude at a 10% significance level that *age of voter* and *party they intend to vote for* are dependent.

5 **a** *Step 1*: H_0: *reason for travelling* and *rating* are independent
H_1: *reason for travelling* and *rating* are dependent.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\text{df} = (2-1)(4-1) = 3$

Step 4: The 2×4 contingency table is:

Rating

Reason for travelling		*Poor*	*Fair*	*Good*	*Excellent*
	Business	27	25	20	8
	Holiday	9	17	24	30

Using technology, $\chi^2_{\text{calc}} \approx 23.6$.

Step 5: From the screenshots above, the p-value $\approx 0.000\,029\,9$.

Step 6: Since the p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1.

Step 7: Since we have accepted H_1, we conclude that, at a 5% significance level, *reason for travelling* and *rating* are dependent.

b From the contingency table, it appears that guests travelling for a holiday are more likely to give a higher rating than those travelling for business.

6 *Step 1*: H_0: *position* and *injury type* are independent
H_1: *position* and *injury type* are dependent.

Step 2: The significance level is $\alpha = 0.1$.

Step 3: $\text{df} = (3-1)(4-1) = 6$

Step 4: The 3×4 contingency table is:

Position

Injury type		*Forward*	*Midfielder*	*Defender*	*Goalkeeper*
	No injury	23	18	24	7
	Mild injury	14	34	23	11
	Serious injury	10	16	13	7

Using technology, $\chi^2_{\text{calc}} \approx 7.94$.

Step 5: From the screenshots above, the p-value ≈ 0.242.

Step 6: Since p-value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at a 10% significance level. We therefore accept H_0.

Step 7: Since we have accepted H_0, we conclude at a 10% significance level that *position* and *injury type* are independent.

7 **a** The 4×2 contingency table is:

Owns a pet

Age	*Yes*	*No*	*Sum*
0 - 19	5	3	8
20 - 29	32	22	54
30 - 49	42	58	100
50+	39	34	73
Sum	118	117	235

The expected frequency table is:

Owns a pet

Age	*Yes*	*No*
0 - 19	$\frac{8 \times 118}{235} \approx 4.02$	$\frac{8 \times 117}{235} \approx 3.98$
20 - 29	$\frac{54 \times 118}{235} \approx 27.1$	$\frac{54 \times 117}{235} \approx 26.9$
30 - 49	$\frac{100 \times 118}{235} \approx 50.2$	$\frac{100 \times 117}{235} \approx 49.8$
50+	$\frac{73 \times 118}{235} \approx 36.7$	$\frac{73 \times 117}{235} \approx 36.3$

b Yes, both the expected frequencies for ages 0 - 19 are less than 5 (4.02 and 3.98).

c We combine ages 0 - 19 with ages 20 - 29 to give:

The new 3×2 contingency table:

Owns a pet

Age	*Yes*	*No*	*Sum*
0 - 29	$5 + 32 = 37$	$3 + 22 = 25$	$37 + 25 = 62$
30 - 49	42	58	100
50+	39	34	73
Sum	118	117	235

The new expected frequency table:

Owns a pet

Age	*Yes*	*No*
0 - 29	$\frac{62 \times 118}{235} \approx 31.1$	$\frac{62 \times 117}{235} \approx 30.9$
30 - 49	≈ 50.2	≈ 49.8
50+	≈ 36.7	≈ 36.3

d *Step 1*: H_0: *age* and *owning a pet* are independent
H_1: *age* and *owning a pet* are not independent.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\text{df} = (3-1)(2-1) = 2$

Step 4:

Using technology and the tables from **c**, $\chi^2_{\text{calc}} \approx 5.22$.

Step 5: From the screenshots above, the p-value ≈ 0.0735.

Step 6: Since the p-value $> 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0.

Step 7: Since we have accepted H_0, we conclude that there is not a link between *age* and *owning a pet*.

8 **a** The 3×4 contingency table is:

Intelligence level

Smoking habits		*Low*	*Average*	*High*	*Very high*	*Sum*
	Non smoker	279	386	96	2	763
	Medium level smoker	123	201	58	5	387
	Heavy smoker	100	147	64	2	313
	Sum	502	734	218	9	1463

The expected frequency table is:

Intelligence level

Smoking habits		*Low*	*Average*	*High*	*Very high*
	Non smoker	$\frac{763 \times 502}{1463} \approx 262$	$\frac{763 \times 734}{1463} \approx 383$	$\frac{763 \times 218}{1463} \approx 114$	$\frac{763 \times 9}{1463} \approx 4.69$
	Medium level smoker	$\frac{387 \times 502}{1463} \approx 133$	$\frac{387 \times 734}{1463} \approx 194$	$\frac{387 \times 218}{1463} \approx 57.7$	$\frac{387 \times 9}{1463} \approx 2.38$
	Heavy smoker	$\frac{313 \times 502}{1463} \approx 107$	$\frac{313 \times 734}{1463} \approx 157$	$\frac{313 \times 218}{1463} \approx 46.6$	$\frac{313 \times 9}{1463} \approx 1.93$

b *Step 1*: H_0: *smoking habits* and *intelligence level* are independent
H_1: *smoking habits* and *intelligence level* are not independent.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: df $= (3-1)(4-1) = 6$

Step 4:

Using technology and the tables from **a**, $\chi^2_{\text{calc}} \approx 16.9$.

Step 5: From the screenshots above, the p-value $\approx 0.009\,59$.

Step 6: Since the p-value $< 0.01 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 at a 1% significance level. We therefore accept H_1.

Step 7: Since we have accepted H_1, we conclude at a 1% significance level that there is a link between *smoking habits* and *intelligence level*.

c Each expected frequency in the *very high intelligence level* is less than 5, so we should combine it with the *high intelligence level*, which gives:

The new 3×3 contingency table:

Intelligence level

Smoking habits	*Low*	*Average*	*High/Very high*	*Sum*
Non smoker	279	386	$96+2=98$	763
Medium level smoker	123	201	$58+5=63$	387
Heavy smoker	100	147	$64+2=66$	313
Sum	502	734	$98+63+66=227$	1463

The new expected frequency table:

Intelligence level

Smoking habits	*Low*	*Average*	*High/Very high*
Non smoker	≈ 262	≈ 383	$\frac{763 \times 227}{1463} \approx 118.4$
Medium level smoker	≈ 133	≈ 194	$\frac{387 \times 227}{1463} \approx 60.0$
Heavy smoker	≈ 107	≈ 157	$\frac{313 \times 227}{1463} \approx 48.6$

d *Step 1*: H_0: *smoking habits* and *intelligence level* are independent

H_1: *smoking habits* and *intelligence level* are not independent.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: df $= (3-1)(3-1) = 4$

Step 4:

Using technology and the tables from **c**, $\chi^2_{calc} \approx 13.2$.

Step 5: From the screenshots above, the p-value ≈ 0.0104.

Step 6: Since the p-value $> 0.01 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 1% significance level. We therefore accept H_0.

Step 7: Since we have accepted H_0, we conclude that there is not a significant link between *smoking habits* and *intelligence level*, which is different from the conclusion we arrived at in **b**.

EXERCISE 31D.2

1 **a** The 2×2 contingency table is:

Result

		Heads	*Tails*	*Sum*
Guess	*Heads*	54	50	$54 + 50 = 104$
	Tails	41	55	$41 + 55 = 96$
	Sum	$54 + 41 = 95$	$50 + 55 = 105$	$104 + 96 = 200$

The expected frequency table is:

Result

		Heads	*Tails*
Guess	*Heads*	$\frac{104 \times 95}{200} = 49.4$	$\frac{104 \times 105}{200} = 54.6$
	Tails	$\frac{96 \times 95}{200} = 45.6$	$\frac{96 \times 105}{200} = 50.4$

b We find χ^2_{calc} using Yates' continuity correction:

f_{obs}	f_{exp}	$f_{obs} - f_{exp}$	$\vert f_{obs} - f_{exp}\vert - 0.5$	$(\vert f_{obs} - f_{exp}\vert - 0.5)^2$	$\frac{(\vert f_{obs} - f_{exp}\vert - 0.5)^2}{f_{exp}}$
54	49.4	4.6	4.1	16.81	≈ 0.3403
50	54.6	-4.6	4.1	16.81	≈ 0.3079
41	45.6	-4.6	4.1	16.81	≈ 0.3686
55	50.4	4.6	4.1	16.81	≈ 0.3335
				Total	≈ 1.3503

So, $\chi^2_{calc} \approx 1.35$

c H_0: Horace's *guess* and the *result* of the toss are independent
H_1: Horace's *guess* and the *result* of the toss are not independent.

Since $\alpha = 0.05$, $\chi^2_{\text{crit}} = 3.84$, and $\chi^2_{\text{calc}} \approx 1.35$, then $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$. This means that we do not have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We conclude that Horace's *guess* and the *result* of the toss are independent.

d According to the test performed in **c**, we conclude that Horace's claim is not valid.

2 **a** The 2×2 contingency table is:

Result

Country		*Pass*	*Fail*	*Sum*
	France	56	29	$56 + 29 = 85$
	Germany	176	48	$176 + 48 = 224$
	Sum	$56 + 176 = 232$	$29 + 48 = 77$	$85 + 224 = 309$

The expected frequency table is:

Result

Country		*Pass*	*Fail*
	France	$\frac{85 \times 232}{309} \approx 63.8$	$\frac{85 \times 77}{309} \approx 21.2$
	Germany	$\frac{224 \times 232}{309} \approx 168.2$	$\frac{224 \times 77}{309} \approx 55.8$

b At a 10% significance level with $\text{df} = (2-1)(2-1) = 1$, $\chi^2_{\text{crit}} = 2.71$.

c

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5$	$(\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5)^2$	$\frac{(\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5)^2}{f_{\text{exp}}}$
56	≈ 63.8	≈ -7.8	≈ 7.3	≈ 53.29	≈ 0.835
29	≈ 21.2	≈ 7.8	≈ 7.3	≈ 53.29	≈ 2.514
176	≈ 168.2	≈ 7.8	≈ 7.3	≈ 53.29	≈ 0.317
48	≈ 55.8	≈ -7.8	≈ 7.3	≈ 53.29	≈ 0.955
				Total	≈ 4.621

So, $\chi^2_{\text{calc}} \approx 4.62$.

d H_0: the *country* where a motorbike test took place and the *result* are independent
H_1: the *country* where a motorbike test took place and the *result* are not independent.

From **b** and **c** we know that $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, and so we have sufficient evidence to reject H_0 at the 10% level of significance. We therefore conclude that the *country* in which a motorbike test took place is not independent of the *result* at a 10% level of significance.

REVIEW SET 31A

1 **a** Using the table on page **842** of the book, if $\text{df} = 6$ and $\alpha = 0.05$, then $\chi^2_{\text{crit}} = 12.59$.

b $\chi^2_{\text{calc}} \approx 5.71 < 12.59 = \chi^2_{\text{crit}}$, so there is insufficient evidence to reject the null hypothesis at a 5% significance level.

2 **a** **i** Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of sizes Small, Medium, Large, X-Large, and XX-Large respectively.

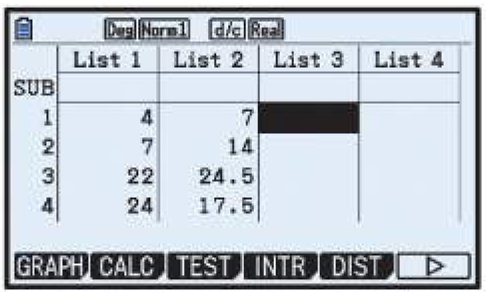
The null hypothesis is therefore:

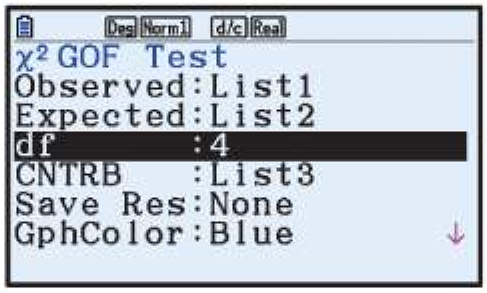
H_0: $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.35$, $p_4 = 0.25$, $p_5 = 0.1$

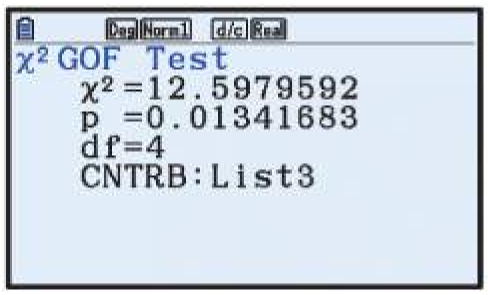
ii There are 5 categories, so df $= 5 - 1 = 4$.

b

Size	f_{obs}	f_{exp}
Small	4	$70 \times 0.1 = 7$
Medium	7	$70 \times 0.2 = 14$
Large	22	$70 \times 0.35 = 24.5$
X-Large	24	$70 \times 0.25 = 17.5$
XX-Large	13	$70 \times 0.1 = 7$







Using technology, the p-value ≈ 0.0134.

c Using significance level $\alpha = 0.05$, the p-value $< \alpha$, and so we reject the null hypothesis at a 5% significance level. We therefore conclude that the proportions of shirts sold in the first week are significantly different from the initial proportions, and so the store should change the distribution of shirt sizes it stocks.

3 *Step 1*: H_0: *age of a driver* and *opinion* are independent

H_1: *age of a driver* and *opinion* are not independent.

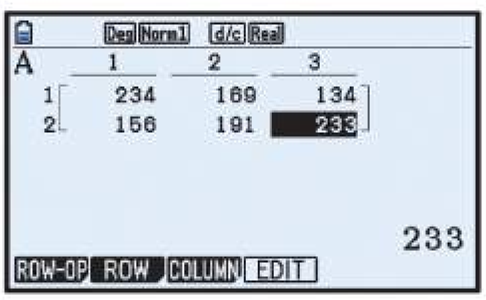
Step 2: The significance level is $\alpha = 0.1$.

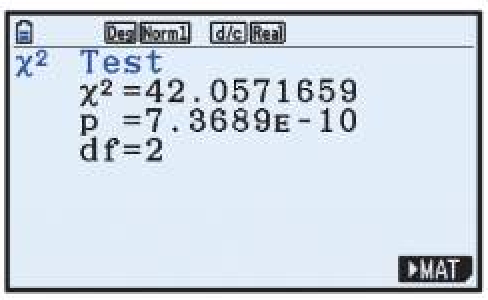
Step 3: df $= (2-1)(3-1) = 2$

Step 4: The 2×3 contingency table is:

Age of driver

	18 to 30	31 to 54	55+
Increase	234	169	134
No increase	156	191	233





Using technology, $\chi^2_{calc} \approx 42.1$.

Step 5: From the screenshots above, the p-value $\approx 7.37 \times 10^{-10}$.

Step 6: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at a 10% significance level. We therefore accept H_1.

Step 7: Since we have accepted H_1, we conclude that there is a significant association between the *age of a driver* and their *opinion* on the speed limit.

4 a

Item rarity	*Expected frequency*
super rare	$250 \times 0.05 = 12.5$
rare	$250 \times 0.1 = 25$
uncommon	$250 \times 0.25 = 62.5$
common	$250 \times 0.6 = 150$

b *Step 1*: Let p_1, p_2, p_3, and p_4 be the population proportions of super rare, rare, uncommon, and common items respectively. The hypotheses to be tested are:

H_0: $p_1 = 0.05,\ \ p_2 = 0.1,\ \ p_3 = 0.25,\ \ p_4 = 0.6$

H_1: at least one of $p_1 \neq 0.05,\ \ p_2 \neq 0.1,\ \ p_3 \neq 0.25,$ or $p_4 \neq 0.6$.

Step 2: The significance level is $\alpha = 0.01$.

Step 3:

Item rarity	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
super rare	5	12.5	4.5
rare	17	25	2.56
uncommon	76	62.5	2.916
common	152	150	≈ 0.0267
		Total	≈ 10.0027

So, $\chi^2_{\text{calc}} \approx 10.0$.

Step 4: $\text{df} = 4 - 1 = 3$, and $\alpha = 0.01$, so, using the table on page **842** of the book, $\chi^2_{\text{crit}} = 11.34$.

Step 5: Since $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, we do not have sufficient evidence to reject H_0 in favour of H_1 with a 1% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that Emmanuel's suspicions are not justified at a 1% level of significance.

5 a

Number of red biros	*Number of packets*
0	1
1	3
2	9
3	17
4	31
5	28
6	11

$$\text{mean} = \overline{x} = \frac{0(1) + 1(3) + ... + 6(11)}{1 + 3 + ... + 11} = \frac{402}{100} = 4.02 \text{ biros per packet}$$

b There are 6 biros in a packet, so $n = 6$.

$$\therefore\ \ p \approx \frac{\overline{x}}{n} = \frac{4.02}{6} = 0.67$$

c *Step 1*: H_1: the data is from a binomial distribution.
H_0: the data is *not* from a binomial distribution.

Step 2: The significance level is $\alpha = 0.1$.

Step 3: From **b**, the proposed distribution is $X \sim \text{B}(6,\ 0.67)$.

Number of red biros (x)	$\text{P}(X = x)$	f_{obs}	f_{exp}	
0	$\approx 0.001\,29$	1	$\approx 0.001\,29 \times 100 \approx 0.1291$	< 5
1	$\approx 0.015\,73$	3	$\approx 0.015\,73 \times 100 \approx 1.5732$	
2	$\approx 0.079\,85$	9	$\approx 0.079\,85 \times 100 \approx 7.9854$	
3	$\approx 0.216\,17$	17	$\approx 0.216\,17 \times 100 \approx 21.6170$	
4	$\approx 0.329\,17$	31	$\approx 0.329\,17 \times 100 \approx 32.9169$	
5	$\approx 0.267\,32$	28	$\approx 0.267\,32 \times 100 \approx 26.7325$	
6	$\approx 0.090\,46$	11	$\approx 0.090\,46 \times 100 \approx 9.0458$	

There are expected frequencies less than 5, so we combine "categories" appropriately:

Number of red biros (x)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
$\leqslant 2$	13	≈ 9.6878	≈ 1.1324
3	17	≈ 21.6170	≈ 0.9861
4	31	≈ 32.9169	≈ 0.1116
5	28	≈ 26.7325	≈ 0.0601
6	11	≈ 9.0458	≈ 0.4222
		Total	≈ 2.7124

So, $\chi^2_{\text{calc}} \approx 2.71$.

Step 4: $\text{df} = 5 - 1 - 1 = 3$

Using technology, p- value ≈ 0.438.

Step 5: Since p-value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 10% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is from a binomial distribution. The manufacturer's claim is therefore valid.

6 *Step 1*: H_0: *social media* and *exercise* are independent.
H_1: *social media* and *exercise* are dependent.

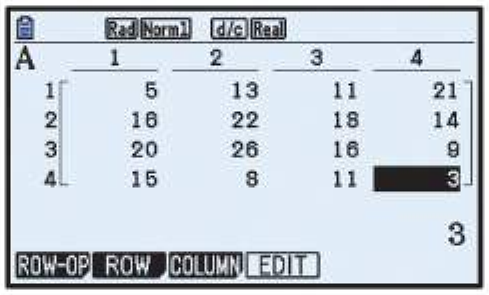
Step 2: The significance level is $\alpha = 0.05$.

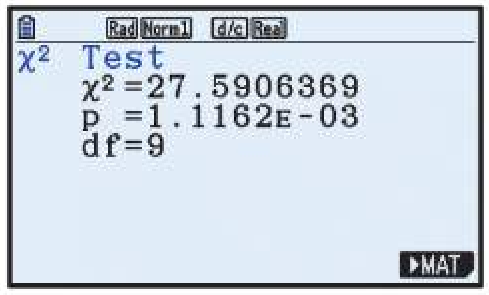
Step 3: $\text{df} = (4-1)(4-1) = 9$

Step 4: The 4×4 contingency table is:

Social media (y hours per week)

Exercise (x hours per week)	$y < 2$	$2 \leqslant y < 6$	$6 \leqslant y < 10$	$y \geqslant 10$
$x < 2$	5	13	11	21
$2 \leqslant x < 4$	16	22	18	14
$4 \leqslant x < 6$	20	26	16	9
$x \geqslant 6$	15	8	11	3





Using technology, $\chi^2_{\text{calc}} \approx 27.6$.

Step 5: From the screenshots above, p-value $\approx 0.001\,12$.

Step 6: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1. We therefore accept H_1.

Step 7: Since we have accepted H_1, we conclude that the variables *social media* and *exercise* are dependent at a 5% level of significance.

REVIEW SET 31B

1 *Step 1*: Let p_1, p_2, p_3, and p_4 be the population proportions of glass, agate, alabaster, and onyx marbles respectively. The hypotheses to be tested are:

H_0: $p_1 = \dfrac{4}{4+2+2+1} = \frac{4}{9}$, $p_2 = \frac{2}{9}$, $p_3 = \frac{2}{9}$, $p_4 = \frac{1}{9}$

H_1: at least one of $p_1 \neq \frac{4}{9}$, $p_2 \neq \frac{2}{9}$, $p_3 \neq \frac{2}{9}$, or $p_4 \neq \frac{1}{9}$.

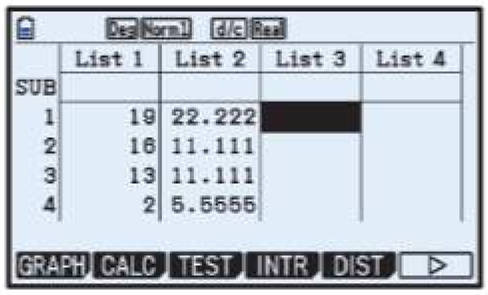
Step 2: The significance level is $\alpha = 0.05$.

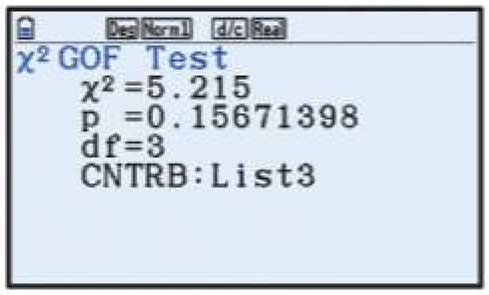
Step 3:

Type	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
glass	19	$50 \times \frac{4}{9} = \frac{200}{9}$	≈ 0.4672
agate	16	$50 \times \frac{2}{9} = \frac{100}{9}$	≈ 2.1511
alabaster	13	$50 \times \frac{2}{9} = \frac{100}{9}$	≈ 0.3211
onyx	2	$50 \times \frac{1}{9} = \frac{50}{9}$	≈ 2.2756
		Total	5.215

So, $\chi^2_{\text{calc}} = 5.215$

Step 4: There are 4 categories, so $\text{df} = 4 - 1 = 3$





Using technology, the p-value ≈ 0.157.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that there is not enough evidence to suggest the company's claim is invalid at the 5% significance level.

2 **a** **i** Assuming that the coin is fair and the coin tosses are independent,

$$\text{P}(X = x) = \underbrace{0.5 \times 0.5 \times .... \times 0.5}_{x-1 \text{ heads}} \times \underbrace{0.5}_{1 \text{ tail}}$$
$$= (0.5)^x$$

ii

Number of tosses required (x)	$\text{P}(X = x)$	*Frequency*	*Expected frequency*
1	0.5	46	$0.5 \times 100 = 50$
2	0.25	20	$0.25 \times 100 = 25$
3	0.125	12	$0.125 \times 100 = 12.5$
4	0.0625	8	$0.0625 \times 100 = 6.25$
5	0.031 25	5	$0.03125 \times 100 = 3.125$
6	0.015 625	3	$0.015625 \times 100 = 1.5625$
7	0.007 812 5	4	$0.0078125 \times 100 = 0.78125$
$\geqslant 8$	0.007 812 5	2	$0.0078125 \times 100 = 0.78125$

b *Step 1*: H_0: the distribution is a suitable model for the data.
H_1: the distribution is *not* a suitable model for the data.

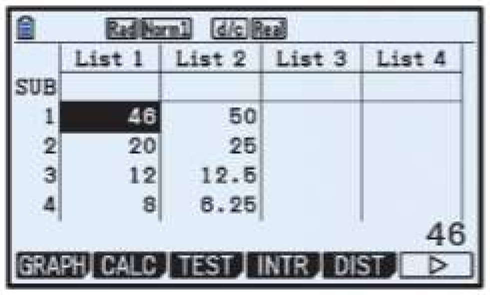
Step 2: The significance level is $\alpha = 0.05$.

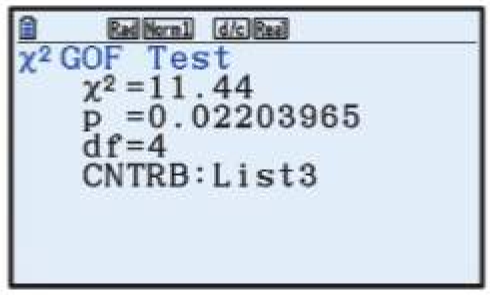
Step 3: From **a ii**, there are expected frequencies less than 5, so we combine "categories" appropriately:

Number of tosses required	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
1	46	50	0.32
2	20	25	1
3	12	12.5	0.02
4	8	6.25	0.49
$\geqslant 5$	14	6.25	9.61
		Total	11.44

So, $\chi^2_{\text{calc}} = 11.44$

Step 4: df $= 5 - 1 = 4$





Using technology, p-value ≈ 0.0220.

Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the distribution is not a suitable model for the data at a 5% significance level.

3 **a** Let X be the test score of a randomly selected student.

$\therefore$ $X \sim N(100, 10^2)$.

Score (x)	*Probability*	*Observed frequencies*	*Expected frequencies*
$x < 70.5$	$\approx 0.001\,59$	10	3
$70.5 \leqslant x < 80.5$	$\approx 0.024\,00$	45	48
$80.5 \leqslant x < 90.5$	$\approx 0.145\,47$	287	$\approx 0.145\,47 \times 2000 \approx 291$
$90.5 \leqslant x < 100.5$	$\approx 0.348\,88$	641	$\approx 0.348\,88 \times 2000 \approx 698$
$100.5 \leqslant x < 110.5$	$\approx 0.333\,20$	725	$2000 - (3 + 48 + ... + 2) = 667$
$110.5 \leqslant x < 120.5$	$\approx 0.126\,68$	250	253
$120.5 \leqslant x < 130.5$	$\approx 0.019\,04$	40	38
$x \geqslant 130.5$	$\approx 0.001\,14$	2	2

b *Step 1*: H_0: the data is from a normal distribution with mean 100 and variance 100.
H_1: the data is *not* from a normal distribution with mean 100 and variance 100.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: From **a**, there are expected frequencies less than 5, so we combine "categories" appropriately:

Score (x)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
$x < 80.5$	55	51	0.3137
$80.5 \leqslant x < 90.5$	287	291	0.0550
$90.5 \leqslant x < 100.5$	641	698	4.6547
$100.5 \leqslant x < 110.5$	725	667	5.0435
$110.5 \leqslant x < 120.5$	250	253	0.0356
$x \geqslant 120.5$	42	40	0.1
		Total	10.2025

So, $\chi^2_{\text{calc}} \approx 10.2$.

Step 4: df $= 6 - 1 = 5$

Using technology, p-value ≈ 0.0697.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is from a normal distribution with mean 100 and variance 100.

4 *Step 1*: H_0: P and Q are independent.
H_1: P and Q are not independent.

Step 2: The significance level in **a** is $\alpha_1 = 0.05$.
The significance level in **b** is $\alpha_2 = 0.01$.

Step 3: df $= (3-1)(4-1) = 6$

Step 4: The 3×4 contingency table is:

	Q_1	Q_2	Q_3	Q_4
P_1	19	23	27	39
P_2	11	20	27	35
P_3	26	39	21	30

Using technology, $\chi^2_{\text{calc}} \approx 13.0$.

Step 5: From the screenshots above, the p-value ≈ 0.0433.

a Since the p-value $< 0.05 = \alpha_1$, we have sufficient evidence to reject H_0 in favour of H_1 at the 5% significance level. We therefore conclude that P and Q are not independent.

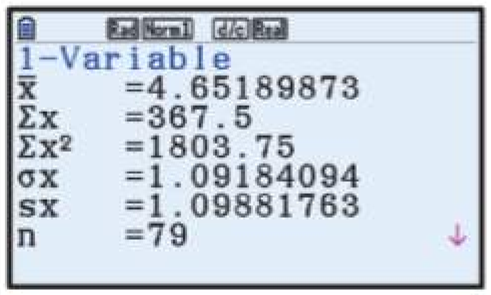
b Since the p-value $> 0.01 = \alpha_2$, we do not have sufficient evidence to reject H_0 in favour of H_1 at the 1% significance level. We therefore conclude that P and Q are independent.

5 *Step 1*: H_0: the data is normally distributed.
H_1: the data is *not* normally distributed.

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

Mass (m kg)	*Midpoint*	*Frequency*
$2 \leqslant m < 3$	2.5	6
$3 \leqslant m < 4$	3.5	11
$4 \leqslant m < 5$	4.5	37
$5 \leqslant m < 6$	5.5	18
$6 \leqslant m < 7$	6.5	4
$7 \leqslant m < 8$	7.5	3
	Total	79



Using technology,

$\mu \approx \overline{x} \approx 4.6519$

$\sigma \approx s \approx 1.0988$

So, the proposed distribution is $M \sim \text{N}(4.6519,\ 1.0988^2)$.

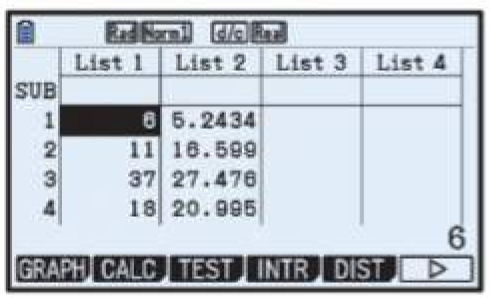
Mass (m kg)	*Probability*	f_{obs}	f_{exp}
$2 \leqslant m < 3$	$\approx 0.066\,38$	6	$\approx 0.066\,38 \times 79 \approx 5.2437$
$3 \leqslant m < 4$	$\approx 0.210\,12$	11	$\approx 0.210\,12 \times 79 \approx 16.5998$
$4 \leqslant m < 5$	$\approx 0.347\,80$	37	$\approx 0.347\,80 \times 79 \approx 27.4763$
$5 \leqslant m < 6$	$\approx 0.265\,76$	18	$\approx 0.265\,76 \times 79 \approx 20.9952$
$6 \leqslant m < 7$	$\approx 0.093\,63$	4	$\approx 0.9363 \times 79 \approx 7.3972$
$7 \leqslant m < 8$	$\approx 0.016\,30$	3	$\approx 0.016\,30 \times 79 \approx 1.2878$ $\leftarrow < 5$

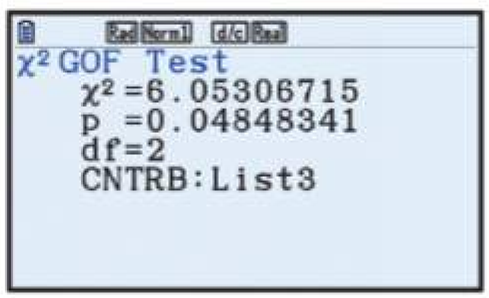
There are expected frequencies less than 5, so we combine "categories" appropriately:

Mass (m kg)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
$2 \leqslant m < 3$	6	≈ 5.2434	≈ 0.1091
$3 \leqslant m < 4$	11	≈ 16.5998	≈ 1.8890
$4 \leqslant m < 5$	37	≈ 27.4767	≈ 3.3010
$5 \leqslant m < 6$	18	≈ 20.9954	≈ 0.4273
$6 \leqslant m < 8$	7	≈ 8.6847	≈ 0.3269
		Total	≈ 6.0534

So, $\chi^2_{\text{calc}} \approx 6.05$.

Step 4: $\text{df} = 5 - 2 - 1 = 2$





Using technology, p-value ≈ 0.0485.

Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude that the data is not normally distributed at a 5% level of significance.

6 We want to test whether two categories are independent, so we conduct a χ^2 test for independence.

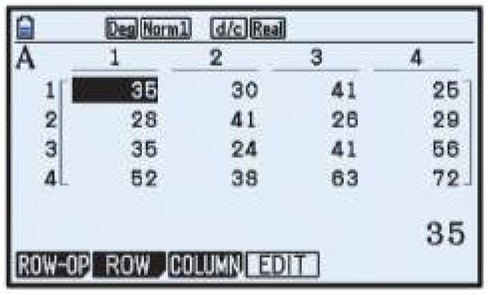
Step 1: H_0: *Business success* and *education level* are independent.
H_1: *Business success* and *education level* are dependent.

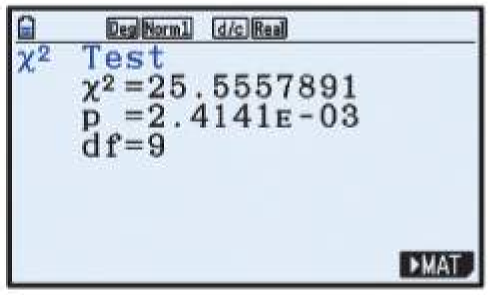
Step 2: The significance level is $\alpha = 0.01$.

Step 3: $\text{df} = (4-1)(4-1) = 9$

Step 4: The 4×4 contingency table is:

		Education level			
		High school	*Graduate certificate*	*Undergraduate degree*	*Postgraduate*
Business success	*No success*	35	30	41	25
	Low success	28	41	26	29
	Success	35	24	41	56
	High success	52	38	63	72





Using technology, $\chi^2_{\text{calc}} \approx 25.6$.

Step 5: $\chi^2_{\text{calc}} \approx 25.6 > \chi^2_{\text{crit}} = 21.67$, so we have enough evidence to reject H_0 in favour of H_1 at a 1% significance level. We therefore accept H_1.

Step 6: Since we have accepted H_1, we conclude at a 1% significance level that there is a link between *education level* and *business success*.